Network Theory and Input-Output-Modeling

Jan David Weber

Objectives and Literature

Vectors and Vector Calculation

Matrices and Special Matrices

Matrix Operations

Eigenvalues and Eigenvectors

Module 1: Basic Linear Algebra

Matrix Operations

Jan David Weber

Institute for Socioeconomics
University Duisburg-Essen

https://doi.org/10.17185/duepublico/84445

Identity of Matrices

A matrix remains “identical” from a mathematical perspective if any of the three operations are performed:

- Interchange any two rows
- Multiply all the elements in a row by a real number which is not *zero*
- Add multiples of the elements of one row to the elements of another row

Note that this does not necessarily hold when applied to Economics

Addition

$\mathbf{A} + \mathbf{B} = \mathbf{C} = \mathbf{B} + \mathbf{A}$ if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ have the same dimension

Addition

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}=\mathbf{C}$$

Scalar Multiplication

$$c\mathbf{A} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}$$

Matrix Multiplication

The product of a row of one matrix with a column of another

$$\mathbf{C_{m\times p}} = \mathbf{A_{m\times n}} \cdot \mathbf{B_{n\times p}}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \dots & \sum_{k=1}^{n} a_{1k}b_{kp} \\ \sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \dots & \sum_{k=1}^{n} a_{2k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \dots & \sum_{k=1}^{n} a_{mk}b_{kp} \end{pmatrix}$$

$\rightarrow$ It is important that $\mathbf{A} \cdot \mathbf{B} = \mathbf{AB} \neq \mathbf{BA} = \mathbf{B} \cdot \mathbf{A}$

Hadamard Multiplication

Matrices must have the same dimension as each element is multiplied with the corresponding element

$$
\begin{aligned}
\mathbf{A_{m\times n}} \odot \mathbf{B_{m\times n}} &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} \times b_{11} & a_{12} \times b_{12} & \dots & a_{1n} \times b_{1n} \\ a_{21} \times b_{21} & a_{22} \times b_{21} & \dots & a_{2n} \times b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \times b_{m1} & a_{m2} \times b_{m2} & \dots & a_{mn} \times b_{mn} \end{pmatrix} \\
&= \mathbf{C_{m\times n}}
\end{aligned}
$$

Transposition

Transposing a matrix means that rows and columns are switched

$$\mathbf{A_{m\times n}}^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}^T$$

$$= \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$= \mathbf{A_{n\times m}}$$

Vector Multiplication

Multiplicate a vector with a matrix:

$$\mathbf{x_{1\times n}}\mathbf{A_{n\times m}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$
$$= \begin{pmatrix} x_1 a_{11} + x_2 a_{21} + \cdots + x_n a_{n1} & x_1 a_{12} + x_2 a_{22} + \cdots + x_n a_{n2} & \cdots & x_1 a_{1m} + x_2 a_{2m} + \cdots \end{pmatrix}$$
$$= \mathbf{B_{1\times m}}$$

Vector Multiplication

Multiplicate a matrix with a vector:

$$\mathbf{A_{m\times n}x_{n\times 1}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

Inverse of a Matrix

Requirements:

- The Matrix must be quadratic and invertible
- The multiplication of a matrix with its inverse returns the identity matrix

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$$

$$\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} \times \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$$

Inverse of a Matrix

Easy calculation of 2×2 matrices with the inverse:

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

- Inversion is possible by using the Cramer's Rule
- Larger matrices can be inverted by using row operations or the adjoint matrix (calculation intensive!)

Determinants of a Matrix

The determinant incorporates all the elements of a matrix and reduces the matrix to a single number.
Requisite for the Calculation:

- The matrix must be **quadratic**
- Determinant can be zero, but shows *by definition* that the matrix is *not* invertible

Determinants of a Matrix

The determinant of a 2×2 and 3×3 matrix is relatively easy to calculate:
For a 2×2 matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(\mathbf{A}) = a_{11} * a_{22} - a_{12} * a_{21}$$

Determinants of a Matrix

For a 3×3 matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(\mathbf{A}) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The process gets more difficult for larger dimensions, see Sterling (2009, p. 189)