UNIVERSITY OF DUISBURG-ESSEN
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DISSERTATION

Analysis and Optimization of the Acoustic Wave Equation

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Preface

This thesis marks the end of a great and incredibly educational chapter in my life, one that I will always remember with joy. Much of this is due to the many people who have accompanied me on this journey, to whom I am deeply grateful.

First and foremost, I would like to thank my family, especially my parents. Your unconditional love and support have carried me through good and difficult times. You have always given me the confidence to pursue all my ambitions.

When I began studying mathematics at the University of Duisburg-Essen, I was fortunate to make great friends among my fellow students right from the start. Attending lectures, solving exercises, and discussing together in the LuDi were crucial to successfully completing my Bachelor's and Master's degrees. I am immensely grateful to all of you.

Thereafter, during my doctoral phase, I was delighted to share my office with Maurice. We constantly motivated and supported each other, spending countless hours discussing and exchanging ideas. Thanks for the great time!

Without any doubt, the most significant influence on my mathematical development was my supervisor, Irwin Yousept. I am deeply thankful for the interest, time, and patience with which you supported me throughout the past years. I am particularly pleased that I had the opportunity to present and discuss our joint research results at various international conferences. It is a great privilege for me to have had you as my supervisor!

Abstract

This dissertation analyzes a nonlinear hyperbolic PDE-constrained optimization problem. Motivated by applications in Full Waveform Inversion, our central goal is to reconstruct the wave speed parameter entering the acoustic wave equation in the coefficient of the second-order time derivative of the acoustic pressure. Starting with the first- and second-order analysis, we prove the well-definedness of the problem and establish corresponding necessary and sufficient optimality conditions. These findings lay the foundation for investigating the application of the Sequential Quadratic Programming method. Here, a broad extension of the parabolic techniques is required due to the hyperbolicity and the bilinear character of the underlying partial differential equation. Based on a two-step estimation process, we show the well-posedness and R-superlinear convergence of the algorithm. Furthermore, the present thesis includes the numerical analysis of a fully discrete approximation of the optimization problem, consisting of a Finite Element discretization in space and a leapfrog time-stepping. Building upon a stability analysis, we prove a convergence result regarding first-order necessary optimality conditions. Moreover, we demonstrate that for every local minimizer of the original problem that satisfies a reasonable growth condition, there is a sequence of locally optimal solutions to the discrete problems that converges to this minimizer. The document concludes with numerical experiments based on synthetic configurations with nonsmooth data, which illustrate the performance and effectiveness of the presented approach.

Zusammenfassung

Diese Dissertation analysiert ein nichtlineares hyperbolisches Optimalsteuerungsproblem. Motiviert durch Anwendungen in der Full Waveform Inversion ist das Ziel, die Wellengeschwindigkeit zu rekonstruieren, welche als Koeffizient der zweiten Zeitableitung des Schalldrucks in die akustische Wellengleichung eingeht. Zunächst beweisen wir die Wohldefiniertheit des Problems, sowie zugehörige notwendige und hinreichende Optimalitätsbedingungen. Diese bilden die Grundlage für die Untersuchung des Verfahrens der Sequentiellen Quadratischen Programmierung. Aufgrund der Hyperbolizität und des bilinearen Charakters der zugrundeliegenden Differentialgleichung erfordert die Analyse eine umfangreiche Erweiterung des parabolischen Falls. Basierend auf einem zweistufigen Abschätzungsverfahren wird die Wohlgestelltheit und R-superlineare Konvergenz des Algorithmus gezeigt. Im Anschluss daran befasst sich die Arbeit mit der numerischen Analyse einer vollständig diskreten Approximation für das Optimalsteuerungsproblem, die aus einer Finite-Elemente-Diskretisierung im Raum und einem Leapfrog-Zeitschrittverfahren besteht. Basierend auf geeigneten Stabilitätsresultaten beweisen wir zunächst ein Konvergenzresultat bezüglich notwendiger Optimalitätsbedingungen erster Ordnung. Darüber hinaus zeigen wir, dass gegen jede lokal optimale Lösung des Ursprungsproblems, die eine geeignete Wachstumsbedingung erfüllt, eine Folge lokal optimaler Lösungen der diskreten Probleme konvergiert. Schließlich demonstrieren numerische Experimente, die auf synthetischen Konfigurationen mit nichtglatten Daten basieren, die Effektivität des vorgestellten Ansatzes.

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1 INTRODUCTION

To myself, I am only a child playing on the beach, while vast oceans of truth lie undiscovered before me.

— Isaac Newton

Primarily, due to its inaccessibility, the Earth's subterranean nature is largely unexplored. In general, observations can only be made indirectly and from a far distance. Natural scientists use measurements of permanent movements and vibrations of our planet to estimate the subsurface structure. This is the basic concept for seismic tomography, whose development stems from a long history of exploration that probably began with observations made by J. Michell in 1760. Michell's pioneering work identified earthquakes with waves traveling through the Earth's crust, a concept later underpinned by the development of the elasticity theory by scientists like A. L. Cauchy and S. D. Poisson. This theoretical framework still forms the basis of modern seismological applications [35].

Traditional seismic tomography models, developed since the mid-20th century, have relied heavily on simplifications such as ray theory, which assumes seismic waves travel along predetermined paths. These methods, constrained by the limited computational resources of their time, often only utilized basic data like travel times and phase velocities, leading to significant limitations in resolution and accuracy. However, the landscape of seismic exploration has dramatically changed with advances in computational power and mathematical modeling. These developments have allowed for the implementation of more sophisticated techniques that can incorporate the full complexity of seismic waves. Among these, Full Waveform Inversion (FWI) has emerged as one of the leading methods due to its ability to utilize the entire waveform content of seismic data, facilitating high-resolution imaging critical for geological exploration and industrial applications [74].

Applied mathematics serves as a bridge between theoretical concepts and applications in various scientific disciplines, including geophysics. From a mathematical point of view, the inversion of the wave propagation can be formulated as a minimization problem governed by an appropriate wave equation. The investigation of such a model is the central focus of this thesis.

In the present dissertation, our goal is to reconstruct the wave speed parameter entering the acoustic wave equation by minimizing the misfit between synthetic and observed seismic data. Understanding the wave speed within an observed domain allows us to reconstruct the domain's overall structure since waves travel at different speeds through different materials. To set the stage for our discussion, we consider the following damped acoustic wave equation:

$$\begin{cases} \nu\partial_t^2 p - \Delta p + \eta\partial_t p = f & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ p(0,\cdot) = p_0 & \text{in } \Omega \\ \partial_t p(0,\cdot) = p_1 & \text{in } \Omega. \end{cases} \tag{1.1}$$

In this context, $\Omega \subset \mathbb{R}^N$ ($N = 2,3$) is a bounded spatial domain, and $I = [0,T] \subset \mathbb{R}$ is a finite time interval. Here, the computational domain Ω can represent a segment or a slice of a segment of the full physical domain, for instance, the Earth. The boundary of Ω consists of the Neumann boundary part $\Gamma_N \subset \partial\Omega$, capable of modeling the Earth's free surface constraining the domain Ω, and the (artificial) Dirichlet boundary part $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Furthermore, the scalar function $p: I \times \Omega \to \mathbb{R}$ denotes the acoustic pressure with the initial value $p_0: \Omega \to \mathbb{R}$ (resp. $p_1: \Omega \to \mathbb{R}$ for the first order time derivative $\partial_t p$). The parameter $\nu: \Omega \to \mathbb{R}$ denotes square slowness, i.e., $\nu \coloneqq c^{-2}$ with $c: \Omega \to \mathbb{R}$ being the acoustic wave speed in the potentially heterogeneous domain Ω. The coefficient $\eta: \Omega \to \mathbb{R}$ is a given damping term employed to absorb and prevent undesired reflections of the wave on the artificial boundary part Γ_D. Given a source term $f: I \times \Omega \to \mathbb{R}$, we aim to minimize the sum of the misfit functionals:

$$\begin{cases} \inf \mathcal{J}(\nu,p) \coloneqq \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} \int_I \int_\Omega a_i (p - p_i^{ob})^2 \,\mathrm{d}x\,\mathrm{d}t + \dfrac{\lambda}{2}\|\nu\|^2_{L^2(\Omega)} \\ \text{s.t. (1.1) and } \nu_-(x) \le \nu(x) \le \nu_+(x) \text{ for a.e. } x \in \Omega. \end{cases} \tag{1.2}$$

Here, $\nu_-: \Omega \to \mathbb{R}$ (resp. $\nu_+: \Omega \to \mathbb{R}$) denotes the lower (resp. upper) bound for ν. Furthermore, for each $i = 1, \dots, m$, the function $p_i^{ob}: I \times \Omega \to \mathbb{R}$ describes given observed wave information that is induced by the signal source f and recorded at receivers modeled through the weight functions $a_i: I \times \Omega \to \mathbb{R}$. The receivers a_i may be characteristic functions of a small region around some receiving points $x_k \in \Omega$. The source term f usually has a small support in space near the Neumann boundary of the domain, representing an external force applied to the domain. More details and examples are included in the remainder of this work. Under a suitable choice of the regularization parameter λ, the first component ν of a solution (ν, p) to (1.2) approximates the underlying true square slowness corresponding to the induced true wave information p_i^{ob}.

We note that the minimization problem (1.2) falls into the class of hyperbolic optimal control problems or hyperbolic PDE-constrained optimization problems. In this field, various similar problems to (1.2) have been studied in the literature. We refer to related publications in the following chapters. The main difficulties in the investigation of (1.2) lie in (i) the hyperbolicity of the underlying PDE system (1.1) and (ii) the bilinear character $\nu\partial_t^2 p$ consisting of the product of the square slowness and the second-order time derivative of the acoustic pressure. Both, (i) and (ii), lead to various challenges throughout the whole (numerical) analysis of (1.2).

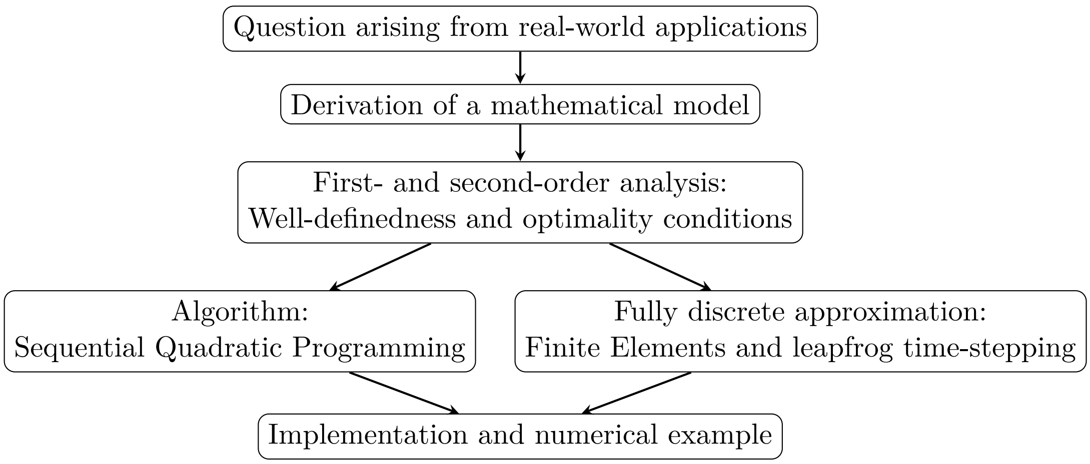


Figure 1.1: Systematic outline

1.1 Outline

In this thesis, the investigation of (1.2) follows the outline in Figure 1.1. More precisely, it is divided into the following chapters:

- In Chapter 2, we provide preparations that pave the way for our later analysis. In the first part, we discuss the derivation of our model formulation (1.2). We deduce the acoustic wave equation (1.1) as a special case of the elastic wave equation that is obtained from classical physics principles. Furthermore, Dirichlet and Neumann boundary conditions are introduced from an applied perspective, justifying their incorporation into our PDE model for real-world seismic applications. Finally, the minimization problem (1.2) is obtained as a capable strategy for reconstructing the true wave speed from observed wave information. The chapter's second part introduces required functional analytical foundations. First, we discuss Lebesgue and Sobolev spaces with and without (partially) vanishing trace conditions. Then, we define the Bochner function spaces which are crucial for the weak solution theory for time-dependent problems such as parabolic and hyperbolic PDEs. At the end of the chapter, we introduce semigroups as a useful concept for solving evolution equations, and in particular, time-dependent PDEs.

- Chapter 3 is devoted to the first- and second-order analysis of (1.2). We develop a novel technique accounting for an auxiliary first-order system. In contrast to the original state equation, the underlying control parameter appears in the auxiliary system not only as the coefficient of the time derivative but also as the initial data under the image of the solution operator for a specific elliptic problem. On this basis, we construct an adjoint state explicitly using the corresponding dual semigroup. This approach leads to necessary optimality conditions with a low adjoint regularity such that no Sobolev smoothing effect occurs in the optimal solution. The final part of the chapter is devoted to the second-order analysis of the optimal control approach. We provide sufficient second-order optimality conditions leading to strict local optimality and a quadratic growth condition. Here, the application of

Stampacchia's method to the hyperbolic case is essential to handle the nonlinearity $\nu\partial_t^2 p$. Note that this chapter is largely based on the author's publication [5].

- In Chapter 4, the Sequential Quadratic Programming (SQP) method is considered. This famous algorithm has been successfully applied to various nonlinear optimization problems in the literature. However, in the application to our model problem (1.2), the aforementioned involved hyperbolic character and second-order bilinear structure lead to undesired effects of regularity loss in the SQP iteration. Therefore, the investigation of the SQP method requires a substantial extension of the developed parabolic techniques. We propose and analyze a novel strategy for the well-posedness and convergence analysis using a smooth-in-time initial condition, a tailored self-mapping operator, and a two-step estimation process along with Stampacchia's method for hyperbolic equations. The chapter's final theoretical result is the SQP method's R-superlinear convergence. Note that the content of this chapter is published in the author's preprint [6].

- In Chapter 5, we explore a fully discrete approximation technique for (1.2). Based on the auxiliary first-order system from Chapter 3, the approximation for (1.1) consists of a Finite Element discretization in space and a leapfrog (Yee) time-stepping. For a fully discrete optimal control problem in finite dimension (P_h), we show its well-definedness and establish corresponding first-order necessary optimality conditions. Building upon a stability analysis of the fully discrete scheme, we demonstrate that the interpolations of solutions to the first-order optimality condition for (P_h) converge up to a subsequence towards a solution satisfying a first-order optimality condition for (1.2). Finally, given a locally optimal solution to (1.2) that satisfies a reasonable growth condition, we verify that there exists a sequence of locally optimal solutions to (P_h) that converges to the locally optimal solution to (1.2). Note that the content of this chapter is published in the author's preprint [7]

- In the final chapter, that is Chapter 6, we discuss the computational implementation of the SQP method from Chapter 4. First, we discuss solving the linear quadratic SQP subproblems using a projected gradient method. Second, using the fully discrete approximation technique from Chapter 5, we present numerical experiments based on synthetic configurations with nonsmooth data. Our algorithm successfully reconstructs the true wave speed parameter from a deterministic noise model. Consequently, the numerical results validate the effectiveness of our methodology, particularly for FWI applications.

BACKGROUND 2

2.1 Derivation of the Model Formulation

In this section, we discuss a formal derivation of the essential PDE-constrained optimization problem considered in this dissertation. In the following, we predicate that all quantities are supposed to be sufficiently smooth. We do not delve into detailed discussions of each physical quantity. Instead, our focus is on their relationships, which are crucial for inferring the desired equations. For a more in-depth view, we refer to the literature (cf. [21, 27, 28, 35, 48]). Notably, we follow the derivation in Dörfler et al. [29].

Elastic and Acoustic Wave Equation

A central goal in continuum mechanics is to have a better understanding of the relationship between stress and deformation. Here, the elastic wave equation provides a realistic model to express how mechanical waves, particularly longitudinal and transverse waves, propagate through solid media when internal stresses and external forces are present. We consider an isotropic material, meaning that physical properties behave uniformly in all directions. For instance, those include mechanical properties, thermal conductivity, and electrical conductivity. Furthermore, let the material be linearly elastic, meaning it responds linearly and reversibly to applied stress. This is valid for most solid materials if the applied stress is in a certain corresponding elastic range.

Let $\Omega \subset \mathbb{R}^3$ be a given spatial domain and let $I = [0, T] \subset \mathbb{R}$ be a finite time interval. We consider an elastic body inside Ω that underlies some stress. Its deformation is then described by the displacement field $\boldsymbol{u}: I \times \Omega \to \mathbb{R}^3$. Further, we define the velocity field $\boldsymbol{v} \coloneqq \partial_t \boldsymbol{u}$, the strain tensor $\epsilon(\boldsymbol{u}) \coloneqq \frac{1}{2}((\nabla \boldsymbol{u})^T + \nabla \boldsymbol{u})$, and analogously the strain rate $\epsilon(\boldsymbol{v})$. We begin our derivation, noting that the considered materials follow *Hooke's Law*, which describes the linear-elastic behavior between stress and strain. In three dimensions, the generalized version can mathematically be expressed in the constitutive relation

$$\sigma = C\epsilon(\boldsymbol{u}), \tag{2.1}$$

where $\sigma: I \times \Omega \to \mathbb{R}^{3\times 3}$ denotes the stress tensor and $C: \Omega \to \mathcal{L}(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$ denotes the elasticity tensor, also known as Hooke's tensor. On the other hand, *Newton's Second Law of Motion* states that the time rate of change of the momentum equals the applied force that acts on the medium. Given the mass density $\rho: \Omega \to (0, \infty)$, the momentum is given by $\rho \boldsymbol{v}$, the internal stress is given by $\nabla \cdot \sigma$, and the external force is given by some vector-valued function $\boldsymbol{f}: I \times \Omega \to \mathbb{R}^3$. Then, the principle above leads to

$$\rho \partial_t \boldsymbol{v} = \nabla \cdot \sigma + \boldsymbol{f}. \tag{2.2}$$

Along with the time derivation and changing of the order of derivatives in (2.1), we obtain the following velocity-stress formulation of the *elastic wave equation*:

$$\begin{cases} \rho \partial_t \boldsymbol{v} - \nabla \cdot \sigma = \boldsymbol{f} & \text{in } I \times \Omega \\ \partial_t \sigma - C\epsilon(\boldsymbol{v}) = 0 & \text{in } I \times \Omega. \end{cases} \tag{2.3}$$

This fundamental equation plays an important role across a spectrum of disciplines within physics and engineering. Since it gives a realistic model for wave propagation phenomena in elastic materials, it also lays a crucial groundwork for methodologies in seismology, solid mechanics, and material science. Notably, its acoustic approximation is commonly embraced for its pragmatic simplicity and favorable analytical properties. We take this as a motivation to discuss its derivation.

To begin with, we take a closer look at the elasticity tensor C. In isotropic media, it can be expressed with only two independent parameters, the Lamé parameters, μ, and λ. Here, μ denotes the shear modulus, and λ is given by $\lambda = \kappa - \frac{2}{3}\mu$ for the bulk modulus κ. Then, it holds that

$$C\epsilon(\boldsymbol{u}) = 2\mu\epsilon(\boldsymbol{u}) + \lambda \operatorname{tr}(\epsilon(\boldsymbol{u}))I_d = 2\mu\epsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u} I_d, \tag{2.4}$$

where tr denotes the trace and I_d is the identity matrix. If we now consider the medium to be a fluid, i.e., a liquid or a gas, the shear modulus μ is essentially zero, such that the constitutive relation (2.1) along with (2.4) leads to

$$\sigma = \kappa \operatorname{div} \boldsymbol{u} I_d. \tag{2.5}$$

By introducing the pressure $p \coloneqq \frac{1}{3}\operatorname{tr}(\sigma)$, we obtain the exact function that we want to use to describe the acoustic wave propagation in a single scalar-valued equation. Using the above representation (2.5) of the stress tensor σ, it holds that

$$p = \kappa \operatorname{div} \boldsymbol{u}.$$

Accordingly, the first-order time derivative of p is given by $\partial_t p = \kappa \operatorname{div} \boldsymbol{v}$ and the gradient is computed by

$$\nabla p = \nabla \cdot (pI_d) = \nabla \cdot (\kappa \operatorname{div} \boldsymbol{u} I_d) = \nabla \cdot \sigma.$$

Along with the first line of the elastic wave equation (2.3), we obtain the first-order formulation of the *acoustic wave equation*, that is

$$\begin{cases} \rho \partial_t \boldsymbol{v} - \nabla p = \boldsymbol{f} & \text{in } I \times \Omega \\ \partial_t p - \kappa \operatorname{div} \boldsymbol{v} = 0 & \text{in } I \times \Omega. \end{cases} \tag{2.6}$$

After time derivation of the second equation in (2.6) and changing the order of derivatives, we obtain that

$$\partial_t^2 p = \kappa \operatorname{div} \partial_t \boldsymbol{v} = \kappa \operatorname{div}(\rho^{-1}(\nabla p + \boldsymbol{f})).$$

For simplicity, we assume that the change in density is much smaller than the change in pressure and the external force to obtain the acoustic wave equation in its second-order formulation:

$$\partial_t^2 p - c^2 \Delta p = c^2 \operatorname{div} \boldsymbol{f}, \tag{2.7}$$

where $c \coloneqq \sqrt{\frac{\kappa}{\rho}}$ is the wave speed and $\Delta = \operatorname{div}(\nabla \cdot)$ is the Laplace operator. Even though, we considered a vanishing shear modulus η, which is only strictly valid in fluid media, the acoustic wave equation is widely used to describe wave propagation phenomena beyond that restriction. For instance, as an approximation of the elastic wave equation, it is a favorable choice for many propagation models of seismic waves inside the Earth [35].

Boundary Conditions and Sponge Layer

In many wave models, the computational domain is usually restricted to a bounded region of the true physical domain. The boundary may be given by some free surface, such as the Earth's surface, but it may also consist of other (artificial) boundary parts.

We denote the boundary of the spatial domain Ω by $\partial\Omega$ and begin considering a subset $\Gamma_N \subset \partial\Omega$. A suitable choice for modeling free surfaces is given by the Neumann boundary condition. It is typically applied to the derivative of the corresponding field variable concerning the normal direction of the boundary. This represents the flux of the field variable across the boundary. Considering the absence of mass or energy transfer through the surface, the condition becomes homogenous. In the case of the acoustic wave equation, this is reflected in the orthogonality of the gradient of pressure ∇p to the normal vector $\mathbf{n}$, i.e.,

$$\partial_n p \coloneqq \nabla p \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \tag{2.8}$$

When the computational domain is not fully surrounded by a free surface, from a mathematical point of view, it is favorable to consider the Dirichlet boundary condition on the remaining boundary part $\Gamma_D = \partial\Omega \setminus \Gamma_N$, that is

$$p = 0 \quad \text{on } \Gamma_D. \tag{2.9}$$

Unfortunately, this condition is artificial and physically unnatural. Its treatment is challenging since an inadequate approach typically causes unfavorable wave reflections in the modeling. Several strategies exist to deal with this issue. Besides absorbing boundary conditions, such as Perfectly Matched Layers (PMLs), Convolutional PMLs, and Stretched Coordinate PMLs, another widely employed approach incorporates an additional absorbing boundary layer, a so-called sponge layer. We choose the latter technique because of its relatively straightforward implementation. In the following, we briefly discuss how it can effectively be designed, such that (2.9) does not cause wave reflections, polluting the solution in the modeling. For better illustration, we consider a two-dimensional rectangular domain, where the upper edge is a free surface and the Neumann boundary condition (2.8) applied. The rest of the boundary is some artificial restriction of the true physical domain. Here, we consider the Dirichlet boundary condition (2.9). Then, in a narrow area ω around that boundary part, we define some scalar-valued damping term η that increases in the direction towards the boundary (see Figure 2.1). A precise example of a suitable implementation can be found in Chapter 6 (or in Münch [64]). The damping

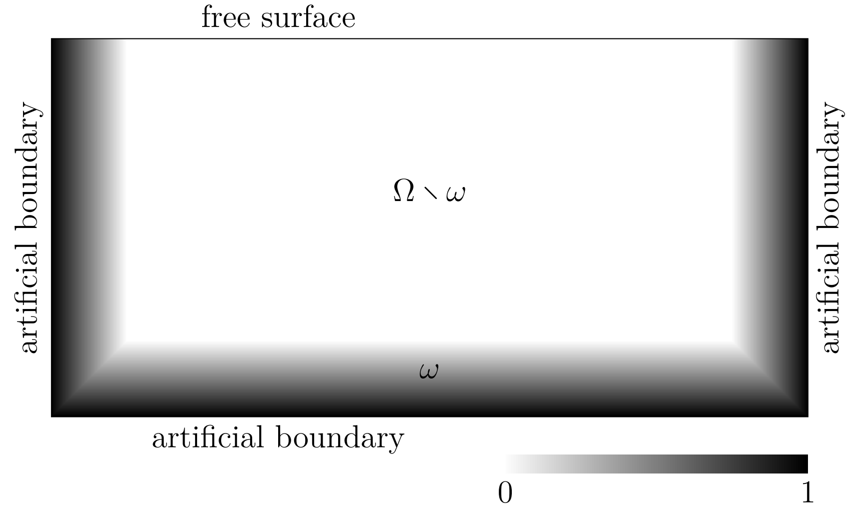


Figure 2.1: Illustration of the damping term η

term is then added to the acoustic wave equation (2.7) as the coefficient for a first-order time derivative, i.e.,

$$\partial_t^2 p - c^2 \Delta p + \eta c^2 \partial_t p = c^2 \operatorname{div} \boldsymbol{f}. \tag{2.10}$$

Along with the Dirichlet and Neumann boundary conditions from above and some arbitrary initial value functions $p_0, p_1 \colon \Omega \to \mathbb{R}$, we arrive at the following hyperbolic second-order PDE with mixed boundary condition:

$$\begin{cases} \nu \partial_t^2 p - \Delta p + \eta \partial_t p = f & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ p(0, \cdot) = p_0 & \text{in } \Omega \\ \partial_t p(0, \cdot) = p_1 & \text{in } \Omega, \end{cases} \tag{2.11}$$

where $\nu \coloneqq c^{-2}$ denotes the square slowness and $f \coloneqq \operatorname{div} \boldsymbol{f}$.

PDE-Constrained Optimization Problem

With knowledge of the true physical parameters, such as ν and f, the solution p to (2.11) gives realistic information of the true acoustic pressure inside the domain Ω. However, in many real-world applications, such as seismic tomography, we are interested in the following *inverse problem*: Given true pressure information, for instance, through measurements, we aim to identify the unknown true square slowness ν. In this scenario, the PDE-model of the acoustic wave equation (2.11) alone is insufficient for the determination of the unknown variable ν. Therefore, we need a more sophisticated model. For $i = 1, \dots, m$ and some $m \in \mathbb{N}$, we denote with a_i some observation patches modeling receivers. These receivers are usually placed at various positions in order to catch sufficient deflections and reflections of the wave caused by material discontinuities inside the domain. A possible configuration of the receiver positions is presented in Figure 2.2. Furthermore, suppose that p_i^{ob} are (typically noisy) wave information corresponding to

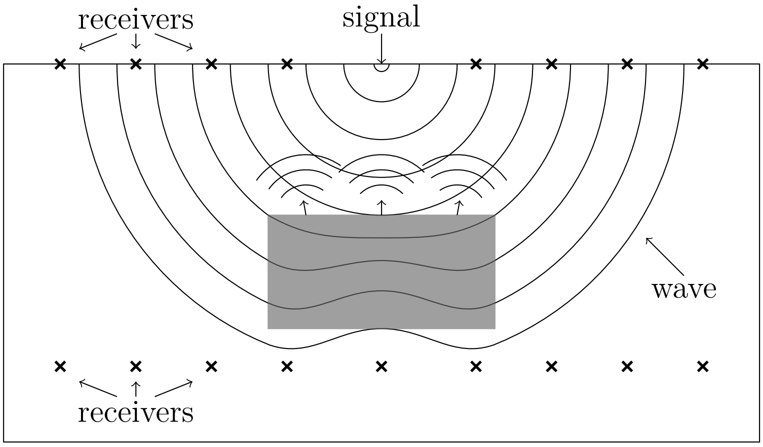


Figure 2.2: Possible configuration for the receiver positions. The grey rectangle represents a material with a smaller wave speed.

the true wave speed ν_d in the domain Ω. Our goal is to compute a suitable pair (ν, p) such that (2.11) is valid and such that the misfit between the synthetic data p and the observation data p_i^{ob} is minimized at the receivers a_i. Incorporating a regularization term with a (small) regularization parameter $\lambda > 0$ and some lower (resp. upper) bound ν_+ (resp. ν_-) for ν, we obtain the following minimization problem:

$$\begin{cases} \inf \mathcal{J}(\nu, p) \coloneqq \dfrac{1}{2} \displaystyle\sum_{i=1}^{m} \int_I \int_\Omega a_i (p - p_i^{ob})^2 \,\mathrm{d}x \,\mathrm{d}t + \dfrac{\lambda}{2} \|\nu\|^2_{L^2(\Omega)} \\ \text{s.t. (2.11) and } \nu_-(x) \le \nu(x) \le \nu_+(x) \text{ for a.e. } x \in \Omega. \end{cases} \tag{2.12}$$

This PDE-constrained optimization problem serves as our strategy for reconstructing the true wave speed parameter ν_d and is the main subject of the present thesis.

2.2 Underlying Function Spaces and Evolution Equations

In this section, we introduce the underlying function spaces that are essential for this thesis. Furthermore, we discuss some fundamental basics of semigroups and evolution equations. We underline that the following introduction does not claim completeness but rather repeats some selected concepts and definitions for the convenience of the reader. For a more extensive overview, we refer to the literature (cf. [13, 26, 32, 34, 36, 66]). In the following, let $N \in \mathbb{N}$ be fixed and let $\Omega \subset \mathbb{R}^N$ be an open set.

Lebesgue Spaces

For some $p \in [1, \infty)$, we denote the space of all equivalence classes of Lebesgue measurable and Lebesgue p-power integrable $\mathbb{R}$-valued functions by

$$L^p(\Omega) \coloneqq \left\{ f \mid f \colon \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |f|^p \,\mathrm{d}x < \infty \right\}.$$

Here, two functions belong to the same equivalence class if they only differ on a set with Lebesgue measure 0. Note that we simply write f instead of $[f]$ where the distinction between an equivalence class and a representant results from the respective context. Furthermore, for conciseness, the variable of integration x is usually omitted in our notation. The space of all equivalence classes of essentially bounded functions is denoted by

$$L^\infty(\Omega) \coloneqq \left\{ f \mid f\colon \Omega \to \mathbb{R} \text{ measurable}, \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty \right\},$$

where ess sup denotes the essential supremum. Endowed with the norm

$$\|f\|_{L^p(\Omega)} \coloneqq \begin{cases} \left(\int_\Omega |f|^p \,\mathrm{d}x \right)^{\frac{1}{p}} & \text{if } p \in [1,\infty) \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| & \text{if } p = \infty \end{cases} \qquad \forall f \in L^p(\Omega),$$

the space $L^p(\Omega)$ is a Banach space. For $p = 2$, the space $L^2(\Omega)$, endowed with the scalar product

$$(f,g)_{L^2(\Omega)} \coloneqq \int_\Omega fg \,\mathrm{d}x \quad \forall f,g \in L^2(\Omega),$$

is a Hilbert space. We indicate functions and function spaces that are $\mathbb{R}^N$-valued with bold letters. In this way, we define $\boldsymbol{L}^2(\Omega) \coloneqq L^2(\Omega)^N$ which is, endowed with the scalar product

$$(\boldsymbol{f},\boldsymbol{g})_{\boldsymbol{L}^2(\Omega)} \coloneqq \int_\Omega \boldsymbol{f}\cdot\boldsymbol{g} \,\mathrm{d}x \quad \forall \boldsymbol{f},\boldsymbol{g} \in \boldsymbol{L}^2(\Omega),$$

also a Hilbert space. Lastly, the space of locally Lebesgue integrable functions is defined by

$$L^1_{\mathrm{loc}}(\Omega) \coloneqq \left\{ f\colon \Omega \to \mathbb{R} \text{ measurable} \,\middle|\, \int_U |f| < \infty \quad \forall U \subset \Omega \text{ open with } \overline{U} \subset \Omega \right\}.$$

Sobolev Spaces

The famous Sobolev spaces provide a fundamental framework for studying the smoothness of functions, particularly those involved in the solutions of PDEs. Let $C_0^\infty(\Omega)$ (resp. $\boldsymbol{C}_0^\infty(\Omega)$) denote the set containing all infinitely differentiable $\mathbb{R}$-valued (resp. $\mathbb{R}^N$-valued) functions with compact support in Ω. Now, given a multi-index α and a function $f \in L^1_{\mathrm{loc}}(\Omega)$, another function $g \in L^1_{\mathrm{loc}}(\Omega)$ is called α-weak derivative of f if and only if

$$\int_\Omega f D^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g\phi \,\mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega).$$

In this case, we write $D^\alpha f = g$. In the special case where $\alpha = e_i$ for a unit vector $e_i \in \mathbb{R}^N$, we write $D_i f \coloneqq D^{e_i} f$ for the i-th partial weak derivative. For $p \in [1,\infty]$ and $k \in \mathbb{N}$, we define the Sobolev space

$$W^{k,p}(\Omega) \coloneqq \{ f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \le k \},$$

where the derivative D^α is understood in the above weak sense. The corresponding norm is defined by

$$\|f\|_{W^{k,p}(\Omega)} \coloneqq \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } p \in [1,\infty) \\ \max_{|\alpha|\le k} \|D^\alpha f\|_{L^\infty(\Omega)} & \text{if } p = \infty \end{cases} \qquad \forall f \in W^{k,p}(\Omega),$$

which makes $W^{k,p}(\Omega)$ a Banach space. In the case $p = 2$, we set $H^k(\Omega) \coloneqq W^{k,2}(\Omega)$ and in the particular case $k = 1$, we endow $H^1(\Omega)$ with the scalar product

$$(f, g)_{H^1(\Omega)} \coloneqq (f, g)_{L^2(\Omega)} + (\nabla f, \nabla g)_{\boldsymbol{L}^2(\Omega)} \quad \forall f, g \in H^1(\Omega),$$

which makes $H^1(\Omega)$ a Hilbert space[1]. Here, $\nabla = (D_1, \dots, D_N)$ denotes the weak gradient operator. For a given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we call $g \in L^2(\Omega)$ the weak divergence of $\boldsymbol{f}$, if and only if

$$\int_\Omega \boldsymbol{f} \cdot \nabla\phi \,\mathrm{d}x = -\int_\Omega g\phi \,\mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega).$$

In that case, we write $\operatorname{div} \boldsymbol{f} \coloneqq g$. The corresponding Hilbert space

$$\boldsymbol{H}(\operatorname{div}, \Omega) \coloneqq \{\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \,|\, \operatorname{div} \boldsymbol{f} \in L^2(\Omega)\}$$

is endowed with the scalar product

$$(\boldsymbol{f}, \boldsymbol{g})_{\boldsymbol{H}(\operatorname{div},\Omega)} \coloneqq (\boldsymbol{f}, \boldsymbol{g})_{\boldsymbol{L}^2(\Omega)} + (\operatorname{div} \boldsymbol{f}, \operatorname{div} \boldsymbol{g})_{L^2(\Omega)} \quad \forall \boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{H}(\operatorname{div}, \Omega)$$

and the induced norm $\|\cdot\|_{\boldsymbol{H}(\operatorname{div},\Omega)}$. Lastly, we define the Banach space

$$D(\Delta) \coloneqq \{f \in H^1(\Omega) \,|\, \nabla f \in \boldsymbol{H}(\operatorname{div}, \Omega)\},$$

that is endowed with the norm $\|\cdot\|_{D(\Delta)} \coloneqq (\|\cdot\|^2_{H^1(\Omega)} + \|\Delta \cdot\|^2_{L^2(\Omega)})^{1/2}$ where $\Delta \colon D(\Delta) \to L^2(\Omega)$, $\Delta \coloneqq \operatorname{div} \nabla$.

(Partially) Vanishing Boundary Conditions

In view of the discussion on boundary conditions from the previous Section 2.1, we define related function spaces, incorporating the Dirichlet and Neumann boundary conditions. The topological closure of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$-norm is denoted by $H_0^1(\Omega)$, i.e.,

$$H_0^1(\Omega) \coloneqq \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$$

and the topological closure of $\boldsymbol{C}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\operatorname{div}, \Omega)$-norm is denoted by $\boldsymbol{H}_0(\operatorname{div}, \Omega)$, i.e.,

$$\boldsymbol{H}_0(\operatorname{div}, \Omega) \coloneqq \overline{\boldsymbol{C}_0^\infty(\Omega)}^{\|\cdot\|_{\boldsymbol{H}(\operatorname{div},\Omega)}}.$$

If we additionally consider Ω to be a bounded Lipschitz domain, trace operators become available. In that case, $H_0^1(\Omega)$ coincides with the set of all $H^1(\Omega)$-functions with vanishing trace, that is

$$H_0^1(\Omega) = \{f \in H^1(\Omega) \,|\, \tau f = 0\},$$

where $\tau \colon H^1(\Omega) \to L^2(\partial\Omega)$ denotes the trace operator. Analogously, $\boldsymbol{H}_0(\operatorname{div}, \Omega)$ coincides with the set of all $\boldsymbol{H}(\operatorname{div}, \Omega)$-functions with vanishing normal trace, that is

$$\begin{aligned} \boldsymbol{H}_0(\operatorname{div}, \Omega) &= \{\boldsymbol{f} \in \boldsymbol{H}(\operatorname{div}, \Omega) \,|\, \gamma_n \boldsymbol{f} = 0\} \\ &= \{\boldsymbol{f} \in \boldsymbol{H}(\operatorname{div}, \Omega) \,|\, (\operatorname{div} \boldsymbol{f}, \phi)_{L^2(\Omega)} = -(\boldsymbol{f}, \nabla\phi)_{\boldsymbol{L}^2(\Omega)} \; \forall \phi \in H^1(\Omega)\} \end{aligned}$$

[1] Analoguesly, higher order spaces, $H^k(\Omega)$ for $k \in \mathbb{N}$, are Hilbert spaces when endowed with suitable scalar products.

where $\gamma_n \colon \boldsymbol{H}(\mathrm{div},\Omega) \to H^{-1/2}(\partial\Omega)$ denotes the normal trace operator and $H^{-1/2}(\partial\Omega)$ is the dual space of the image of τ (cf. [62]). On the one hand, these spaces serve as a generalizing concept, since the trace operator τ generalizes boundary values via $\tau f = f|_{\partial\Omega}$ for every $f \in H^1(\Omega) \cap C(\overline{\Omega})$, and the normal trace operator γ_n generalizes orthogonality conditions on the boundary via $\gamma_n \boldsymbol{f} = \boldsymbol{f}|_{\partial\Omega} \cdot \boldsymbol{n}$ for every $\boldsymbol{f} \in \boldsymbol{C}^1(\overline{\Omega})$. On the other hand, they give rise to an intuitive way of incorporating partially vanishing boundary conditions into suitable function spaces. For some subset $\Gamma_D \subset \partial\Omega$, let us define

$$H^1_D(\Omega) = \overline{C^\infty_D(\Omega)}^{\|\cdot\|_{H^1(\Omega)}} \quad \text{where } C^\infty_D(\Omega) \coloneqq \{v|_\Omega : v \in C^\infty(\mathbb{R}^N), \mathrm{dist}(\mathrm{supp}(v), \Gamma_D) > 0\} \tag{2.13}$$

and

$$\boldsymbol{H}_N(\mathrm{div},\Omega) \coloneqq \{\boldsymbol{f} \in \boldsymbol{H}(\mathrm{div},\Omega) \,|\, (\mathrm{div}\,\boldsymbol{f}, \phi)_{L^2(\Omega)} = -(\boldsymbol{f}, \nabla\phi)_{\boldsymbol{L}^2(\Omega)}\ \forall \phi \in H^1_D(\Omega)\}. \tag{2.14}$$

Lastly, we define

$$D(\Delta_{D,N}) \coloneqq \{\phi \in H^1_D(\Omega) \,|\, \nabla\phi \in \boldsymbol{H}_N(\mathrm{div},\Omega)\}.$$

Abtract Functions

In this subsection, we elaborate on functions having values in a Banach space, so-called abstract functions (or vector-valued functions). In the following, let X be a Banach space and $I \subset \mathbb{R}$ be a finite time interval. An abstract function $f \colon I \to X$ is called a step function if there exists $\{\alpha_i\}_{i=1}^m \subset X$ for some $m \in \mathbb{N}$ and Lebesgue measurable pairwise disjunct subsets $M_i \subset I$ for $i = 1, \ldots, m$ such that

$$I = \bigcup_{i=1,\ldots,m} M_i \quad \text{and} \quad f(t) = \sum_{i=1}^m \alpha_i \chi_{M_i}(t) \quad \forall t \in I.$$

Our goal is to define an integrability concept for abstract functions. For a step function, as above, the Bochner integral is given by

$$\int_I f(t)\,\mathrm{d}t \coloneqq \sum_{i=1}^m f_i |M_i|.$$

Furthermore, an abstract function $f \colon I \to X$ is called

(i) Bochner measurable if there exists a sequence $\{f_k\}_{k=1}^\infty$ of step functions $f_k \colon I \to X$, such that

$$\lim_{k\to\infty} \|f_k(t) - f(t)\|_X \quad \text{for a.e. } t \in I,$$

(ii) Bochner integrable if there exists a sequence $\{f_k\}_{k=1}^\infty$ of step functions $f_k \colon I \to X$, such that

$$\lim_{k\to\infty} \|f_k(t) - f(t)\|_X \quad \text{for a.e. } t \in I \quad \text{and} \quad \lim_{k\to\infty} \int_I \|f_k(t) - f(t)\|_X\,\mathrm{d}t = 0.$$

In that case, we write

$$\int_I f(t)\,\mathrm{d}t \coloneqq \lim_{k\to\infty} \int_I f_k(t)\,\mathrm{d}t.$$

Instead of finding a suitable sequence of step functions for proving the Bochner integrability, there exists another useful equivalent criterion: A Bochner measurable abstract function $f\colon I \to X$ is Bochner integrable if and only if

$$\int_I \|f(t)\|_X \,\mathrm{d}t < \infty.$$

Furthermore, in that case, it holds that

$$\left\| \int_I f(t)\,\mathrm{d}t \right\|_X \le \int_I \|f(t)\|_X \,\mathrm{d}t.$$

For some $p \in [1,\infty]$, we define the following Bochner spaces:

$$\begin{aligned}
L^p(I,X) &\coloneqq \left\{ f\colon I \to X \mid \|f\|_{L^p(I,X)} \coloneqq \left(\int_I \|f(t)\|_X^p \,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\} \quad \text{if } p \in [1,\infty),\\
L^\infty(I,X) &\coloneqq \left\{ f\colon I \to X \mid \|f\|_{L^\infty(I,X)} \coloneqq \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X < \infty \right\},\\
L^1_{\mathrm{loc}}(I,X) &\coloneqq \left\{ f\colon I \to X \mid f \in L^1((a,b),X) \text{ for all } (a,b) \subset (0,T) \text{ such that } [a,b] \subset (0,T) \right\}.
\end{aligned}$$

As in the definition of the Lebesgue and Sobolev spaces, the Bochner spaces are also sets of equivalence classes of abstract functions where two abstract functions belong to the same equivalence class if they differ on a set of Lebesgue measure 0. Nevertheless, again, we write f instead of $[f]$ where the distinction between an equivalence class and a representant results from the context. The notions of continuity and (strong and weak) differentiability can be introduced for abstract functions as well: We call $f\colon I \to X$ continuous in $t \in I$ if

$$\lim_{h\to t} \|f(h) - f(t)\|_X = 0,$$

and we call f continuous if it is continuous in every $t \in I$. We call f (strongly) differentiable in $t \in I$ if the limit

$$\partial_t f(t) \coloneqq \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \in X$$

exists, and we call f differentiable if it is differentiable in every $t \in I$. Then, we define the Banach spaces

$$\begin{aligned}
C(I,X) = C^0(I,X) &\coloneqq \{ f\colon I \to X \mid f \text{ is continuous} \}\\
C^k(I,X) &\coloneqq \{ f\colon I \to X \mid \partial_t^l f \text{ is differentiable for every}\\
&\qquad l = 0,\dots,k-1 \text{ and } \partial_t^k f \text{ is continuous} \} \quad \forall k \in \mathbb{N}
\end{aligned}$$

with the corresponding norms

$$\|f\|_{C^k(I,X)} \coloneqq \sum_{l=0}^{k} \max_{t\in I} \|\partial_t^l f(t)\|_X \quad \forall f \in C^k(I,X), k \in \mathbb{N} \cup \{0\}.$$

Furthermore, let

$$C^\infty(I,X) \coloneqq \bigcup_{k=1}^{\infty} C^k(I,X).$$

For $l \in \mathbb{N}$, we call an abstract function $f \in L^1_{\text{loc}}(I, X)$ l-times weakly differentiable with l-th weak derivative $\partial_t^l f = g$ if

$$\int_I f(t)\partial_t^l \phi(t)\,\mathrm{d}t = (-1)^l \int_I g(t)\phi(t)\,\mathrm{d}t \quad \forall \phi \in C_0^\infty(I).$$

In the vector-valued case, the Sobolev spaces are generalized as follows: For every $p \in [1, \infty]$ and $k \in \mathbb{N}$, we define

$$W^{k,p}(I, X) := \{f \in L^p(I, X) \,|\, \partial_t^l f \in L^p(I, X) \text{ for every } l = 1, \dots, k\}$$

where ∂_t^l is understood in the above weak sense. Endowed with the norm

$$\|f\|_{W^{k,p}(I,X)} := \begin{cases} \left(\sum_{l=0}^{k} \|\partial_t^l f\|_{L^p(I,X)}^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty) \\ \max_{l=0,\dots,k} \|\partial_t^l f\|_{L^\infty(I,X)} & \text{if } p = \infty \end{cases} \qquad \forall f \in W^{k,p}(I, X)$$

the space $W^{k,p}(I, X)$ is a Banach space. If $X = H$ is a Hilbert space, the Bochner space $H^k(I, H) := W^{k,2}(I, H)$ is also a Hilbert space when endowed with the scalar product

$$(f, g)_{H^k(I,H)} := \sum_{l=0}^{k} \int_I (\partial_t^l f(t), \partial_t^l g(t))_H \,\mathrm{d}t \quad \forall f, g \in H^k(I, H).$$

Evolution Equations and Semigroups

Considering a real Banach space X and a time interval $I = [0, T] \subset \mathbb{R}$, we aim to find a solution $u: I \to X$ to the evolution equation given by the abstract Cauchy problem

$$\begin{cases} \partial_t u(t) - Au(t) = F(t) & \forall t \in I \\ u(0) = u_0 \end{cases} \tag{2.15}$$

for a linear operator $A: D(A) \subset X \to X$, an abstract function $F: I \to X$, and some $u_0 \in X$. In the case where $X = \mathbb{R}$, $Ax = ax$ for some $a \in \mathbb{R}$ and for all $x \in \mathbb{R}$, and $F \in C(I)$, the solution is given by

$$u: I \to \mathbb{R}, \quad t \mapsto u_0 e^{at} + \int_0^t e^{a(t-s)} F(s)\,\mathrm{d}s.$$

This motivates the concept of semigroups as a generalization of the exponential function in the context of abstract functions.

Definition 2.1. *Let X be a real Banach space and let $\{\mathbb{T}(t)\}_{t \geq 0} \subset L(X)$ be a familiy of linear and bounded operators. Then, $\{\mathbb{T}(t)\}_{t \geq 0}$ is called a semigroup if $\mathbb{T}(0) = I_d$ and $\mathbb{T}(t + s) = \mathbb{T}(t)\mathbb{T}(s)$ for all $t, s \geq 0$. Furthermore, the semigroup is called strongly continuous if $\lim_{t \searrow 0} \mathbb{T}(t)x = x$ for all $x \in X$. The strongly continuous semigroup is called a contraction semigroup if $\|\mathbb{T}(t)\|_{L(X)} \leq 1$ for every $t \geq 0$.*

Definition 2.2. *Let X be a real Banach space and let $\{\mathbb{T}(t)\}_{t \geq 0} \subset L(X)$ be a semigroup. Then, the linear operator $A: D(A) \to X$, where*

$$Ax := \lim_{t \searrow 0} \frac{\mathbb{T}(t)x - x}{t} \quad \forall x \in D(A) := \left\{ x \in X : \lim_{t \searrow 0} \frac{\mathbb{T}(t)x - x}{t} \text{ exists} \right\}$$

is called the (infinitesimal) generator of $\{\mathbb{T}(t)\}_{t \geq 0} \subset L(X)$.

The concept of semigroups is particularly tailored for solving the abstract Cauchy problem (2.15) for suitable operators A, more precisely, for those that generate an appropriate semigroup. Suppose that the strongly continuous semigroup $\{\mathbb{T}(t)\}_{t\geq 0} \subset L(X)$ and its corresponding generator A is given. Then, the following properties hold (cf. [32, Chapter II, Lemma 1.3]):

(i) $\mathbb{T}(t)x \in D(A)$ for every $x \in D(A)$ and $t \geq 0$,

(ii) $\partial_t \mathbb{T}(t)x = \mathbb{T}(t)Ax = A\mathbb{T}(t)x$ for every $x \in D(A)$ and $t \geq 0$.

From the property (ii) along with $\mathbb{T}(0) = I_d$, we obtain that $u: I \to X, t \mapsto \mathbb{T}(t)u_0$ solves the Cauchy problem (2.15) for $F \equiv 0$ and $u_0 \in D(A)$. In the presence of a source term $F \in W^{1,1}(I, X)$, the (classical) solution is given by

$$u: I \to X, \quad t \mapsto \mathbb{T}(t)u_0 + \int_0^t \mathbb{T}(t-s)F(s)\,\mathrm{d}s \tag{2.16}$$

(cf. [32, Corollary 7.6]). Typically, when considering a problem of the type (2.15), only the operator A is known. Therefore, it is of primary interest whether A is a generator of some unknown semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$. The famous Lumer-Phillips theorem provides a satisfactory answer (cf. [66, Chapter 1, Theorem 4.3]):

Theorem 2.3 (Lumer–Phillips)**.** *Let X be a real Banach space, $D(A) \subset X$ be dense, and $A: D(A) \to X$ be a linear operator. Furthermore, let A be dissipative, i.e.,*

$$\forall x \in D(A) \quad \exists x^* \in \{x^* \in X^* : x^*(x) = \|x\|_X^2 = \|x^*\|_{X^*}^2\} \;:\; x^*(Ax) \leq 0$$

and let $R(\lambda I - A) = X$ for some $\lambda > 0$. Then, A is the infinitesimal generator of a contraction semigroup.

Suppose that an operator A satisfies the assumptions of Theorem 2.3, $F \in W^{1,1}(I, X)$ and $u_0 \in D(A)$. Then, the formula (2.16) provides a solution to the corresponding Cauchy problem (2.15) where $\{\mathbb{T}(t)\}_{t\geq 0}$ is the contraction semigroup generated by A. However, the solution is only implicitly given by (2.16) since typically no explicit representation of the semigroup is $\{\mathbb{T}(t)\}_{t\geq 0}$ is available. Nevertheless, Theorem 2.3 provides the existence of a solution, and from the formula (2.16), useful properties of the solution can be extracted. Note that, in the case where $X = H$ is a Hilbert space, a linear operator $A: D(A) \subset H \to H$ is dissipative if $(Ax, x)_H \leq 0$ for all $x \in D(A)$. We refer to the remainder of this thesis for an application.

ANALYSIS OF THE OPTIMAL CONTROL PROBLEM 3

This chapter is devoted to the first- and second-order analysis of (1.2). We develop a novel strategy for (1.2) based on the use of an auxiliary first-order hyperbolic system (3.9), which is shown to be well-posed and serves as the mild notion for the state equation (1.1) (see Theorem 3.5). However, compared with the original system (1.1), (3.9) features a more involved control-to-state structure: The parameter ν appears in (3.9) not only as the coefficient of the time-derivative but also as initial data under the image of the solution operator $\Phi\colon L^\infty(\Omega) \to \boldsymbol{H}_N(\mathrm{div}, \Omega)$ for a specific elliptic variational problem (see (3.4)-(3.6)). Taking this distinctive control-to-state character into account, we develop necessary optimality conditions for the weak form of (1.2) based on adjoint techniques (see Lemma 3.10 and Theorem 3.12). It is possible to perform a direct analysis of the optimal control based on the original second-order formulation (1.1). This ansatz, however, calls for a higher regularity assumption on the initial values and the right-hand side of (1.1). First, to guarantee the existence of a unique solution $p \in C^2(I, L^2(\Omega)) \cap C^1(I, H^1_D(\Omega))$ to (1.1), we require the higher regularity condition: $p_1 \in H^1_D(\Omega), p_0 \in D(\Delta_{D,N})$, and $f \in W^{1,1}(I, L^2(\Omega))$. Furthermore, the regularity property $\sum_{i=1}^m \int_I \int_\Omega a_i(\overline{p} - p_i^{ob}) \in W^{1,1}(I, L^2(\Omega))$ with $\overline{p}$ being the optimal state is also required for the well-posedness of the corresponding second-order adjoint equation. These regularity assumptions are not needed for the derivation of necessary optimality conditions based on the proposed first-order auxiliary system (3.9). Our first-order analysis relies solely on a lower regularity requirement (see Assumption 3.1). Moreover, as a further advantage, our approach leads to the low adjoint state regularity $\overline{q} \in C(I, L^2(\Omega))$ such that no higher Sobolev regularity can be extracted for the optimal solution $\overline{\nu}$ from the corresponding projection formula (Remark 3.13). Nevertheless, assuming the above-mentioned higher regularity conditions, we also follow first-order necessary optimality conditions based on the second-order formulation (1.2) as a direct consequence of the one based on the auxiliary first-order formulation (see Corollary 3.14).

The final goal of this chapter is to establish second-order sufficient optimality conditions for (1.2). Our second-order analysis is performed based on the Lagrangian functional (3.101) involving the non-reduced objective functional and the first-order auxiliary system (3.9). By the control-to-state structure in (3.9), the Lagrangian (3.101) contains the term $(\nu \partial_t p, q)_{L^2(I, L^2(\Omega))}$, i.e., the product of three quantities related to the control, the adjoint state, and the time-derivative of the state. The treatment of this term turns out to be rather challenging. In the author's paper [5], this issue is tackled by restricting the second-order analysis to the case where the reconstruction is considered only in an open set strictly contained in the hold-all domain Ω. More precisely, the existence of an open set $\omega \subset \Omega$ such that $\overline{\omega} \subset \Omega$ and $\nu_-(x) = \nu_+(x)$ for a.e. $x \in \Omega \setminus \omega$ is assumed. In this case, the reconstruction process is reduced to the subregion ω. Further, the application of an elliptic inner regularity result provides a crucial Lipschitz $C^2(I, C(\overline{\omega}))$-regularity result (see [5, Lemma 4.3]). However, in this chapter, we present an extension of the results in [5]. It turns out that the inner regularity ansatz can be improved by applying Stampacchia's method to the hyperbolic case. In this way, global essential boundedness can be obtained for the corresponding state. Furthermore, in the two- and three-dimensional case, under some additional regularity and compatibility assumption on the data (Assumption 3.18), we establish a Lipschitz $C^2(I, L^\infty(\Omega))$-regularity result (Lemma 3.19). With Lemma 3.19 at hand, we manage to extend the contradiction argument from [18] (cf. [19, 20]) to our case and derive a second-order sufficient optimality condition (SSC) in the form of the positivity of the second-order derivative of the quadratic Lagrangian functional involving the strongly active set (Theorem 3.22). The corresponding SSC yields a quadratic growth condition and local optimality in an L^2-neighborhood. We follow the SSC result based on the original second-order formulation (1.2) as a direct consequence from Theorem 3.22 (see Corollary 3.25).

Let us provide a brief overview of existing contributions related to this chapter. Optimal control approaches for time-domain FWI have been recently discussed and explored in Boehm and Ulbrich [11] and Clason et al. [24]. Our approach is based on the first-order auxiliary system (3.9) and is quite different from the aforementioned contributions. In particular, we solely apply the L^2-penalty for the Tikhonov regularization in (1.2), which is based on the proposed first-order approach readily sufficient for establishing existence theory (Theorem 3.7) and optimality conditions (Theorem 3.12). We refer to Kirsch and Rieder [52–54] for the mathematical analysis of inverse problems related to time-domain FWI. Moreover, level set-based and shape optimization approaches for time-domain FWI have been quite recently proposed and analyzed in Albuquerque et al. [1]. Last but not least, we mention [46, 61, 68, 80, 83] for previous contributions towards FWI in the frequency domain based on Helmholtz and eddy current equation.

The rest of the chapter is organized as follows. We start with presenting our notation and the mathematical assumptions for the data involved in (1.2), including the weak formulation for (1.1). In Section 3.1, we propose and analyze the first-order auxiliary system (3.9), leading to an existence result for (1.2). Based on the developed results for (3.9), we derive our main results regarding the first-order necessary and second-order sufficient optimality conditions for (1.2), respectively, in Section 3.3 and Section 3.5.

Most content of this chapter is available in the author's [5]. Consequently, direct quotations from this work will not be explicitly highlighted.

Assumption 3.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_N \subsetneq \partial\Omega$ satisfying $|\Gamma_N| \neq 0$ and $\Gamma_D \subset \partial\Omega$ beeing closed. Let $p_0 \in H^1_D(\Omega)$, $p_1 \in L^2(\Omega)$, $p_i^{ob} \in L^2(I, L^2(\Omega))$ for all $i = 1, \ldots, m$, and $f \in L^1(I, L^2(\Omega))$ be given data for some $m \in \mathbb{N}$. The coefficients $a_i \in L^\infty(I \times \Omega)$ and $\eta \in L^\infty(\Omega)$ are also given data and assumed to be nonnegative for all $i = 1, \ldots, m$. Furthermore, let $\nu_{\max}, \nu_{\min} > 0$ and $\nu_-, \nu_+ \in L^\infty(\Omega)$ satisfy $\nu_{\min} < \nu_-(x) < \nu_+(x) < \nu_{\max}$ for a.e. $x \in \Omega$.*

Associated with (1.2), we introduce the admissible set

$$\mathcal{V}_{ad} = \{\nu \in L^2(\Omega) : \nu_-(x) \leq \nu(x) \leq \nu_+(x) \text{ for a.e. } x \in \Omega\}. \tag{3.1}$$

For later use, we also introduce a larger and open set

$$\mathcal{V} := \{\nu \in L^\infty(\Omega) : \nu_{\min} < \nu(x) \text{ for a.e. } x \in \Omega\} \supset \mathcal{V}_{ad}. \tag{3.2}$$

Definition 3.2. *Let Assumption 3.1 be satisfied and let $\nu \in \mathcal{V}$ be given. Then, a function $p \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ is called a mild solution to (1.1) if the mapping $t \mapsto (\nu\partial_t p(t), \phi)_{L^2(\Omega)}$ is for every $\phi \in H^1_D(\Omega)$ absolutely continuous in I and*

$$\begin{cases} \partial_t \int_\Omega \nu \partial_t p(t)\phi \,\mathrm{d}x + \int_\Omega \nabla p(t)\cdot\nabla\phi + \eta\partial_t p(t)\phi \,\mathrm{d}x = \int_\Omega f(t)\phi \,\mathrm{d}x \\ \qquad\qquad \forall \phi \in H^1_D(\Omega) \text{ and a.e. } t \in I \\ p(0) = p_0 \quad \text{a.e. in } \Omega \\ \partial_t p(0) = p_1 \quad \text{a.e. in } \Omega. \end{cases} \tag{3.3}$$

3.1 Auxiliary First-Order System

In this section, we propose and analyze an auxiliary first-order system serving as an equivalent formulation of (1.1). The proposed first-order system is the key fundament for our optimal control approach, in particular for the derivation of the adjoint system. Let us begin by considering the following elliptic variational problem: Given $\nu \in L^\infty(\Omega)$, find $y \in H^1_D(\Omega)$ such that

$$\int_\Omega \nabla y \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega (\eta p_0 + \nu p_1)\phi \,\mathrm{d}x \quad \forall \phi \in H^1_D(\Omega). \tag{3.4}$$

Thanks to the Lax-Milgram lemma, the variational problem (3.4) admits for every $\nu \in L^\infty(\Omega)$ a unique solution $y \in H^1_D(\Omega)$. Furthermore, as $C^\infty_0(\Omega) \subset H^1_D(\Omega)$, the definition of the weak divergence applied to (3.4) implies that the solution satisfies

$$\begin{aligned} \operatorname{div}(\nabla y) = -\eta p_0 - \nu p_1 \underbrace{\Rightarrow}_{(3.4)} \int_\Omega \nabla y \cdot \nabla \phi \,\mathrm{d}x = -\int_\Omega \operatorname{div}(\nabla y)\phi \,\mathrm{d}x \quad \forall \phi \in H^1_D(\Omega) \\ \underbrace{\Rightarrow}_{(2.14)} \nabla y \in \boldsymbol{H}_N(\operatorname{div}, \Omega). \end{aligned} \tag{3.5}$$

For this reason, the solution operator associated with (3.4)

$$\Phi : L^\infty(\Omega) \to \boldsymbol{H}_N(\operatorname{div}, \Omega), \quad \nu \mapsto \nabla y \tag{3.6}$$

is well-defined, affine linear, and continuous. In particular, Φ fulfils the properties

$$\begin{cases} \|\Phi(\nu)\|_{\boldsymbol{L}^2(\Omega)} \le c_P \|\eta p_0 + \nu p_1\|_{L^2(\Omega)} & \forall \nu \in L^\infty(\Omega) \\ \|\Phi(\nu_1) - \Phi(\nu_2)\|_{\boldsymbol{L}^2(\Omega)} \le c_P \|p_1(\nu_1 - \nu_2)\|_{L^2(\Omega)} & \forall \nu_1, \nu_2 \in L^\infty(\Omega) \end{cases} \tag{3.7}$$

for a Poincaré constant $c_P > 0$. Here, (3.7) is immediately obtained by inserting $\phi = y$ in (3.4) along with the generalized Poincaré inequality [72, Lemma 2.5]. Now, introducing the antiderivative

$$F \in W^{1,1}(I, L^2(\Omega)), \quad F(t) := \int_0^t f(s)\,\mathrm{d}s \quad \forall t \in I, \tag{3.8}$$

we propose for a given $\nu \in \mathcal{V}$ the following first-order system

$$\begin{cases} \nu\partial_t p + \operatorname{div} \boldsymbol{u} + \eta p = F & \text{in } I \times \Omega \\ \partial_t \boldsymbol{u} + \nabla p = \boldsymbol{0} & \text{in } I \times \Omega \\ p = 0 & \text{on } I \times \Gamma_D \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (p, \boldsymbol{u})(0) = (p_0, \Phi(\nu)) & \text{in } \Omega. \end{cases} \tag{3.9}$$

We underline that the parameter ν appears in (3.9) not only as the coefficient for the time-derivative $\partial_t p$ but also as the initial value for $\boldsymbol{u}$ given by the image of ν under the operator Φ. The upcoming theorem proves the well-posedness of the auxiliary system (3.9) and its equivalence to the second-order system (1.1). The proof is based on the semigroup theory (cf. [30, 84] recent works on nonlinear optimal control problems based on the semigroup theory). In the following, for each $\nu \in \mathcal{V}$, we introduce the Hilbert space

$$X_\nu \coloneqq L^2_\nu(\Omega) \times \boldsymbol{L}^2(\Omega),$$

endowed with the weighted scalar product

$$((q, \boldsymbol{v}), (\psi, \boldsymbol{z}))_{X_\nu} \coloneqq (q, \psi)_{L^2_\nu(\Omega)} + (\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{L}^2(\Omega)} = (\nu q, \psi)_{L^2(\Omega)} + (\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{L}^2(\Omega)} \ \forall (q, \boldsymbol{v}), (\psi, \boldsymbol{z}) \in X_\nu \tag{3.10}$$

and the induced norm

$$\|(q, \boldsymbol{v})\|_{X_\nu} \coloneqq \sqrt{\|\nu^{1/2} q\|^2_{L^2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}} \quad \forall (q, \boldsymbol{v}) \in X_\nu. \tag{3.11}$$

Note that, since $\nu \in \mathcal{V}$, the norm in (3.11) is equivalent to the norm

$$\|(q, \boldsymbol{v})\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \coloneqq (\|q\|^2_{L^2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)})^{1/2},$$

since by virtue of (3.2) it holds for all $\nu \in \mathcal{V}$ that

$$\begin{aligned} \min\{1, \sqrt{\nu_{\min}}\} \|(q, v)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} &\leq \|(q, v)\|_{X_\nu} \\ &\leq \max\left\{1, \sqrt{\|\nu\|_{L^\infty(\Omega)}}\right\} \|(q, v)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{3.12}$$

Introducing the unbounded linear operator $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$, defined by

$$D(A_\nu) \coloneqq H^1_D(\Omega) \times \boldsymbol{H}_N(\operatorname{div}, \Omega), \quad A_\nu(p, \boldsymbol{u}) \coloneqq -(\nu^{-1}\eta p + \nu^{-1} \operatorname{div} \boldsymbol{u}, \nabla p), \tag{3.13}$$

the first-order system (3.9) can be equivalently formulated as the following Cauchy problem:

$$\begin{cases} \partial_t (p, \boldsymbol{u})(t) - A_\nu (p, \boldsymbol{u})(t) = (\nu^{-1} F(t), \boldsymbol{0}) & \forall t \in I \\ (p, \boldsymbol{u})(0) = (p_0, \Phi(\nu)). \end{cases} \tag{3.14}$$

Lemma 3.3. *Let Assumption 3.1 hold. Then, for every $\nu \in \mathcal{V}$, the operator $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$, defined in* (3.13)*, generates a contraction semigroup $\{\mathbb{T}_\nu(t)\}_{t\geq 0}$.*

Proof. At first, we show that $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$ is dissipative. Indeed, according to (3.13) and (3.10), it holds that

$$(A_\nu(p,\boldsymbol{u}),(p,\boldsymbol{u}))_{X_\nu} = -\int_\Omega \eta p^2 + \operatorname{div}\boldsymbol{u} p \,\mathrm{d}x - \int_\Omega \nabla p \cdot \boldsymbol{u} \,\mathrm{d}x \underbrace{=}_{(2.14)} -\int_\Omega \eta p^2 \,\mathrm{d}x \le 0$$

for all $(p,\boldsymbol{u}) \in D(A_\nu)$. Since X_ν is a Hilbert space, this proves the dissipativity of A_ν. Now let $(q,\boldsymbol{v}) \in X_\nu$. The Lax-Milgram lemma yields the existence of a unique solution $p \in H^1_D(\Omega)$ to

$$\int_\Omega (\nu+\eta)p\phi + \nabla p \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega \boldsymbol{v}\cdot\nabla\phi + \nu q\phi \,\mathrm{d}x \quad \forall \phi \in H^1_D(\Omega). \tag{3.15}$$

Letting $\boldsymbol{u} \coloneqq \boldsymbol{v} - \nabla p$, it follows from (3.15) that

$$\int_\Omega ((\nu+\eta)p - \nu q)\phi \,\mathrm{d}x = \int_\Omega (\boldsymbol{v}-\nabla p)\cdot\nabla\phi \,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\nabla\phi \,\mathrm{d}x \quad \forall\phi \in H^1_D(\Omega). \tag{3.16}$$

Considering (3.16), the definition of the weak divergence implies that

$$\operatorname{div}\boldsymbol{u} = \nu q - (\nu+\eta)p. \tag{3.17}$$

By the definition (2.14), applying (3.17) to (3.16) ensures that $\boldsymbol{u} \in \boldsymbol{H}_N(\operatorname{div},\Omega)$. Altogether, in view of (3.17) and (3.13), we conclude that

$$\forall\,(q,\boldsymbol{v}) \in X_\nu \;\exists\,(p,\boldsymbol{u}) \in D(A_\nu) \;:\; (I - A_\nu)(p,\boldsymbol{u}) = (q,\boldsymbol{v}) \quad \Rightarrow \quad R(I-A_\nu) = X_\nu.$$

Along with the dissipativity of $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$, due to the Lumer-Phillips theorem (see Theorem 2.3 or [32, Corollary 3.17]), A_ν generates a contraction semigroup $\{\mathbb{T}(t)\}_{t\ge0}$. □

Lemma 3.4. *Let Assumption 3.1 hold and let* $\nu \in \mathcal{V}$, $(p,\boldsymbol{u})_{0,0} \in L^2(\Omega)\times\boldsymbol{L}^2(\Omega)$, *and* $G \in L^1(I,L^2(\Omega))$. *Further, let* $\{\mathbb{T}_\nu(t)\}_{t\ge0}$ *denote the contraction semigroup generated by* A_ν. *Then, the mild solution* $(p,\boldsymbol{u}) \in C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))$ *to the Cauchy problem*

$$\begin{cases} \partial_t(p,\boldsymbol{u})(t) - A_\nu(p,\boldsymbol{u})(t) = (G(t),\boldsymbol{0}) & \forall t \in I \\ (p,\boldsymbol{u})(0) = (p,\boldsymbol{u})_{0,0} \end{cases} \tag{3.18}$$

defined by

$$(p,\boldsymbol{u})(t) \coloneqq \mathbb{T}_\nu(t)(p,\boldsymbol{u})_{0,0} + \int_0^t \mathbb{T}_\nu(t-s)(G(s),\boldsymbol{0})\,\mathrm{d}s \quad \forall t \in I \tag{3.19}$$

satisfies

$$\|(p,\boldsymbol{u})(t)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \le c(\|(p,\boldsymbol{u})_{0,0}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|G\|_{L^1(I,L^2(\Omega))}) \quad \forall t \in I \tag{3.20}$$

with $c \coloneqq \frac{\max\{\sqrt{\nu_{\max}},1\}}{\min\{\sqrt{\nu_{\min}},1\}}$. *If additionally* $G \in W^{k,1}(I,L^2(\Omega))$ *with* $k \in \mathbb{N}$, $(p,\boldsymbol{u})_{0,0} \in H^1_D(\Omega)\times \boldsymbol{H}_N(\operatorname{div},\Omega)$, *and*

$$(p,\boldsymbol{u})_{0,l} \coloneqq A_\nu(p,\boldsymbol{u})_{0,l-1} + (\partial_t^{l-1}G(0),\boldsymbol{0}) \in H^1_D(\Omega)\times\boldsymbol{H}_N(\operatorname{div},\Omega) \quad \forall l = 1,\dots,k-1,$$

then $(p,\boldsymbol{u}) \in C^k(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^{k-1}(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega))$ *solves* (3.18) *in the classical sense and satisfies*

$$\|\partial_t^l(p,\boldsymbol{u})(t)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \le c(\|(p,\boldsymbol{u})_{0,l}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^l G\|_{L^1(I,L^2(\Omega))}) \quad \forall t \in I \quad \forall l \in \{0,\dots,k\}, \tag{3.21}$$

where $(p,\boldsymbol{u})_{0,k} \coloneqq A_\nu(p,\boldsymbol{u})_{0,k-1} + \partial_t^{k-1}(G(0),\boldsymbol{0})$.

Proof. Let $(p,\boldsymbol{u}) \in C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))$ denote the mild solution to (3.18) in the sense of (3.19). Since $\{\mathbb{T}_\nu(t)\}_{t\ge0}$ is a contraction semigroup, along with the norm equivalence (3.12), the estimate (3.20) is directly obtained from (3.19). Now, let additionally $G \in W^{k,1}(I, L^2(\Omega))$ for some $k \in \mathbb{N}$ and $(p,\boldsymbol{u})_{0,l} \in H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega)$ for all $l = 0,\dots,k-1$. We show inductively that $(p,\boldsymbol{u}) \in C^k(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))$ with

$$\partial_t^l(p,\boldsymbol{u})(t) = \mathbb{T}_\nu(t)(p,\boldsymbol{u})_{0,l} + \int_0^t \mathbb{T}_\nu(t-s)(\partial_t^l G(s),\boldsymbol{0})\,\mathrm{d}s \quad \forall l = 0,\dots,k, t \in I. \tag{3.22}$$

The case $k = 0$ follows from the definition (3.19). Let the claim hold for some fixed $l \in \{0,\dots,k-1\}$. Then, it follows that

$$\begin{aligned}
\partial_t^{l+1}(p,\boldsymbol{u})(t) &= \partial_t\left(\mathbb{T}_\nu(t)(p,\boldsymbol{u})_{0,l} + \int_0^t \mathbb{T}_\nu(t-s)(\partial_t^l G(s),\boldsymbol{0})\,\mathrm{d}s\right)\\
&= A_\nu\mathbb{T}_\nu(t)(p,\boldsymbol{u})_{0,l} + \mathbb{T}_\nu(t)(\partial_t^l G(0),\boldsymbol{0}) + \int_0^t \mathbb{T}(t-s)(\partial_t^{l+1}G(s),\boldsymbol{0})\,\mathrm{d}s\\
&= \mathbb{T}_\nu(t)(A_\nu(p,\boldsymbol{u})_{0,l} + (\partial_t^l G(0),\boldsymbol{0})) + \int_0^t \mathbb{T}(t-s)(\partial_t^{l+1}G(s),\boldsymbol{0})\,\mathrm{d}s\\
&= \mathbb{T}_\nu(t)(p,\boldsymbol{u})_{0,l+1} + \int_0^t \mathbb{T}_\nu(t-s)(\partial_t^{l+1}G(s),\boldsymbol{0})\,\mathrm{d}s,
\end{aligned}$$

particularly leading to $\partial_t^{l+1}(p,\boldsymbol{u}) \in C(I, L^2(\Omega)\times\boldsymbol{L}^2(\Omega))$. This finalizes the induction proof. From (3.22), we obtain for every $l \in \{0,\dots,k-1\}$, $t \in I$, and $h > 0$ such that $t + h \in I$ that

$$\begin{aligned}
&\frac{\partial_t^l(p,\boldsymbol{u})(t+h) - \partial_t^l(p,\boldsymbol{u})(t)}{h}\\
&= \frac{\mathbb{T}_\nu(h) - I_d}{h}\mathbb{T}_\nu(t)(p,\boldsymbol{u})_{0,l} + \frac{1}{h}\int_t^{t+h} \mathbb{T}_\nu(t+h-s)(\partial_t^l G(s),\boldsymbol{0})\,\mathrm{d}s\\
&\quad + \frac{\mathbb{T}_\nu(h) - I_d}{h}\int_0^t \mathbb{T}_\nu(t-s)(\partial_t^l G(s),\boldsymbol{0})\,\mathrm{d}s\\
&= \frac{\mathbb{T}_\nu(h) - I_d}{h}(\partial_t^l(p,\boldsymbol{u})(t)) + \frac{1}{h}\int_t^{t+h} \mathbb{T}_\nu(t+h-s)(\partial_t^l G(s),\boldsymbol{0})\,\mathrm{d}s.
\end{aligned}$$

Since the left-hand side and the second term on the right-hand side converges as $h \to 0$, due the definition of the generator A_ν, it follows for every $l \in \{0,\dots,k-1\}$ and $t \in I$ that $\partial_t^l(p,\boldsymbol{u})(t) \in D(A_\nu)$ and

$$\partial_t^{l+1}(p,\boldsymbol{u})(t) = A_\nu(\partial_t^l(p,\boldsymbol{u})(t)) + (\partial_t^l G(t),\boldsymbol{0}).$$

Thus, $(p,\boldsymbol{u}) \in C^{k-1}(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega))$ and (3.18) is satisfied. Note that (3.21) follows from (3.22) along with the contraction semigroup property of $\{\mathbb{T}_\nu(t)\}_{t\ge0}$ and the norm equivalence (3.12). □

Theorem 3.5. *Let Assumption 3.1 hold. Then, for every $\nu \in \mathcal{V}$, the first-order system* (3.9) *admits a unique solution* $(p, \boldsymbol{u}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ *given by*

$$(p, \boldsymbol{u})(t) = \mathbb{T}_\nu(t)(p_0, \Phi(\nu)) + \int_0^t \mathbb{T}_\nu(t-s)(\nu^{-1}F(s), \mathbf{0})\, \mathrm{d}s \quad \forall t \in I, \tag{3.23}$$

where $\{\mathbb{T}_\nu(t)\}_{t\geq 0}$ *denotes the contraction semigroup generated by* A_ν*. The first component* $p \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ *of* (3.23) *is exactly the unique mild solution to the forward system* (1.1) *in the sense of Definition 3.2.*

Proof. Let $\nu \in \mathcal{V}$ be fixed. Since $\nu^{-1}F \in W^{1,1}(I, L^2(\Omega))$ and $(p_0, \Phi(\nu)) \in H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega)$, Lemma 3.4 implies that (3.14) admits a unique solution $(p, \boldsymbol{u}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ given by the formula of variation of constants (3.19) with $G \coloneqq \nu^{-1}F$. It remains to prove that $p \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ is the unique mild solution to the forward system (1.1) in the sense of Definition 3.2. First, in view of (3.8), applying [32, Lemma 1.3] to (3.19) yields

$$\partial_t(p, \boldsymbol{u})(t) = \mathbb{T}_\nu(t)A_\nu(p_0, \Phi(\nu)) + \int_0^t \mathbb{T}_\nu(t-s)(\nu^{-1}f(s), \mathbf{0})\, \mathrm{d}s \quad \forall t \in I. \tag{3.24}$$

Then, making use of [9], it follows from (3.24) that $\partial_t(p, \boldsymbol{u})$ satisfies

$$\begin{cases} \partial_t \left(\partial_t(p, \boldsymbol{u})(t), (\phi, \boldsymbol{w})\right)_{X_\nu} - \left(\partial_t(p, \boldsymbol{u})(t), A^*_\nu(\phi, \boldsymbol{w})\right)_{X_\nu} = \left((\nu^{-1}f(t), \mathbf{0}), (\phi, \boldsymbol{w})\right)_{X_\nu} \\ \forall (\phi, \boldsymbol{w}) \in D(A^*_\nu) = D(A_\nu) \text{ and a.e. } t \in I \\ \partial_t(p, \boldsymbol{u})(0) = A_\nu(p_0, \Phi(\nu)), \end{cases} \tag{3.25}$$

and the mapping $t \mapsto (\partial_t(p, \boldsymbol{u}), (\phi, \boldsymbol{w}))$ is absolutely continuous for every $(\phi, \boldsymbol{w}) \in D(A_\nu)$. Here, the adjoint operator $A^*_\nu : D(A_\nu) \subset X_\nu \to X_\nu$ is given by

$$A^*_\nu(\phi, \boldsymbol{w}) = (-\nu^{-1}\eta\phi + \nu^{-1}\,\mathrm{div}\,\boldsymbol{w}, \nabla\phi) \quad \forall (\phi, \boldsymbol{w}) \in D(A_\nu). \tag{3.26}$$

Applying (3.26) and (3.10) to (3.25), we obtain by inserting $\boldsymbol{w} = \mathbf{0}$ that

$$\begin{aligned} &\partial_t \int_\Omega \nu\partial_t p(t)\phi\, \mathrm{d}x + \int_\Omega \partial_t p(t)\eta\phi - \partial_t \boldsymbol{u}(t)\cdot\nabla\phi\, \mathrm{d}x = \int_\Omega f(t)\phi\, \mathrm{d}x \\ \underbrace{\Rightarrow}_{(3.9)}\ &\partial_t \int_\Omega \nu\partial_t p(t)\phi\, \mathrm{d}x + \int_\Omega \nabla p(t)\cdot\nabla\phi + \eta\partial_t p(t)\phi\, \mathrm{d}x = \int_\Omega f(t)\phi\, \mathrm{d}x \end{aligned} \tag{3.27}$$

for all $\phi \in H^1_D(\Omega)$ and a.e. $t \in I$. Further, the initial condition in (3.25) and the definition (3.13) yield

$$\partial_t p(0) = -\nu^{-1}\eta p_0 - \nu^{-1}\,\mathrm{div}(\Phi(\nu)) \underbrace{=}_{(3.5)} -\nu^{-1}\eta p_0 + \nu^{-1}\eta p_0 + p_1 = p_1. \tag{3.28}$$

By (3.14), (3.27), and (3.28), we conclude that $p \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ is a mild solution to the forward system (1.1) in the sense of Definition 3.2. Now, for the proof of uniqueness, we assume that $\hat{p} \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ is another mild solution to (1.1). Then, the difference $\tilde{p} \coloneqq p - \hat{p}$ satisfies

$$0 = \partial_t \int_\Omega \nu \partial_t \tilde{p}(t) \tilde{p}(t) \,\mathrm{d}x + \int_\Omega \nabla \tilde{p}(t) \cdot \nabla \tilde{p}(t) + \eta \partial_t \tilde{p}(t) \tilde{p}(t) \,\mathrm{d}x \tag{3.29}$$

for a.e. $t \in I$ and $\tilde{p}(0) = \partial_t \tilde{p}(0) = 0$ a.e. in Ω. Consequently, integrating (3.29) over the time interval $[0, t]$ for an arbitrarily fixed $t \in I$ yields

$$\begin{aligned} 0 &= \int_\Omega \nu \partial_t \tilde{p}(t) \tilde{p}(t) \,\mathrm{d}x + \int_0^t \int_\Omega \nabla \tilde{p}(s) \cdot \nabla \tilde{p}(s) + \eta \partial_t \tilde{p}(s) \tilde{p}(s) \,\mathrm{d}x \,\mathrm{d}s \\ &= \frac{1}{2} \partial_t \|\sqrt{\nu} \tilde{p}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla \tilde{p}(s)\|^2_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}s + \frac{1}{2} \|\sqrt{\eta} \tilde{p}(t)\|^2_{L^2(\Omega)} \\ &\Rightarrow \frac{1}{2} \partial_t \|\sqrt{\nu} \tilde{p}(t)\|^2_{L^2(\Omega)} \leq 0, \end{aligned}$$

and so, together with $\tilde{p}(0) = 0$ and $\nu(x) > \nu_{\min} > 0$ a.e. in Ω, it follows that $\tilde{p}(t) = 0$. □

Corollary 3.6. *Let Assumption 3.1 be satisfied. Furthermore, let $f \in W^{1,1}(I, L^2(\Omega))$, $p_1 \in H^1_D(\Omega)$, and $p_0 \in D(\Delta_{D,N})$. Then, for every $\nu \in \mathcal{V}$, the first component p of (3.23) satisfies $p \in C^2(I, L^2(\Omega)) \cap C^1(I, H^1_D(\Omega)) \cap C(I, D(\Delta_{D,N}))$ and is the unique solution to the second-order wave equation*

$$\begin{cases} \nu \partial_{tt} p - \Delta p + \eta \partial_t p = f & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ (p, \partial_t p)(0) = (p_0, p_1) & \text{in } \Omega. \end{cases} \tag{3.30}$$

Proof. Due to Theorem 3.5, p is the first component of the unique solution $(p, \boldsymbol{u}) \in C^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^1(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ to (3.9). According to the second line in (3.9), it holds that

$$\nabla p = -\partial_t \boldsymbol{u} \in C(I, \boldsymbol{H}_N(\mathrm{div}, \Omega)) \quad \Rightarrow \quad p \in C(I, D(\Delta_{N,D})).$$

Furthermore, by time derivation of the first line in (3.9), we obtain that

$$\nu \partial_t^2 p - \mathrm{div}(\partial_t \boldsymbol{u}) + \eta \partial_t p = \partial_t F \quad \text{in } I \times \Omega.$$

Making use of the second line in (3.9) and the definition of F (see (3.8)), we obtain the first line in (3.30). The last line in (3.30) follows since p is the unique mild solution to (1.1) in the sense of Definition 3.2 due to Theorem 3.5. The uniqueness of the mild solution to (3.30) (see Theorem 3.5) implies the uniqueness of the classical solution to (3.30). □

3.2 Existence of Optimal Solutions

Let us introduce the solution operator associated with the first-order system (3.9) by

$$S \colon \mathcal{V} \to C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega)), \quad \nu \mapsto (p, \boldsymbol{u}), \tag{3.31}$$

that assigns to every parameter $\nu \in \mathcal{V}$ the unique solution $(p, \boldsymbol{u}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ to (3.9). Denoting the first component of the above mapping by

$$S_p: \mathcal{V} \to C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega)), \quad \nu \mapsto p,$$

Theorem 3.5 allows us to formulate the mild (weak) form of the optimal control problem (1.2) as

$$\min_{\nu \in \mathcal{V}_{ad}} J(\nu) := \mathcal{J}(\nu, S_p(\nu)) = \frac{1}{2} \sum_{i=1}^{m} \int_I \int_\Omega a_i (S_p(\nu) - p_i^{ob})^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{\lambda}{2} \|\nu\|^2_{L^2(\Omega)}. \tag{P}$$

Theorem 3.7. *Let Assumption 3.1 hold. Then, the minimization problem* (P) *admits a solution* $\overline{\nu} \in \mathcal{V}_{ad}$.

Proof. As the functional J is positive, there exists a sequence $\{\nu_n\}_{n=1}^\infty \subset \mathcal{V}_{ad}$ such that

$$\lim_{n \to \infty} J(\nu_n) = \inf_{\nu \in \mathcal{V}_{ad}} J(\nu) \geq 0.$$

Moreover, $\mathcal{V}_{ad}$ is bounded, closed, and convex in $L^2(\Omega)$, and consequently $\mathcal{V}_{ad}$ is weakly compact in $L^2(\Omega)$. Therefore, $\{\nu_n\}_{n=1}^\infty$ has a subsequence, still denoted by $\{\nu_n\}_{n=1}^\infty$, such that

$$\nu_n \rightharpoonup \overline{\nu} \text{ weakly in } L^2(\Omega) \text{ as } n \to \infty \tag{3.32}$$

for some $\overline{\nu} \in \mathcal{V}_{ad}$. For every $n \in \mathbb{N}$, let us set $(p_n, \boldsymbol{u}_n) := S(\nu_n)$. By virtue of Lemma 3.4, it holds

$$\|(p_n, \boldsymbol{u}_n)\|_{L^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} \underbrace{\leq}_{(3.20)} \hat{c}(\|(p_0, \Phi(\nu_n))\|_{L^2(\Omega) \times \boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1} \|F\|_{L^1(I, L^2(\Omega))}) \tag{3.33}$$

with $\hat{c} := \sqrt{T} \frac{\max\{\sqrt{\nu_{\max}}, 1\}}{\min\{\sqrt{\nu_{\min}}, 1\}}$ for every $n \in \mathbb{N}$. Thanks to the boundedness of Φ and the definition of $\mathcal{V}_{ad}$ (see (3.7) and (3.1)), (3.33) implies that the sequences $\{p_n\}_{n=1}^\infty \subset L^2(I, L^2(\Omega))$ and $\{\boldsymbol{u}_n\}_{n=1}^\infty \subset L^2(I, \boldsymbol{L}^2(\Omega))$ are bounded. Furthermore, from (3.13) and (3.5), we derive

$$\begin{aligned} A_{\nu_n}(p_0, \Phi(\nu_n)) &= -(\nu_n^{-1} \eta p_0 + \nu_n^{-1} \mathrm{div}(\Phi(\nu_n)), \nabla p_0) \\ &= -(\nu_n^{-1} \eta p_0 + \nu_n^{-1}(-\eta p_0 - \nu_n p_1), \nabla p_0) = (p_1, -\nabla p_0). \end{aligned} \tag{3.34}$$

In view of the above identity, Lemma 3.4 along with $G(0) = \nu^{-1} F(0) = 0$ (see (3.8)) yields for every $n \in \mathbb{N}$ that

$$\begin{aligned} \|\partial_t (p_n, \boldsymbol{u}_n)\|_{L^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} &\underbrace{\leq}_{(3.21)} \hat{c}\left(\|A_{\nu_n}(p_0, \Phi(\nu_n))\|_{L^2(\Omega) \times \boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1} \|f\|_{L^1(I, L^2(\Omega))}\right) \\ &\underbrace{=}_{(3.34)} \hat{c}\left(\|(p_1, -\nabla p_0)\|_{L^2(\Omega) \times \boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1} \|f\|_{L^1(I, L^2(\Omega))}\right). \end{aligned} \tag{3.35}$$

Thus, $\{\partial_t p_n\}_{n=1}^\infty \subset L^2(I, L^2(\Omega))$ and $\{\partial_t \boldsymbol{u}_n\}_{n=1}^\infty \subset L^2(I, \boldsymbol{L}^2(\Omega))$ are bounded. As $(p_n, \boldsymbol{u}_n)$ solves the Cauchy problem (3.14) associated with ν_n, respectively for every $n \in \mathbb{N}$, it follows that $\{\mathrm{div}\, \boldsymbol{u}_n\}_{n=1}^\infty \subset L^2(I, L^2(\Omega))$ and $\{\nabla p_n\}_{n=1}^\infty \subset L^2(I, \boldsymbol{L}^2(\Omega))$ are bounded as well. Altogether, introducing the Hilbert spaces

$$W_p := H^1(I, L^2(\Omega)) \cap L^2(I, H^1_D(\Omega)) \tag{3.36}$$

$$\boldsymbol{W}_u \coloneqq H^1(I, \boldsymbol{L}^2(\Omega)) \cap L^2(I, \boldsymbol{H}_N(\mathrm{div}, \Omega)),$$

we can choose subsequences (still denoted by the same symbol) such that

$$p_n \rightharpoonup \overline{p} \text{ weakly in } W_p \text{ as } n \to \infty \tag{3.37}$$

$$\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}} \text{ weakly in } \boldsymbol{W}_u \text{ as } n \to \infty \tag{3.38}$$

for some $\overline{p} \in W_p$ and some $\overline{\boldsymbol{u}} \in \boldsymbol{W}_u$. By the Aubin–Lions lemma, the embedding $W_p \hookrightarrow L^2(I, L^2(\Omega))$ is compact, and so

$$p_n \to \overline{p} \text{ in } L^2(I, L^2(\Omega)) \text{ as } n \to \infty. \tag{3.39}$$

Let us now prove that $\{\nu_n \partial_t p_n\}_{n=1}^\infty$ converges weakly in $L^2(I, L^2(\Omega))$ to $\overline{\nu}\partial_t\overline{p}$. By (3.32), (3.39), and since $\{\nu_n\}_{n=1}^\infty$ is bounded in $L^\infty(\Omega)$, we obtain the weak convergence of the product $\{p_n\nu_n\}_{n=1}^\infty$ to $\overline{p}\,\overline{\nu}$ in $L^2(I, L^2(\Omega))$. This yields for every $\varphi \in C_0^\infty(I, L^2(\Omega))$ that

$$\begin{aligned}\int_I (\nu_n \partial_t p_n(t), \varphi(t))_{L^2(\Omega)}\,\mathrm{d}t &= -\int_I (\nu_n p_n(t), \partial_t \varphi(t))_{L^2(\Omega)}\,\mathrm{d}t \\ &\to -\int_I (\overline{\nu}\,\overline{p}(t), \partial_t\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t = \int_I (\overline{\nu}\partial_t\overline{p}(t), \varphi(t))_{L^2(\Omega)}\,\mathrm{d}t \text{ as } n \to \infty,\end{aligned} \tag{3.40}$$

where we used the integration by parts formula for $H^1(I, L^2(\Omega))$-functions [36, Proposition 2.2.34]. Now, let $\tilde{\epsilon} > 0$ and $v \in L^2(I, L^2(\Omega))$. Due to the boundedness of $\{\nu_n\partial_t p_n\}_{n=1}^\infty$, it holds that

$$c \coloneqq \sup_{n\in\mathbb{N}} \|\nu_n\partial_t p_n - \overline{\nu}\partial_t\overline{p}\|_{L^2(I,L^2(\Omega))} < \infty. \tag{3.41}$$

As $C_0^\infty(I, L^2(\Omega))$ is dense in $L^2(I, L^2(\Omega))$, we find $\varphi_{\tilde{\epsilon}} \in C_0^\infty(I, L^2(\Omega))$ such that

$$\|v - \varphi_{\tilde{\epsilon}}\|_{L^2(I,L^2(\Omega))} < \frac{\tilde{\epsilon}}{2}c^{-1}.$$

On the other hand, (3.40) implies the existence of $n_0 \in \mathbb{N}$ such that

$$\left|\int_I (\nu_n\partial_t p_n(t) - \overline{\nu}\partial_t\overline{p}(t), \varphi_{\tilde{\epsilon}}(t))_{L^2(\Omega)}\,\mathrm{d}t\right| < \frac{\tilde{\epsilon}}{2} \quad \forall n \geq n_0.$$

Combining these two inequalities leads to

$$\begin{aligned}&\left|\int_I (\nu_n\partial_t p_n(t) - \overline{\nu}\partial_t\overline{p}(t), v(t))_{L^2(\Omega)}\,\mathrm{d}t\right| \\ &\underbrace{\leq}_{(3.41)} c\|v - \varphi_{\tilde{\epsilon}}\|_{L^2(I,L^2(\Omega))} + \left|\int_I (\nu_n\partial_t p_n(t) - \overline{\nu}\partial_t\overline{p}(t), \varphi_{\tilde{\epsilon}}(t))_{L^2(\Omega)}\,\mathrm{d}t\right| < \tilde{\epsilon} \;\; \forall n \geq n_0.\end{aligned}$$

Therefore, $\{\nu_n\partial_t p_n\}_{n=1}^\infty$ converges weakly to $\overline{\nu}\partial_t\overline{p}$ in $L^2(I, L^2(\Omega))$, and hence, together with (3.37)-(3.38), it follows that

$$\begin{aligned}\nu_n\partial_t p_n + \mathrm{div}\,\boldsymbol{u}_n + \eta p_n &\rightharpoonup \overline{\nu}\partial_t\overline{p} + \mathrm{div}\,\overline{\boldsymbol{u}} + \eta\overline{p} && \text{weakly in } L^2(I, L^2(\Omega)) \\ \partial_t\boldsymbol{u}_n + \nabla p_n &\rightharpoonup \partial_t\overline{\boldsymbol{u}} + \nabla\overline{p} && \text{weakly in } L^2(I, L^2(\Omega))\end{aligned}$$

as $n \to \infty$. In conclusion, it holds that

$$\begin{cases} \nu\partial_t\overline{p} + \operatorname{div}\overline{\boldsymbol{u}} + \eta\overline{p} = F & \text{in } I \times \Omega \\ \partial_t\overline{\boldsymbol{u}} + \nabla\overline{p} = \mathbf{0} & \text{in } I \times \Omega. \end{cases}$$

It remains to prove that $(\overline{p}, \overline{\boldsymbol{u}})$ satisfies the desired initial value conditions. We choose an arbitrarily fixed $w \in L^2(\Omega)$ and some $\xi \in C^\infty(I)$ such that $\xi(T) = 0$ and $\xi(0) = 1$. Then, we define $[w\xi] \in C^\infty(I, L^2(\Omega))$ by $[w\xi](t)(x) = w(x)\xi(t)$ for all $t \in I$ and a.e. $x \in \Omega$. By the integration by parts formula, we obtain that

$$L_n \coloneqq \int_I (p_n(t), w)_{L^2(\Omega)}\partial_t\xi(t)\,\mathrm{d}t = -\int_I (\partial_t p_n(t), w)_{L^2(\Omega)}\xi(t)\,\mathrm{d}t - (p_0, w)_{L^2(\Omega)} \eqqcolon R_n.$$

In view of (3.37), it holds that

$$\begin{aligned} \lim_{n\to\infty} L_n &= \int_I (\overline{p}(t), w)_{L^2(\Omega)}\partial_t\xi(t)\,\mathrm{d}t = -\int_I (\partial_t\overline{p}(t), w)_{L^2(\Omega)}\xi(t)\,\mathrm{d}t - (\overline{p}(0), w)_{L^2(\Omega)} \\ \lim_{n\to\infty} R_n &= -\int_I (\partial_t\overline{p}(t), w)_{L^2(\Omega)}\xi(t)\,\mathrm{d}t - (p_0, w)_{L^2(\Omega)}. \end{aligned}$$

As a consequence,

$$(\overline{p}(0), w)_{L^2(\Omega)} = (p_0, w)_{L^2(\Omega)} \quad \forall w \in L^2(\Omega) \quad \Rightarrow \quad \overline{p}(0) = p_0.$$

Similarly, we show $\overline{\boldsymbol{u}}(0) = \Phi(\nu)$. Therefore, $(\overline{p}, \overline{\boldsymbol{u}}) \in W_p \times W_u$ is the (strong) solution to (3.9) associated with $\overline{\nu}$. By well-known results (cf. [66, section 4.2]), the strong solution is also a mild solution, which is uniquely given by the formula (3.23). Thus, $(\overline{p}, \overline{\boldsymbol{u}})$ coincides with unique (classical) solution to (3.9) according to Theorem 3.5, i.e., $(\overline{p}, \overline{\boldsymbol{u}}) = S(\overline{\nu}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\operatorname{div}, \Omega))$ and $\overline{p} = S_p(\overline{\nu})$. Finally, as the objective function $\mathcal{J} : L^2(\Omega) \times L^2(I, L^2(\Omega)) \to \mathbb{R}$ is continuous and convex, the weak convergence properties (3.32) and (3.37) imply that $\overline{\nu}$ is a solution to (P). □

3.3 First-Order Necessary Optimality Conditions

This section develops an adjoint technique via (3.9) and eventually first-order necessary optimality conditions for (P) based on the auxiliary system (3.9). By the explicit use of the dual semigroup $\{\mathbb{T}^*_\nu(t)\}_{t\geq 0}$ of $\{\mathbb{T}_\nu(t)\}_{t\geq 0}$ from Lemma 3.3, we propose the notion of adjoint states for (P) as follows:

Definition 3.8. *Let Assumption 3.1 hold. Further, let $\nu \in \mathcal{V}$, and let $\{\mathbb{T}_\nu(t)\}_{t\geq 0}$ denote the contraction semigroup generated by A_ν according to Lemma 3.3. For every $t \geq 0$, let $\mathbb{T}^*_\nu(t) : X_\nu \to X_\nu$ denote the adjoint operator associated with $\mathbb{T}_\nu(t)$, respectively. Then, the function $(q, \boldsymbol{v}) \in C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))$, defined by*

$$(q, \boldsymbol{v})(t) \coloneqq \sum_{i=1}^m \int_t^T \mathbb{T}^*_\nu(s - t)(\nu^{-1}a_i(s)(p_i^{ob}(s) - S_p(\nu)(s)), \mathbf{0})\,\mathrm{d}s \tag{3.42}$$

for all $t \in I$, is called the adjoint state associated with ν.

Recalling from (3.26), for a given $\nu \in \mathcal{V}$, the adjoint operator of $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$ is given by

$$A_\nu^* : D(A_\nu^*) = D(A_\nu) \subset X_\nu \to X_\nu, \quad (\phi, \boldsymbol{w}) \mapsto (-\nu^{-1}\eta\phi + \nu^{-1}\operatorname{div}\boldsymbol{w}, \nabla\phi).$$

Similarly to Lemma 3.3, the adjoint operator $A_\nu^* : D(A_\nu) \subset X_\nu \to X_\nu$ is dissipative and fulfils $R(I - A_\nu^*) = X_\nu$, and so the Lumer–Phillips theorem (see Theorem 2.3) implies that A_ν^* generates a contraction semigroup $\{\mathbb{S}_\nu(t)\}_{t\geq 0}$. For this reason, a well-known result [32, chapter II, section 2.5] implies that $\mathbb{T}_\nu^*(t) = \mathbb{S}_\nu(t)$ holds for all $t \geq 0$. In other words, the adjoint operator A_ν^* is precisely the infinitesimal generator of the contraction semigroup $\{\mathbb{T}_\nu^*(t)\}_{t\geq 0}$. Introducing

$$g \in L^1(I, L^2(\Omega)), \quad g(t) \coloneqq \sum_{i=1}^{m} a_i(t)(p_i^{ob}(t) - S_p(\nu)(t)) \quad \forall t \in I, \tag{3.43}$$

we note that $(\tilde{q}, \tilde{\boldsymbol{v}}) \coloneqq (q, \boldsymbol{v})(T - \cdot)$, where $(q, \boldsymbol{v})$ is defined by (3.42), satisfies

$$(\tilde{q}, \tilde{\boldsymbol{v}})(t) = \int_{T-t}^{T} \mathbb{T}_\nu^*(s - T + t)(\nu^{-1}g(s))\,\mathrm{d}s = \int_0^t \mathbb{T}_\nu^*(t - s)(\nu^{-1}g(T - s))\,\mathrm{d}s \quad \forall t \in I.$$

In other words, $(\tilde{q}, \tilde{\boldsymbol{v}})$ is the mild solution to

$$\begin{cases} \partial_t(\tilde{q}, \tilde{\boldsymbol{v}})(t) - A_\nu^*(\tilde{q}, \tilde{\boldsymbol{v}})(t) = (\nu^{-1}g(T - t), \mathbf{0}) & \forall t \in I \\ (\tilde{q}, \tilde{\boldsymbol{v}})(0) = (0, \mathbf{0}). \end{cases} \tag{3.44}$$

Therefore, by the time transformation, the strong PDE formulation for $(q, \boldsymbol{v})$ is obtained as follows:

$$\begin{cases} \partial_t(q, \boldsymbol{v})(t) + A_\nu^*(q, \boldsymbol{v})(t) = (-\nu^{-1}g(t), \mathbf{0}) & \forall t \in I \\ (q, \boldsymbol{v})(T) = (0, \mathbf{0}), \end{cases} \tag{3.45}$$

that is nothing but

$$\begin{cases} \nu\partial_t q + \operatorname{div}\boldsymbol{v} - \eta q = \sum_{i=1}^{m} a_i(S_p(\nu) - p_i^{ob}) & \text{in } I \times \Omega \\ \partial_t \boldsymbol{v} + \nabla q = \mathbf{0} & \text{in } I \times \Omega \\ q = 0 & \text{on } I \times \Gamma_D \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (q, \boldsymbol{v})(T) = (0, \mathbf{0}) & \text{in } \Omega. \end{cases} \tag{3.46}$$

Our goal now is to deduce necessary optimality conditions for (P) based on Definition 3.8. To this aim, let us first examine the continuity and differentiability properties for the solution operator $S : L^\infty(\Omega) \supset \mathcal{V} \to C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))$.

Lemma 3.9. *Let Assumption 3.1 hold. Then, the solution operator* $S : L^\infty(\Omega) \supset \mathcal{V} \to C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))$ *is continuous.*

Proof. The claim is obtained by following the argumentation in [53, Theorem 3.5]. Let $\nu \in \mathcal{V}$ and $h \in L^\infty(\Omega)$ such that $\nu + h \in \mathcal{V}$. Further, let $(p_h, \boldsymbol{u}_h) \coloneqq S(\nu + h)$ and $(p, \boldsymbol{u}) \coloneqq S(\nu)$.

As $Q \coloneqq D(\Delta_{D,N}) \times H^1_D(\Omega) \times \{F \in W^{2,1}(I, L^2(\Omega)) : F(0) = 0\}$ is dense in $H^1_D(\Omega) \times L^2(\Omega) \times \{F \in W^{1,1}(I, L^2(\Omega)) : F(0) = 0\}$, there exists a sequence $\{p_0^n, p_1^n, F^n\}_{n=1}^{\infty} \subset Q$ such that

$$(p_0^n, p_1^n, F^n) \to (p_0, p_1, F) \quad \text{in } H^1(\Omega) \times L^2(\Omega) \times W^{1,1}(I, L^2(\Omega)) \text{ as } n \to \infty. \tag{3.47}$$

Now, let $n \in \mathbb{N}$ be arbitrarily fixed. With $(p_h^n, \boldsymbol{u}_h^n) \in C^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^1(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\text{div}, \Omega))$, we denote the solution to (3.9) associated with the parameter $\nu + h$, the initial value $(p_0^n, \Phi^n(\nu + h))$, and the source term F^n. With $(p^n, \boldsymbol{u}^n) \in C^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^1(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\text{div}, \Omega))$, we denote the solution to (3.9) associated with the parameter ν, the initial value $(p_0^n, \Phi^n(\nu))$, and the source term F^n. Here, Φ^n is defined analogously to (3.6), but we replace (p_0, p_1) with (p_0^n, p_1^n). Now, let $\epsilon > 0$. Then, applying Lemma 3.4 to the difference of the corresponding Cauchy problems of $(p_h^n, \boldsymbol{u}_h^n)$ and $(p_h, \boldsymbol{u}_h)$, it follows that

$$\begin{aligned}
&\|(p_h^n - p_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h)\|_{C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} \\
&\le c\big(\|(p_0^n - p_0, \Phi^n(\nu + h) - \Phi(\nu + h))\|_{L^2(\Omega) \times \boldsymbol{L}^2(\Omega)} \\
&\qquad + \|A_{\nu+h}(p_0^n - p_0, \Phi^n(\nu + h) - \Phi(\nu + h))\|_{L^2(\Omega) \times \boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|F^n - F\|_{W^{1,1}(I, L^2(\Omega))}\big) \\
&\le c\big(\|p_0^n - p_0\|_{L^2(\Omega)} + \|\Phi^n(\nu + h) - \Phi(\nu + h)\|_{\boldsymbol{L}^2(\Omega)} + \|p_1^n - p_1\|_{L^2(\Omega)} + \|\nabla(p_0^n - p_0)\|_{\boldsymbol{L}^2(\Omega)} \\
&\qquad + \nu_{\min}^{-1}\|F^n - F\|_{W^{1,1}(I, L^2(\Omega))}\big),
\end{aligned} \tag{3.48}$$

where $c \coloneqq \frac{\max\{\sqrt{\nu_{\max}}, 1\}}{\min\{\sqrt{\nu_{\min}}, 1\}}$. Due to the definition (3.6) of Φ and Φ^n (with (p_0^n, p_1^n) instead of (p_0, p_1)) and the generalized Poincaré inequality, it holds that

$$\|\Phi^n(\nu+h) - \Phi(\nu+h)\|_{\boldsymbol{L}^2(\Omega)} \le c_P\big(\|\eta\|_{L^\infty(\Omega)}\|p_0^n - p_0\|_{L^2(\Omega)} + (\|\nu\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)})\|p_1^n - p_1\|_{L^2(\Omega)}\big). \tag{3.49}$$

Therefore, by (3.47), (3.48), and (3.49), we obtain the existence of an $n_0 \in \mathbb{N}$ such that

$$\|(p_h^{n_0} - p_h, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}_h)\|_{C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} < \frac{\epsilon}{3} \tag{3.50}$$

for every $h \in L^\infty(\Omega)$ such that $\nu + h \in \mathcal{V}$ and $\|h\|_{L^\infty(\Omega)} < 1$. Analogously, eventually by increasing n_0,

$$\|(p^{n_0} - p, \boldsymbol{u}^{n_0} - \boldsymbol{u})\|_{C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} < \frac{\epsilon}{3}. \tag{3.51}$$

Further, $(p_h^{n_0} - p^{n_0}, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0})$ satisfies

$$\begin{cases}
\nu\partial_t(p_h^{n_0} - p^{n_0}) + \text{div}(\boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0}) + \eta(p_h^{n_0} - p^{n_0}) = -h\partial_t p_h^{n_0} & \text{in } I \times \Omega \\
\partial_t(\boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0}) + \nabla(p_h^{n_0} - p^{n_0}) = \boldsymbol{0} & \text{in } I \times \Omega \\
p_h^{n_0} - p^{n_0} = 0 & \text{on } I \times \Gamma_D \\
(\boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0}) \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\
(p_h^{n_0} - p^{n_0}, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0})(0) = (0, \Phi^{n_0}(\nu + h) - \Phi^{n_0}(\nu)) & \text{in } \Omega.
\end{cases} \tag{3.52}$$

Applying Lemma 3.4 to (3.52) yields that

$$\begin{aligned}
&\|(p_h^{n_0} - p^{n_0}, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0})\|_{C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} \\
&\le c\big(\|\Phi^{n_0}(\nu + h) - \Phi^{n_0}(\nu)\|_{\boldsymbol{L}^2(\Omega)}
\end{aligned} \tag{3.53}$$

$$+ \|A_\nu(0, \Phi^{n_0}(\nu+h) - \Phi^{n_0}(\nu)) - (\nu^{-1} h \partial_t p_h^{n_0}(0), \mathbf{0})\|_{L^2(\Omega)\times \boldsymbol{L}^2(\Omega)}$$
$$+ \nu_{\min}^{-1} \|h\|_{L^\infty(\Omega)} \|\partial_t p_h^{n_0}\|_{W^{1,1}(I,L^2(\Omega))}).$$

Due to the property (3.7) of Φ^{n_0} (with $p_1^{n_0}$ instead of p_1), it holds that

$$\|\Phi^{n_0}(\nu+h) - \Phi^{n_0}(\nu)\|_{\boldsymbol{L}^2(\Omega)} \le c_P \|h\|_{L^\infty(\Omega)} \|p_1^{n_0}\|_{L^2(\Omega)} \tag{3.54}$$

for a Poincaré constant $c_P > 0$. As in the proof of Theorem 3.5, we have that $\partial_t p_h^{n_0}(0) = p_1^{n_0}$ a.e. in Ω. By the definition of A_ν (see (3.13)) and the definition of Φ^{n_0} (see (3.6) with $(p_0^{n_0}, p_1^{n_0})$ instead of (p_0, p_1)), this yields

$$A_\nu(0, \Phi^{n_0}(\nu+h) - \Phi^{n_0}(\nu)) - (\nu^{-1} h \partial_t p_h^{n_0}(0), \mathbf{0}) = (\nu^{-1} h p_1^{n_0} - \nu^{-1} h \partial_t p_h^{n_0}(0), \mathbf{0}) = (0, \mathbf{0}). \tag{3.55}$$

Furthermore, by Lemma 3.4, it holds that

$$\begin{aligned} \|\partial_t p_h^{n_0}\|_{L^1(I,L^2(\Omega))} &\le Tc(\|A_{\nu+h}(p_0^{n_0}, \Phi^{n_0}(\nu+h))\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|(\nu+h)^{-1}\partial_t F^{n_0}\|_{L^1(I,L^2(\Omega))}) \\ &\le Tc(\|(p_1^{n_0}, -\nabla p_0^{n_0})\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|\partial_t F^{n_0}\|_{L^1(I,L^2(\Omega))}) \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} &\|\partial_t^2 p_h^{n_0}\|_{L^1(I,L^2(\Omega))} \\ &\le Tc(\|A_{\nu+h}(p_1^{n_0}, -\nabla p_0^{n_0}) + ((\nu+h)^{-1}\partial_t F^{n_0}(0), \mathbf{0})\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|(\nu+h)^{-1}\partial_t^2 F^n\|_{L^1(I,L^2(\Omega))}) \\ &\le Tc(\|((\nu+h)^{-1}(\Delta p_0^{n_0} - \eta p_1^{n_0}), -\nabla p_1^{n_0}) + ((\nu+h)^{-1}\partial_t F^{n_0}(0), \mathbf{0})\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\ &\qquad + \nu_{\min}^{-1}\|\partial_t^2 F^n\|_{L^1(I,L^2(\Omega))}) \\ &\le Tc(\nu_{\min}^{-1}\|\Delta p_0^{n_0} - \eta p_1^{n_0}\|_{L^2(\Omega)} + \|\nabla p_1^{n_0}\|_{\boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|\partial_t F^{n_0}(0)\|_{L^2(\Omega)} + \nu_{\min}^{-1}\|\partial_t^2 F^n\|_{L^1(I,L^2(\Omega))}). \end{aligned} \tag{3.57}$$

Since the right-hand sides in (3.56) and (3.57) are independent of $h > 0$, there exists a constat $\tilde{c} > 0$ such that

$$\|\partial_t p_h^{n_0}\|_{W^{1,1}(I,L^2(\Omega))} \le \tilde{c} \tag{3.58}$$

for all $h \in L^\infty(\Omega)$ such that $\nu + h \in \mathcal{V}$. Thus, (3.53) to (3.55) and (3.58) yield that

$$\|(p_h^{n_0} - p^{n_0}, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0})\|_{C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} < c(c_P\|p_1^{n_0}\|_{L^2(\Omega)} + \nu_{\min}^{-1}\tilde{c})\|h\|_{L^\infty(\Omega)}.$$

Then, defining $\delta \coloneqq \min\{\epsilon(3c(c_P(\|p_1^{n_0}\|_{L^2(\Omega)}) + \nu_{\min}^{-1}\tilde{c}))^{-1}, 1\}$, it holds that

$$\|(p_h^{n_0} - p^{n_0}, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0})\|_{C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} < \frac{\epsilon}{3} \tag{3.59}$$

for all $h \in L^\infty(\Omega)$ such that $\|h\|_{L^\infty(\Omega)} < \delta$ and $\nu + h \in \mathcal{V}$. Together with (3.50), (3.51), and (3.59), we conclude that

$$\begin{aligned} &\|(p_h - p, \boldsymbol{u}_h - \boldsymbol{u})\|_{C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \le \|(p_h - p_h^{n_0}, \boldsymbol{u}_h - \boldsymbol{u}_h^{n_0})\|_{C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \\ &+ \|(p_h^{n_0} - p^{n_0}, \boldsymbol{u}_h^{n_0} - \boldsymbol{u}^{n_0})\|_{C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} + \|(p^{n_0} - p, \boldsymbol{u}^{n_0} - \boldsymbol{u})\|_{C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} < \epsilon \end{aligned}$$

for all $h \in L^\infty(\Omega)$ with $\|h\|_{L^\infty(\Omega)} < \delta$ and $\nu + h \in \mathcal{V}$. □

Lemma 3.10. *Let Assumption 3.1 hold. Then, the solution operator* $S : L^\infty(\Omega) \supset \mathcal{V} \to C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))$ *is Fréchet differentiable. For all* $\nu \in \mathcal{V}$ *and* $h \in L^\infty(\Omega)$*, it holds that*

$$[S'(\nu)h](t) = \mathbb{T}_\nu(t)(0, \Phi'(\nu)h) - \int_0^t \mathbb{T}_\nu(t-s)(\nu^{-1} h \partial_t S_p(\nu)(s), \mathbf{0})\,\mathrm{d}s \quad \forall t \in I, \tag{3.60}$$

where $\{\mathbb{T}_\nu(t)\}_{t\geq 0}$ *denotes the contraction semigroup generated by* A_ν*.*

Proof. Let $\nu \in \mathcal{V}$ and $h \in L^\infty(\Omega)$ such that $\nu + h \in \mathcal{V}$, and we define $S'(\nu)h$ as in (3.60). Firstly, note that the mapping $h \mapsto S'(\nu)h$ is linear and continuous. Further, by subtracting the corresponding Cauchy problems of $(p_h, \boldsymbol{u}_h) \coloneqq S(\nu + h)$ and $(p, \boldsymbol{u}) \coloneqq S(\nu)$, the difference $(p_h - p, \boldsymbol{u}_h - \boldsymbol{u}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ satisfies

$$\begin{cases} \nu \partial_t (p_h - p) + \mathrm{div}(\boldsymbol{u}_h - \boldsymbol{u}) + \eta(p_h - p) = -h \partial_t p_h & \text{in } I \times \Omega \\ \partial_t(\boldsymbol{u}_h - \boldsymbol{u}) + \nabla(p_h - p) = \mathbf{0} & \text{in } I \times \Omega \\ p_h - p = 0 & \text{on } I \times \Gamma_D \\ (\boldsymbol{u}_h - \boldsymbol{u}) \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (p_h - p, \boldsymbol{u}_h - \boldsymbol{u})(0) = (0, \Phi(\nu + h) - \Phi(\nu)) & \text{in } \Omega. \end{cases}$$

Therefore, $S(\nu + h) - S(\nu)$ reads as

$$\begin{aligned} &S(\nu + h)(t) - S(\nu)(t) \\ &= \mathbb{T}_\nu(0, \Phi(\nu + h) - \Phi(\nu)) - \int_0^t \mathbb{T}_\nu(t-s)(\nu^{-1} h \partial_t S_p(\nu + h)(s), \mathbf{0})\,\mathrm{d}s \quad \forall t \in I. \end{aligned} \tag{3.61}$$

Together with the affine linearity of Φ, it follows that

$$\begin{aligned} &\frac{1}{\|h\|_{L^\infty(\Omega)}} \|S(\nu + h)(t) - S(\nu)(t) - [S'(\nu)h](t)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\ &\underbrace{=}_{(3.60),(3.61)} \frac{1}{\|h\|_{L^\infty(\Omega)}} \left\| \int_0^t \mathbb{T}_\nu(t-s)(\nu^{-1} h(\partial_t S_p(\nu + h)(s) - \partial_t S_p(\nu)(s)), \mathbf{0})\,\mathrm{d}s \right\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\ &\leq T c \nu_{\min}^{-1} \|\partial_t S_p(\nu + h) - \partial_t S_p(\nu)\|_{C(I, L^2(\Omega))} \quad \forall t \in I, \end{aligned}$$

where $c \coloneqq \frac{\max\{\sqrt{\nu_{\max}}, 1\}}{\min\{\sqrt{\nu_{\min}}, 1\}}$. Thus, applying Lemma 3.9, the assertion holds. ☐

Remark 3.11. In view of Lemma 3.10 along with Lemma 3.4, the Fréchet derivative $S'(\nu)h = (\tilde{p}, \tilde{\boldsymbol{u}})$ is the mild solution to

$$\begin{cases} \nu \partial_t \tilde{p} + \mathrm{div}\,\tilde{\boldsymbol{u}} + \eta \tilde{p} = -h \partial_t S_p(\nu) & \text{in } I \times \Omega \\ \partial_t \tilde{\boldsymbol{u}} + \nabla \tilde{p} = \mathbf{0} & \text{in } I \times \Omega \\ \tilde{p} = 0 & \text{on } I \times \Gamma_D \\ \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\tilde{p}, \tilde{\boldsymbol{u}})(0) = (0, \Phi'(\nu)h) & \text{in } \Omega. \end{cases}$$

Theorem 3.12. *Let Assumption 3.1 hold. Further, let* $\overline{\nu} \in \mathcal{V}_{ad}$ *be a minimizer to* (P)*, and let* $(\overline{q}, \overline{\boldsymbol{v}})$ *be the adjoint state associated with* $\overline{\nu}$ *given by Definition 3.8. Then, the following variational inequality holds:*

$$\left(\int_I (\partial_t S_p(\overline{\nu})(t) - p_1)\overline{q}(t)\,\mathrm{d}t + \lambda \overline{\nu}, \nu - \overline{\nu} \right)_{L^2(\Omega)} \geq 0 \quad \forall \nu \in \mathcal{V}_{ad}. \tag{3.62}$$

Proof. By standard arguments, we only need to show that (3.62) is equivalent to

$$J'(\overline{\nu})(\nu - \overline{\nu}) \geq 0 \quad \forall \nu \in \mathcal{V}_{ad}. \tag{3.63}$$

Let $\nu \in \mathcal{V}_{ad}$ be arbitrarily fixed. Due to the Fréchet differentiability of S (Lemma 3.10), we have

$$\begin{aligned} &J'(\overline{\nu})(\nu - \overline{\nu}) \\ &= \sum_{i=1}^{m} \int_I ((\overline{\nu}^{-1} a_i(t)(S_p(\overline{\nu})(t) - p_i^{ob}(t)), \mathbf{0}), S'(\overline{\nu})(\nu - \overline{\nu})(t))_{X_{\overline{\nu}}} \,\mathrm{d}t + \lambda(\overline{\nu}, \nu - \overline{\nu})_{L^2(\Omega)}. \end{aligned} \tag{3.64}$$

By means of definition of the adjoint state $(\overline{q}, \overline{\boldsymbol{v}})$ associated with $\overline{\nu}$ (see Definition 3.8), and with the function g from (3.43), we get

$$\begin{aligned} &\sum_{i=1}^{m} \int_I ((\overline{\nu}^{-1} a_i(t)(S_p(\overline{\nu})(t) - p_i^{ob}(t), \mathbf{0}), S'(\overline{\nu})(\nu - \overline{\nu})(t))_{X_{\overline{\nu}}} \,\mathrm{d}t \\ &\underbrace{=}_{(3.43),(3.60)} - \int_I \Bigg((\overline{\nu}^{-1} g(t), \mathbf{0}), \mathbb{T}_{\overline{\nu}}(t)(0, \Phi'(\overline{\nu})(\nu - \overline{\nu})) \\ &\qquad - \int_0^t \mathbb{T}_{\overline{\nu}}(t-s)(\overline{\nu}^{-1}(\nu - \overline{\nu})\partial_t S_p(\overline{\nu})(s), \mathbf{0}) \,\mathrm{d}s \Bigg)_{X_{\overline{\nu}}} \mathrm{d}t \\ &= -\left(\int_I \mathbb{T}_{\overline{\nu}}^*(t)(\overline{\nu}^{-1} g(t), \mathbf{0}) \,\mathrm{d}t, (0, \Phi'(\overline{\nu})(\nu - \overline{\nu})) \right)_{X_{\overline{\nu}}} \\ &\quad + \int_I \left(\int_s^T \mathbb{T}_{\overline{\nu}}^*(t-s)(\overline{\nu}^{-1} g(t), \mathbf{0}) \,\mathrm{d}t, (\overline{\nu}^{-1}(\nu - \overline{\nu})\partial_t S_p(\overline{\nu})(s), \mathbf{0}) \right)_{X_{\overline{\nu}}} \mathrm{d}s \\ &\underbrace{=}_{(3.42),(3.43)} -(\overline{\boldsymbol{v}}(0), \Phi'(\overline{\nu})(\nu - \overline{\nu}))_{\boldsymbol{L}^2(\Omega)} + \int_I (\overline{q}(s), (\nu - \overline{\nu})\partial_t S_p(\overline{\nu})(s))_{L^2(\Omega)} \,\mathrm{d}s. \end{aligned} \tag{3.65}$$

As $W^{1,1}(I, L^2(\Omega))$ lies dense in $L^1(I, L^2(\Omega))$, we can choose $\{g_n\}_{n=1}^{\infty} \subset W^{1,1}(I, L^2(\Omega))$ such that

$$g_n \to g \text{ in } L^1(I, L^2(\Omega)) \text{ as } n \to \infty. \tag{3.66}$$

Now, let $\{(q_n, \boldsymbol{v}_n)\}_{n=1}^{\infty} \subset C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H_D^1(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ be defined by

$$(q_n, \boldsymbol{v}_n)(t) \coloneqq \int_t^T \mathbb{T}_{\overline{\nu}}^*(s-t)(g_n(s), \mathbf{0}) \,\mathrm{d}s \quad \forall t \in I, n \in \mathbb{N}. \tag{3.67}$$

Subtracting (3.67) from the definition of the adjoint state (3.42) implies

$$\|(q_n - \overline{q}, \boldsymbol{v}_n - \overline{\boldsymbol{v}})\|_{C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} \leq c \|g_n - g\|_{L^1(I, L^2(\Omega))} \to 0 \text{ as } n \to \infty, \tag{3.68}$$

where $c \coloneqq \frac{\max\{\sqrt{\nu_{\max}}, 1\}}{\min\{\sqrt{\nu_{\min}}, 1\}}$. Further, due to the higher regularity properties of the source terms g_n, by Lemma 3.4, the functions $(q_n, \boldsymbol{v}_n)$ solve corresponding PDEs (3.45) where g is replaced with g_n, respectively for every $n \in \mathbb{N}$. Therefore, by the property (3.5) of Φ it holds that

$$(\boldsymbol{v}_n(0), \Phi'(\overline{\nu})(\nu - \nu))_{\boldsymbol{L}^2(\Omega)} = -\left(\int_I \partial_t \boldsymbol{v}_n(t) \,\mathrm{d}t, \Phi'(\overline{\nu})(\nu - \overline{\nu}) \right)_{\boldsymbol{L}^2(\Omega)}$$

$$= \Big(\int_I \nabla q_n(t)\, \mathrm{d}t, \Phi'(\overline{\nu})(\nu - \overline{\nu}) \Big)_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(3.5),(3.6)} \int_I (q_n(t), (\nu - \overline{\nu}) p_1)_{L^2(\Omega)}\, \mathrm{d}t \quad \forall n \in \mathbb{N},$$

and together with (3.66) and (3.68) this yields

$$(\overline{\boldsymbol{v}}(0), \Phi'(\overline{\nu})(\nu - \overline{\nu}))_{\boldsymbol{L}^2(\Omega)} = \int_I (\overline{q}(t), (\nu - \overline{\nu}) p_1)_{L^2(\Omega)}\, \mathrm{d}t. \tag{3.69}$$

Applying (3.65) and (3.69) to (3.64), we conclude the equivalence of (3.62) and (3.63). □

Remark 3.13. By well-known arguments, the optimality condition (3.62) is valid if and only if the projection formula

$$\overline{\nu}(x) = \mathbb{P}_{[\nu_-(x), \nu_+(x)]} \Big[-\frac{1}{\lambda} \int_I (\partial_t S_p(\overline{\nu})(t, x) - p_1(x)) \overline{q}(t, x)\, \mathrm{d}t \Big] \tag{3.70}$$

holds true for a.e. $x \in \Omega$ (cf. [72, section 2.8]). Let us also note that $\partial_t S_p(\overline{\nu})$ and $\overline{q}$ are of class $C(I, L^2(\Omega))$. Therefore, their product is only well-defined in $C(I, L^1(\Omega))$, and we cannot extract any Sobolev regularity property for the optimal solution $\overline{\nu} \in L^\infty(\Omega)$ from the projection formula (3.70).

As mentioned above, under higher regularity assumptions, we also obtain necessary first-order conditions based on the second-order formulation of Theorem 3.12:

Corollary 3.14. *Let Assumption 3.1 hold, $f \in W^{1,1}(I, L^2(\Omega))$, $p_1 \in H^1_D(\Omega)$, and $p_0 \in D(\Delta_{D,N})$. Furthermore, let $\overline{\nu} \in \mathcal{V}_{ad}$ be an minimizer to* (P) *with corresponding solution $\overline{p} \in C^2(I, L^2(\Omega)) \cap C^1(I, H^1_D(\Omega)) \cap C(I, D(\Delta_{D,N}))$ to* (1.1) *such that $\sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) \in W^{1,1}(I, L^2(\Omega))$. Then, it holds the variational inequality*

$$\Big(\int_I (\partial_t \overline{p}(t) - p_1) \partial_t \overline{q}(t)\, \mathrm{d}t + \lambda \overline{\nu}, \nu - \overline{\nu} \Big)_{L^2(\Omega)} \geq 0 \quad \forall \nu \in \mathcal{V}_{ad}, \tag{3.71}$$

where $\overline{q} \in C^2(I, L^2(\Omega)) \cap C^1(I, H^1_D(\Omega)) \cap C(I, D(\Delta_{D,N}))$ satisfies the second-order adjoint equation

$$\begin{cases} \overline{\nu} \partial_t^2 \overline{q} - \Delta \overline{q} - \eta \partial_t \overline{q} = \sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) & \text{in } I \times \Omega \\ \partial_n \overline{q} = 0 & \text{on } I \times \Gamma_N \\ \overline{q} = 0 & \text{on } I \times \Gamma_D \\ (\overline{q}, \partial_t \overline{q})(T) = (0, 0) & \text{in } \Omega. \end{cases} \tag{3.72}$$

Remark 3.15. According to Corollary 3.6, the solution $\overline{p}$ to (3.30) associated with $\overline{\nu}$ exists, is unique, and satisfies $\overline{p} \in C^2(I, L^2(\Omega))$. Thus, the assumption $\sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) \in W^{1,1}(I, L^2(\Omega))$ is realistic for suitable a_i and p_i^{ob} for all $i = 1, \ldots, m$.

Proof. Due to Lemma 3.4, there exists a unique solution $(\hat{q}, \hat{\boldsymbol{v}}) \in C^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^1(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ to

$$\begin{cases} \overline{\nu} \partial_t \hat{q} + \mathrm{div}(\hat{\boldsymbol{v}}) + \eta \hat{q} = G & \text{in } I \times \Omega \\ \partial_t \hat{\boldsymbol{v}} + \nabla \hat{p} = 0 & \text{in } I \times \Omega \\ \hat{q} = 0 & \text{on } I \times \Gamma_D \\ \hat{\boldsymbol{v}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\hat{q}, \hat{\boldsymbol{v}})(0) = (0, \boldsymbol{0}) & \text{in } \Omega. \end{cases}$$

where $G \in W^{2,1}(I, L^2(\Omega))$ is given by $G(t) \coloneqq \int_0^t \sum_{i=1}^m a_i(T-s)(\overline{p} - p_i^{ob})(T-s)\,\mathrm{d}s$ for all $t \in I$. Similar to the proof of Corollary 3.6, it follows that $\hat{q} \in C(I, D(\Delta_{N,D}))$ and

$$\begin{cases} \overline{\nu}\partial_t^2\hat{q} - \Delta\hat{q} + \eta\partial_t\hat{q} = \partial_t G & \text{in } I \times \Omega \\ \partial_n\hat{q} = 0 & \text{on } I \times \Gamma_N \\ \hat{q} = 0 & \text{on } I \times \Gamma_D \\ (\hat{q}, \partial_t\hat{q})(0) = (0,0) & \text{in } \Omega. \end{cases}$$

Thus, $\overline{q} \coloneqq \hat{q}(T - \cdot)$ satisfies (3.72). Now, we define $q \coloneqq \partial_t\overline{q}$ and $\boldsymbol{v} \coloneqq \nabla\overline{q}$. Then, due to (3.72), $(q, \boldsymbol{v})$ is the unique solution to the first-order adjoint equation (3.46) associated with $\overline{\nu}$. Therefore, $(q, \boldsymbol{v}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H_D^1(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ coincides with the adjoint state associated with $\overline{\nu}$ in the sense of Definition 3.8 by the uniqueness of the solution to (3.46). Consequently, the variational inequality (3.62) in Theorem 3.12 implies (3.71). □

3.4 Stampacchia's Method for Hyperbolic PDEs

In order to handle the bilinear character $\nu\partial_t^2 p$ in the second-order analysis of (P), we apply Stampacchia's method to the hyperbolic case. As a preparation, in the following lemma, we present the global $L^\infty(\Omega)$-boundedness of the solution to the Poisson equation with mixed boundary conditions. Note that the proof follows closely the well-known arguments of Stampacchia's method (cf. [72, Theorem 4.5]). However, we provide the proof for the reader's convenience.

Lemma 3.16. *Let $g \in L^r(\Omega)$ for some $r > \frac{N}{2}$. Then, the weak solution $y \in H_D^1(\Omega)$ to*

$$\begin{cases} -\Delta y = g & \text{in } \Omega \\ \partial_n y = 0 & \text{on } \Gamma_N \\ y = 0 & \text{on } \Gamma_D \end{cases} \tag{3.73}$$

is in $L^\infty(\Omega)$ and satisfies

$$\|y\|_{L^\infty(\Omega)} \le \tilde{c}\|g\|_{L^r(\Omega)}$$

for a constant $\tilde{c} > 0$ depending on r, N, and Ω.

Proof. For an arbitrarily fixed $k > 0$, we define

$$v_k \colon \Omega \to \mathbb{R}, \quad x \mapsto \max\{y(x) - k, 0\} + \min\{y(x) + k, 0\} = \begin{cases} y(x) - k & \text{if } y(x) \ge k \\ 0 & \text{if } |y(x)| = k \\ y(x) + k & \text{in } y(x) \le -k. \end{cases}$$

Since $v \in H_D^1(\Omega)$ if and only $|v| \in H_D^1(\Omega)$ (cf. [31, Corollary 2.4]), and $\max\{v, 0\} = \frac{|v|+v}{2}$ for every $v \in H_D^1(\Omega)$, it holds that $v_k \in H_D^1(\Omega)$. The weak formulation of (3.73) reads as

$$(\nabla y, \nabla v)_{\boldsymbol{L}^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_D^1(\Omega).$$

Testing with v_k leads to

$$(f, v_k)_{L^2(\Omega)} = (\nabla y, \nabla v_k)_{\boldsymbol{L}^2(\Omega)} = (\nabla y, \nabla v_k)_{\boldsymbol{L}^2(\Omega(k))} = (\nabla v_k, \nabla v_k)_{\boldsymbol{L}^2(\Omega(k))} = (\nabla v_k, \nabla v_k)_{\boldsymbol{L}^2(\Omega)}.$$

Let us define

$$\Omega(k) \coloneqq \{x \in \Omega : |y(x)| > k\}, \quad \Gamma_D(k) \coloneqq \{x \in \Gamma_D : |y(x)| > k\}.$$

Then, along with Hölder's and Young's inequality, we obtain that

$$\begin{aligned}(f, v_k)_{L^2(\Omega)} &\le \|f\|_{L^r(\Omega)} \|v_k\|_{L^{r'}(\Omega)} \\ &\le \|f\|_{L^r(\Omega)} \left(\left(\int_{\Omega(k)} |v_k|^{2r'} \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\Omega(k)} 1 \,\mathrm{d}x \right)^{\frac{1}{2}} \right)^{\frac{1}{r'}} \\ &= \|f\|_{L^r(\Omega)} \|v_k\|_{L^{2r'}(\Omega)} |\Omega(k)|^{\frac{1}{2r'}},\end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Since

$$2r' = \frac{2}{1 - \frac{1}{r}} < \frac{2}{1 - \frac{2}{N}} = \frac{2N}{N-2},$$

the embedding $H^1(\Omega) \hookrightarrow L^{2r'}(\Omega)$ is continuous and it follows that

$$(f, v_k)_{L^2(\Omega)} \le c \|f\|_{L^r(\Omega)} \|v_k\|_{H^1(\Omega)} |\Omega(k)|^{\frac{1}{2r'}}$$

for a constant $c > 0$. Along with the generalized Poincaré inequality [72, Lemma 2.5], we obtain that

$$\begin{aligned}&\|v_k\|^2_{H^1(\Omega)} \le c_p \|\nabla v_k\|^2_{L^2(\Omega)} = c_p (f, v_k)_{L^2(\Omega)} \le c_p c \|f\|_{L^r(\Omega)} \|v_k\|_{H^1(\Omega)} |\Omega(k)|^{\frac{1}{2r'}} \\ \Rightarrow\ &\|v_k\|_{H^1(\Omega)} \le c_p c \|f\|_{L^r(\Omega)} |\Omega(k)|^{\frac{1}{2r'}}.\end{aligned}$$

Now, for every $h > k$, it holds that $\Omega(h) \subset \Omega(k)$. Since, $|v| = |y| - k$ a.e. in Ω, with the definition of v_k, for every $p \in (2r', \frac{2N}{N-2})$, it follows that

$$\begin{aligned}(h-k)^2 |\Omega(h)|^{\frac{2}{p}} &= \left(\int_{\Omega(h)} (h-k)^p \,\mathrm{d}x \right)^{\frac{2}{p}} \le \left(\int_{\Omega(h)} (|y| - k)^p \,\mathrm{d}x \right)^{\frac{2}{p}} \\ &\le \left(\int_{\Omega(k)} (|y| - k)^p \,\mathrm{d}x \right)^{\frac{2}{p}} = \|v_k\|^2_{L^p(\Omega)} \quad \forall h > k.\end{aligned}$$

Since the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is also continuous, there exists another constant $\hat{c} > 0$ such that

$$(h-k) |\Omega(h)|^{\frac{1}{p}} \le \hat{c} \|v_k\|_{L^p(\Omega)} \le \hat{c} c_p c \|f\|_{L^r(\Omega)} |\Omega(k)|^{\frac{1}{2r'}} \quad \forall h > k.$$

This can be written as

$$\phi(h) \le \frac{C}{(h-k)^a} \phi(k)^b \quad \forall h > k$$

where

$$\phi(h) = |\Omega(h)|^{\frac{1}{p}}, \qquad a = 1,$$

$$C = \hat{c}c_p c\|f\|_{L^r(\Omega)}, \qquad b = \frac{p}{2r'}.$$

Now, Stampacchia's auxiliary result (cf. [72, Lemma 7.5]) provides that $\phi(\delta) = 0$ where

$$\delta := C\phi^{b-1}2^{\frac{ab}{b-1}} = \hat{c}c_p c\|f\|_{L^r(\Omega)}|\Omega|^{\frac{p-2r'}{2r'}}2^{\frac{p}{p-2r'}}.$$

That means $y \in L^\infty(\Omega)$ and

$$|y(x)| \leq \delta \quad \text{for a.e. } x \in \Omega. \qquad \square$$

Lemma 3.17. *Let Assumption 3.1 hold, $N \leq 3$, and $\nu \in \mathcal{V}_{ad}$. Furthermore, let $G \in W^{k+1,1}(I, L^2(\Omega))$ with $k \in \mathbb{N}$, $(p, \boldsymbol{u})_{0,0} \in H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega)$, and*

$$(p, \boldsymbol{u})_{0,l} := A_\nu(p, \boldsymbol{u})_{0,l-1} + \partial_t^{l-1}(G(0), \boldsymbol{0}) \in H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega) \quad \forall l = 1, \dots, k.$$

Then, the first component p of the unique solution $(p, \boldsymbol{u}) \in C^{k+1}(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^k(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ to

$$\begin{cases} \partial_t(p, \boldsymbol{u})(t) - A_\nu(p, \boldsymbol{u})(t) = (G(t), \boldsymbol{0}) & \forall t \in I \\ (p, \boldsymbol{u})(0) = (p, \boldsymbol{u})_{0,0} \end{cases} \tag{3.74}$$

satisfies $p \in C^{k-1}(I, L^\infty(\Omega))$ and there exists a constant $\hat{c} > 0$ such that

$$\begin{aligned} \|\partial_t^l p\|_{L^2(I, L^\infty(\Omega))} \leq \hat{c}(&\|(p, \boldsymbol{u})_{0,l+1}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+1}G\|_{L^2(I,L^2(\Omega))} + \|(p, \boldsymbol{u})_{0,l+2}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\ &+ \|\partial_t^{l+2}G\|_{L^1(I,L^2(\Omega))}) \quad \forall l = 0, \dots, k-1 \end{aligned} \tag{3.75}$$

$$\begin{aligned} \|\partial_t^l p\|_{C(I, L^\infty(\Omega))} \leq \hat{c}(&\|(p, \boldsymbol{u})_{0,l+1}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+1}G\|_{C(I,L^2(\Omega))} + \|(p, \boldsymbol{u})_{0,l+2}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\ &+ \|\partial_t^{l+2}G\|_{L^1(I,L^2(\Omega))}) \quad \forall l = 0, \dots, k-1. \end{aligned} \tag{3.76}$$

Proof. By Lemma 3.4, the system (3.74) admits a unique solution $(p, \boldsymbol{u}) \in C^{k+1}(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^k(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$. Furthermore, by the definition of A_ν (see (3.13)), the system (3.74) is nothing but

$$\begin{cases} \nu\partial_t p + \mathrm{div}(\boldsymbol{u}) + \eta p = \nu G & \text{in } I \times \Omega \\ \partial_t \boldsymbol{u} + \nabla p = \boldsymbol{0} & \text{in } I \times \Omega \\ p = 0 & \text{on } I \times \Gamma_D \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (p, \boldsymbol{u})(0) = (p, \boldsymbol{u})_{0,0} & \text{in } \Omega. \end{cases} \tag{3.77}$$

By differentiating the first equation in (3.77) in time and inserting the second equation, it follows for all $t \in I$ and $l = 0, \dots, k-1$ that

$$\begin{aligned} -\Delta\partial_t^l p(t) &= \nu\partial_t^{l+1}G(t) - \nu\partial_t^{l+2}p(t) - \eta\partial_t^{l+1}p(t) \quad \text{in } \Omega, \\ \partial_n(\partial_t^l p(t)) &= 0 \quad \text{on } \Gamma_N, \quad \tau(\partial_t^l p(t)) = 0 \quad \text{on } \Gamma_D. \end{aligned} \tag{3.78}$$

Therefore, by Lemma 3.16, we obtain that

$$\|\partial_t^l p(t)\|_{L^\infty(\Omega)} \leq \tilde{c}\|\nu\partial_t^{l+1}G(t) - \nu\partial_t^{l+2}p(t) - \eta\partial_t^{l+1}p(t)\|_{L^2(\Omega)} \quad \forall t \in I \quad \forall l = 0, \dots, k-1. \tag{3.79}$$

Let us prove that $p \in C^{k-1}(I, L^\infty(\Omega))$. To this aim, let $l \in \{0, \dots, k-1\}$ and $\{t_n\}_{n=1}^\infty \subset I$ with $\lim_{n\to\infty} t_n = t$. Applying the superposition principle to (3.78) yields that

$$
\begin{aligned}
&-\Delta\partial_t^l(p(t)-p(t_n)) = \nu\partial_t^{l+1}(G(t)-G(t_n)) - \nu\partial_t^{l+2}(p(t)-p(t_n)) - \eta\partial_t^{l+1}(p(t)-p(t_n)) \quad \text{in } \Omega,\\
&\partial_n(\partial_t^l(p(t)-p(t_n))) = 0 \quad \text{on } \Gamma_N, \quad \tau(\partial_t^l(p(t)-p(t_n))) = 0 \quad \text{on } \Gamma_D.
\end{aligned}
$$

Then, using again Lemma 3.16, it follows that

$$
\begin{aligned}
&\|\partial_t^l(p(t)-p(t_n))\|_{L^\infty(\Omega)} \\
&\leq \tilde{c}(\|\nu\partial_t^{l+1}(G(t)-G(t_n))\|_{L^2(\Omega)} + \|\nu\partial_t^{l+2}(p(t)-p(t_n))\|_{L^2(\Omega)} + \|\eta\partial_t^{l+1}(p(t)-p(t_n))\|_{L^2(\Omega)}).
\end{aligned} \tag{3.80}
$$

Since $\partial_t^{l+1}G, \partial_t^l p, \partial_t^{l+1}p \in C(I, L^2(\Omega))$, the right-hand side in (3.80) vanishes as $n \to \infty$. In conlusion, the regularity property $p \in C^{k-1}(I, L^\infty(\Omega))$ is valid. Now, we integrate (3.79) and make use of Lemma 3.4, to deduce for every $l = 0, \dots, k-1$ that

$$
\begin{aligned}
&\|\partial_t^l p\|_{L^2(I,L^\infty(\Omega))} \\
&\leq \tilde{c}\|\nu\partial_t^{l+1}G - \nu\partial_t^{l+2}p - \eta\partial_t^{l+1}p\|_{L^2(I,L^2(\Omega))} \\
&\leq \tilde{c}(\nu_{\max}\|\partial_t^{l+1}G\|_{L^2(I,L^2(\Omega))} + c\nu_{\max}(\|(p,\boldsymbol{u})_{0,l+2}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+2}G\|_{L^1(I,L^2(\Omega))}) \\
&\qquad + c\|\eta\|_{L^\infty}(\|(p,\boldsymbol{u})_{0,l+1}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+1}G\|_{L^1(I,L^2(\Omega))}))
\end{aligned}
$$

with $c = \nu_{\min}^{-1}\frac{\max\{\sqrt{\nu_{\max}},1\}}{\min\{\sqrt{\nu_{\min}},1\}}$. This leads to the desired estimate (3.75) with $\hat{c} > 0$ depending on $\tilde{c}$, T, $\nu_{\min}$, $\nu_{\max}$, and η. To obtain (3.76), we take the supremum on both sides of (3.79) and again make use of Lemma 3.4. Then, it follows for every $l = 0, \dots, k-1$ that

$$
\begin{aligned}
&\|\partial_t^l p\|_{C(I,L^\infty(\Omega))} \\
&\leq \tilde{c}\|\partial_t^{l+1}G - \nu\partial_t^{l+2}p - \eta\partial_t^{l+1}p\|_{C(I,L^2(\Omega))} \\
&\leq \tilde{c}(\|\partial_t^{l+1}G\|_{C(I,L^2(\Omega))} + c\nu_{\max}(\|(p,\boldsymbol{u})_{0,l+2}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+2}G\|_{L^1(I,L^2(\Omega))}) \\
&\qquad + c\|\eta\|_{L^\infty}(\|(p,\boldsymbol{u})_{0,l+1}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+1}G\|_{L^1(I,L^2(\Omega))})) \\
&\leq \hat{c}(\|(p,\boldsymbol{u})_{0,l+1}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\partial_t^{l+1}G\|_{C(I,L^2(\Omega))} + \|(p,\boldsymbol{u})_{0,l+2}\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\
&\qquad + \|\partial_t^{l+2}G\|_{L^1(I,L^2(\Omega))})
\end{aligned}
$$

with $\hat{c} > 0$ depending on $\tilde{c}$, T, $\nu_{\min}$, $\nu_{\max}$, and η. □

3.5 Second-Order Sufficient Optimality Conditions

Assumption 3.18. *Let Assumption 3.1 and $N \leq 3$ hold. Furthermore, let $\overline{\nu} \in \mathcal{V}_{ad}$ satisfy the variational inequality* (3.62) *with the corresponding state $(\overline{p}, \overline{\boldsymbol{u}}) \coloneqq S(\overline{\nu})$ and the adjoint state $(\overline{q}, \overline{\boldsymbol{v}})$ ((3.42) for $\nu = \overline{\nu}$). We assume the higher regularity property*

$$
\begin{cases}
f \in W^{6,1}(I, L^2(\Omega)) \\
p_0, p_1 \in D(\Delta_{D,N}) \\
\sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) \in W^{2,1}(I, L^2(\Omega)),
\end{cases} \tag{3.81}
$$

and the compatibility assumption

$$\begin{cases} f(0) = -\Delta p_0 + \eta p_1 \\ \partial_t f(0) = -\Delta p_1 \\ \partial_t^l f(0) = 0 \quad \text{for } l = 2,3,4 \\ a_i(T) = 0 \quad \text{for } i = 1,\dots,m. \end{cases} \tag{3.82}$$

In view of Assumption 3.1, (3.6), and (3.81), the following quantities for any given $\nu \in \mathcal{V}_{ad}$ belong to $H_D^1(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega)$:

$$\begin{aligned}
(p,\boldsymbol{u})_{0,0} &\coloneqq (p_0, \Phi(\nu)) \\
(p,\boldsymbol{u})_{0,1} &\coloneqq A_\nu((p,\boldsymbol{u})_{0,0}) + (\nu^{-1}F(0),\boldsymbol{0}) \underbrace{=}_{(3.13),(3.8)} (p_1, -\nabla p_0) \\
(p,\boldsymbol{u})_{0,2} &\coloneqq A_\nu((p,\boldsymbol{u})_{0,1}) + (\nu^{-1}\partial_t F(0),\boldsymbol{0}) \underbrace{=}_{(3.13),(3.8)} (\nu^{-1}(\Delta p_0 - \eta p_1 + f(0)), -\nabla p_1) \underbrace{=}_{(3.82)} (0, -\nabla p_1) \\
(p,\boldsymbol{u})_{0,3} &\coloneqq A_\nu((p,\boldsymbol{u})_{0,2}) + (\nu^{-1}\partial_t^2 F(0),\boldsymbol{0}) \underbrace{=}_{(3.13),(3.8)} (\nu^{-1}(\Delta p_1 + \partial_t f(0)),\boldsymbol{0}) \underbrace{=}_{(3.82)} (0,\boldsymbol{0}) \\
(p,\boldsymbol{u})_{0,l} &\coloneqq A_\nu((p,\boldsymbol{u})_{0,l-1}) + (\nu^{-1}\partial_t^{l-1} F(0),\boldsymbol{0}) \underbrace{=}_{(3.13),(3.8)} (\nu^{-1}\partial_t^{l-2} f(0),\boldsymbol{0}) \underbrace{=}_{(3.82)} (0,\boldsymbol{0}) \quad \text{for } l = 4,5,6
\end{aligned} \tag{3.83}$$

Along with $F \in W^{7,1}(I, L^2(\Omega))$ (by (3.8) and $f \in W^{6,1}(I, L^2(\Omega))$ due to (3.81)), Lemma 3.4 implies that

$$S(\nu) \in C^7(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^6(I, H_D^1(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega)) \quad \forall \nu \in \mathcal{V}_{ad} \tag{3.84}$$

and

$$\begin{aligned}
\|S(\nu)\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} &\le c(\|(p_0,\Phi(\nu))\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \|\nu^{-1}F\|_{L^1(I,L^2(\Omega))}) \\
&\le c\big((1 + c_P\|\eta\|_{L^\infty(\Omega)})\|p_0\|_{L^2(\Omega)} + c_P\nu_{\max}\|p_1\|_{L^2(\Omega)} \\
&\quad + \nu_{\min}^{-1}\|F\|_{L^1(I,L^2(\Omega))}\big) \\
\|\partial_t S(\nu)\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} &\le c(\|(p_1,-\nabla p_0)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|f\|_{L^1(I,L^2(\Omega))}) \\
\|\partial_t^2 S(\nu)\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} &\le c(\|\nabla p_1\|_{\boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|\partial_t^2 f\|_{L^1(I,L^2(\Omega))}) \\
\|\partial_t^l S(\nu)\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} &\le c\nu_{\min}^{-1}\|\partial_t^l f\|_{L^1(I,L^2(\Omega))} \quad \text{for } l = 3,\dots,7
\end{aligned} \tag{3.85}$$

for all $\nu \in \mathcal{V}_{ad}$ with $c \coloneqq \frac{\max\{\sqrt{\nu_{\max}},1\}}{\min\{\sqrt{\nu_{\min}},1\}}$. Note that the compatibility assumption (3.82) guarantees that no additional regularity property has to be assumed for the control space $\mathcal{V}_{ad}$ to ensure that all quantities in (3.83) belong to $H_D^1(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega)$. Moreover, by Lemma 3.17, the first component of $S(\nu)$ satisfies

$$S_p(\nu) \in C^5(I, L^\infty(\Omega))$$

and

$$\|S_p(\nu)\|_{C(I,L^\infty(\Omega))} \le \hat{c}(\|(p_1,-\nabla p_0)\|_{L^2(\Omega)\times\boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|f\|_{C(I,L^2(\Omega))} + \|\nabla p_1\|_{\boldsymbol{L}^2(\Omega)}$$

$$
\begin{aligned}
&+ \nu_{\min}^{-1} \|\partial_t f\|_{L^1(I,L^2(\Omega))}) \\
\|\partial_t S_p(\nu)\|_{C(I,L^\infty(\Omega))} &\le \hat{c}(\|\nabla p_1\|_{\boldsymbol{L}^2(\Omega)} + \nu_{\min}^{-1}\|\partial_t f\|_{C(I,L^2(\Omega))} + \nu_{\min}^{-1}\|\partial_t^2 f\|_{L^1(I,L^2(\Omega))}) \\
\|\partial_t^l S_p(\nu)\|_{C(I,L^\infty(\Omega))} &\le \hat{c}\nu_{\min}^{-1}(\|\partial_t^l f\|_{C(I,L^2(\Omega))} + \|\partial_t^{l+1} f\|_{L^1(I,L^2(\Omega))}) \quad \forall l = 2,\ldots,5.
\end{aligned}
$$

In view of Lemma 3.4, both the last properties in (3.81) and (3.82) imply for the adjoint state that

$$
(\overline{q},\overline{\boldsymbol{v}}) \in C^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^1(I, H_D^1(\Omega) \times \boldsymbol{H}_N(\mathrm{div},\Omega)) \tag{3.86}
$$

and $(\overline{q},\overline{\boldsymbol{v}})$ solves the adjoint equation in the classical sense:

$$
\begin{cases}
\overline{\nu}\partial_t\overline{q} + \mathrm{div}\,\overline{\boldsymbol{v}} - \eta\overline{q} = \sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) & \text{in } I \times \Omega \\
\partial_t\overline{\boldsymbol{v}} + \nabla\overline{q} = \boldsymbol{0} & \text{in } I \times \Omega \\
\overline{q} = 0 & \text{on } I \times \Gamma_D \\
\overline{\boldsymbol{v}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\
(\overline{q},\overline{\boldsymbol{v}})(T) = (0,\boldsymbol{0}) & \text{in } \Omega.
\end{cases} \tag{3.87}
$$

Lemma 3.19. *Let Assumption 3.18 hold. Then, there exists a constant $L > 0$ such that*

$$
\|S(\nu) - S(\hat{\nu})\|_{C^4(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \le L\|\nu - \hat{\nu}\|_{L^2(\Omega)} \quad \forall \nu,\hat{\nu} \in \mathcal{V}_{ad} \tag{3.88}
$$

$$
\|S_p(\nu) - S_p(\hat{\nu})\|_{C^2(I,L^\infty(\Omega))} \le L\|\nu - \hat{\nu}\|_{L^2(\Omega)} \quad \forall \nu,\hat{\nu} \in \mathcal{V}_{ad}. \tag{3.89}
$$

Proof. Let $\nu,\hat{\nu} \in \mathcal{V}_{ad}$ be arbitrarily fixed, $(p,\boldsymbol{u}) \coloneqq S(\nu)$, and $(\hat{p},\hat{\boldsymbol{u}}) \coloneqq S(\hat{\nu})$. As discussed above, we obtain the regularity properties $p,\hat{p} \in C^5(I, L^\infty(\Omega))$. In particular, there exists a constant $\tilde{c} > 0$, independent of ν and p, such that

$$
\|\partial_t^l p\|_{C(I,L^\infty(\Omega))} \le \tilde{c} \quad \text{for } l = 0,\ldots,5. \tag{3.90}
$$

Now, subtracting the corresponding Cauchy problems of $(p,\boldsymbol{u})$ and $(\hat{p},\hat{\boldsymbol{u}})$ (see (3.9)) provides that $(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})$ satisfies

$$
\begin{cases}
\hat{\nu}\partial_t(p-\hat{p}) + \mathrm{div}(\boldsymbol{u}-\hat{\boldsymbol{u}}) + \eta(p-\hat{p}) = -(\nu-\hat{\nu})\partial_t p & \text{in } I \times \Omega \\
\partial_t(\boldsymbol{u}-\hat{\boldsymbol{u}}) + \nabla(p-\hat{p}) = \boldsymbol{0} & \text{in } I \times \Omega \\
p - \hat{p} = 0 & \text{on } I \times \Gamma_D \\
(\boldsymbol{u}-\hat{\boldsymbol{u}}) \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\
(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})(0) = (0,\Phi(\nu)-\Phi(\hat{\nu})) & \text{in } \Omega.
\end{cases} \tag{3.91}
$$

Note that $(\nu-\hat{\nu})\partial_t p \in C^6(I, L^2(\Omega))$ (see (3.84)) and

$$
(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,0} \coloneqq (0,\Phi(\nu)-\Phi(\hat{\nu})) \tag{3.92}
$$

$$
\begin{aligned}
(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,1} &\coloneqq A_{\hat{\nu}}(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,0} - (\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t p(0),\boldsymbol{0}) \\
&\underbrace{=}_{(3.5),(3.13)} (\hat{\nu}^{-1}(\nu-\hat{\nu})(p_1 - \partial_t p(0)),\boldsymbol{0}) \underbrace{=}_{(3.28)} (0,\boldsymbol{0})
\end{aligned} \tag{3.93}
$$

$$
\begin{aligned}
(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,2} &\coloneqq A_{\hat{\nu}}(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,1} - (\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^2 p(0),\boldsymbol{0}) \\
&= (\nu^{-1}\hat{\nu}^{-1}(\nu-\hat{\nu})(\Delta p_0 - \eta p_1 + f(0)),\boldsymbol{0}) \underbrace{=}_{(3.82)} (0,\boldsymbol{0})
\end{aligned} \tag{3.94}
$$

$$(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,3} \coloneqq A_{\hat{\nu}}(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,2} - (\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^3 p(0),\boldsymbol{0}) \tag{3.95}$$
$$\underbrace{=}_{(3.28)} (\nu^{-1}\hat{\nu}^{-1}(\nu-\hat{\nu})(\Delta p_1 - \eta\partial_t^2 p(0) + \partial_t f(0)),\boldsymbol{0}) \underbrace{=}_{(3.94),(3.82)} (0,\boldsymbol{0})$$
$$(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,4} \coloneqq A_{\hat{\nu}}(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})_{0,3} - (\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^4 p(0),\boldsymbol{0}) = -(\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^4 p(0),\boldsymbol{0}). \tag{3.96}$$

Applying Lemma 3.4 to (3.91), we obtain that

$$\begin{aligned}
&\|(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \\
&\le c(\|\Phi(\nu)-\Phi(\hat{\nu})\|_{\boldsymbol{L}^2(\Omega)} + \|\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t p\|_{L^1(I,L^2(\Omega))}) \\
&\underbrace{\le}_{(3.7)} c(c_P\|p_1(\nu-\hat{\nu})\|_{L^2(\Omega)} + \nu_{\min}^{-1}\|\partial_t p\|_{L^1(I,L^\infty(\Omega))}\|\nu-\hat{\nu}\|_{L^2(\Omega)}) \\
&\underbrace{\le}_{(3.90)} c(c_P\|p_1\|_{L^\infty(\Omega)} + \nu_{\min}^{-1}T\tilde{c})\|\nu-\hat{\nu}\|_{L^2(\Omega)},
\end{aligned} \tag{3.97}$$

where the regularity property $p_1 \in L^\infty(\Omega)$ is satisfied due to (3.82) and Lemma 3.16. Analogously, taking the initial conditions (3.93)-(3.96) into account, applying Lemma 3.4 to (3.91) implies that

$$\begin{aligned}
&\|\partial_t^l(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \le c\|\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^{l+1}p\|_{L^1(I,L^2(\Omega))} \\
&\quad \le c\nu_{\min}^{-1}\|\partial_t^{l+1}p\|_{L^1(I,L^\infty(\Omega))}\|\nu-\hat{\nu}\|_{L^2(\Omega)} \underbrace{\le}_{(3.90)} c\nu_{\min}^{-1}T\tilde{c}\|\nu-\hat{\nu}\|_{L^2(\Omega)} \quad \text{for } l=1,2,3,
\end{aligned} \tag{3.98}$$

and

$$\begin{aligned}
&\|\partial_t^4(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \\
&\le c(\|\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^4 p(0)\|_{L^2(\Omega)} + \|\hat{\nu}^{-1}(\nu-\hat{\nu})\partial_t^5 p\|_{L^1(I,L^2(\Omega))}) \\
&\le c\nu_{\min}^{-1}(\|\partial_t^4 p(0)\|_{L^\infty(\Omega)} + \|\partial_t^5 p\|_{L^1(I,L^\infty(\Omega))})\|\nu-\hat{\nu}\|_{L^2(\Omega)} \underbrace{\le}_{(3.90)} c\nu_{\min}^{-1}(1+T)\tilde{c}\|\nu-\hat{\nu}\|_{L^2(\Omega)}.
\end{aligned} \tag{3.99}$$

Combining (3.97), (3.98), and (3.99), we find a constant $C>0$, independent of p, $\hat{p}$, $\boldsymbol{u}$, $\hat{\boldsymbol{u}}$, ν, and $\hat{\nu}$, such that

$$\|\partial_t^l(p-\hat{p},\boldsymbol{u}-\hat{\boldsymbol{u}})\|_{C(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))} \le C\|\nu-\hat{\nu}\|_{L^2(\Omega)} \quad \text{for } l=0,\dots,4. \tag{3.100}$$

Therefore, the Lipschitz property (3.88) is a direct consequence of (3.100). Applying Lemma 3.17 to (3.91), we complete the proof by

$$\begin{aligned}
\|\partial_t^l p-\hat{p}\|_{C(I,L^\infty(\Omega))} &\le \hat{c}(\|(\nu-\hat{\nu})\partial_t^{l+2}p\|_{C(I,L^2(\Omega))} + \|(\nu-\hat{\nu})\partial_t^{l+3}p\|_{L^1(I,L^2(\Omega))}) \\
&\underbrace{\le}_{(3.90)} \hat{c}(1+T)\|\nu-\hat{\nu}\|_{L^2(\Omega)} \quad \text{for } l=0,1,2.
\end{aligned}$$
□

Let us define the Lagrangian functional associated with (P) by $\mathcal{L}\colon L^\infty(\Omega)\times W_p\times\boldsymbol{W}_u\times C(I,L^2(\Omega))\times C(I,\boldsymbol{L}^2(\Omega))\to\mathbb{R}$ where

$$\begin{aligned}
\mathcal{L}(\nu,p,\boldsymbol{u},q,\boldsymbol{v}) &\coloneqq \mathcal{J}(\nu,p) + (\nu\partial_t p + \operatorname{div}\boldsymbol{u} + \eta p - F, q)_{L^2(I,L^2(\Omega))} \\
&\quad + (\partial_t\boldsymbol{u} + \nabla p,\boldsymbol{v})_{L^2(I,\boldsymbol{L}^2(\Omega))} + (p(0)-p_0,q(0))_{L^2(\Omega)} + (\boldsymbol{u}(0)-\Phi(\nu),\boldsymbol{v}(0))_{\boldsymbol{L}^2(\Omega)}.
\end{aligned} \tag{3.101}$$

Here, W_p and $\boldsymbol{W}_u$ are defined as in (3.36), and $\mathcal{J}$ denotes the objective function (1.2).

Lemma 3.20. *Let Assumption 3.1 hold and let $\overline{\nu} \in \mathcal{V}_{ad}$ satisfy the variational inequality* (3.62) *with the corresponding state $(\overline{p}, \overline{\boldsymbol{u}}) = S(\overline{\nu})$ and the adjoint state $(\overline{q}, \overline{\boldsymbol{v}})$ (see Definition 3.8). Additionally, we assume that $\sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) \in W^{2,1}(I, L^2(\Omega))$. Then, for every $\nu \in \mathcal{V}_{ad}$ with the corresponding state $(p, \boldsymbol{u}) = S(\nu)$, it holds that*

$$\begin{cases} \mathcal{L}(\nu, p, \boldsymbol{u}, \overline{q}, \overline{\boldsymbol{v}}) = \mathcal{J}(\nu, p) \\ D_\nu \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu - \overline{\nu}) \geq 0 \\ D_{(p,\boldsymbol{u})} \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(p - \overline{p}, \boldsymbol{u} - \overline{\boldsymbol{u}}) = 0. \end{cases} \tag{3.102}$$

Proof. Let $\nu \in \mathcal{V}_{ad}$ be arbitrarily given, and $(p, \boldsymbol{u}) := S(\nu)$. Then, since $(p, \boldsymbol{u})$ solves the state equation (3.9), the first assertion in (3.102) is immediately obtained as follows:

$$\begin{aligned} \mathcal{L}(\nu, p, \boldsymbol{u}, \overline{q}, \overline{\boldsymbol{v}}) \underbrace{=}_{(3.101)} & \mathcal{J}(\nu, p) + (\underbrace{\nu \partial_t p + \operatorname{div} \boldsymbol{u} + \eta p - F}_{=0 \text{ by } (3.9)}, \overline{q})_{L^2(I, L^2(\Omega))} \\ & + (\underbrace{\partial_t \boldsymbol{u} + \nabla p}_{=\boldsymbol{0} \text{ by } (3.9)}, \overline{\boldsymbol{v}})_{L^2(I, \boldsymbol{L}^2(\Omega))} + (\underbrace{p(0) - p_0}_{=0 \text{ by } (3.9)}, \overline{q}(0))_{L^2(\Omega)} + (\underbrace{\boldsymbol{u}(0) - \Phi(\nu)}_{=\boldsymbol{0} \text{ by } (3.9)}, \overline{\boldsymbol{v}}(0))_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

Furthermore, for the first-order derivative of $\mathcal{L}$ with respect to ν, we conclude that

$$\begin{aligned} & D_\nu \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu - \overline{\nu}) \\ & \underbrace{=}_{(3.101)} D_\nu \mathcal{J}(\overline{\nu}, \overline{p})(\nu - \overline{\nu}) + ((\nu - \overline{\nu}) \partial_t \overline{p}, \overline{q})_{L^2(I, L^2(\Omega))} - (\Phi'(\overline{\nu})(\nu - \overline{\nu}), \overline{\boldsymbol{v}}(0))_{\boldsymbol{L}^2(\Omega)} \\ & \underbrace{=}_{(3.69)} \lambda(\overline{\nu}, \nu - \overline{\nu}) + ((\nu - \overline{\nu}) \partial_t \overline{p}, \overline{q})_{L^2(I, L^2(\Omega))} - ((\nu - \overline{\nu}) p_1, \overline{q})_{L^2(I, L^2(\Omega))} \\ & = \left(\int_I (\partial_t \overline{p}(t) - p_1) \overline{q}(t) \, \mathrm{d}t + \lambda \overline{\nu}, \nu - \overline{\nu} \right)_{L^2(\Omega)} \underbrace{\geq}_{(3.62)} 0. \end{aligned} \tag{3.103}$$

Making use of (3.86)-(3.87), for the derivative of $\mathcal{L}$ with respect to $(p, \boldsymbol{u})$, we obtain that

$$\begin{aligned} & D_{(p,\boldsymbol{u})} \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(p - \overline{p}, \boldsymbol{u} - \overline{\boldsymbol{u}}) \\ & \underbrace{=}_{(3.101)} D_p \mathcal{J}(\overline{\nu}, \overline{p})(p - \overline{p}) + (\overline{\nu} \partial_t (p - \overline{p}) + \operatorname{div}(\boldsymbol{u} - \overline{\boldsymbol{u}}) + \eta(p - \overline{p}), \overline{q})_{L^2(I, L^2(\Omega))} \\ & \quad + ((p - \overline{p})(0), \overline{q}(0))_{L^2(\Omega)} + (\partial_t (\boldsymbol{u} - \overline{\boldsymbol{u}}) + \nabla(p - \overline{p}), \overline{\boldsymbol{v}})_{L^2(I, \boldsymbol{L}^2(\Omega))} \\ & \quad + ((\boldsymbol{u} - \overline{\boldsymbol{u}})(0), \overline{\boldsymbol{v}}(0))_{\boldsymbol{L}^2(\Omega)} \\ & = \sum_{i=1}^m (a_i(p - \overline{p}), \overline{p} - p_i^{ob})_{L^2(I, L^2(\Omega))} + (p - \overline{p}, -\overline{\nu} \partial_t \overline{q} - \operatorname{div} \overline{\boldsymbol{v}} + \eta \overline{q}))_{L^2(I, L^2(\Omega))} \\ & \quad + (\overline{\nu}(p - \overline{p})(T), \underbrace{\overline{q}(T)}_{=0 \text{ by } (3.87)})_{L^2(\Omega)} + (\boldsymbol{u} - \overline{\boldsymbol{u}}, -\partial_t \overline{\boldsymbol{v}} - \nabla \overline{q})_{L^2(I, \boldsymbol{L}^2(\Omega))} \\ & \quad + ((\boldsymbol{u} - \overline{\boldsymbol{u}})(T), \underbrace{\overline{\boldsymbol{v}}(T)}_{=\boldsymbol{0} \text{ by } (3.87)})_{L^2(\Omega)} \\ & = (p - \overline{p}, \underbrace{\sum_{i=1}^m a_i(\overline{p} - p_i^{ob}) - \overline{\nu} \partial_t \overline{q} - \operatorname{div} \overline{\boldsymbol{v}} + \eta \overline{q}}_{=0 \text{ by } (3.87)})_{L^2(I, L^2(\Omega))} + (\boldsymbol{u} - \overline{\boldsymbol{u}}, \underbrace{-\partial_t \overline{\boldsymbol{v}} - \nabla \overline{q}}_{=\boldsymbol{0} \text{ by } (3.87)})_{L^2(I, \boldsymbol{L}^2(\Omega))} \\ & = 0. \end{aligned}$$

□

Under Assumption 3.18, for $h \in L^2(\Omega)$, we introduce the linearized state equation at $(\overline{\nu}, \overline{p})$ as follows:

$$\begin{cases} \overline{\nu}\partial_t \tilde{p} + \operatorname{div} \tilde{\boldsymbol{u}} + \eta \tilde{p} = -h\partial_t \overline{p} & \text{in } I \times \Omega \\ \partial_t \tilde{\boldsymbol{u}} + \nabla \tilde{p} = \boldsymbol{0} & \text{in } I \times \Omega \\ \tilde{p} = 0 & \text{on } I \times \Gamma_D \\ \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\tilde{p}, \tilde{\boldsymbol{u}})(0) = (0, \Phi'(\overline{\nu})h) & \text{in } \Omega. \end{cases} \tag{3.104}$$

Remark 3.21. Thanks to Assumption 3.18, Lemma 3.17 implies that, for $h \in L^2(\Omega)$, the right-hand side of (3.104) satisfies

$$h\partial_t \overline{p} \in C^1(I, L^2(\Omega)). \tag{3.105}$$

On the other hand, in view of (3.6), $\Phi'(\overline{\nu})h$ is well-defined in $\boldsymbol{H}_N(\operatorname{div}, \Omega)$ for any $h \in L^\infty(\Omega)$. Nevertheless, as $p_1 \in L^\infty(\Omega)$ holds due to (3.82) and Lemma 3.16, the elliptic variational problem (3.4) also admits a unique solution for every $\nu \in L^2(\Omega)$. Thus, we can expand the domain of Φ as follows:

$$\Phi \colon L^2(\Omega) \to \boldsymbol{H}_N(\operatorname{div}, \Omega), \quad \nu \mapsto \nabla y,$$

where $y \in H^1_D(\Omega)$ denotes the unique solution to (3.4). The extended operator Φ remains well-defined, affine linear, and continuous. In particular, it holds that $\Phi'(\overline{\nu})h \in \boldsymbol{H}_N(\operatorname{div}, \Omega)$ for every $h \in L^2(\Omega)$. For this reason, along with (3.105), Lemma 3.4 yields the existence of a unique solution $(\tilde{p}, \tilde{\boldsymbol{u}}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\operatorname{div}, \Omega))$ to the linearised state equation (3.104).

Motivated by the necessary optimality condition (Theorem 3.12), we introduce the set of strongly active constraints

$$\mathscr{A}_0(\overline{\nu}) \coloneqq \{x \in \Omega \colon \varphi_{\overline{\nu}}(x) \neq 0\}, \tag{3.106}$$

where

$$\varphi_{\overline{\nu}} \in L^1(\Omega), \quad \varphi_{\overline{\nu}}(x) \coloneqq \int_I (\partial_t S_p(\overline{\nu})(t,x) - p_1(x))\overline{q}(t,x)\,\mathrm{d}t + \lambda\overline{\nu}(x) \text{ a.e. in } \Omega. \tag{3.107}$$

The associated critical cone is denoted by

$$\begin{aligned} C_{\overline{\nu}} \coloneqq \{h \in L^2(\Omega) \colon h_{|\mathscr{A}_0(\overline{\nu})} \equiv 0 \text{ and for a.e. } x \in \Omega \text{ it holds that} \\ h(x) \geq 0 \text{ if } \overline{\nu}(x) = \nu_-(x) \text{ and } h(x) \leq 0 \text{ if } \overline{\nu}(x) = \nu_+(x)\}. \end{aligned} \tag{3.108}$$

Theorem 3.22. *Let Assumption 3.18 hold. Further, assume that*

$$D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(h, \tilde{p}, \tilde{\boldsymbol{u}})^2 > 0 \tag{SSC}$$

holds for every $h \in C_{\overline{\nu}} \setminus \{0\}$ where $(\tilde{p}, \tilde{\boldsymbol{u}}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\operatorname{div}, \Omega))$ denotes the unique solution to the linearized state equation (3.104) *associated with h. Then, there exist $\sigma > 0$ and $\delta > 0$ such that the quadratic growth condition*

$$J(\nu) \geq J(\overline{\nu}) + \delta\|\nu - \overline{\nu}\|^2_{L^2(\Omega)} \tag{3.109}$$

holds true for every $\nu \in \mathcal{V}_{ad}$ with $\|\nu - \overline{\nu}\|_{L^2(\Omega)} \leq \sigma$. In particular, $\overline{\nu}$ is a locally optimal solution to (P).

Proof. We prove the claim by a careful combination of Lemma 3.19 and the contradiction argument [18, Theorem 2.3]. Assume that there exists a sequence $\{\nu_n\}_{n=1}^\infty \subset \mathcal{V}_{ad}$ such that

$$J(\overline{\nu}) + \frac{1}{n}\|\nu_n - \overline{\nu}\|^2_{L^2(\Omega)} > J(\nu_n) \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} \|\nu_n - \overline{\nu}\|_{L^2(\Omega)} = 0. \tag{3.110}$$

We define

$$h_n := \frac{1}{\alpha_n}(\nu_n - \overline{\nu}) \quad \text{with} \quad \alpha_n := \|\nu_n - \overline{\nu}\|_{L^2(\Omega)} \quad \forall n \in \mathbb{N}. \tag{3.111}$$

Since $\|h_n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}$, the corresponding sequence $\{h_n\}_{n=1}^\infty$ is bounded in $L^2(\Omega)$ and thus admits a weakly converging subsequence (still denoted by $\{h_n\}_{n=1}^\infty$), i.e.,

$$\exists\, \overline{h} \in L^2(\Omega) : h_n \rightharpoonup \overline{h} \quad \text{weakly in } L^2(\Omega) \text{ as } n \to \infty. \tag{3.112}$$

Let us first show that $\overline{h}$ lies inside the critical cone $C_{\overline{\nu}}$ defined as in (3.108). The set

$$\begin{aligned}\widetilde{C}_{\overline{\nu}} := \{h \in L^2(\Omega) \colon &\text{For a.e. } x \in \Omega \text{ it holds that}\\ &h(x) \geq 0 \text{ if } \overline{\nu}(x) = \nu_-(x) \text{ and } h(x) \leq 0 \text{ if } \overline{\nu}(x) = \nu_+(x)\}\end{aligned}$$

is closed and convex and consequently weakly sequentially closed in $L^2(\Omega)$. Therefore, the weak limit $\overline{h}$ of $\{h_n\}_{n=1}^\infty \subset \widetilde{C}_{\overline{\nu}}$ lies in $\widetilde{C}_{\overline{\nu}}$. By the inclusion $C_{\overline{\nu}} \subset \widetilde{C}_{\overline{\nu}}$, it remains to show that $\overline{h}\big|_{\mathscr{A}_0(\overline{\nu})} \equiv 0$. To this aim, we first note that applying (3.84) and (3.86) to (3.107) implies that $\varphi_{\overline{\nu}} \in L^2(\Omega)$ as defined in (3.107). As a consequence, we obtain that

$$\int_\Omega \varphi_{\overline{\nu}} \overline{h}\, \mathrm{d}x \underbrace{=}_{(3.112)} \lim_{n\to\infty} \int_\Omega \varphi_{\overline{\nu}} h_n\, \mathrm{d}x \underbrace{=}_{(3.103),(3.107)} \lim_{n\to\infty} \underbrace{D_\nu \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(h_n)}_{\geq 0 \text{ by Lemma 3.20}} \geq 0. \tag{3.113}$$

Let us set $(p_n, \boldsymbol{u}_n) := S(\nu_n)$ for all $n \in \mathbb{N}$. The second-order Taylor expansion

$$\begin{aligned}\mathcal{L}(\nu_n, p_n, \boldsymbol{u}_n, \overline{q}, \overline{\boldsymbol{v}}) = \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}}) &+ D_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu_n - \overline{\nu}, p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})\\ &+ \frac{1}{2} D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu_n - \overline{\nu}, p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})^2\end{aligned} \tag{3.114}$$

is exact due to the quadratic structure of $\mathcal{L}$ with respect to $\nu, p, \boldsymbol{u}$. Rearranging (3.114) and dividing by α_n yields that

$$\begin{aligned}D_\nu \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(h_n) &\underbrace{=}_{(3.111)} \frac{1}{\alpha_n} D_\nu \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu_n - \overline{\nu})\\
\underbrace{=}_{(3.114)} &-\frac{1}{\alpha_n} \underbrace{D_{(p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})}_{=0 \text{ by Lemma 3.20}}\\
&+ \frac{1}{\alpha_n} \underbrace{\big(\mathcal{L}(\nu_n, p_n, \boldsymbol{u}_n, \overline{q}, \overline{\boldsymbol{v}}) - \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})\big)}_{<\frac{1}{n}\|\nu_n - \overline{\nu}\|^2_{L^2(\Omega)} \text{ by (3.110) and Lemma 3.20}}\\
&- \frac{1}{2\alpha_n} D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu_n - \overline{\nu}, p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})^2\\
&< \frac{1}{n}\|\nu_n - \overline{\nu}\|_{L^2(\Omega)} + \frac{1}{2\alpha_n}\left|D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu_n - \overline{\nu}, p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})^2\right|.\end{aligned} \tag{3.115}$$

We examine the second term on the right-hand side of (3.115) as follows: By the definition of the Lagrangian (3.101), it holds that

$$\begin{aligned}
&\left|D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu},\overline{p},\overline{\boldsymbol{u}},\overline{q},\overline{\boldsymbol{v}})(\nu_n-\overline{\nu},p_n-\overline{p},\boldsymbol{u}_n-\overline{\boldsymbol{u}})^2\right| \qquad (3.116)\\
&=\left|\sum_{i=1}^m\int_I\int_\Omega a_i(p_n-\overline{p})^2\,\mathrm{d}x\,\mathrm{d}t+\lambda\|\nu_n-\overline{\nu}\|^2_{L^2(\Omega)}+2\big((\nu_n-\overline{\nu})\partial_t(p_n-\overline{p}),\overline{q}\big)_{L^2(I,L^2(\Omega))}\right|\\
&\underbrace{\le}_{(3.86)}\sum_{i=1}^m\|a_i\|_{L^\infty(I\times\Omega)}\|p_n-\overline{p}\|^2_{L^2(I,L^2(\Omega))}+\lambda\|\nu_n-\overline{\nu}\|^2_{L^2(\Omega)}\\
&\qquad+2\|\nu_n-\overline{\nu}\|_{L^2(\Omega)}\|\partial_t(p_n-\overline{p})\|_{L^2(I,L^\infty(\Omega))}\|\overline{q}\|_{L^2(I,L^2(\Omega))}.
\end{aligned}$$

By virtue of Lemma 3.19 and (3.110), the right-hand side in (3.115) converges to 0, which implies that

$$0\underbrace{\le}_{(3.113)}\int_\Omega\varphi_{\overline{\nu}}\overline{h}\,\mathrm{d}x=\lim_{n\to\infty}\int_\Omega\varphi_{\overline{\nu}}h_n\,\mathrm{d}x=\lim_{n\to\infty}D_\nu\mathcal{L}(\overline{\nu},\overline{p},\overline{\boldsymbol{u}},\overline{q},\overline{\boldsymbol{v}})(h_n)\underbrace{\le}_{(3.115)}0 \qquad (3.117)$$
$$\Rightarrow\int_\Omega\varphi_{\overline{\nu}}\overline{h}\,\mathrm{d}x=0.$$

On the other hand, since $\overline{\nu}$ satisfies the variational inequality (3.62) and $\overline{h}\in\widetilde{C}_{\overline{\nu}}$, it holds that

$$\varphi_{\overline{\nu}}\overline{h}\ge 0\text{ a.e. in }\Omega\quad\underbrace{\Rightarrow}_{(3.117)}\quad\varphi_{\overline{\nu}}\overline{h}=0\text{ a.e. in }\Omega\quad\underbrace{\Rightarrow}_{\overline{h}\in\widetilde{C}_{\overline{\nu}}}\quad\overline{h}\in C_{\overline{\nu}}.$$

The next goal is to show that it holds $\overline{h}=0$ a.e. in Ω. To this aim, let $(\tilde{p},\tilde{\boldsymbol{u}})\in C^1(I,L^2(\Omega)\times\boldsymbol{L}^2(\Omega))\cap C(I,H^1_D(\Omega)\times\boldsymbol{H}_N(\mathrm{div},\Omega))$ denote the solution of the linearized state equation (3.104) with h replaced by $\overline{h}$. In view of (SSC), we obtain that $\overline{h}=0$ if we can show that

$$D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu},\overline{p},\overline{\boldsymbol{u}},\overline{q},\overline{\boldsymbol{v}})(\overline{h},\tilde{p},\tilde{\boldsymbol{u}})^2\le 0.$$

To verify this, we first introduce

$$(\tilde{p}_n,\tilde{\boldsymbol{u}}_n)\coloneqq\frac{1}{\alpha_n}(p_n-\overline{p},\boldsymbol{u}_n-\overline{\boldsymbol{u}})\quad\forall n\in\mathbb{N}$$

and demonstrate the following convergence:

$$\tilde{p}_n\rightharpoonup\tilde{p}\text{ weakly in }C^1(I,L^2(\Omega))\cap C(I,H^1(\Omega))\text{ as }n\to\infty. \qquad (3.118)$$

In Remark 3.23, we explain how (3.118) implies that the remainder term related to the linearization of S_p at $\overline{\nu}$ in the direction $\nu_n-\overline{\nu}$ vanishes as $n\to\infty$. We consider the second-order Taylor expansion (3.114) of the Lagrangian and divide it by α_n^2 to get

$$\begin{aligned}
&\frac{1}{2}D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu},\overline{p},\overline{\boldsymbol{u}},\overline{q},\overline{\boldsymbol{v}})(h_n,\tilde{p}_n,\tilde{\boldsymbol{u}}_n)^2 \qquad (3.119)\\
&\underbrace{=}_{(3.116)}\frac{1}{2\alpha_n^2}D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu},\overline{p},\overline{\boldsymbol{u}},\overline{q},\overline{\boldsymbol{v}})(\nu_n-\overline{\nu},p_n-\overline{p},\boldsymbol{u}_n-\overline{\boldsymbol{u}})^2
\end{aligned}$$

$$\underbrace{=}_{(3.114)} \frac{1}{\alpha_n^2} \underbrace{\left(\mathcal{L}(\nu_n, p_n, \boldsymbol{u}_n, \overline{q}, \overline{\boldsymbol{v}}) - \mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})\right)}_{<\frac{1}{n}\|\nu_n - \overline{\nu}\|^2_{L^2(\Omega)} \text{ by Lemma 3.20 \& (3.110)}}$$
$$- \frac{1}{\alpha_n^2} \underbrace{D_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\nu_n - \overline{\nu}, p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})}_{\geq 0 \text{ by Lemma 3.20}} \leq \frac{1}{n}.$$

Subtracting the corresponding Cauchy-problems of $(p_n, \boldsymbol{u}_n)$ and $(\overline{p}, \overline{\boldsymbol{u}})$ yields that

$$\begin{cases} \overline{\nu}\partial_t(p_n - \overline{p}) + \operatorname{div}(\boldsymbol{u}_n - \overline{\boldsymbol{u}}) + \eta(p_n - \overline{p}) = -(\nu_n - \overline{\nu})\partial_t p_n & \text{in } I \times \Omega \\ \partial_t(\boldsymbol{u}_n - \overline{\boldsymbol{u}}) + \nabla(p_n - \overline{p}) = \boldsymbol{0} & \text{in } I \times \Omega \\ p_n - \overline{p} = 0 & \text{on } I \times \Gamma_D \\ (\boldsymbol{u}_n - \overline{\boldsymbol{u}}) \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})(0) = (0, \Phi(\nu_n) - \Phi(\overline{\nu})) & \text{in } \Omega. \end{cases} \tag{3.120}$$

Dividing (3.120) by α_n shows that $(\tilde{p}_n, \tilde{\boldsymbol{u}}_n) = \frac{1}{\alpha_n}(p_n - \overline{p}, \boldsymbol{u}_n - \overline{\boldsymbol{u}})$ solves

$$\begin{cases} \overline{\nu}\partial_t \tilde{p}_n + \operatorname{div} \tilde{\boldsymbol{u}}_n + \eta \tilde{p}_n = -h_n \partial_t p_n & \text{in } I \times \Omega \\ \partial_t \tilde{\boldsymbol{u}}_n + \nabla \tilde{p}_n = \boldsymbol{0} & \text{in } I \times \Omega \\ \tilde{p}_n = 0 & \text{on } I \times \Gamma_D \\ \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\tilde{p}_n, \tilde{\boldsymbol{u}}_n)(0) = \alpha_n^{-1}(0, \Phi(\nu_n) - \Phi(\overline{\nu})) = (0, \Phi'(\overline{\nu})h_n) & \text{in } \Omega. \end{cases} \tag{3.121}$$

Substracting the linearized state equation (3.104) (with h replaced by $\overline{h}$) from (3.121) yields that

$$\begin{cases} \overline{\nu}\partial_t(\tilde{p}_n - \tilde{p}) + \operatorname{div}(\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}) + \eta(\tilde{p}_n - \tilde{p}) = -(h_n \partial_t p_n - \overline{h}\partial_t \overline{p}) & \text{in } I \times \Omega \\ \partial_t(\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}) + \nabla(\tilde{p}_n - \tilde{p}) = \boldsymbol{0} & \text{in } I \times \Omega \\ \tilde{p}_n - \tilde{p} = 0 & \text{on } I \times \Gamma_D \\ (\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}) \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\tilde{p}_n - \tilde{p}, \tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}})(0) = (0, \Phi'(\overline{\nu})(h_n - \overline{h})) & \text{in } \Omega. \end{cases} \tag{3.122}$$

In view of Lemma 3.19, there exists a constant $L > 0$, independent of $n \in \mathbb{N}$, such that

$$\|p_n - \overline{p}\|_{C^2(I, L^\infty(\Omega))} \leq L\|\nu_n - \overline{\nu}\|_{L^2(\Omega)} \quad \forall n \in \mathbb{N}. \tag{3.123}$$

Consequently, we obtain for every $\phi \in H^1(I, L^2(\Omega))$ that

$$\begin{aligned} &|(h_n \partial_t p_n - \overline{h}\partial_t \overline{p}, \phi)_{H^1(I, L^2(\Omega))}| \\ &= |(h_n \partial_t p_n - h_n \partial_t \overline{p}, \phi)_{H^1(I, L^2(\Omega))} + (h_n \partial_t \overline{p} - \overline{h}\partial_t \overline{p}, \phi)_{H^1(I, L^2(\Omega))}| \\ &= |(h_n \partial_t p_n - h_n \partial_t \overline{p}, \phi)_{H^1(I, L^2(\Omega))} + (h_n - \overline{h}, \underbrace{\int_I \partial_t \overline{p}\phi + \partial_t^2 \overline{p}\partial_t \phi \,\mathrm{d}t}_{=:\psi \in L^2(\Omega)})_{L^2(\Omega)}| \\ &\leq \sup_{n \in \mathbb{N}} \|h_n\|_{L^2(\Omega)} \|\partial_t p_n - \partial_t \overline{p}\|_{H^1(I, L^\infty(\Omega))} \|\phi\|_{H^1(I, L^2(\Omega))} + |(h_n - \overline{h}, \psi)_{L^2(\Omega)}| \end{aligned}$$

$$\underbrace{\rightarrow}_{(3.110),(3.112),(3.123)} 0.$$

This implies the following weak convergence:

$$h_n \partial_t p_n \rightharpoonup \overline{h} \partial_t \overline{p} \text{ weakly in } H^1(I, L^2(\Omega)). \tag{3.124}$$

Let us introduce the mapping $G_p : L^2(\Omega) \times W^{1,1}(I, L^2(\Omega)) \to C^1(I, L^2(\Omega)) \cap C(I, H^1(\Omega))$, $(\hat{h}, \hat{F}) \mapsto \hat{p}$, where $(\hat{p}, \hat{\boldsymbol{u}})$ solves

$$\begin{cases} \overline{\nu} \partial_t \hat{p} + \operatorname{div}(\hat{\boldsymbol{u}}) + \eta \hat{p} = \hat{F} & \text{in } I \times \Omega \\ \partial_t \hat{\boldsymbol{u}} + \nabla \hat{p} = \mathbf{0} & \text{in } I \times \Omega \\ \hat{p} = 0 & \text{on } I \times \Gamma_D \\ \hat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\hat{p}, \hat{\boldsymbol{u}})(0) = (0, \Phi'(\overline{\nu}) \hat{h}) & \text{in } \Omega. \end{cases} \tag{3.125}$$

Recalling from the proof of Lemma 3.19, $p_1 \in L^\infty(\Omega)$ is satisfied due to (3.82) and Lemma 3.16. Thus, applying Lemma 3.4 to (3.125) yields for all $(\hat{h}, \hat{F}) \in L^2(\Omega) \times W^{1,1}(I, L^2(\Omega))$ that

$$\begin{aligned} \|(\hat{p}, \hat{\boldsymbol{u}})\|_{C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} &\leq c(\|\Phi'(\overline{\nu}) \hat{h}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\nu}^{-1} \hat{F}\|_{L^1(I, L^2(\Omega))}) \\ &\underbrace{\leq}_{(3.7)} c(c_P \|p_1\|_{L^\infty(\Omega)} \|\hat{h}\|_{L^2(\Omega)} + \nu_{\min}^{-1} \|\hat{F}\|_{L^1(I, L^2(\Omega))}) \end{aligned}$$

and

$$\begin{aligned} &\|\partial_t (\hat{p}, \hat{\boldsymbol{u}})\|_{C(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega))} \\ &\leq c(\|A_{\overline{\nu}}(0, \Phi'(\overline{\nu}) \hat{h}) + (\overline{\nu}^{-1} \hat{F}(0), \mathbf{0})\|_{L^2(\Omega) \times \boldsymbol{L}^2(\Omega)} + \|\overline{\nu}^{-1} \hat{\partial}_t F\|_{L^1(I, L^2(\Omega))}) \\ &\underbrace{=}_{(3.13)} c(\|\overline{\nu}^{-1} (\hat{F}(0) - \operatorname{div}(\Phi'(\overline{\nu}) \hat{h}))\|_{L^2(\Omega)} + \|\overline{\nu}^{-1} \partial_t \hat{F}\|_{L^1(I, L^2(\Omega))}) \\ &\underbrace{\leq}_{(3.5)} c \nu_{\min}^{-1} (\|p_1\|_{L^\infty(\Omega)} \|\hat{h}\|_{L^2(\Omega)} + \|\hat{F}(0)\|_{L^2(\Omega)} + \|\partial_t \hat{F}\|_{L^1(I, L^2(\Omega))}) \end{aligned}$$

with $c := \frac{\max\{\sqrt{\nu_{\max}}, 1\}}{\min\{\sqrt{\nu_{\min}}, 1\}}$. As a consequence, the mapping G_p is linear and continuous. Thus, it is sequentially weakly continuous, and since (3.122) implies

$$\tilde{p}_n - \tilde{p} = G_p(h_n - \overline{h}, -h_n \partial_t p_n + \overline{h} \partial_t \overline{p}) \quad \forall n \in \mathbb{N},$$

the desired convergence (3.118) is obtained from (3.112) and (3.124). By the Aubin–Lions lemma and since $\dim(\Omega) \leq 3$, the embedding $C^1(I, L^2(\Omega)) \cap C(I, H^1(\Omega)) \hookrightarrow C(I, L^3(\Omega))$ is compact. For this reason,

$$\tilde{p}_n \to \tilde{p} \text{ in } C(I, L^3(\Omega)) \text{ as } n \to \infty \tag{3.126}$$

$$\underbrace{\Rightarrow}_{(3.112)} h_n \tilde{p}_n \rightharpoonup \overline{h} \tilde{p} \text{ weakly in } L^2(I, L^{6/5}(\Omega)) \text{ as } n \to \infty. \tag{3.127}$$

Furthermore, as the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is continuous (again due to $\dim(\Omega) \leq 3$), we obtain from (3.86) that $\partial_t \overline{q} \in C(I, L^6(\Omega))$. Thus, from (3.127), it follows that

$$(h_n \tilde{p}_n, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} \to (\overline{h}\tilde{p}, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} \quad \text{as } n \to \infty. \tag{3.128}$$

Along with the weak lower semicontinuity of the squared norms, by (3.112), (3.126), and (3.128), we conclude that

$$\begin{aligned}
&D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(\overline{h}, \tilde{p}, \tilde{\boldsymbol{u}})^2 \\
&= \sum_{i=1}^m \int_I \int_\Omega a_i \tilde{p}^2 \, \mathrm{d}x \, \mathrm{d}t + \lambda \|\overline{h}\|^2_{L^2(\Omega)} - 2(\overline{h}\tilde{p}, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} \\
&\leq \liminf_{n\to\infty} \left(\sum_{i=1}^m \int_I \int_\Omega a_i \tilde{p}_n^2 \, \mathrm{d}x \, \mathrm{d}t + \lambda \|h_n\|^2_{L^2(\Omega)} - 2(h_n \tilde{p}_n, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} \right) \\
&= \liminf_{n\to\infty} D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(h_n, \tilde{p}_n, \tilde{\boldsymbol{u}}_n)^2 \underbrace{\leq}_{(3.119)} 0.
\end{aligned} \tag{3.129}$$

By (SSC), we obtain that $\overline{h}(x) = 0$ for a.e. $x \in \Omega$, which implies in view of (3.104) that $(\tilde{p}, \tilde{\boldsymbol{u}}) = (0, \boldsymbol{0})$. As a consequence,

$$\begin{aligned}
0 < \lambda &= \lambda \limsup_{n\to\infty} \underbrace{\|h_n\|^2_{L^2(\Omega)}}_{=1 \text{ by } (3.110)} \\
&= \limsup_{n\to\infty} \Big(D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu}, \overline{p}, \overline{\boldsymbol{u}}, \overline{q}, \overline{\boldsymbol{v}})(h_n, \tilde{p}_n, \tilde{\boldsymbol{u}}_n)^2 - \sum_{i=1}^m \int_I \int_\Omega a_i (\tilde{p}_n)^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - 2(h_n \tilde{p}_n, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} \Big) \\
&\underbrace{\leq}_{(3.119),(3.126),(3.128)} - \sum_{i=1}^m \int_I \int_\Omega a_i \tilde{p}^2 \, \mathrm{d}x \, \mathrm{d}t - 2(\overline{h}\tilde{p}, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} = 0,
\end{aligned}$$

which is a contradiction. □

Remark 3.23. In the construction of the above proof, we have that

$$\tilde{p}_n = \frac{1}{\alpha_n}(p_n - \overline{p}) = \frac{S_p(\nu_n) - S_p(\overline{\nu})}{\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}} \quad \forall n \in \mathbb{N}.$$

Therefore, the convergence property (3.118) is nothing but

$$\frac{S_p(\nu_n) - S_p(\overline{\nu})}{\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}} \rightharpoonup \tilde{p} \text{ weakly in } C^1(I, L^2(\Omega)) \cap C(I, H^1(\Omega)) \text{ as } n \to \infty. \tag{3.130}$$

Here, $\tilde{p}$ denotes the first component of the unique solution $(\tilde{p}, \tilde{\boldsymbol{u}}) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\mathrm{div}, \Omega))$ to the linearized equation

$$\begin{cases}
\overline{\nu}\partial_t \tilde{p} + \mathrm{div}\, \tilde{\boldsymbol{u}} + \eta \tilde{p} = -\overline{h}\partial_t \overline{p} & \text{in } I \times \Omega \\
\partial_t \tilde{\boldsymbol{u}} + \nabla \tilde{p} = \boldsymbol{0} & \text{in } I \times \Omega \\
\tilde{p} = 0 & \text{on } I \times \Gamma_D \\
\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\
(\tilde{p}, \tilde{\boldsymbol{u}})(0) = (0, \Phi'(\overline{\nu})\overline{h}) & \text{in } \Omega.
\end{cases} \tag{3.131}$$

Let us now consider the remainder term for the linearization of the control-to-state operator S_p at $\overline{\nu}$ in the direction $\nu_n - \nu$ that is given by

$$r(\overline{\nu}, \nu_n - \overline{\nu}) := S_p(\nu_n) - S_p(\overline{\nu}) - \varrho_n \quad \forall n \in \mathbb{N}, \tag{3.132}$$

where, for every $n \in \mathbb{N}$, ϱ_n denotes the first component of the unique solution $(\varrho_n, \boldsymbol{\sigma}_n) \in C^1(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C(I, H_D^1(\Omega) \times \boldsymbol{H}_N(\operatorname{div}, \Omega))$ to the associated linearized problem

$$\begin{cases} \overline{\nu}\partial_t \varrho_n + \operatorname{div} \boldsymbol{\sigma_n} + \eta \varrho_n = -(\nu_n - \overline{\nu})\partial_t \overline{p} & \text{in } I \times \Omega \\ \partial_t \boldsymbol{\sigma_n} + \nabla \varrho_n = \boldsymbol{0} & \text{in } I \times \Omega \\ \varrho_n = 0 & \text{on } I \times \Gamma_D \\ \boldsymbol{\sigma_n} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (\varrho_n, \boldsymbol{\sigma_n})(0) = (0, \Phi'(\overline{\nu})(\nu_n - \overline{\nu})) & \text{in } \Omega. \end{cases} \tag{3.133}$$

Now, since

$$\frac{\nu_n - \overline{\nu}}{\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}} = h_n \rightharpoonup \overline{h} \quad \text{weakly in } L^2(\Omega) \text{ as } n \to \infty,$$

we obtain by dividing (3.133) by $\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}$ and subtracting the resulting system from (3.131) that

$$\frac{\varrho_n}{\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}} \rightharpoonup \tilde{p} \text{ weakly in } C^1(I, L^2(\Omega)) \cap C(I, H^1(\Omega)) \text{ as } n \to \infty. \tag{3.134}$$

Combining (3.130), (3.132) and (3.134), it follows that

$$\frac{r(\overline{\nu}, \nu_n - \overline{\nu})}{\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}} \rightharpoonup 0 \text{ weakly in } C^1(I, L^2(\Omega)) \cap C(I, H^1(\Omega)) \text{ as } n \to \infty.$$

In particular, the Aubin–Lions lemma implies the strong convergence

$$\lim_{n\to\infty} \frac{r(\overline{\nu}, \nu_n - \overline{\nu})}{\|\nu_n - \overline{\nu}\|_{L^2(\Omega)}} = 0 \text{ in } C(I, L^{\xi}(\Omega)) \text{ for any } \xi \in [2, 6).$$

As for the first-order necessary optimality condition (see Theorem 3.12), we also derive the second-order formulation for the second-order sufficient optimality condition.

Assumption 3.24. *Let Assumption 3.1 and* $\dim(\Omega) \leq 3$ *hold. Furthermore, let* $(\overline{\nu}, \overline{p}, \overline{q})$ *satisfy the first-order optimality system*

$$\begin{cases} \overline{\nu}\partial_t^2 \overline{p} - \Delta \overline{p} + \eta \partial_t \overline{p} = f & \textit{in } I \times \Omega \\ \partial_n \overline{p} = 0 & \textit{on } I \times \Gamma_N \\ \overline{p} = 0 & \textit{on } I \times \Gamma_D \\ (\overline{p}, \partial_t \overline{p})(0) = (0, 0) & \textit{in } \Omega \\ \overline{\nu}\partial_t^2 \overline{q} - \Delta \overline{q} - \eta \partial_t \overline{q} = \sum_{i=1}^m a_i (\overline{p} - p_i^{ob}) & \textit{in } I \times \Omega \\ \partial_n \overline{q} = 0 & \textit{on } I \times \Gamma_N \\ \overline{q} = 0 & \textit{on } I \times \Gamma_D \\ (\overline{q}, \partial_t \overline{q})(T) = (0, 0) & \textit{in } \Omega \\ \left(\int_I (\partial_t \overline{p}(t) - p_1) \overline{q}(t) \, \mathrm{d}t + \lambda \overline{\nu}, \nu - \overline{\nu} \right)_{L^2(\Omega)} \geq 0 & \forall \nu \in \mathcal{V}_{ad}, \end{cases} \tag{3.135}$$

and let the regularity assumptions (3.81) *and the compatibility assumption* (3.82) *hold.*

Corollary 3.25. *Let Assumption 3.24 hold. We assume that for the (second-order) Lagrangian*

$$\begin{aligned}\mathcal{L}(\nu,p,q) \coloneqq \mathcal{J}(\nu,p) &- (\nu\partial_t^2 p - \nabla p + \eta\partial_t p - f, q)_{L^2(I,L^2(\Omega))} - (p(0) - p_0, q(0))_{L^2(\Omega)} \\ &- (\partial_t p - p_1, \partial_t q)_{\boldsymbol{L}^2(\Omega)},\end{aligned} \tag{3.136}$$

it holds that

$$D^2_{(\nu,p)}\mathcal{L}(\overline{\nu},\overline{p},\overline{q})(h,\tilde{p})^2 > 0 \quad \forall h \in C_{\overline{\nu}} \setminus \{0\} \tag{$\widehat{\text{SSC}}$}$$

where $\tilde{p} \in C^2(I, L^2(\Omega)) \cap C^1(I, H^1_D(\Omega))$ *denotes the solution to the linearized state equation*

$$\begin{cases} \overline{\nu}\partial_t^2\tilde{p} - \Delta\tilde{p} + \eta\partial_t\tilde{p} = -h\partial_t^2\overline{p} & \text{in } I \times \Omega \\ \partial_n\tilde{p} = 0 & \text{on } I \times \Gamma_N \\ \tilde{p} = 0 & \text{on } I \times \Gamma_D \\ (\tilde{p}, \partial_t\tilde{p})(0) = (0,0) & \text{in } \Omega. \end{cases} \tag{3.137}$$

Then, there exist $\sigma > 0$ *and* $\delta > 0$ *such that the quadratic growth condition*

$$J(\nu) \geq J(\overline{\nu}) + \delta\|\nu - \overline{\nu}\|^2_{L^2(\Omega)} \tag{3.138}$$

holds for every $\nu \in \mathcal{V}_{ad}$ *with* $\|\nu - \overline{\nu}\|_{L^2(\Omega)} \leq \sigma$*. In particular,* $\overline{\nu}$ *is a locally optimal solution to* (P)*.*

Remark 3.26. In contrast to the first-order necessary optimality conditions in its second-order formulation (see Corollary 3.14), the second-order sufficient optimality conditions in its second-order formulation do not rely on stronger assumptions on the data than its first-order formulation (see Theorem 3.22). Furthermore, it turns out that conditions (SSC) and $\widehat{\text{SSC}}$ are equivalent. However, the availability of both the first-order and second-order formulations allows for a highly flexible application.

Proof. Let $(q, \boldsymbol{v}) \coloneqq (\partial_t\overline{q}, \nabla\overline{q})$. Then, as in the proof of Corollary 3.14, $(q, \boldsymbol{v})$ is the unique solution to the (first-order) adjoint equation (3.46) associated with $\overline{\nu}$. In particular $(q, \boldsymbol{v})$ is the adjoint state in the sense of Definition 3.8 associated with $\overline{\nu}$. Furthermore, (3.135), implies that (3.62) is valid where $\overline{q}$ is replaced with q. Therefore, due to Theorem 3.22, in order to obtain (3.138), it remains to show that ($\widehat{\text{SSC}}$) implies (SSC) where $(\overline{q}, \overline{\boldsymbol{v}})$ is replaced with $(q, \boldsymbol{v})$. For this purpose, let $h \in C_{\overline{\nu}} \setminus \{0\}$. Then, due to Lemma 3.4, the unique solution $(\tilde{p}, \tilde{\boldsymbol{u}})$ to the linearized (first-order) state equation (3.104) enjoys the higher regularity property $(\tilde{p}, \tilde{\boldsymbol{u}}) \in C^2(I, L^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap C^1(I, H^1_D(\Omega) \times \boldsymbol{H}_N(\text{div}, \Omega))$ since $h\partial_t\overline{p} \in W^{2,1}(I, L^2(\Omega))$ and

$$\begin{aligned}(\tilde{p}, \tilde{\boldsymbol{u}})_{0,0} &= (0, \Phi'(\overline{\nu})h) \in H^1_D(\Omega) \times \boldsymbol{H}_N(\text{div}, \Omega) \\ (\tilde{p}, \tilde{\boldsymbol{u}})_{0,1} &= A_{\overline{\nu}}(0, \Phi'(\overline{\nu})h) - (\overline{\nu}^{-1}h\partial_t\overline{p}(0), \boldsymbol{0}) \underbrace{=}_{(3.13)} (-\overline{\nu}^{-1}\,\text{div}(\Phi'(\overline{\nu})h) - \overline{\nu}^{-1}h\partial_t\overline{p}(0), \boldsymbol{0}) \\ &\underbrace{=}_{(3.5)} (\overline{\nu}h(p_1 - \partial_t\overline{p}(0)), \boldsymbol{0}) = (0, \boldsymbol{0}) \in H^1_D(\Omega) \times \boldsymbol{H}_N(\text{div}, \Omega).\end{aligned}$$

Therefore, by time derivation of the first line in (3.104) and inserting the second line in (3.104), we obtain that $\tilde{p}$ satisfies the first line in (3.137). Furthermore, it holds that

$$\partial_t \tilde{p}(0) \underbrace{=}_{(3.104)} \overline{\nu}^{-1}(-h\partial_t \overline{p}(0) - \operatorname{div}(\tilde{\boldsymbol{u}}(0)) - \eta \tilde{p}(0)) \underbrace{=}_{(3.3),(3.104)} \overline{\nu}^{-1}(-hp_1 - \operatorname{div}(\Phi'(\overline{\nu})h)) \underbrace{=}_{(3.5)} 0.$$

In conclusion, $\tilde{p}$ solves the (second-order) linearised state equation (3.137). Therefore, we have that

$$\begin{aligned}
&D^2_{(\nu,p,\boldsymbol{u})}\mathcal{L}(\overline{\nu},\overline{p},\overline{\boldsymbol{u}},\hat{q},\hat{\boldsymbol{v}})(h,p,\boldsymbol{u})^2 \\
&= \sum_{i=1}^m \int_I \int_\Omega a_i \tilde{p}^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda \|\overline{h}\|^2_{L^2(\Omega)} + 2(\overline{h}\partial_t p, \hat{q})_{L^2(I,L^2(\Omega))} \\
&= \sum_{i=1}^m \int_I \int_\Omega a_i \tilde{p}^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda \|\overline{h}\|^2_{L^2(\Omega)} + 2(\overline{h}\partial_t \tilde{p}, \partial_t \overline{q})_{L^2(I,L^2(\Omega))} \\
&= \sum_{i=1}^m \int_I \int_\Omega a_i \tilde{p}^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda \|\overline{h}\|^2_{L^2(\Omega)} - 2(\overline{h}\partial_t^2 \tilde{p}, \overline{q})_{L^2(I,L^2(\Omega))} \\
&= D^2_{(\nu,p)}\mathcal{L}(\overline{\nu},\overline{p},\overline{q})(h,\tilde{p})^2 \underbrace{>}_{(\widehat{\text{SSC}})} 0.
\end{aligned}$$

Therefore, (SSC) is satisfied and the claim follows from Theorem 3.22. □

SEQUENTIAL QUADRATIC PROGRAMMING 4

The SQP method is a celebrated technique in finite and infinite dimensional optimization, particularly in the context of optimal control problems. We refer to the earlier contributions by Alt [2, 3] and Alt, Sontag, and Tröltzsch [4] for SQP methods of optimization problems with ODE or integral equations constraints. From among many other related works in the context of PDE-constrained problems, we mention the contributions by Ito and Kunisch [50, 51], Tröltzsch et al. [37, 42, 70, 71, 73], Heinkenschloss [41], Hintermüller and Hinze [44, 45], Volkwein [75], Wachsmuth [77], Griesse et al. [39, 40], Hinze and Kunisch [47], and Hoppe and Neitzel [49]. Even though the investigations of SQP methods are highly problem-specified, they mainly follow the same methodology: Reformulation of the SQP method as a generalized Newton method and exploitation of Robinson's concept of *strong regularity* [69]. This unified ansatz leads to well-posedness and quadratic convergence of SQP iterations. Eventually, one verifies the strong regularity condition using suitable second-order sufficient optimality conditions.

Note that the works mentioned above only focus on elliptic and parabolic PDEs. To the best of the author's knowledge, there is no contribution to the analysis of SQP methods in hyperbolic PDE-constrained optimization, apart from the author's preprint [6]. For our model problem (P), the SQP analysis results in a challenging task due to the underlying hyperbolicity and the second-order bilinear structure $\nu\partial_t^2 p$. This character leads to an undesired effect of *loss of regularity* in the SQP method (see Algorithm 1) causing two substantial difficulties (see Remark 4.6):

(i) In general, Algorithm 1 is only executable for a limited number of iterations, i.e., the well-definedness of Algorithm 1 may fail.

(ii) The ansatz through the notion of strong regularity, as done in the parabolic case (cf. [70]), cannot be directly transferred to our case and requires a substantial extension.

This chapter develops a strategy for analyzing Algorithm 1 and consists of three primary steps. First of all, we propose the use of a smooth-in-time initial guess for the state p_0 and the adjoint state q_0 satisfying $\partial_t^l p_0(0) = \partial_t^l q_0(T) = 0$ for all $l \in \mathbb{N}$ (Assumption 4.7). Under these regularity conditions, we manage to prove the well-definedness of Algorithm 1 (see Proposition 4.3). As the second step, for every given SQP iteration (ν_k, p_k, q_k), we construct a suitable self-mapping operator (4.44). Based on a perturbation analysis (see Theorem 4.4) using Stampacchia's method (see Lemma 3.17), it turns out that the contraction principle can be applied to the operator (4.44) (see Proposition 4.14). The resulting fixed point ν_{k+1} is exactly the control component of the solution to the SQP iteration $(\mathbb{P}_k)$ (see Proposition 4.16). The final step comprises a *two-step* estimation process: We estimate $\|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)}$ by the total error of the previous step $\|\nu_k-\overline{\nu}\|_{L^2(\Omega)}$, $\|p_k-\overline{p}\|_{L^2(I,L^2(\Omega))}$, and $\|q_k-\overline{q}\|_{L^2(I,L^2(\Omega))}$. Then, the error in the state $\|p_k-\overline{p}\|_{L^2(I,L^2(\Omega))}$ and adjoint state $\|q_k-\overline{q}\|_{L^2(I,L^2(\Omega))}$ are estimated towards $\|\nu_{k-1}-\overline{\nu}\|_{L^2(\Omega)}$. This process results in the quadratic two-step estimation

$$\|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)} \le \delta\big(\|\overline{\nu}-\nu_k\|_{L^2(\Omega)}+\|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)}\big)^2,$$

which eventually allows us to prove our main result on the R-superlinear convergence of Algorithm 1 (see Theorem 4.17).

The content of this chapter is available in the author's preprint [6]. Consequently, direct quotations from this work will not be explicitly highlighted.

4.1 Perturbed Optimality System

This section is devoted to the analysis of perturbed and linearized optimality systems associated with (P), which play an essential role in the analysis of the SQP method (see Section 4.2). We want the sufficient second-order optimality result (see Corollary 3.25) to be available. For simplicity, we assume in this chapter that the initial values p_0 and p_1 vanish. Therefore, the standing assumptions for this chapter read as follows:

Assumption 4.1. *Let Assumption 3.24 hold with $p_0=p_1=0$. Furthermore, let $p_i^{ob}\in W^{4,1}(I,L^2(\Omega))$, and let $a_i\in C^4(I,L^\infty(\Omega))$ assumed to be nonnegative with $\partial_t^l a_i(T)=0$ for all $i=1,\dots,m$ and $l=0,1,2,3$.*

To begin with, recalling Lemma 3.4 and Lemma 3.17, we deduce the following lemma:

Lemma 4.2. *Let Assumption 4.1 hold and let $\nu\in\mathcal{V}_{ad}$. Further, let $g\in W^{k,1}(I,L^2(\Omega))$ for some $k\in\mathbb{N}$ and $\partial_t^l g(0)=0$ for $l=0,\dots,k-1$. Then, there exists a unique solution $p\in C^{k+1}(I,L^2(\Omega))\cap C^k(I,H_D^1(\Omega))\cap C^{k-1}(I,D(\Delta_{D,N}))\cap C^{k-1}(I,L^\infty(\Omega))$ to*

$$\begin{cases}\nu\partial_t^2 p-\Delta p+\eta\partial_t p=g & \text{in } I\times\Omega\\ \partial_n p=0 & \text{on } I\times\Gamma_N\\ p=0 & \text{on } I\times\Gamma_D\\ (p,\partial_t p)(0)=(0,0) & \text{in } \Omega.\end{cases}\tag{4.1}$$

Furthermore, it holds that

$$\begin{aligned}\|p(t)\|_{L^2(\Omega)} &\le c\|G\|_{L^1(I,L^2(\Omega))} && \forall t\in I\\ \|\partial_t^l p(t)\|_{L^2(\Omega)} &\le c\|\partial_t^{l-1}g\|_{L^1(I,L^2(\Omega))} && \forall t\in I\ \forall l=1,\dots,k+1\\ \|\partial_t^l p\|_{L^2(I,L^\infty(\Omega))} &\le \hat{c}\|\partial_t^l g\|_{L^2(I,L^2(\Omega))}+\|\partial_t^{l+1}g\|_{L^2(I,L^2(\Omega))} && \forall t\in I\ \forall l=0,\dots,k-1\end{aligned}$$

with $c:=\nu_{\min}^{-1}\frac{\max\{\sqrt{\nu_{\max}},1\}}{\min\{\sqrt{\nu_{\min}},1\}}$, $G(t):=\int_0^t g(s)\,\mathrm{d}s$ for all $t\in I$, and for a constant $\hat{c}>0$.

Proof. Writing $G(t) = \nu^{-1}\int_0^t g(s)\,\mathrm{d}s$ for all $t \in I$, due to Lemma 3.4, the system

$$\begin{cases} \nu\partial_t p + \operatorname{div}\boldsymbol{u} + \eta p = G & \text{in } I\times\Omega \\ \partial_t \boldsymbol{u} + \nabla p = \boldsymbol{0} & \text{in } I\times\Omega \\ p = 0 & \text{on } I\times\Gamma_D \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } I\times\Gamma_N \\ (p,\boldsymbol{u})(0) = (0,\boldsymbol{0}) & \text{in } \Omega \end{cases} \tag{4.2}$$

admits a unique solution $(p,\boldsymbol{u}) \in C^{k+1}(I, L^2(\Omega)\times \boldsymbol{L}^2(\Omega)) \cap C^k(I, H^1_D(\Omega)\times \boldsymbol{H}_N(\operatorname{div},\Omega))$. According to (4.2), we have that

$$\nabla p = -\partial_t \boldsymbol{u} \in C^{k-1}(I, \boldsymbol{H}_N(\operatorname{div},\Omega)) \quad\Rightarrow\quad p \in C^{k-1}(I, D(\Delta_{D,N})).$$

Therefore, taking the time derivative of the first equation in (4.2), and inserting the second equation in (4.2), it follows that p satisfies the first equation (4.1). The initial value conditions follow as in the proof of Theorem 3.5. The first two estimates follow from Lemma 3.4. Furthermore, Lemma 3.17 implies the regularity property $p \in C^{k-1}(I, L^\infty(\Omega))$ and the last desired estimate. □

Given some perturbation term $(\rho^{VI}, \rho^{st}, \rho^{adj}) \in L^\infty(\Omega)\times H^1(I,L^2(\Omega))\times H^1(I,L^2(\Omega))$ with $\rho^{st}(0) = \rho^{adj}(T) = 0$, we consider the system

$$\begin{cases} \overline{\nu}\partial_t^2 p - \Delta p + \eta\partial_t p = f - (\nu-\overline{\nu})\partial_t^2\overline{p} + \rho^{st} & \text{in } I\times\Omega \\ \partial_n p = 0 & \text{on } I\times\Gamma_N \\ p = 0 & \text{on } I\times\Gamma_D \\ (p,\partial_t p)(0) = (0,0) & \text{in } \Omega \\ \overline{\nu}\partial_t^2 q - \Delta q - \eta\partial_t q = \sum_{i=1}^m a_i(p-p_i^{ob}) - (\nu-\overline{\nu})\partial_t^2\overline{q} + \rho^{adj} & \text{in } I\times\Omega \\ \partial_n q = 0 & \text{on } I\times\Gamma_N \\ q = 0 & \text{on } I\times\Gamma_D \\ (q,\partial_t q)(T) = (0,0) & \text{in } \Omega \\ (-\int_I \partial_t^2\overline{p}(t)q(t) + \partial_t^2(p(t)-\overline{p}(t))\overline{q}(t)\,\mathrm{d}t + \lambda\nu, \tilde{\nu}-\nu)_{L^2(\Omega)} & \\ \geq (\rho^{VI}, \tilde{\nu}-\nu)_{L^2(\Omega)} & \text{for all } \tilde{\nu}\in\mathcal{V}_{ad}. \end{cases} \tag{OS}$$

Sufficient second-order optimality conditions for (P) are the main ingredients for the analysis of (OS). Unfortunately, the proposed ($\widehat{\text{SSC}}$) in Corollary 3.25 is too weak for our purposes, as the involved critical cone $C^0_{\overline{\nu}} \setminus \{0\}$ is too restrictive. Thus, for a fixed $\tau > 0$, we introduce the enlarged critical cone

$$C^\tau_{\overline{\nu}} := \{h \in L^2(\Omega) : h(x) = 0 \text{ for a.e. } x \in \mathscr{A}_\tau(\overline{\nu})\}, \tag{4.3}$$

where the set of τ-uniform strongly active constraints is given by

$$\mathscr{A}_\tau(\overline{\nu}) := \left\{x\in\Omega : \left| -\int_0^T \partial_t^2\overline{p}(t,x)\overline{q}(t,x)\,\mathrm{d}t + \lambda\overline{\nu}(x)\right| > \tau\right\}. \tag{4.4}$$

Then, the strengthened sufficient second-order optimality condition reads as follows:

$$\begin{cases}\text{There exists } \alpha > 0 \text{ such that } D^2_{(\nu,p)}\mathcal{L}(\overline{\nu},\overline{p},\overline{q})(h,\tilde{p})^2 \geq \alpha\|h\|^2_{L^2(\Omega)} \text{ for every } h \in C^\tau_{\overline{\nu}} \\ \text{where } \tilde{p} \in C^2(I,L^2(\Omega)) \cap C^1(I,H^1_D(\Omega)) \cap C(I,D(\Delta_{D,N})) \text{ solves (3.104).}\end{cases} \tag{SSC$^\tau$}$$

Note that $\mathscr{A}_\tau(\overline{\nu}) \subset \mathscr{A}_0(\overline{\nu})$ (compare (3.106) and (4.4)) and therefore the new critical cone $C^\tau_{\overline{\nu}}$ is in fact enlarged, i.e., $C^0_{\overline{\nu}} \subset C^\tau_{\overline{\nu}}$ (compare (3.108) and (4.3)). Therefore, the strengthened sufficient second-order condition (SSC$^\tau$) particularly implies the original condition ($\widehat{\text{SSC}}$). As a consequence, it also guarantees the optimality of $\overline{\nu}$ along with the quadratic growth condition (3.138) under the same assumptions as in Corollary 3.25. Still, with the strengthened condition (SSC$^\tau$) the following difficulty appears: Given two controls $\nu_1,\nu_2 \in \mathcal{V}_{ad}$, the difference $h = \nu - \overline{\nu}$ for $\nu \in \mathcal{V}_{ad}$ does not belong to the enlarged critical cone $C^\tau_{\overline{\nu}}$. Therefore, it does not satisfy the assumptions of (SSC$^\tau$). To circumvent this difficulty, we extend well-known techniques (see [49,70,75,77]) to our hyperbolic case. We consider an auxiliary problem by replacing the admissible set $\mathcal{V}_{ad}$ with

$$\mathcal{V}^\tau_{ad} := \{\nu \in \mathcal{V}_{ad} \mid \nu = \overline{\nu} \text{ a.e. in } \mathscr{A}_\tau(\overline{\nu})\}. \tag{4.5}$$

Now, given two controls $\nu,\widetilde{\nu} \in \mathcal{V}^\tau_{ad}$, the difference $h = \nu - \widetilde{\nu}$ fullfils $h \in C^\tau_{\overline{\nu}}$. We define the following modification of the perturbed linearized optimality system (OS):

$$\text{(OS) where } \mathcal{V}_{ad} \text{ is replaced with } \mathcal{V}^\tau_{ad}. \tag{OS$^\tau$}$$

Proposition 4.3. *Let Assumption 4.1 and* (SSC$^\tau$) *hold. Then, for all* $(\rho^{st},\rho^{adj},\rho^{VI}) \in H^1(I,L^2(\Omega)) \times H^1(I,L^2(\Omega)) \times L^2(\Omega)$ *with* $\rho^{st}(0) = \rho^{adj}(T) = 0$, *the system* (OS$^\tau$) *admits a unique solution* $(\nu,p,q) \in \mathcal{V}^\tau_{ad} \times (C^2(I,L^2(\Omega)) \cap C^1(I,H^1_D(\Omega)) \cap C(I,D(\Delta_{D,N})))^2$.

Proof. Let $(\rho^{st},\rho^{adj},\rho^{VI}) \in H^1(I,L^2(\Omega)) \times H^1(I,L^2(\Omega)) \times L^2(\Omega)$ with $\rho^{st}(0) = \rho^{adj}(T) = 0$ be given. Thanks to (3.84), Lemma 4.2 implies that

$$\begin{cases}\overline{\nu}\partial_t^2 p - \Delta p + \eta\partial_t p = f - (\nu - \overline{\nu})\partial_t^2\overline{p} + \rho^{st} & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ (p,\partial_t p)(0) = (0,0) & \text{in } \Omega\end{cases} \tag{4.6}$$

admits a unique solution $p \in C^2(I,L^2(\Omega)) \cap C^1(I,H^1_D(\Omega)) \cap C(I,D(\Delta_{D,N}))$ for every $\nu \in L^2(\Omega)$. Denoting by $S_\rho\colon L^2(\Omega) \to C^2(I,L^2(\Omega)) \cap C^1(I,H^1_D(\Omega)), \nu \mapsto p$ the affine linear and continuous solution operator to (4.6), we consider the minimization problem

$$\begin{aligned}\min_{\nu\in\mathcal{V}^\tau_{ad}} J_\rho(\nu) := \mathcal{J}(\nu,S_\rho(\nu)) + \Big(-\int_I \partial_t^2(S_\rho(\nu) - \overline{p})\overline{q}\,\mathrm{d}t - \rho^{VI}, \nu - \overline{\nu}\Big)_{L^2(\Omega)} \\ + (\rho^{adj}, S_\rho(\nu))_{L^2(I,L^2(\Omega))},\end{aligned} \tag{4.7}$$

where $\mathcal{J}$ is defined as in (P). By the quadratic structure of J_ρ, we have that

$$J_\rho(\tilde{\nu}) = J_\rho(\nu) + J_\rho'(\nu)(\tilde{\nu} - \nu) + \frac{1}{2}J_\rho''(\nu)(\tilde{\nu} - \nu)^2 \quad \forall \nu,\tilde{\nu} \in \mathcal{V}^\tau_{ad}. \tag{4.8}$$

Further, for any $\nu, \tilde{\nu} \in \mathcal{V}_{ad}^{\tau}$ with $\tilde{\nu} \neq \nu$, it holds that $h \coloneqq \tilde{\nu} - \nu \in C_{\overline{\nu}}^{\tau} \setminus \{0\}$ (see (4.3)) and $\tilde{p} \coloneqq S_\rho(\tilde{\nu}) - S_\rho(\nu)$ solves the linearized state equation (3.104) such that (SSC$^\tau$) yields that

$$0 < D^2_{(\nu,p)}\mathcal{L}(\overline{\nu},\overline{p},\overline{q})(h,\tilde{p})^2 \underbrace{=}_{(3.136)} \sum_{i=1}^{m}(a_i\tilde{p},\tilde{p})_{L^2(I,L^2(\Omega))} + \lambda\|h\|^2_{L^2(\Omega)} - 2(h\partial_t^2\tilde{p},\overline{q})_{L^2(I,L^2(\Omega))}$$
$$\underbrace{=}_{(4.7)} J_\rho''(\nu)h^2.$$

Thus, for every $t \in (0,1)$, we have that

$$\begin{aligned}
J_\rho(\tilde{\nu} + t(\nu - \tilde{\nu})) &\underbrace{=}_{(4.8)} J_\rho(\nu) + J_\rho'(\nu)((1-t)(\tilde{\nu}-\nu)) + \frac{1}{2}J_\rho''(\nu)((1-t)\tilde{\nu}-\nu)^2 \\
&= tJ_\rho(\nu) + (1-t)(J_\rho(\nu) + J_\rho'(\nu)(\tilde{\nu}-\nu) + \frac{(1-t)}{2}\underbrace{J_\rho''(\nu)(\tilde{\nu}-\nu)^2}_{>0}) \\
&< tJ_\rho(\nu) + (1-t)(J_\rho(\nu) + J_\rho'(\nu)(\tilde{\nu}-\nu) + \frac{1}{2}J_\rho''(\nu)(\tilde{\nu}-\nu)^2) \\
&\underbrace{=}_{(4.8)} tJ_\rho(\nu) + (1-t)J_\rho(\tilde{\nu}).
\end{aligned}$$

Therefore, J_ρ in $\mathcal{V}_{ad}^{\tau}$ is strictly convex. In conclusion, together with the continuity of $J_\rho \colon L^2(\Omega) \to \mathbb{R}$, (4.7) admits a unique solution $\nu \in \mathcal{V}_{ad}^{\tau}$. Moreover, the necessary and sufficient optimality condition for (4.7) is given by $J_\rho'(\nu)(\tilde{\nu} - \nu) \geq 0$ for every $\tilde{\nu} \in \mathcal{V}_{ad}^{\tau}$, which is equivalent to (OS$^\tau$) due to standard arguments. Thus, the claim is valid. □

Theorem 4.4 provides the crucial stability result for the solution to (OS$^\tau$) regarding the perturbation terms. In the proof, we extend known ideas incorporating the first- and second-order optimality conditions (cf. [73, 75]) to our given case.

Theorem 4.4. *Let Assumption 4.1 and* (SSC$^\tau$) *hold. Then, there exist constants* $L > 0, L_p > 0$, *and* $L_q > 0$ *such that for all perturbation terms* $(\rho^{st}, \rho^{adj}, \rho^{VI}), (\widetilde{\rho}^{st}, \widetilde{\rho}^{adj}, \widetilde{\rho}^{VI}) \in H^1(I, L^2(\Omega)) \times H^1(I, L^2(\Omega)) \times L^2(\Omega)$ *with* $\rho^{st}(0) = \widetilde{\rho}^{st}(0) = \rho^{adj}(T) = \widetilde{\rho}^{adj}(T) = 0$, *the corresponding solutions* $(\nu_\rho, p_\rho, q_\rho)$ *and* $(\nu_{\widetilde{\rho}}, p_{\widetilde{\rho}}, q_{\widetilde{\rho}})$ *to* (OS$^\tau$) *satisfy*

$$\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \leq L(\|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} + \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))} + \|\rho^{VI} - \widetilde{\rho}^{VI}\|_{L^2(\Omega)}) \tag{4.9}$$
$$\|p_\rho - \overline{p}\|_{L^2(I,L^\infty(\Omega))} \leq L_p(\|\rho^{st}\|_{H^1(I,L^2(\Omega))} + \|\rho^{adj}\|_{L^2(I,L^2(\Omega))} + \|\rho^{VI}\|_{L^2(\Omega)}) \tag{4.10}$$
$$\|q_\rho - \overline{q}\|_{L^2(I,L^\infty(\Omega))} \leq L_q(\|\rho^{st}\|_{H^1(I,L^2(\Omega))} + \|\rho^{adj}\|_{H^1(I,L^2(\Omega))} + \|\rho^{VI}\|_{L^2(\Omega)}). \tag{4.11}$$

Proof. Let $(\nu_\rho, p_\rho, q_\rho), (\nu_{\widetilde{\rho}}, p_{\widetilde{\rho}}, q_{\widetilde{\rho}}) \in \mathcal{V}_{ad}^{\tau} \times (C^2(I, L^2(\Omega)) \cap C^1(I, H_D^1(\Omega)) \cap C(I, D(\Delta_{D,N})))^2$ denote the unique solutions to (OS$^\tau$) with respect to $(\rho^{st}, \rho^{adj}, \rho^{VI})$ and $(\widetilde{\rho}^{VI}, \widetilde{\rho}^{st}, \widetilde{\rho}^{adj})$ according to Proposition 4.3. Subtracting the corresponding PDE-systems (see (OS)), we obtain that

$$\begin{cases}
\overline{\nu}\partial_t^2(p_\rho - p_{\widetilde{\rho}}) - \Delta(p_\rho - p_{\widetilde{\rho}}) + \eta\partial_t(p_\rho - p_{\widetilde{\rho}}) = -(\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2\overline{p} + \rho^{st} - \widetilde{\rho}^{st} & \text{in } I \times \Omega \\
\partial_n(p_\rho - p_{\widetilde{\rho}}) = 0 & \text{on } I \times \Gamma_N \\
p_\rho - p_{\widetilde{\rho}} = 0 & \text{on } I \times \Gamma_D \\
(p_\rho - p_{\widetilde{\rho}}, \partial_t(p_\rho - p_{\widetilde{\rho}}))(0) = (0,0) & \text{in } \Omega
\end{cases} \tag{4.12}$$

$$\begin{cases} \overline{\nu}\partial_t^2(q_\rho - q_{\widetilde{\rho}}) - \Delta(q_\rho - q_{\widetilde{\rho}}) - \eta\partial_t(q_\rho - q_{\widetilde{\rho}}) & \\ \quad = \sum_{i=1}^m a_i(p_\rho - p_{\widetilde{\rho}}) - (\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2\overline{q} + \rho^{adj} - \widetilde{\rho}^{adj} & \text{in } I \times \Omega \\ \partial_n(q_\rho - q_{\widetilde{\rho}}) = 0 & \text{on } I \times \Gamma_N \\ q_\rho - q_{\widetilde{\rho}} = 0 & \text{on } I \times \Gamma_D \\ (q_\rho - q_{\widetilde{\rho}}, \partial_t(q_\rho - q_{\widetilde{\rho}}))(T) = (0,0) & \text{in } \Omega. \end{cases} \tag{4.13}$$

We begin by elaborating on the control parameter. By the construction of $\mathcal{V}_{ad}^\tau$ (see (4.3)), the quantity $h \coloneqq \nu_\rho - \nu_{\widetilde{\rho}}$ lies in the critical cone C_ν^τ, and $p_\rho - p_{\widetilde{\rho}} - \hat{p}_{\rho,\widetilde{\rho}}$ solves the associated linearized state equation (3.104) where $\hat{p}_{\rho,\widetilde{\rho}}$ denotes the solution to

$$\begin{cases} \overline{\nu}\partial_t^2\hat{p}_{\rho,\widetilde{\rho}} - \Delta\hat{p}_{\rho,\widetilde{\rho}} + \eta\partial_t\hat{p}_{\rho,\widetilde{\rho}} = \rho^{st} - \widetilde{\rho}^{st} & \text{in } I \times \Omega \\ \partial_n\hat{p}_{\rho,\widetilde{\rho}} = 0 & \text{on } I \times \Gamma_N \\ \hat{p}_{\rho,\widetilde{\rho}} = 0 & \text{on } I \times \Gamma_D \\ (\hat{p}_{\rho,\widetilde{\rho}}, \partial_t\hat{p}_{\rho,\widetilde{\rho}})(0) = (0,0) & \text{in } \Omega. \end{cases} \tag{4.14}$$

Thus, (SSC^τ) yields that

$$\begin{aligned} \alpha\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}^2 &\le D^2_{(\nu,p)}\mathcal{L}(\overline{\nu},\overline{p},\overline{q})(\nu_\rho - \nu_{\widetilde{\rho}}, p_\rho - p_{\widetilde{\rho}} - \hat{p}_{\rho,\widetilde{\rho}})^2 \qquad (4.15) \\ &\underbrace{=}_{(3.136)} \sum_{i=1}^m (a_i(p_\rho - p_{\widetilde{\rho}}), p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} + \lambda\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}^2 \\ &\quad - 2((\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2(p_\rho - p_{\widetilde{\rho}}), \overline{q})_{L^2(I,L^2(\Omega))} + \sum_{i=1}^m (a_i\hat{p}_{\rho,\widetilde{\rho}}, \hat{p}_{\rho,\widetilde{\rho}})_{L^2(I,L^2(\Omega))} \\ &\quad - 2\sum_{i=1}^m (a_i\hat{p}_{\rho,\widetilde{\rho}}, (p_\rho - p_{\widetilde{\rho}}))_{L^2(I,L^2(\Omega))} + 2((\nu_\rho - \nu_{\widetilde{\rho}})\hat{p}_{\rho,\widetilde{\rho}}, \partial_t^2\overline{q})_{L^2(I,L^2(\Omega))}. \qquad (4.16) \end{aligned}$$

For the first term on the right-hand side of (4.15), it holds that

$$\begin{aligned} &\sum_{i=1}^m (a_i(p_\rho - p_{\widetilde{\rho}}), p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} \\ &\underbrace{=}_{(4.13)} (\overline{\nu}\partial_t^2(q_\rho - q_{\widetilde{\rho}}) - \Delta(q_\rho - q_{\widetilde{\rho}}) - \eta\partial_t(q_\rho - q_{\widetilde{\rho}}) + (\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2\overline{q} - \rho^{adj} + \widetilde{\rho}^{adj}, p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} \\ &= (q_\rho - q_{\widetilde{\rho}}, \overline{\nu}\partial_t^2(p_\rho - p_{\widetilde{\rho}}) - \Delta(p_\rho - p_{\widetilde{\rho}}) + \eta\partial_t(p_\rho - p_{\widetilde{\rho}}))_{L^2(I,L^2(\Omega))} \\ &\quad + (\overline{q}, (\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2(p_\rho - p_{\widetilde{\rho}}))_{L^2(I,L^2(\Omega))} - (\rho^{adj} - \widetilde{\rho}^{adj}, p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} \\ &\underbrace{=}_{(4.12)} (q_\rho - q_{\widetilde{\rho}}, -(\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2\overline{p} + \rho^{st} - \widetilde{\rho}^{st}) + (\overline{q}, (\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2(p_\rho - p_{\widetilde{\rho}}))_{L^2(I,L^2(\Omega))} \\ &\quad - (\rho^{adj} - \widetilde{\rho}^{adj}, p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))}. \end{aligned}$$

Applying this identity to (4.15), we obtain that

$$\begin{aligned} &\alpha\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}^2 \qquad (4.17) \\ &\le -((\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2\overline{p}, q_\rho - q_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} + (q_\rho - q_{\widetilde{\rho}}, \rho^{st} - \widetilde{\rho}^{st})_{L^2(I,L^2(\Omega))} \\ &\quad - (\rho^{adj} - \widetilde{\rho}^{adj}, p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} + \lambda\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}^2 - ((\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2(p_\rho - p_{\widetilde{\rho}}), \overline{q})_{L^2(I,L^2(\Omega))} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{m} (a_i \hat{p}_{\rho,\widetilde{\rho}}, \hat{p}_{\rho,\widetilde{\rho}})_{L^2(I,L^2(\Omega))} - 2 \sum_{i=1}^{m} (a_i \hat{p}_{\rho,\widetilde{\rho}}, (p_\rho - p_{\widetilde{\rho}}))_{L^2(I,L^2(\Omega))} \\
&+ 2((\nu_\rho - \nu_{\widetilde{\rho}})\hat{p}_{\rho,\widetilde{\rho}}, \partial_t^2 \overline{q})_{L^2(I,L^2(\Omega))}.
\end{aligned}
$$

Testing the variational inequality in (OS^τ) for $(\nu_\rho, p_\rho, q_\rho)$ (resp. $(\nu_{\widetilde{\rho}}, p_{\widetilde{\rho}}, q_{\widetilde{\rho}})$) with $\tilde{\nu} = \nu_{\widetilde{\rho}}$ (resp. $\tilde{\nu} = \nu_\rho$) and adding the resulting two inequalites leads to

$$
\begin{aligned}
&\left(- \int_I \partial_t^2 \overline{p}(t)(q_\rho(t) - q_{\widetilde{\rho}}(t)) + \partial_t^2 (p_\rho(t) - p_{\widetilde{\rho}}(t)) \overline{q}(t) \, \mathrm{d}t + \lambda(\nu_\rho - \nu_{\widetilde{\rho}}), \nu_{\widetilde{\rho}} - \nu_\rho \right)_{L^2(\Omega)} \\
&\geq (\rho^{VI} - \widetilde{\rho}^{VI}, \nu_{\widetilde{\rho}} - \nu_\rho)_{L^2(\Omega)}.
\end{aligned}
$$

Rearranging yields that

$$
\begin{aligned}
&\lambda \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}^2 - ((\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2 \overline{p}, q_\rho - q_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} - ((\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2 (p_\rho - p_{\widetilde{\rho}}), \overline{q})_{L^2(I,L^2(\Omega))} \\
&\leq (\rho^{VI} - \widetilde{\rho}^{VI}, \nu_\rho - \nu_{\widetilde{\rho}})_{L^2(\Omega)}.
\end{aligned} \tag{4.18}
$$

Combining (4.17) and (4.18), we obtain that

$$
\begin{aligned}
&\alpha \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}^2 \\
&\leq (q_\rho - q_{\widetilde{\rho}}, \rho^{st} - \widetilde{\rho}^{st})_{L^2(I,L^2(\Omega))} - (\rho^{adj} - \widetilde{\rho}^{adj}, p_\rho - p_{\widetilde{\rho}})_{L^2(I,L^2(\Omega))} + (\rho^{VI} - \widetilde{\rho}^{VI}, \nu_\rho - \nu_{\widetilde{\rho}})_{L^2(\Omega)} \\
&\quad + \sum_{i=1}^{m} (a_i \hat{p}_{\rho,\widetilde{\rho}}, \hat{p}_{\rho,\widetilde{\rho}})_{L^2(I,L^2(\Omega))} - 2 \sum_{i=1}^{m} (a_i \hat{p}_{\rho,\widetilde{\rho}}, (p_\rho - p_{\widetilde{\rho}}))_{L^2(I,L^2(\Omega))} \\
&\quad + 2((\nu_\rho - \nu_{\widetilde{\rho}})\partial_t^2 \hat{p}_{\rho,\widetilde{\rho}}, \overline{q})_{L^2(I,L^2(\Omega))} \\
&\leq \|q_\rho - q_{\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} + \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))} \|p_\rho - p_{\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \\
&\quad + \|\rho^{VI} - \widetilde{\rho}^{VI}\|_{L^2(\Omega)} \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} + \sum_{i=1}^{m} \|a_i\|_{L^\infty(I,L^\infty(\Omega))} \|\hat{p}_{\rho,\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))}^2 \\
&\quad + 2 \sum_{i=1}^{m} \|a_i\|_{L^\infty(I,L^\infty(\Omega))} \|\hat{p}_{\rho,\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \|p_\rho - p_{\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \\
&\quad + 2 \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\hat{p}_{\rho,\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))}.
\end{aligned} \tag{4.19}
$$

Applying Lemma 4.2 to (4.14) yields for $G(t) \coloneqq \int_0^t \rho^{st}(s) - \widetilde{\rho}^{st}(s) ds$ that

$$
\|\hat{p}_{\rho,\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \leq \sqrt{T} c \|G\|_{L^1(I,L^2(\Omega))} \leq T^{\frac{3}{2}} c \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^1(I,L^2(\Omega))} \leq T^2 c \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} \tag{4.20}
$$

with $c \coloneqq \nu_{\min}^{-1} \frac{\max\{\sqrt{\nu_{\max}},1\}}{\min\{\sqrt{\nu_{\min}},1\}}$. Analogously, applying Lemma 4.2 to (4.12) and (4.13), we obtain that

$$
\|\partial_t^l (p_\rho - p_{\widetilde{\rho}})\|_{L^2(I,L^2(\Omega))} \leq T^2 c \big(\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^{l+2} \overline{p}\|_{L^2(I,L^\infty(\Omega))} + \|\partial_t^l (\rho^{st} - \widetilde{\rho}^{st})\|_{L^2(I,L^2(\Omega))} \big) \tag{4.21}
$$

for $l = 0, 1$ and

$$
\|q_\rho - q_{\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \leq T^2 c \Bigg(\sum_{i=1}^{m} \|a_i\|_{L^\infty(I,L^\infty(\Omega))} \|p_\rho - p_{\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} \tag{4.22}
$$

$$+ \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))}\Bigg)$$

$$\underbrace{\leq}_{(4.21)} T^2 c\Bigg(\sum_{i=1}^m \|a_i\|_{L^\infty(I,L^\infty(\Omega))} T^2 c\big(\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))}\big)$$

$$+ \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))}\Bigg).$$

Therefore, applying (4.20)-(4.22) to (4.19) provides that

$$\begin{aligned}
&\alpha \|\nu_\rho - \nu_{\widetilde{\rho}}\|^2_{L^2(\Omega)} \\
&\leq T^2 c\Bigg(\sum_{i=1}^m \|a_i\|_{L^\infty(I,L^\infty(\Omega))} T^2 c\left(\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))}\right) \\
&\qquad + \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))}\Bigg) \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} \\
&\quad + \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))} T^2 c\big(\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))}\big) \\
&\quad + \|\rho^{VI} - \widetilde{\rho}^{VI}\|_{L^2(\Omega)} \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} + T^4 c^2 \sum_{i=1}^m \|a_i\|_{L^\infty(I,L^\infty(\Omega))} \|\rho^{st} - \widetilde{\rho}^{st}\|^2_{L^2(I,L^2(\Omega))} \\
&\quad + 2 \sum_{i=1}^m \|a_i\|_{L^\infty(I,L^\infty(\Omega))} T^4 c^2 \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} \big(\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))} \\
&\quad + \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))}\big) + 2\|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(I,L^2(\Omega))} T^2 c \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} \|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))} \\
&\leq c_1 \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} + c_2 \|\rho^{st} - \widetilde{\rho}^{st}\|^2_{L^2(I,L^2(\Omega))} \\
&\quad + c_3 \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))} \|\rho^{st} - \widetilde{\rho}^{st}\|_{L^2(I,L^2(\Omega))} + c_4 \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)} \|\rho^{adj} - \widetilde{\rho}^{adj}\|_{L^2(I,L^2(\Omega))} \\
&\quad + \|\rho^{VI} - \widetilde{\rho}^{VI}\|_{L^2(\Omega)} \|\nu_\rho - \nu_{\widetilde{\rho}}\|_{L^2(\Omega)}
\end{aligned}$$

with the constants

$$\begin{aligned}
c_1 &\coloneqq 3T^4 c^2 \sum_{i=1}^m \|a_i\|_{L^\infty(I,L^\infty(\Omega))} \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))} + 3T^2 c \|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))}, \\
c_2 &\coloneqq 4T^4 c^2 \sum_{i=1}^m \|a_i\|_{L^\infty(I,L^\infty(\Omega))}, \quad c_3 \coloneqq 2T^2 c, \quad c_4 \coloneqq T^2 c \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))}.
\end{aligned}$$

Using Young's inequality, we obtain that

$$\begin{aligned}
\alpha \|\nu_\rho - \nu_{\widetilde{\rho}}\|^2_{L^2(\Omega)} \leq \frac{\alpha}{4} \|\nu_\rho - \nu_{\widetilde{\rho}}\|^2_{L^2(\Omega)} &+ \left(\frac{c_1^2}{\alpha} + c_2\right) \|\rho^{st} - \widetilde{\rho}^{st}\|^2_{L^2(I,L^2(\Omega))} + \frac{c_3}{2} \|\rho^{adj} - \widetilde{\rho}^{adj}\|^2_{L^2(I,L^2(\Omega))} \\
&+ \frac{c_3}{2} \|\rho^{st} - \widetilde{\rho}^{st}\|^2_{L^2(I,L^2(\Omega))} + \frac{\alpha}{4} \|\nu_\rho - \nu_{\widetilde{\rho}}\|^2_{L^2(\Omega)} + \frac{1}{\alpha} c_4^2 \|\rho^{adj} - \widetilde{\rho}^{adj}\|^2_{L^2(I,L^2(\Omega))} \\
&+ \frac{1}{\alpha} \|\rho^{VI} - \widetilde{\rho}^{VI}\|^2_{L^2(\Omega)} + \frac{\alpha}{4} \|\nu_\rho - \nu_{\widetilde{\rho}}\|^2_{L^2(\Omega)},
\end{aligned}$$

leading to

$$\frac{\alpha}{4} \|\nu_\rho - \nu_{\widetilde{\rho}}\|^2_{L^2(\Omega)} \leq \left(\frac{c_1^2}{\alpha} + c_2 + \frac{c_3}{2}\right) \|\rho^{st} - \widetilde{\rho}^{st}\|^2_{L^2(I,L^2(\Omega))} + \left(\frac{c_3}{2} + \frac{c_4^2}{\alpha}\right) \|\rho^{adj} - \widetilde{\rho}^{adj}\|^2_{L^2(I,L^2(\Omega))}$$

$$+\frac{1}{\alpha}\|\rho^{VI}-\widetilde{\rho}^{VI}\|^2_{L^2(\Omega)}. \tag{4.23}$$

Therefore, (4.9) is valid. To prove (4.10), note that $p_\rho - \overline{p}$ is the solution to

$$\begin{cases}\overline{\nu}\partial_t^2(p_\rho-\overline{p})-\Delta(p_\rho-\overline{p})+\eta\partial_t(p_\rho-\overline{p})=-(\nu_\rho-\overline{\nu})\partial_t^2\overline{p}+\rho^{st} & \text{in } I\times\Omega\\ \partial_n(p_\rho-\overline{p})=0 & \text{on } I\times\Gamma_N\\ p_\rho-\overline{p}=0 & \text{on } I\times\Gamma_D\\ (p_\rho-\overline{p},\partial_t(p_\rho-\overline{p}))(0)=(0,0) & \text{in } \Omega.\end{cases}$$

Applying Lemma 4.2 to the above system yields that

$$\begin{aligned}&\|p_\rho-\overline{p}\|_{L^2(I,L^\infty(\Omega))} \\ &\le \hat{c}(\|(\nu_\rho-\overline{\nu})\partial_t^2\overline{p}\|_{L^2(I,L^2(\Omega))}+\|\rho^{st}\|_{L^2(I,L^2(\Omega))}+\|(\nu_\rho-\overline{\nu})\partial_t^3\overline{p}\|_{L^2(I,L^2(\Omega))}+\|\partial_t\rho^{st}\|_{L^2(I,L^2(\Omega))})\\ &\le \hat{c}((\|\partial_t^2\overline{p}\|_{L^2(I,L^\infty(\Omega))}+\|\partial_t^3\overline{p}\|_{L^2(I,L^\infty(\Omega))})\|\nu_\rho-\overline{\nu}\|_{L^2(\Omega)}+\|\rho^{st}\|_{L^2(I,L^2(\Omega))}+\|\partial_t\rho^{st}\|_{L^2(I,L^2(\Omega))}).\end{aligned} \tag{4.24}$$

Since $(\overline{\nu},\overline{p},\overline{q})$ solves (OS$^\tau$) with $(\rho^{st},\rho^{adj},\rho^{vi})=0$, applying (4.9) to (4.24) leads to (4.10). Since $q_\rho-\overline{q}$ solves

$$\begin{cases}\overline{\nu}\partial_t^2(q_\rho-\overline{q})-\Delta(q_\rho-\overline{q})-\eta\partial_t(q_\rho-\overline{q})=\sum_{i=1}^m a_i(p_\rho-\overline{p})-(\nu_\rho-\overline{\nu})\partial_t^2\overline{q}+\rho^{adj} & \text{in } I\times\Omega\\ \partial_n(q_\rho-\overline{q})=0 & \text{on } I\times\Gamma_N\\ q_\rho-\overline{q}=0 & \text{on } I\times\Gamma_D\\ (q_\rho-\overline{q},\partial_t(q_\rho-\overline{q}))(T)=(0,0) & \text{in } \Omega,\end{cases}$$

Lemma 4.2 implies that

$$\begin{aligned}\|q_\rho-\overline{q}\|_{L^2(I,L^\infty(\Omega))}\le \hat{c}(&\sum_{i=1}^m\|a_i\|_{L^\infty(I,L^\infty(\Omega))}(\|p_\rho-\overline{p}\|_{L^2(I,L^2(\Omega))}+\|\partial_t(p_\rho-\overline{p})\|_{L^2(I,L^2(\Omega))})\\ &+\sum_{i=1}^m\|\partial_t a_i\|_{L^\infty(I,L^\infty(\Omega))}\|p_\rho-\overline{p}\|_{L^2(I,L^2(\Omega))}\\ &+(\|\partial_t^2\overline{q}\|_{L^2(I,L^\infty(\Omega))}+\|\partial_t^3\overline{q}\|_{L^2(I,L^\infty(\Omega))})\|\nu_\rho-\overline{\nu}\|_{L^2(\Omega)}\\ &+\|\rho^{adj}\|_{L^2(I,L^2(\Omega))}+\|\partial_t\rho^{adj}\|_{L^2(I,L^2(\Omega))}).\end{aligned}$$

Applying (4.9) and (4.21) with $(\widetilde{\rho}^{st},\widetilde{\rho}^{adj},\widetilde{\rho}^{vi})=0$, we obtain (4.11). □

With the following lemma, we will abandon the modification $\mathcal{V}_{ad}^\tau$ of the admissible set $\mathcal{V}_{ad}$. The proof follows the argumentation from [77, Corollary 5.3] with a careful modification.

Lemma 4.5. *Let Assumption 4.1 and* (SSC$^\tau$) *hold. Let* $(\rho^{st},\rho^{adj},\rho^{VI})\in H^1(I,L^2(\Omega))\times H^1(I,L^2(\Omega))\times L^\infty(\Omega)$ *such that* $\rho^{st}(0)=\rho^{adj}(T)=0$ *and*

$$\|\rho^{st}\|_{H^1(I,L^2(\Omega))}+\|\rho^{adj}\|_{H^1(I,L^2(\Omega))}+\|\rho^{VI}\|_{L^\infty(\Omega)}\le\frac{\tau}{c_L} \tag{4.25}$$

with $c_L := \max\{L_p\|\partial_t^2\overline{q}\|_{L^2(I,L^\infty(\Omega))}, L_q\|\partial_t^2\overline{p}\|_{L^2(I,L^\infty(\Omega))},1\}$. *Then, the unique solution to* (OS$^\tau$) *satisfies* (OS).

Proof. Let $(\nu_\rho, p_\rho, q_\rho)$ denote the solution to (OS^τ). Since the equations in (OS) and (OS^τ) coincide, it remains to show that the variational inequality in (OS) is valid. By (OS^τ), it holds that

$$(-\int_I \partial_t^2 \overline{p}(t) q_\rho(t) + \partial_t^2 (p_\rho(t) - \overline{p}(t))\overline{q}(t)\,\mathrm{d}t + \lambda \nu_\rho, \widetilde{\nu} - \nu_\rho)_{L^2(\Omega)} \geq (\rho^{VI}, \widetilde{\nu} - \nu_\rho)_{L^2(\Omega)} \quad \forall \widetilde{\nu} \in \mathcal{V}_{ad}^\tau. \tag{4.26}$$

By the definition of $\mathcal{V}_{ad}^\tau$ (see (4.5)) and since $\nu_\rho \in \mathcal{V}_{ad}^\tau$, it holds for every $\widetilde{\nu} \in \mathcal{V}_{ad}^\tau$ that $\widetilde{\nu} - \nu_\rho = 0$ a.e. in $\mathscr{A}_\tau(\overline{\nu})$. As a consequence, (4.26) implies that

$$\begin{aligned}&(-\int_I \partial_t^2 \overline{p}(t) q_\rho(t) + \partial_t^2 (p_\rho(t) - \overline{p}(t))\overline{q}(t)\,\mathrm{d}t + \lambda \nu_\rho, \widetilde{\nu} - \nu_\rho)_{L^2(\Omega \setminus \mathscr{A}_\tau(\overline{\nu}))} \\ &\geq (\rho^{VI}, \widetilde{\nu} - \nu_\rho)_{L^2(\Omega \setminus \mathscr{A}_\tau(\overline{\nu}))} \quad \forall \widetilde{\nu} \in \mathcal{V}_{ad}^\tau.\end{aligned} \tag{4.27}$$

For every $\nu \in \mathcal{V}_{ad}$, we set $\widetilde{\nu} := \chi_{\mathscr{A}_\tau(\overline{\nu})}\overline{\nu} + \chi_{(\Omega \setminus \mathscr{A}_\tau(\overline{\nu}))}\nu \in \mathcal{V}_{ad}^\tau$ in (4.27). Since $\widetilde{\nu}$ and ν coincide in $\Omega \setminus \mathscr{A}_\tau(\overline{\nu})$, it follows that (4.27) holds for every $\nu \in \mathcal{V}_{ad}$, i.e.,

$$\begin{aligned}&(-\int_I \partial_t^2 \overline{p}(t) q_\rho(t) + \partial_t^2 (p_\rho(t) - \overline{p}(t))\overline{q}(t)\,\mathrm{d}t + \lambda \nu_\rho, \nu - \nu_\rho)_{L^2(\Omega \setminus \mathscr{A}_\tau(\overline{\nu}))} \\ &\geq (\rho^{VI}, \nu - \nu_\rho)_{L^2(\Omega \setminus \mathscr{A}_\tau(\overline{\nu}))} \ \forall \nu \in \mathcal{V}_{ad}.\end{aligned} \tag{4.28}$$

Let $\mathscr{A}_\tau^+(\overline{\nu}) := \{x \in \Omega : -\int_0^T \partial_t^2 \overline{p}(t,x)\overline{q}(t,x)\,\mathrm{d}t + \lambda \overline{\nu}(x) > \tau\}$ and $\mathscr{A}_\tau^-(\overline{\nu}) := \{x \in \Omega : -\int_0^T \partial_t^2 \overline{p}(t,x)\overline{q}(t,x)\,\mathrm{d}t + \lambda \overline{\nu}(x) < -\tau\}$. Then, by (4.4), it holds $\mathscr{A}_\tau(\overline{\nu}) = \mathscr{A}_\tau^+(\overline{\nu}) \cup \mathscr{A}_\tau^-(\overline{\nu})$ and it follows for a.e. $x \in \mathscr{A}_\tau^+(\overline{\nu})$ that

$$\begin{aligned}\tau &< -\int_0^T \partial_t^2 \overline{p}(t,x)\overline{q}(t,x)\,\mathrm{d}t + \lambda \overline{\nu}(x) \\ &= -\int_I \partial_t^2 \overline{p}(t,x) q_\rho(t,x) + \partial_t^2 (p_\rho(t,x) + \overline{p}(t,x))\overline{q}(t,x)\,\mathrm{d}t + \lambda \nu_\rho(x) - \rho^{VI}(x) \\ &\quad + \int_I \partial_t^2 \overline{p}(t,x)(q_\rho(t,x) - \overline{q}(t,x)) + \partial_t^2 (p_\rho(t,x) - \overline{p}(t,x))\overline{q}(t,x)\,\mathrm{d}t + \rho^{VI}(x) \\ &\leq -\int_I \partial_t^2 \overline{p}(t,x) q_\rho(t,x) + \partial_t^2 (p_\rho(t,x) + \overline{p}(t,x))\overline{q}(t,x)\,\mathrm{d}t + \lambda \nu_\rho(x) - \rho^{VI}(x) \\ &\quad + \|\partial_t^2 \overline{p}\|_{L^2(I,L^\infty(\Omega))}\|q_\rho - \overline{q}\|_{L^2(I,L^\infty(\Omega))} + \|p_\rho - \overline{p}\|_{L^2(I,L^\infty(\Omega))}\|\partial_t^2 \overline{q}\|_{L^2(I,L^\infty(\Omega))} + \|\rho^{VI}\|_{L^\infty(\Omega)} \\ &\underbrace{\leq}_{\text{Thm. 4.4}} -\int_I \partial_t^2 \overline{p}(t,x) q_\rho(t,x) + \partial_t^2 (p_\rho(t,x) + \overline{p}(t,x))\overline{q}(t,x)\,\mathrm{d}t + \lambda \nu_\rho(x) - \rho^{VI}(x) \\ &\quad + c_L(\|\rho^{st}\|_{H^1(I,L^2(\Omega))} + \|\rho^{adj}\|_{H^1(I,L^2(\Omega))} + \|\rho^{VI}\|_{L^\infty(\Omega)}).\end{aligned} \tag{4.29}$$

Consequently, we obtain for a.e. $x \in \mathscr{A}_\tau^+(\overline{\nu})$ that

$$\begin{aligned}0 &\underbrace{\leq}_{(4.25)} \tau - c_L(\|\rho^{st}\|_{H^1(I,L^2(\Omega))} + \|\rho^{adj}\|_{H^1(I,L^2(\Omega))} + \|\rho^{VI}\|_{L^\infty(\Omega)}) \\ &\underbrace{\leq}_{(4.29)} -\int_I \partial_t^2 \overline{p}(t,x) q_\rho(t,x) + \partial_t^2 (p_\rho(t,x) + \overline{p}(t,x))\overline{q}(t,x)\,\mathrm{d}t + \lambda \nu_\rho(x) - \rho^{VI}(x).\end{aligned} \tag{4.30}$$

Analogously, it follows for a.e. $x \in \mathscr{A}_\tau^-(\overline{\nu})$ that

$$0 \geq -\int_I \partial_t^2 \overline{p}(t,x) q_\rho(t,x) + \partial_t^2 (p_\rho(t,x) + \overline{p}(t,x))\overline{q}(t,x)\,\mathrm{d}t + \lambda \nu_\rho(x) - \rho^{VI}(x). \tag{4.31}$$

On the other hand, by a standard argumentation, due to (3.71), the pointwise inequality

$$\Big(-\int_I \partial_t^2\overline{p}(t,x)\overline{q}(t,x)\,\mathrm{d}t+\lambda\overline{\nu}(x)\Big)(v-\overline{\nu}(x))\geq 0 \text{ for all } v\in[\nu_-(x),\nu_+(x)] \text{ and a.e. } x\in\Omega$$

holds, implying that $\overline{\nu}=\nu_-$ in $\mathscr{A}_\tau^+(\overline{\nu})$ and $\overline{\nu}=\nu_+$ in $\mathscr{A}_\tau^-(\overline{\nu})$. Therefore, since $\nu_\rho=\overline{\nu}$ in $\mathscr{A}_\tau(\overline{\nu})$, along with (4.30) and (4.31), we obtain that

$$\begin{aligned}
&\Big(-\int_I \partial_t^2\overline{p}(t)q_\rho(t)+\partial_t^2(p_\rho(t)-\overline{p}(t))\overline{q}(t)\,\mathrm{d}t+\lambda\nu_\rho-\rho^{VI},\nu-\nu_\rho\Big)_{L^2(\mathscr{A}_\tau(\overline{\nu}))} \qquad (4.32)\\
&=\Big(\underbrace{-\int_I \partial_t^2\overline{p}(t)q_\rho(t)+\partial_t^2(p_\rho(t)-\overline{p}(t))\overline{q}(t)\,\mathrm{d}t+\lambda\nu_\rho-\rho^{VI}}_{\geq 0 \text{ a.e. in } \mathscr{A}_\tau^+(\overline{\nu}) \text{ due to (4.30)}},\underbrace{\nu-\nu_-}_{\geq 0 \text{ a.e.}}\Big)_{L^2(\mathscr{A}_\tau^+(\overline{\nu}))}\\
&\quad+\Big(\underbrace{-\int_I \partial_t^2\overline{p}(t)q_\rho(t)+\partial_t^2(p_\rho(t)-\overline{p}(t))\overline{q}(t)\,\mathrm{d}t+\lambda\nu_\rho-\rho^{VI}}_{\leq 0 \text{ a.e. in } \mathscr{A}_\tau^-(\overline{\nu}) \text{ due to (4.31)}},\underbrace{\nu-\nu_+}_{\leq 0 \text{ a.e.}}\Big)_{L^2(\mathscr{A}_\tau^-(\overline{\nu}))}\geq 0\quad\forall\nu\in\mathcal{V}_{ad}.
\end{aligned}$$

Combining (4.28) and (4.32) proves the assertion. □

4.2 Formulation of the SQP Method

The SQP method (cf. [72, Section 4.11]) approximates (P) by a sequence of coupled systems arising from a suitable linearization process of the optimality system (3.135):

Algorithm 1 Sequential Quadratic Programming

1: Choose (ν_0,p_0,q_0) and set $k=0$.

2: Find $\nu\in\mathcal{V}_{ad}$ and $p,q\in C^2(I,L^2(\Omega))\cap C^1(I,H_D^1(\Omega))\cap C(I,D(\Delta_{D,N}))$ such that

$$\begin{cases}
\nu_k\partial_t^2 p-\Delta p+\eta\partial_t p=f-(\nu-\nu_k)\partial_t^2 p_k & \text{in } I\times\Omega\\
\partial_n p=0 & \text{on } I\times\Gamma_N\\
p=0 & \text{on } I\times\Gamma_D\\
(p,\partial_t p)(0)=(0,0) & \text{in } \Omega\\
\nu_k\partial_t^2 q-\Delta q-\eta\partial_t q=\sum_{i=1}^m a_i(p-p_i^{ob})-(\nu-\nu_k)\partial_t^2 q_k & \text{in } I\times\Omega\\
\partial_n q=0 & \text{on } I\times\Gamma_N\\
q=0 & \text{on } I\times\Gamma_D\\
(q,\partial_t q)(T)=(0,0) & \text{in } \Omega\\
(-\int_I \partial_t^2 p_k(t)q(t)+\partial_t^2(p(t)-p_k(t))q_k(t)\,\mathrm{d}t+\lambda\nu,\tilde{\nu}-\nu)_{L^2(\Omega)}\geq 0 & \text{for all } \tilde{\nu}\in\mathcal{V}_{ad},
\end{cases} \qquad (\mathbb{P}_k)$$

and set $(\nu_{k+1},p_{k+1},q_{k+1}):=(\nu,p,q)$.

3: Stop or set $k=k+1$ and go back to step 2.

Remark 4.6. The hyperbolicity and the second-order bilinear character of the PDEs in $(\mathbb{P}_k)$ lead to an undesired effect of loss of regularity, causing two major challenges:

(i) For a given iterate $(\nu_k, p_k, q_k) \in \mathcal{V}_{ad} \times C^l(I, L^2(\Omega)) \times C^l(I, L^2(\Omega))$ for some $l > 2$, the solutions p_{k+1}, q_{k+1} to $(\mathbb{P}_k)$ are in general only $l-1$-times continuously differentiable. This can be inferred from Lemma 4.2 due to the regularity of the source terms $(\nu - \nu_k)\partial_t^2 p_k, (\nu - \nu_k)\partial_t^2 q_k \in C^{l-2}(I, L^2(\Omega))$ in the PDEs of $(\mathbb{P}_k)$. For this reason, Algorithm 1 is generally only executable for a limited number of iterations. To tackle this issue, we propose using a smooth-in-time regularity condition (see Assumption 4.7).

(ii) In the parabolic case (cf. [45, 49, 70, 77]), the convergence analysis strongly relies on Robinson's notion of *strong regularity* (see [69]). However, the regularity results and estimation for the hyperbolic case (see Lemma 3.4 or [34, p. 410]) are weaker than those for the parabolic one. Consequently, the developed strategies for parabolic scenarios cannot be directly transferred to our case and require a substantial extension.

Assumption 4.7. *Let Assumption 4.1 hold. Furthermore, let $f \in C^\infty(I, L^2(\Omega))$ with $\partial_t^l f(0) = 0$ for all $l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $a_i \in C^\infty(I, L^\infty(\Omega))$ for all $i = 1, \ldots, m$ with $\partial_t^l a_i(T) = 0$ for all $l \in \mathbb{N}_0$, let $p_i^{ob} \in C^\infty(I, L^2(\Omega))$ for all $i = 1, \ldots m$, and let $(\nu_0, p_0, q_0) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega))$ with $\partial_t^l p_0(0) = \partial_t^l q_0(T) = 0$ for all $l \in \mathbb{N}_0$.*

Assumption 4.7 implies that $\overline{p}, \overline{q} \in C^\infty(I, L^\infty(\Omega))$ with $\partial_t^l \overline{p}(0) = \partial_t^l \overline{q}(T) = 0$ for all $l \in \mathbb{N}_0$. Further, in practice, observation data are typically available through measurements at various time points. Accordingly, their usual extrapolations are smooth in time. Therefore, Assumption 4.1 is reasonable since smoothness is only considered in time, whereas the data are allowed to be non-smooth with respect to the space variable.

Theorem 4.8. *Let Assumption 4.7 hold. Then, for every $k \in \mathbb{N}$, the system $(\mathbb{P}_k)$ admits at least one solution $(\nu_{k+1}, p_{k+1}, q_{k+1}) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega))$ satisfying $\partial_t^l p_k(0) = \partial_t^l q_k(T) = 0$ for all $l \in \mathbb{N}_0$. In particular, Algorithm 1 is well-defined.*

Proof. Let $(\nu_k, p_k, q_k) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega))$ with $\partial_t^l p_k(0) = \partial_t^l q_k(T) = 0$ for all $l \in \mathbb{N}_0$ be given for some $k \in \mathbb{N}_0$. By $G_k : L^2(\Omega) \to C^3(I, H_D^1(\Omega))$ we denote the affine-linear and continuous solution operator that maps every ν to the unique solution p to

$$\begin{cases} \nu_k \partial_t^2 p - \Delta p + \eta \partial_t p = f - (\nu - \nu_k)\partial_t^2 p_k & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ (p, \partial_t p)(0) = (0, 0) & \text{in } \Omega. \end{cases}$$

Note that the well-definedness of G_k follows from Assumption 4.7 and Lemma 4.2. Making use of G_k, we consider the following minimization problem:

$$\inf_{\nu \in \mathcal{V}_{ad}} J_k(\nu) := \mathcal{J}(\nu, G_k \nu) - ((\nu - \nu_k)\partial_t^2 (G_k \nu - p_k), q_k)_{L^2(I, L^2(\Omega))}. \tag{4.33}$$

To prove the existence of a minimizer to (4.33), it remains to show that $J_k : L^2(\Omega) \to \mathbb{R}$ is lower sequentially semicontinuous. The lower sequential semicontinuity of the first term is obvious since it is convex and continuous. For the second term, we note that the embedding $C^1(I, H^1_D(\Omega)) \hookrightarrow C(I, L^2(\Omega))$ is compact due to the Aubin-Lions lemma. Then, along with the continuity and affine-linearity of $\partial_t^2 G_k \colon L^2(\Omega) \to C^1(I, H^1_D(\Omega))$, we obtain the following implication:

$$\nu_n \rightharpoonup \nu \quad \text{weakly in } L^2(\Omega) \quad \Rightarrow \quad (\nu_n \partial_t^2 G_k \nu_n, q_k)_{L^2(I,L^2(\Omega))} \to (\nu \partial_t^2 G_k \nu, q_k)_{L^2(I,L^2(\Omega))}.$$

In conclusion, J_k is lower sequentially semicontinuous, and therefore (4.33) admits at least one minimizer $\nu_{k+1} \in \mathcal{V}_{ad}$. On the other hand, the iteration system $(\mathbb{P}_k)$ is equivalent to the condition that $J_k'(\nu)(\tilde{\nu} - \nu) \geq 0$ for every $\tilde{\nu} \in \mathcal{V}_{ad}$ which is nothing but the necessary optimality condition to (4.33). Therefore, $(\mathbb{P}_k)$ admits at least one solution $(\nu_{k+1}, p_{k+1}, q_{k+1})$. Applying Assumption 4.7 and Lemma 4.2 to the PDE systems in $(\mathbb{P}_k)$ yields $p_{k+1}, q_{k+1} \in C^\infty(I, L^\infty(\Omega))$ and $\partial_t^l p_{k+1}(0) = \partial_t^l q_{k+1}(T) = 0$ for all $l \in \mathbb{N}_0$. The claim follows inductively. □

4.3 Auxiliary Estimates

Assumption 4.9. *Let Assumption 4.7 and* (SSC$^\tau$) *hold. Furthermore, suppose for every* $l \in \mathbb{N}_0$ *that*

$$\begin{aligned}
&\|\partial_t^l (p_0 - \overline{p})\|_{L^2(I,L^\infty(\Omega))} \leq C_0 l! \|\overline{\nu} - \nu_0\|_{L^2(\Omega)}, \quad \|\partial_t^l (q_0 - \overline{q})\|_{L^2(I,L^\infty(\Omega))} \leq C_0 l! \|\overline{\nu} - \nu_0\|_{L^2(\Omega)},\\
&\sum_{i=1}^m \|\partial_t^l a_i\|_{L^\infty(I,L^\infty(\Omega))} \leq C_a l!, \qquad \sum_{i=1}^m \|\partial_t^l (a_i(\overline{p} - p_i^{ob}))\|_{L^2(I,L^2(\Omega))} \leq C_a l!,\\
&\max\left\{\|\partial_t^l p_0\|_{L^2(I,L^\infty(\Omega))}, \|\partial_t^l q_0\|_{L^2(I,L^\infty(\Omega))}\right\} \leq C_0 l!,\\
&\max\left\{\|\partial_t^l \overline{p}\|_{L^2(I,L^\infty(\Omega))}, \|\partial_t^l \overline{q}\|_{L^2(I,L^\infty(\Omega))}\right\} \leq \overline{C} l!, \qquad \|\partial_t^l f\|_{L^2(I,L^2(\Omega))} \leq C_f l!
\end{aligned}$$

and

$$\begin{aligned}
&\|\nu_0 - \overline{\nu}\|_{L^2(\Omega)} \qquad\qquad (4.34)\\
&\leq \min\left\{\frac{\gamma}{4\delta}, \frac{1}{2L(2c_1 5! + 4C_0 + 2\sqrt{|\Omega|}5! C_0 c_1)}, \sqrt{\frac{8!^2 \gamma^{\sqrt{2}}}{8\delta L(2c_0 5! + \sqrt{|\Omega|} C_0 (2c_0 3! + 2C_0 + c_0 5!))}}\right\}
\end{aligned}$$

hold for some constants $C_f, C_a, C_0, \overline{C}, \gamma > 0$, *satisfying*

$$\gamma \prod_{l=1}^\infty (3l+5)!^{\sqrt{2}^{6-l}} =: \overline{\gamma} < \min\left\{1, \frac{\delta}{L(4c_1 + 2c_1^2)}, \frac{2\delta\tau}{c_L(8C_0 + 4\overline{C} + 3c_1 C_0 + c_1 \overline{C})}\right\}, \qquad (4.35)$$

$$2\hat{c} C_f + \hat{c} C_0 \frac{\overline{\gamma}}{\delta} \leq C_0, \qquad \hat{c}(2C_a + C_a c T(\overline{C} + C_0)\frac{\overline{\gamma}}{\delta} + C_0 \frac{\overline{\gamma}}{\delta}) \leq C_0, \qquad (4.36)$$

where

$$\delta := 2L(2c_1 + 3\sqrt{|\Omega|} c_1^2), \qquad c_0 := C_0 \max\{\hat{c}\frac{\gamma}{2\delta} + 1, 2\hat{c}\frac{\gamma}{4\delta}(C_a c T + 1) + 1\}, \qquad (4.37)$$

$$\begin{aligned}
c_1 := \max\{&2\hat{c} C_0, 2\hat{c} C_0 (C_a c T + 1), 2\hat{c}(\overline{C} + C_0), \hat{c}(2C_a c T(\overline{C} + C_0) + 2\overline{C} + 2C_0), 4\hat{c} C_0,\\
&\hat{c} C_0 (2C_a c T + 4)\} \qquad (4.38)
\end{aligned}$$

with $\tau, c, \hat{c}, L > 0$ *as in* (SSC$^\tau$), *Lemma 4.2, and Theorem 4.4, respectively.*

Remark 4.10. Notice that (4.35) and (4.36) can always be guaranteed by choosing γ sufficiently small and C_0 sufficiently large. Furthermore, it is straightforward to check that $\prod_{l=1}^{\infty}(3l+5)!^{\sqrt{2}^{6-l}}$ exists. Indeed, it holds for every $k \in \mathbb{N}$ that

$$\begin{aligned}\ln\left(\prod_{l=1}^{k}(3l+5)!^{\sqrt{2}^{6-l}}\right) &= \sum_{l=1}^{k}\sqrt{2}^{6-l}\ln((3l+5)!) = \sum_{l=1}^{k}\sqrt{2}^{6-l}\sum_{s=1}^{3l+5}\ln(s) \le \sum_{l=1}^{k}\sqrt{2}^{6-l}\sum_{s=1}^{3l+5}s \\ &= \sum_{l=1}^{k}\sqrt{2}^{6-l}\frac{(3l+5)(3l+6)}{2} =: \sum_{l=1}^{k}d_l\end{aligned}$$

and

$$\left|\frac{d_{l+1}}{d_l}\right| = \frac{\sqrt{2}^{5-l}\frac{(3l+8)(3l+9)}{2}}{\sqrt{2}^{6-l}\frac{(3l+5)(3l+6)}{2}} = \sqrt{2}^{-1}\frac{(3l+8)(3l+9)}{(3l+5)(3l+6)} \le \sqrt{2}^{-1}\frac{(12+8)(12+9)}{(12+5)(12+6)} < 1 \quad \forall l \ge 4.$$

Thus, by the ratio test, $\{\sum_{l=1}^{k} d_l\}_{k\in\mathbb{N}}$ is convergent. Consequently, as $\exp \in C(\mathbb{R})$, the limit $\prod_{l=1}^{\infty}(3l+5)!^{\sqrt{2}^{6-l}}$ exists.

Associated with a given $(\nu_k, p_k, q_k) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega))$ satisfying $\partial_t^l p_k(0) = \partial_t^l q_k(T) = 0$ for all $l \in \mathbb{N}_0$ and some $k \in \mathbb{N}_0$, we introduce the mapping

$$S_k\colon L^2(\Omega) \times X_0 \times X_T \to L^2(\Omega) \times X_0 \times X_T, \quad (\hat\nu, \hat p, \hat q) \mapsto (\nu, p, q)$$

with $X_t := \{p \in C^\infty(I, L^\infty(\Omega)) : \partial_t^l p(t) = 0 \text{ for all } l \in \mathbb{N}_0\}$ for $t \in \{0, T\}$, that assigns to every $(\hat\nu, \hat p, \hat q) \in L^2(\Omega) \times X_0 \times X_T$ the solution (ν, p, q) to

$$\begin{cases}\overline{\nu}\partial_t^2 p - \Delta p + \eta\partial_t p = f - (\nu_k - \overline{\nu})\partial_t^2\hat p - (\hat\nu - \nu_k)\partial_t^2 p_k - (\nu - \hat\nu)\partial_t^2\overline{p} & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ (p, \partial_t p)(0) = (0,0) & \text{in } \Omega \\ \overline{\nu}\partial_t^2 q - \Delta q - \eta\partial_t q = \sum_{i=1}^{m} a_i(p - p_i^{ob}) - (\nu_k - \overline{\nu})\partial_t^2\hat q - (\hat\nu - \nu_k)\partial_t^2 q_k - (\nu - \hat\nu)\partial_t^2\overline{q} & \text{in } I \times \Omega \\ \partial_n q = 0 & \text{on } I \times \Gamma_N \\ q = 0 & \text{on } I \times \Gamma_D \\ (q, \partial_t q)(T) = (0,0) & \text{in } \Omega \\ (-\int_I \partial_t^2\overline{p}(t)q(t) + \partial_t^2(p_k(t) - \overline{p}(t))\hat q(t) & \\ \quad + \partial_t^2(\hat p(t) - p_k(t))q_k(t) + \partial_t^2(p(t) - \hat p(t))\overline{q}(t)\,\mathrm{d}t + \lambda\nu, \hat\nu - \nu)_{L^2(\Omega)} \ge 0 & \forall\hat\nu \in \mathcal{V}_{ad}^\tau.\end{cases} \tag{4.39}$$

Remark 4.11. The system (4.39) is nothing but (OS^τ) with the perturbation terms

$$\begin{aligned}\rho^{st} &= -(\nu_k - \overline{\nu})\partial_t^2\hat p - (\hat\nu - \nu_k)\partial_t^2 p_k - (\overline{\nu} - \hat\nu)\partial_t^2\overline{p} && \in H^1(I, L^2(\Omega)) \qquad (4.40)\\ \rho^{adj} &= -(\nu_k - \overline{\nu})\partial_t^2\hat q - (\hat\nu - \nu_k)\partial_t^2 q_k - (\overline{\nu} - \hat\nu)\partial_t^2\overline{q} && \in H^1(I, L^2(\Omega)) \\ \rho^{VI} &= \int_I \partial_t^2(p_k(t) - \overline{p}(t))\hat q(t) + \partial_t^2(\hat p(t) - p_k(t))q_k(t) + \partial_t^2(\overline{p}(t) - \hat p(t))\overline{q}(t)\,\mathrm{d}t && \in L^2(\Omega)\end{aligned}$$

satisfying $\rho^{st}(0) = \rho^{adj}(T) = 0$ such that the well-definedness of S_k follows by Proposition 4.3 and Lemma 4.2, which also imply that the first component ν of $S_k(\hat{\nu}, \hat{p}, \hat{q})$ lies in $\mathcal{V}_{ad}^{\tau}$.

We aim to show that S_k admits a unique fixed point. According to $(\mathbb{P}_k)$, every fixed point to S_k exactly solves the iteration in Algorithm 1 with $\mathcal{V}_{ad}^{\tau}$ instead of $\mathcal{V}_{ad}$. Unfortunately, due to the nature of the hyperbolic PDEs and the second-order time derivatives in the source terms, S_k cannot be defined as a self-map in an appropriate Banach space (see Lemma 4.2). As a consequence, the contraction principle is not applicable directly to S_k. As mentioned in the introduction, we establish a suitable self-map to overcome this issue. First, we define for every $k \in \mathbb{N}_0$ the mapping

$$T_k\colon L^2(\Omega) \to L^2(\Omega) \times X_0 \times X_T, \quad \hat{\nu} \mapsto (\hat{\nu}, \hat{p}, \hat{q}) \tag{4.41}$$

where $\hat{p}, \hat{q}$ solve

$$\begin{cases} \nu_k \partial_t^2 \hat{p} - \Delta \hat{p} + \eta \partial_t \hat{p} = f - (\hat{\nu} - \nu_k)\partial_t^2 p_k & \text{in } I \times \Omega \\ \partial_n \hat{p} = 0 & \text{on } I \times \Gamma_N \\ \hat{p} = 0 & \text{on } I \times \Gamma_D \\ (\hat{p}, \partial_t \hat{p})(0) = (0,0) & \text{in } \Omega \end{cases} \tag{4.42}$$

and

$$\begin{cases} \nu_k \partial_t^2 \hat{q} - \Delta \hat{q} - \eta \partial_t \hat{q} = \sum_{i=1}^{m} a_i(\hat{p} - p_i^{ob}) - (\hat{\nu} - \nu_k)\partial_t^2 q_k & \text{in } I \times \Omega \\ \partial_n \hat{q} = 0 & \text{on } I \times \Gamma_N \\ \hat{q} = 0 & \text{on } I \times \Gamma_D \\ (\hat{q}, \partial_t \hat{q})(T) = (0,0) & \text{in } \Omega. \end{cases} \tag{4.43}$$

The well-definedness of T_k follows from Lemma 4.2. Then, the desired self-mapping operator reads

$$(I_\nu \circ S_k \circ T_k)\colon L^2(\Omega) \to L^2(\Omega) \quad \text{with} \quad I_\nu\colon (\nu, p, q) \mapsto \nu. \tag{4.44}$$

As we will see later, the operator (4.44) is constructed suitably such that every fixed point ν_{k+1} of (4.44) is exactly the first (control) component of the solution to $(\mathbb{P}_k)$. Furthermore, the quantity $T_k(\nu_{k+1})$ is a fixed point of S_k and solves the iteration $(\mathbb{P}_k)$. To prove these results, let us start with the following auxiliary lemmata:

Lemma 4.12. *Let* $\gamma > 0$ *such that*

$$\gamma \prod_{l=1}^{\infty} (3l+5)!^{\sqrt{2}^{6-l}} =: \overline{\gamma} \in (0,1). \tag{4.45}$$

Then, the sequence $\{b_k\}_{k \in \mathbb{N}_0} \subset \mathbb{R}_+$ *defined by*

$$b_0 := \gamma, \quad b_k := \prod_{l=1}^{k} (3l+5)!^{\sqrt{2}^{2+k-l}} \gamma^{\sqrt{2}^k} \quad \forall k \in \mathbb{N}$$

is decreasing and converges R-superlinearly to 0*. Furthermore, it holds that*

$$b_k \le \overline{\gamma}^{\sqrt{2}^k} \qquad \forall k \in \mathbb{N}_0 \tag{4.46}$$

$$(3k+5)!^2 b_{k-1} \le \overline{\gamma} \qquad \forall k \in \mathbb{N}. \tag{4.47}$$

Additionally, suppose that $\{x_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_+$ *satisfies for some* $k \in \mathbb{N}$ *and some* $\delta > 0$ *that*

$$x_{k-1} \le \frac{1}{4\delta} b_{k-1}, \quad x_k \le \frac{1}{4\delta} b_k, \quad x_{k+1} \le \delta (3k+5)!^2 (x_k + x_{k-1})^2. \tag{4.48}$$

Then

$$x_{k+1} \le \frac{1}{4\delta} b_{k+1}. \tag{4.49}$$

Proof. First, we notice that (4.45) implies that $b_0 = \gamma \le \overline{\gamma}$. Furthermore, for every $k \in \mathbb{N}$, we have that

$$b_k = \prod_{l=1}^{k} (3l+5)!^{\sqrt{2}^{2+k-l}} \gamma^{\sqrt{2}^k} = \left(\prod_{l=1}^{k} (3l+5)!^{\sqrt{2}^{2-l}} \gamma \right)^{\sqrt{2}^k} \underbrace{\le}_{(4.45)} \overline{\gamma}^{\sqrt{2}^k} \quad \forall k \in \mathbb{N},$$

and therefore (4.46) is valid. Since $\{\overline{\gamma}^{\sqrt{2}^k}\}_{k\in\mathbb{N}_0}$ converges (Q-)superlinearly to 0, the sequence $\{b_k\}_{k\in\mathbb{N}_0}$ converges R-superlinearly to 0. To prove the monotonicity, notice that $b_1 = 8!^2 \gamma^{\sqrt{2}} = 8!^2 b_0^{\sqrt{2}-1} b_0$ and

$$b_k = (3k+5)!^2 \left(\prod_{l=1}^{k-1} (3l+5)!^{\sqrt{2}^{1+k-l}} \gamma^{\sqrt{2}^{k-1}} \right)^{\sqrt{2}} = (3k+5)!^2 b_{k-1}^{\sqrt{2}} = (3k+5)!^2 b_{k-1}^{\sqrt{2}-1} b_{k-1} \quad \forall k \ge 2.$$

Thus, $\{b_k\}_{k\in\mathbb{N}_0}$ is decreasing if we can show that $(3k+5)!^2 b_{k-1}^{\sqrt{2}-1} \in (0,1)$ for all $k \in \mathbb{N}$. Indeed, this holds since

$$0 < \left(8!^2 b_0^{\sqrt{2}-1} \right)^{\sqrt{2}+1} \le 8!^{\sqrt{2}^5} b_0 = 8!^{\sqrt{2}^5} \gamma \underbrace{\le}_{(4.45)} \overline{\gamma} < 1$$

and

$$0 < \left((3k+5)!^2 b_{k-1}^{\sqrt{2}-1} \right)^{\sqrt{2}+1} \le (3k+5)!^{\sqrt{2}^5} b_{k-1} = (3k+5)!^{\sqrt{2}^5} \prod_{l=1}^{k-1} (3l+5)!^{\sqrt{2}^{1+k-l}} \gamma^{\sqrt{2}^{k-1}} \tag{4.50}$$

$$\le \prod_{l=1}^{k} (3l+5)!^{\sqrt{2}^{5+k-l}} \gamma^{\sqrt{2}^{k-1}} = \left(\prod_{l=1}^{k} (3l+5)!^{\sqrt{2}^{6-l}} \gamma \right)^{\sqrt{2}^{k-1}} \underbrace{\le}_{(4.45)} \overline{\gamma}^{\sqrt{2}^{k-1}} < 1 \quad \forall k \ge 2.$$

The claim (4.47) for $k = 1$ follows immediately from (4.45). For $k \ge 2$, the claim (4.47) is obtained as follows:

$$(3k+5)!^2 b_{k-1} \le (3k+5)!^{\sqrt{2}^5} b_{k-1} \underbrace{\le}_{(4.50)} \overline{\gamma}^{\sqrt{2}^{k-1}} \le \overline{\gamma} \quad \forall k \ge 2.$$

Now, suppose that $\{x_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_+$ satisfies (4.48) for some $k \in \mathbb{N}$ and some $\delta > 0$. If $k = 1$, using the monotonicity $b_1 \le b_0$, we obtain that

$$x_2 \underbrace{\le}_{(4.48)} \delta 8!^2(x_1 + x_0)^2 \le \frac{1}{16\delta} 8!^2(b_1 + b_0)^2 \le \frac{1}{4\delta} 8!^2 b_0^2 = \frac{1}{4\delta} 8!^2\gamma^2 \le \frac{1}{4\delta} 8!^{\sqrt{2}^3} 11!^2\gamma^2 = \frac{1}{4\delta} b_2.$$

If $k \ge 2$, again using the monotonicity $b_k \le b_{k-1}$, we obtain that

$$\begin{aligned} x_{k+1} &\underbrace{\le}_{(4.48)} \delta(3k+5)!^2(x_k + x_{k-1})^2 \le \frac{1}{16\delta}(3k+5)!^2(b_k + b_{k-1})^2 \le \frac{1}{4\delta}(3k+5)!^2 b_{k-1}^2 \\ &= \frac{1}{4\delta}(3k+5)!^2 \prod_{l=1}^{k-1}(3l+5)!^{\sqrt{2}^{3+k-l}} \gamma^{\sqrt{2}^{k+1}} \le \frac{1}{4\delta} \prod_{l=1}^{k+1}(3l+5)!^{\sqrt{2}^{3+k-l}} \gamma^{\sqrt{2}^{k+1}} = \frac{1}{4\delta} b_{k+1}. \end{aligned}$$

This completes the proof. □

Lemma 4.13. *Let Assumption 4.9 hold. Then, for $\nu, \tilde{\nu} \in \mathcal{V}_{ad}$, $(\overline{\nu}, \check{p}, \check{q}) := T_0(\overline{\nu})$, $(\nu, p, q) := T_0(\nu)$, and $(\tilde{\nu}, \tilde{p}, \tilde{q}) := T_0(\tilde{\nu})$, it holds for all $l \in \mathbb{N}_0$ that*

$$\|\partial_t^l(\check{p} - p_0)\|_{L^2(I,L^\infty(\Omega))} \le c_0(l+3)!\|\overline{\nu} - \nu_0\|_{L^2(\Omega)} \tag{4.51}$$
$$\|\partial_t^l(\check{q} - q_0)\|_{L^2(I,L^\infty(\Omega))} \le c_0(l+3)!\|\overline{\nu} - \nu_0\|_{L^2(\Omega)} \tag{4.52}$$
$$\|\partial_t^l(p - \tilde{p})\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3)!\|\nu - \tilde{\nu}\|_{L^2(\Omega)} \tag{4.53}$$
$$\|\partial_t^l(q - \tilde{q})\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3)!\|\nu - \tilde{\nu}\|_{L^2(\Omega)} \tag{4.54}$$

with $c_0, c_1 > 0$ as in Assumption 4.9. Let additionally ν_k and $(\nu_{k-1}, p_{k-1}, q_{k-1})$ for some $k \in \mathbb{N}$ be given such that

$$\begin{cases} \nu_k, \nu_{k-1} \in \mathcal{V}_{ad}, p_{k-1}, q_{k-1} \in C^\infty(I, L^\infty(\Omega)), \partial_t^l p_{k-1}(0) = \partial_t^l q_{k-1}(T) = 0 \quad \forall l \in \mathbb{N}_0 \\ \max\{\|\partial_t^l p_{k-1}\|_{L^2(I,L^\infty(\Omega))}, \|\partial_t^l q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\} \le C_0(l+3k-3)! \quad \forall l \in \mathbb{N}_0 \\ \|\nu_k - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta} b_k, \quad \|\nu_{k-1} - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta} b_{k-1} \end{cases} \tag{A$_k$}$$

with $C_0, \delta > 0$ as in Assumption 4.9 and b_k, b_{k-1} as in Lemma 4.12. Then, for $\nu, \tilde{\nu} \in \mathcal{V}_{ad}$, $(\nu, p, q) := T_k(\nu)$, $(\tilde{\nu}, \tilde{p}, \tilde{q}) := T_k(\tilde{\nu})$, and p_k, q_k being the unique solutions to

$$\begin{cases} \nu_{k-1}\partial_t^2 p_k - \Delta p_k + \eta\partial_t p_k = f - (\nu_k - \nu_{k-1})\partial_t^2 p_{k-1} & \text{in } I \times \Omega \\ \partial_n p_k = 0 & \text{on } I \times \Gamma_N \\ p_k = 0 & \text{on } I \times \Gamma_D \\ (p_k, \partial_t p_k)(0) = (0,0) & \text{in } \Omega \end{cases} \tag{4.55}$$

$$\begin{cases} \nu_{k-1}\partial_t^2 q_k - \Delta q_k - \eta\partial_t q_k = \sum_{i=1}^m a_i(p_k - p_i^{ob}) - (\nu_k - \nu_{k-1})\partial_t^2 q_{k-1} & \text{in } I \times \Omega \\ \partial_n q_k = 0 & \text{on } I \times \Gamma_N \\ q_k = 0 & \text{on } I \times \Gamma_D \\ (q_k, \partial_t q_k)(T) = (0,0) & \text{in } \Omega, \end{cases} \tag{4.56}$$

it holds that $p_k, q_k \in C^\infty(I, L^\infty(\Omega))$, $\partial_t^l p_k(0) = \partial_t^l q_k(T) = 0$ *for all* $l \in \mathbb{N}_0$, *and*

$$\max\{\|\partial_t^l p_k\|_{L^2(I,L^\infty(\Omega))}, \|\partial_t^l q_k\|_{L^2(I,L^\infty(\Omega))}\} \le C_0(l+3k)! \quad \forall l \in \mathbb{N}_0. \tag{4.57}$$

Furthermore, it holds for all $l \in \mathbb{N}_0$ *that*

$$\|\partial_t^l (p - \tilde{p})\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3k+3)!\|\nu - \tilde{\nu}\|_{L^2(\Omega)} \tag{4.58}$$

$$\|\partial_t^l (q - \tilde{q})\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3k+3)!\|\nu - \tilde{\nu}\|_{L^2(\Omega)} \tag{4.59}$$

$$\|\partial_t^l (\overline{p} - p_k)\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3k)!(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)}) \tag{4.60}$$

$$\|\partial_t^l (\overline{q} - q_k)\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3k)!(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)}) \tag{4.61}$$

$$\|\partial_t^l (p - p_k)\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3k+3)!(\|\nu - \nu_k\|_{L^2(\Omega)} + \|\nu - \nu_{k-1}\|_{L^2(\Omega)}) \tag{4.62}$$

$$\|\partial_t^l (q - q_k)\|_{L^2(I,L^\infty(\Omega))} \le c_1(l+3k+3)!(\|\nu - \nu_k\|_{L^2(\Omega)} + \|\nu - \nu_{k-1}\|_{L^2(\Omega)}). \tag{4.63}$$

Proof. Let $\nu, \tilde{\nu} \in \mathcal{V}_{ad}$, $(\overline{\nu}, \check{p}, \check{q}) \coloneqq T_0(\overline{\nu})$, $(\nu, p, q) \coloneqq T_0(\nu)$, and $(\tilde{\nu}, \tilde{p}, \tilde{q}) \coloneqq T_0(\tilde{\nu})$. Furthermore, let $l \in \mathbb{N}_0$. We first note from (4.42) that $\check{p} - \overline{p}$ solves

$$\begin{cases} \nu_0 \partial_t^2(\check{p} - \overline{p}) - \Delta(\check{p} - \overline{p}) + \eta\partial_t(\check{p} - \overline{p}) = -(\overline{\nu} - \nu_0)\partial_t^2(p_0 - \overline{p}) & \text{in } I \times \Omega \\ \partial_n(\check{p} - \overline{p}) = 0 & \text{on } I \times \Gamma_N \\ \check{p} - \overline{p} = 0 & \text{on } I \times \Gamma_D \\ (\check{p} - \overline{p}, \partial_t(\check{p} - \overline{p}))(0) = (0,0) & \text{in } \Omega, \end{cases}$$

such that Lemma 4.2 and Assumption 4.9 yield that

$$\begin{aligned} \|\partial_t^l(\check{p} - \overline{p})\|_{L^2(I,L^2(\Omega))} &\le c\sqrt{T}\|\overline{\nu} - \nu_0\|_{L^2(\Omega)}\|\partial_t^{l+1}(p_0 - \overline{p})\|_{L^1(I,L^\infty(\Omega))} \\ &\le cT\|\overline{\nu} - \nu_0\|_{L^2(\Omega)}\|\partial_t^{l+1}(p_0 - \overline{p})\|_{L^2(I,L^\infty(\Omega))} \\ &\le cTC_0(l+1)!\|\overline{\nu} - \nu_0\|^2_{L^2(\Omega)} \underbrace{\le}_{(4.34)} cTC_0\frac{\gamma}{4\delta}(l+1)!\|\nu_0 - \overline{\nu}\|_{L^2(\Omega)} \end{aligned} \tag{4.64}$$

and

$$\begin{aligned} &\|\partial_t^l(\check{p} - \overline{p})\|_{L^2(I,L^\infty(\Omega))} \\ &\le \hat{c}(\|\overline{\nu} - \nu_0\|_{L^2(\Omega)}\|\partial_t^{l+2}(p_0 - \overline{p})\|_{L^2(I,L^\infty(\Omega))} + \|\overline{\nu} - \nu_0\|_{L^2(\Omega)}\|\partial_t^{l+3}(p_0 - \overline{p})\|_{L^2(I,L^\infty(\Omega))}) \\ &\le 2\hat{c}\frac{\gamma}{4\delta}C_0(l+3)!\|\nu_0 - \overline{\nu}\|_{L^2(\Omega)}. \end{aligned} \tag{4.65}$$

Consequently, by the triangular inequality and Assumption 4.9 along with (4.65), we obtain that

$$\begin{aligned} \|\partial_t^l(\check{p} - p_0)\|_{L^2(I,L^\infty(\Omega))} &\le \|\partial_t^l(\check{p} - \overline{p})\|_{L^2(I,L^\infty(\Omega))} + \|\partial_t^l(\overline{p} - p_0)\|_{L^2(I,L^\infty(\Omega))} \\ &\le (\hat{c}\frac{\gamma}{2\delta} + 1)C_0(l+3)!\|\nu_0 - \overline{\nu}\|_{L^2(\Omega)}. \end{aligned}$$

From the definition of c_0 (see Assumption 4.9), this implies (4.51). Similarly, due to (3.135) and (4.43), $\check{q} - \overline{q}$ satisfies

$$\begin{cases} \nu_0 \partial_t^2(\check{q} - \overline{q}) - \Delta(\check{q} - \overline{q}) - \eta\partial_t(\check{q} - \overline{q}) = \sum_{i=1}^m a_i(\check{p} - \overline{p}) - (\overline{\nu} - \nu_0)\partial_t^2(q_0 - \overline{q}) & \text{in } I \times \Omega \\ \partial_n(\check{q} - \overline{q}) = 0 & \text{on } I \times \Gamma_N \\ \check{q} - \overline{q} = 0 & \text{on } I \times \Gamma_D \\ (\check{q} - \overline{q}, \partial_t(\check{q} - \overline{q}))(T) = (0,0) & \text{in } \Omega, \end{cases}$$

such that Lemma 4.2, Assumption 4.9, and (4.64) yield that

$$\begin{aligned}
&\|\partial_t^l(\check{q}-\overline{q})\|_{L^2(I,L^\infty(\Omega))} \qquad (4.66)\\
&\le \hat{c}\Big(\sum_{j=0}^{l}\binom{l}{j}\sum_{i=1}^{m}\|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l-j}(\check{p}-\overline{p})\|_{L^2(I,L^2(\Omega))}\\
&\quad + \|\overline{\nu}-\nu_0\|_{L^2(\Omega)}\|\partial_t^{l+2}(q_0-\overline{q})\|_{L^2(I,L^\infty(\Omega))}\\
&\quad + \sum_{j=0}^{l+1}\binom{l+1}{j}\sum_{i=1}^{m}\|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l+1-j}(\check{p}-\overline{p})\|_{L^2(I,L^2(\Omega))}\\
&\quad + \|\overline{\nu}-\nu_0\|_{L^2(\Omega)}\|\partial_t^{l+3}(q_0-\overline{q})\|_{L^2(I,L^\infty(\Omega))}\Big)\\
&\le \hat{c}\Big(C_a c T C_0\frac{\gamma}{4\delta}\sum_{j=0}^{l}\binom{l}{j}j!(l-j+1)!\|\overline{\nu}-\nu_0\|_{L^2(\Omega)} + \frac{\gamma}{4\delta}\|\overline{\nu}-\nu_0\|_{L^2(\Omega)}C_0(l+2)!\\
&\quad + C_a c T C_0\frac{\gamma}{4\delta}\sum_{j=0}^{l+1}\binom{l+1}{j}j!(l-j+2)!\|\overline{\nu}-\nu_0\|_{L^2(\Omega)} + \frac{\gamma}{4\delta}\|\overline{\nu}-\nu_0\|_{L^2(\Omega)}C_0(l+3)!\Big)\\
&\underbrace{\le}_{(4.67)} 2\hat{c}C_0\frac{\gamma}{4\delta}(C_a c T+1)(l+3)!\|\overline{\nu}-\nu_0\|_{L^2(\Omega)},
\end{aligned}$$

where we have used that

$$\sum_{j=0}^{l+1}\binom{l+1}{j}j!(l-j+2)! = \sum_{j=0}^{l+1}\frac{(l+1)!}{(l+1-j)!j!}j!(l-j+2)! = \sum_{j=0}^{l+1}(l+1)!(l-j+2) \le (l+3)!. \qquad (4.67)$$

Again, with the triangular inequality and Assumption 4.9 along with (4.66), we obtain that

$$\begin{aligned}
\|\partial_t^l(\check{q}-q_0)\|_{L^2(I,L^\infty(\Omega))} &\le \|\partial_t^l(\check{q}-\overline{q})\|_{L^2(I,L^\infty(\Omega))} + \|\partial_t^l(\overline{q}-q_0)\|_{L^2(I,L^\infty(\Omega))}\\
&\le C_0\Big(2\hat{c}\frac{\gamma}{4\delta}(C_a c T+1)+1\Big)(l+3)!\|\overline{\nu}-\nu_0\|_{L^2(\Omega)},
\end{aligned}$$

leading (4.52). By (4.42), $p-\tilde{p}$ satisfies

$$\begin{cases}
\nu_0\partial_t^2(p-\tilde{p}) - \Delta(p-\tilde{p}) + \eta\partial_t(p-\tilde{p}) = -(\nu-\tilde{\nu})\partial_t^2 p_0 & \text{in } I\times\Omega\\
\partial_n(p-\tilde{p}) = 0 & \text{on } I\times\Gamma_N\\
p-\tilde{p} = 0 & \text{on } I\times\Gamma_D\\
(p-\tilde{p},\partial_t(p-\tilde{p}))(0) = (0,0) & \text{in } \Omega.
\end{cases}$$

Applying Lemma 4.2, and Assumption 4.9 gives that

$$\|\partial_t^l(p-\tilde{p})\|_{L^2(I,L^2(\Omega))} \le c\sqrt{T}\|\nu-\tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^{l+1}p_0\|_{L^1(I,L^\infty(\Omega))} \le cC_0T(l+1)!\|\nu-\tilde{\nu}\|_{L^2(\Omega)} \qquad (4.68)$$

and

$$\begin{aligned}
\|\partial_t^l(p-\tilde{p})\|_{L^2(I,L^\infty(\Omega))} &\le \hat{c}\big(\|\nu-\tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^{l+2}p_0\|_{L^2(I,L^\infty(\Omega))} + \|\nu-\tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^{l+3}p_0\|_{L^2(I,L^\infty(\Omega))}\big)\\
&\le 2\hat{c}C_0(l+3)!\|\nu-\tilde{\nu}\|_{L^2(\Omega)},
\end{aligned}$$

leading to (4.53). By (4.43), the difference $q-\tilde{q}$ solves

$$\begin{cases} \nu_0\partial_t^2(q-\tilde{q})-\Delta(q-\tilde{q})-\eta\partial_t(q-\tilde{q})=\sum_{i=1}^m a_i(p-\tilde{p})-(\nu-\tilde{\nu})\partial_t^2 q_0 & \text{in } I\times\Omega \\ \partial_n(q-\tilde{q})=0 & \text{on } I\times\Gamma_N \\ q-\tilde{q}=0 & \text{on } I\times\Gamma_D \\ (q-\tilde{q},\partial_t(q-\tilde{q}))(T)=(0,0) & \text{in } \Omega, \end{cases}$$

such that Lemma 4.2, Assumption 4.9, and (4.68) provide that

$$\begin{aligned} &\|\partial_t^l(q-\tilde{q})\|_{L^2(I,L^\infty(\Omega))} \\ &\le \hat{c}\Big(\sum_{j=0}^{l}\binom{l}{j}\sum_{i=1}^m\|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l-j}(p-\tilde{p})\|_{L^2(I,L^2(\Omega))}+\|\nu-\tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_0\|_{L^2(I,L^\infty(\Omega))} \\ &\quad+\sum_{j=0}^{l+1}\binom{l+1}{j}\sum_{i=1}^m\|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l+1-j}(p-\tilde{p})\|_{L^2(I,L^2(\Omega))}+\|\nu-\tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^{l+3}q_0\|_{L^2(I,L^\infty(\Omega))}\Big) \\ &\le \hat{c}\Big(C_a c C_0 T\sum_{j=0}^{l}\binom{l}{j}j!(l-j+1)!\|\nu-\tilde{\nu}\|_{L^2(\Omega)}+\|\nu-\tilde{\nu}\|_{L^2(\Omega)}C_0(l+2)! \\ &\quad+C_a c C_0 T\sum_{j=0}^{l+1}\binom{l+1}{j}j!(l-j+2)!\|\nu-\tilde{\nu}\|_{L^2(\Omega)}+\|\nu-\tilde{\nu}\|_{L^2(\Omega)}C_0(l+3)!\Big) \\ &\underbrace{\le}_{(4.67)} 2\hat{c}C_0(C_a cT+1)(l+3)!\|\nu-\tilde{\nu}\|_{L^2(\Omega)}, \end{aligned}$$

leading to (4.54). Now, let $k\in\mathbb{N}$ and we redefine $(\nu,p,q)\coloneqq T_k(\nu)$ and $(\tilde{\nu},\tilde{p},\tilde{q})\coloneqq T_k(\tilde{\nu})$. Furthermore, let ν_k and $(\nu_{k-1},p_{k-1},q_{k-1})$ satisfy (A_k) and p_k,q_k being the unique solutions to (4.55) and (4.56). Due to Assumption 4.9 and (A_k), applying Lemma 4.2 to (4.55) yields that $p_k\in C^\infty(I,L^\infty(\Omega))$, $\partial_t^l p_k(0)=0$ for all $l\in\mathbb{N}_0$, and

$$\begin{aligned} \|\partial_t^l p_k\|_{L^2(I,L^\infty(\Omega))} &\le \hat{c}\big(\|\partial_t^l f\|_{L^2(I,L^2(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}p_{k-1}\|_{L^2(I,L^\infty(\Omega))} \\ &\qquad+\|\partial_t^{l+1}f\|_{L^2(I,L^2(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}p_{k-1}\|_{L^2(I,L^\infty(\Omega))}\big) \\ &\underbrace{\le}_{(4.34),(\mathrm{A}_k)} 2\hat{c}C_f(l+1)!+\hat{c}C_0\frac{1}{2\delta}(b_k+b_{k-1})(l+3k)! \\ &\underbrace{\le}_{(4.47)} 2\hat{c}C_f(l+1)!+\hat{c}C_0\frac{\overline{\gamma}}{\delta}(l+3k)! \underbrace{\le}_{(4.36)} C_0(l+3k)!. \end{aligned} \tag{4.69}$$

From (4.55), we obtain that

$$\begin{cases} \nu_{k-1}\partial_t^2(\overline{p}-p_k)-\Delta(\overline{p}-p_k)+\eta\partial_t(\overline{p}-p_k)=-(\overline{\nu}-\nu_{k-1})\partial_t^2\overline{p}+(\nu_k-\nu_{k-1})\partial_t^2 p_{k-1} & \text{in } I\times\Omega \\ \partial_n(\overline{p}-p_k)=0 & \text{on } I\times\Gamma_N \\ \overline{p}-p_k=0 & \text{on } I\times\Gamma_D \\ (\overline{p}-p_k,\partial_t(\overline{p}-p_k))(0)=(0,0) & \text{in } \Omega. \end{cases} \tag{4.70}$$

Thus, Lemma 4.2, Assumption 4.9, and (A_k) yield

$$\begin{aligned}
&\|\partial_t^l(\overline{p}-p_k)\|_{L^2(I,L^2(\Omega))} \qquad (4.71)\\
&\le c\sqrt{T}\big(\|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+1}\overline{p}\|_{L^1(I,L^\infty(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+1}p_{k-1}\|_{L^1(I,L^\infty(\Omega))}\big)\\
&\le cT(\overline{C}+C_0)(l+3k-2)!\big(\|\overline{\nu}-\nu_k\|_{L^2(\Omega)}+\|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_t^l(\overline{p}-p_k)\|_{L^2(I,L^\infty(\Omega))} \qquad (4.72)\\
&\le \hat{c}\big(\|(\overline{\nu}-\nu_{k-1})\partial_t^{l+2}\overline{p}\|_{L^2(I,L^2(\Omega))}+\|(\overline{\nu}-\nu_{k-1})\partial_t^{l+3}\overline{p}\|_{L^2(I,L^2(\Omega))}\\
&\qquad+\|(\nu_k-\nu_{k-1})\partial_t^{l+2}p_{k-1}\|_{L^2(I,L^2(\Omega))}+\|(\nu_k-\nu_{k-1})\partial_t^{l+3}p_{k-1}\|_{L^2(I,L^2(\Omega))}\big)\\
&\le 2\hat{c}(\overline{C}+C_0)(l+3k)!\big(\|\overline{\nu}-\nu_k\|_{L^2(\Omega)}+\|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)}\big),
\end{aligned}$$

leading to (4.60). Applying Lemma 4.2 to (4.56), it follow that $q_k\in C^\infty(I,L^\infty(\Omega))$, $\partial_t^l q_k(T)=0$ for all $l\in\mathbb{N}_0$, and along with Assumption 4.9 and (4.71), we obtain that

$$\begin{aligned}
&\|\partial_t^l q_k\|_{L^2(I,L^\infty(\Omega))}\\
&\le \hat{c}\Big(\sum_{i=1}^m\|\partial_t^l(a_i(p_k-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\\
&\qquad+\sum_{i=1}^m\|\partial_t^{l+1}(a_i(p_k-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\Big)\\
&\le \hat{c}\Big(\sum_{i=1}^m\|\partial_t^l(a_i(\overline{p}-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}+\sum_{i=1}^m\|\partial_t^{l+1}(a_i(\overline{p}-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}\\
&\qquad+\sum_{i=1}^m\|\partial_t^l(a_i(p_k-\overline{p}))\|_{L^2(I,L^2(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\\
&\qquad+\sum_{i=1}^m\|\partial_t^{l+1}(a_i(p_k-\overline{p}))\|_{L^2(I,L^2(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\Big)\\
&\le \hat{c}\Big(\sum_{i=1}^m\|\partial_t^l(a_i(\overline{p}-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}+\sum_{i=1}^m\|\partial_t^{l+1}(a_i(\overline{p}-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}\\
&\qquad+\sum_{i=1}^m\sum_{j=0}^l\binom{l}{j}\|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l-j}(p_k-\overline{p})\|_{L^2(I,L^2(\Omega))}\\
&\qquad+\sum_{i=1}^m\sum_{j=0}^{l+1}\binom{l+1}{j}\|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l+1-j}(p_k-\overline{p})\|_{L^2(I,L^2(\Omega))}\\
&\qquad+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\Big)\\
&\le \hat{c}\Big(\sum_{i=1}^m\|\partial_t^l(a_i(\overline{p}-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}+\sum_{i=1}^m\|\partial_t^{l+1}(a_i(\overline{p}-p_i^{ob}))\|_{L^2(I,L^2(\Omega))}\\
&\qquad+C_acT(\overline{C}+C_0)\sum_{j=0}^l\binom{l}{j}j!(l-j+3k-2)!\big(\|\overline{\nu}-\nu_k\|_{L^2(\Omega)}+\|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)}\big)\\
&\qquad+\|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}\\
&\qquad+C_acT(\overline{C}+C_0)\sum_{j=0}^{l+1}\binom{l+1}{j}j!(l-j+3k-1)!\big(\|\overline{\nu}-\nu_k\|_{L^2(\Omega)}+\|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)}\big)
\end{aligned}$$

$$+ \|\nu_k - \nu_{k-1}\|_{L^2(\Omega)} \|\partial_t^{l+3} q_{k-1}\|_{L^2(I,L^\infty(\Omega))})$$

$$\underbrace{\leq}_{(4.34),(\mathrm{A}_k)} \hat{c}(2C_a(l+1)! + C_a cT(\overline{C} + C_0)\frac{1}{2\delta}(b_k + b_{k+1})(l+3k)! + C_0\frac{1}{2\delta}(b_k + b_{k+1})(l+3k)!)$$

$$\underbrace{\leq}_{(4.47)} \hat{c}(2C_a(l+1)! + C_a cT(\overline{C} + C_0)\frac{\overline{\gamma}}{\delta}(l+3k)! + C_0\frac{\overline{\gamma}}{\delta}(l+3k)!) \underbrace{\leq}_{(4.36)} C_0(l+3k)!.$$

Along with (4.69), the above estimate implies that (4.57) is valid. Now, (4.58) and (4.59) follow analog to (4.53) and (4.54). By (3.135) and (4.56), we obtain that

$$\begin{cases} \nu_{k-1}\partial_t^2(\overline{q} - q_k) - \Delta(\overline{q} - q_k) - \eta\partial_t(\overline{q} - q_k) & \\ \quad = \sum_{i=1}^m a_i(\overline{p} - p_k) - (\overline{\nu} - \nu_{k-1})\partial_t^2\overline{q} + (\nu_k - \nu_{k-1})\partial_t^2 q_{k-1} & \text{in } I \times \Omega \\ \partial_n(\overline{q} - q_k) = 0 & \text{on } I \times \Gamma_N \\ \overline{q} - q_k = 0 & \text{on } I \times \Gamma_D \\ (\overline{q} - q_k, \partial_t(\overline{q} - q_k))(T) = (0,0) & \text{in } \Omega, \end{cases}$$

such that Lemma 4.2, Assumption 4.9, (A_k), and (4.71) provide

$$\begin{aligned} &\|\partial_t^l(\overline{q} - q_k)\|_{L^2(I,L^\infty(\Omega))} \\ &\leq \hat{c}(\sum_{j=0}^l \binom{l}{j} \sum_{i=1}^m \|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))} \|\partial_t^{l-j}(\overline{p} - p_k)\|_{L^2(I,L^2(\Omega))} \\ &\quad + \sum_{j=0}^{l+1} \binom{l+1}{j} \sum_{i=1}^m \|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))} \|\partial_t^{l+1-j}(\overline{p} - p_k)\|_{L^2(I,L^2(\Omega))} \\ &\quad + \|(\overline{\nu} - \nu_{k-1})\partial_t^{l+2}\overline{q}\|_{L^2(I,L^2(\Omega))} + \|(\overline{\nu} - \nu_{k-1})\partial_t^{l+3}\overline{q}\|_{L^2(I,L^2(\Omega))} \\ &\quad + \|(\nu_k - \nu_{k-1})\partial_t^{l+2} q_{k-1}\|_{L^2(I,L^2(\Omega))} + \|(\nu_k - \nu_{k-1})\partial_t^{l+3} q_{k-1}\|_{L^2(I,L^2(\Omega))}) \\ &\leq \hat{c}(C_a cT(\overline{C} + C_0)\sum_{j=0}^l \binom{l}{j} j!(l+3k-j-2)! + \overline{C}(l+2)! + C_0(l+3k-1)! \\ &\quad + C_a cT(\overline{C} + C_0)\sum_{j=0}^{l+1} \binom{l+1}{j} j!(l+3k-j-1)! + \overline{C}(l+3)! \\ &\quad + C_0(l+3k)!)(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)}) \\ &\leq \hat{c}(2C_a cT(\overline{C} + C_0) + 2\overline{C} + 2C_0)(l+3k)!(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)}), \end{aligned}$$

leading to (4.61). To prove (4.62), due to (4.42) and (4.55), note that $p - p_k$ solves

$$\begin{cases} \nu_k\partial_t^2(p - p_k) - \Delta(p - p_k) + \eta\partial_t(p - p_k) = -(\nu - \nu_{k-1})\partial_t^2 p_k + (\nu_k - \nu_{k-1})\partial_t^2 p_{k-1} & \text{in } I \times \Omega \\ \partial_n(p - p_k) = 0 & \text{on } I \times \Gamma_N \\ p - p_k = 0 & \text{on } I \times \Gamma_D \\ (p - p_k, \partial_t(p - p_k))(0) = (0,0) & \text{in } \Omega, \end{cases} \tag{4.73}$$

such that from Lemma 4.2, (4.57), and (A_k), it follows that

$$\|\partial_t^l(p - p_k)\|_{L^2(I,L^2(\Omega))} \tag{4.74}$$

$$\leq c\sqrt{T}(\|\nu - \nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+1}p_k\|_{L^1(I,L^\infty(\Omega))} + \|\nu_k - \nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+1}p_{k-1}\|_{L^1(I,L^\infty(\Omega))})$$
$$\leq cTC_0(l+3k+1)!(\|\nu - \nu_k\|_{L^2(\Omega)} + \|\nu - \nu_{k-1}\|_{L^2(\Omega)})$$

and

$$\begin{aligned}
&\|\partial_t^l(p-p_k)\|_{L^2(I,L^\infty(\Omega))} \\
&\leq \hat{c}(\|\nu - \nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}p_k\|_{L^2(I,L^\infty(\Omega))} + \|\nu_k - \nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}p_{k-1}\|_{L^2(I,L^\infty(\Omega))} \\
&\quad + \|\nu - \nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}p_k\|_{L^2(I,L^\infty(\Omega))} + \|\nu_k - \nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}p_{k-1}\|_{L^2(I,L^\infty(\Omega))}) \\
&\leq 4\hat{c}C_0(l+3k+3)!(\|\nu - \nu_k\|_{L^2(\Omega)} + \|\nu - \nu_{k-1}\|_{L^2(\Omega)}),
\end{aligned} \tag{4.75}$$

leading to (4.62). Finally, due to (4.43) and (4.56), it holds that

$$\begin{cases}
\nu_k\partial_t^2(q-q_k) - \Delta(q-q_k) - \eta\partial_t(q-q_k) & \\
\quad = \sum_{i=1}^m a_i(p-p_k) - (\nu-\nu_{k-1})\partial_t^2 q_k + (\nu_k - \nu_{k-1})\partial_t^2 q_{k-1} & \text{in } I\times\Omega \\
\partial_n(q-q_k) = 0 & \text{on } I\times\Gamma_N \\
q - q_k = 0 & \text{on } I\times\Gamma_D \\
(q-q_k, \partial_t(q-q_k))(T) = (0,0) & \text{in } \Omega,
\end{cases}$$

such that Lemma 4.2, Assumption 4.9, (A_k), (4.57), and (4.74) yield that

$$\begin{aligned}
&\|\partial_t^l(q-q_k)\|_{L^2(I,L^\infty(\Omega))} \\
&\leq \hat{c}(\sum_{j=0}^{l}\binom{l}{j}\sum_{i=1}^m \|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l-j}(p-p_k)\|_{L^2(I,L^2(\Omega))} \\
&\quad + \sum_{j=0}^{l+1}\binom{l+1}{j}\sum_{i=1}^m \|\partial_t^j a_i\|_{L^\infty(I,L^\infty(\Omega))}\|\partial_t^{l+1-j}(p-p_k)\|_{L^2(I,L^2(\Omega))} \\
&\quad + \|\nu-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_k\|_{L^2(I,L^\infty(\Omega))} + \|\nu-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}q_k\|_{L^2(I,L^\infty(\Omega))} \\
&\quad + \|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+2}q_{k-1}\|_{L^2(I,L^\infty(\Omega))} + \|\nu_k-\nu_{k-1}\|_{L^2(\Omega)}\|\partial_t^{l+3}q_{k-1}\|_{L^2(I,L^\infty(\Omega))}) \\
&\leq \hat{c}(C_a cTC_0\sum_{j=0}^{l}\binom{l}{j}j!(l-j+3k+1)! + 4C_0(l+3k+3)! \\
&\quad + C_a cTC_0\sum_{j=0}^{l+1}\binom{l+1}{j}j!(l-j+3k+2)!)(\|\nu-\nu_k\|_{L^2(\Omega)} + \|\nu-\nu_{k-1}\|_{L^2(\Omega)}) \\
&\leq \hat{c}(2C_a cTC_0 + 4C_0)(l+3k+3)!(\|\nu-\nu_k\|_{L^2(\Omega)} + \|\nu-\nu_{k-1}\|_{L^2(\Omega)}),
\end{aligned}$$

leading to (4.63). □

4.4 Convergence

Under Assumption 4.7, we know that Algorithm 1 is well-defined (see Theorem 4.8), but the iteration step ($\mathbb{P}_k$) may have multiple possible solutions. In the following, we prove that under Assumption 4.9, the solution to the iteration step ($\mathbb{P}_k$) is unique. More importantly, under a two-step estimation process (4.77), we establish the R-superlinear convergence of the unique sequence of iterations towards the solution to (P).

Proposition 4.14. *Let Assumption 4.9 be satisfied and* $\{b_k\}_{k\in\mathbb{N}_0}$ *as in Lemma 4.12. Then, the mapping* $(I_\nu \circ S_0 \circ T_0)\colon L^2(\Omega) \to L^2(\Omega)$ *associated with* (ν_0, p_0, q_0) *is a contraction and admits a unique fixed point* $\nu_1 \in \mathcal{V}_{ad}$ *satisfying*

$$\|\nu_1 - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta} b_1, \tag{4.76}$$

where $\overline{\gamma}$ *and* δ *are as in Assumption 4.9. Let additionally* ν_k *and* $(\nu_{k-1}, p_{k-1}, q_{k-1})$ *satisfy* (A_k) *for some* $k \in \mathbb{N}$. *Then, the mapping* $(I_\nu \circ S_k \circ T_k)\colon L^2(\Omega) \to L^2(\Omega)$ *associated with* (ν_k, p_k, q_k), *with* p_k *and* q_k *being the unique solutions to* (4.55) *and* (4.56), *is a contraction and admits a unique fixed point* $\nu_{k+1} \in \mathcal{V}_{ad}$ *satisfying*

$$\|\nu_{k+1} - \overline{\nu}\|_{L^2(\Omega)} \le \delta (3k+5)!^2 \big(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)}\big)^2 \tag{4.77}$$

$$\|\nu_{k+1} - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta} b_{k+1}. \tag{4.78}$$

Remark 4.15. Notice that the positive constant δ appearing in (4.76)-(4.78) is independent of k. More precisely, it depends only on the given data as defined in Assumption 4.9.

Proof. Let $\nu, \tilde{\nu} \in L^2(\Omega)$ and $k \in \mathbb{N}_0$. Further, if $k \in \mathbb{N}$, suppose that ν_k and $(\nu_{k-1}, p_{k-1}, q_{k-1})$ satisfy (A_k) and p_k, q_k denote the unique solutions to (4.55) and (4.56), respectively. According to (4.41), we may write $(\nu, p, q) \coloneqq T_k(\nu)$ and $(\tilde{\nu}, \tilde{p}, \tilde{q}) \coloneqq T_k(\tilde{\nu})$. As pointed out in Remark 4.11, $(S_k \circ T_k)(\nu)$ and $(S_k \circ T_k)(\tilde{\nu})$ solve (OS^τ) with the perturbation terms (4.40) for $(\hat{\nu}, \hat{p}, \hat{q}) = (\nu, p, q)$ and $(\hat{\nu}, \hat{p}, \hat{q}) = (\tilde{\nu}, \tilde{p}, \tilde{q})$, respectively. Then, by Theorem 4.4, it holds that

$$\begin{aligned}
&\|(I_\nu \circ S_k \circ T_k)(\nu) - (I_\nu \circ S_k \circ T_k)(\tilde{\nu})\|_{L^2(\Omega)} \\
&\le L\big(\|(\nu_k - \overline{\nu})\partial_t^2(p - \tilde{p}) + (\nu - \tilde{\nu})\partial_t^2(p_k - \overline{p})\|_{L^2(I,L^2(\Omega))} \\
&\qquad + \|(\nu_k - \overline{\nu})\partial_t^2(q - \tilde{q}) + (\nu - \tilde{\nu})\partial_t^2(q_k - \overline{q})\|_{L^2(I,L^2(\Omega))} \\
&\qquad + \Big\| \int_I \partial_t^2(p_k(t) - \overline{p}(t))(q(t) - \tilde{q}(t)) + \partial_t^2(p(t) - \tilde{p}(t))(q_k(t) - \overline{q}(t))\,\mathrm{d}t \Big\|_{L^2(\Omega)}\big) \\
&\le L\big(\|\nu_k - \overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2(p - \tilde{p})\|_{L^2(I,L^\infty(\Omega))} + \|\nu - \tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^2(p_k - \overline{p})\|_{L^2(I,L^\infty(\Omega))} \\
&\qquad + \|\nu_k - \overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2(q - \tilde{q})\|_{L^2(I,L^\infty(\Omega))} + \|\nu - \tilde{\nu}\|_{L^2(\Omega)}\|\partial_t^2(q_k - \overline{q})\|_{L^2(I,L^\infty(\Omega))} \\
&\qquad + \|\partial_t^2(p_k - \overline{p})\|_{L^2(I,L^\infty(\Omega))}\|q - \tilde{q}\|_{L^2(I,L^2(\Omega))} + \|\partial_t^2(p - \tilde{p})\|_{L^2(I,L^2(\Omega))}\|q_k - \overline{q}\|_{L^2(I,L^\infty(\Omega))}\big).
\end{aligned} \tag{4.79}$$

According to Assumption 4.9 and Lemma 4.13 (see (4.53) and (4.54)), the above inequality implies for $k = 0$ that

$$\begin{aligned}
&\|(I_\nu \circ S_0 \circ T_0)(\nu) - (I_\nu \circ S_0 \circ T_0)(\tilde{\nu})\|_{L^2(\Omega)} \\
&\le L(2c_1 5! + 4C_0 + 2\sqrt{|\Omega|}5! C_0 c_1)\|\nu_0 - \overline{\nu}\|_{L^2(\Omega)}\|\nu - \tilde{\nu}\|_{L^2(\Omega)} \underbrace{\le}_{(4.34)} \frac{1}{2}\|\nu - \tilde{\nu}\|_{L^2(\Omega)}.
\end{aligned}$$

Similarly, for $k \in \mathbb{N}$, we obtain, using the monotonicity $b_k \le b_{k-1}$ (see Lemma 4.12), that

$$\begin{aligned}
&\|(I_\nu \circ S_k \circ T_k)(\nu) - (I_\nu \circ S_k \circ T_k)(\tilde{\nu})\|_{L^2(\Omega)} \\
&\underbrace{\le}_{(4.58)-(4.61)} L(4c_1 + 2c_1^2)(3k+5)!^2\big(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)}\big)\|\nu - \tilde{\nu}\|_{L^2(\Omega)}
\end{aligned}$$

$$\underbrace{\leq}_{(\mathrm{A}_k)} \frac{1}{2\delta}L(4c_1+2c_1^2)(3k+5)!^2 b_{k-1}\|\nu-\tilde{\nu}\|_{L^2(\Omega)} \underbrace{\leq}_{(4.47)} \frac{1}{2\delta}L(4c_1+2c_1^2)\overline{\gamma}\|\nu-\tilde{\nu}\|_{L^2(\Omega)}$$
$$\underbrace{\leq}_{(4.35)} \frac{1}{2}\|\nu-\tilde{\nu}\|_{L^2(\Omega)}.$$

Therefore, the mapping $(I_\nu \circ S_k \circ T_k)\colon L^2(\Omega) \to L^2(\Omega)$ is a contraction and consequently admits a unique fixed point $\nu_{k+1} \in L^2(\Omega)$ due to Banach's fixed point theorem. Also, according to Remark 4.11, $I_\nu \circ S_k$ maps into $\mathcal{V}_{ad}^\tau$ such that $\nu_{k+1} \in \mathcal{V}_{ad}^\tau$. Now, let us prove (4.76) and (4.77). From the above contraction property, it holds that

$$\frac{1}{2}\|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)} + \|(I_\nu \circ S_k \circ T_k)(\nu_{k+1}) - (I_\nu \circ S_k \circ T_k)(\overline{\nu})\|_{L^2(\Omega)} \leq \|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)}.$$

Thus, since $\nu_{k+1} = (I_\nu \circ S_k \circ T_k)\nu_{k+1}$, it follows that

$$\begin{aligned}\frac{1}{2}\|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)} &\leq \|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)} - \|(I_\nu \circ S_k \circ T_k)(\nu_{k+1}) - (I_\nu \circ S_k \circ T_k)(\overline{\nu})\|_{L^2(\Omega)} \\ &\leq \|(I_\nu \circ S_k \circ T_k)(\overline{\nu}) - \overline{\nu}\|_{L^2(\Omega)}.\end{aligned} \tag{4.80}$$

Furthermore, we set $(\overline{\nu},\check{p},\check{q}) \coloneqq T_k(\overline{\nu})$. As above, according to Remark 4.11, $S_k(T_k(\overline{\nu}))$ solves (OS^τ) with the perturbation terms (4.40) with $(\hat{\nu},\hat{p},\hat{q}) = (\overline{\nu},\check{p},\check{q})$, and $(\overline{\nu},\overline{p},\overline{q})$ solves (OS^τ) with the perturbation $(\rho^{st},\rho^{adj},\rho^{VI}) = (0,0,0)$. Thus, Theorem 4.4 implies that

$$\begin{aligned}&\|(I_\nu \circ S_k \circ T_k)(\overline{\nu}) - \overline{\nu}\|_{L^2(\Omega)} \\ &\leq L\Big(\|(\nu_k-\overline{\nu})\partial_t^2(\check{p}-p_k)\|_{L^2(I,L^2(\Omega))} + \|(\nu_k-\overline{\nu})\partial_t^2(\check{q}-q_k)\|_{L^2(I,L^2(\Omega))} \\ &\quad + \Big\|\int_I \partial_t^2(p_k(t)-\overline{p}(t))\check{q}(t) + \partial_t^2(\check{p}(t)-p_k(t))q_k(t) + \partial_t^2(\overline{p}(t)-\check{p}(t))\overline{q}(t)\,\mathrm{d}t\Big\|_{L^2(\Omega)}\Big) \\ &\leq L\Big(\|\nu_k-\overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2(\check{p}-p_k)\|_{L^2(I,L^\infty(\Omega))} + \|\nu_k-\overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2(\check{q}-q_k)\|_{L^2(I,L^\infty(\Omega))} \\ &\quad + \|\partial_t^2(p_k-\overline{p})\|_{L^2(I,L^\infty(\Omega))}\|\check{q}-\overline{q}\|_{L^2(I,L^2(\Omega))} + \|\partial_t^2(\check{p}-p_k)\|_{L^2(I,L^\infty(\Omega))}\|q_k-\overline{q}\|_{L^2(I,L^2(\Omega))}\Big) \\ &\leq L\Big(\|\nu_k-\overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2(\check{p}-p_k)\|_{L^2(I,L^\infty(\Omega))} + \|\nu_k-\overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2(\check{q}-q_k)\|_{L^2(I,L^\infty(\Omega))} \\ &\quad + \|\partial_t^2(p_k-\overline{p})\|_{L^2(I,L^\infty(\Omega))}\|\check{q}-q_k\|_{L^2(I,L^2(\Omega))} + \|\partial_t^2(p_k-\overline{p})\|_{L^2(I,L^\infty(\Omega))}\|q_k-\overline{q}\|_{L^2(I,L^2(\Omega))} \\ &\quad + \|\partial_t^2(\check{p}-p_k)\|_{L^2(I,L^\infty(\Omega))}\|q_k-\overline{q}\|_{L^2(I,L^2(\Omega))}\Big).\end{aligned} \tag{4.81}$$

For $k = 0$, applying (4.81) to (4.80) and making use of Lemma 4.13 and Assumption 4.9 yield that

$$\|\nu_1-\overline{\nu}\|_{L^2(\Omega)} \leq 2L(2c_0 5! + \sqrt{|\Omega|}C_0(2c_0 3! + 2C_0 + c_0 5!))\|\nu_0-\overline{\nu}\|^2_{L^2(\Omega)} \underbrace{\leq}_{(4.34)} \frac{1}{4\delta}8!^2\gamma^{\sqrt{2}} = \frac{1}{4\delta}b_1.$$

Analogously, for $k \in \mathbb{N}$, Lemma 4.13 implies that

$$\|\nu_{k+1}-\overline{\nu}\|_{L^2(\Omega)} \underbrace{\leq}_{(4.60)-(4.63)} \underbrace{2L(2c_1+3\sqrt{|\Omega|}c_1^2)}_{=\delta}(3k+5)!^2(\|\overline{\nu}-\nu_k\|_{L^2(\Omega)} + \|\overline{\nu}-\nu_{k-1}\|_{L^2(\Omega)})^2.$$

In conclusion, (4.76) and (4.77) are valid. Finally, due to (A_k) and (4.77), we may apply Lemma 4.12 with $x_k \coloneqq \|\nu_k-\overline{\nu}\|_{L^2(\Omega)}$ to obtain (4.78). □

Proposition 4.16. *Let Assumption 4.9 be satisfied and $\nu_1 \in \mathcal{V}_{ad}$ denote the unique fixed point of $(I_\nu \circ S_0 \circ T_0)$ associated with (ν_0, p_0, q_0). Then, $(\nu_1, p_1, q_1) := T_0(\nu_1)$ is the unique solution to the iteration $(\mathbb{P}_k)$ for $k = 0$. Let additionally ν_k and $(\nu_{k-1}, p_{k-1}, q_{k-1})$ satisfy (A_k) for some $k \in \mathbb{N}$ and $\nu_{k+1} \in \mathcal{V}_{ad}$ denote the unique fixed point of $(I_\nu \circ S_k \circ T_k)$ associated with (ν_k, p_k, q_k), with p_k and q_k being the unique solutions to (4.55) and (4.56). Then, $(\nu_{k+1}, p_{k+1}, q_{k+1}) := T_k(\nu_{k+1})$ is the unique solution to $(\mathbb{P}_k)$.*

Proof. Let $k \in \mathbb{N}_0$. Since ν_{k+1} is a fixed-point of $(I_\nu \circ S_k \circ T_k)$, it holds by the definition of S_k (see (4.39) for $(\hat{\nu}, \hat{p}, \hat{q}) = (\nu_{k+1}, p_{k+1}, q_{k+1})$) that $S_k(\nu_{k+1}, p_{k+1}, q_{k+1}) = (\nu_{k+1}, p, q)$ where p, q are the unique solutions to

$$\begin{cases} \overline{\nu}\partial_t^2 p - \Delta p + \eta\partial_t p = f - (\nu_k - \overline{\nu})\partial_t^2 p_{k+1} - (\nu_{k+1} - \nu_k)\partial_t^2 p_k & \text{in } I \times \Omega \\ \partial_n p = 0 & \text{on } I \times \Gamma_N \\ p = 0 & \text{on } I \times \Gamma_D \\ (p, \partial_t p)(0) = (0,0) & \text{in } \Omega \end{cases} \tag{4.82}$$

and

$$\begin{cases} \overline{\nu}\partial_t^2 q - \Delta q - \eta\partial_t q = \sum_{i=1}^m a_i(p - p_i^{ob}) - (\nu_k - \overline{\nu})\partial_t^2 q_{k+1} - (\nu_{k+1} - \nu_k)\partial_t^2 q_k & \text{in } I \times \Omega \\ \partial_n q = 0 & \text{on } I \times \Gamma_N \\ q = 0 & \text{on } I \times \Gamma_D \\ (q, \partial_t q)(T) = (0,0) & \text{in } \Omega. \end{cases} \tag{4.83}$$

Furthermore, according to the definition of T_k (see (4.41) for $(\hat{\nu}, \hat{p}, \hat{q}) = (\nu_{k+1}, p_{k+1}, q_{k+1})$), p_{k+1} and q_{k+1}, respectively, solve the same systems (4.82) and (4.83). Therefore, we obtain $p = p_{k+1}$ and $q = q_{k+1}$, and consequently $T_k(\nu_{k+1})$ is a fixed point of S_k and satisfies the PDEs in $(\mathbb{P}_k)$. Let us prove that $T_k(\nu_{k+1})$ satisfies the variational inequality in $(\mathbb{P}_k)$. Note that the assumptions of Lemma 4.13 are fulfilled. On the other hand, in view of Remark 4.11, the fixed point $(\nu_{k+1}, p_{k+1}, q_{k+1})$ of S_k solves (OS^τ) with the perturbation terms

$$\begin{aligned} \rho^{st} &= -(\nu_k - \overline{\nu})\partial_t^2 p_{k+1} - (\nu_{k+1} - \nu_k)\partial_t^2 p_k - (\overline{\nu} - \nu_{k+1})\partial_t^2 \overline{p} && \in H^1(I, L^2(\Omega)) \\ \rho^{adj} &= -(\nu_k - \overline{\nu})\partial_t^2 q_{k+1} - (\nu_{k+1} - \nu_k)\partial_t^2 q_k - (\overline{\nu} - \nu_{k+1})\partial_t^2 \overline{q} && \in H^1(I, L^2(\Omega)) \\ \rho^{VI} &= \int_I \partial_t^2(p_k(t) - \overline{p}(t))q_{k+1}(t) + \partial_t^2(p_{k+1}(t) - p_k(t))q_k(t) \\ &\quad + \partial_t^2(\overline{p}(t) - p_{k+1}(t))\overline{q}(t)\,\mathrm{d}t && \in L^\infty(\Omega), \end{aligned} \tag{4.84}$$

satisfying $\rho^{st}(0) = \rho^{adj}(T) = 0$. Using Assumption 4.9, Lemma 4.13 for k and $k+1$, we obtain that

$$\begin{aligned} &\|\rho^{st}\|_{H^1(I,L^2(\Omega))} + \|\rho^{adj}\|_{H^1(I,L^2(\Omega))} + \|\rho^{VI}\|_{L^\infty(\Omega)} \\ &\leq \|\nu_k - \overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2 p_{k+1}\|_{H^1(I,L^\infty(\Omega))} + \|\nu_{k+1} - \nu_k\|_{L^2(\Omega)}\|\partial_t^2 p_k\|_{H^1(I,L^\infty(\Omega))} \\ &\quad + \|\overline{\nu} - \nu_{k+1}\|_{L^2(\Omega)}\|\partial_t^2 \overline{p}\|_{H^1(I,L^\infty(\Omega))} + \|\nu_k - \overline{\nu}\|_{L^2(\Omega)}\|\partial_t^2 q_{k+1}\|_{H^1(I,L^\infty(\Omega))} \\ &\quad + \|\nu_{k+1} - \nu_k\|_{L^2(\Omega)}\|\partial_t^2 q_k\|_{H^1(I,L^\infty(\Omega))} + \|\overline{\nu} - \nu_{k+1}\|_{L^2(\Omega)}\|\partial_t^2 \overline{q}\|_{H^1(I,L^\infty(\Omega))} \end{aligned}$$

$$
\begin{aligned}
&+ \|\partial_t^2(p_k - \overline{p})\|_{L^2(I,L^\infty(\Omega))}\|q_{k+1}\|_{L^2(I,L^\infty(\Omega))} + \|\partial_t^2(p_{k+1} - p_k)\|_{L^2(I,L^\infty(\Omega))}\|q_k\|_{L^2(I,L^\infty(\Omega))} \\
&+ \|\partial_t^2(\overline{p} - p_{k+1})\|_{L^2(I,L^\infty(\Omega))}\|\overline{q}\|_{L^2(I,L^\infty(\Omega))} \\
\le{}& ((8C_0 + 4\overline{C})(3k+6)! + (3c_1C_0 + c_1\overline{C})(3k+5)!^2)(\|\overline{\nu} - \nu_{k+1}\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_k\|_{L^2(\Omega)}) \\
\underbrace{\le}_{(\mathrm{A}_k),(4.78)}{}& \frac{1}{4\delta}(8C_0 + 4\overline{C} + 3c_1C_0 + c_1\overline{C})(3k+5)!^2(\underbrace{b_{k+1} + b_k}_{\le 2b_{k-1}}) \\
\le{}& \frac{1}{2\delta}(8C_0 + 4\overline{C} + 3c_1C_0 + c_1\overline{C})(3k+5)!^2 b_{k-1} \underbrace{\le}_{(4.47)} \frac{1}{2\delta}(8C_0 + 4\overline{C} + 3c_1C_0 + c_1\overline{C})\overline{\gamma} \underbrace{<}_{(4.35)} \frac{\tau}{c_L}.
\end{aligned}
$$

Therefore, by Lemma 4.5, $(\nu_{k+1}, p_{k+1}, q_{k+1})$ solves (OS) with the perturbation terms (4.84). As a consequence, $(\nu_{k+1}, p_{k+1}, q_{k+1})$ satisfies $(\mathbb{P}_k)$. Assume that $(\tilde{\nu}_{k+1}, \tilde{p}_{k+1}, \tilde{q}_{k+1})$ is another solution to $(\mathbb{P}_k)$. Then, $(\tilde{\nu}_{k+1}, \tilde{p}_{k+1}, \tilde{q}_{k+1})$ is also a fixed point of S_k, and consequently $\tilde{\nu}_{k+1}$ is a fixed point of $(I_\nu \circ S_k \circ T_k)$. By the uniqueness of the fixed point of $(I_\nu \circ S_k \circ T_k)$ (see Proposition 4.14), it follows that $\tilde{\nu}_{k+1} = \nu_{k+1}$. Furthermore, by the uniqueness of the solutions to the PDEs in $(\mathbb{P}_k)$, we obtain $\tilde{p}_{k+1} = p_{k+1}$ and $\tilde{q}_{k+1} = q_{k+1}$. Therefore, $(\nu_{k+1}, p_{k+1}, q_{k+1})$ is the unique solution to $(\mathbb{P}_k)$. □

Differently from the parabolic case, we cannot prove the quadratic (Q-)convergence of Algorithm 1. As a remedy, the proposed two-step estimation process (4.77) eventually enables us to prove R-superlinear convergence, i.e., the error is dominated by some scalar-valued sequence converging superlinearly to zero [65, page 620].

Theorem 4.17. *Let Assumption 4.9 be satisfied. Then, for every $k \in \mathbb{N}$, the iteration step $(\mathbb{P}_k)$ of Algorithm 1 admits a unique solution $(\nu_{k+1}, p_{k+1}, q_{k+1}) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega))$ satisfying*

$$\|\nu_{k+1} - \overline{\nu}\|_{L^2(\Omega)} \le \delta(3k+5)!^2(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)})^2.$$

Furthermore, Algorithm 1 converges R-superlinearly towards the solution $\overline{\nu}$ to (P) *with*

$$\|\nu_k - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta}\overline{\gamma}^{\sqrt{2}^k} \quad \forall k \in \mathbb{N}. \tag{4.85}$$

Proof. For every $k \in \mathbb{N}_0$, let $(\nu_{k+1}, p_{k+1}, q_{k+1}) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega))$ denote a solution to $(\mathbb{P}_k)$ according to Theorem 4.8. We combine Proposition 4.14 and Proposition 4.16 to prove by induction that

$$
\begin{cases}
(\nu_{k+1}, p_{k+1}, q_{k+1}) \in \mathcal{V}_{ad} \times C^\infty(I, L^\infty(\Omega)) \times C^\infty(I, L^\infty(\Omega)) \text{ is the unique solution to } (\mathbb{P}_k) \\
\|\nu_{k+1} - \overline{\nu}\|_{L^2(\Omega)} \le \delta(3k+5)!^2(\|\overline{\nu} - \nu_k\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_{k-1}\|_{L^2(\Omega)})^2 \\
\|\nu_k - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta}b_k \le \frac{1}{4\delta}\overline{\gamma}^{\sqrt{2}^k}, \quad \|\nu_{k+1} - \overline{\nu}\|_{L^2(\Omega)} \le \frac{1}{4\delta}b_{k+1} \le \frac{1}{4\delta}\overline{\gamma}^{\sqrt{2}^{k+1}} \\
p_k, q_k \in C^\infty(I, L^\infty(\Omega)) \text{ with } \partial_t^l p_k(0) = \partial_t^l q_k(T) = 0 \quad \forall l \in \mathbb{N}_0 \\
\|\partial_t^l p_k\|_{L^2(I,L^\infty(\Omega))} \le C_0(l+3k)!, \quad \|\partial_t^l q_k\|_{L^2(I,L^\infty(\Omega))} \le C_0(l+3k)! \quad \forall l \in \mathbb{N}_0
\end{cases}
\tag{4.86}
$$

for all $k \in \mathbb{N}$ and b_k, b_{k+1} as in Lemma 4.12. Due to Proposition 4.14 and Proposition 4.16, the solution (ν_1, p_1, q_1) to $(\mathbb{P}_k)$ for $k = 0$ is unique and satisfies

$$\|\overline{\nu} - \nu_1\|_{L^2(\Omega)} \le \frac{1}{4\delta}b_1 \underbrace{\le}_{(4.46)} \frac{1}{4\delta}\overline{\gamma}^{\sqrt{2}}. \tag{4.87}$$

Thus, along with Assumption 4.9, ν_1 and (ν_0, p_0, q_0) satisfy (A_k) for $k = 1$ such that Proposition 4.14 and Proposition 4.16 imply that the solution (ν_2, p_2, q_2) to $(\mathbb{P}_k)$ for $k = 1$ is unique and satisfies

$$\|\overline{\nu} - \nu_2\|_{L^2(\Omega)} \leq \delta 8!^2 (\|\overline{\nu} - \nu_1\|_{L^2(\Omega)} + \|\overline{\nu} - \nu_0\|_{L^2(\Omega)})^2, \qquad \|\overline{\nu} - \nu_2\|_{L^2(\Omega)} \leq \frac{1}{4\delta} b_2 \underbrace{\leq}_{(4.46)} \frac{1}{4\delta} \overline{\gamma}^{\sqrt{2}^2}.$$

Moreover, Lemma 4.13 implies $p_1, q_1 \in C^\infty(I, L^\infty(\Omega))$ with $\partial_t^l p_1(0) = 0$, $\partial_t^l q_1(T) = 0$, $\|\partial_t^l p_1\|_{L^2(I,L^\infty(\Omega))} \leq C_0(l+3)!$, and $\|\partial_t^l q_1\|_{L^2(I,L^\infty(\Omega))} \leq C_0(l+3)!$ for all $l \in \mathbb{N}_0$. In conclusion, (4.86) is fulfilled for $k = 1$. Now, let $k \geq 2$ be fixed, and assume that (4.86) is satisfied for $k - 1$. Then, ν_k and $(\nu_{k-1}, p_{k-1}, q_{k-1})$ satisfy (A_k) such that Proposition 4.14 and Proposition 4.16 imply that the solution $(\nu_{k+1}, p_{k+1}, q_{k+1})$ to $(\mathbb{P}_k)$ is unique and it holds that

$$\begin{aligned} &\|\overline{\nu} - \nu_{k+1}\|_{L^2(\Omega)} \leq \delta(3k+5)! (\|\nu_k - \overline{\nu}\|_{L^2(\Omega)} + \|\nu_{k-1} - \overline{\nu}\|_{L^2(\Omega)})^2, \\ &\|\overline{\nu} - \nu_{k+1}\|_{L^2(\Omega)} \leq \frac{1}{4\delta} b_{k+1} \underbrace{\leq}_{(4.46)} \frac{1}{4\delta} \overline{\gamma}^{\sqrt{2}^{k+1}}. \end{aligned}$$

Again, Lemma 4.13 implies that $p_k, q_k \in C^\infty(I, L^\infty(\Omega))$ with $\partial_t^l p_k(0) = \partial_t^l q_k(T) = 0$, $\|\partial_t^l p_k\|_{L^2(I,L^\infty(\Omega))} \leq C_0(l+3k)!$, and $\|\partial_t^l q_k\|_{L^2(I,L^\infty(\Omega))} \leq C_0(l+3k)!$ for all $l \in \mathbb{N}_0$. This completes the induction proof. □

NUMERICAL ANALYSIS OF A FULLY DISCRETE APPROXIMATION 5

This chapter is devoted to the numerical analysis of a fully discrete approximation for (1.2). For numerous optimal control problems governed by elliptic and parabolic PDEs, the numerical analysis, including the convergence of discrete schemes, is explored in the literature. We mention the publications by Casas et al. [8,14–17], Vexler et al. [10,55,60], Yousept [81,82], Gong et al. [38], and von Daniels et al. [76]. As we are primarily interested in the hyperbolic case, we refer to the recent contribution by Peralta and Kunisch [67], where a source control problem governed by the linear wave equation is approximated by proposing a Petrov–Galerkin scheme. Notably, in the realm of Maxwell's equations, a leapfrog (Yee) time-stepping method [79] has been successfully applied. We refer to Li [56], Li et al. [57], Cohen and Monk [25], Monk [63], Winckler and Yousept [78], and Hensel and Yousept [43]. To the best of the author's knowledge, we are the first to investigate a leapfrog time-stepping method for (1.2).

Based on a Finite Element method in space, a leapfrog (Yee) time-stepping, and the auxiliary first-order system (3.9), we derive a fully discrete approximation for (1.2) (see (5.10)). We show well-definedness of (5.10) and necessary first-order optimality condition (see Theorem 5.4) following the argumentation from the continuous case (see Theorem 3.12). Furthermore, based on the techniques in [43, 78], we prove the stability for the solutions of the discrete state and adjoint equations concerning the approximation parameter h (see Theorem 5.6). In the final section of the chapter, that is Section 5.4, we define suitable interpolations of the discrete solutions and prove two different convergence results: First, interpolations of solutions to the discrete first-order optimality systems converge up to a subsequence towards a solution to the first-order optimality system of the original problem (see Theorem 5.10). Second, for every local minimizer to (1.2) satisfying a reasonable growth condition, we prove the existence of a sequence of local minimizers to the fully discrete approximation for (1.2) converging to the local minimizer to (1.2).

The content of this chapter is available in the author's preprint [7]. Consequently, direct quotations from this work will not be explicitly highlighted.

5.1 Fully Discrete Scheme

The essential assumptions for this chapter read as follows:

Assumption 5.1. *Let Assumption 3.1 hold and let $\Omega \subset \mathbb{R}^N$ be a polyhedral Lipschitz domain. Furthermore, let $p_1 \in L^\infty(\Omega)$, $p_i^{ob} \in W^{1,1}(I, L^2(\Omega))$, and let $a_i \in C^1(I, L^\infty(\Omega))$ assumed to be nonnegative for all $i = 1, \dots, m$. Moreover, let $\nu_-, \nu_+ \in \mathbb{R}$ satisfy $0 < \nu_- \leq \nu_+$.*

Let us now establish the fully discrete approximation for the auxiliary first-order system (3.9). For this purpose, let $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of triangulations of Ω such that every edge E of every element $T \in \mathcal{T}_h$ satisfying $E \cap \partial\Omega \neq \varnothing$ belongs either to Γ_D or $\overline{\Gamma_N}$, i.e., either $E \subset \Gamma_D$ or $E \subset \overline{\Gamma_N}$. Here, $h > 0$ denotes the largest diameter of all $K \in \mathcal{T}_h$. Furthermore, with P_1^h, we denote the set of all continuous piecewise linear functions associated with $\mathcal{T}_h$, i.e.,

$$\mathrm{P}_1^h \coloneqq \{p \in C(\overline{\Omega}) \,|\, p|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\},$$

and with DG_0^h (resp. $\mathbf{DG}_0^h$), we denote the set of all scalar-valued (resp. vector-valued) piecewise constant functions associated with $\mathcal{T}_h$. Incorporating the partial Dirichlet boundary condition, let

$$\mathrm{P}_{1,D}^h \coloneqq \mathrm{P}_1^h \cap H_D^1(\Omega) = \{\phi_h \in \mathrm{P}_1^h : \phi_h = 0 \text{ on } \Gamma_D\}.$$

In the following, let Assumption 5.1 hold. Towards discretizing the governing PDE in time, given some $N \in \mathbb{N}$, we choose the equidistant discretization

$$0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_N = T,$$

where $t_l - t_{l-1} = \tau = \frac{1}{N}$ for all $l = 1, \dots, N$. Furthermore, let us employ the middle points of the constructed subintervals in the time discretization, i.e., $t_{l+\frac{1}{2}} \coloneqq t_l + \frac{\tau}{2}$ for all $l = 0, \dots, N-1$. We aim to define an approximation for (3.9), motivated by the leapfrog (Yee) time-stepping [79], where we evaluate the first equation in (3.9) at the discretization nodes while the second one is evaluated at the middle points. This approach leads to approximations $\{p_h^l\}_{l=0}^N \subset \mathrm{P}_{1,D}^h$ and $\{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1} \subset \mathbf{DG}_0^h$ where $p_h^l \approx p(t_l)$ and $\boldsymbol{u}_h^{l+\frac{1}{2}} \approx \boldsymbol{u}(t_{l+\frac{1}{2}})$ for the solution $(p, \boldsymbol{u})$ to (3.9). To arrive at a concise formulation, we introduce the notation

$$\delta p_h^{l+\frac{1}{2}} \coloneqq \frac{p_h^{l+1} - p_h^l}{\tau}, \quad p_h^{l+\frac{1}{2}} \coloneqq \frac{p_h^{l+1} + p_h^l}{2} \quad \forall l = 0, \dots, N-1 \tag{5.1}$$

and

$$\delta \boldsymbol{u}_h^l \coloneqq \frac{\boldsymbol{u}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l-\frac{1}{2}}}{\tau} \quad \forall l = 1, \dots, N-1. \tag{5.2}$$

Furthermore, let $F^{l+\frac{1}{2}} \coloneqq F(t_{l+\frac{1}{2}})$ for all $l = 0, \dots, N-1$ and $\nu_h \in \mathcal{V}_{ad}$ be given. We denote the P_1^h-interpolation operator with

$$\mathcal{I}_h : C(\overline{\Omega}) \to \mathrm{P}_1^h, \quad v \mapsto \sum_{j=1}^{M_h} v(x_j)\phi_j, \tag{5.3}$$

where $\{\phi_j\}_{j=1}^{M_h} \subset \mathrm{P}_1^h$ denotes the nodal basis of P_1^h and $\{x_j\}_{j=1}^{M_h} \subset \overline{\Omega}$ denote the corresponding nodal points. The operator $\mathcal{I}_h$ satisfies

$$\mathcal{I}_h\phi \to \phi \quad \text{in } L^\infty(\Omega) \qquad \text{as } h \to 0 \quad \forall \phi \in W^{1,p}(\Omega), p > N \tag{5.4}$$

$$\mathcal{I}_h\phi \to \phi \quad \text{in } H^1(\Omega) \qquad\qquad \text{as } h \to 0 \quad \forall \phi \in H^2(\Omega). \tag{5.5}$$

(cf. [12, Section 4.4]) and $\mathcal{I}_h(C_D^\infty(\Omega)) \subset \mathrm{P}_{1,D}^h$. Furthermore, Ψ_h denotes the $\mathrm{P}_{1,D}^h$ projection operator, i.e., $\Psi_h\colon H_D^1(\Omega) \to \mathrm{P}_{1,D}^h, p \mapsto y_h$ where y_h solves

$$\int_\Omega \nabla y_h \cdot \nabla \phi_h + y_h\phi_h \,\mathrm{d}x = \int_\Omega \nabla p \cdot \nabla \phi_h + p\phi_h \,\mathrm{d}x \quad \forall \phi_h \in \mathrm{P}_{1,D}^h.$$

For all $p \in H_D^1(\Omega)$, due to the Céa Lemma, it holds that

$$\|\Psi_h p - p\|_{H^1(\Omega)} \le \inf_{\phi_h \in \mathrm{P}_{1,D}^h} \|\phi_h - p\|_{H^1(\Omega)}. \tag{5.6}$$

Furthermore, since $H_D^1(\Omega) = \overline{C_D^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$, for every $p \in H_D^1(\Omega)$ and for every $\epsilon > 0$, there exists a $p_\epsilon \in C_D^\infty(\Omega)$ such that $\|p - p_\epsilon\|_{H^1(\Omega)} \le \frac{\epsilon}{2}$, and due to (5.5), there exists $\overline{h} > 0$ such that $\|\mathcal{I}_h p_\epsilon - p_\epsilon\|_{H^1(\Omega)} \le \frac{\epsilon}{2}$ for all $h \in (0, \overline{h}]$. Then, along with (5.6) and since $\mathcal{I}_h p_\epsilon \in \mathrm{P}_{1,D}^h$, it follows that

$$\|p - \Psi_h p\|_{H^1(\Omega)} \le \|\mathcal{I}_h p_\epsilon - p\|_{H^1(\Omega)} \le \|\mathcal{I}_h p_\epsilon - p_\epsilon\|_{H^1(\Omega)} + \|p_\epsilon - p\|_{H^1(\Omega)} \le \epsilon.$$

This implies that

$$\Psi_h p \to p \quad \text{in } H^1(\Omega) \quad \text{as } h \to 0 \quad \forall p \in H_D^1(\Omega). \tag{5.7}$$

Evaluating the first line in (3.9) at $t_{l+\frac{1}{2}}$ and the second line of (3.9) at t_l, and incorporating a Finite Element discretization in space, we obtain the following fully discrete scheme:

$$\begin{cases} \int_\Omega (\nu_h \delta p_h^{l+\frac{1}{2}} + \eta p_h^{l+\frac{1}{2}})\phi_h - \boldsymbol{u}_h^{l+\frac{1}{2}} \cdot \nabla \phi_h \,\mathrm{d}x = \int_\Omega F^{l+\frac{1}{2}} \phi_h \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \phi_h \in \mathrm{P}_{1,D}^h, l = 0, \dots, N-1 \\ \delta \boldsymbol{u}_h^l + \nabla p_h^l = 0 \qquad\qquad\qquad \forall l = 1, \dots, N-1 \\ p_h^0 \coloneqq \Psi_h p_0, \quad \boldsymbol{u}_h^{\frac{1}{2}} \coloneqq \Phi_h(\nu_h). \end{cases} \tag{5.8}$$

Here, $\Phi_h\colon L^\infty(\Omega) \to \mathbf{DG}_0^h$ maps every $\nu \in L^\infty(\Omega)$ to ∇y_h where $y_h \in \mathrm{P}_{1,D}^h$ denotes the unique solution to

$$\int_\Omega \nabla y_h \cdot \nabla \phi_h \,\mathrm{d}x = \int_\Omega (\eta p_0 + \nu p_1)\phi_h \,\mathrm{d}x \quad \forall \phi_h \in \mathrm{P}_{1,D}^h. \tag{5.9}$$

The system (5.8) can be solved by alternately solving the variational problem and computing the second line as follows: Suppose that $p_h^l \in \mathrm{P}_{1,D}^h$ and $\boldsymbol{u}_h^{l+\frac{1}{2}} \in \mathbf{DG}_0^h$ are already given for some $l = 0, \dots, N-2$. Then, applying (5.1), the finite-dimensional variational problem in (5.8) admits a unique solution $p_h^{l+1} \in \mathrm{P}_{1,D}^h$ due to the Lax-Milgram lemma. Furthermore, due to (5.2) and the second line in (5.8), it follows that $\boldsymbol{u}_h^{l+\frac{3}{2}} = \boldsymbol{u}_h^{l+\frac{1}{2}} - \tau \nabla p_h^{l+1}$. Consequently, the iteration scheme (5.8) is well-defined.

5.2 Discrete Optimal Control Problem

We consider the following discrete admissible set

$$\mathcal{V}_{ad}^h \coloneqq \{\nu \in \mathrm{DG}_0^h : \nu_- \le \nu(x) \le \nu_+ \text{ for every } x \in \Omega\} \subset \mathcal{V}_{ad}.$$

Making use of the midpoint rule for the numerical integration in time and the leapfrog scheme (5.8), our fully discrete optimal control problem reads as

$$\begin{cases} \min \mathcal{J}_h(\nu_h, \{p_h^l\}_{l=0}^N) \coloneqq \dfrac{\tau}{2} \displaystyle\sum_{i=1}^m \sum_{l=0}^{N-1} \int_\Omega a_i(t_{l+\frac{1}{2}})(p_h^{l+\frac{1}{2}} - p_i^{ob}(t_{l+\frac{1}{2}}))^2 \,\mathrm{d}x + \dfrac{\lambda}{2}\|\nu_h\|_{L^2(\Omega)}^2 \\ \text{s.t. } \nu_h \in \mathcal{V}_{ad}^h \text{ and } (\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N \text{ solves (5.8).} \end{cases} \tag{5.10}$$

Since for every $\nu_h \in \mathcal{V}_{ad}^h$, the scheme (5.8) admits a unique solution $(\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$, the associated solution mapping

$$S_h : \mathcal{V}_{ad}^h \to L^2(\Omega)^{N+1} \times \boldsymbol{L}^2(\Omega)^N, \nu_h \mapsto (\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$$

is well-defined. Denoting the first component of S_h with $S_{h,p} : \nu_h \mapsto \{p_h^l\}_{l=0}^N$, the reduced problem reads

$$\min_{\nu \in \mathcal{V}_{ad}^h} J_h(\nu_h) \coloneqq \mathcal{J}_h(\nu_h, S_{h,p}(\nu_h)). \tag{P_h}$$

For proving the existence of a minimizer to (P_h) in Theorem 5.3, we make use of the following inverse estimate:

Lemma 5.2 (Inverse estimate, [22, Theorem 3.2.6])**.** *There exists a constant $c_{inv} > 0$, independent of h, such that*

$$\|\nabla p_h\|_{\boldsymbol{L}^2(\Omega)} \le \frac{c_{inv}}{h}\|p_h\|_{L^2(\Omega)} \quad \forall p_h \in \mathrm{P}_{1,D}^h.$$

Theorem 5.3. *Let Assumption 5.1 hold. Then, for every $h > 0$, the problem* (P_h) *admits at least one global minimizer $\nu_h \in \mathcal{V}_{ad}^h$.*

Proof. Let $h > 0$ be given. First, we show that the above solution operator $S_{h,p} : L^2(\Omega) \supset \mathcal{V}_{ad}^h \to L^2(\Omega)^{N+1}, \nu_h \mapsto \{p_h^l\}_{l=0}^N$ is continuous. For this purpose, let $\nu_h, \tilde{\nu}_h \in \mathcal{V}_{ad}^h$ and $(\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}), (\{\tilde{p}_h^l\}_{l=0}^N, \{\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$ be the solutions to (5.8) associated with ν_h and $\tilde{\nu}_h$, respectively. Then, we have that

$$\begin{cases} \displaystyle\int_\Omega (\tilde{\nu}_h(\delta\tilde{p}_h^{l+\frac{1}{2}} - \delta p_h^{l+\frac{1}{2}}) + \eta(\tilde{p}_h^{l+\frac{1}{2}} - p_h^{l+\frac{1}{2}}))\phi_h - (\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l+\frac{1}{2}}) \cdot \nabla\phi_h \,\mathrm{d}x = \int_\Omega (\tilde{\nu}_h - \nu_h)\delta p_h^{l+\frac{1}{2}}\phi_h \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \phi_h \in \mathrm{P}_{1,D}^h, l = 0, \ldots, N-1 \\ \delta\tilde{\boldsymbol{u}}_h^l - \delta\boldsymbol{u}_h^l + \nabla(\tilde{p}_h^l - p_h^l) = 0 \qquad \forall l = 1, \ldots, N-1 \\ \tilde{p}_h^0 - p_h^0 = 0, \quad \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} - \boldsymbol{u}_h^{\frac{1}{2}} = \Phi_h(\tilde{\nu}_h) - \Phi_h(\nu_h). \end{cases} \tag{5.11}$$

We show by induction that for every $l = 1, \ldots, N$, there exists a constant $c > 0$ that is independent of $\tilde{\nu}_h$ such that

$$\begin{cases} \|\tilde{p}_h^l - p_h^l\|_{L^2(\Omega)} \leq c\|\tilde{\nu}_h - \nu_h\|_{L^2(\Omega)} \\ \|\tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}} - \boldsymbol{u}_h^{l-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq c\|\tilde{\nu}_h - \nu_h\|_{L^2(\Omega)}. \end{cases} \tag{5.12}$$

From (5.11), we obtain for all $l = 0, \ldots, N-1$ that

$$\begin{aligned} &\int_\Omega \left(\frac{\tilde{\nu}_h}{\tau} + \frac{\eta}{2}\right)(\tilde{p}_h^{l+1} - p_h^{l+1})\phi_h \, \mathrm{d}x \\ &= \int_\Omega \left(\frac{\tilde{\nu}_h}{\tau} - \frac{\eta}{2}\right)(\tilde{p}_h^l - p_h^l)\phi_h + (\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l+\frac{1}{2}}) \cdot \nabla\phi_h + (\tilde{\nu}_h - \nu_h)\delta p_h^{l+\frac{1}{2}}\phi_h \, \mathrm{d}x \quad \forall \phi_h \in \mathrm{P}_{1,D}^h . \end{aligned} \tag{5.13}$$

The equation (5.13) with $l = 0$ implies that

$$\begin{aligned} \int_\Omega \left(\frac{\tilde{\nu}_h}{\tau} + \frac{\eta}{2}\right)(\tilde{p}_h^1 - p_h^1)\phi_h \, \mathrm{d}x \underbrace{=}_{(5.11)} &\int_\Omega (\Phi_h(\tilde{\nu}_h) - \Phi_h(\nu_h)) \cdot \nabla\phi_h + (\tilde{\nu}_h - \nu_h)\delta p_h^{\frac{1}{2}}\phi_h \, \mathrm{d}x \\ \underbrace{=}_{(5.9)} &\int_\Omega (\tilde{\nu}_h - \nu_h)(p_1 + \delta p_h^{\frac{1}{2}})\phi_h \, \mathrm{d}x \quad \forall \phi_h \in \mathrm{P}_{1,D}^h . \end{aligned} \tag{5.14}$$

Thus, testing (5.14) with $\phi_h = (\tilde{p}_h^1 - p_h^1)$ and using Hölder's inequality, we obtain that

$$\|\tilde{p}_h^1 - p_h^1\|_{L^2(\Omega)} \leq \nu_{\min^{-1}}\tau(\|p_1\|_{L^\infty(\Omega)} + \|\delta p_h^{\frac{1}{2}}\|_{C(\overline{\Omega})})\|\tilde{\nu}_h - \nu_h\|_{L^2(\Omega)}.$$

Furthermore, we have that

$$\|\tilde{\boldsymbol{u}}_h^{\frac{1}{2}} - \boldsymbol{u}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(5.11)} \|\Phi_h(\tilde{\nu}_h) - \Phi(\nu_h)\|_{\boldsymbol{L}^2(\Omega)} \leq c_P\|p_1\|_{L^\infty(\Omega)}\|\tilde{\nu}_h - \nu_h\|_{L^2(\Omega)},$$

where the last inequality is obtained by inserting $\phi_h = y_h$ in (5.9) and $c_P > 0$ denotes a Poincaré constant. Therefore, (5.12) is valid for $l = 1$. Now, assume that (5.12) holds for a fixed $l \in \{1, \ldots, N-1\}$. Since

$$\|\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \underbrace{\leq}_{(5.11)} \|\tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}} - \boldsymbol{u}_h^{l-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \tau\|\nabla(\tilde{p}_h^l - p_h^l)\|_{\boldsymbol{L}^2(\Omega)}, \tag{5.15}$$

applying the induction assumption and the inverse estimate from Lemma 5.2 to (5.15), we obtain the second inequality in (5.12) for $l+1$. Further, testing (5.13) with $\phi_h = \tilde{p}_h^{l+1} - p_h^{l+1}$ and again using the inverse estimate in Lemma 5.2 yields that

$$\begin{aligned} \|\tilde{p}_h^{l+1} - p_h^{l+1}\|_{L^2(\Omega)} \leq \nu_-^{-1}\tau\Bigg(&\left(\frac{\nu_+}{\tau} + \frac{1}{2}\|\eta\|_{L^\infty(\Omega)}\right)\|\tilde{p}_h^l - p_h^l\|_{L^2(\Omega)} + \frac{c_{inv}}{h}\|\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\ &+ \|\delta p_h^{l+\frac{1}{2}}\|_{C(\overline{\Omega})}\|\tilde{\nu}_h - \nu_h\|_{L^2(\Omega)}\Bigg). \end{aligned}$$

Applying the induction assumption and (5.15), we obtain the first inequality in (5.12) for $l+1$. This completes the induction proof. Consequently, by (5.12) and since $\tilde{p}_h^0 = p_h^0$, the mapping $S_{h,p}\colon L^2(\Omega) \supset \mathcal{V}_{ad}^h \to L^2(\Omega)^{N+1}, \nu \mapsto \{p_h^l\}_{l=0}^N$ is continuous and thus $J_h\colon L^2(\Omega) \supset \mathcal{V}_{ad}^h \to \mathbb{R}$ is continuous. On the other hand, $\mathcal{V}_{ad}^h \subset \mathrm{DG}_0^h$ is compact since it is a closed and bounded subset of the finite-dimensional space DG_0^h. Thus, the minimization problem (P_h) admits at least one global minimizer due to the Weierstrass theorem. □

The following theorem is the discrete version of Theorem 3.12 providing a first-order necessary optimality condition for (P_h).

Theorem 5.4. *Let Assumption 5.1 hold and $h > 0$ be given. Furthermore, let $\overline{\nu}_h$ be a minimizer to (P_h) and $(\{\overline{p}_h^l\}_{l=0}^N, \{\overline{\boldsymbol{u}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$ be the associated unique solution to (5.8). Then, there exists a unique solution $(\{\overline{q}_h^l\}_{l=0}^N, \{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$ to*

$$\begin{cases} \int_\Omega (\overline{\nu}_h \delta \overline{q}_h^{l+\frac{1}{2}} - \eta \overline{q}_h^{l+\frac{1}{2}})\phi_h - \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla \phi_h \,\mathrm{d}x = \sum_{i=1}^m \int_\Omega a_i(t_{l+\frac{1}{2}})(\overline{p}_h^{l+\frac{1}{2}} - p_i^{ob}(t_{l+\frac{1}{2}}))\phi_h \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \forall \phi_h \in \mathrm{P}_{1,D}^h, l = 0, \ldots, N-1 \\ \delta \overline{\boldsymbol{v}}_h^l + \nabla \overline{q}_h^l = 0 \qquad\qquad \forall l = 1, \ldots, N-1 \\ \overline{q}_h^N = 0, \overline{\boldsymbol{v}}_h^{N-\frac{1}{2}} = \mathbf{0}, \end{cases} \tag{5.16}$$

where

$$\delta q_h^{l+\frac{1}{2}} := \frac{q_h^{l+1} - q_h^l}{\tau}, \quad q_h^{l+\frac{1}{2}} := \frac{q_h^{l+1} + q_h^l}{2} \quad \forall l = 0, \ldots, N-1 \tag{5.17}$$

and

$$\delta \boldsymbol{v}_h^l := \frac{\boldsymbol{v}_h^{l+\frac{1}{2}} - \boldsymbol{v}_h^{l-\frac{1}{2}}}{\tau} \quad \forall l = 1, \ldots, N-1. \tag{5.18}$$

Furthermore, it holds that

$$\left(\tau \sum_{l=0}^{N-1} (\delta \overline{p}_h^{l+\frac{1}{2}} - p_1)\overline{q}_h^{l+\frac{1}{2}} + \lambda \overline{\nu}_h, \nu_h - \overline{\nu}_h\right)_{L^2(\Omega)} \geq 0 \quad \forall \nu_h \in \mathcal{V}_{ad}^h. \tag{5.19}$$

Remark 5.5. Note that the first-order necessary optimality condition (5.19) for the discrete optimization problem (P_h) is nothing but the discretization of the first-order necessary optimality condition (3.62) for the continuous problem (P). In particular, our finite element method for (P), based on the leapfrog time-stepping, is consistent in the sense that both ansatzes 'first-discretize-then-optimize' and 'first-optimize-then-discretize' are equivalent.

Proof. As for (5.8), the well-posedness of (5.16) follows with the Lax-Milgram lemma. By standard arguments, it is sufficient to show that (5.19) is equivalent to the condition $DJ_h(\overline{\nu}_h, \nu_h - \overline{\nu}_h) \geq 0$ for all $\nu_h \in \mathcal{V}_{ad}^h$. First, we show for every $\nu_h \in \mathcal{V}_{ad}^h$ that $S_{h,p}\colon L^\infty(\Omega) \supset \mathcal{V}_{ad}^h \to L^2(\Omega)^{N+1}$ is directional differentiable in $\overline{\nu}_h$ into the direction $(\nu_h - \overline{\nu}_h)$ and it holds that $DS_{h,p}(\overline{\nu}_h, \nu_h - \overline{\nu}_h) = \{\tilde{p}_h^l\}_{l=0}^N$ where $(\{\tilde{p}_h^l\}_{l=0}^N, \{\tilde{\boldsymbol{u}}_h^l\}_{l=0}^N)$ is the unique solution to

$$\begin{cases} \int_\Omega (\overline{\nu}_h \delta \tilde{p}_h^{l+\frac{1}{2}} + \eta \tilde{p}_h^{l+\frac{1}{2}})\phi_h - \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} \cdot \nabla \phi_h \,\mathrm{d}x = -\int_\Omega (\nu_h - \overline{\nu}_h)\delta \overline{p}_h^{l+\frac{1}{2}} \phi_h \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \forall \phi_h \in \mathrm{P}_{1,D}^h, l = 0, \ldots, N-1 \\ \delta \tilde{\boldsymbol{u}}_h^l + \nabla \tilde{p}_h^l = 0 \qquad\qquad \forall l = 1, \ldots, N-1 \\ \tilde{p}_h^0 := 0, \quad \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} := \Phi_h'(\overline{\nu}_h)(\nu_h - \overline{\nu}_h). \end{cases} \tag{5.20}$$

Here, $\{\delta\tilde{p}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}$, $\{\tilde{p}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}$, and $\{\delta\tilde{\boldsymbol{u}}_h^l\}_{l=1}^{N-1}$ are defined analog to (5.1) and (5.2). Let $t > 0$ and $\nu_h \in \mathcal{V}_{ad}^h$ be given. Then, subtracting the corresponding system (5.8) for $S_{h,p}(\overline{\nu}_h + t(\nu_h - \overline{\nu}_h)) = \{p_h^l\}_{l=0}^N$ and $tDS_{h,p}(\overline{\nu}_h, \nu_h - \overline{\nu}_h) = t\{\tilde{p}_h^l\}_{l=0}^N$ (see (5.20)) from the system (5.8) for $S_{h,p}(\overline{\nu}_h) = \{\overline{p}_h^l\}_{l=0}^N$, we obtain that

$$\begin{cases} \int_\Omega (\overline{\nu}_h(\delta\overline{p}_h^{l+\frac{1}{2}} - \delta p_h^{l+\frac{1}{2}} - t\delta\tilde{p}_h^{l+\frac{1}{2}}) + \eta(\overline{p}_h^{l+\frac{1}{2}} - p_h^{l+\frac{1}{2}} - t\tilde{p}_h^{l+\frac{1}{2}}))\phi_h - (\overline{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l+\frac{1}{2}} - t\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}}) \cdot \nabla\phi_h \,\mathrm{d}x \\ = -\int_\Omega t(\nu_h - \overline{\nu}_h)(\delta p_h^{l+\frac{1}{2}} - \delta\overline{p}_h^{l+\frac{1}{2}})\phi_h \,\mathrm{d}x \qquad \forall \phi_h \in \mathrm{P}_{1,D}^h, l = 0, \dots, N-1 \\ \delta\overline{\boldsymbol{u}}_h^l - \delta\boldsymbol{u}_h^l - t\delta\tilde{\boldsymbol{u}}_h^l + \nabla(\overline{p}_h^l - p_h^l - t\tilde{p}_h^l) = 0 \qquad \forall l = 1, \dots, N-1 \\ \overline{p}_h^0 - p_h^0 - t\tilde{p}_h^0 = 0, \quad \overline{\boldsymbol{u}}_h^{\frac{1}{2}} - \boldsymbol{u}_h^{\frac{1}{2}} - t\tilde{\boldsymbol{u}}_h^{\frac{1}{2}} = \boldsymbol{0}. \end{cases} \tag{5.21}$$

Following the induction argument in the proof of Theorem 5.3, for every $l = 1, \dots, N$, we obtain the existence of a constant $c > 0$, independent of ν_h, such that

$$\begin{aligned} \|\overline{p}_h^l - p_h^l - t\tilde{p}_h^l\|_{L^2(\Omega)} &\le ct\|\nu_h - \overline{\nu}_h\|_{L^\infty(\Omega)}\|\delta\overline{p}_h^{l-\frac{1}{2}} - \delta p_h^{l-\frac{1}{2}}\|_{L^2(\Omega)} \\ &\le \frac{c}{\tau}t\|\nu_h - \overline{\nu}_h\|_{L^\infty(\Omega)}(\|\overline{p}_h^{l+1} - p_h^{l+1}\|_{L^2(\Omega)} + \|\overline{p}_h^l - p_h^l\|_{L^2(\Omega)}). \end{aligned}$$

Again, by the same induction argument, there exists a constant $\tilde{c} > 0$, independent of ν_h, such that

$$\|\overline{p}_h^l - p_h^l\|_{L^2(\Omega)} \le \tilde{c}t\|\nu_h - \overline{\nu}_h\|_{L^\infty(\Omega)} \quad \forall l = 0, \dots, N.$$

Consequently,

$$\begin{aligned} &\left\| \frac{S_{h,p}(\overline{\nu}_h) - S_{h,p}(\overline{\nu}_h + t(\nu_h - \overline{\nu}_h))}{t} - DS_{h,p}(\overline{\nu}_h, \nu_h - \overline{\nu}_h) \right\|_{L^2(\Omega)^{N+1}}^2 \\ &= \sum_{l=0}^N \frac{\|\overline{p}_h^l - p_h^l - t\tilde{p}_h^l\|_{L^2(\Omega)}^2}{t^2} \le (N+1)\frac{c^2}{\tau^2}\tilde{c}^2 t^2\|\nu_h - \overline{\nu}_h\|_{L^\infty(\Omega)}^4 \to 0 \quad \text{as } t \searrow 0, \end{aligned}$$

which proves the directional differentiability of $S_{h,p}$ at $\overline{\nu}_h$ into the direction $(\nu_h - \overline{\nu}_h)$. By standard argumentation, it follows that

$$DJ_h(\overline{\nu}_h, \nu_h - \overline{\nu}_h) = \tau \sum_{l=0}^{N-1} \sum_{i=1}^m \int_\Omega a_i(t_{l+\frac{1}{2}})(p_h^{l+\frac{1}{2}} - p_i^{ob}(t_{l+\frac{1}{2}}))\tilde{p}_h^{l+\frac{1}{2}} \,\mathrm{d}x + \lambda(\overline{\nu}_h, \nu_h - \overline{\nu}_h)_{L^2(\Omega)}.$$

Testing the first equation in (5.16) with $\phi_h = \tilde{p}_h^{l+\frac{1}{2}}$ for every $l = 0, \dots, N-1$ implies that

$$DJ_h(\overline{\nu}_h, \nu_h - \overline{\nu}_h) = \tau \sum_{l=0}^{N-1} \int_\Omega (\overline{\nu}_h\delta\overline{q}_h^{l+\frac{1}{2}} - \eta\overline{q}_h^{l+\frac{1}{2}})\tilde{p}_h^{l+\frac{1}{2}} - \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla\tilde{p}_h^{l+\frac{1}{2}} \,\mathrm{d}x + \lambda(\overline{\nu}_h, \nu_h - \overline{\nu}_h)_{L^2(\Omega)}. \tag{5.22}$$

In more detail, we elaborate on the right-hand side of (5.22). First, due to (5.1), it holds that

$$\sum_{l=0}^{N-1} \delta\overline{q}_h^{l+\frac{1}{2}}\tilde{p}_h^{l+\frac{1}{2}} = \sum_{l=0}^{N-1} \frac{\overline{q}_h^{l+1} - \overline{q}_h^l}{\tau}\frac{\tilde{p}_h^{l+1} + \tilde{p}_h^l}{2} = \sum_{l=0}^{N-1} \left(-\frac{\overline{q}_h^{l+1} + \overline{q}_h^l}{2}\frac{\tilde{p}_h^{l+1} - \tilde{p}_h^l}{\tau} + \frac{\overline{q}_h^{l+1}\tilde{p}_h^{l+1}}{\tau} - \frac{\overline{q}_h^l\tilde{p}_h^l}{\tau} \right) \tag{5.23}$$

$$= -\sum_{l=0}^{N-1} \overline{q}_h^{l+\frac{1}{2}} \delta\tilde{p}_h^{l+\frac{1}{2}} + \frac{1}{\tau} \underbrace{\overline{q}_h^N}_{=0} \tilde{p}_h^N - \frac{1}{\tau}\overline{q}_h^0 \underbrace{\tilde{p}_h^0}_{=0} = -\sum_{l=0}^{N-1} \overline{q}_h^{l+\frac{1}{2}} \delta\tilde{p}_h^{l+\frac{1}{2}}.$$

Second, it holds that

$$\sum_{l=0}^{N-1} \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla\tilde{p}_h^{l+\frac{1}{2}} \underbrace{=}_{(5.1)} \frac{1}{2}\sum_{l=0}^{N-1} \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla\tilde{p}_h^{l+1} + \frac{1}{2}\sum_{l=1}^{N-1} \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla\tilde{p}_h^{l} + \frac{1}{2}\overline{\boldsymbol{v}}_h^{\frac{1}{2}} \cdot \nabla \underbrace{\tilde{p}_h^0}_{=0} \tag{5.24}$$

$$= \frac{1}{2}\sum_{l=1}^{N} \overline{\boldsymbol{v}}_h^{l-\frac{1}{2}} \cdot \nabla\tilde{p}_h^{l} + \frac{1}{2}\sum_{l=1}^{N-1} \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla\tilde{p}_h^{l} = \sum_{l=1}^{N-1} \frac{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} + \overline{\boldsymbol{v}}_h^{l-\frac{1}{2}}}{2} \cdot \nabla\tilde{p}_h^{l} + \underbrace{\overline{\boldsymbol{v}}_h^{N-\frac{1}{2}}}_{=0} \cdot \nabla\tilde{p}_h^{N}$$

$$\underbrace{=}_{(5.20)} -\sum_{l=1}^{N-1} \frac{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} + \overline{\boldsymbol{v}}_h^{l-\frac{1}{2}}}{2} \cdot \frac{\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}}}{\tau}$$

$$= \sum_{l=1}^{N-1} \left(\frac{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} - \overline{\boldsymbol{v}}_h^{l-\frac{1}{2}}}{\tau} \cdot \frac{\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}}}{2} - \frac{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}}}{\tau} + \frac{\overline{\boldsymbol{v}}_h^{l-\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}}}{\tau} \right)$$

$$= \sum_{l=1}^{N-1} \delta\overline{\boldsymbol{v}}_h^{l} \cdot \frac{\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}}}{2} - \frac{1}{\tau} \underbrace{\overline{\boldsymbol{v}}_h^{N-\frac{1}{2}}}_{=0} \cdot \tilde{\boldsymbol{u}}_h^{N-\frac{1}{2}} + \frac{1}{\tau}\overline{\boldsymbol{v}}_h^{\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}}$$

$$\underbrace{=}_{(5.16)} -\sum_{l=1}^{N-1} \nabla\overline{q}_h^{l} \cdot \frac{\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}}}{2} + \frac{1}{\tau}\overline{\boldsymbol{v}}_h^{\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}}.$$

For the first term of the right-hand side in (5.24), note that

$$-\sum_{l=1}^{N-1} \nabla\overline{q}_h^{l} \cdot \frac{\tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \tilde{\boldsymbol{u}}_h^{l-\frac{1}{2}}}{2} = -\frac{1}{2}\sum_{l=1}^{N-1} \nabla\overline{q}_h^{l} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \frac{1}{2}\sum_{l=0}^{N-2} \nabla\overline{q}_h^{l+1} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} \tag{5.25}$$

$$= -\frac{1}{2}\sum_{l=0}^{N-1} \nabla\overline{q}_h^{l} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \frac{1}{2}\sum_{l=0}^{N-1} \nabla\overline{q}_h^{l+1} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \frac{1}{2}\nabla\overline{q}_h^{0} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} + \frac{1}{2} \underbrace{\nabla\overline{q}_h^{N}}_{=0} \cdot \tilde{\boldsymbol{u}}_h^{N-\frac{1}{2}}$$

$$\underbrace{=}_{(5.17)} -\sum_{l=0}^{N-1} \nabla\overline{q}_h^{l+\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \frac{1}{2}\nabla\overline{q}_h^{0} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}}.$$

Integrating the second term in the right-hand side of (5.24) over Ω, we obtain that

$$\frac{1}{\tau}\int_\Omega \overline{\boldsymbol{v}}_h^{\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} \,\mathrm{d}x = \int_\Omega -\sum_{l=1}^{N-1} \frac{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} - \overline{\boldsymbol{v}}_h^{l-\frac{1}{2}}}{\tau} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} + \frac{1}{\tau} \underbrace{\overline{\boldsymbol{v}}_h^{N-\frac{1}{2}}}_{=0} \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} \,\mathrm{d}x \tag{5.26}$$

$$\underbrace{=}_{(5.16),(5.20)} \int_\Omega \sum_{l=1}^{N-1} \nabla\overline{q}_h^{l} \cdot \Phi_h'(\overline{\nu}_h)(\nu_h - \overline{\nu}_h) \,\mathrm{d}x \underbrace{=}_{(5.9)} \int_\Omega \sum_{l=1}^{N-1} \overline{q}_h^{l}(\nu_h - \overline{\nu}_h)p_1 \,\mathrm{d}x$$

$$\underbrace{=}_{(5.17)} \int_\Omega \sum_{l=0}^{N-1} \overline{q}_h^{l+\frac{1}{2}}(\nu_h - \overline{\nu}_h)p_1 - \frac{1}{2}\overline{q}_h^{0}(\nu_h - \overline{\nu}_h)p_1 - \frac{1}{2} \underbrace{\overline{q}_h^{N}}_{=0}(\nu_h - \overline{\nu}_h)p_1 \,\mathrm{d}x$$

$$\underbrace{=}_{(5.9),(5.20)} \int_\Omega \sum_{l=0}^{N-1} \overline{q}_h^{l+\frac{1}{2}} (\nu_h - \overline{\nu}_h) p_1 - \frac{1}{2} \nabla \overline{q}_h^0 \cdot \tilde{\boldsymbol{u}}_h^{\frac{1}{2}} \, \mathrm{d}x.$$

Integrating (5.24) over Ω and applying (5.25) and (5.26) to (5.24), it follows that

$$\sum_{l=0}^{N-1} \int_\Omega \overline{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla \tilde{p}_h^{l+\frac{1}{2}} \, \mathrm{d}x = \sum_{l=0}^{N-1} \int_\Omega -\nabla \overline{q}_h^{l+\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} + \overline{q}_h^{l+\frac{1}{2}} (\nu_h - \overline{\nu}_h) p_1 \, \mathrm{d}x. \tag{5.27}$$

Applying (5.23) and (5.27) to (5.22), we conclude that

$$\begin{aligned} DJ_h(\overline{\nu}_h, \nu_h - \overline{\nu}_h) &= \tau \sum_{l=0}^{N-1} \int_\Omega -(\overline{\nu}_h \delta \tilde{p}_h^{l+\frac{1}{2}} + \eta \tilde{p}_h^{l+\frac{1}{2}}) \overline{q}_h^{l+\frac{1}{2}} + \nabla \overline{q}_h^{l+\frac{1}{2}} \cdot \tilde{\boldsymbol{u}}_h^{l+\frac{1}{2}} - \overline{q}_h^{l+\frac{1}{2}} (\nu_h - \overline{\nu}_h) p_1 \, \mathrm{d}x \\ &\quad + \lambda(\overline{\nu}_h, \nu_h - \overline{\nu}_h)_{L^2(\Omega)} \\ &\underbrace{=}_{(5.20)} \tau \sum_{l=0}^{N-1} \int_\Omega (\nu_h - \overline{\nu}_h)(\delta \overline{p}_h^{l+\frac{1}{2}} - p_1) \overline{q}_h^{l+\frac{1}{2}} \, \mathrm{d}x + \lambda(\overline{\nu}_h, \nu_h - \overline{\nu}_h)_{L^2(\Omega)}. \end{aligned}$$ □

5.3 Stability

In this subsection, under a suitable CFL-condition, we provide stability results for the solutions of the discrete state equation and discrete adjoint equation. The proof extends the argumentation in [43, 78] to the present case.

Theorem 5.6. *Let Assumption 5.1 hold. Furthermore, let $h > 0$ and $N \in \mathbb{N}$ satisfy the CFL-condition*

$$\frac{1}{Nh} = \frac{\tau}{h} \le c_{cfl} := \frac{\sqrt{\nu_-}}{\sqrt{2} c_{inv}}. \tag{5.28}$$

Then, there exists a constant $C > 0$, independent h and N, such that for every $\nu_h \in \mathcal{V}_{ad}^h$, the associated unique solution $(\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$ to the leapfrog scheme (5.8) *satisfies*

$$\max_{l \in \{0,\dots,N-1\}} \|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \max_{l \in \{1,\dots,N-1\}} \|\delta \boldsymbol{u}_h^l\|_{\boldsymbol{L}^2(\Omega)} + \max_{l \in \{1,\dots,N-1\}} \|\nabla p_h^l\|_{\boldsymbol{L}^2(\Omega)} \le C \tag{5.29}$$

$$\max_{l \in \{0,\dots,N\}} \|p_h^l\|_{L^2(\Omega)} + \max_{l \in \{0,\dots,N-1\}} \|\boldsymbol{u}_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} \le C. \tag{5.30}$$

Proof. To begin with, we investigate the initial approximations. Testing the first line in (5.8) for $l = 0$ with $\phi_h = p_h^{\frac{1}{2}}$ leads to

$$\begin{aligned} &\int_\Omega (\nu_h \delta p_h^{\frac{1}{2}} + \eta p_h^{\frac{1}{2}}) p_h^{\frac{1}{2}} - \boldsymbol{u}_h^{\frac{1}{2}} \cdot \nabla p_h^{\frac{1}{2}} \, \mathrm{d}x = \int_\Omega F^{\frac{1}{2}} p_h^{\frac{1}{2}} \, \mathrm{d}x \\ \underbrace{\Leftrightarrow}_{(5.1)} &\int_\Omega \frac{\nu_h}{2\tau} ((p_h^1)^2 - (p_h^0)^2) + \eta (p_h^{\frac{1}{2}})^2 - \boldsymbol{u}_h^{\frac{1}{2}} \cdot \nabla p_h^{\frac{1}{2}} \, \mathrm{d}x = \int_\Omega F^{\frac{1}{2}} p_h^{\frac{1}{2}} \, \mathrm{d}x. \end{aligned}$$

Therefore, along with the inverse estimate from Lemma 5.2, it follows that

$$\frac{\nu_-}{\tau} \|p_h^{\frac{1}{2}}\|_{L^2(\Omega)}^2 \le \int_\Omega \frac{\nu_h}{\tau} (p_h^{\frac{1}{2}})^2 \, \mathrm{d}x = \int_\Omega \frac{\nu_h}{\tau} \left(\frac{p_h^1 + p_h^0}{2} \right)^2 \mathrm{d}x \le \int_\Omega \frac{\nu_h}{2\tau} ((p_h^1)^2 + (p_h^0)^2) \, \mathrm{d}x$$

$$\leq \int_\Omega \frac{\nu_h}{\tau}(p_h^0)^2 + \boldsymbol{u}_h^{\frac{1}{2}} \cdot \nabla p_h^{\frac{1}{2}} \,\mathrm{d}x + F^{\frac{1}{2}} p_h^{\frac{1}{2}} \,\mathrm{d}x$$
$$\leq \frac{\nu_+}{\tau}\|p_h^0\|^2_{L^2(\Omega)} + \left(\frac{c_{inv}}{h}\|\boldsymbol{u}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|F^{\frac{1}{2}}\|_{L^2(\Omega)}\right)\|p_h^{\frac{1}{2}}\|_{L^2(\Omega)}.$$

Then, applying Young's inequality, we obtain that

$$\frac{\nu_-}{\tau}\|p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)} \leq \frac{\nu_+}{\tau}\|p_h^0\|^2_{L^2(\Omega)} + \frac{\tau}{2\nu_-}\left(\frac{c_{inv}}{h}\|\boldsymbol{u}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|F^{\frac{1}{2}}\|_{L^2(\Omega)}\right)^2 + \frac{\nu_-}{2\tau}\|p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)},$$

such that

$$\frac{\nu_-}{2\tau}\|p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)} \leq \frac{\nu_+}{\tau}\|p_h^0\|^2_{L^2(\Omega)} + \frac{\tau}{2\nu_-}\left(\frac{c_{inv}}{h}\|\boldsymbol{u}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|F^{\frac{1}{2}}\|_{L^2(\Omega)}\right)^2$$
$$\leq \frac{\nu_+}{\tau}\|p_h^0\|^2_{L^2(\Omega)} + \frac{\tau c_{inv}^2}{\nu_- h^2}\|\boldsymbol{u}_h^{\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\tau}{\nu_-}\|F^{\frac{1}{2}}\|^2_{L^2(\Omega)}.$$

Multiplying both sides with $\frac{2\tau}{\nu_-}$ and making use of the CFL-condition (5.28) yields that

$$\|p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)} \leq \frac{2\nu_+}{\nu_-}\|p_h^0\|^2_{L^2(\Omega)} + \frac{2\tau^2 c_{inv}^2}{\nu_-^2 h^2}\|\boldsymbol{u}_h^{\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{2\tau^2}{\nu_-^2}\|F^{\frac{1}{2}}\|^2_{L^2(\Omega)} \tag{5.31}$$
$$\underbrace{\leq}_{(5.8),(5.28)} \frac{2\nu_+}{\nu_-}\|\Psi_h p_0\|^2_{L^2(\Omega)} + \frac{1}{\nu_-}\|\Phi_h(\nu_h)\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{2T^2}{\nu_-^2}\|f\|^2_{L^1(I,L^2(\Omega))},$$

where the inequality $\|F^{\frac{1}{2}}\|_{L^2(\Omega)} \leq \|f\|_{L^1(I,L^2(\Omega))}$ follows from $F^{\frac{1}{2}} = F(t_{\frac{1}{2}}) = \int_0^{t_{\frac{1}{2}}} f(s)\,\mathrm{d}s$. Testing (5.9) with $\phi_h = \nabla y_h$ and utilizing the Poincaré inequality, we obtain the boundedness of $\{\Phi_h(\nu_h)\}_{h>0} \subset \boldsymbol{L}^2(\Omega)$. Furthermore, $\{\Psi_h p_0\}_{h>0} \subset L^2(\Omega)$ is bounded due to (5.7). Altogether, concluding from (5.31), we obtain the boundedness of $\{p_h^{\frac{1}{2}}\}_{h>0} \subset L^2(\Omega)$. Now, testing the first line in (5.8) for $l = 0$ with $\phi_h = \delta p_h^{\frac{1}{2}}$, it holds that

$$\nu_-\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)} \leq \int_\Omega \nu_h \delta(p_h^{\frac{1}{2}})^2 \,\mathrm{d}x \underbrace{=}_{(5.8)} \int_\Omega (F^{\frac{1}{2}} - \eta p_h^{\frac{1}{2}})\delta p_h^{\frac{1}{2}} + \Phi_h(\nu_h)\cdot\nabla\delta p_h^{\frac{1}{2}} \,\mathrm{d}x$$
$$\underbrace{=}_{(5.9)} \int_\Omega (F^{\frac{1}{2}} - \eta p_h^{\frac{1}{2}})\delta p_h^{\frac{1}{2}} + (\eta p_0 + \nu_h p_1)\delta p_h^{\frac{1}{2}} \,\mathrm{d}x$$
$$\leq (\|f\|_{L^1(I,L^2(\Omega))} + \|\eta\|_{L^\infty(\Omega)}(\|p_h^{\frac{1}{2}}\|_{L^2(\Omega)} + \|p_0\|_{L^2(\Omega)}) + \nu_+\|p_1\|_{L^2(\Omega)})\|\delta p_h^{\frac{1}{2}}\|_{L^2(\Omega)},$$

which leads to the boundedness of $\{\delta p_h^{\frac{1}{2}}\}_{h>0} \subset L^2(\Omega)$. Furthermore, it holds that

$$\|\delta\boldsymbol{u}_h^1\|_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(5.8)} \|\nabla p_h^1\|_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(5.1)} \|\nabla(p_h^0 + \tau\delta p_h^{\frac{1}{2}})\|_{\boldsymbol{L}^2(\Omega)} \underbrace{\leq}_{\text{Lem.5.2}} \|\nabla\Psi_h(p_0)\|_{\boldsymbol{L}^2(\Omega)} + \frac{c_{inv}\tau}{h}\|\delta p_h^{\frac{1}{2}}\|_{L^2(\Omega)}$$
$$\underbrace{\leq}_{(5.28)} \|\nabla\Psi_h(p_0)\|_{\boldsymbol{L}^2(\Omega)} + c_{inv}c_{cfl}\|\delta p_h^{\frac{1}{2}}\|_{L^2(\Omega)}.$$

Along with (5.7), the above inequality implies the boundedness of $\{\delta \boldsymbol{u}_h^1\}_{h>0} \subset \boldsymbol{L}^2(\Omega)$. Now, let $l \in \{1, \dots, N-1\}$ be arbitrarily fixed. Testing the first line in (5.8) for l and $l-1$ with $\phi_h = \delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}}$ leads to

$$\int_\Omega (\nu_h \delta p_h^{l+\frac{1}{2}} + \eta p_h^{l+\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}}) - \boldsymbol{u}_h^{l+\frac{1}{2}} \cdot \nabla(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x = \int_\Omega F^{l+\frac{1}{2}}(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \tag{5.32}$$

and

$$\int_\Omega (\nu_h \delta p_h^{l-\frac{1}{2}} + \eta p_h^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}}) - \boldsymbol{u}_h^{l-\frac{1}{2}} \cdot \nabla(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x = \int_\Omega F^{l-\frac{1}{2}}(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x. \tag{5.33}$$

Subtracting (5.33) from (5.32) provides that

$$\begin{aligned}
&\int_\Omega \nu_h(\delta p_h^{l+\frac{1}{2}} - \delta p_h^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}}) + \eta(p_h^{l+\frac{1}{2}} - p_h^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \\
&= \int_\Omega (\boldsymbol{u}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l-\frac{1}{2}}) \cdot \nabla(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}}) + (F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x.
\end{aligned} \tag{5.34}$$

Now, for some arbitrary $N_0 \in \{1, \dots, N-1\}$, we sum up the equation (5.34) for $k = 1, \dots, N_0$ and investigate the resulting terms seperately. For the first term, we obtain that

$$\begin{aligned}
\sum_{l=1}^{N_0} \int_\Omega \nu_h(\delta p_h^{l+\frac{1}{2}} - \delta p_h^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x &= \|\sqrt{\nu_h}\delta p_h^{N_0+\frac{1}{2}}\|_{L^2(\Omega)}^2 - \|\sqrt{\nu_h}\delta p_h^{\frac{1}{2}}\|_{L^2(\Omega)}^2 \\
&\geq \nu_-\|\delta p_h^{N_0+\frac{1}{2}}\|_{L^2(\Omega)}^2 - \nu_+\|\delta p_h^{\frac{1}{2}}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{5.35}$$

Furthermore, it holds that

$$\sum_{l=1}^{N_0} \int_\Omega \eta(p_h^{l+\frac{1}{2}} - p_h^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \underbrace{=}_{(5.1)} \frac{\tau}{2}\sum_{l=1}^{N_0} \int_\Omega \eta(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})^2\,\mathrm{d}x \geq 0. \tag{5.36}$$

Moreover, we have that

$$\begin{aligned}
&\sum_{l=1}^{N_0} \int_\Omega (\boldsymbol{u}_h^{l+\frac{1}{2}} - \boldsymbol{u}_h^{l-\frac{1}{2}}) \cdot \nabla(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \underbrace{=}_{(5.2)} \tau\sum_{l=1}^{N_0} \int_\Omega \delta\boldsymbol{u}_h^l \cdot \nabla(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \\
&\underbrace{=}_{(5.1)} \sum_{l=1}^{N_0} \int_\Omega \delta\boldsymbol{u}_h^l \cdot \nabla(p_h^{l+1} - p_h^l) + \delta\boldsymbol{u}_h^l \cdot \nabla(p_h^l - p_h^{l-1})\,\mathrm{d}x \\
&\underbrace{=}_{(5.8)} \int_\Omega -\sum_{l=1}^{N_0-1} \delta\boldsymbol{u}_h^l \cdot (\delta\boldsymbol{u}_h^{l+1} - \delta\boldsymbol{u}_h^l) - \sum_{l=2}^{N_0} \delta\boldsymbol{u}_h^l \cdot (\delta\boldsymbol{u}_h^l - \delta\boldsymbol{u}_h^{l-1})\,\mathrm{d}x \\
&\quad + \int_\Omega \delta\boldsymbol{u}_h^{N_0} \cdot \nabla(p_h^{N_0+1} - p_h^{N_0}) + \delta\boldsymbol{u}_h^1 \cdot \nabla(p_h^1 - p_h^0)\,\mathrm{d}x \\
&= \int_\Omega -\sum_{l=1}^{N_0-1} \delta\boldsymbol{u}_h^l \cdot (\delta\boldsymbol{u}_h^{l+1} - \delta\boldsymbol{u}_h^l) - \sum_{l=1}^{N_0-1} \delta\boldsymbol{u}_h^{l+1} \cdot (\delta\boldsymbol{u}_h^{l+1} - \delta\boldsymbol{u}_h^l)\,\mathrm{d}x \\
&\quad + \tau\int_\Omega \delta\boldsymbol{u}_h^{N_0} \cdot \nabla\delta p_h^{N_0+\frac{1}{2}} + \delta\boldsymbol{u}_h^1 \nabla\delta p_h^{\frac{1}{2}}\,\mathrm{d}x
\end{aligned} \tag{5.37}$$

$$= \int_\Omega - \sum_{l=1}^{N_0-1} (\delta\boldsymbol{u}_h^{l+1} + \delta\boldsymbol{u}_h^l) \cdot (\delta\boldsymbol{u}_h^{l+1} - \delta\boldsymbol{u}_h^l)\,\mathrm{d}x + \tau \int_\Omega \delta\boldsymbol{u}_h^{N_0} \cdot \nabla\delta p_h^{N_0+\frac{1}{2}} + \delta\boldsymbol{u}_h^1 \nabla\delta p_h^{\frac{1}{2}}\,\mathrm{d}x$$
$$= -\|\delta\boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\delta\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} + \tau \int_\Omega \delta\boldsymbol{u}_h^{N_0} \cdot \nabla\delta p_h^{N_0+\frac{1}{2}} + \delta\boldsymbol{u}_h^1 \nabla\delta p_h^{\frac{1}{2}}\,\mathrm{d}x$$
$$\le -\|\delta\boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\delta\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\|\delta\boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\tau^2}{2}\|\nabla\delta p_h^{N_0+\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\|\delta\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$+ \frac{\tau^2}{2}\|\nabla\delta p_h^{\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\underbrace{\le}_{\text{Lem.5.2}} -\frac{1}{2}\|\delta\boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{3}{2}\|\delta\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{c_{inv}^2\tau^2}{2h^2}\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{c_{inv}^2\tau^2}{2h^2}\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$\underbrace{\le}_{(5.28)} -\frac{1}{2}\|\delta\boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{3}{4}\|\delta\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\nu_-}{4}\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{\nu_-}{4}\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)}.$$

Taking the last term from (5.34) into account, we observe that

$$\sum_{l=1}^{N_0} \int_\Omega (F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \le \sum_{l=1}^{N_0} \|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)}(\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \|\delta p_h^{l-\frac{1}{2}}\|_{L^2(\Omega)})$$
$$= \sum_{l=1}^{N_0} \|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)}\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \sum_{l=0}^{N_0-1} \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)}\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} \tag{5.38}$$
$$= \sum_{l=1}^{N_0-1} (\|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)})\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)}$$
$$+ \|F^{N_0+\frac{1}{2}} - F^{N_0-\frac{1}{2}}\|_{L^2(\Omega)}\|\delta p_h^{N_0+\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{\frac{3}{2}} - F^{\frac{1}{2}}\|_{L^2(\Omega)}\|\delta p_h^{\frac{1}{2}}\|_{L^2(\Omega)}.$$

Since $F(t) = \int_0^t f(s)\,\mathrm{d}s$ for every $t \in I$, it holds for all $l \in \{1, \ldots, N-1\}$ that

$$\|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} = \|F(t_{l+\frac{1}{2}}) - F(t_{l-\frac{1}{2}})\|_{L^2(\Omega)} = \left\| \int_{t_{l-\frac{1}{2}}}^{t_{l+\frac{1}{2}}} f(s)\,\mathrm{d}s \right\| \le \|f\|_{L^1(I,L^2(\Omega))}. \tag{5.39}$$

Applying (5.39) to (5.38) along with Young's inequality, it follows that

$$\sum_{l=1}^{N_0} \int_\Omega (F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}})(\delta p_h^{l+\frac{1}{2}} + \delta p_h^{l-\frac{1}{2}})\,\mathrm{d}x \tag{5.40}$$
$$\le \sum_{l=1}^{N_0-1} (\|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)})\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \frac{2}{\nu_-}\|f\|^2_{L^1(I,L^2(\Omega))}$$
$$+ \frac{\nu_-}{4}\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{\nu_-}{4}\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)}.$$

Now, applying (5.35)-(5.37), and (5.40) to (5.34), we obtain that

$$\nu_-\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} - \nu_+\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$\le -\frac{1}{2}\|\delta\boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{3}{2}\|\delta\boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\nu_-}{2}\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{\nu_-}{2}\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$+ \sum_{l=1}^{N_0-1} (\|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)})\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \frac{2}{\nu_-}\|f\|^2_{L^1(I,L^2(\Omega))}.$$

Rearranging yields that

$$\begin{aligned}
&\frac{\nu_-}{2}\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\delta \boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\leq \left(\nu_+ + \frac{\nu_-}{2}\right)\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{3}{2}\|\delta \boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \sum_{l=1}^{N_0-1}(\|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)})\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \frac{2}{\nu_-}\|f\|^2_{L^1(I,L^2(\Omega))}.
\end{aligned} \tag{5.41}$$

Defining

$$\begin{aligned}
\gamma &\coloneqq 2\min\{\nu_-, 1\}^{-1}, \\
\alpha &\coloneqq \left(\nu_+ + \frac{\nu_-}{2}\right)\|\delta p_h^{\frac{1}{2}}\|^2_{L^2(\Omega)} + \frac{3}{2}\|\delta \boldsymbol{u}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{2}{\nu_-}\|f\|^2_{L^1(I,L^2(\Omega))}, \\
h_l &\coloneqq \|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)} \quad \forall l \in \{1, \dots, N-2\}, \\
I_0 &\coloneqq \{l \in \{1, \dots, N_0 - 1\} : \|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} \geq 1\}, \quad J_0 \coloneqq \{1, \dots, N_0 - 1\} \setminus I_0,
\end{aligned}$$

the inequality (5.41) is equivalent to

$$\begin{aligned}
\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \|\delta \boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \gamma\alpha + \gamma\sum_{l\in J_0} h_l\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \gamma\sum_{l\in I_0} h_l\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} \\
&\leq \gamma\alpha + \gamma\sum_{l\in J_0} h_l + \gamma\sum_{l\in I_0} h_l(\|\delta p_h^{l+\frac{1}{2}}\|^2_{L^2(\Omega)} + \|\delta \boldsymbol{u}_h^l\|^2_{\boldsymbol{L}^2(\Omega)}).
\end{aligned} \tag{5.42}$$

By definition, it holds that

$$\begin{aligned}
\sum_{l=1}^{N_0-1} h_l &= \sum_{l=1}^{N_0-1}(\|F^{l+\frac{1}{2}} - F^{l-\frac{1}{2}}\|_{L^2(\Omega)} + \|F^{l+\frac{3}{2}} - F^{l+\frac{1}{2}}\|_{L^2(\Omega)}) \\
&= \sum_{l=1}^{N_0-1}\left(\left\|\int_{t_l-\tau/2}^{t_l+\tau/2} f(s)\,\mathrm{d}s\right\|_{L^2(\Omega)} + \left\|\int_{t_{l+1}-\tau/2}^{t_{l+1}+\tau/2} f(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\right) \\
&\leq \sum_{l=1}^{N_0-1}\int_{t_l-\tau/2}^{t_l+\tau/2}\|f(s)\|_{L^2(\Omega)}\,\mathrm{d}s + \sum_{l=1}^{N_0-1}\int_{t_{l+1}-\tau/2}^{t_{l+1}+\tau/2}\|f(s)\|_{L^2(\Omega)}\,\mathrm{d}s \leq 2\|f\|_{L^1(I,L^2(\Omega))}.
\end{aligned} \tag{5.43}$$

Applying (5.43) to (5.42), it follows that

$$\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \|\delta \boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \gamma(\alpha + 2\|f\|_{L^1(I,L^2(\Omega))}) + \gamma\sum_{l\in I_0} h_l(\|\delta p_h^{l+\frac{1}{2}}\|^2_{L^2(\Omega)} + \|\delta \boldsymbol{u}_h^l\|^2_{\boldsymbol{L}^2(\Omega)}).$$

Now, the discrete Gronwall lemma (cf. [23]) implies that

$$\begin{aligned}
\|\delta p_h^{N_0+\frac{1}{2}}\|^2_{L^2(\Omega)} + \|\delta \boldsymbol{u}_h^{N_0}\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \gamma(\alpha + 2\|f\|_{L^1(I,L^2(\Omega))})\exp\left(\gamma\sum_{l=1}^{N_0-1} h_l\right) \\
&\underbrace{\leq}_{(5.43)} \gamma(\alpha + 2\|f\|_{L^1(I,L^2(\Omega))})\exp\left(2\gamma\|f\|_{L^1(I,L^2(\Omega))}\right).
\end{aligned}$$

Since this inequality holds for all $N_0 \in \{1, \dots, N-1\}$, along with the boundedness of $\{\delta p_h^{\frac{1}{2}}\}_{h>0} \subset L^2(\Omega)$ and (5.8), we conclude that the claim (5.29) is valid. Furthermore, by (5.1) along with the triangular inequality, we obtain that

$$\|p_h^{l+1}\|_{L^2(\Omega)} \leq \|p_h^l\|_{L^2(\Omega)} + \tau\|\delta p_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} \leq \|p_h^l\|_{L^2(\Omega)} + \tau C \quad \forall l \in \{0, \dots, N-1\}.$$

By induction and since $\tau = \frac{1}{N}$, this leads to

$$\|p_h^{l+1}\|_{L^2(\Omega)} \leq (l+1)\tau C + \|p_h^0\|_{L^2(\Omega)} \leq C + \|p_h^0\|_{L^2(\Omega)} \quad \forall l \in \{0, \dots, N-1\}.$$

Analogously, we conclude

$$\|\boldsymbol{u}_h^{l+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C + \|\boldsymbol{u}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall l \in \{0, \dots, N-1\}.$$

Thus, the second claim (5.30) is also valid. □

Corollary 5.7. *Let Assumption 5.1 hold, and for every $h > 0$, let $\frac{1}{\tau} = N = N(h) \in \mathbb{N}$ satisfy the CFL-condition* (5.28). *Furthermore, let $(\{\overline{q}_h^l\}_{l=0}^N, \{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$ denote the unique solution to the discrete adjoint system* (5.16) *associated with $\overline{\nu}_h \in \mathcal{V}_{ad}^h$ and $(\{\overline{p}_h^l\}_{l=0}^N, \{\overline{\boldsymbol{u}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$. Then, there exists a constant $C > 0$, independent h and N, such that*

$$\max_{l\in\{0,\dots,N-1\}} \|\delta\overline{q}_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} + \max_{l\in\{1,\dots,N-1\}} \|\delta\overline{\boldsymbol{v}}_h^l\|_{\boldsymbol{L}^2(\Omega)} + \max_{l\in\{1,\dots,N-1\}} \|\nabla\overline{q}_h^l\|_{\boldsymbol{L}^2(\Omega)} \leq C$$
$$\max_{l\in\{0,\dots,N\}} \|\overline{q}_h^l\|_{L^2(\Omega)} + \max_{l\in\{0,\dots,N-1\}} \|\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}}\|_{L^2(\Omega)} \leq C.$$

Proof. We define $(\{\widehat{q}_h^l\}_{l=0}^N, \{\widehat{\boldsymbol{v}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) := (\{\overline{q}_h^{N-l}\}_{l=0}^N, \{-\overline{\boldsymbol{v}}_h^{N-l-\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$. Then, (5.16) implies that

$$\begin{cases} \int_\Omega (\widehat{\nu}_h \delta\widehat{q}_h^{l+\frac{1}{2}} + \eta\widehat{q}_h^{l+\frac{1}{2}})\phi_h - \widehat{\boldsymbol{v}}_h^{l+\frac{1}{2}} \cdot \nabla\phi_h \, \mathrm{d}x = -\sum_{i=1}^m \int_\Omega a_i(t_{N-l-\frac{1}{2}})(\overline{p}_h^{N-l-\frac{1}{2}} - p_i^{ob}(t_{N-l-\frac{1}{2}}))\phi_h \, \mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \phi_h \in \mathrm{P}_{1,D}^h, l = 0, \dots, N-1 \\ \delta\widehat{\boldsymbol{v}}_h^l + \nabla\widehat{q}_h^l = 0 \qquad\qquad\qquad\qquad \forall l = 1, \dots, N-1 \\ \widehat{q}_h^0 = 0, \widehat{\boldsymbol{v}}_h^{\frac{1}{2}} = \mathbf{0}, \end{cases}$$

where $\{\delta\widehat{q}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}$, $\{\widehat{q}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}$, and $\{\delta\widehat{\boldsymbol{v}}_h^l\}_{l=1}^{N-1}$ are defined analog to (5.17) and (5.18). Furthermore, we have that

$$\begin{aligned} h_l &:= \left\| \sum_{i=1}^m \left(a_i(t_{N-l-\frac{1}{2}}) \left(\overline{p}_h^{N-l-\frac{1}{2}} - p_i^{ob}(t_{N-l-\frac{1}{2}}) \right) - a_i(t_{N-l+\frac{1}{2}}) \left(\overline{p}_h^{N-l+\frac{1}{2}} - p_i^{ob}(t_{N-l+\frac{1}{2}}) \right) \right) \right\|_{L^2(\Omega)} \\ &\quad + \left\| \sum_{i=1}^m \left(a_i(t_{N-l-\frac{3}{2}}) \left(\overline{p}_h^{N-l-\frac{3}{2}} - p_i^{ob}(t_{N-l-\frac{3}{2}}) \right) - a_i(t_{N-l-\frac{1}{2}}) \left(\overline{p}_h^{N-l-\frac{1}{2}} - p_i^{ob}(t_{N-l-\frac{1}{2}}) \right) \right) \right\|_{L^2(\Omega)} \\ &\leq \left\| \sum_{i=1}^m \left(a_i(t_{N-l-\frac{1}{2}}) \left(\overline{p}_h^{N-l-\frac{1}{2}} - p_i^{ob}(t_{N-l-\frac{1}{2}}) - \overline{p}_h^{N-l+\frac{1}{2}} + p_i^{ob}(t_{N-l+\frac{1}{2}}) \right) \right) \right\|_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&+\left\|\sum_{i=1}^{m}\left(a_i(t_{N-l-\frac{1}{2}})-a_i(t_{N-l+\frac{1}{2}})\right)\left(\overline{p}_h^{N-l+\frac{1}{2}}-p_i^{ob}(t_{N-l+\frac{1}{2}})\right)\right\|_{L^2(\Omega)}\\
&+\left\|\sum_{i=1}^{m}\left(a_i(t_{N-l-\frac{3}{2}})\left(\overline{p}_h^{N-l-\frac{3}{2}}-p_i^{ob}(t_{N-l-\frac{3}{2}})-\overline{p}_h^{N-l-\frac{1}{2}}+p_i^{ob}(t_{N-l-\frac{1}{2}})\right)\right)\right\|_{L^2(\Omega)}\\
&+\left\|\sum_{i=1}^{m}\left(a_i(t_{N-l-\frac{3}{2}})-a_i(t_{N-l-\frac{1}{2}})\right)\left(\overline{p}_h^{N-l-\frac{1}{2}}-p_i^{ob}(t_{N-l-\frac{1}{2}})\right)\right\|_{L^2(\Omega)}\\
\underbrace{\le}_{(5.1),(5.30)}&\sum_{i=1}^{m}\|a_i\|_{C(I,L^\infty(\Omega))}\left(\frac{\tau}{2}\|\delta\overline{p}_h^{N-l-\frac{1}{2}}\|_{L^2(\Omega)}+\frac{\tau}{2}\|\delta\overline{p}_h^{N-l+\frac{1}{2}}\|_{L^2(\Omega)}+\left\|\int_{t_{N-l-\frac{1}{2}}}^{t_{N-l+\frac{1}{2}}}\partial_t p_i^{ob}(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\right)\\
&+\sum_{i=1}^{m}\|a_i\|_{C(I,L^\infty(\Omega))}\left(\frac{\tau}{2}\|\delta\overline{p}_h^{N-l-\frac{3}{2}}\|_{L^2(\Omega)}+\frac{\tau}{2}\|\delta\overline{p}_h^{N-l-\frac{1}{2}}\|_{L^2(\Omega)}+\left\|\int_{t_{N-l-\frac{3}{2}}}^{t_{N-l-\frac{1}{2}}}\partial_t p_i^{ob}(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\right)\\
&+\sum_{i=1}^{m}\left(\left\|\int_{t_{N-l-\frac{1}{2}}}^{t_{N-l+\frac{1}{2}}}\partial_t a_i(s)\,\mathrm{d}s\right\|_{L^\infty(\Omega)}+\left\|\int_{t_{N-l-\frac{3}{2}}}^{t_{N-l-\frac{1}{2}}}\partial_t a_i(s)\,\mathrm{d}s\right\|_{L^\infty(\Omega)}\right)(C+\|p_i^{ob}\|_{C(I,L^2(\Omega))})
\end{aligned}
$$

for every $l \in \{0,\cdots,N-2\}$. Summing the inequalities and using (5.29), it follows that

$$
\sum_{l=0}^{N-2} h_l \le 2\sum_{i=1}^{m}\|a_i\|_{C(I,L^\infty(\Omega))}\left(C+\|\partial_t p_i^{ob}\|_{L^1(I,L^2(\Omega))}\right)+2\sum_{i=1}^{m}\|\partial_t a_i\|_{L^1(I,L^\infty(\Omega))}\left(C+\|p_i^{ob}\|_{C(I,L^2(\Omega))}\right).
$$

Since the right-hand side is independent of h and N, the claim follows by the exact argumentation as in the proof of Theorem 5.6. □

5.4 Convergence

Given $h>0$, $N\in\mathbb{N}$, and $(\{p_h^l\}_{l=0}^N,\{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})\in(\mathrm{P}_{1,D}^h)^{N+1}\times(\mathbf{DG}_0^h)^N$, let us define the interpolations

$$
\begin{aligned}
\Lambda_{N,h}^p,\Pi_{N,h}^p,\Theta_{N,h}^p&: I\to\mathrm{P}_{1,D}^h, & \Lambda_{N,h}^{\boldsymbol{u}},\Pi_{N,h}^{\boldsymbol{u}}&: I\to\mathbf{DG}_0^h,\\
F_{N,h},p_{i,N,h}^{ob}&: I\to L^2(\Omega), & a_{i,N,h}&: I\to L^\infty(\Omega),
\end{aligned}
$$

by

$$
\Lambda_{N,h}^p(t) := \begin{cases} p_h^0 & \text{if } t=0\\ p_h^l+(t-t_l)\delta p_h^{l+\frac{1}{2}} & \text{if } t\in(t_l,t_{l+1}] \text{ for some } l\in\{0,\dots,N-1\}\end{cases} \tag{5.44}
$$

$$
\Pi_{N,h}^p(t) := \begin{cases} p_h^0 & \text{if } t=0\\ p_h^{l+\frac{1}{2}} & \text{if } t\in(t_l,t_{l+1}] \text{ for some } l\in\{0,\dots,N-1\}\end{cases} \tag{5.45}
$$

$$
\Theta_{N,h}^p(t) := \begin{cases} p_h^0 & \text{if } t=0\\ p_h^l & \text{if } t\in(t_l,t_{l+1}] \text{ for some } l\in\{0,\dots,N-1\}\end{cases} \tag{5.46}
$$

$$
\Lambda_{N,h}^{\boldsymbol{u}}(t) := \begin{cases} \boldsymbol{u}_h^{\frac{1}{2}} & \text{if } t=0\\ \boldsymbol{u}_h^{l-\frac{1}{2}}+(t-t_l)\delta\boldsymbol{u}_h^l & \text{if } t\in(t_l,t_{l+1}] \text{ for some } l\in\{0,\dots,N-1\}\end{cases} \tag{5.47}
$$

$$\Pi^{\boldsymbol{u}}_{N,h}(t) \coloneqq \begin{cases} \boldsymbol{u}_h^{\frac{1}{2}} & \text{if } t = 0 \\ \boldsymbol{u}_h^{l+\frac{1}{2}} & \text{if } t \in (t_l, t_{l+1}] \text{ for some } l \in \{0, \dots, N-1\} \end{cases} \tag{5.48}$$

$$F_{N,h}(t) \coloneqq \begin{cases} F(0) & \text{if } t = 0 \\ F(t_{l+\frac{1}{2}}) & \text{if } t \in (t_l, t_{l+1}] \text{ for some } l \in \{0, \dots, N-1\} \end{cases} \tag{5.49}$$

$$a_{i,N,h}(t) \coloneqq \begin{cases} a_i(0) & \text{if } t = 0 \\ a_i(t_{l+\frac{1}{2}}) & \text{if } t \in (t_l, t_{l+1}] \text{ for some } l \in \{0, \dots, N-1\} \end{cases} \tag{5.50}$$

$$p^{ob}_{i,N,h}(t) \coloneqq \begin{cases} p^{ob}_i(0) & \text{if } t = 0 \\ p^{ob}_i(t_{l+\frac{1}{2}}) & \text{if } t \in (t_l, t_{l+1}] \text{ for some } l \in \{0, \dots, N-1\} \end{cases} \tag{5.51}$$

for all $i = 1, \dots, m$. In a completely analog manner, the corresponding interpolations of the discrete adjoint $(\{q_h^l\}_{l=0}^N, \{\boldsymbol{v}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$ variables are denoted by

$$\Lambda^q_{N,h}, \Pi^q_{N,h}, \Theta^q_{N,h} \colon I \to \mathrm{P}^h_{1,D}, \qquad \Lambda^{\boldsymbol{v}}_{N,h}, \Pi^{\boldsymbol{v}}_{N,h} \colon I \to \mathbf{DG}^h_0 .$$

Suppose that $(\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$ solves the leapfrog scheme (5.8) associated with ν_h. Then, the corresponding interpolations satisfy

$$\begin{cases} \displaystyle\int_\Omega (\nu_h \partial_t \Lambda^p_{N,h}(t) + \eta \Pi^p_{N,h}(t))\phi_h - \Pi^{\boldsymbol{u}}_{N,h}(t) \cdot \nabla\phi_h \,\mathrm{d}x = \int_\Omega F_{N,h}\phi_h \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \text{for all } \phi_h \in \mathrm{P}^h_{1,D} \text{ and a.e. } t \in I \\ \partial_t \Lambda^{\boldsymbol{u}}_{N,h}(t) + \nabla\Theta^p_{N,h}(t) = 0 \qquad \text{for a.e. } t \in I \\ (\Lambda^p_{N,h}, \Lambda^{\boldsymbol{u}}_{N,h})(0) = (\Psi_h(p_0), \Phi_h(\nu_h)). \end{cases} \tag{5.52}$$

Moreover, if $\overline{\nu}_h$, $(\{\overline{p}_h^l\}_{l=0}^N, \{\overline{\boldsymbol{u}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$, and $(\{\overline{q}_h^l\}_{l=0}^N, \{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$ satisfy the first-order necessary optimality condition for (P_h) (see (5.8), (5.16), and (5.19)), the corresponding interpolations fulfill

$$\begin{cases} \displaystyle\int_\Omega (\overline{\nu}_h \partial_t \Lambda^{\overline{p}}_{N,h}(t) + \eta \Pi^{\overline{p}}_{N,h}(t))\phi_h - \Pi^{\overline{\boldsymbol{u}}}_{N,h}(t) \cdot \nabla\phi_h \,\mathrm{d}x = \int_\Omega F_{N,h}(t)\phi_h \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \text{for all } \phi_h \in \mathrm{P}^h_{1,D} \text{ and a.e. } t \in I \\ \partial_t \Lambda^{\overline{\boldsymbol{u}}}_{N,h}(t) + \nabla\Theta^{\overline{p}}_{N,h}(t) = 0 \qquad \text{for a.e. } t \in I \\ (\Lambda^{\overline{p}}_{N,h}, \Lambda^{\overline{\boldsymbol{u}}}_{N,h})(0) = (\Psi_h(p_0), \Phi_h(\overline{\nu}_h)) \end{cases} \tag{5.53}$$

$$\begin{cases} \displaystyle\int_\Omega (\overline{\nu}_h \partial_t \Lambda^{\overline{q}}_{N,h}(t) - \eta \Pi^{\overline{q}}_{N,h}(t))\phi_h - \Pi^{\overline{\boldsymbol{v}}}_{N,h}(t) \cdot \nabla\phi_h \,\mathrm{d}x \\ \displaystyle = \sum_{i=1}^m \int_\Omega a_{i,N,h}(t)(\Pi^{\overline{p}}_{N,h}(t) - p^{ob}_{i,N,h}(t))\phi_h \,\mathrm{d}x \qquad \text{for all } \phi_h \in \mathrm{P}^h_{1,D} \text{ and a.e. } t \in I \\ \partial_t \Lambda^{\overline{\boldsymbol{v}}}_{N,h}(t) + \nabla\Theta^{\overline{q}}_{N,h}(t) = 0 \qquad \text{for all a.e. } t \in I \\ (\Lambda^{\overline{q}}_{N,h}, \Lambda^{\overline{\boldsymbol{v}}}_{N,h})(T) = (0, \mathbf{0}) \end{cases} \tag{5.54}$$

$$\left(\int_I (\partial_t \Lambda^{\overline{p}}_{N,h}(t) - p_1)\Pi^{\overline{q}}_{N,h}(t) \,\mathrm{d}t + \lambda\overline{\nu}_h, \nu_h - \overline{\nu}_h \right)_{L^2(\Omega)} \geq 0 \quad \text{for all } \nu_h \in \mathcal{V}^h_{ad}. \tag{5.55}$$

Lemma 5.8. *Let Assumption 5.1 hold. Furthermore, let $\{\upsilon_h\}_{h>0}$ with $\nu_h \in \mathcal{V}_{ad}^h$ for all $h > 0$ converge weakly in $L^2(\Omega)$ towards some $\nu \in \mathcal{V}_{ad}$ for $h \to 0$. Then, it holds that*

$$\Phi_h(\nu_h) \rightharpoonup \Phi(\nu) \quad \textit{weakly in } \boldsymbol{L}^2(\Omega) \textit{ as } h \to 0.$$

Proof. For every $h > 0$, we set $\Phi_h(\nu_h) = \nabla y_h$ where $y_h \in \mathrm{P}_{1,D}^h$ is according to (5.9) the unique solution to

$$\int_\Omega \nabla y_h \cdot \nabla \phi_h \,\mathrm{d}x = \int_\Omega (\eta p_0 + \nu p_1)\phi_h \,\mathrm{d}x \quad \forall \phi_h \in \mathrm{P}_{1,D}^h. \tag{5.56}$$

Testing (5.56) with $\phi_h = y_h$, along with the Poincaré inequality, we obtain the boundedness of $\{y_h\}_{h>0} \subset H_D^1(\Omega)$. Therefore, there exists a subsequence (still denoted by the same symbol) and a $y \in H_D^1(\Omega)$ such that

$$y_h \rightharpoonup y \quad \text{weakly in } H^1(\Omega) \quad \text{as } h \to 0. \tag{5.57}$$

Now, let $\phi \in C_D^\infty(\Omega)$ be arbitrarily fixed. Then, testing (5.56) with $\phi_h = \mathcal{I}_h\phi \in \mathrm{P}_{1,D}^h$ (see (5.3)) and passing to the limit $h \to 0$, we obtain that

$$\int_\Omega \nabla y\cdot\nabla\phi \,\mathrm{d}x = \lim_{h\to 0}\int_\Omega \nabla y_h\cdot\nabla\mathcal{I}_h\phi \,\mathrm{d}x = \lim_{h\to 0}\int_\Omega (\eta p_0+\nu_h p_1)\mathcal{I}_h\phi \,\mathrm{d}x = \int_\Omega (\eta p_0+\nu p_1)\phi \,\mathrm{d}x, \tag{5.58}$$

where we have used (5.4), (5.5), (5.57), and the weak convergence $\nu_h \rightharpoonup \nu$ in $L^2(\Omega)$ as $h \to 0$. In conclusion

$$\int_\Omega \nabla y \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega (\eta p_0 + \nu p_1)\phi \,\mathrm{d}x \quad \forall \phi \in C_D^\infty(\Omega). \tag{5.59}$$

Since $H_D^1(\Omega) = \overline{C_D^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$, it follows that

$$\int_\Omega \nabla y \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega (\eta p_0 + \nu p_1)\phi \,\mathrm{d}x \quad \forall \phi \in H_D^1(\Omega) \quad \underbrace{\Rightarrow}_{(3.6)} \quad \nabla y = \Phi(\nu).$$ □

Theorem 5.9. *Let Assumption 5.1 hold and for every $h > 0$, let $\frac{1}{\tau} = N = N(h) \in \mathbb{N}$ satisfy the CFL-condition (5.28). Furthermore, let $\{\nu_h\}_{h>0}$ with $\nu_h \in \mathcal{V}_{ad}^h$ for all $h > 0$ converge weakly in $L^2(\Omega)$ towards $\nu \in \mathcal{V}_{ad}$. Then, the interpolations of the solution $(\{p_h^l\}_{l=0}^N, \{\boldsymbol{u}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}_{1,D}^h)^{N+1} \times (\mathbf{DG}_0^h)^N$ to (5.8) satisfy*

$$\begin{aligned} \Lambda_{N,h}^p &\to p \quad \textit{in } C(I, L^2(\Omega)) && \textit{as } h \to 0 \\ \Theta_{N,h}^p, \Pi_{N,h}^p &\to p \quad \textit{in } L^\infty(I, L^2(\Omega)) && \textit{as } h \to 0, \end{aligned}$$

where $p \in C^1(I, L^2(\Omega)) \cap C(I, H_D^1(\Omega))$ is the first component of the unique solution to (3.9) associated with ν.

Proof. By Theorem 5.6, along with the definitions of the interpolations (see (5.44)-(5.48)), the families

$$\{\Lambda_{N,h}^p\}_{h>0}, \{\Pi_{N,h}^p\}_{h>0}, \{\Theta_{N,h}^p\}_{h>0}, \{\partial_t\Lambda_{N,h}^p\}_{h>0} \subset L^\infty(I, L^2(\Omega))$$

$$\{\nabla\Theta^p_{N,h}\}_{h>0}, \{\Lambda^{\boldsymbol{u}}_{N,h}\}_{h>0}, \{\Pi^{\boldsymbol{u}}_{N,h}\}_{h>0}, \{\partial_t\Lambda^{\boldsymbol{u}}_{N,h}\}_{h>0} \subset L^\infty(I, \boldsymbol{L}^2(\Omega))$$

are bounded. Furthermore, it holds for every $t \in (t_l, t_{l+1}]$ and $l = 0, \ldots, N-1$ that

$$\|\nabla\Lambda^p_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(5.44)} \|\nabla(p^l_h + (t-t_l)\delta p^{l+\frac{1}{2}}_h)\|_{\boldsymbol{L}^2(\Omega)} \underbrace{\leq}_{(5.1)} \|\nabla p^l_h\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(p^{l+1}_h - p^l_h)\|_{\boldsymbol{L}^2(\Omega)},$$

such that $\{\nabla\Lambda^p_{N,h}\}_{h>0} \subset L^\infty(I, \boldsymbol{L}^2(\Omega))$ is also bounded due to Theorem 5.6. Moreover, by the Aubin-Lions lemma, the following embedding is compact:

$$\{p \in L^\infty(I, H^1(\Omega)) : \partial_t p \in L^\infty(I, L^2(\Omega))\} \overset{c}{\hookrightarrow} C(I, L^2(\Omega)).$$

Therefore, we find a subsequence (denoted by the same symbol) and a $p \in C(I, L^2(\Omega))$, such that

$$\Lambda^p_{N,h} \to p \quad \text{in } C(I, L^2(\Omega)) \text{ as } h \to 0. \tag{5.60}$$

Furthermore, it holds that

$$\|\Lambda^p_{N,h} - \Theta^p_{N,h}\|_{L^\infty(I,L^2(\Omega))} \underbrace{\leq}_{(5.44),(5.46)} \max_{l\in\{0,\ldots,N-1\}} \tau\|\delta p^{l+\frac{1}{2}}_h\|_{L^2(\Omega)} \underbrace{\leq}_{\text{Thm.5.6}} \tau C \underbrace{\leq}_{(5.28)} c_{cfl}hC \to 0 \text{ as } h \to 0$$

$$\|\Theta^p_{N,h} - \Pi^p_{N,h}\|_{L^\infty(I,L^2(\Omega))} \underbrace{\leq}_{(5.45),(5.46)} \max_{l\in\{0,\ldots,N-1\}} \|p^l_h - p^{l+\frac{1}{2}}_h\|_{L^2(\Omega)} \underbrace{=}_{(5.1)} \frac{1}{2}\max_{l\in\{0,\ldots,N-1\}} \|p^{l+1}_h - p^l_h\|_{L^2(\Omega)}$$

$$\underbrace{=}_{(5.1)} \max_{l\in\{0,\ldots,N-1\}} \frac{\tau}{2}\|\delta p^{l+\frac{1}{2}}_h\|_{L^2(\Omega)} \underbrace{\leq}_{\text{Thm.5.6}} \frac{\tau C}{2} \underbrace{\leq}_{(5.28)} \frac{c_{cfl}hC}{2} \to 0 \quad \text{as } h \to 0,$$

such that, along with (5.60), it follows that

$$\Theta^p_{N,h}, \Pi^p_{N,h} \to p \quad \text{in } L^\infty(I, L^2(\Omega)) \text{ as } h \to 0. \tag{5.61}$$

On the other hand, due to the Banach-Alaoglu theorem, there exist subsequences (still denoted by the same indices), $\widetilde{p} \in L^\infty(I, L^2(\Omega))$, and $\boldsymbol{v}, \boldsymbol{u}, \widehat{\boldsymbol{u}}, \widetilde{\boldsymbol{u}} \in L^\infty(I, \boldsymbol{L}^2(\Omega))$ such that

$$\partial_t\Lambda^p_{N,h} \overset{*}{\rightharpoonup} \widetilde{p} \quad \text{weakly-* in } L^\infty(I, L^2(\Omega)) \tag{5.62}$$

$$\nabla\Theta^p_{N,h} \overset{*}{\rightharpoonup} \boldsymbol{v}, \quad \Lambda^{\boldsymbol{u}}_{N,h} \overset{*}{\rightharpoonup} \boldsymbol{u}, \quad \Pi^{\boldsymbol{u}}_{N,h} \overset{*}{\rightharpoonup} \widehat{\boldsymbol{u}}, \quad \partial_t\Lambda^{\boldsymbol{u}}_{N,h} \overset{*}{\rightharpoonup} \widetilde{\boldsymbol{u}} \quad \text{weakly-* in } L^\infty(I, \boldsymbol{L}^2(\Omega)). \tag{5.63}$$

By standard argumentation, it follows that $\widetilde{p} = \partial_t p$, $\boldsymbol{v} = \nabla p$, and $\widetilde{\boldsymbol{u}} = \partial_t\boldsymbol{u}$. As above, we obtain that

$$\|\Lambda^{\boldsymbol{u}}_{N,h} - \Pi^{\boldsymbol{u}}_{N,h}\|_{L^\infty(I,L^2(\Omega))} \underbrace{\leq}_{(5.47),(5.48)} \max_{l\in\{0,\ldots,N-1\}} \Big(\|\boldsymbol{u}^{l-\frac{1}{2}}_h - \boldsymbol{u}^{l+\frac{1}{2}}_h\|_{L^2(\Omega)} + \tau\|\delta\boldsymbol{u}^l_h\|_{L^2(\Omega)}\Big) \tag{5.64}$$

$$\underbrace{=}_{(5.2)} \max_{l\in\{0,\ldots,N-1\}} 2\tau\|\delta\boldsymbol{u}^l_h\|_{L^2(\Omega)} \underbrace{\leq}_{\text{Thm.5.6}} 2\tau C \underbrace{\leq}_{(5.28)} 2c_{cfl}hC \to 0, \text{ as } h \to 0,$$

such that, along with (5.63), it follows $\boldsymbol{u} = \widehat{\boldsymbol{u}}$. Next, let us prove that $(p, \boldsymbol{u})$ is the unique solution to (3.9) associated with ν. Let $t \in I \setminus \{0\}$ be arbitrarily given. Then, for every

$h > 0$, there exists some $l(h) \in \{0, \ldots, N(h)-1\}$ such that $t \in (t_{l(h)}, t_{l(h)+1}]$. In particular, we have that $t_{l(h)+\frac{1}{2}} \to t$ as $h \to 0$, and since $F \in W^{1,1}(I, L^2(\Omega))$, this implies that

$$\|F_{N,h}^{ob}(t) - F(t)\|_{L^2(\Omega)} \underbrace{=}_{(5.49)} \|F(t_{l(h)+\frac{1}{2}}) - F(t)\|_{L^2(\Omega)} \le \int_t^{t_{l(h)+\frac{1}{2}}} \|\partial_t F(s)\|_{L^2(\Omega)} \,\mathrm{d}s \to 0 \quad (5.65)$$

as $h \to 0$. Along with $\|F_{N,h}(t)\|_{L^2(\Omega)} \le \|F\|_{C(I,L^2(\Omega))}$ for every $t \in I$ (see (5.49)), Lebesgue's dominated convergence theorem implies that

$$F_{N,h} \to F \text{ in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0. \quad (5.66)$$

Furthermore, since $\{\nu_h\}_{h>0}$ converges weakly in $L^2(\Omega)$ towards ν, along with (5.60), it holds that for every $\phi \in L^2(I, C^\infty(\Omega))$ that

$$\begin{aligned}
&|(\nu_h \Lambda_{N,h}^p - \nu p, \phi)_{L^2(I,L^2(\Omega))}| \quad (5.67)\\
&\le |(\nu_h(\Lambda_{N,h}^p - p), \phi)_{L^2(I,L^2(\Omega))}| + |((\nu_h - \nu)\phi, p)_{L^2(I,L^2(\Omega))}|\\
&\le |((\nu_h - \nu)\phi, (\Lambda_{N,h}^p - p))_{L^2(I,L^2(\Omega))}| + |((\Lambda_{N,h}^p - p), \nu\phi)_{L^2(I,L^2(\Omega))}| + |((\nu_h - \nu)\phi, p)_{L^2(I,L^2(\Omega))}|\\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Due to (5.60) and $\nu_h \in \mathcal{V}_{ad}$ for every $h > 0$, the sequence $\{\nu_h \Lambda_{N,h}^p\}_{h>0} \subset L^2(I, L^2(\Omega))$ is bounded. Thus, along with the density $C^\infty(I, L^2(\Omega)) \subset L^2(I, L^2(\Omega))$, (5.67) implies that

$$\nu_h \Lambda_{N,h}^p \rightharpoonup \nu p \quad \text{weakly in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0. \quad (5.68)$$

Moreover, for every $\phi \in C_0^\infty(I, L^2(\Omega))$, it holds for $h \to 0$ that

$$\begin{aligned}
(\nu_h \partial_t \Lambda_{N,h}^p, \phi)_{L^2(I,L^2(\Omega))} &= -(\nu_h \Lambda_{N,h}^p, \partial_t \phi)_{L^2(I,L^2(\Omega))} \quad (5.69)\\
\underbrace{\to}_{(5.68)} -(\nu p, \partial_t \phi)_{L^2(I,L^2(\Omega))} &= (\nu \partial_t p, \phi)_{L^2(I,L^2(\Omega))}.
\end{aligned}$$

Due to (5.62) and again $\nu_h \in \mathcal{V}_{ad}$ for every $h > 0$, the sequence $\{\nu_h \partial_t \Lambda_{N,h}^p\}_{h>0} \subset L^2(I, L^2(\Omega))$ is also bounded. Thus, along with the density $C_0^\infty(I, L^2(\Omega) \subset L^2(I, L^2(\Omega))$, (5.69) implies that

$$\nu_h \partial_t \Lambda_{N,h}^p \rightharpoonup \nu \partial_t p \quad \text{weakly in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0. \quad (5.70)$$

Using the P_1^h-interpolation operator $\mathcal{I}_h$ (see (5.3)), we deduce for every $\phi \in C_D^\infty(\Omega)$ and $\psi \in C_0^\infty(I)$ that

$$\begin{aligned}
&\int_I \int_\Omega (\nu \partial_t p(t) + \eta p(t))\phi - \boldsymbol{u}(t) \cdot \nabla \phi \,\mathrm{d}x \psi(t) \,\mathrm{d}t\\
&\underbrace{=}_{(5.5),(5.61),(5.63),(5.70)} \lim_{h\to 0} \int_I \int_\Omega (\nu_h \partial_t \Lambda_{N,h}^p(t) + \eta \Pi_{N,h}^p(t))\mathcal{I}_h \phi - \Pi_{N,h}^{\boldsymbol{u}}(t) \cdot \nabla(\mathcal{I}_h \phi) \,\mathrm{d}x \psi(t) \,\mathrm{d}t\\
&\underbrace{=}_{(5.52)} \lim_{h\to 0} \int_I \int_\Omega F_{N,h}(t) \mathcal{I}_h \phi \,\mathrm{d}x \psi(t) \,\mathrm{d}t \underbrace{=}_{(5.5),(5.66)} \int_I \int_\Omega F(t) \phi \,\mathrm{d}x \psi(t) \,\mathrm{d}t.
\end{aligned}$$

Since $H_D^1(\Omega) = \overline{C_D^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$, it follows that

$$\int_\Omega (\nu\partial_t p(t) + \eta p(t))\phi - \boldsymbol{u}(t)\cdot\nabla\phi \,\mathrm{d}x = \int_\Omega F(t)\phi \,\mathrm{d}x \quad \text{for all } \phi \in H_D^1(\Omega) \text{ and a.e. } t \in I. \tag{5.71}$$

Furthermore, by (5.63) and (5.52), we obtain that

$$\partial_t \boldsymbol{u} = -\nabla p. \tag{5.72}$$

Regarding the initial conditions, it holds that

$$p(0) \underbrace{=}_{(5.60)} \lim_{h\to 0} \Lambda^p_{N,h}(0) \underbrace{=}_{(5.52)} \lim_{h\to 0} \Psi_h(p_0) \underbrace{=}_{(5.7)} p_0 \quad \text{in } L^2(\Omega). \tag{5.73}$$

Let $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)$ be arbitrarily fixed and $\xi \in C^\infty(I)$ such that $\xi(T) = 0$ and $\xi(0) = 1$. We define $[\boldsymbol{\phi}\xi] \in C^\infty(I, \boldsymbol{L}^2(\Omega))$ by $[\boldsymbol{\phi}\xi](t)(x) = \boldsymbol{\phi}(x)\xi(t)$ for all $t \in I$ and a.e. $x \in \Omega$. Then, along with the integration by parts formula, it follows that

$$\begin{aligned}
(\boldsymbol{\phi}, \Phi(\nu))_{\boldsymbol{L}^2(\Omega)} &\underbrace{=}_{\text{Lem.5.8}} \lim_{h\to 0} (\boldsymbol{\phi}, \Phi_h(\nu_h))_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(5.52)} \lim_{h\to 0} (\boldsymbol{\phi}, \Lambda^{\boldsymbol{u}}_{N,h}(0))_{\boldsymbol{L}^2(\Omega)} \\
&= \lim_{h\to 0}\Big(-\int_I (\boldsymbol{\phi}, \partial_t \Lambda^{\boldsymbol{u}}_{N,h}(t))_{\boldsymbol{L}^2(\Omega)} \xi(t)\,\mathrm{d}t - \int_I (\boldsymbol{\phi}, \Lambda^{\boldsymbol{u}}_{N,h}(t))_{\boldsymbol{L}^2(\Omega)} \partial_t\xi(t)\,\mathrm{d}t \Big) \\
&\underbrace{=}_{(5.63)} -\int_I (\boldsymbol{\phi}, \partial_t \boldsymbol{u}(t))_{\boldsymbol{L}^2(\Omega)} \xi(t)\,\mathrm{d}t - \int_I (\boldsymbol{\phi}, \boldsymbol{u}(t))_{L^2(\Omega)} \partial_t\xi(t)\,\mathrm{d}t \\
&= (\boldsymbol{\phi}, \boldsymbol{u}(0))_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Since $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)$ was chosen arbitrarily, this implies $\boldsymbol{u}(0) = \Phi(\nu)$. Along with (5.71)-(5.73), $(p, \boldsymbol{u}) \in W^{1,\infty}(I, L^2(\Omega)) \cap L^2(I, H_D^1(\Omega)) \times W^{1,\infty}(I, \boldsymbol{L}^2(\Omega)) \cap L^2(I, \boldsymbol{H}_N(\mathrm{div}, \Omega))$ satisfies

$$\begin{cases}
\nu\partial_t p + \eta p + \operatorname{div} \boldsymbol{u} = F & \text{in } I \times \Omega \\
\partial_t \boldsymbol{u} + \nabla p = 0 & \text{in } I \times \Omega \\
p = 0 & \text{on } I \times \Gamma_D \\
\boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\
(p, \boldsymbol{u})(0) = (p_0, \Phi(\nu)) & \text{in } \Omega.
\end{cases} \tag{5.74}$$

Thus, $(p, \boldsymbol{u})$ is the strong solution to (3.9) associated with ν. Since the unique classical and strong solutions to (3.9) coincide, the claim follows. □

Theorem 5.10. *Let Assumption 5.1 hold and for every $h > 0$, let $\frac{1}{\tau} = N = N(h) \in \mathbb{N}$ satisfy the CFL-condition* (5.28)*. Furthermore, for every $h > 0$, let $\overline{\nu}_h \in \mathcal{V}^h_{ad}$, $(\{\overline{p}^l_h\}_{l=0}^N, \{\overline{\boldsymbol{u}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1})$, $(\{\overline{q}^l_h\}_{l=0}^N, \{\overline{\boldsymbol{v}}_h^{l+\frac{1}{2}}\}_{l=0}^{N-1}) \in (\mathrm{P}^h_{1,D})^{N+1} \times (\mathbf{DG}^h_0)^N$ satisfy the first-order necessary optimality condition for* (P_h) *(see* (5.8), (5.16), *and* (5.19)*). Then, there exist a subsequence of $\{\overline{\nu}_h\}_{h>0}$ (still denoted by the same symbol) such that*

$$\overline{\nu}_h \rightharpoonup \overline{\nu} \quad \textit{weakly in } L^2(\Omega) \qquad \textit{as } h \to 0 \tag{5.75}$$

$$\Lambda^{\overline{p}}_{N,h} \to \overline{p} \quad \textit{in } C(I, L^2(\Omega)) \qquad \textit{as } h \to 0 \tag{5.76}$$

$$\Theta^{\overline{p}}_{N,h}, \Pi^{\overline{p}}_{N,h} \to \overline{p} \quad \textit{in } L^\infty(I, L^2(\Omega)) \qquad \textit{as } h \to 0 \tag{5.77}$$

$$\Lambda^{\overline{q}}_{N,h} \to \overline{q} \quad \textit{in } C(I, L^2(\Omega)) \qquad \textit{as } h \to 0 \tag{5.78}$$

$$\Theta^{\overline{q}}_{N,h}, \Pi^{\overline{q}}_{N,h} \to \overline{q} \quad \textit{in } L^\infty(I, L^2(\Omega)) \qquad \textit{as } h \to 0, \tag{5.79}$$

where $(\overline{\nu}, \overline{p}, \overline{q}) \in \mathcal{V}_{ad} \times (C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega)))^2$ *is a solution to the first-order optimality system to* (P) *(see* (3.9)*,* (3.46)*, and* (3.62)

Proof. Note that $\mathcal{V}^h_{ad} \subset \mathcal{V}_{ad}$. Furthermore, $\mathcal{V}_{ad}$ is bounded, closed, and convex in $L^2(\Omega)$, and thus, it is weakly sequentially compact in $L^2(\Omega)$. Consequently, there exists a subsequence of $\{\overline{\nu}_h\}_{h>0}$ (still denoted by the same symbol) and a $\overline{\nu} \in \mathcal{V}_{ad}$ such that (5.75) is valid. Then, by the previous Theorem 5.9, we obtain the convergence properties (5.76) and (5.77), where $\overline{p} \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ is the first component of the unique solution to (3.9) associated with $\overline{\nu}$. Due to the stability results in Corollary 5.7, analogously to Theorem 5.9, we obtain the convergence properties (5.78) and (5.79) where $\overline{q} \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ is the first component of the unique solution to (3.46) associated with $\overline{\nu}$ and $\overline{p}$. It remains to prove that $(\overline{v}, \overline{p}, \overline{q})$ satisfies the variational inequality (3.62). So, let $\nu \in \mathcal{V}_{ad}$ be arbitrarily given. With $Q_h : L^2(\Omega) \to \mathrm{DG}^h_0$ we denote the standard $L^2(\Omega)$ orthogonal projection operator onto DG^h_0 that is given by

$$(Q_h v)(x) = \sum_{T \in \mathcal{T}_h} \chi_T(x) \frac{1}{|T|} \int_T v(y) \, \mathrm{d}y \quad \forall v \in L^2(\Omega) \quad \forall x \in \overline{\Omega}, \tag{5.80}$$

where χ_T denotes the characteristic function of T. Then, there exists a constant $c > 0$, independent of h, such that

$$\|v - Q_h v\|_{L^2(\Omega)} \le ch \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

(see [33, Prop. 1.135]). Now, let $v \in L^2(\Omega)$ and $\epsilon > 0$ be arbitrarily given. By the density $H^1(\Omega) \subset L^2(\Omega)$, there exists $v_\epsilon \in H^1(\Omega)$ with $\|v_\epsilon - v\|_{L^2(\Omega)} \le \frac{\epsilon}{3}$. Then, it holds that

$$\begin{aligned} \|v - Q_h v\|_{L^2(\Omega)} &\le \|v - v_\epsilon\|_{L^2(\Omega)} + \|v_\epsilon - Q_h v_\epsilon\|_{L^2(\Omega)} + \|Q_h v_\epsilon - Q_h v\|_{L^2(\Omega)} \\ &\le 2\|v - v_\epsilon\|_{L^2(\Omega)} + ch\|v_\epsilon\|_{H^1(\Omega)} \le \epsilon \quad \forall h \in \left(0, \frac{\epsilon}{3c\|v_\epsilon\|_{H^1(\Omega)}}\right), \end{aligned}$$

which implies that $Q_h v \to v$ in $L^2(\Omega)$ as $h \to 0$ for every $v \in L^2(\Omega)$. Furthermore, Q_h maps $\mathcal{V}_{ad}$ into $\mathcal{V}^h_{ad}$ since

$$\nu_- = \sum_{T \in \mathcal{T}_h} \chi_T(x) \frac{1}{|T|} \int_T v_- \, \mathrm{d}y \le (Q_h v)(x) \le \sum_{T \in \mathcal{T}_h} \chi_T(x) \frac{1}{|T|} \int_T v_+ \, \mathrm{d}y = \nu_+ \quad \forall v \in \mathcal{V}_{ad} \quad \forall x \in \overline{\Omega}.$$

Therefore, the sequence $\{\nu_h\}_{h>0} \coloneqq \{Q_h \nu\}_{h>0}$ satisfies

$$\nu_h \in \mathcal{V}^h_{ad} \quad \forall h > 0 \quad \text{and} \quad \nu_h \to \nu \quad \text{in } L^2(\Omega) \quad \text{as } h \to 0. \tag{5.81}$$

As in the proof of Theorem 5.9, it follows that

$$\nu_h \partial_t \Lambda^{\overline{p}}_{N,h} \rightharpoonup \nu \partial_t \overline{p} \quad \text{weakly in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0 \tag{5.82}$$

$$\overline{\nu}_h \partial_t \Lambda^{\overline{p}}_{N,h} \rightharpoonup \overline{\nu}\partial_t \overline{p} \quad \text{weakly in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0. \tag{5.83}$$

Finally, along with the weak sequential lower semicontinuity of the squared norm, we conclude that

$$\begin{aligned}
\lambda \|\overline{\nu}\|^2_{L^2(\Omega)} &\underbrace{\leq}_{(5.75)} \liminf_{h\to 0} \lambda \|\overline{\nu}_h\|^2_{L^2(\Omega)} \\
&\underbrace{\leq}_{(5.55)} \liminf_{h\to 0} \Bigg(\Big(\int_I (\partial_t \Lambda^{\overline{p}}_{N,h}(t) - p_1)\Pi^{\overline{q}}_{N,h}(t)\,\mathrm{d}t + \lambda\overline{\nu}_h, \nu_h \Big)_{L^2(\Omega)} \\
&\qquad - \Big(\int_I (\partial_t \Lambda^{\overline{p}}_{N,h}(t) - p_1)\Pi^{\overline{q}}_{N,h}(t)\,\mathrm{d}t, \overline{\nu}_h \Big)_{L^2(\Omega)} \Bigg) \\
&\underbrace{=}_{(5.79),(5.75),(5.81)-(5.83)} \Big(\int_I (\partial_t \overline{p}(t) - p_1)\overline{q}(t)\,\mathrm{d}t + \lambda\overline{\nu}, \nu \Big)_{L^2(\Omega)} - \Big(\int_I (\partial_t \overline{p}(t) - p_1)\overline{q}(t)\,\mathrm{d}t, \overline{\nu} \Big)_{L^2(\Omega)},
\end{aligned}$$

that implies

$$\Big(\int_I (\partial_t \overline{p}(\overline{\nu})(t) - p_1)\overline{q}(t)\,\mathrm{d}t + \lambda\overline{\nu}, \nu - \overline{\nu} \Big)_{L^2(\Omega)} \geq 0.$$

Since $\nu \in \mathcal{V}_{ad}$ was arbitrary, (3.62) is valid. □

To prove our final result in Theorem 5.12, we need the following lemma:

Lemma 5.11. *Let Assumption 5.1 hold and for every $h > 0$, let $\frac{1}{\tau} = N = N(h) \in \mathbb{N}$ satisfy the CFL-condition* (5.28). *Furthermore, let $\{\nu_h\}_{h>0}$ with $\nu_h \in \mathcal{V}^h_{ad}$ for all $h > 0$ be given. Then, it holds that*

$$|J(\nu_h) - J_h(\nu_h)| \to 0 \quad \text{as } h \to 0. \tag{5.84}$$

If additionally $\{\nu_h\}_{h>0}$ converges (strongly) in $L^2(\Omega)$ towards a $\nu \in \mathcal{V}_{ad}$ as $h \to 0$, then it holds that

$$|J(\nu) - J_h(\nu_h)| \to 0 \quad \text{as } h \to 0. \tag{5.85}$$

Proof. Let $p_h \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ denote the first component of the unique solution to (3.9) associated with ν_h. Furthermore, for every $h > 0$, let $(\{p^l_h\}^N_{l=0}, \{\boldsymbol{u}^{l+\frac{1}{2}}_h\}^{N-1}_{l=0}) \in (\mathrm{P}^h_{1,D})^{N+1} \times (\mathbf{DG}^h_0)^N$ denote the unique solution to (5.8) and let $\Pi^p_{N,h}$ denote the corresponding interpolate as in (5.45). Then, as in the proof of Theorem 5.10, there exists a subsequence of $\{\nu_h\}_{h>0}$ (still denoted by the same symbol) and a $\widetilde{\nu} \in \mathcal{V}_{ad}$ such that $\{\nu_h\}_{h>0}$ convergences weakly in $L^2(\Omega)$ towards $\widetilde{\nu}$ as $h \to 0$. Thus, Theorem 5.9 implies that

$$\Pi^p_{N,h} \to \widetilde{p} \quad \text{in } L^\infty(I, L^2(\Omega)) \text{ as } h \to 0, \tag{5.86}$$

where $\widetilde{p} \in C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$ denotes first component of the unique solution to (3.9) associated with $\widetilde{\nu}$. Furthermore, the solution operator $S_p\colon L^2(\Omega) \to L^2(I, L^2(\Omega)), \nu \mapsto p$ to (3.9) is weak-strong continuous (see the proof of Theorem 3.7) such that

$$p_h \to \widetilde{p} \quad \text{in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0. \tag{5.87}$$

Since $p_i^{ob} \in W^{1,1}(I, L^2(\Omega))$, as in (5.65), it follows that $p_{i,N,h}^{ob}(t) \to p_i^{ob}(t)$ in $L^2(\Omega)$ as $h \to 0$ for every $t \in I$ and every $i = 1, \ldots, m$. Along with $\|p_{i,N,h}^{ob}(t)\|_{L^2(\Omega)} \le \|p_i^{ob}\|_{C(I,L^2(\Omega))}$ for every $t \in I$ (see (5.51)), Lebesgue's dominated convergence theorem implies that

$$p_{i,N,h}^{ob} \to p_i^{ob} \quad \text{in } L^2(I, L^2(\Omega)) \quad \text{as } h \to 0 \quad \forall i \in \{1, \ldots, m\}. \tag{5.88}$$

We conclude that

$$\begin{aligned}
&|J(\nu_h) - J_h(\nu_h)| \qquad (5.89)\\
&\underbrace{=}_{(\mathrm{P}),(\mathrm{P}_h)} \left| \frac{1}{2} \sum_{i=1}^m \int_I \int_\Omega a_i(t)(p_h(t) - p_i^{ob}(t))^2 \, \mathrm{d}x \, \mathrm{d}t - \frac{\tau}{2} \sum_{i=1}^m \sum_{l=0}^{N-1} \int_\Omega a_i(t_{l+\frac{1}{2}})(p_h^{l+\frac{1}{2}} - p_i^{ob}(t_{l+\frac{1}{2}}))^2 \, \mathrm{d}x \right| \\
&\underbrace{=}_{(5.45),(5.51)} \left| \frac{1}{2} \sum_{i=1}^m \int_I \int_\Omega a_i(t)(p_h(t) - p_i^{ob}(t))^2 \, \mathrm{d}x \, \mathrm{d}t \right. \\
&\qquad \left. - \frac{1}{2} \sum_{i=1}^m \int_I \int_\Omega a_{i,N,h}(t)(\Pi_{N,h}^p(t) - p_{i,N,h}^{ob}(t))^2 \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\le \frac{1}{2} \sum_{i=1}^m \|a_i\|_{C(I,L^\infty(\Omega))} \int_I \int_\Omega |(p_h(t) - p_i^{ob}(t))^2 - (\Pi_{N,h}^p(t) - p_{i,N,h}^{ob}(t))^2| \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \sum_{i=1}^m \|a_i - a_{i,N,h}\|_{L^1(I,L^\infty(\Omega))} \big(\|\Pi_{N,h}^p\|_{L^\infty(I,L^2(\Omega))}^2 + \|p_{i,N,h}^{ob}\|_{L^\infty(I,L^2(\Omega))}^2 \big) \\
&= \frac{1}{2} \sum_{i=1}^m \|a_i\|_{C(I,L^\infty(\Omega))} \int_I \int_\Omega |(p_h(t) - p_i^{ob}(t) - \Pi_{N,h}^p(t) + p_{i,N,h}^{ob}(t)) \\
&\qquad\qquad (p_h(t) - p_i^{ob}(t) + \Pi_{N,h}^p(t) - p_{i,N,h}^{ob}(t))| \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \sum_{i=1}^m \|a_i - a_{i,N,h}\|_{L^1(I,L^\infty(\Omega))} \big(\|\Pi_{N,h}^p\|_{L^\infty(I,L^2(\Omega))}^2 + \|p_{i,N,h}^{ob}\|_{L^\infty(I,L^2(\Omega))}^2 \big) \\
&\le \frac{1}{2} \sum_{i=1}^m \Big(\|a_i\|_{C(I,L^\infty(\Omega))} \big(\|p_h - \Pi_{N,h}^p\|_{L^2(I,L^2(\Omega))} + \|p_i^{ob} - p_{i,N,h}^{ob}\|_{L^2(I,L^2(\Omega))} \big) \\
&\qquad \big(\|p_h\|_{L^2(I,L^2(\Omega))} + \|p_i^{ob}\|_{L^2(I,L^2(\Omega))} + \|\Pi_{N,h}^p\|_{L^2(I,L^2(\Omega))} + \|p_{i,N,h}^{ob}\|_{L^2(I,L^2(\Omega))} \big) \Big) \\
&\quad + \sum_{i=1}^m \|a_i - a_{i,N,h}\|_{L^1(I,L^\infty(\Omega))} \big(\|\Pi_{N,h}^p\|_{L^2(I,L^2(\Omega))}^2 + \|p_{i,N,h}^{ob}\|_{L^2(I,L^2(\Omega))}^2 \big).
\end{aligned}$$

Since $a_i \in C^1(I, L^\infty(\Omega))$, as in (5.65), it follows that $a_{i,N,h}(t) \to a_i(t)$ in $L^\infty(\Omega)$ as $h \to 0$ for every $t \in I$ and every $i = 1, \ldots, m$. Furthermore, $\|a_{i,N,h}(t)\|_{L^\infty(\Omega)} \le \|a_i\|_{C(I,L^\infty(\Omega))}$ for every $t \in I$ and every $i = 1, \ldots, m$. Thus, Lebesgue's dominated convergence theorem yields that

$$a_{i,N,h} \to a_i \quad \text{in } L^1(I, L^\infty(\Omega)) \quad \text{as } h \to 0 \quad \forall i \in \{1, \ldots, m\}. \tag{5.90}$$

Moreover, since $\{\nu_h\}_{h>0} \subset L^\infty(\Omega)$ is bounded, $\{p_h\}_{h>0} \subset L^2(I, L^2(\Omega))$ is bounded due to Lemma 3.4. Furthermore, $\{\Pi_{N,h}^p\}_{h>0} \subset L^\infty(I, L^2(\Omega))$ is bounded due to (5.86) and $\{p_{i,N,h}^{ob}\}_{h>0} \subset L^\infty(I, L^2(\Omega))$ is bounded due to (5.88). Consequently (5.86)-(5.90) imply (5.84). Now, let additionally $\{\nu_h\}_{h>0}$ converge strongly in $L^2(\Omega)$ towards a $\nu \in \mathcal{V}_{ad}$. Then, due to Theorem 5.12, we have that

$$\Pi_{N,h}^p \to p \quad \text{in } L^\infty(I, L^2(\Omega)) \text{ as } h \to 0, \tag{5.91}$$

where $p \in C^1(I, L^2(\Omega)) \cap C(I, H_D^1(\Omega))$ is the first component of the unique solution to (3.9) associated with ν. Analogously to (5.89), it follows that

$$
\begin{aligned}
&|J(\nu) - J_h(\nu_h)| \\
&\underbrace{=}_{(\mathrm{P}),(\mathrm{P}_h)} \Bigg| \frac{1}{2}\sum_{i=1}^{m} \int_I \int_\Omega a_i (p - p_i^{ob})^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{\lambda}{2}\|\nu\|^2_{L^2(\Omega)} - \frac{\tau}{2}\sum_{i=1}^{m}\sum_{l=0}^{N-1} \int_\Omega a_i(t_{l+\frac{1}{2}})(p_h^{l+\frac{1}{2}} - p_i^{ob}(t_{l+\frac{1}{2}}))^2 \,\mathrm{d}x \\
&\qquad - \frac{\lambda}{2}\|\nu_h\|^2_{L^2(\Omega)} \Bigg| \\
&\le \frac{1}{2}\sum_{i=1}^{m} \Big(\|a_i\|_{C(I,L^\infty(\Omega))} (\|p - \Pi^p_{N,h}\|_{L^2(I,L^2(\Omega))} + \|p_i^{ob} - p^{ob}_{i,N,h}\|_{L^2(I,L^2(\Omega))}) \\
&\qquad (\|p\|_{L^2(I,L^2(\Omega))} + \|p_i^{ob}\|_{L^2(I,L^2(\Omega))} + \|\Pi^p_{N,h}\|_{L^2(I,L^2(\Omega))} + \|p^{ob}_{i,N,h}\|_{L^2(I,L^2(\Omega))}) \Big) \\
&\quad + \sum_{i=1}^{m} \|a_i - a_{i,N,h}\|_{L^1(I,L^\infty(\Omega))} (\|\Pi^p_{N,h}\|^2_{L^2(I,L^2(\Omega))} + \|p^{ob}_{i,N,h}\|^2_{L^2(I,L^2(\Omega))}) \\
&\quad + \frac{\lambda}{2}\big|\|\nu\|^2_{L^2(\Omega)} - \|\nu_h\|^2_{L^2(\Omega)}\big| \\
&\underbrace{\to}_{(5.88),(5.90),(5.91)} \quad 0 \quad \text{as } h \to 0.
\end{aligned}
$$

□

Theorem 5.12. *Let Assumption 5.1 hold and for every $h > 0$, let $\frac{1}{\tau} = N = N(h) \in \mathbb{N}$ satisfy the CFL-condition* (5.28)*. Furthermore, let $\overline{\nu} \in \mathcal{V}_{ad}$ be a locally optimal solution to* (P) *such that the following quadratic growth condition holds:*

$$
\exists\, \sigma, \delta > 0 : \quad J(\nu) \ge J(\overline{\nu}) + \delta\|\nu - \overline{\nu}\|^2_{L^2(\Omega)} \quad \forall \nu \in \mathcal{V}_{ad} \quad \textit{with} \quad \|\nu - \overline{\nu}\|^2_{L^2(\Omega)} \le \sigma. \tag{5.92}
$$

Then, there exists a sequence $\{\overline{\nu}_h\}_{h>0}$ such that

$$
\nu_h \in \mathcal{V}^h_{ad} \quad \forall h > 0 \quad \textit{and} \quad \overline{\nu}_h \to \overline{\nu} \quad \textit{in } L^2(\Omega) \textit{ as } h \to 0.
$$

Furthermore, $\overline{\nu}_h \in \mathcal{V}^h_{ad}$ is a locally optimal solution to (P_h) *for all sufficiently small $h > 0$.*

Remark 5.13. Note that the quadratic growth condition (5.92) is reasonable since it can be obtained by assuming suitable regularity and compatibility conditions and a sufficient second-order optimality condition (see Theorem 3.22).

Proof. For $\mathcal{V}^{h,\sigma}_{ad} \coloneqq \{\nu_h \in \mathcal{V}^h_{ad} \mid \|\nu_h - \overline{\nu}\|_{L^2(\Omega)} \le \sigma\}$, we consider the minimization problem

$$
\min_{\nu_h \in \mathcal{V}^{h,\sigma}_{ad}} J_h(\nu_h), \tag{P^σ_h}
$$

where the discrete reduced cost functional J_h is given as in (P_h). As in the proof of Theorem 5.9, there exists a sequence $\{\tilde{\nu}_h\}_{h>0}$ with $\tilde{\nu}_h \in \mathcal{V}^h_{ad}$ for all $h > 0$ such that $\tilde{\nu}_h \to \overline{\nu}$ in $L^2(\Omega)$ as $h \to 0$. Consequently, there exists $\overline{h} > 0$ such that $\tilde{\nu}_h \in \mathcal{V}^{h,\sigma}_{ad}$ for all $h \in (0, \overline{h}]$. In particular, $\mathcal{V}^{h,\sigma}_{ad}$ is non-empty for all $h \in (0, \overline{h}]$. Since $\mathcal{V}^{h,\sigma}_{ad} \subset \mathrm{DG}^h_0$ is compact and $J_h\colon L^2(\Omega) \supset \mathcal{V}^{h,\sigma}_{ad} \to \mathbb{R}$ is continuous (see the proof of Theorem 5.3), (P^σ_h) admits at least one minimizer $\overline{\nu}_h \in \mathcal{V}_{ad}^{h,\sigma}$ due to the Weierstrass theorem for all $h \in (0, \overline{h}]$. Then, by (5.92), we obtain for all $h \in (0, \overline{h}]$ that

$$\delta \|\overline{\nu}_h - \overline{\nu}\|_{L^2(\Omega)}^2 \le J(\overline{\nu}_h) - J(\overline{\nu}) = [J(\overline{\nu}_h) - J_h(\overline{\nu}_h)] + [J_h(\overline{\nu}_h) - J_h(\tilde{\nu}_h)] + [J_h(\tilde{\nu}_h) - J(\overline{\nu})]. \quad (5.93)$$

The first and the third summand at the right-hand side of (5.93) converge to zero due to Lemma 5.11 as $h \to 0$. The second summand in (5.93) is non-positive since $\overline{\nu}_h \in \mathcal{V}_{ad}^{h,\sigma}$ is a minimizer of (P_h^σ) and $\tilde{\nu}_h \in \mathcal{V}_{ad}^{h,\sigma}$ for all $h \in (0, \overline{h}]$. Therefore, (5.93) implies that

$$\overline{\nu}_h \to \overline{\nu} \quad \text{strongly in } L^2(\Omega) \text{ as } h \to 0. \quad (5.94)$$

It remains to show that $\overline{\nu}_h \in \mathcal{V}_{ad}^h$ is a minimizer to (P_h). For this purpose, let $\nu_h \in \mathcal{V}_{ad}^h$ with $\|\nu_h - \overline{\nu}_h\|_{L^2(\Omega)} \le \frac{\sigma}{2}$ be arbitrarily given. According to (5.94), there exists $\widehat{h} > 0$ such that $\|\overline{\nu}_h - \overline{\nu}\|_{L^2(\Omega)} \le \frac{\sigma}{2}$ for all $h \in (0, \min\{\overline{h}, \widehat{h}\}]$. Therefore, it follows that

$$\|\nu_h - \overline{\nu}\|_{L^2(\Omega)} \le \|\nu_h - \overline{\nu}_h\|_{L^2(\Omega)} + \|\overline{\nu}_h - \overline{\nu}\|_{L^2(\Omega)} \le \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma \quad \forall h \in (0, \min\{\overline{h}, \widehat{h}\}].$$

In other words, $\nu_h \in \mathcal{V}_{ad}^{h,\sigma}$ for all $h \in (0, \min\{\overline{h}, \widehat{h}\}]$. Since $\overline{\nu}_h$ is a minimizer to (P_h^σ), it follows that $J_h(\overline{\nu}_h) \le J_h(\nu_h)$ for all $\nu_h \in \mathcal{V}_{ad}^h$ with $\|\nu_h - \overline{\nu}_h\|_{L^2(\Omega)} \le \frac{\sigma}{2}$. In other words, $\overline{\nu}_h$ is a local minimizer to (P_h) for all $h \in (0, \min\{\overline{h}, \widehat{h}\}]$. □

6 IMPLEMENTATION

6.1 Projected Gradient Method for SQP Subproblems

In this section, we present how the iteration step ($\mathbb{P}_k$) of the discussed SQP method (see Algorithm 1 in Chapter 4) can be solved effectively using the projected gradient method. The method is chosen primarily due to its straightforward implementation and popularity for solving PDE-constrained optimization problems. For its later computational implementation, we want to make use of the investigated discretization strategy in Chapter 5. Since it is based on the auxiliary first-order formulation of the wave equation (see (5.8)), we begin with establishing the first-order formulation of the iteration step ($\mathbb{P}_k$): Let Assumption 4.9 and (SSC$^{\mathcal{T}}$) be satisfied and suppose that the k-th iterate (ν_k, p_k, q_k) is given. Then, the next iterate $(\nu_{k+1}, p_{k+1}, q_{k+1})$, along with the auxiliary variables $\boldsymbol{u}_{k+1}$ and $\boldsymbol{v}_{k+1}$, solves the coupled system consisting of the state equation

$$\begin{cases} \nu_k \partial_t p + \operatorname{div} \boldsymbol{u} + \eta p = F - (\nu - \nu_k)\partial_t p_k & \text{in } I \times \Omega \\ \partial_t \boldsymbol{u} + \nabla p = 0 & \text{in } I \times \Omega \\ p = 0 & \text{on } I \times \Gamma_D \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (p, \boldsymbol{u})(0) = (0, \boldsymbol{0}) & \text{in } \Omega, \end{cases} \tag{6.1}$$

the adjoint equation

$$\begin{cases} \nu_k \partial_t q + \operatorname{div} \boldsymbol{v} - \eta q = \sum_{i=1}^{m} a_i (p - p_i^{ob}) - (\nu - \nu_k)\partial_t q_k & \text{in } I \times \Omega \\ \partial_t \boldsymbol{v} + \nabla q = 0 & \text{in } I \times \Omega \\ q = 0 & \text{on } I \times \Gamma_D \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on } I \times \Gamma_N \\ (q, \boldsymbol{v})(T) = (0, \boldsymbol{0}) & \text{in } \Omega, \end{cases} \tag{6.2}$$

and the variational inequality

$$\left(\int_I \partial_t p_k(t) q(t) + \partial_t (p(t) - p_k(t)) q_k(t) \, \mathrm{d}t + \lambda \nu, \tilde{\nu} - \nu \right)_{L^2(\Omega)} \geq 0 \quad \forall \tilde{\nu} \in \mathcal{V}_{ad}. \tag{6.3}$$

By the substitution $q = \partial_t \hat{q}$, where $\hat{q}$ denotes the adjoint variable in the iteration step ($\mathbb{P}_k$) of Algorithm 1, we note that ($\mathbb{P}_k$) is equivalent to (6.1)-(6.3). Furthermore, (6.1)-(6.3) serves as a first-order necessary optimality condition to the optimal control problem

$$\begin{cases} \min \mathcal{J}_k(\nu,p) \coloneqq D_{(\nu,p)}\mathcal{J}(\nu_k,p_k)(\nu-\nu_k,p-p_k) + \frac{1}{2}D^2_{(\nu,p)}\mathcal{L}(\nu_k,p_k,\boldsymbol{u}_k,q_k,\boldsymbol{v}_k)(\nu-\nu_k,p-p_k)^2 \\ \text{s.t. } \nu \in \mathcal{V}_{ad} \text{ and } p \text{ solves (6.1).} \end{cases} \tag{6.4}$$

where $\mathcal{J}$ and $\mathcal{L}$ are given as in (1.2) and (3.101). Then, the cost functional reads as

$$\begin{aligned} \mathcal{J}_k(\nu,p) = \sum_{i=1}^m (a_i(p-p_k), p_k - p_i^{ob})_{L^2(I,L^2(\Omega))} + \frac{1}{2}\sum_{i=1}^m (a_i(p-p_k), p-p_k)_{L^2(I,L^2(\Omega))} \\ + \lambda(\nu_k, \nu-\nu_k)_{L^2(\Omega)} + \frac{\lambda}{2}\|\nu-\nu_k\|^2_{L^2(\Omega)} + ((\nu-\nu_k)\partial_t(p-p_k), q_k)_{L^2(I,L^2(\Omega))}. \end{aligned}$$

Incorporating the solution operator $S_p\colon L^2(\Omega) \to C^1(I, L^2(\Omega)) \cap C(I, H^1_D(\Omega))$, that maps every $\nu \in L^2(\Omega)$ to first component p of the unique solution to (6.1), the reduced formulation to (6.4) is given by $\min_{\nu\in\mathcal{V}_{ad}} J_k(\nu) \coloneqq \mathcal{J}_k(\nu, S_p(\nu))$. Then, the system (6.1)-(6.3) is equivalent to the condition $J_k'(\overline{\nu})(\nu-\overline{\nu}) \geq 0$ for all $\nu \in \mathcal{V}_{ad}$. The idea of the projected gradient method is to choose the negative gradient at the previous iterate as the descent direction and project the result into the admissible set. In our case, because of (6.1)-(6.3), for $l \in \mathbb{N}$, we write

$$h^l = \int_I \partial_t p_k(t) q^l(t) + \partial_t(p^l(t) - p_k(t))q_k(t)\,\mathrm{d}t + \lambda\nu^l. \tag{6.5}$$

Here, p^l denotes the solution to (6.1) associated with ν^l, and q^l denotes the solution to (6.2) associated with ν^l and p^l. One challenge comes with the choice of a suitable step size s. Unfortunately, one generally does not expect to find the optimal step size since the composition of the reduced cost function $\mathcal{J}_k$ involving the orthogonal projection $\mathbb{P}_{\mathcal{V}_{ad}}$ is no longer quadratic. Therefore, an explicit formula for the optimal step size is not available. Instead, various strategies exist to approximate the optimal step size or at least to find a step size that guarantees a descent for the objective function. The most prominent one is the *bisection method*: After choosing an initial step size s, one reduces the value until a desired abort criterion is fulfilled. In order to prevent the step size from becoming too small in the course of multiple iterations, we additionally increase the step length s again after a successful iteration step. Then, applied to the SQP subproblems, along with the step size strategy, the projected gradient method reads as follows:

Algorithm 2 Projected Gradient Method for k-th SQP Iteration

1: Choose ν^0 and $s > 0$. Set $l = 0$.
2: Compute the solution p^l to (6.1) associated with $\nu = \nu^l$.
3: Compute the solution q^l to (6.2) associated with $\nu = \nu^l$ and $p = p^l$.
4: Compute h^l by (6.5).
5: **while** $J_k(\mathbb{P}_{\mathcal{V}_{ad}}(\nu^l + sh^l)) \geq J_k(\nu^l)$ **do**
6: Set $s = \frac{s}{2}$.
7: **end while**
8: Set $\nu^{l+1} \coloneqq \mathbb{P}_{\mathcal{V}_{ad}}(\nu^l + sh^l)$.
9: Set $s = \frac{3s}{2}$. Stop or set $n = n + 1$ and go back to step 2.

Suppose Algorithm 2 is terminated. Then, the last iterate (ν^l, p^l, q^l) approximates the solution $(\nu_{k+1}, p_{k+1}, q_{k+1})$ to the k-th SQP subproblem (6.4) (resp. (6.1)-(6.3)). Finally, with Algorithm 2 at hand, we arrive at the following formulation of the SQP algorithm.

Algorithm 3 Sequential Quadratic Programming

1: Choose (ν_0, p_0, q_0) and set $k = 0$.
2: Approximate the k-th solution $(\nu_{k+1}, p_{k+1}, q_{k+1})$ to the SQP subproblem (6.4) (resp. (6.1)-(6.3)) using Algorithm 2.
3: Stop or set $k = k + 1$ and go back to step 2.

6.2 Numerical Experiments

In this section, we present numerical experiments to illustrate the performance of the proposed SQP algorithm (see Algorithm 3). For our experiments, we set $\Omega \coloneqq (0,2) \times (0,1)$ with the Neumann boundary part $\Gamma_N \coloneqq (0,2) \times \{1\}$ and the Dirichlet part $\Gamma_D \coloneqq \partial\Omega \setminus \Gamma_N$. Furthermore, the damping term η in the forward model (3.30) is specified as a boundary layer suppressing reflections caused through the artificial Dirichlet boundary condition. Inside the layer, the damping term η increases in the direction towards Γ_D, whereas η vanishes outside the layer ω (also see Figure 2.1 in Section 2.1). More precisely, $\eta\colon \Omega \to \mathbb{R}$ is defined by $\eta \coloneqq \chi_{\omega_1}\eta_1 + \chi_{\omega_2}\eta_2 + \chi_{\omega_3}\eta_3$, where χ_{ω_i} denotes the characteristic function of ω_i for $i = 1, 2, 3$, and

$$
\begin{aligned}
\omega_1 &\coloneqq \{(x_1,x_2) \in \Omega : x_1 \le \gamma, x_1 \le x_2\}, & \eta_1(x_1,x_2) &\coloneqq \beta(1 - x_1/\gamma)^\rho \\
\omega_2 &\coloneqq \{(x_1,x_2) \in \Omega : x_2 \le \gamma, x_2 < x_1, x_2 \le 2 - x_1\}, & \eta_2(x_1,x_2) &\coloneqq \beta(1 - x_2/\gamma)^\rho \\
\omega_3 &\coloneqq \{(x_1,x_2) \in \Omega : 2 - \gamma \le x_1, x_2 > 2 - x_1\}, & \eta_3(x_1,x_2) &\coloneqq \beta(1 + (x_1 - 2)/\gamma)^\rho
\end{aligned}
$$

for all $(x_1, x_2) \in \Omega$. Here, $\gamma \coloneqq 1/6$ is the width of the layer, $\beta \coloneqq 100$ denotes the damping value at the boundary, and $\rho \coloneqq 2$ is the degree of growth towards the boundary. Note that the sponge layer $\omega \coloneqq \omega_1 \cup \omega_2 \cup \omega_3$ surrounds the entire Dirichlet boundary part Γ_D, i.e., $\Gamma_D \subset \overline{\omega}$. Due to the choice of η, outside the sponge layer, the wave propagation stays reasonable as the boundary layer causes no significant additional reflections. Our goal is to reconstruct the non-smooth parameter $\nu_d \in \mathcal{V}_{ad}$ by solving (P) under acoustic source signals given by Ricker wavelets. For this purpose, given characteristic functions $\chi_i\colon \Omega \to \{1, 0\}$ with respect to small subsets surrounding three different point sources $(0.5, 1), (1, 1), (1.5, 1)$, respectively, the Ricker wavelet is defined by

$$
f(t,x) \coloneqq \alpha(\chi_1 + \chi_2 + \chi_3)(x)\left(1 - 2(\sigma\pi(t - t_0))^2\right) e^{-(\sigma\pi(t-t_0))^2} \quad \forall (t,x) \in I \times \Omega,
$$

where $t_0 = 0.1$ is the time of the wavelet's peak, $\alpha = 10^5$ is a scaling factor and $\sigma = 5$ is the central frequency. Then, the source term $F(t,x) = \int_0^t f(s,x)\,\mathrm{d}s$ of the auxiliary first-order problem (3.9) is given by

$$
F(t,x) = \alpha(\chi_1 + \chi_2 + \chi_3)(x)\left((t - t_0)e^{-(\sigma\pi(t-t_0))^2} - t_0 e^{-(\sigma\pi t_0)^2}\right) \quad \forall (t,x) \in I \times \Omega.
$$

For the observed wave information, we consider the solution to the forward problem (3.30) for $\nu = \nu_d$ under the deterministic noise model

$$p^{ob} := S_p(\nu_d) + \mu.$$

Here, $S_p(\nu_d)$ represents noiseless data and $\mu \in L^2(I, L^2(\Omega))$ a random perturbation of the observed data caused through background noise. The noise level is then computed by the quotient

$$l := \frac{\|\mu\|_{L^2(I,L^2(\Omega))}}{\|S_p(\nu_d)\|_{L^2(I,L^2(\Omega))}}.$$

We record the observed wave information at $m = 30$ different receivers modeled through the weight functions $a_i := \chi_{R_i}$ and $p_i^{ob} := p^{ob}$ for all $i = 1, \ldots, m$ with the rectangular observation patches $R_i \subset \overline{\Omega}$ given by

$$R_i := \begin{cases} \left[\frac{6+8(i-1)}{64}, \frac{10+8(i-1)}{64}\right] \times \left[\frac{60}{64}, \frac{64}{64}\right] & \text{for } i = 1, \ldots, 15 \\ \left[\frac{6+8(i-16)}{64}, \frac{10+8(i-16)}{64}\right] \times \left[\frac{24}{64}, \frac{28}{64}\right] & \text{for } i = 16, \ldots, 30. \end{cases}$$

For three different choices of ν_d, we perform reconstruction experiments. For all of those experiments, under the above synthetic configurations with noise level $l \approx 2\%$, we compute numerical solutions to (P) perform 16 iterations of the above SQP method (see Algorithm 3). For the second step inside Algorithm 3, we choose a maximum of 64 iterations of the projected gradient method (see Algorithm 2). The involved PDEs are solved using the fully discrete approximation strategy from Chapter 5. For our computational implementation, we choose the programming language `Python` (version 3.6.9) and utilize the package `DOLFIN` from the open-source computing platform `FEniCS` (cf. [58, 59]) to generate the triangulation of Ω and solve the involved variational problems in the finite element space. We set the regularization parameter $\lambda := 0.001$, and the lower and upper bounds $\nu_{\min} := 1$ and $\nu_{\max} := 1.6$.

(i) For the first experiment, let us consider a piecewise constant function $\nu_d : \overline{\Omega} \to \{1, 1.4, 1.6\}$ featuring a checkerboard structure (see Figure 6.1b). We choose the initial value ν_0 (see Figure 6.1a), which is moderately far from the true solution ν_d (see Figure 6.1b). In the first test, we set $T = 1$. After 16 SQP iterations, we note that our approach manages to partially reconstruct and detect the checkerboard structure (see Figure 6.1c). For the second test, we consider a longer wave evolution process with $T = 2$. Compared with the first experiment, the longer operating time leads to a significant improvement in the reconstruction by the SQP algorithm again after 16 iterations (see Figure 6.1d). For each computed iteration ν_k, the error $\|\nu_k - \nu_d\|_{L^2(\Omega)}$ and the value of the objective function $J(\nu_k)$ is presented in Table 6.1. In the third test, we enlarge the operating time to $T = 3$, which, surprisingly, does not significantly improve the reconstruction. It turns out that the reconstruction quality is close to the one for $T = 2$, but the corresponding numerical computation for $T = 3$ is less efficient since more time steps have to be solved.

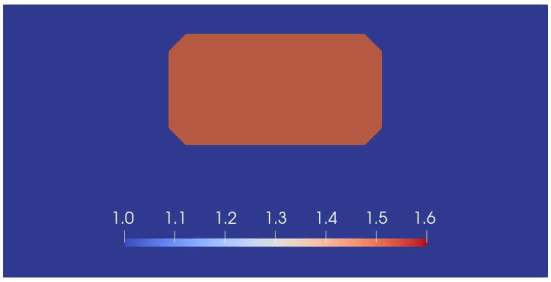
(ii) In the second numerical experiment, with $T = 2$, we aim to reconstruct a piecewise constant function ν_d featuring three distinct rectangles (see Figure 6.2a). In contrast to the first numerical experiment described in (i), we select a constant initial value $\nu_0 \equiv 1$. Notably, this choice indicates no prior information about the three rectangles featured in ν_d. However, we successfully reconstruct ν_d after 16 SQP iterations (see Figure 6.2b).

(iii) In a third experiment, we present the approach's ability to reconstruct a true wave speed parameter ν_d that does not have piecewise constant values in Ω. For this purpose, we define $\nu_d(x_1, x_1) \coloneqq \chi_\omega(x_1, x_2)\sin(x_1)$ where $\omega \coloneqq [0.6, 1.4] \times [0.4, 0.6]$. Again, we select a constant initial value of $\nu_0 \equiv 1$ and $T = 2$. As in the first two experiments, our approach reconstructs the parameter ν_d after 16 iterations of the SQP algorithm reasonably well (see Figure 6.3).

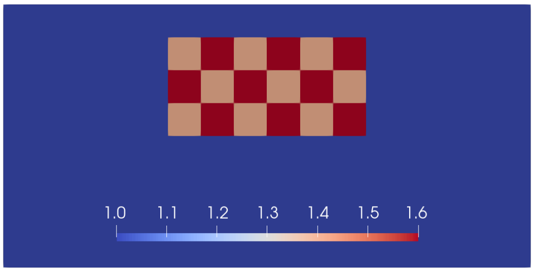
For all three experiments, (i) to (iii), Figures 6.1 to 6.3 show the effectiveness of the presented approach. However, we also want to acknowledge some limitations. Apparently, the error misfit between the iteration ν_k and ν_d diminishes comparably slowly (see Tables 6.1 to 6.3). Safely, this does not disprove the effectiveness of the SQP approach. Instead, the discrepancy between the numerical results and the analytical investigations has multiple reasons. For instance, the subproblems are solved with the projected gradient method due to its straightforward implementation, which does not fully exploit their linear quadratic nature. Furthermore, the available limited computational resources do not allow for a significantly finer discretization. Therefore, motivated by the promising results concerning the SQP algorithm (see Chapter 4) and the discretization strategy (see Chapter 5), we expect a potential for improvements in the efficiency of the implementation.

Code and Data Availability

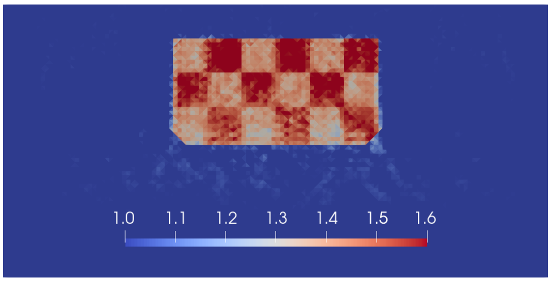
All code, datasets, and related materials generated for this thesis are available from the author upon request to ensure reproducibility and transparency.



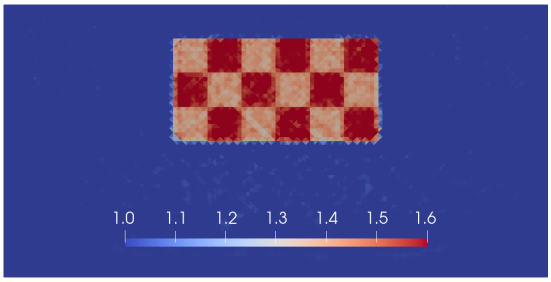
(a) Initial value ν_0.



(b) Reference solution ν_d.

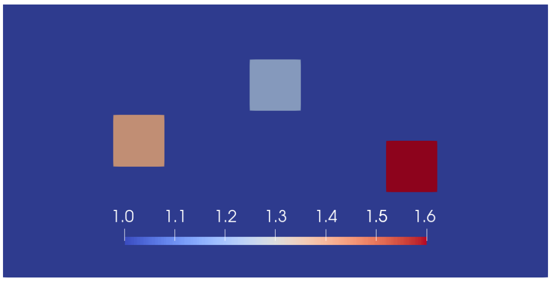


(c) Computed solution ν_{16} for $T = 1$.

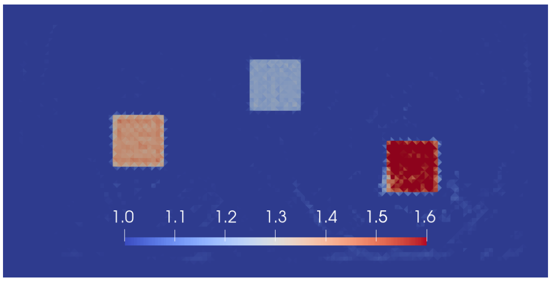


(d) Computed solution ν_{16} for $T = 2$.

Figure 6.1: Reconstruction in the numerical experiment (i).

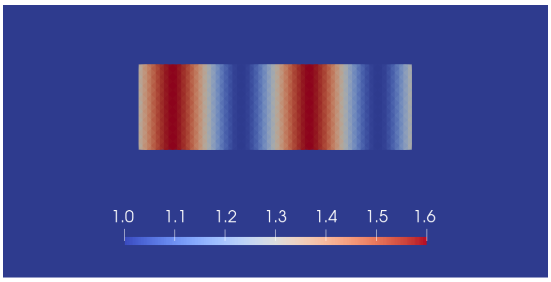


(a) Reference solution ν_d.

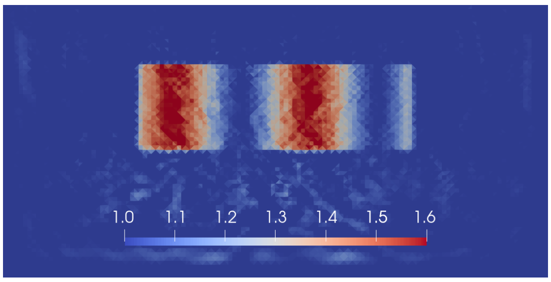


(b) Computed solution ν_{16}.

Figure 6.2: Reconstruction in the numerical experiment (ii).



(a) Reference solution ν_d.



(b) Computed solution ν_{16}.

Figure 6.3: Reconstruction in the numerical experiment (iii).

	T=1		T=2	
k	$\|\nu_k - \nu_d\|_{L^2(\Omega)}$	$J(\nu_k)$	$\|\nu_k - \nu_d\|_{L^2(\Omega)}$	$J(\nu_k)$
1	7.7749e-02	2.8236e+00	6.8861e-02	7.4472e+00
2	6.6873e-02	1.7123e-01	4.8954e-02	4.3834e-01
3	6.5276e-02	3.4345e-02	4.3366e-02	6.4354e-02
4	6.3189e-02	2.9373e-02	3.9880e-02	3.8583e-02
5	6.1432e-02	2.1429e-02	3.7219e-02	2.8104e-02
6	5.9974e-02	1.7264e-02	3.5015e-02	2.2006e-02
7	5.8942e-02	1.4594e-02	3.3164e-02	1.8019e-02
8	5.7994e-02	1.2975e-02	3.1586e-02	1.5276e-02
9	5.7184e-02	1.1715e-02	3.0159e-02	1.3232e-02
10	5.6519e-02	1.0774e-02	2.8940e-02	1.1642e-02
11	5.5929e-02	1.0080e-02	2.7812e-02	1.0457e-02
12	5.5368e-02	9.5238e-03	2.6806e-02	9.4963e-03
13	5.4819e-02	9.0373e-03	2.5906e-02	8.7174e-03
14	5.4369e-02	8.6113e-03	2.5091e-02	8.1003e-03
15	5.3925e-02	8.2743e-03	2.4328e-02	7.6122e-03
16	5.3489e-02	7.9753e-03	2.3619e-02	7.1832e-03

Table 6.1: Errors and costs of the numerical experiment (i).

k	$\|\nu_k - \nu_d\|_{L^2(\Omega)}$	$J(\nu_k)$
1	1.0369e-01	1.3650e+01
2	6.5096e-02	1.8194e+00
3	5.1108e-02	1.9452e-01
4	4.3163e-02	7.1416e-02
5	3.7802e-02	3.8529e-02
6	3.3577e-02	2.4383e-02
7	3.0404e-02	1.6841e-02
8	2.9701e-02	1.2815e-02
9	2.7382e-02	1.2361e-02
10	2.5147e-02	9.9623e-03
11	2.2798e-02	8.3619e-03
12	2.1124e-02	7.0747e-03
13	1.9442e-02	6.3241e-03
14	1.8933e-02	5.7117e-03
15	1.7704e-02	5.6036e-03
16	1.6353e-02	5.2029e-03

Table 6.2: Errors and costs of the numerical experiment (ii).

k	$\|\nu_k - \nu_d\|_{L^2(\Omega)}$	$J(\nu_k)$
1	1.5551e-01	5.8419e+01
2	1.0549e-01	1.3407e+01
3	6.9323e-02	1.4868e+00
4	5.7827e-02	1.8480e-01
5	5.1268e-02	8.9082e-02
6	4.6535e-02	5.6190e-02
7	4.2692e-02	3.9143e-02
8	3.9570e-02	2.8704e-02
9	3.7054e-02	2.2082e-02
10	3.5032e-02	1.7800e-02
11	3.3265e-02	1.4943e-02
12	3.1742e-02	1.2841e-02
13	3.0419e-02	1.1278e-02
14	2.9300e-02	1.0102e-02
15	2.8216e-02	9.2097e-03
16	2.7236e-02	8.4574e-03

Table 6.3: Errors and costs of the numerical experiment (iii).

CONCLUSION AND OUTLOOK

This thesis explores a hyperbolic PDE-constrained optimization problem (P) with applications in Full Waveform Inversion. Insights are presented into the analytical and numerical properties. We prove that the problem is well-defined and that corresponding first-order necessary and second-order sufficient optimality conditions are available. These conditions also lay the foundation for both the convergence analysis of the SQP method and the investigation of a fully discrete approximation strategy. In the final part of this thesis, we present a practical implementation for solving (P) numerically. In the numerical experiments, the algorithm effectively reconstructs the wave speed parameter, utilizing noisy wave information recorded at distinct receiving points. Notably, the numerical results demonstrate that our method is well-suited for FWI applications.

While we provide a comprehensive investigation of the analytical and numerical treatment of the optimal control problem (P), open questions are particularly arising from the practical point of view. Due to our numerical experiments, we can safely conclude that the presented SQP algorithm performs reasonably well. However, the quality of reconstruction highly depends on the choice of the size of the observation interval T, the location and size of the receivers a_i, as well as the central frequency σ of the Ricker wavelet source signals f generating the observation data p_i^{ob}. For instance, our numerical test demonstrates that a better reconstruction can be obtained by specifying a not-too-small value of T. Choosing T too large is unfavorable due to the arising computation costs. While these phenomena are observed in the numerical tests, theoretical investigations on the best choice of the parameters are still open. Furthermore, the presented implementation approach, which solves the SQP subproblems by the projected gradient method, does not fully utilize the linear-quadratic nature of these subproblems. We hypothesize that more sophisticated algorithmic techniques can be applied to solve these subproblems in order to improve efficiency and exploit the full potential of the SQP algorithm.

In conclusion, the promising analytical and numerical results, along with the numerical experiments, pave the way for future investigations into the presented approach.

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