Nonlinear Diffusion Equations with and without Memory and Stochastic Perturbation: Theory and Numerical Approximation

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Contents

Glossary

General notations

$\mathbb{N}$	natural numbers, $0 \notin \mathbb{N}$
$\mathbb{R}$	real numbers
∇u	the gradient of u
$\operatorname{div} \mathbf{u}$	divergence of $\mathbf{u}$
$\partial_t u$	the time-derivative of a function u
sign	sign function, $\operatorname{sign}(r) := 1$ if $r > 0$ and $\operatorname{sign}(r) := -1$ for $r < 0$
sign_0	sign function extended by $\operatorname{sign}_0(0) = 0$
T_K	truncation function, for $K > 0$ defined by $T_K(r) := r$ if $\lvert r\rvert \leq K$ and $T_K(r) := \operatorname{sign}(r)K$ for $\lvert r\rvert > K$
$\rightharpoonup$	weak convergence
$\hookrightarrow$	continuous embedding
$\operatorname{supp} u$	support of a function u
Δu	Laplace operator, $\Delta u := \operatorname{div}(\nabla u)$
$\mathbb{1}$	characteristic function

For a set D in $\mathbb{R}^d$ with $d \in \mathbb{N}$ and a separable Banach space X

∂D	boundary of D
$C(D; X)$	space of continuous functions $\varphi : D \to X$
$C(D)$	$C(D; \mathbb{R})$
$C^1(D)$	space of continuously differentiable functions $\varphi : D \to \mathbb{R}$
$C_c^\infty(D)$	space of infinite times continuously differentiable functions $\varphi : D \to \mathbb{R}$ with compact support
$\mathcal{D}(D)$	$C_c^\infty(D)$

For a bounded domain $D \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$, a time interval $(0, T)$ with $T > 0$, a separable Banach space X, $1 \leq p < \infty$, $m \in \mathbb{N},\ m \geq 2$, and $\beta \in (0, 1)$

p'	$p' = \frac{p}{p-1}$ for $1 < p < \infty$ and $p' = \infty$ for $p = 1$
$L^p(D)$	$\{\varphi : D \to \mathbb{R} : \varphi$ is measurable and $\int_D \lvert\varphi\rvert^p < \infty\}$
$L^p(0,T;X)$	$\{\varphi : (0,T) \to X : \varphi$ is measurable and $\int_0^T \lVert\varphi\rVert_X^p < \infty\}$
$L^\infty(D)$	$\{\varphi : D \to \mathbb{R} : \varphi$ is measurable and there is a constant C, such that $\lvert\varphi(x)\rvert \le C$ a.e. in $D\}$
$L^\infty(0,T;X)$	$\{\varphi : (0,T) \to X : \varphi$ is measurable and there is a constant C, such that $\lVert\varphi(t)\rVert_X \le C$ a.e. in $(0,T)\}$
$W^{1,p}(D)$	$\{\varphi \in L^p(D) : \exists \psi_1, \dots, \psi_d \in L^p(D)$ s.t. $\int_D \varphi \frac{\partial \xi}{\partial x_i} = -\int_D \psi_i \xi \ \forall \xi \in C_c^\infty(D) \ \forall i = 1, \dots, d\}$
$W_0^{1,p}(D)$	closure of $C_c^1(D)$ in $W^{1,p}(D)$
$W^{-1,p'}(D)$	dual space of $W_0^{1,p}(D)$
$H^1(D)$	$W^{1,2}(D)$
$H_0^1(D)$	$W_0^{1,2}(D)$
$W^{m,p}(D)$	$\{\varphi \in W^{m-1,p}(D) : \frac{\partial \varphi}{\partial x_i} \in W^{m-1,p}(D) \ \forall i = 1, \dots, d\}$
$W_0^{m,p}(D)$	closure of $C_c^\infty(D)$ in $W^{m,p}(D)$
$W^{-m,p'}(D)$	dual space of $W_0^{m,p}(D)$
$H^m(D)$	$W^{m,2}(D)$
$H_0^m(D)$	$W_0^{m,2}(D)$
$H^{-m}(D)$	dual space of $H_0^m(D)$
$W^{1,p}(0,T;X)$	$\{\varphi \in L^p(0,T;X) : \partial_t \varphi \in L^p(0,T;X)\}$, where $\partial_t \varphi$ denotes the weak derivative of φ
$W^{\beta,2}(D)$	$\{\varphi \in L^2(D) : \int_D \int_D \frac{\lvert\varphi(x)-\varphi(y)\rvert^2}{\lvert x-y\rvert^{2\beta+d}} \, dx \, dy < \infty\}$
$W^{\beta,2}(0,T;X)$	$\{\varphi \in L^2(0,T;X) : \int_0^T \int_0^T \frac{\lVert\varphi(s)-\varphi(t)\rVert_X^2}{\lvert s-t\rvert^{2\beta+d}} \, ds \, dt < \infty\}$

Chapter 2

d	space dimension, $d \in \mathbb{N}$
Ω	bounded domain in $\mathbb{R}^d$
$(0,T)$	time interval for $T > 0$
Q_T	$Q_T := (0,T) \times \Omega$
Σ_T	$\Sigma_T := (0,T) \times \partial\Omega$
$g_1 * g_2$	convolution, $(g_1 * g_2)(t) := \int_0^t g_1(t-s) g_2(s) \, ds$ for $t \ge 0$
(x,y)	scalar product of x and y in $\mathbb{R}^d$
${}_0V$	the space of all $\varphi \in V$ that vanish at $t = 0$; for $V \subseteq W^{1,1}(0,T;X)$, X Banach space
x^+	$x^+ := \max\{x, 0\}$
x^-	$y^- := -\min\{x, 0\}$

Chapter 3

d	space dimension, $d \in \mathbb{N}$
D	bounded domain in $\mathbb{R}^d$
$(0,T)$	time interval for $T > 0$
$(\Omega, \mathcal{A}, \mathbb{P})$	probability space
$(\Omega', \mathcal{A}', \mathbb{P}')$	probability space
$(\mathcal{F}_t)_{t\in[0,T]}$	right-continuous, complete filtration on $(\Omega, \mathcal{A}, \mathbb{P})$
U	separable Hilbert space such that $U \supseteq L^2(D)$
Q	symmetric, non-negative trace class operator on U
$(W_t)_{t\in[0,T]}$	$(\mathcal{F}_t)_{t\in[0,T]}$-adapted Q-Wiener process
q	$q := \max\{2, p, 2p(p-1), p'\}$
$\mathrm{HS}(L^2(D))$	space of Hilbert-Schmidt operators from $L^2(D)$ to $L^2(D)$
$\lVert \cdot \rVert_{\mathrm{HS}}$	Hilbert-Schmidt norm on $\mathrm{HS}(L^2(D))$
$\lVert \cdot \rVert_r$	norm in $L^r(D)$ for $1 \le r \le \infty$
$\lVert \cdot \rVert_{H_0^m}$	norm in $H_0^m(D)$
$\lVert \cdot \rVert_{W_0^{m,q}}$	norm in $W_0^{m,q}(D)$
$\langle \cdot, \cdot \rangle_{L^2}$	dual pairing in $L^2(D)$
$\langle \cdot, \cdot \rangle_{q',q}$	duality bracket $\langle \cdot, \cdot \rangle_{W^{-m,q'}(D), W_0^{m,q}(D)}$
$x \cdot y$	scalar product of x and y in $\mathbb{R}^d$
$\mathbb{E}$	expectation with respect to Ω
$\mathbb{E}'$	expectation with respect to Ω'
C_E	constants arriving from continuous embeddings
$\mathcal{L}(Y)$	the law of a stochastic process $(Y_t)_t$
Tr	trace operator

Chapter 4

Λ	bounded, open, connected, and polygonal set in $\mathbb{R}^2$
$(0,T)$	time interval for $T > 0$
$(\Omega, \mathcal{A}, \mathbb{P})$	probability space
$(\Omega', \mathcal{A}', \mathbb{P}')$	probability space
$(\mathcal{F}_t)_{t\in[0,T]}$	right-continuous, complete filtration on $(\Omega, \mathcal{A}, \mathbb{P})$
$(W(t))_{t\in[0,T]}$	one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$

$\mathbf{n}$	outer unit normal vector on $\partial\Lambda$
$\mathcal{T}$	admissible finite-volume mesh of Λ
h	mesh size
d_h	number of control volumes of the mesh
$\mathcal{E}$	set of edges of the mesh
$\mathcal{E}_{\text{int}}$	set of interior edges of the mesh
$\mathcal{E}_{\text{ext}}$	set of exterior edges of the mesh
$K\|L$	edge between two neighbouring control volumes K and L
m_K	two-dimensional Lebesgue measure of a control volume K
m_σ	length of an edge σ
x_K	center point of a control volume K
$d_{K\|L}$	$\|x_K - x_L\|$, for neighbouring control volumes K and L
$\mathbf{n}_{KL}$	orthonormal vector on $K\|L$ pointing from K to L
N	integer for time discretization
Δt	$\Delta t := \frac{T}{N}$
t_n	$t_n := n\Delta t$ for $n \in \{0, \dots, N\}$
$\Delta_{n+1}W$	$\Delta_{n+1}W := W(t_{n+1}) - W(t_n)$ for $n \in \{0, \dots, N-1\}$
$H^1(\Lambda)^*$	dual space of $H^1(\Lambda)$
∇^h	discrete gradient
$\|\cdot\|_{1,h}$	discrete H^1-seminorm
$\langle\cdot,\cdot\rangle_{L^2(\Lambda)}$	inner product in $L^2(\Lambda)$
$\langle\cdot,\cdot\rangle_{H^1}$	$H^1(\Lambda)$-$H^1(\Lambda)^*$ duality bracket
$x \cdot y$	scalar product of x and y in $\mathbb{R}^2$
$[\,w\,]_{W^{\alpha,2}(\Lambda)}$	Gagliardo seminorm in $W^{\alpha,2}(\Lambda)$, for $\alpha \in (0,1)$
$\mathbb{E}$	expectation with respect to Ω
$\mathbb{E}'$	expectation with respect to Ω'
$\mathbb{P} \circ Y^{-1}$	the law of a random variable Y

Chapter 1

Introduction

1.1 General Introduction

This thesis is devoted to nonlinear diffusion equations, that are used to model physical phenomena in many fields like filtration, phase transition, heat propagation, or the dynamics of biological groups.
We will study the existence of entropy solutions for a nonlinear deterministic diffusion equation with memory as well as the existence and pathwise uniqueness of probabilistically strong solutions for a nonlinear stochastic diffusion equation with Hölder continuous noise. In addition to the theory of (stochastic) diffusion equations, we will have a look at the numerical analysis of a stochastic diffusion equation. To be precise, we will show the convergence of a finite-volume scheme for a heat equation with a nonlinear multiplicative noise.

We start by studying time-fractional porous medium type equations that are used to model dynamic processes with memory like heat conduction with memory (see [94, 103]) and diffusion of fluids in porous media with memory (see [36, 68]).
The classical porous medium equation is motivated by combining the mass conservation equation and the classical empirical Darcy law, which states that the fluid mass flow rate in a porous medium is proportional to the pore pressure gradient in the same direction. However, this classical porous medium equation arising from the empirical Darcy law does not take into account variations of the permeability of the porous medium. These can occur, for example, when the fluid reacts chemically with the medium or contains particles that obstruct some of the pores. Temperature variations can also influence the permeability. To represent decreasing permeability,

Caputo introduced a Darcy law with memory in [36, 37]. The arising porous medium equation is

$$\partial_t^\alpha u - \operatorname{div}(A\nabla u) = 0,$$

where ∂_t^α represents the fractional time-derivative in the sense of Caputo for $0 < \alpha \leq 1$ and A depends on the permeability of the medium, that may vary in time.
Instead of the fractional time-derivative in the sense of Caputo, we choose the Riemann-Liouville fractional time-derivative, that can be represented by $\partial_t^\alpha u = \partial_t(k * u)$, where k is the Riemann-Liouville kernel (see [86]). We can consider more general $\partial_t(k * u)$ for $\mathcal{PC}$-kernels k, that include the Riemann-Liouville kernel. These kernels are used in applications to model subdiffusion processes (see [88, 89]). For an introduction to $\mathcal{PC}$-kernels, we refer to [127] and the references therein.
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $d \in \mathbb{N}$, $T > 0$, and $k \in L^1(0,T)$ a kernel of type $\mathcal{PC}$. We consider the time-fractional porous medium type equation

$$\partial_t[k * (u - u_0)] - \operatorname{div}(A(t,x)\nabla\varphi(u)) = f \quad \text{in } Q_T = (0,T) \times \Omega \tag{1.1}$$

with homogeneous Dirichlet boundary conditions, where $\varphi \in C^1(\mathbb{R})$ is a strictly increasing function and $A \in L^\infty((0,T) \times \Omega)$ satisfies a coercivity property. In case of bounded data $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q_T)$, the existence of weak solutions has been shown in [126].
In Chapter 2, we consider data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$. In this case, we cannot expect weak solutions. Even in the case of elliptic and parabolic equations, existence and uniqueness of weak solutions is not necessarily given in case of L^1-data (see [23, 101, 102]). These problems carry over to the history-dependent problem (see [68, 69]). To overcome the difficulties of non-existence and non-uniqueness of weak solutions, two new solution concepts have been introduced: renormalized solutions (see, e.g., [27–30, 38, 47]) and entropy solutions.
The idea of an entropy condition was firstly formulated by Kružkov in [73] to guarantee uniqueness of solutions in the theory of conservation laws and then, Bénilan et al. introduced the notion of entropy solutions for elliptic equations in [23]. The notion has been extended to parabolic equations in [7]. In case of elliptic and parabolic equations, entropy and renormalized solutions are equivalent.
For history-dependent problems, to the best of our knowledge, only the notion of entropy solutions was extended (see [40, 69]). The reason lies in the fact that, for renormalized solutions, the integration by parts formula plays an important role. Considering the time-fractional derivation operator $Lv := \partial_t(k * v)$ for $\mathcal{PC}$-kernels k, we have a fundamental identity if

$k \in W^{1,1}(0,T)$ (see [128]), but only an inequality for kernels $k \in L^1(0,T)$ (see [68, 69]). In [69, 114], the authors prove existence and uniqueness of entropy solutions to doubly nonlinear elliptic parabolic integro-differential equations driven by a time-independent Leray-Lions operator for L^1-data by using Kružkov's method of doubling variables (see [73]) and the approach of generalized solutions introduced by Gripenberg in [65]. Since the porous medium operator in (1.1) depends on the time t, the approach of generalized solutions in the sense of Gripenberg is not applicable. Instead, we approximate the given L^1-data by L^∞-data and use the known existence result for bounded data in [126].
For information on the state of the art, we refer to Section 2.1.

In Chapter 3 and Chapter 4, we study stochastic diffusion equations, that have become of great interest in many applications in physics, biology, and climatology (see [22, 41, 98] and the references therein).
One way to take random influences into account is to add a stochastic forcing to the driving diffusion equation. We want to do this in form of a stochastic integral on the right-hand side of the equation. For an introduction to stochastic integrals, we refer to [10, 41, 84].
Let $T > 0$, $D \subset \mathbb{R}^d$ be a bounded domain, $d \in \mathbb{N}$, $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a right-continuous filtration, and $(W(t))_{t\in[0,T]}$ an adapted Q-Wiener process. Typically, stochastic evolution equations are written in an integral form. A classical example of a diffusion equation with stochastic perturbation and random initial data u_0 is, a.s. in Ω,

$$u(t) - u_0 - \int_0^t \operatorname{div} a(x, u, \nabla u)\, ds = \int_0^t \Phi\, dW(s) \quad \text{in } L^2(D),\ \forall t \in [0,T], \tag{1.2}$$

with Dirichlet boundary condition $u = 0$ on $\Omega \times (0,T) \times \partial D$, or, equivalently,

$$\begin{cases} du - \operatorname{div} a(x, u, \nabla u)\, dt = \Phi\, dW(t) & \text{in } \Omega \times (0,T) \times D \\ u(0) = u_0 & \text{in } \Omega \times D \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D \end{cases}$$

for a possibly nonlinear Carathéodory function $a : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfying the classical Leray-Lions conditions, i.e., a satisfies certain monotonicity, coercivity, and growth conditions. The stochastic integral $\int_0^t \Phi\, dW(s)$ is understood in the sense of Itô. If Φ is independent of u or any derivative of u, the noise term is called additive, otherwise, it is called multiplicative.
To show well-posedness of nonlinear stochastic partial differential equations (SPDEs), in particular of the form (1.2), the variational monotonicity method,

that was initially introduced for deterministic evolution equations by Lions in [77], has been extended to SPDEs in [24, 74, 84, 97]. Key properties to apply this method are (local) monotonicity, coercivity, and growth assumptions of the driving operator in combination with the noise term. In case of square-integrable initial data $u_0 \in L^2(\Omega; L^2(D))$ and Lipschitz continuous multiplicative noise, existence and uniqueness of variational solutions have been shown in [84]. Variational solutions, that are also called probabilistically strong or pathwise solutions, are adapted stochastic processes with respect to the initial stochastic basis and satisfy the integral equation (1.2) for all $t \in [0,T]$, in $L^2(D)$, and $\mathbb{P}$-a.s. in Ω. In contrast, for martingale solutions, the stochastic basis is not fixed in advance and becomes part of the solution. For more information and a formal definition of variational and martingale solutions to SPDEs, we refer to [31].
In Chapter 3, we study the well-posedness of the following evolution problem with a Hölder continuous multiplicative noise

$$\begin{cases} du - \operatorname{div} a(x,u,\nabla u)\,dt + f(u)\,dt = B(t,u)\,dW(t) & \text{in } \Omega \times (0,T) \times D \\ u(0,\cdot) = u_0 & \text{in } \Omega \times D \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D, \end{cases} \tag{1.3}$$

where $a : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying the usual Leray-Lions conditions, and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. The operator $B : (0,T) \times L^2(D) \to \mathrm{HS}(L^2(D))$ is assumed to be Hölder continuous but not necessarily Lipschitz continuous, where $\mathrm{HS}(L^2(D))$ denotes the space of Hilbert-Schmidt operators from $L^2(D)$ to $L^2(D)$. In this setting of a Hölder continuous noise term and a pseudomonotone operator, the well-posedness result in [84] is not applicable. For further information on the state of the art, please refer to Section 3.1.1.
To show the existence of probabilistically strong solutions to (1.3), we approximate the noise term by a Lipschitz continuous noise and add a higher order perturbation on the left-hand side to the equation. Then, the result in [84] provides the existence of a unique probabilistically strong solution to the approximated equation. To pass to the limit, we use a stochastic compactness argument based on the theorems of Prokhorov and Skorokhod (see [26, 31]). With this approach, we show the existence of a martingale solution to (1.3). Since we are able to show pathwise uniqueness of solutions to (1.3), we finally get the existence of a probabilistically strong solution by an argument of Gyöngy and Krylov (see [66]).

In the last decades, the study of numerical schemes for SPDEs has also attracted a lot of attention. An overview and a list of references are given

in [8, 44, 95]. In Chapter 4, we consider a parabolic SPDE and use a finite-volume method for the spatial discretization combined with a semi-implicit discretization in time. The finite-volume method relies on the conservation form of the partial differential equation. Integrating the balance equation on a discretization cell, called control volume, and applying Stokes formula, we obtain an integral equation for the fluxes over the boundary of the control volume. Then, the idea is to approximate the flux on the boundary of the control volumes instead of the operator itself. An important feature of the finite-volume method is the local conservativity of the fluxes, i.e., the flux is conserved from one control volume to its neighbour. This feature makes the finite-volume method quite attractive for applications in which the fluxes play an important role like in fluid mechanics, semi-conductor device simulation, and mass and heat transfer. For further information on the finite-volume method, we refer to [54, 58].
In Chapter 4, we consider a stochastic heat equation which is a special case of the stochastic diffusion equation (1.2). Let $T > 0$, $\Lambda \subset \mathbb{R}^2$ be a bounded, open, connected, and polygonal set and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration, and let $(W(t))_{t\in[0,T]}$ be a standard, one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. We consider the stochastic heat equation

$$\begin{cases} du - \Delta u\, dt = g(u)\, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda, \end{cases} \tag{1.4}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $u_0 \in L^2(0,T; H^1(\Omega))$. The finite-element method has often been used as spatial discretization for the numerical analysis of parabolic SPDEs (see, e.g, [22, 32, 64, 67] and the references therein). Instead, for the finite-volume method, there are many results known for stochastic conservation laws but only a few for parabolic SPDEs, see, e.g., [4, 13–17, 50, 51, 55, 60, 87].
Even though the stochastic heat equation (1.4) is driven by a linear operator, we do not use the semigroup approach, because we would like to consider more complicated (nonlinear) operators in the future such as the p-Laplace operator or the porous medium operator. For information on the semigroup approach for linear stochastic equations, we refer to [41]. Instead, we use the variational approach for SPDEs that has been developed in [74, 84, 97].
As already mentioned, in the framework of a finite-volume discretization, we approximate the flux instead of the operator itself. In case of the Laplace operator in (1.4), we use the two-point flux approximation (TPFA) that is derived from the Taylor expansion and an orthogonality condition that is assumed for the finite-volume mesh (see [54, 58]).

In case of a linear multiplicative noise term, i.e., $g(u) = \lambda u$ for $\lambda \in \mathbb{R}$, convergence of a finite-volume scheme with a TPFA for the Laplace operator has been shown in [17]. In case of a nonlinear multiplicative noise term, the weak convergences, that we obtain by *a priori* estimates on the approximated solutions, are not sufficient to pass to the limit in the nonlinear noise term and to identify the limit. Therefore, we apply a stochastic compactness argument based on the theorems of Prokhorov and Skorokhod (see [26, 31]) to show convergence of the scheme to a martingale solution. As in Chapter 3, we use a pathwise uniqueness argument of Gyöngy and Krylov (see [66]) to show convergence of the scheme to the unique variational solution of (1.4) with respect to the initial stochastic basis.

1.2 Outline

In Chapter 2, we prove the existence of entropy solutions to the time-fractional porous medium type equation (1.1) for data $u_0 \in L^1(\Omega)$, $f \in L^1(Q_T)$, and homogeneous Dirichlet boundary conditions.
Since the driving porous medium operator depends on the time t, the approach of generalized solutions in the sense of Gripenberg (see [65]) is not applicable. Instead, we use the existence result of weak solutions to (1.1) for bounded data $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q_T)$ in [126].
First, we show that if u is a weak solution to (1.1), then $v = \varphi(u)$ is an entropy solution to (1.1) by using the fundamental identity (see [70, Lemma 6.1, Corollary 6.1]). Additionally, we extend the contraction principle formulated in [126] for the weak solutions.
Then, we approximate the L^1-data u_0 and f by monotone converging sequences of functions $u_0^{m,n} \in L^\infty(\Omega)$ and $f^{m,n} \in L^\infty(Q_T)$. We already know that the approximated equation admits a weak solution $u_{m,n}$ and that $v_{m,n} := \varphi(u_{m,n})$ is an entropy solution to the approximated equation. By using the extended contraction principle, we obtain convergence of $u_{m,n}$ in $L^1(Q_T)$ to an element $u \in L^1(Q_T)$.
To pass to the limit in the approximated equation, we apply the fundamental identity, make use of the coercivity property of A, and use the fact that φ is increasing to take advantage of the monotone convergences of the approximations.

In Chapter 3, we prove the existence and pathwise uniqueness of probabilistically strong solutions to (1.3). Therefore, we use an integral representation of the Hilbert-Schmidt operator B (see [85, 124]) in form of

$$B(t,v)\varphi(x) = \sigma(t,v(x)) \int_D k(x,y)\varphi(y)\,dy$$

for $(t,v) \in (0,T)\times L^2(D)$, $\varphi \in L^2(D)$, and a.e. $x \in D$, where $\sigma : (0,T)\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is Hölder continuous with respect to the second variable, and $k \in L^2(D \times D)$ is a symmetric kernel. We approximate σ by a Lipschitz continuous function σ_n using inf-convolution and define B_n by using the integral form of Hilbert-Schmidt operators with σ_n instead of σ. Doing so, we receive a Lipschitz continuous multiplicative noise.
Additionally, we add a higher-order operator to the equation in order to apply the existence and uniqueness result on probabilistically strong solutions in [84] to the approximated equation.
To pass to the limit, we use the stochastic compactness argument based on Prokhorov's and Skorokhod's theorems. Thereby, we get a martingale solution to (1.3). Since we can show pathwise uniqueness, we obtain the existence of a probabilistically strong solution to (1.3).

In Chapter 4, we propose a discretization scheme for (1.4) that is semi-implicit in time and uses a finite-volume scheme in space, or, to be more precise, we use the two-point flux approximation for the Laplace operator.
We start by deriving some stability estimates that provide weak convergence of the finite-volume approximations. However, the weak convergence is not sufficient to pass to the limit and, in particular, to identify the limit in the nonlinear noise term. Therefore, we apply a stochastic compactness argument.
By the theorem of Prokhorov, we get convergence in law (up to subsequences) of our finite-volume approximations. At the cost of changing the probability space, Skorokhod's representation theorem provides almost sure convergence of the proposed finite-volume scheme.
We are then able to pass to the limit in the stochastic integral and to identify the limit by using a martingale identification argument. This allows us, to show convergence of our finite-volume scheme to a martingale solution of (1.4), i.e., the stochastic basis is not fixed but enters an unknown in the equation.
However, as in Chapter 3, we are able to show pathwise uniqueness of solutions to (1.4). This, together with a classical argument of Gyöngy and Krylov (see [66]), allows us to deduce convergence in probability of the scheme with respect to the initial stochastic basis.

In Chapter 5, we present briefly three ideas for future works in the field of nonlinear (stochastic) diffusion equations.
Firstly, we present an idea for a time-fractional obstacle problem. The existence and uniqueness of solutions have been studied in the elliptic and parabolic case (see [6, 125]) but, to the best of our knowledge, not in the history-dependent case. One may combine techniques used for time-fractional problems and for obstacle problems to show existence of entropy solutions.
Secondly, we have a look on a stochastic Allen-Cahn equation. In case of a driving linear Laplace operator, well-posedness has been shown in [12]. Combining the used monotonicity arguments and the techniques used in [112], we propose to choose a more general p-Laplace operator.
Thirdly, we propose to study a finite-volume scheme for a parabolic p-Laplace or porous medium type equation with stochastic perturbation. This idea arises from the fact that we do not use the semigroup approach in Chapter 4.

Some of the results presented in this thesis have already led to publications in scientific journals and to a preprint, see [18, 111, 113].

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Chapter 2

Entropy Solutions for Time-Fractional Porous Medium Type Equations

2.1 Introduction, State of the Art, and Outline

We consider for $T > 0$ and a bounded domain $\Omega \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$ the problem

$$\begin{cases} \partial_t[k * (u - u_0)] - \operatorname{div}(A(t,x)\nabla\varphi(u)) = f & \text{in } Q_T \\ u = 0 & \text{in } \Sigma_T \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \qquad (P(u_0, f))$$

where $Q_T := (0,T) \times \Omega,\ \Sigma_T := (0,T) \times \partial\Omega$, and, for $t \in [0,T]$, we define

$$(k * v)(t) := \int_0^t k(t-\tau)v(\tau)\,d\tau.$$

We make the following assumptions:

(Hk) $k \in L^1(0,T)$ is non-negative, non-increasing, and there exists $l \in L^p(0,T)$ with $p > 1$ such that $k * l = 1$ in $(0,T)$.

(HA) $A \in L^\infty((0,T) \times \Omega; \mathbb{R}^{d\times d})$ and there exists $\nu > 0$ such that

$$(A(t,x)\xi, \xi) \geq \nu|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \text{ and a.e. } (t,x) \in Q_T.$$

(Hφ) $\varphi \in C^1(\mathbb{R})$, $\varphi'(r) \geq 0$ for all $r \in \mathbb{R}$, $\varphi(0) = 0$, φ is strictly increasing in $\mathbb{R}$, and there exist $\mu, R > 0$ such that

$$0 < \mu \leq \varphi'(r) \quad \forall r \in \mathbb{R} \text{ with } |r| > R.$$

(Hd) $u_0 \in L^1(\Omega)$, $f \in L^1(Q_T)$.

Note that kernels k satisfying (Hk) are in particular kernels of type $\mathcal{PC}$, which have been studied by many authors, see, e.g., [70,72,121,122]. Kernels of type $\mathcal{PC}$ are used in applications to model subdiffusion processes. Subdiffusion is a special case of anomalous diffusive behaviour which is in the force free-limit slower than Brownian motion. For more information, see [88, 89]. An important example is given by $(k, l) := (g_{1-\alpha}, g_\alpha)$ for $\alpha \in (0, 1)$, where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)} \quad \text{for } t > 0,\ \beta > 0.$$

In this case $\partial_t(k * v)$ represents the Riemann-Liouville fractional derivative of order α and $k * \partial_t v$ the Caputo derivative if v is sufficient smooth. Note, that in this case the condition $l \in L^p(0, T)$ is satisfied. Some further examples of kernels satisfying (Hk) are the time-fractional case with exponential weight

$$k(t) := e^{-\gamma t} g_{1-\alpha}(t), \quad l(t) := e^{-\gamma t} g_\alpha(t) + \gamma[1 * (g_\alpha e^{-\gamma \cdot})](t)$$

for $\alpha \in (0, 1)$ and $\gamma > 0$, and the ultra-slow diffusion case

$$k(t) := \int_0^1 g_\beta(t)\, d\beta, \quad l(t) := \int_0^\infty \frac{e^{-st}}{1+s}\, ds,$$

which is considered in [70–72, 76].

By assuming (Hφ), we cover degenerated time-fractional equations. For example, we can choose $\varphi(r) := |r|^{m-1} r$ for $m > 1$, so that $P(u_0, f)$ becomes a porous medium equation, which has been studied in, e.g., [1,2,48,49,100,121].

In applications, $P(u_0, f)$ appears in the modelling of dynamic processes with memory, for example, to model heat conduction with memory (see [94, 103]) and diffusion of fluids in porous media with memory [36, 68].

Existence of weak solutions to $P(u_0, f)$ and, additionally, a contraction principle for weak solutions were shown for more regular data u_0, f in [126]. In the linear case, existence and uniqueness of weak solutions were shown in [83, 122, 129]. For the porous medium operator, L^1 is a natural space guaranteeing the monotonicity property and, furthermore, from the physical point of view, L^1 is a useful space for several evolution problems, e.g., the transport of fluids in porous media, and heat conduction. In the setting of L^1-data, we cannot expect weak solutions. Therefore, we work with entropy solutions.

For the doubly-nonlinear history-dependent (degenerated) problem with a time-independent operator, existence and uniqueness of entropy solutions (also in the case of L^1-data) were shown in [69, 107, 114]. Here, the theory about generalized solutions for integro-differential equations (see [65]), using the m-accretivity of the time-independent operator, is applied. In the case of a time-dependent operator, we cannot apply this approach. Note that, even in the linear case, i.e., $\varphi = id$, to the best of our knowledge no existence results for L^1-data are known.

Note that there are several articles dealing with decay estimates for time-fractional (porous medium type) equations, see, e.g., [48, 70, 121]. Furthermore, we remark that in [82,83], the authors study time-fractional stochastic partial differential equations with additive noise, including time-fractional stochastic porous medium type equations.

This chapter is structured as follows: In Section 2.2, we consider bounded data $u_0 \in L^\infty(\Omega), f \in L^\infty(Q_T)$. In this case, existence of weak solutions was shown in [126]. We prove that a weak solution to $P(u_0, f)$ is also an entropy solution to $P(u_0, f)$ by using the fundamental identity (see [70, Lemma 6.1, Corollary 6.1]).
Afterwards, in Section 2.3 we formulate a contraction principle for weak solutions, which is a technical extension of the contraction principle formulated in [126].
In Section 2.4, we consider general data $u_0 \in L^1(\Omega), f \in L^1(Q_T)$ and approximate them by functions $u_0^{m,n} \in L^\infty(\Omega), f^{m,n} \in L^\infty(Q_T)$. We know, that there exists an entropy solution to the approximated equation $P(u_0^{m,n}, f_{m,n})$, and we can show by the contraction principle that $u_{m,n}$ converges to a function $u \in L^1(Q_T)$.
In Section 2.5, we then pass to the limit in the equation. Here, we use the coercivity condition (HA) of the operator A and, furthermore, the fact that φ is increasing to take advantage of the monotone convergences of the approximations.

2.2 Entropy Solutions in the Case of L^∞-Data

The idea is to approximate the data u_0 and f by bounded data in $L^\infty(\Omega)$ and $L^\infty(Q_T)$, respectively. By [126, Theorem 6.1], we know that $P(u_0, f)$ then admits a weak solution. We will show that any weak solution to $P(u_0, f)$ is an entropy solution.
For a space $V \subseteq W^{1,1}(0, T; X)$, where X is a Banach space, we denote by ${}_0V$

the space of all $\psi \in V$ that vanish at $t = 0$. We set

$$W_\varphi(T, u_0) := \big\{ w \in L^2(0,T;L^2(\Omega)) : k * (w - u_0) \in {}_0W^{1,1}(0,T;H^{-1}(\Omega)) \\ \text{and } \varphi(w) \in L^2(0,T;H_0^1(\Omega)) \big\}.$$

Definition 2.2.1. *Let (Hk), (HA), (Hφ), and (Hd) be satisfied. A function $u \in W_\varphi(T, u_0)$ is a weak solution to $P(u_0, f)$ if for any test function $\eta \in W^{1,1}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ with $\eta(T, \cdot) = 0$*

$$\int_0^T \int_\Omega -\eta_t [k * (u - u_0)] + \int_0^T \int_\Omega (A \nabla \varphi(u), \nabla \eta) = \int_0^T \int_\Omega f \eta.$$

Under the regularity condition

$$k * (u - u_0) \in {}_0W^{1,1}(0,T;L^1(\Omega)), \tag{2.1}$$

one can show by an approximation argument that, for a weak solution u,

$$\int_0^T \int_\Omega \eta \partial_t [k * (u - u_0)] + \int_0^T \int_\Omega (A \nabla \varphi(u), \nabla \eta) = \int_0^T \int_\Omega f \eta \tag{2.2}$$

is satisfied for all $\eta \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$, and, by a cut-off function argument, this is equivalent to

$$\int_0^{t_1} \int_\Omega \eta \partial_t [k * (u - u_0)] + \int_0^{t_1} \int_\Omega (A \nabla \varphi(u), \nabla \eta) = \int_0^{t_1} \int_\Omega f \eta \tag{2.3}$$

for all $t_1 \in (0, T]$ and all $\eta \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$.
Since $\varphi \in C^1(\mathbb{R})$ is strictly increasing and $\varphi(0) = 0$, we can define the function $b := \varphi^{-1}$, which is continuous, strictly increasing, and satisfies $b(0) = 0$. If we define $v := \varphi(u)$ and $v_0 := \varphi(u_0)$, then $P(u_0, f)$ is equivalent to

$$\begin{cases} \partial_t [k * (b(v) - b(v_0))] - \operatorname{div}(A(t,x) \nabla v) = f & \text{in } Q_T \\ v = 0 & \text{in } \Sigma_T \\ v(0, \cdot) = v_0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

We define an entropy solution to $P(u_0, f)$ based on the definition of entropy solutions for history-dependent elliptic-parabolic equations in [69]. Therefore, we set

$$\mathcal{P} := \big\{ S \in C^1(\mathbb{R}) : 0 \le S' \le 1, \ \operatorname{supp} S' \text{ compact}, \ S(0) = 0 \big\}$$

and denote for $K > 0$ by T_K the cut-off function defined on $\mathbb{R}$ by $T_K(r) := r$ if $|r| \le K$ and $T_K(r) := \operatorname{sign}(r) K$ for $|r| > K$.

Definition 2.2.2. *Let (Hk), (HA), (Hφ) and (Hd) be satisfied. A measurable function $v : Q_T \to \mathbb{R}$ is called an entropy solution to $P(u_0, f)$ if $b(v) \in L^1(Q_T), T_K(v) \in L^2(0,T; H_0^1(\Omega))$ for all $K > 0$, and*

$$-\int_{Q_T} \zeta_t \left[k_1 * \int_{v_0}^{v} S(\sigma - \phi)\, db(\sigma)\right] + \int_{Q_T} \zeta \partial_t [k_2 * (b(v) - b(v_0))] S(v - \phi)$$
$$+ \int_{Q_T} \zeta (A \nabla v, \nabla S(v - \phi)) \leq \int_{Q_T} \zeta f S(v - \phi)$$

for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\zeta \in \mathcal{D}([0,T)), \zeta \geq 0$, $S \in \mathcal{P}$, and $k_1, k_2 \in L^1(0,T)$ non-increasing and non-negative with $k = k_1 + k_2$ and $k_2(0^+) < \infty$.

In order to show that a weak solution u to $P(u_0, f)$ is also an entropy solution to $P(u_0, f)$, we will use $S(v - \phi)\zeta$ as a test function with $S \in \mathcal{P}, \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and $\zeta \in \mathcal{D}([0,T))$ with $\zeta \geq 0$. Since $S(v - \phi)\zeta$ is an element of $L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$, but not in $W^{1,1}(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, we have to assume (2.1).

Lemma 2.2.3. *Let (Hk), (HA), (Hφ), and (Hd) be satisfied. If u is a weak solution to $P(u_0, f)$ which satisfies* (2.1)*, then $v = \varphi(u)$ is an entropy solution to $P(u_0, f)$.*

Proof. Let u be a weak solution to $P(u_0, f)$ and $S \in \mathcal{P}$, $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\zeta \in \mathcal{D}([0,T))$, and $v := \varphi(u)$, $v_0 := \varphi(u_0)$. Since we assume that u satisfies (2.1), we can use $S(v - \varphi)\zeta$ as a test function in (2.2) and obtain

$$\begin{aligned} &\int_{Q_T} \zeta S(v - \phi) \partial_t [k * (b(v) - b(v_0))] + \int_{Q_T} \zeta (A \nabla v, \nabla S(v - \phi)) \\ &= \int_{Q_T} f S(v - \phi) \zeta. \end{aligned} \tag{2.5}$$

Now we choose arbitrary $k_1, k_2 \in L^1(0,T)$ non-increasing and non-negative with $k_2(0^+) < \infty$, such that $k = k_1 + k_2$. To apply the fundamental identity (see [70, Lemma 6.1, Corollary 6.1]), we have to approximate the kernel k_1 by a more regular kernel $k_{1,\lambda} \in W^{1,1}(0,T)$ for $\lambda > 0$. Therefore, we define the operator $L_1 : D(L_1) \subseteq L^1(0,T; L^1(\Omega)) \to L^1(0,T; L^1(\Omega))$ by

$$\begin{aligned} D(L_1) &:= \{w \in L^1(0,T; L^1(\Omega)) : k_1 * w \in {}_0W^{1,1}(0,T; L^1(\Omega))\} \\ L_1 w &:= \partial_t (k_1 * w). \end{aligned}$$

The operator L_1 is m-accretive (see [39, 65]). For $\lambda > 0$, let $L_{1,\lambda}$ be the Yosida approximation of L_1, then we know for each $w \in D(L_1)$

$$L_{1,\lambda} w \to L_1 w \quad \text{in } L^1(0,T; L^1(\Omega)) \text{ for } \lambda \to 0.$$

Furthermore, we know that

$$L_{1,\lambda}w = \partial_t(k_{1,\lambda} * w), \quad w \in L^1(0,T;L^1(\Omega)),\ \lambda > 0$$

for a non-negative and non-increasing kernel $k_{1,\lambda} \in W^{1,1}(0,T)$, see [120]. Note that we have $k_{1,\lambda} \to k_1$ in $L^1(0,T)$ for $\lambda \to 0$, see [120, 127]. Using the approximation kernel $k_{1,\lambda}$, we get from (2.5)

$$\begin{aligned}
&\int_{Q_T} \zeta S(v-\phi)\partial_t[k_{1,\lambda} * (b(v)-b(v_0))] + \int_{Q_T} \zeta \partial_t[k_2 * (b(v)-b(v_0))]S(v-\phi) \\
&+ \int_{Q_T} \zeta (A\nabla v, \nabla S(v-\phi)) \\
&= \int_{Q_T} fS(v-\phi)\zeta + \int_{Q_T} \zeta S(v-\phi)\partial_t[(k_{1,\lambda}-k_1) * (b(v)-b(v_0))].
\end{aligned}$$

Applying the fundamental identity (see [70, Corollary 6.1]) to the first term in the above equation, we get

$$\begin{aligned}
&-\int_{Q_T} \zeta_t \left(k_{1,\lambda} * \int_{v_0}^{v} S(\sigma-\phi)\, db(\sigma) \right) + \int_{Q_T} \zeta \partial_t[k_2 * (b(v)-b(v_0))]S(v-\phi) \\
&+ \int_{Q_T} \zeta (A\nabla v, \nabla S(v-\phi)) \\
&\le \int_{Q_T} \zeta f S(v-\phi) + \int_{Q_T} \zeta S(v-\phi)\partial_t[(k_{1,\lambda}-k_1) * (b(v)-b(v_0))].
\end{aligned}$$

Since $k_{1,\lambda} \to k_1$ in $L^1(0,T)$ and we have $b(v)-b(v_0) \in D(L_1)$ by (2.1), we obtain

$$\partial_t[k_{1,\lambda} * (b(v)-b(v_0))] \to \partial_t[k_1 * (b(v)-b(v_0))] \text{ in } L^1(Q_T) \text{ for } \lambda \to 0.$$

Therefore, by passing to the limit in the above equation, v is an entropy solution to $P(u_0,f)$. $\square$

2.3 Contraction Principle

In this section, we want to extend the contraction principle shown in [126]. It will be useful to obtain convergence of the weak solution to the approximated equation in the next section.

Lemma 2.3.1. *Let $(k,l) \in \mathcal{PC}$, (HA) be satisfied and $\varphi \in C^1(\mathbb{R})$ be a strictly increasing function in $\mathbb{R}$. For $i = 1,2$, let $u_i \in W_\varphi(u_{0,i}, f_i)$ be a* *weak solution to $P(u_{0,i}, f_i)$ in the sense of Definition 2.2.1 with $u_{0,i} \in L^1(\Omega)$ and $f = f_i \in L^1(Q_T)$, and let (2.1) be satisfied for $u = u_i$, $i = 1, 2$, such that in particular (2.3) holds true for all $\eta \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ and $u = u_i$, $i = 1, 2$. Then, we have*

$$\|u_1 - u_2\|_{L^1(Q_T)} \leq T\|u_{0,1} - u_{0,2}\|_{L^1(\Omega)} + \|l\|_{L^1(0,T)}\|f_1 - f_2\|_{L^1(Q_T)} \tag{2.6}$$

$$\int_{Q_T} (u_1 - u_2)^+ \leq T \int_\Omega (u_{0,1} - u_{0,2})^+ + \|l\|_{L^1(0,T)} \int_{Q_T} (f_1 - f_2)^+ \tag{2.7}$$

$$\int_{Q_T} (u_1 - u_2)^- \leq T \int_\Omega (u_{0,1} - u_{0,2})^- + \|l\|_{L^1(0,T)} \int_{Q_T} (f_1 - f_2)^-. \tag{2.8}$$

Proof. Inequality (2.6) was shown in [126, Theorem 7.1]. The proofs of (2.7) and (2.8) are analogous to that of Theorem 7.1 in [126] with the only difference that, in case of (2.7) we approximate $\mathbb{R} \ni y \mapsto y^+$ by $H_\varepsilon(y) := \sqrt{(y^+)^2 + \varepsilon^2} - \varepsilon$ for $\varepsilon > 0$ and, in case of (2.8), we approximate $\mathbb{R} \ni y \mapsto y^-$ by $H_\varepsilon(y) := \sqrt{(y^-)^2 + \varepsilon^2} - \varepsilon$ for $\varepsilon > 0$. □

2.4 Approximation

Let (Hk), (HA), (Hφ), and (Hd) be satisfied. For $m, n \in \mathbb{N}$, we define

$$u_0^{m,n} := \begin{cases} m, & \text{if } u_0 > m \\ u_0, & \text{if } -n \leq u_0 \leq m \\ -n, & \text{if } u_0 < -n \end{cases} \quad \text{and} \quad f^{m,n} := \begin{cases} m, & \text{if } f > m \\ f, & \text{if } -n \leq f \leq m \\ -n, & \text{if } f < -n. \end{cases}$$

By [126, Theorem 6.1], $P(u_0^{m,n}, f^{m,n})$ admits a weak solution $u_{m,n} \in W_\varphi(T, u_0^{m,n}) \cap L^\infty(Q_T)$ for any $m, n \in \mathbb{N}$.

Lemma 2.4.1. *Let $u_{m,n} \in W_\varphi(T, u_0^{m,n}) \cap L^\infty(Q_T)$ be a weak solution to $P(u_0^{m,n}, f^{m,n})$ for any $m, n \in \mathbb{N}$. For fixed $n \in \mathbb{N}$, there exists an element $u_{\infty,n} \in L^1(Q_T)$ such that*

$$u_{m,n} \to u_{\infty,n} \text{ a.e. in } Q_T \text{ for } m \to \infty.$$

Moreover, there exists a function $u \in L^1(Q_T)$ such that

$$u_{\infty,n} \to u \text{ a.e. in } L^1(Q_T) \text{ for } n \to \infty.$$

Proof. Using (2.7) and (2.8), we know that for all $m, n \in \mathbb{N}$, a.e. in Q_T

$$u_{m,n} \leq u_{m+1,n} \quad \text{and} \quad u_{m,n} \geq u_{m,n+1}. \tag{2.9}$$

From inequality (2.6), we further obtain

$$\begin{aligned}\sup_{m,n\in\mathbb{N}} \|u_{m,n}\|_{L^1(Q_T)} &\le \sup_{m,n\in\mathbb{N}} \big(T\|u_0^{m,n}\|_{L^1(\Omega)} + \|l\|_{L^1(0,T)}\|f^{m,n}\|_{L^1(Q_T)}\big) \\ &\le T\|u_0\|_{L^1(\Omega)} + \|l\|_{L^1(0,T)}\|f\|_{L^1(Q_T)}.\end{aligned}$$

As a consequence, we know that the increasing sequence $(u_{m,n})_{m\in\mathbb{N}}$, for fixed $n \in \mathbb{N}$, converges a.e. in Q_T towards an element $u_{\infty,n}$ for $m \to \infty$. From (2.9) it follows $u_{\infty,n} \ge u_{\infty,n+1}$ for all $n \in \mathbb{N}$ and, therefore, we obtain, by the same argumentation, that $u_{\infty,n}$ converges a.e. in Q_T for $n \to \infty$ towards an element u. Using (2.6) and Fatou's Lemma, we get for any $n \in \mathbb{N}$

$$\begin{aligned}\int_{Q_T} |u_{\infty,n}| &\le \liminf_{m\to\infty} \int_{Q_T} |u_{m,n}| \\ &\le \liminf_{m\to\infty} \big(T\|u_0^{m,n}\|_{L^1(\Omega)} + \|l\|_{L^1(0,T)}\|f^{m,n}\|_{L^1(Q_T)}\big) \\ &\le T\|u_0\|_{L^1(\Omega)} + \|l\|_{L^1(0,T)}\|f\|_{L^1(Q_T)}.\end{aligned}$$

Consequently, we know

$$\|u\|_{L^1(Q_T)} \le \liminf_{n\to\infty} \int_{Q_T} |u_{\infty,n}| \le T\|u_0\|_{L^1(\Omega)} + \|l\|_{L^1(0,T)}\|f\|_{L^1(Q_T)}, \quad (2.10)$$

which implies $u_{\infty,n} \in L^1(Q_T)$ and $u \in L^1(Q_T)$. □

Lemma 2.4.2. *There exists, for any $n \in \mathbb{N}$, a function $g^n \in L^1(Q_T)$ and, moreover, there exists a function $g \in L^1(Q_T)$ which is independent of n, such that a.e. in Q_T*

$$|u_{m,n}| \le g^n \quad \forall m, n \in \mathbb{N} \qquad \textit{and} \qquad |u_{\infty,n}| \le g \quad \forall n \in \mathbb{N}.$$

Proof. First let $n \in \mathbb{N}$ be fixed. Since $(u_{m,n})_{m\in\mathbb{N}}$ is an increasing function which converges to $u_{\infty,n}$ a.e. in Q_T, we know that $u_{m,n} \le u_{\infty,n}$ for all $m \in \mathbb{N}$ and a.e. in Q_T. In particular, we know $(u_{m,n})^+ \le (u_{\infty,n})^+$ for all $m \in \mathbb{N}$ and a.e. in Q_T. Additionally, we have for all $m \in \mathbb{N}$, a.e. in Q_T,

$$(u_{m,n})^- = \max\{0, -u_{m,n}\} \le \max\{0, -u_{n,1}\} = (u_{n,1})^-.$$

Consequently, we know for all $m \in \mathbb{N}$, a.e. in Q_T,

$$|u_{m,n}| = (u_{m,n})^+ + (u_{m,n})^- \le |u_{\infty,n}| + |u_{1,n}|.$$

Analogously, we obtain for arbitrary $n \in \mathbb{N}$, a.e. in Q_T,

$$|u_{\infty,n}| = (u_{\infty,n})^+ + (u_{\infty,n})^- \le |u| + |u_{\infty,1}|.$$

□

Lemma 2.4.3. *For fixed $n \in \mathbb{N}$, we have*

$$u_{m,n} \to u_{\infty,n} \text{ in } L^1(Q_T) \text{ for } m \to \infty.$$

Furthermore,

$$u_{\infty,n} \to u \text{ in } L^1(Q_T) \text{ for } n \to \infty.$$

Proof. The convergences in $L^1(Q_T)$ follow by Lebesgue's dominated convergence theorem from the almost sure convergences stated in Lemma 2.4.1 and the bounds in Lemma 2.4.2. □

2.5 Passage to the Limit

Let (Hk), (HA), (Hφ), and (Hd) be satisfied and $u_{m,n}$ for $m, n \in \mathbb{N}$ the weak solution to $P(u_0^{m,n}, f^{m,n})$ as defined in Section 2.4, such that

$$k * (u_{m,n} - u_0^{m,n}) \in {}_0W^{1,1}(0, T; L^1(\Omega)) \tag{2.11}$$

is satisfied. By Lemma 2.2.3, we know that $v_{m,n} := \varphi(u_{m,n})$ is an entropy solution to $P(u_0^{m,n}, f^{m,n})$. Since φ is continuous, we know by Lemma 2.4.1 that

$$v_{m,n} = \varphi(u_{m,n}) \overset{m\to\infty}{\longrightarrow} \varphi(u_{\infty,n}) =: v_{\infty,n} \overset{n\to\infty}{\longrightarrow} \varphi(u) =: v \quad \text{a.e. in } Q_T.$$

Analogously, we get the convergences

$$T_K(v_{m,n}) \overset{m\to\infty}{\longrightarrow} T_K(v_{\infty,n}) \overset{n\to\infty}{\longrightarrow} T_K(v) \text{ a.e. in } Q_T,\ \forall K > 0. \tag{2.12}$$

Lemma 2.5.1. *For all $K > 0$ and $n \in \mathbb{N}$, we have*

$$T_K(v_{m,n}) \rightharpoonup T_K(v_{\infty,n}) \text{ in } L^2(0, T; H_0^1(\Omega)) \text{ for } m \to \infty$$

and, for all $K > 0$, we have

$$T_K(v_{\infty,n}) \rightharpoonup T_K(v) \text{ in } L^2(0, T; H_0^1(\Omega)) \text{ for } n \to \infty.$$

Proof. We fix $K > 0$. Obviously, we have

$$\|T_K(v_{m,n})\|^2_{L^2(Q_T)} \le T|\Omega|K^2 \quad \forall m, n \in \mathbb{N}.$$

Hence, we know by (2.12) that there exist (not relabelled) subsequences of $(T_K(v_{m,n}))_{m\in\mathbb{N}}$ and $(T_K(v_{\infty,n}))_{n\in\mathbb{N}}$ such that

$$\begin{aligned} &T_K(v_{m,n}) \rightharpoonup T_K(v_{\infty,n}) && \text{in } L^2(0, T; L^2(\Omega)) \text{ for } m \to \infty. \\ \text{and} \quad &T_K(v_{\infty,n}) \rightharpoonup T_K(v) && \text{in } L^2(0, T; L^2(\Omega)) \text{ for } n \to \infty. \end{aligned} \tag{2.13}$$

Now, we fix $m, n \in \mathbb{N}$. Since $u_{m,n} = b(v_{m,n})$ is a weak solution to $P(u_0^{m,n}, f^{m,n})$ and (2.11) is satisfied, we can use $T_K(v_{m,n})$ as a test function to get

$$\int_{Q_T} T_K(v_{m,n})\partial_t[k * (b(v_{m,n}) - b(v_0^{m,n}))] + \int_{Q_T} (A\nabla v_{m,n}, \nabla T_K(v_{m,n}))$$
$$= \int_{Q_T} f^{m,n} T_K(v_{m,n}).$$

For $\lambda > 0$, let k_λ be the kernel associated to the Yosida approximation of the operator

$$D(L) := \left\{ w \in L^1(0,T; L^1(\Omega)) : k * w \in {}_0W^{1,1}(0,T; L^1(\Omega)) \right\},$$
$$Lw := \partial_t(k * w).$$

By using (HA), we obtain

$$\begin{aligned}
&\int_{Q_T} T_K(v_{m,n})\partial_t[k_\lambda * (b(v_{m,n}) - b(v_0^{m,n}))] + \nu \int_{Q_T} |\nabla T_K(v_{m,n})|^2 \\
&\le \int_{Q_T} f^{m,n} T_K(v_{m,n}) + \int_{Q_T} T_K(v_{m,n})\partial_t[(k_\lambda - k) * (b(v_{m,n}) - b(v_0^{m,n}))].
\end{aligned} \tag{2.14}$$

Due to (2.11), we have $b(v_{m,n}) - b(v_0^{m,n}) \in D(L)$ and, therefore, we know that

$$\int_{Q_T} T_K(v_{m,n})\partial_t[(k_\lambda - k) * (b(v_{m,n}) - b(v_0^{m,n}))] \to 0 \quad \text{for } \lambda \to 0.$$

The fundamental identity provides

$$\begin{aligned}
\int_{Q_T} T_K(v_{m,n})\partial_t[k_\lambda * (b(v_{m,n}) - b(v_0^{m,n}))] &\ge \int_{Q_T} \partial_t \left[k_\lambda * \int_{v_0^{m,n}}^{v_{m,n}} T_K(\sigma)\, db(\sigma) \right] \\
&= \int_\Omega \left[k_\lambda * \int_{v_0^{m,n}}^{v_{m,n}} T_K(\sigma)\, db(\sigma) \right](T).
\end{aligned}$$

Letting $\lambda \to 0$ in (2.14), we obtain, since $k_\lambda \to k$ in $L^1(0,T)$,

$$\int_\Omega \left[k * \int_{v_0^{m,n}}^{v_{m,n}} T_K(\sigma)\, db(\sigma) \right](T) + \nu \int_{Q_T} |\nabla T_K(v_{m,n})|^2 \le \int_{Q_T} f^{m,n} T_K(v_{m,n}).$$

Since k is non-negative and b non-decreasing, we know that

$$\int_\Omega \left[k * \int_{v_0^{m,n}}^{v_{m,n}} T_K(\sigma)\, db(\sigma) \right](T) \ge 0.$$

Using the fact that $|f^{m,n}| \le |f|$ a.e. in Q_T and $\nu > 0$, we get

$$\|\nabla T_K(v_{m,n})\|^2_{L^2(Q_T)} \le \frac{K}{\nu}\|f\|_{L^1(Q_T)}.$$

Together with (2.12) and (2.13) we conclude that $T_K(v_{m,n}) \rightharpoonup T_K(v_{\infty,n})$ in $L^2(0,T;H^1_0(\Omega))$ for $m \to \infty$ and $T_K(v_{\infty,n}) \rightharpoonup T_K(v)$ in $L^2(0,T;H^1_0(\Omega))$ for $n \to \infty$. □

Theorem 2.5.2. *Let (Hk), (HA), (Hφ), and (Hd) be satisfied. For any $m,n \in \mathbb{N}$, let $u_{m,n}$ be a weak solution to $P(u_{m,n}, f^{m,n})$ such that* (2.11) *holds. Then $v := \lim_{m,n\to\infty} \varphi(u_{m,n})$ is an entropy solution to $P(u_0, f)$.*

Proof. Let $m,n \in \mathbb{N}$. Since $u_{m,n}$ is a weak solution to $P(u_0^{m,n}, f^{m,n})$, we know by Lemma 2.2.3 that $v_{m,n} = \varphi(u_{m,n})$ is an entropy solution to $P(u_0^{m,n}, f^{m,n})$. Therefore, for any $S \in \mathcal{P}$, $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, and $\zeta \in \mathcal{D}([0,T)), \zeta \ge 0$, we have

$$\begin{aligned}
&-\int_{Q_T} \zeta_t \left[k_1 * \int_{v_0^{m,n}}^{v_{m,n}} S(\sigma - \phi)\, db(\sigma)\right]\\
&+\int_{Q_T} \zeta\partial_t[k_2 * (b(v_{m,n}) - b(v_0^{m,n}))]S(v_{m,n} - \phi) \qquad\qquad (2.15)\\
&+\int_{Q_T} \zeta(A\nabla v_{m,n}, \nabla S(v_{m,n} - \phi)) \le \int_{Q_T} \zeta f^{m,n} S(v_{m,n} - \phi),
\end{aligned}$$

where $k_1, k_2 \in L^1(0,T)$ are non-increasing and non-negative with $k = k_1 + k_2$ and $k_2(0^+) < \infty$.
Since $\operatorname{supp} S'$ is compact, there exists a constant $L \ge 0$ such that $\operatorname{supp} S' \subseteq [-L, L]$ and, therefore, for $M := L + \|\phi\|_{L^\infty(\Omega)}$ we obtain

$$\begin{aligned}
&\int_{Q_T} \zeta(A\nabla v_{m,n}, \nabla S(v_{m,n} - \phi))\\
&= \int_{Q_T} \zeta S'(T_M(v_{m,n}) - \phi)(A\nabla T_M(v_{m,n}), \nabla(T_M(v_{m,n}) - \phi)).
\end{aligned}$$

Indeed, if $|v_{m,n}| \ge M$, we have

$$|v_{m,n} - \phi| \ge |v_{m,n}| - |\phi| \ge L + \|\phi\|_{L^\infty(\Omega)} - |\phi| \ge L$$

and, therefore, $S'(v_{m,n} - \phi) = 0$ for $|v_{m,n}| \ge M$. To pass to the limit, we write

$$\int_{Q_T} \zeta S'(T_M(v_{m,n}) - \phi)(A\nabla T_M(v_{m,n}), \nabla(T_M(v_{m,n}) - \phi)) = \int_{Q_T} I_1 + I_2 + I_3,$$

where

$$I_1 = \zeta S'(T_M(v_{m,n}) - \phi)\Big(A\nabla(T_M(v_{m,n}) - T_M(v)), \nabla(T_M(v_{m,n}) - T_M(v))\Big)$$
$$I_2 = \zeta\Big(A\nabla T_M(v), S'(T_M(v_{m,n}) - \phi)\nabla(T_M(v_{m,n}) - T_M(v))\Big)$$
$$I_3 = \zeta\Big(A\nabla T_M(v_{m,n}), S'(T_M(v_{m,n}) - \phi)\nabla T_M(v)\Big).$$

Since S' is continuous, by (2.12), we get

$$S'(T_M(v_{m,n}) - \phi) \overset{m\to\infty}{\longrightarrow} S'(T_M(v_{\infty,n}) - \phi) \overset{n\to\infty}{\longrightarrow} S'(T_M(v) - \phi) \text{ a.e. in } Q_T.$$

Additionally, S' has compact support and, therefore, by using Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} &S'(T_M(v_{m,n}) - \phi) \to S'(T_M(v_{\infty,n}) - \phi) && \text{in } L^1(Q_T) \text{ for } m \to \infty \\ \text{and}\quad &S'(T_M(v_{\infty,n}) - \phi) \to S'(T_M(v) - \phi) && \text{in } L^1(Q_T) \text{ for } n \to \infty. \end{aligned} \tag{2.16}$$

Having in mind that $\zeta \geq 0$ and $S' \geq 0$, the coercivity condition (HA) implies

$$\int_{Q_T} I_1 \geq \nu \int_{Q_T} \zeta S'(T_M(v_{m,n}) - \phi)|\nabla(T_M(v_{m,n}) - T_M(v))|^2 \geq 0 \quad \forall m, n \in \mathbb{N}.$$

Using Lemma 2.5.1 and the convergence (2.16), we know

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q_T} I_2 = 0$$

and

$$\begin{aligned} \lim_{n\to\infty}\lim_{m\to\infty}\int_{Q_T} I_3 &= \int_{Q_T} \zeta(A\nabla T_M(v), S'(T_M(v) - \phi)\nabla T_M(v)) \\ &= \int_{Q_T} \zeta(A\nabla v, S'(v - \phi)\nabla v) \end{aligned}$$

by the definition of M. It follows that

$$\begin{aligned} &\liminf_{n\to\infty}\liminf_{m\to\infty}\int_{Q_T} \zeta\big(A\nabla T_M(v_{m,n}), S'(T_M(v_{m,n}) - \phi)\nabla(T_M(v_{m,n}) - \phi)\big) \\ &\geq \int_{Q_T} \zeta(A\nabla v, \nabla S(v - \phi)). \end{aligned} \tag{2.17}$$

According to the first term in (2.15), we know that a.e. in Q_T

$$\int_{v_0^{m,n}}^{v_{m,n}} S(\sigma - \phi)\, db(\sigma) \overset{m\to\infty}{\longrightarrow} \int_{v_0^{\infty,n}}^{v_{\infty,n}} S(\sigma - \phi)\, db(\sigma) \overset{n\to\infty}{\longrightarrow} \int_{v_0}^{v} S(\sigma - \phi)\, db(\sigma),$$

where $v^0_{\infty,n} := \varphi(u^0_{\infty,n})$ and $u^0_{\infty,n} := \lim_{m\to\infty} u_0^{m,n}$. Since S is bounded, there exists a constant $C \geq 0$ (independent of m and n) such that a.e. in Q_T

$$\begin{aligned}\left|\int_{v_0^{m,n}}^{v_{m,n}} S(\sigma - \phi)\, db(\sigma)\right| &\leq C(|b(v_{m,n})| + |b(v_0^{m,n})|)\\ &\leq C(g^n + |b(v_0)|) \quad \forall m, n \in \mathbb{N}\end{aligned}$$

by Lemma 2.4.2. Analogously, we obtain by Lemma 2.4.2 a.e. in Q_T

$$\left|\int_{v_0^{\infty,n}}^{v_{\infty,n}} S(\sigma - \phi)\, db(\sigma)\right| \leq C(g + |b(v_0)|) \quad \forall n \in \mathbb{N}.$$

Hence, Lebesgue's dominated convergence theorem implies the convergence in $L^1(Q_T)$ and, therefore,

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\left(-\int_{Q_T} \zeta_t \left[k_1 * \int_{v_0^{m,n}}^{v_{m,n}} S(\sigma - \phi)\, db(\sigma)\right]\right)\\ &= -\int_{Q_T} \zeta_t \left[k_1 * \int_{v_0}^{v} S(\sigma - \phi)\, db(\sigma)\right].\end{aligned} \tag{2.18}$$

It remains to show the convergence for

$$\int_{Q_T} \zeta \partial_t[k_2 * (b(v_{m,n}) - b(v_0^{m,n}))] S(v_{m,n} - \phi).$$

Using Lemma 2.4.3, we have for a.e. $t \in (0, T)$

$$\begin{aligned}&\partial_t[k_2 * (b(v_{m,n}) - b(v_0^{m,n}))](t)\\ &= k_2(0^+)(b(v_{m,n}(t) - b(v_0^{m,n})) + \int_0^t b(v_{m,n}(t-s)) - b(v_0^{m,n})\, dk_2(s)\\ &\overset{m\to\infty}{\longrightarrow} k_2(0^+)(b(v_{\infty,n}(t)) - b(v^0_{\infty,n})) + \int_0^t b(v_{\infty,n}(t-s)) - b(v^0_{\infty,n})\, dk_2(s)\\ &\overset{n\to\infty}{\longrightarrow} k_2(0^+)(b(v(t)) - b(v_0)) + \int_0^t b(v(t-s)) - b(v_0)\, dk_2(s),\end{aligned}$$

where the convergences hold in $L^1(Q_T)$. Consequently,

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q_T}\zeta\partial_t[k_2*(b(v_{m,n})-b(v_0^{m,n}))]S(v_{m,n}-\phi)\\&=\int_{Q_T}\zeta\partial_t[k_2*(b(v)-b(v_0))]S(v-\phi).\end{aligned}\tag{2.19}$$

Using (2.17), (2.18), and (2.19), we get from (2.15)

$$\begin{aligned}&-\int_{Q_T}\zeta_t\left[k_1*\int_{v_0}^{v}S(\sigma-\phi)\,db(\sigma)\right]+\int_{Q_T}\zeta\partial_t[k_2*(b(v)-b(v_0))]S(v-\phi)\\&+\int_{Q_T}\zeta(A\nabla v,\nabla S(v-\phi))\\&\leq\lim_{n\to\infty}\lim_{m\to\infty}\int_{Q_T}\zeta f^{m,n}S(v_{m,n}-\phi)=\int_{Q_T}\zeta fS(v-\phi)\end{aligned}$$

and, hence, v is an entropy solution to $P(u_0,f)$. $\square$

Remark 2.5.3. *For $i=1,2$, let $u_{0,i}\in L^1(\Omega)$, $f_i\in L^1(Q_T)$, and v_i be an entropy solution to $P(u_{0,i},f_i)$, such that $v_i=\lim_{n\to\infty}\lim_{m\to\infty}\varphi(u^i_{m,n})$, where $u^i_{m,n}$ is a weak solution to $P(u_{0,i}^{m,n},f_i^{m,n})$ for any $m,n\in\mathbb{N}$. Here, $u_{0,i}^{m,n}$ and $f_i^{m,n}$ are the bounded approximations of $u_{0,i}$ and f_i defined analogously to their definition in Section 2.4. Then, the contraction principle*

$$\|b(v_1)-b(v_2)\|_{L^1(Q_T)}\leq T\|u_{0,1}-u_{0,2}\|_{L^1(\Omega)}+\|l\|_{L^1(0,T)}\|f_1-f_2\|_{L^1(Q_T)}$$

holds. The proof is a consequence of the contraction principle (2.6) *for weak solutions and the convergences of the approximated solutions that can be obtained analogously as in Lemma 2.4.1 and Lemma 2.4.3.*

Chapter 3

Well-Posedness of Stochastic Evolution Equations with Hölder Continuous Noise

3.1 Introduction

3.1.1 Statement of the Problem, Motivation, and Former Results

Let $T > 0$, $D \subseteq \mathbb{R}^d$ be a bounded domain, $d \in \mathbb{N}$, and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We want to show existence and pathwise uniqueness of probabilistically strong solutions to stochastic evolution equations of the form

$$\begin{cases} du - \operatorname{div} a(x, u, \nabla u)\, dt + f(u)\, dt = B(t, u)\, dW_t & \text{in } \Omega \times (0, T) \times D \\ u = 0 & \text{on } \Omega \times (0, T) \times \partial D \\ u(0, \cdot) = u_0 & \text{in } \Omega \times D, \end{cases} \tag{3.1}$$

where u_0 is assumed to be in $L^2(\Omega; L^2(D))$ and $\mathcal{F}_0$-measurable. We fix a separable Hilbert space U such that $U \supseteq L^2(D)$ and a symmetric, non-negative trace class operator $Q : U \to U$ with $Q^{\frac{1}{2}}(U) = L^2(D)$. We endow U with an orthonormal basis of eigenvectors of Q. In the following let $(W_t)_{t\in[0,T]}$ be a $(\mathcal{F}_t)_{t\in[0,T]}$-adapted Q-Wiener process with values in U. The integral on the right-hand side of (3.1) is understood in the sense of Itô. The function $f \in L^\infty(\mathbb{R})$ is assumed to be Lipschitz continuous with Lipschitz constant $L_f \geq 0$ and satisfies $f(0) = 0$.
For $a : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, we assume, that it is a Carathéodory function and

satisfies for $\max\{1, \frac{2d}{d+2}\} < p < \infty$ the usual Leray-Lions conditions which will be specified in Section 3.1.2.

Our aim is to show existence and pathwise uniqueness of probabilistically strong solutions to (3.1). The classical monotonicity method to show well-posedness is originated in [77] for deterministic equations. This method was extended to stochastic partial differential equations by Pardoux (see [96]) and was generalized by Krylov and Rozovskii (see [75]) and Liu and Röckner in [84]. Key properties for these well-posedness results are certain monotonicity, coercivity, and growth conditions of the (locally) monotone operator in combination with the noise term. These assumptions have been applied and extended by many authors (see, e.g., [11, 62, 63, 78–81]).

The main problem we want to tackle in this study is the presence of a pseudomonotone operator and a merely Hölder continuous multiplicative noise term. Precisely, we assume that the operator $B : (0, T) \times L^2(D) \to \mathrm{HS}(L^2(D))$ is Hölder continuous but not necessarily Lipschitz continuous in its second variable, where $\mathrm{HS}(L^2(D))$ denotes the space of Hilbert-Schmidt operators from $L^2(D)$ to $L^2(D)$.

There are many results that address linear SPDEs with Hölder continuous noise or, more generally, nonlinear SPDEs with Lipschitz noise and Hölder continuous coefficients in the literature. Let us mention the results on existence and uniqueness of solutions to stochastic Volterra equations with non-Lipschitz coefficients and on stochastic evolution equations with non-Lipschitz coefficients in [123] and [130]. The existence of mild solutions to the stochastic heat equation with Hölder diffusion coefficients is well-known and has been studied in [90, 92, 93, 115]. The question of uniqueness of solutions to the stochastic heat equation with non-Lipschitz diffusion coefficient and space-time white noise, as well as colored noise, was studied in [90–92]. In these contributions, a semigroup approach is available and allows to use the framework of mild solutions. Motivated by these results, our aim is to study the existence and uniqueness for evolution equations driven by nonlinear pseudomonotone operators and non-Lipschitz multiplicative noise in the variational framework.

In the recent contribution [106], well-posedness of SPDEs driven by multiplicative noise with fully local monotone coefficients has been considered. The authors use Galerkin approximations for the proof of existence of probabilistically weak solutions and a refined L^2-technique for the proof of pathwise uniqueness. The results in our contribution differ from the results in [106] in two ways. Firstly, we use different techniques, namely the simultaneous perturbation with a higher-order operator and regularization by inf-convolution in the noise. Secondly, our operator is rather pseudomonotone than locally monotone and may therefore not satisfy the local monotonicity conditions

from [106], see Remark 3.1.1 for more details.
To show existence of strong solutions to (3.1), we approximate the non-Lipschitz operator B by a Lipschitz continuous operator. In addition, we adapt the ideas proposed in [108] and add a singular perturbation in form of a higher order operator to the equation. This enables us to apply the well-posedness result stated in [84] to get a variational solution to the approximated equation. To obtain then a martingale solution to (3.1), we use a stochastic compactness argument of Prokhorov and Skorokhod which is classical in the framework of SPDEs and has been used in, e.g., [21, 22, 25, 42, 43, 55, 56, 59, 106, 109, 118, 119], see also [31] for a more extensive list of references. Existence of a probabilistically strong solution to (3.1) follows from a pathwise uniqueness argument of Gyöngy and Krylov (see [66]).

3.1.2 Hypotheses

For $a : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, we assume that it is a Carathéodory function, i.e., $D \ni x \mapsto a(x, \lambda, \xi)$ is measurable for all $(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^d$ and $\mathbb{R} \times \mathbb{R}^d \ni (\lambda, \xi) \mapsto a(x, \lambda, \xi)$ is continuous for a.e. $x \in D$. Moreover, a satisfies, for $\max\{1, \frac{2d}{d+2}\} < p < \infty$, the following properties:

(A1) For all $\xi, \eta \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, and a.e. $x \in D$,

$$(a(x, \lambda, \xi) - a(x, \lambda, \eta)) \cdot (\xi - \eta) \geq 0.$$

(A2) There exist $\kappa \in L^1(D)$, constants $C_1 > 0$, $C_2, C_3, C_4 \geq 0$, $1 \leq \nu < p$, and a non-negative function $g \in L^{p'}(D)$ such that for all $(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^d$ and a.e. $x \in D$,

$$a(x, \lambda, \xi) \cdot \xi \geq \kappa(x) + C_1 |\xi|^p - C_2 |\lambda|^\nu$$

and

$$|a(x, \lambda, \xi)| \leq C_3 |\xi|^{p-1} + C_4 |\lambda|^{p-1} + g(x).$$

(A3) There exist a constant $C_5 \geq 0$ and a non-negative function $h \in L^{p'}(D)$, such that, for all $\lambda_1, \lambda_2 \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, and a.e. $x \in D$,

$$|a(x, \lambda_1, \xi) - a(x, \lambda_2, \xi)| \leq (C_5 |\xi|^{p-1} + h(x)) |\lambda_1 - \lambda_2|.$$

Remark 3.1.1. *For $p \geq 2$, an operator induced by a Carathéodory function satisfying (A1)-(A3) is a slight generalization of the operator*

$$\mathcal{A} : W_0^{1,p}(D) \to W^{-1,p'}(D),\ u \mapsto \mathcal{A}(u) = \Delta_p(u) + \operatorname{div} F(u),$$

where $\Delta_p(u) = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ and $F : \mathbb{R} \to \mathbb{R}^d$ is Lipschitz continuous with $F(0) = 0$.

For $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$, we assume that

(S1) For a.e. $t \in (0,T)$,
$$\mathbb{R} \ni \lambda \mapsto \sigma(t,\lambda)$$
is continuous and
$$(0,T) \ni t \mapsto \sigma(t,\lambda)$$
is measurable for every $\lambda \in \mathbb{R}$.

(S2a) σ is α-Hölder continuous, i.e., there exists an $\alpha \in (0,1]$ and a constant $L_\alpha > 0$, such that, for all $\lambda, \mu \in \mathbb{R}$, and a.e. $t \in (0,T)$,
$$|\sigma(t,\lambda) - \sigma(t,\mu)| \leq L_\alpha |\lambda - \mu|^\alpha.$$

(S2b) We assume $\sigma(t,0) = 0$ for almost all $t \in (0,T)$.

(S3) σ has a sublinear growth, i.e., there exists $C_\sigma > 0$, such that
$$|\sigma(t,\lambda)|^2 \leq C_\sigma(1 + |\lambda|^2)$$
for all $\lambda \in \mathbb{R}$, and a.e. $t \in (0,T)$.

In the following, we introduce the notion of infinite dimensional Hölder noise. Let $\mathrm{HS}(L^2(D))$ denote the space of Hilbert-Schmidt operators from $L^2(D)$ to $L^2(D)$. We consider an operator $B : (0,T) \times L^2(D) \to \mathrm{HS}(L^2(D))$ that is, for $(t,v) \in (0,T) \times L^2(D)$, $\varphi \in L^2(D)$, of the form
$$B(t,v)\varphi(x) = \sigma(t,v(x)) \int_D k(x,y)\varphi(y)\,dy \tag{3.2}$$
for a.e. $x \in D$, with a symmetric kernel $k \in L^2(D \times D)$ which satisfies
$$\operatorname*{ess\,sup}_{y \in D} \|k(\cdot,y)\|^2_{L^2(D)} = \operatorname*{ess\,sup}_{x \in D} \|k(x,\cdot)\|^2_{L^2(D)} \leq C_k$$
for a constant $C_k \geq 0$.

Remark 3.1.2. *Let X, Y be two Hilbert spaces. A bounded operator $K : X \to Y$ is a Hilbert-Schmidt operator iff it is a Hilbert-Schmidt integral operator, i.e., there exists a kernel $l \in L^2(X \times Y)$ such that for all $\varphi \in L^2(D)$ and a.e. $x \in D$*
$$K\varphi(x) = \int_D l(x,y)\varphi(y)\,dy.$$
Moreover, there holds $\|K\|_{HS} = \|l\|_{L^2(D)}$, see [124, Satz 3.19] and [85, p.93].

Notation. *In the following, we will denote by* $\|\cdot\|_r$ *the norm in* $L^r(D)$ *for* $1 \le r \le \infty$*, by* $\langle \cdot, \cdot \rangle_{L^2}$ *the dual pairing in* $L^2(D)$*, and by* $\|\cdot\|_{HS}$ *the Hilbert-Schmidt norm on* $HS(L^2(D))$*.*

The operator B is well defined on $L^2(D)$. Indeed, for $v, \varphi \in L^2(D)$ and a.e. $t \in (0,T)$, we have, by using Cauchy-Schwarz inequality, (S3), and Fubini's theorem,

$$\begin{aligned}
\|B(t,v)(\varphi)\|_2^2 &= \int_D |\sigma(t,v(x))|^2 \left| \int_D k(x,y)\varphi(y)\,dy \right|^2 dx \\
&\le \int_D |\sigma(t,v(x))|^2 \left(\int_D |k(x,y)|^2\,dy \right) \left(\int_D |\varphi(y)|^2\,dy \right) dx \\
&\le \|\varphi\|_2^2 \int_D C_\sigma (1 + |v(x)|^2) \|k(x,\cdot)\|_2^2\,dx \\
&\le \|\varphi\|_2^2 \left(C_\sigma \|k\|_{L^2(D\times D)}^2 + C_\sigma \operatorname*{ess\,sup}_{x\in D} \|k(x,\cdot)\|_2^2 \|v\|_2^2 \right) \\
&\le \|\varphi\|_2^2 C_\sigma \left(\|k\|_{L^2(D\times D)}^2 + C_k \|v\|_2^2 \right).
\end{aligned}$$

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(D)$. Using Parseval's identity and (S3), we obtain for $v \in L^2(D)$ and a.e. $t \in (0,T)$, by an analogous argumentation as above,

$$\begin{aligned}
\|B(t,v)\|_{\mathrm{HS}}^2 &= \sum_{n\in\mathbb{N}} \|B(t,v)(e_n)\|_2^2 \\
&= \sum_{n\in\mathbb{N}} \int_D |\sigma(t,v(x))|^2 \left|\langle k(x,\cdot), e_n(\cdot)\rangle_{L^2}\right|^2\,dx \\
&= \int_D |\sigma(t,v(x))|^2 \sum_{n\in\mathbb{N}} \left|\langle k(x,\cdot), e_n(\cdot)\rangle_{L^2}\right|^2\,dx \\
&= \int_D |\sigma(t,v(x))|^2 \|k(x,\cdot)\|_2^2\,dx \\
&\le \int_D C_\sigma(1 + |v(x)|^2)\|k(x,\cdot)\|_2^2\,dx \\
&\le C_\sigma(\|k\|_{L^2(D\times D)}^2 + C_k\|v\|_2^2).
\end{aligned} \tag{3.3}$$

Using (S2) instead of (S3) in the same manner, we get for $v, w \in L^2(D)$ and a.e. $t \in (0,T)$,

$$\begin{aligned}\|B(t,v)-B(t,w)\|_{\mathrm{HS}}^2 &\le \int_D |\sigma(t,v(x))-\sigma(t,w(x))|^2 \int_D |k(x,y)|^2\,dy\,dx \\ &\le L_\alpha^2 \int_D |v(x)-w(x)|^{2\alpha}\|k(x,\cdot)\|_2^2\,dx \\ &\le C_k L_\alpha^2 \|v-w\|_{2\alpha}^{2\alpha} \\ &\le C_k L_\alpha^2 C(\alpha)\|v-w\|_2^{2\alpha},\end{aligned} \tag{3.4}$$

where $C(\alpha) > 0$ is a constant arising from the continuous embedding $L^2(D) \hookrightarrow L^{2\alpha}(D)$ for $\alpha \in (0,1)$.

3.1.3 Main Results and Outline

Let the assumptions of Section 3.1.1 and Section 3.1.2 hold.

Definition 3.1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a given Lipschitz continuous function, and let $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$ fulfill (S1)-(S3) for $\alpha \in (0,1]$.*

i) A stochastic process u is called a probabilistically strong solution, if

1. u is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process and

$$u \in L^2(\Omega; C([0,T]; L^2(D))) \cap L^p(\Omega; L^p(0,T; W_0^{1,p}(D)))$$

2. $u(0) = u_0$, $\mathbb{P}$-a.s. in Ω

3. for all $t \in [0,T]$, in $L^2(D)$, $\mathbb{P}$-a.s. in Ω,

$$u(t) - u_0 - \int_0^t \operatorname{div} a(x,u(s),\nabla u(s))\,ds + \int_0^t f(u(s))\,ds$$
$$= \int_0^t B(s,u(s))\,dW_s.$$

ii) A triple $\Big((\Omega', \mathcal{A}', (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathbb{P}'), \widetilde{u}, (\mathcal{W}_t)_{t\in[0,T]}\Big)$ is called a martingale solution to (3.1) *with initial value v_0, if*

1. $(\Omega', \mathcal{A}', (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, \mathbb{P}')$ is a stochastic basis with a complete, right-continuous filtration

2. $(\mathcal{W}_t)_{t\in[0,T]}$ is an $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$-adapted Q-Wiener process on $(\Omega', \mathcal{A}', \mathbb{P}')$

3. $\widetilde{u}$ is an $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$-adapted stochastic process and

$$\widetilde{u} \in L^2(\Omega'; C([0,T]; L^2(D))) \cap L^p(\Omega'; L^p(0,T; W_0^{1,p}(D)))$$

4. $v_0 \in L^2(\Omega'; L^2(D))$ has the same law as u_0

5. for all $t \in [0,T]$, in $L^2(D)$, $\mathbb{P}'$-a.s. in Ω',

$$\begin{aligned} &\widetilde{u}(t) - v_0 - \int_0^t \operatorname{div} a(x, \widetilde{u}(s), \nabla \widetilde{u}(s))\, ds + \int_0^t f(\widetilde{u}(s))\, ds \\ &= \int_0^t B(s, \widetilde{u}(s))\, d\mathcal{W}_s. \end{aligned}$$

Theorem 3.1.4. *Assume that $f \in L^\infty(\mathbb{R})$ is a given Lipschitz continuous function with $f(0) = 0$, and $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$ fulfills (S1)-(S3) for an arbitrary $\alpha \in (0,1)$. Then,* (3.1) *admits a martingale solution in the sense of Definition 3.1.3 ii).*

Theorem 3.1.5. *Assume that $(W_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t)_{t\in[0,T]}$-adapted Q-Wiener process with values in U with respect to the stochastic basis $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $f : \mathbb{R} \to \mathbb{R}$ is a given Lipschitz continuous function with Lipschitz constant L_f, and $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$ fulfills (S1)-(S3) for $\alpha \in \left[\frac{1}{2}, 1\right)$. If u_1, u_2 are both probabilistically strong solutions to* (3.1) *in the sense of Definition 3.1.3 i) with initial values u_0^1, u_0^2 in $L^2(D)$, respectively, then, for any $t \in [0,T]$,*

$$\mathbb{E}\left[\|u_1(t) - u_2(t)\|_1\right] \le e^{L_f T} \mathbb{E}\left[\|u_0^1 - u_0^2\|_1\right].$$

Theorem 3.1.6. *Assume that $f \in L^\infty(\mathbb{R})$ is a given Lipschitz continuous function with $f(0) = 0$ and $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$ fulfills (S1)-(S3) for $\alpha \in \left[\frac{1}{2}, 1\right)$. Then,* (3.1) *admits a unique probabilistically strong solution u in the sense of Definition 3.1.3 i).*

Remark 3.1.7.
1. *Theorem 3.1.6 is a direct consequence of Theorem 3.1.4 and Theorem 3.1.5 by an argument of Gyöngy and Krylov, see [66, Lemma 1.1].*

2. *We only need the assumption $f \in L^\infty(\mathbb{R})$ for the identification argument in Lemma 3.3.21. If $f : \mathbb{R} \to \mathbb{R}$ is a linear function such that $f(\lambda) = c\lambda$ for $c \in \mathbb{R}$, we can avoid the boundedness assumption $f \in L^\infty(\mathbb{R})$ in Theorem 3.1.4 and Theorem 3.1.6 by using an Itô formula with exponential weight in the proof of Lemma 3.3.21.*

To prove this result we proceed in the following way: At first, in Section 3.2.1, we approximate the Hölder continuous operator B by a Lipschitz continuous operator B_n by using the representation (3.2) and an inf-convolution for σ. Then, in Section 3.2.2, we add a higher order operator to the equation which will disappear in the limit afterwards and that allows us to apply the existence and uniqueness result on variational solutions in [84] to the approximated equation. We will first show some *a priori* estimates in Section 3.3.1 that allow us then to show tightness of the approximations in Section 3.3.2. By using these tightness results, we will pass to the limit in the approximated equation in Section 3.3.3 by applying a stochastic compactness argument based on Prokhorov's and Skorokhod's theorems. Thereby, we obtain a martingale solution to (3.1). In Section 3.4, we show pathwise uniqueness of solutions to (3.1) and, therefore, obtain a probabilistically strong solution to (3.1) by [66].

3.2 Existence of Approximate Solutions

3.2.1 Lipschitz Continuous Approximation of the Noise

We start by approximating the function σ to get a Lipschitz continuous approximation of the operator B.

Definition 3.2.1. *i) Let $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$ satisfy (S1)-(S3). For any $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and a.e. $t \in (0,T)$, we introduce the Lipschitz regularization of σ via inf-convolution:*

$$\sigma_n(t,\lambda) := \inf_{\mu \in \mathbb{R}} (\sigma(t,\mu) + n|\lambda - \mu|), \tag{3.5}$$

see, e.g., [9, Theorem 9.2.1].

ii) Let B be defined by (3.2). Then, we define, for any $n \in \mathbb{N}$, $v, \varphi \in L^2(D)$, and a.e. $(t,x) \in (0,T) \times D$,

$$B_n(t,v)\varphi(x) := \sigma_n(t,v(x)) \int_D k(x,y)\varphi(y)\,dy.$$

Proposition 3.2.2. *Let $\sigma : (0,T) \times \mathbb{R} \to \mathbb{R}$ satisfy (S1)-(S3), C_σ be the constant given by ($S3$), and*

$$n_0 := \left\lceil \sqrt{C_\sigma} \right\rceil = \min\left\{ n \in \mathbb{N} : n \geq \sqrt{C_\sigma} \right\}.$$

Then, there exists a full-measure set $U \subseteq (0,T)$ such that, for any $t \in U$, the Lipschitz regularization via inf-convolution σ_n of σ has the following properties:

i) $\sigma_n(t,\lambda) > -\infty$ *for all* $\lambda \in \mathbb{R}$.

ii) *For all* $\lambda \in \mathbb{R}$ *and all* $n \in \mathbb{N}$ *such that* $n \geq n_0$

$$\sigma_n(t,\lambda) \leq \sigma(t,\lambda).$$

iii) σ_n *is Lipschitz continuous: there holds*

$$|\sigma_n(t,\lambda_1) - \sigma_n(t,\lambda_2)| \leq n|\lambda_1 - \lambda_2|,$$

for all $n \in \mathbb{N}$ *such that* $n \geq n_0$*, and all* $\lambda_1, \lambda_2 \in \mathbb{R}$.

iv) σ_n *is uniformly bounded with respect to* $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}$*: There exists a constant* $C_\alpha > 0$*, only depending on the Hölder exponent* $\alpha \in (0,1)$ *and the Hölder constant* $L_\alpha > 0$ *of* σ*, such that*

$$|\sigma_n(t,\lambda) - \sigma(t,\lambda)| \leq C_\alpha \tag{3.6}$$

for all $n \in \mathbb{N}$ *such that* $n \geq n_0$ *and all* $\lambda \in \mathbb{R}$*. Moreover,*

$$|\sigma_n(t,\lambda)|^2 \leq 2(C_\alpha^2 + C_\sigma(1+|\lambda|^2)) \tag{3.7}$$

for all $\lambda \in \mathbb{R}$.

v) σ_n *converges uniformly to* σ*, i.e.,*

$$\lim_{n\to\infty} \sup_{t\in U} \sup_{\lambda\in\mathbb{R}} |\sigma_n(t,\lambda) - \sigma(t,\lambda)| = 0.$$

Proof. We choose the full-measure set $U \subseteq (0,T)$ such that (S1)-(S3) hold true for all $t \in U$.
i) From (S1)-(S3), it follows that $\lambda \mapsto \sigma(t,\lambda)$ is continuous for all $t \in U$ and

$$\sigma(t,\lambda) \geq -\sqrt{C_\sigma(1+|\lambda|^2)} \geq -\sqrt{C_\sigma}(1+|\lambda|)$$

for all $t \in U$ and $\lambda \in \mathbb{R}$. Now, the result follows from [9, Theorem 9.2.1].
ii) follows immediately by discarding the infimum and plugging $\mu = \lambda$ in (3.5).
iii) For any $t \in U$, any $n \in \mathbb{N}$ such that $n \geq n_0$, and any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&\sigma_n(t,\lambda_1) - \sigma_n(t,\lambda_2) \\
&\leq \inf_{\mu\in\mathbb{R}}(\sigma(t,\mu) + n|\lambda_1 - \lambda_2| + n|\lambda_2 - \mu|) - \inf_{\mu\in\mathbb{R}}(\sigma(t,\mu) + n|\lambda_2 - \mu|) \\
&= n|\lambda_1 - \lambda_2| + \inf_{\mu\in\mathbb{R}}\{\sigma(t,\mu) + n|\lambda_2 - \mu|\} - \inf_{\mu\in\mathbb{R}}(\sigma(t,\mu) + n|\lambda_2 - \mu|) \\
&= n|\lambda_1 - \lambda_2|.
\end{aligned}$$

With the same argument, we obtain

$$\sigma_n(t,\lambda_2) - \sigma_n(t,\lambda_1) \le n|\lambda_1 - \lambda_2|.$$

iv) For any $t \in U$, any $n \in \mathbb{N}$ such that $n \ge n_0$, and any $\lambda \in \mathbb{R}$, using *ii)* and $(S2)$, we get

$$\begin{aligned}
|\sigma(t,\lambda) - \sigma_n(t,\lambda)| &= \sigma(t,\lambda) - \inf_{\mu\in\mathbb{R}}(\sigma(t,\mu) + n|\lambda-\mu|) \\
&= \sigma(t,\lambda) + \sup_{\mu\in\mathbb{R}}(-\sigma(t,\mu) - n|\lambda-\mu|) \\
&= \sup_{\mu\in\mathbb{R}}(\sigma(t,\lambda) - \sigma(t,\mu) - n|\lambda-\mu|) \\
&\le \sup_{\mu\in\mathbb{R}}(|\sigma(t,\lambda) - \sigma(t,\mu)| - n|\lambda-\mu|) \\
&\le \sup_{\mu\in\mathbb{R}}(L_\alpha|\lambda-\mu|^\alpha - n|\lambda-\mu|) \\
&\le \max_{r\in[0,\infty)} h_n(r),
\end{aligned} \tag{3.8}$$

where $h_n : [0,\infty) \to \mathbb{R}$ is defined by $h_n(r) := L_\alpha r^\alpha - nr$. For any $n \in \mathbb{N}$, we have $h_n(0) = 0$ and $h_n'(r) = 0$ iff $r = r_0^n$, where

$$r_0^n := \left(\frac{n}{L_\alpha \alpha}\right)^{\frac{1}{\alpha-1}}.$$

Since $h_n'(r) > 0$ for all $0 < r < r_0^n$, $h_n'(r) < 0$ for all $r > r_0^n$, and

$$h_n(r_0^n) = \frac{n^{\frac{\alpha}{\alpha-1}}(1-\alpha)}{L_\alpha^{\frac{1}{\alpha-1}}\alpha^{\frac{\alpha}{\alpha-1}}} > 0$$

for $\alpha \in (0,1)$, it follows that

$$\max_{r\in[0,\infty)} h_n(r) = h_n(r_0^n) = \frac{n^{\frac{\alpha}{\alpha-1}}(1-\alpha)}{L_\alpha^{\frac{1}{\alpha-1}}\alpha^{\frac{\alpha}{\alpha-1}}}. \tag{3.9}$$

Since $\frac{\alpha}{\alpha-1} < 0$, we have

$$\frac{n^{\frac{\alpha}{\alpha-1}}(1-\alpha)}{L_\alpha^{\frac{1}{\alpha-1}}\alpha^{\frac{\alpha}{\alpha-1}}} \le \frac{1-\alpha}{L_\alpha^{\frac{1}{\alpha-1}}\alpha^{\frac{\alpha}{\alpha-1}}} =: C_\alpha, \tag{3.10}$$

for all $n \in \mathbb{N}$, and (3.6) holds true. Now, using (3.6) and $(S3)$, we know

$$\begin{aligned}
|\sigma_n(t,\lambda)|^2 &\le 2(|\sigma_n(t,\lambda) - \sigma(t,\lambda)|^2 + |\sigma(t,\lambda)|^2) \\
&\le 2(C_\alpha + C_\sigma(1+|\lambda|^2)),
\end{aligned}$$

and obtain (3.7).
v) We recall that, for all $t \in U$, all $n \geq n_0$, and all $\lambda \in \mathbb{R}$, (3.9) holds true, where $\frac{\alpha}{\alpha-1} < 0$ for $\alpha \in (0,1)$. Therefore, we get

$$\lim_{n\to\infty} \sup_{t\in U} \sup_{\lambda\in\mathbb{R}} |\sigma_n(t,\lambda) - \sigma(t,\lambda)| \leq \lim_{n\to\infty} \frac{n^{\frac{\alpha}{\alpha-1}}(1-\alpha)}{L_\alpha^{\frac{1}{\alpha-1}} \alpha^{\frac{\alpha}{\alpha-1}}} = 0.$$

□

From Proposition 3.2.2, we get the following consequences:

Corollary 3.2.3. *Let $\sigma : (0,T)\times\mathbb{R} \to \mathbb{R}$ satisfy (S1)-(S3), C_σ be the constant given by (S3), and $n_0 := \lceil \sqrt{C_\sigma} \rceil$.*

i) For a.e. $t \in (0,T)$, the mapping $L^2(D) \ni v \mapsto B_n(t,v)$ is Lipschitz continuous from $L^2(D)$ to $HS(L^2(D))$ with Lipschitz constant $L_{B_n} = \sqrt{C_k} n$ for all $n \geq n_0$.

ii) For any $u \in L^2(\Omega, L^2(0,T; L^2(D)))$

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \|B_n(t,u) - B(t,u)\|_{HS}^2 \, dt\right] = 0.$$

iii) For all $n \geq n_0$, all $v \in L^2(D)$, and a.e. $t \in (0,T)$, we have

$$\|B_n(t,v)\|_{HS}^2 \leq 2\left[(C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2 + C_\sigma C_k \|v\|_2^2\right].$$

Proof. i) Recalling (3.4) and using Proposition 3.2.2 iii), for any $v, w \in L^2(D)$ and a.e. $t \in (0,T)$, we get

$$\begin{aligned}\|B_n(t,v) - B_n(t,w)\|_{\mathrm{HS}}^2 &\leq \int_D |\sigma_n(t,v(x)) - \sigma_n(t,w(x))|^2 \int_D |k(x,y)|^2 \, dy \, dx \\ &\leq C_k n^2 \|v-w\|_2^2.\end{aligned}$$

ii) With similar arguments as in (3.3), (3.4) and Proposition 3.2.2 v), we get, for all $n \geq n_0$,

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \|B_n(t,u) - B(t,u)\|_{HS}^2 \, dt\right] \\ &\leq \mathbb{E}\left[\int_0^T \int_D |\sigma_n(t,u(\omega,t,x)) - \sigma(t,u(\omega,t,x))|^2 \int_D |k(x,y)|^2 \, dy \, dx \, dt\right] \\ &\leq C_k \mathbb{E}\left[\int_0^T \int_D |\sigma_n(t,u(\omega,t,x)) - \sigma(t,u(\omega,t,x))|^2 \, dx \, dt\right] \\ &\leq C_k T |D| h_n^2(r_0^n)\end{aligned}$$

and the last term on the right-hand side converges to 0 for $n \to \infty$.
iii) is a direct consequence of (3.3) and of Proposition 3.2.2 iv). □

3.2.2 A Higher Order Perturbation

Let $m \in \mathbb{N}$ be chosen such that

$$H_0^m(D) \hookrightarrow W_0^{1,2p}(D) \cap L^\infty(D). \tag{3.11}$$

For $q := \max\{2, p, 2p(p-1), p'\}$, we consider the Gelfand triple

$$W_0^{m,q}(D) \hookrightarrow L^2(D) \hookrightarrow W^{-m,q'}(D)$$

and define, for $n \in \mathbb{N}$, the operator $A_n : W_0^{m,q}(D) \to W^{-m,q'}(D)$ by

$$\langle A_n(u), v\rangle_{q',q} := \int_D a(x, u, \nabla u) \cdot \nabla v \, dx + \frac{1}{n} j(u, v) + \int_D f(u) v \, dx$$

for $u, v \in W_0^{m,q}(D)$, where $\langle \cdot, \cdot \rangle_{q',q}$ denotes the duality bracket $\langle \cdot, \cdot \rangle_{W^{-m,q'}(D), W_0^{m,q}(D)}$ and

$$j(u, v) := (u, v)_{H_0^m(D)} + \int_D \sum_{|\gamma| \leq m} |\nabla^\gamma u|^{q-2} \nabla^\gamma u \cdot \nabla^\gamma v \, dx, \quad u, v \in W_0^{m,q}(D)$$

denotes the variational formulation of the maximal monotone operator associated to the Gâteaux derivative of

$$J : W_0^{m,q}(D) \to \mathbb{R}, \quad J(v) := \frac{1}{q} \|v\|_{W_0^{m,q}}^q + \frac{1}{2} \|v\|_{H_0^m}^2.$$

For $n \in \mathbb{N},\ n \geq n_0 := \lceil \sqrt{C_\sigma} \rceil$, we consider the approximated equation

$$\begin{cases} du_n + A_n(u_n) \, dt = B_n(t, u_n) \, dW_t & \text{in } \Omega \times (0,T) \times D \\ u_n = 0 & \text{on } \Omega \times (0,T) \times \partial D \\ u_n(0, \cdot) = u_0 & \text{in } \Omega \times D, \end{cases} \tag{3.12}$$

3.2.3 Well-Posedness of the Approximated Equation

In the following, we denote by C_E constants arising from embeddings, and let $n_0 := \lceil \sqrt{C_\sigma} \rceil$.

Lemma 3.2.4. *For fixed $n \in \mathbb{N},\ n \geq n_0$, there exists a constant $C_{3.2.4} \in \mathbb{R}$ and a function $\rho : W_0^{m,q}(D) \to [0, \infty)$ which is measurable, hemi-continuous, and locally bounded in $W_0^{m,q}(D)$, such that, for all $u, v \in W_0^{m,q}(D)$ and a.e. $t \in (0,T)$,*

$$\begin{aligned} &- 2\langle A_n(u) - A_n(v), u - v\rangle_{q',q} + \|B_n(t, u) - B_n(t, v)\|_{HS}^2 \\ &\leq (C_{3.2.4} + \rho(v)) \|u - v\|_2^2. \end{aligned}$$

Proof. Let $u, v \in W_0^{m,q}(D)$ be arbitrary and $n \in \mathbb{N},\ n \geq n_0$, be fixed. We know for a.e. $t \in (0,T)$,

$$\begin{aligned}
&-2\langle A_n(u) - A_n(v), u - v\rangle_{q',q} + \|B_n(t,u) - B_n(t,v)\|_{\mathrm{HS}}^2 \\
&= -2\int_D (a(x,u,\nabla u) - a(x,v,\nabla v)) \cdot \nabla(u-v)\,dx - \frac{2}{n}\|u-v\|_{H_0^m}^2 \\
&\quad - \frac{2}{n}\int_D \sum_{|\gamma|\leq m} (|\nabla^\gamma u|^{q-2}\nabla^\gamma u - |\nabla^\gamma v|^{q-2}\nabla^\gamma v)\cdot \nabla^\gamma(u-v)\,dx \\
&\quad - 2\int_D (f(u)-f(v))(u-v)\,dx + \|B_n(t,u) - B_n(t,v)\|_{\mathrm{HS}}^2.
\end{aligned} \tag{3.13}$$

By using (A1), (A3), and Hölder's inequality, we obtain

$$\begin{aligned}
&-2\int_D (a(x,u,\nabla u) - a(x,v,\nabla v)) \cdot \nabla(u-v)\,dx \\
&= -2\int_D (a(x,u,\nabla u) - a(x,u,\nabla v)) \cdot \nabla(u-v)\,dx \\
&\quad - 2\int_D (a(x,u,\nabla v) - a(x,v,\nabla v)) \cdot \nabla(u-v)\,dx \\
&\leq 2\int_D (C_5|\nabla v|^{p-1} + h(x))|u-v||\nabla(u-v)|\,dx \\
&\leq 2^{p'+1}(C_5\|\nabla v\|_p^{p-1} + \|h\|_{p'})\|u-v\|_{2p}\|\nabla(u-v)\|_{2p}.
\end{aligned}$$

Note that $\|u-v\|_{2p}^{2p} \leq \|u-v\|_\infty^{2p-2}\|u-v\|_2^2$. Thanks to the continuous embedding (3.11) and Young's inequality, we have for $\eta > 0$

$$\begin{aligned}
&-2\int_D (a(x,u,\nabla u) - a(x,v,\nabla v)) \cdot \nabla(u-v)\,dx \\
&\leq 2^{p'+1}(C_5\|\nabla v\|_p^{p-1} + \|h\|_{p'})\|u-v\|_\infty^{\frac{2p-2}{2p}}\|u-v\|_2^{\frac{1}{p}} C_E\|u-v\|_{H_0^m} \\
&\leq 2^{p'+1}(C_5\|\nabla v\|_p^{p-1} + \|h\|_{p'})C_E^{\frac{1}{p'}}\|u-v\|_{H_0^m}^{\frac{1}{p'}}\|u-v\|_2^{\frac{1}{p}} C_E\|u-v\|_{H_0^m} \\
&= K_1(\|\nabla v\|_p^{p-1} + 1)\|u-v\|_2^{\frac{1}{p}}\|u-v\|_{H_0^m}^{1+\frac{1}{p'}} \\
&\leq \frac{1}{2p\eta}\left(K_1^{2p}(\|\nabla v\|_p^{p-1}+1)^{2p}\|u-v\|_2^2\right) + \frac{\eta}{(2p)'}\|u-v\|_{H_0^m}^2
\end{aligned}$$

for a constant $K_1 \geq 0$ not depending on n. This estimate implies

$$\begin{aligned}
&-2\int_D (a(x,u,\nabla u) - a(x,v,\nabla v)) \cdot \nabla(u-v)\,dx - \frac{2}{n}\|u-v\|_{H_0^m}^2 \\
&\leq \frac{2^{2p-1}}{\eta p}K_1^{2p}\left(1 + \|\nabla v\|_p^{2p(p-1)}\right)\|u-v\|_2^2 + \left(\frac{2p-1}{2p}\eta - \frac{2}{n}\right)\|u-v\|_{H_0^m}^2.
\end{aligned} \tag{3.14}$$

In the following, we choose $\eta > 0$ small enough such that $\frac{2p-1}{2p}\eta - \frac{2}{n} < 0$. Note that

$$\int_D \sum_{|\gamma|\le m} (|\nabla^\gamma u|^{q-2}\nabla^\gamma u - |\nabla^\gamma v|^{q-2}\nabla^\gamma v) \cdot \nabla^\gamma (u-v)\, dx \ge 0.$$

Therefore, we obtain, for a.e. $t \in (0,T)$, from (3.13), by using (3.14), Corollary 3.2.3 *iii*), and the Lipschitz continuity of f,

$$\begin{aligned} &- 2\langle A_n(u) - A_n(v), u - v\rangle_{q',q} + \|B_n(t,u) - B_n(t,v)\|_{\mathrm{HS}}^2 \\ &\le (C_{3.2.4} + K_2\|\nabla v\|_p^{2p-2})\|u-v\|_2^2 \end{aligned}$$

for constants $C_{3.2.4} \in \mathbb{R}, K_2 \ge 0$ both depending on n. Hence, for

$$\rho(v) := K_2\|\nabla v\|_p^{2p-2},$$

the assertion is satisfied. Note that, by using $q \ge 2p(p-1)$ and the embedding $W_0^{m,q}(D) \hookrightarrow W_0^{1,p}(D)$, we have for any $v \in W_0^{m,q}(D)$

$$\rho(v) = K_2\|\nabla v\|_p^{2p-2} \le K_2 2^q(1 + \|\nabla v\|_p^q) \le K_2 2^q(1 + C_E^q\|v\|_{W_0^{m,q}}^q). \tag{3.15}$$

□

Lemma 3.2.5. *For $n \in \mathbb{N}$ large enough, there exist constants $C^1_{3.2.5}, C^2_{3.2.5} \in \mathbb{R}$ and $\theta \in (0,\infty)$, such that for all $u \in W_0^{m,q}(D)$ and a.e. $t \in (0,T)$*

$$-2\langle A_n(u), u\rangle_{q',q} + \|B_n(t,u)\|_{HS}^2 \le C^1_{3.2.5}\|u\|_2^2 - \theta\|u\|_{W_0^{m,q}}^q + C^2_{3.2.5}.$$

Proof. Let $n \in \mathbb{N}$ with $n \ge n_0$ be arbitrary but fixed. By (A2), the Lipschitz continuity of f, and Corollary 3.2.3 *iii*), we obtain for all $u \in W_0^{m,q}(D)$ and a.e. $t \in (0,T)$

$$\begin{aligned} &- 2\int_D a(x,u,\nabla u)\cdot\nabla u\, dx - \frac{2}{n}\|u\|_{H_0^m}^2 - \frac{2}{n}\|u\|_{W_0^{m,q}}^q \\ &- 2\int_D f(u)u\, dx + \|B_n(t,u)\|_{\mathrm{HS}}^2 \\ &\le -2\int_D (\kappa(x) + C_1|\nabla u|^p - C_2|u|^\nu)\, dx - \frac{2}{n}\|u\|_{W_0^{m,q}}^q + 2L_f\|u\|_2^2 \\ &\quad + 2(C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2 + 2C_kC_\sigma\|u\|_2^2 \\ &\le 2\|\kappa\|_1 + C_2\int_D |u|^\nu\, dx - \frac{2}{n}\|u\|_{W_0^{m,q}}^q + 2(L_f + C_\sigma C_k)\|u\|_2^2 \\ &\quad + 2(C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2. \end{aligned} \tag{3.16}$$

By using the continuous embeddings $L^{2p}(D) \hookrightarrow L^\nu(D)$ and $W_0^{m,q}(D) \hookrightarrow W_0^{1,2p}(D)$, Poincaré's inequality (with constant C_{Poin}), and the fact that $q \geq p > \nu$, we obtain

$$\begin{aligned}\int_D |u(x)|^\nu \, dx &\leq C_E^\nu \|u\|_{2p}^\nu \leq C_E^\nu C_{\text{Poin}}^\nu \|\nabla u\|_{2p}^\nu \leq C_E^\nu C_E^\nu C_{\text{Poin}}^\nu \|u\|_{W_0^{m,q}}^\nu \\ &\leq C_E^\nu C_E^\nu C_{\text{Poin}}^\nu 2^q \left(1 + \|u\|_{W_0^{m,q}}^q\right).\end{aligned}$$

Consequently, we get from (3.16), for all $u \in W_0^{m,q}(D)$, a.e. $t \in (0,T)$, and for $n \in \mathbb{N}$ large enough such that $\theta := 2^q C_2 C_E^\nu C_E^\nu C_{\text{Poin}}^\nu - \frac{2}{n} > 0$,

$$-2\langle A_n(u), u\rangle_{q',q} + \|B_n(t,u)\|_{\text{HS}}^2 \leq C_{3.2.5}^1 \|u\|_2^2 - \theta \|u\|_{W_0^{m,q}}^q + C_{3.2.5}^2$$

for constants $C_{3.2.5}^1, C_{3.2.5}^2 \geq 0$. □

Lemma 3.2.6. *For fixed $n \in \mathbb{N}, n \geq n_0$, there exist $C_{3.2.6}^1, C_{3.2.6}^2 \in \mathbb{R}$, such that for all $u \in W_0^{m,q}(D)$*

$$\|A_n(u)\|_{W^{-m,q'}(D)}^{q'} \leq C_{3.2.6}^1 + C_{3.2.6}^2 \|u\|_{W_0^{m,q}}^q.$$

Proof. Let $n \in \mathbb{N}, n \geq n_0$, be fixed. For $u, v \in W_0^{m,q}(D)$, we obtain by using Hölder's inequality, (A2), and the Lipschitz continuity of f,

$$\begin{aligned}\langle A_n(u), v\rangle_{q',q} &\leq \|a(x,u,\nabla u)\|_{p'} \|\nabla v\|_p + \frac{1}{n}\|u\|_{H_0^m}\|v\|_{H_0^m} \\ &\quad + \frac{1}{n}\|u\|_{W_0^{m,q}}^{q-1}\|v\|_{W_0^{m,q}} + L_f\|u\|_2\|v\|_2 \\ &\leq 2(C_3\|\nabla u\|_p^{p-1} + C_4\|u\|_p^{p-1} + \|g\|_{p'})\|\nabla v\|_p + \frac{1}{n}\|u\|_{H_0^m}\|v\|_{H_0^m} \\ &\quad + \frac{1}{n}\|u\|_{W_0^{m,q}}^{q-1}\|v\|_{W_0^{m,q}} + L_f\|u\|_2\|v\|_2.\end{aligned}$$

Note that $W_0^{m,q}(D)$ is continuously embedded into $W_0^{1,p}(D) \cap H_0^m(D)$, which implies with Poincaré's inequality, for any $u, v \in W_0^{m,q}(D)$,

$$\begin{aligned}\langle A_n(u), v\rangle_{q',q} &\leq 2\left[C_E(C_3 + C_{\text{Poin}}^{p-1}C_4)\|u\|_{W_0^{m,q}}^{p-1} + \|g\|_{p'}\right] C_E\|v\|_{W_0^{m,q}} \\ &\quad + \frac{C_E^2}{n}\|u\|_{W_0^{m,q}}\|v\|_{W_0^{m,q}} + \frac{1}{n}\|u\|_{W_0^{m,q}}^{q-1}\|v\|_{W_0^{m,q}} \\ &\quad + L_f C_E^2\|u\|_{W_0^{m,q}}\|v\|_{W_0^{m,q}}.\end{aligned}$$

Consequently, there exist constants $K_3, K_4 \geq 0$ depending on n, such that

$$\|A_n(u)\|_{W^{-m,q'}}^{q'} \leq K_3\left(\|u\|_{W_0^{m,q}}^{q'(p-1)} + \|u\|_{W_0^{m,q}}^{q'} + \|u\|_{W_0^{m,q}}^q\right) + K_4.$$

By definition of q, we know that $q' < q$ and $q'(p-1) \leq q$ and, hence,

$$\|A_n(u)\|_{W^{-m,q'}}^{q'} \leq C_{3.2.6}^1 + C_{3.2.6}^2 \|u\|_{W_0^{m,q}}^q \quad \forall u \in W_0^{m,q}(D)$$

for constants $C_{3.2.6}^1, C_{3.2.6}^2 \in \mathbb{R}$ depending on n. □

In the following, let

$$N_0 := \max\left\{\lceil \sqrt{C_\sigma}\rceil, \left(2^{q-1} C_2 C_E^{2\nu} C_{\text{Poin}}^{\nu}\right)^{-1}\right\}, \tag{3.17}$$

i.e., if we choose $n > N_0$, then n is large enough such that Lemma 3.2.5 holds.

Proposition 3.2.7. *For any $n \in \mathbb{N},\ n > N_0$, there exists a unique probabilistically strong solution u_n to the approximated equation* (3.12)*, i.e., $u_n \in L^2(\Omega; C([0,T]; L^2(D))) \cap L^q(\Omega; L^q(0,T; W_0^{m,q}(D)))$ is a $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process which satisfies $u_n(0,\cdot) = u_0$ in $L^2(\Omega; L^2(D))$ and, for all $t \in [0,T]$, in $L^2(D)$, $\mathbb{P}$-a.s. in Ω,*

$$u_n(t) = u_0 + \int_0^t A_n(u_n(s))\, ds + \int_0^t B_n(s, u_n(s))\, dW_s.$$

Proof. Using Lemma 3.2.4, Lemma 3.2.5, and Lemma 3.2.6 in connection with (3.15), the result follows from [84, Theorem 5.1.3]. □

Since the case $\alpha = 1$ is already known (see, e.g., [85], [99]), we only consider $\alpha \in (0,1)$ in the following.

3.3 Existence of a Martingale Solution

3.3.1 *A priori* Estimates

In the following, for $n \in \mathbb{N}$ with $n > N_0$, let u_n be the solution function to (3.12) found in Proposition 3.2.7, where N_0 is defined in (3.17).

Lemma 3.3.1. *There exists a constant $C_{3.3.1} \geq 0$ not depending on $n \in \mathbb{N}$, such that for all $n \in \mathbb{N},\ n > N_0$, and $t \in [0,T]$*

$$\mathbb{E}\left[\|u_n(t)\|_2^2\right] + \mathbb{E}\left[\int_0^t \|\nabla u_n(s)\|_p^p\, ds\right] + \frac{1}{n}\mathbb{E}\left[\int_0^t \|u_n(s)\|_{H_0^m}^2\, ds\right]$$
$$+ \frac{1}{n}\mathbb{E}\left[\int_0^t \|u_n(s)\|_{W_0^{m,q}}^q\, ds\right] \leq C_{3.3.1}.$$

Proof. Let $n \in \mathbb{N}$, $n > N_0$, and $t \in [0,T]$ be arbitrary. We get from Itô's formula, $\mathbb{P}$-a.s. in Ω,

$$\begin{aligned}\|u_n(t)\|_2^2 = \|u_0\|_2^2 &+ \int_0^t 2\langle \operatorname{div} a(\cdot, u_n, \nabla u_n), u_n\rangle_{q',q}\, ds - \frac{2}{n}\int_0^t \|u_n\|_{H_0^m}^2\, ds \\ &- \frac{2}{n}\int_0^t \int_D \sum_{|\gamma|\le m} |\nabla^\gamma u_n|^q\, dx\, ds - 2\int_0^t \int_D f(u_n) u_n\, dx\, ds \\ &+ \int_0^t \|B_n(s,u_n)\|_{\mathrm{HS}}^2\, ds + 2\int_0^t \langle B_n(s,u_n)(\cdot), u_n\rangle_{L^2}\, dW_s.\end{aligned}$$

Taking the expectation provides, by using the Lipschitz continuity of f and Corollary 3.2.3 *iii)*,

$$\begin{aligned}&\mathbb{E}\left[\|u_n(t)\|_2^2\right] + 2\mathbb{E}\left[\int_0^t \int_D a(x,u_n,\nabla u_n)\cdot \nabla u_n\, dx\, ds\right] \\ &+ \frac{2}{n}\mathbb{E}\left[\int_0^t \|u_n\|_{H_0^m}^2\, ds\right] + \frac{2}{n}\mathbb{E}\left[\int_0^t \|u_n\|_{W_0^{m,q}}^q\, ds\right] \\ &\le \mathbb{E}\left[\|u_0\|_2^2\right] + 2L_f \mathbb{E}\left[\int_0^t \|u_n\|_2^2\, ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t 2\left((C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2 + C_\sigma C_k \|u_n\|_2^2\right)\, ds\right] \\ &= \mathbb{E}\left[\|u_0\|_2^2\right] + 2(C_\alpha^2 + C_\sigma)t\|k\|_{L^2(D\times D)}^2 + 2(C_\sigma C_k + L_f)\mathbb{E}\left[\int_0^t \|u_n\|_2^2\, ds\right].\end{aligned} \tag{3.18}$$

By using (A2), we obtain

$$\begin{aligned}&\mathbb{E}\left[\|u_n(t)\|_2^2\right] + 2\mathbb{E}\left[\int_0^t \int_D \kappa(x) + C_1|\nabla u_n|^p - C_2|u_n|^\nu\, dx\, ds\right] \\ &\le \mathbb{E}\left[\|u_0\|_2^2\right] + 2(C_\alpha^2 + C_\sigma)t\|k\|_{L^2(D\times D)}^2 + 2(C_\sigma C_k + L_f)\mathbb{E}\left[\int_0^t \|u_n\|_2^2\, ds\right].\end{aligned}$$

Let $\eta > 0$. Since $\nu < p$, we can use Young's inequality in the following way:

$$\mathbb{E}\left[\int_0^t \int_D |u_n|^\nu\, dx\, ds\right] \le \eta\frac{\nu}{p}\mathbb{E}\left[\int_0^t \int_D |u_n|^p\, dx\, ds\right] + \frac{1}{\eta}\frac{p}{p-\nu}t|D|.$$

Applying Poincaré's inequality, we obtain by (3.18)

$$
\begin{aligned}
&\mathbb{E}\left[\|u_n(t)\|_2^2\right] + 2\left(C_1 - C_2 C_{\mathrm{Poin}}^p \frac{\nu}{p}\eta\right)\mathbb{E}\left[\int_0^t \|\nabla u_n\|_p^p \, ds\right] \\
&\le \mathbb{E}\left[\|u_0\|_2^2\right] + \left(2(C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2 + 2\|\kappa\|_1 + 2C_2 \frac{p}{\eta(p-\nu)}|D|\right) T \\
&\quad + 2(C_k C_\sigma + L_f)\mathbb{E}\left[\int_0^t \|u_n\|_2^2 \, ds\right],
\end{aligned}
$$

where we can choose $\eta > 0$ small enough, such that $C_1 - 2C_2 C_{\mathrm{Poin}}^p \frac{\nu}{p}\eta > 0$. Using Gronwall's lemma, we get for all $t \in [0,T]$ and $n > N_0$

$$
\mathbb{E}\left[\|u_n(t)\|_2^2\right] \le K_5\left(1 + K_6 T e^{K_6 T}\right),
$$

for constants $K_5, K_6 \ge 0$ independent of n and t. Consequently, we get from (3.18) by using (A2)

$$
\begin{aligned}
&\mathbb{E}\left[\|u_n(t)\|_2^2\right] + 2\left(C_1 - C_2 C_{\mathrm{Poin}}^p \frac{\nu}{p}\eta\right)\mathbb{E}\left[\int_0^t \|\nabla u_n\|_p^p \, ds\right] \\
&+ \frac{2}{n}\mathbb{E}\left[\int_0^t \|u\|_{H_0^m}^2 \, ds\right] + \frac{2}{n}\mathbb{E}\left[\int_0^t \|u_n\|_{W_0^{m,q}}^q \, ds\right] \\
&\le \mathbb{E}\left[\|u_0\|_2^2\right] + 2\|\kappa\|_1 + 2(C_\alpha^2 + C_\sigma)t\|k\|_{L^2(D\times D)} \\
&\quad + 2(C_\sigma C_k + L_f)T K_5\left(1 + K_6 T e^{K_6 T}\right)
\end{aligned}
$$

for all $t \in [0,T]$ and $n > N_0$. □

Lemma 3.3.2. *There exists a constant $C_{3.3.2} \ge 0$ not depending on $n \in \mathbb{N}$, such that*

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_2^2\right] \le C_{3.3.2} \quad \forall n > N_0.
$$

Proof. Let $n \in \mathbb{N}$, $n > N_0$, be arbitrary. Using Itô's formula, (A2), and

Corollary 3.2.3 *iii*), we get for all $t \in [0,T]$, $\mathbb{P}$-a.s. in Ω,

$$
\begin{aligned}
\|u_n(t)\|_2^2 \leq \|u_0\|_2^2 &- 2\int_0^t \int_D a(x,u_n,\nabla u_n)\cdot \nabla u_n \,dx\,ds \\
&- 2\int_0^t \int_D f(u_n)u_n\,dx\,ds + \int_0^t \|B_n(s,u_n)\|_{\mathrm{HS}}^2\,ds \\
&+ 2\int_0^t \langle B_n(s,u_n)(\cdot),u_n\rangle_{L^2}\,dW_s \\
&\leq \|u_0\|_2^2 + 2\|\kappa\|_1 + C_2\int_0^t\int_D |u_n|^\nu\,dx\,ds \\
&+ 2(L_f + C_\sigma C_k)\int_0^t \|u_n\|_2^2\,ds + 2(C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2 t \\
&+ 2\int_0^t \langle B_n(s,u_n)(\cdot),u_n\rangle_{L^2}\,dW_s.
\end{aligned}
$$

Since $\nu < p$ and, therefore, $L^p(D) \hookrightarrow L^\nu(D)$, we obtain, by applying Poincaré's inequality (with constant C_{Poin}), for all $t \in [0,T]$, $\mathbb{P}$-a.s. in Ω,

$$
\begin{aligned}
\int_0^t \|u_n\|_\nu^\nu\,ds \leq C_E^\nu \int_0^t \|u_n\|_p^\nu\,ds &\leq C_E^\nu C_{\mathrm{Poin}}^\nu \int_0^t \|\nabla u_n\|_p^\nu\,ds \\
&\leq C_E^\nu C_{\mathrm{Poin}}^\nu 2^p \int_0^t \left(1 + \|\nabla u_n\|_p^p\right)\,ds.
\end{aligned}
$$

Taking first the supremum over all $t \in [0,T]$ and then the expectation provides

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\|u_n(t)\|_2^2\right] \leq \mathbb{E}\left[\|u_0\|_2^2\right] &+ 2\|\kappa\|_1 T + C_2 C_E^\nu C_{\mathrm{Poin}}^\nu \mathbb{E}\left[\int_0^T \|\nabla u_n\|_p^p\,ds\right] \\
&+ 2(L_f + C_\sigma C_k)\mathbb{E}\left[\int_0^T \|u_n\|_2^2\,ds\right] \\
&+ \Big(2(C_\alpha^2 + C_\sigma)\|k\|_{L^2(D\times D)}^2 + C_2 C_E^\nu C_{\mathrm{Poin}}^\nu\Big)T \\
&+ 2\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \langle B_n(s,u_n)(\cdot),u_n\rangle_{L^2}\,dW_s\right|\right].
\end{aligned}
\tag{3.19}
$$

Using the Burkholder-Davis-Gundy inequality (see [85, Theorem 1.1.7]) and

Young's inequality with $\beta > 0$ such that $1 - 2C_{BDG}\beta > 0$, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \langle B_n(s,u_n(s))(\cdot),u_n(s)\rangle_{L^2}\,dW_s\right|\right]\\
&\le C_{BDG}\mathbb{E}\left[\left(\int_0^T |\langle B_n(s,u_n(s))(\cdot),u_n(s)\rangle_{L^2}|^2 ds\right)^{\frac{1}{2}}\right]\\
&\le C_{BGD}\mathbb{E}\left[\left(\sup_{t\in[0,T]}\|u_n(t)\|_2^2\int_0^T \|B_n(s,u_n(s))\|_{\mathrm{HS}}^2\,ds\right)^{\frac{1}{2}}\right]\\
&\le C_{BDG}\mathbb{E}\left[\beta\sup_{t\in[0,T]}\|u_n(t)\|_2^2\right] + C_{BDG}\mathbb{E}\left[\beta^{-1}\int_0^T \|B_n(s,u_n(s))\|_{\mathrm{HS}}^2\,ds\right].
\end{aligned}$$

Using this inequality in (3.19) implies

$$\begin{aligned}
&(1-2C_{BDG}\beta)\mathbb{E}\left[\sup_{t\in[0,T]}\|u_n(t)\|_2^2\right]\\
&\le C_2C_E^\nu C_{\mathrm{Poin}}^\nu\mathbb{E}\left[\int_0^T \|\nabla u_n\|_p^p\,ds\right] + 2(L_f + C_\sigma C_k)\mathbb{E}\left[\int_0^T \|u_n\|_2^2\,ds\right]\\
&\quad + 2C_{BDG}\beta^{-1}\mathbb{E}\left[\int_0^T \|B_n(s,u_n)\|_{\mathrm{HS}}^2\,ds\right] + K_7
\end{aligned}$$

for a constant $K_7 \ge 0$ not depending on n and t. The assertion follows from Lemma 3.3.1 and Corollary 3.2.3 *iii*). □

Lemma 3.3.3. *The sequence*

$$(a(\cdot,u_n,\nabla u_n))_{n>N_0} \text{ is bounded in } L^{p'}(\Omega;L^{p'}(0,T;L^{p'}(D)^d)).$$

Proof. The boundedness follows from (A2), Poincaré's inequality, and Lemma 3.3.1. □

Lemma 3.3.4. *The sequence*

$$\left(\partial_t\left(u_n(t) - \int_0^t B_n(s,u_n(s))dW_s\right)\right)_{n>N_0}$$

is bounded in $L^{q'}(\Omega;L^{q'}(0,T;W^{-m,q'}(D)))$.

Proof. Let $n\in\mathbb{N},\ n>N_0$, be arbitrary. Since $u_n \in L^2(\Omega;C([0,T];L^2(D)))\cap L^q(\Omega;L^q(0,T;W_0^{m,q}(D)))$ is a solution to (3.12) by Proposition 3.2.7, we get for $t \in [0,T], \varphi \in W_0^{m,q}(D)$, $\mathbb{P}$-a.s. in Ω,

$$\begin{aligned}
&\langle \partial_t \left(u_n(t) - \int_0^t B_n(s, u_n(s))\, dW_s \right), \varphi \rangle_{q',q} \\
&= - \int_D a(x, u_n(t), \nabla u_n(t)) \cdot \nabla \varphi \, dx - \frac{1}{n} (u_n(t), \varphi)_{H_0^m} \\
&\quad - \frac{1}{n} \int_D \sum_{|\gamma| \leq m} |\nabla^\gamma u_n(t)|^{q-2} \nabla^\gamma u_n(t) \cdot \nabla^\gamma \varphi \, dx - \int_D f(u_n(t)) \varphi \, dx \\
&\leq \|a(\cdot, u_n(t), \nabla u_n(t))\|_{p'} \|\nabla \varphi\|_p + \frac{1}{n} \|u_n(t)\|_{H_0^m} \|\varphi\|_{H_0^m} \\
&\quad + \frac{1}{n} \|u_n(t)\|_{W_0^{m,q}}^{q-1} \|\varphi\|_{W_0^{m,q}} + L_f \|u_n(t)\|_2 \|\varphi\|_2.
\end{aligned}$$

Since we have the continuous embedding

$$W_0^{m,q}(D) \hookrightarrow W_0^{1,p}(D) \cap H_0^m(D) \cap L^2(D),$$

we know, that there exists a constant $C_E \geq 0$, such that

$$(\|\varphi\|_{H_0^m} + \|\nabla \varphi\|_p + \|\varphi\|_2) \leq C_E \|\varphi\|_{W_0^{m,q}}.$$

Hence, we obtain, by taking the supremum over all $\varphi \in W_0^{m,q}(D)$ with $\|\varphi\|_{W_0^{m,q}} = 1$,

$$\begin{aligned}
&\left\| \partial_t \left(u_n(t) - \int_0^t B_n(s, u_n(s))\, dW_s \right) \right\|_{W^{-m,q'}} \\
&\leq C_E \|a(x, u_n(t), \nabla u_n(t))\|_{p'} + \frac{C_E}{n} \|u_n(t)\|_{H_0^m} \\
&\quad + \frac{1}{n} \|u_n(t)\|_{W_0^{m,q}}^{q-1} + C_E L_f \|u_n(t)\|_2 \\
&\leq C_E \|a(x, u_n(t), \nabla u_n(t))\|_{p'} + \frac{C_E^2}{n} \|u_n(t)\|_{W_0^{m,q}} \\
&\quad + \frac{1}{n} \|u_n(t)\|_{W_0^{m,q}}^{q-1} + C_E L_f \|u_n(t)\|_2 \\
&\leq C_E \|a(x, u_n(t), \nabla u_n(t))\|_{p'} + 2^{q-1} \frac{C_E^2}{n} \\
&\quad + \frac{1}{n} (2^{q-1} C_E^2 + 1) \|u_n(t)\|_{W_0^{m,q}}^{q-1} + C_E L_f \|u_n(t)\|_2,
\end{aligned}$$

for all $t \in [0,T]$, $\mathbb{P}$-a.s. in Ω, where we used the fact that $q \geq 2$. Because

$q' \leq \min\{p', 2\}$, there holds, for all $t \in [0,T]$, $\mathbb{P}$-a.s. in Ω,

$$
\begin{aligned}
&\left\| \partial_t \left(u_n(t) - \int_0^t B_n(s, u_n(s))\, dW_s \right) \right\|_{W^{-m,q'}}^{q'} \\
&\leq 8^{q'} \Bigg[C_E^{q'} \|a(x, u_n(t), \nabla u_n(t))\|_{p'}^{q'} + 2^q \left(\frac{C_E^2}{n} \right)^{q'} \\
&\quad + \frac{1}{n^{q'}} (2^{q-1} C_E^2 + 1)^{q'} \|u_n(t)\|_{W_0^{m,q}}^q + C_E^{q'} L_f^{q'} \|u_n(t)\|_2^{q'} \Bigg] \\
&\leq 8^{q'} \Big[2^{q'} C_E^{q'} \left(1 + \|a(x, u_n(t), \nabla u_n(t))\|_{p'}^{p'} \right) + 2^q C_E^{2q'} \\
&\quad + \frac{1}{n} (2^{q-1} C_E^2 + 1)^{q'} \|u_n(t)\|_{W_0^{m,q}}^q + 2 C_E^{q'} L_f^{q'} (1 + \|u_n(t)\|_2^2) \Big].
\end{aligned}
$$

Integrating on $[0,T]$ and taking the expectation provide the boundedness by Lemma 3.3.1 and Lemma 3.3.3. $\square$

Lemma 3.3.5. *The sequence $(B_n(\cdot, u_n))_{n>N_0}$ is bounded in $L^2(\Omega; L^2(0,T; HS(L^2(D))))$ and*

$$
\left(\int_0^\cdot B_n(s, u_n(s))\, dW_s \right)_{n>N_0} \quad \text{is bounded in } L^2(\Omega; C([0,T]; L^2(D))),
$$

both for a constant $C_{3.3.5} \geq 0$ not depending on n.

Proof. By Burkholder-Davis-Gundy inequality and Corollary 3.2.3 *iii*), we obtain for any $n \in \mathbb{N},\ n > N_0$,

$$
\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0,T]} \left\| \int_0^t B_n(s, u_n(s)) dW_s \right\|_2^2 \right] \\
&\leq C_{BDG} \mathbb{E} \left[\int_0^T \|B_n(s, u_n(s))\|_{\mathrm{HS}}^2\, ds \right] \\
&\leq C_{BDG} 2 \left((C_\alpha^2 + C_\sigma) \|k\|_{L^2(D \times D)}^2 T + C_\sigma C_k \mathbb{E} \left[\int_0^T \|u_n(s)\|_2^2\, ds \right] \right)
\end{aligned}
$$

and, by Lemma 3.3.1, this expression is bounded. $\square$

3.3.2 Tightness Results

Lemma 3.3.6. *The sequence*

$$
\left(u_n - \int_0^\cdot B_n(s, u_n(s))\, dW_s \right)_{n>N_0}
$$

is bounded in $L^{q'}(\Omega; W^{\beta,2}(0,T; W^{-m,q'}(D)))$ for all $\beta \in (0, \frac{1}{2})$.

Proof. We know that

$$\mathfrak{V} := \Big\{ v \in L^2(0,T;L^2(D)) : \partial_t v \in L^{q'}(0,T;W^{-m,q'}(D)) \Big\} \tag{3.20}$$

is compactly embedded into $W^{1,q'}(0,T;W^{-m,q'}(D))$ and, by [117, Corollary 19], also in $W^{\beta,2}(0,T;W^{-m,q'}(D))$ for any $\beta \in (0,\frac{1}{2})$. It follows from Lemma 3.3.2, Lemma 3.3.4, and Lemma 3.3.5, that

$$\left(u_n - \int_0^{\cdot} B_n(s,u_n(s))\,dW_s \right)_{n>N_0} \quad \text{is bounded in } L^{q'}(\Omega;\mathfrak{V}) \tag{3.21}$$

and, therefore, also in $L^{q'}(\Omega;W^{\beta,2}(0,T;W^{-m,q'}(D)))$. The assertion follows from a standard Markov inequality. □

Lemma 3.3.7. *The sequence*

$$\left(\int_0^{\cdot} B_n(s,u_n(s))\,dW_s \right)_{n>N_0} \quad \textit{is bounded in } L^2(\Omega;W^{\beta,2}(0,T;L^2(D))).$$

Proof. From Lemma 3.3.5, we know that $(B_n(\cdot,u_n))_{n>N_0}$ is bounded in $L^2(\Omega;L^2(0,T;\mathrm{HS}(L^2(D))))$. Using [59, Lemma 2.1, p.369], we get for any $\beta \in (0,\frac{1}{2})$ and $n \in \mathbb{N}, n > N_0$,

$$\begin{aligned} \mathbb{E}\left[\left\| \int_0^{\cdot} B_n(s,u_n(s))\,dW_s \right\|^2_{W^{\beta,2}(0,T;L^2(D))} \right] &\le C(\beta)\mathbb{E}\left[\int_0^T \|B_n(s,u_n(s))\|^2_{\mathrm{HS}}\,dt \right] \\ &\le C(\beta) C_{3.3.5}. \end{aligned} \tag{3.22}$$

□

Lemma 3.3.8. *For all $R > 0$ and $1 \le s < \infty$,*

$$K_R := \Big\{ v \in L^p(0,T;W_0^{1,p}(D)) \cap W^{\beta,2}(0,T;W^{-m,q'}(D)) \cap C([0,T];L^2(D)) : |||v||| < R \Big\}$$

is relatively compact in $L^s(0,T;L^2(D))$, where

$$|||v||| := \|v\|_{L^p(0,T;W_0^{1,p}(D))} + \|v\|_{W^{\beta,2}(0,T;W^{-m,q'}(D))} + \|v\|_{C([0,T];L^2(D))}.$$

Proof. Using the compact embeddings $W_0^{1,p}(D) \hookrightarrow L^2(D) \hookrightarrow W^{-m,q'}(D)$, we obtain from [116, Corollary 7], that K_R is relatively compact in $L^s(0,T;L^2(D))$ for all $1 \le s < \infty$. □

Lemma 3.3.9. *The sequence of laws of $(u_n)_{n>N_0}$ is tight on $L^s(0,T;L^2(D))$ for any $1 \leq s < \infty$.*

Proof. Using Lemma 3.3.6 and Lemma 3.3.7 with the knowledge that $q' \leq 2$, we know, that the sequence

$$\left(u_n = u_n - \int_0^\cdot B_n(s,u_n(s))\,dW_s + \int_0^\cdot B_n(s,u_n(s))\,dW_s\right)_{n>N_0}$$

is bounded in $L^{q'}(\Omega; W^{\beta,2}(0,T;W^{-m,q'}(D)))$ for any $\beta \in (0,\frac{1}{2})$. Furthermore, we obtain by Lemma 3.18, that $(u_n)_{n>N_0}$ is bounded in $L^p(\Omega;L^p(0,T;W_0^{1,p}(D))) \cap L^2(\Omega;L^2(0,T;L^2(D)))$. By Lemma 3.3.8, K_R is relatively compact in $L^s(0,T;L^2(D))$ for all $R > 0$ and $1 \leq s < \infty$. For all $n \in \mathbb{N}, n > N_0$, and an appropriate $R > 0$,

$$\begin{aligned}
\mu_{u_n}(L^s(0,T;L^2(D)) \setminus K_R) &= \int_{\{v \in L^s(0,T;L^2(D)) : |||v||| \geq R\}} 1\,d\mu_{u_n} \\
&= \int_{\{\omega \in \Omega : |||u_n(\omega)||| \geq R\}} 1\,d\mathbb{P} \\
&= \frac{1}{R^{q'}} \int_{\{\omega \in \Omega : |||u_n(\omega)||| \geq R\}} R^{q'}\,d\mathbb{P} \\
&\leq \frac{1}{R^{q'}} \int_\Omega |||u_n|||^{q'}\,d\mathbb{P}.
\end{aligned}$$

□

3.3.3 Passage to the Limit

For $n \in \mathbb{N}$ with $n > N_0$, we consider the vector

$$Y_n := (u_n, W, u_0) \quad \text{in} \quad \mathcal{X} = L^s(0,T;L^2(D)) \times C([0,T];U) \times L^2(D).$$

By Lemma 3.3.9 and Prokhorov's theorem, a not relabeled subsequence of $(u_n)_{n>N_0}$ converges in law for $n \to \infty$ to a probability measure μ_∞ with respect to $L^s(0,T;L^2(D))$ for all $1 \leq s < \infty$. Skorokhod's theorem implies the existence of

- a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ (which always can be chosen as $([0,1], \mathcal{B}([0,1]), \lambda)$ with $\mathcal{B}([0,1])$ the set of all Borel measures on $[0,1]$ and λ the one-dimensional Lebesgue-measure, see [31, Theorem 2.6.3])
- a family of random variables $\mathcal{Y}_n = (v_n, \mathcal{W}, v_0)$ on $(\Omega', \mathcal{A}', \mathbb{P}')$ with values in $\mathcal{X}$ having the same law as Y_n

- a random variable u_∞ with values in $L^2(0,T;L^2(D))$ such that the law of u_∞ is equal to the law of u_n and for $n \to \infty$

$$v_n \to u_\infty \text{ in } L^s(0,T;L^2(D)),\ \mathbb{P}'\text{-a.s. in } \Omega'. \tag{3.23}$$

Remark 3.3.10. *By [35, Theorem C.1], the random variables $\mathcal{W}$ and v_0 are independent of n.*

Lemma 3.3.11. *We have $v_n \in L^2(\Omega';C([0,T];L^2(D)))$, $v_n(0) = v_0$ a.e. in $\Omega' \times D$, $\mathcal{W}(0) = 0$, $\mathbb{P}'$-a.s. in Ω', and the following convergences hold true for $n \to \infty$ after passing to a not relabeled subsequence if necessary:*

i) $v_n \to u_\infty$ *in* $L^\varrho(\Omega', L^s(0,T;L^2(D)))$ *for all* $\varrho < 2$ *and all* $1 \le s < \infty$

ii) $\nabla v_n \rightharpoonup \nabla u_\infty$ *in* $L^p(\Omega';L^p(0,T;L^p(D)^d))$

iii) $a(\cdot, v_n, \nabla v_n) \rightharpoonup G$ *in* $L^{p'}(\Omega';L^{p'}(0,T;L^{p'}(D)^d))$ *for an element* $G \in L^{p'}(\Omega';L^{p'}(0,T;L^{p'}(D)^d))$

iv) $f(v_n) \rightharpoonup f(u_\infty)$ *in* $L^2(\Omega';L^2(0,T;L^2(D)))$ *and* $f(v_n) \to f(u_\infty)$ *in* $L^\varrho(\Omega';L^s(0,T;L^2(D)))$ *for all* $\varrho < 2$ *and* $1 \le s < \infty$.

Proof. By equality in law, we know $v_n \in L^2(\Omega';C([0,T];L^2(D)))$, $v_n(0) = 0$, and $\mathcal{W}(0) = 0$, $\mathbb{P}'$-a.s. in Ω' (see [119, Lemma A.3]).
i) Since $(u_n)_{n>N_0}$ is bounded in $L^2(\Omega;L^2(0,T;L^2(D)))$ by Lemma 3.3.2,

$$(v_n)_{n>N_0} \text{ is bounded in } L^2(\Omega';L^2(0,T;L^2(D))) \tag{3.24}$$

by equality in law. Using (3.23), we obtain by Vitali's theorem (see [52, Corollaire 1.3.3]) the claimed convergence.
ii) By Lemma 3.3.1 and equality in law,

$$(\nabla v_n)_{n>N_0} \text{ is bounded in } L^p(\Omega';L^p(0,T;L^p(D)^d)). \tag{3.25}$$

Hence, there exists a not relabeled subsequence, such that $\nabla v_n \rightharpoonup \varphi$ in $L^p(\Omega';L^p(0,T;L^p(D)^d))$ for $n \to \infty$ and an element φ which can be verified as ∇u_∞.
iii) Using (3.24) and (3.25), we can show, by an analogous argumentation as in Lemma 3.3.3, that $(a(\cdot, v_n, \nabla v_n))_{n>N_0}$ is bounded in $L^{p'}(\Omega';L^{p'}(0,T;L^{p'}(D)^d))$ and is hence weak convergent.
iv) The convergences are a consequence of the Lipschitz continuity of f. $\square$

Definition 3.3.12. *For $t \in [0,T]$ and $n \in \mathbb{N}$, $n > N_0$, we define $(F_t^n)_{t\in[0,T]}$ to be the smallest sub-σ-field of $\mathcal{A}'$ generated by v_0, $v_n(s)$, and $\mathcal{W}(s)$ for $0 \leq s \leq t$. The right-continuous, $\mathbb{P}'$-augmented filtration of $(F_t^n)_{t\in[0,T]}$ denoted by $(\mathcal{F}_t^n)_{t\in[0,T]}$ is, for any $t \in [0,T]$, defined by*

$$\mathcal{F}_t^n := \bigcap_{s>t} \sigma[F_s^n \cup \{\mathcal{N} \in \mathcal{A}' : \mathbb{P}'(\mathcal{N}) = 0\}].$$

Remark 3.3.13. *From the previous definition, it immediately follows that v_0 is $\mathcal{F}_0^n$-measurable for all $n > N_0$.*

Lemma 3.3.14. *For each $n \in \mathbb{N}$ with $n > N_0$, v_n is adapted to $(\mathcal{F}_t^n)_{t\in[0,T]}$ and $\mathcal{W} = (\mathcal{W}(t))_{t\in[0,T]}$ is a $(\mathcal{F}_t^n)_{t\in[0,T]}$-adapted Q-Wiener process with values in U.*

Proof. Obviously, v_n and $\mathcal{W}$ are $(\mathcal{F}_t^n)_{t\in[0,T]}$-adapted for any $n > N_0$ by construction of the filtration. By equality in law, we know $\mathcal{W}(0) = 0$, $\mathbb{P}'$-a.s. in Ω', and, using Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}'\left[\sup_{t\in[0,T]} |\mathcal{W}(t)|_U^2\right] = \mathbb{E}\left[\sup_{t\in[0,T]} |W(t)|_U^2\right] \leq C_{BDG}\operatorname{Tr}(Q)T < \infty.$$

Let $n \in \mathbb{N}, n > N_0$, be arbitrary. For all $k \in \mathbb{N}$, $0 \leq s \leq t \leq T$, and all bounded and continuous functions $\psi : \mathcal{X} \to \mathbb{R}$, we obtain, by equality in law,

$$\begin{aligned} \mathbb{E}'\left[\langle \mathcal{W}(t) - \mathcal{W}(s), e_k\rangle_U \psi\left((\mathcal{Y}_n)_{|[0,s]}\right)\right] &= \mathbb{E}\left[\langle W(t) - W(s), e_k\rangle_U \psi\left((Y_n)_{|[0,s]}\right)\right] \\ &= 0, \end{aligned} \tag{3.26}$$

where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of U. The real-valued random variable $\omega' \ni \Omega' \mapsto \psi\left((\mathcal{Y}_n)_{|[0,s]}(\omega')\right)$ is $(F_t^n)_{t\in[0,T]}$-measurable by definition. Using (3.26), we obtain for all $k \in \mathbb{N}$, $0 \leq s \leq t \leq T$, and all bounded and continuous functions $\psi : \mathcal{X} \to \mathbb{R}$

$$\begin{aligned} 0 &= \mathbb{E}'\left[\langle \mathcal{W}(t) - \mathcal{W}(s), e_k\rangle_U \psi\left((\mathcal{Y}_n)_{|[0,s]}\right)\right] \\ &= \mathbb{E}'\left[\mathbb{E}'\left[\langle \mathcal{W}(t) - \mathcal{W}(s), e_k\rangle_U \psi\left((\mathcal{Y}_n)_{|[0,s]}\right)\right] |F_s^n\right] \\ &= \mathbb{E}'\left[\psi\left((\mathcal{Y}_n)_{|[0,s]}\right) \mathbb{E}'\left[\langle \mathcal{W}(t) - \mathcal{W}(s), e_k\rangle_U | F_s^n\right]\right]. \end{aligned}$$

The Doob-Dynkin lemma (see, e.g., [104, Proposition 3]) implies

$$\mathbb{E}'\left[\mathbb{1}_A \mathbb{E}'\left[\langle \mathcal{W}(t) - \mathcal{W}(s), e_k\rangle_U | F_s^n\right]\right] = 0$$

for all $k \in \mathbb{N}$, $0 \leq s \leq t \leq T$, and all F_s^n-measurable sets $A \in \mathcal{A}'$. Consequently, there holds

$$\mathbb{E}'\left[\langle \mathcal{W}(t) - \mathcal{W}(s), e_k \rangle_U | F_s^n\right] = 0, \quad \mathbb{P}'\text{-a.s. in } \Omega',$$

for all $0 \leq s \leq t \leq T$, and $k \in \mathbb{N}$. Hence, $\mathcal{W}$ is a $(F_t^n)_{t\in[0,T]}$-martingale, and, by [45, p.75] $\mathcal{W}$ is a martingale with respect to the augmented filtration $(\mathcal{F}_t^n)_{t\in[0,T]}$. Using equality in law of $\mathcal{W}$ and W, we obtain for all $0 \leq s \leq t \leq T$ and $k, j \in \mathbb{N}$

$$\begin{aligned}
0 =& \mathbb{E}\big[\big(\langle W(t) - W(s), e_k \rangle_U \langle W(t) - W(s), e_j \rangle_U \\
&\quad - \langle (t-s)Q(e_k), e_j \rangle_U\big)\psi\left((Y_n)_{|[0,s]}\right)\big] \\
=& \mathbb{E}\big[\big(\langle \mathcal{W}(t) - \mathcal{W}(s), e_k \rangle_U \langle \mathcal{W}(t) - \mathcal{W}(s), e_j \rangle_U \\
&\quad - \langle (t-s)Q(e_k), e_j \rangle_U\big)\psi\left((\mathcal{Y}_n)_{|[0,s]}\right)\big].
\end{aligned}$$

With similar arguments as before, we get $\langle\langle \mathcal{W} \rangle\rangle_t = tQ$ for all $t \in [0,T]$, see [41, p.75], where $\langle\langle \mathcal{W} \rangle\rangle$ denotes the quadratic variation process of $\mathcal{W}$. By a generalized Levy's theorem (see [41, Theorem 4.6]) $\mathcal{W}$ is a Q-Wiener process with values in U. □

Lemma 3.3.15. *For any $n \in \mathbb{N}$, $n > N_0$, and $t \in [0,T]$, we define*

$$M_n(t) := v_n(t) - v_0 + \int_0^t \left[\frac{1}{n} j(v_n, \cdot) - \operatorname{div} a(\cdot, v_n, \nabla v_n) + f(v_n)\right] ds.$$

The stochastic process $(M_n(t))_{t\in[0,T]}$ is a square-integrable, continuous $(\mathcal{F}_t^n)_{t\in[0,T]}$-martingale with values in $L^2(D)$, such that, for each $t \in [0,T]$,

$$\langle\langle M_n \rangle\rangle_t = \int_0^t \left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right) \circ \left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right)^* ds \tag{3.27}$$

$$\langle\langle \mathcal{W}, M_n \rangle\rangle_t = \int_0^t Q \circ B_n(s, v_n(s))\, ds. \tag{3.28}$$

Proof. By Proposition 3.2.7 and equality in law, we know, that the stochastic process $(M_n(t))_{t\in[0,T]}$ has values in $L^2(D)$. Moreover, we get by definition of M_n and equality in law, for all $n > N_0$,

$$\begin{aligned}
\mathcal{L}(M_n) &= \mathcal{L}\left(v_n - v_0 + \int_0^{\cdot} \left[\frac{1}{n} j(v_n, \cdot) - \operatorname{div} a(\cdot, v_n, \nabla v_n) + f(v_n)\right] ds\right) \\
&= \mathcal{L}\left(u_n - u_0 + \int_0^{\cdot} \left[\frac{1}{n} j(u_n, \cdot) - \operatorname{div} a(\cdot, u_n, \nabla u_n) + f(u_n)\right] ds\right) \\
&= \mathcal{L}\left(\int_0^{\cdot} B_n(s, u_n)\, dW_s\right),
\end{aligned} \tag{3.29}$$

where $\mathcal{L}(\cdot)$ denotes the law. Therefore, $(M_n(t))_{t\in[0,T]}$ is a martingale with respect to $(\mathcal{F}_t^n)_{t\in[0,T]}$ for all $n > N_0$, that can be shown with similar arguments as in the proof of Lemma 3.3.14.
Let $n \in \mathbb{N}, n > N_0$, be arbitrary. Because the mapping $(t, v) \mapsto B_n(t, v)$ is measurable on $(0, T) \times L^2(D)$ by Corollary 3.2.3 *i)* and $\mathcal{L}(u_n) = \mathcal{L}(v_n)$, we know $\mathcal{L}(B_n(\cdot, u_n)) = \mathcal{L}(B_n(\cdot, v_n))$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(D)$. For all $k, j \in \mathbb{N}$, $0 \le s \le t \le T$, and all bounded and continuous functions $\psi : \mathcal{X} \to \mathbb{R}$, we get

$$\begin{aligned}
&\mathbb{E}'\Bigg[\Bigg(\langle M_n(t) - M_n(s), e_k\rangle_{L^2}\langle M_n(t) - M_n(s), e_j\rangle_{L^2} \\
&\quad - \langle \int_s^t \left(B_n(r, v_n)Q^{\frac{1}{2}}\right)\left(B_n(r, v_n)Q^{\frac{1}{2}}\right)^* (e_k)\, dr, e_j\rangle_{L^2}\Bigg)\psi\left((\mathcal{Y}_n)_{|[0,s]}\right)\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg(\langle \int_s^t B_n(r, u_n)\, dW_r, e_k\rangle_{L^2}\langle \int_s^t B_n(r, u_n)\, dW_r, e_j\rangle_{L^2} \\
&\quad - \langle \int_s^t \left(B_n(r, u_n)Q^{\frac{1}{2}}\right)\left(B_n(r, u_n)Q^{\frac{1}{2}}\right)^* (e_k)\, dr, e_j\rangle_{L^2}\Bigg)\psi\left((Y_n)_{|[0,s]}\right)\Bigg] \\
&= 0.
\end{aligned}$$

Consequently, we know, for any $t \in [0, T]$,

$$\langle\langle M_n\rangle\rangle_t = \int_0^t \left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right) \circ \left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right)^* ds.$$

Using (3.29) and $\mathcal{L}(\mathcal{W}) = \mathcal{L}(W)$, we obtain, for $t \in [0, T]$,

$$\langle\langle \mathcal{W}, M_n\rangle\rangle_t = \int_0^t Q \circ B_n(s, v_n(s))\, ds.$$

□

Lemma 3.3.16. *For all $n \in \mathbb{N}$, $n > N_0$, and all $t \in [0, T]$,*

$$M_n(t) = \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s \text{ in } L^2(\Omega'; L^2(D)).$$

In particular, we have for all $t \in [0, T]$, in $L^2(D)$, $\mathbb{P}'$-a.s. in Ω',

$$\begin{aligned}
&v_n(t) + \int_0^t \frac{1}{n}\partial J(v_n(s))\, ds - \int_0^t \operatorname{div} a(\cdot, v_n(s), \nabla v_n(s))\, ds \\
&+ \int_0^t f(v_n(s))\, ds = \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s
\end{aligned}$$

Proof. Let $n \in \mathbb{N}, n > N_0$, be arbitrary. Then, for all $t \in [0,T]$, we have

$$\begin{aligned}
&\mathbb{E}'\left[\left\|M_n(t) - \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s\right\|_2^2\right] \\
&= \sum_{k\in\mathbb{N}} \mathbb{E}'\left[\langle M_n(t) - \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s, e_k\rangle_{L^2}^2\right],
\end{aligned} \tag{3.30}$$

where, for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}'\left[\langle M_n(t) - \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s, e_k\rangle_{L^2}^2\right] \\
&= \mathbb{E}'\left[\langle M_n(t), e_k\rangle_{L^2}^2\right] - 2\mathbb{E}'\left[\langle M_n(t), e_k\rangle_{L^2}\langle \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s, e_k\rangle_{L^2}\right] \\
&\quad + \mathbb{E}'\left[\langle \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s, e_k\rangle_{L^2}^2\right].
\end{aligned} \tag{3.31}$$

By (3.27) and (3.29), we know for all $t \in [0,T]$

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \mathbb{E}'\left[\langle \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s, e_k\rangle_{L^2}^2\right] \\
&= \sum_{k\in\mathbb{N}} \mathbb{E}'\left[\langle M_n(t), e_k\rangle_{L^2}^2\right] \\
&= \mathbb{E}'\left[\int_0^t \mathrm{Tr}\left[\left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right) \circ \left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right)^*\right]\, ds\right].
\end{aligned}$$

Using (3.28), we further obtain for $t \in [0,T]$

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \mathbb{E}'\left[\langle M_n(t), e_k\rangle_{L^2}\langle \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s, e_k\rangle_{L^2}\right] \\
&= \mathbb{E}'\left[\mathrm{Tr}\left[\langle\langle M_n(\cdot), \int_0^{\cdot} B_n(s, v_n(s))\, d\mathcal{W}_s\rangle\rangle_t\right]\right] \\
&= \mathbb{E}'\left[\int_0^t \mathrm{Tr}\left[\left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right) \circ \left(B_n(s, v_n(s))Q^{\frac{1}{2}}\right)^*\right]\, ds\right].
\end{aligned}$$

Therefore, we get from (3.30) and (3.31), for all $t \in [0,T]$,

$$\mathbb{E}'\left[\left\|M_n(t) - \int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s\right\|_2^2\right] = 0.$$

□

Lemma 3.3.17. *The filtration $(\mathcal{F}_t^n)_{t\in[0,T]}$ can be chosen independently of n.*

Proof. Let $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ be the smallest filtration in $\mathcal{A}'$ generated by v_0 and $\mathcal{W}(s)$ for $0 \le s \le t \le T$ and augmented in order to satisfy the usual assumptions. One may show as before, that $\mathcal{W}$ is a Q-Wiener process with respect to $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$. Applying the arguments of Section 3.2.2 to the stochastic basis $(\Omega', \mathcal{A}', \mathbb{P}, (\widetilde{\mathcal{F}}_t)_{t\in[0,T]})$ associated with $\mathcal{W}$, there exists, for any $n \in \mathbb{N}$, a unique solution $\widetilde{u}_n$ to the approximated equation with the $\widetilde{\mathcal{F}}_0$-measurable initial datum v_0. By uniqueness $\widetilde{u}_n = v_n$. □

Lemma 3.3.18. *For all $t \in [0,T]$, we have, for $n \to \infty$, the convergence*

$$\int_0^t B_n(s, v_n(s))\, d\mathcal{W}_s \to \int_0^t B(s, u_\infty(s))\, d\mathcal{W}_s \quad \text{in } L^2(\Omega'; L^2(D)).$$

Proof. For all $n \in \mathbb{N}$, $n > N_0$, a.e. $t \in (0,T)$, $\mathbb{P}'$-a.s. in Ω', we know by applying the Parseval identity

$$\begin{aligned}
&\|B_n(t, v_n(t)) - B(t, v_n(t))\|_{\mathrm{HS}}^2 \\
&= \sum_{k\in\mathbb{N}} \int_D |\sigma_n(t, v_n(t,x)) - \sigma(t, v_n(t,x))|^2 \left| \int_D k(x,y) e_k(y)\, dy \right|^2 dx \\
&= \int_D |\sigma_n(t, v_n(t,x)) - \sigma(t, v_n(t,x))|^2 \sum_{k\in\mathbb{N}} |\langle k(x,\cdot), e_k\rangle_{L^2}|^2\, dx \\
&= \int_D |\sigma_n(t, v_n(t,x)) - \sigma(t, v_n(t,x))|^2 \|k(x,\cdot)\|_2^2\, dx \\
&\le C_k \|\sigma_n(t, v_n(t)) - \sigma(t, v_n(t))\|_2^2.
\end{aligned}$$

Note that, for all $n \in \mathbb{N}, n > N_0$, and a.e. $(t,x) \in (0,T) \times D$, $\mathbb{P}'$-a.s. in Ω', by using (S2a) and Proposition 3.2.2 *ii*),

$$\begin{aligned}
|\sigma_n(t, v_n(t,x)) - \sigma(t, v_n(t,x))| &= \sup_{\mu\in\mathbb{R}} \left(\sigma(t, v_n(t,x)) - \sigma(t,\mu) - n|v_n(t,x) - \mu|\right) \\
&\le \sup_{\mu\in\mathbb{R}} \left(L_\alpha |v_n(t,x) - \mu|^\alpha - n|v_n(t,x) - \mu|\right) \\
&\le \sup_{r\in[0,\infty)} \left(L_\alpha r^\alpha - nr\right),
\end{aligned}$$

where this last term converges to zero as shown in the proof of Proposition 3.2.2 *iv*). Moreover, we know, for all $n > N_0$, a.e. $(t,x) \in (0,T) \times D$, $\mathbb{P}'$-a.s. in Ω',

$$|\sigma_n(t, v_n(t,x)) - \sigma(t, v_n(t,x))|^2 \le \max_{r\in[0,\infty)} (L_\alpha r^\alpha - nr)^2 \le \frac{(1-\alpha)^2}{L^\alpha \alpha^{\frac{1+\alpha}{\alpha-1}}},$$

see proof of Proposition 3.2.2 *iv*). From Lebesgue's dominated convergence theorem, we obtain for $n \to \infty$

$$\mathbb{E}'\left[\int_0^T \int_D |\sigma_n(t, v_n(t,x)) - \sigma(t, v_n(t,x))|^2 \, dx \, dt\right] \to 0. \tag{3.32}$$

Since, for any $\alpha \in (0,1)$, there exist $\varrho < 2$ and $1 \leq s < \infty$ such that $L^\varrho(\Omega'; L^s(0,T; L^2(D))) \hookrightarrow L^{2\alpha}(\Omega'; L^{2\alpha}(0,T; L^{2\alpha}(D)))$, we have

$$\begin{aligned}
&\mathbb{E}'\left[\int_0^T \|B(t, v_n(t)) - B(t, u_\infty(t))\|_{\mathrm{HS}}^2 \, dt\right] \\
&\leq C_k L_\alpha^2 \mathbb{E}'\left[\int_0^T \|v_n(t) - u_\infty(t)\|_{2\alpha}^{2\alpha} \, dt\right] \to 0
\end{aligned} \tag{3.33}$$

for $n \to \infty$ by Lemma 3.3.11 *i*). Therefore, we obtain by (3.32) and (3.33)

$$B_n(\cdot, v_n) \to B(\cdot, u_\infty) \quad \text{in } L^2(\Omega'; L^2(0,T; \mathrm{HS}(L^2(D)))). \tag{3.34}$$

Using Burkholder-Davis-Gundy inequality, we get for $n \to \infty$

$$\int_0^\cdot B_n(s, v_n(s)) \, d\mathcal{W}_s \to \int_0^\cdot B(s, u_\infty(s)) \, d\mathcal{W}_s \quad \text{in } L^2(\Omega'; C([0,T]; L^2(D)))$$

and, for all $t \in [0,T]$,

$$\int_0^t B_n(s, v_n(s)) \, d\mathcal{W}_s \to \int_0^t B(s, u_\infty(s)) \, d\mathcal{W}_s \quad \text{in } L^2(\Omega'; L^2(D)).$$

$\square$

Proposition 3.3.19. *The function u_∞ is a $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$-adapted, square-integrable stochastic process with continuous paths in $L^2(D)$, such that $u_\infty(0) = v_0$. Moreover, $u_\infty \in L^p(\Omega'; L^p(0,T; W_0^{1,p}(D)))$ and*

$$\partial_t \left(u_\infty - \int_0^\cdot B(s, u_\infty(s)) \, d\mathcal{W}_s\right) - \operatorname{div} G + f(u_\infty) = 0$$

in $L^{p'}(\Omega; L^{p'}(0,T; W^{-1,p'}(D)))$.

Remark 3.3.20. *If u_∞ is given as in Proposition 3.3.19, in particular, we have for all $t \in [0,T]$, in $L^2(D)$, $\mathbb{P}'$-a.s. in Ω',*

$$u_\infty(t) - v_0 - \int_0^t \operatorname{div} G(s) \, ds + \int_0^t f(u_\infty(s)) \, ds = \int_0^t B(s, u_\infty(s)) \, d\mathcal{W}_s.$$

Proof. By (3.21) and equality in law, we know, that there exists a not relabeled subsequence, such that for $n \to \infty$

$$v_n - \int_0^{\cdot} B_n(s, v_n(s))\, d\mathcal{W}_s \rightharpoonup u_\infty - \int_0^{\cdot} B(s, u_\infty(s))\, d\mathcal{W}_s \quad \text{in } L^{q'}(\Omega'; \mathfrak{V}), \tag{3.35}$$

where $\mathfrak{V}$ is defined in (3.20). Since $\mathfrak{V}$ is continuously and densely embedded into $C([0,T]; W^{-m,q'}(D))$, the weak convergence holds true in $L^{q'}(\Omega'; C([0,T]; W^{-m,q'}(D)))$. In particular, by Lemma 3.3.18,

$$v_n \rightharpoonup u_\infty \quad \text{in } L^{q'}(\Omega'; C([0,T]; W^{-m,q'}(D))).$$

Because $(v_n)_{n>N_0}$ is bounded in $L^2(\Omega'; C([0,T]; L^2(D)))$ by Lemma 3.3.2 and equality in law, and since $L^2(\Omega'; C([0,T]; L^2(D))) \hookrightarrow C([0,T]; L^2(\Omega'; L^2(D)))$,

$$v_n(t) \rightharpoonup u_\infty(t) \quad \text{in } L^2(\Omega'; L^2(D)) \text{ for all } t \in [0,T].$$

Therefore, we have $u_\infty(0) = v_0$.
Let $A \in \mathcal{A}', \xi \in C_c^\infty((0,T))$, and $\varphi \in C_c^\infty(D)$, then for all $n \in \mathbb{N}, n > N_0$, there holds by Lemma 3.3.16

$$\begin{aligned}
0 &= \int_A \int_0^T \xi(t) \langle \partial_t \left(v_n(t) - \int_0^t B_n(s, v_n)\, d\mathcal{W}_s \right), \varphi \rangle_{q',q}\, dt\, d\mathbb{P}' \\
&\quad + \int_A \int_0^T \xi(t) \frac{1}{n} j(v_n, \varphi)\, dt\, d\mathbb{P}' + \int_A \int_0^T \int_D \xi(t) f(v_n) \varphi\, dt\, d\mathbb{P}' \\
&\quad + \int_A \int_0^T \int_D \xi(t) a(x, v_n, \nabla v_n) \cdot \nabla \varphi\, dx\, dt\, d\mathbb{P}' \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using partial integration (see [52, Proposition 2.5.2]), Lemma 3.3.11 i), and (3.35) we get for $n \to \infty$

$$\begin{aligned}
I_1 &= \int_A \langle \int_0^T \xi(t) \partial_t \left(v_n(t) - \int_0^t B_n(s, v_n)\, d\mathcal{W}_s \right) dt, \varphi \rangle_{q',q}\, d\mathbb{P}' \\
&= - \int_A \int_D \int_0^T \xi'(t) \left(v_n(t) - \int_0^t B_n(s, v_n)\, d\mathcal{W}_s \right) \varphi\, dt\, dx\, d\mathbb{P}' \\
&\to - \int_A \int_D \int_0^T \xi'(t) \left(u_\infty(t) - \int_0^t B(s, u_\infty)\, d\mathcal{W}_s \right) \varphi\, dt\, dx\, d\mathbb{P}' \\
&= \int_A \int_0^T \xi(t) \langle \partial_t \left(u_\infty(t) - \int_0^t B(s, u_\infty)\, d\mathcal{W}_s \right), \varphi \rangle_{q',q}\, dt\, d\mathbb{P}'.
\end{aligned} \tag{3.36}$$

Moreover, for all $n > N_0$,

$$\begin{aligned}
I_2 &= \int_A \int_0^T \xi(t) \frac{1}{n} \left[(v_n, \varphi)_{H_0^m} + \int_D \sum_{|\gamma| \le m} |\nabla^\gamma v_n|^{q-2} \nabla^\gamma v_n \cdot \nabla^\gamma \varphi \, dx \right] dt \, d\mathbb{P}' \\
&\le \|\xi\|_\infty \int_A \int_0^T \frac{1}{n} \left[\|v_n(t)\|_{H_0^m} \|\varphi\|_{H_0^m} + \|v_n(t)\|_{W_0^{m,q}}^{q-1} \|\varphi\|_{W_0^{m,q}} \right] dt \, d\mathbb{P}' \\
&\le \|\xi\|_\infty \|\varphi\|_{W_0^{m,q}} \int_A \int_0^T C_E \frac{1}{n^{\frac{1}{2}}} \frac{1}{n^{\frac{1}{2}}} \|v_n(t)\|_{H_0^m} + \frac{1}{n^{\frac{1}{q}}} \frac{1}{n^{\frac{1}{q'}}} \|v_n(t)\|_{W_0^{m,q}}^{q-1} \, dt \, d\mathbb{P}' \\
&\le \|\xi\|_\infty \|\varphi\|_{W_0^{m,q}} \left[C_E \frac{1}{n^{\frac{1}{2}}} \left(\mathbb{E}' \left[\int_0^T \frac{1}{n} \|v_n(t)\|_{H_0^m}^2 \, dt \right] \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \frac{1}{n^{\frac{1}{q}}} \left(\mathbb{E}' \left[\int_0^T \frac{1}{n} \|v_n(t)\|_{W_0^{m,q}}^q \, dt \right] \right)^{\frac{1}{q'}} \right].
\end{aligned}$$

By Lemma 3.3.1 and equality in law, $\mathbb{E}' \left[\int_0^T \frac{1}{n} \|v_n(t)\|_{H_0^m}^2 \, dt \right]$ and $\mathbb{E}' \left[\int_0^T \frac{1}{n} \|v_n(t)\|_{W_0^{m,q}}^q \, dt \right]$ are bounded by a constant independent of n. Hence, we obtain

$$\lim_{n \to \infty} I_2 = 0. \tag{3.37}$$

Lemma 3.3.11 *iii*), *iv*), (3.36), and (3.37) provide for all $A \in \mathcal{A}', \xi \in C_c^\infty((0,T))$, and $\varphi \in C_c^\infty(D)$

$$\begin{aligned}
0 &= \int_A \int_0^T \xi(t) \langle \partial_t \left(u_\infty(t) - \int_0^t B(s, u_\infty) \, d\mathcal{W}_s \right), \varphi \rangle_{q',q} \, dt \, d\mathbb{P}' \\
&\quad + \int_A \int_0^T \int_D \xi(t) G \cdot \nabla \varphi \, dx \, dt \, d\mathbb{P}' + \int_A \int_0^T \int_D \xi(t) f(u_\infty) \varphi \, dt \, d\mathbb{P}'.
\end{aligned} \tag{3.38}$$

We already know, that $\nabla v_n \rightharpoonup \nabla u_\infty$ in $L^p(\Omega'; L^p(0,T; L^p(D)^d))$ and, in particular, $u_\infty \in L^p(\Omega'; L^p(0,T; W_0^{1,p}(D)))$. Now, from equation (3.38) it follows that $u_\infty \in L^{\min\{p',2\}}(\Omega'; C([0,T]; W^{-1,p'}(D)))$ and, that

$$u_\infty(t) - u_0 - \int_0^t \operatorname{div} G \, ds + \int_0^t f(u_\infty) \, ds = \int_0^t B(s, u_\infty) \, d\mathcal{W}_s \tag{3.39}$$

in $W^{-1,p'}(D)$, $\mathbb{P}'$-a.s. in Ω', for all $t \in [0,T]$. From (3.39), we obtain by [84, Theorem 4.2.5], that u_∞ is a $(\widetilde{\mathcal{F}}_t)_{t \in [0,T]}$-adapted, square-integrable stochastic process with continuous paths in $L^2(D)$ and Itô's formula holds. □

Lemma 3.3.21. *There holds*

$$G = a(\cdot, u_\infty, \nabla u_\infty) \text{ in } L^{p'}(\Omega'; L^{p'}(0,T; L^{p'}(D)^d)).$$

Proof. Let $t \in [0,T]$ be arbitrary. Applying Itô's formula and taking the expectation, we obtain from Lemma 3.3.16

$$\begin{aligned} &\frac{1}{2}\mathbb{E}'\left[\|v_n(t)\|_2^2\right] - \frac{1}{2}\mathbb{E}'\left[\|v_0\|_2^2\right] + \mathbb{E}'\left[\int_0^t \int_D a(x, v_n, \nabla v_n) \cdot \nabla v_n \, dx \, ds\right] \\ &+ \frac{1}{n}\mathbb{E}'\left[\int_0^t j(v_n, v_n) \, ds\right] + \mathbb{E}'\left[\int_0^t \int_D f(v_n) v_n \, dx \, ds\right] \\ &= \frac{1}{2}\mathbb{E}'\left[\int_0^t \|B_n(s, v_n)\|_{\mathrm{HS}}^2 \, ds\right]. \end{aligned} \tag{3.40}$$

On the other hand, we obtain from Proposition 3.3.19 by applying Itô's formula and taking the expectation

$$\begin{aligned} &\frac{1}{2}\mathbb{E}'\left[\|u_\infty(t)\|_2^2\right] - \frac{1}{2}\mathbb{E}'\left[\|v_0\|_2^2\right] + \mathbb{E}'\left[\int_0^t \int_D G \cdot \nabla u_\infty \, dx \, ds\right] \\ &+ \mathbb{E}'\left[\int_0^t \int_D f(u_\infty) u_\infty \, dx \, ds\right] = \frac{1}{2}\mathbb{E}'\left[\int_0^t \|B(s, u_\infty)\|_{\mathrm{HS}}^2 \, ds\right]. \end{aligned} \tag{3.41}$$

Taking the difference (3.40)-(3.41), we get

$$\begin{aligned} &\frac{1}{2}\mathbb{E}'\left[\|v_n(t)\|_2^2\right] - \frac{1}{2}\mathbb{E}'\left[\|u_\infty(t)\|_2^2\right] \\ &+ \mathbb{E}'\left[\int_0^t \int_D a(x, v_n, \nabla v_n) \cdot \nabla v_n - G \cdot \nabla u_\infty \, dx \, ds\right] \\ &+ \mathbb{E}'\left[\int_0^t \int_D f(v_n) v_n - f(u_\infty) u_\infty \, dx \, ds\right] \\ &\leq \mathbb{E}'\left[\int_0^t \|B_n(s, v_n)\|_{\mathrm{HS}}^2 - \|B(s, u_\infty)\|_{\mathrm{HS}}^2 \, ds\right]. \end{aligned}$$

Using Lemma 3.3.11 *i*), the fact that $f \in L^\infty(\mathbb{R})$, and (3.34), we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \Bigg(\frac{1}{2}\mathbb{E}'\left[\|v_n(t)\|_2^2\right] - \frac{1}{2}\mathbb{E}'\left[\|u_\infty(t)\|_2^2\right] \\ &\qquad + \mathbb{E}'\left[\int_0^t \int_D a(x, v_n, \nabla v_n) \cdot \nabla v_n - G \cdot \nabla u_\infty \, dx \, ds\right]\Bigg) \\ &\leq 0. \end{aligned}$$

Therefore, using the lower semi-continuity of the norm, there holds

$$\limsup_{n\to\infty} \mathbb{E}'\left[\int_0^t \int_D a(x, v_n, \nabla v_n) \cdot \nabla v_n \, dx \, ds\right] \le \mathbb{E}'\left[\int_0^t \int_D G \cdot \nabla u_\infty \, dx \, ds\right].$$

Applying a stochastic version of Minty's trick (see [105, Lemma 8.8]) provides $G = a(\cdot, u_\infty, \nabla u_\infty)$. □

3.4 Pathwise Uniqueness of Solutions

In this section, we prove Theorem 3.1.5.

Proof. Let $\varepsilon > 0$ and η_ε be a non-decreasing, Lipschitz continuous approximation of the sign-function defined by

$$\eta_\varepsilon(r) := 2\int_0^r \frac{1}{\varepsilon}\rho\left(\frac{s}{\varepsilon}\right) \, ds \quad \text{for } r \in \mathbb{R},$$

where $\rho(s) := c\exp\left(\frac{1}{s^2-1}\right)\mathbb{1}_{\{|s|\le 1\}}$ such that $\int_{\mathbb{R}} \rho(s)\, ds = 1$ is a classical mollifier approximation of the Dirac measure with support on $[-\varepsilon, \varepsilon]$ (see [118, p.195]). We define for $r \in \mathbb{R}$ and $u \in L^2(D)$

$$N_\varepsilon(r) := \int_0^r \eta_\varepsilon(s)\, ds \quad \text{and} \quad F_\varepsilon(u) := \int_D N_\varepsilon(u(x))\, dx.$$

Note that one can show by easy calculation

$$|N''_\varepsilon(r)| \le \frac{2c}{\varepsilon}\mathbb{1}_{\{|r|\le\varepsilon\}}.$$

Because u_1 and u_2 are both solutions to (3.1), we have for any $t \in [0, T]$

$$\begin{aligned} &u_1(t) - u_2(t) - (u_0^1 - u_0^2) - \int_0^t \operatorname{div}(a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2))\, ds \\ &+ \int_0^t f(u_1) - f(u_2)\, ds = \int_0^t B(s, u_1) - B(s, u_2)\, dW_s. \end{aligned}$$

Applying Itô's formula to this stochastic process by using F_ε (see [99, p.78]) provides, for any $t \in [0,T]$, after taking the expectation,

$$
\begin{aligned}
&\mathbb{E}\left[F_\varepsilon(u_1(t)-u_2(t))\right] - \mathbb{E}\left[F_\varepsilon(u_0^1-u_0^2)\right] \\
&- \mathbb{E}\left[\int_0^t \langle \operatorname{div}\left(a(\cdot,u_1,\nabla u_1)-a(\cdot,u_2,\nabla u_2)\right), N_\varepsilon'(u_1-u_2)\rangle_{p',p}\,ds\right] \\
&+ \mathbb{E}\left[\int_0^t \int_D (f(u_1)-f(u_2))N_\varepsilon'(u_1-u_2)\,dx\,ds\right] \\
&= \frac{1}{2}\mathbb{E}\Bigg[\int_0^t \operatorname{Tr}\Big[F_\varepsilon''((u_1-u_2)(s))(B(s,u_1(s))-B(s,u_2(s)))Q \\
&\qquad (B(s,u_1(s))-B(s,u_2(s)))^*\Big]\,ds\Bigg] \\
&\Leftrightarrow I_1+I_2+I_3+I_4 = I_5.
\end{aligned}
\tag{3.42}
$$

By using (A1), we know that, for any $t \in [0,T]$,

$$
\begin{aligned}
I_2 &= \mathbb{E}\left[\int_0^t \int_D (a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_2))\cdot\nabla(u_1-u_2)N_\varepsilon''(u_1-u_2)\,dx\,ds\right] \\
&\geq \mathbb{E}\left[\int_0^t \int_D (a(x,u_1,\nabla u_2)-a(x,u_2,\nabla u_2))\cdot\nabla(u_1-u_2)N_\varepsilon''(u_1-u_2)\,dx\,ds\right].
\end{aligned}
$$

By using (A3) and Hölder inequality, there holds for any $t \in [0,T]$

$$
\begin{aligned}
&\left|\mathbb{E}\left[\int_0^t \int_D (a(x,u_1,\nabla u_2)-a(x,u_2,\nabla u_2))\cdot\nabla(u_1-u_2)N_\varepsilon''(u_1-u_2)\,dx\,ds\right]\right| \\
&\leq \mathbb{E}\left[\int_0^t \int_D (C_5|\nabla u_2|^{p-1}+h(x))|u_1-u_2||\nabla(u_1-u_2)|N_\varepsilon''(u_1-u_2)\,dx\,ds\right] \\
&\leq 2c\mathbb{E}\left[\int_0^t \int_D (C_5|\nabla u_2|^{p-1}+h(x))|\nabla(u_1-u_2)|\mathbb{1}_{\{|u_1-u_2|\leq\varepsilon\}}\,dx\,ds\right] \\
&\leq 2cC_5\left(\mathbb{E}\left[\int_0^t \|\nabla u_2\|_p^p\,ds\right]\right)^{p-1}\left(\mathbb{E}\left[\int_0^t\int_D \mathbb{1}_{\{|u_1-u_2|\leq\varepsilon\}}|\nabla(u_1-u_2)|^p\,dx\,ds\right]\right)^{\frac{1}{p}} \\
&\quad + 2c\|h\|_{p'}\left(\mathbb{E}\left[\int_0^t\int_D \mathbb{1}_{\{|u_1-u_2|\leq\varepsilon\}}|\nabla(u_1-u_2)|^p\,dx\,ds\right]\right)^{\frac{1}{p}} \\
&\to 0 \quad \text{for } \varepsilon\downarrow 0.
\end{aligned}
$$

Therefore, we have

$$
\liminf_{\varepsilon\downarrow 0} I_2 \geq 0. \tag{3.43}
$$

Since η_ε is an approximation of the sign function, we obtain by the Lipschitz continuity of f, for all $t \in [0,T]$,

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} I_3 &= \lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\int_0^t \int_D (f(u_1) - f(u_2))\eta_\varepsilon(u_1 - u_2)\,dx\,ds\right] \\
&= \mathbb{E}\left[\int_0^t \int_D (f(u_1) - f(u_2))\,\mathrm{sign}(u_1 - u_2)\,dx\,ds\right] \\
&\le \mathbb{E}\left[\int_0^t \int_D L_f|u_1 - u_2|\,dx\,ds\right].
\end{aligned} \tag{3.44}$$

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of U consisting of eigenvectors of Q. By using [84, Proposition B.0.10], there holds, for all $t \in [0,T]$,

$$\begin{aligned}
I_5 &= \frac{1}{2}\mathbb{E}\bigg[\int_0^t \mathrm{Tr}\Big[\Big((B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}\Big)^* F_\varepsilon''(u_1 - u_2) \\
&\qquad (B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}\Big]\,ds\bigg] \\
&= \frac{1}{2}\mathbb{E}\bigg[\int_0^t \sum_{k\in\mathbb{N}} \langle\Big((B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}\Big)^* F_\varepsilon''(u_1 - u_2) \\
&\qquad (B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}(e_k), e_k\rangle_U\,ds\bigg] \\
&= \frac{1}{2}\mathbb{E}\bigg[\int_0^t \sum_{k\in\mathbb{N}} \langle F_\varepsilon''(u_1 - u_2)(B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}(e_k), \\
&\qquad (B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}(e_k)\rangle_{L^2}\,ds\bigg] \\
&= \frac{1}{2}\mathbb{E}\left[\int_0^t \int_D N_\varepsilon''(u_1 - u_2)\sum_{k\in\mathbb{N}} \left|(B(s,u_1) - B(s,u_2))Q^{\frac{1}{2}}(e_k)\right|^2\,dx\,ds\right].
\end{aligned}$$

Consequently, for all $t \in [0,T]$,

$$
\begin{aligned}
|I_5| &\le \frac{1}{2}\mathbb{E}\Big[\int_0^t \int_D N_\varepsilon''(u_1-u_2) \\
&\qquad \cdot \sum_{k\in\mathbb{N}} \left| \int_D (\sigma(s,u_1(x)) - \sigma(s,u_2(x))k(x,y)Q^{\frac{1}{2}}(e_k)(y)\,dy \right|^2 dx\,ds\Big] \\
&\le \frac{1}{2}\mathbb{E}\Big[\int_0^t \int_D N_\varepsilon''(u_1-u_2)L_\sigma^2|u_1-u_2|^{2\alpha} \\
&\qquad \cdot \sum_{k\in\mathbb{N}} \left(\int_D |k(x,y)||Q^{\frac{1}{2}}(e_k)(y)|\,dy \right)^2 dx\,ds\Big] \\
&\le \frac{L_\sigma^2}{2}\mathbb{E}\Big[\int_0^t \int_D N_\varepsilon''(u_1-u_2)|u_1-u_2|^{2\alpha} \\
&\qquad \cdot \|k(x,\cdot)\|_2^2 \left(\sum_{k\in\mathbb{N}} \|Q^{\frac{1}{2}}(e_k)\|_2^2 \right) dx\,ds\Big].
\end{aligned}
$$

Since $(e_k)_{k\in\mathbb{N}}$ are eigenvectors of Q, there exist $(\lambda_k)_{k\in\mathbb{N}}$, such that

$$
\sum_{k\in\mathbb{N}} \|Q^{\frac{1}{2}}(e_k)\|_2^2 = \sum_{k\in\mathbb{N}} \|\lambda_k e_k\|_2^2 = \sum_{k\in\mathbb{N}} |\lambda_k| \le C
$$

for a constant $C \ge 0$. Therefore, we obtain for any $t \in [0,T]$

$$
\begin{aligned}
|I_5| &\le \frac{CL_\sigma^2}{2}\mathbb{E}\left[\int_0^t \int_D N_\varepsilon''(u_1-u_2)|u_1-u_2|^{2\alpha}\|k(x,\cdot)\|_2^2\,dx\,ds\right] \\
&\le \frac{CL_\sigma^2 c}{\varepsilon}\mathbb{E}\left[\int_0^t \int_D \mathbb{1}_{\{|u_1-u_2|\le\varepsilon\}}|u_1-u_2|^{2\alpha}\|k(x,\cdot)\|_2^2\,dx\,ds\right]
\end{aligned}
\tag{3.45}
$$

If $\alpha \in (\frac{1}{2},1)$, we can estimate, by (3.45),

$$
|I_5| \le CL_\sigma^2 cT\|k\|_{L^2(D\times D)}\varepsilon^{2\alpha-1} \to 0 \quad \text{for } \varepsilon \downarrow 0.
$$

For $\alpha = \frac{1}{2}$, we find by (3.45)

$$
\begin{aligned}
|I_5| &\le \frac{CL_\sigma^2 c}{\varepsilon}\mathbb{E}\left[\int_0^t \int_D \mathbb{1}_{\{|u_1-u_2|\le\varepsilon\}}\mathbb{1}_{\{u_1\neq u_2\}}|u_1-u_2|\|k(x,\cdot)\|_2^2\,dx\,ds\right] \\
&\le CL_\sigma^2 c\mathbb{E}\left[\int_0^t \int_D \mathbb{1}_{\{|u_1-u_2|\le\varepsilon\}}\mathbb{1}_{\{u_1\neq u_2\}}\|k(x,\cdot)\|_2^2\,dx\,ds\right] \\
&\to 0 \text{ for } \varepsilon \downarrow 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left(\mathbb{E}\left[F_\varepsilon(u_1(t)-u_2(t))\right]-\mathbb{E}\left[F_\varepsilon(u_0^1-u_0^2)\right]\right)\\
&=\mathbb{E}\left[\int_D |u_1(t)-u_2(t)|\,dx\right]-\mathbb{E}\left[\int_D |u_0^1-u_0^2|\,dx\right],
\end{aligned}
$$

we obtain from (3.42), by using (3.43), (3.44), and (3.45),

$$
\begin{aligned}
\mathbb{E}\left[\int_D |u_1(t)-u_2(t)|\,dx\right] \le\, & \mathbb{E}\left[\int_D |u_0^1-u_0^2|\,dx\right]\\
&+L_f\mathbb{E}\left[\int_0^t\int_D |u_1(s)-u_2(s)|\,dx\,ds\right]
\end{aligned}
$$

for all $t\in[0,T]$. Using Gronwalls lemma, we get

$$
\mathbb{E}\left[\int_D |u_1(t)-u_2(t)|\,dx\right]\le e^{L_f t}\mathbb{E}\left[\int_D |u_0^1-u_0^2|\,dx\right].
$$

□

Chapter 4

Convergence of a Finite-Volume Scheme for a Heat Equation with a Multiplicative Lipschitz Noise

4.1 Introduction

Let $\Lambda \subset \mathbb{R}^2$ be a bounded, open, connected, and polygonal set. Moreover, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $(W(t))_{t\geq 0}$ be a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.
For $T > 0$, we consider a nonlinear stochastic heat equation under Neumann boundary conditions:

$$\begin{cases} du - \Delta u\, dt = g(u)\, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases} \tag{4.1}$$

where $\mathbf{n}$ denotes the unit normal vector to $\partial\Lambda$ outward to Λ. We assume the following hypotheses on the data:

H_1: $u_0 \in L^2(\Omega; H^1(\Lambda))$ is $\mathcal{F}_0$-measurable.

H_2: $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L \geq 0$.

Remark 4.1.1. *1. Assumption H_1 is used to obtain a bound on the finite-volume approximations in a discrete H^1-seminorm. This estimate is necessary to obtain a tightness result on the finite-volume approximations that is essential to apply the stochastic compactness argument.*

2. *In this work, we restrict ourselves to dimension two. For the definition of a finite-volume mesh in dimension $d \in \{1, 2, 3\}$, we refer to [58]. For applications, dimensions two and three are the most interesting ones. Later, we need the definition of a discrete gradient which requires a dual mesh that is hard to imagine for dimensions higher than three, thus, for simplicity, we consider dimension two. We remark that in [19, 20] we study finite-volume schemes for both dimensions two and three.*

3. *H_2 implies*
$$|g(r)|^2 \leq C_L(1 + |r|^2) \tag{4.2}$$
for all $r \in \mathbb{R}$ and a constant $C_L \geq 0$ only depending on the Lipschitz constant $L \geq 0$ of g and on $g(0)$.

4.1.1 Concept of Solution and Main Result

The theoretical framework associated with Problem (4.1) is well established in the literature. Indeed, we can find many existence and uniqueness results for various concepts of solutions associated with this problem such as mild solutions, variational solutions, pathwise solutions, and weak solutions, see, e.g., [41, 84]. We are interested in the concept of solution as defined below, which we will call a variational solution:

Definition 4.1.2. *A variational solution to Problem* (4.1) *is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process*
$$u \in L^2(\Omega; C([0, T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0, T; H^1(\Lambda))),$$
such that, for all $t \in [0, T]$,
$$u(t) - u_0 - \int_0^t \Delta u(s)\, ds = \int_0^t g(u(s))\, dW(s),$$
in $L^2(\Lambda)$, and a.s. in Ω.

Existence, uniqueness, and regularity of this variational solution is well-known in the literature, see, e.g., [74, 84, 97]. The main result of this chapter is to propose a finite-volume scheme for the approximation of such a variational solution and to show its stochastically strong convergence by passing to the limit with respect to the time and space discretization parameters. This is stated in the following convergence result:

Theorem 4.1.3. *Assume that hypotheses H_1 and H_2 hold. Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of admissible finite-volume meshes of Λ in the sense of Definition 4.2.1 such that the mesh size h_m tends to 0 and $(N_m)_{m\in\mathbb{N}} \subseteq \mathbb{N}$ is a sequence of positive numbers which tends to infinity. For a fixed $m \in \mathbb{N}$, let $u^r_{h_m,N_m}$ and $u^l_{h_m,N_m}$ be the right and left in time finite-volume approximations defined by (4.4), (4.7)-(4.8) with $\mathcal{T} = \mathcal{T}_m$ and $N = N_m$, respectively. Then, $(u^r_{h_m,N_m})_{m\in\mathbb{N}}$ and $(u^l_{h_m,N_m})_{m\in\mathbb{N}}$ converge in $L^p(\Omega; L^2(0,T; L^2(\Lambda)))$ for any $p \in [1,2)$ to the variational solution of Problem (4.1) in the sense of Definition 4.1.2.*

4.1.2 State of the Art

The study of numerical schemes for stochastic partial differential equations (SPDEs) has attracted a lot of attention in the last decades and there exists extensive literature on this topic. A list of references for the numerical analysis of SPDEs and an overview of the state of the art is given in [8, 44, 95].
Regarding the theoretical and numerical study of stochastic heat equations, semigroup techniques may be used to construct mild solutions (see, e.g., [41]). However, from the point of view of applications and mathematical modeling, it is often interesting to consider first-order perturbations of the stochastic heat equation and more complicated, nonlinear second order operators, such as the p-Laplacian or the porous medium operator. For these nonlinear SPDEs the semigroup approach is not available and variational techniques have been developed in [74, 84, 97].
In the numerical analysis of variational solutions to parabolic SPDEs, spatial discretizations of finite-element type have been frequently used (see, e.g., [22, 32] and the references therein). On the other hand, for stochastic scalar conservation laws, finite-volume schemes have been studied in [13–16, 50, 51, 60, 87]. To the best of our knowledge, there are only a few results on finite-volume schemes for parabolic SPDEs. Let us mention the work of [17] where the authors proposed a convergence result of a finite-volume scheme for the approximation of a stochastic heat equation with linear multiplicative noise.

4.1.3 Aim of the Study

In this chapter, we want to extend the finite-volume approximation results in the hyperbolic case to the stochastic heat equation with Lipschitz continuous multiplicative noise. Having applications to nonlinear operators and also to degenerate parabolic-hyperbolic problems with stochastic force in mind for the future, we propose a method for the convergence of the scheme that does

not rely on mild solutions or on results from semigroup theory. Additionally, we may include a discrete gradient in the right-hand side of our scheme (4.8) in the future. Hence, further studies may be devoted to the convergence analysis of finite-volume schemes for equations with multiplicative noise involving first order spatial derivatives of the solution.
The main technical challenge is the nonlinear multiplicative noise. Indeed, from the *a priori* estimates, we get up to subsequences weak convergence results in several functional spaces for our finite-volume approximations and this mode of convergence is not enough to identify the weak limit of the nonlinear term in the stochastic integral. Therefore, we first show the convergence towards a martingale solution by adapting the stochastic compactness method based on Skorokhod's representation theorem. Then, using a famous argument of pathwise uniqueness (see, e.g., [66]), we obtain the stochastically strong convergence result stated in Theorem 4.1.3.

4.1.4 Outline

This chapter is organized as follows. The next section contains the introduction of the finite-volume framework: the definition of an admissible finite-volume mesh on Λ and the associated notations of discrete unknowns. Then, the notions of the discrete gradient and the discrete H^1-seminorm will be introduced. In the last subsection, we will introduce our finite-volume scheme together with the associated finite-volume approximations.
The remainder of this chapter is then devoted to the proof of the convergence of these approximations towards the variational solution of (4.1). To do so, in Section 4.3, we will prove several stability estimates satisfied by these approximations, but also a boundedness result on the approximation of the stochastic integral. These estimates will allow us to pass the limit in the numerical scheme in Section 4.4. More precisely, we apply the classical stochastic compactness argument (see, e.g., [31]). By the theorem of Prokhorov, we will get convergence in law (up to subsequences) of our finite-volume approximations. At the cost of a change of probability space, Skorokhod's representation theorem will allow us to obtain almost sure convergence of the proposed finite-volume scheme. Then, a martingale identification argument will help us in order to recover at the limit the desired stochastic integral.
In this way, we show that our finite-volume scheme converges to a martingale solution of (4.1), i.e., the stochastic basis is not fixed but enters an unknown in the equation. Next, we show pathwise uniqueness of solutions to (4.1). This, together with a classical argument of Gyöngy and Krylov (see [66]) allows us to deduce convergence in probability of the scheme with respect to the initial stochastic basis.

4.2 The Finite-Volume Framework

4.2.1 Admissible Finite-Volume Meshes and Notations

In order to perform a finite-volume approximation of the variational solution of Problem (4.1) on $[0,T] \times \Lambda$, we need to set a choice for the temporal and spatial discretization. For the time discretization, let $N \in \mathbb{N}$ be given. We define the fixed time step $\Delta t = \frac{T}{N}$ and divide the interval $[0,T]$ in $0 = t_0 < t_1 < .... < t_N = T$ equidistantly with $t_n = n\Delta t$ for all $n \in \{0, ..., N-1\}$. For the space discretization, we refer to [58] and consider finite-volume admissible meshes in the sense of

Definition 4.2.1 (Admissible finite-volume mesh)**.** *An admissible finite-volume mesh $\mathcal{T}$ of Λ (see Fig. 4.1) is given by a family of open, polygonal, and convex subsets K, called control volumes of $\mathcal{T}$, satisfying the following properties:*

- $\overline{\Lambda} = \bigcup_{K \in \mathcal{T}} \overline{K}$*.*
- *If $K, L \in \mathcal{T}$ with $K \neq L$ then* $\operatorname{int} K \cap \operatorname{int} L = \emptyset$*.*
- *If $K, L \in \mathcal{T}$ with $K \neq L$, then either the one-dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is* 0 *or $\overline{K} \cap \overline{L}$ is the edge of the mesh, denoted by $\sigma = K|L$, separating the control volumes K and L.*
- *To each control volume $K \in \mathcal{T}$, we associate a point $x_K \in \overline{K}$ (called the center of K) such that: If $K, L \in \mathcal{T}$ are two neighbouring control volumes the straight line between the centers x_K and x_L is orthogonal to the edge $\sigma = K|L$.*

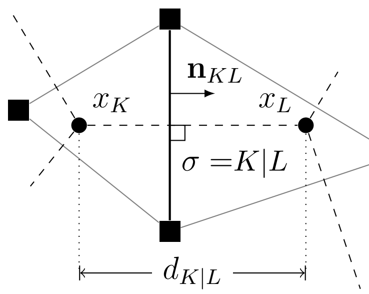

Figure 4.1: Notations of the mesh $\mathcal{T}$ associated with Λ

Once an admissible finite-volume mesh $\mathcal{T}$ of Λ is fixed, we will use the following notations.

Notation. • $h = \mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K) : K \in \mathcal{T}\}$ *the mesh size.*

- $d_h \in \mathbb{N}$ *the number of control volumes* $K \in \mathcal{T}$ *with* $h = \mathrm{size}(\mathcal{T})$.
- $\mathcal{E}$ *is the set of the edges of the mesh* $\mathcal{T}$ *and we define* $\mathcal{E}_{\mathrm{int}} := \{\sigma \in \mathcal{E} : \sigma \nsubseteq \partial\Lambda\}$, $\mathcal{E}_{\mathrm{ext}} := \{\sigma \in \mathcal{E} : \sigma \subseteq \partial\Lambda\}$.
- *For* $K \in \mathcal{T}$, $\mathcal{E}_K$ *is the set of edges of* K *and* m_K *is the two-dimensional Lebesgue measure of* K.
- *Let* $K, L \in \mathcal{T}$ *be two neighbouring control volumes. For* $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, *let* m_σ *be the length of* σ *and* $d_{K|L}$ *the distance between* x_K *and* x_L.
- *For neighbouring control volumes* $K, L \in \mathcal{T}$, *we denote by* $\mathbf{n}_{KL}$ *the unit vector on the edge* $\sigma = K|L$ *pointing from* K *to* L.
- *For* $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, *the diamond* D_σ *(see Fig. 4.2) is the open quadrangle whose diagonals are the edge* σ *and the segment* $[x_K, x_L]$. *For* $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, *we define* $D_\sigma := K$. *Then,* $\overline{\Lambda} = \bigcup_{\sigma\in\mathcal{E}} \overline{D_\sigma}$.
- m_{D_σ} *is the two-dimensional Lebesgue measure of the diamond* D_σ. *Note that for* $\sigma \in \mathcal{E}_{\mathrm{int}}$, *we have* $m_{D_\sigma} = \dfrac{m_\sigma d_{K|L}}{2}$.

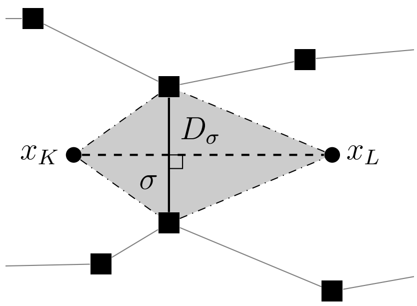

Figure 4.2: Notations on a diamond cell D_σ for $\sigma \in \mathcal{E}_{\mathrm{int}}$

Using these notations, we introduce a positive number

$$\mathrm{reg}(\mathcal{T}) = \max\left(\mathcal{N}, \max_{\substack{K\in\mathcal{T}\\ \sigma\in\mathcal{E}_K}} \frac{\mathrm{diam}(K)}{d(x_K,\sigma)}\right) \tag{4.3}$$

(where $\mathcal{N}$ is the maximum of edges incident to any vertex) that measures the regularity of a given mesh and is useful to perform the convergence analysis of finite-volume schemes. This number should be uniformly bounded when the mesh size tends to 0 for the convergence results to hold.

4.2.2 Discrete Unknowns and Piecewise Constant Functions

From now on and unless otherwise specified we consider $N \in \mathbb{N}$, $\Delta t = \frac{T}{N}$, and $\mathcal{T}$ an admissible finite-volume mesh of Λ in the sense of Definition 4.2.1 with a mesh size h.

For $n \in \{0, \ldots, N-1\}$ given, the idea of a finite-volume scheme for the approximation of Problem (4.1) is the following one: We associate to each control volume $K \in \mathcal{T}$ and time t_n a discrete unknown value denoted by $u_K^n \in \mathbb{R}$ that is expected to be an approximation of $u(t_n, x_K)$, where u is the variational solution of (4.1).

Before presenting the numerical scheme satisfied by the discrete unknowns $\{u_K^n : K \in \mathcal{T}, n \in \{0, \ldots, N-1\}\}$, let us introduce some general notations.

For any arbitrary vector $(w_K^n)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$, we define the piecewise constant function $w_h^n : \Lambda \to \mathbb{R}$ by

$$w_h^n(x) := \sum_{K\in\mathcal{T}} w_K^n \mathbb{1}_K(x) \quad \forall x \in \Lambda.$$

Note that, since the mesh $\mathcal{T}$ is fixed, the space $\mathbb{R}^{d_h}$ can be considered as a finite-dimensional subspace of $L^2(\Lambda)$ by the continuous mapping defined from $\mathbb{R}^{d_h}$ to $L^2(\Lambda)$ by

$$(w_K^n)_{K\in\mathcal{T}} \mapsto \sum_{K\in\mathcal{T}} \mathbb{1}_K w_K^n,$$

and, therefore, we may naturally identify the function and the vector

$$w_h^n \equiv (w_K^n)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}.$$

Knowing for all $n \in \{0, \ldots, N\}$ the function w_h^n, we can define the piecewise constant functions in time and space $w_{h,N}^r, w_{h,N}^l : [0,T] \times \Lambda \to \mathbb{R}$ by

$$\begin{aligned} w_{h,N}^r(t,x) &:= \sum_{n=0}^{N-1} w_h^{n+1}(x)\mathbb{1}_{[t_n,t_{n+1})}(t) \text{ if } t \in [0,T) \text{ and } w_{h,N}^r(T,x) := w_h^N(x), \\ w_{h,N}^l(t,x) &:= \sum_{n=0}^{N-1} w_h^n(x)\mathbb{1}_{[t_n,t_{n+1})}(t) \text{ if } t \in (0,T] \text{ and } w_{h,N}^l(0,x) := w_h^0(x). \end{aligned} \tag{4.4}$$

Remark 4.2.2. *The superscripts r and l in* (4.4) *do not refer to the continuity properties of the associated functions (which may be chosen either càdlàg or càglàd). The difference is that $w_{h,N}^l$ is adapted whereas $w_{h,N}^r$ is not adapted.*

As for the piecewise constant function in space, since $\mathcal{T}$ and N are fixed, the space $\mathbb{R}^{d_h\times N}$ can be considered as a finite-dimensional subspace of $L^2(0,T;L^2(\Lambda))$ by the continuous mapping defined from $\mathbb{R}^{d_h\times N}$ to $L^2(0,T;L^2(\Lambda))$ by

$$(w_K^n)_{\substack{K\in\mathcal{T}\\ n\in\{0,\dots,N-1\}}} \mapsto \sum_{\substack{K\in\mathcal{T}\\ n\in\{0,\dots,N-1\}}} \mathbb{1}_K \mathbb{1}_{[t_n,t_{n+1})} w_K^n,$$

and we may naturally identify

$$\begin{aligned} w_{h,N}^l &\equiv (w_K^n)_{\substack{K\in\mathcal{T}\\ n\in\{0,\dots,N-1\}}} \in \mathbb{R}^{d_h\times N},\\ w_{h,N}^r &\equiv (w_K^{n+1})_{\substack{K\in\mathcal{T}\\ n\in\{0,\dots,N-1\}}} \in \mathbb{R}^{d_h\times N}. \end{aligned}$$

We can also define the piecewise affine, continuous in time and piecewise constant in space reconstruction $\widehat{w}_{h,N} : [0,T]\times\Lambda \to \mathbb{R}$ by

$$\widehat{w}_{h,N}(t,x) := \sum_{n=0}^{N-1} \mathbb{1}_{[t_n,t_{n+1})}(t) \left(\frac{w_h^{n+1}(x) - w_h^n(x)}{\Delta t}(t-t_n) + w_h^n(x) \right). \tag{4.5}$$

Remark 4.2.3. *Note that, in the following, when we will consider a time and space function $\alpha : [0,T]\times\Lambda \to \mathbb{R}$ on all the space Λ (respectively the time interval $[0,T]$) at a fixed time $t\in[0,T]$ (respectively at a fixed $x\in\Lambda$) we will omit the space (respectively time) variable in the notations and write $\alpha(t)$ (respectively $\alpha(x)$) instead of $\alpha(t,\cdot)$ (respectively $\alpha(\cdot,x)$).*

4.2.3 Discrete Norms and Discrete Gradient

We fix $n\in\{0,\dots,N-1\}$ and consider for the remainder of this subsection an arbitrary vector $(w_K^n)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$ and use its natural identification with the piecewise constant function in space $w_h^n \equiv (w_K^n)_{K\in\mathcal{T}}$. In the following, we introduce the notions of a discrete gradient and of discrete norms for such a function w_h^n.

Definition 4.2.4 (Discrete L^2-norm)**.** *We define the L^2-norm of $w_h^n \in \mathbb{R}^{d_h}$ by*

$$||w_h^n||_{L^2(\Lambda)} = \left(\sum_{K\in\mathcal{T}} m_K |w_K^n|^2 \right)^{\frac{1}{2}}.$$

Definition 4.2.5 (Discrete gradient)**.** *We define the gradient operator ∇^h that maps scalar fields $w_h^n \in \mathbb{R}^{d_h}$ into vector fields of $(\mathbb{R}^2)^{e_h}$ (where e_h is the number of edges in the mesh $\mathcal{T}$) by $\nabla^h w_h^n := (\nabla_\sigma^h w_h^n)_{\sigma\in\mathcal{E}}$ with*

$$\nabla_\sigma^h w_h^n := \begin{cases} 2\dfrac{w_L^n - w_K^n}{d_{K|L}} \mathbf{n}_{KL}, & \textit{if } \sigma = K|L \in \mathcal{E}_{\text{int}}\\ 0, & \textit{if } \sigma\in\mathcal{E}_{\text{ext}}. \end{cases}$$

We remark that $\nabla^h w_n^h$ is considered as a piecewise constant function, which is constant on the diamonds D_σ, $\sigma \in \mathcal{E}$.

Definition 4.2.6 (Discrete H^1-seminorm)**.** *We define the H^1-seminorm of $w_h^n \in \mathbb{R}^{d_h}$ by*

$$|w_h^n|_{1,h} := \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |w_K^n - w_L^n|^2 \right)^{\frac{1}{2}}.$$

Notation. *If not marked otherwise, for an edge $\sigma \in \mathcal{E}_{\text{int}}$, we denote by K and L the neighbouring control volumes, i.e., $\sigma = K|L$. In particular, we use this notation in sums.*

Remark 4.2.7. *Note that, in particular,*

$$\|\nabla^h w_h^n\|^2_{(L^2(\Lambda))^2} = \sum_{\sigma \in \mathcal{E}_{\text{int}}} m_{D_\sigma} |\nabla_\sigma^h w_h^n|^2 = 2 \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |w_K^n - w_L^n|^2 = 2|w_h^n|^2_{1,h},$$

where the constant 2 corresponds to the space dimension $d = 2$.

Remark 4.2.8. *If we consider another arbitrary vector $\widetilde{w}_h^n \equiv (\widetilde{w}_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$, by summing over the edges we may rearrange the sum on the left-hand side and get the following rule of "discrete partial integration"*

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (w_K^n - w_L^n) \widetilde{w}_K^n = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (w_K^n - w_L^n)(\widetilde{w}_K^n - \widetilde{w}_L^n). \quad (4.6)$$

4.2.4 The Finite-Volume Scheme

Firstly, we define the vector $u_h^0 \equiv (u_K^0)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ by the discretization of the initial condition u_0 of Problem (4.1) over each control volume:

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x)\, dx, \quad \forall K \in \mathcal{T}. \quad (4.7)$$

The finite-volume scheme we propose reads for this given initial $\mathcal{F}_0$-measurable random vector $u_h^0 \in \mathbb{R}^{d_h}$:
For any $n \in \{0, \dots, N-1\}$, knowing $u_h^n \equiv (u_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$, we search for $u_h^{n+1} \equiv (u_K^{n+1})_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$, such that, for almost every $\omega \in \Omega$, the vector u_h^{n+1} is a solution to the following random equations

$$\frac{m_K}{\Delta t}(u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) = \frac{m_K}{\Delta t} g(u_K^n) \Delta_{n+1} W, \quad \forall K \in \mathcal{T}, \quad (4.8)$$

where $\Delta_{n+1}W$ denotes the increments of the Brownian motion between t_{n+1} and t_n:

$$\Delta_{n+1}W := W(t_{n+1}) - W(t_n) \quad \text{for } n \in \{0, \dots, N-1\}.$$

Remark 4.2.9. *1. The second term on the left-hand side of* (4.8) *is the classical two-point flux approximation of the Laplace operator, see [58, Section 10].*

2. The time-implicit discretization of the Laplace operator has several analytic advantages: First of all, calculations in the a priori estimates are simplified. Secondly, we omit the use of a CFL-condition. Last but not least, for more general nonlinear operators such as the p-Laplace operator, an implicit time discretization is more appropriate. However, an explicit time discretization of the noise is crucial and can not be omitted due to the non-anticipative character of the Itô stochastic integral.

We note that by multiplying equation (4.8) by w_K, summing over $K \in \mathcal{T}$, and using equality (4.6), the numerical scheme can be rewritten as: For any $n \in \{0, \dots, N-1\}$ find $u_h^{n+1} \in \mathbb{R}^{d_h}$, such that for any $w_h \in \mathbb{R}^{d_h}$,

$$\begin{aligned} &\sum_{K\in\mathcal{T}} m_K \left(u_K^{n+1} - u_K^n\right) w_K + \Delta t \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1})(w_K - w_L) \\ &= \sum_{K\in\mathcal{T}} m_K g(u_K^n) w_K \Delta_{n+1} W. \end{aligned} \tag{4.9}$$

The two formulations are equivalent but this "variational" formulation will be more useful for the analysis to follow.

Proposition 4.2.10 (Existence of a discrete solution)**.** *Assume that hypotheses H_1 and H_2 hold. Let $\mathcal{T}$ be an admissible finite-volume mesh of Λ in the sense of Definition 4.2.1 with a mesh size h and $N \in \mathbb{N}$. Then, there exists a unique solution $(u_h^n)_{1\le n\le N} \in (\mathbb{R}^{d_h})^N$ to Problem* (4.8) *associated with the initial vector u_h^0 defined by* (4.7)*. Additionally, for any $n \in \{0, \dots, N\}$, u_h^n is a $\mathcal{F}_{t_n}$-measurable random vector.*

The solution $(u_h^n)_{1\le n\le N} \in (\mathbb{R}^{d_h})^N$ of the scheme (4.7)-(4.8) is then used to build the right and left finite-volume approximations $u_{h,N}^r$ and $u_{h,N}^l$ defined by (4.4) for the variational solution u of Problem (4.1).

Proof of Proposition 4.2.10. Set $n \in \{0, \dots, N-1\}$. For $K \in \mathcal{T}$ and a.s. in Ω, note that (4.9) can be rewritten in the following way:

$$\sum_{K\in\mathcal{T}} m_K \left(u_K^{n+1} - f_K^n\right) w_K + \Delta t \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1})(w_K - w_L) = 0, \tag{4.10}$$

where $f_K^n := g(u_K^n)\Delta_{n+1}W + u_K^n$. For $f_h^n \equiv (f_K^n)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$ and a.e. $\omega \in \Omega$, we define the functional $J_h^n : \mathbb{R}^{d_h} \to \mathbb{R}$ by

$$J_h^n(w_h) := \frac{1}{2}a(w_h, w_h) - \int_\Lambda w_h f_h^n \, dx$$

where the bilinear form $a : \mathbb{R}^{d_h} \times \mathbb{R}^{d_h} \to \mathbb{R}$ is given by

$$a(v_h, w_h) := \int_\Lambda v_h w_h \, dx + \Delta t \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}}(u_K - u_L)(w_K - w_L).$$

From a straightforward calculation, it is easy to see that the bilinear form a is symmetric, continuous, and coercive.
Thus, from the theorem of Stampacchia (see e.g. [34, Theorem 5.6]), J_h^n admits a unique minimizer $u_h^{n+1} \in \mathbb{R}^{d_h}$ and the associated sequence $(u_h^n)_{1\leq n\leq N} \in (\mathbb{R}^{d_h})^N$ is the unique solution of (4.10) a.s. in Ω. If we assume that u_h^n is $\mathcal{F}_{t_n}$-measurable, then f_h^n is $\mathcal{F}_{t_{n+1}}$-measurable and, consequently, the random variable $\omega \mapsto J_h^n(w_h)(\omega)$ is $\mathcal{F}_{t_{n+1}}$-measurable for any $w_h \in \mathbb{R}^{d_h}$. Hence,

$$\omega \mapsto u_h^{n+1}(\omega) = \min_{w_h\in\mathbb{R}^{d_h}} J_h^n(w_h)(\omega)$$

is $\mathcal{F}_{t_{n+1}}$-measurable. By iteration, it follows that, for a given $\mathcal{F}_0$-measurable random variable $u_h^0 \in \mathbb{R}^{d_h}$, there exists for any $n \in \{0, \ldots, N-1\}$ a $\mathcal{F}_{t_{n+1}}$-measurable function $u_h^{n+1} \in \mathbb{R}^{d_h}$, such that $(u_h^n)_{1\leq n\leq N} \in (\mathbb{R}^{d_h})^N$ is a solution to Problem (4.8) associated with the initial vector u_h^0. $\square$

4.3 Stability Estimates

We will derive in this section several stability estimates satisfied by the discrete solution $(u_h^n)_{1\leq n\leq N} \in (\mathbb{R}^{d_h})^N$ of the scheme (4.7)-(4.8) given by Proposition 4.2.10, and also by the associated right and left finite-volume approximations $u_{h,N}^r$ and $u_{h,N}^l$ defined by (4.4).

4.3.1 Bounds on the Finite-Volume Approximations

We start by giving a bound on the discrete initial data.

Lemma 4.3.1. *Let u_0 be a given function satisfying the assumption H_1. Then, the associated discrete initial data $u_h^0 \in \mathbb{R}^{d_h}$ defined by* (4.7) *satisfies, $\mathbb{P}$-a.s. in Ω,*

$$\|u_h^0\|_{L^2(\Lambda)} \leq \|u_0\|_{L^2(\Lambda)}.$$

The proof is a direct consequence of the definition of u_h^0 and the Cauchy-Schwarz inequality.

Now, we can give the bounds on the discrete solutions which is one of the key points of the proof of the convergence theorem.

Proposition 4.3.2 (Bounds on the discrete solutions)**.** *There exists a constant $C_1 > 0$ depending only on u_0, C_L, $|\Lambda|$, and T, such that for all $n \in \mathbb{N}$*

$$\mathbb{E}\left[\|u_h^n\|_{L^2(\Lambda)}^2\right] + \mathbb{E}\left[\sum_{k=0}^{n-1}\|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2\right] + 2\Delta t\sum_{k=0}^{n-1}\mathbb{E}\left[|u_h^{k+1}|_{1,h}^2\right] \leq C_1.$$

Proof. We fix $n \in \{1, \ldots, N\}$. For any $k \in \{0, \ldots, n-1\}$, choosing $w_h = u_h^{k+1}$ as test function in (4.9), we obtain

$$\begin{aligned}&\sum_{K\in\mathcal{T}}\frac{m_K}{\Delta t}(u_K^{k+1} - u_K^k)u_K^{k+1} + \sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}|u_K^{k+1} - u_L^{k+1}|^2\\ &= \sum_{K\in\mathcal{T}}\frac{m_K}{\Delta t}g(u_K^k)u_K^k\Delta_{k+1}W + \sum_{K\in\mathcal{T}}\frac{m_K}{\Delta t}g(u_K^k)(u_K^{k+1} - u_K^k)\Delta_{k+1}W.\end{aligned} \tag{4.11}$$

We consider the terms separately: For the first term on the left-hand side we find

$$\sum_{K\in\mathcal{T}}\frac{m_K}{\Delta t}(u_K^{k+1} - u_K^k)u_K^{k+1} = \frac{1}{2}\sum_{K\in\mathcal{T}}\frac{m_K}{\Delta t}(|u_K^{k+1}|^2 - |u_K^k|^2 + |u_K^{k+1} - u_K^k|^2).$$

Taking expectation in (4.11), the first expression on the right-hand side of (4.11) vanishes, since u_K^k and $\Delta_{k+1}W$ are independent and, therefore,

$$\mathbb{E}\left[g(u_K^k)u_K^k\Delta_{k+1}W\right] = 0.$$

In the second term we apply Young's inequality in order to keep all necessary terms. Then, taking expectation and using Itô isometry we obtain

$$\begin{aligned}\mathbb{E}\left[g(u_K^k)(u_K^{k+1} - u_K^k)\Delta_{k+1}W\right] &\leq \mathbb{E}\left[|g(u_K^k)\Delta_{k+1}W|^2\right] + \frac{1}{4}\mathbb{E}\left[|u_K^{k+1} - u_K^k|^2\right]\\ &\leq \Delta t\mathbb{E}\left[|g(u_K^k)|^2\right] + \frac{1}{4}\mathbb{E}\left[|u_K^{k+1} - u_K^k|^2\right]\end{aligned}$$

for any $K \in \mathcal{T}$. Altogether we find

$$\begin{aligned}&\frac{1}{2\Delta t}\int_\Lambda\mathbb{E}\left[|u_h^{k+1}|^2 - |u_h^k|^2\right]dx + \frac{1}{4\Delta t}\int_\Lambda\mathbb{E}\left[|u_h^{k+1} - u_h^k|^2\right]dx + \mathbb{E}\left[|u_h^{k+1}|_{1,h}^2\right]\\ &\leq \int_\Lambda\mathbb{E}\left[|g(u_h^k)|^2\right]dx.\end{aligned}$$

Summing over $k \in \{0, \dots, n-1\}$ and multiplying with $2\Delta t$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|u_h^n\|_{L^2(\Lambda)}^2 - \|u_h^0\|_{L^2(\Lambda)}^2\right] + \frac{1}{2}\sum_{k=0}^{n-1}\mathbb{E}\left[\|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2\right] \\
&+ 2\Delta t \sum_{k=0}^{n-1}\mathbb{E}\left[|u_h^{k+1}|_{1,h}^2\right] \le 2\Delta t \sum_{k=0}^{n-1}\mathbb{E}\left[\|g(u_h^k)\|_{L^2(\Lambda)}^2\right].
\end{aligned} \tag{4.12}$$

Since the second and third term in (4.12) are non-negative, from H_2 and (4.2), it follows that

$$\mathbb{E}\left[\|u_h^n\|_{L^2(\Lambda)}^2\right] \le \mathbb{E}\left[\|u_h^0\|_{L^2(\Lambda)}^2\right] + 2C_L\Delta t \sum_{k=0}^{n-1}\mathbb{E}\left[\|u_h^k\|_{L^2(\Lambda)}^2\right] + 2C_L|\Lambda|T.$$

Applying the discrete Gronwall lemma provides

$$\mathbb{E}\left[\|u_h^n\|_{L^2(\Lambda)}^2\right] \le \left((1 + 2C_LT)\mathbb{E}\left[\|u_h^0\|_{L^2(\Lambda)}^2\right] + 2C_L|\Lambda|T\right) e^{2C_LT}.$$

From (4.13) and Lemma 4.3.1, we may conclude that there exists a constant $\Upsilon > 0$, such that

$$\sup_{n\in\{1,\dots,N\}} \mathbb{E}\left[\|u_h^n\|_{L^2(\Lambda)}^2\right] \le \Upsilon. \tag{4.13}$$

Applying (4.13) and (4.2), it follows that

$$\begin{aligned}
\sum_{k=0}^{n-1}\mathbb{E}\left[\|g(u_h^k)\|_{L^2(\Lambda)}^2\right] &\le C_L\left(|\Lambda|n + \sum_{k=0}^{n-1}\mathbb{E}\left[\|u_h^k\|_{L^2(\Lambda)}^2\right]\right) \\
&\le C_LN(|\Lambda| + \Upsilon)
\end{aligned} \tag{4.14}$$

for all $n \in \{1, \dots N\}$. From (4.12), Lemma 4.3.1 and (4.14), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|u_h^n\|_{L^2(\Lambda)}^2\right] + \frac{1}{2}\sum_{k=0}^{n-1}\mathbb{E}\left[\|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2\right] + 2\Delta t\sum_{k=0}^{n-1}\mathbb{E}\left[|u_h^{k+1}|_{1,h}^2\right] \\
&\le \mathbb{E}\left[\|u_0\|_{L^2(\Lambda)}^2\right] + 2C_LT(|\Lambda| + \Upsilon) =: C_1
\end{aligned}$$

for all $n \in \{1, \dots, N\}$. □

We are now interested in the bounds on the right and left finite-volume approximations defined by (4.4). As a direct consequence of Proposition 4.3.2 we get a $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$-bound on these approximations.

Lemma 4.3.3. *The sequences $(u^r_{h,N})_{h,N}$ and $(u^l_{h,N})_{h,N}$ are bounded independently of the discretization parameters $N \in \mathbb{N}$ and h in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$.*

Thanks to Proposition 4.3.2, we can also obtain a $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$-bound on the discrete gradients of the finite-volume approximations.

Lemma 4.3.4. *There exist a constant $K_1 \geq 0$ depending only on u_0, C_L, $|\Lambda|$, and T, and a constant $K_2 \geq 0$ additionally depending on the mesh regularity* $\mathrm{reg}(\mathcal{T})$ *(defined by (4.3)), such that*

$$\int_0^T \mathbb{E}\left[|u^r_{h,N}(t)|^2_{1,h}\right] dt \leq K_1 \tag{4.15}$$

and

$$\int_0^T \mathbb{E}\left[|u^l_{h,N}(t)|^2_{1,h}\right] dt \leq K_2. \tag{4.16}$$

Proof. Since

$$\int_0^T \mathbb{E}\left[|u^r_{h,N}(t)|^2_{1,h}\right] dt = \Delta t \sum_{k=0}^{N-1} \mathbb{E}\left[|u^{k+1}_h|^2_{1,h}\right],$$

estimate (4.15) follows directly from Proposition 4.3.2. Using the definition of $u^l_{h,N}$ and (4.15), we get

$$\begin{aligned}\int_0^T \mathbb{E}\left[|u^l_{h,N}(t)|^2_{1,h}\right] dt &\leq \Delta t \mathbb{E}\left[|u^0_h|^2_{1,h}\right] + \Delta t \sum_{k=0}^{N-1} \mathbb{E}\left[|u^{k+1}_h|^2_{1,h}\right] \\ &\leq \Delta t \mathbb{E}\left[|u^0_h|^2_{1,h}\right] + K_1.\end{aligned}$$

Since u_0 is assumed to be in $L^2(\Omega; H^1(\Lambda))$, by [58, Lemma 9.4], there exists $C_\Lambda \geq 0$ depending on the mesh regularity $\mathrm{reg}(\mathcal{T})$, such that

$$\mathbb{E}\left[|u^0_h|^2_{1,h}\right] \leq C_\Lambda \mathbb{E}\left[\|\nabla u_0\|^2_{L^2(\Lambda)}\right]$$

and, therefore, (4.16) follows. □

We end this section with a bound for the discrete solution, which will be useful for obtaining the time translate estimate and bounds for the Gagliardo seminorm. Note that the difficulty here is to have the maximum inside the expectation.

Lemma 4.3.5. *There exists a constant $K_3 \geq 0$, which is independent of the discretization parameters $N \in \mathbb{N}$ and h, such that*

$$\mathbb{E}\left[\max_{n\in\{0,\dots,N\}} \|u_h^n\|^2_{L^2(\Lambda)}\right] \leq K_3.$$

Proof. For $N \in \mathbb{N}$, we choose an arbitrary $k \in \{0,\dots,N-1\}$ and an arbitrary $K \in \mathcal{T}$. Testing the implicit scheme (4.9) with u_K^{k+1} provides

$$\begin{aligned}&\sum_{K\in\mathcal{T}} \frac{m_K}{\Delta t}(u_K^{k+1}-u_K^k)u_K^{k+1} + \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}}|u_K^{k+1}-u_L^{k+1}|^2 \\ &= \sum_{K\in\mathcal{T}} \frac{m_K}{\Delta t} g(u_K^k)u_K^{k+1}\Delta_{k+1}W.\end{aligned}$$

This implies, with the use of the Cauchy-Schwarz and Young inequalities,

$$\begin{aligned}&\frac{1}{2}\left(\|u_h^{k+1}\|^2_{L^2(\Lambda)} - \|u_h^k\|^2_{L^2(\Lambda)} + \|u_h^{k+1}-u_h^k\|^2_{L^2(\Lambda)}\right) \\ &\leq \langle \int_{t_k}^{t_{k+1}} g(u_h^k)dW(s), u_h^{k+1}-u_h^k\rangle_{L^2(\Lambda)} + \langle \int_{t_k}^{t_{k+1}} g(u_h^k)dW(s), u_h^k\rangle_{L^2(\Lambda)} \\ &\leq \frac{1}{2}\left\|\int_{t_k}^{t_{k+1}} g(u_h^k)dW(s)\right\|^2_{L^2(\Lambda)} + \frac{1}{2}\left\|u_h^{k+1}-u_h^k\right\|^2_{L^2(\Lambda)} \\ &\quad + \langle \int_{t_k}^{t_{k+1}} g(u_h^k)dW(s), u_h^k\rangle_{L^2(\Lambda)}.\end{aligned}$$

We obtain

$$\begin{aligned}\|u_h^{k+1}\|^2_{L^2(\Lambda)} - \|u_h^k\|^2_{L^2(\Lambda)} \leq &\left\|\int_{t_k}^{t_{k+1}} g(u_h^k)dW(s)\right\|^2_{L^2(\Lambda)} \\ &+ 2\langle \int_{t_k}^{t_{k+1}} g(u_h^k)dW(s), u_h^k\rangle_{L^2(\Lambda)}.\end{aligned}$$

For $n \in \{1,\dots,N\}$ fixed, we sum over $k = \{0,\dots,n-1\}$ to obtain

$$\begin{aligned}\|u_h^n\|^2_{L^2(\Lambda)} \leq &\|u_h^0\|^2_{L^2(\Lambda)} + \sum_{k=0}^{n-1}\left\|\int_{t_k}^{t_{k+1}} g(u_h^k)dW(s)\right\|^2_{L^2(\Lambda)} \\ &+ 2\sum_{k=0}^{n-1}\langle \int_{t_k}^{t_{k+1}} g(u_h^k)dW(s), u_h^k\rangle_{L^2(\Lambda)}.\end{aligned}$$

Taking firstly the maximum over $n \in \{1, \dots, N\}$ and, secondly, the expectation, Itô isometry implies

$$\begin{aligned}
\mathbb{E}\Big[\max_{n=1,\dots,N} \|u_h^n\|^2_{L^2(\Lambda)}\Big] &\leq \mathbb{E}\left[\|u_h^0\|^2_{L^2(\Lambda)}\right] + \mathbb{E}\left[\sum_{k=0}^{N-1}\left\|\int_{t_k}^{t_{k+1}} g(u_h^k)dW(s)\right\|^2_{L^2(\Lambda)}\right] \\
&\quad + 2\mathbb{E}\left[\max_{n=1,\dots,N}\sum_{k=0}^{n-1}\langle\int_{t_k}^{t_{k+1}} g(u_h^k)dW(s), u_h^k\rangle_{L^2(\Lambda)}\right] \\
&\leq \mathbb{E}\left[\|u_h^0\|^2_{L^2(\Lambda)}\right] + \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\|g(u_h^k)\|^2_{L^2(\Lambda)}\right] ds \\
&\quad + 2\mathbb{E}\left[\max_{n=1,\dots,N}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \langle g(u_h^k), u_h^k\rangle_{L^2(\Lambda)} dW(s)\right].
\end{aligned} \tag{4.17}$$

We can estimate the second term by the Burkholder-Davis-Gundy inequality:

$$\begin{aligned}
&2\mathbb{E}\left[\max_{n=1,\dots,N}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \langle g(u_h^k), u_h^k\rangle_{L^2(\Lambda)} dW(s)\right] \\
&\leq 2\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \langle g(u^l_{h,N}(s)), u^l_{h,N}(s)\rangle_{L^2(\Lambda)} dW(s)\right|\right] \\
&\leq 2C_B\mathbb{E}\left[\left(\int_0^T |\langle g(u^l_{h,N}(s)), u^l_{h,N}(s)\rangle_{L^2(\Lambda)}|^2 ds\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Now, we apply Cauchy-Schwarz and Young inequalities (with $\alpha > 0$), and H_2 with (4.2) to estimate

$$\begin{aligned}
&2C_B\mathbb{E}\left[\left(\int_0^T |\langle g(u^l_{h,N}(s)), u^l_{h,N}(s)\rangle_{L^2(\Lambda)}|^2 ds\right)^{\frac{1}{2}}\right] \\
&\leq 2C_B\mathbb{E}\left[\left(\sup_{t\in[0,T]}\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\int_0^T \|g(u^l_{h,N}(s))\|^2_{L^2(\Lambda)} ds\right)^{\frac{1}{2}}\right] \\
&\leq 2C_B\mathbb{E}\left[\frac{\alpha}{2}\sup_{t\in[0,T]}\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)} + \frac{1}{2\alpha}\int_0^T \|g(u^l_{h,N}(s))\|^2_{L^2(\Lambda)} ds\right] \\
&\leq C_B\alpha\mathbb{E}\left[\max_{n=1,\dots,N}\|u_h^n\|^2_{L^2(\Lambda)}\right] + \frac{C_B C_L}{\alpha}\left(T|\Lambda| + \mathbb{E}\left[\int_0^T \|u^l_{h,N}(s)\|^2_{L^2(\Lambda)} ds\right]\right).
\end{aligned}$$

Plugging the above estimate in (4.17) and again using H_2 with (4.2), we arrive at

$$\mathbb{E}\left[\max_{n=1,\dots,N}\|u_h^n\|^2_{L^2(\Lambda)}\right] \le C_B\alpha\mathbb{E}\left[\max_{n=1,\dots,N}\|u_h^n\|^2_{L^2(\Lambda)}\right] + \mathbb{E}\left[\|u_h^0\|^2_{L^2(\Lambda)}\right]$$
$$+ C_L\left(\frac{C_B}{\alpha}+1\right)\int_0^T \mathbb{E}\left[\|u^l_{h,N}(s)\|^2_{L^2(\Lambda)}\right] ds$$
$$+ C_L|\Lambda|T\left(\frac{C_B}{\alpha}+1\right).$$

Choosing $\alpha > 0$, such that $1 - C_B\alpha > 0$, we find a constant $C(\alpha, L) > 0$, such that

$$\mathbb{E}\left[\max_{n=1,\dots,N}\|u_h^n\|^2_{L^2(\Lambda)}\right]$$
$$\le C(\alpha,L)\left(\int_0^T \mathbb{E}\left[\|u^l_{h,N}(s)\|^2_{L^2(\Lambda)}\right] ds + \mathbb{E}\left[\|u_h^0\|^2_{L^2(\Lambda)}\right] + 1\right).$$

Now, the assertion follows by Lemma 4.3.1 and Lemma 4.3.3. □

4.3.2 Time and Space Translate Estimates

For the stochastic compactness argument in Subsection 4.4.2, we need a uniform bound on $(u^l_{h,N})_{h,N}$ in the spaces $L^2(\Omega; L^2(0,T; W^{\alpha,2}(\Lambda)))$ and $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$ for $\alpha \in (0, \frac{1}{2})$.
In order to prove the bound in $L^2(\Omega; L^2(0,T; W^{\alpha,2}(\Lambda)))$, we establish a uniform estimate on the space translates of $(u^l_{h,N})_{h,N}$ in Lemma 4.3.6.
The proof of the bound in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$ is more complicated. To do this, we introduce the following intermediate quantity: For any $(t,x) \in [0,T]\times\Lambda$, we define

$$M_{h,N}(t,x) := \int_0^t g(u^l_{h,N}(s,x))dW(s). \tag{4.18}$$

Then, Lemma 4.3.7 is a technical result for the proof of Lemma 4.3.8, where we show a uniform estimate on time translates of $(u^l_{h,N} - M_{h,N})_{h,N}$. Thanks to Lemma 4.3.8, we may conclude a uniform bound on $(u^l_{h,N} - M_{h,N})_{h,N}$ in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$ in Lemma 4.3.10. Then, the desired bound on $(u^l_{h,N})_{h,N}$ is obtained in Lemma 4.3.11 by using the additional information that $(M_{h,N})_{h,N}$ is bounded in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$.

We start with an estimate of the space translate. The proof is similar to the one given in [58, Theorem 10.3] and is done in Appendix A.

Lemma 4.3.6. *Let $\bar{u}^l_{h,N}$ be $d\mathbb{P}\otimes dt\otimes dx$-a.s. defined by*

$$\bar{u}^l_{h,N} := \begin{cases} u^l_{h,N}, & \text{on } \Omega\times(0,T)\times\Lambda \\ 0, & \text{on } \Omega\times(\mathbb{R}^3\setminus((0,T)\times\Lambda)). \end{cases}$$

Then, there exists a constant $C\geq 0$ only depending on Λ, such that for all $\eta\in\mathbb{R}^2$ with $|\eta|\leq R,\ R>0$, and almost every $t\in[0,T]$, $\mathbb{P}$-a.s in Ω,

$$\int_{\mathbb{R}^2} |\bar{u}^l_{h,N}(t,x+\eta)-\bar{u}^l_{h,N}(t,x)|^2dx \leq C|\eta|\left(|u^l_{h,N}(t)|^2_{1,h}+\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\right).$$

Lemma 4.3.7. *There exists a constant $K_4>0$, which is independent of the discretization parameters $N\in\mathbb{N}$ and h, such that for all $\tau\in(0,T)$ there holds*

$$\mathbb{E}\left[\int_0^{T-\tau}\left\|u^l_{h,N}(t+\tau)-M^l_{h,N}(t+\tau)-(u^l_{h,N}(t)-M^l_{h,N}(t))\right\|^2_{L^2(\Lambda)}dt\right]\leq K_4\tau, \tag{4.19}$$

where $M^l_{h,N}$ is defined for any $(t,x)\in[0,T]\times\Lambda$ by

$$M^l_{h,N}(t,x) := \sum_{n=0}^{N-1}\mathbb{1}_{[t_n,t_{n+1})}(t)\int_0^{t_n} g(u^l_{h,N}(s,x))\,dW(s),$$
$$M^l_{h,N}(T,x) := \int_0^T g(u^l_{h,N}(s,x))\,dW(s).$$

Proof. Let $\tau\in(0,T)$ be fixed. In the following, we set

$$\varphi^N_h(t,x) := u^l_{h,N}(t,x)-M^l_{h,N}(t,x) \quad \text{for } t\in[0,T],\ x\in\Lambda.$$

Furthermore, for $n\in\{0,\dots,N\}$ and $K\in\mathcal{T}$, we set $M^n_K := M^l_{h,N}(t_n,x_K)$ and $\varphi^n_K := u^n_K - M^n_K$. For $t\in(0,T-\tau)$, let $n_0(t),n_1(t)\in\{0,\dots,N-1\}$ be the unique non-negative integers satisfying

$$n_0(t)\Delta t\leq t<(n_0(t)+1)\Delta t \quad\text{and}\quad n_1(t)\Delta t\leq t+\tau<(n_1(t)+1)\Delta t.$$

There holds, $\mathbb{P}$-a.s in Ω,

$$\begin{aligned}&\int_0^{T-\tau}\|u^l_{h,N}(t+\tau)-M^l_{h,N}(t+\tau)-(u^l_{h,N}(t)-M^l_{h,N}(t))\|^2_{L^2(\Lambda)}dt\\ &=\int_0^{T-\tau}\sum_{K\in\mathcal{T}} m_K|\varphi^{n_1(t)}_K-\varphi^{n_0(t)}_K|^2dt =: \int_0^{T-\tau}A(t)dt.\end{aligned}$$

Since $\tau > 0$, we necessarily have $n_0(t) \le n_1(t)$. If $n_0(t) = n_1(t)$, then, we know $A(t) = 0$. Thus, we only consider $t \in (0, T-\tau)$ with $n_1(t) > n_0(t)$. Using the notation

$$\chi_{n+1}(t, t+\tau) := \begin{cases} 1, & \text{if } (n+1)\Delta t \in [t, t+\tau) \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 0, \dots, N-1$, we get

$$\begin{aligned} A(t) &= \sum_{K\in\mathcal{T}} m_K(\varphi_K^{n_1(t)} - \varphi_K^{n_0(t)})(\varphi_K^{n_1(t)} - \varphi_K^{n_0(t)}) \\ &= \sum_{K\in\mathcal{T}} m_K(\varphi_K^{n_1(t)} - \varphi_K^{n_0(t)}) \sum_{n=n_0(t)}^{n_1(t)-1} (\varphi_K^{n+1} - \varphi_K^{n}) \\ &= \sum_{K\in\mathcal{T}} m_K(\varphi_K^{n_1(t)} - \varphi_K^{n_0(t)}) \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau)(\varphi_K^{n+1} - \varphi_K^{n}) \\ &= \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{K\in\mathcal{T}} (\varphi_K^{n_1(t)} - \varphi_K^{n_0(t)}) m_K(\varphi_K^{n+1} - \varphi_K^{n}). \end{aligned}$$

Using (4.8), we know for any $n \in \{0, \dots, N-1\}$ and $K \in \mathcal{T}$

$$\begin{aligned} \varphi_K^{n+1} - \varphi_K^{n} &= u_K^{n+1} - u_K^{n} - \int_{t_n}^{t_{n+1}} g(u_{h,N}^{l}(s, x_K))\, dW(s) \\ &= -\frac{\Delta t}{m_K} \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(u_K^{n+1} - u_L^{n+1}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} A(t) = -\Delta t \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{K\in\mathcal{T}} (\varphi_K^{n_1(t)} - \varphi_K^{n_0(t)}) \\ \cdot \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(u_K^{n+1} - u_L^{n+1}). \end{aligned}$$

Rearranging the sum in the same way as for discrete partial integration (see Remark 4.2.8), using the definition of φ_h^N and the notation $u_K^{N,l} := u_{h,N}^{l}(x_K)$

for $K \in \mathcal{T}$, we get

$$\begin{aligned}
A(t) = &-\Delta t \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \cdot \\
&\cdot \left(\varphi_K^{n_1(t)} - \varphi_L^{n_1(t)} - (\varphi_K^{n_0(t)} - \varphi_L^{n_0(t)})\right) \\
= &-\Delta t \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \cdot \\
&\cdot \left(u_K^{n_1(t)} - u_L^{n_1(t)} - u_K^{n_0(t)} + u_L^{n_0(t)}\right) \\
&+ \Delta t \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \cdot \\
&\cdot \int_{n_0(t)\Delta t}^{n_1(t)\Delta t} \left(g(u_K^{N,l}) - g(u_L^{N,l})\right) dW(s) \\
=: &\, A_1(t) + A_2(t),
\end{aligned}$$

By Cauchy-Schwarz and Young inequalities we get

$$\begin{aligned}
A_1(t) \le &\frac{\Delta t}{2} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) |u_h^{n+1}|_{1,h}^2 \\
&+ \frac{\Delta t}{2} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) |u_h^{n_1(t)} - u_h^{n_0(t)}|_{1,h}^2 \\
\le &\frac{\Delta t}{2} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) |u_h^{n+1}|_{1,h}^2 \\
&+ \Delta t \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) (|u_h^{n_1(t)}|_{1,h}^2 + |u_h^{n_0(t)}|_{1,h}^2).
\end{aligned}$$

Consequently, we know

$$\mathbb{E}\left[\int_0^{T-\tau} A_1(t)\, dt\right] \le I_1 + I_2,$$

where

$$I_1 = \frac{1}{2} \int_0^{T-\tau} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \mathbb{E}\left[\Delta t |u_h^{n+1}|_{1,h}^2\right] dt$$

and

$$I_2 = \int_0^{T-\tau} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \Delta t \mathbb{E} \left[|u_h^{n_0(t)}|_{1,h}^2 + |u_h^{n_1(t)}|_{1,h}^2 \right] dt.$$

Since

$$\begin{aligned} \chi_{n+1}(t, t+\tau) = 1 \quad &\Leftrightarrow \quad (n+1)\Delta t \in [t, t+\tau) \\ &\Leftrightarrow \quad t - \tau \leq (n+1)\Delta t - \tau < t \leq (n+1)\Delta t, \end{aligned}$$

we have

$$\int_0^{T-\tau} \chi_{n+1}(t, t+\tau) dt = \int_{(n+1)\Delta t - \tau}^{(n+1)\Delta t} 1 \, dt = \tau. \tag{4.20}$$

Using this and (4.15), we have

$$I_1 = \frac{\tau}{2} \mathbb{E} \left[\int_0^T |u_{h,N}^r(s)|_{1,h}^2 \, ds \right] \leq \frac{K_1 \tau}{2}.$$

To estimate I_2, we write $I_2 = I_{2,1} + I_{2,2}$, where

$$\begin{aligned} I_{2,1} &= \int_0^{T-\tau} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \Delta t \mathbb{E} \left[|u_h^{n_0(t)}|_{1,h}^2 \right] dt, \\ I_{2,2} &= \int_0^{T-\tau} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \Delta t \mathbb{E} \left[|u_h^{n_1(t)}|_{1,h}^2 \right] dt. \end{aligned}$$

We note that, for any $m \in \{0, \ldots, N-1\}$, if $t \in [t_m, t_{m+1})$ then the definition of n_0 implies $n_0(t) = m$ and, therefore,

$$I_{2,1} \leq \sum_{m=0}^{N-1} \left(\int_{t_m}^{t_{m+1}} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \, dt \right) \Delta t \mathbb{E} \left[|u_h^m|_{1,h}^2 \right].$$

Now, we proceed as in [61, Lemma 6.2]. For all $m \in \{0, \ldots, N-1\}$, we have

$$\int_{t_m}^{t_{m+1}} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \, dt = \sum_{n=0}^{N-1} \int_{t_m - t_{n+1}}^{t_{m+1} - t_{n+1}} \chi_{n+1}(t + t_{n+1}, t + t_{n+1} + \tau) \, dt.$$

Note that, for any $n \in \{0, \ldots, N-1\}$,

$$\begin{aligned} &\chi_{n+1}(t + t_{n+1}, t + t_{n+1} + \tau) = 1 \\ &\Leftrightarrow \ (n+1)\Delta t = t_{n+1} \in [t + t_{n+1}, t + t_{n+1} + \tau) \\ &\Leftrightarrow \ t \in (-\tau, 0]. \end{aligned}$$

Hence, we have for any $m \in \{0, \dots, N-1\}$

$$\int_{t_m}^{t_{m+1}} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau)\, dt \le \int_{\mathbb{R}} \mathbb{1}_{(-\tau,0]}(t)\, dt = \tau$$

Therefore, thanks to Lemma 4.3.4, we arrive at

$$I_{2,1} \le \tau \Delta t \sum_{m=0}^{N-1} \mathbb{E}\left[|u_h^m|_{1,h}^2\right] = \tau \int_0^T \mathbb{E}\left[|u_{h,N}^l(t)|_{1,h}^2\right] dt \le K_2 \tau.$$

Analogously, for any $m \in \{0, \dots, N-1\}$, if $t \in [t_m - \tau, t_{m+1} - \tau)$, then the definition of n_1 implies $n_1(t) = m$ and, therefore,

$$I_{2,2} \le \sum_{m=0}^{N-1} \Delta t \mathbb{E}\left[|u_h^m|_{1,h}^2\right] \int_{t_m-\tau}^{t_{m+1}-\tau} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau)\, dt \le K_2 \tau$$

by [61, Lemma 6.2] and Lemma 4.3.4, where $\chi_{n+1}(t, t+\tau) = 0$ for $t < 0$. Combining the previous estimates we arrive at

$$\mathbb{E}\left[\int_0^{T-\tau} A_1(t)\, dt\right] \le \left(\frac{K_1}{2} + 2K_2\right) \tau. \tag{4.21}$$

Now, we consider A_2. Applying Young's inequality, we find

$$\begin{aligned}
A_2(t) \le{}& \frac{\Delta t}{2} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1})^2 \\
&+ \frac{\Delta t}{2} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} \cdot \\
&\cdot \left| \int_{n_0(t)\Delta t}^{n_1(t)\Delta t} \left(g(u_K^{N,l}) - g(u_L^{N,l}) \right) dW(s) \right|^2 \\
=:{}& A_{2,1}(t) + A_{2,2}(t).
\end{aligned}$$

There holds

$$\mathbb{E}\left[\int_0^{T-\tau} A_{2,1}(t) dt\right] = \frac{\Delta t}{2} \mathbb{E}\left[\sum_{n=0}^{N-1} |u_h^{n+1}|_{1,h}^2 \int_0^{T-\tau} \chi_{n+1}(t, t+\tau) dt\right].$$

By (4.20) and (4.15), we may conclude

$$\mathbb{E}\left[\int_0^{T-\tau} A_{2,1}(t) dt\right] = \frac{\tau}{2} \mathbb{E}\left[\int_0^T |u_{h,N}^r(s)|_{1,h}^2\, ds\right] \le \frac{K_1 \tau}{2}. \tag{4.22}$$

For the study of the term $A_{2,2}$, we recall the notation $u_K^{N,l} := u_{h,N}^l(x_K)$ for $K \in \mathcal{T}$. From Itô isometry it follows that, for any $t \in (0, T-\tau)$ with $n_0(t) < n_1(t)$,

$$\mathbb{E}\left[\left|\int_{n_0(t)\Delta t}^{n_1(t)\Delta t} \left(g(u_K^{N,l}) - g(u_L^{N,l})\right)\, dW(s)\right|^2\right] \le \mathbb{E}\left[\int_0^T |g(u_K^{N,l}) - g(u_L^{N,l})|^2\, ds\right].$$

Therefore, we have by using H_2

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\tau} A_{2,2}(t)dt\right] \\
&= \frac{\Delta t}{2}\int_0^{T-\tau}\sum_{n=0}^{N-1}\chi_{n+1}(t,t+\tau)\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}\cdot \\
&\quad\cdot \mathbb{E}\left[\left|\int_{n_0(t)\Delta t}^{n_1(t)\Delta t}\left(g(u_K^{N,l}) - g(u_L^{N,l})\right) dW(s)\right|^2\right] dt \\
&\le \frac{\Delta t}{2}\int_0^{T-\tau}\sum_{n=0}^{N-1}\chi_{n+1}(t,t+\tau)\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}\mathbb{E}\left[\int_0^T |g(u_K^{N,l}) - g(u_L^{N,l})|^2 ds\right] dt \\
&\le L^2\frac{\Delta t}{2}\int_0^{T-\tau}\sum_{n=0}^{N-1}\chi_{n+1}(t,t+\tau)\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}\mathbb{E}\left[\int_0^T |u_K^{N,l} - u_L^{N,l}|^2 ds\right] dt \\
&= L^2\frac{\Delta t}{2}\int_0^{T-\tau}\sum_{n=0}^{N-1}\chi_{n+1}(t,t+\tau)\, dt\int_0^T \mathbb{E}\left[|u_{h,N}^l(s)|_{1,h}^2\right] ds.
\end{aligned}$$

Because of (4.20), there holds

$$\int_0^{T-\tau}\sum_{n=0}^{N-1}\chi_{n+1}(t,t+\tau)dt = \sum_{n=0}^{N-1}\tau = N\tau.$$

Therefore, (4.16) implies

$$\mathbb{E}\left[\int_0^{T-\tau} A_{2,2}(t)dt\right] \le \frac{1}{2}L^2 T\tau K_2. \tag{4.23}$$

Finally, (4.19) follows from (4.21), (4.22), and (4.23). $\square$

Lemma 4.3.8. *There exists a constant $K_5 > 0$, which is independent of the discretization parameters $N \in \mathbb{N}$ and h, such that for all $\tau \in (0,T)$*

$$\mathbb{E}\left[\int_0^{T-\tau}\|u_{h,N}^l(t+\tau) - M_{h,N}(t+\tau) - (u_{h,N}^l(t) - M_{h,N}(t))\|_{L^2(\Lambda)}^2 dt\right] \le K_5\tau.$$

Proof. Let $0 < \tau < T$. Using the fact that, for any $a, b, c \in \mathbb{R}$, we have $|a+b+c|^2 \leqslant 3(|a|^2+|b|^2+|c|^2)$, we know

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\tau} \|u_{h,N}^l(t+\tau) - M_{h,N}(t+\tau) - (u_{h,N}^l(t) - M_{h,N}(t))\|_{L^2(\Lambda)}^2 dt\right] \\
&\leq 3\mathbb{E}\left[\int_0^{T-\tau} \|u_{h,N}^l(t+\tau) - M_{h,N}^l(t+\tau) - (u_{h,N}^l(t) - M_{h,N}^l(t))\|_{L^2(\Lambda)}^2 dt\right] \\
&\quad + 3\mathbb{E}\left[\int_0^{T-\tau} \|M_{h,N}(t+\tau) - M_{h,N}(t)\|_{L^2(\Lambda)}^2 dt\right] \\
&\quad + 3\mathbb{E}\left[\int_0^{T-\tau} \|M_{h,N}^l(t+\tau) - M_{h,N}^l(t)\|_{L^2(\Lambda)}^2 dt\right] \\
&=: 3(I_1 + I_2 + I_3).
\end{aligned}$$

From Lemma 4.3.7 we know that $I_1 \leq K_4\tau$. By using Itô isometry, H_2 with (4.2), and Lemma 4.3.5, we get

$$\begin{aligned}
I_2 &= \int_0^{T-\tau}\int_t^{t+\tau} \mathbb{E}\left[\|g(u_{h,N}^l(s))\|_{L^2(\Lambda)}^2\right] ds\, dt \\
&\leq C_L \int_0^{T-\tau}\int_t^{t+\tau} \left(|\Lambda| + \mathbb{E}\left[\|u_{h,N}^l(s)\|_{L^2(\Lambda)}^2\right]\right) ds\, dt \\
&\leq C_L|\Lambda|T\tau + C_L \int_0^{T-\tau}\int_t^{t+\tau} \mathbb{E}\left[\max_{n\in\{0,\dots,N\}} \|u_h^n\|_{L^2(\Lambda)}^2\right] ds\, dt \\
&\leq C_L T(|\Lambda| + K_3)\tau.
\end{aligned}$$

For $t \in [0, T]$, let $n_0(t), n_1(t) \in \{0, \dots, N-1\}$ be defined as in the proof of Lemma 4.3.7. From Itô isometry, H_2 with (4.2), and Lemma 4.3.5, we obtain

$$\begin{aligned}
I_3 &= \int_0^{T-\tau}\int_{n_0(t)\Delta t}^{n_1(t)\Delta t} \mathbb{E}\left[\|g(u_{h,N}^l(s))\|_{L^2(\Lambda)}^2\right] ds\, dt \\
&\leq C_L(|\Lambda| + K_3)\int_0^{T-\tau}\int_{n_0(t)\Delta t}^{n_1(t)\Delta t} 1\, ds\, dt.
\end{aligned}$$

As in the proof of Lemma 4.3.7, let χ_n for $n \in \mathbb{N}$ and $t \in [0, T-\tau]$ be defined by $\chi_n(t, t+\tau) := 1$ if $n\Delta t \in (t, t+\tau]$ and 0 otherwise. Taking (4.20) into account, we can continue the above estimate by

$$
\begin{aligned}
I_3 &\leq C_L(|\Lambda| + K_3) \int_0^{T-\tau} \sum_{n=0}^{N-1} \chi_{n+1}(t, t+\tau) \Delta t \, dt \\
&= C_L(|\Lambda| + K_3) \Delta t \sum_{n=0}^{N-1} \int_0^{T-\tau} \chi_{n+1}(t, t+\tau) \, dt \\
&= C_L T(|\Lambda| + K_3)\tau,
\end{aligned}
$$

and the assertion follows. □

4.3.3 Bound on the Gagliardo Seminorm

In this subsection we give bounds on the approximate solutions which will be used in the stochastic compactness argument in Subsection 4.4.2. We denote by $[\,\cdot\,]_{W^{\alpha,2}(\Lambda)}$ the Gagliardo seminorm, i.e., for any function $w : \Lambda \to \mathbb{R}$ one has,

$$
[\, w \,]_{W^{\alpha,2}(\Lambda)} = \left(\int_\Lambda \int_\Lambda \frac{|w(x) - w(y)|^2}{|x-y|^{2+2\alpha}} dx \, dy \right)^{\frac{1}{2}}.
$$

Note that $W^{\alpha,2}(\Lambda) = \{w \in L^2(\Lambda) : [\, w \,]_{W^{\alpha,2}(\Lambda)} < \infty\}$.

Lemma 4.3.9. *For any fixed $\alpha \in (0, \frac{1}{2})$, the sequences $(u^l_{h,N})_{h,N}$ and $(u^r_{h,N})_{h,N}$ are bounded in $L^2(\Omega; L^2(0,T; W^{\alpha,2}(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}$ and h.*

Proof. We fix $0 < \alpha < \frac{1}{2}$, $R > 0$, and define $\bar{u}^l_{h,N}$ as in Lemma 4.3.6. For almost every $t \in (0,T)$, we can write

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,y)|^2}{|x-y|^{2+2\alpha}} \, dx \, dy \\
&= \int_{|\eta|>R} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}} \, dx \, d\eta \\
&\quad + \int_{|\eta|<R} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}} \, dx \, d\eta.
\end{aligned}
$$

Obviously, we can estimate for almost every $t \in (0,T)$

$$\begin{aligned}
&\int_{|\eta|>R} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}}\, dx\, d\eta \\
&\leq 4\|\bar{u}^l_{h,N}(t)\|^2_{L^2(\mathbb{R}^2)} \int_{|\eta|>R} |\eta|^{-2(1+\alpha)}\, d\eta \\
&= 4\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)} \int_0^{2\pi} \int_R^\infty r^{-2(1+\alpha)} r\, dr\, d\varphi
\end{aligned}$$

and, by Lemma 4.3.6,

$$\begin{aligned}
&\int_{|\eta|<R} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}}\, dx\, d\eta \\
&\leq C\left(|u^l_{h,N}(t)|^2_{1,h} + \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\right) \int_{|\eta|<R} |\eta|^{-2(1+\alpha)+1} d\eta \\
&= C\left(|u^l_{h,N}(t)|^2_{1,h} + \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\right) \int_0^{2\pi} \int_0^R r^{-2\alpha-1} r\, dr d\varphi.
\end{aligned}$$

Hence, there exist constants $\tilde{C}_1, \tilde{C}_2 \geq 0$ only depending on Λ and $R > 0$, such that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,y)|^2}{|x-y|^{2+2\alpha}}\, dx\, dy \\
&\leq \tilde{C}_1 \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)} + \tilde{C}_2 |u^l_{h,N}(t)|^2_{1,h}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_0^T [u^l_{h,N}(t)]^2_{W^{\alpha,2}(\Lambda)} dt &\leq \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,y)|^2}{|x-y|^{2+2\alpha}} dx\, dy\, dt \\
&\leq \int_0^T \left(\tilde{C}_1 \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)} + \tilde{C}_2 |u^l_{h,N}(t)|^2_{1,h}\right) dt.
\end{aligned}$$

Therefore, thanks to Lemma 4.3.4 and 4.3.5, we get

$$\begin{aligned}
&\mathbb{E}\left[\|u^l_{h,N}\|_{L^2(0,T;W^{\alpha,2}(\Lambda))}\right] \\
&= \mathbb{E}\left[\int_0^T \left(\|u^l_{h,N}(t)\|_{L^2(\Lambda)} + [u^l_{h,N}(t)]_{W^{\alpha,2}(\Lambda)}\right)^2 dt\right] \\
&\leq 2(1+\tilde{C}_1) \int_0^T \mathbb{E}\left[\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\right] dt + 2\tilde{C}_2 \int_0^T \mathbb{E}\left[|u^l_{h,N}(t)|^2_{1,h}\right] dt \\
&\leq 2TK_3(1+\tilde{C}_1) + 2\tilde{C}_2 K_2.
\end{aligned}$$

Using similar arguments, we obtain the boundedness of $(u^r_{h,N})_{h,N}$ in $L^2(\Omega; L^2(0,T;W^{\alpha,2}(\Lambda)))$. □

In order to establish the $L^2(\Omega; W^{\alpha,2}(0,T;L^2(\Lambda)))$-bound on the discrete solutions, we give the following auxiliary result:

Lemma 4.3.10. *For any fixed $\alpha \in (0, \frac{1}{2})$, the sequence $(u^l_{h,N} - M_{h,N})_{h,N}$ defined by (4.18) is bounded in $L^2(\Omega; W^{\alpha,2}(0,T;L^2(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}$ and h.*

Proof. For any $x \in \Lambda$, let $\bar{\varphi}_{h,N}(t,x) := u^l_{h,N}(t,x) - M_{h,N}(t,x)$ for $t \in [0,T]$ and $\bar{\varphi}_{h,N}(t,x) = 0$ for $t \in \mathbb{R} \setminus [0,T]$. We have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_0^T \frac{\|\bar{\varphi}_{h,N}(s) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|t-s|^{1+2\alpha}}\, ds\, dt\right] \\
&= \mathbb{E}\left[\int_0^T \int_0^t \frac{\|\bar{\varphi}_{h,N}(s) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|t-s|^{1+2\alpha}}\, ds\, dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \int_t^T \frac{\|\bar{\varphi}_{h,N}(s) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|t-s|^{1+2\alpha}}\, ds\, dt\right] \\
&= \mathbb{E}\left[\int_0^T \int_0^t \frac{\|\bar{\varphi}_{h,N}(t-\tau) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|\tau|^{1+2\alpha}}\, d\tau\, dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \int_0^{T-t} \frac{\|\bar{\varphi}_{h,N}(t+\tau) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|\tau|^{1+2\alpha}}\, d\tau\, dt\right] \\
&=: I_1 + I_2.
\end{aligned} \tag{4.24}$$

Using Funbini's theorem, we know

$$\begin{aligned}
I_1 &= \mathbb{E}\left[\int_0^T \int_\tau^T \frac{\|\bar{\varphi}_{h,N}(t-\tau) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|\tau|^{1+2\alpha}}\, dt\, d\tau\right] \\
&= \mathbb{E}\left[\int_0^T \int_0^{T-\tau} \frac{\|\bar{\varphi}_{h,N}(s) - \bar{\varphi}_{h,N}(s+\tau)\|^2_{L^2(\Lambda)}}{|\tau|^{1+2\alpha}}\, ds\, d\tau\right] \\
&= \int_0^T |\tau|^{-1-2\alpha} \int_0^{T-\tau} \mathbb{E}\left[\|\bar{\varphi}_{h,N}(s+\tau) - \bar{\varphi}_{h,N}(s)\|^2_{L^2(\Lambda)}\right] ds\, d\tau.
\end{aligned} \tag{4.25}$$

Applying Fubinis theorem to I_2, we also obtain

$$\begin{aligned}
I_2 &= \mathbb{E}\left[\int_0^T \int_0^{T-\tau} \frac{\|\bar{\varphi}_{h,N}(t+\tau) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|\tau|^{1+2\alpha}}\, dt\, d\tau\right] \\
&= \int_0^T |\tau|^{-1-2\alpha} \int_0^{T-\tau} \mathbb{E}\left[\|\bar{\varphi}_{h,N}(t+\tau) - \bar{\varphi}_{h,N}(t)\|^2_{L^2(\Lambda)}\right] dt\, d\tau.
\end{aligned} \tag{4.26}$$

By Lemma 4.3.8, we get

$$I_1 + I_2 \leq 2K_5 \int_0^T |\tau|^{-2\alpha}\, d\tau$$

and the integral is finite for $\alpha \in \left(0, \frac{1}{2}\right)$. $\square$

Lemma 4.3.11. *For any fixed $\alpha \in (0, \frac{1}{2})$, the sequence $(u^l_{h,N})_{h,N}$ is bounded in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}$ and h.*

Proof. From Lemma 4.3.10 we know that $(u^l_{h,N} - M_{h,N})_{h,N}$ is bounded in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$. By Lemma 4.3.5, we know

$$\int_0^T \mathbb{E}\left[\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\right] dt \leq T\mathbb{E}\left[\sup_{t\in[0,T]} \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\right] \leq TK_3.$$

Thus, by applying [59, Lemma 2.1], we obtain that $(M_{h,N})_{h,N}$ is bounded in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda)))$. Now, since $u^l_{h,N} = (u^l_{h,N} - M^l_{h,N}) + M^l_{h,N}$, the assertion follows. $\square$

4.4 Convergence of the Finite-Volume Scheme

We now have all the necessary material to pass to the limit in the numerical scheme.

In the sequel, for $m \in \mathbb{N}$, let $(\mathcal{T}_m)_m$ be a sequence of admissible meshes of Λ in the sense of Definition 4.2.1 such that the mesh size h_m tends to 0 when m tends to ∞ and let $(N_m)_m \subset \mathbb{N}$ be a sequence with $\lim_{m\to\infty} N_m = \infty$ and $\Delta t_m := \frac{T}{N_m}$.

For the sake of simplicity, we shall use the notations $\mathcal{T} = \mathcal{T}_m$, $h = \text{size}(\mathcal{T}_m)$, $\Delta t = \Delta t_m$, and $N = N_m$, when the m-dependency is not useful for the understanding of the reader.

4.4.1 Weak Convergence of Finite-Volume Approximations

First, thanks to the bounds on the discrete solutions, we obtain the following weak convergences.

Lemma 4.4.1. *There exist not relabeled subsequences of $(u^r_{h,N})_m$ and of $(u^l_{h,N})_m$ and a function $u \in L^2(\Omega; L^2(0,T; H^1(\Lambda)))$, such that*

$$u^l_{h,N} \rightharpoonup u \quad \textit{and} \quad u^r_{h,N} \rightharpoonup u$$

for $m \to \infty$ in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$.

Proof. From Lemma 4.3.3 it follows that the sequences $(u^r_{h,N})_m$ and $(u^l_{h,N})_m$ are bounded in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$, thus, up to a not relabeled subsequence, they are weak convergent in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$ towards possibly distinct elements u, $\tilde{u}$, respectively. Moreover, by Lemma 4.3.4 and Remark 4.2.7, we know that

$$\|\nabla^h u^r_{h,N}\|^2_{L^2(\Omega\times(0,T)\times\Lambda)} \leq 2K_1.$$

Consequently, there exists $\chi \in L^2(\Omega; L^2(0,T; L^2(\Lambda)))$ such that, passing to a not relabeled subsequence if necessary,

$$\nabla^h u^r_{h,N} \rightharpoonup \chi \quad \text{in } L^2(\Omega; L^2(0,T; L^2(\Lambda))) \text{ for } m \to \infty.$$

With similar arguments as in [57, Lemma 2] and [58, Theorem 14.3], we get the additional regularity $u \in L^2(\Omega; L^2(0,T; H^1(\Lambda)))$ and $\chi = \nabla u$. Since, by Proposition 4.3.2, we have

$$\mathbb{E}\left[\|u^r_{h,N} - u^l_{h,N}\|^2_{L^2(0,T;L^2(\Lambda))}\right] = \Delta t \mathbb{E}\left[\sum_{n=0}^{N-1} \|u^{n+1}_h - u^n_h\|^2_{L^2(\Lambda)}\right] \leq C_1 \Delta t, \tag{4.27}$$

we know that $(u^r_{h,N} - u^l_{h,N})_m$ converges strongly to 0 in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$ for $m \to \infty$, and, hence, also weakly. Therefore, we obtain $u = \tilde{u}$. □

Our aim is to show that u is the unique solution to (4.1). But weak convergence is not enough to pass to the limit in the nonlinear noise term of our finite-volume scheme. Therefore, we will apply the method of stochastic compactness.

4.4.2 The Stochastic Compactness Argument

For better readability, we define $V := L^2(0,T; L^2(\Lambda))$ and

$$\mathcal{W} := W^{\alpha,2}(0,T; L^2(\Lambda)) \cap L^2(0,T; W^{\alpha,2}(\Lambda)).$$

From Lemmas 4.3.9 and 4.3.11 we get immediately the following bound.

Lemma 4.4.2. *For any fixed $\alpha \in (0, \frac{1}{2})$, there exists a constant $K_6 \geq 0$ depending on u_0 and the mesh regularity* reg$(\mathcal{T})$ *but not depending on the discretization parameter $m \in \mathbb{N}$, such that*

$$\mathbb{E}\left[\|u^l_{h,N}\|^2_{\mathcal{W}}\right] \leq K_6.$$

In the following, for a random variable X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ the law of X will be denoted by $\mathbb{P} \circ X^{-1}$.

Lemma 4.4.3. *The sequence of laws $(\mathbb{P} \circ (u^l_{h,N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight.*

Proof. By [59, Theorem 2.1], we know that $\mathcal{W}$ is compactly embedded in V. Let $\varepsilon > 0$ be arbitrary. For any $R > 0$, the ball $B_{\mathcal{W}}(0,R) := \{v \in \mathcal{W} : \|v\|_{\mathcal{W}} \leq R\}$ is compact in V. There holds

$$\begin{aligned}[\mathbb{P} \circ (u^l_{h,N})^{-1}](B_{\mathcal{W}}(0,R)) &= 1 - [\mathbb{P} \circ (u^l_{h,N})^{-1}](B_{\mathcal{W}}(0,R)^c) \\ &= 1 - \int_{\{\|u^l_{h,N}\|_{\mathcal{W}} > R\}} 1 \, d\mathbb{P}.\end{aligned}$$

Then, by using Markov inequality

$$\int_{\{\|u^l_{h,N}\|_{\mathcal{W}} > R\}} 1 \, d\mathbb{P} \leq \frac{1}{R^2} \int_{\{\|u^l_{h,N}\|_{\mathcal{W}} > R\}} \|u^l_{h,N}\|^2_{\mathcal{W}} \, d\mathbb{P} \leq \frac{1}{R^2} \mathbb{E}\left[\|u^l_{h,N}\|^2_{\mathcal{W}}\right].$$

Since $(u^l_{h,N})_{h,N}$ is bounded in $L^2(\Omega;\mathcal{W})$, thanks to Lemma 4.4.2, we get

$$[\mathbb{P} \circ (u^l_{h,N})^{-1}](B_{\mathcal{W}}(0,R)) \geq 1 - \frac{K_6}{R^2}.$$

If we choose an appropriate R, the assertion follows. $\square$

For the next lemmas, we recall that the initial value u_0 of Problem (4.1) is $\mathcal{F}_0$-measurable and belongs to $L^2(\Omega; H^1(\Lambda))$. Moreover, its spatial discretization, denoted by u^0_h, is defined by (4.7).
In the following, we will write $(W(t))_{t\geq 0} =: W$ whenever the t-dependence is not relevant for the argumentation.
In order to apply Skorokhod's theorem and to obtain almost sure convergence, we begin by proving convergence in law.

Lemma 4.4.4. *For $m \in \mathbb{N}$, we consider the sequence of random vectors*

$$Y_m = ((u^l_{h_m,N_m}, u^r_{h_m,N_m} - u^l_{h_m,N_m}, W, u^0_{h_m})$$

with values in

$$\mathcal{X} := L^2(0,T;L^2(\Lambda)) \times L^2(0,T;L^2(\Lambda)) \times C([0,T]) \times L^2(\Lambda).$$

There exists a not relabeled subsequence of $(Y_m)_m$ converging in law, i.e., there exists a probability measure μ_∞ on $\mathcal{X}$ with marginal laws $\mu^1_\infty, \delta_0, \mathbb{P} \circ W^{-1}, \mathbb{P} \circ (u_0)^{-1}$, such that

$$\mathbb{E}\left[f(Y_m)\right] \overset{m\to\infty}{\longrightarrow} \int_{\mathcal{X}} f \, d\mu_\infty$$

for all bounded, continuous functions $f : \mathcal{X} \to \mathbb{R}$.

Proof. We recall that a subsequence of $(Y_m)_m$ is tight if and only if all its components are tight. The tightness of laws of $(u^l_{h_m,N_m})_m$ was shown in Lemma 4.4.3. Then, from Prokhorov's theorem (see [26, Theorem 5.1]) it follows that, passing to a not relabeled subsequence if necessary, $(u^l_{h_m,N_m})_m$ converges in law towards a probability measure μ^1_∞ defined on $L^2(0,T;L^2(\Lambda))$. Clearly, as a constant sequence, the Brownian motion W converges in law towards $\mathbb{P}\circ W^{-1}$. Since $(u^0_{h_m})_m$ converges to u_0 in $L^2(\Lambda)$ for $m\to\infty$, a.s. in Ω, (see [5, Proposition 3.5]), we know that $(u^0_{h_m})_m$ converges in law towards $\mathbb{P}\circ(u_0)^{-1}$. From (4.27) we obtain that $(u^r_{h_m,N_m}-u^l_{h_m,N_m})_m$ converges to 0 for $m\to\infty$ in $L^2(\Omega;L^2(0,T;L^2(\Lambda)))$ and this convergence implies for all bounded, continuous functions $f: L^2(0,T;L^2(\Lambda))\to\mathbb{R}$

$$\begin{aligned}\int_{L^2(0,T;L^2(\Lambda))} f\, d(\mathbb{P}\circ(u^r_{h_m,N_m}-u^l_{h_m,N_m})^{-1}) &= \mathbb{E}\left[f(u^r_{h_m,N_m}-u^l_{h_m,N_m})\right]\\ &\overset{m\to\infty}{\longrightarrow} \mathbb{E}\left[f(0)\right].\end{aligned}$$

Therefore, we obtain the convergence in law of $(u^r_{h_m,N_m}-u^l_{h_m,N_m})_m$ towards δ_0. $\square$

Thanks to Lemma 4.4.4, we can apply Skorokhod's representation theorem (see [26, Theorem 6.7]): There exist

- a probability space $(\Omega',\mathcal{A}',\mathbb{P}')$
- a family of random variables $Y'_m=(v_m,z_m,B_m,v^0_m)$ on $(\Omega',\mathcal{A}',\mathbb{P}')$ with values in $\mathcal{X}$ having the same law as Y_m for all $m\in\mathbb{N}$
- random variables u_∞ with values in $L^2(0,T;L^2(\Lambda))$ and $\mathbb{P}'\circ(u_\infty)^{-1}=\mu^1_\infty$, W_∞ with values in $C([0,T])$ having the same law as W, and v_0 with values in $L^2(\Lambda)$ having the same law as u_0, such that for $m\to\infty$

$$\begin{aligned} v_m &\to u_\infty && \text{in } L^2(0,T;L^2(\Lambda)),\ \mathbb{P}'\text{-a.s. in } \Omega'\\ z_m &\to 0 && \text{in } L^2(0,T;L^2(\Lambda)),\ \mathbb{P}'\text{-a.s. in } \Omega'\\ B_m &\to W_\infty && \text{in } C([0,T]),\ \mathbb{P}'\text{-a.s. in } \Omega'\\ v^0_m &\to v_0 && \text{in } L^2(\Lambda),\ \mathbb{P}'\text{-a.s. in } \Omega'. \end{aligned}\tag{4.28}$$

In Lemmas 4.4.5 and 4.4.6, we will show that, thanks to equality in law, v_m and z_m are in fact finite-volume functions with the same piecewise constant structure as $u^l_{h_m,N_m}$ and $u^r_{h_m,N_m}-u^l_{h_m,N_m}$, respectively.

Lemma 4.4.5. *For $m\in\mathbb{N}$ fixed, v_m is a step function with respect to time and space in the sense that there exists $v^l_{h_m,N_m}\in\mathbb{R}^{d_{h_m}\times N_m}$, such that $v_m=v^l_{h_m,N_m}$, $\mathbb{P}'$-a.s. in Ω'. Moreover, $v^l_{h_m,N_m}(0,x):=v^0_{h_m}(x)=v^0_m(x)$ for all $x\in\Lambda$ and, in particular, $v^0_m=v^0_{h_m}$ is a spatial step function.*

Proof. By [119, Lemma A3] with $E = L^2(0,T;L^2(\Lambda))$ and $F = \mathbb{R}^{d_{h_m}\times N_m}$, there exists $(v_K^n)_{\substack{K\in\mathcal{T}_m\\ n\in\{0,\dots,N_m-1\}}}$ in $\mathbb{R}^{d_{h_m}\times N_m}$, such that

$$v_m \equiv (v_K^n)_{\substack{K\in\mathcal{T}_m\\ n\in\{0,\dots,N_m-1\}}}, \quad \mathbb{P}'\text{−a.s. in } \Omega'.$$

In the same manner with $E = L^2(\Lambda)$ and $F = \mathbb{R}^{d_{h_m}}$, it follows that there exists $(\tilde{v}_K^0)_{K\in\mathcal{T}_m}$ in $\mathbb{R}^{d_{h_m}}$, such that

$$v_m^0 \equiv (\tilde{v}_K^0)_{K\in\mathcal{T}_m}, \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

We recall the notation of Subsection 4.2.2 and, in particular, that

$$u^l_{h_m,N_m} \equiv (u_K^n)_{\substack{K\in\mathcal{T}_m\\ n\in\{0,\dots,N_m-1\}}}, \quad \mathbb{P}\text{−a.s. in } \Omega.$$

For any $K \in \mathcal{T}_m$, we consider the non-negative, Borel measurable mapping

$$\begin{aligned} \xi_K^0 : \mathbb{R}^{d_{h_m}} \times \mathbb{R}^{d_{h_m}\times N_m} &\to \mathbb{R} \\ ((a_M)_M, (b_M^k)_{M,k}) &\mapsto |a_K - b_K^0|. \end{aligned}$$

Since

$$\mathbb{P}\circ((u_M^0)_M, (u_M^k)_{M,k}))^{-1} = \mathbb{P}'\circ((\tilde{v}_M^0)_M, (v_M^k)_{M,k})^{-1},$$

we have

$$0 = \mathbb{E}\left[\xi_K^0((u_M^0)_M, (u_M^k)_{M,k})\right] = \mathbb{E}'\left[\xi_K^0((\tilde{v}_M^0)_M, (v_M^k)_{M,k})\right] = \mathbb{E}'\left[|\tilde{v}_K^0 - v_K^0|\right]$$

and, therefore, for all $x \in K$ and all $K \in \mathcal{T}_m$,

$$v_m(0,x) = v_K^0 = \tilde{v}_K^0 = v_m^0(x), \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

□

Lemma 4.4.6. *For $m \in \mathbb{N}$ fixed, $z_m(t,x) = v_K^{n+1} - v_K^n$ for all $(t,x) \in (t_n,t_{n+1}] \times K$ and $\mathbb{P}'$-a.s. in Ω', for any $K \in \mathcal{T}_m$, and $n \in \{0,\dots,N_m-1\}$, where $(v_K^n)_{\substack{K\in\mathcal{T}_m\\ n\in\{0,\dots,N_m-1\}}}$ is defined as in the proof of Lemma 4.4.5.*

Proof. Using similar arguments as in Lemma 4.4.5, we know that there exists $(z_K^n)_{\substack{K\in\mathcal{T}_m\\ n\in\{0,\dots,N_m-1\}}} \in \mathbb{R}^{d_{h_m}\times N_m}$, such that

$$z_m \equiv (z_K^n)_{\substack{K\in\mathcal{T}_m\\ n\in\{0,\dots,N_m-1\}}}, \quad \mathbb{P}'\text{−a.s. in } \Omega'.$$

For any fixed $K \in \mathcal{T}_m$, $n \in \{0, \ldots, N_m - 1\}$, the mapping

$$\Phi_K^n : \mathbb{R}^{d_{h_m} \times N_m} \times \mathbb{R}^{d_{h_m} \times N_m} \to \mathbb{R},\ ((a_M^k)_{M,k}, (b_M^k)_{M,k}) \mapsto |a_K^{n+1} - a_K^n - b_K^n|$$

is non-negative and Borel measurable. Since

$$\mathbb{P} \circ ((u_M^k)_{M,k}, (u_M^{k+1} - u_M^k)_{M,k})^{-1} = \mathbb{P}' \circ ((v_M^k)_{M,k}, (z_M^k)_{M,k})^{-1},$$

we know that for any $K \in \mathcal{T}_m$ and all $n \in \{0, \ldots, N_m - 1\}$

$$\begin{aligned} 0 &= \mathbb{E}\left[\Phi_K^n((u_M^k)_{M,k}, (u_M^{k+1} - u_M^k)_{M,k})\right] = \mathbb{E}'\left[\Phi_K^n((v_M^k)_{M,k}, (z_M^k)_{M,k})\right] \\ &= \mathbb{E}'\left[|v_K^{n+1} - v_K^n - z_K^n|\right]. \end{aligned}$$

Therefore, for all $K \in \mathcal{T}_m$ and all $n \in \{0, \ldots, N_m - 1\}$, there holds $z_K^n = v_K^{n+1} - v_K^n$, $\mathbb{P}'$-a.s. in Ω'. □

Next, we prove that the finite-volume function $(v_h^n)_{1 \le n \le N}$ we have just constructed verifies the following numerical scheme.

Lemma 4.4.7. *For $m \in \mathbb{N}$ fixed, any $n \in \{0, \ldots, N_m - 1\}$, and any $K \in \mathcal{T}_m$, v_K^{n+1} satisfies the semi-implicit equation*

$$\frac{m_K}{\Delta t}(v_K^{n+1} - v_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1}) - \frac{m_K}{\Delta t} g(v_K^n) \Delta_{n+1} B_m = 0, \tag{4.29}$$

$\mathbb{P}'$-a.s. in Ω', where $\Delta_{n+1} B_m := B_m(t_{n+1}) - B_m(t_n)$.

Proof. By Lemma 4.4.6, $z_K^n = v_K^{n+1} - v_K^n$, $\mathbb{P}'$-a.s. in Ω', for all $K \in \mathcal{T}_m$, and all $n \in \{0, \ldots, N_m - 1\}$. For arbitrary $K \in \mathcal{T}_m$ and $n \in \{0, \ldots, N_m - 1\}$, the mapping

$$\begin{aligned} \Psi_K^n &: \mathbb{R}^{d_{h_m} \times N_m} \times \mathbb{R}^{d_{h_m} \times N_m} \times C([0,T]) \to \mathbb{R}, \\ &((a_M^k)_{M,k}, (b_M^k)_{M,k}, f) \mapsto \left| \frac{m_K}{\Delta t} b_K^n + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (b_K^n + a_K^n) - (b_L^n + a_L^n) \right. \\ &\qquad\qquad \left. - \frac{m_K}{\Delta t} g(a_K^n)(f(t_{n+1}) - f(t_n)) \right| \end{aligned}$$

is non-negative and Borel measurable. Since we know

$$\mathbb{P} \circ ((u_M^k)_{M,k}, (u_M^{k+1} - u_M^k)_{M,k}, W)^{-1} = \mathbb{P}' \circ ((v_M^k)_{M,k}, (z_M^k)_{M,k}, B_m)^{-1},$$

we get from Proposition 4.2.10

$$\begin{aligned}
0 &= \mathbb{E}\left[\Psi_K^n((u_M^k)_{M,k},(u_M^{k+1}-u_M^k)_{M,k},W)\right]\\
&= \mathbb{E}'\left[\Psi_K^n((v_M^k)_{M,k},(z_M^k)_{M,k},B_m)\right]\\
&= \mathbb{E}\Bigg[\Bigg|\frac{m_K}{\Delta t}(v_K^{n+1}-v_K^n)\\
&\qquad + \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(v_K^{n+1}-v_L^{n+1}) - \frac{m_K}{\Delta t}g(v_K^n)\Delta_{n+1}B_m\Bigg|\Bigg].
\end{aligned}$$

Therefore, we obtain for all $K\in\mathcal{T}_m$, $n\in\{0,\ldots,N_m-1\}$, and $\mathbb{P}'$-a.s. in Ω',

$$0 = \frac{m_K}{\Delta t}(v_K^{n+1}-v_K^n) + \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(v_K^{n+1}-v_L^{n+1}) - \frac{m_K}{\Delta t}g(v_K^n)\Delta_{n+1}B_m.$$

□

4.4.3 Identification of the Stochastic Integral

In this subsection, we adapt ideas taken from [31, 42, 95] and adjust the arguments to our specific situation. We show that, for each $m\in\mathbb{N}$, B_m is a Brownian motion with respect to a filtration generated by v_m^0 and B_m, which we define in Definition 4.4.8. With this result at hand, we may show that $(W_\infty(t))_{t\geq 0}$ is a Brownian motion with respect to a filtration given in Definition 4.4.11. In Lemma 4.4.13, we then prove that u_∞ has a $d\mathbb{P}'\otimes dt$-representative that is predictable with respect to the filtration given in Definition 4.4.11 and is, therefore, admissible for the stochastic Itô integral with respect to $(W_\infty(t))_{t\geq 0}$. Finally, in Lemma 4.4.14, we provide an approximation result for the stochastic Itô integrals.

Definition 4.4.8. *For $t\in[0,T]$, we define $\mathcal{F}_t^m$ to be the smallest sub-σ-field of $\mathcal{A}'$ generated by v_m^0 and $B_m(s)$ for $0\leq s\leq t$. The right-continuous, $\mathbb{P}'$-augmented filtration of $(\mathcal{F}_t^m)_{t\in[0,T]}$ denoted by $(\mathfrak{F}_t^m)_{t\in[0,T]}$ is, for any $t\in[0,T]$, defined by*

$$\mathfrak{F}_t^m := \bigcap_{s>t}\sigma\left[\mathcal{F}_s^m\cup\{\mathcal{N}\in\mathcal{A}' : \mathbb{P}'(\mathcal{N})=0\}\right].$$

Remark 4.4.9. *We recall that, for the augmented filtration and for given processes $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$, such that $(X_t)_{t\geq 0}$ is adapted and $Y_t = X_t$ holds a.s. for all t, it holds true that $(Y_t)_{t\geq 0}$ is also adapted (see, e.g., [10, p.35]).*

Lemma 4.4.10. *$(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\in[0,T]}$ and $(B_m(t))_{t\in[0,T]}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\in[0,T]}$.*

Proof. Since $(\mathfrak{F}_t^m)_{t\in[0,T]}$ is a filtration induced by v_m^0 and B_m, in particular v_m^0 is $\mathfrak{F}_0^m$-measurable. Thus, applying the same arguments as in the proof of Proposition 4.2.10, from (4.29) it follows that v_m is adapted to $(\mathfrak{F}_t^m)_{t\in[0,T]}$. Since $\mathbb{P}'\circ(B_m)^{-1}=\mathbb{P}\circ W^{-1}$, we get the following results:

- $\mathbb{E}'\left[|B_m(0)|\right]=\mathbb{E}\left[|W(0)|\right]=0$, hence $B_m(0)=0$ $\mathbb{P}'$-a.s. in Ω'.
- By Burkholder-Davis-Gundy inequality, there exists a constant $C_B>0$, such that
$$\mathbb{E}'\left[\sup_{t\in[0,T]}|B_m(t)|^2\right]=\mathbb{E}\left[\sup_{t\in[0,T]}|W(t)|^2\right]\le C_BT^{\frac{1}{2}}<\infty. \tag{4.30}$$
- For all $0\le s\le t\le T$ and all bounded, continuous functions $\psi: C_b(L^2(\Lambda)\times C([0,s]))\to\mathbb{R}$,
$$\begin{aligned}0&=\mathbb{E}\left[(W(t)-W(s))\psi(u_{h_m}^0,W|_{[0,s]})\right]\\&=\mathbb{E}'\left[(B_m(t)-B_m(s))\psi(v_m^0,B_m|_{[0,s]})\right],\end{aligned}\tag{4.31}$$
and
$$\begin{aligned}0&=\mathbb{E}\left[(W^2(t)-W^2(s)-(t-s))\psi(u_{h_m}^0,W|_{[0,s]})\right]\\&=\mathbb{E}'\left[(B_m^2(t)-B_m^2(s)-(t-s))\psi(v_m^0,B_m|_{[0,s]})\right].\end{aligned}\tag{4.32}$$

Recalling Definition 4.4.8, $\mathcal{F}_t^m=\sigma_t(v_m^0,B_m)$ for $t\in[0,T]$. The real-valued random variable
$$\Omega'\ni\omega'\mapsto\psi(v_m^0(\omega'),B_m|_{[0,s]}(\omega'))$$
is $\mathcal{F}_s^m$-measurable. Using the properties of conditional expectation from (4.31), we get
$$\begin{aligned}0=&\mathbb{E}'\left[(B_m(t)-B_m(s))\psi(v_m^0,B_m|_{[0,s]})\right]\\=&\mathbb{E}'\left[\mathbb{E}'\left((B_m(t)-B_m(s))\psi(v_m^0,B_m|_{[0,s]})|\,\mathcal{F}_s^m\right)\right]\\=&\mathbb{E}'\left[\psi(v_m^0,B_m|_{[0,s]})\mathbb{E}'\left(B_m(t)-B_m(s)|\,\mathcal{F}_s^m\right)\right].\end{aligned}\tag{4.33}$$

Since (4.33) applies to every bounded and continuous function $\psi: C_b(L^2(\Lambda)\times C([0,s]))\to\mathbb{R}$, we obtain from the Lemma of Doob-Dynkin (see, e.g., [104, Proposition 3])
$$0=\mathbb{E}'\left[\mathbb{1}_A\mathbb{E}'\left(B_m(t)-B_m(s)|\,\mathcal{F}_s^m\right)\right]$$
for all $\mathcal{F}_s^m$-measurable subsets $A\in\mathcal{A}'$ and for all $0\le s\le t\le T$. This implies
$$\mathbb{E}'\left(B_m(t)-B_m(s)|\,\mathcal{F}_s^m\right)=0,\quad \mathbb{P}'\text{-a.s. in }\Omega',$$

for all $0 \leq s \leq t \leq T$ and, therefore, $(B_m(t))_{t\in[0,T]}$ is a martingale with respect to $(\mathcal{F}_t^m)_{t\in[0,T]}$. Using [45, p.75], we may conclude that $(B_m(t))_{t\in[0,T]}$ is also a martingale with respect to the augmented filtration $(\mathfrak{F}_t^m)_{t\in[0,T]}$. With similar arguments, we obtain from (4.32) that $((B_m(t))^2 - t)_{t\in[0,T]}$ is a martingale with respect to $(\mathfrak{F}_t^m)_{t\in[0,T]}$ and, consequently, the quadratic variation process $\langle\langle B_m\rangle\rangle_t$ of $(B_m(t))_{t\in[0,T]}$ is given by t for all $t \in [0,T]$ (for the Definition of the quadratic variation of a stochastic process see [22, Definition 2.19]). Summarizing the above results, $(B_m(t))_{t\in[0,T]}$ is a square-integrable martingale with respect to $(\mathfrak{F}_t^m)_{t\in[0,T]}$ starting in 0 with almost surely continuous paths and quadratic variation $\langle\langle B_m\rangle\rangle_t = t$. By [41, Theorem 3.11], $(B_m(t))_{t\in[0,T]}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\in[0,T]}$. □

In the following, we want to show firstly that the stochastic process $W_\infty =: (W_\infty(t))_{t\in[0,T]}$ is a Brownian motion and, secondly, that a filtration may be chosen in order to have compatibility of u_∞ with stochastic integration in the sense of Itô with respect to W_∞. Since u_∞ is a random variable taking values in $L^2(0,T;L^2(\Lambda))$, $u_\infty(t,\cdot)$ is only defined for a.e. $t \in [0,T]$ and the construction of an appropriate filtration induced by u_∞ becomes delicate.

Definition 4.4.11. *For $t \in [0,T]$, let $\mathcal{F}_t^\infty$ be the smallest sub-σ-field of $\mathcal{A}'$ generated by v_0, $W_\infty(s)$, and $\int_0^s u_\infty(r)\,dr$ for $0 \leq s \leq t$. The right-continuous, $\mathbb{P}'$-augmented filtration of $(\mathcal{F}_t^\infty)_{t\in[0,T]}$ denoted by $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$ is, for any $t \in [0,T]$, defined by*

$$\mathfrak{F}_t^\infty := \bigcap_{s>t} \sigma\left[\mathcal{F}_s^\infty \cup \{\mathcal{N} \in \mathcal{A}' : \mathbb{P}'(\mathcal{N}) = 0\}\right].$$

In the following, we will show that W_∞ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$ and u_∞ admits a $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$-predictable representative.

Lemma 4.4.12. *There holds $B_m \to W_\infty$ in $L^2(\Omega';C([0,T]))$ for $m \to \infty$ and $(W_\infty(t))_{t\in[0,T]}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$.*

Proof. Combining (4.30) with $\mathbb{P}' \circ (W_\infty)^{-1} = \mathbb{P} \circ W^{-1}$, we have

$$\mathbb{E}'\left[\sup_{t\in[0,T]} |W_\infty(t)|^2\right] = \mathbb{E}\left[\sup_{t\in[0,T]} |W(t)|^2\right] < \infty$$

and, consequently, $W_\infty \in L^2(\Omega';C([0,T]))$. Moreover, since $\mathbb{P}' \circ B_m^{-1} = \mathbb{P} \circ W^{-1}$,

$$\mathbb{E}'\left[\sup_{t\in[0,T]} |B_m(t)|^2\right] = \mathbb{E}'\left[\sup_{t\in[0,T]} |W_\infty(t)|^2\right], \quad \forall m \in \mathbb{N}.$$

We already know, that, for $m \to \infty$, B_m converges to W_∞ in $C([0,T])$ a.s. in Ω'. Therefore, a version of the Lemma of Brézis and Lieb (see [119, Lemma A2]) provides the desired convergence result in $L^2(\Omega'; C([0,T]))$. From

$$\mathbb{P}' \circ (v_m^0, B_m, v_m)^{-1} = \mathbb{P} \circ (u_{h_m}^0, W, u_{h_m,N_m}^l)^{-1},$$

it follows that, for any $0 \leq s \leq t \leq T$ and every bounded and continuous function $\psi : L^2(\Lambda) \times C([0,s]) \times C([0,s]; L^2(\Lambda)) \to \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{E}' \left[(B_m(t) - B_m(s)) \psi \left(v_m^0, B_m|_{[0,s]}, \int_0^\cdot v_m(r)\, dr|_{[0,s]} \right) \right] \\ &= \mathbb{E} \left[(W(t) - W(s)) \psi \left(u_{h_m}^0, W|_{[0,s]}, \int_0^\cdot u_{h_m,N_m}^l(r)\, dr|_{[0,s]} \right) \right]. \end{aligned} \tag{4.34}$$

Now, using the fact that $u_{h_m}^0$ is $\mathcal{F}_0$-measurable, by construction $\int_0^s u_{h_m,N_m}^l(r)\, dr$ is $\mathcal{F}_s$-measurable for all $m \in \mathbb{N}$, and that $(W(t))_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, one gets that

$$\mathbb{E} \left[(W(t) - W(s)) \psi \left(u_{h_m}^0, W|_{[0,s]}, \int_0^\cdot u_{h_m,N_m}^l(r)\, dr|_{[0,s]} \right) \right] = 0. \tag{4.35}$$

We recall that, $\mathbb{P}'$-a.s. in Ω', $v_m^0 \to v_0$ in $L^2(\Lambda)$ and that $B_m \to W_\infty$ in $C([0,T])$ and, therefore, also in $C([0,s])$ for all $0 \leq s \leq t \leq T$. Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left\| \int_0^\cdot v_m(r)\, dr - \int_0^\cdot u_\infty(r)\, dr \right\|_{C([0,s];L^2(\Lambda))}^2 \\ &= \sup_{z\in[0,s]} \left\| \int_0^z (v_m(r) - u_\infty(r))\, dr \right\|_{L^2(\Lambda)}^2 \\ &\leq \sup_{z\in[0,s]} \left(\int_0^z \|v_m(r) - u_\infty(r)\|_{L^2(\Lambda)}\, dr \right)^2 \\ &\leq T \int_0^T \|v_m(r) - u_\infty(r)\|_{L^2(\Lambda)}^2\, dr. \end{aligned}$$

Since $v_m \overset{m\to\infty}{\longrightarrow} u_\infty$ in $L^2(0,T; L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω', it follows that

$$\int_0^\cdot v_m(r)\, dr \overset{m\to\infty}{\longrightarrow} \int_0^\cdot u_\infty(r)\, dr$$

in $C([0,s]; L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'. Using the convergence of B_m towards W_∞ in $L^2(\Omega'; C([0,T]))$, the convergence results from above, and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}\lim_{m\to\infty} \mathbb{E}' &\left[(B_m(t)-B_m(s))\psi\left(v_m^0, B_m|_{[0,s]}, \int_0^\cdot v_m(r)\,dr|_{[0,s]}\right)\right] \\ = \mathbb{E}' &\left[(W_\infty(t)-W_\infty(s))\psi\left(v_0, W_\infty|_{[0,s]}, \int_0^\cdot u_\infty(r)\,dr|_{[0,s]}\right)\right].\end{aligned} \tag{4.36}$$

Now, combining (4.34), (4.35), and (4.36), we obtain

$$\mathbb{E}'\left[(W_\infty(t)-W_\infty(s))\psi\left(v_0, W_\infty|_{[0,s]}, \int_0^\cdot u_\infty(r)\,dr|_{[0,s]}\right)\right] = 0. \tag{4.37}$$

From (4.37) it follows that

$$\mathbb{E}'\left(W_\infty(t)-W_\infty(s)|\,\mathcal{F}_s^\infty\right) = 0,$$

$\mathbb{P}'$-a.s. in Ω', for all $0 \le s \le t \le T$. With similar arguments as used for equation (4.37), we also get

$$\mathbb{E}'\left[(W_\infty^2(t)-W_\infty^2(s)-(t-s))\psi\left(v_0, W_\infty|_{[0,s]}, \int_0^\cdot u_\infty(r)\,dr|_{[0,s]}\right)\right] = 0.$$

Now, using a similar argumentation as in the end of the proof of Lemma 4.4.10, we obtain that $(W_\infty(t))_{t\in[0,T]}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$. □

By [31, Theorem 2.6.3], it is always possible to choose $(\Omega', \mathcal{A}', \mathbb{P}') = ([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ denotes the Borel sets on $[0,1]$ and λ denotes the Lebesgue measure on $[0,1]$. We will need this particular choice of the new probability space in the proof of the following lemma.
We recall that, for a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathcal{F} = (\mathcal{F}_t)_{t\ge 0}$ and $T > 0$, the predictable σ-field on $\Omega \times [0,T]$ is the σ-field generated by the sets

$$(s,t]\times F_s,\ 0\le s<t\le T,\ F_s\in\mathcal{F}_s \quad \text{and} \quad \{0\}\times F_0,\ F_0\in\mathcal{F}_0.$$

For more details on stochastic integration in infinite dimension, we refer to [41].

Lemma 4.4.13. *There exists a $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$-predictable, $d\mathbb{P}'\otimes dt$-representative of u_∞.*

Proof. For $\delta > 0$, we define $u_\infty^\delta : \Omega' \times [0,T] \to L^2(\Lambda)$ by

$$u_\infty^\delta(t) := \frac{1}{\delta}\int_{(t-\delta)^+}^t u_\infty(s)\,ds = \frac{1}{\delta}\left(\int_0^t u_\infty(s)\,ds - \int_0^{(t-\delta)^+} u_\infty(s)\,ds\right),$$

where the integrals on the right-hand side are understood as Bochner integrals with values in $L^2(\Lambda)$. Since u_∞^δ is an $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$-adapted stochastic process with a.s. continuous paths, it is predictable with respect to $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$. For fixed $k \in \mathbb{N}$, the cut-off function $T_k : \mathbb{R} \to [-k,k]$, defined by $T_k(r) := r$ if $|r| < k$ and $T_k(r) := \operatorname{sign}(r)k$ if $|r| \geq k$, induces a continuous operator $L^2(\Lambda) \ni v \mapsto T_k(v) \in L^2(\Lambda)$. Hence, the stochastic process

$$\Omega' \times [0,T] \ni (\omega', t) \mapsto T_k(u_\infty^\delta(\omega', t)) \in L^2(\Lambda)$$

is $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$-predictable. Again, we recall that, $\mathbb{P}'$-a.s. in Ω', $v_m \to u_\infty$ for $m \to \infty$ in $L^2(0,T;L^2(\Lambda))$ and, thus, also in $L^1(0,T;L^2(\Lambda))$. Therefore, we have

$$\lim_{m\to\infty}\int_0^T \|v_m(t)\|_{L^2(\Lambda)}\,dt = \int_0^T \|u_\infty(t)\|_{L^2(\Lambda)}\,dt, \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

Using Fatou's lemma, $\mathbb{P}' \circ (v_m)^{-1} = \mathbb{P} \circ (u^l_{h_m,N_m})^{-1}$, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathbb{E}'\left[\int_0^T \|u_\infty(t)\|_{L^2(\Lambda)}\,dt\right] &\leq \liminf_{m\to\infty}\mathbb{E}'\left[\int_0^T \|v_m(t)\|_{L^2(\Lambda)}\,dt\right] \\
&= \liminf_{m\to\infty}\mathbb{E}\left[\int_0^T \|u^l_{h_m,N_m}(t)\|_{L^2(\Lambda)}\,dt\right] \\
&\leq \sqrt{T}\liminf_{m\to\infty}\|u^l_{h_m,N_m}\|^2_{L^2(\Omega;L^2(0,T;L^2(\Lambda)))}.
\end{aligned}$$

From Lemma 4.3.3 it follows that the right-hand side of the equation is uniformly bounded and, consequently, $u_\infty \in L^1(\Omega';L^1(0,T;L^2(\Lambda)))$. In particular, $u_\infty \in L^1(\Omega';L^1(0,T;L^1(\Lambda)))$. Since $(\Omega',\mathcal{A}',\mathbb{P}') = ([0,1],\mathcal{B}([0,1]);\lambda)$ according to [52, Remark after Proposition 1.8.1] we have

$$L^1(\Omega';L^1(0,T;L^1(\Lambda))) \cong L^1(\Omega' \times (0,T);L^1(\Lambda)) \cong L^1(0,T;L^1(\Omega';L^1(\Lambda))).$$

For almost every $t \in (0,T)$ and $0 < \delta < t$, we know

$$\begin{aligned}
\left\|u_\infty^\delta(t) - u_\infty(t)\right\|_{L^1(\Omega';L^1(\Lambda))} &= \left\|\frac{1}{\delta}\int_{(t-\delta)^+}^t (u_\infty(s) - u_\infty(t))\,ds\right\|_{L^1(\Omega';L^1(\Lambda))} \\
&\leq \frac{1}{\delta}\int_{(t-\delta)^+}^t \|u_\infty(s) - u_\infty(t)\|_{L^1(\Omega';L^1(\Lambda))}\,ds.
\end{aligned}$$

By a generalisation of Lebesgue differentiation theorem for vector-valued functions (see, e.g. [46, Theorem 9, Chapter II]), the right-hand side of the above inequality goes to 0 for $\delta \downarrow 0$ for almost every $t \in (0,T)$ and, therefore, $u^\delta_\infty(t) \to u_\infty(t)$ a.e in $L^1(\Omega'; L^1(\Lambda))$ for $\delta \downarrow 0$. Then, Lebesgue's dominated convergence theorem provides

$$\lim_{\delta \downarrow 0} T_k(u^\delta_\infty) = T_k(u_\infty)$$

in $L^1(0,T; L^1(\Omega'; L^1(\Lambda)))$, thus also in $L^1(\Omega' \times (0,T); L^1(\Lambda))$. Therefore, passing to a not relabeled subsequence if necessary, we obtain for $\delta \downarrow 0$

$$T_k(u^\delta_\infty(\omega', t)) \to T_k(u_\infty(\omega', t)) \text{ in } L^1(\Lambda) \text{ for a.e. } (\omega', t) \in \Omega' \times (0,T).$$

Hence, $T_k(u_\infty(\omega', t))$ has a $d\mathbb{P}' \otimes dt$-representative which is $(\mathfrak{F}^\infty_t)_{t\in[0,T]}$-predictable for every $k \in \mathbb{N}$. Obviously, there holds

$$u_\infty(\omega', t) = \sup_{k\in\mathbb{N}} T_k(u_\infty(\omega', t)) \text{ in } L^1(\Lambda) \text{ for a.e. } (\omega', t) \text{ in } \Omega' \times (0,T),$$

where the set of measure zero can be chosen independently of $k \in \mathbb{N}$. This provides the existence of a $d\mathbb{P}' \otimes dt$-representative of u_∞ that is $(\mathfrak{F}^\infty_t)_{t\in[0,T]}$-predictable. □

Lemma 4.4.14. *For $t \in [0,T]$, $x \in \Lambda$, and $\mathbb{P}'$-a.s. in Ω', we define the stochastic processes*

$$\mathcal{M}_{h_m,N_m}(t,x) := \int_0^t g(v^l_{h_m,N_m}(s,x))\, dB_m(s)$$
$$\mathcal{M}_\infty(t,x) := \int_0^t g(u_\infty(s,x))\, dW_\infty(s).$$

Then, passing to a not relabeled subsequence if necessary,

$$\mathcal{M}_{h_m,N_m} \overset{m\to\infty}{\longrightarrow} \mathcal{M}_\infty \text{ in } L^2(0,T; L^2(\Lambda)),\ \mathbb{P}'\text{-a.s. in } \Omega'. \tag{4.38}$$

Proof. From Lemma 4.4.12, we know that $(B_m)_m$ converges in $L^2(\Omega'; C([0,T]))$ towards W_∞ which is a Brownian motion with respect to $(\mathfrak{F}^\infty_t)_{t\in[0,T]}$. Particularly, this convergence result also holds in probability in $C([0,T])$. Moreover, from the convergence (4.28) and Lemma 4.4.5, we know that $(v^l_{h_m,N_m})_m$ converges towards u_∞ in $L^2(0,T; L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'. Thus, up to a subsequence denoted in the same way, using the Lipschitz property of g, $(g(v^l_{h_m,N_m}))_m$ converges to $g(u_\infty)$ in probability in $L^2(0,T; L^2(\Lambda))$. Now, we can apply Lemma 2.1 in [42] and conclude that the convergence in (4.38) holds true in probability in $L^2(0,T; L^2(\Lambda))$ and, therefore, passing to a subsequence if necessary, the assertion follows. □

4.4.4 Convergence towards a Martingale Solution

For the sake of simplicity, we use the notations $\mathcal{T} = \mathcal{T}_m$, $h = h_m$, $\Delta t = \Delta t_m$, and $N = N_m$. For any $n \in \{0, \ldots, N\}$ and $K \in \mathcal{T}$, setting $\mathcal{M}_K^n := \mathcal{M}_{h,N}(t_n, x_K)$, we define $\widehat{\mathcal{M}}_{h,N}$ using the definition in (4.5) and we obtain the following strong convergence result in $L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$.

Lemma 4.4.15. *Passing to a not relabeled subsequence if necessary, we have the following convergence results for any $p \in [1,2)$:*

$$\begin{aligned} v_{h,N}^l,\ v_{h,N}^r \text{ and } \widehat{v}_{h,N} &\overset{m\to\infty}{\longrightarrow} u_\infty && \text{in } L^p(\Omega'; L^2(0,T; L^2(\Lambda))), \\ \mathcal{M}_{h,N} \text{ and } \widehat{\mathcal{M}}_{h,N} &\overset{m\to\infty}{\longrightarrow} \mathcal{M}_\infty && \text{in } L^p(\Omega'; L^2(0,T; L^2(\Lambda))) \end{aligned}$$

and

$$v_h^0 \overset{m\to\infty}{\longrightarrow} v_0 \quad \text{in } L^p(\Omega'; L^2(\Lambda)).$$

Moreover, $u_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)))$ and $\mathcal{M}_\infty \in L^2(\Omega'; C([0,T]; L^2(\Lambda)))$.

Proof. We recall that, thanks to the convergence (4.28), $(v_{h,N}^l)_m$ converges to u_∞ for $m \to \infty$ in $L^2(0,T; L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'. Since $\mathbb{P}' \circ (v_{h,N}^l)^{-1} = \mathbb{P} \circ (u_{h,N}^l)^{-1}$, from Lemma 4.3.3 it follows that there exists a constant $C \geq 0$, such that

$$\mathbb{E}' \left[\|v_{h,N}^l\|_{L^2(0,T;L^2(\Lambda))}^2 \right] \leq C \tag{4.39}$$

for all $m \in \mathbb{N}$. From Fatou's lemma we obtain $u_\infty \in L^2(\Omega'; L^2(0,T; L^2(\Lambda)))$. The convergence of $(v_{h,N}^l)_m$ towards u_∞ in $L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$ is a consequence of (4.39) and of the theorem of Vitali (see, e.g., [52, Corollaire 1.3.3]). Now, using (4.27), we get

$$\mathbb{E}' \left[\|v_{h,N}^r - v_{h,N}^l\|_{L^2(0,T;L^2(\Lambda))}^2 \right] = \mathbb{E} \left[\|u_{h,N}^r - u_{h,N}^l\|_{L^2(0,T;L^2(\Lambda))}^2 \right] \overset{m\to\infty}{\longrightarrow} 0,$$

and, therefore, $(v_{h,N}^r - v_{h,N}^l) \to 0$ in $L^2(\Omega'; L^2(0,T; L^2(\Lambda)))$ for $m \to \infty$. Thanks to the continuous embedding $L^2(\Omega'; L^2(0,T; L^2(\Lambda))) \hookrightarrow L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$, the convergence holds also true in $L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$ for all $1 \leq p < 2$. Therefore, we have, for all $1 \leq p < 2$,

$$v_{h,N}^r \to u_\infty \text{ in } L^p(\Omega'; L^2(0,T; L^2(\Lambda))) \text{ for } m \to \infty.$$

Now, we apply a similar argumentation to $(\widehat{v}_{h,N})_m$. We have

$$\begin{aligned}
&\mathbb{E}'\left[\|v^l_{h,N}-\widehat{v}_{h,N}\|^2_{L^2(0,T;L^2(\Lambda))}\right]\\
&=\mathbb{E}'\left[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left\|v_h^n-\left[\frac{v_h^{n+1}-v_h^n}{\Delta t}(t-t_n)+v_h^n\right]\right\|^2_{L^2(\Lambda)}dt\right]\\
&=\mathbb{E}'\left[\sum_{n=0}^{N-1}\|v_h^{n+1}-v_h^n\|^2_{L^2(\Lambda)}\int_{t_n}^{t_{n+1}}\left(\frac{t-t_n}{\Delta t}\right)^2dt\right]\\
&=\frac{\Delta t}{3}\mathbb{E}'\left[\sum_{n=0}^{N-1}\|v_h^{n+1}-v_h^n\|^2_{L^2(\Lambda)}\right].
\end{aligned}\tag{4.40}$$

Repeating the arguments of Proposition 4.3.2 on (4.29), we know, that there exists a constant $C_1'\geq 0$, such that

$$\mathbb{E}'\left[\int_0^T|v^r_{h,N}|^2_{1,h}\,dt\right]+\mathbb{E}'\left[\sum_{n=0}^{N-1}\|v_h^{n+1}-v_h^n\|^2_{L^2(\Lambda)}\right]\leq C_1'.\tag{4.41}$$

Combining (4.40) and (4.41), it follows that

$$(v^l_{h,N}-\widehat{v}_{h,N})\to 0 \text{ in } L^2(\Omega';L^2(0,T;L^2(\Lambda))) \text{ for } m\to\infty$$

and we may conclude that $\widehat{v}_{h,N}\to u_\infty$ in $L^p(\Omega';L^2(0,T;L^2(\Lambda)))$ for all $1\leq p<2$. Using (4.41) and the same arguments as in the proof of Lemma 4.4.1 on the discrete gradient $\nabla^h v^r_{h,N}$ of $v^r_{h,N}$, we obtain that, passing to a not relabeled subsequence if necessary,

$$\nabla^h v^r_{h,N}\rightharpoonup\nabla u_\infty \text{ in } L^2(\Omega';L^2(0,T;L^2(\Lambda)^2)) \text{ for } m\to\infty,.$$

Therefore, we know $u_\infty\in L^2(\Omega';L^2(0,T;H^1(\Lambda)))$.
We recall that, according to Lemma 4.4.14, $\mathcal{M}_{h,N}\to\mathcal{M}_\infty$ for $m\to\infty$ in $L^2(0,T;L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'. Using H_2, (4.2), the Burkholder-Davis-Gundy inequality with constant $C_B\geq 0$, and Lemma 4.3.5, we get

$$\begin{aligned}
\mathbb{E}'\left[\sup_{t\in[0,T]}\|\mathcal{M}_{h,N}(t)\|^2_{L^2(\Lambda)}\right]&\leq C_B\mathbb{E}'\left[\int_0^T\|g(v^l_{h,N}(t))\|^2_{L^2(\Lambda)}\,dt\right]\\
&\leq C_BC_L\left(|\Lambda|T+\mathbb{E}'\left[\int_0^T\|v^l_{h,N}(t)\|^2_{L^2(\Lambda)}\,dt\right]\right)\\
&=C_BC_L\left(|\Lambda|T+\mathbb{E}\left[\int_0^T\|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}\,dt\right]\right)\\
&=C_BC_LT(|\Lambda|+K_3).
\end{aligned}\tag{4.42}$$

Now, the convergence of $(\mathcal{M}_{h,N})_m$ towards $\mathcal{M}_\infty$ in $L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$ for all $1 \leq p < 2$ follows from (4.42) and the theorem of Vitali (see, e.g., [52, Corollaire 1.3.3]). Using Itô isometry, H_2 with (4.2), and Lemma 4.3.5, we know that there exists a constant $C_3 \geq 0$, such that

$$
\begin{aligned}
&\mathbb{E}'\left[\int_0^T \|\mathcal{M}_{h,N}(t) - \widehat{\mathcal{M}}_{h,N}(t)\|^2_{L^2(\Lambda)}\,dt\right] \\
&= \mathbb{E}\Bigg[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \Bigg\| \int_{t_n}^{t} g(v^l_{h,N}(s))\,dB_m(s) \\
&\qquad - \frac{t-t_n}{\Delta t}\int_{t_n}^{t_{n+1}} g(v^l_{h,N}(s))\,dB_m(s)\Bigg\|^2_{L^2(\Lambda)} dt\Bigg] \\
&\leq 2\mathbb{E}\Bigg[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \Bigg(\int_{t_n}^{t} \|g(v^l_{h,N}(s))\|^2_{L^2(\Lambda)}\,ds \\
&\qquad + \left(\frac{t-t_n}{\Delta t}\right)^2 \int_{t_n}^{t_{n+1}} \|g(v^l_{h,N}(s))\|^2_{L^2(\Lambda)}\,ds\Bigg)\,dt\Bigg] \\
&\leq 2\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} C_L(t-t_n)\left(|\Lambda| + \mathbb{E}'\left[\sup_{s\in[0,T]}\|v^l_{h,N}(s)\|^2_{L^2(\Lambda)}\right]\right) \\
&\qquad \cdot \left(1 + \frac{(t-t_n)}{\Delta t}\right)\,dt \\
&\leq 2\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} C_L(t-t_n)\left(|\Lambda| + \mathbb{E}\left[\sup_{s\in[0,T]}\|u^l_{h,N}(s)\|^2_{L^2(\Lambda)}\right]\right) \\
&\qquad \cdot \left(1 + \frac{(t-t_n)}{\Delta t}\right)\,dt \\
&\leq \frac{5}{3}C_L T(|\Lambda| + K_3)\Delta t \to 0 \quad \text{for } m \to \infty.
\end{aligned}
$$

Since $L^2(\Omega'; L^2(0,T; L^2(\Lambda))) \hookrightarrow L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$ for $p \in [1,2)$, we obtain the convergence

$$
\widehat{\mathcal{M}}_{h,N} \to \mathcal{M}_\infty \text{ in } L^p(\Omega'; L^2(0,T; L^2(\Lambda))) \text{ for } m \to \infty,\ p \in [1,2).
$$

Recalling that $\mathcal{M}_\infty$ is a stochastic Itô integral with respect to the Brownian motion $(W_\infty(t))_{t\geq 0}$, we may conclude that $\mathcal{M}_\infty$ has $\mathbb{P}'$-a.s. continuous paths in $L^2(\Lambda)$. From (4.42) and Fatou's lemma, we know that $\mathcal{M}_\infty \in L^2(\Omega'; C([0,T]; L^2(\Lambda)))$. Since $\mathbb{P}' \circ (v^0_h)^{-1} = \mathbb{P} \circ (u^0_h)^{-1}$, we obtain by Lemma 4.3.1

$$
\mathbb{E}'\left[\|v^0_h\|^2_{L^2(\Lambda)}\right] = \mathbb{E}\left[\|u^0_h\|^2_{L^2(\Lambda)}\right] \leq \mathbb{E}\left[\|u_0\|^2_{L^2(\Lambda)}\right].
$$

Because we already know that $(v_h^0)_m$ converges $\mathbb{P}'$-a.s. in Ω' to v_0, the last assertion is a consequence of Vitali's theorem (see, e.g., [52, Corollaire 1.3.3]). $\square$

Now, we have all the necessary tools to pass to the limit in the scheme.

Proposition 4.4.16. *There exists a subsequence of $(\widehat{v}_{h,N})_m$, still denoted by $(\widehat{v}_{h,N})_m$, converging in $L^p(\Omega'; L^2(0,T; L^2(\Lambda)))$ (for any $p \in [1,2)$) for $m \to \infty$ to a $(\mathfrak{F}_t^\infty)_{t\in[0,T]}$-adapted stochastic process u_∞ with values in $L^2(\Lambda)$ and having $\mathbb{P}'$-a.s. continuous paths. Moreover, $u_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)))$ and satisfies, for all $t \in [0,T]$,*

$$u_\infty(t) - v_0 - \int_0^t \Delta u_\infty \, ds = \int_0^t g(u_\infty) \, dW_\infty, \quad \text{in } L^2(\Lambda), \ \mathbb{P}'\text{-a.s. in } \Omega'.$$

Proof. Let $A \in \mathcal{A}'$, $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$, and $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\nabla\varphi \cdot \mathbf{n} = 0$ on $\partial\Lambda$, where we denote $\mathcal{D}(D) := C_c^\infty(D)$ for any open subset $D \subseteq \mathbb{R}^j$, $j \in \mathbb{N}$. Moreover, we define the piecewise constant function $\varphi_h(x) := \varphi(x_K)$ for $x \in K$, $K \in \mathcal{T}$.
For $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$, and $t \in [t_n, t_{n+1})$, we multiply (4.29) by $\mathbb{1}_A \xi(t) \varphi(x_K)$ to obtain

$$\begin{aligned} &\mathbb{1}_A \xi(t) \frac{m_K}{\Delta t} [v_K^{n+1} - v_K^n - g(v_K^n) \Delta_{n+1} B_m] \varphi(x_K) \\ &+ \mathbb{1}_A \xi(t) \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (v_K^{n+1} - v_L^{n+1}) \varphi(x_K) = 0. \end{aligned} \tag{4.43}$$

First, we sum (4.43) over each control volume $K \in \mathcal{T}$, then we integrate over each time interval $[t_n, t_{n+1}]$ for fixed $n = 0, \dots, N-1$, then we sum over $n = 0, \dots, N-1$, and, finally, we take the expectation to obtain

$$\begin{aligned} 0 = &\,\mathbb{E}' \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} m_K \mathbb{1}_A \xi(t) \frac{1}{\Delta t} [v_K^{n+1} - v_K^n - g(v_K^n) \Delta_{n+1} B_m] \varphi(x_K) \, dt \right] \\ &+ \mathbb{E}' \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A \xi(t) \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (v_K^{n+1} - v_L^{n+1}) \varphi(x_K) \, dt \right] \\ =: &\, T_{1,m} + T_{2,m}. \end{aligned} \tag{4.44}$$

In the following, we will pass to the limit for $m \to \infty$ on the right-hand side of (4.44). Using partial integration, we obtain

$$\begin{aligned}
T_{1,m} &= \mathbb{E}'\left[\mathbb{1}_A \int_0^T \int_\Lambda \partial_t[\widehat{v}_{h,N} - \widehat{\mathcal{M}}_{h,N}](t,x)\xi(t)\varphi_h(x)\,dx\,dt\right] \\
&= -\mathbb{E}'\left[\mathbb{1}_A \int_0^T \int_\Lambda [\widehat{v}_{h,N} - \widehat{\mathcal{M}}_{h,N}](t,x)\xi'(t)\varphi_h(x)\,dx\,dt\right] \\
&\quad - \mathbb{E}'\left[\mathbb{1}_A \int_\Lambda v_h^0(x)\xi(0)\varphi_h(x)\,dx\right].
\end{aligned}$$

Thanks to the convergence results of Lemma 4.4.15, passing to a not relabeled subsequence if necessary, we can pass to the limit for $m \to \infty$ and obtain

$$\begin{aligned}
&- \mathbb{E}'\left[\mathbb{1}_A \int_0^T \int_\Lambda [\widehat{v}_{h,N} - \widehat{\mathcal{M}}_{h,N}](t,x)\xi'(t)\varphi_h(x)\,dx\,dt\right] \\
&- \mathbb{E}'\left[\mathbb{1}_A \int_\Lambda v_h^0(x)\xi(0)\varphi_h(x)\,dx\right] \\
&\to -\mathbb{E}'\left[\mathbb{1}_A \int_0^T \int_\Lambda [u_\infty - \mathcal{M}_\infty](t,x)\xi'(t)\varphi(x)\,dx\,dt\right] \\
&\quad - \mathbb{E}'\left[\mathbb{1}_A \int_\Lambda v_0(x)\xi(0)\varphi(x)\,dx\right].
\end{aligned}$$

Our aim is to show the following convergence result for $m \to \infty$:

$$T_{2,m} \to -\mathbb{E}'\left[\mathbb{1}_A \int_0^T \int_\Lambda \xi(t)\Delta\varphi(x)u_\infty(t,x)\,dx\,dt\right].$$

First, we note that by rearranging the sum in (4.44) the term $T_{2,m}$ can be rewritten as

$$T_{2,m} = \mathbb{E}'\left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A \xi(t) \sum_{K\in\mathcal{T}} v_K^{n+1} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}_K} m_\sigma \left(\frac{\varphi(x_K) - \varphi(x_L)}{d_{K|L}}\right) dt\right].$$

Since $\nabla\varphi \cdot \mathbf{n} = 0$ on $\partial\Lambda$, thanks to the Stokes formula, one has, for any $K \in \mathcal{T}$,

$$\int_K \Delta\varphi(x)\,dx = \int_{\partial K} \nabla\varphi(x)\cdot\mathbf{n}\,d\gamma(x) = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}_K} \int_\sigma \nabla\varphi(x)\cdot\mathbf{n}_{KL}\,d\gamma(x), \tag{4.45}$$

where γ denotes the one-dimensional Lebesgue measure. Note that, since $\mathbf{n}_{KL} = -\mathbf{n}_{LK}$, we have by rearranging the sum

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} v_K^{n+1} \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} \int_\sigma \nabla\varphi(x)\cdot\mathbf{n}_{KL}\, d\gamma(x) \\
&= \sum_{\sigma\in\mathcal{E}_{\text{int}}} (v_K^{n+1} - v_L^{n+1}) \int_\sigma \nabla\varphi(x)\cdot\mathbf{n}_{KL}\, d\gamma(x).
\end{aligned} \tag{4.46}$$

Therefore, we obtain by (4.45), (4.46), and (4.6),

$$\begin{aligned}
T_{2,m} =& -\mathbb{E}'\left[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{1}_A\xi(t)\sum_{K\in\mathcal{T}} v_K^{n+1}\int_K \Delta\varphi(x)\,dx\right] \\
&+\mathbb{E}'\Bigg[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{1}_A\xi(t)\sum_{K\in\mathcal{T}} v_K^{n+1} \\
&\quad\cdot\sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K}\left(m_\sigma\frac{\varphi(x_K)-\varphi(x_L)}{d_{K|L}} + \int_\sigma \nabla\varphi(x)\cdot\mathbf{n}_{KL}\,d\gamma(x)\,dt\right)\Bigg] \\
=& -\mathbb{E}'\left[\int_0^T \mathbb{1}_A\xi(t)\int_\Lambda v_{h,N}^r(t,x)\Delta\varphi(x)\,dx\,dt\right] \\
&+\mathbb{E}'\left[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{1}_A\xi(t)\sum_{\sigma\in\mathcal{E}_{\text{int}}} m_\sigma(v_K^{n+1}-v_L^{n+1})R_\sigma^\varphi\,dt\right] \\
=&: T_{2,m}^1 + T_{2,m}^2,
\end{aligned}$$

where

$$R_\sigma^\varphi = \frac{1}{m_\sigma}\int_\sigma \nabla\varphi(x)\cdot\mathbf{n}_{KL}\,d\gamma(x) - \frac{\varphi(x_L)-\varphi(x_K)}{d_{K|L}}.$$

Using Lemma 4.4.15 and passing to a not relabeled subsequence if necessary, we get for $m\to\infty$

$$T_{2,m}^1 \to -\mathbb{E}'\left[\int_0^T\int_\Lambda \mathbb{1}_A\xi(t)u_\infty(t,x)\Delta\varphi(x)\,dx\,dt\right].$$

Using the orthogonality condition of the mesh, i.e., $x_L - x_K = d_{K|L}\mathbf{n}_{KL}$ for two neighbouring control volumes $K, L\in\mathcal{T}$, we know, thanks to the Taylor formula,

$$\nabla\varphi(x)\cdot\mathbf{n}_{KL} = \frac{\varphi(x_L)-\varphi(x_K)}{d_{K|L}} + \mathcal{O}(h) \quad \text{for } x\in\sigma = K|L\in\mathcal{E}_{\text{int}}.$$

Consequently, we obtain

$$R_\sigma^\varphi \leq C_\varphi h \quad \text{for any } \sigma \in \mathcal{E}_{\text{int}},$$

and for a constant $C_\varphi \geq 0$ only depending on φ. Therefore, thanks to Cauchy-Schwarz inequality and Inequality (4.41)

$$\begin{aligned}
|T_{2,m}^2| &\leq C_\varphi h \mathbb{E}'\Bigg[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{1}_A|\xi(t)|\Bigg(\sum_{\sigma\in\mathcal{E}_{\text{int}}} m_\sigma d_{K|L}\Bigg)^{\frac{1}{2}} \\
&\qquad \cdot \Bigg(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}}|v_K^{n+1}-v_L^{n+1}|^2\Bigg)^{\frac{1}{2}}\,dt\Bigg] \\
&\leq C_\varphi h\sqrt{2|\Lambda|}\,\mathbb{E}'\left[\int_0^T \mathbb{1}_A|\xi(t)||v_{h,N}^r(t)|_{1,h}\,dt\right] \\
&\leq C_\varphi h\sqrt{2|\Lambda|}\|\xi\mathbb{1}_A\|_{L^2(\Omega'\times(0,T))}\left(\mathbb{E}'\left[\int_0^T |v_{h,N}^r(t)|_{1,h}^2\,dt\right]\right)^{\frac{1}{2}} \\
&\to 0 \quad \text{for } m\to\infty.
\end{aligned}$$

Thus, we have shown that for all $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ such that $\nabla\varphi\cdot\mathbf{n} = 0$ on $\partial\Lambda$, there holds, $\mathbb{P}'$-a.s. in Ω',

$$\begin{aligned}
&-\int_0^T\int_\Lambda\left(u_\infty(t,x)-\int_0^t g(u_\infty(s,x))\,dW_\infty(s)\right)\xi'(t)\varphi(x)\,dx\,dt \\
&-\int_\Lambda v_0(x)\xi(0)\varphi(x)\,dx \\
&=\int_0^T\int_\Lambda u_\infty(t,x)\Delta\varphi(x)\xi(t)\,dx\,dt \\
&=-\int_0^T\int_\Lambda \nabla u_\infty(t,x)\cdot\nabla\varphi(x)\xi(t)\,dx\,dt.
\end{aligned} \tag{4.47}$$

By [53, Theorem 1.1], the set $\{\varphi\in\mathcal{D}(\mathbb{R}^2)\mid \nabla\varphi\cdot\mathbf{n}=0 \text{ on } \partial\Lambda\}$ is dense in $H^1(\Lambda)$ and, therefore, (4.47) applies to all $\varphi\in H^1(\Lambda)$.
In the following, we denote the dual space of $H^1(\Lambda)$ by $H^1(\Lambda)^*$, recall that

$$H^1(\Lambda)\hookrightarrow L^2(\Lambda)\hookrightarrow H^1(\Lambda)^*$$

with continuous and dense embeddings and we will denote the $H^1(\Lambda)$-$H^1(\Lambda)^*$ duality bracket by $\langle\cdot,\cdot\rangle_{H^1}$. The additional information

$$u_\infty \in L^2(\Omega'; L^2(0,T;H^1(\Lambda)))$$

in Lemma 4.4.15 provides

$$\Delta u_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)^*))$$

and

$$-\int_0^T \int_\Lambda \nabla u_\infty(t,x) \cdot \nabla \varphi(x) \xi(t)\, dx\, dt = \int_0^T \langle \Delta u_\infty(t,\cdot), \varphi \rangle_{H^1} \xi(t)\, dt, \quad (4.48)$$

$\mathbb{P}'$-a.s. in Ω', for all $\xi \in \mathcal{D}(\mathbb{R})$ such that $\xi(T) = 0$, and all $\varphi \in H^1(\Lambda)$. Combining (4.47) and (4.48), and using the identity

$$-\int_\Lambda v_0(x)\varphi(x)\xi(0)\, dx = \int_0^T \int_\Lambda v_0(x)\varphi(x)\xi'(t)\, dx\, dt \quad (4.49)$$

(see, [105, Lemma 7.3]), Fubini's theorem provides for all $\xi \in \mathcal{D}(\mathbb{R})$ such that $\xi(T) = 0$, and all $\varphi \in H^1(\Lambda)$, $\mathbb{P}'$-a.s. in Ω',

$$\begin{aligned}&\langle -\int_0^T \left(u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty - v_0 \right) \xi'(t)\, dt, \varphi \rangle_{H^1} \\ &= \langle \int_0^T \Delta u_\infty(t)\xi(t)\, dt, \varphi \rangle_{H^1}.\end{aligned}$$

By a separability argument, the exceptional set in Ω' may be chosen independently of φ, and, therefore, we have

$$-\int_0^T \left(u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty - v_0 \right) \xi'(t)\, dt = \int_0^T \Delta u_\infty(t)\xi(t)\, dt$$

in $H^1(\Lambda)^*$, for all $\xi \in \mathcal{D}(\mathbb{R})$ such that $\xi(T) = 0$, $\mathbb{P}'$-a.s. in Ω'. Consequently, (see, e.g. [33, Proposition A6])

$$u_\infty - \int_0^\cdot g(u_\infty)\, dW_\infty - v_0 \in W^{1,2}(0,T; H^1(\Lambda)^*) \quad \mathbb{P}'\text{-a.s. in } \Omega'$$

and

$$\frac{d}{dt}\left(u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty - v_0 \right) = \Delta u_\infty \quad \text{in } L^2(\Omega'; L^2(0,T; H^1(\Lambda)^*). \quad (4.50)$$

Because g is Lipschitz continuous, the chain rule for Sobolev functions implies $g(u_\infty) \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)))$ and

$$\nabla \left(\int_0^t g(u_\infty)\, dW_\infty \right) = \int_0^t g'(u_\infty) \nabla u_\infty\, dW_\infty.$$

Hence, we have

$$u_\infty - \int_0^\cdot g(u_\infty)\,dW_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda))).$$

From [105, Lemma 7.3], we obtain $u_\infty \in L^2(\Omega'; C([0,T]; L^2(\Lambda))$ and, by (4.50), the following rule of partial integration:

$$\begin{aligned}
&\langle u_\infty(t) - \int_0^t g(u_\infty)\,dW_\infty - v_0, \zeta(t)\rangle_{L^2(\Lambda)} - \langle u_\infty(0) - v_0, \zeta(0)\rangle_{L^2(\Lambda)} \\
&= \int_0^t \langle \Delta u_\infty(s), \zeta(s)\rangle_{H^1}\,ds + \int_0^t \langle \zeta'(s), u_\infty(s) - \int_0^s g(u_\infty)\,dW_\infty - v_0\rangle_{H^1}\,ds,
\end{aligned} \tag{4.51}$$

$\mathbb{P}'$-a.s. in Ω', for all $0 \le t \le T$, and $\zeta \in L^2(0,T; H^1(\Lambda))$ with $\zeta' \in L^2(0,T; H^1(\Lambda)^*))$. Choosing $\zeta(t,x) = \xi(t)\varphi(x)$ in (4.51), where $\varphi \in H^1(\Lambda)$ and $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$, we get, $\mathbb{P}'$-a.s. in Ω',

$$\begin{aligned}
&\langle u_\infty(t) - \int_0^t g(u_\infty)\,dW_\infty - v_0, \varphi\rangle_{L^2(\Lambda)}\xi(t) - \langle u_\infty(0) - v_0, \varphi\rangle_{L^2(\Lambda)}\xi(0) \\
&= \int_0^t \xi(s)\langle \Delta u_\infty(s), \varphi\rangle_{H^1}\,ds \qquad (4.52)\\
&\quad + \int_0^t \xi'(s)\langle u_\infty(s) - \int_0^s g(u_\infty)\,dW_\infty - v_0, \varphi\rangle_{L^2(\Lambda)}\,ds.
\end{aligned}$$

The particular choice of $t = T$ and $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and $\xi(0) = 1$ in (4.52) combined with (4.47), (4.48), and (4.49) provides

$$\langle u_\infty(0) - v_0, \varphi\rangle_{L^2(\Lambda)} = 0 \quad \text{for all } \varphi \in H^1(\Lambda),\ \mathbb{P}'\text{-a.s. in } \Omega'$$

and, therefore, $u_\infty(0) = v_0$, $\mathbb{P}'$-a.s. in Ω'.
Now, we fix $t \in [0,T)$ and choose $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and $\xi(s) = 1$ for all $s \in [0,t]$. With this choice, from (4.52) we obtain

$$\langle u_\infty(t) - \int_0^t g(u_\infty)\,dW_\infty - u_\infty(0), \varphi\rangle_{L^2(\Lambda)} = \int_0^t \langle \Delta u_\infty(s), \varphi\rangle_{H^1}\,ds, \tag{4.53}$$

$\mathbb{P}'$-a.s. in Ω', for all $\varphi \in H^1(\Lambda)$. Since, for fixed $\varphi \in H^1(\Lambda)$,

$$\begin{aligned}
&t \mapsto \langle u_\infty(t) - \int_0^t g(u_\infty)\,dW_\infty - u_\infty(0), \varphi\rangle_{L^2(\Lambda)} \\
\text{and}\quad &t \mapsto \int_0^t \langle \Delta u_\infty(s), \varphi\rangle_{H^1}\,ds
\end{aligned}$$

are continuous in $[0,T]$, $\mathbb{P}'$-a.s. in Ω', the exceptional set in Ω' in (4.53) may be chosen independently of $t \in [0,T)$ and (4.53) holds also true for $t = T$. This implies

$$u_\infty(t) - u_\infty(0) - \int_0^t g(u_\infty)\, dW_\infty = \int_0^t \Delta u_\infty(s)\, ds \text{ in } H^1(\Lambda)^*, \ \mathbb{P}'\text{-a.s. in } \Omega'$$

and, since the left-hand side of the above equation is in $L^2(\Lambda)$, the equation holds also true in $L^2(\Lambda)$. □

Remark 4.4.17. *Applying the chain rule in* (4.51) *for* $t = T$ *and* $\zeta = \Psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^2)$ *such that* $\Psi(T, \cdot) = 0$*, we immediately get that* u_∞ *is a weak solution to*

$$\begin{cases} du_\infty - \Delta u_\infty\, dt = g(u_\infty)\, dW_\infty(t) & \text{in } \Omega' \times (0,T) \times \Lambda \\ u_\infty(0,\cdot) = v_0 & \text{in } \Omega' \times \Lambda \\ u_\infty = 0 & \text{on } \Omega' \times (0,T) \times \partial\Lambda, \end{cases}$$

i.e., $u_\infty \in L^2(\Omega'; C([0,T]; L^2(\Lambda))) \cap L^2(\Omega'; L^2(0,T; H^1(\Lambda)))$ *and*

$$\begin{aligned} &\int_0^T \int_\Lambda u_\infty(t,x) \partial_t \Psi(t,x)\, dx\, dt - \int_0^T \int_\Lambda \nabla u_\infty(t,x) \cdot \nabla \Psi(t,x)\, dx\, dt \\ &+ \int_\Lambda v_0(x) \Psi(0,x)\, dx = \int_0^T \int_\Lambda \int_0^t g(u_\infty(s,x))\, dW_\infty(s) \partial_t \Psi(t,x)\, dx\, dt, \end{aligned}$$

$\mathbb{P}'$*-a.s. in* Ω'*, for all* $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^2)$ *with* $\psi(T,\cdot) = 0$*. In particular, convergence in distribution has been achieved.*

4.4.5 Strong Convergence of Finite-Volume Approximations

In the previous subsections, we have shown that our finite-volume approximations converge towards a *martingale solution* of (4.1), i.e., the stochastic basis

$$(\Omega', \mathcal{A}', \mathbb{P}', (\mathfrak{F}_t^\infty)_{t\in[0,T]}, (W_\infty(t))_{t\in[0,T]})$$

is not a priori given but part of the solution. In this subsection, we want to show convergence of our finite-volume approximations with respect to the initially given stochastic basis

$$(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W(t))_{t\in[0,T]}).$$

To do so, we will proceed in several steps. First, pathwise uniqueness of the heat equation with multiplicative Lipschitz noise is a consequence of

Proposition 4.4.18: Roughly speaking, martingale solutions of (4.1) on a joint stochastic basis and with respect to the same initial datum coincide. In the proof of Proposition 4.4.20, we construct two convergent finite-volume approximations with respect to a joint stochastic basis, namely $(v^l_{\nu_k})$ and $(v^l_{\rho_k})$, from the function $(u^l_{h,N})$ of our original finite-volume scheme using the theorems of Prokhorov and Skorokhod. Then, as a consequence of pathwise uniqueness, the limits coincide and we may apply [66, Lemma 1.1] in order to obtain convergence in probability of $(u^l_{h,N})$. Thanks to our previous result, we can improve the convergence and pass to the limit in the originally given finite-volume scheme (see Lemma 4.4.21).

Proposition 4.4.18. *Let $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W(t))_{t\in[0,T]})$ be a stochastic basis and u_1, u_2 be solutions to* (4.1) *with respect to the $\mathcal{F}_0$-measurable initial values u_0^1 and u_0^2 in $L^2(\Omega; L^2(\Lambda))$, respectively, on $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W(t))_{t\in[0,T]})$. Then, there exists a constant $C \geq 0$, such that*

$$\mathbb{E}\left[\|u_1(t) - u_2(t)\|^2_{L^2(\Lambda)}\right] \leq C\mathbb{E}\left[\|u_0^1 - u_0^2\|^2_{L^2(\Lambda)}\right] \quad \forall t \in [0,T].$$

Proof. We apply the Itô formula (see [84, Theorem 4.2.5]) to the process $u_1 - u_2$, discard the non-negative term on the left-hand side of the resulting equation, and take expectation. Then, the assertion is a straightforward consequence of Gronwall's inequality, see [84, Proposition 2.4.10]. □

Remark 4.4.19. *If u_1, u_2 are both solutions to* (4.1) *on $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}, (W(t))_{t\in[0,T]})$ with respect to the same initial value u_0, Proposition 4.4.18 provides $u_1(t) = u_2(t)$ in $L^2(\Lambda)$, for all $t \in [0,T]$, $\mathbb{P}$-a.s. in Ω. Since u_1 and u_2 have continuous paths in $L^2(\Lambda)$, the exceptional set in Ω may be chosen independently of $t \in [0,T]$ and it follows that $u_1 = u_2$ in $L^2(0,T; L^2(\Lambda))$, $\mathbb{P}$-a.s. in Ω.*

Proposition 4.4.20. *Let $(u^l_{h,N})_m$ be given by Proposition 4.2.10. Then, there exists a subsequence of $(u^l_{h,N})_m$, still denoted by $(u^l_{h,N})_m$, such that*

$$u^l_{h,N} \to u \text{ in } L^p(\Omega; L^2(0,T; L^2(\Lambda))) \text{ for } m \to \infty$$

for any $p \in [1,2)$, where u is the stochastic process with values in $L^2(\Lambda)$ introduced in Lemma 4.4.1. Moreover, u has $\mathbb{P}$-a.s. continuous paths and belongs to $L^2(\Omega; L^2(0,T; H^1(\Lambda)))$.

Proof. For the sake of simplicity, we will write $u^l_m := u^l_{h_m,N_m}$ and $u^r_m := u^r_{h_m,N_m}$ in the following. We consider an arbitrary pair of subsequences $(u^l_\nu)_\nu$, $(u^l_\rho)_\rho$ of $(u^l_m)_m$. Our aim is to apply [66, Lemma 1.1]. Therefore, we show that there exists a joint subsequence $(u^l_{\nu_k}, u^l_{\rho_k})_k$ converging in law to a probability measure η on $L^2(0,T;L^2(\Lambda))^2$, such that

$$\eta(\{(x,y) \in L^2(0,T;L^2(\Lambda))^2 \mid x = y\}) = 1.$$

We define the random vector-valued sequence $(Y_{\nu,\rho})_{\nu,\rho}$ by

$$Y_{\nu,\rho} := \left(u^l_\nu, u^l_\rho, (u^r_\nu - u^l_\nu), (u^r_\rho - u^l_\rho), W, u^0_\nu, u^0_\rho, (u^0_\nu - u^0_\rho)\right),$$

for any $\nu, \rho \in \mathbb{N}$ and extract a joint subsequence

$$Y_k := \left(u^l_{\nu_k}, u^l_{\rho_k}, (u^r_{\nu_k} - u^l_{\nu_k}), (u^r_{\rho_k} - u^l_{\rho_k}), W, u^0_{\nu_k}, u^0_{\rho_k}, (u^0_{\nu_k} - u^0_{\rho_k})\right)$$

for any $k \in \mathbb{N}$ that converges in law towards a probability measure η_∞ with marginals η^1_∞,η^2_∞, δ_0, δ_0, $\mathbb{P} \circ W^{-1}$, $\mathbb{P} \circ (u_0)^{-1}$, $\mathbb{P} \circ (u_0)^{-1}$, δ_0. Note that we include the difference of the random initial data v^0_ν and u^0_ρ into the vector $Y_{\nu,\rho}$ to ensure that $u^0_{\nu^k}$ and $u^0_{\rho^k}$ converge for $k \to \infty$ to the same limit. With straightforward modifications of the arguments of Subsections 4.4.2-4.4.4, we find

- a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$
- a random vector

  $$Y'_k = (v^l_{\nu_k}, v^l_{\rho_k}, z_{\nu_k}, z_{\rho_k}, W_k, v^0_{\nu_k}, v^0_{\rho_k}, (v^0_{\nu_k} - v^0_{\rho_k}))$$

  having the same law as Y_k for all $k \in \mathbb{N}$
- random elements u^1_∞, u^2_∞, and v_0 with $\mathbb{P}' \circ (u^1_\infty)^{-1} = \eta^1_\infty$, $\mathbb{P}' \circ (u^2_\infty)^{-1} = \eta^2_\infty$, and $\mathbb{P} \circ (v_0)^{-1} = \mathbb{P} \circ (u_0)^{-1}$
- a filtration $(\mathfrak{F}^\infty_t)_{t\in[0,T]}$ and a Brownian motion $(W_\infty(t))_{t\in[0,T]})$, such that u^1_∞ and u^2_∞ are both solutions to (4.1) with initial value v_0 on the joint stochastic basis $(\Omega', \mathcal{A}', \mathbb{P}', (\mathfrak{F}^\infty_t)_{t\in[0,T]}, (W_\infty(t))_{t\in[0,T]})$.

Thus, by Proposition 4.4.18 and Remark 4.4.19, we obtain for $\eta = (\eta^1_\infty, \eta^2_\infty)$

$$\begin{aligned} 1 &= \mathbb{P}'(\{u^1_\infty = u^2_\infty\}) \\ &= \mathbb{P}' \circ (u^1_\infty, u^2_\infty)^{-1}(\{(x,y) \in L^2(0,T;L^2(\Lambda))^2 \mid x = y\}) \\ &= \eta(\{(x,y) \in L^2(0,T;L^2(\Lambda))^2 \mid x = y\}). \end{aligned}$$

Then, by [66, Lemma 1.1] we get convergence of $(u^l_m)_m$ in probability to a random element $\tilde{u}$ in $L^2(0,T,L^2(\Lambda))$. Obviously, by Lemma 4.1, we have

$u = \tilde{u}$. The convergence in probability of $(u_m^l)_m$ allows us to extract a not relabeled subsequence of $(u_m^l)_m$, such that

$$u_m^l \to u \quad \text{in } L^2(0,T;L^2(\Lambda)),\ \mathbb{P}\text{-a.s. in } \Omega,\ \text{for } m \to \infty.$$

Because $(u_m^l)_m$ is bounded in $L^2(\Omega; L^2(0,T;L^2(\Lambda)))$ by Lemma 4.3.3, Vitali's theorem implies the strong convergence of $(u_m^l)_m$ in $L^p(\Omega; L^2(0,T;L^2(\Lambda)))$ for any $1 \le p < 2$. □

Finally, to conclude the proof of Theorem 4.1.3, it remains to show, that the obtained limit u is a solution of the Problem (4.1) in the sense of Definition 4.1.2. This is the aim of the following last lemma:

Lemma 4.4.21. *The stochastic process u introduced in Lemma 4.4.1 is the unique solution of Problem* (4.1) *in the sense of Definition 4.1.2.*

Proof. Let $p \in [1,2)$. With similar arguments as in the proof of Proposition 4.4.20, one can show that $(u_{h,N}^r)_m$ and $(\widehat{u}_{h,N})_m$ converge for $m \to \infty$ to u in $L^p(\Omega; L^2(0,T;L^2(\Lambda)))$. Moreover, there holds $g(u_{h,N}^l) \to g(u)$ in $L^p(\Omega; L^2(0,T;L^2(\Lambda)))$ for $m \to \infty$. Therefore, we know

$$M_{h,N} = \int_0^{\cdot} g(u_{h,N}^l)\, dW \overset{m\to\infty}{\longrightarrow} \int_0^{\cdot} g(u)\, dW \quad \text{in } L^p(\Omega; C([0,T];L^2(\Lambda))).$$

As shown in Lemma 4.4.15, we get

$$\widehat{M}_{h,N} \to \int_0^{\cdot} g(u)\, dW \quad \text{in } L^p(\Omega; L^2(0,T;L^2(\Lambda))) \text{ for } m \to \infty.$$

Now, we consider the semi-implicit finite-volume scheme (4.8). Let $A \in \mathcal{A}$, $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$, and $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\nabla\varphi \cdot \mathbf{n} = 0$ be arbitrary. Multiplying (4.8) with $\mathbb{1}_A \xi\varphi$, summing over $K \in \mathcal{T}$, integrating over $[t_n, t_{n+1})$, and summing over $n = 0, \ldots, N-1$, we get

$$\begin{aligned}
0 =& \mathbb{E}\left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K\in\mathcal{T}} m_K \mathbb{1}_A \xi(t) \frac{1}{\Delta t}[u_K^{n+1} - u_K^n - g(u_K^n)\Delta_{n+1}W]\varphi(x_K)\, dt\right] \\
&+ \mathbb{E}\left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K\in\mathcal{T}} \mathbb{1}_A \xi(t) \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1})\varphi(x_K)\, dt\right] \\
=:& T_{1,m} + T_{2,m}.
\end{aligned}$$

If we define $\varphi_h(x) := \varphi(x_K)$ for $x \in K$, $K \in \mathcal{T}$, there holds

$$\begin{aligned} T_{1,m} &= \mathbb{E}\left[\mathbb{1}_A \int_0^T \int_\Lambda \partial_t[\widehat{u}_{h,N} - \widehat{M}_{h,N}](t,x)\xi(t)\varphi_h(x)\,dx\,dt\right] \\ &= -\mathbb{E}\left[\mathbb{1}_A \int_0^T \int_\Lambda (\widehat{u}_{h,N} - \widehat{M}_{h,N})(t,x)\xi'(t)\varphi_h(x)\,dx\,dt\right] \\ &\quad - \mathbb{E}\left[\mathbb{1}_A \int_\Lambda u_h^0(x)\xi(0)\varphi_h(x)\,dx\right]. \end{aligned}$$

From [5, Proposition 3.5] we know that $u_h^0 \to u_0$ in $L^2(\Lambda)$, $\mathbb{P}$-a.s. in Ω, and, thanks to Lemma 4.3.1, we can apply Lebesgue's dominated convergence theorem. The passage to the limit is analogous to that on Ω'. $\square$

Chapter 5

Work in Progress and Outlooks

In this chapter, we present three research ideas for the future in the field of nonlinear (stochastic) diffusion equations.

5.1 Existence of Entropy Solutions for Time-Fractional Obstacle Problems

Let Ω be an open and bounded set in $\mathbb{R}^d$ with $d \in \mathbb{N}$, $T > 0$, and for a.e. $x \in \Omega$ let $\beta(x, \cdot) := \partial j(x, \cdot)$ be the subdifferential of a function $j : \Omega \times \mathbb{R} \to [0, \infty]$ which is measurable for a.e. $x \in \Omega$, convex, and lower semi-continuous in $r \in \mathbb{R}$ with $j(\cdot, 0) = 0$.
As in Chapter 2, we want to study a time-fractional nonlinear diffusion problem. Here, we propose to consider the p-Laplace operator $\Delta_p(u) := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$ and add an obstacle term $\beta(x, u)$, to be precise, we want to consider the time-fractional obstacle problem

$$\begin{cases} \partial_t^\alpha (u - u_0) + \Delta_p(u) + \beta(x, u) \ni f & \text{in } Q_T = (0, T) \times \Omega \\ u = 0 & \text{on } \Sigma_T = (0, T) \times \partial\Omega, \end{cases} \tag{5.1}$$

where ∂_t^α denotes the time-fractional derivative of order $\alpha \in (0, 1)$ in the sense of Riemann-Liouville. We know, that for the elliptic diffusion-absorption problem

$$\begin{cases} u + \Delta_p(u) + \beta(x, u) \ni f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$, there exists a unique generalized solution, see [125]. Moreover, we know that the parabolic diffusion-absorption problem

$$\begin{cases} \partial_t (u - u_0) + \Delta_p(u) + \beta(x, u) \ni f & \text{in } Q_T \\ u = 0 & \text{in } \Sigma_T, \end{cases}$$

for $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$, admits a unique entropy solution, see [6].
To study the existence of solutions to (5.1), we may start by assuming $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q_T)$. Adapting the ideas in [6], we can approximate the obstacle β by its Yosida approximation β_λ. The approximated equation admits a unique strong solution u_λ by [65, Theorem 4].
To show boundedness of $(\beta_\lambda(\cdot, u_\lambda))_{\lambda>0}$, we may add a bi-monotone perturbation $\psi_{m,n}(u)$ on the left-hand side of (5.1) and adopt the arguments in [6] combined with the use of a Kato inequality, see [69]. Then, we may pass to the limit firstly for $\lambda \downarrow 0$ and, secondly, for $m, n \to \infty$.
The main challenge will be to identify the weak limit of the nonlinear diffusion term arising from the p-Laplace operator in the passage to the limit for $\lambda \downarrow 0$. To apply a pseudo-monotonicity argument, we have to combine arguments used for obstacle problems like in [6] and arguments used for time-fractional problems like in [69, 114].

5.2 Well-Posedness of a Stochastic Allen-Cahn Equation with Constraint

Let D be a smooth and bounded domain in $\mathbb{R}^d$ with $d \in \mathbb{N}$, $T > 0$, $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(W(t))_{t\in[0,T]}$ an one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$. We define $I_{[0,1]} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$I_{[0,1]}(x) := \begin{cases} 0 & \text{if } x \in [0,1], \\ +\infty & \text{else} \end{cases}$$

and denote by $\partial I_{[0,1]}$ its subdifferential. In [12] the authors proved existence and uniqueness of solutions to the stochastic problem

$$w_s(u) + f - \partial_t\Big(u - \int_0^t h(u)dW\Big) + \Delta u \in \partial I_{[0,1]}(u) \text{ in } \Omega \times (0,T) \times D \quad (5.2)$$

with homogeneous Neumann boundary conditions, initial condition $u_0 \in H^1(D)$ with $0 \le u_0 \le 1$ a.e. in D, random data $f \in L^2(\Omega \times (0,T) \times D)$, and $w_s : \mathbb{R} \to [0,\infty)$ a Lipschitz continuous function with $w_s(0) = 0$. The stochastic integral $\int_0^t h(u)dW$ is understood in the sense of Itô, where $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function satisfying $h(0) = h(1) = 0$.
From the physical point of view, the equation is motivated by describing the evolution of damage in a continuum medium. More precisely, our solution u represents the local proportion of intact bonds within the considered material. Thanks to the subdifferential $\partial I_{[0,1]}$, the solution u will be a quantity between 0 and 1, where the case $u = 0$ corresponds to a totally damaged

material and the case $u = 1$ to a material without damage. The function w_s is related to the internal cohesion of the material while the function f represents an external source of damage (mechanical or chemical). The stochastic perturbation is motivated by the consideration of changes at the microscopic scale of the material structure, such as the formation of cavities during damage.
An idea for a future work is to replace the Laplace operator in the stochastic Allen-Cahn equation (5.2) by a p-Laplace operator for $p \geq 2$ and show similar existence and uniqueness results for solutions.
To show the existence of solutions, we may combine the techniques used in [12] with the arguments that we have already used in [112] to show the well-posedness of a stochastic p-Laplace equation on $\mathbb{R}^d$. More precisely, we may approximate the maximal monotone operator $\partial I_{[0,1]}$ by its Yosida approximation and do a time discretization which is implicit in the deterministic part and explicit in the stochastic part.
The techniques used in [12] for the passage to the limit are based on the monotonicity of the Laplace operator and should be therefore adaptable in case of the p-Laplace operator.
Since the proof of the uniqueness of solutions to (5.2) is also based on the monotonicity of the Laplace operator, we may also obtain uniqueness in the case of the p-Laplace operator.

5.3 Convergence of a Finite-Volume Scheme for a p-Laplace Equation with Multiplicative Noise

In Chapter 4, we considered a finite-volume scheme for the following heat equation with a nonlinear multiplicative noise:

$$du - \Delta u\, dt = g(u)\, dW(t) \quad \text{in } \Omega \times (0,T) \times \Lambda \tag{5.3}$$

with homogeneous Neumann boundary conditions and an initial value $u_0 \in L^2(\Omega; H^1(\Lambda))$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, $T > 0$, and Λ a bounded, connected, and polygonal domain in $\mathbb{R}^2$. We assumed $g : \mathbb{R} \to \mathbb{R}$ to be a Lipschitz continuous function, and $(W(t))_{t\in[0,T]}$ a standard one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$.
In the study of the finite-volume scheme for (5.3) in Chapter 4, we did not use the approach of semigroup theory to make it possible to consider more general operators such as a porous medium operator, a p-Laplace operator for $1 < p < \infty$, or even a general Leray-Lions operator. To do so, we have to

approximate the gradient ∇u itself instead of "just" $\nabla u \cdot \mathbf{n}$. This has been done in [3] by using a discrete duality finite-volume method. We also refer to [4] for a finite-volume scheme for a deterministic nonlinear degenerate diffusion equation, and furthermore, we refer to [55] for gradient discretization methods for nonlinear stochastic evolution equations.
We mention that we already proposed a finite-volume scheme for a diffusion-convection equation with a nonlinear multiplicative noise in [19], precisely, we considered

$$du - \Delta u \, dt + \operatorname{div}(\mathbf{v} f(u)) = g(u) \, dW(t) + \beta(u) \, dt \quad \text{in } \Omega \times (0,T) \times \Lambda \quad (5.4)$$

with homogeneous Neumann boundary conditions and an initial value $u_0 \in L^2(\Omega; L^2(\Lambda))$, where $\mathbf{v} \in C^1([0,T] \times \overline{\Lambda}; \mathbb{R}^d)$ is divergence free and satisfies $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary, $f, \beta : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions, and f is non-decreasing.
For $f = id$, we proved in [20] strong convergence of the scheme to the unique variational solution of (5.4) by using well-known methods for the time discretization of stochastic PDEs instead of using the stochastic compactness method. Note that the approach in [20] is not less complicated or technical.

We remark that, recently, convergence rates for the finite-volume scheme for the stochastic heat equation with multiplicative Lipschitz noise, that we presented in Chapter 4, have been studied in [110].

Appendix A

An Estimate on Space Translations of Finite-Volume Approximations

We want to give a detailed proof of Lemma 4.3.6 and orientate ourselves on the proof given for Theorem 10.3 in [58].

Let $\eta \in \mathbb{R}^2 \setminus \{0\}$ with $|\eta| \leq R$ be arbitrary. In the following, for $x_1, x_2 \in \mathbb{R}^2$, we denote by $[x_1, x_2]$ the completed line segment between x_1 and x_2, i.e., $[x_1, x_2] := \{\lambda x_1 + (1-\lambda)x_2 : \lambda \in [0,1]\}$ and define, for any $\sigma \in \mathcal{E}$, the function $\chi_\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \{0,1\}$ by

$$\chi_\sigma(x_1, x_2) := \begin{cases} 1, & \text{if } [x_1, x_2] \cap \sigma \neq \emptyset \\ 0, & \text{if } [x_1, x_2] \cap \sigma = \emptyset. \end{cases}$$

For a.e. $t \in (0,T)$ and a.e. $x \in \mathbb{R}^2$, we obtain, by applying the triangle inequality,

$$\begin{aligned} |\bar{u}^l_{h,N}(t, x+\eta) - \bar{u}^l_{h,N}(t,x)| \leq & \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \chi_\sigma(x, x+\eta) |u^l_{h,N}(t, x_K) - u^l_{h,N}(t, x_L)| \\ & + \sum_{\sigma \in \mathcal{E}_{\mathrm{ext}}} \chi_\sigma(x, x+\eta) |u^l_{h,N}(t, x_K)|, \end{aligned}$$

where we recall that, if not marked otherwise, we denote for $\sigma \in \mathcal{E}_{\mathrm{int}}$ the neighbouring control volumes by K and L, i.e., $\sigma = K|L$, and for $\sigma \in \mathcal{E}_{\mathrm{ext}}$ we assume by default $\sigma \in \mathcal{E}_K$. Because Λ is a bounded, connected, and polygonal set, Λ has a finite number of sides and there exists a finite number

$M \in \mathbb{N}$ of exterior edges $\sigma \in \mathcal{E}_{\text{ext}}$. Therefore, we obtain

$$\begin{aligned}
&|\bar{u}^l_{h,N}(t,x) - \bar{u}^l_{h,N}(t,x+\eta)|^2 \\
&\leq (M+1)^2 \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) |u^l_{h,N}(t,x_K) - u^l_{h,N}(t,x_L)| \right)^2 \\
&\quad + (M+1)^2 \sum_{\sigma \in \mathcal{E}_{\text{ext}}} \chi_\sigma(x,x+\eta) |u^l_{h,N}(t,x_K)|^2.
\end{aligned} \tag{A.1}$$

Using the Cauchy-Schwarz inequality, we know

$$\begin{aligned}
&\left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) |u^l_{h,N}(t,x_K) - u^l_{h,N}(t,x_L)| \right)^2 \\
&\leq \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) \frac{|u^l_{h,N}(t,x_K) - u^l_{h,N}(t,x_L)|^2}{d_{K|L} c_{K|L}} \right) \\
&\quad \cdot \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) d_{K|L} c_{K|L} \right),
\end{aligned}$$

where $c_{K|L} := \left| \mathbf{n}_{K|L} \frac{\eta}{|\eta|} \right|$ for $\sigma = K|L \in \mathcal{E}_{\text{int}}$. Let us assume for the moment, that there exists a constant $C_1 \geq 0$ only depending on Λ, such that

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) d_{K|L} c_{K|L} \leq |\eta| + C_1 h. \tag{A.2}$$

Then, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) |u^l_{h,N}(t,x_K) - u^l_{h,N}(t,x_L)| \right)^2 dx \\
&\leq \int_{\mathbb{R}^2} \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta) \frac{|u^l_{h,N}(t,x_K) - u^l_{h,N}(t,x_L)|^2}{d_{K|L}} \right) (|\eta| + C_1 h)\, dx \\
&= (|\eta| + C_1 h) \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left(\frac{|u^l_{h,N}(t,x_K) - u^l_{h,N}(t,x_L)|^2}{d_{K|L}} \int_{\mathbb{R}^2} \chi_\sigma(x,x+\eta)\, dx \right).
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^2} \chi_\sigma(x,x+\eta)\, dx \leq m_\sigma |\eta|, \tag{A.3}$$

because $\chi_\sigma(x, x+\eta) \neq 0$ iff $x \in \{y - \lambda\eta : y \in \sigma, \lambda \in [0,1]\}$ and $|\{y - \lambda\eta : y \in \sigma, \lambda \in [0,1]\}| = m_\sigma|\eta|$. Using this estimate, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} \chi_\sigma(x, x+\eta) |u^l_{h,N}(t, x_K) - u^l_{h,N}(t, x_L)| \right)^2 dx \\
&\leq (|\eta| + C_1 h)|\eta| \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |u^l_{h,N}(t, x_K) - u^l_{h,N}(t, x_L)|^2 \\
&= (|\eta| + C_1 h)|\eta| |u^l_{h,N}(t)|^2_{1,h}.
\end{aligned} \tag{A.4}$$

Considering the sum of exterior edges in (A.1), we obtain by using (A.3) and [58, Lemma 10.5]

$$\begin{aligned}
\int_{\mathbb{R}^2} \sum_{\sigma \in \mathcal{E}_{\text{ext}}} \chi_\sigma(x, x+\eta) |u^l_{h,N}(t, x_K)|^2 \, dx &\leq |\eta| \sum_{\sigma \in \mathcal{E}_{\text{ext}}} m_\sigma |u^l_{h,N}(t, x_K)|^2 \\
&\leq |\eta| \|\overline{\gamma}(u^l_{h,N}(t))\|^2_{L^2(\partial\Lambda)} \\
&\leq |\eta| C_2 (|u^l_{h,N}(t)|^2_{1,h} + \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)}),
\end{aligned} \tag{A.5}$$

for a constant $C_2 \geq 0$ only depending on Λ, where $\overline{\gamma}(u^l_{h,N}(t)) := u^l_{h,N}(t, x_K)$ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$. From (A.1), (A.4), and (A.5), we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\bar{u}^l_{h,N}(t, x_K) - \bar{u}^l_{h,N}(t, x_L)|^2 \, dx \\
&\leq \int_{\mathbb{R}^2} (M+1)^2 |\eta| \left((R + C_1 h + C_2) |u^l_{h,N}(t)|^2_{1,h} + C_2 \|u^l_{h,N}(t)\|^2_{L^2(\Lambda)} \right).
\end{aligned}$$

We recall that $h = \sup\{\text{diam}(K) : K \in \mathcal{T}\}$ is bounded.
It remains to show (A.2). Let $x \in \mathbb{R}^2$ be chosen such that $[x, x+\eta] \cap \sigma$ contains at most one point for all edges $\sigma \in \mathcal{E}$. Note that the two-dimensional Lebesgue measure of

$$\{x \in \mathbb{R}^2 : \exists \sigma \in \mathcal{E} \text{ s.t. } [x, x+\eta] \cap \sigma \text{ contains more than one point}\}$$

is zero, because the number of edges is finite. Moreover, we assume that the line segment $[x, x+\eta]$ does not contain any vertex of $\mathcal{T}$. This assumption is also satisfied for a.e. $x \in \mathbb{R}^2$, because $\mathcal{T}$ has a finite number of vertexes.
If $[x, x+\eta] \in \mathbb{R}^2 \setminus \Lambda$, then, obviously, $\chi_\sigma(x, x+\eta) = 0$ for all $\sigma \in \mathcal{E}_{\text{int}}$. Thus, let $[x, x+\eta] \cap \Lambda \neq \emptyset$. Because Λ is not assumed to be convex, it may happen, that the line segment $[x, x+\eta]$ is not completely included in Λ. Therefore, let $y, z \in [x, x+\eta]$, $y \neq z$, be chosen such that $[y, z] \subseteq \Lambda$. Since $y, z \in \Lambda$,

there exist control volumes $L_y, L_z \in \mathcal{T}$, such that $y \in \overline{L_y}$ and $z \in \overline{L_z}$. Then, there holds

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(y,z) d_{K|L} c_{K|L} &= \sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(y,z) |\eta|^{-1} \left| |x_K - x_L| \mathbf{n}_{KL} \cdot \eta \right| \\ &= |\eta|^{-1} \sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(y,z) |(x_L - x_K)\cdot\eta|\end{aligned} \tag{A.6}$$

Now, we rearrange the sum: We start by choosing $L_1 = L_y$ and $\sigma_1 = L_1|L_2 \in \mathcal{E}_{\text{int}}$, such that $\chi_{\sigma_1}(y,z) = 1$. For $i \in \{2,\dots,j\},\ j \in \mathbb{N}$, we define by iteration $\sigma_i = L_i|L_{i+1} \in \mathcal{E}_{\text{int}}$, such that $\chi_{\sigma_i}(y,z) = 1$ and $L_{i+1} \neq L_{i-1}$. This choice is unique, because $[y,z]$ does not intersect with any vertexes. We choose $j \in \mathbb{N}$, such that $L_{j+1} = L_z$ and

$$\{\sigma_i : i \in \{1,\dots,j\}\} = \{\sigma \in \mathcal{E}_{\text{int}} : \chi_\sigma(y,z) = 1\}.$$

Then, either

$$\operatorname{sign}_0(\cos(\mathbf{n}_{L_iL_{i+1}} \sphericalangle \eta) = 1 \quad \forall i \in \{1,\dots,j\}$$

or

$$\operatorname{sign}_0(\cos(\mathbf{n}_{L_iL_{i+1}} \sphericalangle \eta) \in \{-1,0\} \quad \forall i \in \{1,\dots,j\},$$

since either $\mathbf{n}_{L_iL_{i+1}} \sphericalangle \eta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for all $i \in \{1,\dots,j\}$ or $\mathbf{n}_{L_iL_{i+1}} \sphericalangle \eta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for all $i \in \{1,\dots,j\}$ by the choice of $(L_i)_{i\in\{1,\dots,j+1\}}$. Since

$$\begin{aligned}(x_L - x_K)\cdot\eta &= |x_K - x_L||\eta| \cos((x_L - x_K)\sphericalangle\eta) \\ &= |x_K - x_L||\eta| \cos(\mathbf{n}_{KL}\sphericalangle\eta)\end{aligned}$$

for $\sigma = K|L \in \mathcal{E}_{\text{int}}$, we obtain

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(y,z)|(x_L - x_K)\cdot\eta| &= \sum_{i=1}^{j} |(x_{L_{i+1}} - x_{L_i})\cdot\eta| \\ &= \left| \sum_{i=1}^{j} (x_{L_{i+1}} - x_{L_i})\cdot\eta \right| \\ &= |\eta||x_{L_y} - x_{L_z}|.\end{aligned}$$

Plugging this into (A.6), we obtain

$$\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(y,z) d_{K|L} c_{K|L} \le |x_{L_y} - x_{L_z}|. \tag{A.7}$$

Since $y \in \overline{L_y}$ and $z \in \overline{L_z}$, we know

$$|x_{L_y} - x_{L_z}| \le |x_{L_y} - y| + |y - z| + |z - x_{L_z}| \le |y - z| + 2h$$

and obtain, therefore, from (A.7)

$$\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(y,z) d_{K|L} c_{K|L} \leq |y-z| + 2h. \tag{A.8}$$

Since Λ is a polygon with a finite number $\widetilde{M}$ of sides, the line segment $[x, x+\eta]$ intersects the boundary $\partial\Lambda$ of Λ at most $\widetilde{M}$ times. Hence, there exist $x_1, \dots, x_m \in (\Lambda \cap [x, x+\eta]), m \in \mathbb{N}, m \leq \widetilde{M}$, such that $[x_i, x_{i+1}] \subseteq \Lambda$ for all $i \in \{1, \dots, m-1\}$ and

$$\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x, x+\eta) d_{K|L} c_{K|L} = \sum_{i=1}^{m-1} \sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x_i, x_{i+1}) d_{K|L} c_{K|L}.$$

We obtain by (A.8)

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x, x+\eta) d_{K|L} c_{K|L} &= \sum_{i=1}^{m-1} \sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x_i, x_{i+1}) d_{K|L} c_{K|L} \\
&= \sum_{i=1}^{m-1} (|x_i - x_{i+1}| + 2h) \\
&= 2(m-1)h + |\eta|^{-1} \sum_{i=1}^{m-1} |(x_i - x_{i+1}) \cdot \eta| \\
&\leq 2(\widetilde{M}-1)h + |\eta|.
\end{aligned}$$

This concludes the proof of (A.2), which was still to be shown.

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