Modular representation theory and sheaves on the Bruhat–Tits building

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Abstract

Let G denote the group of rational points of a split connected reductive group over a nonarchimedean local field. Furthermore, let R denote a quasi-Frobenius ring and let H denote the pro-p Iwahori-Hecke algebra of G over R. Inspired by the work of Schneider and Stuhler [SS97], Schneider [Sch98b], and Kohlhaase [Koh22] we construct a fully faithful functor from the category of H-modules into that of G-equivariant sheaves of R-modules on the Bruhat-Tits building $\mathscr{X}$ of G. We study the cohomology (with compact support) of our sheaves in terms of the homology of G-equivariant coefficient systems, using Verdier duality on the building.

Zusammenfassung

Sei G die Gruppe der rationalen Punkte einer zerfallenden, zusammenhängenden, reduktiven Gruppe über einem nicharchimedischen lokalen Körper. Sei außerdem R ein Quasi-Frobenius-Ring und H die pro-p Iwahori-Hecke-Algebra von G über R. Aufbauend auf den Arbeiten von Schneider und Stuhler [SS97], Schneider [Sch98b] und Kohlhaase [Koh22] konstruieren wir einen volltreuen Funktor von der Kategorie der H-Moduln in die Kategorie der G-äquivarianten Garben von R-Moduln auf dem Bruhat-Tits-Gebäude $\mathscr{X}$ von G. Wir studieren die Kohomologie (mit kompaktem Träger) unserer Garben mit Hilfe der Homologie G-äquivarianter Koeffizientensysteme und der Verdier-Dualität des Gebäudes.

Contents

Introduction

Combining representation theory and algebraic number theory, the Langlands program is a mathematical pursuit that has significantly shaped the research landscape for nearly seven decades. Its profound impact on mathematics is exemplified by the historic proof of Fermat's Last Theorem by Richard Taylor and Andrew Wiles in [Wil95; TW95].

Originally rooted in the study of representations defined over the field of complex numbers, the Langlands program was catalyzed by Robert Langlands's Local Langlands Conjectures in the late 1960s. Generalizing local class field theory, it predicted a correspondence between complex representations of the Weil group of a local field K and the irreducible smooth representations of $\mathbb{G}(K)$ where $\mathbb{G}$ is a reductive group defined over K.

We will use the same notations for the rest of this introduction, namely K will denote a non-archimedean local field of residue characteristic p, for a prime p, $\mathbb{G}$ will denote a reductive group (for example, GL_n) over K and $G := \mathbb{G}(K)$ will denote its group of K-points.

Over time, the Langlands program has evolved to encompass various settings beyond complex coefficients. The study of complex smooth representations of p-adic groups has been ongoing since the mid-1960s. In the 2000s, works by Breuil [Bre03a; Bre03b] and by Khare and Wintenberger [KW09], among others, spurred investigations into the cases where $\mathbb{C}$ is replaced by fields of positive characteristic ℓ. This is the so-called mod-ℓ setting.

More complete results have been made in this direction in the easier case $\ell \neq p$. Vignéras extensively studied the split case in [Vig96], while Minguez and Sécherre handled the non-split case in [MS13; MS14a; MS14b], paving the way for deeper insights into the intricate interplay between representations of algebraic groups and Galois representations.

This thesis is essentially concerned with the much more difficult and largely mysterious case where $\ell = p$, thus focusing on the study of smooth representations of G on vector spaces over fields of characteristic p.

The realm of smooth mod-p representation theory has been a dynamic and fertile area of research for nearly three decades. Originating from the seminal work of Barthel and Livné in the mid-1990s [BL94; BL95], this field shows striking differences from complex representation theory. Even for seemingly simple cases like $GL_2(K)$ over finite extensions of the p-adic numbers, the differences in behavior were profound, as highlighted in the aforementioned work.

Let us discuss one example of the problems that arise when we work in the mod-p setting. Given a commutative unital ring R and a compact open subgroup I of G, one can define the Hecke R-algebra of G by $H := R[I\backslash G/I]$. Let Mod_H denote the category of left H-modules and $\mathrm{Rep}_R^\infty(G)$ denote the category of smooth R-linear G-representations, then there is an adjunction between these two categories given by the following functors:

$$\mathrm{Rep}_R^\infty(G) \underset{\mathrm{ind}_I^G(R)\otimes_H(.)}{\overset{(.)^I}{\rightleftarrows}} \mathrm{Mod}_H\,.$$

Here $\mathrm{ind}_I^G(R)$ denotes the compact induction of the trivial representation from I to G.

Classically, when R is the field of complex numbers and I is a (pro-p) Iwahori subgroup, then Bernstein proved, in [Ber84], that the above functors give an equivalence of categories between Mod_H and the full subcategory $\mathrm{Rep}_R^I(G)$ of $\mathrm{Rep}_R^\infty(G)$ of smooth G-representations generated by their I-invariants vectors. When R is of characteristic p, I is a pro-p Iwahori subgroup of G, and H is the associated pro-p Iwahori–Hecke algebra one does not know, in general, the precise relation between Mod_H and $\mathrm{Rep}_R^I(G)$. In the case of $G = GL_2(\mathbb{Q}_p)$ or $G = SL_2(\mathbb{Q}_p)$, Ollivier and Koziol [Oll09; Koz15] proved that the adjunction gives again an equivalence of categories. In the same papers, they also proved that for $K \neq \mathbb{Q}_p$ it is, in general, not the case anymore. We will discuss a little more this adjunction in Section 1.3.

Various techniques have been developed over the years, drawing inspiration from the complex setting. One of the commonly used tools is the theory of Bruhat–Tits buildings, as well as G-equivariant objects on it. For instance, the groundbreaking work of Schneider and Stuhler [SS97] gives a detailed study of the relation between complex smooth G-representations, G-equivariant coefficient systems on the Bruhat–Tits building of G and G-equivariant sheaves on it.

In [Pas04], Paskunas uses the notion of coefficients systems in the p-modular setting to construct supersingular irreducible smooth representations of $GL_2(F)$ over arbitrary finite extensions F of p-adic fields. Note that supersingular (or supercuspidal) representations often play a distinguished role in the representation theory of p-adic groups and in the Langlands correspondences.

Later, Kohlhaase, in [Koh22], took up the work of Schneider and Stuhler and clarified the relation between G-equivariant coefficient systems and modules over a certain Hecke algebra H (the pro-p Iwahori-Hecke algebra of G). In the p-modular setting the latter seems more accessible than the category of smooth G-representations. More precisely, building on ideas of Cabanes, Ollivier and Schneider, he constructed an embedding of categories $\mathcal{F}(.)$ from pro-p Iwahori-Hecke modules to G-equivariant coefficient systems over a rather general base ring R. It allowed him to obtain a new proof of results in [SS97], and to clarify some of the specificities of the p-modular setting.

In this thesis, we complement the work [Koh22] of Kohlhaase by extending the cohomological theory developed by Schneider and Stuhler in [SS97] to the p-modular setting. In fact, we work over a much more general class of base rings R. Building on ideas from [SS97] and [Sch98b] we study the relation between pro-p Iwahori-Hecke modules and G-equivariant sheaves on the Bruhat–Tits building $\mathscr{X}$ of G. As an intermediate tool, we pass through G-equivariant coefficient systems.

These notions will be defined respectively in Section 1.2 and in Section 1.4 after a short introduction on the theory of Bruhat–Tits buildings in Section 1.1. The goal of the first chapter is to introduce the necessary concepts for the subsequent discussion. We also take the time to explain the construction of the functor $\mathcal{F}(.)$ from [Koh22].

In Chapter 2, we take up an idea from [Sch98b] and associate to a G-equivariant coefficient system $\mathcal{F}$ a suitable G-equivariant sheaf of R-modules, for R a fixed commutative unital ring. We prove in Lemma 2.1 that the obtained sheaf, that we denote $\mathbb{S}(\mathcal{F})$, is characterized by the fact that its restriction to any face F is the constant sheaf with value $(\mathcal{F}_F)^* = \mathrm{Hom}_R(\mathcal{F}_F, R)$. In [SS97], Schneider and Stuhler already associated a G-equivariant sheaf $\underset{\approx}{V}$ to a complex smooth G-representation V. This construction can be generalized to other settings but, as discussed in Section 1.4, the obtained sheaf in our case will generally be the zero sheaf. However, for any pro-p Iwahori-Hecke module M whose underlying R-module is finitely generated, we prove in Corollary 2.13 that the sheaf $\mathbb{S}(\mathcal{F}(M))$, restricted to the star of any face, is of the form $\underset{\approx}{V}$ for a certain representation V. The category of such H-modules is denoted by $\mathrm{Mod}_H^{\mathrm{fg}}$.

In Section 2.2 and Section 2.3 we establish that the sheaf $\mathbb{S}(\mathcal{F})$ is G-equivariant. Thus, its cohomology with compact support gives smooth R-linear G-representations that are hoped to be of general interest. Moreover we prove that the construction is functorial thereby giving a functor

$$\mathbb{S} : \mathrm{Coeff}_G(\mathscr{X}) \to \mathrm{Sh}_G(\mathscr{X})$$

from the category of G-equivariant coefficient systems on $\mathscr{X}$ to the category of G-equivariant sheaves on $\mathscr{X}$.

As an initial result, we are able to prove Proposition 2.7 which states that the functor $\mathbb{S}$ is fully faithful when restricted to $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$, the full subcategory of coefficient systems whose terms at all faces are finitely generated R-modules. It was already observed by Schneider in [Sch98b] that on $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$ the essential image of $\mathbb{S}$ is the category of constructible G-equivariant sheaves (see Definition 1.17). One of our key arguments to get back $\mathcal{F}$ from $\mathbb{S}(\mathcal{F})$ is that any finitely generated module over a quasi-Frobenius ring is reflexive. Using results of [Koh22] (especially [Koh22, Theorem 3.21]) together with $\mathbb{S}(.)$, we prove Theorem 2.11 which is stated as follows :

Theorem 1. *The functor $\mathbb{S}\circ\mathcal{F}(.) : \mathrm{Mod}_H^{\mathrm{fg}} \to \mathrm{Sh}_G(\mathscr{X})$ is fully faithful. Moreover, the essential image of the functor $\mathbb{S} : \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X}) \to \mathrm{Sh}_G(\mathscr{X})$ is the full subcategory of constructible G-equivariant sheaves on $\mathscr{X}$.*

Since one of the main result of [SS97] is the computation of the cohomology with compact support of the sheaves constructed there (see [SS97, Proposition IV.1.3]), we also studied the cohomology (with compact support) of the sheaf $\mathbb{S}(\mathcal{F})$. In particular, we show how the cohomology of $\mathbb{S}(\mathcal{F})$ depends on the homology of $\mathcal{F}$ defined in Section 1.2. In fact, we prove that there is an isomorphism of R-linear G-representations between the cohomology of $\mathbb{S}(\mathcal{F})$ and the dual of the homology of $\mathcal{F}$ (see Proposition 3 below). More precisely, let $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F}), \partial_\bullet)$ be the oriented chain complex of $\mathcal{F}$. Its homology is what we call the homology of $\mathcal{F}$. In Section 2.5 we explain how to fix orientations and make this complex isomorphic to the complex $\bigoplus_{F\in\mathscr{X}_\bullet} \mathcal{F}_F$ endowed with a certain differential $\delta_\bullet$. In Lemma 2.16, we demonstrate that the complex $(\bigoplus_{F\in\mathscr{X}_\bullet} (\mathcal{F}_F)^*, \delta_\bullet^*)$ is nothing more than a subcomplex of the dual complex $(\mathcal{C}_c^{or}(\mathscr{X}_\bullet, \mathcal{F})^*, \partial_\bullet^*) \cong (\prod_{F\in\mathscr{X}_\bullet}(\mathcal{F}_F)^*, \delta_\bullet^*)$. We then prove Proposition 2.14, which is an analog of [SS97, Proposition VI.1.3] in the mod-p setting :

Proposition 2. *For all $q \geq 0$ and all $\mathcal{F} \in \mathrm{Coeff}_G(\mathscr{X})$, we have the equality*

$$H_c^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) = H^q(\bigoplus_{F\in\mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*).$$

In fact, the proof follows the same strategy as in [SS97], using a spectral sequence coming from the filtration of $\mathscr{X}$ by its q-skeleta.

Regrettably, we possess so far limited information about the cohomology of this subcomplex, although we are able to obtain a first vanishing result in Section 2.6. More precisely we study the case $G = GL_2(\mathbb{Q}_p)$ and $R = \bar{\mathbb{F}}_p$ and make use of the higher smooth duals introduced in [Koh17] by Kohlhaase. Our analysis indicates that there is a functorial relation between the higher smooth duals of V and the cohomology with compact support of $\mathbb{S}(\mathcal{F}(V^I))$, where I is a pro-p Iwahori subgroup of G. If V is an admissible irreducible smooth $GL_2(\mathbb{Q}_p)$-representation of infinite dimension over $\bar{\mathbb{F}}_p$ then the smooth dual and the cohomology with compact support agree in degree 0 (see Section 2.6). However, this is not always true as is shown by the trivial representation (see Example 2.17). Nonetheless, we showed that if we ask V to be an admissible smooth $GL_2(\mathbb{Q}_p)$-representation, then there is an injection of the 0-th group of cohomology of compact support of $\mathbb{S}(\mathcal{F}(V^I))$ into the smooth dual.

In the last chapter of our thesis, we change notation and denote our base ring by A instead of R. We focus on the usual cohomology groups of the sheaf $\mathbb{S}(\mathcal{F})$. Our main tool is the Verdier duality on the building $\mathscr{X}$ as developed in [Sch98b] and relying on foundational results from [KS90]. In Proposition 3.5, we obtain the following isomorphism :

Proposition 3. *For all $q \geq 0$ and all $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$, there is an isomorphism of A-linear G-representations*

$$H^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) \cong H_q(\mathscr{X}, \mathcal{F})^*.$$

We note that the G-representations in Proposition 3 are generally not smooth.

Notations

Let R denote a fixed commutative unital ring. Let K be a non-archimedean local field of residue characteristic p. We denote by $\mathfrak{o}$ its valuation ring and by k its residue class field. Let $\mathbb{G}$ denote a split connected reductive group over K, d its semisimple rank and $G = \mathbb{G}(K)$ the group of K-rational points of $\mathbb{G}$. We will denote by $\mathrm{Rep}_R^\infty(G)$ the category of R-linear smooth G-representations.

Chapter 1

Equivariant objects on the Bruhat–Tits building

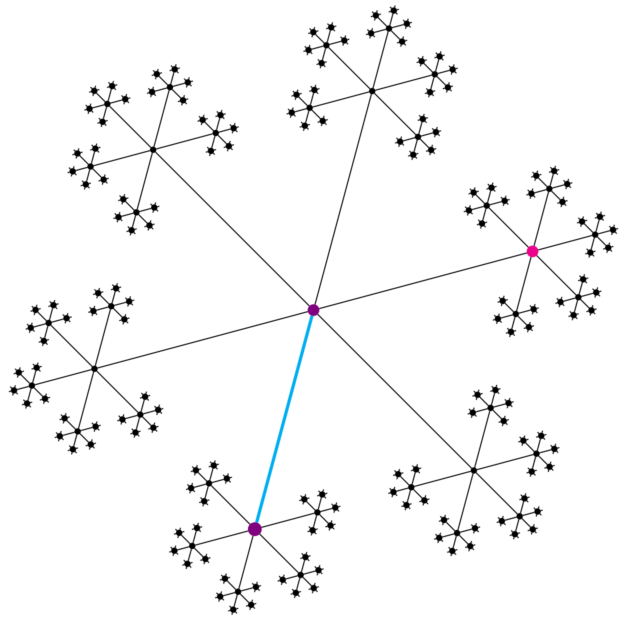
The primary objective of this chapter is to consolidate foundational knowledge drawn from the theory of Bruhat-Tits buildings, alongside an exploration of G-equivariant coefficient systems and G-equivariant sheaves defined on these buildings. These concepts serve as indispensable tools for our subsequent investigations into p-adic representation theory. Furthermore, we present key constructions from [Koh22] in Section 1.3 and [SS97] in Section 1.4, which offer crucial insights into the interplay between smooth representations of p-adic reductive groups and the geometric structures associated with their Bruhat-Tits buildings. By synthesizing these theoretical underpinnings, we establish a solid groundwork for our ensuing work.

1.1 The Bruhat–Tits building $\mathscr{X}$

We will denote by $\mathscr{X}$ the semi-simple Bruhat–Tits building of G constructed in [BT72] and [BT84]. The reader can find a brief summary of the necessary theory in [SS97, §I.1]. In short, $\mathscr{X}$ is a contractible d-dimensional polysimplicial complex with a simplicial action of G. We usually refer to its 0-dimensional (resp. d-dimensional) faces as the vertices (resp. the chambers) of $\mathscr{X}$ and we will denote by $\mathscr{X}_i$ the set of all i-dimensional faces of $\mathscr{X}$, with $0 \leq i \leq d$. Being polysimplices, the faces of $\mathscr{X}$ are connected topological spaces. Given a face F of $\mathscr{X}$, we will denote by $P_F^\dagger = \{g \in G | gF = F\}$ the stabilizer of F in G, we will write $\bar{F}$ for the closure of a face F and for any $0 \leq i \leq d$ we will denote by $\mathscr{X}^i = \bigcup_{F \in \mathscr{X}_i} \bar{F}$ the i-skeleton of $\mathscr{X}$.

Examples 1.1.

- Let p be a prime. The Bruhat–Tits building of $G = SL_2(\mathbb{Q}_p)$ is a tree. Let us call it T_p. A face F of T_p is either a vertex or an edge, with the edges of T_p being its chambers since T_p is of dimension 1. Bruhat–Tits trees are easily drawn. We fix $p = 5$. To draw T_5, the Bruhat–Tits building of $SL_2(\mathbb{Q}_5)$, we need to keep in mind that it is connected, it is a tree and each vertex has 6 neighbours. This tree looks like this :



We highlighted a pink vertex, a blue edge as well as the two purple vertices in its closure.

- The Bruhat–Tits building of a direct product of groups is the cartesian product of the Bruhat–Tits buildings of the factors. Therefore, the Bruhat–Tits building of $SL_2(\mathbb{Q}_p) \times SL_2(\mathbb{Q}_q)$, with p and q two primes, is the product $T_p \times T_q$ of two trees.

Let pr be the restriction map from the enlarged Bruhat–Tits building of G (in the sense of [BT84, §4.2.16]) to $\mathscr{X}$. For any face F of $\mathscr{X}$, we note that the pointwise stabilizer of $pr^{-1}(F)$ in G is the group of $\mathfrak{o}$-rational points of a smooth group scheme $\mathbb{G}_F$ over $\mathfrak{o}$ with generic fiber $\mathbb{G}$. We call the parahoric subgroup of G associated with F the group of $\mathfrak{o}$-rational points

$$P_F = \mathring{\mathbb{G}}_F(\mathfrak{o})$$

where $\mathring{\mathbb{G}}_F$ is the connected component of $\mathbb{G}_F$.

Let $\pi_F : P_F = \mathring{\mathbb{G}}_F(\mathfrak{o}) \to \mathring{\mathbb{G}}_F(k)$ be the group homomorphism induced by the residue class map $\mathfrak{o} \to k$, and let $R^u(\mathring{\mathbb{G}}_{F,k})$ be the unipotent radical of the special fiber $\mathring{\mathbb{G}}_{F,k}$ of $\mathring{\mathbb{G}}_F$. We obtain a pro-p group

$$I_F = \pi_F^{-1}(R^u(\mathring{\mathbb{G}}_{F,k})(k)) \subseteq P_F,$$

which is in fact the pro-p radical of P_F.

In particular, if we let $F = x$ be a vertex of $\mathscr{X}$ we can define P_x and I_x. Similarly, if we let $F = C$ be a chamber then the subgroup P_C and I_C are respectively called an Iwahori subgroup and a pro-p Iwahori subgroup of G.

Remark 1.2. If we take two faces F and F' of $\mathscr{X}$ such that $F' \subseteq \bar{F}$ then, by [SS97, Proposition I.2.11(i)] and [Tit79, §3.4.3], we have the following chain of inclusions

$$I_{F'} \subseteq I_F \subseteq P_F \subseteq P_{F'}.$$

Example 1.3. Let us exhibit the parahoric subgroups and their pro-p radical in the case of $G = GL_2(\mathbb{Q}_p)$. Firstly, the Bruhat–Tits building associated to $GL_2(\mathbb{Q}_p)$ is the same tree as $SL_2(\mathbb{Q}_p)$ therefore we only have vertices and chambers to study.
There is a vertex x such that,

$$P_x = GL_2(\mathbb{Z}_p).$$

The morphism $\pi_x : P_x = GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{F}_p)$ is the reduction modulo p. Since $GL_{2,\mathbb{F}_p}$ is a reductive group scheme, its unipotent radical $R^u(\overset{\circ}{GL}_{2,\mathbb{F}_p})$ is trivial. Therefore, we have

$$\begin{aligned} I_x &= \pi_x^{-1}(R^u(\overset{\circ}{GL}_{2,\mathbb{F}_p})(\mathbb{F}_p)), \\ &= \ker(\pi_x), \\ &= \left\{ \begin{pmatrix} 1+pa & pb \\ pc & 1+pd \end{pmatrix} \in GL_2(\mathbb{Z}_p) \,\middle|\, a,b,c,d \in \mathbb{Z}_p \right\}. \end{aligned}$$

For a chamber C,

$$\begin{aligned} P_C &= \{M \in GL_2(\mathbb{Z}_p) \mid \ M \text{ mod } p \text{ is upper triangular}\}, \\ &= \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \,\middle|\, a,b,c,d \in \mathbb{Z}_p \right\}. \end{aligned}$$

Its pro-p radical is

$$\begin{aligned} I_C &= \{M \in GL_2(\mathbb{Z}_p) \mid \ M \text{ mod } p \text{ is upper triangular and unipotent}\}, \\ &= \left\{ \begin{pmatrix} 1+pa & b \\ pc & 1+pd \end{pmatrix} \in GL_2(\mathbb{Z}_p) \,\middle|\, a,b,c,d \in \mathbb{Z}_p \right\}. \end{aligned}$$

For any face $F \subseteq \mathscr{X}$, we can also define

$$\mathrm{St}(F) = \bigcup_{F \subseteq \overline{F'}} F',$$

which is an open neighborhood of F in $\mathscr{X}$ called the star of F. Here the union is over all faces of $\mathscr{X}$ containing F in their closure. In fact we can define the star of any point of $\mathscr{X}$. Given any point $z \in \mathscr{X}$ we denote by F_z the unique face of $\mathscr{X}$ containing z and we define the star of z as $\mathrm{St}(z) := \mathrm{St}(F_z)$.

Remark 1.4. Once again, if we take two faces F and F' of $\mathscr{X}$ such that $F' \subseteq \bar{F}$ then we have

$$\mathrm{St}(F) \subseteq \mathrm{St}(F').$$

Indeed, let F'' be a face in $\mathscr{X}$ such that $F'' \subseteq \mathrm{St}(F)$, then we have $F \subseteq \bar{F''}$. This together with the assumption that F' is contained in the closure of F gives us $F'' \subseteq \mathrm{St}(F')$ and therefore the inclusion between the stars we announced.

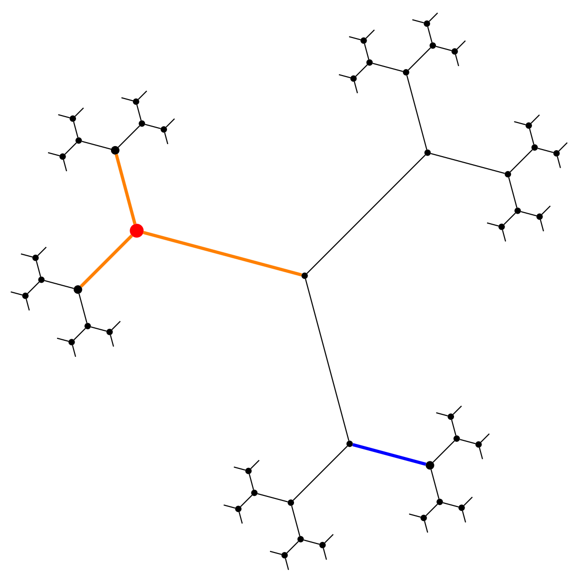
Finally, for $0 \leq i \leq d$, let $\mathscr{X}_{(i)}$ be the set of oriented i-dimensional faces of $\mathscr{X}$ as defined in [SS97, §II.1]. An element of $\mathscr{X}_{(i)}$ is of the form (F, c) where F is a face of $\mathscr{X}_i$ and the orientation c is a generator of $H_i(\mathscr{X}^i, \mathscr{X}^i \backslash F; \mathbb{Z})$. Note that, for any $i \geq 0$, we have $H_i(\mathscr{X}^i, \mathscr{X}^i \backslash F; \mathbb{Z}) \cong \mathbb{Z}$ which means that, for any i-dimensional face, there is only two possible orientations and they are opposite to each other. For an orientation c, we will denote $-c$ its unique opposite orientation. Therefore, let $(F, c) \in \mathscr{X}_{(i)}$ for $i > 0$, then we also have $(F, -c) \in \mathscr{X}_{(i)}$. By convention, the 0-dimensional faces are endowed with the trivial orientation, i.e. $\mathscr{X}_{(0)} = \mathscr{X}_0$.

For any $(F, c) \in \mathscr{X}_{(i)}$ and any $(i-1)$-dimensional face $F' \subseteq \bar{F}$ of $\mathscr{X}$ there is an isomorphism

$$\partial^F_{F'} : H_i(\mathscr{X}^i, \mathscr{X}^i \backslash F; \mathbb{Z}) \to H_{i-1}(\mathscr{X}^{i-1}, \mathscr{X}^{i-1} \backslash F'; \mathbb{Z}),$$

constructed in [SS97, §II.1] using results from [Dol95, §V.6]. We will call $\partial^F_{F'}(c)$ the induced orientation on F'.

Example 1.5. We will use the tree of $SL_2(\mathbb{Q}_2)$ to exhibit the latest introduced notions. We highlighted the orange star of the red vertex. In any tree any edge (1-dimensional face) has two possible orientations. For example, the blue edge can either be oriented from right to left or from left to right. These two orientations will be opposite to each other.



1.2 G-equivariant coefficient systems on $\mathscr{X}$

One of the main objects studied in this thesis are coefficient systems of R-modules on the Bruhat–Tits building $\mathscr{X}$. The goal of this section is to present this notion to the reader and recall a few necessary facts. We start by giving their definition.

Definition 1.6.

- A *coefficient system of R-modules on $\mathscr{X}$* is a family

$$\mathcal{F} = \big((\mathcal{F}_F)_F, (r^F_{F'})_{F' \subseteq \overline{F}}\big)$$

indexed by the faces of $\mathscr{X}$ where, for any faces F' and F of $\mathscr{X}$ such that $F' \subseteq \overline{F}$, the object $\mathcal{F}_F$ is an R-module and $r_{F'}^F : \mathcal{F}_F \to \mathcal{F}_{F'}$ is an R-linear map. These morphisms must moreover satisfy the two following properties:

$$r_F^F = Id_{\mathcal{F}_F} \text{ and } r_{F''}^F = r_{F''}^{F'} \circ r_{F'}^F,$$

for any faces $F'' \subseteq \overline{F'} \subseteq \overline{F}$ of $\mathscr{X}$.

- Given $\mathcal{F} = \big((\mathcal{F}_F)_F, (r_{F'}^F)_{F' \subseteq \overline{F}}\big)$ and $\mathcal{G} = \big((\mathcal{G}_F)_F, (\rho_{F'}^F)_{F' \subseteq \overline{F}}\big)$ two coefficient systems of R-modules on $\mathscr{X}$, a *morphism of coefficient systems of R-modules* $f : \mathcal{F} \to \mathcal{G}$ is a family of R-linear morphisms $(f_F : \mathcal{F}_F \to \mathcal{G}_F)$ indexed by the faces F of $\mathscr{X}$ such that, for any faces $F' \subseteq \overline{F}$ of $\mathscr{X}$, the diagram

$$\begin{array}{ccc} \mathcal{F}_F & \xrightarrow{f_F} & \mathcal{G}_F \\ {\scriptstyle r_{F'}^F}\downarrow & & \downarrow{\scriptstyle \rho_{F'}^F} \\ \mathcal{F}_{F'} & \xrightarrow[f_{F'}]{} & \mathcal{G}_{F'} \end{array}$$

commutes, i.e. such that $\rho_{F'}^F \circ f_F = f_{F'} \circ r_{F'}^F$.

Such objects form the category $\mathrm{Coeff}(\mathscr{X})$ of coefficient systems of R-modules on $\mathscr{X}$.

Remark 1.7. To simplify our terminology, we will refer to coefficient systems of R-modules on $\mathscr{X}$ simply as 'coefficient systems'.

Example 1.8.
- *Constant coefficient system associated to a module* : Let M be an R-module. A natural example of a coefficient system is the constant coefficient system associated to the module M, defined as the family $\mathcal{K}_M := \big((M)_F, (Id_M)_{F' \subseteq \overline{F}}\big)$.
- *Coefficient system associated to a representation* : Let V be a smooth R-linear G-representation. For any faces F' and F of $\mathscr{X}$ such that $F' \subseteq \overline{F}$, let $\underline{\underline{V}}_F$ be the R-module of I_F-invariants V^{I_F} and $i_{F'}^F : V^{I_F} \hookrightarrow V^{I_{F'}}$ be the inclusion map induced by Remark 1.2. The family $\underline{\underline{V}} := \Big((\underline{\underline{V}}_F)_F, (i_{F'}^F)_{F' \subseteq \overline{F}}\Big)$ forms a coefficient system on $\mathscr{X}$.
- *Action of G on coefficient systems* : Let $g \in G$. Since the group G has an action on the Bruhat–Tits building $\mathscr{X}$, starting with a coefficient system $\mathcal{F}$, we can construct a coefficient system $g_* \mathcal{F} = \big((g_*\mathcal{F}_F)_F, (g_* r_{F'}^F)_{F' \subseteq \bar{F}}\big)$ where $g_*\mathcal{F}_F := \mathcal{F}_{gF}$ and $g_* r_{F'}^F := r_{gF'}^{gF}$. In the same way, let $f : \mathcal{F} \to \mathcal{G} \in \mathrm{Coeff}(\mathscr{X})$, we denote by $g_* f$ the morphism of coefficient systems such that $(g_* f)_F = f_{gF}$.

Coefficient systems on $\mathscr{X}$ come with the natural G-action described in the third point of Example 1.8. Using this action, we can introduce an equivariance condition on the objects of $\mathrm{Coeff}(\mathscr{X})$.

Definition 1.9.
- An object $\mathcal{F}$ of $\mathrm{Coeff}(\mathscr{X})$ is called *G-equivariant* if it comes with a family of morphisms of coefficient systems $(c_g : \mathcal{F} \to g_* \mathcal{F})_{g \in G}$ such that $c_1 = Id_{\mathcal{F}}$ and such that they satisfy the following cocycle relation:

$$\forall g, h \in G, c_{gh} = h_* c_g \circ c_h.$$

- Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Coeff}(\mathscr{X})$. If $\mathcal{F}$ and $\mathcal{G}$ are G-equivariant, with $(c_g)_{g\in G}$ and $(\gamma_g)_{g\in G}$ their respective G-actions, then f is called G-*equivariant* if it commutes with these morphisms, i.e. if for any $g \in G$, $g_* f \circ c_g = \gamma_g \circ f$.

For a G-equivariant coefficient system $\mathcal{F}$, the morphisms $(c_g : \mathcal{F}_F \to \mathcal{F}_F)_{g\in P_F^\dagger}$ endow the R-module $\mathcal{F}_F$ with a structure of $P_F^\dagger$-representation, for any $F \subseteq \mathscr{X}$.

We denote by $\mathrm{Coeff}_G(\mathscr{X})$ the category of G-equivariant coefficient systems on $\mathscr{X}$ for which $\mathcal{F}_F$ is a smooth $P_F^\dagger$-representation, for each face F of $\mathscr{X}$. We will adopt the designation 'G-equivariant coefficient systems' for objects within $\mathrm{Coeff}_G(\mathscr{X})$, presupposing this condition of smoothness without explicit mention.

Notation 1.10. For the rest of this thesis, when we will work with coefficient systems we will generally denote them $\mathcal{F} = \big((\mathcal{F}_F)_F, (r_{F'}^F)_{F'\subseteq \bar{F}}\big)$ and $\mathcal{G} = \big((\mathcal{G}_F)_F, (\rho_{F'}^F)_{F'\subseteq \bar{F}}\big)$. When we will make the assumptions that $\mathcal{F}$ or $\mathcal{G}$ are G-equivariant we will write $(c_g)_{g\in G}$ and $(\gamma_g)_{g\in G}$ for their respective G-actions.

Example 1.11. Let V be a smooth R-linear G-representation. For $g \in G$ and $F \subseteq \mathscr{X}$, we define the map $c_{g,F} : V^{I_F} \to V^{I_{gF}} = V^{gI_Fg^{-1}}$ by $c_{g,F}(v) = gv$, for any $v \in V$. It turns the coefficient system $\underline{\underline{V}}$ introduced in Example 1.8 into a G-equivariant coefficient system as per Definition 1.9. In fact, since we assumed that V is smooth, $\underline{\underline{V}} = \Big((\underline{\underline{V}}_F)_F, (i_{F'}^F)_{F'\subseteq \overline{F}}, (c_g)_{g\in G}\Big)$ is an element of $\mathrm{Coeff}_G(\mathscr{X})$.

We recall that to any coefficient system $\mathcal{F}$ we can associate the R-module $\mathcal{C}_c^{or}(\mathscr{X}_{(i)}, \mathcal{F})$, for any $0 \le i \le d$, of finitely supported maps

$$f : \mathscr{X}_{(i)} \to \coprod_{(F,c)\in \mathscr{X}_{(i)}} \mathcal{F}_F$$

such that for any i-dimensional oriented face (F, c) we have $f((F, c)) \in \mathcal{F}_F$ and such that, if $i > 0$, we have that $f((F, -c)) = -f((F, c))$. If we take $\mathcal{F}$ to be in $\mathrm{Coeff}_G(\mathscr{X})$ then the group G acts on these modules via

$$(g.f)((F, c)) = c_g \circ f((g^{-1}F, g^{-1}c)),$$

for any $i \in I$, any $g \in G$, any $f \in \mathcal{C}_c^{or}(\mathscr{X}_{(i)}, \mathcal{F})$ and any $(F, c) \in \mathscr{X}_{(i)}$.

We denote by $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F}), \partial_\bullet)$ the oriented chain complex associated to the G-equivariant coefficient system $\mathcal{F}$

$$0 \to \mathcal{C}_c^{or}(\mathscr{X}_{(d)}, \mathcal{F}) \xrightarrow{\partial_{d-1}} \ldots \xrightarrow{\partial_1} \mathcal{C}_c^{or}(\mathscr{X}_{(1)}, \mathcal{F}) \xrightarrow{\partial_0} \mathcal{C}_c^{or}(\mathscr{X}_{(0)}, \mathcal{F}) \to 0.$$

It is a complex of smooth G-representations where the differentials are defined by the following equation : for any $0 \le i \le d-1$, for any $f \in \mathcal{C}_c^{or}(\mathscr{X}_{(i+1)}, \mathcal{F})$ and for any $(F', c') \in \mathscr{X}_{(i)}$ oriented face,

$$\partial_i(f)((F', c')) = \sum_{\substack{(F,c)\in \mathscr{X}_{(i+1)} \\ F'\subseteq \bar{F}, \partial_{F'}^F(c)=c'}} r_{F'}^F(f((F, c))).$$

Later on, more precisely in Section 2.5, we will describe this oriented chain complex a little differently by fixing for each face an orientation.

Coefficient systems on $\mathscr{X}$ proved to be useful tools for the study of G-representations. In the case of $R = \mathbb{C}$, the relation between the category of G-equivariant coefficient systems and the category $\mathrm{Rep}^\infty_{\mathbb{C}}(G)$ was thoroughly studied by Schneider and Stuhler in [SS97]. More precisely, let V be a smooth complex G-representation, in [SS97, §II.2] they associated to V multiple coefficient systems $\gamma_e(V)$ as follows : for $e \geq 0$ an integer,

$$\gamma_e(V) \coloneqq \left((V^{U_F^{(e)}})_F, (V^{U_F^{(e)}} \hookrightarrow V^{U_{F'}^{(e)}})_{F' \subseteq \overline{F}} \right),$$

where $U_F^{(e)}$ is a certain compact open subgroup of G satisfying, in particular, that for any two faces $F' \subseteq \overline{F}$, $U_{F'}^{(e)} \subseteq U_F^{(e)}$. In the case $e = 0$, $U_F^{(0)} = I_F$ and $\gamma_0(V) = \underline{\underline{V}}$ (see Example 1.8). From there, they studied, for any $e \geq 0$, the chain complex $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \gamma_e(V)), \partial_\bullet)$ associated to the coefficient system $\gamma_e(V)$. Under the assumption that V is generated by its $U_x^{(e)}$, for a fixed vertex x, they proved that the augmented complex $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \gamma_e(V)), \partial_\bullet)) \to V$ is an exact resolution of V in $\mathrm{Rep}^\infty_{\mathbb{C}}(G)$ (see [SS97, Proposition II.3.1])

Remark 1.12. Without any restriction on R, except that it is an arbitrary field, and assuming that I is a pro-p Iwahori subgroup of G, Ollivier and Schneider proved the exactness of the augmented complex $0 \to (\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \underline{\underline{V}})^I, \partial_\bullet)) \to V^I \to 0$ of I-invariants in [OS14, Proposition 3.4]. It is important to note that the augmented complex $0 \to (\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \underline{\underline{V}}), \partial_\bullet)) \to V \to 0$ itself is not always exact (see [OS14, Remark 3.2] for an example). This result motivated the study of the exactness of the complex of I-invariants $\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F})^I$ for more general $\mathcal{F} \in \mathrm{Coeff}_G(\mathscr{X})$. This is what Kohlhaase does in [Koh22] while extending the theory developed in [SS97] to the modular setting.

1.3 The functor $\mathcal{F}(.) : \mathrm{Mod}_H \to \mathcal{C}$

Given a compact open subgroup I of G, one can define the corresponding Hecke R-algebra of G by $H \coloneqq \mathrm{End}_G(X)^{\mathrm{op}}$, where the right H-module $X \coloneqq \mathrm{ind}_I^G(R)$ is the compactly induced smooth G-representation of the trivial I-representation over R as in [OS14, §2]. Let Mod_H denote the category of left H-modules and $\mathrm{Rep}^\infty_R(G)$ denote, as previously, the category of smooth R-linear G-representations. There is an adjunction between these two categories given by the following functors:

$$\mathrm{Rep}^\infty_R(G) \underset{\mathrm{ind}_I^G(R) \otimes_H (.)}{\overset{(.)^I}{\rightleftarrows}} \mathrm{Mod}_H \,.$$

Classically, when $R = \mathbb{C}$ and I is a (pro-p) Iwahori subgroup, the above functors induce an equivalence of categories between Mod_H and the full subcategory $\mathrm{Rep}^I_R(G)$ of $\mathrm{Rep}^\infty_R(G)$ of smooth G-representations generated by their I-invariants vectors, see [Ber84]. In particular, such a representation is irreducible if and only if the associated Hecke module is simple. When R is of characteristic p, any non-zero representation over R has non-zero vectors fixed by the pro-p radical of an Iwahori subgroup of G. This was proved for $R = \bar{\mathbb{F}}_p$ in [Pas04, Lemma 2.1] and for R a quasi-Frobenius ring in which p is nilpotent in [Koh22, Lemma 4.13]. Because of this, we still have that a representation generated by its I-invariants is irreducible if the associated Hecke module is simple. Indeed, let W be a non-zero subrepresentation of $V \in \mathrm{Rep}_R^I(G)$. The module of I-invariants of W is a non-zero submodule of V^I which is simple so $W^I = V^I$. But, V is generated by its I-invariants so $V = W$. It means that V is irreducible as claimed. However, not all Hecke modules associated to an irreducible representation are simple. Nonetheless, these comments suggest that the above adjunction gives a strong link between G-representations and H-modules. For example, if $G = GL_2(\mathbb{Q}_p)$ or $G = SL_2(\mathbb{Q}_p)$, the adjunction again gives an equivalence of categories [Oll09; Koz15]. Note that, when $K \neq \mathbb{Q}_p$, the same authors proved that, in general, it is not an equivalence anymore [Oll09; Koz15]. In general, one does not know the precise relation between Mod_H and $\mathrm{Rep}_R^I(G)$. In the rest of this thesis, I will be a pro-p Iwahori subgroup of G and we will denote by H the corresponding Hecke algebra, called a *pro-p Iwahori–Hecke algebra*.

In [Koh22], Kohlhaase studies the relation between Mod_H and $\mathrm{Coeff}_G(\mathscr{X})$. To do so he introduces a certain full subcategory $\mathcal{C}$ of $\mathrm{Coeff}_G(\mathscr{X})$ and constructs an equivalence of categories between $\mathcal{C}$ and Mod_H. This equivalence of categories will be of the utmost importance in this thesis, therefore, we would like to recall its construction.

In this section, we will assume that R is a quasi-Frobenius ring, meaning that it is noetherian and self-injective (see [Lam99, §15]. This latter property, implying that the functor $\mathrm{Hom}_R(-, R)$ is exact, will be of great use in Section 2.4.

Definition 1.13 ([Koh22, Definition 3.1 (i)])**.** Let F be a face of $\mathscr{X}$. We say that an object $V \in \mathrm{Rep}_R^\infty(P_F)$ *satisfies condition (H)* if $V \cong \varinjlim_{j\in J} V_j$ is isomorphic to the inductive limit of objects $V_j \in \mathrm{Rep}_R^\infty(P_F)$ such that the transition maps of the inductive system are injective and such that for each $j \in J$ there is a non-negative integer n_j and an element $\phi_j \in \mathrm{End}_{P_F}(X_F^{n_j})$ with $V_j \cong \mathrm{im}(\phi_j)$ in $\mathrm{Rep}_R^\infty(P_F)$. We denote by $\mathrm{Rep}_R^H(P_F)$ the full subcategory of $\mathrm{Rep}_R^\infty(P_F)$ consisting of all representations satisfying condition (H).

Let I_C be the pro-p Iwahori subgroup associated to a chamber C and let F be a face of $\mathscr{X}$ contained in C. We can define $H_F = R[I_C \backslash P_F / I_C]$ the Hecke algebra at F which may be viewed as a subalgebra of H (see [Koh22, §1.2]). There exists a similar adjunction as the one discussed before between $\mathrm{Rep}_R^\infty(P_F)$ and Mod_{H_F}. By [Koh22, Lemma 3.5(i)], we have $\mathrm{Rep}_R^H(P_F) \subseteq \mathrm{Rep}_R^{I_C}(P_F)$. In fact, Kohlhaase proved that the functor $(.)^{I_C}$ restricts to an equivalence of categories between $\mathrm{Rep}_R^H(P_F)$ and Mod_{H_F} (see [Koh22, Theorem 3.8] essentially due to [Cab90, Theorem 2]).

Definition 1.14 ([Koh22, Proposition 3.18])**.** Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Coeff}_G(\mathscr{X})$ consisting of all objects $\mathcal{F}$ such that, for every vertex $x \in \mathscr{X}_0$ there is a representation $V_x \in \mathrm{Rep}_R^H(P_x)$ and an isomorphism $\mathcal{F}|_{\mathrm{St}(x)} = \mathcal{F}_{V_x}$ in $\mathrm{Coeff}_{P_x}(\mathrm{St}(x))$.

As recalled before, for a representation $V \in \mathrm{Rep}_R^\infty(G)$, the augmented complex of I-invariant $0 \to \mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F}_V)^I \to V^I \to 0$ is exact (Remark 1.12) which makes it natural to study the functor

$$\begin{array}{rccc} M(.): & \mathcal{C} & \longrightarrow & \mathrm{Mod}_H \\ & \mathcal{F} & \mapsto & H_0(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F})^I). \end{array}$$

One of the main results of [Koh22] is that this functor is an equivalence of categories (see [Koh22, Theorem 3.20]). To construct a quasi-inverse, Kohlhaase made use of an idea of [OS18, §1.2]. The functor $\mathcal{F}(.)$ is then defined to be the functor sending $M \in \mathrm{Mod}_H$ to the G-equivariant coefficient system $\mathcal{F}(M)$ defined by, for $F \subseteq \mathscr{X}$ a face,

$$\mathcal{F}(M)_F = \mathrm{im}(X^{I_F} \otimes_H M \xrightarrow{\tau_{M,F}} \mathrm{Hom}_H(\mathrm{Hom}_H(X^{I_F}, H), M)).$$

The map $\tau_{M,F}$ is described in [Koh22]. It is the unique homomorphism sending $x \otimes m$ to the H-linear map $(\phi \to \phi(x).m)$. By [Koh22, Theorem 3.21] we know that the image of $\mathcal{F}(M)$ is contained in the category $\mathcal{C}$ and is the quasi-inverse of the functor $M(.)$ previously mentioned.

It is worth noting that, if we take R to be a field of characteristic 0, Kohlhaase reproved in [Koh22, Corollary 4.9] a special case of the previously mentioned result [SS97, Proposition II.3.1] concerning the exactness of $\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \underline{\underline{V}})$.

Later in this thesis we will be interested in focusing our study to the full subcategory of G-equivariant coefficient systems on $\mathscr{X}$ such that the R-module $\mathcal{F}_F$ is finitely generated for all faces F of $\mathscr{X}$. We will denote this full subcategory by $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$.

Remark 1.15. Another result of Kohlhaase, [Koh22, Proposition 1.7(ii)] together with [Koh22, Equation 1.12], implies that if we take a H-module whose underlying R-module is finitely generated then its image by the functor $\mathcal{F}(.)$ will be an object of $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$.

1.4 G-equivariant sheaves on $\mathscr{X}$

This section concludes the introduction of the essential notions for this thesis. It will focus on defining one of the main objects we will be working with, namely G-equivariant sheaves on $\mathscr{X}$.

Definition 1.16. A *G-equivariant sheaf of R-modules $\mathcal{H}$ on $\mathscr{X}$* is a sheaf of R-modules on $\mathscr{X}$ together with a family of isomorphisms of sheaves of R-modules $(d_g : g^*\mathcal{H} \to \mathcal{H})_{g \in G}$ satisfying the following cocycle relation

$$\forall g, h \in G, d_{gh} = d_h \circ h^* d_g.$$

We will write $\mathrm{Sh}_G(\mathscr{X})$ for the category of G-equivariant sheaves on $\mathscr{X}$.

Definition 1.17 (c.f. [KS90, Definition 8.1.3])**.** A sheaf of R-modules on $\mathscr{X}$ is called weakly constructible if for any face F of $\mathscr{X}$, its restriction to F is constant. A weakly constructible sheaf of R-modules on $\mathscr{X}$ is called constructible if its stalks are finitely generated R-modules.

Remark 1.18. Over the complex numbers, Schneider gives a more restrictive definition of a G-equivariant sheaf of $\mathbb{C}$-vector spaces on $\mathscr{X}$ in [Sch98a, §1]. This definition can be generalized to sheaves of R-modules and a G-equivariant sheaf in our sense only satisfies the condition (2) of Schneider's definition. He calls such a sheaf a G^{dis}-equivariant sheaf. For a weakly-constructible G^{dis}-equivariant sheaf $\mathcal{H}$ to be G-equivariant in the sense of Schneider it is enough to ask that for any face F of $\mathscr{X}$, the action of P_F on $\mathcal{H}(\mathrm{St}(F))$ is smooth (as explained in [Sch98b, §4]).

Over the complex numbers, the relation between G-equivariant sheaves on $\mathscr{X}$ and smooth G-representations has already been thoroughly studied in [SS97]. More precisely, in [SS97, §IV.1], the authors associate to a smooth G-representation V a G-equivariant sheaf on the Bruhat–Tits building $\mathscr{X}$ that they denote by $\underset{\approx}{V}$. As the aim of this thesis is to partly generalize to the modular setting the cohomological theory developed in [SS97], we explain some of the constructions and results from op. cit. which are of interest for us.

Let V be a complex smooth G-representation. Then the sheaf $\underset{\approx}{V}$ is define as follows : For any open subset $\Omega \subseteq \mathscr{X}$ we write

$$\underset{\approx}{V}(\Omega) := \mathbb{C}\text{-vector space of all maps } s : \Omega \to \coprod_{z\in\Omega} V_{I_{F_z}} \text{ such that}$$
$$(i)\ \forall z \in \Omega, s(z) \in V_{I_{F_z}},$$
$$(ii) \text{ there is an open covering } \Omega = \cup_{i\in I}\Omega_i \text{ and vectors } v_i \in V \text{ such that}$$
$$\text{for all } i \in I \text{ and } z \in \Omega_i, \text{ we have } s(z) \equiv v_i \bmod I_{F_z}.$$

Here $V_{I_{F_z}}$ denotes the space of coinvariants for the action of I_{F_z} and $s(z) \equiv v_i \bmod I_{F_z}$ means that the image of v_i by the surjection $V \twoheadrightarrow V_{I_{F_z}}$ is $s(z)$.

This sheaf has the following nice properties.

Lemma 1.19 ([SS97, Lemma IV.1.1])**.** *1.* $(\underset{\approx}{V})_z = V_{I_{F_z}}$ *for any* $z \in \mathscr{X}$*;*

2. The restriction of $\underset{\approx}{V}$ *to any facet* F *of* $\mathscr{X}$ *is the constant sheaf with value* V_{I_F}*.*

Remark 1.20.
- In [SS97], Schneider and Stuhler define the notations $U_z^{(e)}$ and $U_F^{(e)}$. In the case we are interested in, we take $e = 0$ and we have that $U_z^{(0)} = I_{F_z}$ and $U_F^{(0)} = I_F$. They also write X for the Bruhat–Tits building when we denote it by $\mathscr{X}$.
- The previous construction can be generalized to other settings. Let R be a ring and V be an R-linear smooth G-representation. Then the definition of $\underset{\approx}{V}$ still make sense and gives a G-equivariant sheaf on $\mathscr{X}$. Moreover, let F be a face of $\mathscr{X}$. If we take V to be, instead of an R-linear smooth G-representation, an R-linear smooth P_F-representation (resp. $P_F^\dagger$-representation), then we can imitate this construction to obtain a P_F-equivariant (resp. $P_F^\dagger$-equivariant) sheaf on $\mathrm{St}(F)$, which we will also denote $\underset{\approx}{V}$.

In [SS97], Schneider and Stuhler aimed at the computation of the compactly supported cohomology of the sheaf $\underset{\approx}{V}$ they constructed. Their first result in this direction is the following proposition.

Proposition 1.21 ([SS97, Proposition IV.1.3]). *We have the equality*

$$H_c^*(\mathscr{X}, \underset{\approx}{V}) = H^*(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \gamma_e(V)), d_\bullet),$$

where the differentials $(d_i)_{0\leq i\leq d-1}$ *defined in [SS97, §III.1] make the complex in question a cochain complex.*

All of these results were obtained over the complex numbers. However, a significant challenge emerges when attempting to generalize them to the modular setting. In fact, when V is an R-linear smooth G-representation, with R being a field of characteristic p, the sheaf $\underset{\approx}{V}$ is often the zero sheaf. To see why, consider a face F of $\mathscr{X}$. We have that $(V_{I_F})^*$ is isomorphic to $(V^*)^{I_F}$, which is a subset of the smooth dual of the representation V. Herein lies the crux of the issue: the smooth dual of such a representation is trivial in many important cases. For example, it is the case for V irreducible, admissible and satisfying $\dim_R(V) = \infty$. If K has characteristic zero then this is [Koh17, Proposition 3.9]. In general, this was shown by Abe, Henniart and Vignéras in [AHV18].

Instead, inspired by a construction in [Sch98b], we will associate to a G-equivariant coefficient system $\mathcal{F}$ a sheaf $\mathbb{S}(\mathcal{F})$ of R-modules on $\mathscr{X}$. In Lemma 2.1, we establish that the constructed sheaf shares important properties with $\underset{\approx}{V}$, if $\mathcal{F} = \mathcal{F}(M)$ is a coefficient system associated with an H-module M. In fact, we will emphasize the relation between the two constructions in Corollary 2.13.

Remark 1.22. Later on in [SS97] the authors succeed to give an explicit computation of $H_c^*(\mathscr{X}, \underset{\approx}{V})$ by extending the sheaf to the Borel–Serre compactification of $\mathscr{X}$. Even though it is not something we do in this thesis, we believe that extending our sheaf to the Borel–Serre compactification of $\mathscr{X}$ and computing its stalks at the boundary points will be interesting.

Chapter 2

The functor $\mathbb{S}$ and its properties

In this chapter we will study a functor $\mathbb{S}$ which goes from the category $\mathrm{Coeff}_G(\mathscr{X})$ of G-equivariant coefficient systems on $\mathscr{X}$ to the category $\mathrm{Sh}_G(\mathscr{X})$ of G-equivariant sheaves of R-modules on $\mathscr{X}$. This functor also appears in the proof of [Sch98b, Proposition 3.3], but the details of its construction are omitted. Firstly, in Section 2.1, we will give the detailed construction of the sheaf $\mathbb{S}(\mathcal{F})$ associated to a G-equivariant coefficient system $\mathcal{F}$. Then in the other sections we will discuss the G-equivariance of the sheaf constructed and the functoriality of this construction. In Section 2.4, we will prove that the obtained functor is fully faithful on $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. Finally, in Section 2.5, we will study the cohomology with compact support of $\mathbb{S}(\mathcal{F})$.

2.1 Construction

Let $\mathcal{F} = ((\mathcal{F}_F)_F, (r^F_{F'})_{F' \subseteq \bar{F}})$ be a coefficient system of R-modules on $\mathscr{X}$. For a face $F \subseteq \mathscr{X}$, we define the dual of the R-module $\mathcal{F}_F$ as the R-module $\mathcal{F}_F^* := \mathrm{Hom}_R(\mathcal{F}_F, R)$. We recall that, F_z denotes the unique face of $\mathscr{X}$ containing the element z of $\mathscr{X}$. We now define the sheaf $\mathbb{S}(\mathcal{F})$ of R-modules on $\mathscr{X}$ as follows : For any open subset $\Omega \subseteq \mathscr{X}$ we define

$$\mathbb{S}(\mathcal{F})(\Omega) := R\text{-module of all maps } s : \Omega \to \coprod_{z \in \Omega} \mathcal{F}^*_{F_z} \text{ such that}$$

$$(i)\ \forall z \in \Omega, s(z) \in \mathcal{F}^*_{F_z},$$
$$(ii)\ \forall z \in \Omega, \exists V \subseteq \Omega \cap \mathrm{St}(F_z) \text{ open neighborhood of } z \text{ such that}$$
$$\forall z' \in V, s(z') = (r^{F_{z'}}_{F_z})^*(s(z)).$$

Let us point out that, for two points z and z' of $\mathscr{X}$, if $z' \in \mathrm{St}(F_z)$ then $F_z \subseteq \bar{F}_{z'}$ and hence the restriction map $r^{F_{z'}}_{F_z}$ is available. The following lemma is the exact analog of Lemma 1.19. For the convenience of the reader we will give a detailed proof.

Lemma 2.1. *1. For any $z \in \mathscr{X}$, the map* $\mathrm{ev}_z : \begin{array}{ccc} \mathbb{S}(\mathcal{F})_z & \xrightarrow{\sim} & \mathcal{F}^*_{F_z} \\ [s] & \mapsto & s(z) \end{array}$ *is an isomorphism of R-modules.*

2. *The restriction of* $\mathbb{S}(\mathcal{F})$ *to any face* F *of* $\mathscr{X}$ *is the constant sheaf with value* $\mathcal{F}_F^*$.

Remark 2.2. By the second point of Lemma 2.1 we can conclude that, for any $\mathcal{F} \in \mathrm{Coeff}_G(\mathscr{X})$, the sheaf $\mathbb{S}(\mathcal{F})$ is weakly constructible (see Definition 1.17 for the definition).

Proof. Fixing $z \in \mathscr{X}$ we can prove the first point of the lemma :

- Injectivity : We take a germ sent to 0. We take a representative s of this germ. This representative is an element of $\mathbb{S}(\mathcal{F})(\Omega)$ for some Ω open subset of $\mathscr{X}$ containing z. This means by (ii) that there exists $V \subseteq \Omega \cap \mathrm{St}(F_z)$ an open neighborhood of z such that $\forall z' \in V$,
$$s(z') = (r_{F_z}^{F_{z'}})^*(s(z)) = s(z) \circ r_{F_z}^{F_{z'}}.$$
Since $s(z) = 0$ in $\mathcal{F}_{F_z}^*$, we get $s|_V = 0$ and therefore the injectivity needed.

- Surjectivity : We fix an element $t \in \mathcal{F}_{F_z}^*$. We define the map
$$\begin{array}{rccl} s: & \mathrm{St}(F_z) & \to & \coprod_{z' \in \mathrm{St}(F_z)} \mathcal{F}_{F_{z'}}^* \\ & z' & \mapsto & (r_{F_z}^{F_{z'}})^*(t). \end{array}$$
This is a well defined section of $\mathrm{St}(F_z) \subseteq \mathscr{X}$, its evaluation at z is t and therefore its germ is as required and we can conclude the surjectivity.

By definition, $\mathbb{S}(\mathcal{F})\,|_F$ is the sheafification of a certain presheaf H. The R-modules of sections of this presheaf over an open $\Sigma \subseteq F$ is the colimit of $\mathbb{S}(\mathcal{F})(\Omega)$ taken over every open $\Omega \subseteq \mathscr{X}$ verifying that $\Omega \cap F = \Sigma$. By cofinality we may restrict to all such Ω which are contained in $\mathrm{St}(F)$.

To prove the second point of the lemma, we first observe that the restriction of functions to an open $\Sigma \subseteq F$ induces a well-defined morphism of presheaves $H \to \underline{\mathcal{F}_F^*}$ on F. Indeed, let Σ be an open in F. Let Ω be an open in $\mathrm{St}(F)$ such that $\Omega \cap F = \Sigma$. Let s be an element of $\mathbb{S}(\mathcal{F})(\Omega)$.

Any element y of Ω is such that $F_y \subseteq \mathrm{St}(F)$. Using this remark, we can define for all $t \in \mathcal{F}_F^*$ the set Ω_t as
$$\Omega_t := \left\{ x \in \Omega \,\middle|\, s(x) = (r_F^{F_x})^*(t) \right\}.$$
The union of these sets contains Σ. In particular, for any element y of Σ, there exist a unique element $t \in \mathcal{F}_F^*$ such that $y \in \Omega_t$, namely $t := s(y)$. Moreover, we deduce from the local constancy condition on s that each of the sets Ω_t are open in $\mathrm{St}(F)$. It implies that, each of the sets $\Sigma \cap \Omega_t$ are open in Σ. As a consequence, we can write Σ as the following disjoint union, $\Sigma = \coprod_{t \in \mathcal{F}_F^*} (\Sigma \cap \Omega_t)$. Therefore, the restriction of the section s to Σ is, as required, a locally constant $\mathcal{F}_F^*$-valued function on Σ (it is constant of value $t \in \mathcal{F}_F^*$ on $\Sigma \cap \Omega_t$) and the morphism is well-defined.

From this morphism of presheaves we get a morphism of sheaves between $\mathbb{S}(\mathcal{F})\,|_F$ and $\underline{\mathcal{F}_F^*}$. Stalkwise, it is in fact the isomorphism constructed in the first part of this lemma. Thus, we get an isomorphism of sheaves and conclude that the restriction of $\mathbb{S}(\mathcal{F})$ to any face F of $\mathscr{X}$ is the constant sheaf with value $\mathcal{F}_F^*$. □

Remark 2.3. In the previous proof we showed that for any open $\Sigma \subseteq F$, elements of $\mathbb{S}(\mathcal{F})(\Omega)$ restrict to locally constant $\mathcal{F}_F^*$-valued functions on Σ for Ω an open in $\mathrm{St}(F)$ such that $\Omega \cap F = \Sigma$. What if we look at $\Omega = \mathrm{St}(F)$ and $s \in \mathbb{S}(\mathcal{F})(\mathrm{St}(F))$? In this case, $\Sigma = F$ and we obtain that s is not only a locally constant function but a constant function on F. Indeed, as earlier we can define the open sets $\Omega_t \subseteq \mathrm{St}(F)$ for $t \in \mathcal{F}_F^*$ and write F as the following disjoint union, $F = \coprod_{t \in \mathcal{F}_F^*} (\Omega_t \cap F)$. This time, since F is connected, we get that $F = \Omega_t \cap F$ for a unique t. So $F \subseteq \Omega_t$. Therefore, we conclude that s is constant on F (of value t).

2.2 Functoriality

Let $\mathcal{F} = ((\mathcal{F}_F)_F, (r_{F'}^F)_{F' \subseteq \bar{F}})$ and $\mathcal{G} = ((\mathcal{G}_F)_F, (\rho_{F'}^F)_{F' \subseteq \bar{F}})$ be two coefficient systems and $f : \mathcal{F} \to \mathcal{G}$ be a morphism of coefficient systems.

Having the first point of Lemma 2.1 in mind we look at the dual family

$$(f_F^* : \mathcal{G}_F^* \to \mathcal{F}_F^*)_{\substack{F \subseteq \mathscr{X} \\ \text{face}}}$$

of morphisms of R-modules. Let Ω be an open subset of $\mathscr{X}$. Using pointwise compositions with the maps f_F we get R-linear maps

$$\mathbb{S}(f)(\Omega) : \mathbb{S}(\mathcal{G})(\Omega) \to \mathbb{S}(\mathcal{F})(\Omega)$$

which preserve the local constancy condition. Indeed, let s be an element of $\mathbb{S}(\mathcal{G})(\Omega)$. For any $z \in \Omega$, there exists $V \subseteq \Omega \cap \mathrm{St}(F_z)$ an open neighborhood of z such that for all $z' \in V$, $s(z') = (\rho_{F_z}^{F_{z'}})^*(s(z)) = s(z) \circ \rho_{F_z}^{F_{z'}}$. Now we would like that the image of s by $\mathbb{S}(f)(\Omega)$ satisfies the same property. Taking the same open neighborhood V of z and any element z' of this open, we have that

$$\begin{aligned}
\mathbb{S}(f)(\Omega)(s)(z') &= s(z') \circ f_{F_{z'}} \\
&= s(z) \circ \rho_{F_z}^{F_{z'}} \circ f_{F_{z'}} \\
&= s(z) \circ f_{F_z} \circ r_{F_z}^{F_{z'}} \\
&= \mathbb{S}(f)(\Omega)(s)(z) \circ r_{F_z}^{F_{z'}}
\end{aligned}$$

as required.
Moreover, these morphisms $\mathbb{S}(f)(\Omega)$ are compatible with restrictions on open subsets of $\mathscr{X}$.

Altogether, taking a morphism of coefficient systems $f : \mathcal{F} \to \mathcal{G}$, we have constructed a morphism of sheaves

$$\mathbb{S}(f) : \mathbb{S}(\mathcal{G}) \to \mathbb{S}(\mathcal{F}).$$

This construction clearly satisfies $\mathbb{S}(\mathrm{Id}_{\mathcal{F}}) = \mathrm{Id}_{\mathbb{S}(\mathcal{F})}$ and $\mathbb{S}(f_1 \circ f_2) = \mathbb{S}(f_2) \circ \mathbb{S}(f_1)$ for f_1, f_2 morphisms of coefficient systems, i.e. we have constructed a contravariant functor $\mathbb{S}$ from the category of coefficient systems on $\mathscr{X}$ to the category of sheaves of R-modules on $\mathscr{X}$.

2.3 G-equivariance

We are interested in G-equivariant coefficient systems. It is natural to wonder if the G-equivariance is preserved by the functor $\mathbb{S}$. We will show that, indeed, the sheaf $\mathbb{S}(\mathcal{F})$ associated to a G-equivariant coefficient system $\mathcal{F}$ naturally carries the structure of a G-equivariant sheaf.

Lemma 2.4. *For $g \in G$, we have the following statements.*

1. *The sheaves $g^* \mathbb{S}(\mathcal{F})$ and $\mathbb{S}(g_* \mathcal{F})$ are isomorphic. An isomorphism on the level of sections over any open $\Omega \subseteq \mathscr{X}$ is given as follows.*

$$\begin{array}{rccc} \alpha_{g,\mathcal{F}}(\Omega): & \mathbb{S}(g_* \mathcal{F})(\Omega) & \xrightarrow{\sim} & \mathbb{S}(\mathcal{F})(g\Omega) \\ & s & \mapsto & (z \in g\Omega \mapsto s(g^{-1}z)). \end{array}$$

2. *For any morphism $f : \mathcal{F} \to \mathcal{G}$ of G-equivariant coefficient systems and any $g \in G$, the diagram*

$$\begin{array}{ccc} \mathbb{S}(g_* \mathcal{G}) & \xrightarrow{\mathbb{S}(g_* f)} & \mathbb{S}(g_* \mathcal{F}) \\ {\scriptstyle \alpha_{g,\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle \alpha_{g,\mathcal{F}}} \\ g^*\mathbb{S}(\mathcal{G}) & \xrightarrow[g^*\mathbb{S}(f)]{} & g^*\mathbb{S}(\mathcal{F}) \end{array}$$

   commutes.

Proof. That the square is commutative can be deduced from direct computations once the first part of the lemma has been proved. Let $g \in G$ and $\Omega \subseteq \mathscr{X}$. We have,

$$g^* \mathbb{S}(\mathcal{F})(\Omega) = \varinjlim_{\substack{V \supseteq g\Omega \\ V \subseteq \mathscr{X} \text{ open}}} \mathbb{S}(\mathcal{F})(V) = \mathbb{S}(\mathcal{F})(g\Omega).$$

Firstly we should verify that the morphism $\alpha_{g,\mathcal{F}}(\Omega)$ we are studying is well-defined. Let $s \in \mathbb{S}(g_* \mathcal{F})(\Omega)$, for $z \in g\Omega$,

$$\alpha_{g,\mathcal{F}}(\Omega)(s)(z) = s(g^{-1}z) \in (g_* \mathcal{F})_{F_{g^{-1}z}} = \mathcal{F}_{F_z}.$$

Moreover, let $z = g\bar{z} \in g\Omega$. Because s is a section of $\mathbb{S}(g_* \mathcal{F})$ on Ω, there exists an open neighborhood $V \subseteq \Omega \cap \mathrm{St}(F_{\bar{z}})$ of $\bar{z}$ such that for any $\bar{z}' \in V$,

$$s(\bar{z}') = s(\bar{z}) \circ g_* r_{F_{\bar{z}}}^{F_{\bar{z}'}} = s(\bar{z}) \circ r_{F_{g\bar{z}}}^{F_{g\bar{z}'}}.$$

Therefore, $gV \in g\Omega \cap \mathrm{St}(F_z)$ is an open neighborhood of z such that for any $z' \in gV$,

$$\alpha_{g,\mathcal{F}}(\Omega)(s)(z') = s(g^{-1}z') = s(\bar{z}) \circ r_{F_{g\bar{z}}}^{F_{gg^{-1}z'}} = \alpha_{g,\mathcal{F}}(\Omega)(s)(z) \circ r_{F_z}^{F_{z'}}.$$

This makes s a section of $\mathbb{S}(\mathcal{F})(g\Omega)$ and proves that $\alpha_{g,\mathcal{F}}(\Omega)$ is well-defined.

- Injectivity : This property easily comes from the definition of the morphism. Indeed, let $s \in \mathbb{S}(g_* \mathcal{F})(\Omega)$ be sent to 0 and let z be an element of Ω. Then we have that gz is an element of $g\Omega$ and $s(z) = s(g^{-1}gz) = 0$.

- Surjectivity : Let $s : g\Omega \to \coprod_{z \in g\Omega} \mathcal{F}^*_{F_z}$ be an element of $\mathbb{S}(\mathcal{F})(g\Omega)$. We construct the following map:

$$\begin{array}{rccc} s' : & \Omega & \to & \coprod_{z \in \Omega} \mathcal{F}^*_{F_{gz}} \\ & z & \mapsto & s(gz). \end{array}$$

  We need to verify that s' is an element of $\mathbb{S}(g_* \mathcal{F})(\Omega)$. Let $z \in \Omega$, $s'(z) = s(gz)$ is in $\mathcal{F}^*_{F_{gz}}$. Moreover, because $s \in \mathbb{S}(\mathcal{F})(g\Omega)$, there is an open neighborhood $V \subseteq g\Omega \cap \mathrm{St}(F_{gz})$ of gz such that for any $z'' \in V$,

$$s(z'') = s(gz) \circ r^{F_{gz}}_{F_{z''}}.$$

  We can now take $W := g^{-1}V \subseteq \Omega \cap \mathrm{St}(F_z)$. It is an open neighborhood of z such that for any $z' \in W$,

$$s'(z') = s(gz') = s(gz) \circ r^{F_{gz}}_{F_{gz'}} = s'(z) r^{F_{gz}}_{F_{gz'}}.$$

  The map s' is therefore a well-defined section of $\mathbb{S}(g_* \mathcal{F})$ and satisfy $\alpha_{g,\mathcal{F}}(\Omega)(s') = s$. This proves the surjectivity of the $\alpha_{g,\mathcal{F}}(\Omega)$.

□

Since the sheaves $g^* \mathbb{S}(\mathcal{F})$ and $\mathbb{S}(g_* \mathcal{F})$ are canonically isomorphic and that, for any morphism f of G-equivariant coefficient systems, the morphisms of sheaves $\mathbb{S}(g_* f)$ and $g^*\mathbb{S}(f)$ are also canonically isomorphic, we can verify that the family of isomorphisms

$$\left(d_g := \mathbb{S}(c_g) \circ \alpha^{-1}_{g,\mathcal{F}} : g^*\mathbb{S}(\mathcal{F}) \to \mathbb{S}(g_* \mathcal{F}) \to \mathbb{S}(\mathcal{F})\right)_{g \in G}$$

makes $\mathbb{S}(\mathcal{F})$ a G-equivariant sheaf.

Indeed, for $g, h \in G$, the diagram

$$\begin{array}{ccccc} & & h^*\mathbb{S}(g_* \mathcal{F}) & & \\ & \overset{h^*\alpha^{-1}_{g,\mathcal{F}}}{\nearrow} & & \overset{\alpha^{-1}_{h,g_* \mathcal{F}}}{\searrow} & \\ (gh)^* \mathbb{S}(\mathcal{F}) = h^* g^* \mathbb{S}(\mathcal{F}) & & \xrightarrow[\alpha^{-1}_{gh,\mathcal{F}}]{} & & \mathbb{S}((gh)_* \mathcal{F}) = \mathbb{S}(h_* g_* \mathcal{F}) \end{array}$$

commutes, as can be verified by a direct computation.

Then,

$$\begin{aligned}
d_{gh} &= \mathbb{S}(c_{gh}) \circ \alpha_{gh,\mathcal{F}}^{-1}, \\
&= \mathbb{S}(h_* c_g \circ c_h) \circ \alpha_{gh,\mathcal{F}}^{-1}, \\
&= \mathbb{S}(c_h) \circ \mathbb{S}(h_* c_g) \circ \alpha_{gh,\mathcal{F}}^{-1}, \\
&= \mathbb{S}(c_h) \circ \alpha_{h,\mathcal{F}}^{-1} \circ \alpha_{h,\mathcal{F}} \circ \mathbb{S}(h_* c_g) \circ \alpha_{gh,\mathcal{F}}^{-1}, \\
&= d_h \circ \alpha_{h,\mathcal{F}} \circ \mathbb{S}(h_* c_g) \circ \alpha_{gh,\mathcal{F}}^{-1}, \\
&= d_h \circ h^* \mathbb{S}(c_g) \circ \alpha_{h,g_* \mathcal{F}} \circ \alpha_{gh,\mathcal{F}}^{-1}, \text{ by Lemma 2.4 for } f = c_g \\
&= d_h \circ h^* \mathbb{S}(c_g) \circ h^* \alpha_{g,\mathcal{F}}, \\
&= d_h \circ h^* (\mathbb{S}(c_g) \circ \alpha_{g,\mathcal{F}}), \\
&= d_h \circ h^* d_g.
\end{aligned}$$

To make things easier for the rest of the thesis, we will use the isomorphisms $\alpha_{g,\mathcal{F}}$ as identifications and omit them from the notations.

2.4 Full faithfulness

To get the nice property of full faithfulness for our functor, see Theorem 2.11, we need to restrict the category we are working with. We recall that $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$ stands for the full subcategory of G-equivariant coefficient systems on $\mathscr{X}$ such that the R-module $\mathcal{F}_F$ is a smooth $P_F^\dagger$-representation and is finitely generated for all faces F of $\mathscr{X}$.

From now on we will work with the functor

$$\begin{array}{rccc}
\mathbb{S}: & \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X}) & \longrightarrow & \mathrm{Sh}_G(\mathscr{X}) \\
& \mathcal{F} & \mapsto & \mathbb{S}(\mathcal{F}) \\
& f : \mathcal{F} \to \mathcal{G} & \mapsto & \mathbb{S}(f) : \mathbb{S}(\mathcal{G}) \to \mathbb{S}(\mathcal{F}) .
\end{array}$$

We will also request R to be a quasi-Frobenius ring. By [Lam99, Theorem 15.11], this means that any finitely generated R-modules M is reflexive (i.e. the map $M \to M^{**}$ is an isomorphism of R-modules). This property will be essential in proving Theorem 2.11 since it will allow us to construct a quasi-inverse to $\mathbb{S}$.

As previously mentioned in Remark 2.2, for any coefficient system $\mathcal{F}$ the sheaf $\mathbb{S}(\mathcal{F})$ is weakly constructible. With the assumption that $\mathcal{F}$ is an object of $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$ we have that $\mathbb{S}(\mathcal{F})$ is in fact constructible and by the following lemma we also have that it is a G-equivariant sheaf in the sense of Schneider in [Sch98b, §4] (as discussed in Remark 1.18).

Lemma 2.5. *Let F be a face of $\mathscr{X}$ and z an element of F. Then, the map*

$$\mathrm{ev}_F : \ \mathbb{S}(\mathcal{F})(\mathrm{St}(F)) \ \to \ \mathcal{F}_F^*,$$

which evaluates the sections on z, is bijective, $P_F^\dagger$-equivariant and independent of the choice of z.

Proof. We already know that the map ev_F is independent of the choice of $z \in F$. In fact, this is exactly the content of Remark 2.3. It implies the $P_F^\dagger$-equivariance as for $g \in P_F^\dagger$, the product gz remains in the face F. Moreover, the surjectivity of ev_F can be proven exactly the same way as we proved the one of ev_z in Lemma 2.1. The only difference is that we are not interested in the germ but directly in the section constructed. Let me quickly recall the construction. We fix an element $t \in \mathcal{F}_F^*$. We define the map

$$\begin{array}{rccl} s: & \mathrm{St}(F) & \to & \coprod_{z \in \mathrm{St}(F)} \mathcal{F}_{F_z}^* \\ & z & \mapsto & (r_F^{F_z})^*(t). \end{array}$$

This is a well defined element of $\mathbb{S}(\mathcal{F})(\mathrm{St}(F))$. Over F, this is the constant section with value t which implies the required surjectivity.

It remains to prove the injectivity. Let $s \in \mathbb{S}(\mathcal{F})(\mathrm{St}(F))$ be a section which maps to 0. It means that for any $z \in F$, $s(z) = 0$. Let F' be another face of $\mathscr{X}$ such that $F \subseteq \bar{F}'$. Then there exists an element $z' \in F'$ such that $s(z') = 0$, by the local constancy of s. The independence of the choice of $z' \in F'$ of the map

$$\begin{array}{ccccc} \mathbb{S}(\mathcal{F})(\mathrm{St}(F)) & \xrightarrow{\mathrm{res}} & \mathbb{S}(\mathcal{F})(\mathrm{St}(F')) & \xrightarrow{ev_{F'}} & \mathcal{F}_{F'}^* \\ s & \longrightarrow & s|_{\mathrm{St}(F')} & \longrightarrow & s(z') \end{array}$$

implies that the section s is equal to 0 over F'. Since such result is true for all $F' \in \mathrm{St}(F)$, we get that $s = 0$ and therefore we conclude the proof of the lemma. □

Remark 2.6. We should remark that, if we fix a face $F \subseteq \mathscr{X}$ and $z \in F$, there is a canonical relation between the map ev_F and the map ev_z. Indeed, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{S}(\mathcal{F})(\mathrm{St}(F)) & \xrightarrow[\sim]{\mathrm{ev}_F} & \mathcal{F}_F^* \\ & {\scriptstyle \mathrm{can}}\searrow\sim \quad \sim\nearrow{\scriptstyle \mathrm{ev}_z} & \\ & \mathbb{S}(\mathcal{F})_z & \end{array} .$$

Proposition 2.7. *The contravariant functor*

$$\begin{array}{rccc} \mathbb{S}: & \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X}) & \longrightarrow & \mathrm{Sh}_G(\mathscr{X}) \\ & \mathcal{F} & \mapsto & \mathbb{S}(\mathcal{F}) \\ & f: \mathcal{F} \to \mathcal{G} & \mapsto & \mathbb{S}(f): \mathbb{S}(\mathcal{G}) \to \mathbb{S}(\mathcal{F}) \end{array}$$

is fully faithful.

Proof. Let $(\mathcal{F}, (r_{F'}^F)_{F' \subseteq \bar{F} \subseteq \mathscr{X}}, (c_g)_{g \in G})$ and $(\mathcal{G}, (\rho_{F'}^F)_{F' \subseteq \bar{F} \subseteq \mathscr{X}}, (\gamma_g)_{g \in G})$ be two objects of the category $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. We need to prove that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathbb{S}(\mathcal{G}), \mathbb{S}(\mathcal{F}))$. Let $(\mathrm{ev}_F)_{F \subseteq \mathscr{X}}$ (resp. $(\mathrm{ev}'_F)_{F \subseteq \mathscr{X}}$) be the isomorphisms introduced in Lemma 2.5 corresponding to the sheaf $\mathbb{S}(\mathcal{F})$ (resp. $\mathbb{S}(\mathcal{G})$). Let $\Lambda : \mathbb{S}(\mathcal{G}) \to \mathbb{S}(\mathcal{F})$ be a morphism in $\mathrm{Sh}_G(\mathscr{X})$. For any face $F \subseteq \mathscr{X}$, let f_F^* be the unique R-linear map such that the diagram

$$\begin{array}{ccc} \mathbb{S}(\mathcal{G})(\mathrm{St}(F)) & \xrightarrow[\sim]{\mathrm{ev}'_F} & \mathcal{G}_F^* \\ {\scriptstyle \Lambda(\mathrm{St}(F))}\downarrow & & \downarrow{\scriptstyle f_F^*} \\ \mathbb{S}(\mathcal{F})(\mathrm{St}(F)) & \xrightarrow[\mathrm{ev}_F]{\sim} & \mathcal{F}_F^* \end{array}$$

commutes. Since R is quasi-Frobenius and we are working with elements in $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$, the dual of $\mathcal{F}_F^*$ (resp. $\mathcal{G}_F^*$) is $\mathcal{F}_F$ (resp. $\mathcal{G}_F$) itself. We can therefore define the family of morphisms $f = (f_F)_{F\subseteq\mathscr{X}}$ such that for any face F in $\mathscr{X}$, $f_F : \mathcal{F}_F \to \mathcal{G}_F$ is the dual of the map f_F^*.

For now we will assume that f is a morphism in $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. We will prove this later in Lemma 2.10, i.e., we will show that the construction of f respects the restriction maps and the G-actions. Assuming this, to prove the full faithfulness of the functor $\mathbb{S}$, we need to prove that the morphism f as constructed is such that $\mathbb{S}(f) = \Lambda$ and that it is the only one satisfying this property.

Let us first prove the second statement. Let h be a morphism in $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. For $t \in \mathcal{G}_F^*$, we have,

$$\begin{aligned} (\mathrm{ev}'_F)^{-1}(t) &= (z \mapsto (r_F^{F_z})^*(t)), \\ \mathbb{S}(h)\circ(\mathrm{ev}'_F)^{-1}(t) &= (z \mapsto h_{F_z}^* \circ (r_F^{F_z})^*(t)), \\ \mathrm{ev}_F \circ \mathbb{S}(h)\circ(\mathrm{ev}'_F)^{-1}(t) &= h_F^*(t). \end{aligned}$$

This proves that $\mathbb{S}(h)$ is such that the diagram

$$\begin{array}{ccc} \mathbb{S}(\mathcal{G})(\mathrm{St}(F)) & \xrightarrow[\sim]{\mathrm{ev}'_F} & \mathcal{G}_F^* \\ {\scriptstyle \mathbb{S}(h)(\mathrm{St}(F))}\downarrow & & \downarrow{\scriptstyle h_F^*} \\ \mathbb{S}(\mathcal{F})(\mathrm{St}(F)) & \xrightarrow[\mathrm{ev}_F]{\sim} & \mathcal{F}_F^* \end{array}$$

is commutative. If we assume $\mathbb{S}(h) = \Lambda$ we get $h_F^* = f_F^*$ for all F and hence $h = f$. This implies the faithfulness of $\mathbb{S}$.

To prove that $\mathbb{S}(f) = \Lambda$ it is enough to prove it on the stalks. In other words, we want the diagram

$$\begin{array}{ccccc} \mathbb{S}(\mathcal{G})_z & \xrightarrow[\sim]{\mathrm{ev}'_z} & \mathcal{G}_{F_z}^* & \xleftarrow[\sim]{\mathrm{ev}'_z} & \mathbb{S}(\mathcal{G})_z \\ {\scriptstyle \mathbb{S}(f)_z}\downarrow & & \downarrow{\scriptstyle f_{F_z}^*} & & \downarrow{\scriptstyle \Lambda_z} \\ \mathbb{S}(\mathcal{F})_z & \xrightarrow[\mathrm{ev}_z]{\sim} & \mathcal{F}_{F_z}^* & \xleftarrow[\mathrm{ev}_z]{\sim} & \mathbb{S}(\mathcal{F})_z \end{array}$$

to commute, for any $z \in \mathscr{X}$. By Remark 2.6 we can deduce the commutativity of each of the squares from the commutativity of the previous diagrams in the proof. Moreover, the composition of the horizontal maps are both the identity which means that we have all the arguments to conclude the equality needed.

We proved the surjectivity and the injectivity of $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathbb{S}(\mathcal{G}),\mathbb{S}(\mathcal{F}))$ and so the full faithfulness of the functor $\mathbb{S}$. □

Remark 2.8. Due to the injectivity of the duality map $M \to M^{**}$ for any R-module M (see [Lam99, Theorem 15.11(1)]), the functor $\mathbb{S}(.)$ is faithful even on the category $\mathrm{Coeff}_G(\mathscr{X})$.

Remark 2.9. We need to make some quick remarks about the G-actions and the restriction maps before proving the next lemma. The previous proof gives us the equality

$$c_{g,F}^* = \mathrm{ev}_F \circ \mathbb{S}(c_g)(\mathrm{St}(F)) \circ (\mathrm{ev}_{gF})^{-1},$$

for $F \subseteq \mathscr{X}$ and $g \in G$. Fixing $F' \subseteq \bar{F} \subseteq \mathscr{X}$, we also have an analogous equality for the restriction maps :

$$(r_{F'}^F)^* = \mathrm{ev}_F \circ \mathrm{res} \circ (\mathrm{ev}_{F'})^{-1},$$

where res stands for the restriction map from $\mathbb{S}(\mathcal{F})(\mathrm{St}(F'))$ to $\mathbb{S}(\mathcal{F})(\mathrm{St}(F))$. Indeed, for $t \in \mathcal{G}_F^*$, we have,

$$\begin{aligned}(\mathrm{ev}_{F'})^{-1}(t) &= (z \mapsto (r_{F'}^{F_z})^*(t)),\\ \mathrm{res} \circ (\mathrm{ev}_{F'})^{-1}(t) &= (z \mapsto (r_{F'}^{F_z})^*(t)),\\ \mathrm{ev}_F \circ \mathrm{res} \circ (\mathrm{ev}_{F'})^{-1}(t) &= (r_{F'}^F)^*(t).\end{aligned}$$

Lemma 2.10. *We keep the notations introduced in the proof of Proposition 2.7. Then, for any $F' \subseteq \bar{F} \subseteq \mathscr{X}$ and any $g \in G$, the following two diagrams are commutative :*

$$\begin{array}{ccc} \mathcal{F}_F & \xrightarrow{r_{F'}^F} & \mathcal{F}_{F'} \\ {\scriptstyle f_F}\downarrow & & \downarrow{\scriptstyle f_{F'}} \\ \mathcal{G}_F & \xrightarrow[\rho_{F'}^F]{} & \mathcal{G}_{F'}, \end{array} \qquad \begin{array}{ccc} \mathcal{F}_F & \xrightarrow{c_g} & (g_*\mathcal{F})_F = \mathcal{F}_{gF} \\ {\scriptstyle f_F}\downarrow & & \downarrow{\scriptstyle (g_*f)_F = f_{gF}} \\ \mathcal{G}_F & \xrightarrow[\gamma_g]{} & (g_*\mathcal{G})_F = \mathcal{G}_{gF}. \end{array}$$

Proof. To prove these statements we will show the commutativity of the dual diagrams. To do so, we will use the construction of the map f and Remark 2.9. The commutativity of the first dual diagram then follows from the following equalities :

$$\begin{aligned} f_F^* \circ (\rho_{F'}^F)^* &= \mathrm{ev}_F \circ \Lambda(\mathrm{St}(F)) \circ (\mathrm{ev}'_F)^{-1} \circ (\rho_{F'}^F)^*,\\ &= \mathrm{ev}_F \circ \Lambda(\mathrm{St}(F)) \circ (\mathrm{ev}'_F)^{-1} \circ \mathrm{ev}'_F \circ \mathrm{res} \circ (\mathrm{ev}'_{F'})^{-1},\\ &= \mathrm{ev}_F \circ \mathrm{res} \circ \Lambda(\mathrm{St}(F')) \circ (\mathrm{ev}'_{F'})^{-1},\\ &= \mathrm{ev}_F \circ \mathrm{res} \circ (\mathrm{ev}_{F'})^{-1} \circ f_{F'}^*,\\ &= (r_{F'}^F)^* \circ f_{F'}^*. \end{aligned}$$

The commutativity of f with the G-actions follows from similar calculations and arguments, replacing the restriction maps by the corresponding G-action maps. □

We now turn back to the functor $\mathcal{F}(.) : \mathrm{Mod}_H \to \mathcal{C}$ introduced in section 1.3.

Let us write $\mathrm{Mod}_H^{\mathrm{fg}}$ for the subcategory of Mod_H consisting of the H-modules with underlying finitely generated R-modules. Since $\mathcal{F}(\mathrm{Mod}_H^{\mathrm{fg}}) \subseteq \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$ (see remark 1.15), we can look at the composition of functors $\mathbb{S} \circ \mathcal{F}(.)$.

Theorem 2.11. *The functor $\mathbb{S} \circ \mathcal{F}(.) : \mathrm{Mod}_H^{\mathrm{fg}} \to \mathrm{Sh}_G(\mathscr{X})$ is fully faithful. Moreover, the essential image of the functor $\mathbb{S} : \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X}) \to \mathrm{Sh}_G(\mathscr{X})$ is the full subcategory of constructible G-equivariant sheaves on $\mathscr{X}$.*

Proof. The first part follows directly from the full faithfulness of $\mathcal{F}(.)$ (see [Koh22, Theorem 3.21]) and of $\mathbb{S}$ (see Proposition 2.7). For the second part, let S be an arbitrary sheaf of R-modules on $\mathscr{X}$, we can associate to it a coefficient system whose module associated to a face F is $S(\mathrm{St}(F))^*$ and whose restriction map associated to faces $F' \subseteq \bar{F}$ are the dual of the restriction map $S(\mathrm{St}(F')) \to S(\mathrm{St}(F))$. Starting with this construction, the arguments in Chapter 2 can be generalized to show the following : We have a contravariant functor $\mathrm{Sh}_G(\mathscr{X}) \longrightarrow \mathrm{Coeff}_G(\mathscr{X})$ which restricts to a contravariant functor from the category of constructible G-equivariant sheaves of R-modules on $\mathscr{X}$ to $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. It is a quasi-inverse to $\mathbb{S}$, viewed as a functor from $\mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$ to the category of constructible G-equivariant sheaves of R-modules on $\mathscr{X}$. In fact, this observation is due to Schneider who claims it without proof in [Sch98b, Proposition 3.3]. □

2.5 Cohomology with compact support

Following the example of [SS97] and using results from [Koh22] we will associate to any coefficient system $\mathcal{F}$ a cochain complex and prove that the cohomology of a specific subcomplex is the same as the cohomology with compact support of $\mathbb{S}(\mathcal{F})$. This result and its proof are analogous to [SS97, Proposition VI.1.3], mentioned earlier in Proposition 1.21. In fact, before going on with the study of the functor $\mathbb{S}$ we would like to emphasize the parallels between our construction and the one introduced by Schneider and Stuhler in [SS97] (see Section 1.4). To clarify the relation between their construction and ours, we need the following lemma.

Lemma 2.12. *Let $F' \subseteq \bar{F}$ be faces of $\mathscr{X}$ and let V be a P_F-representation whose underlying R-module is finitely generated. Then the dual of the map $V^{I_F} \hookrightarrow V$ induces an isomorphism $(V^*)_{I_F} \cong (V^{I_F})^*$ of P_F-representations.*

Proof. The map $V^{I_F} \hookrightarrow V$ dualises to a P_F-equivariant map $V^* \to (V^{I_F})^*$. Since the I_F-action on $(V^{I_F})^*$ is trivial, the previous map induces a P_F-equivariant map $(V^*)_{I_F} \to (V^{I_F})^*$. To show that this map is in fact an isomorphism it is enough to show it for its dual map

$(V^{I_F})^{**} \to ((V^*)_{I_F})^*$. We have that

$$\begin{aligned}(V^{I_F})^{**} &\cong V^{I_F}.\\ ((V^*)_{I_F})^* &= \mathrm{Hom}_R((V^*)_{I_F}, R),\\ &\cong \mathrm{Hom}_{R[I_F]}(V^*, R),\\ &\cong (\mathrm{Hom}_R(V^*, R))^{I_F},\\ &= (V^{**})^{I_F},\\ &\cong V^{I_F}.\end{aligned}$$

The dual map is in fact the identity map on V^{I_F} and we can conclude $(V^*)_{I_F} \cong (V^{I_F})^*$. □

By choosing the right setting, i.e. working in the category $\mathcal{C}$ introduced in [Koh22], we can establish the following result.

Corollary 2.13. *If $M \in \mathrm{Mod}_H^{\mathrm{fg}}$, if $x \in \mathscr{X}$ is a vertex and if we set $\mathcal{F} := \mathcal{F}(M)$ then $\mathbb{S}(\mathcal{F})\,|_{\mathrm{St}(x)}$ is the sheaf $\underset{\approx}{\mathcal{F}_x^*}$ associated with the P_x-representation $\mathcal{F}_x^*$ (see Remark 1.20).*

Proof. Any coefficient system of the form $\mathcal{F}(M)$ for $M \in \mathrm{Mod}_H^{\mathrm{fg}}$ is in the category $\mathcal{C}$. Therefore, by [Koh22, Proposition 3.18], we know that the restriction $\mathcal{F}\,|_{\mathrm{St}(x)}$ is isomorphic to the fixed point coefficient system $\mathcal{F}_W = (W^{I_F})_{x\in\overline{F}}$ of the P_x-representation $W := \mathcal{F}_x$. For $F' \subseteq \bar{F} \subseteq \mathrm{St}(x)$, the restriction map $r_{F'}^F$ corresponds to the inclusion $W^{I_F} \hookrightarrow W^{I_{F'}}$. Using the previous lemma, the dual map $(r_{F'}^F)^* : (W^{I_{F'}})^* \to (W^{I_F})^*$ can be identified with the canonical map $(W^*)_{I_{F'}} \to (W^*)_{I_F}$. Therefore, the sections of $\mathbb{S}(\mathcal{F})$ on an open subset Ω of $\mathrm{St}(x)$ are given by

$$\begin{aligned}\mathbb{S}(\mathcal{F})\,|_{\mathrm{St}(x)}(\Omega) := \text{ maps } s : \Omega \to \coprod_{z\in\Omega} (W^*)_{I_{F_z}} \text{ such that}\\ (i)\ \forall z \in \Omega, s(z) \in (W^*)_{I_{F_z}},\\ (ii)\ \forall z \in \Omega, \exists V \subseteq \Omega \cap \mathrm{St}(F_z) \text{ open neighborhood of } z \text{ such that}\\ \forall z' \in V, s(z') \equiv s(z) \bmod I_{F_z},\end{aligned}$$

where $s(z') \equiv s(z) \bmod I_{F_z}$ means that the image of $s(z)$ by the surjection $(W^*)_{I_{F_z}} \twoheadrightarrow (W^*)_{I_{F_{z'}}}$ is $s(z')$. This shows that the restriction of $\mathbb{S}(\mathcal{F})$ to $\mathrm{St}(x)$ is $\underset{\approx}{W^*} = \underset{\approx}{\mathcal{F}_x^*}$. □

As mentioned previously, the goal of this section is to prove the following analog of [SS97, Proposition IV.1.3].

Proposition 2.14. *For all $q \geq 0$ and all $\mathcal{F} \in \mathrm{Coeff}_G(\mathscr{X})$, we have the equality*

$$H_c^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) = H^q(\bigoplus_{F\in\mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*).$$

One can see in the previous statement that we hope to compute the cohomology of the sheaf $\mathbb{S}(F)$ using the cohomology of a certain complex. Before going through the proof we need to explain in more detail what this complex is. We will start by giving an alternative definition of the complex $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F}), \partial_\bullet)$ introduced in Section 1.2. Keeping in mind that $\partial_{F'}^F(-c) = -\partial_{F'}^F(c)$ for (F, c) an oriented face and $F' \subseteq \bar{F}$, we can see the oriented chain complex a little differently. Let us fix for each face F of $\mathscr{X}$ an orientation that we will note c_F. For any $0 \leq i \leq d$, this gives an isomorphism of R-modules

$$\begin{array}{rccc} \Delta_i : & \mathcal{C}_c^{or}(\mathscr{X}_{(i)}, \mathcal{F}) & \longrightarrow & \bigoplus_{F \in \mathscr{X}_i} \mathcal{F}_F \\ & f & \mapsto & (f(F, c_F))_{F \in \mathscr{X}_i}. \end{array}$$

We claim that they form an isomorphism of complexes between $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F}), \partial_\bullet)$ and $\bigoplus_{F \in \mathscr{X}_\bullet} \mathcal{F}_F$ endowed with the differential $\delta_\bullet$ defined by

$$\delta_i((y_F)_{F \in \mathscr{X}_{i+1}}) = \left(\sum_{F' \subseteq \bar{F}} \epsilon_{F'}^F r_{F'}^F (y_F) \right)_{F' \in \mathscr{X}_i},$$

for any $0 \leq i \leq d-1$, for any $(y_F)_{F \in \mathscr{X}_{i+1}} \in \bigoplus_{F \in \mathscr{X}_{i+1}} \mathcal{F}_F$ and for $\epsilon_{F'}^F := \begin{cases} 1, & \text{if } \partial_{F'}^F(c_F) = c_{F'} \\ -1, & \text{if } \partial_{F'}^F(c_F) = -c_{F'} \end{cases}$.

Remark 2.15.
- These differentials are well defined. Indeed if $(y_F)_{F \in \mathscr{X}_{i+1}}$ is an element in the $(i+1)$-th component of the complex then there are only finitely many $y_F \neq 0$. Moreover, for any F such that $y_F \neq 0$, there are only finitely many $F' \subseteq \bar{F}$. This means that there are only finitely many $F' \in \mathscr{X}_i$ such that $\sum_{F' \subseteq \bar{F}} \epsilon_{F'}^F r_{F'}^F (y_F) \neq 0$.
- These differentials fulfill $\delta_{i-1} \circ \delta_i = 0$, this follows from the definition of $\delta_\bullet$ and the corresponding property of $\partial_\bullet$.

The next lemma will show in which way the complex appearing in Proposition 2.14 is well defined.

Lemma 2.16. *Under the identification*

$$\mathcal{C}_c^{or}(\mathscr{X}_{(i)}, \mathcal{F})^* \cong (\bigoplus_{F \in \mathscr{X}_i} \mathcal{F}_F)^* \cong \prod_{F \in \mathscr{X}_i} \mathcal{F}_F^*,$$

we have

$$\delta_i^*(\bigoplus_{F \in \mathscr{X}_i} \mathcal{F}_F^*) \subseteq \bigoplus_{F \in \mathscr{X}_{i+1}} \mathcal{F}_F^*.$$

Thus, $(\bigoplus_{F \in \mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*)$ *is a subcomplex of* $(\prod_{F \in \mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*)$ *and may be viewed, via the isomorphism* $\Delta_\bullet$*, as a subcomplex of* $(\mathcal{C}_c^{or}(\mathscr{X}_{(\bullet)}, \mathcal{F})^*, \partial_\bullet^*)$.

Proof. Let $(y_F)_{F \in \mathscr{X}_i}$ be an element of $\bigoplus_{F \in \mathscr{X}_i} \mathcal{F}_F^*$. Since we have that

$$\bigoplus_{F \in \mathscr{X}_i} \mathcal{F}_F^* \subseteq \prod_{F \in \mathscr{X}_i} \mathcal{F}_F^* \cong (\bigoplus_{F \in \mathscr{X}_i} \mathcal{F}_F)^* \cong \mathrm{Hom}_R(\bigoplus_{F \in \mathscr{X}_i} (\mathcal{F}_F), R),$$

we can view $(y_F)_{F\in\mathscr{X}_i}$ as a function $l \in \mathrm{Hom}_R(\bigoplus_{F\in\mathscr{X}_i}(\mathcal{F}_F), R)$ such that $l(\mathcal{F}_F) \neq 0$ only for finitely many $F \in \mathscr{X}_i$.
Set $l' = \delta_i^*(l) = l \circ \delta_i \in \mathrm{Hom}_R(\bigoplus_{F'\in\mathscr{X}_{i+1}}(\mathcal{F}_{F'}), R)$. We need to prove that $l'(\mathcal{F}_{F'}) = 0$ for almost all $F' \in \mathscr{X}_{i+1}$. But

$$S := \{F' \in \mathscr{X}_{i+1} | \exists F \in \mathscr{X}_i \text{ such that } F \subseteq \bar{F}' \text{ and } l(\mathcal{F}_F) \neq 0\}$$

is finite and if $F \notin S$, then $l'(\mathcal{F}_{F'}) = 0$. Therefore $\delta_i^*((y_F)_{F\in\mathscr{X}_i})$ is an element of $\bigoplus_{F\in\mathscr{X}_{i+1}} \mathcal{F}_F^*$ □

With these preparations at hand, we can finally prove Proposition 2.14. Our proof is essentially identical to the proof of [SS97, Proposition VI.1.3].

Proof of Proposition 2.14. Recall that we denote by $\mathscr{X}^q := \bigcup_{F\in\mathscr{X}_q} \bar{F}$ the q-skeleton of $\mathscr{X}$. We have a filtration

$$\mathscr{X} = \Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^d$$

of $\mathscr{X}$ by the open subsets $\Omega^{q+1} := \mathscr{X} \setminus \mathscr{X}^q$, setting $\Omega^0 := \mathscr{X}$.
Let $\mathcal{G}$ be a sheaf of abelian groups on the topological space $\mathscr{X}$ and let U be an open subset of $\mathscr{X}$. We note $i : U \hookrightarrow \mathscr{X}$ the inclusion map and $(i)_!$ the extension by zero from U to $\mathscr{X}$. By [God97, Equation 4.10 (4)], we have

$$H_c^\bullet(\mathscr{X}, i_! i^{-1}\mathcal{G}) = H_c^\bullet(U, \mathcal{G}|_U),$$

(in [God97], the notation is $\mathcal{G}_{\mathscr{X}\setminus A}$ for $i_! i^{-1}\mathcal{G}$, with $A = \mathscr{X} \setminus U$).
If $i_q : \Omega^q \hookrightarrow \mathscr{X}$ denotes the inclusion map, then we have a filtration of $\mathcal{G}$ by subsheaves:

$$\mathcal{G} \supseteq i_{1!} i_1^{-1}\mathcal{G} \supseteq i_{2!} i_2^{-1}\mathcal{G} \supseteq \ldots \supseteq i_{d!} i_d^{-1}\mathcal{G}.$$

On the level of cohomology with compact support, the spectral sequence of this filtration computes $H_c^\bullet(\mathscr{X}, \mathcal{G})$ and its E_1-terms are the cohomology with compact support of the subquotients of the filtration. By exactness of $i_!$ and the above result of [God97], the term $E_1^{n,m}$ is :

$$H_c^{n+m}(A^n, \mathcal{G}|_{A^n}) = \bigoplus_{F\in\mathscr{X}_n} H_c^{n+m}(F, \mathcal{G}|_F),$$

where $A^n = \Omega^{n-1} \setminus \Omega^n = \coprod_{F\in\mathscr{X}_n} F$. Therefore, if we apply this to $\mathcal{G} = \mathbb{S}(\mathcal{F})$, the spectral sequence reads

$$E_1^{n,m} = \bigoplus_{F\in\mathscr{X}_n} H_c^{n+m}(F, \mathbb{S}(\mathcal{F})|_F) \Rightarrow H_c^{n+m}(\mathscr{X}, \mathbb{S}(\mathcal{F})).$$

Since the restriction of the sheaf $\mathbb{S}(\mathcal{F})$ to any face F of $\mathscr{X}$ is constant with value $\mathcal{F}_F^*$ (see Lemma 2.1), we have

$$H_c^q(F, \mathbb{S}(\mathcal{F})|_F) \cong \begin{cases} \mathcal{F}_F^*, & \text{if } q = \dim(F) \\ 0, & \text{otherwise} \end{cases}.$$

Inserting this into the spectral sequence we obtain

$$H^q_c(\mathscr{X}, \mathbb{S}(\mathcal{F})) = H^q \left\{ \bigoplus_{F\in\mathscr{X}_0} \mathcal{F}^*_F \to \ldots \to \bigoplus_{F\in\mathscr{X}_d} \mathcal{F}^*_F \right\}.$$

Going through the constructions of the previous isomorphisms, one sees that the differentials on the right hand side are given by the $\delta^*_\bullet$, as claimed. □

2.6 First examples

Let us study the case $G = GL_2(\mathbb{Q}_p)$ and $R = \bar{\mathbb{F}}_p$.

In this setting, I is a pro-p Iwahori subgroup of $GL_2(\mathbb{Q}_p)$, H denotes the corresponding pro-p Iwahori–Hecke algebra and we have $X := \mathrm{ind}_I^{GL_2(\mathbb{Q}_p)}(\bar{\mathbb{F}}_p)$.

As mentionned in Section 1.3, the functor

$$(.)^I : \mathrm{Rep}^I_{\bar{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p)) \to \mathrm{Mod}_H$$

is an equivalence of categories, where $\mathrm{Rep}^I_{\bar{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$ denotes the full subcategory of all smooth $\bar{\mathbb{F}}_p$-linear $GL_2(\mathbb{Q}_p)$-representations generated by their pro-p-invariants. This is a deep result that Ollivier obtained in [Oll09].

Its quasi-inverse is given by the functor

$$X \otimes_H (.) : \mathrm{Mod}_H \to \mathrm{Rep}^I_{\bar{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p)).$$

In particular, this means that for any $V \in \mathrm{Rep}^I_{\bar{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$, we have $V \cong X \otimes_H M$ with $M := V^I$ a left H-module. In this case and as explained in [OS11, §6.3] (see also [Koh22, Proposition 4.14 and Proposition 4.16(iii)]), the oriented chain complex associated to $\mathcal{F}(M)$ gives an exact resolution

$$0 \to \mathcal{C}^{or}_c(\mathscr{X}_{(1)}, \mathcal{F}(M)) \xrightarrow{\partial_0} \mathcal{C}^{or}_c(\mathscr{X}_{(0)}, \mathcal{F}(M)) \to V \to 0$$

of V.

We now consider an admissible smooth irreducible $GL_2(\mathbb{Q}_p)$-representation V which is infinite dimensional over $\bar{\mathbb{F}}_p$. Let $\mathcal{C}_q$ denote $\mathcal{C}^{or}_c(\mathscr{X}_{(q)}, \mathcal{F}(M))$ for $q \in \{0; 1\}$. Passing to the long exact sequence in higher smooth duality as constructed by Kohlhaase in [Koh17, Definition 3.12], we obtain the exact sequence

$$0 \to S^0(V) \to S^0(\mathcal{C}_0) \xrightarrow{S^0(\partial_0)} S^0(\mathcal{C}_1) \xrightarrow{d} S^1(V)$$

As we took V to be irreducible and not of finite dimension on $\bar{\mathbb{F}}_p$, by [Koh17, Proposition 3.9], we have $S^0(V) = 0$. Which means that the map $S^0(\partial_0)$ is injective.

By Proposition 2.14, the complex $(\bigoplus_{F\in\mathscr{X}_\bullet}\mathcal{F}(M)^*_F,\delta^*_\bullet)$ computes the cohomology of the sheaf $\mathbb{S}(\mathcal{F}(M))$. We proved that this complex is a subcomplex of the dual of $(\mathcal{C}_\bullet,\partial_\bullet)$ (see Lemma 2.16). Since the representations $\bigoplus_{F\in\mathscr{X}_q}\mathcal{F}(M)^*_F$ are smooth for $q\in\{0;1\}$, it is also a subcomplex of the smooth dual complex $(S^0(\mathcal{C}_\bullet),S^0(\partial_\bullet))$. Therefore, we obtain the following vanishing result,

$$H^0_c(\mathscr{X},\mathbb{S}(\mathcal{F}(M)))=\ker(\delta^*_0)\subseteq\ker(S^0(\partial_0))=0.$$

Moreover, the connecting morphism d induces a morphism

$$H^1_c(\mathscr{X},\mathbb{S}(\mathcal{F}(M)))=\operatorname{coker}(\partial^*_0)\to\operatorname{coker}(S^0(\partial_0))\xrightarrow{\bar{d}}S^1(V)$$

of smooth $GL_2(\mathbb{Q}_p)$-representations, where $\bar{d}$ is an injection. Since $S^1(V)$ is irreducible (see [Koh17]) it seems very likely to us that $H^1_c(\mathscr{X},\mathbb{S}(\mathcal{F}(M)))\cong S^1(V)$.

This last comment raises the question of a more general and conceptual relation between the functors $H^q_c(\mathscr{X},\mathbb{S}(\mathcal{F}((.)^I)))$ and $S^q(.)$ for other representations as well as other groups.

Let us continue with the case of smooth admissible $\bar{\mathbb{F}}_p$-linear $GL_2(\mathbb{Q}_p)$-representations. In the case $q=0$ we would like to compare the functors $H^0_c(\mathscr{X},\mathbb{S}(\mathcal{F}((.)^I)))$ and $S^0(.)$. Let $V\in\operatorname{Rep}^I_{\bar{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$,

- $H^0_c(\mathscr{X},\mathbb{S}(\mathcal{F}(V^I)))$ is defined as the subspace of functions with compact support inside $\mathbb{S}(\mathcal{F}(V^I))(\mathscr{X})$. The following equalities makes $H^0_c(\mathscr{X},\mathbb{S}(\mathcal{F}(V^I)))$ a subrepresentation of $(V)^*$:

$$\begin{aligned}\mathbb{S}(\mathcal{F}(V^I))(\mathscr{X})&=H^0(\mathscr{X},\mathbb{S}(\mathcal{F}(V^I)));\\&\cong H_0(\mathscr{X},\mathcal{F}(V^I))^*,\ \text{by Proposition 3.5;}\\&\cong(X\otimes_H V^I)^*,\ \text{by [Koh22, Proposition 4.16(iii)];}\\&\cong V^*,\ \text{by [Oll09];}\end{aligned}$$

- $S^0(V)=\{\ell\in V^*|\ell\text{ is fixed by an open subgroup of }G\}$, it is the G-subrepresentation of V^* consisting of smooth vectors.

We wish to compare these two subfunctors of the dual functor $(.)^*$. To do so, we need to understand how elements of V^* can be viewed as global sections of the sheaf $\mathbb{S}(\mathcal{F}(V^I))$.

By [Koh22, Proposition 4.16(i)], the $\bar{\mathbb{F}}_p$-module $\mathcal{F}(V^I)_F$ associated to a face F of $\mathscr{X}$ is isomorphic to the module $X^{I_F}\otimes_H V^I$. Moreover, we have the map $\varpi_F:\mathcal{F}(V^I)_F\to V^{I_F}$. Now let $\ell\in V^*$. The corresponding global section of $\mathbb{S}(\mathcal{F}(V^I))$ is the map

$$\begin{array}{cccc}f_\ell: & \mathscr{X} & \longrightarrow & \coprod_{z\in\mathscr{X}}(X^{I_{F_z}}\otimes_H V^I)^*\\ & z & \mapsto & \ell\circ\iota_{F_z}\circ\varpi_{F_z},\end{array}$$

where $\iota_{F_z} : V^{I_{F_z}} \to V$ is the inclusion map. The result we obtain later in Chapter 3, more precisely Proposition 3.5, states that any global section of $\mathbb{S}(\mathcal{F}(V^I))$ is in fact equal to a map f_ℓ for a unique $\ell \in V^*$.

If we ask f_ℓ to be an element of $H_c^0(\mathscr{X}, \mathbb{S}(\mathcal{F}(V^I)))$ then its support is compact which means that there are finitely many faces $F_1, ..., F_n$ for which there exist $z \in \mathscr{X}$ with $F_z = F_i$ for some $1 \leq i \leq n$ and $f_\ell(z) \neq 0$. Therefore, f_ℓ is fixed by the open subgroup $\bigcap_{i=1}^n I_{F_i}$ of G and ℓ, which is fixed by the same subgroup, is smooth. It means that, in the case $q = 0$ we have that $H_c^0(\mathscr{X}, \mathbb{S}(\mathcal{F}(V^I)))$ is a subfunctor of the smooth dual functor $S^0(V)$.

We already have examples where these two functors are equal, namely when V is admissible, irreducible and infinite dimensional over $\bar{\mathbb{F}}_p$. However the injection

$$H_c^0(\mathscr{X}, \mathbb{S}(\mathcal{F}(V^I))) \hookrightarrow S^0(V)$$

is not always an equality. In fact we have the following counter-example.

Example 2.17. Let $V = \bar{\mathbb{F}}_p$ be the trivial $GL_2(\mathbb{Q}_p)$-representation. In this case, we have that the (smooth) dual representation $V^* = \bar{\mathbb{F}}_p = S^0(V)$ is again the trivial $GL_2(\mathbb{Q}_p)$-representation. Moreover, for any face F of $\mathscr{X}$, the map ϖ_F is surjective and the map ι_F is an isomorphism (using $V^I = V^{I_F} = V$). Therefore, if $f \in V^*$ is non-zero then it is also the case of $f_\ell(z)$ for all $z \in \mathscr{X}$. But in this case $supp(f_\ell) = \mathscr{X}$ is not compact, which means that $H_c^0(\mathscr{X}, \mathbb{S}(\mathcal{F}(V^I))) \hookrightarrow S^0(V)$ is strictly an injection. In fact, since V is one-dimensional we have

$$0 = H_c^0(\mathscr{X}, \mathbb{S}(\mathcal{F}(V^I))) \subsetneq S^0(V) = V^* = \bar{\mathbb{F}}_p.$$

Chapter 3

Verdier duality

Our objective in Section 2.5 was to establish a connection between the cohomology with compact support of the sheaf $\mathbb{S}(\mathcal{F})$ and the homology of the coefficient system $\mathcal{F}$. The outcome may not be entirely satisfactory, however, as the cohomology of the cochain complex $(\bigoplus_{F \in \mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*)$ seems hard to compute in general.

In the present chapter, we will focus on the cohomology $H^\bullet(\mathscr{X}, \mathbb{S}(\mathcal{F}))$ of $\mathbb{S}(\mathcal{F})$ and its relationship with the homology of $\mathcal{F}$. Drawing inspiration from Schneider's work in [Sch98a] and [Sch98b], our primary tool will be Verdier duality. To help the reader, we will provide an introduction to Verdier duality in Section 3.1 before presenting and proving our result in Section 3.2. This chapter has been greatly influenced by [KS90] and [Sch98b].

For the sake of clarity in this chapter, we have opted to use the notation A to represent the quasi-Frobenius ring we are working with, instead of R.

3.1 Reminders

For X a locally compact space, let $D(X)$ denote the derived category of the category $\mathrm{Sh}(X)$ of sheaves of A-modules on X. Let $D^b(X)$ (resp. $D^+(X)$, resp. $D^-(X)$) denote the full subcategory of $D(X)$ consisting of objects S such that $H^n(S) = 0$ for $|n| >> 0$ (resp. $n << 0$, resp. $n >> 0$). In particular, let $D^b(\mathscr{X})$ (resp. $D^+(\mathscr{X})$, resp. $D^-(\mathscr{X})$) denote the bounded (resp. bounded below, resp. bounded above) derived category of $\mathrm{Sh}(\mathscr{X})$. Finally, let $D^b(A)$, $D^+(A)$ and $D^-(A)$ denote the same for the category Mod_A of A-modules. Let us recall the following functors and relations.

Notations 3.1. Let X and Y be locally compact spaces and let $f : X \to Y$ be a continuous map. Let S and S' be two sheaves of A-modules on X and let T be a sheaf of A-modules on Y. We have the following functors:

- $\Gamma(X, -) : \mathrm{Sh}(X) \to \mathrm{Mod}_A$, the global section functor;
- $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, the direct image functor;
- $\mathrm{Hom}(-, -) : \mathrm{Sh}(X) \times \mathrm{Sh}(X) \to \mathrm{Mod}_A$, the external-Hom bifunctor;

- $\underline{\mathrm{Hom}}(-,-) : \mathrm{Sh}(X) \times \mathrm{Sh}(X) \to \mathrm{Sh}(X)$, the internal-Hom bifunctor. The image of (S, S') is defined by $\underline{\mathrm{Hom}}(S, S')(U) = \mathrm{Hom}(S|_U, S'|_U)$ for any $U \subseteq X$ open. We have that $\mathrm{Hom}(S, T) = \Gamma(X, \underline{\mathrm{Hom}}(S, T))$;

- $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, the proper direct image functor. The image of S under $f_!$ is the subsheaf of f_*S defined by $(f_!S)(V) = \{s \in S(f^{-1}(V)) | f : \mathrm{supp}(s) \to Y \text{ is proper}\}$ for any open subset V of Y (see [KS90, Equation 2.5.1]);

- $\Gamma_c(X, -) : \mathrm{Sh}(X) \to \mathrm{Mod}_A$ defined by $\Gamma_c(X, S) = \{s \in \Gamma(X, S) | \ \mathrm{supp}(s) \text{ is compact}\}$ (see [KS90, Equation 2.5.2]). We call this the global sections with compact supports of S on X.

As in [KS90, §2.6], we denote their corresponding derived functors as follows:

- $R\Gamma(X, -) : D^+(X) \to D^+(A)$,

- $Rf_* : D^+(X) \to D^+(Y)$,

- $R\,\mathrm{Hom}(-,-) : D^-(X)^{\mathrm{op}} \times D^+(X) \to D^+(A)$,

- $R\underline{\mathrm{Hom}}(-,-) : D^-(X)^{\mathrm{op}} \times D^+(X) \to D^+(X)$, note that $R\,\mathrm{Hom}(-,-)$ is the composition of $R\underline{\mathrm{Hom}}(-,-)$ and $R\Gamma(X, -)$.

- $Rf_! : D^+(X) \to D^+(Y)$.

In the case where we take Y to be the set with one element $\{\mathrm{pt}\}$ and f the map $a_X : X \to \{\mathrm{pt}\}$ then we can identify the category $\mathrm{Sh}(Y)$ with Mod_A and the category $D^b(Y)$ with $D^b(A)$. It brings us to the following identifications of functors.

$$a_{X*}(-) = \Gamma(X, -) \text{ and } a_{X!} = \Gamma_c(X, -).$$

Therefore their derived functors can also be identified.

The functors $\underline{\mathrm{Hom}}(-,-)$, f_*, and $\Gamma(X, -)$ are left exact. Since A is a quasi-Frobenius ring and, therefore, is injective as a right A-module, we have that the functor $\mathrm{Hom}_A(-, A)$ is exact. It is also the case for the functor dual $(-)^*$.

In [KS90, §3.1], Kashiwara and Schapira follow ideas of Verdier's (see [Ver66, §3]) to show that if $f_!$ has finite cohomological dimension then $Rf_! : D^+(X) \to D^+(Y)$ has a right adjoint $f^!$ (see [KS90, Theorem 3.1.5]). In particular, if $S \in D^b(X)$ and $T \in D^+(Y)$ then there is an isomorphism $R\,\mathrm{Hom}(S, f^!T) = R\,\mathrm{Hom}(Rf_!S, T)$ (see [KS90, Proposition 3.1.10]).

With this functor defined they constructed the Verdier dual $D_X S$ of a sheaf $S \in \mathrm{Sh}(X)$ as follows.

Definition 3.2 ([KS90]Definition 3.1.16). Let X be a locally compact space with finite c-soft dimension (in the sense of [KS90, Exercice II.9]) and S be an element of $D^b(X)$. Then $\omega_X = a_X^! A$ is called the dualizing complex of X and $D_X S = R\underline{\mathrm{Hom}}(S, \omega_X)$ is called the (Verdier) dual of S.

As stated in [Sch98b, §3], the duality functor D_X comes with a natural transformation of biduality $S \to D_X D_X S$.

Since the semi-simple Bruhat–Tits building $\mathscr{X}$ of G has the required properties, the formalism of Verdier duality applies and for $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$ we will write $D_{\mathscr{X}}\, \mathbb{S}(\mathcal{F})$ for the Verdier dual of the sheaf $\mathbb{S}(\mathcal{F})$.

It is important to note that, even though Schneider worked over $\mathbb{C}$, the results from [Sch98b, §1-3] hold over a more general setting, in particular they hold in our case. Let us recall some of his results since we will need them later on.

Let $D_c^b(\mathscr{X})$ denote the full triangulated subcategory of $D^b(\mathscr{X})$ consisting of all complexes whose cohomology sheaves are constructible (for the definition of constructible sheaves, see Remark 2.2). When restricted to this category, the natural transformation of biduality is in fact an isomorphism. This is the first statement of [Sch98b, Proposition 3.3]. We recall that, in particular, $\mathbb{S}(\mathcal{F})$ is constructible for $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. Therefore, we have the following equation,

$$D_{\mathscr{X}} D_{\mathscr{X}}\, \mathbb{S}(\mathcal{F}) \cong \mathbb{S}(\mathcal{F})\,.$$

Let $D^b(\mathrm{Coeff}(\mathscr{X}))$ denote the bounded derived category of $\mathrm{Coeff}(\mathscr{X})$. For a coefficient system $\mathcal{F} = ((\mathcal{F}_F)_F, (r_{F'}^F)_{F' \subseteq \bar{F}})$, we will denote by $\underline{\mathcal{F}_F}$ the constant sheaf on $\mathscr{X}$ with value $\mathcal{F}_F$. Let $\mathcal{F}_q$ denote the sheaf on $\mathscr{X}$ defined as follows : For all open subsets $\Omega \subseteq \mathscr{X}$ we have

$$\begin{aligned} \mathcal{F}_q(\Omega) := A\text{-module of all maps } s : \mathscr{X}_{(q)} \to \bigcup_{F \in \mathscr{X}_q} \underline{\mathcal{F}_F}(\Omega) \text{ such that} \\ (i)\ \forall (F,c) \in \mathscr{X}_{(q)}, s((F,c)) \in \underline{\mathcal{F}_F}(\Omega), \\ (ii)\ \forall (F,c) \in \mathscr{X}_{(q)}, s((F,-c)) = -s((F,c)). \end{aligned}$$

We have the following functor.

$$\begin{array}{rccc} \sigma: & D^b(\mathrm{Coeff}(\mathscr{X})) & \longrightarrow & D_c^b(\mathscr{X}) \\ & \mathcal{F} & \mapsto & [\mathcal{F}_d \xrightarrow{\delta_{d-1}} \ldots \xrightarrow{\delta_0} \mathcal{F}_0], \end{array}$$

where the complex of sheaves $[\mathcal{F}_d \xrightarrow{\delta_{d-1}} \ldots \xrightarrow{\delta_0} \mathcal{F}_0]$ obtained is put in degrees $-d$ through 0. The functor σ is, by [KS90, Theorem 8.1.11] for R noetherian (see also [Sch98b, Proposition 2.2] for $R = \mathbb{C}$), an equivalence of categories. Schneider studied the cohomology with compact support of $\sigma(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Coeff}(\mathscr{X})$. He obtained the following result.

Proposition 3.3 ([Sch98b, Corollary 2.4]). *For any coefficient system $\mathcal{F}$ on $\mathscr{X}$ we have*

$$H_\bullet(\mathscr{X}, \mathcal{F}) = H_c^{-\bullet}(\mathscr{X}, \sigma(\mathcal{F})).$$

Finally, in [Sch98b], Schneider introduced two functors that he both denoted by $(-)^*$:

$$\begin{array}{ccc} \mathrm{Coeff}(\mathscr{X}) & \rightleftarrows & w-Cons(\mathscr{X}) \\ \mathcal{F} & \mapsto & \mathcal{F}^* \text{ such that, } \mathcal{F}^*(\mathrm{St}(F)) := \mathcal{F}_F^* \\ S^* := (S(\mathrm{St}(F))^*)_{F\subseteq\mathscr{X}} & \leftarrow\!\shortmid & S. \end{array}$$

That the recipe from left to right really gives a functor on G-equivariant objects is what we worked out in detail in Chapter 2. In fact, the functor in the other direction gives a quasi-inverse on constructible sheaves. This means we have $\mathbb{S}(\mathcal{F})^* \cong \mathcal{F}$ for any $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. Using this construction, Schneider was able to establish a relation between σ and the Verdier duality functor.

Proposition 3.4 ([Sch98b, Proposition 3.2])**.** *For $S \in D_c^b(\mathscr{X})$, we have*

$$D_{\mathscr{X}}(S) = \sigma(S^*).$$

In particular, let $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$, we can apply this proposition for $S = \mathbb{S}(\mathcal{F})$ and we get

$$D_{\mathscr{X}}(\mathbb{S}(\mathcal{F})) = \sigma(\mathcal{F}).$$

We can now, start the study of the cohomology of $\mathbb{S}(\mathcal{F})$).

3.2 Cohomology

As previously mentioned, the goal of this section will be to relate the cohomology of $\mathbb{S}(\mathcal{F})$ to the homology of $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$. In fact, using the formalism of Verdier duality on $\mathscr{X}$, we will show the following proposition.

Proposition 3.5. *For all $q \geq 0$ and all $\mathcal{F} \in \mathrm{Coeff}_G^{\mathrm{fg}}(\mathscr{X})$, there is an isomorphism of (not necessarily smooth) A-linear G-representations*

$$H^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) \cong H_q(\mathscr{X}, \mathcal{F})^*.$$

Remark 3.6. By Proposition 2.14 we have a natural map

$$\begin{aligned} H_c^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) = H^q(\bigoplus_{F\in\mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*) \longrightarrow & H^q(\prod_{F\in\mathscr{X}_\bullet} \mathcal{F}_F^*, \delta_\bullet^*), \\ & = H^q((\bigoplus_{F\in\mathscr{X}_\bullet} \mathcal{F}_F, \delta_\bullet)^*), \\ & = (H_q(\bigoplus_{F\in\mathscr{X}_\bullet} \mathcal{F}_F, \delta_\bullet))^*, \text{ because } A \text{ is selfinjective,} \\ & = H_q(\mathscr{X}, \mathcal{F})^*. \end{aligned}$$

Once Proposition 3.5 is proved, we will be able to identify this map with the natural transformation $H_c^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) \to H^q(\mathscr{X}, \mathbb{S}(\mathcal{F}))$.

Before starting the proof of the proposition let us prove the following lemma.

Lemma 3.7. *Let J be an element of* $\mathrm{Sh}(\mathscr{X})$*. Then the cohomology of the Verdier dual of J is the dual of the cohomology with compact support of J. In other words, we have the following,*

$$H^q(\mathscr{X}, D_{\mathscr{X}} J) \cong \mathrm{Hom}_A(H^q_c(\mathscr{X}, J), A).$$

Proof. Let $J \in \mathrm{Sh}(\mathscr{X})$, then for $q \geq 0$, using the identifications mentionned in Section 3.1 we have

$$\begin{aligned} H^q(\mathscr{X}, D_{\mathscr{X}} J) &\cong H^q R\Gamma R\underline{\mathrm{Hom}}(J, a^!_{\mathscr{X}} A), \\ &\cong H^q R\,\mathrm{Hom}(R\Gamma_c J, A), \\ &\cong H^q\,\mathrm{Hom}(R\Gamma_c J, A), \\ &\cong \mathrm{Hom}_A(H^q R\Gamma_c J, A), \\ &\cong \mathrm{Hom}_A(H^q_c(\mathscr{X}, J), A). \end{aligned}$$

We will briefly explain each step. The first and last isomorphisms come from definitions and notations. More precisely, the definition of $D_{\mathscr{X}} J := R\underline{\mathrm{Hom}}(J, a^!_{\mathscr{X}} A)$ and the fact that for $q \geq 0$ and $F \in \mathrm{Sh}(\mathscr{X})$, $H^q(\mathscr{X}, F) = H^q R\Gamma(\mathscr{X}, F)$ and $H^q_c(\mathscr{X}, F) = H^q R\Gamma_c(\mathscr{X}, F)$. The third and fourth isomorphisms come from the property of selfinjectivity of A. Finally, the second isomorphism comes from the fact that $a^!_{\mathscr{X}}$ is right adjoint to $R\Gamma_c$ and from the fact that for $F, G \in D^b(\mathscr{X})$, $R\Gamma(\mathscr{X}, R\underline{\mathrm{Hom}}(F, G)) = R\,\mathrm{Hom}(F, G)$ (see [KS90, Equation 2.6.4]). □

We can go on with the proof of Proposition 3.5.

Proof. Using the previous section and applying the previous lemma to $J = D_{\mathscr{X}}\, \mathbb{S}(\mathcal{F})$, we get

$$\begin{aligned} H^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) &\cong H^q(\mathscr{X}, D_{\mathscr{X}} D_{\mathscr{X}}\, \mathbb{S}(\mathcal{F})), \\ &\cong \mathrm{Hom}_A(H^q_c(\mathscr{X}, D_{\mathscr{X}}\, \mathbb{S}(\mathcal{F})), A), \\ &\cong \mathrm{Hom}_A(H^q_c(\mathscr{X}, \sigma(\mathcal{F})), A), \\ &\cong \mathrm{Hom}_A(H_q(\mathscr{X}, \mathcal{F}), A), \\ &\cong H_q(\mathscr{X}, \mathcal{F})^*. \end{aligned}$$

In particular, the fourth isomorphism comes from [Sch98b, Corollary.2.4], see Proposition 3.3. □

As an application of Proposition 3.5 one gets the following result.

Corollary 3.8. *Assume that p is nilpotent in A and that the semisimple rank of $\mathbb{G}$ is equal to one. If $\mathcal{F} = \mathcal{F}(M)$ for some $M \in \mathrm{Mod}^*(H)$, then*

$$H^q(\mathscr{X}, \mathbb{S}(\mathcal{F})) \cong \begin{cases} H_0(\mathscr{X}, \mathcal{F})^* & \textit{if } q = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Follows from Proposition 3.5 and [Koh22, Proposition 4.14]. □

Bibliography

[AHV18] Noriyuki Abe, Guy Henniart, and Marie-France Vignéras. "On pro-p-Iwahori invariants of R-representations of reductive p-adic groups". In: *Represent. Theory* 22 (2018), pp. 119–159. ISSN: 1088-4165. DOI: 10.1090/ert/518. URL: https://doi.org/10.1090/ert/518.

[BL94] Laure Barthel and Ron Livné. "Irreducible modular representations of $GL(2)$ of a local field". In: *Duke Math. J.* 75.2 (1994), pp. 261–292. DOI: 10.1215/S0012-7094-94-07508-X. URL: https://doi.org/10.1215/S0012-7094-94-07508-X.

[BL95] Laure Barthel and Ron Livné. "Modular representations of $GL(2)$ of a local field : the ordinary, unramified case". In: *J. Number Theory* 55 (1995), pp. 1–27. URL: https://doi.org/10.1006/jnth.1995.1124.

[Ber84] Joseph Bernstein. "Le centre de Bernstein". In: *Travaux en Cours, Representations of reductive groups over a local field*. Ed. by Pierre Deligne. Paris: Hermann, 1984, pp. 1–32.

[Bre03a] Christophe Breuil. "Sur quelques représentations modulaires et p-adiques de $GL_2(Qp)$: I". In: *Compositio Mathematica* 138.2 (2003), pp. 165–188. DOI: 10.1023/A:1026191928449.

[Bre03b] Christophe Breuil. "Sur quelques représentations modulaires et p-adiques de $GL_2(Qp)$: II". In: *Journal of the Institute of Mathematics of Jussieu* 2.1 (2003), pp. 23–58. DOI: 10.1017/S1474748003000021.

[BT72] François Bruhat and Jacques Tits. *Groupes réductifs sur un corps local : I. Schémas en groupes. Données radicielles valuées.* Math. IHÉS, 1972.

[BT84] François Bruhat and Jacques Tits. *Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée*. Math. IHÉS, 1984.

[Cab90] Marc Cabanes. "A criterion for complete reducibility and some applications". In: *Astérisque* 181-182 (1990), pp. 93–112. ISSN: 0303-1179,2492-5926.

[Dol95] Albrecht Dold. *Lectures on Algebraic Topology*. Springer Berlin, Heidelberg, 1995.

[God97] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann Éditeurs, 1997.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*. Springer Berlin, Heidelberg, 1990.

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger. "Serre's modularity conjecture (I)". In: *Inventiones mathematicae* 178 (2009), pp. 485–504. DOI: 10.1007/s00222-009-0205-7.

[Koh17] Jan Kohlhaase. "Smooth duality in natural characteristic". In: *Adv. Math.* 317 (2017), pp. 1–49. ISSN: 0001-8708. DOI: 10.1016/j.aim.2017.06.038. URL: https://www.sciencedirect.com/science/article/pii/S0001870817301810.

[Koh22] Jan Kohlhaase. "Coefficient systems on the Bruhat-Tits building and pro-p Iwahori-Hecke modules". In: *Mem. Amer. Math. Soc.* 279.1374 (2022), pp. v+69. ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/1374. URL: https://doi.org/10.1090/memo/1374.

[Koz15] Karol Kozioł. "Pro-p-Iwahori Invariants for SL_2 and L-Packets of Hecke Modules". In: *Int. Math. Res. Not.* 2016.4 (2015), pp. 1090–1125. ISSN: 1073-7928. DOI: 10.1093/imrn/rnv162. URL: https://doi.org/10.1093/imrn/rnv162.

[Lam99] Tsit Yuen Lam. *Lectures on Modules and Rings*. Springer New York, NY, 1999.

[MS13] Alberto Mínguez and Vincent Sécherre. "Représentations banales de $GL_m(D)$". In: *Compositio Math.* 149.4 (2013), pp. 679–704. DOI: 10.1112/S0010437X12000590.

[MS14a] Alberto Mínguez and Vincent Sécherre. "Représentations lisses modulo ℓ de $GL_m(D)$". In: *Duke Math. J.* 163.4 (2014), pp. 795–887. DOI: 10.1215/00127094-2430025.

[MS14b] Alberto Mínguez and Vincent Sécherre. "Unramified ℓ-modular representations of $GL(n; F)$ and its inner forms". In: *Int. Math. Res. Not.* 163.8 (2014), pp. 2090–2118. DOI: 10.1093/imrn/rns278.

[Oll09] Rachel Ollivier. "Le foncteur des invariants sous l'action du pro-p-Iwahori de $GL_2(F)$". In: *J. reine angew. Math.* 2009.635 (2009), pp. 149–185. ISSN: 1073-7928. DOI: 10.1515/CRELLE.2009.078. URL: https://doi.org/10.1515/CRELLE.2009.078.

[OS14] Rachel Ollivier and Peter Schneider. "Pro-p Iwahori-Hecke algebras are Gorenstein". In: *J. Inst. Math. Jussieu* 13.4 (2014), pp. 753–809. ISSN: 1474-7480,1475-3030. DOI: 10.1017/S1474748013000303. URL: https://doi.org/10.1017/S1474748013000303.

[OS18] Rachel Ollivier and Peter Schneider. "A canonical torsion theory for pro-p Iwahori-Hecke modules". In: *Adv. Math.* 327 (2018), pp. 52–127. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2017.06.013. URL: https://doi.org/10.1016/j.aim.2017.06.013.

[OS11] Rachel Ollivier and Vincent Sécherre. *Modules universels de GL(3) sur un corps p-adique en caractéristique p*. 2011. arXiv: 1105.2957 [math.RT].

[Pas04] Vytautas Paskunas. "Coefficient systems and supersingular representations of $GL_2(F)$". In: *Mémoires Soc. Math. France* 99 (2004), pp. 1–90. ISSN: 0249-633-X. DOI: 10.24033/msmf.412.

[Sch98a] Peter Schneider. "Equivariant Homology for Totally Disconnected Groups". In: *J. of Alg.* 203.1 (1998), pp. 50–68. ISSN: 0021-8693. DOI: 10.1006/jabr.1997.7309.

[Sch98b] Peter Schneider. "Verdier duality on the building". In: *J. reine angew. Math.* 494 (1998), pp. 205–218. DOI: 10.1515/crll.1998.008.

[SS97] Peter Schneider and Ulrich Stuhler. "Representation theory and sheaves on the Bruhat Tits building". In: *Inst. Hautes Études Sci. Publ. Math.* 85 (1997), pp. 97–191. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1997__85__97_0.

[TW95] Richard Taylor and Andrew Wiles. "Ring-Theoretic Properties of Certain Hecke Algebras." In: *Annals of Math.* 141.3 (1995), pp. 553–572. DOI: 10.2307/2118560. URL: https://www.jstor.org/stable/2118560.

[Tit79] Jacques Tits. "Reductive groups over local fields". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Vol. XXXIII. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1979, pp. 29–69. ISBN: 0-8218-1435-4.

[Ver66] Jean-Louis Verdier. "Dualité dans la cohomologie des espaces localement compacts". fr. In: *Séminaire Bourbaki : années 1964/65 1965/66, exposés 277-312*. Séminaire Bourbaki 9. talk:300. Société mathématique de France, 1966. URL: http://www.numdam.org/item/SB_1964-1966__9__337_0/.

[Vig96] Marie-France Vignéras. *Représentations ℓ-modulaires d'un groupe réductif p-adique avec $\ell \neq p$*. Vol. 137. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, xviii and 233. ISBN: 0-8176-3929-2.

[Wil95] Andrew Wiles. "Modular elliptic curves and Fermat's Last Theorem". In: *Annals of Math.* 141.3 (1995), pp. 443–551. DOI: 10.2307/2118559. URL: https://www.jstor.org/stable/2118559.