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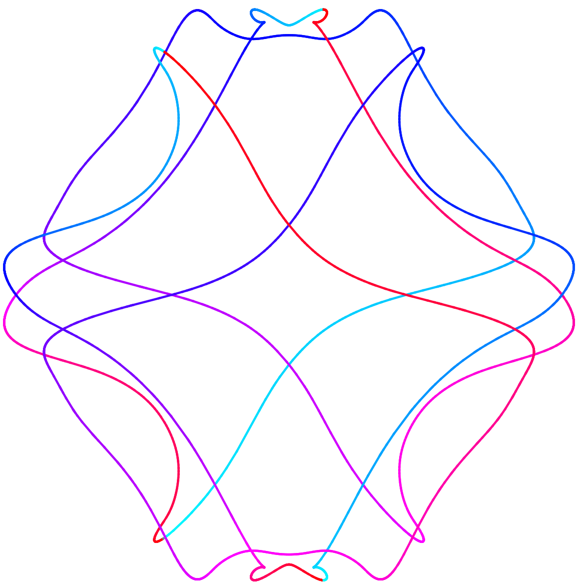
Statistical Topology

The Winding Number in One-Dimensional Chiral Systems



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Abstract

Topological insulators and topological superconductors are characterized by edge states that evade disorder-induced localization. The number of these states is determined by a topological invariant. Whether a topologically non-trivial phase is possible depends on the symmetries and the dimension of the system. The topological invariant relevant for one-dimensional systems with chiral symmetry is the winding number.

In this work, we perform a statistical analysis of the winding number in the framework of random matrix theory. Random matrix theory is known to produce universal results for systems with a sufficient degree of complexity in the limit of large matrix dimensions. In the context of solid state physics, this complexity corresponds to disorder, i.e. spatially inhomogeneous perturbations of the system parameters. In order to conduct our study, we first set up a parametric random matrix model with chiral symmetry for the Bloch Hamiltonian. In addition to chiral symmetry, we also classify our model based on the presence or absence of time reversal invariance. Specifically, we calculate the correlations of the winding number density, which yield the statistical moments of the winding number upon integration, as well as the distribution of the winding number.

On a technical level, we trace the topological problem back to a spectral one, which renders the toolbox of random matrix theory applicable. In doing so, we encounter the spherical ensemble of random matrices, which, unlike the classical ensembles of random matrix theory, does not follow a Gaussian matrix probability distribution. We employ different methods of random matrix theory to carry out the ensemble averages. In particular, we work with a technique that is related to the supersymmetry method of random matrix theory. It exploits supersymmetry structures without reformulating the problem in superspace and is therefore also referred to as supersymmetry without supersymmetry.

Zusammenfassung

Topologische Isolatoren und topologische Supraleiter zeichnen sich durch Randzustände aus, die sich der durch Unordnung verursachten Lokalisierung entziehen. Die Anzahl dieser Zustände wird durch eine topologische Invariante bestimmt. Ob eine topologisch nicht-triviale Phase möglich ist, hängt von den Symmetrien und der Dimension des Systems ab. Die für eindimensionale Systeme mit chiraler Symmetrie relevante topologische Invariante ist die Windungszahl.

In dieser Arbeit führen wir eine statistische Analyse der Windungszahl im Rahmen der Zufallsmatrixtheorie durch. Zufallsmatrixtheorie ist in der Lage universelle Ergebnisse für Systeme mit einem ausreichenden Grad an Komplexität im Limes großer Matrixdimensionen zu liefern. Im Falle der Festkörperphysik entspricht diese Komplexität der Unordnung, d.h. räumlich inhomogenen Störungen der Systemparameter. Zur Durchführung unserer Analyse stellen wir zunächst ein parametrisches Zufallsmatrixmodell mit chiraler Symmetrie für den Bloch-Hamiltonian auf. Neben der chiralen Symmetrie klassifizieren wir unser Modell auch anhand der Zeitumkehrinvarianz bzw. der Abwesenheit dieser. Wir berechnen die Korrelationen der Windungszahldichte, deren Integrale die statistischen Momente der Windungszahl ergeben, sowie die Verteilung der Windungszahl.

Auf technischer Ebene führen wir das topologische Problem auf ein spektrales Problem zurück, so dass die Methoden der Zufallsmatrixtheorie anwendbar werden. Dabei stoßen wir auf das sphärische Ensemble von Zufallsmatrizen, welches anders als die klassischen Ensembles der Zufallsmatrixtheorie keiner Gaußschen Matrixwahrscheinlichkeitsverteilung folgt. Wir verwenden verschiedene Methoden der Zufallsmatrixtheorie um die Ensemblemittelwerte zu berechnen. Insbesondere arbeiten wir mit einer Methode, die mit der Supersymmetriemethode der Zufallsmatrixtheorie verwandt ist. Diese Methode nutzt Strukturen der Supersymmetrie aus, ohne das Problem auf den Superraum abzubilden, und wird daher auch als Supersymmetrie ohne Supersymmetrie bezeichnet.

List of publications

Parts of this thesis are included in the following contributions:

[1] P. Braun, N. Hahn, D. Waltner, O. Gat and T. Guhr. *Winding Number Statistics of a Parametric Chiral Unitary Random Matrix Ensemble.* J. Phys. A: Math. Theor. **55** 224011 (2022)

[2] N. Hahn, M. Kieburg, O. Gat and T. Guhr. *Winding Number Statistics for Chiral Random Matrices: Averaging Ratios of Determinants With Parametric Dependence*. J. Math. Phys. **64** 021901 (2023)

[3] N. Hahn, M. Kieburg, O. Gat and T. Guhr. *Winding Number Statistics for Chiral Random Matrices: Averaging Ratios of Parametric Determinants in the Orthogonal Case*. J. Math. Phys. **64** 111902 (2023)

Contents

Chapter 1

Introduction

TOPOLOGY is the branch of mathematics that studies spatial objects beyond geometric concepts like angles and lengths [4]. Instead, it is concerned with properties that do not change under continuous deformations, called topological invariants. Figuratively speaking, these transformations correspond to bending or stretching the object, whereas tearing or gluing are not allowed. In figure 1.1 an example of such a transformation is depicted. A conventional cup with one handle can be continuously deformed into a torus. In this case, the topological invariant is the number of holes in the object.

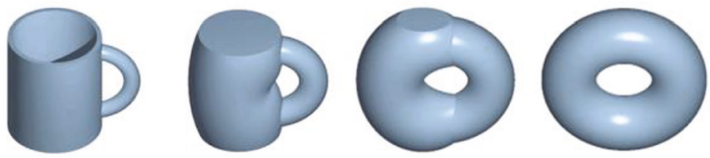


Figure 1.1: The prime example of topological invariance: The amount of holes is unchanged when deforming the torus into a cup or vice versa. Figure taken from [5] via [6].

Topology appears in various physical theories. This ranges from Hamiltonian mechanics, where the topology of the phase space provides information about the dynamics of the system [7], to yet unsolved questions in cosmology about the topology of spacetime [8, 9]. In recent years, topology has experienced a wave of renewed attention, originating from the discovery of topological condensed matter [10–15]. This is also the motivation for the present work.

The starting point for this field was the discovery of the quantum Hall effect [16–19]. The Hall resistance of a two-dimensional electron gas at low temperature does not exhibit the classically expected proportional growth in the external magnetic field, but is quantized instead. This peculiarity is explained by the formation of electronic states, that are localized to the edges of the sample [20–22]. While the edge states allow for dissipationless electronic transport, the interior states cannot contribute to a current. One speaks of a topological insulator, a material that is conducting on its boundary, but insulating in its interior. Varying the magnetic field may change the number of edge states, which can be measured as an increase or decrease in Hall resistance.

But where is the link to topology? To uncover this link, one has to distinguish between the system with open boundary conditions, in which edge states can occur, and the corresponding system with closed boundary conditions, commonly referred to as the bulk. As it turns out, the number of edge states in the open system is determined by a topological invariant, which is defined for the bulk system. Although real space representations for the topological invariant exist [23–26], the bulk Hamiltonian is commonly considered in momentum space. In the presence of discrete translation invariance, Bloch's theorem applies, and the Hamiltonian depends parametrically on the crystal momenta in the Brillouin zone. The eigenstates of the Bloch Hamiltonian form a vector bundle over the Brillouin zone, which may be topologically non-trivial. The topological characterization of vector bundles has been a long-standing achievement of mathematics, culminating in general methods for computing topological invariants, such as Chern-Weil theory [4]. Once topological non-triviality is established, one finally has to argue that the topological invariant, defined for the bulk Hamiltonian, is related to the number of edge states in the open system. This relation is known as bulk-boundary correspondence, which has only recently been shown to hold under general conditions [27, 28].

Symmetry considerations play a pivotal role for the existence of a non-trivial topology. In the quantum Hall effect, time reversal invariance is broken by the external magnetic field. In contrast, for a related topological phenomenon, the quantum spin Hall effect, in which a quantized spin current, but a vanishing net electronic current is observed, time reversal invariance must be preserved [29, 30]. Excluding, for the time being, all spatial symmetries like inversion or discrete rotations, we are left with three fundamental symmetries: time reversal invariance, the particle-hole constraint and chiral symmetry. The particle-hole constraint is present for instance in superconductors. Chiral symmetry arises as a combination of the former with time reversal invariance. The Altland-Zirnbauer classification distinguishes between ten symmetry classes on the basis of these fundamental symmetries and is therefore also known as the tenfold way [31, 32]. In this framework, a periodic table of topological insulators and topological superconductors can be established [33–36], which predicts whether a non-trivial topology is possible, depending on the symmetry class and the dimension of the system, and which is valid for Fermionic systems that are non-interacting except for a mean-field theory. While topological insulators are characterized by electronic edge states, their superconducting counterparts host collective electron-hole excitations, so-called Bogolyubov quasiparticles, on their boundary [13, 36–38].

Generally, topological invariants may change when varying the system parameters, such as the external magnetic field in the quantum Hall effect. Assuming that the symmetries are preserved, this coincides with a closing of the insulating energy gap. This is a topological phase transition, which differs from an ordinary one in that there is no symmetry breaking involved [39]. In other words, two Hamiltonians with identical symmetries belong to the same topological phase if they can be deformed into each other while maintaining the gap. However, the topological invariant may also change while maintaining the gap. To do this, one has to break the symmetry, temporarily destroying the topological state, and then restoring it, obtaining another topological phase [14].

In light of this, a heuristic explanation for bulk-boundary correspondence can be given. The energy gap necessarily has to close at the interface of two materials in different topological phases, for instance between a topological insulator and air (which is a trivial insulator). This leads to the emergence of states at the Fermi level that are localized to the interface, i.e. the boundary of the topological phase. These are precisely the aforementioned topological edge states.

Disorder plays a special role in topological condensed matter. Spatially inhomogeneous perturbations of the system parameters break all spatial symmetries, rendering the Altland-Zirnbauer classification exhaustive. At the same time, it prevents the definition of the topological invariant via the Bloch Hamiltonian. Furthermore, disorder usually leads to localization [40,41], putting the previously established conductive properties of topological matter into question. Nevertheless, topological edge states turn out to be stable against disorder as long as symmetries are protected, and the energy gap stays finite [27,42,43].

The latter property is also what makes topological matter interesting for novel technological applications, most notably in quantum information [44–48]. Thus, the field of topological matter extends beyond theoretical considerations. Topological insulators and their electronic edge states have been studied extensively in numerous experiments [49–53]. Surprisingly, these materials have even been found to occur naturally in the Earth's crust [54]. However, in the case of topological superconductors and their quasiparticle edge states, it seems to be more difficult to obtain direct evidence, and research here is still ongoing [55,56].

The present work contributes to the field of disordered topological matter. We consider one-dimensional systems with chiral symmetry, for which the above-mentioned periodic table predicts a non-trivial topology. The relevant topological invariant in these symmetry classes is the winding number [14, 26, 35, 36]. In general, the topological invariant is sensitive to the disorder configuration, which motivates a statistical description in which the invariant becomes a random variable. We realize the disorder by a random matrix model for the Bloch Hamiltonian. Our justification to stick to the Bloch Hamiltonian despite the presence of disorder is that we may assume that the disorder is itself periodic with a large period, so that the disordered system remains periodic with a large disordered unit cell. Under this assumption, the winding number can still be defined as a property of the Bloch Hamiltonian, eventually taking the limit of a large disorder period.

The idea of random matrix theory is to substitute the Hamiltonian (or other operators relevant for the description of the system) with an ensemble of random matrices. The only information about the system entering the model are its symmetries. It is a versatile tool for the description of complex systems and the long-standing experience is that it is often capable of modeling universal statistical properties in the limit of large matrix dimensions [57, 58]. It found application in various areas of physics such as condensed matter physics [31, 37, 59, 60], quantum chaos [61] and quantum chromodynamics [62–68]. Since the Bloch Hamiltonian depends on the crystal momenta we employ a parametric random matrix model. Such random matrix models have been considered before in the context of disordered systems under the influence of a magnetic flux, where spectral properties instead of topological ones were examined [69–72]. Furthermore, there are studies, some of which were partly developed in parallel and in close communication with us, which carry out a statistical analysis of the Chern number, the topological invariant relevant to the quantum Hall effect [73–75]. To our knowledge, a statistical analysis of the winding number in the context of random matrix theory has not been performed before.

The outline of this thesis is as follows. Chapter 2 covers the aspects of topological matter required for the later chapters. In detail, this is the tenfold way symmetry classification of topological insulators and topological superconductors and the winding number as the topological invariant for one-dimensional chiral symmetric systems. We illustrate the concepts developed using the example of the Kitaev chain, a toy model of a one-dimensional superconductor. Chapter 3 is an introduction to random matrix theory. We present its role in physics and define some basic concepts. We develop these using the example of random matrix ensembles, which we encounter in the later chapters. In addition, we discuss random matrix models with parametric dependence and a method that we use to evaluate certain ensemble averages later on. Chapter 4 is based on [1]. We define our random matrix model and map the topological problem of the winding number statistics to a spectral problem concerning a certain random matrix ensemble, referred to as the spherical ensemble. This ensemble has been analyzed in several works and does not follow a Gaussian matrix probability distribution like the ensembles of classical random matrix theory [76–79]. We calculate the probability distribution of the winding number and the correlation functions of the winding number density, which yield the moments of the winding number upon integration over the Brillouin zone. Chapter 5 is based on [2] and [3]. We generalize the random matrix model from chapter 4 to two further symmetry classes with time reversal invariance. We formulate a generating function for the correlation functions of the winding number density, which allows us to map the topological problem to a spectral one in the same fashion as before. We achieve exact expressions for the ensemble averages by employing a method that is related to the supersymmetry method of random matrix theory [80, 81]. It exploits supersymmetry structures without mapping the integrals to superspace, and thus has been coined supersymmetry without supersymmetry [82, 83]. In chapter 6 we provide a conclusion of the new developments and put them into context with contemporary work and future research objectives.

Chapter 2

Symmetry and Topology

THIS chapter offers a brief review on the tenfold symmetry classification and the periodic table of topological insulators and topological superconductors. In section 2.1 we specify the systems we consider, for which the prediction of non-trivial topological phases via the symmetry classification is valid. Before we turn to the tenfold way in section 2.2, it is necessary to introduce the symmetry operations in which it is formulated. In section 2.3 we define the winding number, which is the relevant topological invariant for one-dimensional chiral systems. Finally, in section 2.4 we illustrate the concepts introduced by means of a prominent toy model of a topological superconductor with chiral symmetry, the Kitaev chain.

2.1 Free Fermion Hamiltonian

A common situation in condensed matter physics is that of electrons subject to the spatially periodic potential of a crystal. Furthermore, the electrons interact with each other. This can be due to Coloumb interaction or due to indirect mechanisms such as phonon coupling. We assume that the latter is well described by a mean-field approximation, i.e. the electrons are free of many-particle interactions except for the mean field of all electrons. This leads to the concept of a free Fermion model [31, 33, 36]. The general second quantized Hamiltonian is

$$\hat{H} = \sum_{\mu,\nu} \left(\Xi_{\mu\nu} \hat{c}_\mu^\dagger \hat{c}_\nu + \frac{1}{2} \Delta_{\mu\nu} \hat{c}_\mu^\dagger \hat{c}_\nu^\dagger - \frac{1}{2} \Delta_{\mu\nu}^* \hat{c}_\mu \hat{c}_\nu \right), \tag{2.1}$$

where $\hat{c}_\mu^\dagger$, $\hat{c}_\mu$ are the creation- and annihilation operators for electrons with the combined index μ, containing all relevant quantum numbers. The normal part $\Xi_{\mu\nu}$ describes all single-particle terms such as a hopping between sites or an on-site potential and the pairing potential $\Delta_{\mu\nu}$ results from the mean-field approximation. Due to the Hermiticity of $\hat{H}$, the following must apply

$$\Xi_{\mu\nu} = \Xi_{\nu\mu}^*, \qquad \Delta_{\mu\nu} = -\Delta_{\nu\mu}. \tag{2.2}$$

Since the Hamiltonian (2.1) contains the creation- and annihilation operators only in second order, it can be written as a sesquilinear form

$$\hat{H} = \frac{1}{2}\hat{\Psi}^\dagger H \hat{\Psi} \tag{2.3}$$

in the vectors

$$\hat{\Psi} = \begin{bmatrix} \hat{c}_1 & \cdots & \hat{c}_N & \hat{c}_1^\dagger & \cdots & \hat{c}_N^\dagger \end{bmatrix}^T, \qquad \hat{\Psi}^\dagger = \begin{bmatrix} \hat{c}_1^\dagger & \cdots & \hat{c}_N^\dagger & \hat{c}_1 & \cdots & \hat{c}_N \end{bmatrix}. \tag{2.4}$$

The Hermitian 2×2-block matrix

$$H = \begin{bmatrix} \Xi & \Delta \\ -\Delta^* & -\Xi^T \end{bmatrix} \tag{2.5}$$

is referred to as the Bogolyubov-de Gennes Hamiltonian or first quantized Hamiltonian. In case there is no pairing potential, we omit the 2×2-structure and work with the normal part alone.

We may represent the Hamiltonian in momentum space by a discrete Fourier transform from the crystal lattice to the reciprocal lattice [15]. In the presence of discrete translation symmetry this leads to the Bloch Hamiltonian

$$H(p) = \sum_r e^{-ipr} \begin{bmatrix} \Xi(r) & \Delta(r) \\ -\Delta^*(r) & -\Xi^T(r) \end{bmatrix}, \tag{2.6}$$

where we write the site indices as a functional argument, because they are affected by the transformation. We want to consider them on the scale of the lattice constant, such that $r \in \mathbb{Z}^d$ for a d-dimensional system. The crystal momentum is denoted by p instead of k as is usual due to an overlap of notational conventions appearing in the later chapters. We assume a continuity limit, such that $p \in [0, 2\pi)^d$ is on the Brillouin torus.

2.2 Symmetry Classification

Since we want to consider disordered systems, it is reasonable to exclude all spatial symmetries. In this section we first want to shed some light upon the three remaining fundamental symmetries: time reversal invariance, the particle-hole constraint and chiral symmetry. Subsequently, we introduce the Altland-Zirnbauer symmetry classes and the periodic table of topological insulators and topological superconductors.

To begin with, let us clarify which operators are suitable for describing symmetries. A symmetry operation $\mathcal{V}$ should leave the overlap between quantum states unchanged

$$\left|\langle\varphi|\mathcal{V}^\dagger\mathcal{V}|\chi\rangle\right|^2 = |\langle\varphi|\chi\rangle|^2 . \tag{2.7}$$

A theorem of Wigner states that there are only two possibilities [84]. Either $\mathcal{V}$ is unitary

$$\langle\varphi|\mathcal{V}^\dagger\mathcal{V}|\chi\rangle = \langle\varphi|\chi\rangle \tag{2.8}$$

or antiunitary

$$\langle\varphi|\mathcal{V}^\dagger\mathcal{V}|\chi\rangle = \langle\varphi|\chi\rangle^* . \tag{2.9}$$

An antiunitary operator can be written as

$$\mathcal{V} = U_\mathcal{V}\mathcal{K}, \tag{2.10}$$

where $U_\mathcal{V}$ is unitary and $\mathcal{K}$ is the complex conjugation of all objects to its right. Next, we will introduce the above-mentioned symmetry operations, place them in one of these two classes, and examine their effect on the Hamiltonian.

2.2.1 Time Reversal

A reversal of the arrow of time is accompanied by a reversal of various other quantities, such as the velocity, while others, such as the position are left unchanged. This naturally carries over to quantum theory, although we have to pay special attention to the spin, which has no classical analogue. Nevertheless, it can be argued, for instance when considering the Einstein-de Haas effect, that the spin, just like the angular momentum, is odd under time reversal. In equations, we may denote for position, momentum and spin

$$\mathcal{T}\hat{r}\mathcal{T}^{-1} = \hat{r}, \qquad \mathcal{T}\hat{p}\mathcal{T}^{-1} = -\hat{p}, \qquad \mathcal{T}\hat{S}\mathcal{T}^{-1} = -\hat{S} \tag{2.11}$$

with the time reversal operator $\mathcal{T}$. These should be understood as a general set of rules, valid also for the operators creating or annihilating a particle at position r or with momentum p. The Hamiltonian of a time reversal invariant system commutes with this operator

$$\mathcal{T}\hat{H}\mathcal{T}^{-1} = \hat{H} . \tag{2.12}$$

Applying (2.11) to the commutation relation between position and momentum

$$\mathcal{T}[\hat{r},\hat{p}]\mathcal{T}^{-1} = \mathcal{T}i\mathcal{T}^{-1} = -i \tag{2.13}$$

it is quickly inferred that $\mathcal{T}$ is antiunitary. Furthermore, we require that $\mathcal{T}$ is an involution on the projective Hilbert space, which means applying it twice can yield at most a phase

$$\mathcal{T}^2 = e^{i\varphi}. \tag{2.14}$$

Since $\mathcal{T}$ is antiunitary, applying it once from the left and once from the right shows that the phase is real

$$\mathcal{T}^3 = e^{-i\varphi}\mathcal{T} = e^{i\varphi}\mathcal{T}, \tag{2.15}$$

so that there are only two possibilities for its square

$$\mathcal{T}^2 = \pm 1. \tag{2.16}$$

The case, where $\mathcal{T}$ squares to positive unity contains systems without spin. Here, we have only a complex conjugation

$$\mathcal{T} = \mathcal{K}, \qquad\qquad \mathcal{T}^2 = +1. \tag{2.17}$$

According to the commutation relation (2.12), this results in a real Hamiltonian. Taking its Hermiticity into account, it is consequently real symmetric. In spinful systems, we have to add a unitary rotation in spin space

$$\mathcal{T} = U_{\mathcal{T}}\mathcal{K}, \qquad\qquad U_{\mathcal{T}} = \mathbb{1} \otimes \exp\left(i\pi S_y\right) \tag{2.18}$$

for the reversal of the spin. The square of $\mathcal{T}$ depends on the Fermion parity. For an odd number of Fermions it squares to negative, and for an even number to positive unity. The tensor product with the unit matrix is a reminder on the remaining degrees of freedom, which are not affected by time reversal. The choice of the canonical y-axis as the rotational axis is indeed not arbitrary, as can be shown by the requirement (2.11), see [61]. In the case of Fermionic single-particle systems like the free Fermion models (2.1), we obtain

$$\mathcal{T} = \mathbb{1} \otimes \exp\left(i\frac{\pi}{2}\sigma_y\right)\mathcal{K} = \left(\mathbb{1} \otimes i\sigma_y\right)\mathcal{K}, \qquad\qquad \mathcal{T}^2 = -1, \tag{2.19}$$

where σ_y is the second Pauli matrix acting on spin space. Each 2×2-block of a matrix commuting with this operator can be parametrized by

$$\begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} = \operatorname{Re} z\, \mathbb{1}_2 + i \operatorname{Im} w\, \sigma_x + i \operatorname{Re} w\, \sigma_y + i \operatorname{Im} z\, \sigma_z \tag{2.20}$$

with $z, w \in \mathbb{C}$ and is referred to as a real quaternion. Indeed, the anti-Hermitian Pauli matrices together with the unit matrix $\{\mathbb{1}_2, i\sigma_x, i\sigma_y, i\sigma_z\}$ satisfy the multiplication table of the quaternions, and the prefactors $\{\operatorname{Re} z, \operatorname{Im} w, \operatorname{Re} w, \operatorname{Im} z\}$ are real numbers. In combination with its Hermiticity, a real quaternion Hamiltonian is referred to as a quaternion self-dual matrix [58, 61, 85]. An important property of such a Hamiltonian is that each eigenenergy is evenfold degenerate, which is the well-known Kramers' degeneracy.

Since we carry out our investigation in momentum space, it remains to consider the action of the time reversal operator on the Bloch Hamiltonian (2.6). We see directly that due to its antiunitarity the crystal momentum p is reversed

$$\mathcal{T}H(p)\mathcal{T}^{-1} = H(-p). \tag{2.21}$$

Consequently, a time reversal invariant Bloch Hamiltonian does not have a real structure in general. Only for the time reversal invariant momenta, where p corresponds to $-p$ due to the periodicity of the Brillouin zone, we find $H(p)$ to be real symmetric or quaternion self-dual. We will return to this in chapter 5 when defining our random matrix model, where this is seen more clearly.

2.2.2 Particle-Hole Conjugation

The Bogolyubov-de Gennes Hamiltonian (2.5) satisfies the constraint

$$\mathcal{P} H \mathcal{P}^{-1} = -H \tag{2.22}$$

with the particle-hole conjugation

$$\mathcal{P} = (\mathbb{1} \otimes \tau_x)\,\mathcal{K}, \qquad \mathcal{P}^2 = +1. \tag{2.23}$$

In the presence of spin rotation symmetry, the Hamiltonian decomposes into two equivalent subblocks, giving rise to a new structure [31, 36], obeying the anticommutation under

$$\mathcal{P} = (\mathbb{1} \otimes i\tau_y)\,\mathcal{K}, \qquad \mathcal{P}^2 = -1, \tag{2.24}$$

where τ_x and τ_y are the first and second Pauli matrix now acting on particle-hole space, i.e. the 2×2-structure of the Hamiltonian (2.5). Just like the time reversal operator, particle-hole conjugation is antiunitary and therefore it squares to either positive or negative unity. The eigenstates of a Hamiltonian fulfilling the anticommutation relation (2.22) form pairs of positive and negative energies

$$H\left|\varphi_j\right\rangle = E_j \left|\varphi_j\right\rangle \Rightarrow H\mathcal{P}\left|\varphi_j\right\rangle = -\mathcal{P}H\left|\varphi_j\right\rangle = -E_j\mathcal{P}\left|\varphi_j\right\rangle, \tag{2.25}$$

rendering its spectrum symmetric around zero energy. On the Bloch Hamiltonian particle-hole conjugation acts as

$$\mathcal{P} H(p) \mathcal{P}^{-1} = -H(-p), \tag{2.26}$$

where, as in the case of time reversal, the crystal momentum is reversed due to the antiunitary of the operator.

Prior, when discussing time reversal, we started from physical requirements on the observables and then deduced the time reversal operator from them. Here, we instead identified particle-hole conjugation from the structure of the Bogolyubov-de Gennes Hamiltonian. However, particle-hole conjugation can as well be defined ad hoc as an exchange of creation operators, the "particles", and annihilation operators, the "holes"

$$\mathcal{P}\hat{c}_\mu^\dagger\mathcal{P}^{-1} = \hat{c}_\mu, \qquad \mathcal{P}\hat{c}_\mu\mathcal{P}^{-1} = \hat{c}_\mu^\dagger. \tag{2.27}$$

Naturally, any traceless Fermionic Hamiltonian anticommutes with this operation. Therefore, what (2.22) and (2.26) encode is simply the Fermionic anticommutation relation. There have been discussions about the terminology [86], as it is not uncommon in contemporary literature to refer to this property as particle-hole symmetry [34–36]. However, it is clear that it is a property of the Bogolyubov-de Gennes formalism for free Fermions rather than a symmetry of the underlying physical system. Other terminological suggestions are Fermi constraint or particle-hole constraint and particle-hole conjugation for the operator $\mathcal{P}$, where the latter is what we adopted in this work.

2.2.3 Chiral Symmetry

Finally, we obtain the chiral operator as a combination of time reversal and particle-hole conjugation

$$\mathcal{C} \sim \mathcal{P}\mathcal{T} \sim \mathcal{T}\mathcal{P}. \tag{2.28}$$

As a combination of two antiunitary operators, the chiral operator itself is unitary. The order of $\mathcal{T}$ and $\mathcal{P}$ does not matter as they act in different subspaces and thus commute up to a phase factor. Any unitary involution can be rescaled such that it squares to positive unity

$$\mathcal{C}^2 = +1, \tag{2.29}$$

which is the usual convention for the chiral operator. A chiral symmetric Hamiltonian obeys the anticommutation relation

$$\mathcal{C} H \mathcal{C}^{-1} = -H. \tag{2.30}$$

We point out that such a Hamiltonian does not have to be invariant under time reversal and particle-hole conjugation individually. It may also be the case that only their combination is invariant. Since $\mathcal{C}$ squares to positive unity, its eigenvalues are ± 1. Thus, its diagonal form is

$$\mathcal{C} = \begin{bmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_M \end{bmatrix} \tag{2.31}$$

with N positive and M negative eigenvalues. In the diagonal basis of the chiral operator, the Hamiltonian assumes a block off-diagonal form

$$H = \begin{bmatrix} 0 & K \\ K^\dagger & 0 \end{bmatrix}. \tag{2.32}$$

The $N \times M$-matrix K is a priori arbitrary, since Hermiticity is ensured by the 2×2-block structure. Similar to the particle-hole constraint, the spectrum of such a

Hamiltonian is, due to the anticommutation relation (2.30), symmetric around zero energy. In the case of rectangular K, the Hamiltonian necessarily has pairs of zero energy eigenvalues, whose number is given by the rectangularity

$$\nu = |N - M| . \tag{2.33}$$

In quantum chromodynamics, this parameter is commonly referred to as the topological charge [87].

In some systems, chiral symmetry can be understood from a geometrical perspective and is then also referred to as sublattice symmetry. One example of such a system is graphene with nearest-neighbour hopping. Generally, the Hilbert space of a chiral system consists of two subspaces (e.g. sublattices) with eigenvalues ± 1 of the chiral operator. The Hamiltonian swaps states between these two blocks. The anticommutation relation (2.30) transfers directly to the Bloch Hamiltonian

$$\mathcal{C} H(p) \mathcal{C}^{-1} = -H(p), \tag{2.34}$$

and therefore it assumes a block off-diagonal form like (2.32) as well.

2.2.4 The Tenfold Way

We addressed three fundamental symmetry operations and are now in the position to formulate the Altland-Zirnbauer symmetry classes, also known as the tenfold way. There are three possibilities for behaviour under time reversal and particle-hole conjugation: invariances are either absent or their operators square to positive or negative unity. Chiral symmetry, on the other hand, comes only in one flavour and is present when the system is both $\mathcal{T}$- and $\mathcal{P}$-invariant, but may also appear by itself. This results in a total of ten symmetry classes, which are summarized in table 2.1.

The symmetry classes are grouped therein according to historical context. Interestingly, all of them were discovered in works on random matrix theory. Dyson's threefold way distinguishes three symmetry classes based only on the behaviour under time reversal [58, 61, 85]. As mentioned above, time reversal invariance assigns a real structure to the Hamiltonian. It is either symmetric with real entries ($\beta = 1$) or quaternion self-dual ($\beta = 4$). In the absence of time reversal invariance the Hamiltonian is Hermitian with complex entries ($\beta = 2$). The Dyson index β is the dimension of the corresponding number field and is used to label these three cases in random matrix theory.

The chiral classes have been identified in works on quantum chromodynamics [88], where the chiral symmetry of the anti-Hermitian Dirac operator is broken by the quark masses and is restored in the high temperature limit. The so emerging chiral

	Cartan label	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	Symmetric space
Dyson classes	A				U(N)
	AI			1	U(N)/O(N)
	AII			-1	U($2N$)/Sp(N)
Chiral classes	AIII	1			U($N+M$)/U(N)$\times$ U(M)
	BDI	1	1	1	O($N+M$)/O(N)$\times$ O(M)
	CII	1	-1	-1	Sp($N+M$)/Sp(N)$\times$ Sp(M)
Superconducting classes	D		1		SO($2N$)
	C		-1		Sp(N)
	DIII	1	1	-1	SO($2N$)/U(N)
	CI	1	-1	1	Sp(N)/U(N)

Table 2.1: The Altland-Zirnbauer symmetry classes, also known as the tenfold way.

random matrix theory turned out to be a fruitful approach in quantum chromodynamics [62–68]. Again, one distinguishes between three classes with real, complex or quaternion entries, which amounts to the threefold classification of Hamiltonians in terms of time reversal.

The remaining four classes were introduced in works on normal-superconductor interfaces [31, 89, 90], which yield the possibility for Andreev reflection [91]. An electron in the normal conductor is not ideally reflected at the normal-superconductor interface, instead it passes into the superconductor, forming a Cooper pair, and a hole is retroreflected into the normal conductor, obeying charge conservation.

It is more common in the condensed matter community to use the Cartan label than the Dyson index, as it comprises all ten symmetry classes. It stems from a classification of symmetric spaces, a task already undertaken by Cartan in the 1920s [92, 93]. Simply put, these spaces contain the time evolution operators $\exp(-itH)$ of the corresponding Hamiltonians. As an example, consider the classes A and AI. A class A Hamiltonian has no symmetries at all and is therefore a complex Hermitian $N \times N$-matrix. The complex exponential maps these matrices to the unitary group U(N), being the symmetric space of class A. Imposing spinless time reversal invariance invokes the real symmetric Hamiltonians of class AI, which are mapped onto the coset space U(N)/O(N) by the complex exponential, which is the symmetric space of class AI. In addition to the symmetric spaces found in the tenfold way, Cartan's classification includes spaces involving exceptional Lie groups [94], which feature a fixed dimension and are therefore not suited to describe Hamiltonians.

The periodic table of topological insulators and topological superconductors is shown in table 2.2. It is formulated in the tenfold symmetry classification and predicts whether a topological phase is possible in dependence of the symmetry class

Symmetry class	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	d=0	1	2	3	4	5	6	7
A				$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
AIII	1				$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AI			1	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$
D		1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$	
DIII	1	1	-1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$
AII			-1	$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
CII	1	-1	-1		$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
C		-1				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
CI	1	-1	1				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 2.2: The periodic table of topological insulators and topological superconductors.

and the dimension d of the parameter manifold. In the case of a Bloch Hamiltonian, this manifold is the Brillouin zone and d is equal to the real space dimension of the system. The entries indicate whether the topological invariant is binary $\mathbb{Z}_2$, integer $\mathbb{Z}$, or an even integer $2\mathbb{Z}$. An empty field means that a topological phase is not possible.

The periodic table is constructed in the framework of topological k-theory [33,95] or by a homotopy classification of Dirac Hamiltonians [36,96] describing phase transition points. It is periodic in the truest sense of the word. The complex classes A and AIII, whose Hamiltonians are not constrained by an antiunitary operator, have a periodicity of two in the dimension d, while the remaining real classes have a periodicity of eight in d. This property is known as Bott periodicity. Topological phases with $d > 3$ are not fictitious, because parameter spaces with additional dimensions exist [97], for example in quasicrystals [98]. Furthermore, the symmetry classes are arranged in a different order than in table 2.1, uncovering a diagonal structure. The reason for this structure is the Bott clock mechanism [36, 99]. Topological nontriviality can be maintained while making specific changes to the symmetries and adding a dimension to the parameter space.

Taking spatial symmetries into account leads to a wealth of new symmetry classes to which the periodic table can be extended [36, 100–103]. In this context, a distinction is made between strong and weak topological insulators and topological superconductors. The former are stable against disorder and their phases are categorized in table 2.2, while the latter are not stable against disorder and thus rely on additional spatial symmetries.

2.3 Winding Number

The topological invariant relevant for the one-dimensional chiral classes AIII, BDI and CII is termed winding number. The Bloch Hamiltonian of a chiral symmetric system obeys the anticommutation relation (2.34) and therefore can be cast into a block off-diagonal form

$$H(p) = \begin{bmatrix} 0 & K(p) \\ K^\dagger(p) & 0 \end{bmatrix}. \tag{2.35}$$

From now on, we assume that $K(p)$ is a square matrix of dimension N. In the case of rectangular $K(p)$ the number of zero modes is given by the rectangularity parameter, see (2.33). Due to the periodicity in p, the determinant of $K(p)$ is a closed curve in the complex plane. The winding number is the number of times this curve winds around the origin, where counterclockwise revolutions are counted positively and clockwise revolutions negatively. It is given by

$$W = \frac{1}{2\pi i} \oint\limits_{\det K(p)} \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_0^{2\pi} dp\, w(p). \tag{2.36}$$

This expression evaluates to an integer as can be verified quickly by invoking Cauchy's argument principle [104]. The winding number density is the logarithmic derivative of the determinant

$$w(p) = \frac{d}{dp} \ln \det K(p) = \frac{1}{\det K(p)} \frac{d}{dp} \det K(p) = \operatorname{tr} K^{-1}(p) \frac{d}{dp} K(p), \tag{2.37}$$

which is well defined only for invertible $K(p)$. In the case of non-invertibility for some p the system undergoes a topological phase transition during which the winding number changes its value. Indeed, this coincides with a closing of the band gap. If $\det K(p) = 0$, it must also be $\det H(p) = 0$, and because of chiral symmetry, there must be at least one pair of zero energy modes at this point.

In the following chapters we work with expression (2.37), as it is particularly suitable for calculations within random matrix theory. Nevertheless, it is worthwhile to consider also an alternative expression for the same quantity. We define the projection on the N eigenstates with positive resp. negative energy

$$P_\pm(p) = \sum_{n=1}^{N} |n_\pm(p)\rangle \langle n_\pm(p)| \tag{2.38}$$

and the flat band Hamiltonian

$$Q(p) = \mathbb{1}_{2N} - 2P_-(p) = P_+(p) - P_-(p) = \begin{bmatrix} 0 & q(p) \\ q^\dagger(p) & 0 \end{bmatrix}, \tag{2.39}$$

for which the eigenvalues $\pm E_n(p)$ are deformed to ± 1. It has the same eigenbasis as $H(p)$ and thus assumes a block off-diagonal form as well. The matrix $q(p)$ is unitary because $Q^2(p) = \mathbb{1}_{2N}$. The eigenstates of $H(p)$ and $Q(p)$ have the block form

$$|n_\pm(p)\rangle = \begin{bmatrix} u_n(p) \\ \pm v_n(p) \end{bmatrix} \tag{2.40}$$

and the chiral operator $\mathcal{C}$ maps an eigenstate to its chiral partner with energy of opposite sign

$$\mathcal{C}\,|n_\pm(p)\rangle = |n_\mp(p)\rangle\,. \tag{2.41}$$

The eigenstates form an orthonormal system

$$\begin{aligned} \langle n_\pm(p)|m_\pm(p)\rangle &= \langle u_n(p)|u_m(p)\rangle + \langle v_n(p)|v_m(p)\rangle = \delta_{mn}, \\ \langle n_\pm(p)|\mathcal{C}|m_\pm(p)\rangle &= \langle u_n(p)|u_m(p)\rangle - \langle v_n(p)|v_m(p)\rangle = 0, \end{aligned} \tag{2.42}$$

which yields

$$\langle u_n(p)|u_m(p)\rangle = \langle v_n(p)|v_m(p)\rangle = \frac{\delta_{nm}}{2}. \tag{2.43}$$

By considering the eigenvalue equations

$$H(p)\,|n_\pm(p)\rangle = \pm E_n(p)\,|n_\pm(p)\rangle\,, \qquad Q(p)\,|n_\pm(p)\rangle = \pm\,|n_\pm(p)\rangle \tag{2.44}$$

we find

$$K(p) = 2\sum_{n=1}^{N} E_n(p)\,|u_n(p)\rangle\,\langle v_n(p)| = \varepsilon(p)q(p) \tag{2.45}$$

with

$$q(p) = 2\sum_{n=1}^{N} |u_n(p)\rangle\,\langle v_n(p)|\,, \qquad \varepsilon(p) = 2\sum_{n=1}^{N} E_n(p)\,|u_n(p)\rangle\,\langle u_n(p)|\,. \tag{2.46}$$

Inserting this into (2.37), we obtain two contributions to the winding number density

$$w(p) = \operatorname{tr} K^{-1}(p)\frac{d}{dp}K(p) = \operatorname{tr}\varepsilon^{-1}(p)\frac{d}{dp}\varepsilon(p) + \operatorname{tr} q^{-1}(p)\frac{d}{dp}q(p). \tag{2.47}$$

The first contribution is the real part of the winding number density and corresponds to the radial part of the determinant. Thus, it integrates to zero over a whole period

$$\int_0^{2\pi} dp\,\operatorname{tr}\varepsilon^{-1}(p)\frac{d}{dp}\varepsilon(p) = \int_0^{2\pi} dp \sum_{n=1}^{N} \frac{1}{E_n(p)}\frac{d}{dp}E_n(p) = \sum_{n=1}^{N} \ln E_n(p)\Bigg|_{p=0}^{p=2\pi} = 0. \tag{2.48}$$

The second is the imaginary part and corresponds to the angular part of the determinant. It becomes

$$\operatorname{tr} q^{-1}(p)\frac{d}{dp}q(p) = 2\sum_{n=1}^{N}\langle u_n(p)|\frac{d}{dp}|u_n(p)\rangle - \langle v_n(p)|\frac{d}{dp}|v_n(p)\rangle = \sum_{n=1}^{2N}\langle n(p)|\mathcal{C}\frac{d}{dp}|n(p)\rangle \tag{2.49}$$

and the winding number may be achieved as an integral over this quantity instead of (2.37). From this expression it is clear that the winding number is indeed a property of the eigenvector bundle of the Bloch Hamiltonian and not of its bands. Furthermore, similarities to other topological invariants like the Chern number and the Chern-Simons invariant are revealed, see [14, 15, 36].

2.4 Example: Kitaev Chain

We want to illustrate the concepts introduced in the last section by means of a well-studied toy model of a one-dimensional superconductor. It is a chain of N electrons located at sites j with a chemical potential μ, a superconducting pairing potential Δ and nearest-neighbour hopping with amplitude t [105]. The electrons are considered "spinless". Physically, this corresponds to a strongly polarized chain for which we project the Hamiltonian onto the lowest energy subspace, in which all spins are parallel [37]. The second quantized Hamiltonian is

$$\hat{H} = \sum_{j=1}^{N}\left[\frac{t}{2}\left(\hat{c}_j^\dagger\hat{c}_{j+1} + \hat{c}_{j+1}^\dagger\hat{c}_j\right) - \mu\,\hat{c}_j^\dagger\hat{c}_j + \frac{1}{2}\left(\Delta^*\hat{c}_j^\dagger\hat{c}_{j+1}^\dagger - \Delta\hat{c}_j\hat{c}_{j+1}\right)\right]. \tag{2.50}$$

If the chain has closed boundary conditions, j is defined modulo N, and if it has open boundary conditions, terms containing operators at the (non-existent) site $N+1$ are omitted. The pairing potential Δ can be assumed real without loss of generality, since one can always apply the gauge transformation $\hat{c}_j \to e^{i\varphi}\hat{c}_j$ with an appropriate phase. We additionally assume $t \in \mathbb{R}$ so that the system is time reversal invariant. Consequently, its Bogolyubov-de Gennes Hamiltonian is real and there is chiral symmetry as well. Applying (2.6) we find the Bloch Hamiltonian. In the eigenbasis of the chiral operator it reads

$$H(p) = \begin{bmatrix} 0 & \mu - t\cos p + i\Delta\sin p \\ \mu - t\cos p - i\Delta\sin p & 0 \end{bmatrix}. \tag{2.51}$$

The off-diagonal block describes an ellipse in the complex plane

$$K(p) = \mu - t\cos p + i\Delta\sin p. \tag{2.52}$$

For a higher-dimensional Bloch Hamiltonian we would have to take its determinant to obtain a closed curve and define a winding number. The center point of the ellipse is determined by μ and the semiaxes by Δ and t. In figure 2.1 we depict $K(p)$ and the eigenvalues of $H(p)$ as functions of p for $t = \Delta = 1$, so that $K(p)$ is a circle, and $\mu \in \{1/2, 1, 3/2\}$. We find the winding number to be $W = 1$ for $|\mu| < 1$ and $W = 0$ for $|\mu| > 1$. For $|\mu| = 1$ the winding number is undefined as $K(p)$ crosses the origin and the energy gap closes, marking the topological phase transition.

In figure 2.2 we depict the eigenvalues of the Bogolyubov-de Gennes Hamiltonian for a chain with $N = 20$ sites and closed resp. open boundary conditions as functions of the chemical potential μ. Again, we fix $t = \Delta = 1$. The spectra are symmetric around zero due to invariance under particle-hole conjugation resp. chiral symmetry. For the closed chain a pair of modes crosses zero energy at $\mu = 1$, where the gap has to close due to the topological phase transition. For the open chain, on the other hand, a pair of zero modes persists until some value $\mu < 1$. These are the edge states of the topological superconductor. They do not persist exactly until $\mu = 1$ due to the finite size of the chain.

Next, we want to consider the wave functions of the edge states. For this we need to provide some more details. Just like in (2.40), but now sticking to the position basis, we write the eigenstates of the Bogolyubov-de Gennes Hamiltonian in a block form

$$|n\rangle = \begin{bmatrix} u_n \\ v_n \end{bmatrix} . \tag{2.53}$$

When diagonalizing the Bogolyubov-de Gennes Hamiltonian the second quantized Hamiltonian (2.3) can be written as

$$\hat{H} = \frac{1}{2} \sum_{n=1}^{2N} E_n \hat{\gamma}_n^\dagger \hat{\gamma}_n \tag{2.54}$$

with the eigenenergies E_n and new Fermionic operators

$$\hat{\gamma}_n = \sum_{j=1}^{N} u_{nj} \hat{c}_n + v_{nj} \hat{c}_n^\dagger, \tag{2.55}$$

which embody particle-hole excitations and are referred to as Bogolyubov quasiparticles. Note that, unlike the electrons in (2.50), their number is conserved. The probability to find a particle in state n at site j is

$$|u_{nj}|^2 + |v_{nj}|^2 . \tag{2.56}$$

In figure 2.3 we plot the discrete probability distribution for finding the Bogolyubov quasiparticle belonging to the eigenstate of one of the topological zero modes at

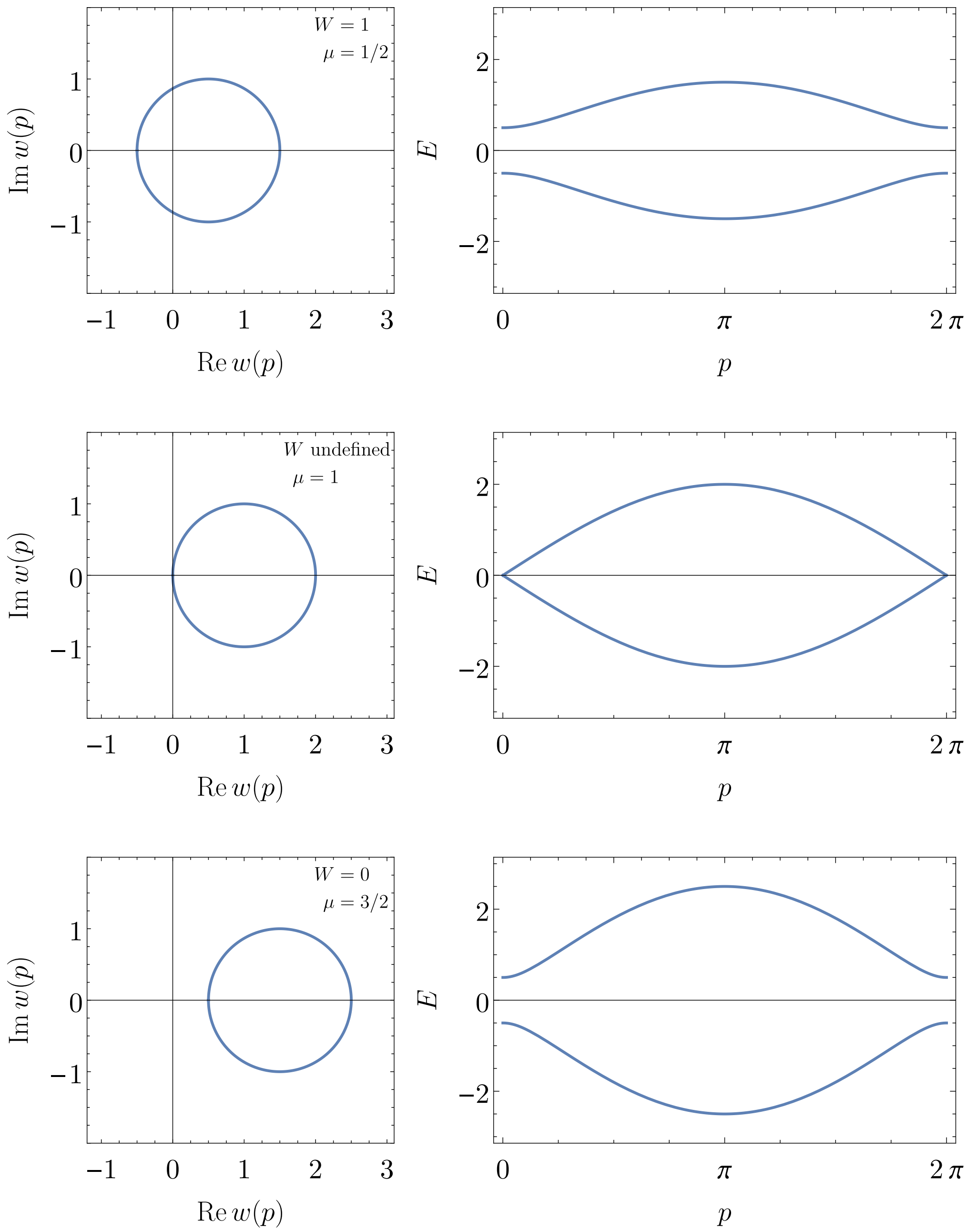


Figure 2.1: The curve $K(p)$ (left) describing a circle for $t = \Delta = 1$ and corresponding dispersion relation $E(p)$ (right). Topological phase with $W = 1$ for $\mu = 1/2$ (top), topological phase transition with undefined W for $\mu = 1$ (center) and trivial phase with $W = 0$ for $\mu = 3/2$ (bottom).

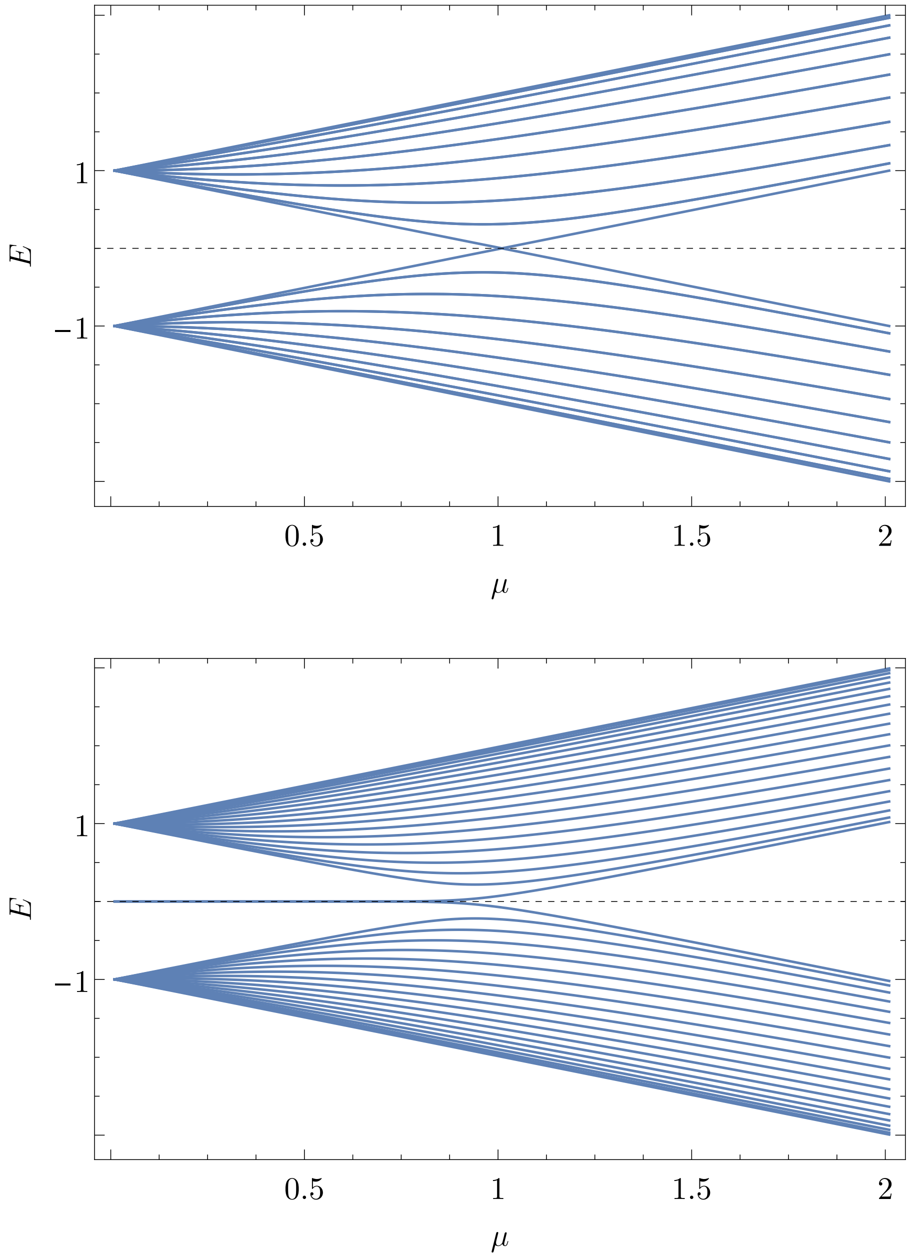


Figure 2.2: Energy spectrum of a Kitaev chain with closed (top) and open (bottom) boundary conditions in dependence of the chemical potential μ. The chain has $N = 20$ sites and the remaining system parameters are set to $t = \Delta = 1$.

site j. In the topological regime ($\mu = 1/2$) it is localized to the edges, whereas in the trivial regime ($\mu = 3/2$) the wave function is delocalized over the whole chain. This is in accordance with bulk-boundary correspondence: the winding number, defined for the chain with closed boundary conditions, determines the number of states localized to one of the edges in the open chain.

Bogolyubov quasiparticles have the Majorana property, i.e. they are their own antiparticles [106]. Majorana Fermions have been proposed as elementary particles already in the 1930s, but so far no such particle has been identified. However, Majorana quasiparticles turned out to be a rewarding concept in condensed matter physics [13,36,37,105]. Due to their real wave function they are particularly resistant to decoherence and therefore of interest for quantum information applications [38, 44, 45, 47, 48].

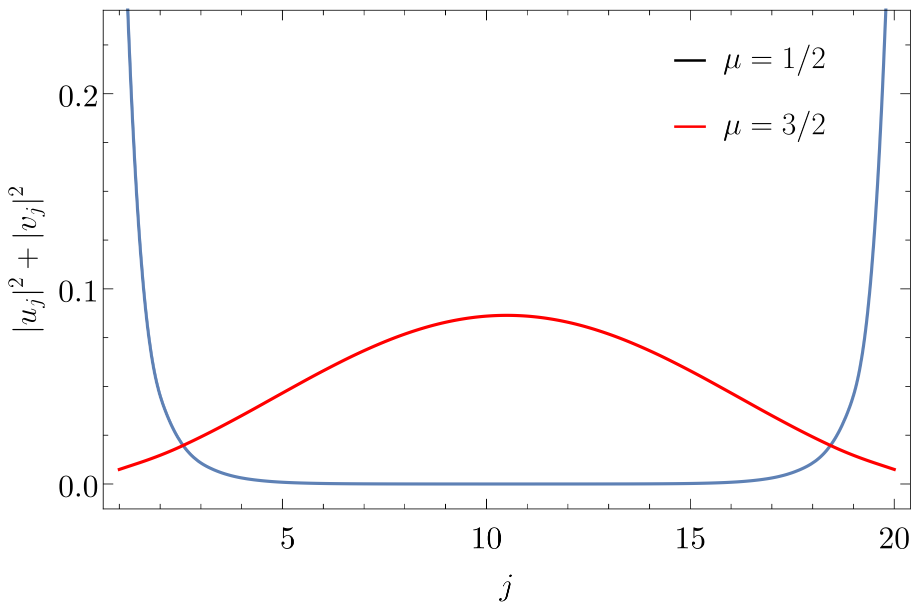


Figure 2.3: Position probability distribution of a Bogolyubov quasiparticle eigenstate in the Kitaev chain with open boundary conditions in the topological regime ($\mu = 1/2$) and in the trivial regime ($\mu = 3/2$). It is the eigenstate that corresponds to one of the zero modes in the topological regime. The probability distribution of its chiral partner is identical and therefore not represented. The chain has $N = 20$ sites and the remaining system parameters are set to $t = \Delta = 1$. The discrete plot is smoothed for visual reasons.

Chapter 3

Random Matrix Theory

This chapter serves as an introduction to random matrix theory. Section 3.1 covers the main aspects and motivates the use of random matrices in physics. In section 3.2 we introduce three families of random matrix ensembles that we will encounter in the later chapters. In section 3.3 we briefly discuss applications and universality aspects of random matrix models with an additional parametric dependence. In section 3.4 we introduce the supersymmetry without supersymmetry technique, which we use in chapter 5.

3.1 Main Aspects

A random matrix ensemble is a collection of matrices whose entries are random variables [58]. Such ensembles were first considered in mathematics [107] and became popular in physics through Wigner, who used them to describe the spectra of nuclear reactions involving heavy nuclei [108–110]. Over the years, random matrix theory spread to other areas of physics [57, 111], such as quantum chaos [61, 112], quantum chromodynamics [62–68] and condensed matter [31, 37, 59, 60].

In the study of many-body systems, the overwhelming complexity of the interactions renders the Hamiltonian virtually unknown. Therefore, complexity is often approached by statistical methods, such as disorder averaging. The problem to be faced is the exponential growth of the Hilbert space dimension with the particle number, making the diagonalization of the Hamiltonian a difficult task.

The idea of random matrix theory is to replace the Hamiltonian by a random matrix so that the only information about the system entering the model are its symmetries. The trade-off is that it is only able to describe universal properties that appear in the random matrix model as well as in the physical reality. This, however, turns out to work surprisingly well. On a technical level, the main difficulty is to evaluate high-dimensional integrals, which appear as averages over the matrix ensemble. Accordingly, random matrix theory has developed a diverse toolbox of methods to deal with this problem [57,58,61]. During the calculation, a finite dimensional Hilbert space is assumed, eventually taking the limit of an infinite dimension at the end. Although in the present context we only work with the Hamiltonian as a random matrix, we want to mention that any other matrix important for the description of the system can be modeled by random matrices as well, for example the scattering matrix or the Floquet operator of a time-discrete system [61,113–115], both being unitary rather than Hermitian.

The simplest and most ubiquitous symmetry classification is Dyson's threefold way, which is based on time reversal invariance [58, 61, 85]. It has been discussed in more detail in section 2.2. There are two symmetry classes with time reversal invariance featuring real symmetric ($\beta = 1$) and quaternion self-dual ($\beta = 4$) Hamiltonians. On the other hand, when time reversal invariance is broken, the Hamiltonian is complex Hermitian ($\beta = 2$). The Dyson index β is the dimension of the corresponding number field and is used to label these three cases in random matrix theory. In the tenfold way classification, see section 2.2, the classes are labeled AI ($\beta = 1$), A ($\beta = 2$) and AII ($\beta = 4$). The dimension of the Hilbert space is N resp. $2N$ in the presence of Kramers' degeneracy ($\beta = 4$).

Since we are looking for universal properties, there is a certain freedom of choice regarding the distribution of the matrix entries [116]. It is convenient to choose a Gaussian distribution. The joint probability distribution of all independent entries can be summarized as a matrix probability distribution [57, 58, 61]

$$\widetilde{P}^{(\beta,N)}_{\text{Gaussian}}(H) = \left(2\pi\sigma^2\right)^{-N/2}\left(\pi\sigma^2\right)^{-\beta N(N-1)/4}\exp\left(-\frac{1}{2\sigma^2\gamma}\operatorname{tr} H^2\right). \tag{3.1}$$

The parameter

$$\gamma = \begin{cases} 1 & \beta \in \{1,2\} \\ 2 & \beta = 4 \end{cases} \tag{3.2}$$

takes care of a double counting of variables in the quaternion case. The random variables are centered and σ controls their variance. It is the only parameter of the model, but as we will see below, it is fixed when considering scales where the universal properties of the matrix model are visible. These three ensembles are also called the Gaussian orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensembles, abbreviated as GOE, GUE and GSE, respectively. The adjective refers to the group of matrices U which diagonalize H and under which the symmetries of the Hamiltonian are preserved. For example, if H is real symmetric and U orthogonal, then the matrix $UHU^\dagger$ is also real symmetric. The matrix distributions (3.1) are invariant under these transformations and therefore basis independent.

Let $f(H)$ be some quantity that depends on the Hamiltonian. Its ensemble average is

$$\langle f(H)\rangle = \int d[H]\widetilde{P}^{(N,\beta)}_{\text{Gaussian}}(H)f(H), \tag{3.3}$$

where $d[H]$ is the flat measure of all independent matrix entries. If $f(H)$ is also basis independent, i.e. depends only on the eigenvalues of H, it may be preferable to integrate over the eigenvalues instead of the matrix elements. This amounts to the diagonalization of the Hamiltonian

$$H = UEU^{\dagger}, \tag{3.4}$$

where E is the diagonal matrix of all eigenvalues and U is a member of the diagonalizing group. Upon this variable transformation, the measure changes according to

$$d[H] = \Delta_N^{\beta}(E) d\mu(U) d[E] \tag{3.5}$$

with the Vandermonde determinant

$$\Delta_N(E) = \prod_{1\leq n<m\leq N} (E_m - E_n) = \det\left[E_m^{n-1}\right]_{1\leq m,n\leq N} \tag{3.6}$$

and the invariant Haar measure of the diagonalizing group $d\mu(U)$, see [58]. If the integrand is invariant under transformations in this group, integration over the Haar measure simply gives the group volume. Thus, we may write the expectation value (3.3) as

$$\langle f(H)\rangle = \int d[E] P_{\text{Gaussian}}^{(N,\beta)}(E) f(E) \tag{3.7}$$

with the joint probability distribution of eigenvalues

$$\begin{aligned} P_{\text{Gaussian}}^{(\beta,N)}(E) &= C_{N\beta}\Delta_N^{\beta}(E)\exp\left(-\frac{1}{2\sigma^2}\sum_{n=1}^{N} E_n^2\right), \\ C_{N\beta} &= \frac{\sigma^{-N-\beta N(N-1)/2}}{(2\pi)^{N/2}} \frac{\Gamma^N(1+\beta/2)}{\prod_{n=1}^{N}\Gamma(1+\beta n/2)}. \end{aligned} \tag{3.8}$$

The Vandermonde determinant (3.6) introduces a repulsion between levels, whose strength is governed by the Dyson index β. This is, in fact, the only source of correlation between the eigenvalues. Since the Vandermonde determinant results from the diagonalization of the Hamiltonian, we find this level repulsion in any ensemble, which is invariant under the respective group. The level repulsion suggests that the eigenvalues can be thought of as a gas of charged particles. This Coloumb gas analogy can be helpful in the calculation of various quantities in invariant random matrix ensembles [117, 118].

Since the advent of random matrix theory, spectral correlations have been the focus of interest. However, different physical systems may have different energy scales, so their spectra cannot be directly compared. Furthermore, the level density of a physical system

$$\rho(E) = \sum_{n=1}^{N} \delta(E - E_n) \tag{3.9}$$

often grows with increasing energy, while the level density of the Gaussian ensembles (3.1) for large N is given by

$$\rho(E) = \begin{cases} \frac{2}{\pi}\sqrt{\frac{N}{2\beta\sigma^2}}\sqrt{1-\frac{E^2}{2\beta N\sigma^2}} & |E| \leq \sqrt{2\beta N\sigma^2} \\ 0 & |E| > \sqrt{2\beta N\sigma^2} \end{cases}, \tag{3.10}$$

which is known as the Wigner semicircle law (although it does not describe an exact semicircle in general) [57, 58, 61]. The mean level spacing is given by the inverse of the level density $\Delta(E) = 1/\rho(E)$ and is thus of order $\mathcal{O}(N^{-1/2})$. In passing, we would like to point out that there is also another convention for the Gaussian ensembles (3.1), where the energy is scaled with $\sqrt{N}$ so that the Wigner semicircle has finite support even in the limit $N \to \infty$.

To compensate for these differences, we consider the spectra on a new scale

$$\xi(E) = \int_{-\infty}^{E} dE'\,\rho(E') \tag{3.11}$$

for which both the average level density and the mean level spacing are one. This step is referred to as the unfolding of the spectrum and is performed also in the case of experimentally or numerically obtained data, see [57]. Only after this rescaling, universal spectral correlations are revealed.

Next, we will discuss an example of a spectral correlation function. Let

$$s_n = \xi_{n+1} - \xi_n \geq 0 \tag{3.12}$$

be the distance between adjacent levels. The level spacing distribution $p(s)$ is the probability distribution of finding two adjacent levels at distance s. On the standardized scale (3.11) it is normalized to

$$\int_{-\infty}^{\infty} ds\,p(s) = 1, \qquad \int_{-\infty}^{\infty} ds\,s\,p(s) = 1, \tag{3.13}$$

i.e. the mean level spacing is one. There is no exact expression for the level spacing distribution of the Gaussian ensembles (3.1). In the limit $N \to \infty$ they are approximated by

$$p^{(\beta)}(s) = \begin{cases} \frac{\pi}{2}s\exp\left(-\frac{\pi}{4}s^2\right) & \beta = 1 \\ \frac{32}{\pi^2}s^2\exp\left(-\frac{4}{\pi}s^2\right) & \beta = 2 \\ \frac{262144}{729\pi^3}s^4\exp\left(-\frac{64}{9\pi}s^2\right) & \beta = 4 \end{cases}, \tag{3.14}$$

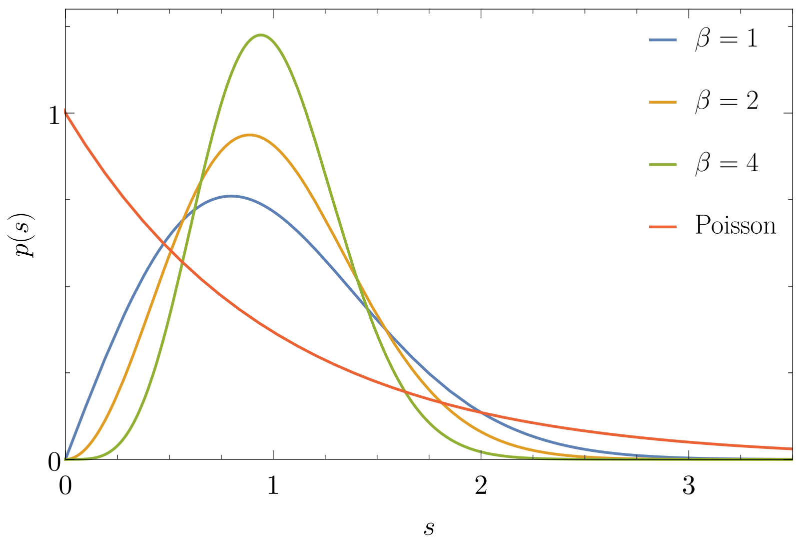


Figure 3.1: Wigner-Dyson distributions $p^{(\beta)}(s)$ and the Poisson distribution $p(s) = \exp(-s)$.

which are also referred to as the Wigner-Dyson distributions. For small spacings s the distributions go as $p^{(\beta)} \sim s^{\beta}$, which corresponds to the strength of the level repulsion that we observed in the eigenvalue distributions (3.8). They are plotted in figure 3.1 together with the Poisson distribution $p(s) = \exp(-s)$, which arises as the level spacing distribution of uncorrelated eigenvalues present in regular, i.e. non-complex, systems.

Figure 3.2 shows the level spacing distributions of various physical systems. The experimentally resp. numerically obtained data show good agreement with the Wigner-Dyson distributions (3.14). Some of the spectra are not of quantum mechanical origin, showing that universality also holds for classical wave equations. What the systems have in common is a certain degree of complexity or irregularity, which can be quantified in particular by the absence of symmetries.

The universal spectral correlations obtained from random matrix theory play a central role in quantum chaos [61, 112]. In classical mechanics, chaos is defined by the sensitivity of phase space trajectories to their initial conditions. The distance between two phase space trajectories that differ only by a small perturbation in their initial conditions, initially grows exponentially in time. The lack of trajectories in quantum mechanics demands an alternative definition. This definition is based on spectral statistics. A quantum system that exhibits the spectral statistics of random matrix theory is said to be chaotic. Conversely, the spectrum of a quantum system with chaotic classical analogue shows universal statistics as well. The last state-

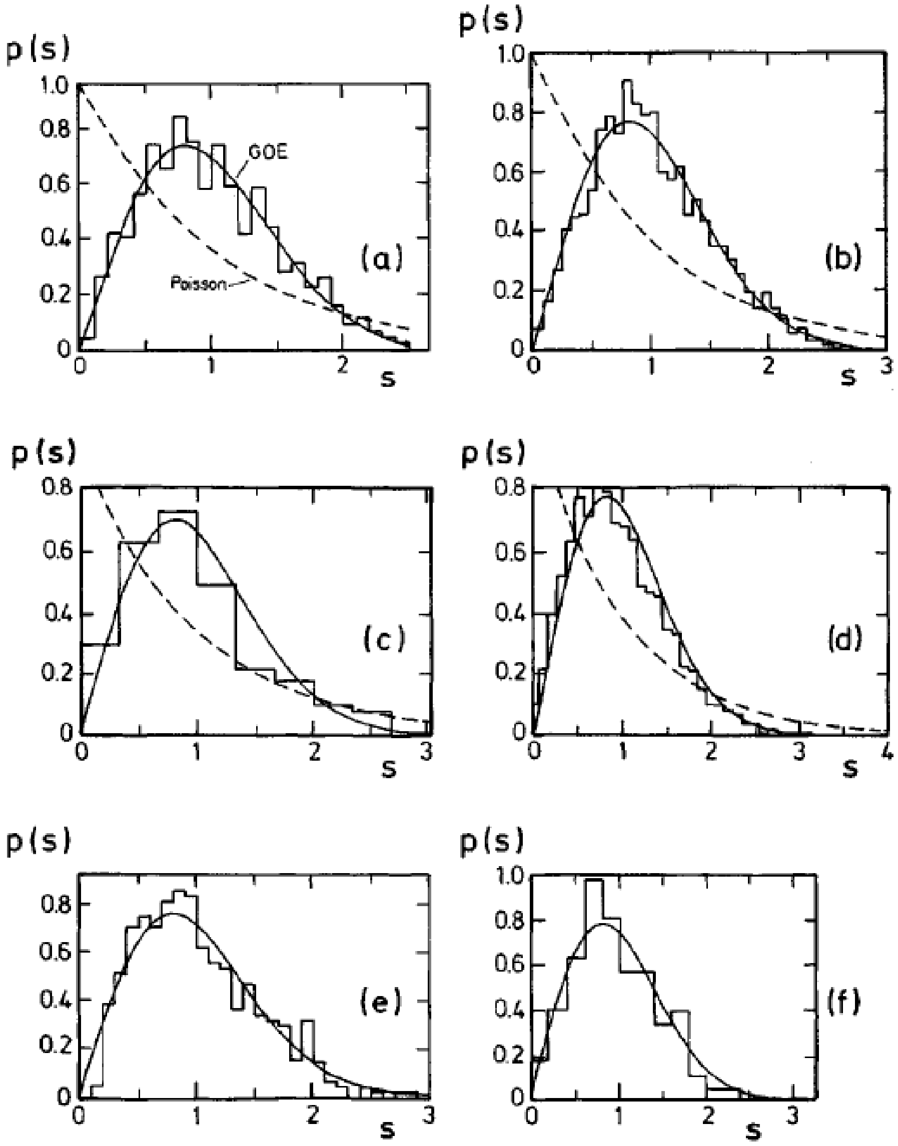


Figure 3.2: Level spacing distributions of various physical systems: (a) chaotic quantum billiard [119], (b) hydrogen atom in a strong magnetic field [120], (c) excitation spectrum of a NO_2 molecule [121], (d) resonance spectrum of an irregular shaped quartz block [122], (e) spectrum of a chaotic microwave cavity [123], (f) vibration spectrum of an irregular shaped plate [124]. The histograms represent experimental resp. numerical data and show well agreement with the Wigner-Dyson distribution in all cases. Figure taken from [112].

ment is the content of the Bohigas-Giannoni-Schmit conjecture [119]. In contrast, the spectrum of a quantum system with an integrable classical analogue, i.e. the presence of symmetries imposes some regularity on the phase space trajectories [7], shows no correlation and therefore its level spacing distribution is of Poisson type.

3.2 Three Families of Random Matrix Ensembles

In the last section we discussed the Gaussian ensembles of Hermitian matrices, which are divided into three symmetry classes based on their behaviour under time reversal. In this section we introduce further random matrix ensembles that we will encounter in the course of the next chapters. Let us first venture into systems with an additional chiral symmetry. According to our discussion in section 2.2, the Hamiltonian of a chiral symmetric system reads

$$H = \begin{bmatrix} 0 & K \\ K^\dagger & 0 \end{bmatrix} \tag{3.15}$$

in the diagonal basis of the chiral operator

$$\mathcal{C} = \begin{bmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_M \end{bmatrix}. \tag{3.16}$$

Since Hermiticity is ensured by the block off-diagonal structure, the matrices K are generic with real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$) entries. Once again choosing a Gaussian distribution gives rise to the Ginibre ensembles [58, 125–127]

$$\widetilde{P}^{(\beta,N)}_{\mathrm{Ginibre}}(K) = \left(2\pi\sigma^2\right)^{-\beta N^2/2} \exp\left(-\frac{1}{2\sigma^2\gamma}\mathrm{tr}\, KK^\dagger\right), \tag{3.17}$$

where, for the sake of simplicity, we assume square matrices, i.e. the rectangularity $\nu = |N - M|$ is zero. The ensembles of such Hamiltonians are referred to as the chiral Gaussian orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensemble, commonly abbreviated as chGOE, chGUE and chGSE, respectively.

Chiral symmetry imposes a reflection symmetry on the spectrum of the Hamiltonian around zero energy, which affects the spectral statistics. Put simply, the zero energy acts as a mirror, introducing an additional level repulsion. Away from this chiral anomaly, in the bulk of the spectrum, the spectral statistics resemble those of the classical Gaussian ensembles. The joint probability distribution of eigenvalues in the chiral Gaussian ensembles is

$$P^{(\beta,N)}_{\mathrm{Chiral}}(E) \sim \Delta^\beta_N(E) \prod_{n=1}^{N} E_n^{\beta-1} \exp\left(-\frac{E_n^2}{2\sigma^2}\right). \tag{3.18}$$

It is actually more refined when considering non-zero rectangularity, see [88].

The eigenvalue joint probability distribution of the Ginibre ensembles is a more difficult matter [125, 128]. Since the matrices are non-Hermitian, the eigenvalues are generally complex. In the complex case, the eigenvalues are independent up to a level repulsion that we have already seen in the Hermitian ensembles (3.8). In the time reversal invariant cases, however, the real structure introduces a pairing between eigenvalues. The eigenvalues of a real quaternion matrix come in complex conjugated pairs, and the eigenvalues of a real matrix are either real or come in complex conjugated pairs as well. This makes the real cases technically more demanding than the complex one. This statement applies in general to ensembles of non-Hermitian matrices. We will revisit this problem in section 3.4 and in chapter 5.

In the last section we mentioned Wigner's semicircle law for the asymptotic level density of the Gaussian ensembles. Its analogue for the level density of the Ginibre ensembles is the circular law. It states that in the limit of large matrix dimensions the eigenvalues are uniformly distributed over a circle in the complex plane. However, there is a caveat in the two real cases. In the quaternion case, as mentioned above, the eigenvalues form complex conjugate pairs. Thus, due to level repulsion, there is a depletion of eigenvalues along the real axis. In the real case, on the other hand, the eigenvalues accumulate along the real axis, which is due to the fact that the average number of real eigenvalues is asymptotically given by $\sqrt{2N/\pi}$, see [129, 130]. In [127] this is picturesquely referred to as the "Saturn effect". The circular law has a wide range of validity and applies to more general ensembles than the Ginibre ensembles [131].

Let us introduce yet another random matrix ensemble. It is the product ensemble of matrices

$$Y = K_1^{-1} K_2, \tag{3.19}$$

where K_1 and K_2 are Ginibre matrices in the same symmetry class and with equal scale parameter σ. These ensembles are referred to as spherical ensembles and they have been analyzed in several works [76–79]. Their matrix probability distribution is given by

$$\widetilde{G}^{(\beta,N)}(Y) = \pi^{-\beta N^2/2} \prod_{j=1}^{N} \frac{\Gamma\left(\beta(N+j)/2\right)}{\Gamma\left(\beta j/2\right)} \frac{1}{\det^{\beta N/\gamma}\left(\mathbb{1}_{\gamma N} + YY^\dagger\right)} \tag{3.20}$$

and is independent of the scale parameter σ. As mentioned before, the eigenvalue joint probability distribution of non-Hermitian matrices tend to be more complicated. We will refrain from stating them here and postpone this to the later chapters, where we will use them to calculate ensemble averages. In appendix A.3, we derive the joint probability distribution of eigenvalues for the real spherical ensemble.

The name of the spherical ensembles originates in the spherical law, which is once again a statement about the level density in the limit of large matrix dimensions. If the complex plane is mapped onto a sphere by stereographic projection, the eigenvalues are uniformly distributed over the sphere in this limit. As in the case of the circular law, there is a "Saturn effect", i.e. a depletion resp. accumulation of eigenvalues along the great circle corresponding to the real axis for the quaternion resp. real case.

Finally, we want to introduce a generalization of the spherical ensemble that will be useful later in simplifying our integrals. This generalization is again the product ensemble of $Y = K_1^{-1} K_2$, but now we draw K_1 and K_2 from a deformed Ginibre ensemble

$$\widetilde{P}_{\mu\nu}^{(\beta,N)}(K) = \pi^{-\beta N^2/2} \prod_{j=1}^{N} \frac{\Gamma\left(\beta j/2\right)}{\Gamma\left(\beta(j+2\mu)/2\right)} \exp\left(-\frac{1}{\gamma}\mathrm{tr}\, KK^\dagger\right) \det{}^{\mu/\gamma} KK^\dagger \tag{3.21}$$

with $\mu \geq 0$. We omit the scale parameter as the spherical ensemble is insensitive to it. These ensembles have been analyzed in [132–134]. Their matrix probability distributions are given by

$$\begin{aligned}\widetilde{G}_{\mu\nu}^{(\beta,N)}(Y) = \pi^{-\beta N^2/2} \prod_{j=1}^{N} & \frac{\Gamma\left(\beta j/2\right)\Gamma\left(\beta(N+2\mu+2\nu+j)/2\right)}{\Gamma\left(\beta(j+2\nu)/2\right)\Gamma\left(\beta(j+2\mu)/2\right)} \\ & \times \frac{\det^{\beta\nu/\gamma} YY^\dagger}{\det^{\beta(N+\mu+\nu/\gamma)}\left(\mathbb{1}_{\gamma N} + YY^\dagger\right)}.\end{aligned} \tag{3.22}$$

The ensembles (3.21) and (3.22) are also referred to as induced Ginibre and induced spherical ensembles respectively, as they can be generated by a specific inducing procedure, see [133–135]. There are also modified versions of the spherical law for the induced spherical ensembles. We refer to the mentioned works for details.

3.3 Parametric Random Matrix Models

In some physical situations, additional parameters are required. We provide examples and show how parametric dependence is implemented in a random matrix model. Furthermore, we discuss universality aspects that we want to tie in later on.

An extensively studied phenomenon is the transition between symmetry classes, e.g. through symmetry breaking. In a random matrix model, this can be realized by

$$H(p) = H_0 + p\, H_1 \tag{3.23}$$

with a parameter $p \in [0, 1]$. In [136,137] the breaking of time reversal invariance was studied. In this case, H_0 is a Gaussian orthogonal matrix and H_1 is imaginary antisymmetric with Gaussian distributed elements, such that $H(p)$ interpolates between the Gaussian orthogonal ensemble ($p = 0$, $\beta = 1$) and the Gaussian unitary ensemble ($p = 1$, $\beta = 2$). Similar models have been used in the study of isospin symmetry breaking induced by Coloumb interaction in nuclei [138, 139]. Unlike time reversal invariance, isospin symmetry is a true symmetry, described by a unitary operator that commutes with the Hamiltonian. Thus, the matrices H_0 and H_1 are set up as block diagonal and block off-diagonal, respectively.

The random matrix model may as well maintain its symmetry class for all parameters. Such a situation arises, for example, when studying the response of a chaotic system to an external perturbation. In [69–72] disordered metals subject to an Aharonov-Bohm flux were considered. A suitable model is

$$H(p) = \cos(p)H_1 + \sin(p)H_2, \tag{3.24}$$

where the parametric dependence is periodic, $p \in [0, 2\pi)$. The parametric dependence of the model is chosen quite arbitrarily. The line of reasoning is, as before, that random matrix theory is only useful to describe universal properties, which justifies the choice of the simplest non-trivial parametric dependence. Equivalently, one can also specify the two-point correlation function of the matrix elements

$$\langle H_{ij}(p_1)H^*_{kl}(p_2)\rangle = S(p_1, p_2)\delta_{ik}\delta_{jl}. \tag{3.25}$$

If the entries of $H(p)$ are Gaussian distributed at all points of the parameter manifold, higher correlations follow by Isserlis' theorem [140].

The spectral statistics of parametric random matrix models include correlations between levels at different points of the parameter manifold [141]. In [69–72, 142] it was shown that parametric correlations become universal when not only the spectrum is considered on a local scale, see section 3.1, but also the parametric dependence

$$\psi = p/\ell. \tag{3.26}$$

This scale is given by the correlation length

$$\ell = \frac{\Delta}{\sqrt{\left\langle \left(\frac{d}{dp} E_n(p) \right)^2 \right\rangle}} \tag{3.27}$$

and can be motivated as follows. On average, perturbing the system by ℓ should change the energy of a level by the mean level spacing Δ. We therefore require that the root mean square of the difference is

$$\sqrt{\left\langle (E_n(p) - E_n(p+\ell))^2 \right\rangle} = \Delta. \tag{3.28}$$

Suppose ℓ becomes small, eventually in a large N limit, (3.27) follows. The mean squared level velocity in (3.27) depends only on the two-point correlation $S(p_1, p_2)$ of the matrix model, see [72]. If this function is N-independent, then the correlation length ℓ is of the same order in N as the mean level spacing Δ, namely $\mathcal{O}(N^{-1/2})$. We will draw on these insights in chapter 4.

Parametric random matrix models such as (3.24) are also suitable for the Bloch Hamiltonian, see chapter 2, allowing an analysis of topological properties. In [73–75] the statistics of the Chern number, the relevant topological invariant of the unitary class A, in random matrix models with two-dimensional parametric dependence were studied. In the next chapters 4 and 5 we aim for a similar study of the winding number.

3.4 Supersymmetry without Supersymmetry

In this section we discuss the supersymmetry without supersymmetry technique, which we will employ in chapter 5 and which was developed in [82, 83]. It aims to compute ensemble averages over ratios of characteristic polynomials. These quantities are of wide interest in physics and mathematics, see the by far non-exhaustive list [82, 83, 143–153]. They play an important role, for example, as generators of spectral correlations [57, 61] or of scattering matrix elements [114]. As the name suggests, this technique is related to the supersymmetry method of random matrix theory. Therefore, we first want to provide a sketch of the latter.

Supersymmetry originated in particle physics as a possible resort to the no-go theorem of Coleman and Mandula [154–156], which states that the symmetries of a physical system and the symmetries of spacetime, described by the Poincaré group, cannot be subgroups of a larger non-trivial group, only their product group is possible. Following the arguments of supersymmetry implies the possibility of Bosons and Fermions transforming into each other. Nowadays, this hypothetical symmetry is notorious for its lack of experimental evidence.

Efetov adopted the mathematical framework of supersymmetry and applied it to the random matrix theory of disordered systems [80, 157]. The supersymmetry method is used to compute ensemble averages such as

$$\left\langle \prod_{j=1}^{k} \frac{\det\left(H - J_{j2}\right)}{\det\left(H - J_{j1}\right)} \right\rangle, \tag{3.29}$$

where H is a Hermitian random matrix and J_{1j} and J_{2j} are real numbers. The limitation is that the determinants in the denominator must be real, which is satisfied by our assumptions.

The supersymmetry method proceeds as follows. First, a finite number of anticommuting variables $\zeta = (\zeta_1, \ldots, \zeta_n)$, also called Grassmann variables, are introduced.

The combination of ζ with the same number of commuting variables $z = (z_1, \ldots, z_n)$, i.e. the tupel

$$\Psi = (z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n) \tag{3.30}$$

is called a supervector and is an element of the $n|n$-dimensional superspace. The vertical bar notation indicates the number of commuting and anticommuting variables, which are also referred to as Bosonic and Fermionic variables, respectively. This is an artefact of the particle physics origin of this description and, in the last consequence, it is due to the commutation resp. anticommutation relations of Bosonic and Fermionic fields. The transformations between commuting and anticommuting variables, i.e. Bosons and Fermions, are represented by supermatrices. A supermatrix has a 2×2-block structure

$$\sigma = \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{bmatrix}. \tag{3.31}$$

The diagonal blocks σ^{11} and σ^{22} have commuting entries and describe transformations within the Bosonic and Fermionic blocks of the superspace, while the off-diagonal blocks σ^{12} and σ^{21} have anticommuting entries and describe transformations between the Bosonic and Fermionic blocks. The definition of further structures such as supertraces, superdeterminants and supergroups is necessary. Accordingly, the field of mathematics dealing with these structures is called supermathematics, see [57, 80, 81, 157] for an overview.

The introduction of the superspace allows a facilitation of the ensemble average (3.29) in the following way. The determinants in the denominator and numerator are mapped to integrals over commuting and anticommuting variables, respectively. Assuming a Gaussian probability distribution in H, the ensemble average can be readily performed and one is left with an integral over a supermatrix model. A major advantage is that the dimension of the superspace is independent of the dimension N of the matrix ensemble. There are different ways to deal with the supermatrix model. Again, we refer to [57, 80, 81, 157] for details.

In section 3.1 we discussed integrals over ordinary random matrix models. When diagonalizing the matrix we obtain the Vandermonde determinant (3.6) as the Jacobian of this transformation. A similar transformation is possible in superspace, where the analogue of the Vandermonde determinant is the Berezinian. In the unitary case [158], it is given by

$$\sqrt{\mathrm{Ber}^{(2)}_{k|l}(\kappa_1; \kappa_2)} = \frac{\Delta_k(\kappa_1)\Delta_l(\kappa_2)}{\prod_{m=1}^{k}\prod_{n=1}^{l}(\kappa_{m1} - \kappa_{n2})}. \tag{3.32}$$

The superscript indicates the Dyson index $\beta = 2$, and the subscript the dimension of the underlying superspace, i.e. the length of the tupels $\kappa_1 = (\kappa_{11}, \ldots, \kappa_{k1})$ and

$\kappa_2 = (\kappa_{12}, \ldots, \kappa_{l2})$. A similar structure arises for $\beta = 1$ and $\beta = 4$, where the superspace has a real structure [82,159]. In the present context, however, we require only $\beta = 2$. Setting $k = 0$ or $l = 0$ recovers the Vandermonde determinant. Setting $k = l$ yields the determinant of a Cauchy matrix

$$\sqrt{\mathrm{Ber}^{(2)}_{k|k}(\kappa_1; \kappa_2)} = (-1)^{k(k-1)/2} \det \left[\frac{1}{\kappa_{m1} - \kappa_{n2}}\right]_{1 \leq m,n \leq k} . \tag{3.33}$$

Generally, the Berezinian can be written as a determinant [82]. For $k \leq l$ one obtains

$$\sqrt{\mathrm{Ber}^{(2)}_{k|l}(\kappa_1; \kappa_2)} = (-1)^{k(k-1)/2 + k(l+1)} \det \begin{bmatrix} \left[\dfrac{1}{\kappa_{a1} - \kappa_{b2}}\right]_{\substack{1 \leq a \leq k \\ 1 \leq b \leq l}} \\ \left[\kappa_{b2}^{a-1}\right]_{\substack{1 \leq a \leq l-k \\ 1 \leq b \leq l}} \end{bmatrix} , \tag{3.34}$$

which is a mixture of a Cauchy- and a Vandermonde matrix.

Just as with the "true" supersymmetry method, the aim of supersymmetry without supersymmetry is to calculate ensemble averages over ratios of characteristic polynomials. In this technique, we use supersymmetry structures, namely the Berezinian just introduced, to reformulate ensemble averages without mapping the integrals to superspace. This coins the term supersymmetry without supersymmetry.

We define

$$Z^{(\beta,N)}_{k|l}(\kappa_1, \kappa_2) = \left\langle \frac{\prod_{j=1}^{l} \det\left(K - \kappa_{j2}\right)}{\prod_{j=1}^{k} \det\left(K - \kappa_{j1}\right)} \right\rangle , \tag{3.35}$$

where the superscript is again the Dyson index β and the matrix dimension N. In this case, there are no constraints concerning the reality of the determinants.

Let us first assume a random matrix ensemble with unitary symmetry, i.e. $\beta = 2$. The joint probability distribution of eigenvalues is

$$P^{(2,N)}(K) = \frac{|\Delta_N(z)|^2}{c^{(2,N)}} \prod_{j=1}^{N} g^{(2,N)}(z_j) \tag{3.36}$$

with an ensemble dependent function $g^{(2,N)}(z)$ and a normalization constant $c^{(2,N)}$. Although arbitrary k and l are possible, we will only discuss the case $k = l$, which is also the one we consider in chapter 5. We refer to the original work [82] for the general case $k \neq l$. In eigenvalue coordinates, the ensemble average becomes

$$Z^{(2,N)}_{k|k}(\kappa_1, \kappa_2) = \frac{1}{c^{(2,N)}} \int\limits_{\mathbb{C}^N} d[z] \, |\Delta_N(z)|^2 \prod_{n=1}^{N} \left[g^{(2,N)}(z_n) \prod_{j=1}^{k} \frac{z_n - \kappa_{j2}}{z_n - \kappa_{j1}} \right] . \tag{3.37}$$

The first crucial step is to identify a part of the integrand as a quotient of Berezinians

$$\frac{\sqrt{\mathrm{Ber}^{(2)}_{k|k+N}(\kappa_1;\kappa_2,z)}}{\sqrt{\mathrm{Ber}^{(2)}_{k|k}(\kappa_1;\kappa_2)}} = (-1)^{Nk}\Delta_N(z)\prod_{n=1}^{N}\prod_{j=1}^{k}\frac{z_n-\kappa_{j2}}{z_n-\kappa_{j1}}. \tag{3.38}$$

This identity can be quickly verified following (3.32). Inserting it yields

$$\begin{aligned} Z^{(2,N)}_{k|k}(\kappa_1,\kappa_2) =& \frac{(-1)^{Nk}}{c^{(2,N)}}\frac{1}{\sqrt{\mathrm{Ber}^{(2)}_{k|k}(\kappa_1;\kappa_2)}} \\ &\times\int\limits_{\mathbb{C}^N} d[z]\det\left[(z_a^*)^{b-1}g^{(2,N)}(z_a)\right]_{1\le a,b\le N}\sqrt{\mathrm{Ber}^{(2)}_{k|k+N}(\kappa_1;\kappa_2,z)}, \end{aligned} \tag{3.39}$$

where we wrote the factors $g^{(2,N)}(z_n)$ into the remaining Vandermonde determinant. Writing the Berezinian in the integrand in the determinant form (3.34) leaves us with an integral over the product of two determinants in which the integration variables z_n are separated row by row (or column by column). According to Andréief's theorem, the solution is again a determinant [160,161]. In this case, we need a generalized version of this theorem because some rows are independent of the integration variables [82]. This yields

$$Z^{(2,N)}_{k|k}(\kappa_1,\kappa_2) = \frac{(-1)^{Nk}N!}{c^{(2,N)}}\frac{1}{\sqrt{\mathrm{Ber}^{(2)}_{k|k}(\kappa_1;\kappa_2)}}\det\begin{bmatrix}[C(\kappa_{b1},\kappa_{a2})]_{1\le a,b\le k} & [V(\kappa_{a2})]_{1\le a\le k}\\ [F(\kappa_{b1})]_{1\le b\le k} & M\end{bmatrix}, \tag{3.40}$$

which is an $(N+k)\times(N+k)$-determinant with the blocks

$$\begin{aligned} C(\kappa_{b1},\kappa_{a2}) &= \frac{1}{\kappa_{b1}-\kappa_{a2}}, & F(\kappa_{b1}) &= \left[\int\limits_{\mathbb{C}} d[z]\frac{g^{(2,N)}(z)(z^*)^{a-1}}{\kappa_{b1}-z}\right]_{1\le a\le N}, \\ V(\kappa_{a2}) &= \left[\kappa_{a2}^{b-1}\right]_{1\le b\le N}, & M &= \left[\int\limits_{\mathbb{C}} d[z]g^{(2,N)}(z)(z^*)^{a-1}z^{b-1}\right]_{1\le a,b\le N}. \end{aligned} \tag{3.41}$$

Thus, the ensemble average is reduced to calculating the integrals over the complex plane in (3.41) and evaluating the determinant. Often, this task is not feasible when N is large. However, we may apply the identity

$$\det\begin{bmatrix}A & B\\ C & D\end{bmatrix} = \det[D]\det\left[A-BD^{-1}C\right] \tag{3.42}$$

in order to reduce the dimension of the determinant. This yields a $k \times k$-determinant for the ensemble average

$$\begin{aligned} Z_{k|k}^{(2,N)}(\kappa_1,\kappa_2) =& \frac{(-1)^{Nk} N!}{c^{(2,N)}} \frac{\det[M]}{\sqrt{\mathrm{Ber}_{k|k}^{(2)}(\kappa_1;\kappa_2)}} \\ & \times \det\left[C(\kappa_{b1},\kappa_{a2}) - V(\kappa_{a2}) M^{-1} F(\kappa_{b1})\right]_{1\leq a,b\leq k} \\ =& \frac{(-1)^{Nk}}{\sqrt{\mathrm{Ber}_{k|k}^{(2)}(\kappa_1;\kappa_2)}} \det\left[C(\kappa_{b1},\kappa_{a2}) - V(\kappa_{a2}) M^{-1} F(\kappa_{b1})\right]_{1\leq a,b\leq k} . \end{aligned} \tag{3.43}$$

In the second step we inserted

$$c^{(2,N)} = N! \det[M], \tag{3.44}$$

which is a consequence of Andréief's theorem as well. An evaluation of the function inside the determinant might be possible in certain situations, e.g. if M is diagonal. However, we can also identify this function with the left-hand side of the equation by setting $k = 1$

$$C(\kappa_1,\kappa_2) - V(\kappa_2) M^{-1} F(\kappa_1) = (-1)^N \sqrt{\mathrm{Ber}_{1|1}^{(2)}(\kappa_1;\kappa_2)} Z_{1|1}^{(2,N)}(\kappa) \tag{3.45}$$

and write the ensemble average as

$$Z_{k|k}^{(2,N)}(\kappa_1,\kappa_2) = \frac{\det\left[\mathrm{K}(\kappa_{b1},\kappa_{a2})\right]_{1\leq a,b\leq k}}{\sqrt{\mathrm{Ber}_{k|k}^{(2)}(\kappa_1;\kappa_2)}} \tag{3.46}$$

with the determinant kernel

$$\mathrm{K}(\kappa_1,\kappa_2) = \frac{Z_{1|1}^{(2,N)}(\kappa)}{\kappa_1 - \kappa_2}, \tag{3.47}$$

where we inserted the Berezinian for $k = 1$. This is the final expression resulting from supersymmetry without supersymmetry for $k = l$ in the unitary case. The problem of calculating the ensemble average for general k is reduced to $k = 1$. This function can often be evaluated with other methods of random matrix theory. In chapter 5 we use a certain symmetry of our random matrix model to simplify this average.

Let us now move on to the symplectic and the orthogonal case. The eigenvalue joint probability distribution of a random matrix ensemble with symplectic symmetry ($\beta = 4$) is

$$P^{(4,N)}(z) = \frac{\Delta_{2N}(z)}{c^{(4,N)}} \prod_{j=1}^{N} g^{(4,N)}(z_{2j-1}, z_{2j}). \tag{3.48}$$

There are $2N$ eigenvalues which come in complex conjugate pairs. This is described by the antisymmetric two-point measure

$$g^{(4,N)}(z_1, z_2) = g_{\mathbb{C}}^{(4,N)}(z_1, z_1^*)\delta(z_1^* - z_2), \tag{3.49}$$

where we use a Dirac delta function for complex numbers. Its antisymmetry guarantees the permutation invariance of the eigenvalues. In the orthogonal case ($\beta = 1$) we have to distinguish between even and odd N

$$\begin{aligned} P^{(1,N)}(z) &= \frac{\Delta_N(z)}{c^{(1,N)}} \prod_{j=1}^{N/2} g^{(1,N)}(z_{2j-1}, z_{2j}), \\ P^{(1,N)}(z) &= \frac{\Delta_N(z)}{c^{(1,N)}} h(z_N) \prod_{j=1}^{(N-1)/2} g^{(1,N)}(z_{2j-1}, z_{2j}). \end{aligned} \tag{3.50}$$

Here, the eigenvalues are either real or complex conjugate pairs. Therefore the antisymmetric two-point measure is

$$g^{(1,N)}(z_1, z_2) = g_{\mathbb{R}}^{(1,N)}(z_1, z_2)\delta(y_1)\delta(y_2) + g_{\mathbb{C}}^{(1,N)}(z_1, z_1^*)\delta(z_1^* - z_2). \tag{3.51}$$

We denote the complex eigenvalues as $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$. In the odd case, there is an unpaired eigenvalue which is always real and is described by the function $h(z)$.

For the sake of simplicity, we assume N even when dealing with the orthogonal case. We refer to [83] for the general case. Although we need only the case $k = l$, we have to consider $k \neq l$ here. The reason for this will become clear later on. In eigenvalue coordinates, the ensemble average is

$$\begin{aligned} Z_{k|l}^{(\beta,N)}(\kappa_1, \kappa_2) &= \left\langle \frac{\prod_{j=1}^{l} \det(K - \kappa_{j2})}{\prod_{j=1}^{k} \det(K - \kappa_{j1})} \right\rangle \\ &= \frac{1}{c^{(\beta,N)}} \int_{\mathbb{C}^N} d[z]\Delta_N(z) \prod_{n=1}^{N/2} g^{(\beta,N)}(z_{2n-1}, z_{2n}) \prod_{n=1}^{N} \frac{\prod_{j=1}^{l}(z_n - \kappa_{j2})}{\prod_{j=1}^{k}(z_n - \kappa_{j1})} \\ &= \frac{(-1)^{Nk}}{c^{(\beta,N)}} \frac{1}{\sqrt{\mathrm{Ber}_{k|l}^{(2)}(\kappa_1; \kappa_2)}} \int_{\mathbb{C}^N} d[z] \prod_{n=1}^{N/2} g^{(\beta,N)}(z_{2n-1}, z_{2n}) \sqrt{\mathrm{Ber}_{k|l+N}^{(2)}(\kappa_1; \kappa_2, z)}. \end{aligned} \tag{3.52}$$

In the last step we inserted the identity for $k \neq l$

$$\frac{\sqrt{\mathrm{Ber}_{k|l+N}^{(2)}(\kappa_1;\kappa_2,z)}}{\sqrt{\mathrm{Ber}_{k|l}^{(2)}(\kappa_1;\kappa_2)}} = (-1)^{Nk}\Delta_N(z)\prod_{n=1}^{N}\frac{\prod_{j=1}^{l}(z_n-\kappa_{j2})}{\prod_{j=1}^{k}(z_n-\kappa_{j1})}, \tag{3.53}$$

which can again be verified following (3.32). The Berezinian can again be written in the determinant form (3.34), while tacitly assuming $l+N \geq k$. In this determinant all integration variables are separated row by row (or column by column). However, the function $g^{(\beta,N)}(z_1,z_2)$ couples two rows, making a row wise integration not possible. The solution of this integral is given by de Bruijn's theorem [162]. In this case, too, we have to resort to a generalized form of this theorem since the determinant has rows that do not dependent on the eigenvalues [82]. The result is a Pfaffian

$$\begin{aligned} Z_{k|l}^{(\beta,N)}(\kappa_1,\kappa_2) =& \frac{(-1)^{k(k-1)/2+l(l-1)/2+k(l+1)}(N/2)!}{c^{(\beta,N)}}\frac{1}{\sqrt{\mathrm{Ber}_{k|l}^{(2)}(\kappa_1;\kappa_2)}} \\ &\times \mathrm{Pf}\begin{bmatrix} 0 & [C(\kappa_{b1},\kappa_{a2})]_{\substack{1\leq a\leq l\\1\leq b\leq k}} & \left[V^T(\kappa_{a2})\right]_{1\leq a\leq l} \\ [-C(\kappa_{a1},\kappa_{b2})]_{\substack{1\leq a\leq k\\1\leq b\leq l}} & [G(\kappa_{a1},\kappa_{b1})]_{1\leq a,b\leq k} & \left[F^T(\kappa_{a1})\right]_{1\leq a\leq k} \\ [-V(\kappa_{b2})]_{1\leq b\leq l} & [-F(\kappa_{b1})]_{1\leq b\leq k} & D^{(d)}, \end{bmatrix} \end{aligned} \tag{3.54}$$

of an $(N+2l)\times(N+2l)$-matrix. The blocks are

$$\begin{aligned} C(\kappa_{b1},\kappa_{a1}) &= \frac{1}{\kappa_{b1}-\kappa_{a2}}, \qquad\qquad V(\kappa_{b2}) = \left[\kappa_{b2}^{a-1}\right]_{1\leq a\leq d}, \\ G(\kappa_{a1},\kappa_{b1}) &= \int_{\mathbb{C}^2} g^{(\beta,N)}(z_1,z_2)\left(\frac{1}{\kappa_{a1}-z_1}\frac{1}{\kappa_{b1}-z_2}-\frac{1}{\kappa_{b2}-z_1}\frac{1}{\kappa_{a1}-z_2}\right), \\ F(\kappa_{b1}) &= \left[\int_{\mathbb{C}^2} g^{(\beta,N)}(z_1,z_2)\left(\frac{z_2^{a-1}}{\kappa_{b1}-z_1}-\frac{z_1^{a-1}}{\kappa_{b1}-z_2}\right)\right]_{1\leq a\leq d}, \\ D^{(d)} &= \left[\int_{\mathbb{C}^2} d[z]g^{(\beta,N)}(z_1,z_2)(z_1^{a-1}z_2^{b-1}-z_1^{b-1}z_2^{a-1})\right]_{1\leq a,b\leq d} \end{aligned} \tag{3.55}$$

and we define the parameter $d = N+l-k$.

The Pfaffian is ubiquitous in random matrix theory and important to our results in chapter 5. It is a function of an antisymmetric matrix with even dimension. Let A be an antisymmetric $2n\times 2n$-matrix. Its Pfaffian is given by

$$\mathrm{Pf}[A] = \frac{1}{2^n n!}\sum_{\sigma\in\mathbb{S}_{2n}}\operatorname{sgn}\sigma\prod_{j=1}^{n}A_{\sigma(2j-1)\sigma(2j)}, \tag{3.56}$$

which is a polynomial in the matrix entries and takes the value

$$\operatorname{Pf} A = \sqrt{\det A}. \tag{3.57}$$

In other words, the determinant of an antisymmetric even-dimensional matrix is the square of a polynomial in the matrix entries. This polynomial is given by (3.56). Because of the square, it remains to fix a sign convention. For (3.56) we have

$$\operatorname{Pf}\left[\mathbb{1}_n \otimes i\tau_2\right] = +1 \tag{3.58}$$

with the second Pauli matrix τ_2.

We continue by employing the decomposition

$$\operatorname{Pf}\begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} = \operatorname{Pf}[C]\operatorname{Pf}[A + BC^{-1}B^T] \tag{3.59}$$

in (3.54). Just as in the unitary case, we do this to reduce the dimension. For this step we need to assume that d is even, which is satisfied when $k = l$. We obtain a $(k+l)$-Pfaffian with a 2×2-block structure

$$\begin{aligned} Z_{k|l}^{(\beta,N)}(\kappa_1,\kappa_2) =& \frac{(-1)^{k(k-1)/2+l(l-1)/2+k(l+1)}(N/2)!}{c^{(\beta,N)}} \frac{\operatorname{Pf}[D^{(d)}]}{\sqrt{\operatorname{Ber}_{k|l}^{(2)}(\kappa_1;\kappa_2)}} \\ & \times \operatorname{Pf}\begin{bmatrix} \left[\mathrm{K}_1^{(d)}(\kappa_{a2},\kappa_{b2})\right]_{1\leq a,b\leq l} & \left[\mathrm{K}_2^{(d)}(\kappa_{b1},\kappa_{a2})\right]_{\substack{1\leq a\leq l\\ 1\leq b\leq k}} \\ \left[-\mathrm{K}_2^{(d)}(\kappa_{a1},\kappa_{b2})\right]_{\substack{1\leq a\leq k\\ 1\leq b\leq l}} & \left[\mathrm{K}_3^{(d)}(\kappa_{a1},\kappa_{b1})\right]_{1\leq a,b\leq k} \end{bmatrix}. \end{aligned} \tag{3.60}$$

The kernels are given by

$$\begin{aligned} \mathrm{K}_1^{(d)}(\kappa_{a2},\kappa_{b2}) &= V^T(\kappa_{a2})\left[D^{(d)}\right]^{-1} V(\kappa_{b2}), \\ \mathrm{K}_2^{(d)}(\kappa_{a1},\kappa_{b2}) &= C(\kappa_{a1},\kappa_{b2}) + V^T(\kappa_{b2})\left[D^{(d)}\right]^{-1} F(\kappa_{a1}), \\ \mathrm{K}_3^{(d)}(\kappa_{a1},\kappa_{b1}) &= G(\kappa_{a1},\kappa_{b1}) + F^T(\kappa_{a1})\left[D^{(d)}\right]^{-1} F(\kappa_{b1}). \end{aligned} \tag{3.61}$$

We note that the matrix in the Pfaffian is antisymmetric which can be verified following (3.55) and (3.61). Instead of computing these expressions we determine the kernels by consecutively setting $k = 0, l = 2$ and $k = l = 1$ and $k = 2, l = 0$ and comparing with the left-hand side of the equation. This gives the expressions

$$\begin{aligned} \mathrm{K}_1^{(d)}(\kappa_{a2},\kappa_{b2}) &= -\frac{c^{(\beta,d-2)}}{[(d-2)/2]!}\frac{\kappa_{b2}-\kappa_{a2}}{\operatorname{Pf}[D^{(d)}]} Z_{0|2}^{(\beta,d-2)}(\kappa_{a2},\kappa_{b2}), \\ \mathrm{K}_2^{(d)}(\kappa_{a1},\kappa_{b2}) &= \frac{c^{(\beta,d-2)}}{(d/2)!}\frac{1}{\kappa_{a1}-\kappa_{b2}}\frac{1}{\operatorname{Pf}[D^{(d)}]} Z_{1|1}^{(\beta,d)}(\kappa_{a1},\kappa_{b2}), \\ \mathrm{K}_3^{(d)}(\kappa_{a1},\kappa_{b1}) &= \frac{c^{(\beta,d+2)}}{[(d+2)/2]!}\frac{\kappa_{b1}-\kappa_{a1}}{\operatorname{Pf}[D^{(d)}]} Z_{2|0}^{(\beta,d+2)}(\kappa_{a1},\kappa_{b2}). \end{aligned} \tag{3.62}$$

Therefore, the problem of finding the ensemble average is reduced to averaging only two characteristic polynomials. For $k = l$ we obtain

$$Z_{k|k}^{(\beta,N)}(\kappa_1, \kappa_2) = \frac{1}{\sqrt{\mathrm{Ber}_{k|k}^{(2)}(\kappa_1; \kappa_2)}} \mathrm{Pf} \begin{bmatrix} \hat{\mathrm{K}}_1(\kappa_{a2}, \kappa_{b2}) & \hat{\mathrm{K}}_2(\kappa_{b1}, \kappa_{a2}) \\ -\hat{\mathrm{K}}_2(\kappa_{a1}, \kappa_{b2}) & \hat{\mathrm{K}}_3(\kappa_{a1}, \kappa_{b1}) \end{bmatrix}_{1 \leq a,b \leq k} \tag{3.63}$$

with

$$\begin{aligned} \hat{\mathrm{K}}_1(\kappa_{a2}, \kappa_{b2}) &= -\frac{c^{(\beta,N-2)}}{[(N-2)/2]!} \frac{\kappa_{b2} - \kappa_{a2}}{\mathrm{Pf}[D^{(N)}]} Z_{0|2}^{(\beta,N-2)}(\kappa_{a2}, \kappa_{b2}), \\ \hat{\mathrm{K}}_2(\kappa_{a1}, \kappa_{b2}) &= \frac{c^{(\beta,N-2)}}{(N/2)!} \frac{1}{\kappa_{a1} - \kappa_{b2}} \frac{1}{\mathrm{Pf}[D^{(N)}]} Z_{1|1}^{(\beta,N)}(\kappa_{a1}, \kappa_{b2}), \\ \hat{\mathrm{K}}_3(\kappa_{a1}, \kappa_{b1}) &= \frac{c^{(\beta,N+2)}}{[(N+2)/2]!} \frac{\kappa_{b1} - \kappa_{a1}}{\mathrm{Pf}[D^{(N)}]} Z_{2|0}^{(\beta,N+2)}(\kappa_{a1}, \kappa_{b2}). \end{aligned} \tag{3.64}$$

Here we used

$$c^{(\beta,N)} = (N/2)!\, \mathrm{Pf}\left[D^{(N)}\right], \tag{3.65}$$

which is a consequence of de Bruijn's theorem.

The Pfaffian (3.63) together with the three ensemble dependent kernel functions (3.64) is the final expression of supersymmetry without supersymmetry for $k = l$ in the symplectic case and the orthogonal case for even matrix dimensions. The ensemble average for general k is reduced to $k + l = 2$. We use this result in chapter 5. There we compute the kernel functions using a symmetry of our random matrix model and other methods such as skew-orthogonal polynomials.

There is a deeper reason why we arrive at the determinantal expression (3.46) for the ensemble average in the unitary case. It reflects that the spectrum is described by a determinantal point process, meaning that the eigenvalue correlations can be written as a determinant of an ensemble dependent kernel [58, 163]. On the other hand, the symplectic and the orthogonal case follow a Pfaffian point process, which leads to Pfaffian expressions (3.63) for the ensemble average.

Chapter 4

Winding Number Statistics of a Parametric Chiral Unitary Random Matrix Ensemble

This chapter deals with the winding number statistics in systems with broken time reversal invariance belonging to the chiral unitary class AIII. It is based on [1]. In section 4.1 we deepen our motivation for a statistical analysis of topological invariants in disordered systems. In section 4.2 we set up the random matrix model and define the goals of this chapter. Section 4.3 contains the main idea of our calculations and our results. More involved derivations are relegated to section 4.4. We conclude in section 4.5.

4.1 Introduction

Translationally invariant one-dimensional chiral systems are characterized by the winding number. Systems with non-zero winding number W are topologically non-trivial, and therefore, according to bulk-boundary correspondence, have $|W|$ modes localized to each boundary [27, 164, 165].

When (discrete) translation invariance is broken by disorder obeying chiral symmetry, crystal momentum is no longer a good quantum number. Nevertheless, it is possible to express the winding number in position representation and calculate it for a system with closed boundary conditions. It is then found in weakly disordered systems that the winding number is self-averaging and robust in the thermodynamic limit [25, 166]. On the other hand, if disorder is strong, the winding number may no longer be calculated by spatial averaging. Instead, one can assume that the disorder is itself periodic with a large period, so that the disordered system remains periodic with a large disordered unit cell. With this assumption, eventually taking the limit of a large disorder period, the winding number can be defined as a property of the Bloch Hamiltonian, see section 2.3, and becomes a random variable.

The probability distribution of the winding number in periodic systems with a disordered unit cell depends on that of the disorder, but like the spectral statistics, it may turn out that the winding number statistics becomes universal when the unit cell becomes large, and moreover that the universal distribution can be reproduced by random matrix models. This question has not yet been addressed in the chiral classes, but there are precursors in the unitary class A, where Hamiltonians with two-dimensional parametric dependence are topologically classified by the (first) Chern number. Random matrix models defined on compact two-dimensional parameter spaces were studied in [73–75], showing that the Chern number distribution is Gaussian with a universal covariance.

In [74] the Chern number covariance was calculated as an integral of the correlation function of the adiabatic curvature, the analogue of the winding number density, which is universal as well. The two-point correlation function of the adiabatic curvature follows a scaling form, with a scale parameter equal to the density of states multiplied by the correlation length of the elements of the random matrices in parameter space, and with a universal scaling function. Universal scaling behaviour of this kind has been known for a long time in parametric correlations of spectral properties of random matrices, such as the density and current of energy states [72]. Furthermore, the universal properties of parametric spectral correlations of random matrices agree with those of disordered systems [69–71], motivating the universality hypothesis for the correlations of the adiabatic curvature and its chiral class analogue, the winding number density, and a fortiori the probability distributions of Chern numbers and winding numbers.

In this chapter we first propose a minimal parametric random matrix model for the chiral Hamiltonian. We conjecture that it captures universal properties of the winding number and the winding number density. We next aim to calculate the discrete probability distribution of the winding numbers as well as its first two moments, showing that the width of the winding number distribution grows as the fourth root of the matrix size. We also wish to compute the parametric correlation functions of the winding number density. Furthermore, we discuss aspects of universality for the two-point function by identifying an unfolding procedure.

4.2 Posing the Problem

Following the discussion in chapters 2 and 3, the lack of time reversal invariance causes the Hamiltonian to be a complex Hermitian matrix. Adding chiral symmetry we end up in the symmetry class AIII of the tenfold way, also referred to as the chiral unitary class. In the chiral classes, the Hamiltonian can be brought into a block off-diagonal form, when considered in the diagonal basis of the chiral operator.

This is directly inherited by the Bloch Hamiltonian

$$H(p) = \begin{bmatrix} 0 & K(p) \\ K^\dagger(p) & 0 \end{bmatrix}, \tag{4.1}$$

where we take $K(p)$ as an $N \times N$-matrix. In our interpretation above, N is the dimension of the unit cell. The winding number of such systems is discussed in section 2.3. For ease of reference, we write the relevant expressions once again. The winding number is given by

$$W = \frac{1}{2\pi i} \oint\limits_{\det K(p)} \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_0^{2\pi} dp\, w(p) \tag{4.2}$$

with the winding number density

$$w(p) = \frac{d}{dp} \ln \det K(p) = \frac{1}{\det K(p)} \frac{d}{dp} \det K(p) = \operatorname{tr} K^{-1}(p) \frac{d}{dp} K(p) \tag{4.3}$$

as the logarithmic derivative of the determinant.

To set up a concrete random matrix model for the Bloch Hamiltonian, we recall the discussion in section 3.3 about the universality in the spectra of parametric random matrix ensembles. In random matrix models with N independent scale parameter, the mean level spacing is of order $\mathcal{O}(N^{-1/2})$. Likewise, the correlation length, the scale on which energy levels lose correlation, is also of order $\mathcal{O}(N^{-1/2})$. Considering that universality can only emerge on these local scales, it is justified to study simplest generic models. We therefore choose the parametric dependence in the explicit form

$$K(p) = \cos(p)K_1 + \sin(p)K_2, \tag{4.4}$$

where the matrices K_1 and K_2 are $N \times N$-dimensional complex matrices with independently Gaussian distributed elements, i.e. members of the complex Ginibre ensemble

$$\widetilde{P}^{(2,N)}(K) = \pi^{-N^2} \exp\left(-\operatorname{tr} KK^\dagger\right), \tag{4.5}$$

see section 3.2. The correlation of matrix elements is given by

$$\left\langle K^*_{ij}(p_1) K_{kl}(p_2) \right\rangle = 2\cos\left(p_1 - p_2\right) \delta_{ik}\delta_{jl}, \tag{4.6}$$

which depends only on the distance $|p_1 - p_2|$. We refer to this property as translation invariance on the parameter manifold. The associated Hamiltonians

$$H(p) = \cos(p)H_1 + \sin(p)H_2 \qquad \text{with} \qquad H_m = \begin{bmatrix} 0 & K_m \\ K_m^\dagger & 0 \end{bmatrix},\ m = 1, 2, \tag{4.7}$$

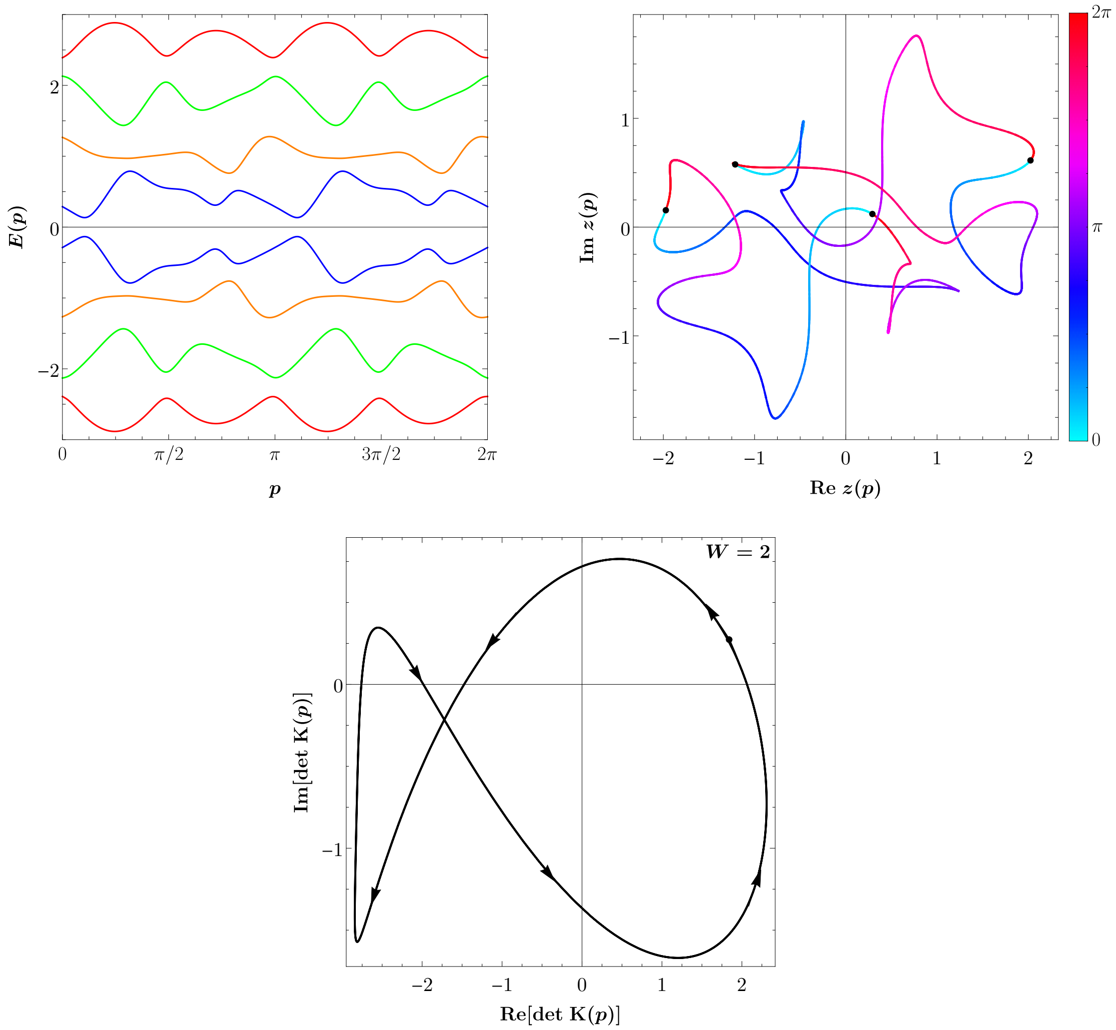


Figure 4.1: A realization of an AIII Hamiltonian with $K(p) = \cos(p)K_1 + \sin(p)K_2$ and some fixed 4×4 complex matrices K_1 and K_2. The top left plot shows the real eigenvalues of $H(p)$, the top right one shows the generically complex eigenvalues of $K(p)$, and the bottom plot depicts the determinant $\det K(p)$. All plots show the parametric dependence in $p \in [0, 2\pi)$, where we have employed the step size $2\pi/100$ and a B-Spline to obtain the curves. In both of the parametric plots the starting points $p = 0$ are marked by black points and the directions are marked by a color gradient resp. arrows.

can be viewed as a parametric combination of two chiral Gaussian unitary ensembles. We also refer to $K(p)$ and $H(p)$ as the random matrix field. Our random matrix model is the chiral version of the model proposed in [73] to study statistics of Chern numbers in the unitary class, justified in a similar manner.

In figure 4.1 we illustrate the spectral flow of the matrix field with generic complex $K_1, K_2 \in \mathbb{C}^{4\times 4}$. We show the eight eigenvalues of $H(p)$, the four complex eigenvalues of $K(p)$, and the determinant $\det K(p)$. Two important observations can be made, which hold true also for matrix fields other than (4.4). The order of the eigenvalues of $H(p)$ remains the same for all p when level repulsion governs the spectral statistics, which implies that each eigenvalue of $H(p)$ is a 2π-periodic function. In contrast, the eigenvalue spectrum of $K(p)$ may experience a permutation, meaning when running once from $p = 0$ to $p = 2\pi$, a chosen eigenvalue can become another one. Thence, the eigenvalues of $K(p)$ might have a different period than 2π and are therefore not suitable for a topological classification. The determinant $\det K(p)$ is more suitable as this quantity must be 2π-periodic. For the specific choice of the parametric dependence (4.4), we have $K(p + \pi) = -K(p)$, which manifests itself as a point symmetry around the origin of the curves describing the eigenvalues of $K(p)$. It also restricts the amount of times $\det K(p)$ winds around the origin to be an even resp. odd number for even resp. odd matrix dimensions.

With the random matrix model prepared, we now define our goals. We calculate the k-point correlation function of the winding number density

$$C_k^{(2,N)}(p_1, \dots, p_k) = \langle w(p_1) \cdots w(p_k) \rangle \tag{4.8}$$

as a random matrix ensemble average. The superscript refers to the Dyson index $\beta = 2$, i.e. it labels the symmetry class, and to the matrix dimension N. The precise meaning of the angular brackets indicating the ensemble average will be given in the next section. The arguments $p = (p_1, \dots, p_k) \in [0, 2\pi)^k$ are the different points on the parameter manifold. Furthermore, we compute the distribution of winding numbers $P(W)$. An exact expression for its moments

$$\left\langle W^k \right\rangle = \sum_{W\in\mathbb{Z}} W^k P(W) = \frac{1}{(2\pi i)^k} \int_0^{2\pi} dp_1 \cdots \int_0^{2\pi} dp_k \; C_k^{(2,N)}(p_1, \dots, p_k) \tag{4.9}$$

is given in terms of the k-point correlation function.

We would like to point out that the random matrix field (4.4) is equally well suited to describe any other complex system in the chiral unitary class. One example are disordered system subject to an Aharanov-Bohm flux, taking the role of the parameter p. The universal spectral statistics of such systems has been studied in [69–72]. Furthermore, chiral random matrix theory is a fruitful approach to low temperature quantum chromodynamics [62–68,88] and our calculations could be relevant in cases where a quantum chromodynamical system depends on a parameter.

4.3 Results

In section 4.3.1 we sketch the strategy for our calculation of the k-point correlation function. The details of the derivations are collected in section 4.4. Quite remarkably, we arrive at closed-form results for arbitrary k and particularly simple expressions for $k = 1$ and $k = 2$. In section 4.3.2 we discuss aspects of universality and unfold the parametric dependence of the two-point function. We conjecture the resulting limit to be universal. The winding number distribution and its moments are addressed in section 4.3.3.

4.3.1 Expressions and Results for the k-Point Correlation Function

For the specific form of our random matrix field (4.4), noting that $\det K_1 \neq 0$ with probability one, we can evaluate the logarithm appearing in (4.3) as

$$\begin{aligned} \ln\det K(p) &= \ln\det K_1 + N \ln\sin p + \ln\det\left(\cot p + K_1^{-1}K_2\right) \\ &= \ln\det K_1 + N\ln\sin p + \sum_{n=1}^{N} \ln\left(\cot p + z_n\right), \end{aligned} \tag{4.10}$$

where z_n are the complex eigenvalues of the matrix $K_1^{-1}K_2$, we also use $z = (z_1, \ldots, z_N)$. Taking the derivative yields the (unaveraged) winding number density

$$w(p) = N\cot p - \frac{1}{\sin^2 p}\sum_{n=1}^{N}\frac{1}{\cot p + z_n}. \tag{4.11}$$

Here and below, intermediate singularities at $p = 0$ and $p = \pi$ cancel to yield analytic correlations functions for all values of p. The matrices $K_1^{-1}K_2$ form the complex spherical ensemble [76, 77], see also our discussion in section 3.2. The corresponding joint eigenvalue distribution is known

$$\begin{aligned} G^{(2,N)}(z) &= \frac{|\Delta_N(z)|^2}{c^{(2,N)}}\prod_{n=1}^{N}\frac{1}{(1+|z_n|^2)^{N+1}}, \\ c^{(2,N)} &= \pi^N N! \prod_{n=1}^{N} \mathrm{B}(n, N-n+1). \end{aligned} \tag{4.12}$$

Again, the superscript refers to the Dyson index $\beta = 2$ and to the matrix dimension N. Furthermore, $\mathrm{B}(n,m) = \Gamma(n)\Gamma(m)/\Gamma(n+m)$ is the Euler Beta function [167] and

$$\Delta_N(z) = \prod_{1\leq n<m\leq N}(z_m - z_n) = \det\left[z_n^{m-1}\right]_{n,m=1,\ldots,N} \tag{4.13}$$

is the Vandermonde determinant. With the volume element over the combined N complex planes

$$d[z] = \prod_{n=1}^{N} d[z_n], \quad \text{where} \quad d[z_n] = d\mathrm{Re}z_n \, d\mathrm{Im}z_n, \tag{4.14}$$

we eventually arrive at a precise definition for the ensemble average of a function $F(z) = F(z_1, \ldots, z_N)$ as

$$\begin{aligned} \langle F(z) \rangle &= \int\limits_{\mathbb{C}^N} d[z] G^{(2,N)}(z) F(z) \\ &= \int\limits_{\mathbb{C}} d[z_1] \cdots \int\limits_{\mathbb{C}} d[z_N] \, G^{(2,N)}(z_1, \ldots, z_N) \, F(z_1, \ldots, z_N). \end{aligned} \tag{4.15}$$

In particular, for (4.8), we find that

$$C_k^{(2,N)}(p_1, \ldots, p_k) = \int\limits_{\mathbb{C}^N} d[z] \, G^{(2,N)}(z) \, w(p_1) \cdots w(p_k) \tag{4.16}$$

is the integral we have to compute. For convenience, we suppress the z-dependence in the argument of the function $w(p)$.

To proceed with the calculation of the integral (4.16), we observe that the winding number density $w(p)$ according to (4.11) features a term independent of the eigenvalues z_n. We subtract this term by defining

$$\begin{aligned} y(p) &= w(p) - Nq = -\frac{1}{\sin^2 p} \sum_{n=1}^{N} \frac{1}{q + z_n}, \\ q &= \cot p \end{aligned} \tag{4.17}$$

and calculate the correlation functions

$$\langle y(p_1) \cdots y(p_k) \rangle = \frac{(-1)^k}{\prod_{i=1}^{k} \sin^2 p_i} \left\langle \prod_{i=1}^{k} \sum_{n=1}^{N} \frac{1}{q_i + z_n} \right\rangle \tag{4.18}$$

from which the correlation functions (4.8) can always be reconstructed. Expanding the k-fold product over the $w(p_i) = y(p_i) + Nq_i$, we arrive at

$$C_k^{(2,N)}(p_1, \ldots, p_k) = \sum_{i=0}^{k} \sum_{\omega \in \mathbb{S}_k} \frac{N^{k-i}}{i!(k-i)!} \left(\prod_{l=1}^{k-i} q_{\omega(l)} \right) \left\langle \prod_{l=k-i+1}^{k} y\left(p_{\omega(l)}\right) \right\rangle. \tag{4.19}$$

The second sum runs over all elements $\omega(l)$ in the permutation group $\mathbb{S}_k$ of k objects. It enters the formula, because the correlation functions (4.18) appear in all orders i

up to k, comprising different subsets of $\{p_1, \dots, p_k\}$ with cardinality i. Thus, the k-point correlation function $C_k^{(2,N)}$ can be determined from all lower order correlation functions (4.18).

Performing the product, the average (4.18) becomes a complicated sum of terms. In some of them, only one of the eigenvalues z_n appears, these are the disconnected parts of the average to be performed. All other terms contain at least two different eigenvalues and may thus be referred to as connected. However, in section 4.4 we will rewrite the ensemble average in (4.18) in such a way that all terms can be obtained from the average of the N-point completely connected average

$$\left\langle \prod_{n=1}^{N} \frac{1}{q_n + z_n} \right\rangle, \tag{4.20}$$

which is, due to its very definition as an average, invariant under all permutations of the N arguments $(q_1, \dots, q_N)$. Our correlation functions, however, only depend on k of those arguments $(q_1, \dots, q_k)$, where we assume $k \leq N$. We find the proper k-point connected average by taking the limit

$$\left\langle \prod_{n=1}^{k} \frac{1}{q_n + z_n} \right\rangle = \lim_{q_{k+1},\dots,q_N \to \infty} \left(\prod_{m=k+1}^{N} q_m \right) \left\langle \prod_{n=1}^{N} \frac{1}{q_n + z_n} \right\rangle, \tag{4.21}$$

over the $N-k$ excess variables $(q_{k+1}, \dots, q_N)$. For this N-point connected average (4.20) we derive in section 4.4 the result

$$\left\langle \prod_{n=1}^{N} \frac{1}{q_n + z_n} \right\rangle = \frac{1}{c^{(2,N)}} \sum_{\omega \in \mathbb{S}_N} \det \left[L_{nm\omega(n)}(q_{\omega(n)}) \right]_{n,m=1,\dots,N} \tag{4.22}$$

with the function

$$L_{nml}(q_l) = \frac{(-1)^{m-n}\pi}{q_l^{m-n+1}} \mathrm{B}(m, N-m+1) \begin{cases} u_m(N, q_l^2) & m \geq n \\ -v_m(N, q_l^2) & m < n \end{cases}. \tag{4.23}$$

The functions $u_m(N, q_l^2)$ and $v_m(N, q_l^2)$ are given by

$$\begin{aligned} u_m(N, q_l^2) &= \frac{2}{\mathrm{B}(m, N-m+1)} \int_0^{q_l} d\rho \frac{\rho^{2m-1}}{(1+\rho^2)^{N+1}} \\ v_m(N, q_l^2) &= \frac{2}{\mathrm{B}(m, N-m+1)} \int_{q_l}^{\infty} d\rho \frac{\rho^{2m-1}}{(1+\rho^2)^{N+1}} \end{aligned} \tag{4.24}$$

and may be viewed as normalized incomplete Beta functions with the property

$$u_m(N, q_l^2) + v_m(N, q_l^2) = 1. \tag{4.25}$$

We will come across these functions also in the distribution of the winding number to be discussed in section 4.3.3. Taking the limit (4.21), the result (4.22) yields the k-point connected average

$$\left\langle \prod_{n=1}^{k} \frac{1}{q_n + z_n} \right\rangle = \frac{\pi^{N-k}}{c^{(2,N)}} \sum_{\omega \in \mathbb{S}_N} \left(\prod_{l=k+1}^{N} \mathrm{B}(\omega(l), N - \omega(l) + 1) \right) \times \det \left[L_{\omega(m)\omega(n)n}(q_n) \right]_{n,m=1,\dots,k}, \tag{4.26}$$

which is a $k \times k$ determinant, as derived in section 4.4.

From the general formulae (4.19) and (4.26), we obtain in section 4.4 for the first two correlation functions

$$\begin{aligned} C_1^{(2,N)}(p_1) &= 0, \\ C_2^{(2,N)}(p_1, p_2) &= -\frac{1 - \cos^{2N}(p_1 - p_2)}{1 - \cos^2(p_1 - p_2)}. \end{aligned} \tag{4.27}$$

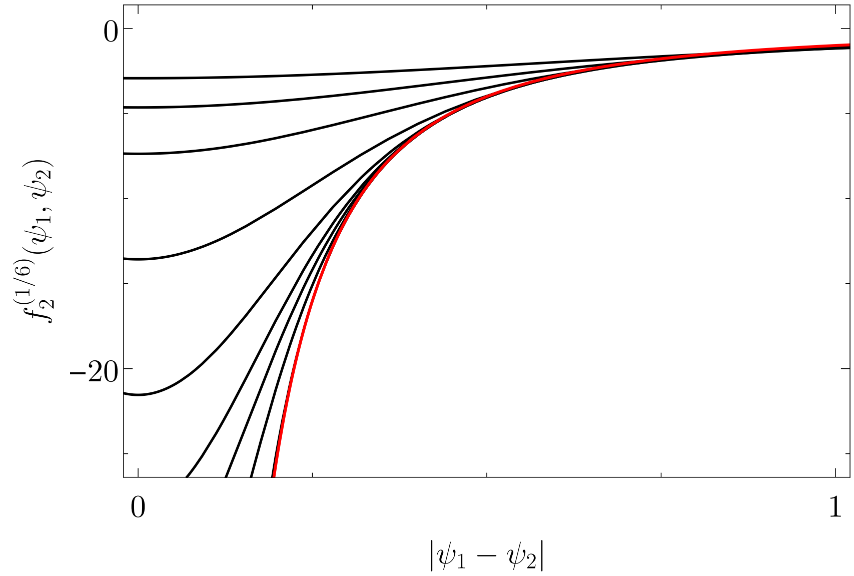
We notice that the two-point function depends only on the distance between the points p_1 and p_2 on the parameter manifold, which is a consequence of the translation invariance of our random matrix field (4.4). It turns out that for all k one of the parameters can be set to zero (or any other arbitrary point) without losing any information.

4.3.2 Universality Aspects and Unfolding of the Two-Point Function

The power of random matrix theory lies in the universality of its statistical predictions in the limit where the matrix dimensions tend to infinity. When studying the spectral properties of a single matrix, the universal statistics emerge when the energy levels are measured on the the local scale of the mean level spacing Δ for all probability distributions of random matrices that do not have scales competing with the mean level spacing [57, 58]. The required rescaling procedure is referred to as unfolding. When studying parameter-dependent matrix ensembles, universality is obtained if energies are still unfolded on the scale of Δ, and the parameter(s) are unfolded using the typical scale in parameter space [72]

$$\ell = \frac{\Delta}{\sqrt{\left\langle \sum_i \left(\frac{\partial}{\partial p_i} E_n(p_i) \right)^2 \right\rangle}}. \tag{4.28}$$

Inspired by these results, we search for universal regimes in our correlation functions. To this end, we rescale the parameters p_i appearing as arguments in the



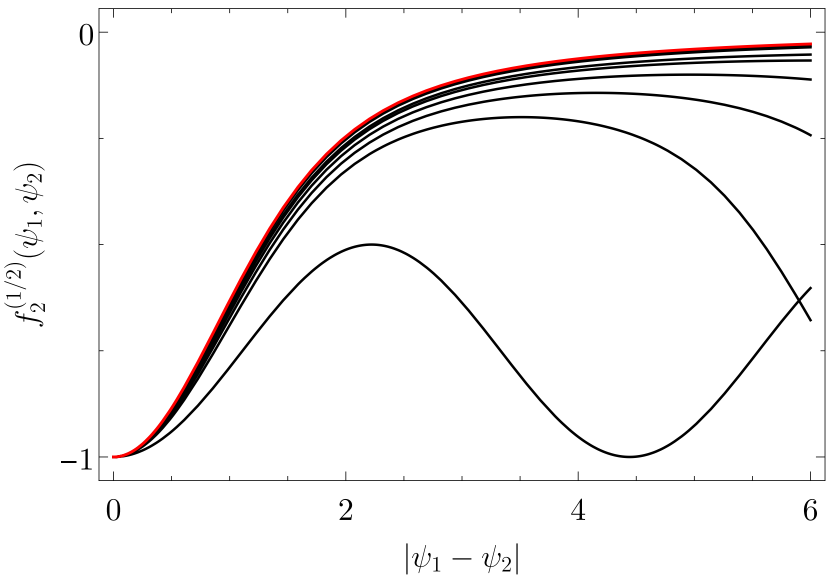


Figure 4.2: Unfolded two-point function after the rescaling (4.29) for different values of N (blue). In the top plot we use $N \in \{5, 10, 20, 50, 100, 150, 200, 300, 1000\}$ and $\alpha = 1/6$, in the bottom plot $N \in \{2, 5, 7, 10, 15, 20, 50, 100\}$ and $\alpha = 1/2$. For comparison the limit (4.30) (red).

correlation functions $C_k^{(2,N)}$ with a positive power of N according to

$$\psi_i = N^\alpha p_i. \tag{4.29}$$

We consider positive powers because we want to zoom into the parametric dependence to observe it on a proper local scale in the limit $N \to \infty$. Naturally, all physical systems that we want to compare with our random matrix theory should be considered on the same scale.

We turn to the two-point function (4.27). In the limit of large N the rescaled arguments ψ_i/N^α become small, allowing us to expand the cosines. We find

$$\lim_{N\to\infty} C_2^{(2,N)}\left(\frac{\psi_1}{N^\alpha}, \frac{\psi_2}{N^\alpha}\right)\frac{d\psi_1}{N^\alpha}\frac{d\psi_2}{N^\alpha} = f_2^{(\alpha)}(\psi_1,\psi_2)d\psi_1 d\psi_2 \tag{4.30}$$

with the function

$$f_2^{(\alpha)}(\psi_1,\psi_2) = \begin{cases} -\dfrac{1}{(\psi_1-\psi_2)^2} & \alpha < \frac{1}{2} \\ -\dfrac{1-\exp\left[-(\psi_1-\psi_2)^2\right]}{(\psi_1-\psi_2)^2} & \alpha = \frac{1}{2} \\ 0 & \alpha > \frac{1}{2} \end{cases}. \tag{4.31}$$

The case $p_1 = p_2$ or $\psi_1 = \psi_2$, respectively, is subject to interpretation. As obvious from (4.27), we have $C_2^{(2,N)}(p_1, p_1) = -1$. Hence, we must assume that the arguments are not equal, $\psi_1 \neq \psi_2$, when taking the limit for arbitrary α.

We observe different regimes in the result (4.31). Since $\langle(\partial E_n(p)/\partial p)^2\rangle = 1$ in our model, as can be shown following [72], the regime with $\alpha = 1/2$ amounts to an unfolding of p with the typical parameter scale (4.28) discovered in the works on parametric level correlations [69,70]. In figure 4.2 we show our result for two choices of α and various values of N. As can be seen, the unfolded two-point function approaches the limit (4.31) when N increases. We conjecture that the function $f_2^{(\alpha)}(\psi_1,\psi_2)$ is universal.

4.3.3 Winding Number Distribution

For the discussion to follow, it is useful to cast the random matrix field (4.4) into an equivalent, but different form. Introducing $s = e^{ip}$ as a complex variable on the unit circle, we have

$$K(s) = \frac{s}{2}(K_1 - iK_2) + \frac{1}{2s}(K_1 + iK_2). \tag{4.32}$$

For the determinant we have

$$\begin{aligned}\det K(s) &= \frac{1}{(2s)^N}\det\Big[K_1 + iK_2 + s^2(K_1 - iK_2)\Big]\\ &= \frac{\det(K_1 - iK_2)}{(2s)^N}\prod_{n=1}^{N}\left(s^2 + z_n'\right),\end{aligned} \tag{4.33}$$

where the z_n' are the solutions of the generalized eigenvalue problem

$$(K_1 + iK_2)v_n = z_n'\,(K_1 - iK_2)\,v_n \tag{4.34}$$

with eigenvectors v_n. The matrices $K_1 \pm iK_2$ are again Ginibre matrices, implying that the joint probability distribution of the z_n' is the one of the spherical ensemble (4.12). In the sequel, we thus always write z_n. The winding number in terms of (4.32) is

$$W = \frac{1}{2\pi i}\oint\limits_{|s|=1} ds\frac{1}{\det K(s)}\frac{d}{ds}\det K(s) = \frac{1}{2\pi i}\oint\limits_{|s|=1} ds\frac{d}{ds}\ln\det K(s). \tag{4.35}$$

Obviously, $\det(K_1 - iK_2)$ drops out in the integrand. The contour integral yields the difference of zeros and poles of $\det K(s)$ inside the unit circle. This result is also known as Cauchy's argument principle [104]. From (4.33) we infer that it has a pole of order N at zero and that its zeros come in pairs, making their number even. Let m be the number of solutions of (4.34) that lie inside the unit circle, then

$$W = 2m - N \tag{4.36}$$

is the winding number. The number m takes values from 0 to N, thus the winding number lies between $-N$ and N. The probability that m eigenvalues are inside the unit circle and the remaining ones outside is

$$r(m) = \int\limits_{|z_1|<1} d[z_1]\cdots\int\limits_{|z_m|<1} d[z_m]\int\limits_{|z_{m+1}|>1} d[z_{m+1}]\cdots\int\limits_{|z_N|>1} d[z_N]\,G^{(2,N)}(z). \tag{4.37}$$

In section 4.4 we show that

$$r(m) = \frac{1}{N!}\sum_{\omega\in\mathbb{S}_N}\left(\prod_{i=1}^{m}u_{\omega(i)}(N,1)\right)\left(\prod_{i=m+1}^{N}v_{\omega(i)}(N,1)\right), \tag{4.38}$$

where the expressions $u_i(N,1)$ and $v_i(N,1)$ follow from the functions (4.24). Taking into account the permutation invariance of the eigenvalues inside, respectively outside, the unit circle and using (4.36), we find the discrete probability distribution

$$P(W) = r\left(\frac{W+N}{2}\right)\binom{N}{(W+N)/2} \tag{4.39}$$

on the integers W between $-N$ and N as the winding number distribution for arbitrary, finite matrix dimension N.

Let us now turn to the moments (4.9) of this distribution. Since the one-point function (4.27) vanishes, the mean winding number is zero

$$\langle W \rangle = 0. \tag{4.40}$$

To arrive at a closed form for $k = 2$ we calculate, instead of directly applying the definition (4.9), the difference in the winding number variance of systems with $(N+1) \times (N+1)$ and $N \times N$ dimensional chiral subblocks. The second moment is given by

$$\left\langle W^2 \right\rangle \Big|_N = -\frac{1}{4\pi^2} \int\limits_0^{2\pi} dp_1 dp_2 \, C_2^{(2,N)}(p_1, p_2) = \frac{1}{2\pi} \int\limits_0^{2\pi} d\varphi \frac{1 - \cos^{2N} \varphi}{1 - \cos^2 \varphi}, \tag{4.41}$$

where we indicate the N dependence. For the difference we find

$$\begin{aligned} \left\langle W^2 \right\rangle \Big|_{(N+1)} - \left\langle W^2 \right\rangle \Big|_N &= \frac{1}{2\pi} \int\limits_0^{2\pi} d\varphi \cos^{2N} \varphi \\ &= \frac{(2N-1)!!}{(2N)!!} = \frac{(2N+1)!!}{(2N)!!} - \frac{(2N-1)!!}{(2N-2)!!}, \end{aligned} \tag{4.42}$$

and with $\langle W^2 \rangle |_1 = 1$ we obtain

$$\left\langle W^2 \right\rangle = \frac{(2N-1)!!}{(2N-2)!!} \simeq 2\sqrt{\frac{N}{\pi}}. \tag{4.43}$$

The last expression holds for large N. Hence, the second moment grows with $\sqrt{N}$. The results (4.40) and (4.43) suggests to look at the distribution of $P(W)$ as a function of $W^2/\sqrt{N}$ for large N. Numerically, we find that it is well described by

$$P(W) = \frac{1}{2(\pi N)^{1/4}} \exp\left(-\frac{1}{4}\sqrt{\frac{\pi}{N}} W^2 \right), \tag{4.44}$$

i.e. by a Gaussian distribution.

4.4 Derivations

In section 4.4.1 we reformulate the quantity to be ensemble averaged in the k-point correlation function (4.18). We calculate the N-point and the k-point connected ensemble averages in sections 4.4.2 and 4.4.3, respectively. The explicit expressions for the one and two-point functions are worked out in section 4.4.4. In section 4.4.5 we compute the probability (4.37) appearing in the discrete winding number distribution (4.39).

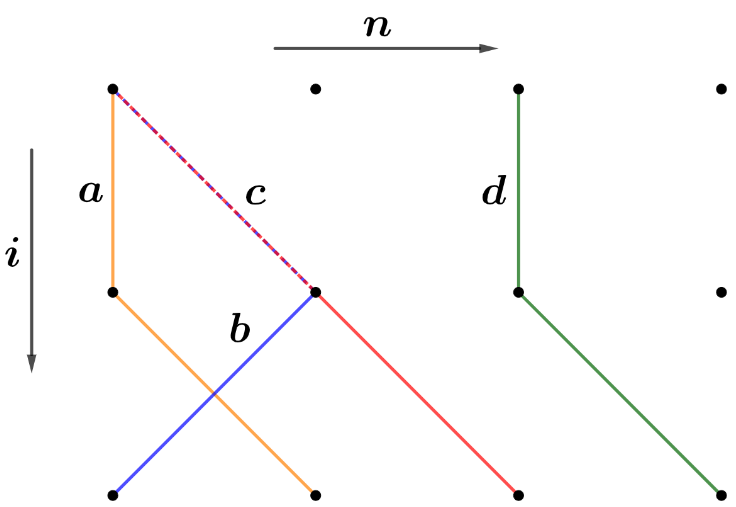


Figure 4.3: Examples for paths on a 3×4 lattice. Paths a and b both have the multiindex $(2, 1, 0, 0)$ and are thus equivalent up to permutations in i. Path $c = (1, 1, 1, 0)$ is the completely connected path along the angle bisector of the lattice. Path $d = (0, 0, 2, 1)$ is equal to path b as the integration variables z_n are permutation invariant.

4.4.1 Reformulation of the Key Expression to be Ensemble Averaged

To perform the calculation of the correlation function (4.18), it is helpful to rewrite the expression to be ensemble averaged, namely

$$\left\langle \prod_{i=1}^{k} \sum_{n=1}^{N} \frac{1}{q_i + z_n} \right\rangle , \tag{4.45}$$

by extracting the sums from the angular brackets, i.e. to cast the average (4.45) into a sum of terms containing only products to be averaged. This requires some work. We use the permutation invariance of the distribution (4.12) and think of the product of sums as a $k \times N$ lattice. Let the rows be labelled by $i = 1, \ldots, k$ and the columns by $n = 1, \ldots, N$. As depicted in figure 4.3 for some examples, each term in the product is a path through the lattice, obeying the following rules:

- Each row is visited once and only once. This amounts to each q_i appearing only once in each of the terms.
- Two paths are considered equal if they visit the same lattice points, irrespective of order. The points on the lattice are coupled via multiplication, which is commutative.

To each path we assign a multiindex $l = (l_1, \ldots, l_N) \in \mathbb{N}_0^N$ of length $|l| = \sum_{n=1}^N l_n = k$. It describes how many times l_n the path has visited the n-th column and therefore how many factors including z_n appear in the associated term. However, this mapping is not unique. In total there are

$$\binom{k}{l} = \frac{k!}{l_1! \cdots l_N!} \tag{4.46}$$

paths sharing the same l. In the matrix average these terms are equal up to permutations in the q_i. We take care of this by setting $q_i \to q_{\omega(i)}$ and summing over all permutations

$$\left\langle \prod_{i=1}^{k} \sum_{n=1}^{N} \frac{1}{q_i + z_n} \right\rangle = \frac{1}{k!} \sum_{\omega \in \mathbb{S}_k} \sum_{|l|=k} \left\langle \prod_{i=1}^{k} \frac{1}{q_{\omega(i)} + z_{g_l(i)}} \right\rangle. \tag{4.47}$$

Here, we introduce the step function

$$g_l(i) = 1 + \sum_{n=1}^{N} \Theta\left(i - \sum_{j=1}^{n} l_j\right), \quad \text{with} \quad \Theta(0) = 0, \tag{4.48}$$

employing the Heaviside unit step function Θ, to select the correct variables z_n for the integration of the corresponding product. We distinguish between different types of paths. In the disconnected paths only one z_n appears, which amounts to $l_n = k$ for one n and $l_n = 0$ for all other n. We refer to all other paths as connected. Out of the connected paths the ones with $l_n \in \{0, 1\}$, where each z_n may appear only once, stand out. To these paths we refer as completely connected and their contributions may be evaluated via (4.21).

Next we consider the permutation invariance of the z_n. Let $h_l(i)$ be the function that tallies up the number of integers i appearing in l. There are

$$\frac{N!}{\prod_{i=1}^{k} h_l(i)!} \tag{4.49}$$

possible ways to permute the z_n without changing the ensemble average. We choose the ordered multiindex l with $l_1 \leq \ldots \leq l_N$ as a representative of all these paths. On the $k \times N$ lattice, this amounts to paths below the angle bisector. We thus finally arrive at

$$\left\langle \prod_{i=1}^{k} \sum_{n=1}^{N} \frac{1}{q_i + z_n} \right\rangle = \frac{1}{k!} \sum_{\omega \in \mathbb{S}_k} \sum_{\substack{l_1 \leq \ldots \leq l_N \\ |l|=k}} \binom{k}{l} \frac{N!}{\prod_{i=1}^{k} h_l(i)!} \left\langle \prod_{i=1}^{k} \frac{1}{q_{\omega(i)} + z_{g_l(i)}} \right\rangle. \tag{4.50}$$

Indeed, this is a sum over ensemble averages of products only. Generally, any z_n may appear l times. To handle this, we use the partial fraction expansion

$$\prod_{i=1}^{l} \frac{1}{q_i + z_n} = \sum_{i=1}^{l} \frac{1}{\prod_{j\neq i}(q_j - q_i)} \frac{1}{q_i + z_n}, \tag{4.51}$$

which reduces the corresponding averages to a sum of completely connected averages. Thus, the resulting expression can again be treated with (4.21).

4.4.2 Calculation of the N-Point Connected Ensemble Average

As already pointed out in section 4.3.1, all connected k-point ensemble averages can be, via proper limits, obtained from the connected N-point average

$$\left\langle \prod_{n=1}^{N} \frac{1}{q_n + z_n} \right\rangle = \int_{\mathbb{C}^N} d[z]\, G^{(2,N)}(z) \prod_{n=1}^{N} \frac{1}{q_n + z_n}, \tag{4.52}$$

where $G^{(2,N)}(z)$ is the joint eigenvalue probability distribution (4.12) of the spherical ensemble. We use

$$|\Delta_N(z)|^2 = \Delta_N(z)\Delta_N^*(z) = \Delta_N(z)\Delta_N(z^*) \tag{4.53}$$

and expand the Vandermonde determinant $\Delta_N(z)$ in the Laplace form. This yields

$$\begin{aligned} \left\langle \prod_{n=1}^{N} \frac{1}{q_n + z_n} \right\rangle &= \frac{1}{c^{(2,N)}} \sum_{\omega\in\mathbb{S}_N} \operatorname{sgn}\omega \int_{\mathbb{C}^N} d[z]\Delta_N(z^*) \prod_{n=1}^{N} \frac{z_n^{\omega(n)-1}}{\left(1+|z_n|^2\right)^{N+1}(q_n+z_n)} \\ &= \frac{1}{c^{(2,N)}} \sum_{\omega\in\mathbb{S}_N} \int_{\mathbb{C}^N} d[z]\Delta_N(z^*) \prod_{n=1}^{N} \frac{z_n^{n-1}}{\left(1+|z_n|^2\right)^{N+1}\left(q_{\omega(n)}+z_n\right)}, \end{aligned} \tag{4.54}$$

where the second equation follows from renaming the integration variables $z_n \to z_{\omega(n)}$ for each permutation $\omega \in \mathbb{S}_N$. The sign $\operatorname{sgn}\omega$ of the permutation ω is canceled by the same sign appearing in $\Delta_N(z^*)$ when changing the integration variables. Inserting the remaining Vandermonde determinant and integrating row by row we obtain (4.22) with the function

$$\begin{aligned} L_{nml}(q_l) &= \int_{\mathbb{C}} d[z] \frac{(z^*)^{m-1} z^{n-1}}{\left(1+|z|^2\right)^{N+1}(q_l+z)} \\ &= \int_0^\infty d\rho \frac{\rho^{m+n-1}}{(1+\rho^2)^{N+1}} \int_0^{2\pi} d\vartheta \frac{e^{i(n-m)\vartheta}}{q_l + \rho\, e^{i\vartheta}}, \end{aligned} \tag{4.55}$$

where we employ polar coordinates $z = \rho\, e^{i\vartheta}$ in the second equation. The angular integral yields, by virtue of the residue theorem,

$$\int\limits_0^{2\pi} d\vartheta \frac{e^{i(n-m)\vartheta}}{q_l + \rho\, e^{i\vartheta}} = \begin{cases} \dfrac{2\pi}{q_l}\left(-\dfrac{\rho}{q_l}\right)^{m-n} & m \geq n, \rho < q_l \\ \dfrac{2\pi}{\rho}\left(-\dfrac{\rho}{q_l}\right)^{m-n+1} & m < n, \rho > q_l \\ 0 & \text{else} \end{cases}. \tag{4.56}$$

Thus, we arrive at (4.23).

4.4.3 Reduction to the k-Point Connected Ensemble Average

To take the limit (4.21) we need as an intermediate result a proper limit involving the function $L_{nml}(q_l)$. As the limit $q_l \to \infty$ of the incomplete Beta functions (4.24) gives either unity or zero, the total limit is only non-vanishing if $m = n$,

$$\lim_{q_l \to \infty} q_l L_{nml}(q_l) = \begin{cases} \pi\, \mathrm{B}(m, N-m+1) & m = n \\ 0 & m \neq n \end{cases}. \tag{4.57}$$

We apply this result to reduce the k-point connected average, which is, according to (4.21) and (4.22), a limit of an $N \times N$-determinant. The limit makes all elements in the $\omega^{-1}(n)$-th row vanish except the diagonal element, which is $\pi\, \mathrm{B}(\omega^{-1}(n), N - \omega^{-1}(n) + 1)$. We expand the determinant in these elements

$$\begin{aligned} \left\langle \prod_{n=1}^{k} \frac{1}{q_n + z_n} \right\rangle =& \frac{\pi^{N-k}}{c^{(2,N)}} \sum_{\omega \in \mathbb{S}_N} \left(\prod_{l=k+1}^{N} \mathrm{B}(\omega^{-1}(l), N - \omega^{-1}(l) + 1) \right) \\ & \times \det\left[L_{nm\omega(n)}(q_{\omega(n)}) \right]_{n,m=1,\ldots,N}^{n,m \neq \omega^{-1}(l), l=k+1,\ldots,N}. \end{aligned} \tag{4.58}$$

Interchanging row n with row $\omega^{-1}(n)$ and column m with column $\omega^{-1}(m)$ yields for the right hand side

$$\begin{aligned} & \frac{\pi^{N-k}}{c^{(2,N)}} \sum_{\omega \in \mathbb{S}_N} \left(\prod_{l=k+1}^{N} \mathrm{B}(\omega^{-1}(l), N - \omega^{-1}(l) + 1) \right) \det\left[L_{\omega^{-1}(n)\omega^{-1}(m)n}(q_n) \right]_{n,m=1,\ldots,k} \\ =& \frac{\pi^{N-k}}{c^{(2,N)}} \sum_{\omega \in \mathbb{S}_N} \left(\prod_{l=k+1}^{N} \mathrm{B}(\omega(l), N - \omega(l) + 1) \right) \det\left[L_{\omega(n)\omega(m)n}(q_n) \right]_{n,m=1,\ldots,k}. \end{aligned} \tag{4.59}$$

We also used that the order in the sum over the permutations ω is invariant due to the group property of $\mathbb{S}_N$. Thus, we arrive at the result (4.26).

4.4.4 Explicit Expressions for the One- and Two-Point Correlation Functions

For $k = 1$ there are no connected terms. According to (4.19) and (4.50) the one-point function is given by

$$C_1^{(2,N)}(p_1) = \langle y(p_1)\rangle + Nq_1 = -\frac{N}{\sin^2 p_1}\left\langle \frac{1}{q_1+z_1}\right\rangle + Nq_1. \tag{4.60}$$

The average follows from (4.26),

$$\left\langle \frac{1}{q_1+z_1}\right\rangle = \frac{1}{Nq_1}\sum_{n=1}^{N} u_n(N,q_1^2). \tag{4.61}$$

The incomplete Beta functions (4.24) may be rewritten using integration by parts, we find

$$v_m(N,q_1^2) = \sum_{l=0}^{m-1}\binom{N-1-l}{m-1-l}\frac{(q_1^2)^{m-l-1}}{(1+q_1^2)^{N-l}}. \tag{4.62}$$

Using the property (4.25), the sum in (4.61) can be evaluated by means of the binomial theorem, implying

$$\left\langle \frac{1}{q_1+z_1}\right\rangle = \frac{1}{q_1} - \frac{1}{q_1(1+q_1^2)} = \sin p_1 \cos p_1. \tag{4.63}$$

In the last step we reinserted $q_1 = \cot p_1$. Altogether we arrive at the first of the results (4.27).

For $k = 2$ we apply (4.19) and (4.50) and use the vanishing of the one-point function

$$\begin{aligned} C_2^{(2,N)}(p_1,p_2) &= \langle y(p_1)y(p_2)\rangle - N^2 q_1 q_2 \\ &= \frac{1}{\sin^2 p_1 \sin^2 p_2}\left\langle \prod_{i=1}^{2}\sum_{n=1}^{N}\frac{1}{q_i+z_n}\right\rangle - N^2 q_1 q_2. \end{aligned} \tag{4.64}$$

With (4.50) we find

$$\left\langle \prod_{i=1}^{2}\sum_{n=1}^{N}\frac{1}{q_i+z_n}\right\rangle = N\left\langle \frac{1}{q_1+z_1}\frac{1}{q_2+z_1}\right\rangle + N(N-1)\left\langle \frac{1}{q_1+z_1}\frac{1}{q_2+z_2}\right\rangle. \tag{4.65}$$

The connected average is given by (4.26) and reads

$$\left\langle \frac{1}{q_1+z_1}\frac{1}{q_2+z_2}\right\rangle = \frac{1}{N(N-1)}\frac{1}{q_1q_2}\left(\sum_{\substack{n,m=1\\ n\neq m}}^{N} u_n(N,q_1^2)u_m(N,q_2^2)\right.$$
$$\left. + \sum_{\substack{n,m=1\\ n>m}}^{N}\left(\frac{q_2}{q_1}\right)^{n-m} u_n(N,q_1^2)v_m(N,q_2^2) + \left(\frac{q_1}{q_2}\right)^{n-m} v_n(N,q_1^2)u_m(N,q_2^2)\right). \tag{4.66}$$

This expression is readily simplified by using the translation invariance on the parameter manifold. We set $p_2 = \pi/2$ which amounts to $q_2 = 0$ and find

$$\begin{aligned}\left\langle \frac{1}{q_1+z_1}\frac{1}{z_2}\right\rangle &= \frac{1}{N(N-1)}\frac{1}{q_1^2}\sum_{n=2}^{N} u_n(N,q_1^2)\\ &= \frac{1}{(N-1)q_1^2}\left(1-\frac{1}{1+q_1^2}+\frac{1}{N(1+q_1^2)^N}-\frac{1}{N}\right).\end{aligned} \tag{4.67}$$

For the disconnected average we employ the partial fraction expansion (4.51) and (4.63),

$$\left\langle \frac{1}{q_1+z_1}\frac{1}{z_1}\right\rangle = -\frac{1}{q_1}\left(\left\langle \frac{1}{q_1+z_1}\right\rangle - \left\langle \frac{1}{z_1}\right\rangle\right) = -\frac{1}{1+q_1^2}. \tag{4.68}$$

Reinserting $q_1 = \cot p_1$ yields

$$C_2^{(2,N)}\left(p_1,\frac{\pi}{2}\right) = -\frac{1-\cos^{2N} p_1}{1-\cos^2 p_1} \tag{4.69}$$

or, equivalently, the second of the results (4.27).

4.4.5 Calculation of the Probability $r(m)$

For the discrete winding number distribution (4.39) we need to compute the probability (4.37). The calculation is similar to the one in section 4.4.2. Inserting the eigenvalue distribution (4.12), we have

$$\begin{aligned}r(m) = \frac{1}{C^{(2,N)}}\sum_{\omega\in\mathbb{S}_N}\int\limits_{|z_{\omega(1)}|<1} d[z_{\omega(1)}]\cdots\int\limits_{|z_{\omega(m)}|<1} d[z_{\omega(m)}]\\ \int\limits_{|z_{\omega(m+1)}|>1} d[z_{\omega(m+1)}]\cdots\int\limits_{|z_{\omega(N)}|>1} d[z_{\omega(N)}]\Delta_N^*(z)\prod_{n=1}^{N}\frac{z_n^{n-1}}{\left(1+|z_n|^2\right)^{N+1}}.\end{aligned} \tag{4.70}$$

Like in section 4.4.2 we expanded one of the Vandermonde determinants and renamed the integration variables $z_n \to z_{\omega(n)}$, which cancels the sign of the first determinant. When integrating in polar coordinates $z_n = \rho_n e^{i\vartheta_n}$ we find Kronecker deltas for the angular integrals,

$$\int_0^{2\pi} d\vartheta_n e^{i(n-\omega(n))\vartheta_n} = 2\pi\delta_{n\omega(n)}. \tag{4.71}$$

Therefore, for the second determinant only the diagonal terms contribute and we obtain

$$\begin{aligned} r(m) &= \frac{(2\pi)^N}{c^{(2,N)}} \sum_{\omega\in\mathbb{S}_N} \int_0^1 d\rho_{\omega(1)} \cdots \int_0^1 d\rho_{\omega(m)} \int_1^\infty d\rho_{\omega(m+1)} \cdots \int_1^\infty d\rho_{\omega(N)} \prod_{n=1}^N \frac{\rho_n^{2n-1}}{(1+\rho_n^2)^{N+1}} \\ &= \frac{1}{N!} \sum_{\omega\in\mathbb{S}_N} \left(\prod_{i=1}^m u_{\omega(i)}(N,1) \right) \left(\prod_{i=m+1}^N v_{\omega(i)}(N,1) \right). \end{aligned} \tag{4.72}$$

The radial integrals are given by the incomplete Beta functions (4.24) for $q_l = 1$. Altogether we arrive at formula (4.38).

4.5 Summary

In this chapter we studied the winding number statistics in the chiral unitary class AIII. We set up an appropriate random matrix model with one-dimensional parametric dependence as a combination of two chiral Gaussian unitary matrices. Within this model, we analytically calculated the discrete probability distribution of the winding number as well as the k-point correlation functions of the winding number density. We derived a closed formula for the former and arrived at explicit determinant expressions for certain correlation functions of arbitrary order for the latter, which allow for a construction of the winding number density correlation functions. We constructed the one- and two-point functions explicitly and used them to calculate the first and second moment of the winding number.

We found that our random matrix model is intimately connected to the complex spherical ensemble of random matrices, which has been analyzed in earlier works [76,77]. We used these results to address the new questions of statistical topology. In our calculations we integrated over the joint probability distribution of eigenvalues of the complex spherical ensemble. In all our results we obtained incomplete Beta functions, which are fairly simple.

Since random matrix theory is widely known to provide universal results for spectral statistics, including parametric spectral statistics, we are confident that our results hold universal information as well. In order to reveal it, we carried out an unfolding procedure similar to the one in the above contexts. Remarkably, we found different scaling limits that we expect to be universal. Furthermore, our result for the discrete probability distribution of the winding number indicates that it becomes Gaussian in the large N limit. These observations are analogous to the ones obtained numerically in the case of the adiabatic curvature and the Chern number [74, 75].

Chapter 5

Averaging Ratios of Parametric Determinants

In this chapter we take a different approach to the winding number statistics involving generating functions. It is based on [2] and [3]. We extend our study to a general two-matrix model, including also the time reversal invariant classes. In section 5.1 we briefly outline our intentions, and in section 5.2 we mathematically define the goals of this chapter. In section 5.3 we summarize our results, while we give their derivation in section 5.4. Some details are postponed to several appendices. We conclude in section 5.5.

5.1 Introduction

It is a fruitful approach in statistical physics to work with a generating function that yields the desired quantities upon taking the derivative with respect to a source parameter. We establish such a function for the k-point correlation of the winding number density. Within this scope, we generalize our examinations to arbitrary two-matrix models. Furthermore, we include also the time reversal invariant chiral classes BDI and CII in our considerations. In doing so, we encounter the real and quaternion spherical ensemble in addition to the complex one.

The generating function is given by the ensemble average over ratios of characteristic polynomials in the corresponding spherical ensemble. In section 3.4 we discussed the supersymmetry without supersymmetry method, which aims to compute such ensemble averages. Using this method we obtain a determinantal expression in the chiral unitary class AIII and Pfaffian expressions in the chiral orthogonal class BDI and the chiral symplectic class CII.

The remaining task is to compute the ensemble dependent kernel functions, which are given by ensemble averages over only two characteristic polynomials. This is by no means an easy endeavour and we have to resort to various techniques to solve these integrals. In particular, we exploit an inherent symmetry of our two-matrix model to reduce the number of characteristic polynomials by one. Subsequently, the averages can be evaluated by methods of classical random matrix theory such as orthogonal and skew-orthogonal polynomials.

5.2 Posing the Problem

As pointed out in chapters 2 and 3, time reversal invariance assigns a real structure to the Hamiltonian. It is then either a real symmetric ($\beta = 1$) or a quaternion self-dual ($\beta = 4$) matrix, whereas in the absence of time reversal invariance it is complex Hermitian ($\beta = 2$). Assuming chiral symmetry as well, we distinguish between the chiral orthogonal class BDI ($\beta = 1$), the chiral symplectic class CII ($\beta = 4$) and the chiral unitary class AIII ($\beta = 2$). The winding number statistics of the latter were considered already in chapter 4. Here, we will first set up a parametric random matrix model, similar to the one used there, which is valid also for the time reversal invariant cases.

In the chiral classes, the Bloch Hamiltonian assumes a block off-diagonal form

$$H(p) = \begin{bmatrix} 0 & K(p) \\ K^\dagger(p) & 0 \end{bmatrix}. \tag{5.1}$$

In the symplectic case the matrix elements are quaternions, effectively doubling the matrix dimension. Therefore, $K(p)$ is of dimension $N \times N$ for AIII and BDI and of dimension $2N \times 2N$ for CII. In chapter 2 we found that the Bloch Hamiltonian, defined as the Fourier transform of the Hamiltonian from the crystal lattice to the reciprocal lattice, behaves like

$$\mathcal{T} H(p) \mathcal{T}^{-1} = H(-p) \tag{5.2}$$

under time reversal. Therefore, by inserting the time reversal operator, $K(p)$ has to fulfill

$$\begin{aligned} \text{BDI}: \quad & \mathcal{T} K(p) \mathcal{T}^{-1} = K^*(p) = K(-p), \\ \text{CII}: \quad & \mathcal{T} K(p) \mathcal{T}^{-1} = [\tau_2 \otimes \mathbb{1}_N] K^*(p) [\tau_2 \otimes \mathbb{1}_N] = K(-p), \end{aligned} \tag{5.3}$$

where τ_2 is the second Pauli matrix. This has the important consequence that, unlike the chiral block form, the real structure is not simply passed on to the Bloch Hamiltonian. Only at the time reversal invariant momenta $p = 0$ and $p = \pi$, where $K(p) = K(-p)$, due to the 2π-periodicity of p, it will be real symmetric resp. quaternion self-dual. Hence, for a general p, we can expect that $K(p)$ is a complex matrix interpolating between real and imaginary numbers resp. real and imaginary quaternions.

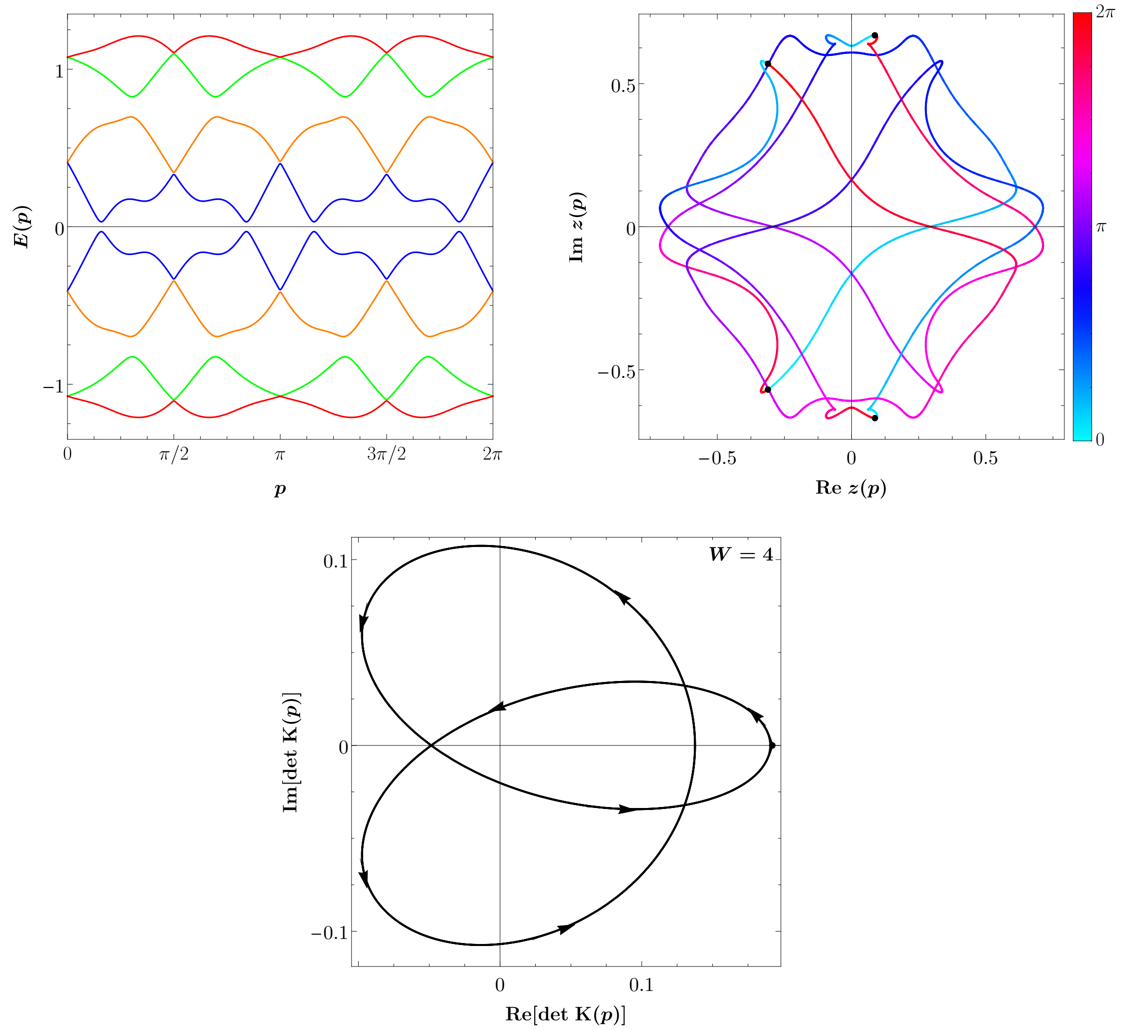


Figure 5.1: A realization of a CII Hamiltonian with $K(p) = \cos(p)K_1 + i\sin(p)K_2$ and some fixed 4×4 real quaternion matrices K_1 and K_2. The top left plot shows the real eigenvalues of $H(p)$, the top right one shows the generically complex eigenvalues of $K(p)$, and the bottom plot depicts the determinant $\det K(p)$. All plots show the parametric dependence in $p \in [0, 2\pi)$, where we have employed the step size $2\pi/100$ and a B-Spline to obtain the curves. In both of the parametric plots the starting points $p = 0$ are marked by black points and the directions are marked by a color gradient resp. arrows.

In figures 5.1 and 5.2 we illustrate the spectral flow of the matrix field $K(p) = \cos(p)K_1 + i\sin(p)K_2$ with generic real quaternion $K_1, K_2 \in \mathbb{H}^{2\times 2} \subset \mathbb{C}^{4\times 4}$ for the case CII and with generic real $K_1, K_2 \in \mathbb{R}^{4\times 4}$ for the case BDI, just as we did for the corresponding AIII matrix field, see figure 4.1. We show the eight eigenvalues of $H(p)$, the four complex eigenvalues of $K(p)$ and the determinant $\det K(p)$. Again, we observe that the order of the eigenvalues of $H(p)$ remains the same for all p, whereas the complex eigenvalues of $K(p)$ may experience a permutation, meaning when running once from $p = 0$ to $p = 2\pi$ a chosen eigenvalue can become another one. Therefore, only the determinant $\det K(p)$ is suitable for a topological classification as it is generally 2π-periodic.

In the symplectic case, see figure 5.1, we find Kramers' degeneracy in the spectrum of $H(p)$ at the time reversal invariant points $p = 0$ and $p = \pi$, where $K(p)$ is real quaternion in general, and at $p = \pi/2$ and $p = 3\pi/2$, where it is imaginary quaternion for our specific choice of $K(p)$. In the unitary case we observed a point symmetry around the origin of the curves describing the eigenvalues of $K(p)$, which was caused by $K(p+\pi) = -K(p)$, specific to our choice of $K(p)$. In the orthogonal and symplectic cases this point symmetry is raised to independent reflection symmetries around the real- and imaginary axis. The reflection symmetry around the imaginary axis exists in general and is due to (5.3). The property $K(p+\pi) = -K(p)$ also restricts the amount of times $\det K(p)$ winds around the origin to be an even resp. odd number for even resp. odd matrix dimensions. This symmetry does not exist in general as we also illustrated in figure 5.2 by means of another random matrix field $K(p) = (a_1 + a_2 e^{ip} + a_3 e^{2ip} + a_4 e^{3ip})K_1 + (b_1 + b_2 e^{ip} + b_3 e^{2ip} + b_4 e^{3ip})K_2$ with generic real $K_1, K_2 \in \mathbb{R}^{4\times 4}$ and real coefficients a_j and b_j, belonging to class BDI. In this example we find an odd winding number. In the symplectic case, however, the winding number is generally even, as predicted by the tenfold way classification, see table 2.2. The reason for this is that the matrix field $K(p)$ is real quaternion at the points $p = 0$ and $p = \pi$, so that its eigenvalues come in complex conjugated pairs and its determinant is real positive, $\det K(0) > 0$ and $\det K(\pi) > 0$. This obstruction forces the winding number to be even and is not present in the orthogonal case, where the determinant can have an arbitrary sign at these points. Physically, this can be understood in terms of the edge states, which must arrive in pairs in the symplectic case due to Kramers' degeneracy.

The topological invariant describing this effect and classifying such subsets of chiral Hamiltonians is the winding number. We discussed it thoroughly in section 2.3, but do not hesitate to state the relevant expressions once again. The winding number is

$$W = \frac{1}{2\pi i} \oint\limits_{\det K(p)} \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_0^{2\pi} dp\, w(p), \tag{5.4}$$

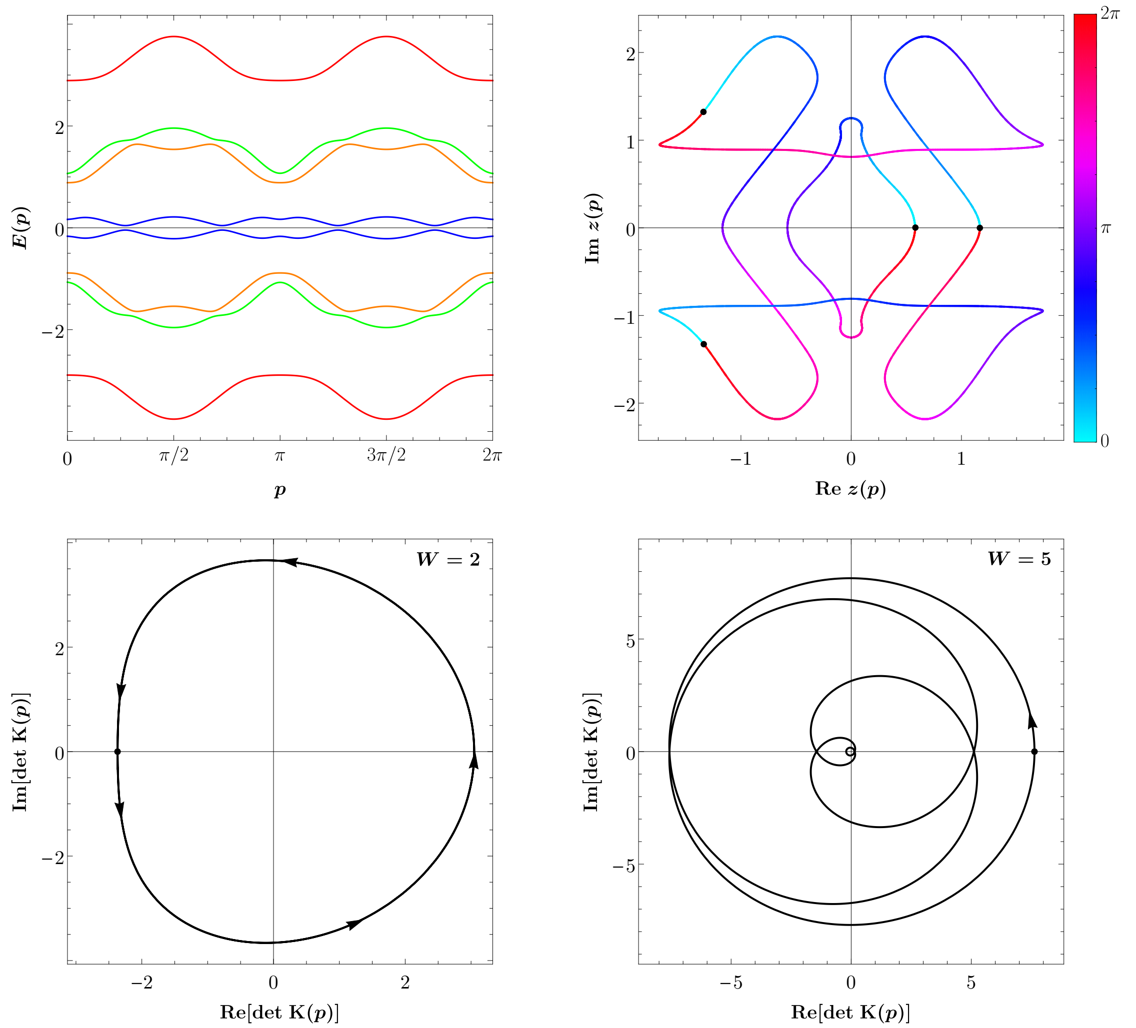


Figure 5.2: A realization of a BDI Hamiltonian with $K(p) = \cos(p)K_1 + i\sin(p)K_2$ and some fixed 4×4 real matrices K_1 and K_2. The top left plot shows the real eigenvalues of $H(p)$, the top right one shows the generically complex eigenvalues of $K(p)$, and the bottom left plot depicts the determinant $\det K(p)$. The bottom right plot shows the determinant of a different random matrix field $K(p) = (a_1 + a_2e^{ip} + a_3e^{2ip} + a_4e^{3ip})K_1 + (b_1 + b_2e^{ip} + b_3e^{2ip} + b_4e^{3ip})K_2$ with fixed 4×4 real matrices K_1 and K_2 and real coefficients a_j and b_j. All plots show the parametric dependence in $p \in [0, 2\pi)$, where we have employed the step size $2\pi/100$ and a B-Spline to obtain the curves. In the parametric plots the starting points $p = 0$ are marked by black points and the directions are marked by a color gradient resp. arrows.

where we integrate over the logarithmic derivative of the determinant

$$w(p) = \frac{d}{dp} \ln \det K(p) = \frac{1}{\det K(p)} \frac{d}{dp} \det K(p) = \operatorname{tr} K^{-1}(p) \frac{d}{dp} K(p), \tag{5.5}$$

called winding number density.

The parametric dependence of the random matrix $K(p)$ describes a random field on $[0, 2\pi)$, which has its values in $\mathrm{Gl}_{\mathbb{C}}(N)$ for AIII and BDI or in $\mathrm{Gl}_{\mathbb{C}}(2N)$ for CII. To have an analytically feasible model we assume a centered Gaussian random field. Thus, the model is fully controlled by its variance, which we assume to have the only non-vanishing covariances

$$\left\langle K_{lj}^*(p) K_{lj}(q) \right\rangle = S(p,q) \neq 0, \qquad S(p,p) \geq 0 \tag{5.6}$$

with $p, q \in [0, 2\pi)$ and any l, j, where $\langle \cdot \rangle$ is the ensemble average. As this choice is independent of the matrix indices l and j, $S(p,q)$ must be a scalar product on a vector space because of

$$\begin{aligned} \langle K_{lj}^*(p) \left[\lambda\, K_{lj}(q_1) + \mu\, K_{lj}(q_2) \right] \rangle =& \lambda\, S(p, q_1) + \mu\, S(p, q_2), \\ S^*(p,q) = \langle K_{lj}^*(p) K_{lj}(q) \rangle^* =& \langle K_{lj}^*(q) K_{lj}(p) \rangle = S(q,p) \end{aligned} \tag{5.7}$$

for any $p, q, q_1, q_2 \in [0, 2\pi)$, $\mu, \lambda \in \mathbb{C}$, and l, j. Hitherto, we considered the most general form for the covariance $S(p,q)$. The easiest non-trivial choice is a scalar product of a two-dimensional complex vector space, which can be realized by setting up the random matrix field as the linear combination

$$K(p) = a(p) K_1 + b(p) K_2 \tag{5.8}$$

with two scalar functions $a(p)$ and $b(p)$ that are smooth and 2π-periodic. Arranging the two functions as a vector

$$v(p) = (a(p), b(p)) \in \mathbb{C}^2, \tag{5.9}$$

the scalar product takes the form $S(p,q) = v^\dagger(p) v(q)$. Furthermore, when interpreting our random matrix model as a Bloch Hamiltonian (i.e. p is the crystal momentum), in the time reversal invariant cases the functions should satisfy

$$\mathcal{T} v(p) \mathcal{T}^{-1} = v^*(p) = v(-p) \tag{5.10}$$

due to (5.3). However, the expressions for the ensemble averages derived in the following hold true for arbitrary functions and are therefore valid also in the case that our random matrix model is interpreted otherwise and condition (5.10) is relaxed.

The matrices K_1 and K_2 are drawn from the complex Ginibre ensemble in the case AIII, from the real Ginibre ensemble in the case BDI and from the real quaternion Ginibre ensemble in the case CII. We state the exact expressions for their joint probability distributions $\widetilde{P}^{(\beta,N)}(K_1,K_2)$ further below, see also section 3.2. As aforementioned, we denote the corresponding ensemble averages of an observable $F(K_1,K_2)$ with angular brackets,

$$\langle F\rangle = \int d[K_1,K_2]\widetilde{P}^{(\beta,N)}(K_1,K_2)F(K_1,K_2), \tag{5.11}$$

where the flat measures $d[K_1,K_2]$ are simply the products of the differentials of all independent real variables. The structure of the random matrix field carries over from $K(p)$ to the Hamiltonian $H(p)$ which becomes

$$H(p) = a(p)H_1 + b(p)H_2, \qquad H_m = \begin{bmatrix} 0 & K_m \\ K_m^\dagger & 0 \end{bmatrix}, \qquad m = 1,2. \tag{5.12}$$

This construction defines parametric combinations of two chiral Gaussian ensembles, having unitary ($\beta = 2$), orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) symmetry, respectively.

Our goal is to calculate the ensemble averages for ratios of determinants with parametric dependence

$$Z_{k|l}^{(\beta,N)}(q,p) = \left\langle \frac{\prod_{j=1}^{l}\det K(p_j)}{\prod_{j=1}^{k}\det K(q_j)} \right\rangle \tag{5.13}$$

for two sets of variables $(p_1,\ldots,p_l)$ and $(q_1,\ldots,q_k)$ in the case $k = l$. We introduce the more general definition (5.13) for k and l being different for reasons that will become clear in the sequel, see also section 3.4. We note that k and l are the numbers of determinants in denominator and numerator, respectively.

These ensemble averages are closely related to averages over ratios of characteristic polynomials, which are mathematically the key objects in the supersymmetry method [80,81,157] since they serve as generators for correlation functions of operator or matrix resolvents. Similarly, we can compute the k-point correlation function

$$C_k^{(\beta,N)}(p_1,\ldots,p_k) = \langle w(p_1)\cdots w(p_k)\rangle \tag{5.14}$$

of the winding number density as the k-fold derivative

$$C_k^{(\beta,N)}(p_1,\ldots,p_k) = \frac{\partial^k}{\prod_{j=1}^{k}\partial p_j} Z_{k|k}^{(\beta,N)}(q,p)\Bigg|_{q=p} \tag{5.15}$$

of the generator (5.13). As pointed out earlier, the k-point correlations yield the moments of the winding number upon integration

$$\left\langle W^k \right\rangle = \frac{1}{(2\pi i)^k}\int_0^{2\pi} dp_1 \cdots \int_0^{2\pi} dp_k\, C_k^{(\beta,N)}(p_1,\ldots,p_k). \tag{5.16}$$

5.3 Results

In sections 5.3.1, 5.3.2 and 5.3.3 we state the results for the unitary, symplectic and orthogonal case, respectively. They are sorted by increasing complexity, with the orthogonal case being the most difficult one.

As mentioned in section 3.4, the unitary case is described by a determinantal point process, while the symplectic and the orthogonal case are described by a Pfaffian point process [58, 163]. This means that the eigenvalue correlations can be written as a determinant resp. a Pfaffian of ensemble dependent kernel functions. Indeed, we find this also in the expressions resulting from the supersymmetry without supersymmetry method.

Regardless of which of the three cases, it is very useful to write the two coefficients $a(p)$ and $b(p)$ in terms of the two-dimensional vector $v(p)$. Only then certain inherent symmetries are properly reflected in the results.

5.3.1 The Unitary Case (AIII)

The generating function, see (5.13), is invariant under the group $\mathrm{SU}(2) \times \mathrm{Gl}_{\mathbb{C}}(1)$. The part $\mathrm{Gl}_{\mathbb{C}}(1)$ corresponds to the invariance under rescaling $v(p) \to s\, v(p)$ for all $s \in \mathrm{Gl}_{\mathbb{C}}(1) = \mathbb{C} \setminus \{0\}$. The scaling factor drops out in the ratio of the determinants. The subgroup $\mathrm{SU}(2)$ reflects an invariance when rotating K_1 and K_2 into each other. This carries over to an invariance for the vector $v(p)$, see section 5.4.1 for more details. Therefore, the result can only depend on the combinations $v^\dagger(p)v(q)$ and $v^T(p)\tau_2 v(q)$ and their complex conjugates. We emphasize that $v^T(p)\tau_2 v(q)$ is also an invariant because $U = \tau_2 U^* \tau_2$ for any $U \in \mathrm{SU}(2)$. Additionally, $Z_{k|l}^{(2,N)}(q,p)$ is a polynomial in $v(p_j)$ while it is quite likely to be not holomorphic in $v(q_j)$.

We derive the following result

$$Z_{k|k}^{(2,N)}(q,p) = \frac{\det\left[\dfrac{1}{v^T(q_m)\tau_2 v(p_n)}\left(\dfrac{v^\dagger(q_m)v(p_n)}{v^\dagger(q_m)v(q_m)}\right)^N\right]_{1\le m,n\le k}}{\det\left[\dfrac{1}{v^T(q_m)\tau_2 v(p_n)}\right]_{1\le m,n\le k}} \tag{5.17}$$

for the unitary case.

5.3.2 The Symplectic Case (CII)

As often, the symplectic and orthogonal cases CII and BDI, respectively, are considerably more demanding and lead to Pfaffian structures. The symplectic case turns out to be simpler in its computation and its results. However, the biggest obstruction is that it respects the smaller invariance group $\mathrm{O}(2) \times \mathrm{Gl}_{\mathbb{R}}(1)$. The $\mathrm{Gl}_{\mathbb{R}}(1)$ part is once more the simple rescaling of the two dimensional vector $v(p) \to s\,v(p)$ with $s \in \mathrm{Gl}_{\mathbb{R}}(1) = \mathbb{R} \setminus \{0\}$. Yet, the condition that the two matrices K_1 and K_2 must be real quaternion only allows a rotation of one matrix into the other one via the real orthogonal group O(2). Again, more details of this symmetry discussion can be found in section 5.4.2.

We derive the following result

$$Z_{k|k}^{(4,N)}(q,p) = \frac{\mathrm{Pf}\begin{bmatrix} \widehat{\mathrm{K}}_1^{(4,N)}(p_m,p_n) & \widehat{\mathrm{K}}_2^{(4,N)}(p_m,q_n) \\ -\widehat{\mathrm{K}}_2^{(4,N)}(p_n,q_m) & \widehat{\mathrm{K}}_3^{(4,N)}(q_m,q_n) \end{bmatrix}_{1\leq m,n\leq k}}{\det\left[\dfrac{1}{iv^T(q_m)\tau_2 v(p_n)}\right]_{1\leq m,n\leq k}} \tag{5.18}$$

with the kernel functions

$$\begin{aligned}
\widehat{\mathrm{K}}_1^{(4,N)}(p_m,p_n) =& 2N(2N+1)[iv^T(p_n)\tau_2 v(p_m)]^{2N-1} q_{2N-2}^{(N)}\left(\frac{v^T(p_m)v(p_n)}{iv^T(p_m)\tau_2 v(p_n)}\right), \\
\widehat{\mathrm{K}}_2^{(4,N)}(p_n,q_m) =& \frac{1}{iv^T(q_m)\tau_2 v(p_n)}\left(\frac{v^T(p_n)v(p_n)}{iv^T(q_m)\tau_2 v(p_n)}\right)^{2N} \\
\times& \left(1 - \frac{v^T(q_m)v(p_n)v^\dagger(q_m)v(p_n)}{v^T(q_m)\tau_2 v(p_n)v^\dagger(q_m)\tau_2 v(p_n)}\right)^{-2N-1} \\
\times& \left[\left(\frac{v^\dagger(q_m)v(p_n)}{iv^\dagger(q_m)\tau_2 v(p_n)}\right)^{2N+1}\frac{v^T(q_m)v(p_n)}{iv^T(q_m)\tau_2 v(p_n)} + (2N+1)q_{2N}^{(N+1)}\left(\frac{v^\dagger(q_m)v(p_n)}{iv^\dagger(q_m)\tau_2 v(p_n)}\right)\right], \\
\widehat{\mathrm{K}}_3^{(4,N)}(q_m,q_n) =& \left(\frac{iv^\dagger(q_n)\tau_2 v^*(q_m)}{v^\dagger(q_m)v(q_m)v^\dagger(q_n)v(q_n)}\right)^{2N+1} q_{2N}^{(N+1)}\left(\frac{v^\dagger(q_n)v^*(q_m)}{iv^\dagger(q_n)\tau_2 v^*(q_m)}\right) \\
-& iv^T(q_m)\tau_2 v(q_n)\left(\frac{v^\dagger(q_m)v^*(q_n)}{v^\dagger(q_m)v(q_m)v^\dagger(q_n)v(q_n)}\right)^{2N+2}\Phi_{2N+2}^{(1)}\left(\frac{|v^T(q_m)v(q_n)|^2}{v^\dagger(q_m)v(q_m)v^\dagger(q_n)v(q_n)}\right).
\end{aligned} \tag{5.19}$$

The block matrix in (5.18) has to be read such that one takes a $k \times k$-matrix with 2×2-matrices of the shown form as matrix entries. We chose the sign of the Pfaffian as

$$\mathrm{Pf}[\mathbb{1}_n \otimes i\tau_2] = +1. \tag{5.20}$$

We find a special kind of Lerch's transcendental function [167]

$$\Phi_{n+1}^{(1)}(z) = -\frac{1}{z^{n+1}}\left[\ln(1-z) + \sum_{j=1}^{n}\frac{z^j}{j}\right] \tag{5.21}$$

as well as the polynomial

$$q_{2n}^{(N)}(x) = \sum_{m=0}^{n} \frac{\mathrm{B}(n+1, N-n+1/2)}{\mathrm{B}(m+1, N-m+1/2)} x^{2m}$$
$$= \frac{2N+1}{2} \mathrm{B}\left(n+1, \frac{2N-2n+1}{2}\right) \left(1+x^2\right)^{n-1/2}$$
$$- \frac{2N-2n-1}{2n+2} x^{2n+2} {}_2\mathrm{F}_1\left(1, \frac{3+2n-2N}{2}; n+2; -x^2\right). \tag{5.22}$$

The function $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is Euler's Beta function with the Gamma function $\Gamma(x)$. The polynomials are essentially truncated binomial series. The second representation involves Gauss' hypergeometric function ${}_2\mathrm{F}_1$. The polynomials are actually the skew-orthogonal polynomials of even order corresponding to the quaternion spherical ensemble, see appendix A.1 for their derivation. Having the result at hand, we again want to point out its invariance under $\mathrm{O}(2)$, which is checked quickly by (5.18) and (5.19).

5.3.3 The Orthogonal Case (BDI)

The orthogonal case is the most challenging of the three cases. Therefore, for technical reasons exclusive to this case, we assume an even matrix dimension N. This does not interfere with our actual goal to uncover universal behaviour in the large N limit as is outlined in section 5.4.3. Just as in the symplectic case, we obtain a Pfaffian structure. The symmetry group is again $\mathrm{O}(2) \times \mathrm{Gl}_{\mathbb{R}}(1)$, where $\mathrm{Gl}_{\mathbb{R}}(1)$ corresponds to a rescaling and $\mathrm{O}(2)$ rotates the real matrices K_1 and K_2 into each other.

We derive the following result

$$Z_{k|k}^{(1,N)}(q,p) = \frac{\mathrm{Pf}\begin{bmatrix} \widehat{\mathrm{K}}_1^{(1,N)}(p_m,p_n) & \widehat{\mathrm{K}}_2^{(1,N)}(p_m,q_n) \\ -\widehat{\mathrm{K}}_2^{(1,N)}(p_n,q_m) & \widehat{\mathrm{K}}_3^{(1,N)}(q_m,q_n) \end{bmatrix}_{1\le m,n\le k}}{\det\left[\dfrac{1}{iv^T(q_m)\tau_2 v(p_n)}\right]_{1\le m,n\le k}} \tag{5.23}$$

with the kernel functions

$$
\begin{aligned}
\widehat{\mathrm{K}}_1^{(1,N)}(p_m,p_n) =&\frac{N(N-1)}{4\pi}\, iv^T(p_n)\tau_2 v(p_m)\left[v^T(p_m)v(p_n)\right]^{N-2},\\
\widehat{\mathrm{K}}_2^{(1,N)}(p_n,q_m) =&\frac{1}{iv^T(p_n)\tau_2 v(q_m)}\frac{N(N-1)}{4\pi}\left(\frac{v^T(p_n)v(p_n)}{iv^T(p_n)\tau_2 v(q_m)}\right)^N\\
\times &\int\limits_{\mathbb{C}^2} d[z]\, z_1^{N-2} g^{(1,N)}(z_1,z_2)\left(z_2+\frac{v^T(p_n)v(q_m)}{iv^T(p_n)\tau_2 v(q_m)}\right)^{-1},\\
\widehat{\mathrm{K}}_3^{(1,N)}(q_m,q_n) =&\frac{1}{[b(q_m)b(q_n)]^N}\left[\int\limits_{\mathbb{C}^2} d[z]\frac{g^{(1,N)}(z_1,z_2)}{(a(q_m)+b(q_m)\,z_1)(a(q_n)+b(q_n)\,z_2)}\right.\\
-&\left.\frac{N(N-1)}{4\pi}\int\limits_{\mathbb{C}^4} d[z]\frac{(z_3-z_1)(z_1z_3+1)^{N-2}}{(a(q_m)+b(q_m)\,z_2)(a(q_n)+b(q_n)\,z_4)}g^{(1,N)}(z_1,z_2)g^{(1,N)}(z_3,z_4)\right].
\end{aligned}
\tag{5.24}
$$

The antisymmetric function

$$
\begin{aligned}
g^{(1,N)}(z_1,z_2) =&\frac{|z_2-z_1|}{z_2-z_1}\\
&\times\frac{\mathrm{B}(1/2,(N+1)/2)\delta(y_1)\delta(y_2)+2\delta(x_1-x_2)\delta(y_1+y_2)Q(z_1,z_1^*)}{[(1+z_1^2)(1+z_2^2)]^{(N+1)/2}}
\end{aligned}
\tag{5.25}
$$

directly emerges from our random matrix problem as the antisymmetric two-point measure in the joint eigenvalue distribution of the real spherical ensemble. For the complex integration variables we use the notation $z_j = x_j + i\,y_j$ with $x_j, y_j \in \mathbb{R}$. Furthermore, a lower incomplete Beta function appears in our results

$$
\begin{aligned}
Q(z,z^*) &= \mathrm{B}\left(\frac{4y^2}{|1+z^2|^2+4y^2};1/2,(N+1)/2\right),\\
\mathrm{B}(x;a,b) &= \int\limits_x^1 dt\, t^{a-1}(1-t)^{b-1} = 2\int\limits_{\sqrt{\frac{x}{1-x}}}^{\infty} dt\frac{t^{2a-1}}{(1+t^2)^{a+b}},\ \text{with } x\in[0,1],\\
\mathrm{B}(0;a,b) &= \mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
\end{aligned}
\tag{5.26}
$$

Inserting (5.25) into the second and third kernel yields the full expressions

$$\begin{aligned}\widehat{\mathrm{K}}_2^{(1,N)}(p_n,q_m) =& \frac{1}{iv^T(p_n)\tau_2 v(q_m)}\frac{N(N-1)}{2\pi}\left(\frac{v^T(p_n)v(p_n)}{iv^T(p_n)\tau_2 v(q_m)}\right)^N\\ &\times\left[\frac{(-1)^{N/2}\pi\,\mathrm{B}\left(1/2,\frac{N+1}{2}\right)}{N-1}\binom{\frac{N-1}{2}}{N}{}_2\mathrm{F}_1\left(1,\frac{N+1}{2};N+1;1+\left(\frac{v^T(p_n)v(q_m)}{iv^T(p_n)\tau_2 v(q_m)}\right)^2\right)\right.\\ &\left.+\,i\int\limits_{\mathbb{C}} d[z]\frac{z^{N-2}\,\mathrm{sgn}(\mathrm{Im}\,z)Q(z,z^*)}{|1+z^2|^{N+1}}\left(z^*+\frac{v^T(p_n)v(q_m)}{iv^T(p_n)\tau_2 v(q_m)}\right)^{-1}\right]\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\widehat{\mathrm{K}}_3^{(1,N)}(q_m,q_n) =& \frac{1}{[b(q_m)b(q_n)]^N}\left[\int\limits_{\mathbb{R}} dx\frac{r(x,v(q_n))}{(a(q_m)+b(q_m)x)}+\int\limits_{\mathbb{C}} d[z]\frac{s(z,z^*,v(q_n))}{(a(q_m)+b(q_m)z)}\right]\\ &-\frac{N(N-1)}{2\pi[b(q_m)b(q_n)]^{N-1}}\left[\int\limits_{\mathbb{R}^2} d[x]r(x_1,v(q_m))r(x_2,v(q_n))(x_2-x_1)(x_1x_2+1)^{N-2}\right.\\ &+\int\limits_{\mathbb{R}} dx\int\limits_{\mathbb{C}} d[z]\det\begin{bmatrix} r(x,v(q_m)) & s(z,z^*,v(q_m))\\ r(x,v(q_n)) & s(z,z^*,v(q_n))\end{bmatrix}(z-x)(z\,x+1)^{N-2}\\ &\left.+\int\limits_{\mathbb{C}^2} d[z]s(z_1,z_1^*,v(q_m))s(z_2,z_2^*,v(q_n))(z_2-z_1)(z_1z_2+1)^{N-2}\right]\end{aligned} \tag{5.28}$$

with the functions

$$\begin{aligned}r(x,v(q)) &= \mathrm{B}(1/2,(N+1)/2)\int\limits_{\mathbb{R}} dx'\frac{\mathrm{sgn}(x'-x)}{(a(q)+b(q)x')[(1+x^2)(1+x'^2)]^{(N+1)/2}},\\ s(z,z^*,v(q)) &= \frac{2i\,\mathrm{sgn}(\mathrm{Im}\,z)Q(z,z^*)}{(a(q)+b(q)z^*)\,|1+z^2|^{N+1}}.\end{aligned} \tag{5.29}$$

These expressions still contain integrals over up to four real dimensions. A further analytic computation of them is difficult due to the singularities appearing at various points in the complex plane. As in the unitary and in the symplectic case, the rotational invariance in $v(p)$ can be checked from the final results (5.23) and (5.24). Only the third kernel evades such an inspection because of its complicated expression.

5.4 Derivations

In sections 5.4.1, 5.4.2 and 5.4.3 we derive the results for the unitary, symplectic and orthogonal class, respectively. We first analyze the symmetries of the generating function (5.13), which become handy when simplifying the computations. Furthermore, we trace the ensemble average over the two independent Ginibre matrices back to an ensemble average over the spherical ensembles that have been studied in [76–79], see also section 3.2.

Using results from [82, 83], we make use of determinantal and Pfaffian structures that reduce the problem of averaging over a ratio of $2k$ characteristic polynomials to averages of only two characteristic polynomials. In combination with the techniques of orthogonal and skew-orthogonal polynomials as well as some complex analysis tools we find the results summarized in section 5.3.

5.4.1 The Unitary Case (AIII)

When the two matrices $K_1, K_2 \in \mathbb{C}^{N\times N}$ are independently drawn from a complex Ginibre ensemble, i.e. their joint probability distribution is

$$\widetilde{P}^{(2,N)}(K_1, K_2) = \pi^{-2N^2} \exp\left(-\mathrm{tr}\, K_1 K_1^\dagger - \mathrm{tr}\, K_2 K_2^\dagger\right), \tag{5.30}$$

it is useful to write the two complex functions $a(p), b(p)$ in terms of the two-dimensional complex vector $v(p)$. The reason is that this ensemble actually satisfies an SU(2) symmetry given by

$$\hat{K} = \left[\begin{array}{c} K_1 \\ K_2 \end{array}\right] \longrightarrow [U \otimes \mathbb{1}_N] \left[\begin{array}{c} K_1 \\ K_2 \end{array}\right] \tag{5.31}$$

with $U \in \mathrm{SU}(2)$ acting on the two components of the matrix valued vector $\hat{K}$. One can readily verify $\widetilde{P}^{(2,N)}(\hat{K}) = \widetilde{P}^{(2,N)}([U \otimes \mathbb{1}_N]\hat{K})$ for any $U \in \mathrm{SU}(2)$. This will become handy when computing the generating function $Z_{k|k}^{(2,N)}(q,p)$ and recognizing that

$$K(p) = a(p)K_1 + b(p)K_2 = v^T(p)\hat{K}. \tag{5.32}$$

Surely this SU(2) invariance will carry over to the vectors $v(p_j)$ and $v(q_j)$.

Before we exploit this symmetry, we would like to draw attention to the relation of this ensemble to the complex spherical ensemble for which we need to rephrase the matrix $K(p)$ as follows

$$K(p) = a(p)K_1 + b(p)K_2 = b(p)K_1\left(\kappa(p)\mathbb{1}_N + K_1^{-1}K_2\right) \tag{5.33}$$

with

$$\kappa(p) = \frac{a(p)}{b(p)}. \tag{5.34}$$

This way of writing is only possible when $b(p) \neq 0$. This is, however, not very restrictive as the limit $b(p) \to 0$ can be readily carried out in the results. The generating function (5.13) for $k = l$, has then the form

$$Z_{k|k}^{(2,N)}(q,p) = \left(\prod_{j=1}^{k} \frac{b(p_j)}{b(q_j)}\right)^N \left\langle \prod_{j=1}^{k} \frac{\det(\kappa(p_j)\mathbb{1}_N + K_1^{-1}K_2)}{\det(\kappa(q_j)\mathbb{1}_N + K_1^{-1}K_2)} \right\rangle. \tag{5.35}$$

The random matrix $Y = K_1^{-1}K_2$ describes the complex spherical ensemble, which has been analyzed in [76, 77]. Its matrix probability distribution is

$$\widetilde{G}^{(2,N)}(Y) = \pi^{-N^2} \prod_{j=0}^{N-1} \frac{(N+j)!}{j!} \frac{1}{\det^{2N}(\mathbb{1}_{2N} + YY^\dagger)} \tag{5.36}$$

and the corresponding joint probability distribution of the N complex eigenvalues $(z_1, \ldots, z_N) \in \mathbb{C}^N$ is

$$\begin{aligned} G^{(2,N)}(z) &= \frac{1}{c^{(2,N)}} \frac{|\Delta_N(z)|^2}{\prod_{j=1}^N (1+|z_j|^2)^{N+1}}, \\ c^{(2,N)} &= \pi^N N! \prod_{j=1}^{N} \mathrm{B}(j, N+1-j). \end{aligned} \tag{5.37}$$

An important remark about the integrability of the generating function is in order. We certainly make use of the fact that a simple pole like $1/(\kappa(q_j) + z)$ is integrable in two dimensions such as the complex plane. However, we need to assume that all $\kappa(q_j)$ are pairwise distinct. In spite of this, it is rather remarkable that the final result can be nonetheless analytically continued to these singular points without any problems.

It is the structure of the joint probability distribution (5.36), which tells us that this ensemble follows a determinantal point process [77, 163]. In particular, the k-point correlation function is a $k \times k$-determinant with a single kernel function. This structure actually applies to the generating function (5.35) as well. In [82, 144] it was shown for more general ensembles than the one we study that

$$\begin{aligned} Z_{k|k}^{(2,N)}(q,p) &= \left(\prod_{j=1}^{k} \frac{b(p_j)}{b(q_j)}\right)^N \frac{\det\left[\left(\frac{b(q_m)}{b(p_n)}\right)^N \frac{Z_{1|1}^{(2,N)}(q_m,p_n)}{\kappa(q_m)-\kappa(p_n)}\right]_{1\le m,n\le k}}{\det\left[\frac{1}{\kappa(q_m)-\kappa(p_n)}\right]_{1\le m,n\le k}} \\ &= \frac{\det\left[\frac{Z_{1|1}^{(2,N)}(q_m,p_n)}{a(q_m)b(p_n)-b(q_m)a(p_n)}\right]_{1\le m,n\le k}}{\det\left[\frac{1}{a(q_m)b(p_n)-b(q_m)a(p_n)}\right]_{1\le m,n\le k}}. \end{aligned} \tag{5.38}$$

The normalization can be checked by the asymptotic behaviour

$$\lim_{a(p),a(q)\to\infty}\left(\prod_{j=1}^{k}\frac{a(q_j)}{a(p_j)}\right)^N Z_{k|k}^{(2,N)}(q,p)=1. \tag{5.39}$$

The denominator in the first line of (5.38) is a Cauchy determinant, see [168]. This factor appears in exactly the same way also in the ensemble averages of the symplectic and orthogonal cases. It can be identified with a Berezinian, see [82] and section 3.4,

$$\sqrt{\mathrm{Ber}_{k|k}^{(2)}(\kappa(q);\kappa(p))}=\det\left[\frac{1}{\kappa(q_m)-\kappa(p_n)}\right]_{1\leq m,n\leq k}, \tag{5.40}$$

which is the superspace analogue to the Vandermonde determinant that is obtained as the Jacobian of a diagonalization. This highlights the intimate link to a supersymmetric formulation of the problem.

The advantage of the determinantal form (5.38) is that we actually need to compute the generating function for $k=1$. For this purpose, we finally make use of the SU(2) symmetry we have mentioned previously. The generating function

$$Z_{1|1}^{(2,N)}(q_m,p_n)=F(v(q_m),v(p_n)) \tag{5.41}$$

can be understood as a function of the two vectors $v(q_m)$ and $v(p_n)$ and the SU(2) symmetry tells us that $F(v(q_m),v(p_n))=F(U^Tv(q_m),U^Tv(p_n))$ for all $U\in\mathrm{SU}(2)$. Therefore, we can choose the unitary matrix

$$U=\frac{1}{\sqrt{|a(q_m)|^2+|b(q_m)|^2}}\begin{bmatrix} a^*(q_m) & -b(q_m)\\ b^*(q_m) & a(q_m)\end{bmatrix}\in\mathrm{SU}(2) \tag{5.42}$$

such that the generating function simplifies to

$$\begin{aligned}Z_{1|1}^{(2,N)}(q_m,p_n)&=\left\langle\det\left(\frac{v^\dagger(q_m)v(p_n)}{v^\dagger(q_m)v(q_m)}\mathbb{1}_N+\tilde{b}K_1^{-1}K_2)\right)\right\rangle\\&=\left\langle\det\left(\frac{v^\dagger(q_m)v(p_n)}{v^\dagger(q_m)v(q_m)}\mathbb{1}_N+\tilde{b}Y)\right)\right\rangle.\end{aligned} \tag{5.43}$$

The coefficient $\tilde{b}=iv^T(q_m)\tau_2v(p_n)/v^\dagger(q_m)v(q_m)\in\mathbb{C}$ is not very important as the U(1) invariance $Y\to e^{i\varphi}Y$ of the complex spherical ensemble tells us that the average of the characteristic polynomial $\det(x\mathbb{1}_N-Y)$ only yields the monomial x^N.

Thus, the final result is

$$Z_{k|k}^{(2,N)}(q,p) = \frac{\det\left[\dfrac{1}{\left(|a(q_m)|^2+|b(q_m)|^2\right)^N}\dfrac{(a^*(q_m)a(p_n)+b^*(q_m)b(p_n))^N}{a(q_m)b(p_n)-b(q_m)a(p_n)}\right]_{1\leq m,n\leq k}}{\det\left[\dfrac{1}{a(q_m)b(p_n)-b(q_m)a(p_n)}\right]_{1\leq m,n\leq k}} \tag{5.44}$$

and expressing it in terms of the vector $v(p)$ yields

$$Z_{k|k}^{(2,N)}(q,p) = \frac{\det\left[\dfrac{1}{v^T(q_m)\tau_2 v(p_n)}\left(\dfrac{v^\dagger(q_m)v(p_n)}{v^\dagger(q_m)v(q_m)}\right)^N\right]_{1\leq m,n\leq k}}{\det\left[\dfrac{1}{v^T(q_m)\tau_2 v(p_n)}\right]_{1\leq m,n\leq k}}. \tag{5.45}$$

This result nicely reflects the SU(2) symmetry as it only depends on the SU(2) invariants $v^\dagger(q)v(q)$, $v^\dagger(q)v(p)$, and $v^T(q)\tau_2 v(p)$.

The SU(2) invariance is actually also reflected in the symmetry of the eigenvalue spectrum of the complex spherical ensemble. In [76] it was pointed out that the complex spectrum is uniformly distributed on a two-dimensional sphere after a stereographic projection. It is the adjoint representation of SU(2), which is the special orthogonal group SO(3) that highlights the uniform distribution as it is the invariance group of a two-dimensional sphere.

5.4.2 The Symplectic Case (CII)

In the symplectic case we cannot exploit an SU(2) invariance. Due to the reality constraint of the real quaternion matrices $K_1, K_2 \in \mathbb{H}^{N\times N} \subset \mathbb{C}^{2N\times 2N}$ in the form

$$K_j = [\tau_2 \otimes 1\!\!1_N]K_j^*[\tau_2 \otimes 1\!\!1_N], \tag{5.46}$$

we can only make use of the smaller invariance group O(2). The symmetry transformation is given by

$$\hat{K} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \longrightarrow [U \otimes 1\!\!1_{2N}] \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{5.47}$$

with $U \in \mathrm{O}(2)$. The joint probability distribution of two independent quaternion Ginibre ensembles, i.e.

$$\widetilde{P}^{(4,N)}(K_1,K_2) = \pi^{-4N^2}\exp\left(-\frac{1}{2}\mathrm{tr}\,K_1K_1^\dagger - \frac{1}{2}\mathrm{tr}\,K_2K_2^\dagger\right), \tag{5.48}$$

respects this symmetry. Thence, it will have some impact in our computations and will be visible in our results.

Once again, we express the expectation value over the two quaternion matrices K_1, K_2 as an expectation value over the random matrix $Y = K_1^{-1}K_2 \in \mathbb{H}^{N\times N}$, namely

$$Z_{k|k}^{(4,N)}(q,p) = \left(\prod_{j=1}^{k} \frac{b(p_j)}{b(q_j)}\right)^{2N} \left\langle \prod_{j=1}^{k} \frac{\det(\kappa(p_j)\mathbb{1}_{2N} + Y)}{\det(\kappa(q_j)\mathbb{1}_{2N} + Y)} \right\rangle, \tag{5.49}$$

with $\kappa(p) = a(p)/b(p)$ defined as before. The matrix Y is now drawn from the quaternion spherical ensemble, which has been studied in [79]. It follows the matrix probability distribution

$$\widetilde{G}^{(4,N)}(Y) = \pi^{-2N^2} \prod_{j=1}^{N} \frac{(2N+2j-1)!}{(2j-1)!} \frac{1}{\det^{2N}(\mathbb{1}_{2N} + YY^\dagger)}. \tag{5.50}$$

Due to being quaternion the eigenvalues of Y come in pairs z and z^*, meaning the complex conjugate of an eigenvalue is also an eigenvalue. The corresponding joint probability distribution of the eigenvalues $z = (z_1, z_1^*, z_2, z_2^*, \ldots, z_N, z_N^*) \in [\mathbb{C}\backslash\{0\}]^{2N}$ is given by

$$\begin{aligned} G^{(4,N)}(z) &= \frac{1}{c^{(4,N)}} \Delta_{2N}(z) \prod_{j=1}^{N} \frac{z_j - z_j^*}{(1+|z_j|^2)^{2N+2}}, \\ c^{(4,N)} &= (2\pi)^N N! \prod_{j=1}^{N} \mathrm{B}(2j, 2N+2-2j). \end{aligned} \tag{5.51}$$

Considering this explicit form, the question of integrability for the considered generating function can be raised anew. It is this time not evident even in the case of pairwise distinct complex pairs $(\kappa(q_j), \kappa^*(q_j))$ as we encounter terms of the form $1/[(\kappa(q_j) + z_j)(\kappa(q_j) + z_j^*)]$. As long as $\kappa(q_j)$ is not real, the singularities are simple poles. However, when $\kappa(q_j)$ is real this term becomes a double pole of the integrand, which is, in general, not integrable even in two dimensions. The fortunate fact that renders also this kind of pole integrable is the factor $|z_j - z_j^*|^2$ as it vanishes like a square when z_j becomes real. Therefore, the combination $|z_j - z_j^*|^2/[(\kappa(q_j) + z_j)(\kappa(q_j) + z_j^*)]$ is absolutely integrable even when $\kappa(q_j)$ becomes real. The condition of pairwise distinct complex pairs $(\kappa(q_j), \kappa^*(q_j))$ can be anew dropped for the final result where the limit $\kappa(q_a) \to \kappa(q_b)$ as well as $\kappa(q_a) \to \kappa^*(q_b)$ is well-defined, see the summary of the results in section 5.3.

It is well known, see [79], that the quaternion spherical ensemble describes a Pfaffian point process, and as before, this structure carries over to the generating function, which becomes, see [83, 144],

$$Z_{k|k}^{(4,N)}(q,p) = \frac{\mathrm{Pf}\begin{bmatrix} \mathrm{K}_1^{(4,N)}(p_m,p_n) & \mathrm{K}_2^{(4,N)}(p_m,q_n) \\ -\mathrm{K}_2^{(4,N)}(p_n,q_m) & \mathrm{K}_3^{(4,N)}(q_m,q_n) \end{bmatrix}_{1\leq m,n\leq k}}{\det\left[\dfrac{1}{\kappa(q_m)-\kappa(p_n)}\right]_{1\leq m,n\leq k}}, \tag{5.52}$$

where the three kernel functions are

$$\begin{aligned} \mathrm{K}_1^{(4,N)}(p_m,p_n) =&[\kappa(p_n)-\kappa(p_m)][b(p_m)b(p_n)]^{2N}\widetilde{Z}_{0|2}^{(4,N-1)}(p_m,p_n), \\ \mathrm{K}_2^{(4,N)}(p_n,q_m) =&\frac{1}{\kappa(q_m)-\kappa(p_n)}Z_{1|1}^{(4,N)}(p_n,q_m), \\ \mathrm{K}_3^{(4,N)}(q_m,q_n) =&\frac{\kappa(q_n)-\kappa(q_m)}{[b(q_m)b(q_n)]^{2N}}\widetilde{Z}_{2|0}^{(4,N+1)}(q_m,q_n). \end{aligned} \tag{5.53}$$

We have employed the following definition for $l-k$ even and $M+(l-k)/2\leq N$

$$\begin{aligned} \widetilde{Z}_{k|l}^{(4,M)}(q,p) =&\frac{1}{(2\pi)^{M+(l-k)/2}M!\prod_{j=1}^{M+(l-k)/2}\mathrm{B}(2j,2N+2-2j)} \\ &\times\int_{\mathbb{C}^M}d[z]\Delta_{2M}(z)\prod_{r=1}^{M}\frac{z_r-z_r^*}{(1+|z_r|^2)^{2N+2}}\prod_{j=1}^{M}\frac{\prod_{n=1}^{l}(\kappa(p_n)+z_j)(\kappa(p_n)+z_j^*)}{\prod_{m=1}^{k}(\kappa(q_m)+z_j)(\kappa(q_m)+z_j^*)}. \end{aligned} \tag{5.54}$$

Let us highlight that the weight function $g^{(4,N)}(z)=(z-z^*)/(1+|z|^2)^{2N+2}$ remains always the same in this definition, while the number M of integration variables varies. The result (5.52) follows from [83] when identifying in a distributional way the weight function $g^{(4,N)}(z)$ with the skew-symmetric two-point weight involving the Dirac delta function for complex numbers

$$\tilde{g}^{(4,N)}(z_1,z_2) = \frac{z_1-z_2}{(1+|z_1|^2)^{N+1}(1+|z_2|^2)^{N+1}}\delta(z_2-z_1^*). \tag{5.55}$$

The integration over every second variable yields the joint probability distribution (5.51). The prefactor in (5.54) actually contains the Pfaffian of a moment matrix

$$\begin{aligned} D^{(4,d)} =& \left[D_{ab}^{(4)}\right]_{1\leq a,b\leq d}, \\ D_{ab}^{(4)} =& \int_{\mathbb{C}}d[z]g^{(4,N)}(z)\left[z^{a-1}(z^*)^{b-1}-z^{b-1}(z^*)^{a-1}\right] = 2\int_{\mathbb{C}}d[z]g^{(4,N)}(z)z^{a-1}(z^*)^{b-1} \\ =& 2\pi\,\mathrm{B}\left(2N+2-\frac{a+b+1}{2},\frac{a+b+1}{2}\right)\left(\delta_{a,b-1}-\delta_{a-1,b}\right), \end{aligned} \tag{5.56}$$

where d is even. Since the matrix is tridiagonal, the Pfaffian can be conveniently evaluated

$$\mathrm{Pf}\, D^{(4,d)} = (2\pi)^{d/2} \prod_{j=1}^{d/2} \mathrm{B}(2j, 2N+2-2j). \tag{5.57}$$

In the ensuing three subsections we compute explicit expressions of the three kernels (5.53).

The Kernel $\mathrm{K}_1^{(4,N)}$

The kernel function $\mathrm{K}_1^{(4,N)}(p_m, p_n)$ is expressed in terms of $\widetilde{Z}_{0|2}^{(4,N-1)}(\kappa(p_m), \kappa(p_n))$. We are in the lucky position that we can relate this function to $Z_{0|2}^{(4,N-1)}(p_m, p_n)$ for which we can exploit the O(2) symmetry. The limits

$$\begin{aligned} \lim_{\kappa(p)\to\infty} \frac{\widetilde{Z}_{0|2}^{(4,N-1)}(p_m, p_n)}{[\kappa(p_m)\kappa(p_n)]^{2N-2}} &= \frac{1}{2\pi\,\mathrm{B}(2N,2)}, \\ \lim_{a(p)\to\infty} \frac{Z_{0|2}^{(4,N-1)}(p_m, p_n)}{[a(p_m)a(p_n)]^{2N-2}} &= \left\langle \det K_1^2 \right\rangle \end{aligned} \tag{5.58}$$

relate the normalization of the two kinds of functions. They are equivalent because $\kappa(p) = a(p)/b(p)$ is directly proportional to $a(p)$. Therefore we obtain

$$\begin{aligned} \widetilde{Z}_{0|2}^{(4,N-1)}(p_m, p_n) =& \frac{1}{2\pi\,\mathrm{B}(2N,2)\left\langle \det K_1^2 \right\rangle} \frac{Z_{0|2}^{(4,N-1)}(p_m, p_n)}{[b(p_m)b(p_n)]^{2N-2}} \\ =& \frac{\left\langle \det(a(p_m)K_1 + b(p_m)K_2) \det(a(p_n)K_1 + b(p_n)K_2) \right\rangle}{2\pi\,\mathrm{B}(2N,2)\left\langle \det K_1^2 \right\rangle [b(p_m)b(p_n)]^{2N-2}}, \end{aligned} \tag{5.59}$$

where we average over two independent $(2N-2)$-dimensional quaternion Ginibre matrices K_1, K_2.

The function $Z_{0|2}^{(4,N-1)}(p_m, p_n)$ is a polynomial in the complex functions $a(p_m)$, $b(p_m)$, $a(p_n)$, and $b(p_n)$. Hence, we can also consider the average

$$\Xi_1^{(4,N-1)} = \frac{\left\langle \det(a_1 K_1 + b_1 K_2) \det(a_2 K_1 + b_2 K_2) \right\rangle}{\left\langle \det K_1^2 \right\rangle} \tag{5.60}$$

with only fixed real $a_1, b_1, a_2, b_2 \in \mathbb{R}$ variables satisfying $b_1 a_2 - a_1 b_2 \neq 0$ and then perform an analytic continuation in the result to the complex functions. We need this detour via analytic continuation because we can only rotate real vectors with the O(2) symmetry similar to what we have done in the unitary case AIII.

We rotate with the special orthogonal matrix

$$U = \frac{1}{\sqrt{a_2^2 + b_2^2}} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} \in \mathrm{SO}(2), \tag{5.61}$$

obtaining for the average

$$\begin{aligned} \Xi_1^{(4,N-1)} =& \frac{\langle \det\left[(a_1a_2 + b_1b_2)K_1 + (b_1a_2 - a_1b_2)K_2\right] \det K_1 \rangle}{\left\langle \det K_1^2 \right\rangle} \\ =& \frac{(b_1a_2 - a_1b_2)^{2N-2}}{\langle \det K_1^2 \rangle} \int d[K_1]d[K_2]\widetilde{P}^{(4,N-1)}(K_1, K_2) \det K_1^2 \\ & \times \det\left(\frac{a_1a_2 + b_1b_2}{b_1a_2 - a_1b_2} + K_1^{-1}K_2\right). \end{aligned} \tag{5.62}$$

Up to the factor $\det K_1^2$ appearing in the integrand this is almost an integral over the spherical ensemble. In fact, this type of measure leads to a generalized form of the spherical ensemble of matrices $K_1^{-1}K_2$, where one or both of K_1 and K_2 are drawn from a deformed Ginibre ensemble

$$P_\mu^{(4,N)}(K) = \pi^{-2N^2} \prod_{j=1}^{N} \frac{\Gamma(2j)}{\Gamma(2\mu + 2j)} \exp\left(-\frac{1}{2}\mathrm{tr}\, KK^\dagger\right) \det{}^\mu KK^\dagger. \tag{5.63}$$

These generalizations of the spherical ensemble are referred to as induced spherical ensembles. They are well studied for $\beta = 1, 2, 4$, see [132–134]. In this case, the presence of the additional factor $\det K_1^2$ causes the weight function to resemble that of a $2N$-dimensional quaternion spherical ensemble $g^{(1,N)}(z) = (z-z^*)/(1+|z|^2)^{2N+2}$, despite dealing with $(2N-2)$-dimensional matrices. Therefore, the average is equal to

$$\begin{aligned} \Xi_1^{(4,N-1)} =& \frac{(b_1a_2 - a_1b_2)^{2N-2}}{(2\pi)^{N-1}(N-1)! \prod_{j=1}^{N-1} \mathrm{B}(2j, 2N+2-2j)} \\ & \times \int_{\mathbb{C}^{N-1}} d[z]\Delta_{2N-2}(z) \prod_{r=1}^{N-1} \frac{z_r - z_r^*}{(1+|z_r|^2)^{2N+2}} \\ & \times \prod_{j=1}^{N-1} \left(\frac{a_1a_2 + b_1b_2}{b_1a_2 - a_1b_2} + z_j\right)\left(\frac{a_1a_2 + b_1b_2}{b_1a_2 - a_1b_2} + z_j^*\right). \end{aligned} \tag{5.64}$$

Apart from the factor $(b_1a_2 - a_1b_2)^{2N-2}$ this integral is the Heine-like formula, see [169] as well as (A.4), for the monic skew-orthogonal polynomial $q_{2N-2}^{(N)}(x)$ of degree

$2N-2$ corresponding to the weight function $g^{(4,N)}$. The skew-orthogonal polynomials have been computed in appendix A.1.

Summarizing, the function $Z_{0|2}^{(4,N-1)}(p_m,p_n)$ has the form

$$\begin{aligned}\frac{Z_{0|2}^{(4,N-1)}(p_m,p_n)}{\left\langle \det K_1^2 \right\rangle} =& [b(p_m)a(p_n)-a(p_m)b(p_n)]^{2N-2} q_{2N-2}^{(N)}\left(\frac{a(p_m)a(p_n)+b(p_m)b(p_n)}{a(p_m)b(p_n)-b(p_m)a(p_n)}\right)\\ =& \sum_{j=0}^{N-1}\frac{\mathrm{B}(N,3/2)}{\mathrm{B}(j+1,N-j+1/2)}[a(p_m)a(p_n)+b(p_m)b(p_n)]^{2j}\\ & \times [b(p_m)a(p_n)-a(p_m)b(p_n)]^{2N-2-2j}.\end{aligned} \tag{5.65}$$

We would like to underline that this formula is also true for the complex functions $a(p)$ and $b(p)$ despite we have derived it for real coefficients due to being a polynomial in these functions. The first kernel function is then

$$\begin{aligned}\mathrm{K}_1^{(4,N)}(p_m,p_n) =& \frac{\kappa(p_n)-\kappa(p_m)}{2\pi\,\mathrm{B}(2N,2)}[b(p_m)b(p_n)]^2[b(p_m)a(p_n)-a(p_m)b(p_n)]^{2N-2}\\ & \times q_{2N-2}^{(N)}\left(\frac{a(p_m)a(p_n)+b(p_m)b(p_n)}{a(p_m)b(p_n)-b(p_m)a(p_n)}\right)\\ =& \frac{b(p_m)b(p_n)}{2\pi}\sum_{j=0}^{N-1}\frac{\mathrm{B}(N,3/2)}{\mathrm{B}(2N,2)\,\mathrm{B}(j+1,N-j+1/2)}\\ & \times [a(p_m)a(p_n)+b(p_m)b(p_n)]^{2j}[b(p_m)a(p_n)-a(p_m)b(p_n)]^{2N-1-2j}\\ =& \frac{b(p_m)b(p_n)}{2\sqrt{\pi}}\sum_{j=0}^{N-1}\frac{N!\,(2N+1)}{j!\,\Gamma(N-j+1/2)}[a(p_m)a(p_n)+b(p_m)b(p_n)]^{2j}\\ & \times [b(p_m)a(p_n)-a(p_m)b(p_n)]^{2N-1-2j}.\end{aligned} \tag{5.66}$$

This sum is apart from a prefactor a truncated binomial series. Expressing it in terms of the vector $v(p)$ yields

$$\mathrm{K}_1^{(4,N)}(p_m,p_n) = \frac{b(p_m)b(p_n)}{2\pi\,\mathrm{B}(2N,2)}[iv^T(p_n)\tau_2 v(p_m)]^{2N-1} q_{2N-2}^{(N)}\left(\frac{v^T(p_m)v(p_n)}{iv^T(p_m)\tau_2 v(p_n)}\right), \tag{5.67}$$

which is not entirely O(2) invariant. In the final result, the factor $b(p_m)b(p_n)$ eventually drops out because of corresponding factors in the other two kernels. Furthermore, the combination $v^T(p_m)\tau_2 v(p_n)$ is only invariant under SO(2) and achieves a sign when $v(p)$ is rotated with $\mathrm{O}(2)\setminus\mathrm{SO}(2)$. However, this quantity appears only in even powers such that the final result (5.23) is indeed O(2) invariant.

The Kernel $\mathbf{K}_2^{(4,N)}$

For the second kernel function we need to evaluate

$$Z_{1|1}^{(4,N)}(\kappa(q_m),\kappa(p_n)) = \left\langle \frac{\det(a(p_n)K_1 + b(p_n)K_2)}{\det(a(q_m)K_1 + b(q_m)K_2)} \right\rangle, \tag{5.68}$$

which is a polynomial in $a(p_n)$ and $b(p_n)$. With the very same arguments as in the previous subsection we can exploit the analyticity in these two variables and replace them by two fixed real variables $a_1, b_1 \in \mathbb{R}$ and analytically continue the result at the end of the day. Unfortunately, we are not allowed to do the same trick for $a(q_n)$ and $b(q_n)$ as the function is not holomorphic in these two variables. Actually, the result will also depend on their complex conjugates such that we only replace them by two generic but fixed complex variables $a_2, b_2 \in \mathbb{C}$.

We are now allowed to apply an O(2) rotation. The average simplifies to

$$\begin{aligned}
\Xi_2^{(4,N)} &= \left\langle \frac{\det(a_1K_1 + b_1K_2)}{\det(a_2K_1 + b_2K_2)} \right\rangle \\
&= \left(a_1^2 + b_1^2\right)^{2N} \left\langle \frac{\det K_1}{\det\left[(a_2a_1 + b_1b_2)K_1 + (b_1a_2 - a_1b_2)K_2\right]} \right\rangle \\
&= \frac{1}{(2\pi)^N N! \prod_{j=1}^N \mathrm{B}(2j, 2N+2-2j)} \left(\frac{a_1^2 + b_1^2}{b_1a_2 - a_1b_2}\right)^{2N} \\
&\quad \times \int\limits_{\mathbb{C}^N} d[z]\Delta_{2N}(z) \prod_{r=1}^N \frac{z_r - z_r^*}{(1+|z_r|^2)^{2N+2}} \\
&\quad \times \prod_{j=1}^N \left(\frac{a_1a_2 + b_1b_2}{b_1a_2 - a_1b_2} + z_j\right)^{-1} \left(\frac{a_1a_2 + b_1b_2}{b_1a_2 - a_1b_2} + z_j^*\right)^{-1},
\end{aligned} \tag{5.69}$$

where we rotated with

$$U = \frac{1}{\sqrt{a_1^2 + b_1^2}} \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \in \mathrm{SO}(2). \tag{5.70}$$

We abbreviate the ratio

$$\widehat{\kappa} = \frac{a_1a_2 + b_1b_2}{b_1a_2 - a_1b_2} \tag{5.71}$$

and identify another Berezinian, see [82],

$$\sqrt{\mathrm{Ber}_{2N|1}^{(2)}(z; -\widehat{\kappa})} = \frac{\Delta_{2N}(z)}{\prod_{j=1}^N (z_j + \widehat{\kappa})(z_j^* + \widehat{\kappa})} = -\det \left[\begin{array}{c|c} z_a^{b-1} & \dfrac{1}{z_a + \widehat{\kappa}} \\ (z_a^*)^{b-1} & \dfrac{1}{z_a^* + \widehat{\kappa}} \end{array}\right]_{\substack{1\leq a\leq N \\ 1\leq b\leq 2N-1}}, \tag{5.72}$$

which is the mixture of a Cauchy determinant and a Vandermonde determinant, see [168]. The notation with the vertical line highlights the last column, which consists of rational functions, while the rows have to be understood in pairs, meaning the odd rows consist of $(z_a^0, \ldots, z_a^{2N-2}, 1/(z_a + \widehat{\kappa}))$ and the even ones are $((z_a^*)^0, \ldots, (z_a^*)^{2N-2}, 1/(z_a^* + \widehat{\kappa}))$.

It is this determinantal form of the Berezinian, which is useful, as we can expand it in the very last column. Due to the permutation symmetry of the integrand in the integration variables z_j as well as their conjugates, each expansion term yields the very same contribution and, hence, an overall factor $2N$ so that we can also write

$$\begin{aligned}\Xi_2^{(4,N)} =& \frac{-2}{(2\pi)^N (N-1)! \prod_{j=1}^N \mathrm{B}(2j, 2N+2-2j)} \left(\frac{a_1^2 + b_1^2}{b_1 a_2 - a_1 b_2} \right)^{2N} \\ \times& \int_{\mathbb{C}^N} d[z] \Delta_{2N-2}(z_1, z_1^*, \ldots, z_{N-1}, z_{N-1}^*) \prod_{r=1}^N \frac{z_r - z_r^*}{(1+|z_r|^2)^{2N+2}} \frac{\prod_{j=1}^{N-1}(z_j - z_N)(z_j^* - z_N)}{z_N^* + \widehat{\kappa}} \\ =& - \frac{2N(2N+1)}{\pi} \left(\frac{a_1^2 + b_1^2}{b_1 a_2 - a_1 b_2} \right)^{2N} \int_{\mathbb{C}} d[z_N] \frac{z_N - z_N^*}{(1+|z_N|^2)^{2N+2}} \frac{q_{2N-2}^{(N)}(z_N)}{z_N^* + \widehat{\kappa}}.\end{aligned} \tag{5.73}$$

In the second equality we have identified the integral over $(z_1, \ldots, z_{N-1})$ with the Heine-formula (A.4) for $q_{2N-2}^{(N)}(z_N)$.

In expression (5.73) it becomes immediate why the function cannot be holomorphic in $a(q_m)$ and $b(q_m)$ anywhere in the complex plane. One can apply the standard formula for the derivative in the complex conjugate $\widehat{\kappa}^*$ on the integral

$$\frac{\partial}{\partial \widehat{\kappa}^*} \int_{\mathbb{C}} d[z] \frac{f(z, z^*)}{z + \widehat{\kappa}} \propto f(-\widehat{\kappa}, -\widehat{\kappa}^*) \tag{5.74}$$

for an arbitrary suitably integrable complex function $f(z, z^*)$. Considering the integrand in (5.73) we notice that, apart from the real line, the integral must be a function of both, $\widehat{\kappa}$ and $\widehat{\kappa}^*$, which is also what we find. Thus, the analyticity of the integral in $\widehat{\kappa}$ is violated everywhere.

With the help of a similar argument, the remaining integral can be carried out, namely by noticing

$$\frac{\partial}{\partial z_N} \frac{z_N^{2N+1} z_N^* + (2N+1) q_{2N}^{(N+1)}(z_N)}{(1+|z_N|^2)^{2N+1}} = 2N(2N+1) \frac{(z_N - z_N^*) q_{2N-2}^{(N)}(z_N)}{(1+|z_N|^2)^{2N+2}} \tag{5.75}$$

as well as exploiting the following identity, which is a consequence of the generalized Stokes' theorem (Dolbeault–Grothendieck lemma in complex analysis),

$$\int_{\mathbb{C}} d[z_N] \frac{\partial}{\partial z_N} \frac{f(z_N, z_N^*)}{\prod_{j=1}^L (z_N^* + \widehat{\kappa}_j)} = -\pi \sum_{l=1}^L \frac{f(-\widehat{\kappa}_l^*, -\widehat{\kappa}_l)}{\prod_{j \neq l}^L (\widehat{\kappa}_j - \widehat{\kappa}_l)} \tag{5.76}$$

for L distinct $\widehat{\kappa}_j \in \mathbb{C}$ and any differentiable measurable function $f(z_1, z_2)$, which vanishes at infinity in both arguments and where $f(z, z^*)$ is singularity free.

Collecting everything, we find

$$\Xi_2^{(4,N)} = \left(\frac{a_1^2 + b_1^2}{b_1 a_2 - a_1 b_2}\right)^{2N} \frac{(\widehat{\kappa}^*)^{2N+1}\,\widehat{\kappa} + (2N+1) q_{2N}^{(N+1)}\,(\widehat{\kappa}^*)}{(1 + |\widehat{\kappa}|^2)^{2N+1}} \tag{5.77}$$

with

$$\widehat{\kappa} = \frac{a_1 a_2 + b_1 b_2}{b_1 a_2 - a_1 b_2}, \qquad \widehat{\kappa}^* = \frac{a_1 a_2^* + b_1 b_2^*}{b_1 a_2^* - a_1 b_2^*}, \tag{5.78}$$

where we have employed the fact that $a_1, b_1 \in \mathbb{R}$ are real while $a_2, b_2 \in \mathbb{C}$ are complex. The point about which parameter is real or complex is crucial when reinserting the complex functions $(a_1, b_1, a_2, b_2) \to (a(p_n), b(p_n), a(q_m), b(q_m))$ because only $a(q_m)$ and $b(q_m)$ can be complex conjugated while $a(p_n)$ and $b(p_n)$ are analytic continuations of a_1 and b_1. Therefore, the second kernel is equal to

$$\begin{aligned}
\mathrm{K}_2^{(4,N)}(p_n, q_m) =& \frac{Z_{1|1}^{(4,N)}(p_n, q_m)}{\kappa(q_m) - \kappa(p_n)} \\
=& \frac{b(p_n) b(q_m)}{a(q_m) b(p_n) - b(q_m) a(p_n)} \left(\frac{a^2(p_n) + b^2(p_n)}{b(p_n) a(q_m) - a(p_n) b(q_m)}\right)^{2N} \\
&\times \frac{\widehat{\kappa}_*^{2N+1}(p_n, q_m) \widehat{\kappa}(p_n, q_m) + (2N+1) q_{2N}^{(N+1)}(\widehat{\kappa}_*(p_n, q_m))}{(1 + \widehat{\kappa}(p_n, q_m) \widehat{\kappa}_*(p_n, q_m))^{2N+1}}
\end{aligned} \tag{5.79}$$

with

$$\begin{aligned}
\widehat{\kappa}(p_n, q_m) &= \frac{a(p_n) a(q_m) + b(p_n) b(q_m)}{b(p_n) a(q_m) - a(p_n) b(q_m)}, \\
\widehat{\kappa}_*(p_n, q_m) &= \frac{a(p_n) a^*(q_m) + b(p_n) b^*(q_m)}{b(p_n) a^*(q_m) - a(p_n) b^*(q_m)}.
\end{aligned} \tag{5.80}$$

We would like to underline that $\widehat{\kappa}_*(p_n, q_m)$ is not the complex conjugate of $\widehat{\kappa}(p_n, q_m)$, in spite of how we have obtained the expression. It is not immediate from expression (5.79) that the function $Z_{1|1}^{(4,N)}(p_n, q_m)$ is a polynomial in $a(p_n)$ and $b(p_n)$. We only know this from the starting expression in terms of averages over a ratio of two characteristic polynomials of the random matrix Y.

Anew, one can express the kernel function in the vector $v(p)$, obtaining

$$\begin{aligned}
\mathrm{K}_2^{(4,N)}(p_n,q_m) =& \frac{b(p_n)b(q_m)}{iv^T(q_m)\tau_2 v(p_n)}\left(\frac{v^T(p_n)v(p_n)}{iv^T(q_m)\tau_2 v(p_n)}\right)^{2N}\\
\times&\left(1-\frac{v^T(q_m)v(p_n)v^\dagger(q_m)v(p_n)}{v^T(q_m)\tau_2 v(p_n)v^\dagger(q_m)\tau_2 v(p_n)}\right)^{-2N-1}\\
\times&\left[\left(\frac{v^\dagger(q_m)v(p_n)}{iv^\dagger(q_m)\tau_2 v(p_n)}\right)^{2N+1}\frac{v^T(q_m)v(p_n)}{iv^T(q_m)\tau_2 v(p_n)}+(2N+1)q_{2N}^{(N+1)}\left(\frac{v^\dagger(q_m)v(p_n)}{iv^\dagger(q_m)\tau_2 v(p_n)}\right)\right],
\end{aligned} \tag{5.81}$$

and check the O(2) invariance. Indeed, we find that $Z_{1|1}^{(4,N)}(p_n,q_m)$ depends only on the O(2) invariants $v^T(p_n)v(p_n)$, $v^T(q_m)v(p_n)$, $v^\dagger(q_m)v(p_n)$ and the SO(2) invariants $v^T(q_m)\tau_2 v(p_n)$ and $v^\dagger(q_m)\tau_2 v(p_n)$, which appear in even powers only.

The Kernel $\mathbf{K}_3^{(4,N)}$

For computing the third kernel function, we need to evaluate the integral

$$\begin{aligned}
\Xi_3^{(4,N+1)} =& \frac{1}{(2\pi)^N(N+1)!\prod_{j=1}^N \mathrm{B}(2j,2N+2-2j)}\\
&\times\int_{\mathbb{C}^{N+1}} d[z]\Delta_{2N+2}(z)\prod_{r=1}^{N+1}\frac{z_r-z_r^*}{(1+|z_r|^2)^{2N+2}}\\
&\times\prod_{j=1}^{N+1}\frac{1}{(\kappa_1+z_j)(\kappa_1+z_j^*)(\kappa_2+z_j)(\kappa_2+z_j^*)}
\end{aligned} \tag{5.82}$$

with two distinct complex numbers $\kappa_1,\kappa_2\in\mathbb{C}$. The Vandermonde determinant and the product involving the κ_j times the difference $\kappa_2-\kappa_1$ can be written in terms of a Berezinian, see [82],

$$\begin{aligned}
\sqrt{\mathrm{Ber}_{2N+2|2}^{(2)}(z;-\kappa)} &= -\frac{(\kappa_2-\kappa_1)\Delta_{2N+2}(z)}{\prod_{j=1}^{N+1}(\kappa_1+z_j)(\kappa_1+z_j^*)(\kappa_2+z_j)(\kappa_2+z_j^*)}\\
&= -\det\left[\begin{array}{c|c|c} z_a^{b-1} & \dfrac{1}{z_a+\kappa_1} & \dfrac{1}{z_a+\kappa_2}\\ (z_a^*)^{b-1} & \dfrac{1}{z_a^*+\kappa_1} & \dfrac{1}{z_a^*+\kappa_2}\end{array}\right]_{\substack{1\le a\le N+1\\ 1\le b\le 2N}}.
\end{aligned} \tag{5.83}$$

As before the vertical lines should highlight the two last columns, while the odd rows only comprise z_a and the even rows z_a^*. We may choose the skew-orthogonal polynomials $q_j(x)$ in the entries of this determinant instead of the monomials,

$$\begin{aligned}
&\det\left[\begin{array}{c|c|c} z_a^{b-1} & \dfrac{1}{z_a+\kappa_1} & \dfrac{1}{z_a+\kappa_2} \\ (z_a^*)^{b-1} & \dfrac{1}{z_a^*+\kappa_1} & \dfrac{1}{z_a^*+\kappa_2} \end{array}\right]_{\substack{1\leq a\leq N+1\\ 1\leq b\leq 2N}} \\
=&\det\left[\begin{array}{c|c|c} q_{b-1}^{(N)}(z_a) & \dfrac{1}{z_a+\kappa_1} & \dfrac{1}{z_a+\kappa_2} \\ q_{b-1}^{(N)}(z_a^*) & \dfrac{1}{z_a^*+\kappa_1} & \dfrac{1}{z_a^*+\kappa_2} \end{array}\right]_{\substack{1\leq a\leq N+1\\ 1\leq b\leq 2N}} .
\end{aligned} \tag{5.84}$$

This allows us to apply the generalized de Bruijn theorem to carry out the integral, see [82], yielding

$$\begin{aligned}
\Xi_3^{(4,N+1)} =& \frac{2}{(\kappa_1-\kappa_2)\,\pi^N\prod_{j=1}^N \mathrm{B}(2j,2N+2-2j)} \\
&\times \mathrm{Pf}\left[\begin{array}{c|c|c} \left\langle q_{a-1}^{(N)}|q_{b-1}^{(N)}\right\rangle & \left\langle q_{a-1}^{(N)}\middle|\dfrac{1}{z+\kappa_1}\right\rangle & \left\langle q_{a-1}^{(N)}\middle|\dfrac{1}{z+\kappa_2}\right\rangle \\ \hline \left\langle\dfrac{1}{z+\kappa_1}\middle|q_{b-1}^{(N)}\right\rangle & 0 & \left\langle\dfrac{1}{z+\kappa_1}\middle|\dfrac{1}{z+\kappa_2}\right\rangle \\ \hline \left\langle\dfrac{1}{z+\kappa_2}\middle|q_{b-1}^{(N)}\right\rangle & \left\langle\dfrac{1}{z+\kappa_2}\middle|\dfrac{1}{z+\kappa_1}\right\rangle & 0 \end{array}\right]_{1\leq a,b\leq 2N} ,
\end{aligned} \tag{5.85}$$

where we have employed the skew-symmetric product

$$\langle f_1|f_2\rangle = \int\limits_{\mathbb{C}} d[z] f_1(z)f_2(z^*)g^{(4,N)}(z) = -\int\limits_{\mathbb{C}} d[z] f_1(z^*)f_2(z)g^{(4,N)}(z) = -\langle f_2|f_1\rangle \tag{5.86}$$

with the weight function

$$g^{(4,N)}(z) = \frac{z-z^*}{(1+|z|^2)^{2N+2}}. \tag{5.87}$$

This time the vertical and horizontal lines in (5.85) emphasize the last two rows and columns. The index a is the row index for the first $2N$ rows and b the column index for the first $2N$ columns. The skew-orthogonality of the polynomials simplifies the upper left $2N\times 2N$ block drastically, which becomes a 2×2 block-diagonal matrix. This can be exploited in combination with the standard identity

$$\mathrm{Pf}\begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} = \mathrm{Pf}[A]\,\mathrm{Pf}[C+B^TA^{-1}B] \tag{5.88}$$

to simplify the expression to

$$
\begin{aligned}
(\kappa_2-\kappa_1)\Xi_3^{(4,N+1)} =& -2\int\limits_{\mathbb{C}} d[z]\frac{z-z^*}{(1+|z|^2)^{2N+2}}\frac{1}{(\kappa_1+z)(\kappa_2+z^*)}\\
&+2\sum_{j=0}^{N-1}\frac{1}{h_j}\left[\left\langle q_{2j}^{(N)}\middle|\frac{1}{z+\kappa_1}\right\rangle\left\langle q_{2j+1}^{(N)}\middle|\frac{1}{z+\kappa_2}\right\rangle-\left\langle q_{2j+1}^{(N)}\middle|\frac{1}{z+\kappa_1}\right\rangle\left\langle q_{2j}^{(N)}\middle|\frac{1}{z+\kappa_2}\right\rangle\right]
\end{aligned}
\tag{5.89}
$$

with $h_j = 1/[\pi\,\mathrm{B}(2j+2, 2N-2j)]$ being the normalization of the skew-orthogonal polynomials. Plugging in the explicit expressions of the skew-symmetric product and the skew-orthogonal polynomials, we have

$$
\begin{aligned}
(\kappa_1-\kappa_2)\Xi_3^{(4,N+1)} =& -2\int\limits_{\mathbb{C}} d[z]\frac{z-z^*}{(1+|z|^2)^{2N+2}}\frac{1}{(\kappa_1+z)(\kappa_2+z^*)}\\
&+\frac{2}{\pi}\int\limits_{\mathbb{C}^2} d[z]\frac{(z_1-z_1^*)(z_2-z_2^*)}{(1+|z_1|^2)^{2N+2}\,(1+|z_2|^2)^{2N+2}}\\
&\times\sum_{j=0}^{N-1}\sum_{m=0}^{j}\frac{(2N+1)!\,j!\,\Gamma(N-j+1/2)}{(2j+1)!\,(2N-2j-1)!\,m!\,\Gamma(N-m+1/2)}\frac{z_1^{2m}z_2^{2j+1}-z_2^{2m}z_1^{2j+1}}{(z_1^*+\kappa_1)(z_2^*+\kappa_2)}.
\end{aligned}
\tag{5.90}
$$

In appendix A.2, we evaluate the complex integrals and find

$$
\begin{aligned}
(\kappa_1-\kappa_2)\Xi_3^{(4,N+1)} =& -2\pi\left(\frac{\kappa_2^*-\kappa_1^*}{(1+|\kappa_1|^2)(1+|\kappa_2|^2)}\right)^{2N+1} q_{2N}^{(N+1)}\left(\frac{\kappa_1^*\kappa_2^*+1}{\kappa_2^*-\kappa_1^*}\right)\\
&+2\pi(\kappa_1-\kappa_2)\left(\frac{1+\kappa_1^*\kappa_2^*}{(1+|\kappa_1|^2)(1+|\kappa_2|^2)}\right)^{2N+2}\Phi_{2N+2}^{(1)}\left(\frac{|1+\kappa_1\kappa_2|^2}{(1+|\kappa_1|^2)(1+|\kappa_2|^2)}\right)
\end{aligned}
\tag{5.91}
$$

with Lerch's trancendent $\Phi_{2N+2}^{(1)}(z)$, see (5.21). Exploiting this result, the third kernel function has the form

$$
\begin{aligned}
\mathrm{K}_3^{(4,N)}(q_m,q_n) =& 2\pi b(q_m)b(q_n)\Bigg[\left(\frac{b^*(q_m)a^*(q_n)-a^*(q_m)b^*(q_n)}{(|a(q_m)|^2+|b(q_m)|^2)(|a(q_n)|^2+|b(q_n)|^2)}\right)^{2N+1}\\
&\times q_{2N}^{(N+1)}\left(\frac{a^*(q_m)a^*(q_n)+b^*(q_m)b^*(q_n)}{b^*(q_m)a^*(q_n)-a^*(q_m)b^*(q_n)}\right)\\
&-\left(\frac{a^*(q_m)a^*(q_n)+b^*(q_m)b^*(q_n)}{(|a(q_m)|^2+|b(q_m)|^2)(|a(q_n)|^2+|b(q_n)|^2)}\right)^{2N+2}\\
&\times[b(q_n)a(q_m)-a(q_n)b(q_m))]\,\Phi_{2N+2}^{(1)}\left(\frac{|a(q_m)a(q_n)+b(q_m)b(q_n)|^2}{(|a(q_m)|^2+|b(q_m)|^2)(|a(q_n)|^2+|b(q_n)|^2)}\right)\Bigg].
\end{aligned}
\tag{5.92}
$$

Rewriting this expression in terms of the vector $v(p)$ yields

$$
\begin{aligned}
&\mathrm{K}_3^{(4,N)}(q_m,q_n) = 2\pi b(q_m)b(q_n) \\
&\times \left[\left(\frac{iv^\dagger(q_n)\tau_2 v^*(q_m)}{v^\dagger(q_m)v(q_m)v^\dagger(q_n)v(q_n)} \right)^{2N+1} q_{2N}^{(N+1)} \left(\frac{v^\dagger(q_n)v^*(q_m)}{iv^\dagger(q_n)\tau_2 v^*(q_m)} \right) \right. \\
&\left. - iv^T(q_m)\tau_2 v(q_n) \left(\frac{v^\dagger(q_m)v^*(q_n)}{v^\dagger(q_m)v(q_m)v^\dagger(q_n)v(q_n)} \right)^{2N+2} \Phi_{2N+2}^{(1)} \left(\frac{\left| v^T(q_m)v(q_n) \right|^2}{v^\dagger(q_m)v(q_m)v^\dagger(q_n)v(q_n)} \right) \right]
\end{aligned}
\tag{5.93}
$$

and underlines the O(2) invariance as before.

5.4.3 The Orthogonal Case (BDI)

In the orthogonal case the matrices $K_1, K_2 \in \mathbb{R}^{N\times N}$ are drawn from the real Ginibre ensemble. Their joint distribution is

$$
\widetilde{P}^{(1,N)}(K_1,K_2) = \pi^{-N^2} \exp\left(-\mathrm{tr}\, K_1 K_1^T - \mathrm{tr}\, K_2 K_2^T \right), \tag{5.94}
$$

which is invariant under O(2) transformations in the same fashion as in the symplectic case.

Again, we rephrase the generating function

$$
Z_{k|k}^{(1,N)}(q,p) = \left(\prod_{j=1}^{k} \frac{b(p_j)}{b(q_j)} \right)^N \left\langle \prod_{j=1}^{k} \frac{\det(\kappa(p_j)\mathbb{1}_N + Y)}{\det(\kappa(q_j)\mathbb{1}_N + Y)} \right\rangle \tag{5.95}
$$

with $\kappa(p) = a(p)/b(p)$ and obtain the random matrix $Y = K_1^{-1}K_2 \in \mathbb{R}^{N\times N}$. These matrices define the real spherical ensemble, which has been analyzed in [78]. It follows the matrix probability distribution

$$
\widetilde{G}^{(1,N)}(Y) = \pi^{-N^2/2} \prod_{j=1}^{N} \frac{\Gamma((N+j)/2)}{\Gamma(j/2)} \frac{1}{\det^N\left(\mathbb{1}_N + YY^T\right)}. \tag{5.96}
$$

Unfortunately, in the literature the joint probability distribution of the eigenvalues $(z_1,\ldots,z_N) \in \mathbb{C}^N$ exists only in a form, that is impractical for our purpose. In appendix A.3 we find the following expression for N even

$$
\begin{aligned}
G^{(1,N)}(z) &= \frac{1}{c^{(1,N)}} \Delta_N(z) \prod_{j=1}^{N/2} g^{(1,N)}(z_{2j-1}, z_{2j}), \\
c^{(1,N)} &= 2^{N/2}\pi^{N/4}(N/2)! \prod_{j=1}^{N} \frac{\Gamma(j/2)}{\Gamma((N+j)/2)} \prod_{j=1}^{N/2} \frac{\Gamma\left(N+1/2-j\right)\Gamma\left(N+1-j\right)}{\Gamma\left((N+1)/2\right)\Gamma\left(N/2+1\right)}
\end{aligned}
\tag{5.97}
$$

with the skew-symmetric two-point weight

$$g^{(1,N)}(z_1,z_2) = \frac{|z_2-z_1|}{z_2-z_1}\frac{\mathrm{B}(1/2,(N+1)/2)\delta(y_1)\delta(y_2)+2\delta(z_2^*-z_1)Q(z_1,z_1^*)}{[(1+z_1^2)(1+z_2^2)]^{(N+1)/2}} \tag{5.98}$$

and the lower incomplete Beta function, see (5.26),

$$Q(z,z^*) = \mathrm{B}\left(\frac{4y^2}{|1+z^2|^2+4y^2}; 1/2, (N+1)/2\right). \tag{5.99}$$

We denote the eigenvalues as $z_j = x_j + i\, y_j$ with $x_j, y_j \in \mathbb{R}$ and make use of the Dirac delta function for complex numbers.

In a distributional sense (5.98) actually describes that the eigenvalues of a real matrix are either real or come in complex conjugate pairs. Although the part of this function concerning the complex eigenvalues is imaginary valued, the full joint eigenvalue probability distribution will be real as the phases drop out when considering the product with the Vandermonde determinant.

Only in the odd dimensional case we have to supplement this with an additional function for the unpaired eigenvalue that must be real. See appendix A.3, where we derive the joint probability distribution of eigenvalues for arbitrary N. This would complicate our already cumbersome expressions even more, which is why we limit ourselves to the even case. This restriction is not an obstacle to our plan to deduce universal statistics in the large N limit. This is also underlined by the fact that the number of expected real eigenvalues is asymptotically given by $\sqrt{\pi N/2}$, regardless of the parity of N [78].

A caveat is in order concerning the integrability of the generating function. We obtain terms of the form $1/[(\kappa(q_j)+z_1)(\kappa(q_j)+z_2)]$, which are not integrable for real $\kappa(q_j)$, irrespective of whether z_1 and z_2 are real or complex. In the following, we treat integrals over singularities on the real axis as principal value integrals.

Like the symplectic case, the orthogonal case follows a Pfaffian point process [78]. This is also manifest in the result for the generating function, obtained by the methods of [82, 83]

$$Z_{k|k}^{(1,N)}(q,p) = \frac{\mathrm{Pf}\begin{bmatrix} \mathrm{K}_1^{(1,N)}(p_m,p_n) & \mathrm{K}_2^{(1,N)}(p_m,q_n) \\ -\mathrm{K}_2^{(1,N)}(p_n,q_m) & \mathrm{K}_3^{(1,N)}(q_m,q_n)\end{bmatrix}_{1\le m,n\le k}}{\det\left[\dfrac{1}{\kappa(q_m)-\kappa(p_n)}\right]_{1\le m,n\le k}}, \tag{5.100}$$

where the three kernel functions are

$$\begin{aligned}
\mathrm{K}_1^{(1,N)}(p_m,p_n) &= [\kappa(p_n)-\kappa(p_m)][b(p_m)b(p_n)]^N\,\widetilde{Z}_{0|2}^{(1,N-2)}(p_m,p_n),\\
\mathrm{K}_2^{(1,N)}(p_n,q_m) &= \left(\frac{b(p_n)}{b(q_m)}\right)^N \frac{\widetilde{Z}_{1|1}^{(1,N)}(q_m,p_n)}{\kappa(q_m)-\kappa(p_n)} = \frac{Z_{1|1}^{(1,N)}(q_m,p_n)}{\kappa(q_m)-\kappa(p_n)},\\
\mathrm{K}_3^{(1,N)}(q_m,q_n) &= \frac{\kappa(q_n)-\kappa(q_m)}{[b(q_m)b(q_n)]^N}\,\widetilde{Z}_{2|0}^{(1,N+2)}(q_m,q_n).
\end{aligned} \tag{5.101}$$

They are completely analogous to the ones in the symplectic case (5.53). Similarly, we define for $l-k$ even and $2M+l-k\leq N$

$$\begin{aligned}
\widetilde{Z}_{k|l}^{(1,2M)}(q,p) =& \frac{1}{M!\,\mathrm{Pf}\,D^{(1,2M+l-k)}}\\
&\times \int\limits_{\mathbb{C}^{2M}} d[z]\Delta_{2M}(z)\prod_{j=1}^{M} g^{(1,N)}(z_{2j-1},z_{2j})\prod_{j=1}^{2M}\frac{\prod_{n=1}^{l}(\kappa(p_n)+z_j)}{\prod_{m=1}^{k}(\kappa(q_m)+z_j)}.
\end{aligned} \tag{5.102}$$

The prefactor contains the Pfaffian of the moment matrix

$$\begin{aligned}
D^{(1,d)} &= \left[D_{ab}^{(1)}\right]_{1\leq a,b\leq d},\\
D_{ab}^{(1)} &= \int\limits_{\mathbb{C}^2} d[z]g^{(1,N)}(z_1,z_2)\,(z_1^{a-1}z_2^{b-1}-z_1^{b-1}z_2^{a-1}) = 2\int\limits_{\mathbb{C}^2} d[z]g^{(1,N)}(z_1,z_2)\,z_1^{a-1}z_2^{b-1},
\end{aligned} \tag{5.103}$$

where d is even. In this case we cannot calculate the Pfaffian straightforwardly. Instead, we have to relate it to the normalization constant of the real induced spherical ensemble, see appendix A.3. In the following we compute explicit expressions for the kernel functions (5.101).

The Kernel $\mathrm{K}_1^{(1,N)}$

We proceed here exactly as in the symplectic case. The first kernel is determined by $\widetilde{Z}_{0|2}^{(1,N-2)}(p_m,p_n)$, which is related to $Z_{0|2}^{(1,N-2)}(p_m,p_n)$. For the latter function we can use the O(2) symmetry to reduce the amount of determinants in the average by one. We obtain the proper normalization via the limits

$$\begin{aligned}
\lim_{\kappa(p)\to\infty}\frac{\widetilde{Z}_{0|2}^{(1,N-2)}(p_m,p_n)}{[\kappa(p_m)\kappa(p_n)]^{N-2}} &= \frac{\mathrm{Pf}\,D^{(1,N-2)}}{\mathrm{Pf}\,D^{(1,N)}},\\
\lim_{a(p)\to\infty}\frac{Z_{0|2}^{(1,N-2)}(p_m,p_n)}{[a(p_m)a(p_n)]^{N-2}} &= \left\langle \det K_1^2\right\rangle.
\end{aligned} \tag{5.104}$$

Therefore the functions are related by

$$
\begin{aligned}
\widetilde{Z}_{0|2}^{(1,N-2)}(p_m,p_n) &= \frac{\operatorname{Pf} D^{(1,N-2)}}{\operatorname{Pf} D^{(1,N)}} \frac{Z_{0|2}^{(1,N-2)}(p_m,p_n)}{\langle \det K_1^2 \rangle \left[b(p_m)b(p_n)\right]^{N-2}} \\
&= \frac{\operatorname{Pf} D^{(1,N-2)}}{\operatorname{Pf} D^{(1,N)}} \frac{\langle \det\left(a(p_m)K_1 + b(p_m)K_2\right) \det\left(a(p_n)K_1 + b(p_n)K_2\right)\rangle}{\langle \det K_1^2 \rangle \left[b(p_m)b(p_n)\right]^{N-2}},
\end{aligned}
\tag{5.105}
$$

where we average over $(N-2)$-dimensional real Ginibre ensembles. Now let $a_1, b_1, a_2, b_2 \in \mathbb{R}$ and

$$
\Xi_1^{(1,N-2)} = \frac{\langle \det\left(a_1 K_1 + b_1 K_2\right) \det\left(a_2 K_1 + b_2 K_2\right)\rangle}{\langle \det K_1^2 \rangle}. \tag{5.106}
$$

This function is a polynomial in a_1, b_1, a_2, b_2 and can thus be analytically continued to the function $Z_{0|2}^{(1,N-2)}(p_m,p_n)$, that has, in general, complex coefficients. We are now finally in the position to exploit the O(2) symmetry as we are allowed to rotate only real vectors. We use the special orthogonal matrix

$$
U = \frac{1}{\sqrt{a_1^2 + b_1^2}} \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \in \mathrm{SO}(2) \tag{5.107}
$$

and obtain

$$
\begin{aligned}
\Xi_1^{(1,N-2)} = \frac{(a_1 b_2 - b_1 a_2)^{N-2}}{\left\langle \det^2 K_1 \right\rangle} &\int d[K_1] d[K_2] \widetilde{P}^{(1,N-2)}(K_1,K_2) \det{}^2 K_1 \\
&\times \det\left(\frac{a_1 a_2 + b_1 b_2}{a_1 b_2 - b_1 a_2} + K_1^{-1} K_2\right).
\end{aligned}
\tag{5.108}
$$

Once again, we find the factor $\det^2 K_1$ in the integrand, which leads to an induced spherical ensemble for $K_1^{-1}K_2$, where K_1 and K_2 are drawn from deformed Ginibre ensembles

$$
\widetilde{P}_\mu^{(1,N)}(K) = \pi^{-N^2/2} \prod_{j=1}^{N} \frac{\Gamma(j/2)}{\Gamma(\mu + j/2)} \exp\left(-\operatorname{tr} K K^T\right) \det{}^\mu K K^T. \tag{5.109}
$$

In appendix A.2 we show that the characteristic polynomial of the real spherical ensemble, induced or not, is given by a monomial. Thus, we find

$$
\Xi_1^{(1,N-2)} = (a_1 a_2 + b_1 b_2)^{N-2} \tag{5.110}
$$

and the function $Z_{0|2}^{(1,N-2)}(p_m, p_n)$ has the simple form

$$\frac{Z_{0|2}^{(1,N-2)}(p_m, p_n)}{\left\langle \det^2 K_1 \right\rangle} = [a(p_m)a(p_n) + b(p_m)b(p_n)]^{N-2}, \tag{5.111}$$

which is indeed O(2) invariant. The Pfaffians of the moment matrix can be related to the normalization of an induced spherical ensemble. This is covered in appendix A.1. We find

$$\frac{\operatorname{Pf} D^{(1,N-2)}}{\operatorname{Pf} D^{(1,N)}} = \frac{N(N-1)}{8\pi} \tag{5.112}$$

and altogether for the first kernel

$$\begin{aligned} \mathrm{K}_1^{(1)}(p_m, p_n) =& \frac{N(N-1)}{8\pi} b(p_m)b(p_n)[a(p_n)b(p_m) - a(p_m)b(p_n)] \\ & \times [a(p_m)a(p_n) + b(p_m)b(p_n)]^{N-2} \\ =& \frac{N(N-1)}{8\pi} b(p_m)b(p_n)\, iv^T(p_n)\tau_2 v(p_m)[v^T(p_m)v(p_n)]^{N-2}. \end{aligned} \tag{5.113}$$

Again, we write this expression in terms of the vector $v(p)$ to check on the O(2) invariance. However, only the functions $Z_{k|l}^{(1,N)}(q,p)$ have this invariance such that the non-invariant factors need to drop out when we compose the final result for $Z_{k|k}^{(1,N)}(q,p)$.

The Kernel $\mathbf{K}_2^{(1,N)}$

The second kernel is essentially the generator $Z_{1|1}^{(1,N)}(q_m, p_n)$ of the one-point function. Once again we want to apply the O(2) symmetry to eliminate one of the determinants. Similar to the symplectic case, we consider

$$\Xi_2^{(1,N)} = \left\langle \frac{\det(a_1 K_1 + b_1 K_2)}{\det(a_2 K_1 + b_2 K_2)} \right\rangle \tag{5.114}$$

for this purpose. This function is a polynomial in a_1, b_1 and non-holomorphic in a_2, b_2. Therefore, we set $a_1, b_1 \in \mathbb{R}$ and $a_2, b_2 \in \mathbb{C}$. We may rotate only in the real variables a_1, b_1, for which we can use analytical continuation to complex values at the end of the calculation.

We rotate with the same matrix (5.107) as for the first kernel and obtain an integral over the spherical ensemble

$$\Xi_2^{(1,N)} = \left(\frac{a_1^2 + b_1^2}{a_1 b_2 - b_1 a_2} \right)^N \int d[Y] \widetilde{G}^{(1,N)}(Y) \frac{1}{\det(\widehat{\kappa} + Y)} \tag{5.115}$$

with

$$\widehat{\kappa} = \frac{a_1 a_2 + b_1 b_2}{a_1 b_2 - b_1 a_2}. \tag{5.116}$$

We perform the integral in the joint probability distribution of the eigenvalues

$$\Xi_2^{(1,N)} = \frac{1}{c^{(1,N)}} \left(\frac{a_1^2 + b_1^2}{a_1 b_2 - b_1 a_2} \right)^N \int\limits_{\mathbb{C}^N} d[z] \Delta_N(z) \prod_{j=1}^{N/2} g^{(1,N)}(z_{2j-1}, z_{2j}) \prod_{j=1}^{N} \frac{1}{\widehat{\kappa} + z_j}. \tag{5.117}$$

First, we identify part of the integrand as a Berezinian

$$\Delta_N(z) \prod_{j=1}^{N} \frac{1}{\widehat{\kappa} + z_j} = \sqrt{\mathrm{Ber}_{N|1}^{(2)}(z; -\widehat{\kappa})} = (-1)^{N+1} \det \left[\begin{array}{c|c} z_a^{b-1} & \dfrac{1}{z_a + \widehat{\kappa}} \end{array} \right]_{\substack{1 \leq a \leq N \\ 1 \leq b \leq N-1}} \tag{5.118}$$

that we write in its determinantal form [82]. We expand this determinant in the last column. Using its skew-symmetry under row permutation and the antisymmetry of $g^{(1,N)}(z_{2j-1}, z_{2j})$ we find that each terms yields the same contribution, which is

$$\begin{aligned} \Xi_2^{(1,N)} &= \frac{(-1)^{N+1} N}{c^{(1,N)}} \left(\frac{a_1^2 + b_1^2}{a_1 b_2 - b_1 a_2} \right)^N \int\limits_{\mathbb{C}^N} d[z] \frac{\Delta_{N-1}(z_1, \ldots, z_{N-1})}{z_N + \widehat{\kappa}} \prod_{j=1}^{N/2} g^{(1,N)}(z_{2j-1}, z_{2j}) \\ &= -\frac{N(N-1)}{4\pi} \left(\frac{a_1^2 + b_1^2}{a_1 b_2 - b_1 a_2} \right)^N \int\limits_{\mathbb{C}^2} d[z] \frac{z_1^{N-2}}{z_2 + \widehat{\kappa}} g^{(1,N)}(z_1, z_2). \end{aligned} \tag{5.119}$$

The integral over $(z_1, \ldots, z_{N-2})$ can be conceived as a characteristic polynomial of an induced spherical ensemble in the variable z_{N-1}, which gives up to a prefactor a monomial. The details of this calculation are covered in appendix A.2.

The integral over the remaining two eigenvalues yields two contributions, one for the case in which they are real and one for the case in which they are a complex conjugate pair

$$\Xi_2^{(1,N)} = -\frac{N(N-1)}{4\pi} \left(\frac{a_1^2 + b_1^2}{a_1 b_2 - b_1 a_2} \right)^N \left[I_R(\widehat{\kappa}) + I_C(\widehat{\kappa}) \right]. \tag{5.120}$$

The contribution of the real eigenvalues is

$$\begin{aligned} I_R(\widehat{\kappa}) &= \mathrm{B}(1/2, (N+1)/2) \int\limits_{\mathbb{R}^2} d[x] \frac{\mathrm{sgn}(x_2 - x_1)}{[(1 + x_1^2)(1 + x_2^2)]^{(N+1)/2}} \frac{x_1^{N-2}}{x_2 + \widehat{\kappa}} \\ &= \mathrm{B}(1/2, (N+1)/2) \int\limits_{x_2 > x_1} d[x] \frac{1}{[(1 + x_1^2)(1 + x_2^2)]^{(N+1)/2}} \left(\frac{x_1^{N-2}}{x_2 + \widehat{\kappa}} - \frac{x_2^{N-2}}{x_1 + \widehat{\kappa}} \right), \end{aligned} \tag{5.121}$$

where we treat the sign function by splitting the integral into two terms, that we calculate separately. In the first term we integrate x_1 over $(-\infty, x_2]$ yielding

$$\begin{aligned}&\int\limits_{-\infty}^{\infty} dx_2 \int\limits_{-\infty}^{x_2} dx_1 \frac{1}{[(1+x_1^2)(1+x_2^2)]^{(N+1)/2}} \frac{x_1^{N-2}}{x_2+\widehat{\kappa}} \\ &= \int\limits_{-\infty}^{\infty} dx_2 \frac{1}{(1+x_2^2)^{(N+1)/2}} \frac{1}{x_2+\widehat{\kappa}} \frac{(-1)^N + x_2^{N-1}(1+x_2^2)^{(1-N)/2}}{N-1}\end{aligned} \tag{5.122}$$

and in the second term we integrate x_2 over $[x_1, \infty)$

$$\begin{aligned}&\int\limits_{-\infty}^{\infty} dx_1 \int\limits_{x_1}^{\infty} dx_2 \frac{1}{[(1+x_1^2)(1+x_2^2)]^{(N+1)/2}} \frac{x_2^{N-2}}{x_1+\widehat{\kappa}} \\ &= \int\limits_{-\infty}^{\infty} dx_1 \frac{1}{(1+x_1^2)^{(N+1)/2}} \frac{1}{x_1+\widehat{\kappa}} \frac{1 - x_1^{N-1}(1+x_1^2)^{(1-N)/2}}{N-1}.\end{aligned} \tag{5.123}$$

For both integrals the antiderivative is

$$\int dx \frac{x^{N-2}}{(1+x^2)^{(N+1)/2}} = \frac{1}{N-1} \frac{x^{N-1}}{(1+x^2)^{(N-1)/2}}. \tag{5.124}$$

Considering N even one of the terms drops out and (5.121) results in an integral over the real axis

$$I_R(\widehat{\kappa}) = \frac{2\,\mathrm{B}(1/2,(N+1)/2)}{N-1} \int\limits_{\mathbb{R}} dx \frac{x^{N-1}}{(1+x^2)^N} \frac{1}{x+\widehat{\kappa}}. \tag{5.125}$$

In the case that $\widehat{\kappa}$ is real it has to be understood as a principal value integral. We continue by symmetrizing the integrand and substituting $t = x^2$

$$\begin{aligned}I_R(\widehat{\kappa}) &= \frac{2\,\mathrm{B}(1/2,(N+1)/2)}{N-1} \int\limits_0^{\infty} dx \frac{x^{N-1}}{(1+x^2)^N} \left(\frac{1}{x+\widehat{\kappa}} + \frac{1}{x-\widehat{\kappa}}\right) \\ &= \frac{2\,\mathrm{B}(1/2,(N+1)/2)}{N-1} \int\limits_0^{\infty} dt \frac{t^{(N-1)/2}}{(1+t)^N} \frac{1}{t-\widehat{\kappa}^2}.\end{aligned} \tag{5.126}$$

The last integrand has a branch cut along the negative real axis. It is equivalent to an integral of the function

$$f(z) = \frac{z^{(N-1)/2}}{(1-z)^N} \frac{-1}{z+\widehat{\kappa}^2} \tag{5.127}$$

over a keyhole contour around the negative real axis. Applying the residue theorem yields a truncated binomial series

$$\begin{aligned} I_R(\widehat{\kappa}) =& \frac{(-1)^{N/2} 2\pi\, \mathrm{B}(1/2,(N+1)/2)}{N-1} \\ &\times \left[\sum_{l=0}^{N-1} \binom{(N-1)/2}{l} \frac{(-1)^{N-1-l}}{(1+\widehat{\kappa}^2)^{N-l}} + \frac{(-\widehat{\kappa}^2)^{(N-1)/2}}{(1+\widehat{\kappa}^2)^N} \right] \\ =& \frac{(-1)^{N/2} 2\pi\, \mathrm{B}(1/2,(N+1)/2)}{N-1} \binom{(N-1)/2}{N} {}_2\mathrm{F}_1\left(1,(N+1)/2;N+1;1+\widehat{\kappa}^2\right). \end{aligned} \tag{5.128}$$

The second representation involves the hypergeometric function [167]. The contribution of the complex conjugated eigenvalues is

$$I_C(\widehat{\kappa}) = 2i \int_{\mathbb{C}} d[z]\, \mathrm{sgn}(\mathrm{Im}\, z) \frac{z^{N-2}}{z^* + \widehat{\kappa}} \frac{Q(z,z^*)}{|1+z^2|^{N+1}}, \tag{5.129}$$

which is difficult to simplify further.

Therefore, we find for the second kernel

$$\begin{aligned} \mathrm{K}_2^{(1)}(p_n,q_m) =& \frac{N(N-1)}{2\pi} \frac{b(p_n)b(q_m)}{iv^T(p_n)\tau_2 v(q_m)} \left(\frac{v^T(p_n)v(p_n)}{iv^T(p_n)\tau_2 v(q_m)} \right)^N \\ \times & \left[\frac{(-1)^{N/2}\pi\, \mathrm{B}\left(1/2,\frac{N+1}{2}\right)}{N-1} \binom{\frac{(N-1)}{2}}{N} {}_2\mathrm{F}_1\left(1,\frac{N+1}{2};N+1;1+\left(\frac{v^T(p_n)v(q_m)}{iv^T(p_n)\tau_2 v(q_m)}\right)^2\right) \right. \\ &\left. + i \int_{\mathbb{C}} d[z] \mathrm{sgn}(\mathrm{Im}\, z) \frac{z^{N-2} Q(z,z^*)}{|1+z^2|^{N+1}} \left(z^* + \frac{v^T(p_n)v(q_m)}{iv^T(p_n)\tau_2 v(q_m)} \right)^{-1} \right]. \end{aligned} \tag{5.130}$$

We immediately expressed it as a function in the vector $v(p) = (a(p), b(p))$ and recover the O(2) invariance up to the prefactor that drops out in the end result (5.100).

The Kernel $\mathbf{K}_3^{(1,N)}$

The third kernel is determined by $\widetilde{Z}_{2|0}^{(1,N+2)}(q_m, q_n)$. Therefore, we need to evaluate the following integral

$$\begin{aligned}\Xi_3^{(1,N+2)} =& \frac{1}{(N/2+1)!\,\mathrm{Pf}\, D^{(1,N)}} \\ & \times \int\limits_{\mathbb{C}^{N+2}} d[z]\Delta_{N+2}(z) \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \prod_{j=1}^{N+2} \frac{1}{(\kappa_1+z_j)(\kappa_2+z_j)}.\end{aligned} \tag{5.131}$$

This time we cannot use the O(2) symmetry to simplify the integral, because it cannot be traced back to the ensemble average (5.13). Instead, we proceed by applying the identity

$$\begin{aligned}\Delta_{N+2}(z) \prod_{j=1}^{N+2} \frac{1}{(\kappa_1+z_j)(\kappa_2+z_j)} &= \frac{1}{\kappa_1-\kappa_2}\sqrt{\mathrm{Ber}_{N+2|2}^{(2)}(z;-\kappa)} \\ &= \frac{1}{\kappa_2-\kappa_1}\det\left[\begin{array}{c|c|c} z_a^{b-1} & \dfrac{1}{z_a+\kappa_1} & \dfrac{1}{z_a+\kappa_2}\end{array}\right]_{\substack{1\le a\le N+2\\ 1\le b\le N}}\end{aligned} \tag{5.132}$$

like in the symplectic case, see also [82], and expand the determinant in the last two columns. Similar to (5.119), we use the antisymmetry of $g^{(1,N)}(z_{2j-1}, z_{2j})$ to reduce the amount of terms appearing in the integral

$$\begin{aligned}\Xi_3^{(1,N+2)} =& \frac{2(\kappa_2-\kappa_1)^{-1}}{c^{(1,N)}(N+2)} \int\limits_{\mathbb{C}^{N+2}} d[z] \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \\ & \times \det\left[\begin{array}{c|c|c} z_a^{b-1} & \dfrac{1}{z_a+\kappa_1} & \dfrac{1}{z_a+\kappa_2}\end{array}\right]_{\substack{1\le a\le N+2\\ 1\le b\le N}} \\ =& \frac{2}{c^{(1,N)}(\kappa_2-\kappa_1)}\Bigg[\int\limits_{\mathbb{C}^{N+2}} d[z] \frac{\Delta_N(z)}{(z_{N+1}+\kappa_1)(z_{N+2}+\kappa_2)} \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \\ & - N \int\limits_{\mathbb{C}^{N+2}} d[z] \frac{\Delta_N(z_1,\ldots,z_{N-1},z_{N+1})}{(z_N+\kappa_1)(z_{N+2}+\kappa_2)} \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j})\Bigg].\end{aligned} \tag{5.133}$$

In the first term the pair of eigenvalues z_{N+1}, z_{N+2} is decoupled from $(z_1, \ldots, z_N)$. Integration over the latter yields only a constant

$$\begin{aligned}&\frac{1}{c^{(1,N)}} \int\limits_{\mathbb{C}^{N+2}} d[z] \frac{\Delta_N(z)}{(z_{N+1}+\kappa_1)(z_{N+2}+\kappa_2)} \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \\ &\quad = \int\limits_{\mathbb{C}^2} d[z] \frac{g^{(1,N)}(z_1, z_2)}{(z_1+\kappa_1)(z_2+\kappa_2)}.\end{aligned} \tag{5.134}$$

In the second term we expand the Vandermonde determinant in the last two variables

$$\begin{aligned}&\frac{1}{c^{(1,N)}} \int\limits_{\mathbb{C}^{N+2}} d[z] \frac{\Delta_N(z_1, \ldots, z_{N-1}, z_{N+1})}{(z_N+\kappa_1)(z_{N+2}+\kappa_2)} \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \\ =&\frac{1}{c^{(1,N)}} \int\limits_{\mathbb{C}^{N+2}} d[z] \frac{\Delta_{N-2}(z)\,(z_{N+1}-z_{N-1})}{(z_N+\kappa_1)(z_{N+2}+\kappa_2)} \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \\ &\qquad \times \prod_{j=1}^{N-2} (z_{N+1}-z_j)(z_{N-1}-z_j).\end{aligned} \tag{5.135}$$

This allows us to identify the integral over $(z_1, \ldots z_{N-2})$ as the function $\Xi_1^{(1,N-2)}$, see (5.106), that we calculated for the first kernel, resulting in

$$\begin{aligned}&\frac{N-1}{4\pi} \int\limits_{\mathbb{C}^4} d[z] \frac{z_3 - z_1}{(z_2+\kappa_1)(z_4+\kappa_2)} g^{(1,N)}(z_1, z_2) g^{(1,N)}(z_3, z_4) \\ &\qquad \times \frac{\langle \det(z_1 K_1 - K_2) \det(z_3 K_1 - K_2) \rangle}{\left\langle \det^2 K_1 \right\rangle} \\ =&\frac{N-1}{4\pi} \int\limits_{\mathbb{C}^4} d[z] \frac{z_3 - z_1}{(z_2+\kappa_1)(z_4+\kappa_2)} g^{(1,N)}(z_1, z_2) g^{(1,N)}(z_3, z_4)\,(z_1 z_3 + 1)^{N-2}.\end{aligned} \tag{5.136}$$

Altogether the third kernel function is given by

$$\begin{aligned}\mathrm{K}_3^{(1)}(q_m, q_n) =& \frac{2}{[b(q_m) b(q_n)]^{N-1}} \Bigg[\int\limits_{\mathbb{C}^2} d[z] \frac{g^{(1,N)}(z_1, z_2)}{(a(q_m) + b(q_m) z_1)(a(q_n) + b(q_n) z_2)} \\ &- \frac{N(N-1)}{4\pi} \int\limits_{\mathbb{C}^4} d[z] \frac{(z_3 - z_1)(z_1 z_3 + 1)^{N-2}}{(a(q_m) + b(q_m) z_2)(a(q_n) + b(q_n) z_4)} g^{(1,N)}(z_1, z_2) g^{(1,N)}(z_3, z_4) \Bigg].\end{aligned} \tag{5.137}$$

Unlike for the first and the second kernel, expressing this result in $v(p)$ is not beneficial. Thus, the O(2) symmetry cannot be checked. However, our final result must have this symmetry. We define the functions

$$
\begin{aligned}
r(x, v(q)) &= \mathrm{B}(1/2, (N+1)/2) \int_{\mathbb{R}} dx' \frac{\operatorname{sgn}(x'-x)}{(a(q)+b(q)x')\,[(1+x^2)(1+x'^2)]^{(N+1)/2}}, \\
s(z, z^*, v(q)) &= \frac{2i \operatorname{sgn}(\operatorname{Im} z) Q(z, z^*)}{(a(q)+b(q)z^*)\,|1+z^2|^{N+1}}.
\end{aligned}
\tag{5.138}
$$

Inserting (5.25) yields the cumbersome expression

$$
\begin{aligned}
\mathrm{K}_3^{(1)}(q_m, q_n) =& \frac{2}{[b(q_m)b(q_n)]^{N-1}} \left[\int_{\mathbb{R}} dx \frac{r(x, v(q_n))}{(a(q_m)+b(q_m)x)} + \int_{\mathbb{C}} d[z] \frac{s(z, z^*, v(q_n))}{(a(q_m)+b(q_m)z)} \right] \\
&- \frac{N(N-1)}{2\pi [b(q_m)b(q_n)]^{N-1}} \\
&\times \left[\int_{\mathbb{R}^2} d[x] r(x_1, v(q_m)) r(x_2, v(q_n)) (x_2 - x_1)(x_1 x_2 + 1)^{N-2} \right. \\
&+ \int_{\mathbb{R}} dx \int_{\mathbb{C}} d[z] \det \begin{bmatrix} r(x, v(q_m)) & s(z, z^*, v(q_m)) \\ r(x, v(q_n)) & s(z, z^*, v(q_n)) \end{bmatrix} (z - x)(z\,x + 1)^{N-2} \\
&+ \left. \int_{\mathbb{C}^2} d[z] s(z_1, z_1^*, v(q_m)) s(z_2, z_2^*, v(q_n)) (z_2 - z_1)(z_1 z_2 + 1)^{N-2} \right]
\end{aligned}
\tag{5.139}
$$

for the third kernel, which contains six integrals over at most four real variables.

5.5 Summary

In this chapter, we generalized the parametric random matrix model that we used in chapter 4. Our generalization includes the time reversal invariant chiral classes, i.e. the chiral symplectic class CII and the chiral orthogonal class BDI, besides the chiral unitary class AIII with broken time reversal invariance. Moreover, it allows for quite arbitrary parametric combinations of the respective Ginibre matrices in the two-matrix model.

Within this model, we studied statistical aspects of the winding number. In particular, ensemble averages of ratios of determinants with parametric dependence were computed and related to the k-point correlation functions of the winding number density. We mapped this problem to averages of ratios of characteristic polynomials for the respective spherical ensembles. We applied the supersymmetry without supersymmetry method to the ensemble averages of $2k$ characteristic polynomials, which yields a determinant in the unitary case and a Pfaffian in the symplectic and the orthogonal case, containing simplified ensemble averages of only two characteristic polynomials as kernel functions. These simplified averages were computed by exploiting the symmetries of the random matrix model and employing techniques from orthogonal and skew-orthogonal polynomial theory. We verified our results with numerical calculations.

The main difficulties occurred in the time reversal invariant classes. In the symplectic case, we obtained ensemble averages over the quaternion spherical ensemble. As such, all its eigenvalues come in complex conjugate pairs, leading to significantly more complicated expressions than in the unitary case. However, we managed to solve all integrals and find a closed-form result. In the orthogonal case, on the other hand, we had to deal with characteristic polynomials of the real spherical ensemble. The eigenvalues of real matrices are either real or come in complex conjugated pairs. Moreover, an odd-dimensional real matrix has an unpaired eigenvalue that is always real. Due to this splitting of eigenvalues, ensembles of real matrices are hard to approach technically. We therefore restricted our calculation to even dimensional matrices. Even though we did not arrive at a closed-form result, we drastically reduced the number of integrals. An evaluation of the k-point correlation functions and the kth moment of the winding number in the large N limit should be feasible also in these cases.

Chapter 6

Conclusion and Outlook

In the present work, we studied statistical aspects of the winding number in chiral random matrix models with one-dimensional parametric dependence. In particular, we derived closed-form expressions for the discrete probability distribution of the winding number as well as for the k-point correlation functions of the winding number density in the chiral unitary class, labeled AIII in the tenfold way classification. The latter give the kth statistical moments of the winding number upon integration over the parameter manifold. Next to the chiral unitary class AIII, we also considered the chiral symplectic class CII and the chiral orthogonal class BDI. These symmetry classes have a real structure, typically caused by the time reversal invariance of the underlying physical system. We set up a generating function for the k-point correlations of the winding number density as the ensemble average of ratios of determinants with parametric dependence. We traced this back to ratios of characteristic polynomials of the corresponding spherical ensemble. Employing the supersymmetry without supersymmetry technique, a method that exploits supersymmetry structures without mapping the integrals to superspace, together with an inherent symmetry of our two-matrix model as well as other methods of random matrix theory, we derived closed-form results for the generating function in the chiral unitary and the chiral symplectic classes, and drastically reduced the number of integrals in the chiral orthogonal class.

The motivation for our investigation lies in condensed matter physics, where the winding number is the relevant topological invariant for one-dimensional chiral symmetric systems, and as such, predicts the number of edge states in a system with open boundary conditions. Our random matrix model realizes the situation where the system is disordered, so that the winding number becomes a random variable. The chiral matrix plays the role of the Bloch Hamiltonian and the parameter that of the crystal momentum in the Brillouin zone. Nevertheless, our results hold true also if the model is interpreted otherwise.

In related works, a statistical analysis of the Chern number in the unitary class A was performed [74, 75]. It was found that the two-point correlation of the adiabatic curvature, the analogue of the winding number density in this class, follows a universal form when the parametric dependence is considered on a local scale. The rescaling procedure established there is similar to the one we used in chapter 4 for the two-point correlation of the winding number density. Furthermore, the distribution of the Chern number was found to be Gaussian in the limit of large matrix dimensions when the correlation length of the matrix elements remains small. Similarly, our results for the winding number distribution in the chiral unitary class also suggest a Gaussian distribution in the large N limit.

In general, the topological invariant is a property of the eigenvector bundle of the parametric Hamiltonian. When considering the statistics of topological invariants, a major advantage of the winding number over the Chern number is that we are able to map the topological problem to a spectral problem. The ensemble averages of ratios of characteristic polynomials that we evaluated in chapter 5 for the chiral classes AIII, CII and BDI contain all the information about the winding number statistics. It remains to take the derivative of the generating function to obtain the k-point correlation functions, and to integrate these functions over the Brillouin zone to obtain the kth moment of the winding number. This approach is more convenient than the recursive approach we followed in chapter 4, because we are able to consider arbitrary k directly.

The success of random matrix theory in various branches of physics is based on the fact that even simple Gaussian matrix models are able to describe universal results in the limit of large matrix dimensions. Likewise, we expect our results to contain universal information as well. However, a rigorous mathematical proof is in order. First results are available for the chiral unitary class. In an unpublished work by our collaborators [170] it was shown that in the limit of large matrix dimensions the model depends only on the three quantities

$$\Gamma_1 = \frac{\partial}{\partial p} \ln S(p,q)\bigg|_{q=p}, \qquad \Gamma_2 = \frac{\partial^2}{\partial p^2} \ln S(p,q)\bigg|_{q=p}, \qquad \Gamma_3 = \frac{\partial^2}{\partial p \partial q} \ln S(p,q)\bigg|_{q=p}, \tag{6.1}$$

which are the logarithmic derivatives up to second order in the two-point correlation function of the matrix elements

$$S(p,q) = \left\langle K_{jl}^*(p) K_{jl}(q) \right\rangle . \tag{6.2}$$

These quantities can be realized by a two-matrix model with coefficient functions $a(p)$ and $b(p)$ as we used it in chapter 5. The universality of our model follows from this. These results were obtained within the framework of the supersymmetry method.

In chapter 5 we used the supersymmetry without supersymmetry technique, developed in [82,83]. As a consequence, our results include supersymmetry structures, namely a Berezinian as a prefactor. This suggests that an evaluation of the ensemble averages within the framework of the "true" supersymmetry method might be feasible. So far, however, our attempts following the standard procedure [81, 114, 171] remain fruitless.

Another future research objective is the experimental verification of a universal winding number statistics. Besides our original motivation, the edge states of topological insulators and topological superconductors, analogue systems may come into play here. In fact, random matrix theory has a rich history of experimental verification in microwave cavities and in elastomechanics [112, 172]. The classical wave equations of these systems correspond to the stationary Schrödinger equation, and their spectra are well described by the Gaussian ensembles, provided that the microwave cavity resp. the solid body are irregularly shaped, i.e. have no geometric symmetries. Furthermore, the spectral correlations of parametric random matrix ensembles [141] were measured in various systems such as microwave cavities [173], resonating quartz blocks [174] and quantum graphs [175–178]. Recently, the chiral Gaussian ensembles were realized in a chain of coupled microwave resonators [179].

A Appendix

The following appendices contain technical details of the calculations in chapter 5. Like the latter, they are based on [2] and [3].

A.1 Skew-Orthogonal Polynomials of the Quaternion Spherical Ensemble

The skew-orthogonal polynomials $q_n^{(N)}$ are defined by choosing them of degree n and the relations

$$\langle q_{2j}^{(N)} | q_{2l}^{(N)} \rangle = \langle q_{2j+1}^{(N)} | q_{2l+1}^{(N)} \rangle = 0, \qquad \langle q_{2j}^{(N)} | q_{2l+1}^{(N)} \rangle = h_j^{(N)} \delta_{jl} \tag{A.1}$$

for all $j, l = 0, \ldots, N-1$, where we employ the skew-symmetric product (5.86). The normalization constants

$$h_n^{(N)} = \pi \, \mathrm{B}(2n+2, 2N-2n) \tag{A.2}$$

are related to the normalization $c^{(4,N)}$ of the joint probability distribution (5.51) in the standard way, see [58, 169], namely by

$$c^{(4,N)} = 2^N N! \prod_{j=0}^{N-1} h_j^{(N)}. \tag{A.3}$$

It is well-known, see [58, 169], that there is some kind of gauging possible for the polynomials of odd degree by adding a multiple of the even ones ($q_{2j+1}^{(N)}(z) \to q_{2j+1}^{(N)}(z) + c_j q_{2j}^{(N)}(z)$ for any $c_j \in \mathbb{C}$) without destroying the skew-orthogonality. This creates an ambiguity even when choosing monic normalization $q_j^{(N)}(x) = x^j + \ldots$ like we will do.

This kind of ambiguity can be fixed by choosing the Heine-like formulae, see [169], for these polynomials, which are

$$
\begin{aligned}
q_{2n}^{(N)}(x) &= \frac{\int\limits_{\mathbb{C}^n} d[z]\Delta_{2n}(z)\prod_{j=1}^n g^{(4,N)}(z_j)\ \prod_{j=1}^n (z_j-x)(z_j^*-x)}{\int\limits_{\mathbb{C}^n} d[z]\Delta_{2n}(z)\prod_{m=1}^n g^{(4,N)}(z_m)},\\
q_{2n+1}^{(N)}(x) &= \frac{\int\limits_{\mathbb{C}^n} d[z]\Delta_{2n}(z)\prod_{j=1}^n g^{(4,N)}(z_j)\ \Big(x+\sum_{j=1}^n [z_j+z_j^*]\Big)\prod_{j=1}^n (z_j-x)(z_j^*-x)}{\int\limits_{\mathbb{C}^n} d[z]\Delta_{2n}(z)\prod_{m=1}^n g^{(4,N)}(z_m)}.
\end{aligned}
\tag{A.4}
$$

The skew-orthogonal polynomials of even degree are evaluated as follows

$$
q_{2n}^{(N)}(x) \propto \int\limits_{\mathbb{C}^n} d[z]\Delta_{2n+1}(x,z,z^*)\prod_{j=1}^n \frac{z_j-z_j^*}{\left(1+|z_j|^2\right)^{2N+2}} \propto \mathrm{Pf}\left[\begin{array}{c|c} 0 & x^{b-1} \\ \hline -x^{a-1} & D_{ab}^{(4)}\end{array}\right]_{\substack{1\le a\le 2n+1\\ 1\le b\le 2n+1}},
\tag{A.5}
$$

where we have employed the generalized form of de Bruijn's theorem, see [82, 162], in the second expression and dropped the normalization, which can be reintroduced at the end by employing the monic normalization. The vertical and horizontal line underline the first row and column and a is the running index for the last $2n+1$ rows and b those of the columns. The Pfaffian involves an antisymmetric $(2n+1)\times(2n+1)$-kernel with the elements

$$
\begin{aligned}
D_{ab}^{(4)} =& 2\int\limits_{\mathbb{C}} d[z]\frac{(z-z^*)z^{a-1}(z^*)^{b-1}}{\left(1+|z|^2\right)^{2N+2}}\\
=& 2\pi\,\mathrm{B}\left(2N+2-\frac{a+b+1}{2},\frac{a+b+1}{2}\right)(\delta_{a,b-1}-\delta_{a-1,b})\,.
\end{aligned}
\tag{A.6}
$$

After expanding the Pfaffian in the last row and column we obtain a recursion relation

$$
\begin{aligned}
&\mathrm{Pf}\left[\begin{array}{c|c} 0 & x^{b-1} \\ \hline -x^{a-1} & D_{ab}^{(4)}\end{array}\right]_{\substack{1\le a,b\le 2n+1\\ 1\le b\le 2n+1}}\\
=&-\mathrm{Pf}\left[D_{ab}^{(4)}\right]_{\substack{1\le a\le 2n\\ 1\le b\le 2n}}\ x^{2n} + D_{2n,2n+1}^{(4)}\mathrm{Pf}\left[\begin{array}{c|c} 0 & x^{b-1} \\ \hline -x^{a-1} & D_{ab}^{(4)}\end{array}\right]_{\substack{1\le a\le 2n-1\\ 1\le b\le 2n-1}}\\
=&-(2\pi)^n\sum_{m=0}^n\prod_{j=1}^m \mathrm{B}(2N+2-2j,2j)\prod_{j=m+1}^n \mathrm{B}(2N-2j+1,2j+1)x^{2m}\\
=&-(2\pi)^n\prod_{j=1}^n \mathrm{B}(2N+2-2j,2j)\ \sum_{m=0}^n\frac{\mathrm{B}(n+1,N-n+1/2)}{\mathrm{B}(m+1,N-m+1/2)}x^{2m},
\end{aligned}
\tag{A.7}
$$

where we have used

$$\mathrm{Pf}\left[D_{ab}^{(4)}\right]_{1\leq a,b\leq 2n} = \prod_{j=0}^{n-1} h_j^{(N)} = (2\pi)^n \prod_{j=1}^{n} \mathrm{B}(2j, 2N+2-2j). \tag{A.8}$$

After proper normalization we find (5.22).

The calculation of the skew-orthogonal polynomials of odd degree works along the same lines with the only difference of the need for the identity

$$\Delta_{2n+1}(x,z,z^*)\left(x+\sum_{j=1}^{n}\left(z_j+z_j^*\right)\right) = \det\left[\begin{array}{c|c} z_a^{b-1} & z_a^{2n+1} \\ (z_a^*)^{b-1} & (z_a^*)^{2n+1} \\ \hline x^{b-1} & x^{2n+1} \end{array}\right]_{\substack{1\leq a\leq n\\ 1\leq b\leq 2n}}, \tag{A.9}$$

where the vertical and horizontal line highlights the last column and row and the first n odd and even rows comprise z_a and z_a^*, respectively. The polynomials of odd degree are then

$$\begin{aligned} q_{2n+1}^{(N)}(x) &\propto \int_{\mathbb{C}^n} d[z]\Delta_{2n+1}(x,z,z^*)\left(x+\sum_{j=1}^{n}\left(z_j+z_j^*\right)\right)\prod_{j=1}^{n}\frac{z_j-z_j^*}{\left(1+|z_j|^2\right)^{2N+2}} \\ &\propto \mathrm{Pf}\left[\begin{array}{c|c|c} 0 & x^{b-1} & x^{2n+1} \\ \hline -x^{a-1} & D_{ab}^{(4)} & 0 \\ \hline -x^{2n+1} & 0 & 0 \end{array}\right]_{\substack{1\leq a\leq 2n\\ 1\leq b\leq 2n}}, \end{aligned} \tag{A.10}$$

where we anew applied the generalized de Bruijn theorem [82, 162]. This time the two vertical and horizontal lines underline the particular role of the first and last columns and rows. The antisymmetric kernel is the same as in the even case (A.5) for $1 \leq a, b \leq 2n$. The integrals in the last row and column are the skew-symmetric product $\langle z^{a-1}|z^{2n+1}\rangle$ with $a = 1, \ldots, 2n$ and, thus, vanish. Expanding the Pfaffian in the last row and column yields the monomial

$$q_{2n+1}^{(N)}(x) = x^{2n+1}. \tag{A.11}$$

These skew-orthogonal polynomials have to be seen in contrast to those derived in [79] where the author has first mapped the spherical ensemble to a different matrix ensemble. This is the reason why the author of [79] has found the monomials also for the polynomials of even degree.

A.2 Evaluating $\mathbf{K}_3^{(4,N)}$

To simplify expression (5.90), we pursue the same ideas as for the second kernel function. One can show

$$
\begin{aligned}
&\frac{\partial^2}{\partial z_1 \partial z_2} \sum_{j=0}^{N} \sum_{m=0}^{j} \left[\frac{(2N+1)!\, j!\, \Gamma(N-j+3/2)}{(2j+1)!\,(2N-2j+1)!\, m!\, \Gamma(N-m+3/2)} \right. \\
&\left. \times \frac{z_1^{2m} z_2^{2j+1} - z_2^{2m} z_1^{2j+1}}{(1+|z_1|^2)^{2N+1}(1+|z_2|^2)^{2N+1}} \right] \\
=&(z_1 - z_1^*)(z_2 - z_2^*) \sum_{j=0}^{N-1} \sum_{m=0}^{j} \left[\frac{(2N+1)!\, j!\, \Gamma(N-j+1/2)}{(2j+1)!\,(2N-2j-1)!\, m!\, \Gamma(N-m+1/2)} \right. \\
&\left. \times \frac{z_1^{2m} z_2^{2j+1} - z_2^{2m} z_1^{2j+1}}{(1+|z_1|^2)^{2N+2}(1+|z_2|^2)^{2N+2}} \right] + (2N+1) \frac{(z_2^* - z_1^*)(1+z_1 z_2)^{2N}}{(1+|z_1|^2)^{2N+2}(1+|z_2|^2)^{2N+2}}.
\end{aligned}
\tag{A.12}
$$

This derivative can be found by recognizing

$$
\begin{aligned}
(1+|z_1|^2)^{2N+2} \frac{\partial}{\partial z_1} \frac{z_1^{2m}}{(1+|z_1|^2)^{2N+1}} =& 2m\, z_1^{2m-1} - (2N-2m+1) z_1^* z_1^{2m}, \\
(1+|z_2|^2)^{2N+2} \frac{\partial}{\partial z_2} \frac{z_2^{2j+1}}{(1+|z_2|^2)^{2N+1}} =& (2j+1) z_2^{2j} - (2N-2j) z_2^* z_2^{2j+1},
\end{aligned}
\tag{A.13}
$$

which leads to telescopic sums when taking the difference of the left hand side and the first term on the right hand side.

The very first term is the integrand of the twofold integral apart from the factor $1/[(z_1^* + \kappa_1)(z_2^* + \kappa_2)]$. Making use of identity (5.76) for both integration variables z_1 and z_2 for the left hand side of the equation above, we find

$$
\begin{aligned}
(\kappa_2 - \kappa_1)\Xi_3^{(4,N+1)} =& -2 \int_{\mathbb{C}} d[z] \frac{z - z^*}{(1+|z|^2)^{2N+2}} \frac{1}{(\kappa_1 + z)(\kappa_2 + z^*)} \\
&- \frac{2(2N+1)}{\pi} \int_{\mathbb{C}^2} d[z] \frac{1}{(z_1^* + \kappa_1)(z_2^* + \kappa_2)} \frac{(z_2^* - z_1^*)(1+z_1 z_2)^{2N}}{(1+|z_1|^2)^{2N+2}(1+|z_2|^2)^{2N+2}} \\
&- 2\pi \sum_{j=0}^{N} \sum_{m=0}^{j} \left[\frac{(2N+1)!\, j!\, \Gamma(N-j+3/2)}{(2j+1)!\,(2N-2j+1)!\, m!\, \Gamma(N-m+3/2)} \right. \\
&\left. \times \frac{(\kappa_1^*)^{2m}(\kappa_2^*)^{2j+1} - (\kappa_2^*)^{2m}(\kappa_1^*)^{2j+1}}{(1+|\kappa_1|^2)^{2N+1}(1+|\kappa_2|^2)^{2N+1}} \right].
\end{aligned}
\tag{A.14}
$$

The double sum is, apart from the factor $1/[(1+|\kappa_1|^2)^{2N+1}(1+|\kappa_2|^2)^{2N+1}]$, equivalent to an expectation value

$$\begin{aligned}
&\sum_{j=0}^{N}\sum_{m=0}^{j}\Bigg(\frac{(2N+1)!\,j!\,\Gamma(N-j+3/2)}{(2j+1)!\,(2N-2j+1)!\,m!\,\Gamma(N-m+3/2)}\\
&\quad\times\left[(\kappa_1^*)^{2m}(\kappa_2^*)^{2j+1}-(\kappa_2^*)^{2m}(\kappa_1^*)^{2j+1}\right]\Bigg)\\
=&\frac{\kappa_2^*-\kappa_1^*}{(2\pi)^N N!\prod_{j=1}^N B(2j,2N+4-2j)}\\
&\times\int\limits_{\mathbb{C}^N} d[z]\Delta_{2N}(z)\prod_{r=1}^{N}\frac{z_r-z_r^*}{(1+|z_r|^2)^{2N+4}}\prod_{j=1}^{N}(\kappa_1^*+z_j)(\kappa_1^*+z_j^*)(\kappa_2^*+z_j)(\kappa_2^*+z_j^*)\\
=&(\kappa_2^*-\kappa_1^*)\frac{\langle\det(\kappa_1^*K_1+K_2)(\kappa_2^*K_1+K_2)\rangle}{\langle\det K_1^2\rangle},
\end{aligned}\tag{A.15}$$

where we average over $2N\times 2N$ real quaternion Ginibre matrices $K_1,K_2\in\mathbb{H}^{N\times N}$. We emphasize that we can exploit the results of the first kernel function $K_1^{(4)}(p_m,p_n)$, see (5.66), with the difference that the matrix dimension is larger. Thus, it is

$$\begin{aligned}
&\sum_{j=0}^{N}\sum_{m=0}^{j}\Bigg(\frac{(2N+1)!\,j!\,\Gamma(N-j+3/2)}{(2j+1)!\,(2N-2j+1)!\,m!\,\Gamma(N-m+3/2)}\\
&\quad\times\left[(\kappa_1^*)^{2m}(\kappa_2^*)^{2j+1}-(\kappa_2^*)^{2m}(\kappa_1^*)^{2j+1}\right]\Bigg)\\
=&(\kappa_2^*-\kappa_1^*)\sum_{j=0}^{N}\frac{\mathrm{B}(N+1,3/2)}{\mathrm{B}(j+1,N-j+3/2)}\left(\kappa_1^*\kappa_2^*+1\right)^{2j}\left(\kappa_2^*-\kappa_1^*\right)^{2N-2j}\\
=&\frac{(\kappa_2^*-\kappa_1^*)^{2N+1}}{2N+1}\left[\left(1+\left(\frac{\kappa_1^*\kappa_2^*+1}{\kappa_2^*-\kappa_1^*}\right)^2\right)q_{2N}^{(N+1)}\left(\frac{\kappa_1^*\kappa_2^*+1}{\kappa_2^*-\kappa_1^*}\right)-\left(\frac{\kappa_1^*\kappa_2^*+1}{\kappa_2^*-\kappa_1^*}\right)^{2N+2}\right]\\
=&(\kappa_2^*-\kappa_1^*)^{2N+1}q_{2N}^{(N+1)}\left(\frac{\kappa_1^*\kappa_2^*+1}{\kappa_2^*-\kappa_1^*}\right).
\end{aligned}\tag{A.16}$$

In addition, the remaining twofold integral can be simplified further. For that purpose, we note that

$$\frac{\partial}{\partial z_1}\frac{(z_2^*-z_1^*)(1+z_1z_2)^{2N+1}}{(z_1^*+\kappa_1)(z_2-z_1^*)\left(1+|z_1|^2\right)^{2N+1}}=(2N+1)\frac{(z_2^*-z_1^*)(1+z_1z_2)^{2N}}{(z_1^*+\kappa_1)\left(1+|z_1|^2\right)^{2N+2}}.\tag{A.17}$$

Therefore, we can also evaluate the respective integral for these derivatives along (5.76) where we need to take into account the two poles at $z_1 = -\kappa_1^*$ and $z_1 = z_2^*$, such that we arrive at

$$\begin{aligned}(\kappa_2 - \kappa_1)\Xi_3^{(4,N+1)} =& -2\pi \left(\frac{\kappa_2^* - \kappa_1^*}{(1+|\kappa_1|^2)\,(1+|\kappa_2|^2)} \right)^{2N+1} q_{2N}^{(N+1)} \left(\frac{\kappa_1^*\kappa_2^* + 1}{\kappa_2^* - \kappa_1^*} \right) \\ &+ 2 \int_{\mathbb{C}} d[z] \frac{1}{(1+|z|^2)^{2N+2}} \frac{z^* + \kappa_1}{(z+\kappa_1)(z^*+\kappa_2)} \left(\frac{1-\kappa_1^* z}{1+|\kappa_1|^2} \right)^{2N+1} .\end{aligned} \tag{A.18}$$

Extending $z^* + \kappa_1 = z^* + \kappa_2 + \kappa_1 - \kappa_2$ in the numerator, it is straightforward to show that the integral

$$\int_{\mathbb{C}} d[z] \frac{1}{(1+|z|^2)^{2N+2}} \frac{1}{(z+\kappa_1)} \left(\frac{1-\kappa_1^* z}{1+|\kappa_1|^2} \right)^{2N+1} = 0 \tag{A.19}$$

vanishes, for instance, with the help of Stokes' theorem with

$$\frac{\partial}{\partial z^*} \frac{(1-\kappa_1^* z)^{2N+1}}{z(z+\kappa_1)\,(1+|z|^2)^{2N+1}} = -\frac{(2N+1)(1-\kappa_1^* z)^{2N+1}}{(z+\kappa_1)\,(1+|z|^2)^{2N+2}}, \tag{A.20}$$

where the contributions at the poles $z = 0$ and $z = -\kappa_1$ cancel each other.

What remains is essentially the integral

$$J = \int_{\mathbb{C}} d[z] \frac{1}{(1+|z|^2)^{2N+2}} \frac{1}{(z+\kappa_1)(z^*+\kappa_2)} \left(\frac{1-\kappa_1^* z}{1+|\kappa_1|^2} \right)^{2N+1} . \tag{A.21}$$

Choosing polar coordinates $z = \sqrt{r} e^{i\varphi}$, we first integrate over the angle $\varphi \in [0, 2\pi)$, exploiting the partial fraction decomposition

$$\frac{1}{(\sqrt{r}e^{i\varphi} + \kappa_1)(\sqrt{r}e^{-i\varphi} + \kappa_2)} = \frac{e^{i\varphi}}{r - \kappa_1\kappa_2} \left[\frac{1}{e^{i\varphi} + \kappa_1/\sqrt{r}} - \frac{1}{e^{i\varphi} + \sqrt{r}/\kappa_2} \right] \tag{A.22}$$

and employing the residue theorem, which leads to

$$J = \pi \int_{|\kappa_1|^2}^{\infty} \frac{dr}{(1+r)^{2N+2}(r-\kappa_1\kappa_2)} - \pi \int_0^{|\kappa_2|^2} \frac{dr}{(1+r)^{2N+2}(r-\kappa_1\kappa_2)} \left(\frac{1 + r\kappa_1^*/\kappa_2}{1+|\kappa_1|^2} \right)^{2N+1} . \tag{A.23}$$

The first integral is explicitly

$$\int_{|\kappa_1|^2}^{\infty} \frac{dr}{(1+r)^{2N+2}(r-\kappa_1\kappa_2)} = -\frac{1}{(1+\kappa_1\kappa_2)^{2N+2}} \times \left[\ln\left(1-\frac{1+\kappa_1\kappa_2}{1+|\kappa_1|^2}\right)+\sum_{j=1}^{2N+1}\frac{1}{j}\left(\frac{1+\kappa_1\kappa_2}{1+|\kappa_1|^2}\right)^j\right], \tag{A.24}$$

which is Lerch's transcendent (5.21). The second integral can be evaluated once one has performed the Möbius transformation

$$s=\frac{(\kappa_2-\kappa_1^*)r}{\kappa_2+\kappa_1^* r} \quad\Leftrightarrow\quad r=\frac{\kappa_2 s}{\kappa_2-\kappa_1^*-\kappa_1^* s}. \tag{A.25}$$

Then, the integral simplifies to

$$\begin{aligned}&\int_0^{|\kappa_2|^2} \frac{dr}{(1+r)^{2N+2}(r-\kappa_1\kappa_2)}\left(\frac{1+r\,\kappa_1^*/\kappa_2}{1+|\kappa_1|^2}\right)^{2N+1}\\ =&\int_0^{(|\kappa_2|^2-\kappa_1^*\kappa_2^*)/(1+\kappa_1^*\kappa_2^*)} \frac{ds}{(1+|\kappa_1|^2)^{2N+1}(1+s)^{2N+2}\left[(1+|\kappa_1|^2)\,s+|\kappa_1|^2-\kappa_1\kappa_2\right]}.\end{aligned} \tag{A.26}$$

This integral can be carried out like the former one, yielding

$$\begin{aligned}&\int_0^{|\kappa_2|^2} \frac{dr}{(1+r)^{2N+2}(r-\kappa_1\kappa_2)}\left(\frac{1+r\,\kappa_1^*/\kappa_2}{1+|\kappa_1|^2}\right)^{2N+1}\\ =&\frac{1}{(1+\kappa_1\kappa_2)^{2N+2}}\left[\ln\left(1-\frac{1+\kappa_1\kappa_2}{1+|\kappa_1|^2}\right)+\sum_{j=1}^{2N+1}\frac{1}{j}\left(\frac{1+\kappa_1\kappa_2}{1+|\kappa_1|^2}\right)^j\right]\\ &-\frac{1}{(1+\kappa_1\kappa_2)^{2N+2}}\left[\ln\left(1-\frac{|1+\kappa_1\kappa_2|^2}{(1+|\kappa_1|^2)\,(1+|\kappa_2|^2)}\right)\right.\\ &\left.+\sum_{j=1}^{2N+1}\frac{1}{j}\left(\frac{|1+\kappa_1\kappa_2|^2}{(1+|\kappa_1|^2)\,(1+|\kappa_2|^2)}\right)^j\right].\end{aligned} \tag{A.27}$$

As can be seen the first logarithm and sum cancel with the one from the first integral of J. Therefore, we arrive at

$$\begin{aligned} J =& -\frac{\pi}{(1+\kappa_1\kappa_2)^{2N+2}} \\ &\times\left[\ln\left(1-\frac{|1+\kappa_1\kappa_2|^2}{(1+|\kappa_1|^2)(1+|\kappa_2|^2)}\right)+\sum_{j=1}^{2N+1}\frac{1}{j}\left(\frac{|1+\kappa_1\kappa_2|^2}{(1+|\kappa_1|^2)\,(1+|\kappa_2|^2)}\right)^j\right] \\ =&\pi\left(\frac{1+\kappa_1^*\kappa_2^*}{(1+|\kappa_1|^2)(1+|\kappa_2|^2)}\right)^{2N+2}\Phi_{2N+2}^{(1)}\left(\frac{|1+\kappa_1\kappa_2|^2}{(1+|\kappa_1|^2)(1+|\kappa_2|^2)}\right), \end{aligned} \tag{A.28}$$

which is anew Lerch's transcendent (5.21) apart from a prefactor. Despite that some expressions of this integral has been in some intermediate steps not obviously symmetric under $\kappa_1 \leftrightarrow \kappa_2$, this final result reflects this symmetry.

A.3 The Joint Eigenvalue Probability Distribution of the Real Induced Spherical Ensemble

We consider the real induced spherical ensemble with the matrix probability distribution

$$\widetilde{G}_{\mu,\nu}^{(1,N)}(Y) = \pi^{-N^2/2}\prod_{j=1}^{N}\frac{\Gamma(j/2)\Gamma(\mu+\nu+(N+j)/2)}{\Gamma(\nu+j/2)\Gamma(\mu+j/2)}\frac{\det^{2\nu}Y}{\det^{N+\mu+\nu}\left(\mathbb{1}_N+YY^T\right)}. \tag{A.29}$$

It is the ensemble of matrices $Y = K_1^{-1}K_2$, where K_1 and K_2 are distributed according to deformed Gaussians

$$P_\mu(K) = \pi^{-N^2/2}\prod_{j=1}^{N}\frac{\Gamma(j/2)}{\Gamma(\mu+j/2)}\exp\left(-\operatorname{tr}KK^T\right)\det{}^{\mu}KK^T \tag{A.30}$$

for the real positive parameters μ resp. ν. In the body of the text we frequently omitted these indices, implying that they are zero, e.g. $\widetilde{G}_{0,0}^{(1,N)}(Y) = \widetilde{G}^{(1,N)}(Y)$, which leads to the ordinary spherical ensemble. The induced spherical ensemble is well-studied for $\beta = 1, 2, 4$, see [132–134]. Unfortunately, in the real case the known joint eigenvalue probability distribution is available only in a form, that is impractical for our purpose. For this reason we will reproduce this result once again. In doing so, we adhere to methods employed in the aforementioned works.

Generally for real matrices, we have to distinguish between even ($\widetilde{N} = N/2$) and odd ($\widetilde{N} = (N-1)/2$) matrix dimensions. We apply a real Schur decomposition

$$Y = U(D+T)U^T, \tag{A.31}$$

where D is a diagonal matrix of $\widetilde{N}$ real 2×2-blocks D_j. In the odd case these are complemented by a real 1×1-block $D_{\widetilde{N}+1}$. The matrix T is a real strict upper triangular matrix and U is orthogonal, $U \in \mathrm{O}(N)/\mathrm{O}(2)^{\widetilde{N}}$ resp. $U \in \mathrm{O}(N)/(\mathrm{O}(2)^{\widetilde{N}} \times \mathrm{O}(1))$ for even resp. odd N. Upon this transformation the measure changes according to [130, 180]

$$d[Y] = \Delta_N(z) \prod_{j=1}^{\widetilde{N}} \frac{1}{z_{-j} - z_{+j}} d[D]d[T]d\mu(U), \tag{A.32}$$

where z_{+j} and z_{-j} are the eigenvalues of D_j and thus also of Y. First, one integrates over the upper triangular matrix T. Only the denominator of the matrix distribution (A.29) depends on T. The determinant is expanded and the free parameters are integrated out column wise. In the even case this yields [78, 133]

$$\begin{aligned}\int d[T] \frac{\det^{2\nu} Y}{\det^{N+\mu+\nu}\left(\mathbb{1}_N + YY^T\right)} =& \\ &\prod_{j=1}^{\widetilde{N}} \frac{\pi^{N-2j}\,\Gamma\left(N/2+\mu+\nu+1/2\right)\Gamma\left(N/2+\mu+\nu+1\right)}{\Gamma\left(N+\mu+\nu+1/2-j\right)\Gamma\left(N+\mu+\nu+1-j\right)} \\ &\times \prod_{j=1}^{\widetilde{N}} \frac{\det^{2\nu} D_j}{\det^{N/2+\mu+\nu+1}(\mathbb{1}_N + D_j D_j^T)}\end{aligned} \tag{A.33}$$

and in the odd case

$$\begin{aligned}\int d[T] \frac{\det^{2\nu} Y}{\det^{N+\mu+\nu}\left(\mathbb{1}_N + YY^T\right)} =& \\ &\prod_{j=1}^{\widetilde{N}} \frac{\pi^{N-2j}\,\Gamma\left(N/2+\mu+\nu+1/2\right)\Gamma\left(N/2+\mu+\nu+1\right)}{\Gamma\left(N+\mu+\nu+1/2-j\right)\Gamma\left(N+\mu+\nu+1-j\right)} \\ &\times h_{\mu,\nu}^{(N)}\left(z_{2\widetilde{N}+1}\right) \prod_{j=1}^{\widetilde{N}} \frac{\det^{2\nu} D_j}{\det^{N/2+\mu+\nu+1}(\mathbb{1}_N + D_j D_j^T)}.\end{aligned} \tag{A.34}$$

Here, we obtain an additional factor

$$h_{\mu,\nu}^{(N)}(z) = \frac{z^{2\nu}}{\left(1+z^2\right)^{N/2+\mu+\nu+1/2}} \delta\left(y\right), \tag{A.35}$$

which effectively renders the variable $z_{2\widetilde{N}+1}$ real. We use the notation $z_j = x_j + i\, y_j$ with $x_j, y_j \in \mathbb{R}$ for the generally complex eigenvalues. Next, we work on the 2×2-blocks. Gathering all factors depending on the eigenvalues of D_j, except those in the Vandermonde determinant, we obtain

$$A_j = \frac{1}{z_{-j} - z_{+j}} \frac{\det^{2\nu} D_j}{\det^{N/2+\mu+\nu+1}(\mathbb{1}_N + D_j D_j^T)}. \tag{A.36}$$

Following [181], we start by diagonalizing the symmetric part

$$O_j^T D_j O_J = \begin{bmatrix} \lambda_{1j} & \rho_j \\ -\rho_j & \lambda_{2j} \end{bmatrix}, \qquad O_j = \begin{bmatrix} \cos\varphi_j & \sin\varphi_j \\ -\sin\varphi_j & \cos\varphi_j \end{bmatrix} \in \mathrm{SO}(2). \tag{A.37}$$

The range of parameters is $\rho_j \in \mathbb{R}$, $\varphi_j \in [0, \pi)$ and $\lambda_{1j}, \lambda_{2j} \in \mathbb{R}$ with $\lambda_{1j} \geq \lambda_{2j}$. The flat measure of the four real independent variables transforms as

$$d[D_j] = 2(\lambda_{1j} - \lambda_{2j}) d\varphi_j d\rho_j d\lambda_{1j} d\lambda_{2j}. \tag{A.38}$$

Next, we want to change coordinates from the eigenvalues of the symmetric part $\lambda_{1j}, \lambda_{2j}$ to the eigenvalues $z_{\pm j}$ of the full matrix. They are related via

$$\begin{aligned} z_{\pm j} &= \frac{\lambda_{1j} + \lambda_{2j}}{2} \pm \sqrt{\left(\frac{\lambda_{1j} - \lambda_{2j}}{2}\right)^2 - \rho_j^2}, \\ \lambda_{1,2j} &= \frac{z_{+j} + z_{-j}}{2} \pm \sqrt{\left(\frac{z_{+j} - z_{-j}}{2}\right)^2 + \rho_j^2}. \end{aligned} \tag{A.39}$$

According to the ordering of λ_1 and λ_2, the eigenvalues $z_{\pm j}$ also have to be ordered

$$\begin{aligned} z_{\pm j} \in \mathbb{R} &: x_{+j} \geq x_{-j}, \\ z_{\pm j} = z_{\mp j}^* &: y_{+j} = -y_{-j} > 0, \end{aligned} \tag{A.40}$$

where we distinguished between the cases of real and complex conjugated eigenvalues. However, in the following we will disregard this ordering, which is be compensated by an overall factor 1/2. We obtain the Jacobian

$$\left| \det \frac{\partial(z_{+j}, z_{-j})}{\partial(\lambda_{1j}, \lambda_{2j})} \right| = \frac{\lambda_{1j} - \lambda_{2j}}{|z_{-j} - z_{+j}|}. \tag{A.41}$$

The determinants in the new coordinates are

$$\begin{aligned} \det\left(\mathbb{1}_2 + D_j D_j^T\right) &= 4\rho_j^2 + (1 + z_{+j}^2)(1 + z_{-j}^2), \\ \det D_j &= z_{+j} z_{-j}. \end{aligned} \tag{A.42}$$

We also need to consider that in the case, where $z_{\pm j}$ are complex conjugates, the integration regime of ρ_j is confined to

$$\rho_j^2 \geq -\left(\frac{z_{+j} - z_{-j}}{2}\right)^2 = y_{\pm j}^2 \Rightarrow |\rho_j| \geq |y_{\pm j}|. \tag{A.43}$$

This is due to the condition $\lambda_{1,2j} \in \mathbb{R}$. Gathering everything and integrating over φ_j and ρ_j we obtain for (A.36)

$$
\begin{aligned}
&2(\lambda_{1j} - \lambda_{2j}) d\lambda_{1j} d\lambda_{2j} \int\limits_{\mathbb{R}} d\rho_j \int\limits_0^\pi d\varphi_j \, A_j \\
&= \frac{\pi}{2} \frac{(z_{+j} z_{-j})^{2\nu}}{[(1+z_+^2)(1+z_-^2)]^{N/2+\nu+\mu+1/2}} \frac{|z_{-j} - z_{+j}|}{z_{-j} - z_{+j}} \\
&\times \Big[\mathrm{B}(1/2, N/2+\mu+\nu+1/2) \Theta(x_{+j} - x_{-j}) \delta(y_{-j}) \delta(y_{+j}) \\
&+ 2\,\Theta(y_{+j}) \delta(x_{+j} - x_{-j}) \delta(y_{+j} + y_{-j}) Q_{\mu,\nu}^{(N)}(z_{+j}, z_{-j}) \Big] dx_{+j} dx_{-j} dy_{+j} dy_{-j},
\end{aligned}
\tag{A.44}
$$

where the function

$$
\begin{aligned}
Q_{\mu,\nu}^{(N)}(z_{+j}, z_{-j}) &= 2 \int\limits_{\frac{2|y_{\pm j}|}{|1+z_{\pm j}^2|}}^{\infty} \frac{d\rho_j}{\left(1+\rho_j^2\right)^{N/2+\mu+\nu+1}} \\
&= \mathrm{B}\left(\frac{4y^2}{|1+z^2|^2 + 4y^2}; 1/2, N/2+\mu+\nu+1/2 \right)
\end{aligned}
\tag{A.45}
$$

emerges. Now only the integral over the Haar measure of the respective cosets remains to be computed. It yields the constants

$$
\begin{aligned}
\mathrm{Vol}\left(\mathrm{O}(N)/\mathrm{O}(2)^{\widetilde{N}}\right) &= \frac{\mathrm{Vol}(\mathrm{O}(N))}{(4\pi)^{N/2}} = \frac{1}{(4\pi)^{N/2}} \prod_{j=1}^N \frac{2\pi^{j/2}}{\Gamma(j/2)}, \\
\mathrm{Vol}\left(\mathrm{O}(N)/(\mathrm{O}(2)^{\widetilde{N}} \times \mathrm{O}(1))\right) &= \frac{\mathrm{Vol}(\mathrm{O}(N))}{2(4\pi)^{(N-1)/2}} = \frac{1}{2(4\pi)^{(N-1)/2}} \prod_{j=1}^N \frac{2\pi^{j/2}}{\Gamma(j/2)}
\end{aligned}
\tag{A.46}
$$

for N even resp. odd. We summarize this under

$$
\int d\mu(U) = \frac{1}{(1+N-2\widetilde{N})(4\pi)^{\widetilde{N}}} \prod_{j=1}^N \frac{2\pi^{j/2}}{\Gamma(j/2)}. \tag{A.47}
$$

As we not only disregard the ordering of the eigenvalues $z_{\pm j}$ in each 2×2-block, but also the ordering of the 2×2-blocks themselves, all of this has to be supplemented with a factor $1/\widetilde{N}!$.

Finally bringing everything together we obtain the correctly normalized joint eigenvalue distribution

$$
\begin{aligned}
G_{\mu,\nu}^{(1,N)}(z) &= \frac{\Delta_N(z)}{c_{\mu,\nu}^{(1,N)}} \prod_{j=1}^{\widetilde{N}} g_{\mu,\nu}^{(1,N)}(z_{2j-1}, z_{2j}), \\
G_{\mu,\nu}^{(1,N)}(z) &= \frac{\Delta_N(z) h_{\mu,\nu}^{(1,N)}(z_N)}{c_{\mu,\nu}^{(1,N)}} \prod_{j=1}^{\widetilde{N}} g_{\mu,\nu}^{(1,N)}(z_{2j-1}, z_{2j})
\end{aligned}
\tag{A.48}
$$

for N even resp. odd with the antisymmetric function

$$
\begin{aligned}
g_{\mu,\nu}^{(1,N)}(z_1, z_2) = &(z_1 z_2)^{2\nu} \frac{|z_2 - z_1|}{z_2 - z_1} \\
&\times \frac{\mathrm{B}(1/2, N/2+\mu+\nu+1/2)\delta(y_1)\delta(y_2) + 2\,\delta(x_1 - x_2)\delta(y_1 + y_2) Q_{\mu,\nu}^{(N)}(z_1, z_1^*)}{[(1+z_1^2)(1+z_2^2)]^{N/2+\mu+\nu+1/2}}.
\end{aligned}
\tag{A.49}
$$

The antisymmetry of this function is due to us disregarding the ordering (A.40). The normalization is

$$
\begin{aligned}
c_{\mu,\nu}^{(1,N)} = &2^{3\widetilde{N}-N} \pi^{\widetilde{N}^2 - \widetilde{N}(N-1) + N(N-1)/4} (1 + N - 2\widetilde{N}) \widetilde{N}! \prod_{j=1}^{N} \frac{\Gamma(\nu + j/2)\Gamma(\mu + j/2)}{\Gamma(\mu + \nu + (N+j)/2)} \\
&\times \prod_{j=1}^{\widetilde{N}} \frac{\Gamma\left(N+\mu+\nu+1/2-j\right)\Gamma\left(N+\mu+\nu+1-j\right)}{\Gamma\left(N/2+\mu+\nu+1/2\right)\Gamma\left(N/2+\mu+\nu+1\right)}.
\end{aligned}
\tag{A.50}
$$

Setting $\mu = \nu = 0$ one immediately finds the joint probability distribution of eigenvalues (5.97) of the ordinary real spherical ensemble.

The Pfaffian of the moment matrix (5.103) can be related to this constant. One finds for the even case [181]

$$
\int_{\mathbb{C}^{2M}} d[z] \Delta_{2M}(z) \prod_{j=1}^{M} g^{(1,N)}(z_{2j-1}, z_{2j}) = M! \, \mathrm{Pf}\, D^{(1,2M)}, \tag{A.51}
$$

where $M \le N/2$. On the other hand by equations (A.48) and (A.49) this integral is

$$
\int_{\mathbb{C}^{2M}} d[z] \Delta_{2M}(z) \prod_{j=1}^{M} g_{(N-2M)/2,0}^{(1,2M)}(z_{2j-1}, z_{2j}) = c_{(N-2M)/2,0}^{(1,2M)} \tag{A.52}
$$

and therefore

$$
\mathrm{Pf}\, D^{(1,2M)} = \frac{c_{(N-2M)/2,0}^{(1,2M)}}{M!}. \tag{A.53}
$$

We use this result at multiple points in the body of the text to evaluate the prefactors of our integrals.

A.4 Characteristic Polynomials of the Real Induced Spherical Ensemble

We are interested in integrals of the type

$$I_M = \int_{\mathbb{C}^{2M}} d[z]\Delta_{2M}(z) \prod_{j=1}^{M} g^{(1,N)}(z_{2j-1}, z_{2j}) \prod_{j=1}^{2M}(x - z_j), \tag{A.54}$$

where $M \leq N/2$. Applying equations (A.48) we map them to integrals over a matrix distribution

$$\begin{aligned} I_M = &\int_{\mathbb{C}^{2M}} d[z]\Delta_{2M}(z) \prod_{j=1}^{M} g^{(1,2M)}_{(N-2M)/2,0}(z_{2j-1}, z_{2j}) \prod_{j=1}^{2M}(x - z_j) \\ =& c^{(1,2M)}_{(N-2M)/2,0} \int d[Y]\widetilde{G}^{(1,2M)}_{(N-2M)/2,0}(Y) \det(x - Y). \end{aligned} \tag{A.55}$$

The matrix distribution $\widetilde{G}^{(1,N)}_{\mu,\nu}(Y)$ is invariant under left and right actions of the orthogonal group, see (A.29),

$$Y \to O_1 Y O_2 \qquad \text{with } O_1, O_2 \in \mathrm{O}(N). \tag{A.56}$$

Let us choose $O_1 = \mathbb{1}_N$ and O_2 and as diagonal with entries ± 1. This effectively changes the sign of Y column-wise, $Y_{jl} \to \pm Y_{jl}$ for all l. Expanding the determinant we see that only one term survives the average

$$\langle \det(x - Y) \rangle = \sum_{\sigma \in \mathbb{S}_N} \operatorname{sgn} \sigma \left\langle \prod_{j=1}^{N} \left(x\delta_{j\sigma(j)} - Y_{j\sigma(j)} \right) \right\rangle = \sum_{\sigma \in \mathbb{S}_N} \operatorname{sgn} \sigma \, x^N \prod_{j=1}^{N} \delta_{j\sigma(j)} = x^N, \tag{A.57}$$

which is a monomial of power N. Therefore we obtain

$$I_M = c^{(1,2M)}_{(N-2M)/2,0} x^{2M}. \tag{A.58}$$

This result corresponds to the skew-orthogonal polynomial of even degree [2, 169].

A.5 Alternative Expression of $\mathbf{K}_3^{(1,N)}$

Unlike in (5.133) we expand the determinant only in the last column

$$\begin{aligned} \Xi_3^{(1,N+2)} =& \frac{1}{c^{(1,N)}} \frac{2}{\kappa_2 - \kappa_1} \\ &\times \int_{\mathbb{C}^{N+2}} d[z] \frac{1}{z_{N+2} + \kappa_2} \prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1}, z_{2j}) \det \left[\, z_a^{b-1} \, \middle| \, \frac{1}{z_a + \kappa_1} \, \right]_{\substack{1 \leq a \leq N+1 \\ 1 \leq b \leq N}} \end{aligned} \tag{A.59}$$

and apply the identity,see (5.118),

$$\det\left[\; z_a^{b-1} \;\middle|\; \frac{1}{z_a+\kappa_1} \;\right]_{\substack{1\le a\le N+1\\ 1\le b\le N}} = (-1)^{N+2}\Delta_{N+1}(z)\prod_{j=1}^{N+1}\frac{1}{z_j+\kappa_1} = (-1)^{N+2}\frac{\Delta_N(z)}{z_{N+1}+\kappa_1}\prod_{j=1}^{N}\frac{z_{N+1}-z_j}{z_j+\kappa_1}. \tag{A.60}$$

This allows us to identify the integral over $(z_1,\ldots,z_N)$ as the function $\Xi_2^{(1,N)}$, see (5.114), that we calculated for the second kernel, resulting in

$$\begin{aligned}\Xi_3^{(1,N+2)} =& \frac{1}{c^{(1,N)}}\frac{2}{\kappa_2-\kappa_1} \\ &\times \int\limits_{\mathbb{C}^{N+2}} d[z]\frac{\Delta_N(z)}{(z_{N+1}+\kappa_1)(z_{N+2}+\kappa_2)}\prod_{j=1}^{N/2+1} g^{(1,N)}(z_{2j-1},z_{2j})\prod_{j=1}^{N}\frac{z_{N+1}-z_j}{z_j+\kappa_1} \\ =& \frac{2}{\kappa_2-\kappa_1}\int\limits_{\mathbb{C}^2} d[z]\frac{g^{(1,N)}(z_1,z_2)}{(z_1+\kappa_1)(z_2+\kappa_2)}\left\langle\frac{\det(-z_1K_1+K_2)}{\det(\kappa_1K_1+K_2)}\right\rangle.\end{aligned} \tag{A.61}$$

Inserting our result (5.120) for the ensemble average we find the following expression for the third kernel

$$\begin{aligned}\mathrm{K}_3^{(1)}(q_m,q_n) =& \frac{2}{[b(q_m)b(q_n)]^{N-1}} \\ &\times \int\limits_{\mathbb{C}^2} d[z]\frac{g^{(1,N)}(z_1,z_2)}{(a(q_m)+b(q_m)z_1)(a(q_n)+b(q_n)z_2)}\left\langle\frac{\det(-z_1K_1+K_2)}{\det(\kappa(q_m)K_1+K_2)}\right\rangle \\ =& \frac{-N(N-1)}{2\pi[b(q_m)b(q_n)]^{N-1}}\int\limits_{\mathbb{C}^2} d[z]\frac{g^{(1,N)}(z_1,z_2)}{(a(q_m)+b(q_m)z_1)(a(q_n)+b(q_n)z_2)} \\ &\times\left[\frac{(-1)^{N/2}2\pi\mathrm{B}(1/2,(N+1)/2)}{N-1}\binom{(N-1)/2}{N}\right. \\ &\times {}_2\mathrm{F}_1\left(1,(N+1)/2;N+1;1+\left(\frac{\kappa(q_m)-z_1}{\kappa(q_m)+z_1}\right)^2\right) \\ &\left.+\,2i\int\limits_{\mathbb{C}} d[z]\frac{z^{N-2}\mathrm{sgn}(\mathrm{Im}\,z)Q(z,z^*)}{|1+z^2|^{N+1}}\left(z^*+\frac{\kappa(q_m)-z_1}{\kappa(q_m)+z_1}\right)^{-1}\right].\end{aligned} \tag{A.62}$$

Furthermore inserting (5.25) yields

$$
\begin{aligned}
\mathrm{K}_3^{(1)}(q_m, q_n) =& \frac{-N(N-1)b(q_m)}{2\pi b^{N-1}(q_n)} \Bigg(\int_{\mathbb{R}} dx \frac{r(x, v(q_n))}{a(q_m)+b(q_m)x} \left(\frac{x^2+1}{a(q_m)+b(q_m)x} \right)^N \\
& \times \Bigg[\frac{(-1)^{N/2} 2\pi \mathrm{B}(1/2, (N+1)/2)}{N-1} \binom{(N-1)/2}{N} \\
& \times {}_2\mathrm{F}_1 \left(1, (N+1)/2; N+1; 1 + \left(\frac{a(q_m)-b(q_m)x}{a(q_m)+b(q_m)x} \right)^2 \right) \\
& + \int_{\mathbb{C}} d[z] s\left(z, z^*, \left(\frac{a(q_m)-b(q_m)x}{a(q_m)+b(q_m)x} \right) \right) z^{N-2} \Bigg] \\
& + \int_{\mathbb{C}} d[z_1] \frac{s(z_1, z_1^*, v(q_n))}{a(q_m)+b(q_m)z_1} \left(\frac{z_1^2+1}{a(q_m)+b(q_m)z_1} \right)^N \\
& \times \Bigg[\frac{(-1)^{N/2} 2\pi \mathrm{B}(1/2, (N+1)/2)}{N-1} \binom{(N-1)/2}{N} \\
& \times {}_2\mathrm{F}_1 \left(1, (N+1)/2; N+1; 1 + \left(\frac{a(q_m)-b(q_m)z_1}{a(q_m)+b(q_m)z_1} \right)^2 \right) \\
& + \int_{\mathbb{C}} d[z_2] s\left(z_2, z_2^*, \left(\frac{a(q_m)-b(q_m)z_1}{a(q_m)+b(q_m)z_1} \right) \right) z_2^{N-2} \Bigg] \Bigg)
\end{aligned}
\tag{A.63}
$$

with the functions (5.29). In this representation also a dependence solely in O(2) invariants cannot be identified.

Bibliography

[1] P. Braun, N. Hahn, D. Waltner, O. Gat and T. Guhr. *Winding Number Statistics of a Parametric Chiral Unitary Random Matrix Ensemble*. J. Phys. A: Math. Theor. **55** 22 224011 (2022)

[2] N. Hahn, M. Kieburg, O. Gat and T. Guhr. *Winding Number Statistics for Chiral Random Matrices: Averaging Ratios of Determinants with Parametric Dependence*. J. Math. Phys. **64** 2 (2023). 021901

[3] N. Hahn, M. Kieburg, O. Gat and T. Guhr. *Winding Number Statistics for Chiral Random Matrices: Averaging Ratios of Parametric Determinants in the Orthogonal Case*. J. Math. Phys. **64** 11 111902 (2023)

[4] M. Nakahara. *Geometry, Topology and Physics*. Institute of Physics Publishing (2003)

[5] User: LucasVB. Wikimedia

[6] T. G. Kalnin, D. A. Ivonin, K. N. Abrosimov, E. A. Grachev and N. V. Sorokina. *Analysis of Tomographic Images of the Soil Pore Space Structure by Integral Geometry Methods*. Eurasian Soil Sci. **54** 9 1400 (2021)

[7] H. Goldstein, C. Poole and J. Safko. *Classical Mechanics*. Addison-Wesley (2002)

[8] I. Nikitin. *Mathematical Modeling and Visualization of Topologically Non-trivial Solutions in General Relativity*. J. Phys. Conf. Ser. **1730** 1 012074 (2021)

[9] S. Carlip. *Spacetime Foam: a Review*. Rep. Prog. Phys. **86** 6 066001 (2023)

[10] M. Z. Hasan and C. L. Kane. *Colloquium: Topological Insulators*. Rev. Mod. Phys. **82** 3045 (2010)

[11] X.-L. Qi and S.-C. Zhang. *Topological Insulators and Superconductors*. Rev. Mod. Phys. **83** 1057 (2011)

[12] X.-G. Wen. *Colloquium: Zoo of Quantum-Topological Phases of Matter*. Rev. Mod. Phys. **89** 041004 (2017)

[13] M. Sato and Y. Ando. *Topological Superconductors: a Review*. Rep. Prog. Phys. **80** 7 076501 (2017)

[14] J. K. Asbóth, L. Oroszlány and A. Pályi. *A Short Course on Topological Insulators*. Springer International Publishing (2016)

[15] B. A. Bernevig and T. L. Hughes. *Topological Insulators and Topological Superconductors*. Princeton University Press (2013)

[16] K. v. Klitzing, G. Dorda and M. Pepper. *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*. Phys. Rev. Lett. **45** 494 (1980)

[17] R. B. Laughlin. *Quantized Hall Conductivity in Two Dimensions*. Phys. Rev. B **23** 5632 (1981)

[18] D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs. *Quantized Hall Conductance in a Two-Dimensional Periodic Potential*. Phys. Rev. Lett. **49** 405 (1982)

[19] M. Kohmoto. *Topological Invariant and the Quantization of the Hall Conductance*. Ann. Phys. **160** 2 343 (1985)

[20] B. I. Halperin. *Quantized Hall Conductance, Current-Carrying Edge States, and the Existence of Extended States in a Two-Dimensional Disordered Potential*. Phys. Rev. B **25** 2185 (1982)

[21] Y. Hatsugai. *Chern Number and Edge States in the Integer Quantum Hall Effect*. Phys. Rev. Lett. **71** 3697 (1993)

[22] Y. Hatsugai. *Edge States in the Integer Quantum Hall Effect and the Riemann Surface of the Bloch Function*. Phys. Rev. B **48** 11851 (1993)

[23] E. Prodan, T. L. Hughes and B. A. Bernevig. *Entanglement Spectrum of a Disordered Topological Chern Insulator*. Phys. Rev. Lett. **105** 115501 (2010)

[24] E. Prodan, B. Leung and J. Bellissard. *The Non-commutative nth-Chern Number ($n \geq 1$)*. J. Phys. A: Math. Theor. **46** 48 485202 (2013)

[25] I. Mondragon-Shem, T. L. Hughes, J. Song and E. Prodan. *Topological Criticality in the Chiral-Symmetric AIII Class at Strong Disorder*. Phys. Rev. Lett. **113** 046802 (2014)

[26] M. Maffei, A. Dauphin, F. Cardano, M. Lewenstein and P. Massignan. *Topological Characterization of Chiral Models through their Long Time Dynamics.* New J. Phys. **20** 1 013023 (2018)

[27] E. Prodan and H. Schulz-Baldes. *Bulk and Boundary Invariants for Complex Topological Insulators: From k-Theory to Physics.* Mathematical Physics Studies. Springer International Publishing, Cham (2016)

[28] A. Alldridge, C. Max and M. R. Zirnbauer. *Bulk-Boundary Correspondence for Disordered Free-Fermion Topological Phases.* Commun. Math. Phys. **377** 3 1761 (2020)

[29] C. L. Kane and E. J. Mele. *Z_2 Topological Order and the Quantum Spin Hall Effect.* Phys. Rev. Lett. **95** 146802 (2005)

[30] C. L. Kane and E. J. Mele. *Quantum Spin Hall Effect in Graphene.* Phys. Rev. Lett. **95** 226801 (2005)

[31] A. Altland and M. R. Zirnbauer. *Nonstandard Symmetry Classes in Mesoscopic Normal-Superconducting Hybrid Structures.* Phys. Rev. B **55** 1142 (1997)

[32] P. Heinzner, A. Huckleberry and M. R. Zirnbauer. *Symmetry Classes of Disordered Fermions.* Commun. Math. Phys. **257** 725 (2005)

[33] A. Kitaev. *Periodic Table for Topological Insulators and Superconductors.* AIP Conf. Proc. **1134** 1 22 (2009)

[34] A. P. Schnyder, S. Ryu, A. Furusaki and A. W. W. Ludwig. *Classification of Topological Insulators and Superconductors in Three Spatial Dimensions.* Phys. Rev. B **78** 195125 (2008)

[35] S. Ryu, A. P. Schnyder, A. Furusaki and A. W. W. Ludwig. *Topological Insulators and Superconductors: Tenfold Way and Dimensional Hierarchy.* New J. Phys. **12** 6 065010 (2010)

[36] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder and S. Ryu. *Classification of Topological Quantum Matter with Symmetries.* Rev. Mod. Phys. **88** 035005 (2016)

[37] C. W. J. Beenakker. *Random-Matrix Theory of Majorana Fermions and Topological Superconductors.* Rev. Mod. Phys. **87** 1037 (2015)

[38] C. Beenakker and L. Kouwenhoven. *A Road to Reality with Topological Superconductors.* Nat. Phys. **12** 7 618 (2016)

[39] S. Sachdev. *Quantum Phase Transitions.* Cambridge University Press, 2 ed. (2011)

[40] P. W. Anderson. *Absence of Diffusion in Certain Random Lattices.* Phys. Rev. **109** 1492 (1958)

[41] T. Brandes and S. Kettemann (eds.). *Anderson Localization and its Ramifications - Disorder, Phase Coherence, and Electron Correlations.* Springer Berlin, Heidelberg (2003)

[42] R. Chaunsali, H. Xu, J. Yang, P. G. Kevrekidis and G. Theocharis. *Stability of Topological Edge States under Strong Nonlinear Effects.* Phys. Rev. B **103** 024106 (2021)

[43] S. F. E. Oliviero, L. Leone, Y. Zhou and A. Hamma. *Stability of Topological Purity under Random Local Unitaries.* SciPost Phys. **12** 096 (2022)

[44] J. Alicea, Y. Oreg, G. Refael, F. von Oppen and M. P. A. Fisher. *Non-Abelian Statistics and Topological Quantum Information Processing in 1D Wire Networks.* Nat. Phys. **7** 5 412 (2011)

[45] S. D. Sarma, M. Freedman and C. Nayak. *Majorana Zero Modes and Topological Quantum Computation.* npj Quantum Inf. **1** 1 15001 (2015)

[46] L. O. Martín. *Topology in Quantum Information.* In *Topological Orders with Spins and Fermions: Quantum Phases and Computation*, pp. 11–15. Springer International Publishing, Cham (2019)

[47] A. Kitaev. *Fault-Tolerant Quantum Computation by Anyons.* Ann. Phys. **303** 1 2 (2003)

[48] C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma. *Non-Abelian Anyons and Topological Quantum Computation.* Rev. Mod. Phys. **80** 1083 (2008)

[49] C.-Z. Chang, J. Zhang, X. Feng, J. Shen, Z. Zhang, M. Guo, K. Li, Y. Ou, P. Wei, L.-L. Wang, Z.-Q. Ji, Y. Feng, S. Ji, X. Chen, J. Jia, X. Dai, Z. Fang, S.-C. Zhang, K. He, Y. Wang, L. Lu, X.-C. Ma and Q.-K. Xue. *Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator.* Science **340** 6129 167 (2013)

[50] S.-B. Zhang, H.-Z. Lu and S.-Q. Shen. *Edge States and Integer Quantum Hall Effect in Topological Insulator Thin Films.* Sci. Rep. **5** 1 13277 (2015)

[51] E. J. Meier, F. A. An and B. Gadway. *Observation of the Topological Soliton State in the Su–Schrieffer–Heeger Model.* Nat. Commun. **7** 1 13986 (2016)

[52] S. Howard, L. Jiao, Z. Wang, N. Morali, R. Batabyal, P. Kumar-Nag, N. Avraham, H. Beidenkopf, P. Vir, E. Liu, C. Shekhar, C. Felser, T. Hughes and V. Madhavan. *Evidence for One-Dimensional Chiral Edge States in a Magnetic Weyl Semimetal $Co_3Sn_2S_2$.* Nat. Commun. **12** 1 4269 (2021)

[53] Y. Ando. *Topological Insulator Materials.* J. Phys. Soc. Japan **82** 10 102001 (2013)

[54] P. Gehring, H. M. Benia, Y. Weng, R. Dinnebier, C. R. Ast, M. Burghard and K. Kern. *A Natural Topological Insulator.* Nano Lett. **13** 3 1179 (2013)

[55] S. Das Sarma. *In Search of Majorana.* Nat. Phys. **19** 2 165 (2023)

[56] A. Yazdani, F. von Oppen, B. I. Halperin and A. Yacoby. *Hunting for Majoranas.* Science **380** 6651 eade0850 (2023)

[57] T. Guhr, A. Müller-Groeling and H. A. Weidenmüller. *Random-Matrix Theories in Quantum Physics: Common Concepts.* Phys. Rep. **299** 4 189 (1998)

[58] M. L. Mehta. *Random Matrices.* Academic Press (2004)

[59] C. W. J. Beenakker. *Random-Matrix Theory of Quantum Transport.* Rev. Mod. Phys. **69** 731 (1997)

[60] Y. Alhassid. *The Statistical Theory of Quantum Dots.* Rev. Mod. Phys. **72** 895 (2000)

[61] F. Haake. *Quantum Signatures of Chaos.* Springer-Verlag Berlin Heidelberg (2010)

[62] J. Verbaarschot and T. Wettig. *Random Matrix Theory and Chiral Symmetry in QCD.* Annu. Rev. Nucl. Sci. **50** 1 343 (2000)

[63] E. Shuryak and J. Verbaarschot. *Random Matrix Theory and Spectral Sum Rules for the Dirac Operator in QCD.* Nucl. Phys. A **560** 1 306 (1993)

[64] T. Wettig, A. Schäfer and H. Weidenmüller. *The Chiral Phase Transition and Random Matrix Models.* Nucl. Phys. A **610** 492 (1996). Quark Matter '96

[65] T. Wettig, A. Schäfer and H. Weidenmüller. *The Chiral Phase Transition in a Random Matrix Model with Molecular Correlations.* Phys. Lett. B **367** 1 28 (1996)

[66] A. D. Jackson and J. J. M. Verbaarschot. *Random Matrix Model for Chiral Symmetry Breaking*. Phys. Rev. D **53** 7223 (1996)

[67] J. J. M. Verbaarschot and I. Zahed. *Spectral Density of the QCD Dirac Operator near Zero Virtuality*. Phys. Rev. Lett. **70** 3852 (1993)

[68] T. Guhr, T. Wilke and H. A. Weidenmüller. *Stochastic Field Theory for a Dirac Particle Propagating in Gauge Field Disorder*. Phys. Rev. Lett. **85** 2252 (2000)

[69] B. D. Simons and B. L. Altshuler. *Universal Velocity Correlations in Disordered and Chaotic Systems*. Phys. Rev. Lett. **70** 4063 (1993)

[70] B. D. Simons and B. L. Altshuler. *Universalities in the Spectra of Disordered and Chaotic Systems*. Phys. Rev. B **48** 5422 (1993)

[71] A. Szafer and B. L. Altshuler. *Universal Correlation in the Spectra of Disordered Systems with an Aharonov-Bohm Flux*. Phys. Rev. Lett. **70** 587 (1993)

[72] C. Beenakker and B. Rejaei. *Random-Matrix Theory of Parametric Correlations in the Spectra of Disordered Metals and Chaotic Billiards*. Phys. A: Stat. Mech. Appl. **203** 1 61 (1994)

[73] P. N. Walker and M. Wilkinson. *Universal Fluctuations of Chern Integers*. Phys. Rev. Lett. **74** 4055 (1995)

[74] O. Gat and M. Wilkinson. *Correlations of Quantum Curvature and Variance of Chern Numbers*. SciPost Phys. **10** 149 (2021)

[75] O. Swartzberg, M. Wilkinson and O. Gat. *Universal Chern Number Statistics in Random Matrix Fields*. SciPost Phys. **15** 015 (2023)

[76] P. J. Forrester and M. Krishnapur. *Derivation of an Eigenvalue Probability Density Function Relating to the Poincaré Disk*. J. Phys. A: Math. Theor. **42** 38 385204 (2009)

[77] M. Krishnapur. *From Random Matrices to Random Analytic Functions*. Ann. Probab. **37** 1 314 (2009)

[78] P. J. Forrester and A. Mays. *Pfaffian Point Process for the Gaussian Real Generalised Eigenvalue Problem*. Probab. Theory Relat. Fields **154** 1 1 (2012)

[79] A. Mays. *A Real Quaternion Spherical Ensemble of Random Matrices*. J. Stat. Phys. **153** 1 48 (2013)

[80] K. B. Efetov. *Supersymmetry and Theory of Disordered Metals.* Adv. Phys. **32** 53 (1983)

[81] T. Guhr. *Supersymmetry.* Oxford Handbooks (2015). Chapter 7 in "The Oxford Handbook of Random Matrix Theory"

[82] M. Kieburg and T. Guhr. *Derivation of Determinantal Structures for Random Matrix Ensembles in a New Way.* J. Phys. A: Math. Theor. **43** 075201 (2010)

[83] M. Kieburg and T. Guhr. *A New Approach to Derive Pfaffian Structures for Random Matrix Ensembles.* J. Phys. A: Math. Theor. **43** 135204 (2010)

[84] E. P. Wigner. *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra.* New York: Academic Press (1959)

[85] F. J. Dyson. *The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics.* J. Math. Phys. **3** 6 1199 (1962)

[86] M. R. Zirnbauer. *Particle–Hole Symmetries in Condensed Matter.* J. Math. Phys. **62** 2 021101 (2021)

[87] C. Alexandrou, A. Athenodorou, K. Cichy, A. Dromard, E. Garcia-Ramos, K. Jansen, U. Wenger and F. Zimmermann. *Comparison of Topological Charge Definitions in Lattice QCD.* Eur. Phys. J. C **80** 5 (2020)

[88] J. Verbaarschot. *Spectrum of the QCD Dirac Operator and Chiral Random Matrix Theory.* Phys. Rev. Lett. **72** 2531 (1994)

[89] R. Oppermann. *Anderson Localization Problems in Gapless Superconducting Phases.* Phys. A: Stat. Mech. Appl. **167** 1 301 (1990)

[90] Keith Slevin, Jean-Louis Pichard and Pier A. Mello. *Quantum Interference and the Spin Orbit Interaction in Mesoscopic Normal-Superconducting Junctions.* J. Phys. I France **6** 4 529 (1996)

[91] C. Beenakker. *Andreev Billiards.* In *Quantum Dots: a Doorway to Nanoscale Physics*, pp. 131–174. Springer Berlin Heidelberg, Berlin, Heidelberg (2005)

[92] M. R. Zirnbauer. *Riemannian Symmetric Superspaces and their Origin in Random-Matrix Theory.* J. Math. Phys. **37** 10 4986 (1996)

[93] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press (1978)

[94] J. C. Baez. *The Octonions.* Bull. Amer. Math. Soc. **39** 145 (2002)

[95] G. C. Thiang. *On the K-Theoretic Classification of Topological Phases of Matter.* Ann. Henri Poincaré **17** 4 757 (2016)

[96] G. Abramovici and P. Kalugin. *Clifford Modules and Symmetries of Topological Insulators.* Int. J. Geom. Methods Mod. Phys. **09** 03 1250023 (2012)

[97] E. Prodan. *Virtual Topological Insulators with Real Quantized Physics.* Phys. Rev. B **91** 245104 (2015)

[98] Y. E. Kraus and O. Zilberberg. *Quasiperiodicity and Topology Transcend Dimensions.* Nat. Phys. **12** 7 624 (2016)

[99] M. Stone, C.-K. Chiu and A. Roy. *Symmetries, Dimensions and Topological Insulators: The Mechanism behind the Face of the Bott Clock.* J. Phys. A: Math. Theor. **44** 4 045001 (2010)

[100] L. Fu and C. L. Kane. *Topological Insulators with Inversion Symmetry.* Phys. Rev. B **76** 045302 (2007)

[101] L. Fu. *Topological Crystalline Insulators.* Phys. Rev. Lett. **106** 106802 (2011)

[102] T. L. Hughes, E. Prodan and B. A. Bernevig. *Inversion-Symmetric Topological Insulators.* Phys. Rev. B **83** 245132 (2011)

[103] E. Cornfeld and S. Carmeli. *Tenfold Topology of Crystals: Unified Classification of Crystalline Topological Insulators and Superconductors.* Phys. Rev. Res. **3** 013052 (2021)

[104] T. Guhr. *Statistical Topology - Distribution and Density Correlations of Winding Numbers in Chiral Systems.* Entropy **25** 2 (2023)

[105] A. Y. Kitaev. *Unpaired Majorana Fermions in Quantum Wires.* Phys.-Uspekhi **44** 10S 131 (2001)

[106] C. W. J. Beenakker. *Annihilation of Colliding Bogoliubov Quasiparticles Reveals their Majorana Nature.* Phys. Rev. Lett. **112** 070604 (2014)

[107] J. Wishart. *The Generalised Product Moment Distribution in Samples From a Normal Multivariate Population.* Biometrika **20A** 1-2 32 (1928)

[108] A. M. Lane, R. G. Thomas and E. P. Wigner. *Giant Resonance Interpretation of the Nucleon-Nucleus Interaction.* Phys. Rev. **98** 693 (1955)

[109] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey and S. S. M. Wong. *Random-Matrix Physics: Spectrum and Strength Fluctuations.* Rev. Mod. Phys. **53** 385 (1981)

[110] H. A. Weidenmüller and G. E. Mitchell. *Random Matrices and Chaos in Nuclear Physics: Nuclear Structure*. Rev. Mod. Phys. **81** 539 (2009)

[111] G. Akemann, J. Baik and P. Di Francesco. *The Oxford Handbook of Random Matrix Theory*. Oxford University Press (2015)

[112] H.-J. Stöckmann. *Quantum Chaos: An Introduction*. Cambridge University Press (1999)

[113] F. J. Dyson. *Statistical Theory of the Energy Levels of Complex Systems. I.* J. Math. Phys. **3** 1 140 (1962)

[114] J. Verbaarschot, H. Weidenmüller and M. Zirnbauer. *Grassmann Integration in Stochastic Quantum Physics: The Case of Compound-Nucleus Scattering*. Phys. Rep. **129** 6 367 (1985)

[115] R. Blümel and U. Smilansky. *Random-Matrix Description of Chaotic Scattering: Semiclassical Approach*. Phys. Rev. Lett. **64** 241 (1990)

[116] A. Kuijlaars. *Universality*. Oxford Handbooks (2015). Chapter 6 in "The Oxford Handbook of Random Matrix Theory"

[117] P. Forrester. *Log-Gases and Random Matrices*. Princeton University Press (2010)

[118] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber. *Planar Diagrams*. Commun. Math. Phys. **59** 1 35 (1978)

[119] O. Bohigas, M. J. Giannoni and C. Schmit. *Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws*. Phys. Rev. Lett. **52** 1 (1984)

[120] A. Hönig and D. Wintgen. *Spectral Properties of Strongly Perturbed Coulomb Systems: Fluctuation Properties*. Phys. Rev. A **39** 5642 (1989)

[121] T. Zimmermann, H. Köppel, L. S. Cederbaum, G. Persch and W. Demtröder. *Confirmation of Random-Matrix Fluctuations in Molecular Spectra*. Phys. Rev. Lett. **61** 3 (1988)

[122] M. Oxborrow and C. Ellegaard. *Quantum Chaology in Quartz*. In *Proceedings of the 3rd Experimental Chaos Conference*. Edinburgh, Scotland, UK (1995)

[123] S. Deus, P. M. Koch and L. Sirko. *Statistical Properties of the Eigenfrequency Distribution of Three-Dimensional Microwave Cavities*. Phys. Rev. E **52** 1146 (1995)

[124] O. Legrand, C. Schmit and D. Sornette. *Quantum Chaos Methods Applied to High-Frequency Plate Vibrations.* Europhys. Lett. **18** 2 101 (1992)

[125] J. Ginibre. *Statistical Ensembles of Complex, Quaternion, and Real Matrices.* J. Math. Phys. **6** 3 440 (1965)

[126] S.-S. Byun and P. J. Forrester. *Progress on the Study of the Ginibre Ensembles I: GinUE* (2023). arXiv: 2211.16223

[127] S.-S. Byun and P. J. Forrester. *Progress on the Study of the Ginibre Ensembles II: GinOE and GinSE* (2023). arXiv: 2301.05022

[128] N. Lehmann and H.-J. Sommers. *Eigenvalue Statistics of Random Real Matrices.* Phys. Rev. Lett. **67** 941 (1991)

[129] A. Edelman, E. Kostlan and M. Shub. *How Many Eigenvalues of a Random Matrix are Real?* J. Am. Math. Soc. **7** 1 247 (1994)

[130] A. Edelman. *The Probability that a Random Real Gaussian Matrix has k Real Eigenvalues, Related Distributions, and the Circular Law.* J. Multivar. Anal. **60** 2 203 (1997)

[131] T. Tao, V. Vu and M. Krishnapur. *Random Matrices: Universality of ESDs and the Circular Law.* Ann. Probab. **38** 5 2023 (2010)

[132] J. Fischmann and P. J. Forrester. *One-Component Plasma on a Spherical Annulus and a Random Matrix Ensemble.* J. Stat. Mech. Theory Exp. **2011** 10 P10003 (2011)

[133] J. A. Fischmann. *Eigenvalue Distributions on a Single Ring.* Ph.D. thesis, Queen Mary University of London (2015)

[134] A. Mays and A. Ponsaing. *An Induced Real Quaternion Spherical Ensemble of Random Matrices.* Random Matrices: Theory Appl. **06** 01 1750001 (2017)

[135] J. Fischmann, W. Bruzda, B. A. Khoruzhenko, H.-J. Sommers and K. Życzkowski. *Induced Ginibre Ensemble of Random Matrices and Quantum Operations.* J. Phys. A: Math. Theor. **45** 7 075203 (2012)

[136] A. Pandey and M. L. Mehta. *Gaussian Ensembles of Random Hermitian Matrices Intermediate Between Orthogonal and Unitary Ones.* Commun. Math. Phys. **87** 4 449 (1983)

[137] M. L. Mehta and A. Pandey. *Spacing Distributions for Some Gaussian Ensembles of Hermitian Matrices.* J. Phys. A: Math. Theor. **16** 16 L601 (1983)

[138] T. Guhr and H. Weidenmüller. *Isospin Mixing and Spectral Fluctuation Properties.* Ann. Phys. **199** 2 412 (1990)

[139] C. I. Barbosa, T. Guhr and H. L. Harney. *Impact of Isospin Breaking on the Distribution of Transition Probabilities.* Phys. Rev. E **62** 1936 (2000)

[140] J. V. Michalowicz, J. M. Nichols, F. Bucholtz and C. C. Olson. *An Isserlis' Theorem for Mixed Gaussian Variables: Application to the Auto-Bispectral Density.* J. Stat. Phys. **136** 1 89 (2009)

[141] J. Zakrzewski. *Quantum Chaos and Level Dynamics.* Entropy **25** 3 (2023)

[142] M. Faas, B. D. Simons, X. Zotos and B. L. Altshuler. *Universality in the Spectra of Strongly Correlated Systems.* Phys. Rev. B **48** 5439 (1993)

[143] Y. V. Fyodorov and E. Strahov. *An Exact Formula for General Spectral Correlation Function of Random Hermitian Matrices.* J. Phys. A: Math. Gen. **36** 3203 (2003)

[144] A. Borodin and E. Strahov. *Averages of Characteristic Polynomials in Random Matrix Theory.* Commun. Pure Appl. Math. **59** 2 161 (2006)

[145] G. Akemann, E. Strahov and T. R. Würfel. *Averages of Products and Ratios of Characteristic Polynomials in Polynomial Ensembles.* Ann. Henri Poincare **21** 3973 (2020)

[146] J. R. Ipsen and P. J. Forrester. *Kac-Rice Fixed Point Analysis for Single- and Multi-Layered Complex Systems.* J. Phys. A **51** 474003 (2018)

[147] C. Webb. *The Characteristic Polynomial of a Random Unitary Matrix and Gaussian Multiplicative Chaos - The L^2-Phase.* Electron. J. Probab. **20** 104 (2015)

[148] M. L. Mehta and J.-M. Normand. *Complexity of Random Energy Landscapes, Glass Transition and Absolute Value of Spectral Determinant of Random Matrices.* J. Phys. A **34** 4627 (2001)

[149] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith. *Autocorrelation of Random Matrix Polynomials.* Commun. Math. Phys. **237** 365 (2003)

[150] P. Bourgade, C. P. Hughes, A. Nikeghbali and M. Yor. *The Characteristic Polynomial of a Random Unitary Matrix: A Probabilistic Approach.* Duke Math. J. **145** 1 45 (2008)

[151] S. Eberhard. *The Characteristic Polynomial of a Random Matrix.* Combinatorica **42** 4 491 (2022)

[152] Y. V. Fyodorov. *Complexity of Random Energy Landscapes, Glass Transition and Absolute Value of Spectral Determinant of Random Matrices.* Phys. Rev. Lett. **92** 240601 (2004)

[153] G. Akemann and A. Pottier. *Ratios of Characteristic Polynomials in Complex Matrix Models.* J. Phys. A: Math. Gen. **37** 37 L453 (2004)

[154] S. Coleman and J. Mandula. *All Possible Symmetries of the S Matrix.* Phys. Rev. **159** 1251 (1967)

[155] J. Wess and B. Zumino. *Supergauge Transformations in Four Dimensions.* Nucl. Phys. B **70** 1 39 (1974)

[156] R. Haag, J. T. Łopuszański and M. Sohnius. *All Possible Generators of Supersymmetries of the S-Matrix.* Nucl. Phys. B **88** 2 257 (1975)

[157] K. Efetov. *Supersymmetry in Disorder and Chaos.* Cambridge University Press (1996)

[158] T. Guhr. *Dyson's Correlation Functions and Graded Symmetry.* J. Math. Phys. **32** 336 (1991)

[159] M. Kieburg, J. Grönqvist and T. Guhr. *Arbitrary Rotation Invariant Random Matrix Ensembles and Supersymmetry: Orthogonal and Unitary-Symplectic Case.* J. Phys. A: Math. Theor. **42** 275205 (2009)

[160] P. J. Forrester. *Meet Andréief, Bordeaux 1886, and Andreev, Kharkov 1882–1883.* Random Matrices: Theory Appl. **08** 02 1930001 (2019)

[161] C. Andréief. *Note sur une Relation entre les Intégrales Définies des Produits des Fonctions.* Mém. Soc. Sci. Phys. Nat. Bordeaux **2** 1 (1886)

[162] N. Bruijn, de. *On Some Multiple Integrals Involving Determinants.* J. Indian Math. Soc. New Series **19** 133 (1955)

[163] A. Borodin. *Determinantal Point Processes.* Oxford Handbooks (2015). Chapter 11 in "The Oxford Handbook of Random Matrix Theory"

[164] B.-H. Chen and D.-W. Chiou. *An Elementary Rigorous Proof of Bulk-Boundary Correspondence in the Generalized Su-Schrieffer-Heeger Model.* Phys. Lett. A **384** 7 126168 (2020)

[165] J. Shapiro. *The Bulk-Edge Correspondence in Three Simple Cases*. Rev. Math. Phys. **32** 03 2030003 (2020)

[166] A. Altland, D. Bagrets and A. Kamenev. *Topology Versus Anderson Localization: Nonperturbative Solutions in One Dimension*. Phys. Rev. B **91** 085429 (2015)

[167] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[168] E. L. Basor and P. J. Forrester. *Formulas for the Evaluation of Toeplitz Determinants with Rational Generating Functions*. Math. Nachrichten **170** 5 (1994)

[169] G. Akemann, M. Kieburg and M. J. Phillips. *Skew-Orthogonal Laguerre Polynomials for Chiral Real Asymmetric Random Matrices*. J. Phys. A **43** 375207 (2010)

[170] M. Yahiaoui and M. Kieburg. Private Communication (2023)

[171] T. Guhr. *Arbitrary Unitarily Invariant Random Matrix Ensembles and Supersymmetry*. J. Phys. A: Math. Gen. **39** 13191 (2006)

[172] H.-J. Stöckmann and U. Kuhl. *Microwave Studies of the Spectral Statistics in Chaotic Systems*. J. Phys. A: Math. Theor. **55** 38 383001 (2022)

[173] B. Dietz, A. Heine, A. Richter, O. Bohigas and P. Leboeuf. *Spectral Statistics in an Open Parametric Billiard System*. Phys. Rev. E **73** 035201 (2006)

[174] P. Bertelsen, C. Ellegaard, T. Guhr, M. Oxborrow and K. Schaadt. *Measurement of Parametric Correlations in Spectra of Resonating Quartz Blocks*. Phys. Rev. Lett. **83** 2171 (1999)

[175] M. Ławniczak, A. Nicolau-Kuklińska, O. Hul, P. Masiak, S. Bauch and L. Sirko. *Experimental and Numerical Determination of the Correlation Function of Level Velocities for Microwave Networks Simulating Quantum Graphs*. Phys. Scr. **2013** T153 014041 (2013)

[176] J. Lu, J. Che, X. Zhang and B. Dietz. *Experimental and Numerical Investigation of Parametric Spectral Properties of Quantum Graphs with Unitary or Symplectic Symmetry*. Phys. Rev. E **102** 022309 (2020)

[177] O. Hul, P. Šeba and L. Sirko. *Departure of Some Parameter-Dependent Spectral Statistics of Irregular Quantum Graphs From Random Matrix Theory Predictions.* Phys. Rev. E **79** 066204 (2009)

[178] O. Hul, P. Šeba and L. Sirko. *Investigation of Parameter-Dependent Properties of Quantum Graphs With and Without Time-Reversal Symmetry.* Phys. Scr. **2009** T135 014048 (2009)

[179] A. Rehemanjiang, M. Richter, U. Kuhl and H.-J. Stöckmann. *Microwave Realization of the Chiral Orthogonal, Unitary, and Symplectic Ensembles.* Phys. Rev. Lett. **124** 116801 (2020)

[180] J. R. Ipsen and M. Kieburg. *Weak Commutation Relations and Eigenvalue Statistics for Products of Rectangular Random Matrices.* Phys. Rev. E **89** 032106 (2014)

[181] H.-J. Sommers and W. Wieczorek. *General Eigenvalue Correlations for the Real Ginibre Ensemble.* J. Phys. A: Math. Theor. **41** 40 405003 (2008)

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