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D I S S E R T A T I O N

ZUR ERLANGUNG DES AKADEMISCHEN GRADES EINES
DOKTORS DER NATURWISSENSCHAFTEN (DR. RER. NAT.)

GENERALIZED TRIPLE PRODUCT p-ADIC L-FUNCTIONS, DIAGONAL CLASSES AND RATIONAL POINTS ON ELLIPTIC CURVES

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Datum der mündlichen Prüfung:

15. März 2024

DOI: 10.17185/duepublico/81772
URN: urn:nbn:de:hbz:465-20240402-094904-8

Abstract. In the first part of this thesis (chapters 1 to 5), we generalize and simplify the constructions of [DR14] and [Hsi21] of an unbalanced triple product p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ attached to a triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ of p-adic families of modular forms, allowing more flexibility for the choice of $\boldsymbol{g}$ and $\boldsymbol{h}$.

Assuming that $\boldsymbol{g}$ and $\boldsymbol{h}$ are families of theta series of infinite p-slope, we prove a factorization of (an improvement of) $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in terms of two anticyclotomic p-adic L-functions. As a corollary, when $\boldsymbol{f}$ specializes in weight 2 to the newform attached to an elliptic curve E over $\mathbb{Q}$ with multiplicative reduction at p, we relate Heegner points on E to p-adic partial derivatives of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ evaluated at the critical triple of weights $(2,1,1)$.

In the second part (chapters 6 to 9, we generalize the p-adic explicit reciprocity laws for balanced diagonal classes appearing in [DR17] and [BSV20] to the case of geometric balanced triples (f,g,h) of modular eigenforms where f is ordinary at p, while g and h are supercuspidal at p. This allows to obtain a geometric interpretation of the specializations of the p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the so-called geometric balanced region, when $\boldsymbol{g}$ and $\boldsymbol{h}$ are families of theta series of infinite p-slope.

Zusammenfassung. Im ersten Teil dieser Arbeit (Kapitel 1 bis 5) und vereinfachen wir die Konstruktionen einer unausgewogenen Tripelprodukt p-adischen L-Funktion $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ von [DR14] und [Hsi21], die einem Tripel $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ von p-adischen Familien von Modulformen zugeordnet ist, und ermöglichen dabei mehr Flexibilität bei der Wahl von $\boldsymbol{g}$ und $\boldsymbol{h}$.

Unter der Annahme, dass $\boldsymbol{g}$ und $\boldsymbol{h}$ Familien von Theta-Reihen unendlicher p-Steigung sind, beweisen wir eine Faktorisierung von (einer Verbesserung von) $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ als Produkt von zwei antizyklotomischen p-adischen L-Funktionen. Falls die Spezialisierung von $\boldsymbol{f}$ in Gewicht 2 der Modulform einer elliptischen Kurve E über $\mathbb{Q}$ mit multiplikativer Reduktion bei p entspricht, erhalten wir als Korollar, dass Heegner-Punkte auf E in Beziehung zu p-adischen partiellen Ableitungen von $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in dem kritischen Tripel von Gewichten $(2,1,1)$ stehen.

Im zweiten Teil (Kapitel 6 bis 9) verallgemeinern wir die p-adischen expliziten Reziprozitätsgesetze für ausgewogene diagonale Klassen, die in [DR17] und [BSV20] auftreten, auf den Fall geometrisch ausgewogener Tripel (f,g,h) von Eigenformen, bei denen f bei p gewöhnlich ist, während g und h bei p superkuspidal sind. Dies ermöglicht eine geometrische Interpretation der Spezialisierungen der p-adischen L-Funktion $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ im sogenannten geometrisch ausgewogenen Bereich, wenn $\boldsymbol{g}$ und $\boldsymbol{h}$ Familien von Theta-Reihen unendlicher p-Steigung sind.

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Introduction and statement of the main results

The unbalanced triple product p-adic L-function. Let $p \geq 3$ be a rational prime. We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ together with an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ extending the canonical inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$. All algebraic extensions of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) are viewed inside the corresponding fixed algebraic closures. We extend the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ (normalized so that $|p|_p = 1/p$) to $\bar{\mathbb{Q}}_p$ in the unique possible way. We denote by $\mathbb{C}_p$ the completion of $\bar{\mathbb{Q}}_p$ with respect to this absolute value. It is well-known that $\mathbb{C}_p$ is itself algebraically closed. We also fix an embedding $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ extending the canonical inclusion $\mathbb{Q} \hookrightarrow \mathbb{C}$ and we often omit the embeddings ι_p and ι_∞ from the notation.

Let $L/\mathbb{Q}_p$ be a finite extension and let $\Lambda := \mathcal{O}_L[[1 + p\mathbb{Z}_p]]$ be the corresponding Iwasawa algebra ($\mathcal{O}_L$ being the ring of integers of L). Consider a new, L-rational, Hida family

$$\boldsymbol{f} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{f}) q^n \in \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}, \Lambda)$$

of tame level $N_{\boldsymbol{f}}$ ($p \nmid N_{\boldsymbol{f}}$) and tame character $\chi_{\boldsymbol{f}}$ of conductor dividing $N_{\boldsymbol{f}}$.

Let also

$$\boldsymbol{g} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{g}) q^n \in \mathbb{S}_{\Omega_1}(M, \chi_{\boldsymbol{g}}, R_{\boldsymbol{g}}) \quad \text{and} \quad \boldsymbol{h} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{h}) q^n \in \mathbb{S}_{\Omega_2}(M, \chi_{\boldsymbol{h}}, R_{\boldsymbol{h}})$$

be two generalized normalized Λ-adic eigenforms with $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}} = \omega^{2a}$ for some integer a, where ω denotes the Teichmüller character modulo p and $N_{\boldsymbol{f}} \mid M$.

Our notion of generalized Λ-adic forms takes inspiration from [DR14, Definition 2.16]. For a precise definition and for the explanation of the notation we refer to chapter 1. Here we just mention that we are not imposing any condition on p-slopes and that we are allowing the rings of coefficients $R_{\boldsymbol{g}}$ and $R_{\boldsymbol{h}}$ to be complete local noetherian flat Λ-algebras (not necessarily finite as Λ-algebras), having the same residue field as $\mathcal{O}_L$.

If $\boldsymbol{g}$ and $\boldsymbol{h}$ are Hida families, the works of Darmon-Rotger [DR14] and Hsieh [Hsi21] attach to the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ a so-called $\boldsymbol{f}$-unbalanced square-root triple product p-adic L-function. It arises as an element

$$\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in R_{\boldsymbol{fgh}} := \Lambda \hat{\otimes}_{\mathcal{O}_L} R_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}_L} R_{\boldsymbol{h}}\,,$$

whose square interpolates the central values of the triple product L-functions attached to the specializations of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at $\boldsymbol{f}$-unbalanced triples of weights.

More precisely, given two primitive Hida families $\boldsymbol{g}^\#$ and $\boldsymbol{h}^\#$ of respective tame level $N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$, Hsieh associates to the triple $(\boldsymbol{f}, \boldsymbol{g}^\#, \boldsymbol{h}^\#)$ a preferred choice of *test vectors* $(\boldsymbol{f}^*, \boldsymbol{g}^*, \boldsymbol{h}^*)$ of tame level $N_{\boldsymbol{fgh}} = \mathrm{lcm}(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$ and then performs the construction of the p-adic L-function for this choice of test vectors, which grants some control on the nonvanishing of the local zeta-integrals at primes dividing $N_{\boldsymbol{fgh}}$ appearing in Ichino's formula (cf. [Ich08, theorem 1.1]). In our applications finding the correct test vector will not be a problem, so the reader is invited to think of our generalized families $\boldsymbol{g}$ and $\boldsymbol{h}$ fixed above as test vectors for families of tame level dividing M.

We show in chapter 2 that the costruction of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ can be extended to our more general setting.

PROPOSITION 0.1 (cf. definition 2.2, proposition 2.6 and proposition 2.11): *Assume that the residual Galois representation $\overline{\mathbb{V}}_{\boldsymbol{f}}$ of the big Galois representation $\mathbb{V}_{\boldsymbol{f}}$ attached to $\boldsymbol{f}$ is absolutely irreducible and p-distinguished. Then there is an element $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in R_{\boldsymbol{fgh}}$ such that for every $\boldsymbol{f}$-unbalanced triple of meaningful weights $w = (x,y,z)$, the following formula holds:*

$$(\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w))^2 = \frac{L^*(\boldsymbol{f}_x \times \boldsymbol{g}_y \times \boldsymbol{h}_z, \frac{k+l+m-2}{2})}{\zeta_{\mathbb{Q}}(2)^2 \cdot \Omega_{\boldsymbol{f}_x}^2} \cdot \mathscr{I}_{w,p}^{unb} \cdot \left(\prod_{\ell \mid M} \mathscr{I}_{w,\ell}\right)$$

where:

(i) L^* denotes the completed L-function (including the archimedean local factor);
(ii) $\Omega_{\boldsymbol{f}_x}$ is a suitable period attached to $\boldsymbol{f}_x$, essentially given by its Petersson norm;
(iii) $\mathscr{I}_{w,p}^{unb}$ (resp. $\mathscr{I}_{w,\ell}$) is a suitable normalized local zeta integral at p (resp. at ℓ).

REMARK 0.2: Here we group some observations elucidating the relations between our construction of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and the existing literature on the subject.

(i) As already pointed out, we adapt Hsieh's construction to our setting, following a method that essentially already appears in [Hid93, chapters 7 and 8]. The theory of generalized Λ-adic forms developed in chapter 1 allows us to simplify the construction. In particular, we show that the theory of ordinary parts carries over in this generalized setting (cf. proposition 1.18) and thus we do not need to prove the equivalent of [Hsi21, lemma 3.4].
(ii) The (only) novelty of our p-adic L-function consists in allowing $\boldsymbol{g}$ and $\boldsymbol{h}$ to be generalized families in the sense described above. A natural question to ask is whether in our generalized setting one can find more naturally families which are not captured by Hida-Coleman theory.
(iii) In [Fuk22] the author provides a similar generalization of Hsieh's work to the case in which $\boldsymbol{g}$ and $\boldsymbol{h}$ are not necessarily Hida families. Yet, Fukunaga's notion of *general p-adic families of modular forms* does not allow our generality for the rings of coefficients. Moreover, in the framework of [Fuk22] one cannot view the Fourier coefficients of such families as continuous/analytic function on a suitable weight space in general.
(iv) It should not be too hard to extend our results to the case where $\boldsymbol{f}$ is a Coleman family (i.e., to the finite p-slope case), adapting the techniques of [AI21] (cf. also the recent preprint [GPJ23]).
(v) As already observed, we do not perform a general and careful *level adjustment* as in [Hsi21]. It is clear that one could mimic Hsieh's recipes to achieve more generality in the construction.

▲

Factorization of triple product p-adic L-functions. In the second part of the paper, we discuss some arithmetic applications in the setting the we now describe.

Assume that $p \geq 5$ and let $\boldsymbol{f}$ be a Hida family of tame level $N_{\boldsymbol{f}}$ with trivial tame character. Fix $K/\mathbb{Q}$ a quadratic imaginary field of odd discriminant $-d_K$ and two ray class characters η_1 and η_2 of K, that we can view as valued in L.

The following assumptions are in force:

(A) p is inert in K;

(B) $N_{\boldsymbol{f}}$ is squarefree, coprime to the discriminant of K and with an even number of prime divisors which are inert in K (*Heegner hypothesis*);

(C) η_i has conductor $cp^r\mathcal{O}_K$, with $r \geq 1$ and $c \in \mathbb{Z}_{\geq 1}$, $(c, p \cdot d_K \cdot N_{\boldsymbol{f}}) = 1$, c not divisible by primes inert in K.

(D) η_1 and η_2 are not induced by Dirichlet characters and the central characters of η_1 and η_2 are inverse to each other, so that $\varphi = \eta_1\eta_2$ and $\psi = \eta_1\eta_2^\sigma$ are ring class characters of K (here $\langle\sigma\rangle = \mathrm{Gal}(K/\mathbb{Q})$).

A classical theorem of Hecke and Shimura attaches to the character η_1 (resp. η_2) a cuspidal newform g (resp. h) of weight 1, namely the theta series attached to η_1 (resp. η_2). In chapter 3 we describe how to realize g (resp. h) as the weight 1 specialization of a p-adic family $\boldsymbol{g}$ (resp. $\boldsymbol{h}$) of theta series of tame level d_K. Note that our notion of generalized Λ-adic form is taylored to include families such as $\boldsymbol{g}$ and $\boldsymbol{h}$ as non-trivial examples and that the specializations of $\boldsymbol{g}$ (resp. $\boldsymbol{h}$) will always be supercuspidal at p (hence of infinite p-slope).

After fixing a choice of test vectors $\boldsymbol{g}^*$ (resp. $\boldsymbol{h}^*$) of tame level $N_{\boldsymbol{f}} \cdot d_K \cdot c^2$, in chapter 4 we define an improved version $\mathcal{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}^*, \boldsymbol{h}^*)$, satisfying a simplified interpolation property. This relies on Hsieh's computations of local zeta integrals (and on Fukunaga's generalizations of Hsieh's results in [Fuk22]).

Let H_n denote the ring class field of K of conductor cp^n for every $n \in \mathbb{Z}_{\geq 0}$ and let H_∞ be the union of all the H_n's. Let $\mathscr{G}_\infty := \mathrm{Gal}(H_\infty/K)$. We can identity the maximal $\mathbb{Z}_p$-free quotient Γ^- of $\mathscr{G}_\infty$ with the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K and there is an exact sequence $0 \to \Delta_c \to \mathscr{G}_\infty \to \Gamma^- \to 0$ of abelian groups with Δ_c a finite group and $\Gamma^- \cong \mathbb{Z}_p$. We fix a non-canonical isomorphism $\mathscr{G}_\infty \cong \Delta_c \times \Gamma^-$ once and for all.

Then φ (resp. ψ) factors through $\mathscr{G}_\infty$ and we write it as (φ_t, φ^-) (resp. (ψ_t, ψ^-)) according to the fixed isomorphism $\mathscr{G}_\infty \cong \Delta_c \times \Gamma^-$.

For $k \in \mathbb{Z}_{\geq 2} \cap 2\mathbb{Z}$, let $\mathfrak{X}_{p,k}^{\mathrm{crit}}$ denote the set of continuous characters $\hat{\nu} : \Gamma^- \to \mathbb{C}_p^\times$ such that the associated algebraic Hecke character $\nu : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ has infinity type $(j, -j)$ with $|j| < k/2$.

The main result of chapter 4 is the following factorization theorem for the *anticyclotomic projection* $\mathcal{L}_{p,ac}^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of $\mathcal{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ (cf. definition 4.24). This factorization is a counterpart of [Hsi21, proposition 8.1] (which assumes p split in K) and an upgrade of [BSV22a, theorem 3.1] to the case of Hecke characters with non-trivial p-part.

Theorem 0.3 (cf. theorem 4.25): *In the above setting, it holds:*

$$\mathcal{L}_{p,ac}^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \pm \mathscr{A}_{\boldsymbol{fgh}} \cdot \left(\varphi^-\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f}, \varphi_t)\right) \hat{\otimes} \psi^-\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f}, \psi_t)\right)\right).$$

This equality takes place in the ring

$$\mathcal{R}^- = (R_{\Gamma^-} \hat{\otimes}_\Lambda R_{\Gamma^-})[1/p], \quad \text{where} \quad R_{\Gamma^-} := \Lambda \hat{\otimes}_{\mathcal{O}_L} \mathcal{O}_L[[\Gamma^-]]$$

and the notation is as follows.

(i) $\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\varphi_t) \in R_{\Gamma^-}$ (resp. $\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\psi_t) \in R_{\Gamma^-}$) is (a slight generalizations of) a so-called *big theta element* constructed by Castella-Longo in [CL16], building up on works by Bertolini-Darmon (cf. [BD96],[BD98],[BD07]) and Chida-Hsieh (cf. [CH18]). These p-adic L-functions interpolate (the square root of the algebraic part of) the special values $L(\boldsymbol{f}_k/K,\varphi_t\nu,k/2)$ (resp. $L(\boldsymbol{f}_k/K,\psi_t\nu,k/2)$) for $k \in \mathbb{Z}_{\geq 2}$ even and $\hat{\nu} \in \mathfrak{X}_{p,k}^{\mathrm{crit}}$.

(ii) $\varphi^-(\tau)$ (resp. $\psi^-(\tau)$) for $\tau \in R_{\Gamma^-}$ denotes the image of the element τ via the $\mathcal{O}_L$-linear automorphism of R_{Γ^-} uniquely determined by the identity on Λ and the assignment $[\gamma] \mapsto \varphi^-(\gamma)[\gamma]$ (resp. $[\gamma] \mapsto \psi^-(\gamma)[\gamma]$) on group-like elements on $\mathcal{O}_L[[\Gamma^-]]$.

(iii) The element $\mathscr{A}_{\boldsymbol{fgh}} \in \mathcal{R}^-$ is defined in proposition 4.23 and satisfies the crucial property that, for all $\hat{\nu},\hat{\mu} \in \mathfrak{X}_{p,2}^{\mathrm{crit}}$, $\mathscr{A}_{\boldsymbol{fgh}}(2,\hat{\nu},\hat{\mu}) \neq 0$.

The proof of theorem 0.3 follows from the decomposition arising in our setting at the level of Galois representations (cf. lemma 4.7) and from a careful comparison of the Euler factors at p (or p-adic multipliers) appearing in the interpolation formulae for the various p-adic L-functions. In particular, this requires an explicit computation of the normalized local zeta integral at p (denoted above by $\mathscr{I}_{w,p}^{unb}$), carried out in proposition 4.14.

p-adic formulas for Heegner points. In chapter 5 we apply theorem 0.3 to the study and the construction of Heegner points on elliptic curves. In what follows, we keep the notation as above and we let $E/\mathbb{Q}$ be an elliptic curve with multiplicative reduction at p. Let $f_E \in S_2(\Gamma_0(N_E))$ be the cuspidal newform of level N_E attached to E via modularity. Note that this implies that $N_E = p \cdot N_E^\circ$ with $p \nmid N_E^\circ$. Assume now that $\boldsymbol{f}$ denotes the unique primitive Hida family in $\mathbb{S}^{\mathrm{ord}}(N_E^\circ,\mathbf{1},\Lambda)$ of tame level N_E° and trivial tame character, such that $\boldsymbol{f}_2 = f_E$.

We also impose an extra condition on the characters η_1,η_2 (cf. assumption 5.1):

(E) $\varphi = \eta_1\eta_2$ has conductor prime to p and $\psi = \eta_1\eta_2^\sigma$ has non-trivial anticyclotomic part (i.e., ψ^- is non-trivial).

In particular it follows that φ^- is trivial and that we can identify $\varphi = \varphi_t$ as a character of the finite group Δ_c. Let H_φ denote the abelian extension of K cut out by φ and observe that p splits completely in H_φ.

Upon fixing a primitive Heegner point $P \in E(H_\varphi)\otimes\mathbb{Q}$ and setting $\alpha := a_p(E) \in \{\pm 1\}$, one can define:

$$P_\varphi := \sum_{\sigma\in\mathrm{Gal}(H_\varphi/K)} \varphi(\sigma)^{-1}P^\sigma \in (E(H_\varphi)\otimes\mathbb{Q})^\varphi$$

$$P_{\varphi,\alpha}^{\pm} := P_\varphi \pm \alpha\cdot P_\varphi^{\mathrm{Frob}_\mathfrak{p}} \in E(H_\varphi)\otimes\mathbb{Q}\,.$$

One can show that $P_{\varphi,\alpha}^{\pm}$ does not depend on the choice of prime $\mathfrak{p}$ of H_φ above p. In what follows we fix the choice induced by our fixed embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and we view the points P_φ and $P_{\varphi,\alpha}^{\pm}$ as elements of $E(\mathbb{Q}_{p^2})\otimes\mathbb{Q}$ under such an embedding.

As E has multiplicative reduction at p, we can take advantage of Tate's parametrization of E to define a logarithm $\log_E : E(\mathbb{Q}_{p^2})\otimes\mathbb{Q} \to \mathbb{Q}_{p^2}$ at the level of $\mathbb{Q}_{p^2}$-rational points.

Relying on theorem 0.3 and on previous results by Bertolini-Darmon (cf. [BD98] and [BD07]), we deduce the results summarized in the following statement.

PROPOSITION 0.4 (cf. corollaries 5.4, 5.8 and 5.9): *In the above setting, assume moreover that $L(E/K,\psi,1) \neq 0$. Then the restriction $\mathcal{L}_p^f(\boldsymbol{f},g,h)$ of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to the* line $(k,1,1)$ *vanishes at $k=2$ and*

$$\frac{d}{dk}\mathcal{L}_p^f(\boldsymbol{f},g,h)_{|k=2} = \frac{c_E}{2}\cdot \log_E(P^+_{\varphi,\alpha})$$

for some explicit constant $c_E \in \bar{\mathbb{Q}}_p^\times$.

Similarly, the restriction $\mathcal{L}_{p,ac}^f(f_E,\boldsymbol{gh})$ of $\mathcal{L}_{p,ac}^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to the line $(2,\hat{\nu},\hat{\nu})$ *vanishes at $\hat{\nu}=1$ (the trivial character) and*

$$\frac{d}{d\hat{\nu}}\mathcal{L}_{p,ac}^f(f_E,\boldsymbol{gh})_{|\hat{\nu}=1} = c_E\cdot \log_E(P^-_{\varphi,\alpha})$$

for the same constant c_E.

In particular, if φ is a quadratic (or genus) character, the following are equivalent:

(i)

$$\left(\frac{d}{dk}\mathcal{L}_p^f(\boldsymbol{f},g,h)_{|k=2}\,,\,\frac{d}{d\hat{\nu}}\mathcal{L}_{p,ac}^f(f_E,\boldsymbol{gh})_{|\hat{\nu}=1}\right) \neq (0,0)$$

(ii) The point P_φ is of infinite order.

REMARK 0.5: In [BSV22a] (cf. also [DR22]) the authors study a setting similar to ours, but require the characters η_1 and η_2 to have conductor coprime to p. As a consequence, the order of vanishing of the restriction $\mathcal{L}_p^f(\boldsymbol{f},g,h)$ to the line $(k,1,1)$ of the corresponding triple product p-adic L-function is at least 2. From a factorization in the style of theorem 0.3, they deduce a formula for the second derivative of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at $k=2$ in terms of the product of logarithms of two Heegner points (respectively related to the characters that we denoted φ and ψ). Our construction allows instead to pin down a single Heegner point from the study of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ around the triple of weights $(2,1,1)$. ▲

The explicit reciprocity law. The second part of this thesis is devoted to the proof of a p-adic explicit reciprocity laws for balanced diagonal classes, which extends those appearing in [DR17] and [BSV20] to the setting that we now describe.

Fix a positive integer M coprime to p and a positive integer t such that $Mp^t \geq 5$. We let $Y_t := Y_1(Mp^t)_{\mathbb{Q}}$ denote the *open* modular curve over $\mathbb{Q}$ of level $\Gamma_1(Mp^t)$ and $X_t := X_1(Mp^t)_{\mathbb{Q}}$ denote the *compactified* modular curve of that level. We consider a triple of cuspidal modular forms

$$f\in S_k(Mp^t,\chi_f),\quad g\in S_l(Mp^t,\chi_g),\quad h\in S_m(Mp^t,\chi_h).$$

We assume that the triple (f,g,h) satisfies the following requirements:

(i) For $\xi\in\{f,g,h\}$, we assume that ξ is a normalized eigenform (for the level Mp^t) and that ξ is an eigenform for the U_p operator.

(ii) The triple (f,g,h) is **self-dual**, i.e., $\chi_f\chi_g\chi_h$ is the trivial character modulo Mp^t (in particular $k+l+m$ is an even integer).

(iii) The triple of weights (k,l,m) is **balanced** and **geometric**, i.e., (k,l,m) are the sizes of the edges of a triangle and $\nu\geq 2$ for $\nu\in\{k,l,m\}$.

We fix a finite and large enough extension L of $\mathbb{Q}_p$ (with ring of integers $\mathcal{O}_L$) containing the Fourier coefficients of f,g,h (and a primitive Mp^t-th root of 1) and we write $\mathbf{r} = (k-2,l-2,m-2)\in(\mathbb{Z}_{\geq 0})^3$ and $r=(k+l+m-6)/2$.

In part 3 of [BSV22b], the authors associate to the triple (f,g,h) a Galois cohomology class $\kappa(f,g,h)\in H^1(\mathbb{Q},V(f,g,h))$ where $V(f,g,h)$ is essentially a suitable twist of the tensor product of the (duals of the) Deligne representations attached to f, g, h (more precisely the direct sum of some copies of this tensor product). These cohomology classes are realized as the (f,g,h)-isotypical projection of the pushforward along the diagonal $d_t : Y_t \hookrightarrow Y_t^3$ (p-adic Abel-Jacobi map) of an invariant $\mathtt{Det}_{\mathbf{r}}^{\text{ét}}$ depending only on the triple of weights (k,l,m) (or equivalently, as shown in [BSV22b, section 3.2], the pushforward of a so-called *generalized Gross-Kudla-Schoen diagonal cycle* on the corresponding product of Kuga-Sato varieties, again only depending on the triple of weights (k,l,m)). The construction of $\kappa(f,g,h)$ is recalled in more detail in chapter 6.

The setting in which we will work is characterized by the two following assumptions.

(f_{ord}) There exist a positive integer $M_1 \mid M$ such that $f \in S_k(M_1 p, \chi_f)$ is the ordinary p-stabilization of a newform of level $M_1 \geq 5$

(SC) The forms g and h are supercuspidal at p and lie in the kernel of U_p.

REMARK 0.6: The above assumption (ii) will be relaxed in chapter 6 (cf. assumption 6.11). Similarly, f will be allowed to be a p-ordinary newform of level $M_1 p^s$ for some $s \leq t$ (not necessarily $s = 1$). In this introduction we impose stronger assumptions in order to obtain cleaner statements and to avoid inessential technicalities. ▲

Note that the reciprocity laws proven in [DR17] and [BSV20] always require some finite slope (or ordinarity) assumption on g and h. To the author's knowledge, the case of g and h supercuspidal has not been addressed in the literature so far.

A first complication introduced by assumption (SC) is that one can only hope that the class $\kappa(f,g,h)$, viewed as a local class in $H^1(\mathbb{Q}_p, V(f,g,h))$, becomes crystalline over a non-trivial finite extension of $\mathbb{Q}_p$. Nevertheless, in chapters 7 and 8 we explain (at least when the weight k of f is at least 3) how to view the Bloch-Kato logarithm of $\kappa(f,g,h)$ as a linear functional

$$\log_{\mathrm{BK}}^{fgh}(\kappa(f,g,h)) : \mathrm{Fil}^0(V_{\mathrm{dR}}^*(f,g,h)) \to L$$

Here $V^*(f,g,h)$ arises as the Kummer dual of $V(f,g,h)$ (and by the self-duality assumption on (f,g,h) it is actually isomorphic to $V(f,g,h)$ itself).

One can find a distinguished element

$$\eta_f^{\varphi=a_p} \otimes \omega_g \otimes \omega_h \otimes t_{r+2} \in \mathrm{Fil}^0(V_{\mathrm{dR}}^*(f,g,h))$$

which is defined more precisely in section 7.2. The explicit reciprocity law alluded to in the title of this section describes the value of $\log_{\mathrm{BK}}^{fgh}(\kappa(f,g,h))$ at $\eta_f^{\varphi=a_p} \otimes \omega_g \otimes \omega_h \otimes t_{r+2}$ as follows.

THEOREM 0.7 (cf. theorem 9.2): *Let (f,g,h) be a triple satisfying assumptions (f_{ord}) and (SC) as above. If, moreover, the weight k of f is at least 3, then*

$$\log_{\mathrm{BK}}^{fgh}(\kappa(f,g,h))(\eta_f^{\varphi=a_p} \otimes \omega_g \otimes \omega_h \otimes t_{r+2})$$

is equal to

$$(-1)^{k-2}(r-k+2)! \cdot a_1(e_{\breve{f}}(\mathrm{Tr}_{Mp^t/M_1p^t}(g \times d^{(k-l-m)/2} h)))\,.$$

The notation appearing in the theorem goes as follows. We let $a_1(\xi)$ denote the first Fourier coefficient of the q-expansion at ∞ of a modular form ξ. The modular form $\breve{f}$ is the normalized eigenform which is a scalar multiple of $w_{M_1}(f)$ (where w_{M_1} is a suitably defined Atkin-Lehner operator) and $e_{\breve{f}}$ denotes $\breve{f}$-isotypical projection.

Finally, d denotes Serre's derivative operator, which acts as $q\frac{d}{dq}$ on q-expansions. Note that for a negative integer t (here $(k-l-m)/2 < 0$ since the triple of weights (k,l,m) is balanced), we define d^t as the p-adic limit of the operators $d^{t+(p-1)p^m}$ for $m \to +\infty$. Then d^t is an operator sending p-adic modular forms of weight ν to p-adic modular forms of weight $\nu + 2t$. In particular one can interpret $g \times d^{(k-l-m)/2}h$ as a p-adic modular form of weight k. Since the operator $e_{\breve{f}}$ includes an ordinary projection, Hida's classicality theorem shows that

$$e_{\breve{f}}(\mathrm{Tr}_{Mp^t/M_1p^t}(g \times d^{(k-l-m)/2}h))$$

is given by the q-expansion of a classical modular form of weight k and level M_1p^t.

The proof of theorem 0.7 follows closely the steps of the proof of theorem A in [BSV20] (namely a p-adic reciprocity law for triples of eigenforms of level coprime to the fixed prime p) and, more generally, the recipe for the computation of the *cohomological triple symbol* described in [BLZ16]. However, in our setting we had to face some further difficulties. In particular, since the forms g and h inevitably have non-trivial level at p, we are forced to work over modular curves (or products of such) which do not have good reduction at p and only admit a semistable model over the ring of integers of a finite (typically ramified) extension of $\mathbb{Q}_p$. In this setting cohomology theories such as Hyodo-Kato cohomology and syntomic cohomology - that typically allow this kind of computations - are more complicated to handle.

In chapter 8 we introduce the necessary facts concerning syntomic and finite-polynomial cohomology for semistable varieties and we describe a syntomic version of the p-adic Abel-Jacobi map. The proof of theorem 0.7 is the subject of chapter 9.

Cohomological description of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and further directions. We complete this introduction by underlying the close link between the p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and the explicit reciprocity law of theorem 0.7. Indeed, when $\boldsymbol{g}$ and $\boldsymbol{h}$ are generalized p-adic families whose classical specializations are supercuspidal at p (and p-depleted) - as it happens in the case of families of theta series of infinite p-slope considered in proposition 0.4 above - we have that, essentially by construction,

$$\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) = \gamma_{\boldsymbol{f}_x} \cdot a_1(e_{\breve{\boldsymbol{f}}_x}(\mathrm{Tr}_{Mp^t/M_1p^t}(\boldsymbol{g}_y \times d^{(k-l-m)/2}\boldsymbol{h}_z)))\,.$$

for every balanced triple of meaningful weights $w = (x,y,z)$, where $\gamma_{\boldsymbol{f}_x}$ denotes the specialization at x of the congruence number $\gamma_{\boldsymbol{f}}$ of the Hida family $\boldsymbol{f}$ (see the discussion in section 2.2). Assuming that $(\boldsymbol{f}_x,\boldsymbol{g}_y,\boldsymbol{h}_z)$ satisfies the self-duality condition (ii) described above, we deduce immediately the equality

$$\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) = \frac{(-1)^{k-2}\cdot\gamma_{\boldsymbol{f}_x}}{(r-k+2)!}\cdot \log_{\mathrm{BK}}^{fgh}(\kappa(\boldsymbol{f}_x,\boldsymbol{g}_y,\boldsymbol{h}_z))(\eta_{\boldsymbol{f}_x}^{\varphi=a_p}\otimes\omega_{\boldsymbol{g}_y}\otimes\omega_{\boldsymbol{h}_z}\otimes t_{r+2}) \tag{0.1}$$

Such a result is consistent with the fact that one expects to interpolate p-adically in a meaningful way the objects appearing in the RHS of formula (0.1), in particular the classes $\kappa(\boldsymbol{f}_x,\boldsymbol{g}_y,\boldsymbol{h}_z)$. Following [DR17] and [BSV22b], there should exist a *big* diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ interpolating p-adically the diagonal classes $\kappa(\boldsymbol{f}_x,\boldsymbol{g}_y,\boldsymbol{h}_z)$.

Motivated by results of [BSV22a], one expects to provide a relation between the specialization at $(2,1,1)$ of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ (or a suitable improvement) to suitable p-adic derivatives of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ evaluated at $(2,1,1)$. In the arithmetic setting of proposition 0.4, this result would provide a link between Heegner points and (p-adic limits of) diagonal classes. The author plans to address these sort of questions in the coming future.

Notation and conventions

If F is any field, we denote by G_F the absolute Galois group of F (defined after fixing a suitable separable closure) and we denote F^{ab} the maximal abelian extension of F (inside such a separable closure).

If Γ is a profinite group and R is a topological ring we denote by $R[[\Gamma]]$ the completed group algebra with coefficients in R (with the profinite topology) and we write $[\gamma]$ for $\gamma \in \Gamma$ to denote the corresponding group element in the ring $R[[\Gamma]]$.

We denote by $\mathbb{A}$ the ring of adéles of $\mathbb{Q}$ and if B is a finite separable $\mathbb{Q}$-algebra we let $\mathbb{A}_B := \mathbb{A} \otimes_{\mathbb{Q}} B$ denote the corresponding ring of adéles of B.

For every number field E, we let the Artin reciprocity map

$$\mathrm{rec}_E : \mathbb{A}_E^\times / E^\times \to \mathrm{Gal}(E^{ab}/E)$$

to be *arithmetically normalized*, i.e., if v is a finite place of E the compatible local Artin reciprocity map

$$\mathrm{rec}_{E_v} : E_v^\times \to D_v \cong \mathrm{Gal}(E_v^{ab}/E_v)$$

is the unique map such that for every uniformizer π of E_v it holds that $\mathrm{rec}_{E_v}(\pi)$ acts as the Frobenius morphism on the maximal unramified extension of E_v (inside E_v^{ab}). We write Frob_v to denote an arithmetic Frobenius element at the place v in G_E.

If K is a quadratic imaginary field and $\eta : G_K \to R^\times$ (here R can be any ring) is a character, we let η^σ to denote the conjugate of η, i.e., $\eta^\sigma(\gamma) = \eta(\sigma\gamma\sigma^{-1})$ for $\gamma \in G_K$, where $\sigma \in G_K$ is any element such that $\sigma|_K$ generates $\mathrm{Gal}(K/\mathbb{Q})$ (one possible explicit choice for σ is the complex conjugation induced by the fixed embdedding ι_∞).

If $\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ is an algebraic Hecke character of K, we say that χ has ∞-type (a,b) if for all $z \in \mathbb{C}^\times$ it holds $\chi(z \otimes 1) = z^{-a}\overline{z}^{-b}$.

Given a smooth function f on the upper-half plane $\mathcal{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ and $\omega = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})^+$ (invertible 2×2 matrices with positive determinant) and $k \in \mathbb{Z}$, we set

$$f|_k\omega(\tau) := \det(\omega)^{k/2} \cdot (c\tau + d)^{-k} \cdot f\left(\tfrac{a\tau+b}{c\tau+d}\right) \qquad \tau \in \mathcal{H}$$

If $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup and $k \in \mathbb{Z}_{\geq 1}$, we let $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) be the $\mathbb{C}$-vector space of (holomorphic) modular forms (resp. cusp forms) of weight k and level Γ. For $\Gamma = \Gamma_1(N)$ for some $N \geq 1$ and χ a Dirichlet character modulo N, we let $M_k(N,\chi)$ (resp. $S_k(N,\chi)$) denote the spaces of modular forms (resp. cusp forms) of weight k, level $\Gamma_1(N)$ and nebentypus χ. Unless otherwise specified, we refer to [Miy06] for the all the basic facts concerning the analytic theory of modular forms which are mentioned freely without proof.

We refer to the notes [BC] for the basic facts concerning p-adic Hodge theory and for the definition of Fontaine's period rings $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{st}}$.

Acknowledgements

First of all, I would like to thank heartily my supervisor Massimo Bertolini, for introducing me to the topics of this thesis and for his support during these last years.

I am also grateful to Jan Vonk for accepting to be the external referee for my thesis defense.

Many thanks go to my second supervisor Vytas Paškūnas, to Johannes Sprang and to Fabrizio Andreatta for their interest in my work and their help in the difficult challenge of finding a postdoctoral position for my future as a mathematician.

I also want to thank the DFG Graduiertenkolleg 2553 for funding my doctoral position in Essen.

A special thank goes to the friends and colleagues from the ESAGA group in Essen. I enjoyed the atmosphere at the department and learned a lot from them about mathematics and life.

Finally, I want to thank my family and loved ones for their support, their patience and their love.

CHAPTER 1

Generalized Λ-adic modular forms and ordinary projection

In this chapter, we define a generalized notion of Λ-adic forms and we extend Hida's theory of the ordinary projector to this setting.

1.1. First definitions and examples

Let L be (as in the introduction) a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformizer ϖ_L and and residue field $\mathbb{F}_L := \mathcal{O}_L/\varpi_L\mathcal{O}_L$.

Recall that $\Lambda := \mathcal{O}_L[[1+p\mathbb{Z}_p]]$ is the completed group algebra for the profinite group $1+p\mathbb{Z}_p$. It is a complete local $\mathcal{O}_L$ algebra of Krull dimension 2, with maximal ideal $\mathfrak{m}_\Lambda = (\varpi_L, T)$ and residue field $\mathbb{F}_L$. We fix once and for all the isomorphism

$$\Lambda \cong \mathcal{O}_L[[T]]$$

uniquely determined by sending $[1+p] \mapsto 1+T$ and sometimes we write Λ to denote directly $\mathcal{O}_L[[T]]$ via this identification.

In this section, we will denote by (R,φ) a complete local noetherian Λ-algebra (here we also mean that $\varphi : \Lambda \to R$ is a continuous local homomorphism of $\mathcal{O}_L$-algebras) with maximal ideal $\mathfrak{m}_R$ (also denoted $\mathfrak{m}$ when it is clear from the context) and residue field $R/\mathfrak{m}_R$ isomorphic to $\mathbb{F}_L$. We let $\hat{\mathcal{C}}_\Lambda$ to be the category of such Λ-algebras, with arrows given by (continuous) homomorphisms of Λ-algebras. Similarly we have a category $\hat{\mathcal{C}}_{\mathcal{O}_L}$ and viewing Λ as $\mathcal{O}_L$-algebra in the obvious way, we get a functor $\hat{\mathcal{C}}_\Lambda \to \hat{\mathcal{C}}_{\mathcal{O}_L}$ by pullback.

Sometimes we just write R instead of (R,φ) to simplify the notation, although the structure morphisms are going to play an important role in what follows.

Definition 1.1: For $R \in \hat{\mathcal{C}}_\Lambda$ and any complete subring $\mathcal{O}_L \subseteq A \subseteq \mathbb{C}_p$, we write

$$\mathcal{W}_R(A) := \mathrm{Hom}^{cont}_{\mathcal{O}_L-alg}(R, A)\,,$$

endowed with the topology of uniform convergence on compact sets (which is essentially the p-adic topology). The elements of $\mathcal{W}_R(A)$ will be called (A-valued) **R-weights** (or **R-specializations**).

Remark 1.2: Let L' be a finite extension of L inside $\mathbb{C}_p$ with ring of integers $\mathcal{O}_{L'}$. Then for every $w \in \mathrm{Hom}^{cont}_{\mathcal{O}_L-alg}(R, L')$ it holds $\mathcal{O}_L \subseteq w(R) \subseteq L'$, but $w(R)$ cannot be a field. This forces $w(R) \subseteq \mathcal{O}_{L'}$, so that we can identify $\mathcal{W}_R(L') = \mathcal{W}_R(\mathcal{O}_{L'}) = \mathrm{Hom}_{\hat{\mathcal{C}}_{\mathcal{O}_L}}(R, \mathcal{O}_{L'})$ in our setting. ▲

We fix an embedding $\mathbb{Z}_p \hookrightarrow \mathcal{W}_\Lambda(L)$, given by sending $k \in \mathbb{Z}_p$ to the unique $\mathcal{O}_L$-algebra homomorphism sending $T \mapsto (1+p)^k - 1$.

Definition 1.3: An element $w \in \mathcal{W}_\Lambda(\mathbb{C}_p)$ is an **arithmetic** weight if it is uniquely determined by the assignment $T \mapsto \varepsilon(1+p)\cdot(1+p)^k - 1$, where $k \in \mathbb{Z}_{\geq 1}$ and $\varepsilon : 1+p\mathbb{Z}_p \to \mu_{p^\infty} \subset \mathbb{C}_p^\times$ is a finite order character. In this case we write $w = (k, \varepsilon)$ and we denote the set of arithmetic weights by $\mathcal{W}_\Lambda^{ar}$.

We say that $w = (k, \varepsilon)$ is **classical** if $k \geq 2$ and we denote the set of classical weights by $\mathcal{W}_\Lambda^{cl}$. Clearly $\mathbb{Z}_p \cap \mathcal{W}_\Lambda^{cl} = \mathbb{Z}_{\geq 2} \subset \mathcal{W}_\Lambda^{cl}$ via the embedding $\mathbb{Z}_p \hookrightarrow \mathcal{W}_\Lambda(L)$.

DEFINITION 1.4: Let $(R, \varphi) \in \hat{\mathcal{C}}_\Lambda$. We define the set of classical R-weights as

$$\mathcal{W}_R^{cl} := \{w \in \mathcal{W}_R(\mathbb{C}_p) \mid w \circ \varphi \in \mathcal{W}_\Lambda^{cl}\}$$

and the set of integral classical R-weights as

$$\mathcal{W}_{R,\mathbb{Z}}^{cl} := \{w \in \mathcal{W}_R(\mathbb{C}_p) \mid w \circ \varphi \in \mathbb{Z}_{\geq 2}\}.$$

For every $w \in \mathcal{W}_R^{cl}$ we define $(k_w, \varepsilon_w) := w \circ \varphi$ and, if $w \circ \varphi \in \mathcal{W}_{R,\mathbb{Z}}^{cl}$, we simply write $w \circ \varphi = k_w$. For any subset $V \subset \mathcal{W}_R(\mathbb{C}_p)$ we set $\varphi^*(V) = \{w \circ \varphi \mid w \in V\}$.

DEFINITION 1.5: We say that a subset $\Omega \subseteq \mathcal{W}_{R,\mathbb{Z}}^{cl}$ is **(Λ, R)-admissible** if the following conditions are satisfied:

(i) the closure of $\varphi^*(\Omega)$ inside $\mathbb{Z}_p \subseteq \mathcal{W}_\Lambda(L)$ contains a non-empty open subset of $\mathbb{Z}_p$;
(ii) the intersection of prime ideals $\bigcap_{w \in \Omega} \operatorname{Ker}(w)$ is the trivial ideal in R.

We will need the following result later.

LEMMA 1.6: *Let $R \in \hat{\mathcal{C}}_\Lambda$ and let $\mathcal{S} \subseteq \mathcal{W}_R^{cl}$ be a countable infinite set. Let $\mathcal{B}$ denote the set of ideals in R that can be written as a finite intersection of pairwise different primes of R of the form $\mathfrak{q} = \operatorname{Ker}(w)$ for $w \in \mathcal{S}$. For every $J \in \mathcal{B}$, consider R/J with the quotient topology. Let $I = \bigcap_{w \in \mathcal{S}} \operatorname{Ker}(w)$ and consider R/I with the quotient topology. Then the natural map $R/I \to \varprojlim_{J \in \mathcal{B}} R/J$ induces an isomorphism of topological rings $R/I \cong \varprojlim_{J \in \mathcal{B}} R/J$.*

PROOF. For every $J \in \mathcal{B}$, R/J is a complete noetherian local ring with maximal ideal $\mathfrak{m}_R/I$. Note that the quotient topology and the $\mathfrak{m}_R/J$-adic topology on R/J coincide and that the natural projection $R \to R/J$ is open and continuous (the same applies to R/I).

We claim that such topology on R/J is the same as the ϖ_L-adic topology. It is clear that for every $n \geq 1$ it holds that $(\varpi_L^n R + J)/J \subseteq (\mathfrak{m}_R^n + J)/J$. We are left to show that, for every $n \geq 1$, $R/(J, \varpi_L^n)$ is a quotient of $R/\mathfrak{m}_R^m$ for $m \gg 1$ (in particular it is a finite ring). Indeed writing $J = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ one checks that

$$\sqrt{(J, \varpi_L^n)} = \sqrt{\bigcap_{i=1}^s (\mathfrak{q}_i, \varpi_L^n)} = \bigcap_{i=1}^s \sqrt{(\mathfrak{q}_i, \varpi_L^n)} = \bigcap_{i=1}^s \sqrt{(\mathfrak{q}_i, \varpi_L)} = \mathfrak{m}_R.$$

The first equality follows from $(\bigcap_{i=1}^s (\mathfrak{q}_i, \varpi_L^n))^s \subseteq (J, \varpi_L^n) \subseteq \bigcap_{i=1}^s (\mathfrak{q}_i, \varpi_L^n)$. The second and the third equalities are obvious. The last one follows from the fact that $\sqrt{(\mathfrak{q}_i, \varpi_L)} = \mathfrak{m}_R$ for all $i = 1, \ldots, s$, since $R/\mathfrak{q}_i$ is (algebraically isomorphic to) a finite extension of $\mathcal{O}_L$ inside $\bar{Q}_p$ and $R/\mathfrak{m}_R = \mathbb{F}_L$ by assumption. In particular it follows that $\mathfrak{m}_R^m \subseteq (J, \varpi_L^n)$ for some $m \geq 1$ large enough, proving our claim. Hence we have natural topological isomorphisms for all $J \in \mathcal{B}$

$$R/J \cong \varprojlim_n R/(J, \varpi_L^n).$$

Arguing as above it also follows that a fundamental system of open neighbourhoods of 0 in R/I is given by the open ideals $\{(\varpi_L^n + J)/I)\}_{n \geq 1, J \in \mathcal{B}}$.

This shows that we can realize the natural map $R/I \to \varprojlim_{J\in\mathcal{B}} R/J$ as a chain of topological isomorphisms

$$R/I \cong \varprojlim_{J\in\mathcal{B},\, n} R/(J, \varpi_L^n) \cong \varprojlim_{J\in\mathcal{B}} R/J$$

proving the proposition. □

We are ready to give the key definition of this section.

DEFINITION 1.7: Let $N \in \mathbb{Z}_{\geq 1}$ be an integer with $p \nmid N$, let χ be a Dirichlet character modulo Np^t for some $t \in \mathbb{Z}_{\geq 1}$ with values in $\mathcal{O}_L^\times$. We say that a **generalized Λ-adic form** of tame level N and character χ is a couple $((R,\varphi),\boldsymbol{\xi})$ where:

(i) (R,φ) is an object of $\hat{\mathcal{C}}_\Lambda$, which is also flat as Λ-algebra and an integral domain,
(ii) $\boldsymbol{\xi} \in R[[q]]$ is a formal q-expansion,

such that the set of integral weights

$$\Omega_{\boldsymbol{\xi},\mathbb{Z}} := \{ w \in \mathcal{W}^{cl}_{R,\mathbb{Z}} \mid \boldsymbol{\xi}_w \in M_{k_w}(Np^t, \chi\omega^{2-k_w}, \mathbb{C}_p) \}$$

is (Λ, R)-admissible in the sense of definition 1.5, where $\boldsymbol{\xi}_w$ denotes the q-expansion obtained applying w to the coefficients of $\boldsymbol{\xi}$. We say that $((R,\varphi),\boldsymbol{\xi})$ is **cuspidal** if, moreover, $\boldsymbol{\xi}_w$ is cuspidal for all $w \in \Omega_{\boldsymbol{\xi},\mathbb{Z}}$.

Given a generalized Λ-adic form $((R,\varphi),\boldsymbol{\xi})$ and a (Λ,R)-admissible set of integral classical weights $\Omega \subseteq \Omega_{\boldsymbol{\xi},\mathbb{Z}}$, we say that $((R,\varphi),\boldsymbol{\xi})$ is **Ω-compatible**. Often we shorten the notation and we simply write $\boldsymbol{\xi}$ to denote the Λ-adic form $((R,\varphi),\boldsymbol{\xi})$.

DEFINITION 1.8: Given a generalized Λ-adic form of tame level N and character χ with coefficients in (R,φ), we set

$$\Omega_{\boldsymbol{\xi}} := \{ w \in \mathcal{W}^{ar}_R \mid \boldsymbol{\xi}_w \in \mathbb{M}_{k_w}(Np^{e_w}, \chi\omega^{2-k_w}\varepsilon_w, \mathbb{C}_p) \}$$

where the exponent $e_w \geq 1$ depends on the p-part of χ and on w.

DEFINITION 1.9: We let $\mathbb{M}_\Omega(N,\chi,(R,\varphi))$ (respectively $\mathbb{S}_\Omega(N,\chi,(R,\varphi))$) denote the R-modules of generalized Λ-adic forms (resp. cuspidal generalized Λ-adic forms) of level N and character χ, with coefficients in (R,φ) and Ω-compatible (where Ω is a (Λ,R)-admissible set of classical integral R-weights). When all the inputs are clear from the context (or when it is not necessary to specify them) we simply write $\mathbb{M}$ and $\mathbb{S}$ to denote such R-modules, which we view as submodules of $R[[q]]$ in the obvious way. We endow all such R-modules with the $\mathfrak{m}$-adic topology.

REMARK 1.10: The noetherianity of R implies that $R[[q]]$ is $\mathfrak{m}$-adically separated and complete. ▲

REMARK 1.11: On $\mathbb{M} = \mathbb{M}_\Omega(N,\chi,(R,\varphi))$ and $\mathbb{S} = \mathbb{S}_\Omega(N,\chi,(R,\varphi))$ there is an action of Hecke operators T_ℓ for $\ell \nmid Np$ prime, U_ℓ for $\ell \mid N$ prime and U_p. Those operators can be defined directly on the q-expansions in such a way that the specialization maps are Hecke-equivariant morphisms. More precisely, there is a character $\langle\cdot\rangle_\Lambda : \mathbb{Z}_p^\times \to \Lambda^\times$ given by $\langle s\rangle = [s\cdot\omega^{-1}(s)]$. For $(R,\varphi) \in \hat{\mathcal{C}}_\Lambda$ we then let $\langle\cdot\rangle_R : \mathbb{Z}_p^\times \to R^\times$ to be the composition of $\langle\cdot\rangle_\Lambda$ with φ. Then, for every $\boldsymbol{\xi} = \sum_{n=0} a_n(\boldsymbol{\xi})q^n \in \mathbb{M}$ and for every prime $\ell \neq p$ the Hecke operator T_ℓ acts as follows

$$T_\ell(\boldsymbol{\xi}) = \sum_{n=0}^{+\infty} a_n(T_\ell(\boldsymbol{\xi}))q^n, \quad \text{where} \quad a_n(T_\ell(\boldsymbol{\xi})) = \sum_{d\mid(n,\ell)} \langle d\rangle_R \cdot \chi(d) d^{-1} a_{n\ell/d^2}(\boldsymbol{\xi}),$$

with the convention that $\chi(\ell) = 0$ if $\ell \mid N$. If $\ell \mid N$, we write U_ℓ to denote the T_ℓ operator

We are particularly interested about the U_p operator, whose action on q-expansions is the familiar one:

$$U_p\left(\sum_{n=0}^{+\infty} a_n q^n\right) = \sum_{n=0}^{+\infty} a_{np} q^n .$$

We end this remark recalling the action of the V_p operator on q-expansions, given by

$$V_p\left(\sum_{n=0}^{+\infty} a_n q^n\right) = \sum_{n=0}^{+\infty} a_p q^{np} .$$

This operator will appear later in the paper. Recall that $U_p \circ V_p$ is the identity on q-expansions, while $1 - V_p \circ U_p$ defines the so-called p-depletion operator. ▲

DEFINITION 1.12: Let N, χ, (R,φ) and Ω be as above. The notation $\mathbb{T}_\Omega(N,\chi,(R,\varphi))$ will denote the R-subalgebra of $\mathrm{End}_R(\mathbb{S}_\Omega(N,\chi,(R,\varphi))$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ prime, U_ℓ for $\ell \mid N$ prime and U_p. When all the inputs are clear from the context we simply write $\mathbb{T}$ or $\mathbb{T}_\Omega$ to denote such Hecke algebra.

DEFINITION 1.13: An element $\boldsymbol{\xi} \in \mathbb{M}$ is called a **generalized Λ-adic eigenform** (of given tame level N, character, branch, coefficients) if it is a simultaneous eigenvector for the Hecke operators T_ℓ ($\ell \nmid Np$ prime) and for the Hecke operator U_p.

EXAMPLE 1.14: Let $\boldsymbol{\xi}_1 \in \mathbb{M}_{\Omega_1}(N,\chi_1,R_1)$ and $\boldsymbol{\xi}_2 \in \mathbb{M}_{\Omega_2}(N,\chi_2,R_2)$. Set $R := R_1\hat{\otimes}_{\mathcal{O}_L}R_2$. If $\mathfrak{m}_i \subset R_i$ denotes the respective maximal ideal for $i = 1,2$, then recall that by definition

$$R = \varprojlim_{m,n}\left(\tfrac{R_1}{\mathfrak{m}_1^n} \otimes_{\mathcal{O}_L} \tfrac{R_2}{\mathfrak{m}_2^m}\right).$$

R is then identified with the $\tilde{\mathfrak{m}}$-adic completion of $R_1 \otimes_{\mathcal{O}_L} R_2$ where

$$\tilde{\mathfrak{m}} = \mathfrak{m}_1 \otimes_{\mathcal{O}_L} R_2 + R_1 \otimes_{\mathcal{O}_L} \mathfrak{m}_2 \subset R_1 \otimes_{\mathcal{O}_L} R_2$$

is a maximal ideal of $R_1 \otimes_{\mathcal{O}_L} R_2$ such that $(R_1 \otimes_{\mathcal{O}_L} R_2)/\tilde{\mathfrak{m}} \cong \mathbb{F}_L$ (thanks to our strict conditions on the residue fields of R_1 and R_2).

For every $a \in R_1, b \in R_2$ we let $a\hat{\otimes}b$ denote the image of $a\otimes b \in R_1 \otimes_{\mathcal{O}_L} R_1$ inside R via the natural map. We endow R with the following canonical Λ-algebra structure $\varphi : \Lambda \to R$ uniquely determined by $\mathcal{O}_L$-linearity and the assignment

$$\varphi(T) := \varphi_1(T)\hat{\otimes}1 + 1\hat{\otimes}\varphi_2(T) + \varphi_1(T)\hat{\otimes}\varphi_2(T)$$

where φ_i are the structure morphisms for R_i, $i = 1,2$ (notice that this is well-defined).

We refer to [GD71, section 0.7.7] for the needed properties of completed tensor products. In particular it follows that $R \in \hat{\mathcal{C}}_\Lambda$ and R is an integral domain. Note that R is a flat Λ-algebra via φ. This can be seen easily factoring φ as composition of flat morphisms as

$$\Lambda \to \Lambda\hat{\otimes}_{\mathcal{O}_L}\Lambda \xrightarrow{(\varphi_1,\varphi_2)} R,$$

where the first arrow sends $T \mapsto T\hat{\otimes}1 + 1\hat{\otimes}T + T\hat{\otimes}T$.

By the universal property of completed tensor product it follows that, for every complete subring A of $\mathbb{C}_p$ containing $\mathcal{O}_L$,$\mathcal{W}_R(A) = \mathcal{W}_{R_1}(A) \times \mathcal{W}_{R_2}(A)$ (also as topological spaces) and, by our definition of φ, it also follows that under this identification we get an inclusion

$$\mathcal{W}^{cl}_{R_1,\mathbb{Z}} \times \mathcal{W}^{cl}_{R_2,\mathbb{Z}} \subset \mathcal{W}^{cl}_{R,\mathbb{Z}}$$

such that

$$k_{(w_1,w_2)} = (w_1, w_2) \circ \varphi = (w_1 \circ \varphi_1) + (w_2 \circ \varphi_2) = k_{w_1} + k_{w_2}$$

Let $\Omega = \Omega_1 \times \Omega_2$, viewed as a subset of $\mathcal{W}^{cl}_{R,\mathbb{Z}}$ as above. It is easy to see that Ω is (Λ, R)-admissible.

It follows that $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 \in \mathbb{M}_\Omega(N, \chi_1\chi_2\omega^2, R)$, where as usual if

$$\boldsymbol{\xi}_1 = \sum_{n=0}^{+\infty} a_n q^n, \qquad \boldsymbol{\xi}_1 = \sum_{n=0}^{+\infty} b_n q^n$$

we let

$$\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \sum_{n=0}^{+\infty} \left(\sum_{j=0}^{n} a_j \hat{\otimes} b_{n-j} \right) q^n \in R[[q]].$$

Indeed it is clear that, for all $(w_1, w_2) \in \Omega_1 \times \Omega_2$, it holds

$$(\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2)_{(w_1,w_2)} = \boldsymbol{\xi}_{1,w_1} \times \boldsymbol{\xi}_{2,w_2} \in M_{k_{(w_1,w_2)}}(Np^t, \chi_1\chi_2 \cdot \omega^{4-k_{(w_1,w_2)}}, \mathcal{O}_L)$$

▲

1.2. The ordinary projector

We want to check that also in our generalized setting one can attach to the operator U_p an idempotent operator e^{ord} obtained as

$$e^{\mathrm{ord}} = \lim_{n \to +\infty} U_p^{n!}$$

where the limit is taken in the $\mathfrak{m}$-adic topology. The theory of locally finite operators developed in [Pil20] simplifies our task.

PROPOSITION 1.15: *There exists a unique ordinary projector* $e^{\mathrm{ord}} \in \mathrm{End}_R(\mathbb{M})$ *attached to the Hecke operator* U_p*, such that*

(i) $e^{\mathrm{ord}}(\boldsymbol{\xi}) = \lim_{n\to+\infty} U_p^{n!}(\boldsymbol{\xi})$ *(limit taken in the* $\mathfrak{m}$*-adic topology)*

(ii) e^{ord} *and* U_p *commute and the module* $\mathbb{M}$ *carries a* U_p*-stable decomposition* $\mathbb{M} = e^{\mathrm{ord}}\mathbb{M} \oplus (1 - e^{\mathrm{ord}})\mathbb{M}$ *where* U_p *is bijective on* $e^{\mathrm{ord}}\mathbb{M}$ *and topologically nilpotent on* $(1 - e^{\mathrm{ord}})\mathbb{M}$.

(iii) e^{ord} *commutes with* T_ℓ *for all* $\ell \nmid Np$ *and is compatible with every meaningful arithmetic specialization.*

(iv) the formation of e^{ord} *is compatible with inclusions* $\mathbb{M}_\Omega \subseteq \mathbb{M}_{\Omega'}$ *induced by inclusions* $\Omega' \subseteq \Omega$ *of* (Λ, R)*-admissible sets of classical integral weights.*

The analogue assertions for $\mathbb{S}$ *hold.*

PROOF. We only give the proof for $\mathbb{M}$ (the proof for $\mathbb{S}$ is identical). Thanks to lemmas 2.1.2 and 2.1.3 of [Pil20], in order to define an ordinary projector $e^{\mathrm{ord}} = e^{\mathrm{ord}}(U_p)$ on $\mathbb{M}$, it suffices to check the following facts:

(a) $\mathbb{M}$ is $\mathfrak{m}$-adically complete and separated.

(b) $\mathbb{M}/\mathfrak{m}\mathbb{M}$ is a finite dimensional $R/\mathfrak{m}$-vector space.

It is clear that $\mathbb{M}$ is $\mathfrak{m}$-adically separated, being a submodule of $R[[q]]$ (which is $\mathfrak{m}$-adically complete and separated by remark 1.10). An element $(\bar{\boldsymbol{\xi}}_n)_{n\geq 1} \in \varprojlim_n \mathbb{M}/\mathfrak{m}^n\mathbb{M}$ defines (by left exactness of $\varprojlim_n$) a unique element

$$\boldsymbol{\xi} \in R[[q]] = \varprojlim_n R[[q]]/\mathfrak{m}^n R[[q]]\,.$$

If for every $n \geq 1$ we fix a lift $\boldsymbol{\xi}_n \in \mathbb{M}$ of $\overline{\boldsymbol{\xi}}_n$ we know that for every $w \in \Omega$ it holds $\boldsymbol{\xi}_{n,w} \in M_{k_w}(Np^t, \chi\omega^{2-k_w}, \mathcal{O}_L)$ and by the continuity of the specializations and the fact that $M_{k_w}(Np^t, \chi\omega^{2-k_w}, \mathcal{O}_L)$ is a finite and free $\mathcal{O}_L$-module (thus complete), we deduce that

$$\boldsymbol{\xi}_w = \lim_{n\to+\infty} \boldsymbol{\xi}_{n,w} \in M_{k_w}(Np^t, \chi\omega^{2-k_w}, \mathcal{O}_L),$$

so that indeed $\boldsymbol{\xi} \in \mathbb{M}$ and (a) follows.

For every $w \in \Omega$, $\mathbb{M}/\mathrm{Ker}(w)\mathbb{M}$ is a submodule of $M_{k_w}(Np^t, \chi, \mathcal{O}_L)$. This shows that $\mathbb{M}/\mathrm{Ker}(w)\mathbb{M}$ is a finite free $\mathcal{O}_L$-module surjecting onto $\mathbb{M}/\mathfrak{m}\mathbb{M}$, which is thus a finite dimensional $R/\mathfrak{m}$-vector space, proving (b). □

We are then led to the following definition:

Definition 1.16: We say that a generalized eigenform $\boldsymbol{\xi} \in \mathbb{M}_\Omega(N, \chi, R)$ (respectively $\boldsymbol{\xi} \in \mathbb{S}_\Omega(N, \chi, R)$) is a **generalized Hida family** (resp. a cuspidal generalized Hida family) if $e^{\mathrm{ord}}(\boldsymbol{\xi}) = \boldsymbol{\xi}$.

We define the R-modules $\mathbb{M}^{\mathrm{ord}}_\Omega(N, \chi, R) := e^{\mathrm{ord}}(\mathbb{M}_\Omega(N, \chi, R))$ (resp. in the cuspidal case $\mathbb{S}^{\mathrm{ord}}_\Omega(N, \chi, R) := e^{\mathrm{ord}}(\mathbb{S}_\Omega(N, \chi, R)))$ to be the submodules of $\mathbb{M}_\Omega(N, \chi, R)$ (resp. $\mathbb{S}_\Omega(N, \chi, R))$ of ordinary generalized Λ-adic forms. When the inputs are clear from the context we simply write $\mathbb{M}^{\mathrm{ord}}$ or $\mathbb{M}^{\mathrm{ord}}_\Omega$ (resp. $\mathbb{S}^{\mathrm{ord}}$ or $\mathbb{S}^{\mathrm{ord}}_\Omega$).

We let $\mathbb{T}^{\mathrm{ord}}_\Omega(N, \chi, R)$ to denote the R-subalgebra of $\mathrm{End}_R(\mathbb{S}^{\mathrm{ord}}_\Omega(N, \chi, R))$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ prime, U_ℓ for $\ell \mid N$ prime and U_p. When all the inputs are clear from the context, we simply write $\mathbb{T}^{\mathrm{ord}}$ or $\mathbb{T}^{\mathrm{ord}}_\Omega$ to denote such Hecke algebra.

Remark 1.17: Equivalently one could define generalized Hida families asking that every meaningful classical specialization is a p-ordinary eigenform in the usual sense. ▲

The following proposition shows that generalized Hida families are actually essentially the same as classical Hida families.

Proposition 1.18: *For any $R \in \hat{\mathcal{C}}_\Lambda$ which is Λ-flat and an integral domain and any (Λ, R)-admissible set of classical integral weights Ω, the R-modules $\mathbb{M}^{\mathrm{ord}}_\Omega(N, \chi, R)$ (resp. $\mathbb{S}^{\mathrm{ord}}_\Omega(N, \chi, R)$) are free R-modules of finite rank. Moreover (assuming that χ takes values in $\mathcal{O}_L^\times$), there are canonical isomorphisms*

$$\mathbb{M}^{\mathrm{ord}}(N, \chi, \Lambda) \otimes_\Lambda R \xrightarrow{\cong} \mathbb{M}^{\mathrm{ord}}_\Omega(N, \chi, R), \quad \mathbb{S}^{\mathrm{ord}}(N, \chi, \Lambda) \otimes_\Lambda R \xrightarrow{\cong} \mathbb{S}^{\mathrm{ord}}_\Omega(N, \chi, R).$$

Proof. We will omit the proof of the cuspidal case because the proof does not change. In this proof, we write $\mathbb{M}^{\mathrm{ord}}_\Lambda = \mathbb{M}^{\mathrm{ord}}(N, \chi, \Lambda)$ and $\mathbb{M}^{\mathrm{ord}}_R = \mathbb{M}^{\mathrm{ord}}(N, \chi, R)$ to simplify the notation. In order to prove that $\mathbb{M}^{\mathrm{ord}}_R$ is R-free of finite rank we adapt Wiles's proof for classical Hida theory (cf. [Hid93, section 7.3]). We recall the main ideas for the convenience of the reader. Let M be a finite free R-submodule of $\mathbb{M}^{\mathrm{ord}}_R$, with R-basis $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r\}$. Write

$$\boldsymbol{\xi}_i = \sum_{n=0}^{+\infty} a_n(\boldsymbol{\xi}_i) q^n$$

for $i = 1, \ldots, r$. Then there is a sequence of integers $0 \leq n_1 < n_2 < \cdots < n_r$ such that the $r \times r$ matrix $\left(a_{n_j}(\boldsymbol{\xi}_i)\right)_{i,j,=1,\ldots r}$ has non-zero determinant $d \in R$. Since by assumption $\cap_{w\in\Omega}\mathrm{Ker}(w) = (0)$, we deduce that there exists $w \in \Omega$ such that $d \neq 0 \mod \mathrm{Ker}(w)$, so that the specializations $\{\boldsymbol{\xi}_{1,w}, \ldots, \boldsymbol{\xi}_{r,w}\}$ would still be $\mathcal{O}_L[w]$-linearly independent in $M^{\mathrm{ord}}_{k_w}(Np^t, \chi\omega^{2-k_w}, \mathcal{O}_L[w])$. It is well-known (and established by Hida) that the rank of

$M_{k_w}^{\mathrm{ord}}(Np^t, \chi\omega^{2-k_w}, \mathcal{O}_L[w])$ is independent on w if $k_w \geq 3$. Hence there exists $r^* \in \mathbb{Z}_{\geq 0}$ such that $\mathbb{M}_R^{\mathrm{ord}}$ admits finite free R-submodules of rank r^*, but not of rank r^*+1. Assume now that M is such a finite free R-submodule of $\mathbb{M}_R^{\mathrm{ord}}$ of rank r^*. One checks easily that, with the notation as above, $d \cdot \mathbb{M}_R^{\mathrm{ord}} \subseteq M$. Hence, by the noetherianity of R, it follows that $\mathbb{M}_R^{\mathrm{ord}}$ is finitely generated as R-module. In particular it is a compact R-module (equivalently a profinite R-module). The topological Nakayama's lemma (cf. [Hid12, lemma 3.2.6] for instance) implies that $\mathbb{M}_R^{\mathrm{ord}}$ is generated by $r := \dim_{\mathbb{F}_L}(\mathbb{M}_R^{\mathrm{ord}}/\mathfrak{m}_R\mathbb{M}_R^{\mathrm{ord}})$ elements (a lift of an $\mathbb{F}_L$-basis of $\mathbb{M}_R^{\mathrm{ord}}/\mathfrak{m}_R\mathbb{M}_R^{\mathrm{ord}}$).

Now note that (using the flatness of R over Λ) $\mathbb{M}_\Lambda^{\mathrm{ord}} \otimes_\Lambda R$ can be naturally seen as an R-free submodule of $\mathbb{M}_R^{\mathrm{ord}}$ of R-rank r. We define the quotient

$$Q := \frac{\mathbb{M}_R^{\mathrm{ord}}}{\mathbb{M}_\Lambda^{\mathrm{ord}} \otimes_\Lambda R}$$

and we claim that $Q = 0$. This would conclude the proof of the proposition, since it is well-known that $\mathbb{M}_\Lambda^{\mathrm{ord}}$ is a free Λ-module of rank r^*.

Picking $w \in \Omega$ with $k_w \geq 3$, one has $Q \otimes_R R/\mathrm{Ker}(w) = 0$, since both $\mathbb{M}_\Lambda^{\mathrm{ord}} \otimes_\Lambda R$ and $\mathbb{M}_R^{\mathrm{ord}}$ project onto $M_{k_w}^{\mathrm{ord}}(Np^t, \chi\omega^{2-k_w}, \mathcal{O}_L[w])$ via w (to see this one uses the trick of twisting with a suitable family of Eisenstein series, cf. [Hid93, pag. 199]). Hence *a fortiori* $Q \otimes_R R/\mathfrak{m} = 0$ and, since also Q is a profinite R-module, it follows again from the topological Nakayama's lemma that $Q = 0$. □

REMARK 1.19: Proposition 1.18 shows that the R-modules $\mathbb{M}_\Omega^{\mathrm{ord}}(N, \chi, R)$ (respectively $\mathbb{S}_\Omega^{\mathrm{ord}}(N, \chi, R)$) actually does not depend on Ω, so that in the ordinary setting we will omit the (Λ, R) admissible set of weights from the notation from now on. ▲

CHAPTER 2

The unbalanced triple product p-adic L-function

In this chapter we carry out the construction of a generalized unbalanced triple product p-adic L-function, closely following the method appearing in [Hsi21]. Having defined the ordinary projector e^{ord} in wider generality and having proved proposition 1.18, the construction simplifies remarkably. For instance we do not need the equivalent of [Hsi21, lemma 3.4].

2.1. Remarks on the Atkin-Lehner involution

Recall that given $\xi \in S_k(M,\chi)$, one has an Atkin-Lehner involution $w_M : S_k(M,\chi) \to S_k(M,\chi^{-1})$ given by $w_M(\xi) = \xi|_k\left(\begin{smallmatrix} 0 & -1 \\ M & 0 \end{smallmatrix}\right)$. For our constructions we will need a Λ-adic version of the Atkin-Lehner involution. This entails considering more general Atkin-Lehner operators.

Let N be a positive integer coprime to p and $t \in \mathbb{Z}_{\geq 1}$. If d is an integer coprime to Np, we write $\langle d\rangle = \langle a; b\rangle$ for the diamond operator corresponding to $d \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$, where the convention is that $d \equiv a \mod N$ and $d \equiv b \mod p^t$.

For $\xi \in S_k(Np^t,\chi)$ we define the Atkin-Lehner operator w_N on ξ as

$$w_N(\xi) := \langle 1; N\rangle(\xi|_k\,\omega_N) \qquad \omega_N := \omega_{N,p^t} := \begin{pmatrix} N & -1 \\ Np^tc & Nd \end{pmatrix}, \tag{2.1}$$

where we require that $\det(\omega_N) = N$. Write $\chi = \chi_{p^t}\chi_N$ in a unique way for χ_{p^t} a character modulo p^t and χ_N a character modulo N.

Then (cf. [AL78, §1], where they define an operator which is the *inverse* of ours) w_N is an operator

$$w_N : S_k(Np^t,\chi) \to S_k(Np^t,\overline{\chi_N}\chi_{p^t})$$

such that for all primes $\ell \nmid N$ it holds that $w_N \circ T_\ell = \chi_N(\ell)(T_\ell \circ w_N)$ and (when $t \geq 1$) that $w_N \circ U_p = \chi_N(p)(U_p \circ w_N)$. One can also check that if $s > r \geq 0$, the action of w_N on $S_k(\Gamma_1(Np^r))$ is the restriction of the action of w_N on $S_k(\Gamma_1(Np^s))$, by our choice of the matrices ω_{N,p^t}, so that it makes sense to drop p^t from the notation.

In particular, if $\xi \in S_k(Np^t,\chi)$ is a normalized newform, then $w_N(\xi) = \lambda_N(\xi)\cdot\breve{\xi}$ where $\lambda_N(\xi)$ is an algebraic number of complex absolute value 1 (a so called *pseudo-eigenvalue*) and $\breve{\xi}$ is a normalized newform such that if

$$\xi = \sum_{n=1}^{+\infty} a_n q^n \qquad \breve{\xi} = \sum_{n=1}^{+\infty} b_n q^n$$

then

$$b_\ell = \begin{cases} \overline{\chi_N}(\ell)a_\ell & \text{if } \ell \nmid N \\ \chi_{p^t}(\ell)a_\ell & \text{if } \ell \mid N. \end{cases}$$

Moreover if $\xi \in S_k(N,\chi)$ is a p-ordinary newform with $k \geq 2$ and $\xi_\alpha \in S_k(Np,\chi)$ is its ordinary p-stabilisation, then $\lambda_N(\xi)^{-1} \cdot w_N(\xi_\alpha)$ coincides with the ordinary p-stabilisation of the newform $\breve{\xi}$, so we will write

$$\breve{\xi}_\alpha := \lambda_N(\xi)^{-1} \cdot w_N(\xi_\alpha).$$

Note that in this case it is well-known that $\breve{\xi}$ is the modular form obtained applying complex conjugation to the Fourier coefficients of ξ.

Now let

$$\boldsymbol{\xi} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{\xi}) q^n \in \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{\xi}}, \chi_{\boldsymbol{\xi}}, \Lambda_{\boldsymbol{\xi}})$$

be a classical new Hida family of tame level $N_{\boldsymbol{\xi}}$ with character $\chi_{\boldsymbol{\xi}}$ of conductor dividing $N_{\boldsymbol{\xi}} \cdot p$, i.e., the classical specializations at integral weights of $\boldsymbol{\xi}$ are either newforms of level $N_{\boldsymbol{\xi}} \cdot p$ or ordinary p-stabilizations of newforms of level $N_{\boldsymbol{\xi}}$. Here $\Lambda_{\boldsymbol{\xi}}$ is a finite flat Λ-algebra in $\hat{\mathcal{C}}_\Lambda$ and we assume that L contains a primitive $N_{\boldsymbol{\xi}}$-th root of unity. We require that $\boldsymbol{\xi}$ is normalized (i.e., $a_1(\boldsymbol{\xi}) = 1$). Note that we can omit the admissible set of integral classical weights in the notation here, since classical Hida theory shows that for classical Hida families it always happens $\Omega_{\boldsymbol{\xi},\mathbb{Z}} = \mathcal{W}^{cl}_{\Lambda_{\boldsymbol{\xi}},\mathbb{Z}}$.

Following [Hsi21, section 3.3], there is a unique new Hida family $\breve{\boldsymbol{\xi}} \in \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{\xi}}, \chi_{\boldsymbol{\xi}}^{-1}, \Lambda_{\boldsymbol{\xi}})$ which is characterised by the fact that, for all $x \in \mathcal{W}^{cl}_{\Lambda_{\boldsymbol{\xi}}}$

$$(\breve{\boldsymbol{\xi}})_x = (\breve{\boldsymbol{\xi}}_x) = \lambda_N(\boldsymbol{\xi}_x)^{-1} \cdot w_N(\boldsymbol{\xi}_x).$$

2.2. Construction of the p-adic L-function

We fix a Hida family $\boldsymbol{f}$

$$\boldsymbol{f} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{f}) q^n \in \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}, \Lambda_{\boldsymbol{f}})$$

primitive of tame level $N_{\boldsymbol{f}}$, tame character $\chi_{\boldsymbol{f}}$ of conductor dividing $N_{\boldsymbol{f}} \cdot p$.

We also let

$$\boldsymbol{g} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{g}) q^n \in \mathbb{S}_{\Omega_1}(M, \chi_{\boldsymbol{g}}, R_{\boldsymbol{g}}) \quad \text{and} \quad \boldsymbol{h} = \sum_{n=1}^{+\infty} a_n(\boldsymbol{h}) q^n \in \mathbb{S}_{\Omega_2}(M, \chi_{\boldsymbol{h}}, R_{\boldsymbol{h}})$$

be two generalized normalized Λ-adic eigenforms with $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}} = \omega^{2a}$ for some integer a, where as usual ω denotes the $\bmod p$ Teichmüller character. Assume that $N_{\boldsymbol{f}} \mid M$. In the language of [Hsi21], we are implicitly thinking about $\boldsymbol{g}$ and $\boldsymbol{h}$ as *test vectors* for families of tame level dividing M. We also assume that L contains a primitive M-th root of unity from now on.

For $s \in \mathbb{Z}_p^\times$ and $R \in \hat{\mathcal{C}}_\Lambda$ we always write $\langle s \rangle_R^{1/2} = \langle \tilde{s} \rangle_R$ where $\tilde{s}$ is the unique root of the polynomial $X^2 - s \cdot \omega^{-1}(s)$ lying in $1 + p\mathbb{Z}_p$. We also write $\langle s \rangle_R^{-1/2} = \langle s^{-1} \rangle_R^{1/2}$ (note that this does not create ambiguity).

Let $R_{\boldsymbol{fgh}} := \Lambda_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}_L} R_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}_L} R_{\boldsymbol{h}}$ and set

$$\Theta_{\boldsymbol{fgh}} := \Theta : \mathbb{Z}_p^\times \to R_{\boldsymbol{fgh}}^\times \qquad \Theta(s) := \omega^{-a-1}(s) \cdot \langle s \rangle_{\Lambda_{\boldsymbol{f}}}^{1/2} \hat{\otimes} \langle s \rangle_{R_{\boldsymbol{g}}}^{-1/2} \hat{\otimes} \langle s \rangle_{R_{\boldsymbol{h}}}^{-1/2}. \tag{2.2}$$

View $R_{\boldsymbol{fgh}}$ as Λ-algebra via $[s] \mapsto \langle s \rangle_{\Lambda_{\boldsymbol{f}}} \hat{\otimes} 1 \hat{\otimes} 1$ for $s \in 1 + p\mathbb{Z}_p$.

We define a Θ-twist operator on q-expansions given by

$$|_{\Theta} : R_{\boldsymbol{fgh}}[[q]] \to R_{\boldsymbol{fgh}}[[q]] \qquad Z = \sum_{n=0}^{+\infty} a_n q^n \mapsto Z|_{\Theta} = \sum_{p \nmid n} \Theta(n) a_n q^n . \tag{2.3}$$

Now let $\Xi := \boldsymbol{g} \times (\boldsymbol{h}|_{\Theta})$ and define

$$\Omega^0_{\boldsymbol{fgh}} := \{w = (x,y,z) \in \Omega_{\boldsymbol{f}} \times \Omega_{\boldsymbol{g}} \times \Omega_{\boldsymbol{h}} \mid k_x = k_y + k_z,\ k_z \geq 2\}$$

One checks that for $w = (x,y,z) \in \Omega^0_{\boldsymbol{fgh}}$ it holds

$$(\boldsymbol{h}|_{\Theta})_w = \boldsymbol{h}_z \otimes \psi_w \in S_{k_z}(Mp^?, \chi_{\boldsymbol{h}} \omega^{2-k_z} \varepsilon_z \psi_w^2, \mathbb{C}_p)\,,$$

where (for $(n,p) = 1$) we set

$$\psi_w(n) = \omega^{-a-1}(n) \cdot \varepsilon_x(n\omega^{-1}(n))^{1/2} \cdot \varepsilon_y(n\omega^{-1}(n))^{-1/2} \cdot \varepsilon_z(n\omega^{-1}(n))^{-1/2} .$$

It follows that

$$\Xi_w = \boldsymbol{g}_y \times (\boldsymbol{h}_z \otimes \psi_w) \in S_{k_x}(Mp^?, \chi_{\boldsymbol{f}}^{-1} \omega^{2-k_x} \varepsilon_x, \mathbb{C}_p)\,.$$

Notice that by our definition of Λ-algebra structure on $R_{\boldsymbol{fgh}}$, for $w = (x,y,z) \in \Omega_{\boldsymbol{f}} \times \Omega_{\boldsymbol{g}} \times \Omega_{\boldsymbol{h}}$ it holds $k_w = k_x$. It follows easily that $\Omega^0_{\boldsymbol{fgh}}$ is a $(\Lambda, R_{\boldsymbol{fgh}})$-admissible set of classical integral weights.

Looking at integral classical weights specializations $w \in \Omega^0_{\boldsymbol{fgh}} \cap (\Omega_{\boldsymbol{f},\mathbb{Z}} \times \Omega_{\boldsymbol{g},\mathbb{Z}} \times \Omega_{\boldsymbol{h},\mathbb{Z}})$ it is easy to deduce that, according to our definitions, it holds

$$\Xi \in \mathbb{S}(M, \chi_{\boldsymbol{f}}^{-1}, R_{\boldsymbol{fgh}})\,.$$

Thanks to proposition 1.15, we can thus consider the ordinary projection

$$\Xi^{\mathrm{ord}} := e(\Xi) \in \mathbb{S}^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}}^{-1}, R_{\boldsymbol{fgh}}) = \mathbb{S}^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}}^{-1}, \Lambda_{\boldsymbol{f}}) \otimes_{\Lambda_{\boldsymbol{f}}} R_{\boldsymbol{fgh}}\,,$$

where the last equality follows easily from proposition 1.18 and we emphasize (again) that the structure of $\Lambda_{\boldsymbol{f}}$-algebra on $R_{\boldsymbol{fgh}}$ is given by $a \mapsto a \hat{\otimes} 1 \hat{\otimes} 1$ for $a \in \Lambda_{\boldsymbol{f}}$.

We can proceed as in [Hsi21] to define the triple product p-adic L-function. We will need an assumption on our $\boldsymbol{f}$.

ASSUMPTION 2.1 (CR): The residual Galois representation $\bar{\mathbb{V}}_{\boldsymbol{f}}$ of the big Galois representation $\mathbb{V}_{\boldsymbol{f}}$ attached to $\boldsymbol{f}$ is absolutely irreducible and p-distinguished.

Let $\mathrm{Tr}_{M/N_{\boldsymbol{f}}} : \mathbb{S}^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}}^{-1}, \Lambda_{\boldsymbol{f}}) \to \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}^{-1}, \Lambda_{\boldsymbol{f}})$ be the usual trace map.

By the primitiveness of $\boldsymbol{f}$ and assumption 2.1, it follows that the so-called congruence ideal $C(\boldsymbol{f}) \subset \Lambda_{\boldsymbol{f}}$ of $\boldsymbol{f}$ is principal, generated by a non-zero element $\eta_{\boldsymbol{f}}$, called the congruence number for $\boldsymbol{f}$ (it is unique up to units). One can prove that $\breve{\boldsymbol{f}}$ is primitive as well and that $\boldsymbol{f}$ and $\breve{\boldsymbol{f}}$ have the same congruence number.

Since $\boldsymbol{f}$ is primitive, we also get an idempotent operator $e_{\boldsymbol{f}}$ lying in $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}_{\boldsymbol{f}}} \otimes_{\Lambda_{\boldsymbol{f}}} \mathrm{Frac}(\Lambda_{\boldsymbol{f}})$, where $\mathfrak{m}_{\boldsymbol{f}}$ the maximal ideal of $\mathbb{T}^{\mathrm{ord}} := \mathbb{T}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}, \Lambda_{\boldsymbol{f}})$ corresponding to $\boldsymbol{f}$ and $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}_{\boldsymbol{f}}}$ is the localization of $\mathbb{T}^{\mathrm{ord}}$ at such maximal ideal. Morally, $e_{\boldsymbol{f}}$ plays the role of a projection to the $\boldsymbol{f}$-Hecke eigenspace. A similar discussion applies to $\breve{\boldsymbol{f}}$.

Then we can let $e_{\breve{\boldsymbol{f}}}$ act on $\mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}^{-1}, \Lambda_{\boldsymbol{f}}) \otimes_{\Lambda_{\boldsymbol{f}}} \mathrm{Frac}(\Lambda_{\boldsymbol{f}})$ and, by definition of congruence number, one has that $\eta_{\boldsymbol{f}} \cdot e_{\breve{\boldsymbol{f}}}(\boldsymbol{\xi}) \in \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}^{-1}, \Lambda_{\boldsymbol{f}})$ for all $\boldsymbol{\xi} \in \mathbb{S}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}^{-1}, \Lambda_{\boldsymbol{f}})$.

We refer to [Hsi21, section 3.3] and to [Col20, section 3.5] for a more detailed discussion concerning congruence numbers and idempotents attached to primitive Hida families.

DEFINITION 2.2: With the above notation, the generalized $\boldsymbol{f}$-unbalanced triple product p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ attached to the triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is defined as

$$\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) := a_1\left(\eta_{\boldsymbol{f}}\cdot e_{\breve{\boldsymbol{f}}}\left(\mathrm{Tr}_{M/N_{\boldsymbol{f}}}\left(\Xi^{\mathrm{ord}}\right)\right)\right) \in R_{\boldsymbol{fgh}}.$$

REMARK 2.3: We view $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ as a function on $\mathcal{W}_{\Lambda_{\boldsymbol{f}}}(\mathbb{C}_p)\times\mathcal{W}_{R_{\boldsymbol{g}}}(\mathbb{C}_p)\times\mathcal{W}_{R_{\boldsymbol{h}}}(\mathbb{C}_p)$. In particular for $w=(x,y,z)\in\Omega_{\boldsymbol{fgh}}$ one gets that the evaluation of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w is given by

$$\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) = \eta_{\boldsymbol{f}_x}\cdot a_1(e_{\breve{f}}(\mathrm{Tr}_{M/N_{\boldsymbol{f}}}(\Xi_w^{\mathrm{ord}}))).$$

Recall that $(\boldsymbol{h}|_\Theta)_w$ is in the image of the $m=(k_x-k_y-k_z)/2$-th power of Serre's derivative operator $d=q\frac{d}{dq}$ acting on p-adic modular forms of weight k_z, where if m is negative one defines the m-th power of d as a p-adic limit. We can conclude that $(\boldsymbol{h}|_\Theta)_w$ is the q-expansion of a p-adic modular form of weight k_x-k_y and tame level M. Hence by Hida's classicality theorem for ordinary forms, we deduce that

$$\Xi_w^{\mathrm{ord}} = e(\boldsymbol{g}_y\times d^m(\boldsymbol{h}_z\otimes\psi_w))\in S_{k_x}^{\mathrm{ord}}(Mp^t,\chi_{\boldsymbol{f}}^{-1}\omega^{2-k_x}\varepsilon_x,\mathbb{C}_p)$$

where $\psi_w=\omega^{-a-1-m}\varepsilon_x^{1/2}\varepsilon_y^{-1/2}\varepsilon_z^{-1/2}$ and $t\geq 1$ depends on w, $\chi_{\boldsymbol{g}}$ and $\chi_{\boldsymbol{h}}$ (and it is always chosen to be large enough). ▲

2.3. The p-adic L-function and Petersson products

DEFINITION 2.4: We set our conventions for the Petersson inner product on the spaces $S_k(N,\chi)$ of complex modular forms of level N and character χ to be

$$\langle\xi_1,\xi_2\rangle_{Pet} := \frac{1}{\mathrm{Vol}(\mathcal{H}/\Gamma_0(N))}\int_{\mathcal{D}_0(N)}\xi_1(\tau)\overline{\xi_2(\tau)}v^k\frac{du\,dv}{v^2}$$

for ξ_1, $\xi_2\in S_k(N,\chi)$ where we write $\tau=u+iv\in\mathcal{H}$ (the upper half-plane) and $\mathcal{D}_0(N)$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$.

REMARK 2.5: Note that by the above definition our Petersson inner product is linear in the first variable and conjugate linear in the second variable. Moreover, it is normalized so that it does not depend on the level N considered. ▲

PROPOSITION 2.6: *Pick $w=(x,y,z)\in\Omega_{\boldsymbol{fgh}}$ and set*

$$C := C_{N_{\boldsymbol{f}},M} := [\Gamma_0(N_{\boldsymbol{f}}):\Gamma_0(M)] = \tfrac{M}{N_{\boldsymbol{f}}}\cdot\prod_{\substack{\ell\mid M\\ \ell\nmid N_{\boldsymbol{f}}}}\left(1+\tfrac{1}{\ell}\right)\in\mathbb{Z}_{\geq 1}.$$

Write $f=\boldsymbol{f}_x$, $\breve{f}=(\breve{\boldsymbol{f}})_x$, $\Xi=\Xi_w^{\mathrm{ord}}\in S_{k_x}^{\mathrm{ord}}(Mp^t,\chi_{\boldsymbol{f}}^{-1}\omega^{2-k_x}\varepsilon_x,\mathbb{C}_p)$ to simplify the notation, so that $\breve{f}=\lambda_N(f)^{-1}\cdot w_N(f)$ as before. Assume that $t\geq 1$ is large enough (in particular larger that the p-order of the exact level of f). Then the evaluation of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w can be described as follows, depending on two mutually exclusive cases.

(A) Assume f is a newform in $S_k(N_{\boldsymbol{f}}p^s,\chi_{\boldsymbol{f}}\omega^{2-k}\varepsilon,L)$. Then:

$$\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) = \frac{\eta_f\cdot C\cdot p^{k(t-s)}}{a_p(\breve{f})^{t-s}}\cdot\frac{\langle\Xi,V_p^{t-s}(\breve{f})\rangle_{Pet}}{\|f\|_{Pet}^2}. \tag{2.4}$$

(B) Assume that f is the ordinary p-stabilization of a newform $f^\circ \in S_k(N_{\boldsymbol{f}}, \chi^\circ_{\boldsymbol{f}}, L)$ (where $\chi^\circ_{\boldsymbol{f}}$ is the $N_{\boldsymbol{f}}$-part of $\chi_{\boldsymbol{f}}$). Set $f^\# := w_{N_{\boldsymbol{f}}p}(\breve{f}^\rho)$, where $\breve{f}^\rho$ is obtained from $\breve{f}$ applying complex conjugation to the Fourier coefficients. Then

$$\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) = \frac{\eta_f \cdot C \cdot p^{k(t-1)}}{a_p(\breve{f})^{t-1}} \cdot \frac{\langle \Xi, V_p^{t-1}(f^\#)\rangle_{Pet}}{\langle \breve{f}, f^\#\rangle_{Pet}}. \tag{2.5}$$

PROOF. This follows directly from [Hid85, proposition 4.5] (note that our conventions for the Petersson inner product differ from those of Hida, so we have to adjust the result accordingly). □

REMARK 2.7: In case (B) of the above proposition (with the notation as above), assume that $t = 1$ and that we can write

$$e_{\breve{f}}(\mathrm{Tr}_{Mp^t/N_{\boldsymbol{f}}p^t}(\Xi)) = \xi - \beta_k \chi^\circ_{\boldsymbol{f}}(p)^{-1} \cdot V_p(\xi)$$

for some $\xi \in S_k(N_{\boldsymbol{f}}, (\chi^\circ_{\boldsymbol{f}})^{-1})$. Then one can check that

$$\frac{\langle \Xi, w_N(f^\#)\rangle_{Pet}}{\langle f, f^\#\rangle_{Pet}} = \frac{\langle \xi, w_N(f^\circ)\rangle_{Pet}}{\langle f^\circ, f^\circ\rangle_{Pet}} = \frac{\langle \Xi, w_N(f)\rangle_{Pet}}{\langle f, f\rangle_{Pet}}.$$

In particular, assume that $\boldsymbol{g}$ and $\boldsymbol{h}$ are classical Hida families of tame level $N_{\boldsymbol{f}}$ with $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \mathbf{1}$ and $w = (k,l,m) \in \Omega_{\boldsymbol{fgh}}$ is a triple of classical integral weights such that $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ are ordinary p-stabilizations of forms $g^\circ \in S_l(N_{\boldsymbol{f}}, \chi^\circ_{\boldsymbol{g}})$ and $h^\circ \in S_m(N_{\boldsymbol{f}}, \chi^\circ_{\boldsymbol{h}})$ respectively. Then the hypothesis made on Ξ is verified (cf. [BSV20, section 4.4]) and we recover the p-adic periods which are denoted by $I_p(f^\circ, h^\circ, g^\circ)$ in [BSV20, section 1.1] and by $\mathscr{L}_p^f(f_\alpha, h_\alpha, g_\alpha)$ in [BSV22b, section 3.1]. Note that we have switched the role of $\boldsymbol{g}$ and $\boldsymbol{h}$ in our construction, compared to what happens in [BSV20] and [BSV22b]. ▲

2.4. Comparison with the complex L-values

In this section we compare the values of our square root triple product L-function with the central values of the Garret-Rankin triple product L-function associated to a triple of modular forms. Most of the material contained in this section is derived from [Hsi21, section 3].

In this section we fix positive integers N, M coprime to p such that $N \mid M$. We consider a triple of cuspidal modular forms

$$f = \sum_{n=1}^{+\infty} a_n(f)q^n, \qquad g = \sum_{n=1}^{+\infty} a_n(g)q^n, \qquad h = \sum_{n=1}^{+\infty} a_n(h)q^n$$

with

$$f \in S_k(Np^{e_1}, \chi_f\omega^{2-k}\varepsilon_1),\ g \in S_l(Mp^{e_2}, \chi_g\omega^{2-l}\varepsilon_2),\ h \in S_m(Mp^{e_3}, \chi_h\omega^{2-m}\varepsilon_3),$$

where $e_i \geq 1$ and ε_i are Dirichlet characters of p-power order for $i = 1,2,3$, while χ_f (resp. χ_ξ for $\xi \in \{g,h\}$) is a Dirichlet character defined modulo Np (resp. Mp).

ASSUMPTION 2.8: (i) f,g,h are normalized eigenforms, i.e., for $\xi \in \{f,g,h\}$ it holds $a_1(\xi) = 1$ and ξ is an eigenform for all the Hecke operators T_ℓ for all primes $\ell \nmid N$ (resp. $\ell \nmid M$ if $\xi \in \{g,h\}$). We also assume that f,g,h are eigenforms for the U_p operator.

(ii) The triple (f,g,h) is tamely self-dual, i.e., $\chi_f \cdot \chi_g \cdot \chi_h = \omega^{2a}$ for some integer a.

(iii) The triple of weights (k, l, m) is arithmetic and f-unbalanced, i.e., $\nu \geq 1$ for $\nu \in \{k, l, m\}$, $k + l + m$ is even and $k \geq l + m$.
(iv) The form f is a p-stabilized ordinary newform, i.e., either the ordinary p-stabilization of a p-ordinary newform f° of level N or an ordinary newform of level Np^{e_1}.
(v) The tame level N is a squarefree integer.

When f is the ordinary p-stabilization of a newform f° of level N, we write α_f, β_f for the roots of the Hecke polynomial at p for f° and we always assume that $|\alpha_f|_p = 1$.

Let $r = (k + l + m)/2$ and let $\chi_{\mathbb{A}}$ be the adèlization of the Dirichlet character

$$\chi := \omega^{a-r}(\varepsilon_1\varepsilon_2\varepsilon_3)^{1/2}.$$

Let $\pi_1 = \pi_f \otimes \chi_{\mathbb{A}}$, $\pi_2 = \pi_g$, $\pi_3 = \pi_h$, where for $\xi \in \{f, g, h\}$ we denote by π_ξ the irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to ξ as in [Bum97, chapter 3].

It is well-known that there is a decomposition $\pi_\xi = \bigotimes_{\ell \leq \infty} \pi_{\xi,\ell}$ into local representations.

Finally let $\Pi := \pi_1 \times \pi_2 \times \pi_3$ denote the corresponding automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ where $E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is the split cubic étale algebra over $\mathbb{Q}$. Thanks to our choices one can verify that the central character of Π is trivial, so that Π is isomorphic to its contragradient.

We let $L(\Pi, s)$ denote the triple product complex L-function attached to Π (cf. for instance [PR87]). It is known (cf. for instance the summary in [Ike92, pagg. 225-228] and the references therein) that $L(\Pi, s)$ is given by a suitable Euler product converging for $\mathrm{Re}(s) \gg 0$ and that it admits analytic continuation to an entire function with a functional equation of the form

$$L^*(\Pi, s) = \varepsilon(\Pi, s) \cdot L^*(\Pi, 1 - s)$$

Here $L^*(\Pi, s) = L(\Pi, s) \cdot L(\Pi, s)_\infty$ with

$$L(\Pi, s)_\infty = \Gamma_{\mathbb{C}}(s + r - 3/2) \cdot \Gamma_{\mathbb{C}}(s - r + k + 1/2) \cdot \Gamma_{\mathbb{C}}(s + r - l - 1/2) \cdot \Gamma_{\mathbb{C}}(s + r - m - 1/2)$$

and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ ($\Gamma(\cdot)$ being Euler's gamma function). This explicit description of the archimedean L-factor is proven in [Ike98].

Moreover, $\varepsilon(\Pi, s) = \prod_{\ell \leq \infty} \varepsilon_\ell(\Pi, s)$ is an invertible function satisfying the property that $\varepsilon_\ell(\Pi, 1/2) \in \{\pm 1\}$ and $\varepsilon_\ell(\Pi, 1/2) = 1$ for almost all ℓ. In particular, it is known that:
(a) $\varepsilon_\infty(\Pi, 1/2) = 1$ in our case (this depends on the fact that the triple of weights (k, l, m) is unbalanced);
(b) $\varepsilon_\ell(\Pi, 1/2) = 1$ if $\ell \nmid pM$.

We are then led to the following further assumption.

ASSUMPTION 2.9: In what follows we assume that $\varepsilon_\ell(\Pi) = 1$ for all $\ell \mid M$.

DEFINITION 2.10: If π is an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for a rational prime ℓ and $\mathcal{V}_\pi$ is a realization of π, we let $c(\pi)$ denote the smallest integer (which exists, by smoothness) such that $\mathcal{V}_\pi^{\mathcal{U}_1(\ell^{c(\pi)})} \neq 0$, where for all $m \in \mathbb{Z}_{\geq 0}$ we set

$$\mathcal{U}_1(\ell^m) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_\ell) \mid \mathrm{ord}_\ell(c) \geq m,\ \mathrm{ord}_\ell(d - 1) \geq m \right\}.$$

Now we connect this discussion to the triple product p-adic L-function, assuming that $f = \boldsymbol{f}_x$, $g = \boldsymbol{g}_y$, $h = \boldsymbol{h}_z$ are suitable specializations of families of the types considered in section 2.2 with $w = (x, y, z) \in \Omega_{\boldsymbol{fgh}}$ so that $k_x = k, k_y = l, k_z = m$ (with $k \geq l + m$ as we have assumed before). Write Π_w for the corresponding automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$.

Following Harris-Kudla ([HK91]) and Ichino ([Ich08]), Hsieh proved in [Hsi21] the following fact.

PROPOSITION 2.11: *Under assumptions 2.8 and 2.9, the following formula holds:*

$$(\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w))^2 = \frac{L^*(\Pi_w,1/2)}{\zeta_{\mathbb{Q}}(2)^2\cdot\Omega_f^2}\cdot\mathscr{I}^{unb}_{\Pi_w,p}\cdot\left(\prod_{\ell\mid M}\mathscr{I}_{\Pi_w,\ell}\right) \tag{2.6}$$

where

(i)

$$\Omega_f := 2^{k+1}\cdot\|f^\circ\|^2_{Pet}\cdot\mathcal{E}_p(f,\mathrm{Ad})\cdot\eta_f^{-1}\cdot[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N_{\boldsymbol{f}})] \tag{2.7}$$

with

$$f^\circ := \begin{cases} f & \text{in case (A) of prop. 2.6}\\ \text{the newform of level } N_{\boldsymbol{f}} \text{ associated to } f & \text{in case (B) of prop. 2.6}\end{cases}$$

and

$$\mathcal{E}_p(f,\mathrm{Ad}) = a_p(f)^{-c(\pi_{f,p})}\cdot p^{c(\pi_{f,p})(k/2-1)}\cdot\varepsilon(\pi_{f,p},1/2)\cdot\sigma_f, \tag{2.8}$$

where

$$\sigma_f := \begin{cases} 1 & \text{in case (A) of prop. 2.6}\\ \left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right) & \text{in case (B) of prop. 2.6, equiv. if } c(\pi_{f,p})=0\end{cases}$$

(ii) $\mathscr{I}^{unb}_{\Pi_w,p}$ *is the normalized local zeta integral defined as [Hsi21, equation 3.28];*
(iii) $\mathscr{I}_{\Pi_w,\ell}$ *is the normalized local zeta integral defined as in [Hsi21, equation 3.29];*
(iv) $\zeta_{\mathbb{Q}}(\cdot) = \pi^{-1}\zeta(\cdot)$ *where* $\zeta(\cdot)$ *is the usual Riemann zeta function, so that* $\zeta_{\mathbb{Q}}(2) = \pi/6$.

PROOF. This is essentially a restatement of proposition 3.10 and corollary 3.13 in [Hsi21]. Note that our normalization for the Petersson inner product is different from Hsieh's. This explains the appearance of the factor $\zeta_{\mathbb{Q}}(2)^2$ in our formula and the slight changes in the definition of the period Ω_f. ☐

REMARK 2.12: One can compute directly that, if we are in case (B) of proposition 2.6, it holds that

$$\|f^\circ\|^2_{Pet}\cdot\sigma_f\cdot\frac{(-1)^k\cdot\alpha_f\cdot\chi^\circ_{\boldsymbol{f}}(p)^{-1}}{\lambda_N(f)\cdot p^{k/2}\cdot(1+1/p)} = \langle\breve{f},f^\#\rangle_{Pet}.$$

We refer [Col20, proposition 5.4.1] for a very similar computation, where the form denoted $h^\natural$ there should be thought as a constant multiple of our $f^\#$. This explains the appearance of the factor σ_f and allows an even more direct comparison (in the $\boldsymbol{f}$-unbalanced region) between the formula given by equation 2.6 and the formulas appearing in the statement of proposition 2.6. ▲

CHAPTER 3

Families of theta series of infinite p-slope

3.1. Setup for the interpolation

We fix an odd prime p and we let K be an imaginary quadratic field where p is inert. Denote by $N_{K/\mathbb{Q}}$ the norm morphism on fractional ideals in K. Let $-d_K$ be the discriminant of K (so that $p \nmid d_K$) and let ε_K denote the central character of K, i.e., more explicitly

$$\varepsilon_K(n) = \left(\frac{-d_K}{n}\right) \qquad \text{if} \quad (n, d_K) = 1$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol.

Definition 3.1: For $\mathfrak{a} \subset \mathcal{O}_K$ an integral ideal in $\mathcal{O}_K$, we let $I_K(\mathfrak{a})$ denote the group of fractional ideals of K prime to $\mathfrak{a}$ and we set

$$P_K(\mathfrak{a}) := \{(\alpha) \in I_K(\mathfrak{a}) \mid \alpha \equiv 1 \mod {}^{\times}\mathfrak{a}\}, \qquad Cl_K(\mathfrak{a}) := I_K(\mathfrak{a})/P_K(\mathfrak{a}).$$

The group $Cl_K(\mathfrak{a})$ is the so-called ray class group modulo $\mathfrak{a}$.

Remark 3.2: It is well-known that $Cl_K(\mathfrak{a})$ is a finite group. ▲

We fix a finite order character $\eta : G_K \to \bar{\mathbb{Q}}^{\times}$ with conductor $\mathfrak{c}$ (a non-trivial proper integral ideal in $\mathcal{O}_K$). Via class field theory we will freely view η as a *ray class character* $\eta : Cl_K(\mathfrak{c}) \to \bar{\mathbb{Q}}^{\times}$ or a finite order character $\eta : \mathbb{A}_K^{\times}/K^{\times} \to \bar{\mathbb{Q}}^{\times}$ (note the slight abuse of notation here). Moreover, we assume that η is not the restriction of a character of $G_{\mathbb{Q}}$.

Denote by $\eta_{|\mathbb{Q}}$ the Dirichlet character defined modulo $N_{K/\mathbb{Q}}(\mathfrak{c})$ and given by

$$\eta_{|\mathbb{Q}}(n) := \eta((n)) \qquad \text{for} \quad (n, N_{K/\mathbb{Q}}(\mathfrak{c})) = 1$$

It is then a classical theorem of Hecke and Shimura (cf. [Miy06], theorem 4.8.2) that the q-expansion (where as usual $q = \exp(2\pi i\tau)$ for $\tau \in \mathcal{H}$)

$$g(\tau) := \theta_\eta(\tau) := \sum_{(\mathfrak{a},\mathfrak{c})=1} \eta(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \tag{3.1}$$

defines a cuspidal modular form of weight 1 (the theta series attached to the character η). Here the sum runs over the integral ideals in $\mathcal{O}_K$ prime to $\mathfrak{c}$.

More precisely, $g \in S_1(d_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}), \varepsilon_K \cdot \eta_{|\mathbb{Q}})$ and since we assume that η is of exact conductor $\mathfrak{c}$, g is also a newform of level $d_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c})$. From now on, we set $N_g := d_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c})$ and $\chi_g := \varepsilon_K \cdot \eta_{|\mathbb{Q}}$.

The Fourier coefficients of g generate a finite extension of $\mathbb{Q}$. We can thus view g as a modular form whose q-expansion at ∞ has coefficients in a finite extension L of $\mathbb{Q}_p$ (via the embedding ι_p), i.e., $g \in S_1(N_g, \chi_g, L)$. As in the previous sections, we assume that L is large enough. In particular, here we assume that L contains the completion of K inside $\mathbb{C}_p$ (which we will denote by K_p with ring of integers $\mathcal{O}_{K,p}$).

We would like to find a p-adic family of modular forms - all with complex multiplication by K - of varying weights (in the sense of Hida-Coleman) having g (or a slight modification of g) as a specialization in weight 1. We will see that this can actually be done explicitly.

REMARK 3.3: Since the fixed prime p is inert in K, $p^r\mathcal{O}_K \mid \mathfrak{c}$ if and only if $p^{2r} \mid N_{K/\mathbb{Q}}(\mathfrak{c})$. Hence we should distinguish two cases:

(a) $(p\mathcal{O}_K, \mathfrak{c}) = 1$, or equivalently $p \nmid N_g$
(b) $\mathrm{ord}_p(N_g) = 2r$ for some $r \in \mathbb{Z}_{\geq 1}$

In both cases it holds that $a_p(g) = 0$, or equivalently that $T_p(g) = 0$ in case (a) (resp. $U_p(g) = 0$ in case (b)). This is usually described as g having infinite p-slope. ▲

REMARK 3.4: While case (a) can be reinterpreted in the realm of Hida theory (as in this case g admits one or two ordinary p-stabilizations), case (b) is instead more genuinely a problem in *infinite slope*. This dichotomy is also reflected in the fact that the local component at p of the automorphic representation associated with g is a principal series in case (a) and a supercuspidal representation in case (b). ▲

ASSUMPTION 3.5: From now on in this section we will always assume that $p\mathcal{O}_K \mid \mathfrak{c}$ and we will write $\mathfrak{c} = \mathfrak{c}_0 \cdot p^r\mathcal{O}_K$ with $\mathfrak{c}_0$ coprime to $p\mathcal{O}_K$ and $r \geq 1$.

REMARK 3.6: When p splits in K one can explicitly write down families of theta series, specializing to (p-stabilizations) of modular forms of the shape described in (3.1). See, for instance, [BDV22, section 4.2] for a discussion about this construction, which - again - is well-understood within Hida theory. ▲

In what follows, we try to adapt such construction to our setting. Notice that $K_p/\mathbb{Q}_p$ is the unique degree two unramified extension of $\mathbb{Q}_p$ inside our fixed algebraic closure $\bar{\mathbb{Q}}_p$, so we will identify $K_p = \mathbb{Q}_{p^2}$ (with ring of integers $\mathbb{Z}_{p^2}$). Moreover we have a decomposition

$$\mathbb{Z}_{p^2}^\times = \mu_{p^2-1} \times (1 + p\mathbb{Z}_{p^2})$$

induced by the Teichmüller lift. Note that $1+p\mathbb{Z}_{p^2}$ does not contain p-power roots of unity.

Let G_p be the subgroup of the idèlic class group $C_K := \mathbb{A}_K^\times/K^\times$ over K defined by

$$G_p := K^\times \cdot (\mathbb{C}^\times \cdot \mu_{p^2-1} \cdot \prod_{\mathfrak{l} \neq p\mathcal{O}_K} \mathcal{O}_\mathfrak{l}^\times)/K^\times .$$

Set moreover $I_{K,\infty} := K^\times \cdot (\mathbb{C}^\times \cdot \prod_\mathfrak{l} \mathcal{O}_\mathfrak{l}^\times)/K^\times$ and let $\mathrm{Pic}(\mathcal{O}_K)$ denote the classical ideal class group of K.

The snake lemma applied to the following diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & G_p \arrow[r] \arrow[d, hook] & C_K \arrow[r] \arrow[d, equal] & C_K/G_p \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & I_{K,\infty} \arrow[r] & C_K \arrow[r] & \mathrm{Pic}(\mathcal{O}_K) \arrow[r] & 0 \end{tikzcd}$$

identifies $1 + p\mathbb{Z}_{p^2} \cong \mathrm{Ker}(C_K/G_p \twoheadrightarrow \mathrm{Pic}(\mathcal{O}_K))$. We can thus consider the diagram

$$\begin{tikzcd} 1 \arrow[r] & 1 + p\mathbb{Z}_{p^2} \arrow[r] \arrow[d, hook, "\iota"] & C_K/G_p \arrow[r] \arrow[dl, dashed] & \mathrm{Pic}(\mathcal{O}_K) \arrow[r] & 1 \\ & \bar{\mathbb{Q}}_p^\times & & & \end{tikzcd}$$

where the horizontal row is an exact sequence of abelian groups, ι is given by $\iota(u) = u^{-1}$ and the dashed arrow is any (continuous) extension of ι to the quotient $\mathbb{A}_K^\times/G_p$, obtained using the divisibility of $\bar{\mathbb{Q}}_p^\times$. Finally we let $\lambda^{(p)}$ to be the following composition:

$$\lambda^{(p)} : \mathbb{A}_K^\times/K^\times \twoheadrightarrow \mathbb{A}_K^\times/G_p \dashrightarrow \bar{\mathbb{Q}}_p^\times.$$

We associate to $\lambda^{(p)}$ an algebraic Hecke character of K of ∞-type $(1,0)$ as follows:

$$\lambda^{(\infty)} : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times \qquad x = [(x_\nu)_\nu] \mapsto \Big(\iota_\infty \circ \iota_p^{-1}(\lambda^{(p)}(x)\cdot x_p)\Big)\cdot x_\infty^{-1}.$$

Finally, writing $\lambda^{(\infty)} = \otimes_v \lambda_v^{(\infty)}$ one gets a character at the level of fractional ideals

$$\lambda : I_K(p\mathcal{O}_K) \to \bar{\mathbb{Q}}^\times \qquad \mathfrak{a} \mapsto \prod_{\mathfrak{l}\mid\mathfrak{a}} \lambda_{\mathfrak{l}}^{(\infty)}(\varpi_{\mathfrak{l}})^{\mathrm{ord}_{\mathfrak{l}}(\mathfrak{a})},$$

where $\varpi_{\mathfrak{l}}$ is a uniformizer at $\mathfrak{l}$. One can verify that $\lambda((\alpha)) = \alpha$ whenever $\alpha \equiv 1 \mod {}^\times p\mathcal{O}_K$.

DEFINITION 3.7: In the above setting, we will say that $\lambda^{(p)}$ is the **p-adic avatar** of λ and that $\lambda^{(\infty)}$ is the **complex avatar** of λ.

REMARK 3.8: We will also look at $\lambda^{(p)}$ as a p-adic Galois character $\lambda^{(p)} : G_K \to \bar{\mathbb{Q}}_p^\times$ via global class field theory. ▲

Up to enlarging L, we can assume that $\lambda(\mathfrak{a}) \in L$ for all $\mathfrak{a} \in I_K(p\mathcal{O}_K)$ and $\eta(\mathfrak{a}) \in L$ for all $\mathfrak{a} \in I_K(\mathfrak{c})$.

DEFINITION 3.9: We let $\langle\cdot\rangle : \mathcal{O}_L^\times \to \mathcal{O}_L^\times$ to be the projection onto the free units (note that now $\mathcal{O}_L^\times$ might contain p-power roots of unity). By slight abuse of notation we will write $\langle\lambda(\mathfrak{a})\rangle$ to denote $\iota_p^{-1}(\langle\iota_p(\lambda(\mathfrak{a}))\rangle)$ (notice that this makes sense).

DEFINITION 3.10: For $k \in \mathbb{Z}_{\geq 1}$, let $\eta_k : I_K(\mathfrak{c}) \to \bar{\mathbb{Q}}^\times$ be the character $\mathfrak{a} \mapsto \eta(\mathfrak{a})\cdot\langle\lambda(\mathfrak{a})\rangle^{k-1}$, so that

$$g_k := \sum_{(\mathfrak{a},\mathfrak{c})=1} \eta_k(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in S_k(N_{\boldsymbol{g}}, \chi_k)$$

where $N_{\boldsymbol{g}} = N_g$ and $\chi_k = \chi_g\cdot\omega^{1-k} = \chi_{\boldsymbol{g}}\cdot\omega^{2-k}$ where ω is the Teichmüller character and clearly $\chi_{\boldsymbol{g}} = \chi_g\cdot\omega^{-1}$. We will also write $N_{\boldsymbol{g}}^\circ := N_{\boldsymbol{g}}/p^{2r}$ in the sequel.

REMARK 3.11: Note that, since p is inert in K, the p-part of the conductor of χ_k is at most p^r for all $k \geq 1$, so that χ_k will never be p-primitive as a Dirichlet character modulo $N_{\boldsymbol{g}}$. This is a typical feature for newforms of infinite p-slope and level divisible by p. It is well-known, on the other hand, that if the p-order of N and of $\mathrm{cond}(\chi)$ of a normalized newform $f \in S_k(N,\chi)$ coincide, then $a_p(f)$ must have euclidean absolute value $p^{(k-1)/2}$ (cf. theorem 4.6.17 of [Miy06]). ▲

REMARK 3.12: Recall the (unique) continuous $\mathbb{Z}_p$-action on $U_1 := \{z \in \mathbb{C}_p \mid |z-1|_p < 1\}$ extending the natural structure of U_1 as a multiplicative abelian group, namely

$$z^s := \sum_{n=0}^{\infty} \binom{s}{n}(z-1)^n \qquad z \in U_1,\ s \in \mathbb{Z}_p.$$

We thus view U_1 as a topological $\mathbb{Z}_p$-module. One can show that $\mu_{p^\infty}(\mathbb{C}_p)$ (i.e., the subgroup of roots of unity of p-power order) is dense inside U_1. It follows that the natural action of $G_{\mathbb{Q}_p}$ on U_1 given by the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}^{(p)} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is compatible with the action of $\mathbb{Z}_p^\times$, in the sense that $\sigma(z) = z^{\varepsilon_{\mathrm{cyc}}^{(p)}(\sigma)}$ for $z \in U_1$, $\sigma \in G_{\mathbb{Q}_p}$. ▲

DEFINITION 3.13: We define W_K to be the smallest closed $\mathbb{Z}_p$-submodule of U_1 containing $\langle\lambda(\mathfrak{a})\rangle$ for all $\mathfrak{a} \in I_K(p\mathcal{O}_K)$.

REMARK 3.14: Note that the notation W_K makes sense, since different choices for λ (i.e., different choices for the dashed arrow in the diagram above) differ by a finite order character, so that W_K only depends on K and not on λ. ▲

LEMMA 3.15: *W_K is a free $\mathbb{Z}_p$-module of rank 2. If $a \in \mathbb{Z}_{\geq 0}$ is such that $p^a = \#(Cl_K(p\mathcal{O}_K) \otimes \mathbb{Z}_p)$, then $w^{p^a} \in 1 + p\mathbb{Z}_{p^2}$ for all $w \in W_K$. In particular, if $p \nmid \#(\mathrm{Pic}(\mathcal{O}_K))$, we have $W_K = 1 + p\mathbb{Z}_{p^2}$.*

PROOF. Let $m = \#Cl_K(p\mathcal{O}_K)$. Since $\lambda((\alpha)) = \alpha$ for all $\alpha \equiv 1 \mod {}^{\times}p\mathcal{O}_K$ we deduce that $\langle\lambda(\mathfrak{a}^m)\rangle \in 1 + p\mathbb{Z}_{p^2}$ for all $\mathfrak{a} \in I_K(p\mathcal{O}_K)$, whence $W_K^{(m)} = \{w^m \mid w \in W_K\} \subseteq 1 + p\mathbb{Z}_{p^2}$.

Raising to the m/p^a-th power is an automorphism of W_K as $\mathbb{Z}_p$-module, hence $W_K^{(p^a)} = \{w^{p^a} \mid w \in W_K\} \subseteq 1 + p\mathbb{Z}_{p^2}$. Finally, it is also clear that $1 + p\mathbb{Z}_{p^2} \subseteq W_K$, which proves the statment concerning the rank of W_K. □

REMARK 3.16: Denote by $\langle\lambda\rangle : G_K \twoheadrightarrow W_K$ the corresponding Galois character (given by the composition $\langle\cdot\rangle \circ \lambda^{(p)}$) and let K_∞ denote the (unique) $\mathbb{Z}_p^2$-extension of K. It follows from the construction that $\langle\lambda\rangle$ factors through $\Gamma_\infty := \mathrm{Gal}(K_\infty/K)$, inducing an isomorphism $\Gamma_\infty \cong W_K$. We will consider W_K as a $G_{\mathbb{Q}}$-module via this isomorphism (and the $G_{\mathbb{Q}}$-action on Γ_∞ by conjugation). In particular we have $\Gamma_\infty = \Gamma^+ \times \Gamma^-$ where

(i) Γ^+ is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of K, denoted by K_∞^+, where complex conjugation acts as the identity;
(ii) Γ^- is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K, denoted by K_∞^-, where complex conjugation acts as taking the inverse.

We will write $W_K = W_K^+ \times W_K^-$ for the corresponding decomposition of W_K. ▲

3.2. Construction *à la Coleman*

As above, let $a \in \mathbb{Z}_{\geq 0}$ be such that $p^a = \#(Cl_K(p\mathcal{O}_K) \otimes \mathbb{Z}_p)$. Since $|\mathbb{Z}_{p^2}|_p = |\mathbb{Z}_p|_p$, we have group isomorphisms

$$\begin{tikzcd} 1 + p\mathbb{Z}_{p^2} \arrow[r, bend left, "\log_p", "\cong"'] & p\mathbb{Z}_{p^2} \arrow[l, bend left, "\exp_p", "\cong"'] & \mathbb{Z}_{p^2} \arrow[l, "\cong", "\cdot\log_p(1+p)"'] \end{tikzcd}.$$

LEMMA 3.17: *For every $\alpha \in \mathbb{Z}_{p^2}$ the formal power series*

$$(1+T)^\alpha = \exp(\alpha\log(1+T)) = \sum_{n=0}^{+\infty} \binom{\alpha}{n} T^n \in \mathbb{Q}_{p^2}[[T]]$$

actually lies in the ring $\mathbb{Z}_{p^2}[\gamma][[\frac{T}{\gamma}]]$, where $\gamma \in \mathbb{C}_p$ is a (fixed) $p-1$-th root of p.

PROOF. It is well-known that for $n \geq 1$ one has $|n!|_p > p^{-\frac{n}{p-1}}$. It follows immediately that $\gamma^n \cdot \binom{\alpha}{n} \in \mathbb{Z}_{p^2}[\gamma]$ for all $n \geq 0$ (the case $n = 0$ being trivially checked). □

For $\mathfrak{a} \in I_K(p\mathcal{O}_K)$, we define

$$s(\mathfrak{a}) := \frac{\log_p(\langle\iota_p(\lambda(\mathfrak{a}))\rangle)}{\log_p(1+p)} \in \mathbb{C}_p \tag{3.2}$$

and observe that, by lemma 3.15, we know that $p^a \cdot s(\mathfrak{a}) \in \mathbb{Z}_{p^2}$ for all $\mathfrak{a} \in I_K(p\mathcal{O}_K)$. As we did above, we define power series

$$(1+T)^{s(\mathfrak{a})} := \exp(s(\mathfrak{a}) \cdot \log(1+T)) \in \mathbb{Q}_{p^2}[[T]]$$

and one can check using the above lemma that $(1+T)^{s(\mathfrak{a})} \in \mathbb{Z}_{p^2}[\gamma][[\frac{T}{p^a\gamma}]]$. In particular, this series converges to $\langle\lambda(\mathfrak{a})\rangle^{hp^a}$ when evaluating at $T = (1+p)^{hp^a} - 1$ for $\in \mathbb{Z}_{\geq 0}$.

DEFINITION 3.18: Assuming $\gamma \in L$, we define

$$\boldsymbol{g}_{\mathrm{Col}} := \sum_{(\mathfrak{a},\mathfrak{c})=1} \eta(\mathfrak{a}) \left(1 + \tfrac{T-p}{p+1}\right)^{s(\mathfrak{a})} \cdot q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \left(\mathcal{O}_L[[\tfrac{T-p}{p^a\gamma}]]\right)[[q]]. \tag{3.3}$$

LEMMA 3.19: *The power series $\boldsymbol{g}_{\mathrm{Col}}$ satisfies the interpolation property*

$$\boldsymbol{g}_{\mathrm{Col}}(1+hp^a) := \boldsymbol{g}_{\mathrm{Col}}((1+p)^{1+hp^a} - 1; q) = g_{1+hp^a} \in S_{1+hp^a}(N_{\boldsymbol{g}}, \chi_{1+hp^a}, L) \tag{3.4}$$

for all $h \in \mathbb{Z}_{\geq 0}$ and moreover $\boldsymbol{g}_{\mathrm{Col}}(1) = g$.
PROOF. This follows immediately from the construction. □

DEFINITION 3.20: We will write $\Lambda_{\mathrm{Col}} := \mathcal{O}_L[[\frac{T-p}{p^a\gamma}]]$ and $\mathcal{O}_{\mathrm{Col}} := \Lambda_{\mathrm{Col}}[\frac{1}{p}]$ in what follows.

REMARK 3.21: The Λ-algebra Λ_{Col} (resp. $\mathcal{O}_{\mathrm{Col}}$) is the ring of analytic functions bounded by 1 (resp. bounded) on the *open* ball of radius $|p^a\gamma|_p$ centered at the weight $w = 1$ in the weight space $\mathcal{W}_\Lambda$ (cf. section 1). ▲

For $k \in \mathbb{Z}_{\geq 1}$ we have Hecke characters η_k as defined above (definition 3.10) and, passing to p-adic avatars and via class field theory, we can consider them as Galois characters $\eta_k^{(p)} : G_K \to L^\times$ unramified outside $\mathfrak{c}$ such that $\eta_k^{(p)}(\mathrm{Frob}_\mathfrak{l}) = \eta_k(\mathfrak{l})$ for all $\mathfrak{l} \subset \mathcal{O}_K$ prime ideals, $\mathfrak{l} \nmid \mathfrak{c}$.

It is well-known that $V_k := \mathrm{Ind}_K^{\mathbb{Q}}(\eta_k^{(p)})$ is a 2-dimensional (over L) Galois representation isomorphic to the dual of the p-adic Galois representation of $G_\mathbb{Q}$ attached (by the work of Shimura and Deligne) to the modular form g_k when $k \geq 2$. More precisely, this means that

$$\det(1 - \mathrm{Frob}_\ell | V_k \cdot X) = 1 - a_\ell(g_k)X + \chi_k(\ell)\ell^{k-1}X^2 \tag{3.5}$$

for every prime number $\ell \nmid N_{\boldsymbol{g}}$, where as usual $a_\ell(g_k)$ denotes the ℓ-th Fourier coefficients of the q-expansion of g_k at ∞.

We also have a big Hecke character

$$\boldsymbol{\eta}_{\mathrm{Col}} : I_K(\mathfrak{c}) \to \mathcal{O}_{\mathrm{Col}}^\times, \quad \mathfrak{a} \mapsto \eta(\mathfrak{a}) \cdot \left(1 + \tfrac{T-p}{p+1}\right)^{s(\mathfrak{a})} \tag{3.6}$$

satisfying for all $k \in \mathbb{Z}_{\geq 1}$, $k \equiv 1 \mod p^a$ the property $(\boldsymbol{\eta}_{\mathrm{Col}}(\mathfrak{a}))((1+p)^k - 1) = \eta_k(\mathfrak{a})$. Then, again via class field theory, one gets a big Galois character $\boldsymbol{\eta}_{\mathrm{Col}} : G_K \to \mathcal{O}_{\mathrm{Col}}^\times$.

DEFINITION 3.22: We set

$$\mathbb{V}_{\boldsymbol{g}_{\mathrm{Col}}} := \mathrm{Ind}_K^{\mathbb{Q}} \boldsymbol{\eta}_{\mathrm{Col}} \tag{3.7}$$

and we call it the big Galois representation associated with the family $\boldsymbol{g}_{\mathrm{Col}}$.

3.3. Construction *à la Hida*

It is possible to realize the families of theta series of infinite p-slope considered above in another way, as suggested in Hida's blue book [Hid93, pagg. 236-237].

DEFINITION 3.23: We define the Λ-algebras $\Lambda_{\mathrm{Hida}} := \mathcal{O}_L[[W_K]]$ and $\mathcal{O}_{\mathrm{Hida}} := \Lambda_{\mathrm{Hida}}[1/p]$, with Λ-algebra structure induced by the natural inclusion $1 + p\mathbb{Z}_p \subset W_K$.

DEFINITION 3.24: We define

$$\boldsymbol{g}_{\mathrm{Hida}} := \sum_{(\mathfrak{a},\mathfrak{c})=1} \frac{\eta(\mathfrak{a})}{\langle\lambda(\mathfrak{a})\rangle}[\langle\lambda(\mathfrak{a})\rangle] \cdot q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \Lambda_{\mathrm{Hida}}[[q]]\,,$$

where recall that $[\,\cdot\,]$ denotes group elements in W_K.

Let $w : \Lambda_{\mathrm{Hida}} \to \mathbb{C}_p$ be a continuous $\mathcal{O}_L$-algebra homomorphism. Assume that there exists integers $a_w \geq 1$ and $k_w \geq 1$ such that w sends group elements in $[u] \in 1+p^{a_w}\mathbb{Z}_{p^2} \subseteq W_K$ to $u^{k_w} \in \mathbb{C}_p$. Then

$$\eta_w : I_K(\mathfrak{c}) \to \mathbb{C}_p^\times \qquad \mathfrak{a} \mapsto \frac{\eta(\mathfrak{a})}{\langle\lambda(\mathfrak{a})\rangle} \cdot w([\langle\lambda(\mathfrak{a})\rangle])$$

is a primitive Hecke character of infinity type $(k_w - 1, 0)$ with conductor $p^{e(w,\eta)}\mathfrak{c}$ for a suitable integer $e(w,\eta) \geq 0$ (depending on a_w and the p-part of η), so that

$$\boldsymbol{g}_{\mathrm{Hida}}(w) := \sum_{(\mathfrak{a},\mathfrak{c})=1} \eta_w(\mathfrak{a}) \cdot q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in S_{k_w}(N_w, \chi_w, \mathcal{O}_L[w])\,, \tag{3.8}$$

where

(i) $N_w = d_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}) \cdot p^{2e(w,\eta)}$

(ii) $\chi_w = \varepsilon_K \cdot \eta_{|\mathbb{Q}} \cdot \omega^{1-k} \cdot \varepsilon_w = \chi_{\boldsymbol{g}} \cdot \omega^{2-k} \cdot \varepsilon_w$, where ε_w is an explicit character valued in $\mu_{p^\infty}(\mathbb{C}_p)$, depending on w.

(iii) $\mathcal{O}_L[w]$ is the finite extension of $\mathcal{O}_L$ generated by the values of w (one can assume that it is a cyclotomic extension of $\mathcal{O}_L$ generated by a p-power root of unity).

When w acts on group elements $[u] \in W_K$ as $w([u]) = u^k$ for some $k \geq 1$, we recover the specialisations $\boldsymbol{g}_{\mathrm{Hida}}(w) = g_k$. If, moreover, $k \equiv 1 \mod p^a$ (notation as in lemma 3.15), we get back all the classical specializations of $\boldsymbol{g}_{\mathrm{Col}}$.

REMARK 3.25: The family $\boldsymbol{g}_{\mathrm{Hida}}$ admits *more general* classical specializations than the family $\boldsymbol{g}_{\mathrm{Col}}$ (in particular ramification at p is allowed), but one has to allow Fourier coefficients in the *larger* ring Λ_{Hida}. ▲

One can then again produce a big Hecke character

$$\boldsymbol{\eta}_{\mathrm{Hida}} : I_K(\mathfrak{c}) \to \mathcal{O}_{\mathrm{Hida}}^\times \qquad \mathfrak{a} \mapsto \frac{\eta(\mathfrak{a})}{\langle\lambda(\mathfrak{a})\rangle} \cdot [\langle\lambda(\mathfrak{a})\rangle] \tag{3.9}$$

with associated Galois character $\boldsymbol{\eta}_{\mathrm{Hida}} : G_K \to \mathcal{O}_{\mathrm{Hida}}^\times$. Note that, by construction, $\boldsymbol{\eta}_{\mathrm{Hida}}$ factors through the Galois group of the ray class field modulo $\mathfrak{c}_0 p^\infty$ over K.

DEFINITION 3.26: We set $\mathbb{V}_{\boldsymbol{g}_{\mathrm{Hida}}} := \mathrm{Ind}_K^{\mathbb{Q}} \boldsymbol{\eta}_{\mathrm{Hida}}$ and we call it the big Galois representation associated with the family $\boldsymbol{g}_{\mathrm{Hida}}$.

REMARK 3.27: By construction, it follows that for any w as above, the 2-dimensional (over $L[w] = \mathrm{Frac}(\mathcal{O}_L[w])$) $G_{\mathbb{Q}}$-representation obtained as

$$\mathbb{V}_{\boldsymbol{g}_{\mathrm{Hida}}}(w) := \mathbb{V}_{\boldsymbol{g}_{\mathrm{Hida}}} \otimes_{\mathcal{O}_{\mathrm{Hida}},w} L[w]$$

is the dual of the Deligne representation attached to the specialization $\boldsymbol{g}_{\mathrm{Hida}}(w)$. ▲

3.4. Families of theta series as generalized Λ-adic eigenforms

Now we are ready to prove that the families of the form $\boldsymbol{g}_{\mathrm{Col}}$ and $\boldsymbol{g}_{\mathrm{Hida}}$ fit in the framework of generalized Λ-adic modlar forms, as defined in section 1.

LEMMA 3.28: *The families $\boldsymbol{g}_{\mathrm{Col}}$ and (resp.) $\boldsymbol{g}_{\mathrm{Hida}}$ constructed as in equations* (3.3) *and (resp.)* (3.24) *satisfy (with the notation introduced in section 1 and above)*

$$\boldsymbol{g}_{\mathrm{Col}} \in \mathbb{S}_{\Omega_{\mathrm{Col}}}(N_{\boldsymbol{g}}^{\circ}, \chi_{\boldsymbol{g}}, \Lambda_{\mathrm{Col}}) \quad \text{and} \quad \boldsymbol{g}_{\mathrm{Hida}} \in \mathbb{S}_{\Omega_{\mathrm{Hida}}}(N_{\boldsymbol{g}}^{\circ}, \chi_{\boldsymbol{g}}, \Lambda_{\mathrm{Hida}}),$$

where $\Omega_{\mathrm{Col}} := \mathcal{W}^{cl}_{\Lambda_{\mathrm{Col}},\mathbb{Z}}$ and $\Omega_{\mathrm{Hida}} := \Omega_{\boldsymbol{g}_{\mathrm{Hida}},\mathbb{Z}}$. Moreover, $\boldsymbol{g}_{\mathrm{Col}}$ and $\boldsymbol{g}_{\mathrm{Hida}}$ are generalized Λ-adic eigenforms, both lying in the kernel of U_p.

PROOF. As far as $\boldsymbol{g}_{\mathrm{Hida}}$ is concerned, it is enough to check that $\Omega_{\boldsymbol{g}_{\mathrm{Hida}},\mathbb{Z}}$ is $(\Lambda, \Lambda_{\mathrm{Hida}})$-admissible. Condition (i) of definition 1.5 is clearly satisfied. For condition (ii), for every $k \geq 2$ let $w_k : \Lambda_{\mathrm{Hida}} \to \mathbb{C}_p$ denote the weight uniquely determined by the assignment $w_k([u]) = u^k - 1$ on group elements. We know that $w_k \in \Omega_{\boldsymbol{g}_{\mathrm{Hida}},\mathbb{Z}}$ and we claim that $I := \bigcap_{k\geq 2} \mathrm{Ker}(w_k) = (0)$. Since $\varpi_L \notin \mathrm{Ker}(w_k)$ for every $k \geq 2$, one can prove the assertion working in $\Lambda_{\mathrm{Hida}}[1/p]$, where it is easy to show that $\bigcap_{k=2}^m \mathrm{Ker}(w_k)[1/p] = \prod_{k=2}^m \mathrm{Ker}(w_k)[1/p]$ for all $m \geq 2$. Using that $\Lambda_{\mathrm{Hida}}[1/p]$ is a UFD (since Λ_{Hida} is such), one concludes that indeed it must be $I = (0)$.

As far as $\boldsymbol{g}_{\mathrm{Col}}$ is concerned, we are left to prove that

$$\Omega_{\boldsymbol{g}_{\mathrm{Col}},\mathbb{Z}} = \mathcal{W}^{cl}_{\Lambda_{\mathrm{Col}},\mathbb{Z}} \xrightarrow[\sim]{\varphi^*_{\mathrm{Col}}} (1 + p^a\mathbb{Z}) \cap \mathbb{Z}_{\geq 2}$$

is a bijection (then the lemma immediately follows). Recall that $(\Lambda_{\mathrm{Col}} = \mathcal{O}_L[[X]], \varphi_{\mathrm{Col}})$ is a Λ-algebra via $\varphi_{\mathrm{Col}}(T) = p^a\gamma X + p$, with γ a fixed $(p-1)$-th root of p. Let $\mathfrak{p}_k = (T + 1 - (1+p)^k) \subset \Lambda$ (for some $k \geq 2$) be the kernel of the specialization to weight k. To give $w \in \mathcal{W}^{cl}_{\Lambda_{\mathrm{Col}},\mathbb{Z}}$ with $w \circ \varphi_{\mathrm{Col}} = k$ is equivalent to give a prime ideal of Λ_{Col} lying over $\mathfrak{p}_k$ and with residue field a finite extension of L, i.e., to give a prime ideal of

$$\Lambda_{\mathrm{Col}} \otimes_{\Lambda} \frac{\Lambda_{\mathfrak{p}_k}}{\mathfrak{p}_k\Lambda_{\mathfrak{p}_k}} \cong \frac{\Lambda_{\mathrm{Col}}[\frac{1}{p}]}{(X - \frac{(1+p)((1+p)^{k-1}-1)}{p^a\gamma})}$$

with residue field a finite extension of L. Given $\alpha \in L$ it is clear (look at the inverse of $X - \alpha$ in $L[[X]]$ when $\alpha \neq 0$) that

$$\frac{\Lambda_{\mathrm{Col}}[\frac{1}{p}]}{(X-\alpha)} \cong \begin{cases} L & \Leftrightarrow \ \alpha \in \varpi_L\mathcal{O}_L \\ 0 & \text{otherwise} \end{cases}$$

so that for us there exists a unique $w \in \mathcal{W}^{cl}_{\Lambda_{\mathrm{Col}},\mathbb{Z}}$ such that $w \circ \varphi_{\mathrm{Col}} = k$ if and only if $p^a \mid k-1$. This proves the claimed bijection. □

REMARK 3.29: The families $\boldsymbol{g}_{\mathrm{Col}}$ and $\boldsymbol{g}_{\mathrm{Hida}}$ are examples of Λ-adic forms admitting classical specializations also for arithmetic weights w with $k_w = 1$. ▲

CHAPTER 4

Factorization of triple product p-adic L-functions

4.1. Remarks on complex L-functions

In this section we recollect some facts concerning Hecke L-functions and Rankin-Selberg convolution that will be needed in the sequel.

Fix $K/\mathbb{Q}$ a quadratic imaginary field and let $\chi_{\mathbb{C}} : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ be an algebraic Hecke character of ∞-type (a,b). Let $|\cdot|_{\mathbb{A}_K}$ denote the adèlic norm. Then $\chi_{\mathbb{C}} = \chi_0 \cdot |\cdot|_{\mathbb{A}_K}^{(a+b)/2}$ is a unitary Hecke character (i.e. taking values in $\{z \in \mathbb{C}^\times \mid |z| = 1\}$) and the completed L-function $L^*(\chi_0, s)$ attached to χ_0 has meromorphic continuation and functional equation with center $s = 1/2$ (cf. Tate's thesis). Note that $L^*(\chi_0, s)$ is actually an entire function if χ_0 is not of the form $\chi_0 = \nu \circ N_{K/\mathbb{Q}}$ for some Dirichlet character ν.

As explained in [JL70, theorem 11.3 and proposition 12.1], one can attach to $\chi_{\mathbb{C}}$ an automorphic representation $\pi(\chi)$ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ such that $L^*(\pi(\chi), s) = L^*(\chi_0, s)$. Note that if $b = 0$ and $a \geq 0$, then $\pi(\chi)$ is the automorphic representation attached to the theta series θ_χ and $L^*(\theta_\chi, s) = L^*(\pi(\chi), s + a/2)$.

Given two automorphic representations π_1 and π_2 of $\mathrm{GL}_2(\mathbb{A})$ with central characters ω_1 and ω_2, one can construct - via the so-called Rankin-Selberg method - an L-function $L^*(\pi_1 \times \pi_2, s)$, prove its meromorphic continuation and functional equation of the form

$$L^*(\pi_1 \times \pi_2, s) = \varepsilon(\pi_1 \times \pi_2, s) \cdot L^*(\tilde{\pi}_1 \times \tilde{\pi}_2, 1 - s)$$

where $\tilde{\pi}$ denotes the contragradient representation of π. The poles of $L^*(\pi_1 \times \pi_2, s)$ are those of $L(\omega_1\omega_2, 2s - 1)$. Moreover, the ε-factor $\varepsilon(\pi_1 \times \pi_2, s)$ is an invertible function.

We refer to the standard reference [Jac72] for this construction and for the definition of the local L-factors and ε-factors of such L-functions. The local theory is also nicely summarized in [GJ78, section 1]). For the definition of the local ε-factors we always use the standard additive character of the corresponding local field and the self-dual Haar measure with respect to the standard character.

Starting from two cuspidal eigenforms $f \in S_k(N_f, \chi_f)$ and $g \in S_l(N_g, \chi_g)$, one can also define the L-function $L(f \times g, s)$ more classically via an Euler product expansion (cf. [Kat04, section 7]). If f and g are newforms and $k \geq l$, it holds

$$L(f \times g, s) \cdot \Gamma_{\mathbb{C}}(s) \cdot \Gamma_{\mathbb{C}}(s - l + 1) = L^*(\pi_f \times \pi_g, s - \tfrac{k+l-2}{2}).$$

Finally, if $f \in S_k(N_f, \chi_f)$ and $\psi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ is an algebraic Hecke character of ∞-type (a,b), we set

$$L^*(f/K, \psi, s) := L^*(\pi_f \times \pi(\psi), s - \tfrac{k-1+a+b}{2}).$$

One can write $L^*(f/K, \psi, s) = L_\infty(f/K, \psi, s) \cdot L(f/K, \psi, s)$ with archimedean L-factor given by

$$L_\infty(f/K, \psi, s) = \Gamma_{\mathbb{C}}(s - \min\{a,b\}) \cdot \Gamma_{\mathbb{C}}(s - \min\{k-1, |a-b|\} - \min\{a,b\}).$$

Assume now that we are given $f \in S_k(N_f, \chi_f)$ a Hecke eigenform and two Hecke characters ψ_1, ψ_2 of K of ∞-type $(l-1,0)$ and $(m-1,0)$ respectively (here $l \geq 1, m \geq 1$), which are not induced by Dirichlet characters. Then $g = \theta_{\psi_1}$ and $h = \theta_{\psi_2}$ are cuspidal newforms, say $g \in S_l(N_l, \chi_g)$ and $h \in S_m(N_h, \chi_h)$. Assume that $\chi_f \cdot \chi_g \cdot \chi_h = 1$ and consider the Garret-Rankin triple product L-function

$$L^*(f \times g \times h, s) = L^*(\pi_f \times \pi(\psi_1) \times \pi(\psi_2), s - \tfrac{k+l+m-3}{2}).$$

If one looks at the corresponding ℓ-adic Galois representations for ℓ any rational prime, one easily deduces the following decomposition

$$\begin{aligned} V_\ell(f) \otimes V_\ell(g) \otimes V_\ell(h) &\cong V_\ell(f) \otimes \left(\mathrm{Ind}_K^{\mathbb{Q}} \psi_1\psi_2 \oplus \mathrm{Ind}_K^{\mathbb{Q}} \psi_1\psi_2^\sigma\right) \cong \\ &\cong \left(V_\ell(f) \otimes \mathrm{Ind}_K^{\mathbb{Q}} \psi_1\psi_2\right) \oplus \left(V_\ell(f) \otimes \mathrm{Ind}_K^{\mathbb{Q}} \psi_1\psi_2^\sigma\right). \end{aligned} \tag{4.1}$$

For the sake of precision, here $V_\ell(\xi)$ denotes the dual of the Deligne representation attached to ξ and we look at ψ_1 and ψ_2 as Galois characters attached to the ℓ-adic avatars of ψ_1 and ψ_2 via class field theory.

The decomposition (4.1) corresponds to the following factorization of L-functions

$$L^*(f \times g \times h, s) = L^*(f/K, \psi_1\psi_2, s) \cdot L^*(f/K, \psi_1\psi_2^\sigma, s). \tag{4.2}$$

4.2. Study of the big Galois representations

As usual, we let L denote a (large enough) finite extension of $\mathbb{Q}_p$, containing all the needed coefficients.

SETTING 4.1: We work in the following setting.

(i) We fix $\boldsymbol{f} \in \mathbb{S}^{ord}(N_{\boldsymbol{f}}, \mathbf{1}, \Lambda_{\boldsymbol{f}})$ a primitive Hida family with trivial tame character, square-free tame level $N_{\boldsymbol{f}}$ and coefficients in $\Lambda_{\boldsymbol{f}}$ (a ring in $\hat{\mathcal{C}}_\Lambda$, which is also finite flat over $\Lambda = \mathcal{O}_L[[1+p\mathbb{Z}_p]]$), satisfying assumption 2.1.

(ii) We let $K/\mathbb{Q}$ denote a quadratic imaginary field of odd discriminant $-d_K$ (i.e. we have $d_K \equiv 3 \mod 4$) coprime to $pN_{\boldsymbol{f}}$ such that the fixed odd prime p is inert in K and does not divide the class number of K. Writing $N_{\boldsymbol{f}} = N_{\boldsymbol{f}}^+ \cdot N_{\boldsymbol{f}}^-$ where $N_{\boldsymbol{f}}^+$ is the product of prime factors of $N_{\boldsymbol{f}}$ which are split in K, we assume that $N_{\boldsymbol{f}}^-$ is the product of an *odd* number of prime factors (*Heegner hypothesis*).

(iii) We fix two ray class characters η_1 and η_2 of G_K, both of conductor $cp^r\mathcal{O}_K$ with c a positive integer with $(c, pN_{\boldsymbol{f}}) = 1$ and $r \geq 1$. We then let $\boldsymbol{g}$ and (respectively) $\boldsymbol{h}$ denote the generalized Λ-adic eigenforms attached to η_1 and (respectively) η_2 via the construction explained in section 3.3.

(iv) We assume that the central characters of η_1 and η_2 are inverse to each other (*self-duality condition*).

(v) We assume that the prime divisors of the integer c are all split in K.

REMARK 4.2: Let $\langle\varepsilon_{\mathrm{cyc}}\rangle : G_{\mathbb{Q}} \to 1 + p\mathbb{Z}_p$ be the character $g \mapsto \varepsilon_{\mathrm{cyc}}(g) \cdot \omega(\varepsilon_{\mathrm{cyc}}(g))^{-1}$, where $\varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ is p-adic cyclotomic character. We then get automatically a universal weight character (cf. remark 1.11 for the notation):

$$\langle\varepsilon_{\mathrm{cyc}}\rangle_\Lambda : G_{\mathbb{Q}} \to \Lambda^\times \qquad g \mapsto [\langle\varepsilon_{\mathrm{cyc}}(g)\rangle] = \langle\cdot\rangle_\Lambda \circ \varepsilon_{\mathrm{cyc}}(g)$$

and, for $(R, \varphi) \in \hat{\mathcal{C}}_\Lambda$, we set $\langle\varepsilon_{\mathrm{cyc}}\rangle_R = \varphi \circ \langle\varepsilon_{\mathrm{cyc}}\rangle_\Lambda = \langle\cdot\rangle_R \circ \varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to R^\times$.

Since we assume that p does not divide the class number of K, we have $W_K = 1 + p\mathbb{Z}_{p^2}$ (cf. lemma 3.15) and moreover (with the notation of remark 3.16)

$$\langle \varepsilon_{\mathrm{cyc}} \rangle|_{G_K} = \langle \lambda \rangle \cdot \langle \lambda \rangle^{\sigma}. \tag{4.3}$$

▲

By the work of Hida and Wiles, it is known that one can attach to $\boldsymbol{f}$ a *big* Galois representation $\mathbb{V}_{\boldsymbol{f}}$, which can be realized as a free module of rank 2 over $\Lambda_{\boldsymbol{f}}[1/p]$ equipped with a continuous action of $G_{\mathbb{Q}}$, specializing for all $x \in \mathcal{W}^{cl}_{\Lambda_{\boldsymbol{f}}}$ to the dual $V_p(\boldsymbol{f}_x)$ of the p-adic Deligne representation attached to $\boldsymbol{f}_x$ (or, in case $\boldsymbol{f}_x$ is the p-stabilization of a newform of level $N_{\boldsymbol{f}}$, to the dual of the representation attached to such newform). In particular it holds that $\det(\mathbb{V}_{\boldsymbol{f}}) = \omega_{\mathrm{cyc}} \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle_{\Lambda_{\boldsymbol{f}}}$. We refer to [BSV22b, section 5] for a detailed discussion concerning such Galois modules.

We defined a big Galois representation $\mathbb{V}_{\boldsymbol{g}}$ (resp. $\mathbb{V}_{\boldsymbol{h}}$) attached to $\boldsymbol{g}$ (resp. $\boldsymbol{h}$) as

$$\mathbb{V}_{\boldsymbol{g}} = \mathrm{Ind}_K^{\mathbb{Q}} \boldsymbol{\eta}_1 \qquad (\text{resp.} \mathbb{V}_{\boldsymbol{h}} = \mathrm{Ind}_K^{\mathbb{Q}} \boldsymbol{\eta}_2),$$

where $\boldsymbol{\eta}_1$ (resp. $\boldsymbol{\eta}_2$) is the big Galois character valued in $\Lambda_{\mathrm{Hida}}[1/p]$ constructed as in section 3.3.

NOTATION 4.3: We will write $R_K := \Lambda_{\mathrm{Hida}}$ in what follows, to simplify the notation. We will also write $\langle \lambda \rangle_{R_K} : G_K \to R_K^{\times}$ for the big Galois character given by $g \mapsto [\langle \lambda(g) \rangle]$.

LEMMA 4.4: *We have*

$$\det(\mathbb{V}_{\boldsymbol{g}}) = \varepsilon_K \cdot \eta_1^{\mathrm{cen}} \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle_{R_K}, \qquad \det(\mathbb{V}_{\boldsymbol{h}}) = \varepsilon_K \cdot \eta_2^{\mathrm{cen}} \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle_{R_K}.$$

PROOF. It follows easily form equation (4.3). □

Consider the Galois representation $\mathbb{V} := \mathbb{V}_{\boldsymbol{f}} \hat{\otimes}_L \mathbb{V}_{\boldsymbol{g}} \hat{\otimes}_L \mathbb{V}_{\boldsymbol{h}}$. It is a free $\mathcal{R} := R_{\boldsymbol{fgh}}[1/p]$-module of rank 2 and it follows immediately from the above discussion and our assumptions that

$$\det(\mathbb{V}) = \omega_{\mathrm{cyc}}^6 \cdot \varepsilon_{\mathrm{cyc}}^{-3} \cdot (\langle \varepsilon_{\mathrm{cyc}} \rangle_{\Lambda_{\boldsymbol{f}}} \hat{\otimes} \langle \varepsilon_{\mathrm{cyc}} \rangle_{R_K} \hat{\otimes} \langle \varepsilon_{\mathrm{cyc}} \rangle_{R_K}).$$

Since p is odd, there exists a character $\boldsymbol{\chi}_{\boldsymbol{fgh}} = \boldsymbol{\chi} : G_{\mathbb{Q}} \to \mathcal{R}^{\times}$ such that $\varepsilon_{\mathrm{cyc}} \cdot \boldsymbol{\chi}^2 = \det(\mathbb{V})$, i.e. we can write

$$\boldsymbol{\chi} = \omega_{\mathrm{cyc}} \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle^{-2} \cdot (\langle \varepsilon_{\mathrm{cyc}} \rangle_{\Lambda_{\boldsymbol{f}}} \hat{\otimes} \langle \varepsilon_{\mathrm{cyc}} \rangle_{R_K} \hat{\otimes} \langle \varepsilon_{\mathrm{cyc}} \rangle_{R_K})^{1/2}.$$

If we define

$$\mathbb{V}^{\dagger} := \mathbb{V} \otimes_{\mathcal{R}} \mathcal{R}(\boldsymbol{\chi}^{-1}),$$

then one checks easily that such representation is Kummer self-dual, i.e.

$$(\mathbb{V}^{\dagger})^{\vee}(1) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{V}^{\dagger}, \mathcal{R})(1) \cong \mathbb{V}^{\dagger}.$$

We want to study the specializations $\mathbb{V}^{\dagger}(w)$ for a suitable $w \in \Omega_{\boldsymbol{fgh}}$.

DEFINITION 4.5: We define the following big Galois characters

$$\boldsymbol{\varphi} := \eta_1 \eta_2 \langle \lambda \rangle^{\sigma} \langle \lambda \rangle^{-1} \cdot \boldsymbol{\lambda}_{\mathrm{ac}} \qquad \text{and} \qquad \boldsymbol{\psi} := \eta_1 \eta_2^{\sigma} \cdot \boldsymbol{\lambda}_{\mathrm{ac}}, \tag{4.4}$$

where

$$\boldsymbol{\lambda}_{\mathrm{ac}} : G_K \to R_K^{\times} \qquad \boldsymbol{\lambda}_{\mathrm{ac}} := \langle \lambda \rangle_{R_K}^{1/2} \cdot (\langle \lambda \rangle_{R_K}^{\sigma})^{-1/2}.$$

REMARK 4.6: Note that $\mathcal{W}_{R_K}(\mathbb{C}_p) \cong \mathrm{Hom}^{cont}_{grp}(W_K, \mathbb{C}_p^{\times})$ has a natural group structure, so it makes sense to multiply or invert weights. ▲

LEMMA 4.7: *Let $w = (x,y,z) \in \Omega_{\boldsymbol{fgh}}$ with $k = k_x$ even and let f° be the newform associated to $\boldsymbol{f}_x$ (as in proposition 2.6). Then there is a decomposition*

$$\mathbb{V}^\dagger(w) \cong \Big(\Big(V_p(\tilde{f}^\circ) \otimes_{L[w]} \mathrm{Ind}_K^{\mathbb{Q}} \boldsymbol{\varphi}_{y\cdot z}\Big) \oplus \Big(V_p(\tilde{f}^\circ) \otimes_{L[w]} \mathrm{Ind}_K^{\mathbb{Q}} \boldsymbol{\psi}_{y/z}\Big)\Big)\left(-\tfrac{k}{2}\right),$$

where $\tilde{f}^\circ := f^\circ \otimes \omega^{k/2-1}\varepsilon_x^{-1/2}$.

Moreover, setting $l = k_y$ and $m = k_z$, the Hecke character of K attached to $\boldsymbol{\varphi}_{y\cdot z}$ (resp. to $\boldsymbol{\psi}_{y/z}$) is anticyclotomic and has ∞-type $(\frac{l+m-2}{2}, \frac{2-l-m}{2})$ (resp. $(\frac{l-m}{2}, \frac{m-l}{2})$).

PROOF. This is an easy computation. The only passage when one has to be slightly careful consists in observing that for a G_K-character η and an even $G_{\mathbb{Q}}$-character χ it holds $\Big((\mathrm{Ind}_K^{\mathbb{Q}}(\eta)\Big)(\chi) = \mathrm{Ind}_K^{\mathbb{Q}}(\eta \cdot \chi|_{G_K})$. □

4.3. Improvement of the triple product p-adic L-function

We let $M := c^2 \cdot d_K \cdot N_{\boldsymbol{f}}$ in what follows.

Inspired by the level adjustment performed by Hsieh in [Hsi21, section 3.4], we are led to consider the following *test vectors* associated to our families $\boldsymbol{g}$ and $\boldsymbol{h}$, namely we set

$$\text{(4.5)} \qquad \boldsymbol{g}^* := \boldsymbol{g}(q^{N_{\boldsymbol{f}}}) \in \mathbb{S}_{\Omega_{\mathrm{Hida}}}(M, \chi_{\boldsymbol{g}}, R_K), \qquad \boldsymbol{h}^* := \boldsymbol{h}(q^{N_{\boldsymbol{f}}}) \in \mathbb{S}_{\Omega_{\mathrm{Hida}}}(M, \chi_{\boldsymbol{h}}, R_K).$$

One can check that our adjustment matches Hsieh's more general version, in view of the following facts concerning the local automorphic types for the specializations of the families $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$.

PROPOSITION 4.8: *Let ℓ be a prime different from p. Let $w = (x,y,z) \in \Omega_{\boldsymbol{fgh}}$ and write $(f,g,h) = (\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z)$. Denote by $\pi_{\xi,\ell}$ the local component at ℓ of the automorphic representation π_ξ attached to $\xi \in \{f,g,h\}$. Then the following facts hold.*

(i) The automorphic type of $\pi_{\xi,\ell}$ does not depend on the chosen specialization for $\xi \in \{f,g,h\}$ (rigidity of automorphic types).
(ii) If $\ell \nmid M$, then $\pi_{\xi,\ell}$ is an unramified principal series representation for $\xi \in \{f,g,h\}$.
(iii) If $\ell \mid N_{\boldsymbol{f}}$, then $\pi_{f,\ell}$ is special, while $\pi_{g,\ell}$ and $\pi_{h,\ell}$ are unramified principal series.
(iv) If $\ell \mid c^2 d_K$, then $\pi_{f,\ell}$ is an unramified principal series, while $\pi_{g,\ell}$ and $\pi_{h,\ell}$ are ramified principal series.

PROOF. All the assertions regarding $\pi_{f,\ell}$ are well-known for Hida families and for the choice of squarefree tame level $N_{\boldsymbol{f}}$ and trivial character in our setting. The assertions regarding $\pi_{g,\ell}$ and $\pi_{h,\ell}$ follow from the explicit description of the Weil-Deligne representations which correspond to them via the local Langlands correspondence. Here we use the assumption that d_K is odd and that the prime divisors of c split in K to grant that the restriction of $V_p(g)$ and $V_p(h)$ to a decomposition group at ℓ is reducible when $\ell \mid d_K$. □

Along the lines of [Hsi21, proposition 6.12], we can thus define the so-called *fudge factors* at the primes dividing M.

PROPOSITION 4.9: *For each $\ell \mid M$, there exists a unique element $\mathfrak{f}_{\boldsymbol{fgh},\ell} \in R_{\boldsymbol{fgh}}^\times$ such that for all $w \in \Omega_{\boldsymbol{fgh}}$ it holds*

$$(\mathfrak{f}_{\boldsymbol{fgh},\ell})_w = \mathscr{I}_{\Pi_w,\ell},$$

with $\mathscr{I}_{\Pi_w,\ell}$ as in proposition 2.6.

PROOF. This is proven (adapting Hsieh's methods) in the same way as in [Fuk22, section 5.1]. □

DEFINITION 4.10: We define the element

$$\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) := \mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g}^*,\boldsymbol{h}^*)\cdot \prod_{\ell\mid M} \mathfrak{f}_{\boldsymbol{fgh},\ell}^{-1/2} \in R_{\boldsymbol{fgh}}$$

and call it the square root f-unbalanced p-adic triple product L-function attached to our triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$.

COROLLARY 4.11: *With the above notation, for all $w \in \Omega_{\boldsymbol{fgh}}$ lying in the $\boldsymbol{f}$-unbalanced region, it holds*

$$(\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w))^2 = \frac{L^*(\Pi,1/2)}{\zeta_{\mathbb{Q}}(2)^2\cdot\Omega_f^2}\cdot \mathscr{I}^{unb}_{\Pi_w,p}\,. \tag{4.6}$$

PROOF. Obvious from the formula (2.6) and the definition of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. □

We are left to find a more explicit description of the local integral $\mathscr{I}^{unb}_{\Pi_w,p}$. We will fix a triple of weights $w=(x,y,z)\in\Omega_{\boldsymbol{fgh}}$ which is $\boldsymbol{f}$-unbalanced. Write $k=k_x, l=k_y, m=k_z$ as usual, so that $k\geq l+m$. Assume furthermore that k is even.

Let $(f,g,h)=(\boldsymbol{f}_x,\boldsymbol{g}_y,\boldsymbol{h}_z)$ as above and, only for this section, set

$$\pi_1 := \pi_{f,p}\otimes\tilde{\chi}_1\,,\qquad \pi_2 := \pi_{g,p}\,,\qquad \pi_3 := \pi_{h,p}.$$

where

$$\tilde{\chi}_1 = \omega^{(k+l+m-6)/2}\cdot(\varepsilon_x\varepsilon_y\varepsilon_z)^{-1/2}$$

Let $\chi_1=\alpha_{f,p}\cdot\tilde{\chi}_1$, where $\alpha_{f,p}$ denotes the unramified character of $\mathbb{Q}_p^\times$ such that $\alpha_{f,p}(p)=a_p(f)p^{(1-k)/2}$.

Then π_i is an irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ for $i=1,2,3$ and, by our assumptions, we know the following.

LEMMA 4.12: *The representations π_2 and π_3 are always supercuspidal. The representation π_1 satisfies one of the following:*

(a) π_1 is the principal series $\pi_1=\chi_1\boxplus\nu_1$ with $\nu_1=\omega_{f,p}\chi_1^{-1}$ where $\omega_{f,p}$ is the p-component of the central character of $\pi_{f,p}$;

(b) π_1 is the special representation $\pi_1=\chi_1|\cdot|^{-1/2}\mathrm{St}$.

The latter case happens if and only if $x=2$ and $f=\boldsymbol{f}_2$ is p-new.

PROOF. All the assertions concerning π_1 are well-known for Hida families. The fact that π_2 and π_3 are always supercuspidal follows from the fact that g and h are theta series attached to a Hecke character of K ramified at p (recall that the prime p is inert in K by assumption). □

PROPOSITION 4.13: *In the above setting, we have that*

$$\mathscr{I}^{unb}_{\Pi_w,p} = \frac{L(\pi_2\otimes\pi_3\otimes\chi_1,1/2)}{\varepsilon(\pi_2\otimes\pi_3\otimes\chi_1,1/2)\cdot L(\pi_2\otimes\pi_3\otimes\nu_1,1/2)\cdot L(\pi_1\otimes\pi_2\otimes\pi_3,1/2)}\,.$$

PROOF. This follows adapting [Hsi21, proposition 5.4] in the same way as it is suggested in [Fuk22, remark 3.4.8]. □

We can give an even more explicit description of $\mathscr{I}^{unb}_{\Pi_w,p}$. Write φ (resp. ψ) to denote - again only in this section - the p-component of $\boldsymbol{\varphi}_{y\cdot z}$ (resp. $\boldsymbol{\psi}_{y/z}$) seen as Hecke character of K. Let also μ_1 and μ_2 denote the characters of $\mathbb{Q}_{p^2}$ given by

$$\mu_1 = (\alpha_{f,p} \cdot \omega_{f,p}^{-1/2}) \circ N_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\,, \qquad \mu_2 = (\alpha_{f,p}^{-1} \cdot \omega_{f,p}^{-1/2}) \circ N_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}$$

and set $\pi_1' = \pi_{f,p} \otimes \omega_{f,p}^{-1/2}$.

PROPOSITION 4.14: *With the above notation, it holds* $\mathscr{I}^{unb}_{\Pi_w,p} = \mathscr{I}_{\varphi,w} \cdot \mathscr{I}_{\psi,w}$, *where for* $\eta \in \{\varphi, \psi\}$ *we set*

$$\mathscr{I}_{\eta,w} := \frac{L(\pi(\eta\mu_1), 1/2)}{\varepsilon(\pi(\eta\mu_1), 1/2) \cdot L(\pi(\eta\mu_2), 1/2) \cdot L(\pi_1' \otimes \pi(\eta), 1/2)}\,. \tag{4.7}$$

Moreover one can compute $\mathscr{I}_{\eta,w}$ *as follows.*

(1) Assume that we are in case (a) of lemma 4.12 and that the character $\eta\mu_1$ *is unramified, then*

$$\mathscr{I}_{\eta,w} = \left(1 - \frac{p^{k-2}}{a_p(f)^2}\right)^2.$$

(2) Assume that we are in case (b) of lemma 4.12 and that the character $\eta\mu_1$ *of* $\mathbb{Q}_{p^2}^\times$ *is unramified, then*

$$\mathscr{I}_{\eta,w} = 1 - \frac{p^{k-2}}{a_p(f)^2} = 1 - a_p(f)^{-2}\,.$$

(3) Assume that the character $\eta\mu_1$ *of* $\mathbb{Q}_{p^2}^\times$ *is ramified of level* n, *then*

$$\mathscr{I}_{\eta,w} = \left(\frac{p}{a_p(f)^2}\right)^n \cdot \frac{p^{n(k-2)}}{W(\tilde\eta)}\,,$$

where $\tilde\eta$ *is the unitary character of* $\mathbb{Q}_{p^2}$ *given by* $\eta\mu_1$ *on* $\mathbb{Z}_{p^2}^\times$ *and such that* $\tilde\eta(p) = 1$ *and* $W(\tilde\eta)$ *denotes the root number of* $\tilde\eta$, *defined as*

$$W(\tilde\eta) = \varepsilon(\tilde\eta, 1/2)\,,$$

which is an algebraic integer of complex absolute value 1.

PROOF. The factorization $\mathscr{I}^{unb}_{\Pi_w,p} = \mathscr{I}_{\varphi,w} \cdot \mathscr{I}_{\psi,w}$ follows directly from the corresponding factorization at the level of Galois representations given in lemma 4.7 and the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence we know that

$$\mathscr{I}_{\eta,w} = \frac{L(\eta\mu_1, 1/2)}{\varepsilon(\eta\mu_1, 1/2) \cdot L(\eta\mu_2, 1/2) \cdot L(\pi_1' \otimes \pi(\eta), 1/2)}\,.$$

Now note that for $\eta \in \{\varphi, \psi\}$ we have that $\eta\mu_1$ is a unitary character $\mathbb{Q}_{p^2}^\times \to \mathbb{C}^\times$, since $\boldsymbol{\varphi}_{y\cdot z}$ and $\boldsymbol{\psi}_{y/z}$ are anticyclotomic and μ_i is unitary for $i = 1, 2$. The fact that $\boldsymbol{\varphi}_{y\cdot z}$ and $\boldsymbol{\psi}_{y/z}$ are anticyclotomic also implies that $\eta(p) = 1$.

We can proceed depending on the three cases, applying the known facts from Tate's thesis for the definition local L-factors and ε-factors attached to Hecke characters.

(1) If $\eta\mu_1$ is unramified, then $\varepsilon(\eta\mu_1, 1/2) = 1$. Moreover, if π_1' is the unramified principal series $\pi_1' = \alpha_{f,p} \cdot \omega_{f,p}^{-1/2} \boxplus \alpha_{f,p}^{-1} \cdot \omega_{f,p}^{-1/2}$, then $L(\pi_1' \times \pi(\eta), s) = L(\eta\mu_1, s) \cdot L(\eta\mu_2, s)$. Hence

$$\mathscr{I}_{\eta,w} = L(\eta\mu_2, 1/2)^{-2} = \left(1 - \eta\mu_2(p)\cdot^{-1}\right)^2 = \left(1 - \frac{p^{k-2}}{a_p(f)^2}\right)^2 .$$

(2) If $\pi_1' = \alpha_{f,p} \cdot \omega_{f,p}^{-1/2} |\cdot|^{-1/2}\mathrm{St}$, then $L(\pi_1' \times \pi(\eta), s) = L(\eta\mu_1, s)$. Hence

$$\mathscr{I}_{\eta,w} = L(\eta\mu_2, 1/2)^{-1} = 1 - \eta\mu_2(p)p^{-1} = 1 - a_p(f)^{-2} ,$$

where we used that this situation can only occur with $x = k = 2$.

(3) If $\eta\mu_1$ is ramified of level n (so that necessarily also $\eta\mu_2$ is ramified), all the L-factors involved are equal to 1, so that $\mathscr{I}_{\eta,w} = \varepsilon(\eta\mu_1, 1/2)^{-1}$ and by Tate's thesis we know $\varepsilon(\eta\mu_1, 1/2) = \eta\mu_1(p)^n \cdot \varepsilon(\tilde{\eta}, 1/2)$. Hence

$$\mathscr{I}_{\eta,w} = \varepsilon(\eta\mu_1, 1/2)^{-1} = \eta\mu_1(p)^{-n} \cdot W(\tilde{\eta})^{-1} = \left(\frac{p}{a_p(f)^2}\right)^n \cdot \frac{p^{n(k-2)}}{W(\tilde{\eta})} .$$

□

REMARK 4.15: We observe that the results of the above computation match perfectly the shape of the modification of the Euler factor at p (for the Galois theoretic side) described in [Coa91, pagg. 162-163], also in the cases of *bad reduction* at p. ▲

We have some control on the root numbers appearing in proposition 4.14 (case (3)).

LEMMA 4.16: *With the notation introduced above, if $x = k \equiv 2 \mod (p-1)$ and the character $\eta \in \{\varphi, \psi\}$ is ramified, then $W(\tilde{\eta}) = W(\eta) \in \{\pm 1\}$. Moreover the sign $W(\varphi)$ (resp. $W(\psi)$) depends only on the parity of $j_1 = (l+m-2)/2$ (resp. $j_2 = (l-m)/2$).*

PROOF. Note that under our assumptions the character denoted μ_1 above is unramified and $\eta = \tilde{\eta}$ is of finite order and trivial on $\mathbb{Q}_p^\times$. We can thus apply [MS00, proposition 3.7] to a suitable twist of η to deduce that $W(\eta) = \eta^{-1}(\alpha)$, where $\alpha \in \mathbb{Q}_{p^2}^\times$ is a primitive $2(p-1)$-th root of unit, so that $1 = \eta(-1) = \eta(\alpha)^{-2}$. In particular this shows that $W(\eta) \in \{\pm 1\}$.

Recall that $\mathbb{Z}_{p^2}^\times = \mu_{p^2-1} \times (1 + p\mathbb{Z}_{p^2})$. Thus the only way one can affect the sign $W(\eta)$ is changing the weights l, m. More precisely, one can check (cf. remark 4.2) that

$$\varphi|_{\mu_{p^2-1}} = \eta_1\eta_2|_{\mu_{p^2-1}} \cdot (-)^{\frac{(p-1)(l+m-2)}{2}}, \qquad \psi|_{\mu_{p^2-1}} = \eta_1\eta_2^\sigma|_{\mu_{p^2-1}} \cdot (-)^{\frac{(p-1)(l-m)}{2}} .$$

Writing $\alpha = \zeta^{(p+1)/2}$ for ζ a primitive (p^2-1)-th root of 1, we see that the sign $W(\varphi)$ (resp. $W(\psi)$) depends only on the parity of $j_1 = (l+m-2)/2$ (resp. $j_2 = (l-m)/2$). □

4.4. Anticyclotomic p-adic L-functions

As in the introduction, let H_n denote the ring class field of K of conductor cp^n and let H_∞ be the union of all the H_n's. It follows that the big characters $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ (defined in equation 4.4) factor through $\mathscr{G}_\infty := \mathrm{Gal}(H_\infty/K)$. With the same notation as in remark 3.16, we can identify $\Gamma^- = \mathrm{Gal}(K_\infty^-/K)$ (the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K) with the maximal $\mathbb{Z}_p$-free quotient of $\mathscr{G}_\infty$, i.e. there is an exact sequence

$$0 \to \Delta_c \to \mathscr{G}_\infty \to \Gamma^- \to 0$$

of abelian groups with Δ_c a finite group and $\Gamma^- \cong \mathbb{Z}_p$. We fix a non-canonical isomorphism $\mathscr{G}_\infty \cong \Delta_c \times \Gamma^-$ once and for all. Notice that $\boldsymbol{\lambda}_{\mathrm{ac}}$ will factor through Γ^-.

As in lemma 4.16, set $j_1 := \frac{l+m-2}{2}$ and $j_2 := \frac{l-m}{2}$. If we assume moreover that the triple of weights $w = (k, y, z)$ is $\boldsymbol{f}$-unbalanced (i.e. $k \geq l + m$), then it follows immediately that $|j_i| < \frac{k}{2}$ for $i = 1, 2$.

Building up on previous work of Bertolini-Darmon ([BD96], [BD98]) and Chida-Hsieh ([CH18]), Castella and Longo in [CL16] have constructed so-called **big theta elements**, denoted

$$\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f}) \in R_{\boldsymbol{f},\Gamma^-} := \Lambda_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}_L} \mathcal{O}_L[[\Gamma^-]] \tag{4.8}$$

attached to the Hida family $\boldsymbol{f}$ and the quadratic imaginary field K (satisfying a suitable Heegner hypothesis relative to the tame level of $\boldsymbol{f}$). The two variables are given by the weight specializations for $\boldsymbol{f}$ and by continuous characters $\hat{\nu} : \Gamma^- \to \mathbb{C}_p^\times$ such that the associated algebraic Hecke character $\nu : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ has infinity type $(j, -j)$ with $|j| < k/2$. We let $\mathfrak{X}_{p,k}^{\mathrm{crit}}$ to denote the set of characters $\hat{\nu}$ satisfying such requirement for a fixed k. The specializations of the square of $\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f})$ at $(k, \hat{\nu})$ with $k \geq 2$ even integer and $\hat{\nu} \in \mathfrak{X}_{p,k}^{\mathrm{crit}}$ interpolate the (algebraic part of the) special values $L(\boldsymbol{f}_k^\circ / K, \nu, k/2)$.

Following the strategy of Castella and Longo applied to the more general construction of Hung ([Hun17]), one can construct a big theta element $\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f}, \chi_t) \in R_{\boldsymbol{f},\Gamma^-}$ associated with the Hida family $\boldsymbol{f}$ and a branch character χ_t of conductor c (i.e. a character of the finite group Δ_c).

REMARK 4.17: The construction of $\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f}, \chi_t)$ depends on the following choices that we fix from now on:

(a) a factorization $N_{\boldsymbol{f}}^+ \mathcal{O}_K = \mathfrak{N}^+ \cdot \overline{\mathfrak{N}^+}$, where recall that $N_{\boldsymbol{f}}^+$ is the product of the prime divisors of $N_{\boldsymbol{f}}$ that split in K;
(b) a family of quaternionic modular forms $\boldsymbol{\Phi}$ associated to $\boldsymbol{f}$, with the property that there exists an open neighbourhood $U_{\boldsymbol{f}}$ of 2 in $\mathcal{W}_{\Lambda_{\boldsymbol{f}}}(\mathcal{O}_L)$ such that for all $k \in U_{\boldsymbol{f}} \cap \mathbb{Z}_{\geq 2}$ it holds
$$\boldsymbol{\Phi}_k = \lambda_{B,k} \cdot \varphi_k \,,$$
where $\lambda_{B,k} \in L^\times$ and φ_k corresponds to $\boldsymbol{f}_k^\circ$ via a version of the Jacquet-Langlands correspondence.

We can (and will) choose the following normalizations for $\boldsymbol{\Phi}$:

(i) $\lambda_{B,2} = 1$;
(ii) $\eta_{\boldsymbol{f}_k^\circ, N^-} = 1$ for $k \in U_{\boldsymbol{f}} \cap \mathbb{Z}_{>}2$.

The period $\eta_{\boldsymbol{f}_k^\circ, N^-}$ (appearing in the following proposition) is defined as a suitable Petersson norm of φ_k, which we can normalize to be 1 (this will determine φ_k up to sign). We refer to [BD07, theorem 2.5] for the existence of $\boldsymbol{\Phi}$ and its properties and to [CH18, equations 3.9 and 4.3] for the description of $\eta_{\boldsymbol{f}_k^\circ, N^-}$ as Petersson norm (Chida-Hsieh's notation is $\langle f_{\pi'}, f_{\pi'} \rangle_R$). ▲

PROPOSITION 4.18: *Fix an even integer $k \in U_{\boldsymbol{f}} \cap \mathbb{Z}_{\geq 2}$ and a character $\hat{\nu} \in \mathfrak{X}_{p,k}^{\mathrm{crit}}$ of conductor p^n. Write $f = \boldsymbol{f}_k$ and $f^\circ = \boldsymbol{f}_k^\circ$ (with the usual conventions). Then:*

$$\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f}/K, \chi_t)\right)^2 (k, \hat{\nu}) = \lambda_B(k)^2 \cdot C_p(f, \chi_t \nu) \cdot e_p(f, \chi_t \nu) \cdot \frac{L(f^\circ / K, \chi_t \nu, k/2)}{\Omega_{f^\circ, N^-}} \tag{4.9}$$

where:

(i) setting $u_K = \frac{\#\mathcal{O}_K^\times}{2}$ *and* $\delta_K := \sqrt{d_K}$, *one has*

$$C_p(f, \chi_t\nu) := (-1)^{\frac{2+2j-k}{2}} \cdot \Gamma(k/2+j) \cdot \Gamma(k/2-j) \cdot c \cdot \delta_K^{k-1} \cdot u_K^2 \cdot \varepsilon(\pi_{f,p}, 1/2) \cdot \chi_t\nu(\mathfrak{N}^+);$$

(ii)

$$e_p(f, \chi_t\nu) = \begin{cases} \left(\dfrac{p}{a_p(f)^2}\right)^n \cdot p^{n(k-2)} & \text{if } n > 0 \\ \left(1 - \dfrac{p^{k-2}}{a_p(f)^2}\right)^2 & \text{if } n = 0 \text{ and } f \text{ is } p\text{-old} \\ 1 - \dfrac{p^{k-2}}{a_p(f)^2} & \text{if } n = 0 \text{ and } f \text{ is } p\text{-new} \end{cases};$$

(iii) Ω_{f°, N^-} *is Gross's period, that we can write as*

$$\Omega_{f^\circ, N^-} = \frac{(4\pi)^k \cdot \|f^\circ\|_{Pet}^2 \cdot \zeta_{\mathbb{Q}}(2) \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_{\boldsymbol{f}})]}{2 \cdot \eta_{f^\circ, N^-}} . \tag{4.10}$$

PROOF. This follows from the work of Chida-Hsieh [CH18], Hung [Hun17] and Castella-Longo [CL16]. We refer in particular to [CL16, section 4.2] and to [Hun17, theorem 5.6] for the interpolation formula. □

REMARK 4.19: We keep the notation of proposition 4.18. The Heegner hypothesis (iii) on $N_{\boldsymbol{f}}$ in assumption 4.1 implies that the sign of the functional equation for $L(f^\circ/K, \chi, s)$ is +1 for every anticyclotomic Hecke character χ of K of conductor coprime to $N_{\boldsymbol{f}} \cdot d_K$ (unless $k = 2$, $\boldsymbol{f}_2$ is p-new and χ is unramified at p), i.e. we are in the so-called *definite setting*. One of the main results of [Hun17] (namely theorem C in the introduction), generalizing work of Vatsal [Vat02] and Chida-Hsieh [CH18], implies that in our setting it holds $L(f^\circ/K, \chi_t\nu, k/2) \neq 0$ for all but finitely many $\hat{\nu} \in \mathfrak{X}_{p,k}^{\mathrm{crit}}$. ▲

4.5. Factorization of the triple product p-adic L-function

We consider the automorphism s of $R_K \hat{\otimes}_{\mathcal{O}_L} R_K$ in $\hat{\mathcal{C}}_{\mathcal{O}_L}$ given by the assignment

$$[\gamma] \otimes [\delta] \mapsto [\gamma^{1/2}\delta^{1/2}] \otimes [\gamma^{1/2}\delta^{-1/2}]$$

on group-like elements (note that again it is important that $p \neq 2$ for this to be a well-defined automorphism).

Let again K_∞ denote the (unique) $\mathbb{Z}_p^2$-extension of K. Recall (remark 3.16) that the character $\langle\lambda\rangle$ induces an isomorphism $\Gamma_\infty \cong W_K$. The natural projection $\Gamma_\infty \twoheadrightarrow \Gamma^-$ can be described as $\gamma \mapsto \gamma^{1/2}(\gamma^\sigma)^{-1/2}$. Accordingly, we get a morphism

$$\tau : R_K \twoheadrightarrow \mathcal{O}_L[[\Gamma^-]] . \tag{4.11}$$

NOTATION 4.20: We set $\varphi_t := \eta_1\eta_2|_{\Delta_c}$ and $\psi_t := \eta_1\eta_2^\sigma|_{\Delta_c}$. With respect to the chosen isomorphism $\mathscr{G}_\infty \cong \Delta_c \times \Gamma^-$, we also define the characters of Γ^- given by $\varphi^- := \eta_1\eta_2|_{\Gamma^-}$ and $\psi^- := \eta_1\eta_2^\sigma|_{\Gamma^-}$.

Note that the assigments $[\gamma] \mapsto \varphi^-(\gamma)[\gamma]$ (resp. $[\gamma] \mapsto \psi^-(\gamma)[\gamma]$) define $\mathcal{O}_L$-linear automorphisms $\varphi^- : \mathcal{O}_L[[\Gamma^-]] \cong \mathcal{O}_L[[\Gamma^-]]$ (resp. $\psi^- : \mathcal{O}_L[[\Gamma^-]] \cong \mathcal{O}_L[[\Gamma^-]]$), since $|\varphi^-(\gamma) - 1|_p < 1$ (resp. $|\psi^-(\gamma) - 1|_p < 1$) for $\gamma \in \Gamma^-$. By slight abuse of denote by φ^- (resp. ψ^-) the automorphism of $R_{\boldsymbol{f},\Gamma^-}$ given by the identity on Λ and φ^- (resp. ψ^-) on $\mathcal{O}_L[[\Gamma^-]]$.

LEMMA 4.21: *Consider the composition*

$$\mathrm{pr}_{ac} : R_{\boldsymbol{fgh}} \xrightarrow[\cong]{1\otimes s} R_{\boldsymbol{fgh}} \xrightarrow{1\otimes\tau\otimes\tau} \Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\Gamma^-]]\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\Gamma^-]]\,.$$

Given a specialization $(k,\hat{\nu},\hat{\mu}) \in \mathcal{W}^{cl}_{\Lambda_{\boldsymbol{f}},\mathbb{Z}} \times \mathfrak{X}^{\mathrm{crit}}_{p,k} \times \mathfrak{X}^{\mathrm{crit}}_{p,k}$ (with $k \geq 2$ even integer), then the specializations in $\Omega_{\boldsymbol{fgh}}$ which lift $(k,\hat{\nu},\hat{\mu})$ are $\boldsymbol{f}$-unbalanced triples $w = (k,y,z)$ with the property that

$$\hat{\nu} = (yz)|_{\Gamma^-}\cdot\langle\lambda\rangle^{\sigma}\langle\lambda\rangle^{-1}\,, \qquad \hat{\mu} = (y/z)|_{\Gamma^-}\,. \tag{4.12}$$

Moreover, we can always find such $y \in \Omega_{\boldsymbol{g}}$ and $z \in \Omega_{\boldsymbol{h}}$ for given $\hat{\nu}$ and $\hat{\mu}$ such that $w = (k,y,z)$ is $\boldsymbol{f}$-unbalanced.

PROOF. This is an easy exercise. □

NOTATION 4.22: Now let $\sigma_{\mathfrak{N}^+} \in \mathscr{G}_\infty$ denote the projection to $\mathscr{G}_\infty$ of the element of G_K corresponding to $\mathfrak{N}^+$ by class field theory. We write $(\sigma_c, \gamma_{\mathfrak{N}^+}^{-2}) := \sigma_{\mathfrak{N}^+} \in \Delta_c \times \Gamma^- \cong \mathscr{G}_\infty$ to denote the components of $\sigma_{\mathfrak{N}^+}$ according to the fixed isomorphism $\Delta_c \times \Gamma^- \cong \mathscr{G}_\infty$ (note that such $\gamma_{\mathfrak{N}^+} \in \Gamma^-$ is well-defined). We also choose an element $\alpha_c \in \bar{\mathbb{Q}}$ such that $\alpha_c^{-2} = \varphi_t(\sigma_c)\cdot\psi_t(\sigma_c)$. We will also write

$$\mathcal{R}^- := \left(\Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\Gamma^-]]\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\Gamma^-]]\right)[1/p]$$

in what follows.

PROPOSITION 4.23: *There exists an element $\mathscr{A}_{\boldsymbol{fgh}} \in \mathcal{R}^-$ such that*

(i) for infinitely many $k \in U_{\boldsymbol{f}} \cap \mathbb{Z}_{>2}$ and for all $\hat{\nu},\hat{\mu} \in \mathfrak{X}^{\mathrm{crit}}_{p,k}$, it holds (with $f = \boldsymbol{f}_k$ as usual)

$$\mathscr{A}_{\boldsymbol{fgh}}(k,\hat{\nu},\hat{\mu}) = \frac{\eta_f}{\lambda_B(k)\cdot\mathcal{E}_p(f,\mathrm{Ad})\cdot\delta_K^{k-1}}\cdot\varphi^-\hat{\nu}(\gamma_{\mathfrak{N}^+})\cdot\psi^-\hat{\mu}(\gamma_{\mathfrak{N}^+})\cdot\frac{\alpha_c}{c\cdot u_K^2}\,,$$

(ii) for all $\hat{\nu},\hat{\mu} \in \mathfrak{X}^{\mathrm{crit}}_{p,2}$, $\mathscr{A}_{\boldsymbol{fgh}}(2,\hat{\nu},\hat{\mu}) \neq 0$.

PROOF. It follows from [BSV22a, lemma 3.3] that there exists an element $\mathscr{A}_{\boldsymbol{f}} \in \Lambda_{\boldsymbol{f}}[1/p]$ such that for infinitely many $k \in U_{\boldsymbol{f}} \cap \mathbb{Z}_{>2}$ it holds

$$\mathscr{A}_{\boldsymbol{f}_k} = \frac{\eta_f}{\lambda_B(k)\cdot\mathcal{E}_p(f,\mathrm{Ad})\cdot\delta_K^{k-1}}\,.$$

and such that $\mathscr{A}_{\boldsymbol{f}}(2) \neq 0$. We now set

$$u := \frac{\alpha_c\cdot\varphi^-(\gamma_{\mathfrak{N}^+})\cdot\psi^-(\gamma_{\mathfrak{N}^+})}{c\cdot u_K^2} \in L^\times\,.$$

Then the element $\mathscr{A}_{\boldsymbol{fgh}} := u\cdot\left(\mathscr{A}_{\boldsymbol{f}}\hat{\otimes}[\gamma_{\mathfrak{N}^+}]\hat{\otimes}[\gamma_{\mathfrak{N}^+}]\right) \in \mathcal{R}^-$ visibly satisfies the required interpolation property (cf. notation 4.22). □

DEFINITION 4.24: In the setting 4.1, the image of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the map pr_{ac} of lemma 4.21 is denoted by $\mathcal{L}_{p,ac}^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and called the **anticyclotomic projection** of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$.

THEOREM 4.25: *Under the natural identification*

$$\mathcal{R}^- = \left(\Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\Gamma^-]]\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L[[\Gamma^-]]\right)[1/p] \cong \left(R_{\boldsymbol{f},\Gamma^-}\hat{\otimes}_{\Lambda_{\boldsymbol{f}}}R_{\boldsymbol{f},\Gamma^-}\right)[1/p]\,,$$

we have that

$$\mathcal{L}_{p,ac}^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \pm\mathscr{A}_{\boldsymbol{fgh}}\cdot\left(\varphi^-\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\varphi_t)\right)\hat{\otimes}\,\psi^-\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\psi_t)\right)\right) \tag{4.13}$$

as elements of $\mathcal{R}^-$.

PROOF. It is enough to check that squares of both sides of equation 4.13 agree, when specialized to $(k, \hat{\nu}, \hat{\mu})$ for infinitely many $k \in U_{\boldsymbol{f}} \cap \mathbb{Z}_{>}2$ and for every $\hat{\nu}$ and $\hat{\mu}$ finite order characters of Γ^- (so that $\varphi^-\hat{\nu}$ and $\psi^-\hat{\mu}$ lie in $\mathfrak{X}^{\mathrm{crit}}_{p,k}$ for every such k).

We have

$$\mathcal{L}^f_{p,ac}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(k, \hat{\nu}, \hat{\mu}) = \mathcal{L}^f_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(k, y, z)$$

for any y, z satisfying condition 4.12.

On the other hand we have that

$$\varphi^- \left(\Theta^{\mathrm{Heeg}}_\infty(\boldsymbol{f}, \varphi_t)\right)(k, \hat{\nu}) = \Theta^{\mathrm{Heeg}}_\infty(\boldsymbol{f}, \varphi_t)(k, \varphi^-\hat{\nu})$$

and

$$\psi^- \left(\Theta^{\mathrm{Heeg}}_\infty(\boldsymbol{f}, \psi_t)\right)(k, \hat{\mu}) = \Theta^{\mathrm{Heeg}}_\infty(\boldsymbol{f}, \psi_t)(k, \psi^-\hat{\mu}) \, .$$

The result follows putting together the following ingredients:

(i) the factorization of the corresponding complex L-functions (cf. equation (4.2) and lemma 4.7);

(ii) the comparison formulas (4.6) and (4.9);

(iii) our explicit computations for the local factor $\mathscr{I}^{unb}_{\Pi_w,p}$ (cf. proposition 4.14 and lemma 4.16);

(iv) the control on the factor $\mathscr{A}_{\boldsymbol{fgh}}$, as described in proposition 4.23.

□

CHAPTER 5

Derivatives of triple product p-adic L-functions and Heegner points

In this chapter we describe some applications of theorem 4.25. We keep the notation as in the previous section (cf. setting 4.1).

5.1. Heegner points and Tate's parametrization

Let $p > 3$ denote our fixed prime and let $E/\mathbb{Q}$ be an elliptic curve with multiplicative reduction at p. This means that the conductor of E is of the form $N_E = N_E^\circ \cdot p$ with $p \nmid N_E^\circ$. We let $f_E \in S_2(\Gamma_0(N_E))$ to denote the cuspidal newform associated to E via modularity, whose q-expansion at ∞ will be denoted

$$f_E = \sum_{n=1}^{+\infty} a_n(E) q^n \, .$$

In particular we have $a_n(E) \in \mathbb{Z}$ for all $n \geq 1$ and $a_p(E) = 1$ (resp. $a_p(E) = -1$) if E has split (resp. non-split) multiplicative reduction at p. We write $\alpha := a_p(E) \in \{\pm 1\}$ in the sequel.

Hida theory shows that there exists a unique primitive Hida family

$$\boldsymbol{f} \in \mathbb{S}^{ord}(N_{\boldsymbol{f}}, \mathbf{1}, \Lambda_{\boldsymbol{f}})$$

of tame level $N_{\boldsymbol{f}} := N_E^\circ$ and trivial tame character, such that $\boldsymbol{f}_2 = f_E$.

This family will play the role of the Hida family $\boldsymbol{f}$ of the previous section. As for the rest, we keep working in the setting 4.1 and, possibly, add further restrictions. In particular, the conductor N_E of our elliptic curve E is squarefree and to satisfies a suitable *Heegner hypothesis* with respect to the fixed quadratic imaginary field K.

For our applications, we are led to impose one further condition throughout this section.

Assumption 5.1: $\varphi = \eta_1\eta_2$ has conductor prime to p and $\psi = \eta_1\eta_2^\sigma$ has non-trivial anticyclotomic part ψ^-.

With the notation of section 4, it follows that φ^- is trivial and that we can identify $\varphi_t = \varphi$.

Following the discussion in [BD07, section 4.3], one can define a Heegner point

$$P_\varphi \in \begin{cases} E(H_\varphi)^\varphi & \text{if } \varphi \neq 1 \\ E(K) \otimes \mathbb{Q} & \text{if } \varphi = 1 \end{cases} \tag{5.1}$$

associated with φ, essentially coming from a parametrisation of E in terms of the Jacobian of a suitable Shimura curve. Here H_φ is the field cut out by φ. Note that, since p is inert in K and H_φ is contained in the Hilbert class field of K, it follows that p splits completely in H_φ, so that we can fix an embedding $H_\varphi \subset \mathbb{Q}_{p^2}$ and view the point P_φ as a point in $E(\mathbb{Q}_{p^2})\otimes\mathbb{Q}$. Under this identification, the Galois actions on P_φ of the Frobenius (as generator of $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$) and of any Frobenius element for the abelian extension $H_\varphi/\mathbb{Q}$ coincide. It follows that the points

$$P^{\pm}_{\varphi,\alpha} := P_\varphi \pm \alpha\cdot P_\varphi^{\mathrm{Frob}_{\mathfrak{p}}} \in E(H_\varphi)\otimes\mathbb{Q}\,.$$

do not depend on the choice of prime $\mathfrak{p}$ of H_φ above p. In what follows, we fix the choice induced by our fixed embedding $\iota_p:\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$ and we view the points P_φ and $P^{\pm}_{\varphi,\alpha}$ as elements of $E(\mathbb{Q}_{p^2})\otimes\mathbb{Q}$ under such an embedding.

Since E has multiplicative reduction at p, it admits a Tate parametrisation, i.e., there is an isomorphism of rigid analyitic varieties

$$\Phi_{\mathrm{Tate}}:\mathbb{G}^{\mathrm{rig}}_{m,\mathbb{Q}_{p^2}}/q_E^{\mathbb{Z}}\xrightarrow{\cong}E^{\mathrm{rig}}_{\mathbb{Q}_{p^2}}. \tag{5.2}$$

One can define the branch $\log_{q_E}:\mathbb{C}_p^\times\to\mathbb{C}_p$ of the p-adic logarithm, uniquely determined by the condition $\log_{q_E}(q_E)=0$, where $q_E\in p\mathbb{Z}_p$ is Tate's p-adic period associated with E. This yields a logarithm

$$\log_E := \log_{q_E}\circ\Phi_{\mathrm{Tate}}^{-1}:E(\mathbb{Q}_{p^2})\to\mathbb{Q}_{p^2} \tag{5.3}$$

at the level of $\mathbb{Q}_{p^2}$-rational points.

5.2. Restriction to the line $(k,1,1)$

We now restrict our attention to the *line* $(k,1,1)$. Recall that $y=1$ (or $z=1$) means that we consider the specializations given by $y([u])=z([u])=u$ on group-like elements $u\in W_K$. For the first variable, we let k vary in $U_{\boldsymbol{f}}\cap\mathbb{Z}_{\ge 2}$ (same notation as in remark 4.17). The corresponding characters of Γ^- via equation 4.12 are clearly both the trivial character 1_{Γ^-}.

An easy check shows that, with this choice of specializations, the square of the element

$$\mathcal{L}_p(\boldsymbol{f}/K,\varphi):=\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\varphi_t)(\,\cdot\,,1_{\Gamma^-})\in\Lambda$$

interpolates the algebraic part of the special values $L(\boldsymbol{f}_k^\circ/K,\varphi,k/2)$, at least when $k>2$. For $k=2$ the p-adic multiplier $e_p(f_E,\varphi)$ (cf. proposition 4.18) vanishes, as a manifestation of a so-called **exceptional zero** for our p-adic L-function.

Moreover, we see that the element $\mathcal{L}_p(\boldsymbol{f}/K,\varphi)$ coincides with the square-root Hida-Rankin p-adic L-function attached to $\boldsymbol{f}$ and φ in [BD07]. This follows comparing the above stated interpolation formula 4.9 and the one of [BD07, theorem 3.8].

We can now state one of the main results of [BD07] (extended to the case of not necessarily quadratic characters $\varphi=\eta_1\eta_2$).

THEOREM 5.2: *([BD07, theorem 4.9]) In the setting described above, it holds*

$$\frac{d}{dk}\mathcal{L}_p(\boldsymbol{f}/K,\varphi)_{|k=2}=\frac{\log_E(P^+_{\varphi,\alpha})}{2}$$

.

DEFINITION 5.3: We set $\mathcal{L}_p(\boldsymbol{f}/K,\psi):=\psi^-\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\psi_t)\right)(\,\cdot\,,1_{\Gamma^-})\in\Lambda_{\boldsymbol{f}}$ and we define the restriction to the line $(k,1,1)$ of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ as

$$\mathcal{L}_p^f(\boldsymbol{f},g,h):=\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(\,\cdot\,,1,1)=\mathcal{L}_{p,ac}^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(\,\cdot\,,1_{\Gamma^-},1_{\Gamma^-})\in\Lambda_{\boldsymbol{f}}\,.$$

COROLLARY 5.4: *In the above setting (in particular under assumption 5.1), assume that* $L(f_E/K,\psi,1)\neq 0$. *Then* $\mathcal{L}_p^f(\boldsymbol{f},g,h)(2)=0$ *and*

$$\frac{d}{dk}\mathcal{L}_p^f(\boldsymbol{f},g,h)_{|k=2}=\frac{c_E}{2}\cdot\log_E(P^+_{\varphi,\alpha}),$$

where $c_E=\pm\mathscr{A}_{\boldsymbol{fgh}}(2,\hat{\nu}_{1,1},\hat{\mu}_{1,1})\cdot\mathcal{L}_p(\boldsymbol{f}/K,\psi)(2)\in\bar{\mathbb{Q}}_p^\times$.

In particular, $\frac{d}{dk}\mathcal{L}_p^f(\boldsymbol{f},g,h)_{|k=2}=0$ *if and only if the point* $P^+_{\varphi,\alpha}$ *is of infinite order.*

PROOF. This follows immediately from the above theorem 5.2, the running hypothesis, lemma 4.23 and the factorization proven in theorem 4.25. Note that

$$\mathcal{L}_p(\boldsymbol{f}/K,\psi)(2)=\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\psi_t)(2,\psi^-)\neq 0\,.$$

Indeed, by assumption 5.1, we have that ψ^- is non-trivial, so that the p-adic multiplier $e_p(\boldsymbol{f}_k,\psi_t\psi^-)$ of the interpolation formula 4.9 never vanishes for $k\in U\cap\mathbb{Z}_{\geq 2}$. □

REMARK 5.5: Note that (cf. remark 4.19) the condition $L(f_E/K,\psi,1)\neq 0$ is *generically* expected to be satisfied. ▲

5.3. Restriction to the line $(2,\nu,\nu)$

In this section we fix the weight $k=2$ and we let the anticyclotomic twists vary along the *diagonal* of $\mathfrak{X}_{p,2}^{\mathrm{crit}}\times\mathfrak{X}_{p,2}^{\mathrm{crit}}$. In this situation, $\mathfrak{X}_{p,2}^{\mathrm{crit}}$ is given by finite order characters of Γ^-.

DEFINITION 5.6: We define the restriction of $\mathcal{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to the line $(2,\nu,\nu)$ as

$$\mathcal{L}_{p,ac}^f(f_E,\boldsymbol{gh}):=\mathcal{L}_{p,ac}^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_{|k=2,\hat{\nu}=\hat{\mu}}\in\mathcal{O}_L[[\Gamma^-]]\,.$$

We also set

$$\theta_\infty(E/K,\varphi):=\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\varphi_t)_{|k=2}\in\mathcal{O}_L[[\Gamma^-]]$$

and

$$\theta_\infty(E/K,\psi):=\psi^-\left(\Theta_\infty^{\mathrm{Heeg}}(\boldsymbol{f},\psi_t)\right)_{|k=2}\in\mathcal{O}_L[[\Gamma^-]]\,.$$

One can check that, under our assumptions, the element $\theta_\infty(E/K,\varphi)$ coincides with the theta-element defined by Bertolini-Darmon (cf. [BD96, section 2.7]) in the case of trivial tame character and in more generality by Chida-Hsieh ([CH18]) and Hung ([Hun17]). Similarly, the element $\theta_\infty(E/K,\psi)$ is essentially a shift a such a theta-element.

Any choice of topological generator $\gamma_0\in\Gamma^-$ gives rise to a topological isomorphism

$$\mathcal{O}_L[[\Gamma^-]]\cong\mathcal{O}_L[[T]] \tag{5.4}$$

sending γ_0 to $1+T$. One of the main results of [BD98] can be stated as follows.

THEOREM 5.7: *(cf. [BD98, theorem B]) The element* $\theta_\infty(E/K,\varphi)$ *lies in the augmentation ideal of* $\mathcal{O}_L[[\Gamma^-]]$. *Equivalently, viewing* $\theta_\infty(E/K,\varphi)$ *as an element of* $\mathcal{O}_L[[T]]$ *via the above identification 5.4, we have*

$$\theta_\infty(E/K,\varphi)\in T\cdot\mathcal{O}_L[[T]]\,.$$

Moreover, taking derivatives we obtain

$$\frac{d}{dT}\theta_\infty(E/K,\varphi)_{|T=0}=\log_E(P^-_{\varphi,\alpha}),$$

This formula does not depend on the choice of a topological generator of Γ^-.

This leads to the following result concerning our triple product p-adic L-function.

COROLLARY 5.8: *In the above setting (in particular under assumption 5.1), assume that* $L(f_E/K,\psi,1)\neq 0$. *View* $\mathcal{L}^f_{p,ac}(f_E,\boldsymbol{gh})$ *as an element of* $\mathcal{O}_L[[T]]$ *via* (5.4). *Then*

$$\mathcal{L}^f_{p,ac}(f_E,\boldsymbol{gh})_{|T=0}=0$$

and

$$\frac{d}{dT}\mathcal{L}^f_{p,ac}(f_E,\boldsymbol{gh})_{|T=0}=c_E\cdot\log_E(P^-_{\varphi,\alpha})\,,$$

where $c_E\in\bar{\mathbb{Q}}_p^\times$ *is the same explicit constant as in corollary 5.4.*

PROOF. This follows essentially from the above theorem 5.7, the factorization of theorem 4.25 and the running hypothesis, in the same way as corollary 5.4. $\square$

5.4. A corollary

Keeping the same setting as in the previous sections (in particular assumption 5.1, we impose moreover that $\varphi=\varphi_t$ is a quadratic (or genus) character of K.

As explained in [BD07, section 3.1], if the quadratic character φ is non-trivial, it cuts out a biquadratic extension $H_\varphi=\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ where d_i is a fundamental discriminant for $i=1,2$ and $d_1d_2=-d_K$. If we define φ_i to be the Dirichlet character attached to the quadratic extension $\mathbb{Q}(\sqrt{d_i})$ for $i=1,2$, one sees that $\varphi_1\varphi_2=\varepsilon_K$. In particular we get

$$\varphi_1(-N_E)\varphi_2(-N_E)=\varepsilon_K(-N_E)=-1$$

where the last equality follows from our Heegner assumption.

When φ is trivial, one sets $H_\varphi=K$ (this situation corresponds to the case $\{d_1,d_2\}=\{1,-d_K\}$).

If $\lambda_E\in\{\pm1\}$ denotes the eigenvalue relative to f_E for the Atkin-Lehner involution w_{N_E}, we can always assume (up to reordering) that

$$\varphi_1(-N_E)=\lambda_{N_E}\,,\qquad \varphi_2(-N_E)=-\lambda_{N_E}\,.$$

Moreover, it follows from [BD07, corollary 4.8] that

$$P_\varphi^{\mathrm{Frob}_\mathfrak{p}}=\varphi_1(p)P_\varphi\,. \tag{5.5}$$

Here is a corollary combining the discussion of the previous sections.

COROLLARY 5.9: *In the setting described by assumptions 4.1 and 5.1, assume that* $\varphi=\varphi_t$ *is quadratic and that* $L(f_E/K,\psi,1)\neq0$. *Then the following facts are equivalent:*

(i)

$$\left(\frac{d}{dk}\mathcal{L}^f_p(\boldsymbol{f},g,h)_{|k=2}\,,\frac{d}{dT}\mathcal{L}^f_{p,ac}(f_E,\boldsymbol{gh})_{|T=0}\right)\neq(0,0)$$

(ii) The point P_φ *is of infinite order.*

PROOF. Equation 5.5 shows that, under our assumptions,

$$P^\pm_{\varphi,\alpha}=\begin{cases}2\cdot P_\varphi & \text{if } \varphi_1(p)\alpha=\pm1\\ 0 & \text{if } \varphi_1(p)\alpha=\mp1\end{cases}$$

Then the result follows immediately from corollaries 5.4 and 5.8 and the fact that the kernel of $\log_E$ is given by finite order points in $E(\mathbb{Q}_{p^2})$. $\square$

CHAPTER 6

Balanced diagonal classes and the étale Abel-Jacobi map

Fix a prime number p. In this chapter we introduce the facts about étale and de Rham cohomology of modular curves with p-adic coefficients which are needed to define the balanced diagonal classes and the étale Abel-Jacobi map alluded to in the title above. Most of the material is covered in more detail in [BDP13, section 1] and/or in [BSV22b, sections 2 and 3] and in the references given therein.

6.1. De Rham of modular curves

For every integer $N \geq 5$, we let $Y_1(N)$ denote the *open* modular curve of level $\Gamma_1(N)$, defined over $\mathbb{Z}[1/N]$, classifying isomorphism classes of pairs (E, P) where E is a family of elliptic curves over a $\mathbb{Z}[1/N]$-scheme S and $P \in E(S)$ is a section of exact order N. The curve $Y_1(N)$ is affine and smooth over $\mathbb{Z}[1/N]$. The universal elliptic curve arising from this moduli problem will be denoted $u_1(N) : \mathscr{E}_1(N) \to Y_1(N)$.

As explained in [KM85, chapter 8], the normalization of the projective j-line in $Y_1(N)$ is a smooth projective curve over $\mathbb{Z}[1/N]$, usually denoted by $X_1(N)$. The curve $X_1(N)$ contains $Y_1(N)$ as an open subscheme and the complement $C_1(N)$ of $Y_1(N)$ in $X_1(N)$, with the reduced subscheme structure, is finite and étale over $\mathbb{Z}[1/N]$. It is usually called the subscheme of cusps. Indeed, over $\mathbb{Z}[1/N, \zeta_N]$ (where ζ_N is a fixed primitive N-th root of unity in $\bar{\mathbb{Q}}$), the scheme $C_1(N)$ is simply given by a disjoint union of points (usually one refers to those as the cusps).

For any $\mathbb{Z}[1/N]$-algebra R, we let $Y_1(N)_R$ (resp. $X_1(N)_R$), $C_1(N)_R$) denote the base change of $Y_1(N)$ (resp. $X_1(N)$, $C_1(N)$) to R.

In what follows, F will be a field of characteristic zero and we write $Y := Y_1(N)_F$, $X := X_1(N)_F$, $C := C_1(N)_F$. We let, moreover, $u : \mathscr{E} \to Y$ denote the universal elliptic curve.

The notation $(\mathrm{Tate}(q), P_{\mathrm{Tate}})_{/F((q^{1/d}))}$ will denote the Tate elliptic curve $\mathbb{G}_m/q^{\mathbb{Z}}$, with a $\Gamma_1(N)$-level structure P_{Tate} defined over $F[\zeta_N]((q^{1/d}))$ for some $d \mid N$. We let ω_{can} denote the canonical differential $\omega_{\mathrm{can}} := dT/T$ over $F((q))$, where T is the parameter on $\mathbb{G}_m$. The level structure P_{Tate} corresponds (over $F[\zeta_N]((q^{1/d}))$) to a point of order N on $\mathrm{Tate}(q)$, i.e., a point of the form $\zeta_N^i q^{j/d}$ for some $i \in \{1, \dots, N-1\}$ and $j \in \{1, \dots, d-1\}$ with $(i, N) = 1$ and $(j, d) = 1$.

We let $\omega := u_* \Omega^1_{\mathscr{E}/Y}$ be the line bundle of relative differentials on $\mathscr{E}/Y$ and we let $\mathscr{H} = R^1 u_*(\Omega^\bullet_{\mathscr{E}/Y})$ be the relative de Rham cohomology sheaf on Y. The latter is a rank 2 vector bundle over Y, equipped with Hodge filtration

$$0 \to \omega \to \mathscr{H} \to \omega^{-1} \to 0\,. \tag{6.1}$$

The vector bundle $\mathscr{H}$ is also equipped with the so-called Gauss-Manin connection

$$\nabla : \mathscr{H} \to \mathscr{H} \otimes \Omega^1_Y . \tag{6.2}$$

The Kodaira-Spencer map is the composite

$$KS : \omega \to \mathscr{H} \xrightarrow{\nabla} \mathscr{H} \otimes \Omega^1_Y \to \omega^{-1} \otimes \Omega^1_Y \tag{6.3}$$

induced by the Gauss-Manin connection and the Hodge filtration. It turns out that KS is an $\mathcal{O}_Y$-linear isomorphism of sheaves, giving rise to an identification $\omega^2 \cong \Omega^1_Y$ of line bundles on Y.

One can extend ω and $\mathscr{H}$ to sheaves on the projective curve X, which will be again denoted (resp.) ω and $\mathscr{H}$. One way to proceed is to view X as moduli space for a suitable moduli problem involving generalised elliptic curves. Since these constructions are well-known and appear often in the literature, we only introduce the facts which are needed in this work.

The sheaf ω on X is essentially characterised by the fact that for $F = \mathbb{C}$ there is an identification $H^0(X, \omega^k) = M_k(\Gamma_1(N))$ (where $M_k(\Gamma_1(N))$ denotes the $\mathbb{C}$-vector space of holomorphic modular forms of level $\Gamma_1(N)$). The local sections of ω in the (formal) neighbourhood $\mathrm{Spec}(F[\zeta_N][[q^{1/d}]])$ of the cusp attached to the pair $(\mathrm{Tate}(q), \zeta_N q^{1/d})$ are expressions of the form $h \cdot \omega_{\mathrm{can}}$ with $h \in F[\zeta_N][[q^{1/d}]]$ and ω_{can} the canonical differential on the Tate curve.

The extension of $\mathscr{H}$ is then determined by the extension of ω and the Hodge filtration (6.1). The Gauss-Manin connection ∇ extends to a connection with logarithmic poles at the cusps

$$\nabla : \mathscr{H} \to \mathscr{H} \otimes \Omega^1_X \langle C \rangle \tag{6.4}$$

The local sections of $\mathscr{H}$ in a neighbourhood of $(\mathrm{Tate}(q), \zeta_N q^{1/d})$ are $F[\zeta_N][[q^{1/d}]]$-linear combinations of ω_{can} and the local section η_{can}, which is defined by the equation

$$\nabla \omega_{\mathrm{can}} = \eta_{\mathrm{can}} \otimes \frac{dq}{q} \tag{6.5}$$

Over $\mathrm{Spec}(F[\zeta_N][[q^{1/d}]])$, the Gauss-Manin connection is completely determined by the above equation and by $\nabla \eta_{\mathrm{can}} = 0$.

The Kodaira-Spencer map gives rise to an isomorphism $\sigma : \omega^2 \xrightarrow{\cong} \Omega^1_X \langle C \rangle$, which over $\mathrm{Spec}(F[\zeta_N][[q^{1/d}]])$ is simply described by $\sigma(\omega^2_{\mathrm{can}}) = \frac{dq}{q}$.

A possible definition of modular forms of weight $k \geq 2$ and level $\Gamma_1(N)$ with F-coefficients is then given by

$$M_k(\Gamma_1(N), F) := H^0(X, \omega^k) = H^0(X, \omega^{k-2} \otimes \Omega^1_X \langle C \rangle)$$

with subspace of cusp forms defined as

$$S_k(\Gamma_1(N), F) := H^0(X, \omega^{k-2} \otimes \Omega^1_X) .$$

We now let, for $r \geq 1$, $\mathscr{H}_k := \mathrm{Sym}^r(\mathscr{H})$ (which we view as a sheaf on Y or X, depending on the context). The sheaf $\mathscr{H}_r$ is endowed with a Hodge filtration

$$\mathscr{H}_r \supset \mathscr{H}_{r-1} \otimes \omega \supset \cdots \supset \omega^r$$

induced by the filtration (6.1). We will set $\mathrm{Fil}^i \mathscr{H}_r := \mathscr{H}_{r-i} \otimes \omega^i$ for $i = 0, \ldots, r$.

The Gauss-Manin connection 6.4 extends to a connection $\nabla_r : \mathscr{H}_r \to \mathscr{H}_r \otimes \Omega^1_X\langle C\rangle$ which satisfies Griffiths transversality

$$\nabla(\mathrm{Fil}^{i+1}\mathscr{H}_r) \subseteq \mathrm{Fil}^i\mathscr{H}_r \otimes \Omega^1_X\langle C\rangle \qquad (0 \le i \le r-1)$$

and induces isomorphisms on the graded pieces

$$\mathrm{Gr}^{i+1}\mathscr{H}_r \cong \mathrm{Gr}^i\mathscr{H}_r \otimes \Omega^1_X\langle C\rangle \qquad (0 \le i \le r-1)\,.$$

We let $\mathscr{L} := \mathcal{H}om_{\mathcal{O}_X}(\mathscr{H}, \mathcal{O}_X)$ be the dual of $\mathscr{H}$ and for $r \ge 1$ we set $\mathscr{L}_r := \mathrm{Tsym}^r(\mathscr{L})$ (the notation Tsym refers to the submodule of symmetric tensors). We use the same notations for the restrictions of these sheaves to Y. Also $\mathscr{L}$ (and consequently $\mathscr{L}_r$) is equipped with an induced Hodge filtration and integrable connection. For $\mathscr{F} \in \{\mathscr{H}, \mathscr{L}\}$ we finally define the de Rham cohomology groups

$$H^i_{\mathrm{dR}}(X, \mathscr{F}_r) := \mathbb{H}^i(X, \mathscr{F}_r \xrightarrow{\nabla_r} \mathscr{F}_r \otimes \Omega^1_X\langle C\rangle)\,, \tag{6.6}$$

where $\mathbb{H}$ denotes hypercohomology. These de Rham cohomology groups are naturally endowed with a two-step descending filtration. In particular, one has an identification

$$\mathrm{Fil}^i H^1_{\mathrm{dR}}(X, \mathscr{H}_r) \otimes_F F[\zeta_N] = M_{r+2}(\Gamma_1(N), F) \otimes_F F[\zeta_N] \qquad i = 1, \ldots, r+1\,.$$

Since we are working with connections with logarithmic poles at the cusps, the de Rham cohomology groups defined above can actually be defined on the open modular curve Y, i.e., there is a natural isomorphism of filtered F-vector spaces

$$H^i_{\mathrm{dR}}(X, \mathscr{F}_r) \cong H^i_{\mathrm{dR}}(Y, \mathscr{F}_r|_Y)\,. \tag{6.7}$$

One can similarly define de Rham cohomology with compact support $H^i_{\mathrm{dR},c}(Y, \mathscr{F}_r)$.

Over Y, there is a perfect pairing (essentially coming from Poincaré duality on the universal elliptic curve $\mathscr{E}$)

$$(\ ,\ ) : \mathscr{H} \otimes_{\mathcal{O}_Y} \mathscr{H} \to \mathcal{O}_Y[-1]\,.$$

Here $\mathcal{O}_Y[n]$ is the sheaf $\mathcal{O}_Y$, with trivial connection and shifted filtration (i.e., $\mathrm{Fil}^j\mathcal{O}_Y[n] = \mathcal{O}_Y$ if $j \le -n$ and $\mathrm{Fil}^j\mathcal{O}_Y[n] = 0$ if $j \ge 1-n$). Such a pairing induces a perfect duality

$$(\ ,\ )_r : \mathscr{H}_r \otimes_{\mathcal{O}_Y} \mathscr{H}_r \to \mathcal{O}_Y[-r]\,, \tag{6.8}$$

which at the level of de Rham cohomology becomes a perfect duality

$$(\ ,\ )_{\mathrm{dR},Y,r} : H^1_{\mathrm{dR}}(Y, \mathscr{H}_r) \otimes_F H^1_{\mathrm{dR},c}(Y, \mathscr{H}_r) \to H^2_{\mathrm{dR},c}(Y, \mathcal{O}_Y[-r]) \cong F[-r-1]\,. \tag{6.9}$$

6.2. Étale cohomology of modular curves

The sheaves $\mathscr{H}_r$ (resp. $\mathscr{L}_r$) admit (Kummer) pro-étale versions. One can work in the more classical setting of [FK88, §12] and obtain locally constant p-adic sheaves $\mathscr{H}_r$ (resp. $\mathscr{L}_r$) on $Y_1(N)$, so that it makes sense to study the cohomology groups $H^j_{\text{ét}}(Y_1(N)_R, \mathscr{H}_r)$ (resp. $H^j_{\text{ét}}(Y_1(N)_R, \mathscr{L}_r)$) for any $\mathbb{Q}$-algebra R (cf. [BSV22b, section 2.3] and the references therein). In this section, we prefer to focus on the rigid analytic (or better adic) setting and to adopt the more modern approach of [Sch13] (and its various generalizations, for instance [Dia+23]).

In this section we denote by F a complete discretely valued field of mixed characteristic $(0, p)$ with perfect residue field and fixed algebraic closure $\bar{F}$.

Notation 6.1: For a rigid analytic variety S over F, we write $\hat{\mathbb{Z}}_{p,S}$ (resp. $\hat{\mathbb{Q}}_{p,S}$) to denote the constant sheaf on $S_{\text{proét}}$ associated to $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) and for a $\mathbb{Z}_p$-local system $\mathbb{L}$ (resp. $\mathbb{Q}_p$-local system) on $S_{\text{ét}}$, we let $\widehat{\mathbb{L}}$ to denote the lisse $\hat{\mathbb{Z}}_{p,S}$-module (resp. $\hat{\mathbb{Q}}_{p,S}$-module) on $S_{\text{proét}}$ associated with $\mathbb{L}$ (cf. the discussion in [Sch13, section 8.2]). We let $H^j_{\text{ét}}(S,\mathbb{L})$ (resp. $H^j_{\text{ét}}(S_{\bar{F}},\mathbb{L})$) denote the p-adic étale cohomology groups.

For A a complete subring of $\mathbb{C}_p$ and $\mathcal{F}$ a $\hat{\mathbb{Z}}_{p,S}$-module, we also set $\mathcal{F}_A \coloneqq \mathcal{F} \otimes_{\hat{\mathbb{Z}}_{p,S}} \hat{A}_S$, where $\hat{A}_S$ is the constant pro-étale sheaf attached to A.

Remark 6.2: According to [Sch13, proposition 8.2], the association $\mathbb{L} \to \widehat{\mathbb{L}}$ defines an equivalence between the category of $\mathbb{Z}_p$-local systems on $S_{\text{ét}}$ and the category of lisse $\hat{\mathbb{Z}}_{p,S}$-modules on $S_{\text{proét}}$. ▲

In what follows we view the algebraic varieties $Y, X, \mathscr{E}$ introduced in section 6.1 as rigid analytic varieties over $\mathbb{Q}_p$ (or better locally noetherian adic spaces over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$), without changing the notation.

Definition 6.3: We let $\mathcal{H}^1(\mathscr{E}) \coloneqq R^1u_*\hat{\mathbb{Z}}_{p,\mathscr{E}}$ to be the first relative pro-étale cohomology sheaf of the family $u : \mathscr{E} \to Y$. We also let $\mathcal{T}_p(\mathscr{E}) \coloneqq \mathcal{H}om_{\hat{\mathbb{Z}}_{p,Y}}(\mathcal{H}^1(\mathscr{E}),\hat{\mathbb{Z}}_{p,Y})$ denote the relative p-adic Tate module of the family $u : \mathscr{E} \to Y$.

The sheaves $\mathcal{H}^1(\mathscr{E})$ and $\mathcal{T}_p(\mathscr{E})$ are (pro-étale versions of) rank 2 $\mathbb{Z}_p$-local systems. The perfect relative p-adic Weil pairing can then be seen as a perfect pairing

$$\mathcal{T}_p(\mathscr{E}) \otimes_{\hat{\mathbb{Z}}_{p,Y}} \mathcal{T}_p(\mathscr{E}) \to \hat{\mathbb{Z}}_{p,Y}(1)\,, \tag{6.10}$$

under which one obtains an isomorphism $\mathcal{T}_p(\mathscr{E}) \cong \mathcal{H}^1(\mathscr{E})(1)$.

Here $\hat{\mathbb{Z}}_{p,Y}(1)$ is the Tate twist of $\hat{\mathbb{Z}}_{p,Y}$ (one can obtain it as usual as $\varprojlim_n \mu_{p^n,Y}$ with Galois action given by the p-adic cyclotomic character). More generally for a lisse $\hat{\mathbb{Z}}_{p,Y}$-module and every $n \in \mathbb{Z}$, we let $\mathcal{F}(n) \coloneqq \mathcal{F} \otimes_{\hat{\mathbb{Z}}_{p,Y}} \hat{\mathbb{Z}}_{p,Y}(n)$ (with the usual conventions on higher Tate twists).

Definition 6.4: For every integer $r \geq 0$, we define $\hat{\mathbb{Z}}_{p,Y}$-modules $\mathscr{H}_r \coloneqq \mathrm{Sym}^r(\mathcal{H}^1(\mathscr{E}))$ and $\mathscr{L}_r \coloneqq \mathrm{Tsym}^r(\mathcal{T}_p(\mathscr{E}))$ on $Y_{\text{proét}}$.

Clearly, the relative Weil pairing (6.10) induces for all integers $r \geq 0$ an isomorphism

$$s_r : \mathscr{H}_{r,\mathbb{Q}_p} \xrightarrow{\cong} \mathscr{L}_{r,\mathbb{Q}_p}(-r) \tag{6.11}$$

(we have to invert p to obtain an isomorphism when we pass to Sym^r and Tsym^r).

The careful reader will immediately notice the abuse of notation for the symbols $\mathscr{H}_r$ and $\mathscr{L}_r$. This is justified in the following remark.

Remark 6.5: Let S be any smooth rigid analytic variety over $\mathbb{Q}_p$ (or any smooth locally noetherian adic space over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$). As explained in [Sch13] (and extended in [Dia+23] to the case of not necessarily proper varieties), when a $\mathbb{Z}_p$-local system $\mathbb{L}$ on $S_{\text{ét}}$ is *de Rham* (cf. [Sch13, definition 8.3], [LZ17, theorem 3.9]), one can attach to it a pair $(\mathcal{E},\nabla)$, where $\mathcal{E}$ is a filtered vector bundle on S and $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_S$ is an integrable connection satisfying Griffiths transversality, in such a way that for all integers $j \geq 0$ there is a canonical isomorphism

$$\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Z}_p} H^j_{\text{ét}}(S_{\bar{\mathbb{Q}}_p},\mathbb{L}) \cong \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^j_{\mathrm{dR}}(S,\mathcal{E})\,, \tag{6.12}$$

implying that

$$\mathbb{D}_{\mathrm{dR}}(H^j_{\text{ét}}(S_{\bar{\mathbb{Q}}_p},\mathbb{L})) := H^0\left(\mathbb{Q}_p, \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Z}_p} H^j_{\text{ét}}(S_{\bar{\mathbb{Q}}_p},\mathbb{L})\right) \cong H^j_{\mathrm{dR}}(S,\mathcal{E})\,.$$

Let us mention here two more features of this kind of results.

(i) One can extend such result to a comparison between Kummer-étale cohomology and de Rham cohomology with poles along a divisor, considering the datum of S as above together with a normal crossing divisor $D \subset S$. More precisely one can define a notation of *de Rham* Kummer-étale $\mathbb{Z}_p$-local system on (S, D) and, given such a local system $\mathbb{L}$, one can attach to it a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a filtered vector bundle on S and ∇ is an integrable log connection (same as 6.4 above) satisfying Griffiths transversality (cf. [DF23, theorem 1.7]). We will need this when $S = X_1(N)_{\mathbb{Q}_p}$ and $D = C_1(N)_{\mathbb{Q}_p}$ (the cusps).

Moreover, the étale analogue of the isomorphism 6.7 holds, i.e., there are canonical identifications

$$H^i_{\text{két}}(X_1(N)_{\bar{\mathbb{Q}}_p}, \mathscr{F}_r(n)) \cong H^i_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}_p}, \mathscr{F}_r(n)) \tag{6.13}$$

for $\mathscr{F} \in \{\mathscr{H}, \mathscr{L}\}$, $r \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$ (note that abusively we do not change the notation for the coefficients on the Kummer étale side).

(ii) The whole picture extends also to the case of cohomology with compact support. We refer to [LLZ23] for this generalization.

It is clear that, under the association $\mathbb{L} \mapsto (\mathcal{E}, \nabla)$, the $\hat{\mathbb{Z}}_p$-local system $\mathcal{H}^1(\mathscr{E})$ is sent to $(\mathscr{H}, \nabla)$, whence the choice of keeping the same notation for $\mathscr{H}_r$ and $\mathscr{L}_r$ for $r \in \mathbb{Z}_{\geq 0}$. ▲

We conclude this section recalling the needed facts concerning the Hecke action on cohomology groups.

(i) For $\mathscr{F} \in \{\mathscr{H}, \mathscr{L}\}$ one can define the action of Hecke operators T_ℓ for primes $\ell \nmid N$ and U_ℓ for $\ell \mid N$ and of dual Hecke operators T'_ℓ for $\ell \nmid N$ and U'_ℓ for $\ell \mid N$ on the cohomology groups $H^j_{\text{ét}}(Y_1(N)_R, \mathscr{F}_r)$ for any $\mathbb{Q}$-algebra R. We refer to [BSV22b, section 2.3] (where $\mathscr{H}$ is denoted $\mathscr{S}$) for the precise definition of these operators (as usual they arise from the suitable Hecke correspondences).

(ii) One can also define, for every unit $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, a diamond operator $\langle d\rangle$ and a dual diamond operator $\langle d\rangle'$ on $H^j_{\text{ét}}(Y_1(N)_R, \mathscr{F}_r)$ for any $\mathbb{Q}$-algebra R. An Atkin-Lehner operator w_N (and its dual w'_N) acts on $H^j_{\text{ét}}(Y_1(N)_R, \mathscr{F}_r)$ for any $\mathbb{Q}[\zeta_N]$-algebra R, where ζ_N is a fixed N-th root of unity in $\bar{\mathbb{Q}}$. We refer to [BSV22b, paragraph 2.3.1] for more details.

(iii) The action of Hecke and diamond operators can also be defined on compactly supported cohomology and on the corresponding de Rham cohomology groups.

(iv) Assume that $N = N^\circ p^n$ with $p \nmid N^\circ$ and $n \in \mathbb{Z}_{\geq 1}$. In the sequel we will also need Atkin-Lehner operators w_{N° and w_{p^n} acting on modular forms and more generally on the cohomology of $Y_1(N^\circ p^n)$ (where $p \nmid N^\circ$). The action of w_{N° on a cuspidal modular form $\xi \in S_k(\Gamma_1(N^\circ p^n))$ has already been described in equation (2.1). Concerning w_{p^n}, we define:

$$w_{p^n}(\xi) := \langle p^n; 1\rangle(\xi|_k \omega_{p^n}) \qquad \omega_{p^n} := \omega_{p^n,N^\circ} := \begin{pmatrix} p^n & -1 \\ N^\circ p^n c & p^n d \end{pmatrix}, \tag{6.14}$$

where we require that $\det(\omega_{p^n}) = p^n$ and the diamond operator $\langle p^n;1\rangle$ is the one corresponding to the unique element of $(\mathbb{Z}/N^\circ p^n)^\times$ which is congruent to 1 modulo p^n and to p^n modulo N°.

The operators w_{N° and w_{p^n} are the *inverses* of the corresponding operators appearing in [AL78]. One can also define a geometric version of such operators (and of their duals w'_{N° and w'_{p^n}) on $H^1_{\text{ét}}(Y_1(N)_R, \mathscr{F}_r)$ for $\mathscr{F} \in \{\mathscr{H}, \mathscr{L}\}$ and for any $\mathbb{Q}[\zeta_N]$-algebra R (where ζ_N is a fixed primitive N-th root of unity in $\bar{\mathbb{Q}}$). We refer again to (cf. [BSV22b, paragraph 2.3.1]) for more details.

(v) The isomorphisms s_r and Poincaré duality induce perfect pairings $\langle\ ,\ \rangle_r$:

$$\begin{array}{c} H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{L}_r(1))_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} H^1_{\text{ét},c}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{H}_r)_{\mathbb{Q}_p} \\ \Big\downarrow \langle\ ,\ \rangle_r \\ H^2_{\text{ét},c}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))_{\mathbb{Q}_p} \cong \mathbb{Q}_p\,. \end{array} \tag{6.15}$$

The operators T_ℓ and T'_ℓ (resp. T'_ℓ and T_ℓ) are adjoint to each other under the pairing (6.15). The same applies to the operators U_ℓ and U'_ℓ for $\ell \mid N$.

(vi) After fixing an algebraic embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$, the classical Eichler-Shimura isomorphism

$$H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{H}_r) \otimes_{\mathbb{Z}_p} \mathbb{C} \cong M_{r+2}(\Gamma_1(N)) \oplus \overline{S_{r+2}(\Gamma_1(N))} \tag{6.16}$$

commutes with the action of Hecke and diamond operators on both sides. If $N = N^\circ p^n$ with $p \nmid N^\circ$, the isomorphism 6.16 commutes also with the action of w_{p^n} on both sides.

REMARK 6.6: In chapter 9 we will need a more explicit geometric description of some of the operators mentioned above. Fix an integer $N \geq 5$ and a prime p. In this remark we write $X_t := X_1(Np^t)_{\mathbb{Q}}$ and we view it as (the compactification of) the curve classifying triples $(E, \iota_N, \iota_{p^t})$ where E is a family of (generalized) elliptic curves over a $\mathbb{Q}$-scheme S and $\iota_N : \mu_{N,S} \to E$ and $\iota_{p^t} : \mu_{p^t,S} \to E$ are embeddings of group schemes.

For every $t \geq 0$, one can define two degeneracy maps:

$$\mathtt{pr}_1^t : X_{t+1} \to X_t\,, \qquad \mathtt{pr}_2^t : X_{t+1} \to X_t\,. \tag{6.17}$$

The map $\mathtt{pr}_1^t$ is described as the map which, at the level of the moduli problem of level $\Gamma_1(Np^{t+1})$ and $\Gamma_1(Np^t)$, sends $(E, \iota_N, \iota_{p^{t+1}})$ to $(E, \iota_N, p\cdot\iota_{p^{t+1}})$. On the other hand, $\mathtt{pr}_2^t$ sends a triple $(E, \iota_N, \iota_{p^{t+1}})$ to $(E, \iota_N, \iota_{p^{t+1}})/C_p$, where $C_p = \iota_{p^{t+1}}(\mu_{p,S})$. One checks that for $i = 1, 2$ the maps $\mathtt{pr}_i^t$ have degree p^2 if $t \geq 1$ and degree $p^2 - 1$ if $t = 0$.

For $t > s \geq 0$ and $i = 1, 2$, we will also write $\mathtt{pr}_i^{t,s} = \mathtt{pr}_i^s \circ \cdots \circ \mathtt{pr}_i^{t-1}$ as morphisms $X_t \to X_s$.

As explained in [DR17, section 1.2], one can show that for every $t \in \mathbb{Z}_{\geq 1}$ one has

$$p \cdot U_p = (\mathtt{pr}_2^t)_* \circ (\mathtt{pr}_1^t)^* = (\mathtt{pr}_1^{t-1})^* \circ (\mathtt{pr}_2^{t-1})_*$$

as operators on $H^1_{\text{dR}}(X_t, \mathscr{H}_r)$. ▲

From now on in this section L will denote a finite extension of $\mathbb{Q}_p$.

NOTATION 6.7: If V is a $\mathbb{Q}_p$-vector space, we write $V_L := V \otimes_{\mathbb{Q}_p} L$ and if M is a $\mathbb{Z}_p$-module, we write $M_L := M \otimes_{\mathbb{Z}_p} L$.

For an L-vector space H endowed with an action of the good Hecke operators of level N and for $\xi \in S_\nu(N, \chi, L)$ an eigenform, we let $H[\xi]$ denote the ξ-Hecke isotypical component of H (i.e., the maximal L-vector subspace where the Hecke operators act with the same eigenvalues as on ξ).

DEFINITION 6.8: Given $\xi \in S_\nu(N,\chi,L)$ with $\nu \geq 2$ a normalized eigenform, one can attach to it two Galois representations.

(i) We let $V_N(\xi)$ be the maximal L-quotient of $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{L}_{\nu-2}(1))_L$ where the dual good Hecke operators and dual diamond operators act with the same eigenvalues as those of ξ.

(ii) We let $V_N^*(\xi) := H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{H}_{\nu-2})_L[\xi]$ (cf. notation 6.7) be the maximal L-submodule of $H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{H}_{\nu-2})_L$ where good Hecke and diamond operators act with the same eigenvalues as those of ξ.

When the level N is understood, we simply write $V(\xi) = V_N(\xi)$ and $V^*(\xi) = V_N^*(\xi)$.

REMARK 6.9: If ξ is new of level N, then $V_N^*(\xi)$ is identified with the p-adic Deligne representation associated to ξ and $V_N(\xi)$ is identified with its dual. In general $V_N(\xi)$ (resp. $V_N^*(\xi)$ is non-canonically isomorphic to finitely many copies of $V_{N_\xi}(\xi^\circ)$ (resp. $V_{N_\xi}^*(\xi^\circ)$), where ξ° is the newform of level dividing N associated to ξ and $N_\xi \mid N$ is the corresponding level. ▲

DEFINITION 6.10: For an eigenform $\xi \in S_\nu(N,\chi,L)$ (with $\nu \geq 2$) and $? \in \{*, \varnothing\}$, we set

$$V^?_{\mathrm{dR},N}(\xi) := \mathbb{D}_{\mathrm{dR}}(V_N^?(\xi)) = H^0(\mathbb{Q}_p, \mathbb{B}_{\mathrm{dR}} \otimes_L V_N^?(\xi))$$

and we simply write $V^?_{\mathrm{dR}}(\xi)$ if the level N is understood.

The comparison isomorphism (6.12) yields canonical isomorphisms

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(\xi) \cong S_\nu(\Gamma_1(N), L)[\xi^w] \qquad \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\xi) \cong S_\nu(\Gamma_1(N), L)[\xi] \tag{6.18}$$

(where $\xi^w := w_N(\xi)$ and we follow notation 6.7) and a perfect duality

$$\langle -, - \rangle_\xi : V_{\mathrm{dR}}(\xi) \otimes_L V^*_{\mathrm{dR}}(\xi) \to \mathbb{D}_{\mathrm{dR}}(L) = L \tag{6.19}$$

under which we get identifications

$$V^*_{\mathrm{dR}}(\xi)/\mathrm{Fil}^1 V^*_{\mathrm{dR}}(\xi) \cong (S_\nu(\Gamma_1(N), L)[\xi^w])^\vee \tag{6.20}$$

and

$$V_{\mathrm{dR}}(\xi)/\mathrm{Fil}^0 V_{\mathrm{dR}}(\xi) \cong (S_\nu(\Gamma_1(N), L)[\xi])^\vee \tag{6.21}$$

where $(-)^\vee$ denotes the L-dual of an L-vector space.

6.3. The étale Abel-Jacobi map

In this section we fix a positive integer M coprime to p and a positive integer t and we assume that $Mp^t \geq 5$. We consider a triple of cuspidal modular forms

$$f = \sum_{n=1}^{+\infty} a_n(f) q^n, \qquad g = \sum_{n=1}^{+\infty} a_n(g) q^n, \qquad h = \sum_{n=1}^{+\infty} a_n(h) q^n$$

with

$$f \in S_k(Mp^t, \chi_f \omega^{2-k+k_0} \varepsilon_f), \qquad g \in S_l(Mp^t, \chi_g \omega^{2-l+l_0} \varepsilon_g), \qquad h \in S_m(Mp^t, \chi_h \omega^{2-m+m_0} \varepsilon_h).$$

Here, ω is the Teichmüller character modulo p, χ_ξ is a character defined modulo M and ε_ξ is a character valued in μ_{p^∞} for $\xi \in \{f, g, h\}$.

ASSUMPTION 6.11: (i) For $\xi \in \{f, g, h\}$, we assume that ξ is a normalized eigenform, i.e., it holds $a_1(\xi) = 1$ and ξ is an eigenform for all the Hecke operators T_ℓ for all primes $\ell \nmid M$ (i.e., the good Hecke operators). We also assume that ξ is an eigenform for the U_p operator.
(ii) There exist a positive integer $M_1 \mid M$ and a non-negative integer $s \leq t$ such that $f \in S_k(M_1 p^s, \chi_f \omega^{2-k} \varepsilon_f, L)$ is a normalized p-ordinary newform of level $M_1 p^s \geq 5$ or the ordinary p-stabilization of a newform of level $M_1 \geq 5$.
(iii) The triple (f, g, h) is *tamely self-dual*, i.e., $\chi_f \chi_g \chi_h$ is the trivial character modulo M and $k_0 + m_0 + l_0 \equiv 0 \mod (p-1)$.
(iv) The triple of weights (k, l, m) is balanced and geometric, i.e., (k, l, m) are the sizes of the edges of a triangle and $\nu \geq 2$ for $\nu \in \{k, l, m\}$.

NOTATION 6.12: (i) We fix a finite extension L of $\mathbb{Q}_p$ containing the Fourier coefficients of f, g, h (and a primitive Mp^t-th root of 1) via the fixed embedding ι_p.
(ii) From now on in this section, we write $Y_t := Y_1(Mp^t)_{\mathbb{Q}}$ (modular curve over $\mathbb{Q}$), with corresponding universal elliptic curve $\mathscr{E}_t \xrightarrow{u_t} Y_t$.

DEFINITION 6.13: With the above notation, we set $r_1 := k-2$, $r_2 := l-2$, $r_3 := m-2$, $r := (r_1 + r_2 + r_3)/2$, $\mathbf{r} := (r_1, r_2, r_3)$. We also define the Dirichlet characters of conductor a power of p given by:

$$\chi_{fgh} := \omega^r \cdot (\varepsilon_f \varepsilon_g \varepsilon_h)^{-1/2}, \qquad \psi_{fgh} := \omega^{(r_2+r_3-r_1-2k_0)/2} \cdot (\varepsilon_f^{-1} \varepsilon_g \varepsilon_h)^{-1/2}. \tag{6.22}$$

Applying [AL78, theorem 3.2]), we can give the following definition.

DEFINITION 6.14: We let f' denote the unique normalized newform of level dividing $M_1 p^{2s}$ such that $f \otimes \omega^{k-2+k_0} \varepsilon_f^{-1} = f'^{[p]}$ (p-depletion, i.e., $f'^{[p]} = f' - V_p \circ U_p(f')$).

REMARK 6.15: It holds that $f' \in S_k(M_1 p^s, \chi_f \omega^{k-2-k_0} \varepsilon_f^{-1}, L)$. More precisely:
(i) if f is the ordinary p-stabilization of a newform f° of level M_1, then $f' = f^\circ$ (we have $k \equiv 2 \mod p-1$ and ε_f trivial in this case);
(ii) if $f \in S_2(M_1 p, \chi_f, L)$ is a newform, then $f' = f$;
(iii) if f is new of level $M_1 p^s$ with $s \geq 1$ and it is p-primitive (i.e., the conductor of $\omega^{k-2+k_0} \varepsilon_f$ is exactly p^s), then f' is the normalized eigenform given by a suitable multiple of $w_{p^s}(f)$ (this follows looking at the action of the good Hecke operators and knowing that $w_{p^s}(f)$ is new of level $M_1 p^s$).

▲

Write $h' := h \otimes \psi_{fgh}$. Since we are not assuming that g and h are newforms, it is harmless to impose that f is new of level $M_1 p^s$ with $s \in \mathbb{Z}_{\geq 0}$ such that $s < t/2$. In this way it follows that $h' \in S_m(Mp^t, \chi_h \omega^{2-m} \varepsilon_h \psi_{fgh}^2)$ (i.e., that the p-part of the level is not increased by the twist).

DEFINITION 6.16: We define

$$V(f, g, h) := V_{Mp^t}(f') \otimes_L V_{Mp^t}(g) \otimes_L V_{Mp^t}(h')(-1-r) \tag{6.23}$$

and

$$V^*(f, g, h) := V^*_{Mp^t}(f') \otimes_L V^*_{Mp^t}(g) \otimes_L V^*_{Mp^t}(h')(r+2). \tag{6.24}$$

REMARK 6.17: (i) The representations $V(f,g,h)$ and $V^*(f,g,h)$ are Kummer self-dual by design and moreover they are canonically isomorphic to the Kummer dual of each other (essentially by the pairing (6.15)).

(ii) Note that if the character of f has trivial p-part (e.g. cases (i) and (ii) in remark 6.15 above) and if we assume that the triple (f,g,h) is self-dual (i.e., the product of the characters of the three forms is the trivial character), then $V^?_{Mp^t}(\xi') = V^?_{Mp^t}(\xi)$ for $\xi \in \{f,h\}$ and $? \in \{*, \varnothing\}$.

▲

In [BSV22b, section 3], the authors associate to the triple (f,g,h) a Galois cohomology class $\kappa(f,g,h) \in H^1(\mathbb{Q}, V(f,g,h))$.

Here is a diagram depicting the situation.

$$\begin{tikzcd}
\mathtt{Det}^{\text{ét}}_{\mathbf{r}} \in H^0_{\text{ét}}(Y_t, \mathscr{H}_{\mathbf{r}}(r)) \arrow[r, "d_{t,*}"] \arrow[dr, "AJ_{\text{ét}}"] \arrow[dd, bend right] & H^4_{\text{ét}}(Y_t^3, \mathscr{H}_{[\mathbf{r}]}(r+2)) \arrow[d, "\mathrm{HS}"] \\
H^1\left(\mathbb{Q}, H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}}, \mathscr{L}_{[\mathbf{r}]}(2-r))\right) \arrow[d, two heads, "pr^{fgh}_{\text{ét}}"'] & H^1\left(\mathbb{Q}, H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}}, \mathscr{H}_{[\mathbf{r}]}(r+2))\right) \arrow[l, "\cong", "s_{\mathbf{r}}"'] \\
\kappa(f,g,h) \in H^1(\mathbb{Q}, V(f,g,h)) &
\end{tikzcd}$$

REMARK 6.18: The following discussion explains the diagram above.

(i) $d_t : Y_t \hookrightarrow Y_t \times Y_t \times Y_t$ is the diagonal embedding and $d_{t,*}$ is the corresponding Gysin map.

(ii) For $\mathscr{F} \in \{\mathscr{H}, \mathscr{L}\}$ and $\mathbf{r} = (r_1, r_2, r_3)$ as above, the $\mathbb{Z}_p$-local system $\mathscr{F}_{\mathbf{r}}$ on Y_t is defined as

$$\mathscr{F}_{\mathbf{r}} := \mathscr{F}_{r_1} \otimes_{\mathbb{Z}_p} \mathscr{F}_{r_2} \otimes_{\mathbb{Z}_p} \mathscr{F}_{r_3}$$

and the $\mathbb{Z}_p$-local system $\mathscr{F}_{[\mathbf{r}]}$ on Y_t^3 is defined as

$$\mathscr{F}_{[\mathbf{r}]} := p_1^*\mathscr{F}_{r_1} \otimes_{\mathbb{Z}_p} p_2^*\mathscr{F}_{r_2} \otimes_{\mathbb{Z}_p} p_3^*\mathscr{F}_{r_3}$$

where $p_j : Y_t \times Y_t \times Y_t \to Y_t$ is the natural projection on the j-th factor. In particular, it follows that $d_t^*\mathscr{F}_{[\mathbf{r}]} = \mathscr{F}_{\mathbf{r}}$.

(iii) The map HS is a morphism coming from the Hochschild-Serre spectral sequence

$$H^i(\mathbb{Q}, H^j_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}}, \mathscr{H}_{[\mathbf{r}]}(r+2))) \Rightarrow H^{i+j}_{\text{ét}}(Y_t^3, \mathscr{H}_{[\mathbf{r}]}(r+2))\,. \tag{6.25}$$

The existence of such a morphism is due to the fact that $H^4_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}}, \mathscr{H}_{[\mathbf{r}]}(r+2)) = 0$ (by Artin vanishing theorem, as Y_t is an affine curve).

(iv) The étale Abel-Jacobi map in this setting is defined as $AJ_{\text{ét}} := d_{t,*} \circ \mathrm{HS}$.

(v) The isomorphism $s_{\mathbf{r}}$ is induced by the isomorphisms s_{r_j} for $j \in \{1,2,3\}$ (cf. equation (6.11)).

(vi) The projection $pr^{fgh}_{\text{ét}}$ is induced in Galois cohomology by viewing $V(f,g,h)$ as a quotient of $H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}}, \mathscr{L}_{[\mathbf{r}]}(2-r))_L$ via Künneth formula and projection to the corresponding Hecke isotypical component.

▲

The class $\kappa(f,g,h)$ is the image under the composition $pr_{\text{ét}}^{fgh} \circ s_{\mathbf{r}} \circ AJ_{\text{ét}}$ of the element

$$\mathtt{Det}_{\mathbf{r}}^{\text{ét}} \in H^0_{\text{ét}}(Y_t, \mathscr{H}_{\mathbf{r}}(r))_L\,, \tag{6.26}$$

which we now describe. Write $Y = Y_t$ till the end of this section.

We fix a geometric point $y : \operatorname{Spec}(\bar{\mathbb{Q}}) \to Y$ and we consider the étale fundamental group $\pi_1^{\text{ét}}(Y,y)$. Passing to the stalk at y induces an equivalence of categories

$$\operatorname{Loc}_Y(\mathbb{Z}_p) \simeq \operatorname{Rep}^{cont}_{\mathbb{Z}_p}(\pi_1^{\text{ét}}(Y,y)),$$

where:

(i) $\operatorname{Loc}_Y(\mathbb{Z}_p)$ is the category of étale $\mathbb{Z}_p$-local systems on Y;

(ii) $\operatorname{Rep}^{cont}_{\mathbb{Z}_p}(\pi_1^{\text{ét}}(Y,y))$ is the category of continuous representations of $\pi_1^{\text{ét}}(Y,y)$ in finite free $\mathbb{Z}_p$-modules.

In particular, the stalk $\mathcal{T}_p(\mathscr{E})_y$ is a rank 2 free $\mathbb{Z}_p$-module. The p-adic Weil pairing $\mathcal{T}_p(\mathscr{E})_y \otimes_{\mathbb{Z}_p} \mathcal{T}_p(\mathscr{E})_y \to \mathbb{Z}_p(1)$ is well-known to be perfect, $\mathbb{Z}_p$-bilinear, alternating and Galois-invariant (for the action of $G_{\mathbb{Q}}$). The construction recalled in appendix A applies to this setting (with $M = \mathcal{T}_p(\mathscr{E})_y$), yielding an element

$$\mathtt{Det}_{\mathbf{r}} \in H^0_{cont}(\operatorname{Aut}_{\mathbb{Z}_p}(\mathcal{T}_p(\mathscr{E})_y), S_{\mathbf{r}} \otimes \mathbb{Z}_p[r])\,.$$

The action of $\pi_1^{\text{ét}}(Y,y)$ on $\mathcal{T}_p(\mathscr{E})_y$ is encoded in a continuous morphism $\pi_1^{\text{ét}}(Y,y) \to \operatorname{Aut}_{\mathbb{Z}_p}(\mathcal{T}_p(\mathscr{E})_y)$. We thus obtain a functor

$$(-)^{\text{ét}} : \operatorname{Rep}^{cont}_{\mathbb{Z}_p}(\operatorname{Aut}_{\mathbb{Z}_p}(\mathcal{T}_p(\mathscr{E})_y) \to \operatorname{Rep}^{cont}_{\mathbb{Z}_p}(\pi_1^{\text{ét}}(Y,y)) \simeq \operatorname{Loc}_Y(\mathbb{Z}_p)\,, \quad N \rightsquigarrow N^{\text{ét}}\,.$$

It follows from the construction that, with the notation of appendix A, $S_n^{\text{ét}} = \mathscr{H}_n$ for all integers $n \geq 0$ and $\mathbb{Z}_p[m]^{\text{ét}} = \mathbb{Z}_p(m)$ for all $m \in \mathbb{Z}$. As a consequence, we obtain an inclusion at the level of invariants

$$H^0_{cont}(\operatorname{Aut}_{\mathbb{Z}_p}(\mathcal{T}_p(\mathscr{E})_y), S_{\mathbf{r}} \otimes \mathbb{Z}_p[r]) \hookrightarrow H^0_{cont}(\pi_1^{\text{ét}}(Y,y), \mathscr{H}_{\mathbf{r}}(r)_y) \cong H^0_{\text{ét}}(Y, \mathscr{H}_{\mathbf{r}}(r))\,.$$

The element $\mathtt{Det}_{\mathbf{r}}^{\text{ét}}$ is defined as the image of the invariant $\mathtt{Det}_{\mathbf{r}}$ inside $H^0_{\text{ét}}(Y, \mathscr{H}_{\mathbf{r}}(r))_L$ via the above inclusion (followed by extension of scalars to L).

CHAPTER 7

Some p-adic Hodge theory

In this chapter we introduce the necessary tools from p-adic Hodge theory and we apply them to the study of the Galois representation $V(f,g,h)$, seen as a representation of $G_{\mathbb{Q}_p}$. The main references are [BLZ16, section 1] and [GM09].

7.1. Filtered (φ, N)-modules and Galois representations

For the generalities on (φ, N)-modules with coefficients we refer to [GM09, section 2]. As in the previous chapters, we let L be a finite and large enough extension of $\mathbb{Q}_p$ and we denote by $\mathbb{Q}_p^{nr}$ the maximal unramified extension of $\mathbb{Q}_p$ (inside the fixed algebraic closure $\bar{\mathbb{Q}}_p$) and we let $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p)$ denote the Frobenius (i.e., the unique lift to $\mathbb{Q}_p^{nr}$ of the Frobenius automorphism $x \mapsto x^p$ of $\bar{\mathbb{F}}_p$)

Definition 7.1: A filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module D is a free $(\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L)$-module of finite rank endowed with:

(i) the Frobenius endomorphism: a σ-semilinear, L-linear, bijective map $\varphi : D \to D$;
(ii) the monodromy operator: a $\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L$-linear, nilpotent endomorphism $N : D \to D$ such that $N \circ \varphi = p \cdot \varphi \circ N$;
(iii) a σ-semilinear, L-linear action of $G_{\mathbb{Q}_p}$, commuting with φ and N;
(iv) a decreasing, separated, exhaustive, $G_{\mathbb{Q}_p}$-stable filtration on $\bar{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p^{nr}} D$ given by free $(\bar{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} L)$-submodules.

There are similar definitions for filtered (φ, N, G_F, L)-modules, where F is any finite extension of $\mathbb{Q}_p$ and one can view filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-modules as filtered (φ, N, G_F, L)-modules by restriction.

Notation 7.2: We fix a filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module D and we let F be a finite extension of $\mathbb{Q}_p$. Denote by F_0 the maximal unramified subextension of F and let $q = p^d$ be the cardinality of the residue field of F. We will always assume that $F \subseteq L$ in what follows. We set

$$D_{\mathrm{st},F_0} := D^{G_F} \quad D_{\mathrm{st},F} := D^{G_F} \otimes_{F_0} F, \quad D_{\mathrm{dR},F} := (D \otimes_{\mathbb{Q}_p^{nr}} \bar{\mathbb{Q}}_p)^{G_F}, \quad D_{\mathrm{cris},F} = D_{\mathrm{st},F}^{N=0}.$$

Assumption 7.3: Every p-adic Galois representation V of $G_{\mathbb{Q}_p}$ appearing in the sequel will be a de Rham (equivalently, potentially semistable) representation.

Definition 7.4: (i) If V is a p-adic Galois representation of $G_{\mathbb{Q}_p}$ with coefficients in L, F is a finite extension of $\mathbb{Q}_p$ and $? \in \{\mathrm{dR}, \mathrm{st}, \mathrm{cris}\}$, we set

$$\mathbb{D}_{?,F}(V) := (\mathbb{B}_? \otimes_{\mathbb{Q}_p} V)^{G_F}.$$

We say that V is F-semistable (resp. F-crystalline) if $\mathbb{D}_{\mathrm{st},F}(V)$ (resp. $\mathbb{D}_{\mathrm{cris},F}(V)$) is a free $F_0 \otimes_{\mathbb{Q}_p} L$-module of rank equal to $\dim_L(V)$.

(ii) For V as in (i), we also define the $(\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L)$-module attached to it as

$$\mathbb{D}_{\mathrm{pst}}(V) := \varinjlim_{\substack{M/\mathbb{Q}_p \\ \text{finite}}} (\mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_M}. \tag{7.1}$$

REMARK 7.5: If V is a p-adic Galois representation of $G_{\mathbb{Q}_p}$ with coefficients in L, the $(\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L)$-module $\mathbb{D}_{\mathrm{pst}}(V)$ inherits a natural structure of filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module. The functor $\mathbb{D}_{\mathrm{pst}}(-)$ provides an equivalence of categories between potentially semistable Galois representations with coefficients in L and *admissible* filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-modules. ▲

For later purposes, we define the so-called Bloch-Kato subspaces in first Galois cohomology group.

DEFINITION 7.6: For V a (de Rham) p-adic Galois representation of $G_{\mathbb{Q}_p}$ and F a finite extension of $\mathbb{Q}_p$, one defines:

$$\begin{aligned}
H^1_g(F, V) &:= \mathrm{Ker}\left(H^1(F, V) \to H^1(F, \mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)\right), \\
H^1_f(F, V) &:= \mathrm{Ker}\left(H^1(F, V) \to H^1(F, \mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)\right), \\
H^1_e(F, V) &:= \mathrm{Ker}\left(H^1(F, V) \to H^1(F, \mathbb{B}_{\mathrm{cris}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V)\right).
\end{aligned}$$

One can study the cohomology of admissible filtered (φ, N)-modules and compare it with the Galois cohomology of the associated Galois representation.

DEFINITION 7.7: Let D be a filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module and $F \subseteq L$ as above. The cohomology groups $H^i_{\mathrm{st}}(F, D)$ are given by the cohomology of the complex

$$\begin{array}{rcccl}
C^\bullet_{\mathrm{st},F}(D): & D_{\mathrm{st},F_0} \oplus \mathrm{Fil}^0 D_{\mathrm{dR}} & \to & D_{\mathrm{st},F_0} \oplus D_{\mathrm{st},F_0} \oplus D_{\mathrm{dR},F} & \longrightarrow D_{\mathrm{st},F_0} \\
& (u, v) & \longmapsto & ((1-\varphi)u, Nu, u - v) & \\
& & & (w, x, y) & \longmapsto Nw - (1 - p\varphi)x
\end{array} \tag{7.2}$$

concentrated in degrees 0, 1 and 2.

THEOREM 7.8: *The complex $C^\bullet_{\mathrm{st},F}(-)$ computes* Ext *groups* $\mathrm{Ext}^i(\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L, -)$ *for $i = 0, 1, 2$ in the category of (*admissible*) filtered (φ, N, G_F, L)-modules. If V is a p-adic representation of $G_{\mathbb{Q}_p}$ with L-coefficients and $D = \mathbb{D}_{\mathrm{pst}}(V)$, the functor $\mathbb{D}_{\mathrm{pst}}$ induces functorial maps $H^i_{\mathrm{st}}(F, D) \to H^i(F, V)$ for every finite extension $F \subseteq L$ of $\mathbb{Q}_p$. This maps are isomorphisms for $i = 0$ and injective for $i = 1$. If V is F-semistable, then for $i = 1$ the image of the map $H^1_{\mathrm{st}}(F, D) \to H^i(F, V)$ coincides with the subspace $H^1_g(F, V)$ of definition 7.6.*

PROOF. This is well-known. Let us just note here that the fact that the image of the inclusion $H^1_{\mathrm{st},F}(D) \hookrightarrow H^1(F, V)$ coincides with $H^1_g(F, V)$ is a consequence of Hyodo's theorem (or a way of stating it). □

DEFINITION 7.9: Let V be an F-semistable representation of $G_{\mathbb{Q}_p}$ with L-coefficients. We define the semistable Bloch-Kato exponential map for V as the isomorphism

$$\exp_{\mathrm{st},V} : H^1_{\mathrm{st}}(F, \mathbb{D}_{\mathrm{pst}}(V)) \xrightarrow{\cong} H^1_g(F, V)$$

afforded by theorem 7.8.

Now we describe slight generalizations/modifications of the complex $C^\bullet_{\mathrm{st},F}(D)$ for D a filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module. We let $\Phi = \varphi^d$, so that Φ is F_0-linear on D^{G_F} and extends to a linear endomorphism of $D_{\mathrm{st},F}$. For every polynomial $Q(T) \in 1+T\cdot L[T]$ one can define two variants of the above complex (7.2) given by

$$
\begin{array}{lrcl}
C^\bullet_{\mathrm{st},F,Q}(D): & D_{\mathrm{st},F_0} \oplus \mathrm{Fil}^0 D_{\mathrm{dR}} \to D_{\mathrm{st},F_0} \oplus D_{\mathrm{st},F_0} \oplus D_{\mathrm{dR},F} & \longrightarrow & D_{\mathrm{st},F_0} \\
(7.3) \qquad & (u,v) \longmapsto (Q(\varphi)u, Nu, u-v) & & \\
 & (w,x,y) & \longmapsto & Nw - Q(p\varphi)x
\end{array}
$$

and

$$
\begin{array}{lrcl}
\tilde{C}^\bullet_{\mathrm{st},F,Q}(D): & D_{\mathrm{st},F} \oplus \mathrm{Fil}^0 D_{\mathrm{dR}} \to D_{\mathrm{st},F} \oplus D_{\mathrm{st},F} \oplus D_{\mathrm{dR},F} & \longrightarrow & D_{\mathrm{st},F} \\
(7.4) \qquad & (u,v) \longmapsto (Q(\Phi)u, Nu, u-v) & & \\
 & (w,x,y) & \longmapsto & Nw - Q(q\Phi)x
\end{array}
$$

and define cohomology groups $H^i_{\mathrm{st},Q}(F,D)$ and $\tilde{H}^i_{\mathrm{st},Q}(F,D)$ (for $i=0,1,2$) accordingly. In particular, note that $H^i_{\mathrm{st},1-T}(F,D) = H^i_{\mathrm{st}}(F,D)$ in our notation.

LEMMA 7.10: *Let $P_1, P_2 \in 1+T\cdot L[T]$ be two polynomials. Then there is a natural morphism of complexes $C^\bullet_{\mathrm{st},F,P_1}(D) \to C^\bullet_{\mathrm{st},F,P_1P_2}(D)$ given by*

$$
\begin{array}{ccccccc}
C^\bullet_{\mathrm{st},F,P_1}(D): & D_{\mathrm{st},F_0} \oplus \mathrm{Fil}^0 D_{\mathrm{dR},F} & \longrightarrow & D_{\mathrm{st},F_0} \oplus D_{\mathrm{st},F_0} \oplus D_{\mathrm{dR},F} & \longrightarrow & D_{\mathrm{st},F_0} \\
\Big\downarrow{\scriptstyle c_{P_2}} & \Big\downarrow{\scriptstyle id\oplus id} & & \Big\downarrow{\scriptstyle P_2(\varphi)\oplus id\oplus id} & & \Big\downarrow{\scriptstyle P_2(p\varphi)} \\
C^\bullet_{\mathrm{st},F,P_1P_2}(D): & D_{\mathrm{st},F_0} \oplus \mathrm{Fil}^0 D_{\mathrm{dR},F} & \longrightarrow & D_{\mathrm{st},F_0} \oplus D_{\mathrm{st},F_0} \oplus D_{\mathrm{dR},F} & \longrightarrow & D_{\mathrm{st},F_0}
\end{array}
$$

which is a quasi-isomorphism if $P_2(\varphi)$ and $P_2(p\varphi)$ are bijective on D_{st,F_0}.

Moreover the morphism c_{P_2} always induces a short exact sequence of the form:

$$
\begin{array}{c}
0 \to H^0_{\mathrm{st},P_2}(F,D) \to D^{P_2(\varphi)=0,\, N=0}_{\mathrm{st},F_0} \to \mathrm{Ker}(H^1_{\mathrm{st},P_1}(F,D) \to H^1_{\mathrm{st},P_1P_2}(F,D)) \to 0 \\
w \mapsto [(w,0,0)]
\end{array}
$$

PROOF. This is also an easy exercise. The only point where one has to be a slightly careful is to show that every class $[(w,x,y)]$ inside $\mathrm{Ker}(H^1_{\mathrm{st},P_1}(F,D) \to H^1_{\mathrm{st},P_1P_2}(F,D))$ can be represented as $[(w',0,0)]$ for a suitable $w' \in D^{P_2(\varphi)=0,\, N=0}_{\mathrm{st},P_2}$.

But if $(P_2(\varphi)w, x, y) = (P_1P_2(\varphi)u, Nu, u-v)$ for some $(u,v) \in D_{\mathrm{st},F_0} \oplus \mathrm{Fil}^0 D_{\mathrm{dR},F}$, then $w' = w - P_1(\varphi)u$ does the job. $\square$

DEFINITION 7.11: An admissible $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module D is (F,Q)**-convenient** if D is F-crystalline (i.e., $D_{\mathrm{dR},F} = D_{\mathrm{st},F} = D_{\mathrm{cris},F}$) and $Q(\Phi)$ and $Q(q\Phi)$ are bijective on $D_{\mathrm{st},F}$.

LEMMA 7.12: *In the above setting, if D is (F,Q)-convenient, then the morphism of complexes $[\mathrm{Fil}^0 D_{\mathrm{dR},F} \to D_{\mathrm{dR},F} \to 0] \to C^\bullet_{\mathrm{st},F,Q}(D)$ given by the obvious inclusions of* $\mathrm{Fil}^0 D_{\mathrm{dR},F}$ *inside $D_{\mathrm{st},F} \oplus \mathrm{Fil}^0 D_{\mathrm{dR},F}$ and of $D_{\mathrm{dR},F}$ inside $D_{\mathrm{st},F} \oplus D_{\mathrm{st},F} \oplus D_{\mathrm{dR},F}$ is actually a quasi-isomorphism. If, moreover, we consider $P(T) \in 1+T\cdot L[T]$ such that $P(T) \mid Q(T^d)$, then* *we can actually identify*

$$\frac{D_{\mathrm{dR},F}}{\mathrm{Fil}^0 D_{\mathrm{dR},F}} \cong \tilde{H}^1_{\mathrm{st},Q}(F,D) \cong H^1_{\mathrm{st},P}(F,D)\,.$$

Proof. This is an easy exercise. One can check that the inverse to the isomorphism

$$\frac{D_{\mathrm{dR},F}}{\mathrm{Fil}^0 D_{\mathrm{dR},F}} \xrightarrow{\cong} H^1_{\mathrm{st},P}(F,D)$$

is given by $[(w,x,y)] \mapsto y - Q(\Phi)^{-1}w \mod \mathrm{Fil}^0 D_{\mathrm{dR},F}$. The identification $H^1_{\mathrm{st},P}(F,D) \cong \tilde{H}^1_{\mathrm{st},Q}(F,D)$ follows immediately from lemma 7.10, taking $P_1(T) = P(T)$ and $P_2(T) = Q(T^d)/P(T)$. $\square$

For $P, Q \in 1 + T \cdot L[T]$, we let $P * Q \in 1 + T \cdot L[T]$ be the polynomial whose roots are $\{\alpha_i \beta_j\}$ if $\{\alpha_i\}$ and $\{\beta_j\}$ are the roots of P and Q respectively. We then have the following proposition.

Proposition 7.13: *Let D_1, D_2 be two filtered $(\varphi, N, G_{\mathbb{Q}_p}, L)$-modules. Then there are cup products*

$$C^\bullet_{\mathrm{st},F,P}(D_1) \times C^\bullet_{\mathrm{st},F,Q}(D_2) \to C^\bullet_{st,F,P*Q}(D_1 \otimes_{\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L} D_2),$$

associative and graded-commutative up to homotopy, hence inducing well-defined products on cohomology groups.

Proof. See [BLZ16], proposition 1.3.2. $\square$

Remark 7.14: Here we include the table (taken from [BLZ16]) giving the recipe to compute the pairing of the above lemma 7.13.

(7.5)

$\times$	(u',v')	(w',x',y')	z'
(u,v)	$(u \otimes u', v \otimes v')$	$\begin{pmatrix} b(\varphi_1,\varphi_2)(u\otimes w'), \\ u\otimes x', \\ (\lambda u+(1-\lambda)v)\otimes y' \end{pmatrix}$	$b(\varphi_1, p\varphi_2)(u \otimes z')$
(w,x,y)	$\begin{pmatrix} a(\varphi_1,\varphi_2)(w\otimes u'), \\ x\otimes u', \\ y\otimes(\lambda v'+(1-\lambda)u') \end{pmatrix}$	$-a(\varphi_1, p\varphi_2)(w \otimes x')$ $+b(p\varphi_1, \varphi_2)(x \otimes w')$	0
z	$a(p\varphi_1, \varphi_2)(z \otimes u')$	0	0

In the table, we have fixed $a(X,Y), b(X,Y) \in L[X,Y]$ such that

$$P * Q(XY) = a(X,Y)P(X) + b(X,Y)Q(Y)$$

and $\lambda \in L$. One can check that changing the polynomials $a(X,Y)$ and $b(X,Y)$ or changing λ will change the product by a chain homotopy (i.e., the induced pairings on cohomology groups are well-defined). ▲

7.2. Study of the relevant filtered (φ, N)-modules

Now we can go back to the setting of section 6.3 to study the filtered (φ, N)-modules attached to the modular forms (f,g,h) more closely. Recall that

$$V^?_{\mathrm{dR}}(f,g,h) := \mathbb{D}_{\mathrm{dR}}(V^?(f,g,h))\,.$$

We let $\breve{f} := \lambda_{M_1}(f)^{-1} \cdot w_{M_1}(f)$, where $\lambda_{M_1}(f)$ is the pseudo-eigenvalue for the action of w_{M_1} on f. It follows from [AL78, theorem 1.1] that $\lambda_{M_1}(f)$ is an algebraic number of complex absolute value 1. We denote by $e_{\breve{f}}$ the idempotent corresponding to $\breve{f}$ by the theory of p-stabilized ordinary newforms (cf. [Hid85, chapter 4]).

DEFINITION 7.15: (i) For $\xi \in \{g, h'\}$ we let $\omega_\xi \in \mathrm{Fil}^1 V^*_{\mathrm{dR},Mp^t}(\xi)$ be the element corresponding to ξ under the isomorphism (6.18).

(ii) Consider the $G_{\mathbb{Q}}$-representation $V^*_{Mp^s}(f')$ (note that we are imposing $s = 1$, i.e., working on $Y_1(Mp)_{\bar{\mathbb{Q}}}$), if f is the ordinary p-stabilisation of a newform of level coprime to p). We let $\eta_{f',s} \in V^*_{\mathrm{dR},Mp^s}(f')/\mathrm{Fil}^1 V^*_{\mathrm{dR},Mp^s}(f')$ be the element corresponding under the isomorphism (6.20) to the linear functional

$$\begin{array}{ccccc} S_k(\Gamma_1(Mp^s), L)[f'^w] & \longrightarrow & S_k(M_1p^s, \chi_f^{-1}\omega^{2-k+k_0}\varepsilon_f, L) & \longrightarrow & L \\ \gamma & \longmapsto & \mathrm{Tr}_{Mp^s/M_1p^s}(\gamma) & & \\ & & \delta & \longmapsto & a_1(e_{\breve{f}}(\delta)) \end{array}$$

Note that this gives rise to a non trivial functional since $\breve{f} \in S_k(\Gamma_1(Mp^s), L)[f'^w]$.

REMARK 7.16: The fact the such a linear functional actually takes values in L follows from the work of Hida (cf. [Hid85, proposition 4.5]). ▲

REMARK 7.17: Since f is p-ordinary, we know that $V^*_{Mp^s}(f)$ is F_1-semistable, where F_1 is a cyclotomic extension of $\mathbb{Q}_p$ generated by a p^n-th root of unity for some $n \geq 0$. More precisely (cf. remark 6.15):

(i) if f is the ordinary p-stabilisation of a newform f° of level M_1, then $V^*_{Mp^s}(f)$ is already crystalline over $\mathbb{Q}_p$;

(ii) if $f \in S_2(M_1p, \chi_f, L)$ is a newform, then $V^*_{Mp^s}(f)$ is semistable (but not crystalline) over $\mathbb{Q}_p$;

(iii) if f is new of level M_1p^s and p-primitive, then $V^*_{Mp^s}(f)$ becomes crystalline over $\mathbb{Q}_p(\zeta_{p^s})$ (where ζ_{p^s} is a primitive p^s-th root of unity).

The same remarks apply to $V^*_{Mp^s}(f')$. ▲

LEMMA 7.18: *The filtered L-vector space $V^*_{\mathrm{dR},Mp^s}(f')$ decomposes, as L-vector space, as*

$$V^*_{\mathrm{dR},Mp^s}(f') = \mathrm{Fil}^1 V^*_{\mathrm{dR},Mp^s}(f') \oplus \left(F_1 \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{st},F_1}\left(V^*_{Mp^s}(f')\right)^{\varphi = a_p(f)}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)}$$

PROOF. In order to simplify the notation, we write $V = V^*_{Mp^s}(f')$, $V_{\mathrm{dR}} = V^*_{\mathrm{dR},Mp^s}(f')$ and $V_{\mathrm{st},F_1} = \mathbb{D}_{\mathrm{st},F_1}\left(V^*_{Mp^s}(f')\right)$ in this proof.

It is possible to describe explicitly the structure of filtered $(\varphi, N, \mathrm{Gal}(F_1/\mathbb{Q}_p), L)$-module of V_{st,F_1}. We refer to [GM09, sections 3.1 and 3.2], where the authors actually describe the duals of our modules.

Combining everything, we can give the following explicit description of V_{st,F_1}. Such module is non-canonically isomorphic to a finite number of copies of a two-dimensional filtered $(\varphi, N, \mathrm{Gal}(F_1/\mathbb{Q}_p), L)$-module with Hodge-Tate weights $\{1-k, 0\}$, which we denote by D. Clearly, it is enough to prove the corresponding statement of the lemma for D.

If V is F_1-crystalline, then D has a basis $\{e_1, e_2\}$ as L-vector space such that

$$\begin{cases} \varphi(e_1) & = & \chi_f(p)p^{k-1}a_p(f)^{-1} \cdot e_1 \\ \varphi(e_2) & = & a_p(f) \cdot e_2 \\ \mathrm{Fil}^1 D & = & (F_1 \otimes_{\mathbb{Q}_p} L)(x_f e_1 + y_f e_2) \\ N & = & 0 \\ g(e_1) & = & \left(\omega^{2-k}\varepsilon_f\right)(g) \cdot e_1 \\ g(e_2) & = & e_2 \\ g & \in & \mathrm{Gal}(F_1/\mathbb{Q}_p) \end{cases}$$

where $x_f, y_f \in F_1 \otimes_{\mathbb{Q}_p} L$ are elements such that either $y_f = 0$ and $x_f \neq 0$ (the *split* case) or they are both non-zero (the *non-split* case). These elements are unique up to multiplication by elements of $L^\times$.

If V is not F_1-crystalline (i.e., $f \in S_2(M_1 p, \chi_f)$ newform, under our assumptions), then we can choose $F_1 = \mathbb{Q}_p$ and D has a basis $\{e_1, e_2\}$ as L-vector space

$$\begin{cases} \varphi(e_1) & = & p \cdot a_p(f) \cdot e_1 \\ \varphi(e_2) & = & a_p(f) \cdot e_2 \\ \mathrm{Fil}^1 D & = & (F_1 \otimes_{\mathbb{Q}_p} L)(e_1 - \mathfrak{L} e_2) \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \end{cases}$$

where $\mathfrak{L} = \mathfrak{L}_p(f)$ is the $\mathfrak{L}$-invariant of f (defined as in [Maz94]).

From the explicit description of the filtration on D, it follows easily in all cases that

$$\mathrm{Fil}^1(D) \cap \left(F_1 \otimes_{\mathbb{Q}_p} D^{\varphi = a_p(f)}\right) = \{0\}$$

and that $D^{\varphi = a_p(f)}$ is one-dimensional (over L). We thus get a decomposition of $F_1 \otimes_{\mathbb{Q}_p} D$ as L-vector space given by

$$F_1 \otimes_{\mathbb{Q}_p} D = \mathrm{Fil}^1 D \oplus \left(F_1 \otimes_{\mathbb{Q}_p} D^{\varphi = a_p(f)}\right)$$

and it follows that such a decomposition is stable for the action of $\mathrm{Gal}(F_1/\mathbb{Q}_p)$ (cf. the discussion in [GM09, section 3.2]) and gives an analogous decomposition for $F_1 \otimes_{\mathbb{Q}_p} V_{\mathrm{st},F_1}$.

Taking $\mathrm{Gal}(F_1/\mathbb{Q}_p)$-invariants yields

$$\begin{aligned} V_{\mathrm{dR}} &= \left(F_1 \otimes_{\mathbb{Q}_p} V_{\mathrm{st},F_1}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)} = \\ &= \left(\mathrm{Fil}^1 V_{\mathrm{dR},F_1}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)} \oplus \left(F_1 \otimes_{\mathbb{Q}_p} D^{\varphi = a_p(f)}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)} = \\ &= \mathrm{Fil}^1 V_{\mathrm{dR}} \oplus \left(F_1 \otimes_{\mathbb{Q}_p} D^{\varphi = a_p(f)}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)}, \end{aligned}$$

as we wished to prove. □

REMARK 7.19: We observe that the decomposition proved in the above lemma 7.18 does not depend on the choice of $F_1 \subset \mathbb{Q}_p(\mu_{p^\infty})$ (Galois over $\mathbb{Q}_p$) such that $V^*_{Mp^s}(f')$ is F_1-semistable, i.e., if F_2 is another such extension we can identify

$$\left(F_1 \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{st},F_1}\left(V^*_{Mp^s}(f')\right)^{\varphi=a_p(f)}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)} = \left(F_2 \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{st},F_2}\left(V^*_{Mp^s}(f')\right)^{\varphi=a_p(f)}\right)^{\mathrm{Gal}(F_2/\mathbb{Q}_p)}.$$

This follows easily from the fact that the action of φ and the Galois action commute and from Hilbert Theorem 90. ▲

DEFINITION 7.20: With the notation introduced above, we define

$$\eta^{\varphi=a_p}_{f',s} \in \left(F_1 \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{st},F_1}\left(V^*_{Mp^s}(f')\right)^{\varphi=a_p(f)}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)} \subset V^*_{\mathrm{dR},Mp^s}(f')$$

as the unique lift of $\eta_{f',s}$ to the subspace $\left(F_1 \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{st},F_1}\left(V^*_{Mp^s}(f')\right)^{\varphi=a_p(f)}\right)^{\mathrm{Gal}(F_1/\mathbb{Q}_p)}$.

DEFINITION 7.21: We let $\eta^{\varphi=a_p}_{f'} := (\mathtt{pr}_2^{t,s})^*(\eta^{\varphi=a_p}_{f',s}) \in V^*_{\mathrm{dR},Mp^t}(f') \subseteq H^1_{\mathrm{dR}}(X_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{k-2})_L$ denote the pullback of the class $\eta^{\varphi=a_p}_{f',s}$ under the degeneracy map $(\mathtt{pr}_2^{t,s})^*$.

REMARK 7.22: Since the triple (k,l,m) is balanced, we have that

$$\eta^{\varphi=a_p}_{f'} \otimes \omega_g \otimes \omega_{h'} \otimes t_{r+2} \in \mathrm{Fil}^0(V^*_{\mathrm{dR}}(f,g,h)). \tag{7.6}$$

Here, for all $n \in \mathbb{Z}$, t_n denotes a (canonical) generator of $\mathbb{Q}_p(n)$ (on which the Frobenius φ acts as multiplication by p^{-n}). ▲

The idea is now to associate to $\kappa(f,g,h)$ an element in $\mathrm{Fil}^0(V_{\mathrm{dR}}(f,g,h))^\vee$, so that we can pair it with $\eta^{\varphi=a_p}_{f'} \otimes \omega_g \otimes \omega_{h'} \otimes t_{r+2}$. We are led to study the Bloch-Kato local conditions for the Galois representation $V(f,g,h)$.

DEFINITION 7.23: A triple (f,g,h) satisfying assumption 6.11 is called **F-exponential** if the equality

$$H^1_e(F, V(f,g,h)) = H^1_f(F, V(f,g,h)) = H^1_g(F, V(f,g,h))$$

holds (for an appropriate F depending on (f,g,h)).

We say that a triple (f,g,h) satisfying assumption 6.11 is **(F,Q)-convenient** for a finite extension $F/\mathbb{Q}_p$ and $Q \in 1 + T\cdot L[T]$ if the $(\varphi, N, G_{\mathbb{Q}_p}, L)$-module $\mathbb{D}_{\mathrm{pst}}(V(f,g,h))$ is (F,Q)-convenient, in the sense of definition 7.11.

REMARK 7.24: It is easy to check that if (f,g,h) is $(F, 1-T)$-convenient, then it is F-exponential. ▲

As already explained in the introduction, in the sequel we will be mostly interested in the following setting.

ASSUMPTION 7.25: The forms g and h are supercuspidal at p and lie in the kernel of U_p. In particular (since p is odd), the Galois representations $V_{Mp^t}(g)$ and $V_{Mp^t}(h)$, seen as local Galois representations of $G_{\mathbb{Q}_p}$, are isomorphic to (finitely many copies of) the induced representations of a character of a quadratic extension of $\mathbb{Q}_p$ (which is not the restriction of a character of $G_{\mathbb{Q}_p}$).

REMARK 7.26: Asking that g and h are supercuspidal at p implies directly that g and h lie in the kernel of U_p^m for some m large enough, since we are not assuming that g and h are new of level Mp^t. The slightly stronger assumption 7.25 will be needed to simplify some arguments in what follows. ▲

REMARK 7.27: As explained in [GM09, sections 3.3 and 3.4], under our assumption we can find a finite Galois extension of $\mathbb{Q}_p$, which we denote by F, such that:

(i) the maximal unramified subextension of F is of the form $F_0 := \mathbb{Q}_{p^{2a}}$ for some $a \in \mathbb{Z}_{\geq 1}$;
(ii) F contains the cyclotomic extension F_1 of remark 7.17;
(iii) F is contained in our field of coefficients L (up to extending L if necessary);
(iv) $V_{Mp^t}(g)$, $V_{Mp^t}(h)$ and $V_{Mp^t}(h')$ are F-crystalline.

Moreover, one can describe the filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p), L)$-modules associated to $V_{Mp^t}(\xi)$ for $\xi \in \{g, h, h'\}$ as follows (always relying on [GM09, sections 3.3 and 3.4]). Such modules are given by (finitely many copies of) a rank 2 free $F_0 \otimes_{\mathbb{Q}_p} L$-module D_ξ with basis $\{v_\xi, w_\xi\}$ and such that

$$\begin{cases} \varphi(v_\xi) & = & \mu_\xi \cdot v_\xi \\ \varphi(w_\xi) & = & \mu_\xi \cdot w_\xi \\ N & = & 0 \\ \mu_\xi & \in & L^\times \\ \mathrm{ord}_p(\mu_\xi) & = & \frac{1-\nu}{2} \end{cases}$$

where ν denotes the weight of the form ξ. Note that $\mu_h = \mu_{h'}$ since h' is a twist of h by a character of order a power of p.

One could also give a more explicit description of the Galois group $\mathrm{Gal}(F/\mathbb{Q}_p)$ and of its action on such modules, but we will not need it for the moment. The same remark applies to the filtration on $F \otimes_{F_0} D_\xi$.

Recall that the Frobenius endomorphism φ is F_0-semilinear, so that if we want to look at it as a linear operator on D_ξ, we have to view D_ξ as an L-vector space of dimension $4a$. ▲

PROPOSITION 7.28: *Under assumptions 6.11 and 7.25, we have that:*

(a) if the form f has weight $k > 2$, then the triple (f, g, h) is $(F, 1-T)$-convenient (for F as in remark 7.27),
(b) if the form f is a newform in $S_2(M_1 p, \chi_f, L)$, then the triple (f, g, h) is F-exponential.

PROOF. The inclusions

$$H^1_e(F, V(f,g,h)) \subseteq H^1_f(F, V(f,g,h)) \subseteq H^1_g(F, V(f,g,h))$$

are always true. Since $V(f,g,h)$ is Kummer self-dual (by design), in order to show that (f,g,h) is F-exponential, it is enough to prove that $H^1_e(F, V(f,g,h)) = H^1_f(F, V(f,g,h))$. As a consequence of [BK90, corollary 3.8.4], we are then reduced to show that

$$\mathbb{D}_{\mathrm{cris},F}(V(f,g,h))^{\varphi=1} = \{0\}\,.$$

We will write

$$D_{fgh} := D_{f'} \otimes_{\tilde{L}} D_g \otimes_{\tilde{L}} D_{h'} \otimes_{\tilde{L}} (\tilde{L} \cdot t_{-1-r})\,,$$

where $\tilde{L} := F_0 \otimes_{\mathbb{Q}_p} L$ and recall that $r = (k+l+m-6)/2$ and that $\varphi(t_{-1-r}) = p^{r+1} t_{-1-r}$.

Here $D_{f'}$ is, up to extending scalars to F_0, the dual of the module D appearing in the proof of lemma 7.18. We can fix an $\tilde{L}$-basis of $D_{f'}$ given by $\{v_f, w_f\}$ (essentially passing to the dual basis of the one described in the proof of lemma 7.18), such that

$$\varphi(v_f) = \chi_f(p)^{-1}p^{1-k}a_p(f)\cdot v_f\,, \qquad \varphi(w_f) = a_p(f)^{-1}\cdot w_f\,, \qquad N = 0$$

if $V(f)$ is F_1-crystalline and

$$\varphi(v_f) = a_p(f)^{-1}\cdot v_f\,, \quad \varphi(w_f) = (pa_p(f))^{-1}\cdot w_f\,, \qquad N(v_f) = -w_f \quad N(w_f) = 0$$

if f is a newform in $S_2(M_1p, \chi_f, L)$.

Since $\mathbb{D}_{\mathrm{st},F}(V(f,g,h))$ is isomorphic to a finite direct sum of copies of D_{fgh}, it is enough to prove that $D_{fgh}^{\varphi=1,N=0} = \{0\}$. We will look at D_{fgh} as an L-vector space of dimension $16a$ where φ-acts L-linearly.

We fix $\alpha \in L$, a primitive $p^{2a}-1$-th root of 1, so that $F_0 = \mathbb{Q}_p(\alpha)$ and the arithmetic Frobenius σ (i.e., the generator of $\mathrm{Gal}(F_0/\mathbb{Q}_p)$ inducing the arithmetic Frobenius modulo p) acts on α simply as $\sigma(\alpha) = \alpha^p$.

By our previous discussion we know that a basis $\mathscr{B}$ of D_{fgh} as L-vector space can be described as follows:

$$\mathscr{B} := \left\{(\alpha^{p^j}\otimes 1)\cdot e_\tau \mid j = 0,\dots,2a-1, \quad \tau \in \{v,w\}^{\{1,2,3\}}\right\}$$

where for $\tau : \{1,2,3\} \to \{v,w\}$ we let

$$e_\tau = \tau(1)_f \otimes \tau(2)_g \otimes \tau(3)_{h'} \otimes t_{-1-r}$$

Now we prove assertion (a) in the statement. Since $k > 2$, $V_{Mp^t}(f)$ is F-crystalline (so that $V(f,g,h)$ is F-crystalline) and we have

$$\varphi((\alpha^{p^j}\otimes 1)\cdot e_\tau) := \begin{cases} \alpha^{(p-1)p^j}\cdot p^{r+2-k}\chi_f(p)^{-1}\cdot a_p(f)\cdot \mu_g\mu_{h'}\cdot((\alpha^{p^j}\otimes 1)\cdot e_\tau) & \text{if } \tau(1) = v,\\ \alpha^{(p-1)p^j}\cdot p^{r+1}\cdot a_p(f)^{-1}\cdot \mu_g\mu_{h'}\cdot((\alpha^{p^j}\otimes 1)\cdot e_\tau) & \text{if } \tau(1) = w.\end{cases}$$

If we let $\lambda_{j,\tau}$ to be the φ-eigenvalue relative to $(\alpha^{p^j}\otimes 1)\cdot e_\tau$ as described above, we can immediately compute that

$$|\lambda_{j,\tau}|_p = \begin{cases} p^{k/2} & \text{if } \tau(1) = v,\\ p^{1-k/2} & \text{if } \tau(1) = w.\end{cases}$$

so that if we assume $k > 2$ it cannot happen that $\lambda_{j,\tau} = 1$ or $\lambda_{j,\tau} = p^{-1}$. We have thus proven that (f,g,h) is $(F, 1-T)$-convenient (and in particular F-exponential).

For assertion (b), note that $N((\alpha^{p^j}\otimes 1)\cdot e_\tau) = 0$ if and only if $\tau(1) = w$. Assuming $\tau(1) = w$ we get

$$\varphi((\alpha^{p^j}\otimes 1)\cdot e_\tau) := \alpha^{(p-1)p^j}\cdot p^r\cdot a_p(f)^{-1}\cdot \mu_g\mu_{h'}\cdot((\alpha^{p^j}\otimes 1)\cdot e_\tau)\,.$$

and one checks that

$$|\alpha^{(p-1)p^j}\cdot p^r\cdot a_p(f)^{-1}\cdot \mu_g\mu_{h'}|_p = p,$$

so that also in this case $D_{fgh}^{\varphi=1,N=0} = \{0\}$ and the proof is complete. □

When (f,g,h) is F-exponential, [BK90, corollary 3.8.4] shows that the Bloch-Kato exponential map

$$\exp_{\mathrm{BK},F} : \frac{\mathbb{D}_{\mathrm{dR},F}(V(f,g,h))}{\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR},F}(V(f,g,h))} \to H^1_e(F,V(f,g,h)) = H^1_g(F,V(f,g,h)) \tag{7.7}$$

is an isomorphism.

The perfect duality (6.19) induces a perfect duality

$$\langle -,-\rangle_{fgh} : V_{\mathrm{dR}}(f,g,h) \otimes_L V^*_{\mathrm{dR}}(f,g,h) \to \mathbb{D}_{\mathrm{dR}}(L(1)) = L, \tag{7.8}$$

under which one has an identification

$$V_{\mathrm{dR}}(f,g,h)/\mathrm{Fil}^0(V_{\mathrm{dR}}(f,g,h)) \cong \mathrm{Fil}^0(V^*_{\mathrm{dR}}(f,g,h))^\vee.$$

Moreover, note that we have identifications

$$\mathbb{D}_{\mathrm{dR},F}(V(f,g,h)) = F \otimes_{\mathbb{Q}_p} V_{\mathrm{dR}}(f,g,h)\,, \qquad \mathrm{Fil}^0\mathbb{D}_{\mathrm{dR},F}(V(f,g,h)) = F \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$$

yielding an isomorphism

$$\frac{\mathbb{D}_{\mathrm{dR},F}(V(f,g,h))}{\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR},F}(V(f,g,h))} \cong F \otimes_{\mathbb{Q}_p} \frac{V_{\mathrm{dR}}(f,g,h)}{\mathrm{Fil}^0(V_{\mathrm{dR}}(f,g,h))}. \tag{7.9}$$

DEFINITION 7.29: Let (f,g,h) be an F-exponential triple (for some finite Galois extension $F/\mathbb{Q}_p$). We define the Bloch-Kato logarithm

$$\log^{fgh}_{\mathrm{BK}} : H^1_g(\mathbb{Q}_p, V(f,g,h)) \to \mathrm{Fil}^0(V^*_{\mathrm{dR}}(f,g,h))^\vee$$

as the following composition (we write $V = V(f,g,h)$ for simplicity):

$$H^1_g(\mathbb{Q}_p, V) \xrightarrow[\cong]{\mathrm{Res}_{F/\mathbb{Q}_p}} H^1_g(F,V)^{\mathrm{Gal}(F/\mathbb{Q}_p)} \xrightarrow{\frac{1}{[F:\mathbb{Q}_p]}\cdot \exp^{-1}_{\mathrm{BK},F}} \frac{V_{\mathrm{dR}}}{\mathrm{Fil}^0(V_{\mathrm{dR}})} \cong \mathrm{Fil}^0(V^*_{\mathrm{dR}})^\vee$$

REMARK 7.30: The first isomorphism $\mathrm{Res}_{F/\mathbb{Q}_p}$ above follows from the Hochschild-Serre spectral sequence in continuous group cohomology and the fact that the representations involved are vector spaces over characteristic zero fields ($\mathbb{B}_{\mathrm{dR}}$ is a field). Note that the map $\log^{fgh}_{BK}$ is actually independent of F (i.e., if F' is another finite Galois extension of $\mathbb{Q}_p$ such that $V(f,g,h)$ is F'-semistable, we obtain the same map via our construction). Note, moreover, that we are crucially using the isomorphism (7.9) and the fact that the exponential map (7.7) is equivariant for the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ (so that it induces and isomorphism between the Galois invariants on both sides).

In the next chapter, we will show that $\kappa(f,g,h) \in H_g(\mathbb{Q}_p, V(f,g,h))$, so that one can indeed view $\log^{fgh}_{\mathrm{BK}}(\kappa(f,g,h))$ as a linear functional on $\mathrm{Fil}^0(V^*_{\mathrm{dR}}(f,g,h))$ and study its value at

$$\eta^{\varphi=a_p}_{f'} \otimes \omega_g \otimes \omega_{h'} \otimes t_{r+2} \in \mathrm{Fil}^0(V^*_{\mathrm{dR}}(f,g,h))\,.$$

▲

CHAPTER 8

The syntomic Abel-Jacobi map

This chapter develops the formalism of syntomic and finite polynomial cohomology, with the aim of giving a syntomic description of the Abel-Jacobi map in our setting. We also recall the needed facts about the Coleman's study of the geometry of modular curves as rigid analytic spaces.

8.1. Syntomic and finite-polynomial cohomology

In this section we recollect some facts concerning syntomic (and finite-polynomial) cohomology and we postulate some properties for syntomic (and finite-polynomial) cohomology with coefficients that will be assumed in the sequel, trying to motivate why we expect such properties to hold.

We will have to consider recent extensions of the original theory (due, among others, to Besser, cf. [Bes00]), since the modular curves $X_1(Mp^t)$ (for $t \geq 1$) only admit a semistable model over the ring of integers of a (large enough) finite extension of $\mathbb{Q}_p$.

As in the previous chapters, let F denote a fixed finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$ and residue field k_F. We fix a choice of a uniformizer $\varpi \in \mathcal{O}_F$. Let $F_0 = W(k_F)[1/p]$ denote the maximal unramified subextension of $\mathbb{Q}_p$ inside F.

The theory of syntomic cohomology for trivial coefficients and general smooth rigid varieties over F (without specific requirements on the existence of semistable models over $\mathcal{O}_F$) is established in [NN16] (cf. also [BLZ16]).

We will refer to the work of Ertl and Yamada (cf. [EY21] and the preprints [EY22], [EY23]) for a version of Hyodo-Kato cohomology for strictly semistable log-schemes over k_F^0 (where k_F^0 denotes the log-scheme $\mathrm{Spec}(k_F)$ with the log structure associated with the monoid homomorphism $\mathbb{N} \to k_F$, $1 \mapsto 0$). Building up on the aforementioned works, the preprints [Yam22] and [HW22]) develop versions of syntomic (and finite polynomial) cohomology with coefficients for strictly semistable log schemes over $\mathcal{O}_F^{\varpi}$ (where $\mathcal{O}_F^{\varpi}$ denotes the scheme $\mathrm{Spec}(\mathcal{O}_F)$ with canonical log structure associated with the monoid homomorphismm $\mathbb{N} \to \mathcal{O}_F$, $1 \mapsto \varpi$).

We let $\mathcal{X}$ denote a proper strictly semistable log scheme over $\mathcal{O}_F^{\varpi}$, with horizontal divisor $\mathcal{D} \subset \mathcal{X}$ (cf. [EY23, definition 4.16]) and open complement $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{D}$. Denote by $\mathcal{X}_0$ the special fiber of $\mathcal{X}$ (which is a strictly semistable log-scheme over k_F^0) and by $\mathcal{D}_0 \subset \mathcal{X}_0$ the corresponding horizontal divisor. Finally, we denote $X = \mathcal{X}_F$, $D = \mathcal{D}_F$, $U = \mathcal{U}_F$ the corresponding generic fibers.

Given a filtered overconvergent isocrystal $(\mathscr{E}, \Phi, \mathrm{Fil})$ over $\mathcal{X}$, [Yam22] constructs a Hyodo-Kato map in the derived category of F_0-vector spaces (dipending on the choice of the uniformizer $\varpi \in \mathcal{O}_F$ and of a branch of the p-adic logarithm $\log : F^\times \to F$)

$$\Psi_{\varpi,\log,F_0} : R\Gamma_{\mathrm{HK}}(\mathcal{X}_0, \mathscr{E}) \to R\Gamma_{\mathrm{dR}}(X, \mathscr{E}_{\mathrm{dR}}) \, .$$

The notations is as follows:

(i) $R\Gamma_{\mathrm{HK}}(X_0, \mathscr{E})$ is a complex of F_0-vector spaces with F_0-semilinear Frobenius φ and F_0-linear monodromy N such that $N\varphi = p\varphi N$ which computes Hyodo-Kato cohomology;

(ii) $R\Gamma_{\mathrm{dR}}(X, \mathscr{E}_{\mathrm{dR}}) = R\Gamma(X, \mathscr{E}_{\mathrm{dR}} \otimes \Omega^\bullet_X\langle D\rangle)$ is the complex of F-vector spaces with filtration which computes de Rham cohomology of X with coefficients in $\mathscr{E}_{\mathrm{dR}}$ (the de Rham realization of $\mathscr{E}$) and log poles along D (or equivalently the de Rham cohomology of U with coefficients in $\mathscr{E}_{\mathrm{dR}}$).

Analogously, [EY23] in the compactly supported case constructs a map

$$\Psi_{\varpi,\log,F_0,c} : R\Gamma_{\mathrm{HK},c}(\mathcal{X}_0, \mathscr{E}) \to R\Gamma_{\mathrm{dR},c}(X, \mathscr{E}_{\mathrm{dR}})$$

Here $R\Gamma_{\mathrm{HK},c}(\mathcal{X}_0, \mathscr{E})$ computes Hyodo-Kato cohomology with compact support and

$$R\Gamma_{\mathrm{dR},c}(X, \mathscr{E}_{\mathrm{dR}}) := R\Gamma(X, \mathscr{E}_{\mathrm{dR}} \otimes \Omega^\bullet_X\langle -D\rangle)$$

computes compactly supported de Rham cohomology of U with coefficients in $\mathscr{E}_{\mathrm{dR}}$.

Assuming that $\mathscr{E}$ is unipotent (i.e., an iterated extension of the trivial isocrystal) makes sure that the map $\Psi_{\varpi,\log,F,?} := \Psi_{\varpi,\log,F_0,?} \otimes_{F_0} F$ (for $? \in \{c, \varnothing\}$) is an isomorphism (cf. [Yam22, proposition 8.8] and [EY23, corollary 3.9]).

On the other hand (cf. [HW22, section 6]), if one was able to prove that $\Psi_{\varpi,\log,F,?}$ (for $? \in \{c, \varnothing\}$) is an isomorphism also for an isocrystal $(\mathscr{E}, \Phi, \mathrm{Fil})$ which is not necessarily unipotent, then the definition of syntomic (and finite polynomial) cohomology via a suitable mapping fiber and the subsequent formalism would work in the same way as in the unipotent case.

From now on we will suppress the dependance on ϖ and of $\log : F^\times \to F$ from the notation and we fix a polynomial $P \in 1 + T \cdot \mathbb{Q}_p[T]$ (one can also make the whole theory L-linear for any finite extension L of $\mathbb{Q}_p$ and work with $P \in 1 + T \cdot L[T]$, but we will avoid it here for simplicity). Moreover, we will assume that $\Psi_{F,?}$ is an isomorphism for $? \in \{c, \varnothing\}$ and for every log overconvergent filtered isocrystal considered. Provided this assumption, one can define the complex computing syntomic P-cohomology of $\mathcal{X}$ with coefficients in $(\mathscr{E}, \Phi, \mathrm{Fil})$ twisted by any $n \in \mathbb{Z}$ formally as in [EY21, definition 4.5], where the authors work with trivial coefficients and $P(T) = 1 - T$.

Since we are assuming that $\mathcal{X}$ is proper, this complex can be defined directly as the homotopy limit (cf. [EY21, section 1.2] for the conventions on the notation):

$$R\Gamma_{\mathrm{syn},P,?}(\mathcal{X}, \mathscr{E}, n)$$

$$:= \left[\begin{tikzcd} R\Gamma_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E}) \arrow[r, "{(P(p^{-n}\varphi),\, \Psi_F^{-1})}"] \arrow[d, "N"] & R\Gamma_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E}) \oplus R\Gamma_{\mathrm{dR},?}(X, \mathscr{E}_{\mathrm{dR}})/\mathrm{Fil}^n \arrow[d, "{(N,0)}"] \\ R\Gamma_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E}) \arrow[r, "P(p^{1-n}\varphi)"] & R\Gamma_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E}) \end{tikzcd}\right] \tag{8.1}$$

We will write simply $R\Gamma_{\mathrm{syn},P,?}(\mathcal{X}, n)$ when $(\mathscr{E}, \Phi, \mathrm{Fil})$ is the trivial filtered isocrystal.

REMARK 8.1: In this remark we axiomatize some features of syntomic P-cohomology that we will need in the sequel (in the setting described above).

(i) Assume that a filtered isocrystal $(\mathscr{E}, \Phi, \mathrm{Fil})$ is such that $\mathscr{E}_{\mathrm{dR}}$ is the de Rham realization of a Kummer étale $\mathbb{Z}_p$-local system $\mathbb{L}$ on (X, D). Then, for $P \in 1 + T \cdot \mathbb{Q}_p[T]$, $n \in \mathbb{Z}$

and $? \in \{c, \varnothing\}$, there is a spectral sequence

$$E_2^{i,j} := H^i_{\mathrm{st},P}\left(F, \mathbb{D}_{\mathrm{pst}}(H^j_{\text{ét},?}(U_{\bar{\mathbb{Q}}_p}, \mathbb{L}(n))\right) \Rightarrow H^{i+j}_{\mathrm{syn},P,?}(\mathcal{X}, \mathscr{E}, n)\,. \tag{8.2}$$

(ii) (cf. [HW22, section 4.2]) Given $P, Q \in 1 + T \cdot \mathbb{Q}_p(T)$ and log overconvergent filtered isocrystals $(\mathscr{E}_1, \Phi_1, \mathrm{Fil}_1)$ and $(\mathscr{E}_2, \Phi_2, \mathrm{Fil}_2)$, one has for every $n, m \in \mathbb{Z}$ a natural map

$$R\Gamma_{\mathrm{syn},P}(\mathcal{X}, \mathscr{E}_1, n) \otimes_{K_0} R\Gamma_{\mathrm{syn},Q,c}(\mathcal{X}, \mathscr{E}_2, m) \to R\Gamma_{\mathrm{syn},P*Q,c}(\mathcal{X}, \mathscr{E}_1 \otimes \mathscr{E}_2, n+m)\,, \tag{8.3}$$

inducing cup products

$$H^i_{\mathrm{syn},P}(\mathcal{X}, \mathscr{E}_1, n) \times H^j_{\mathrm{syn},Q,c}(\mathcal{X}, \mathscr{E}_2, m) \xrightarrow{\cup} H^{i+j}_{\mathrm{syn},P*Q,c}(\mathcal{X}, \mathscr{E}_1 \otimes \mathscr{E}_2, n+m)\,. \tag{8.4}$$

The spectral sequence (8.2) is compatible with this cup product and the cup product of proposition 7.13.

(iii) If $P \in 1 + T \cdot \mathbb{Q}_p(T)$ is such that $P(1) \neq 0 \neq P(p^{-1})$ and X has dimension d_X, then there is a trace map

$$\mathrm{Tr}_{\mathcal{X},\mathrm{syn},P} : H^{2d_X+1}_{\mathrm{syn},P,c}(\mathcal{X}, d_X + 1) \twoheadrightarrow H^1_{\mathrm{st},P}(F, \mathbb{Q}_p^{nr}(1)) \cong F \tag{8.5}$$

whose kernel is the second step Fil^2 of the 3-step filtration induced on the cohomology group $H^{2d_X+1}_{\mathrm{syn},P,c}(\mathcal{X}, d_X+1)$ by the spectral sequence 8.2.

The isomorphism $H^1_{\mathrm{st},P}(F, \mathbb{Q}_p^{nr}(1)) \cong F$ is explicitly given by sending a class $[(w, z, y)] \in H^1_{\mathrm{st},P}(F, \mathbb{Q}_p^{nr}(1))$ to $y - P(p^{-1})^{-1}w$.

(iv) (cf. [HW22, section 4.2]) Assume that $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of proper strictly semistable log schemes of codimension j such that Z (the generic fibre of $\mathcal{Z}$) is smooth over K. Then for every $i \in \mathbb{Z}_{\geq 0}$, $P \in 1 + T \cdot \mathbb{Q}_p(T)$, $n \in \mathbb{Z}$ and every log overconvergent filtered isocrystal $(\mathscr{E}, \Phi, \mathrm{Fil})$ on $\mathcal{X}$, there are pushforward maps

$$\iota_* : H^i_{\mathrm{syn},P}(\mathcal{Z}, \iota^*(\mathscr{E}), n) \to H^{i+2j}_{\mathrm{syn},P}(\mathcal{X}, \mathscr{E}, n+j)\,. \tag{8.6}$$

It follows from the construction of such pushforward maps that the following projection formula holds:

$$\iota_*\alpha \cup \beta = \iota_*(\alpha \cup \iota^*\beta) \tag{8.7}$$

for every $\alpha \in H^i_{\mathrm{syn},P}(\mathcal{Z}, \iota^*(\mathscr{E}_1), n)$ and every $\beta \in H^j_{\mathrm{syn},P,c}(\mathcal{X}, \mathscr{E}_2, m)$, where the cup product is the syntomic cup product defined in (8.4) (on $\mathcal{X}$ and $\mathcal{Z}$ respectively).

▲

REMARK 8.2: The results discussed in (i)-(iv) in the above remark are well-known in the case of trivial coefficients, as established in [NN16] (cf. also [BLZ16] for the P-syntomic case and [EY21, section 5] for the comparison between Nekovář-Nizioł and Ertl-Yamada syntomic cohomology in our setting). Thanks to the so-called *Lieberman's trick* (cf. for instance [BSV22b, section 3.2] and/or [BSV20, paragraph 4.1.3]), we will always be able to assume that the relevant cohomology groups for (products of) modular curves appearing in the sequel are direct summands of cohomology groups of (products of) suitable Kuga-Sato varieties with trivial coefficients. Hence we will use freely the properties (i)-(iv) in the sequel.

On the other hand, it is still an interesting problem (cf. [HW22, section 1.3]) to single out a nice and not too restrictive category of syntomic coefficients for which the properties (i)-(iv) can be proven. As already noted in [HW22, section 1.3], the category of syntomic coefficients introduced in [Yam22] looks a bit too restrictive, since one is allowed to consider only unipotent log overconvergent isocrystals (while for instance the isocrystals associated with the local systems $\mathscr{H}_r$ or $\mathscr{L}_r$ of section 6.1 are not unipotent, but satisfy the property that the Hyodo-Kato map $\Psi_{K,?}$ for $? \in \{c, \varnothing\}$ is an isomorphism, as it was already shown in [CI10]). ▲

REMARK 8.3: As explained in [BLZ16, section 2.4] (cf. also [EY21, section 4.2]), a class $\eta \in H^i_{\mathrm{syn},P,?}(\mathcal{X}, \mathscr{E}, n)$ can be represented by a sextuple $(u, v; w, x, y; z)$, where

$$u \in R\Gamma^i_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E}), \qquad v \in \mathrm{Fil}^n R\Gamma^i_{\mathrm{dR},?}(X, \mathscr{E}_{\mathrm{dR}}),$$
$$w, x \in R\Gamma^{i-1}_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E}), \qquad y \in R\Gamma^{i-1}_{\mathrm{dR},?}(X, \mathscr{E}_{\mathrm{dR}}),$$
$$z \in R\Gamma^{i-2}_{\mathrm{HK},?}(\mathcal{X}_0, \mathscr{E})$$

and the following relations are satisfied (where d denotes the differential in the respective complex):

$$du = 0, \qquad dv = 0,$$
$$dw = P(p^{-n}\varphi)u, \qquad dx = Nu,$$
$$dy = \Psi_K(u) - v, \qquad dz = Nw - P(p^{1-n}\varphi)x.$$

One can give explicit formulas for the cup products (8.4) introduced above in terms of this description. We refer to [BLZ16, proposition 2.4.1] for that. ▲

8.2. De Rham cohomology and overconvergent modular forms

In this section we recall some facts concerning the geometry of $X_1(Mp^t)$ (for $t \geq 1$ and $p \nmid M$) as a rigid analytic variety over $\mathbb{Q}_p$ (or a suitable extension of $\mathbb{Q}_p$). One can find a more detailed treatment in [Col97], [BE10, section 4.4] and [DR17, section 4.1]. Recall that L denotes a large enough finite extension of $\mathbb{Q}_p$ as usual.

Set $K_t := \mathbb{Q}_p(\zeta_{p^t})$ (where ζ_{p^t} denotes a primitive p^t-th root of unity). It is well-known that $X_1(Mp^t)_{K_t}$ admits a proper flat regular model $\mathcal{X}_1(Mp^t)$ over $\mathcal{O}_{K_t} = \mathbb{Z}_p[\zeta_{p^t}]$ (cf. [KM85, chapters 12-13]). If the integer M considered is large enough (which we can always ensure), we can assume that the irreducible components of the special fiber of $\mathcal{X}_1(Mp^t)$ are all smooth of genus at least 2. It is known that exactly two of the irreducible components of $\mathcal{X}_1(Mp^t)_0$ (the special fiber of $\mathcal{X}_1(Mp^t)$) are isomorphic to the Igusa curve usually denoted $\mathrm{Ig}(Mp^t)$ as curves over $\mathbb{F}_p$. Usually this two irreducible components are denoted $\mathrm{Ig}_\infty(p^t)$ and $\mathrm{Ig}_0(p^t)$, as $\mathrm{Ig}_\infty(p^t)$ contains the cusp ∞, while $\mathrm{Ig}_0(p^t)$ contains the cusp 0.

Up to extending scalars to a finite extension $F \subseteq L$ of $\mathbb{Q}_p(\zeta_{p^t})$, one can construct a regular stable model $\mathcal{X}_t$ of $X_{t,F} := X_1(Mp^t)_F$ over $\mathrm{Spec}(\mathcal{O}_F)$, together with a birational map $\mathcal{X}_t \to \mathcal{X}_1(Mp^t)_{\mathcal{O}_F}$. This maps identifies two irreducible components of $\mathcal{X}_{t,0}$ (the special fiber of $\mathcal{X}_t$) with $\mathrm{Ig}_\infty(p^t)_{k_F}$ and $\mathrm{Ig}_0(p^t)_{k_F}$ (where k_F is the residue field of F).

We can look at X_{t,K_t} (resp. $X_{t,F}$) as a rigid analytic space X^{an}_{t,K_t} (resp. $X^{\mathrm{an}}_{t,F}$) over K_t (resp. F). We define two so-called *wide open subspaces* $\mathcal{W}_\infty(p^t)$ and (resp.) $\mathcal{W}_0(p^t)$ of X^{an}_{t,k_T} as the preimages of $\mathrm{Ig}_\infty(p^t)$ and (resp.) $\mathrm{Ig}_0(p^t)$ under the reduction map. The preimage of the smooth locus of $\mathrm{Ig}_\infty(p^t)$ (resp. $\mathrm{Ig}_0(p^t)$) is an affinoid subset of X^{an}_{t,K_t} which will be denoted by $\mathcal{A}_\infty(p^t)$ (resp. $\mathcal{A}_0(p^t)$).

Since X_{t,K_t} is proper over K_t, rigid de Rham cohomology agrees with algebraic de Rham cohomology. In particular we can consider for every $\nu \geq 2$ and for $? \in \{\infty, 0\}$ restriction maps

$$\mathrm{Res}_? : H^1_{\mathrm{dR}}(X_{t,K_t}, \mathscr{H}_{\nu-2})_L \to H^1_{\mathrm{dR}}(\mathcal{W}_?(p^t), \mathscr{H}_{\nu-2})_L\,.$$

According to [BE10, lemme 4.4.1], one can equip the groups $H^1_{\mathrm{dR}}(\mathcal{W}_?(p^t), \mathscr{H}_{\nu-2})$ with an action of Hecke and diamond operators away from Mp in such a way that the restriction maps are Hecke equivariant.

Since $X_{t,F}$ admits a stable model over $\mathcal{O}_F$, we can consider (up to fixing once a for all a uniformizer $\varpi \in \mathcal{O}_F$ and a branch of the p-adic logarithm) the Hyodo-Kato cohomology $H^1_{\mathrm{HK}}(\mathcal{X}_{t,0}, \mathscr{H}_{\nu-2})$.

Recall that the F_0-vector space $H^1_{\mathrm{HK}}(\mathcal{X}_{t,0}, \mathscr{H}_{\nu-2})$ is endowed with the structure of (φ, N)-module. We let $\Phi := \varphi^d$ (where $d = [F_0 : \mathbb{Q}_p]$) so that, extending scalars from F_0 to F, we can endow, via the isomorphism $\Psi_{\varpi,\log,K}$ of the previous section, $H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{\nu-2})$ with an F-linear Frobenius Φ and monodromy operator N such that $N \circ \Phi = p^d \cdot \Phi \circ N$ (cf. also the discussion in remark 8.2).

One can endow the F-vector spaces $H^1_{\mathrm{dR}}(\mathcal{W}_?(p^t)_F, \mathscr{H}_{\nu-2})$ with an F-linear Frobenius $\Phi_?$ such that the restriction map $\mathrm{Res}_?$ is Frobenius equivariant for $? \in \{\infty, 0\}$, as explained in [Col97]. One of the main results of [Col97] (namely theorem 2.1) can be stated as follows. The restriction maps induce an isomorphism

$$H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{\nu-2})^{\mathrm{prim}}_L \xrightarrow{\cong} H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t)_F, \mathscr{H}_{\nu-2})^*_L \oplus H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{\nu-2})^*_L \tag{8.8}$$

Here $H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{\nu-2})^{\mathrm{prim}}_L$ denotes the p-primitive subspace, i.e., the subspace spanned by the common eigenspaces for the action of Hecke and diamond operators relative to systems of eigenvalues coming from p-primitive modular eigenforms of exact level dividing Mp^t (recall that a modular newform is p-primitive if the p-part of the conductor of its nebentypus equals the p-part of its level).

The notation $H^1_{\mathrm{dR}}(\mathcal{W}_?(p^t)_F, \mathscr{H}_{\nu-2})^*_L$ denotes the so-called *pure classes*, i.e., those whose restriction to supersingular annuli is trivial.

One can show that the action of Φ on overconvergent modular forms of weight ν, seen as sections of ω^ν in a wide open neighbourhood of $\mathcal{A}_\infty(p^t)$ inside $\mathcal{W}_\infty(p^t)$, can be described by the d-th power of the operator $p^{(\nu-1)}\langle p\,;1\rangle V$. Here (cf. the end of section 6.1) $\langle a\,;b\rangle$ denotes the diamond operator corresponding to the unique class in $(\mathbb{Z}/Mp^t\mathbb{Z})^\times$ which is congruent to a modulo M and to b modulo p^t, while V acts on q-expansions sending $q \mapsto q^p$ as usual.

This yields an operator Φ_∞ on $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t)_F, \mathscr{H}_{\nu-2})_L$ which is the F-linear extension of the d-th power φ^d_∞ of an operator φ_∞ on $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t), \mathscr{H}_{\nu-2})_L$. One can define the corresponding operator φ_0 on $H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t), \mathscr{H}_{\nu-2})_L$ simply as $\varphi_0 = (w^{-1}_{p^t})^* \circ \varphi_\infty \circ w_{p^t}$.

The isomorphism (8.8) is equivariant for the action of Φ on $H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{\nu-2})^{\mathrm{prim}}_L$ and the action of (Φ_∞, Φ_0) on $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t)_F, \mathscr{H}_{\nu-2})^*_L \oplus H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{\nu-2})^*_L$, where clearly $\Phi_0 := (w^{-1}_{p^t})^* \circ \Phi_\infty \circ w_{p^t}$. One can also show that on $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t), \mathscr{H}_{\nu-2})_L$ it holds $U \circ V = V \circ U = 1$, where U is the usual U_p operator.

For these facts we refer again to [BE10, section 4.4] and to [DR17, section 4.1]

8.3. The syntomic Abel-Jacobi map

In the above section 8.2 - after possibly replacing the field F of remark 7.27 by a finite extension and enlarging M - we fixed stable model $\mathcal{X}_t$ of $X_{t,F}$ over $\mathrm{Spec}(\mathcal{O}_F)$. Using the recipes of [Har01], one can also obtain a regular strictly semistable variety $\mathcal{W}_t$ over $\mathrm{Spec}(\mathcal{O}_F)$, together with a morphism of $\mathcal{O}_F$-schemes (obtained by successively blowing up the products of irreducible components of the special fiber of $\mathcal{X}_t$) $b_t : \mathcal{W}_t \to \mathcal{X}_t \times_{\mathcal{O}_F} \mathcal{X}_t \times_{\mathcal{O}_F} \mathcal{X}_t$. The morphism b_t, after base change to F, induces an identification $\mathcal{W}_{t,F} = X^3_{t,F}$. Moreover, there is a unique factorization

$$\begin{tikzcd} & \mathcal{W}_t \arrow[d, "b_t"] \\ \mathcal{X}_t \arrow[ur, dashed, hook, "\delta_t"] \arrow[r, hook, "d_t"'] & \mathcal{X}_t \times_{\mathcal{O}_F} \mathcal{X}_t \times_{\mathcal{O}_F} \mathcal{X}_t \end{tikzcd}$$

afforded by the universal property of blow-ups, where d_t is the diagonal embedding. By [GW20, proposition 13.96], it also follows that the lift δ_t is still a closed embedding.

The following diagram summarizes the étale and the syntomic versions of the Abel-Jacobi maps over F. Note that, by abuse of notation, we use the same symbol for the syntomic coefficients and their étale counterparts. Here (f,g,h) is again a triple of modular forms satisfying assumption 6.11.

$$\begin{tikzcd} H^0_{\mathrm{syn}}(\mathcal{X}_t, \mathscr{H}_{\mathbf{r}}, r) \cong H^0_{\text{ét}}(Y_{t,F}, \mathscr{H}_{\mathbf{r}}(r)) \arrow[d, "AJ_{\mathrm{syn},F}"'] \arrow[dr, bend left, "AJ_{\text{ét},F}"] & \\ H^1_{\mathrm{st}}\Big(F, \mathbb{D}_{\mathrm{pst}}(H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{[\mathbf{r}]}(r+2))_L)\Big) \arrow[r, "\exp_{\mathrm{st}}", "\cong"'] \arrow[d, "s_{\mathbf{r}}"', "\cong"] & H^1_g\Big(F, H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{[\mathbf{r}]}(r+2))_L\Big) \arrow[d, "\cong"', "s_{\mathbf{r}}"] \\ H^1_{\mathrm{st}}\Big(F, \mathbb{D}_{\mathrm{pst}}(H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}_p}, \mathscr{L}_{[\mathbf{r}]}(2-r))_L)\Big) \arrow[r, "\exp_{\mathrm{st}}", "\cong"'] \arrow[d, two heads, "pr^{fgh}_{\mathrm{st}}"'] & H^1_g\Big(F, H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}_p}, \mathscr{L}_{[\mathbf{r}]}(2-r))_L\Big) \arrow[d, two heads, "pr^{fgh}_{\text{ét}}"] \\ H^1_{\mathrm{st}}(F, \mathbb{D}_{\mathrm{pst}}(V(f,g,h))) \arrow[r, "\exp_{\mathrm{st}}", "\cong"'] & H^1_g(F, V(f,g,h)) \end{tikzcd}$$

REMARK 8.4: Some remarks are in order.

(i) The isomorphism $H^0_{\mathrm{syn}}(\mathcal{X}_t, \mathscr{H}_{\mathbf{r}}, r) \cong H^0_{\text{ét}}(Y_{t,F}, \mathscr{H}_{\mathbf{r}}(r))$ can be realized as the composition of the following canonical isomorphisms:

$$\begin{aligned} H^0_{\mathrm{syn}}(\mathcal{X}_t, \mathscr{H}_{\mathbf{r}}, r) &\cong H^0_{\mathrm{st}}(F, \mathbb{D}_{\mathrm{pst}}(H^0_{\text{ét}}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\mathbf{r}}(r))) \\ &\cong H^0(F, H^0_{\text{ét}}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\mathbf{r}}(r)) \\ &\cong H^0_{\text{ét}}(Y_{t,F}, \mathscr{H}_{\mathbf{r}}(r)) \end{aligned}$$

The first isomorphism follows from the corresponding spectral sequence 8.2, the third isomorphism follows from theorem 7.8 and the third isomorphism is afforded by the Hochschild-Serre spectral sequence.

(ii) The construction of the syntomic Abel-Jacobi map $AJ_{\mathrm{syn},F}$ follows the same ideas as for the étale counterpart (cf. the diagram above remark 6.18 and the remark itself). The fact that δ_t is a closed immersion gives rise to pushforward map

$$\delta_{t,*} : H^0_{\mathrm{syn}}(\mathcal{X}_t, \mathscr{H}_{\mathbf{r}}, r) \to H^4_{\mathrm{syn}}(\mathcal{W}_t, \mathscr{H}_{[\mathbf{r}]}, r+2)\,.$$

Set $D^j := \mathbb{D}_{\mathrm{pst}}\left(H^j_{\text{két}}(X^3_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{[\mathbf{r}]}(r+2)_L)\right)$. Then the spectral sequence

$$E_2^{i,j} := H^i_{\mathrm{st}}(F, D^j) \Rightarrow H^{i+j}_{\mathrm{syn}}(\mathcal{W}_t, \mathscr{H}_{[\mathbf{r}]}, r+2))_L \tag{8.9}$$

induces a surjective morphism $H^4_{\mathrm{syn}}(\mathcal{W}_t, \mathscr{H}_{[\mathbf{r}]}, r+2)_L \twoheadrightarrow H^1_{\mathrm{st}}(F, D^3)$, whose kernel is the second step filtration $\mathrm{Fil}^2(H^4_{\mathrm{syn}}(\mathcal{W}_t, \mathscr{H}_{[\mathbf{r}]}, r+2)_L)$. The syntomic Abel-Jacobi maps is the composition

$$AJ_{\mathrm{syn},F} : H^0_{\mathrm{syn}}(\mathcal{X}_t, \mathscr{H}_{\mathbf{r}}, r) \xrightarrow{\delta_{t,*}} H^4_{\mathrm{syn}}(\mathcal{W}_t, \delta_{t,*}(\mathscr{H}_{\mathbf{r}}), r+2) \twoheadrightarrow H^1_{\mathrm{st}}(F, D^3)$$

▲

COROLLARY 8.5: *The class $\kappa(f,g,h) \in H^1(\mathbb{Q}_p, V(f,g,h))$ defined in section 6.3 always belongs to the subspace $H^1_g(\mathbb{Q}_p, V(f,g,h))$.*

PROOF. The above discussion shows that the image of $\kappa(f,g,h)$ inside $H^1(F, V(f,g,h))$ belongs to the subspace $H^1_g(F, V(f,g,h))$ and it is $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariant. We conclude recalling that restriction induces an isomorphism (cf. remark 7.30)

$$H^1_g(\mathbb{Q}_p, V(f,g,h)) \xrightarrow[\cong]{\mathrm{Res}_{F/\mathbb{Q}_p}} H^1_g(F, V(f,g,h))^{\mathrm{Gal}(F/\mathbb{Q}_p)}.$$

□

CHAPTER 9

The explicit reciprocity law

In this chapter we state and prove the main result of the second part of this thesis.

9.1. Statement and first consequences

We consider as before a triple (f, g, h) of modular forms satisfying assumption 6.11 and which is moreover F-exponential (cf. definition 7.23). In what follows we keep the notation introduced in the previous chapters

DEFINITION 9.1: We define the p-adic period

$$\mathscr{I}_p(f,g,h) := \log_{\mathrm{BK}}^{fgh}(\kappa(f,g,h))(\eta_{f'}^{\varphi=a_p} \otimes \omega_g \otimes \omega_{h'} \otimes t_{r+2}) \in L$$

attached to the F-exponential triple (f, g, h).

Note again that the definition of $\mathscr{I}_p(f,g,h)$ does not depend on F, but only on the triple (f, g, h) (and possibly on the common level Mp^t chosen).

THEOREM 9.2: *If the triple (f, g, h) satisfies assumption 7.25 (i.e., the forms g and h are supercuspidal at p and killed by U_p) and $k > 2$, we have*

$$\mathscr{I}_p(f,g,h) = (-1)^{k-2}(r-k+2)! \cdot a_1(e_{\breve{f}}(\mathrm{Tr}_{Mp^t/M_1p^t}(g \times d^{(k-l-m)/2}h'))) ,$$

where $d = q\frac{d}{dq}$ denotes Serre's derivative operator and $a_1(-)$ denotes the first Fourier coefficient of the q-expansion at ∞.

The rest of this chapter is devoted to the proof of theorem 9.2.

9.2. Reduction to a pairing in de Rham cohomology

For $\xi \in \{g, h'\}$ we have

$$F \otimes_{\mathbb{Q}_p} V^*_{\mathrm{dR},Mp^t}(\xi) \subseteq H^1_{\mathrm{dR},c}(Y_{t,F}, \mathscr{H}_{\nu-2})_L$$

and this inclusion is compatible for the structure of (Φ, N, L)-module on both sides (recall that $\Phi = \varphi^d$, $d = [F_0 : \mathbb{Q}_p]$). In particular we deduce that

$$\omega_\xi \otimes t_{\nu-1} \in \mathrm{Fil}^{\nu-1} H^1_{\mathrm{dR},c}(Y_{t,F}, \mathscr{H}_{\nu-2})_L \cap H^1_{\mathrm{HK},c}(\mathcal{X}_{t,0}, \mathscr{H}_{\nu-2}, \nu-1)_L^{P_\xi(\varphi)=0} , \tag{9.1}$$

where $P_\xi(T) := 1 - \mu_\xi p^{\nu-1} \cdot T \in 1 + T \cdot L[T]$ and (as before) t_n is a (canonical) generator of $\mathbb{Q}_p(n)$.

With the notation introduced in section 7.1, we can rephrase (9.1) as

$$\omega_\xi \otimes t_{\nu-1} \in H^0_{\mathrm{st},P_\xi}\left(F, \mathbb{D}_{\mathrm{pst}}(H^1_{\mathrm{\acute{e}t},c}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\nu-2}(\nu-1))_L)\right) .$$

LEMMA 9.3: *Assume $\nu > 2$. Then for every $P(T) \in 1 + T \cdot L[T]$ and every $n \in \mathbb{Z}$ there are natural isomorphisms*
(9.2)
$$H^1_{\mathrm{syn},P,c}(\mathcal{X}_t, \mathscr{H}_{\nu-2}, n)_L \xrightarrow{\cong} H^0_{\mathrm{st},P}(F, D^1(n)), \quad H^2_{\mathrm{syn},P,c}(\mathcal{X}_t, \mathscr{H}_{\nu-2}, n)_L \xrightarrow{\cong} H^1_{\mathrm{st},P}(F, D^1(n)),$$
where
$$D^j(n) := \mathbb{D}_{\mathrm{pst}}\left(H^j_{\mathrm{\acute{e}t},c}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\nu-2}(n))_L\right)$$

PROOF. Since in this case $H^0_{\mathrm{dR}}(Y_{t,F}, \mathscr{H}_{\nu-2}) = H^0_{\mathrm{dR},c}(Y_{t,F}, \mathscr{H}_{\nu-2}) = 0$ (cf. [BDP13, lemma 2.1]) and (by duality) $H^2_{\mathrm{dR}}(Y_{t,F}, \mathscr{H}_{\nu-2}) = 0 = H^2_{\mathrm{dR},c}(Y_{t,F}, \mathscr{H}_{\nu-2})$, then the spectral sequence (cf. equation (8.2))

$$E_2^{i,j} := H^i_{\mathrm{st},P_\xi}(F, D^j(n)) \Rightarrow H^{i+j}_{\mathrm{syn},P_\xi,c}(\mathcal{X}_t, \mathscr{H}_{\nu-2}, n)_L \tag{9.3}$$

has $E_2^{i,j} = 0$ unless $j = 1$ and $i \in \{0, 1, 2\}$. The required isomorphisms follow directly from the degeneration of this spectral sequence. □

PROPOSITION 9.4: *For $\xi \in \{g, h'\}$, the class $\omega_\xi \otimes t_{\nu-1}$ can be lifted to a class*
$$\tilde{\omega}_{\xi,\nu-1} \in H^1_{\mathrm{syn},P_\xi,c}(\mathcal{X}_t, \mathscr{H}_{\nu-2}, \nu - 1)_L .$$
Such a lift is unique if $\nu > 2$ and it is unique up to an element of $F \otimes_{\mathbb{Q}_p} L$ if $\nu = 2$.

PROOF. This follows directly from lemma 9.2 when $\nu > 2$, so we assume $\nu = 2$.

The first four terms of the 5-term exact sequence associated to the spectral sequence (9.3) (with $n = \nu - 1 = 1$) look like
$$0 \to H^1_{\mathrm{st},P_\xi}(F, D^0(1)) \to H^1_{\mathrm{syn},P_\xi,c}(\mathcal{X}_t, 1)_L \to H^0_{\mathrm{st},P_\xi}(F, D^1(1)) \to H^2_{\mathrm{st},P_\xi}(F, D^0(1)),$$
so that the existence of a lift $\tilde{\omega}_{\xi,1}$ is equivalent to the vanishing of $\omega_\xi \otimes t_1$ under the *knight move* $H^0_{\mathrm{st},P_\xi}(F, D^1(1)) \to H^2_{\mathrm{st},P_\xi}(F, D^0(1))$. One can easily compute that
$$H^2_{\mathrm{st},P_\xi}(F, D^0(1)) \cong H^2_{\mathrm{st},P_\xi}(F, \mathbb{Q}_p^{nr}(1) \otimes_{\mathbb{Q}_p} L) \cong \frac{F_0 \otimes_{\mathbb{Q}_p} L}{P_\xi(1)(F_0 \otimes_{\mathbb{Q}_p} L)} .$$
Looking at p-adic absolute values, one sees that $P_\xi(1) \neq 0$, showing that $H^2_{\mathrm{st},P_\xi}(F, D^0) = 0$ also if $\nu = 2$. Hence in this case the required lift exists and it is unique up to an element of $H^1_{\mathrm{st},P_\xi}(F, D^0(1)) \cong F \otimes_{\mathbb{Q}_p} L$. □

DEFINITION 9.5: For $\xi \in \{g, h'\}$ and with the notation of remark 8.3, we will represent the class $\tilde{\omega}_{\xi,\nu-1}$ afforded by proposition 9.4 as the sextuple $(\omega_\xi \otimes t_{\nu-1}, \omega_\xi \otimes t_{\nu-1}; F_\xi, x_\xi, y_\xi; 0)$, where, by abuse of notation, we let ω_ξ denote the differential form or the section in Hyodo-Kato cohomology affording the corresponding cohomology class ω_ξ.

REMARK 9.6: Note that the natural map
$$H^1_{\mathrm{syn},P_\xi,c}(\mathcal{X}_t, \mathscr{H}_{\nu-2}, n)_L \to H^0_{\mathrm{st},P_\xi}\left(F, \mathbb{D}_{\mathrm{pst}}\left(H^1_{\mathrm{\acute{e}t},c}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\nu-2}(n))_L\right)\right)$$
arising from the spectral sequence (9.3) clearly sends a class $\eta = [(u, v; w, x, y; 0)]$ to the class $[(\bar{u}, \bar{v})]$ (where $\bar{\tau}$ denotes the cohomology class of the section τ).

Moreover, since by construction the differential forms $\omega_\xi \otimes_{\nu-1}$ for $\xi \in \{g, h'\}$ lie already in $\mathrm{Fil}^{\nu-1} R\Gamma^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{\nu-2})_L$, it follows from the discussion in remark 8.3 that the section y_ξ in definition 9.5 is a constant (and actually $y_\xi = 0$ if $\nu > 2$). ▲

REMARK 9.7: Similarly, one lift $\eta_{f'}^{\varphi=a_p} \in V^*_{\mathrm{dR},Mp^t}(f') \subseteq H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2})_L$ to a syntomic class. One has that

$$\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r} \in \mathrm{Fil}^{k-2-r} H^1_{\mathrm{dR},c}(Y_{t,F}, \mathscr{H}_{k-2})_L \cap H^1_{\mathrm{HK},c}(\mathcal{X}_{t,0}, \mathscr{H}_{\nu-2}, k-2-r)_L^{P_f(\varphi)=0},$$

where $P_f(T) = 1 - a_p(f)^{-1} \cdot p^{k-2-r} \cdot T \in 1 + T \cdot L[T]$. One can proceed as above to get a lift

$$\tilde{\eta}^{\varphi=a_p}_{f',k-2-r} \in H^1_{\mathrm{syn},P_f,c}(\mathcal{X}_t, \mathscr{H}_{k-2}, k-2-r) \tag{9.4}$$

of $\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r}$. Such lift is unique for $k > 2$. One can actually show that if $k = 2$ and f' is p-primitive, then one still gets a unique lift, since $P_f(p^r) \neq 0 \neq P_f(p^{r+1})$ (one has to use that the complex absolute value of $a_p(f)$ is $p^{1/2}$ for every embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ in this case). ▲

PROPOSITION 9.8: *Assume that (f,g,h) is $(F, 1-T)$-convenient. Then the p-adic period introduced in definition 9.1 satisfies (with the notation introduced above):*

$$\mathscr{I}_p(f,g,h) = \mathrm{Tr}_{\mathcal{X}_t,\mathrm{syn},P_{fgh}}\left(\tilde{\eta}^{\varphi=a_p}_{f',k-2-r} \bigcup \Upsilon(\mathtt{Det}^{\mathrm{syn}}_{\mathbf{r}} \cup \tilde{\omega}_{g,l-1} \cup \tilde{\omega}_{h',m-1})\right) \tag{9.5}$$

The notation is as follows.

(i) $\mathtt{Det}^{\mathrm{syn}}_{\mathbf{r}} \in H^0_{\mathrm{syn}}(\mathcal{X}_t, \mathscr{H}_{\mathbf{r}}, r)$ is the syntomic incarnation of $\mathtt{Det}^{\text{ét}}_{\mathbf{r}}$ (via the isomorphism discussed in remark 8.4 (i)).

(ii) The cup products are taken in syntomic cohomology and the big cup product $\bigcup$ arises from the pairing

$$\mathscr{H}_{k-2}(k-2-r) \otimes_{\mathcal{O}_{X_{t,F}}} \mathscr{H}_{k-2}(r+2) \to \mathcal{O}_{X_{t,F}}(2),$$

where recall that for all $i \geq 0$ one has pairings $\mathscr{H}_i \otimes_{\mathcal{O}_{X_{t,F}}} \mathscr{H}_i \to \mathcal{O}_{X_{t,F}}(-i)$ (cf. equation 6.8), which we also use to produce the map

$$\Upsilon : \mathscr{H}_{\mathbf{r}}(r) \otimes_{\mathcal{O}_{X_{t,F}}} \mathscr{H}_{l-2}(l-1) \otimes_{\mathcal{O}_{X_{t,F}}} \mathscr{H}_{m-2}(m-1) \to \mathscr{H}_{k-2}(r+2).$$

(iii) The syntomic trace map

$$\mathrm{Tr}_{\mathcal{X}_t,\mathrm{syn},P_{fgh}} : H^3_{\mathrm{syn},P_{fgh},c}(\mathcal{X}_t, 2)_L \twoheadrightarrow F \otimes_{\mathbb{Q}_p} L \xrightarrow{[F:\mathbb{Q}_p]^{-1}\mathrm{Tr}_{F/\mathbb{Q}_p}\otimes 1} L$$

is defined since $P_{fgh} := P_f * P_g * P_{h'}$ satisfies the conditions $P_{fgh}(1) \neq 0 \neq P_{fgh}(p^{-1})$ (by the assumption that (f,g,h) is $(F,1-T)$-convenient).

PROOF. Since we are assuming that $\mathbb{D}_{\mathrm{pst}}(V(f,g,h))$ is an $(F,1-T)$-convenient quotient of $\mathbb{D}_{\mathrm{pst}}(H^3_{\text{ét}}(Y^3_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{[\mathbf{r}]}(r+2))_L)$, the compatibility between cup products in the spectral sequences of type 8.2 implies that

$$\mathscr{I}_p(f,g,h) = \mathrm{Tr}_{\mathcal{W}_t,\mathrm{syn},P_{fgh}}\left(\delta_{t,*}(\mathtt{Det}^{\mathrm{syn}}_{\mathbf{r}}) \cup_{[\mathbf{r}]} \tilde{\omega}_{fgh}\right), \tag{9.6}$$

The notation is as follows.

(i) The trace $\mathrm{Tr}_{\mathcal{W}_t,\mathrm{syn},P_{fgh}} : H^7_{\mathrm{syn},P_{fgh},c}(\mathcal{W}_t, 4)_L \twoheadrightarrow L$ is defined in the same way as $\mathrm{Tr}_{\mathcal{X}_t,\mathrm{syn},P_{fgh}}$.

(ii) The class $\tilde{\omega}_{fgh}$ is any lift of the element

$$(\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r}) \cup (\omega_g \otimes t_{l-1}) \cup (\omega_{h'} \otimes t_{m-1}) \in H^0_{\mathrm{st},P_{fgh}}\left(F, \mathbb{D}_{\mathrm{pst}}(H^3_{\text{ét},c}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\mathbf{r}}(r+2))_L)\right)$$

under the surjective map

$$H^3_{\mathrm{syn},P_{fgh}}(\mathcal{W}_t, \mathscr{H}_{[\mathbf{r}]}, r+2)_L \twoheadrightarrow H^0_{\mathrm{st},P_{fgh}}\left(F, \mathbb{D}_{\mathrm{pst}}(H^3_{\text{ét},c}(Y_{t,\bar{\mathbb{Q}}_p}, \mathscr{H}_{\mathbf{r}}(r+2))_L)\right)$$

(note that we have implicitly used Künneth decomposition).

(iii) The cup product $\cup_{[\mathbf{r}]}$ is induced by the pairing

$$\mathscr{H}_{[\mathbf{r}]}(r+2) \otimes_{\mathcal{O}_{X^3_{t,F}}} \mathscr{H}_{[\mathbf{r}]}(r+2) \to \mathcal{O}_{X^3_{t,F}}(4)\,.$$

Applying the projection formula (8.7), it follows that

$$\mathscr{I}_p(f,g,h) = \mathrm{Tr}_{\mathcal{X}_t,\mathrm{syn},P_{fgh}}\left(\mathtt{Det}^{\mathrm{syn}}_{\mathbf{r}} \cup_{\mathbf{r}} (\tilde{\eta}^{\varphi=a_p}_{f',k-2-r} \cup \tilde{\omega}_{g,l-1} \cup \tilde{\omega}_{h',m-1})\right) \tag{9.7}$$

where the cup product $\cup_{\mathbf{r}}$ arises from the pairing $\mathscr{H}_{\mathbf{r}}(r) \otimes_{\mathcal{O}_{X_{t,F}}} \mathscr{H}_{\mathbf{r}}(r+2) \to \mathcal{O}_{X_{t,F}}(2)$, obtained combining the pairings $\mathscr{H}_i \otimes_{\mathcal{O}_{X_{t,F}}} \mathscr{H}_i \to \mathcal{O}_{X_{t,F}}(-i)$ for $i \in \{r_1, r_2, r_3\}$.

Formula (9.5) in the statement of the proposition is essentially (9.7) after suitably rearranging the pairings. □

Once obtained the formula in proposition 9.8, we move the computation to the rigid analytic setting, where the objects involved can be made more explicit.

ASSUMPTION 9.9: From now on, we assume that the triple (f,g,h) satisfying assumptions 6.11 and 7.25 also satisfies $k > 2$ (in particular it is always $(F, 1-T)$-convenient).

DEFINITION 9.10: We define

$$\xi_{gh'} := [(w_{gh'}, x_{gh'}, y_{gh'})] \in H^1_{\mathrm{st},P_{gh'}}\left(F, \mathbb{D}_{\mathrm{pst}}(H^1_{\mathrm{\acute{e}t}}(Y_{\bar{\mathbb{Q}}_p}, \mathscr{H}_{k-2}(r+2))_L)\right)$$

as the class corresponding to

$$\Upsilon(\mathtt{Det}^{\mathrm{syn}}_{\mathbf{r}} \cup \tilde{\omega}_{g,l-1} \cup \tilde{\omega}_{h',m-1}) \in H^2_{\mathrm{syn},P_{gh'}}(\mathcal{X}_t, \mathscr{H}_{k-2}, r+2)_L$$

under the isomorphism of lemma 9.3.

Fix polynomials $a(X,Y), b(X,Y) \in L[X,Y]$ such that

$$P_{gh'}(XY) = a(X,Y)P_g(X) + b(X,Y)P_{h'}(Y)\,.$$

For instance we can (and will) choose $a(X,Y) := \mu_{h'}p^{m-1} \cdot Y$ and $b(X,Y) := 1$.

LEMMA 9.11: *With the notation of definition 9.5 and definition 9.10, the class* $[y_{gh'}] \in H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2}[r+2])_L$ *can be represented by the differential form* $y_{gh'} = \Upsilon(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes y')$, *where*

$$y' = y_g \cup (\omega_{h'} \otimes t_{m-1}) - (\omega_g \otimes t_{l-1}) \cup y_{h'}\,.$$

In particular $y_{gh'}$ *is trivial in* $H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2}[r+2])_L$ *if* $l > 2$ *and* $m > 2$.

The class $[w_{gh'}] \in H^1_{\mathrm{HK}}(\mathcal{X}_{t,0}, \mathscr{H}_{k-2}, r+2)_L$ *can be represented by the differential form* $w_{gh'} = \Upsilon(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes w')$, *where*

$$w' = a(\varphi,\varphi)(F_g \otimes (\omega_{h'} \otimes t_{m-1})) - b(\varphi,\varphi)((\omega_g \otimes t_{l-1}) \otimes F_{h'})\,.$$

The notation $\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}}$ *stands for the de Rham incarnation of* $\mathtt{Det}^{\mathrm{\acute{e}t}}_{\mathbf{r}}$ *via the natural identifications of remark 8.4 (i).*

PROOF. The proposition follows from the explicit description of the cup product in syntomic cohomology given in [BLZ16, proposition 2.4.1]. A class in $H^2_{\mathrm{syn},P_{gh'}}(\mathcal{X}_t, \mathscr{H}_{k-2}, r+2)_L$ is represented by a sextuple $(u, v; w, x, y; z)$ as in remark 8.3. Analogously, the syntomic classes $\tilde{\omega}_{g,l-1}$ and $\tilde{\omega}_{h',m-1}$ are represented by sextuples described in definition 9.5.

It follows that the cup product $\tilde{\omega}_{g,l-1} \cup \tilde{\omega}_{h',m-1}$ can be represented by the sextuple $(u', v'; w', x', y'; z')$, where in particular

(i) $y' = y_g \cup (\omega_{h'} \otimes t_{m-1}) - (\omega_g \otimes t_{l-1}) \cup y_{h'}$;
(ii) $w' = a(\varphi,\varphi)(F_g \otimes (\omega_{h'} \otimes t_{m-1})) - b(\varphi,\varphi)((\omega_g \otimes t_{l-1}) \otimes F_{h'})$.

The isomorphism of lemma 9.3 used to define $\xi_{gh'}$ sends the class $[(u',v';w',x',y';z')]$ to $[(\bar{w'},\bar{x'},\bar{y'})]$, where $\bar{\tau}$ denotes the cohomology class corresponding to τ. The statement concerning the vanishing of $y_{gh'}$ follows from remark 9.6. □

LEMMA 9.12: *Under assumption 9.9 and with the notation as in* (6.9), *we have*

$$\begin{aligned}\mathscr{I}_p(f,g,h) &= \tfrac{\mathrm{Tr}_{F/\mathbb{Q}_p}}{[F:\mathbb{Q}_p]} \otimes 1 \left(\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r}, y_{gh'} - \tfrac{1}{P_{fgh}(p^{-1})} w_{gh'}\right)_{\mathrm{dR},X_{t,F},k-2} \\ &= \tfrac{\mathrm{Tr}_{F/\mathbb{Q}_p}}{[F:\mathbb{Q}_p]} \otimes 1 \left(\eta_{f',s}^{\varphi=a_p} \otimes t_{k-2-r}, (\mathtt{pr}_2^{t,s})_* \left(y_{gh'} - \tfrac{1}{P_{fgh}(p^{-1})} w_{gh'}\right)\right)_{\mathrm{dR},X_{s,F},k-2},\end{aligned}$$

where we view $w_{gh'}$ as a class in $H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2}[r+2])_L$.

PROOF. The fist equality follows from the compatibility between cup product in syntomic cohomology and the de Rham duality (6.9) (cf. [BLZ16, proposition 1.4.3 and proposition 3.4.1]) and the fact that, since $\varphi(\eta_{f'}^{\varphi=a_p}) = a_p(f) \cdot \eta_{f'}^{\varphi=a_p}$ by design, one has

$$P_{gh'}(p^{-1} \cdot a_p(f)^{-1} p^{k-2-r}) = P_{fgh}(p^{-1}).$$

The second equality follows from the projection formula in de Rham cohomology. □

9.3. End of the proof

This section concludes the proof of theorem 9.2. We split the discussion into two cases:

(♡) f is new of level $M_1 p^s$ for $s \geq 1$, p-ordinary, with $s < t/2$ as prescribed by assumption 7.25;
(◇) f is the ordinary p-stabilization of a newform f° of level M_1 ($p \nmid M_1$).

In case (♡), we know that f' is essentially $w_{p^s}(f)$. Since f is p-ordinary, $w_{p^s} f$ is *anti-ordinary*. In particular the class $\eta_{f',s}^{\varphi=a_p}$ is fixed by the anti-ordinary projector e'_{ord}.

In case (◇), one obtains the same conclusion due to the presence of the idempotent $e_{\breve{f}}$ in the definition of $\eta_{f',1}^{\varphi=a_p}$ as linear functional and the fact that $\breve{f}$ is the ordinary p-stabilization of $\breve{f}^\circ$.

REMARK 9.13: The fact that we manage to reduce our computation to a pairing in de Rham cohomology theoretically allows us to perform the computation over $\mathbb{Q}_p$, since the cohomology classes appearing in the pairing are already defined over $X_{t,\mathbb{Q}_p}$. Hence the trace $\mathrm{Tr}_{F/\mathbb{Q}_p}$ appearing in the formulas of lemma 9.12 does not affect the computation and can be safely removed. ▲

LEMMA 9.14: *We can perform the computation of $\mathscr{I}_p(f,g,h)$ after restricting to $\mathcal{W}_\infty(p^t)$ (as a rigid analytic space over $K_t := \mathbb{Q}_p(\zeta_{p^t})$) and the following formula holds:*

$$\mathscr{I}_p(f,g,h) = \left(\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r}\ ,\ e_{\mathrm{ord}}\left(y_{gh'} - \tfrac{1}{P_{fgh}(p^{-1})} w_{gh'}\right)\right)_{\mathrm{dR},\mathcal{W}_\infty(p^t),k-2}. \tag{9.8}$$

Proof. Since the anti-ordinary projector e'_{ord} is adjoint under the given pairing to the ordinary projector e_{ord}, we can compute (for every $\omega \in H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2}[r+2])_L)$ that

$$\begin{aligned}
&\Big(\eta^{\varphi=a_p}_{f',s} \otimes t_{k-2-r}, (\mathtt{pr}_2^{t,s})_*(\omega)\Big)_{\mathrm{dR},X_{s,F},k-2}\\
&=\Big(e'_{\mathrm{ord}}(\eta^{\varphi=a_p}_{f',s} \otimes t_{k-2-r}), (\mathtt{pr}_2^{t,s})_*(\omega)\Big)_{\mathrm{dR},X_{s,F},k-2}\\
&=\Big(\eta^{\varphi=a_p}_{f'} \otimes t_{k-2-r}, e_{\mathrm{ord}} \circ (\mathtt{pr}_2^{t,s})_*(\omega)\Big)_{\mathrm{dR},X_{s,F},k-2}\\
&=\Big(\eta^{\varphi=a_p}_{f',s} \otimes t_{k-2-r}, (\mathtt{pr}_2^{t,s})_* \circ e_{\mathrm{ord}}(\omega)\Big)_{\mathrm{dR},X_{s,F},k-2}\\
&=\Big(\eta^{\varphi=a_p}_{f'} \otimes t_{k-2-r}, e_{\mathrm{ord}}(\omega)\Big)_{\mathrm{dR},X_{t,F},k-2}\,.
\end{aligned}$$

The chain of equalities follows observing that $e_{\mathrm{ord}} \circ (\mathtt{pr}_2^{t,s})_* = (\mathtt{pr}_2^{t,s})_* \circ\ e_{\mathrm{ord}}$. The latter is a consequence of the fact that $(p \cdot U_p)^{t-s} = (\mathtt{pr}_2^{t,s})_* \circ (\mathtt{pr}_1^{t,s})^*$ on $H^1_{\mathrm{dR}}(X_{s,F}, \mathscr{H}_{k-2})$ and that $(p \cdot U_p)^{t-s} = (\mathtt{pr}_1^{t,s})^* \circ (\mathtt{pr}_2^{t,s})_*$ on $H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2})$ (cf. remark 6.6).

The isomorphism (8.8) induces an isomorphism

$$H^1_{\mathrm{dR}}(X_{t,F}, \mathscr{H}_{k-2})_L[f'] \cong H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t)_F, \mathscr{H}_{k-2})^*_L[f'] \oplus H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{k-2})^*_L[f']\,.$$

In case ($\heartsuit$), we observe that $U_p(w_{p^s}(f)) = \chi_f(p)p^{k-1}a_p(f)^{-1} \cdot w_{p^s}(f)$, so that from the explicit description of the action of Frobenius Φ_∞ on $\mathcal{W}_\infty(p^t)$ it follows that Φ acts as multiplication by $a_p(f)^d$ on $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t)_F, \mathscr{H}_{k-2})^*_L[f']$ (recall $d = [F_0 : \mathbb{Q}_p]$).

The Frobenius action on $H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{k-2})^*$ is given by $\Phi_0 = (w_{p^t}^{-1})^* \circ \Phi_\infty \circ w^*_{p^t}$. One can check that

$$w^*_{p^t} \circ (\mathtt{pr}_2^{t,s})^* = (\mathtt{pr}_1^{t,s})^* \circ \langle p^{s-t};\ 1\rangle \circ w^*_{p^s} \tag{9.9}$$

to deduce that actually

$$(\mathtt{pr}_2^{t,s})^* \circ \Phi_0 = \Phi_0 \circ (\mathtt{pr}_2^{t,s})^* \tag{9.10}$$

as maps

$$H^1_{\mathrm{dR}}(\mathcal{W}_0(p^s)_F, \mathscr{H}_{k-2})^*_L \to H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{k-2})^*_L\,.$$

One checks easily that Φ_0 acts via multiplication by $\beta_p(f)^d$ on $H^1_{\mathrm{dR}}(\mathcal{W}_0(p^s)_F, \mathscr{H}_{k-2})^*[f']$, where $\beta_p(f) \coloneqq \chi_f(p)p^{k-1}a_p(f)^{-1}$, whence Φ_0 will act by multiplication by $\beta_p(f)^d$ also on $H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{k-2})^*_L[f']$. Since $\beta_p(f)$ is not a p-adic unit (as $k \neq 1$), while $a_p(f) \in \mathcal{O}_L^\times$, it follows that the class $\eta^{\varphi=a_p}_{f'}$ restricts trivially to $\mathcal{W}_0(p^t)_F$.

In case ($\diamond$) one has to be more careful, since the Φ-eigenspaces for the eigenvalues $a_p(f)^d$ and $\beta_p(f)^d$ intersect non-trivially with both the subspaces $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t)_F, \mathscr{H}_{k-2})^*_L[f']$ and $H^1_{\mathrm{dR}}(\mathcal{W}_0(p^t)_F, \mathscr{H}_{k-2})^*_L[f']$. Again, looking at the definition of $\eta^{\varphi=a_p}_{f',1}$ in this case, one sees that, since for the dual operator U'_p (which is adjoint to U_p under the pairing $(-,-)_{\mathrm{dR},X_{1,F},k-2}$) we have $\langle p;1\rangle \circ U'_p = w_{p^t} \circ U_p \circ w_{p^t}^{-1}$ (and $w^2_{p^t} = \langle p^t;-1\rangle$), it follows that U'_p acts on such class as multiplication by $a_p(f)\chi_f(p)^{-1}$, so that

$$U_p \circ w_p(\eta^{\varphi=a_p}_{f',1}) = a_p(f) w_p(\eta^{\varphi=a_p}_{f',1})\,.$$

It follows that, restricting our attention to $\mathcal{W}_0(p)$ and keeping track of the various definitions, one obtains:

$$\Phi_0(\eta^{\varphi=a_p}_{f',1}|_{\mathcal{W}_0(p)_F}) = (w_p^{-1})^* \circ \Phi_\infty \circ w_p(\eta^{\varphi=a_p}_{f',1}|_{\mathcal{W}_0(p)_F}) = \beta_p(f)^d \cdot \eta^{\varphi=a_p}_{f',1}|_{\mathcal{W}_0(p)_F}$$

Since by construction $\Phi(\eta_{f',1}^{\varphi=a_p}) = (a_p(f))^d \cdot \eta_{f',1}^{\varphi=a_p}$, we deduce that $\eta_{f',1}^{\varphi=a_p}|_{\mathcal{W}_0(p)_F} = 0$ also in this case and, using again the equality (9.10), we conclude that $\eta_{f'}^{\varphi=a_p}|_{\mathcal{W}_0(p^t)_F} = 0$ as well.

Finally, the fact that we can forget the trace $\mathrm{Tr}_{F/\mathbb{Q}_p}$ and that we can work on $\mathcal{W}_\infty(p^t)$ as a rigid space over K_t follows from the discussion of remark 9.13.

□

Now we turn our attention to the term appearing in the RHS of the pairing in the formula (9.8) above and we try to make it more computable.

LEMMA 9.15: *Fix $\xi \in \{g, h'\}$. Then the restriction $F_\xi|_{\mathcal{W}_\infty(p^t)}$ of the section F_ξ (cf. definition 9.5) can be described around the cusp ∞ as*

$$F_{\xi,\infty} := \sum_{j=0}^{\nu-2} (-1)^j j! \binom{\nu-2}{j} \cdot d^{\nu-2-j}(\Xi) \cdot \omega_{\mathrm{can}}^{\nu-2-j} \otimes \eta_{\mathrm{can}}^j \otimes t_{\nu-1}$$

where $\Xi \in \mathbb{Z}((q)) \otimes L$ satisfies $\xi - \mu_\xi p^{\nu-1}\chi_\xi(p)V\xi = d^{\nu-1}\Xi$ as q-expansions and $d = q\frac{d}{dq}$ is Serre's derivative operator.

PROOF. Let ξ denote either g or h' and write ν for the corresponding weight and write $F_{\xi,\infty}$ for the restriction of F_ξ to a formal neighbourhood of the cusp ∞, where by construction (cf. remark 8.3 and definition 9.5) it satisfies

$$\nabla F_{\xi,\infty} = (\xi(q) - \mu_\xi p^{\nu-1}\chi_\xi(p)\xi(q^p)) \cdot \omega_{\mathrm{can}}^{\nu-2} \otimes \frac{dq}{q} \otimes t_{\nu-1}\,.$$

Then the result follows formally as in [Col94, section 9]. □

PROOF OF THEOREM 9.2. Recall that $U \circ V = V \circ U = 1$ on $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t), \mathscr{H}_r)_L$ for every $r \in \mathbb{Z}_{\geq 0}$. The operator $U = U_p$ kills sections of the form $\xi(q) \cdot \omega_{\mathrm{can}}^{r-j} \otimes \eta^j \otimes \frac{dq}{q}$, where $\xi(q)$ is a p-depleted q-expansion (i.e. a q-expansion where the terms corresponding to integers divible by p vanish). It follows that, when restricting to $\mathcal{W}_\infty(p^t)$, such sections are exact (hence trivial in $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t), \mathscr{H}_r)_L$).

Assumption 7.25 implies that the q-expansions of g and h' are p-depleted, so that the differential forms ω_g and $\omega_{h'}$ are exact when restricted to $\mathcal{W}_\infty(p^t)$. Since (cf. remark 9.6) the sections y_g and $y_{h'}$ are constants, we immediately deduce that (also if $l = 2$ or $m = 2$) $y_{gh'}|_{\mathcal{W}_\infty(p^t)} = 0$ in $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t), \mathscr{H}_{k-2}[r+2])$.

It is easy to check that the product of a p-depleted q-expansion and of a q-expansion of the form $\sum_{n=1}^{+\infty} b_n q^{np}$ is still p-depleted.

Note that the q-expansion G found in lemma 9.15 above can be described explicitly as

$$G = \sum_{n=1}^{+\infty} \frac{a_n(g)}{n^{l-1}} q^n - \sum_{n=1}^{+\infty} \frac{\mu_g \chi_g(p) a_n(g)}{n^{l-1}} q^{np} = d^{1-l}g - \mu_g\chi_g(p)V(d^{1-l}g)\,.$$

Given $t \in \mathbb{Z}_{\geq 1}$, recall that the operator d^{-t} is the p-adic limit of the operators $d^{-t+(p-1)p^k}$ for $k \to +\infty$. Hence the second equality above makes sense because the q-expansion of g is already p-depleted. Similarly $H' = d^{1-m}h' - \mu_h\chi_h(p)V(d^{1-m}h')$.

Define, for $\xi \in \{g, h'\}$, the sections F'_ξ and F''_ξ of $\mathscr{H}_{\nu-2}$ over $\mathcal{W}_\infty(p^t)$ which in a neighbourhood of the cusp ∞ are described by:

$$F'_{\xi,\infty} = \sum_{j=0}^{\nu-2}(-1)^j j! \binom{\nu-2}{j} \cdot d^{-1-j}\xi \cdot \omega_{\mathrm{can}}^{\nu-2-j} \otimes \eta_{\mathrm{can}}^j \otimes t_{\nu-1}$$

and

$$F''_{\xi,\infty} := \sum_{j=0}^{\nu-2}(-1)^j j! \binom{\nu-2}{j} \cdot p^{-1-j}\langle p;1\rangle V(d^{-1-j}\xi) \cdot \omega_{\mathrm{can}}^{\nu-2-j} \otimes \eta_{\mathrm{can}}^j \otimes t_{\nu-1}\,,$$

so that $F_\xi = F'_\xi - \mu_\xi p^{\nu-1} \cdot F''_\xi$, $\nabla F'_\xi = \omega_\xi \otimes t_{\nu-1}$ and $\varphi_\infty(F'_\xi) = F''_\xi$.

The above discussion shows that in the cohomology group $H^1_{\mathrm{dR}}(\mathcal{W}_\infty(p^t), \mathscr{H}_{k-2}[r+2])_L$ we have:

$$\begin{aligned}
&\Upsilon(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes w')|_{W_\infty(p^t)} = \\
&= \Upsilon\left(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes a(\varphi_\infty, \varphi_\infty)(F_g \otimes (\omega_{h'} \otimes t_{m-1})) - b(\varphi_\infty, \varphi_\infty)((\omega_g \otimes t_{l-1}) \otimes F_{h'})\right) \\
&= \Upsilon\left(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes (F_g \otimes \mu_h p^{m-1}\varphi_\infty(\omega_{h'} \otimes t_{m-1}) - (\omega_g \otimes t_{l-1}) \otimes F_{h'})\right) \\
&= \Upsilon\left(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes (\mu_g\mu_h p^{l+m-2}(F''_g \otimes \varphi_\infty(\omega_{h'} \otimes t_{m-1})) + (F'_g \otimes (\omega_{h'} \otimes t_{m-1}))))\right) \\
&= \Upsilon\left(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes P_{gh'}(\varphi_\infty \otimes \varphi_\infty)(F'_g \otimes (\omega_{h'} \otimes t_{m-1})))\right) \\
&= -P_{gh'}(\varphi_\infty)(\Upsilon(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes (\omega_g \otimes t_{l-1}) \otimes F'_{h'}))\,.
\end{aligned}$$

In the same way as in [BSV20, pagg. 1023-1024], over $\mathcal{W}_\infty(p^t)$ one can compute that

$$e_{\mathrm{ord}}(\Upsilon(\mathtt{Det}^{\mathrm{dR}}_{\mathbf{r}} \otimes (\omega_g \otimes t_{l-1}) \otimes F'_{h'})) = (-1)^{k-2}(r-k+2)! \cdot \omega_{\Xi^{\mathrm{ord}}(g,h')} \otimes t_{r+2},$$

where $\Xi^{\mathrm{ord}}(g,h') := e_{\mathrm{ord}}(g \times d^{(k-l-m)/2}h')$. Combining everything we can conclude that

$$\begin{aligned}
&\mathscr{I}_p(f,g,h) = \left(\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r}\ ,\ e_{\mathrm{ord}}\left(-\tfrac{1}{P_{fgh}(p^{-1})} w_{gh'}\right)\right)_{\mathrm{dR},\mathcal{W}_\infty(p^t),k-2} \\
&= (-1)^{k-2}(r-k+2)! \cdot \left(\eta_{f'}^{\varphi=a_p} \otimes t_{k-2-r}\ ,\ \frac{P_{gh'}(p^{-2-r}\varphi_\infty)(\omega_{\Xi^{\mathrm{ord}}(g,h')} \otimes t_{r+2})}{P_{fgh}(p^{-1})}\right)_{\mathrm{dR},\mathcal{W}_\infty(p^t),k-2} \\
&= (-1)^{k-2}(r-k+2)! \cdot \eta_{f'}^{\varphi=a_p}(\Xi^{\mathrm{ord}}(g,h'))\,.
\end{aligned}$$

The first equality follows directly from the previous computations, while the second follows from the fact that the functional $\eta_{f'}^{\varphi=a_p}$ includes a projection to the $\breve{f}$-isotypical component. Indeed, having control on the action of U and $\langle p;1\rangle$ on $\breve{f}$, we can check that on $\mathcal{W}_\infty(p^t)$ it holds

$$P_{gh'}(\varphi_\infty)(\omega_{\breve{f}} \otimes t_{r+2}) = (1 - \mu_g\mu_h p^{l+m-2}\varphi_\infty)(\omega_{\breve{f}} \otimes t_{r+2}) = P_{fgh}(p^{-1}) \cdot (\omega_{\breve{f}} \otimes t_{r+2})\,.$$

This concludes the proof of the theorem. □

APPENDIX A

Construction of the invariant $\mathtt{Det}_{\mathbf{r}}$

In this section we let R denote either a local ring or a principal ideal domain and we let M denote a free R-module of rank 2, with perfect alternating R-bilinear form

$$\langle\ ,\ \rangle_M : M \times M \to R\,.$$

Set $G := \mathrm{Aut}_R(M)$ and let $S := \mathrm{Hom}_R(M,R)$ denote the R-dual of M, which we consider as a left-$R[G]$ module in the usual way, i.e.

$$g * \lambda(v) := \lambda(g^{-1}(v)) \qquad g \in G, \lambda \in S, v \in M\,.$$

For $n \geq 0$ we let $S_n := \mathrm{Sym}^n(S)$ denote the n-th symmetric power of S, defined as the maximal symmetric quotient of $S^{\otimes n}$. We view S_n as a $R[G]$ module with the induced action.

For $m \in \mathbb{Z}$, we let $R[m]$ denote the ring R, with G-action via the m-th power of the determinant. We think of $R[m]$ as the free R-module of rank one, with fixed generator $a_m = 1 \in R$ such that $g * a_m = \det(g)^m \cdot a_m$ for all $g \in G$.

In particular, the pairing $\langle\ ,\ \rangle_M$ can be viewed as a perfect alternating R-bilinear form $\langle\ ,\ \rangle_M : M \times M \to R[1]$, which is also G-equivariant, i.e. $\langle g * v_1, g * v_2\rangle_M = g * \langle v_1, v_2\rangle$.

We fix a triple of non-negative integers $\mathbf{r} := (r_1, r_2, r_3)$ such that:

(i) $r := (r_1 + r_2 + r_3)/2 \in \mathbb{Z}$;
(ii) for every permutation of $\{i,j,k\}$ of $\{1,2,3\}$, it holds $r_i + r_j > r_k$.

We finally set $S_{\mathbf{r}} := S_{r_1} \otimes_R S_{r_2} \otimes_R S_{r_3}$, which we endow with the structure of $R[G]$-module via the diagonal action on the three factors.

The aim of this section is to produce a canonical invariant element

$$\mathtt{Det}_{\mathbf{r}} \in H^0(G, S_{\mathbf{r}} \otimes R[r])\,.$$

In order to proceed more explicitly, we fix a symplectic R-basis $\{e_1, e_2\}$ of M (i.e. $\langle e_1, e_2\rangle_M = 1$, $\langle e_2, e_1\rangle_M = -1$). We let $\{\varepsilon_1, \varepsilon_2\}$ be the R-basis of S which is dual to $\{e_1, e_2\}$.

In this way we identify $M = R \oplus R$ (column vectors), $S = R \oplus R$ (row vectors) and $G = \mathrm{GL}_2(R)$, with the natural action of $\mathrm{GL}_2(R)$ by matrix multiplication on column vectors on M and the action of $\mathrm{GL}_2(R)$ on S given by

$$g * \lambda = \lambda \cdot g^{-1} \qquad g \in \mathrm{GL}_2(R), \lambda \in S\,.$$

For every $n \geq 0$ we can then identify S_n with the R-module of two-variable homogeneous polynomials of degree n with R-coefficients, with $\mathrm{GL}_2(R)$-action given by

$$g * P(x_1, x_2)) = P\left(\left(g^{-1} \cdot \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right)^t\right)\,,$$

where $(\cdot)^t$ denotes transposition.

We view $S_{\mathbf{r}}$ as the R-module of polynomials in six variables $(x_1, x_2, y_1, y_2, z_1, z_2)$ with R-coefficients which are homogeneous of degree r_1 with respect to (x_1, x_2), homogeneous of degree r_2 with respect to (y_1, y_2) and homogeneous of degree r_3 with respect to (z_1, z_2). The action of $\mathrm{GL}_2(R)$ on $S_{\mathbf{r}}$ can then be explicitly described in terms of the variables.

We finally define an element $P_{\mathbf{r}} \in S_{\mathbf{r}}$ as follows

$$P_{\mathbf{r}}(x_1, x_2, y_1, y_2, z_1, z_2) := \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \cdot \det\begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \cdot \det\begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1}$$

and one can easily check that for every $g \in \mathrm{GL}_2(R)$ it holds $g * P_{\mathbf{r}} = \det(g)^{-r} \cdot P_{\mathbf{r}}$, so that clearly $\mathtt{Det}_{\mathbf{r}} := P_{\mathbf{r}} \otimes a_r \in S_{\mathbf{r}} \otimes R[r]$ is an invariant for the $G = \mathrm{GL}_2(R)$-action.

It is also clear that the element $\mathtt{Det}_{\mathbf{r}}$ depends only on M and the pairing $\langle\ ,\ \rangle_M$, but not on the choice of a symplectic basis.

Bibliography

[AI21] Fabrizio Andreatta and Adrian Iovita. "Triple product p-adic L-functions associated to finite slope p-adic families of modular forms". In: *Duke Math. J.* 170.9 (2021), pp. 1989–2083. ISSN: 0012-7094. DOI: 10.1215/00127094-2020-0076. URL: https://doi.org/10.1215/00127094-2020-0076.

[AL78] A. O. L. Atkin and Wen Ch'ing Winnie Li. "Twists of newforms and pseudo-eigenvalues of W-operators". In: *Invent. Math.* 48.3 (1978), pp. 221–243. ISSN: 0020-9910. DOI: 10.1007/BF01390245. URL: https://doi.org/10.1007/BF01390245.

[BC] Olivier Brinon and Brian Conrad. *CMI Summer school notes on p-adic Hodge theory.* URL: https://math.stanford.edu/~conrad/papers/notes.pdf.

[BD07] Massimo Bertolini and Henri Darmon. "Hida families and rational points on elliptic curves". In: *Invent. Math.* 168.2 (2007), pp. 371–431. ISSN: 0020-9910. DOI: 10.1007/s00222-007-0035-4. URL: https://doi.org/10.1007/s00222-007-0035-4.

[BD96] M. Bertolini and H. Darmon. "Heegner points on Mumford-Tate curves". In: *Invent. Math.* 126.3 (1996), pp. 413–456. ISSN: 0020-9910. DOI: 10.1007/s002220050105. URL: https://doi.org/10.1007/s002220050105.

[BD98] Massimo Bertolini and Henri Darmon. "Heegner points, p-adic L-functions, and the Cerednik-Drinfeld uniformization". In: *Invent. Math.* 131.3 (1998), pp. 453–491. ISSN: 0020-9910. DOI: 10.1007/s002220050211. URL: https://doi.org/10.1007/s002220050211.

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. "Generalized Heegner cycles and p-adic Rankin L-series". In: *Duke Math. J.* 162.6 (2013). With an appendix by Brian Conrad, pp. 1033–1148. ISSN: 0012-7094. DOI: 10.1215/00127094-2142056. URL: https://doi.org/10.1215/00127094-2142056.

[BDV22] Massimo Bertolini, Henri Darmon, and Rodolfo Venerucci. "Heegner points and Beilinson-Kato elements: a conjecture of Perrin-Riou". In: *Adv. Math.* 398 (2022), Paper No. 108172, 50. ISSN: 0001-8708. DOI: 10.1016/j.aim.2021.108172. URL: https://doi.org/10.1016/j.aim.2021.108172.

[BE10] Christophe Breuil and Matthew Emerton. "Représentations p-adiques ordinaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ et compatibilité local-global". In: *Astérisque* 331 (2010), pp. 255–315. ISSN: 0303-1179.

[Bes00] Amnon Besser. "A generalization of Coleman's p-adic integration theory". In: *Invent. Math.* 142.2 (2000), pp. 397–434. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s002220000093. URL: https://doi.org/10.1007/s002220000093.

[BK90] Spencer Bloch and Kazuya Kato. "L-functions and Tamagawa numbers of motives". In: *The Grothendieck Festschrift, Vol. I.* Vol. 86. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.

[BLZ16] Amnon Besser, David Loeffler, and Sarah Livia Zerbes. "Finite polynomial cohomology for general varieties". In: *Ann. Math. Qué.* 40.1 (2016), pp. 203–220. ISSN: 2195-4755. DOI: 10.1007/s40316-015-0041-7. URL: https://doi.org/10.1007/s40316-015-0041-7.

[BSV20] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. "Diagonal classes and the Bloch-Kato conjecture". In: *Münster J. Math.* 13.2 (2020), pp. 317–352. ISSN: 1867-5778.

[BSV22a] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. "Balanced diagonal classes and rational points on elliptic curves". In: *Astérisque* 434 (2022), pp. 175–201. ISSN: 0303-1179.

[BSV22b] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. "Reciprocity laws for balanced diagonal classes". In: *Astérisque* 434 (2022), pp. 77–174. ISSN: 0303-1179.

[Bum97] Daniel Bump. *Automorphic forms and representations.* Vol. 55. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997, pp. xiv+574. ISBN: 0-521-55098-X. DOI: 10.1017/CBO9780511609572. URL: https://doi.org/10.1017/CBO9780511609572.

[CH18] Masataka Chida and Ming-Lun Hsieh. "Special values of anticyclotomic L-functions for modular forms". In: *J. Reine Angew. Math.* 741 (2018), pp. 87–131. ISSN: 0075-4102. DOI: 10.1515/crelle-2015-0072. URL: https://doi.org/10.1515/crelle-2015-0072.

[CI10] Robert Coleman and Adrian Iovita. "Hidden structures on semistable curves". In: *Astérisque* 331 (2010), pp. 179–254. ISSN: 0303-1179,2492-5926.

[CL16] Francesc Castella and Matteo Longo. "Big Heegner points and special values of L-series". In: *Ann. Math. Qué.* 40.2 (2016), pp. 303–324. ISSN: 2195-4755. DOI: 10.1007/s40316-015-0045-3. URL: https://doi.org/10.1007/s40316-015-0045-3.

[Coa91] John Coates. "Motivic p-adic L-functions". In: *L-functions and arithmetic (Durham, 1989).* Vol. 153. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1991, pp. 141–172. DOI: 10.1017/CBO9780511526053.006. URL: https://doi.org/10.1017/CBO9780511526053.006.

[Col20] Dan J. Collins. "Anticyclotomic p-adic L-functions and Ichino's formula". In: *Ann. Math. Qué.* 44.1 (2020), pp. 27–89. ISSN: 2195-4755. DOI: 10.1007/s40316-019-00118-1. URL: https://doi.org/10.1007/s40316-019-00118-1.

[Col94] Robert F. Coleman. "A p-adic Shimura isomorphism and p-adic periods of modular forms". In: *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991).* Vol. 165. Contemp. Math. Amer. Math. Soc., Providence, RI, 1994, pp. 21–51. ISBN: 0-8218-5180-2. DOI: 10.1090/conm/165/01602. URL: https://doi.org/10.1090/conm/165/01602.

[Col97] Robert F. Coleman. "Classical and overconvergent modular forms of higher level". In: *J. Théor. Nombres Bordeaux* 9.2 (1997), pp. 395–403. ISSN: 1246-7405. URL: http://jtnb.cedram.org/item?id=JTNB_1997__9_2_395_0.

[DF23] Henri Darmon and Michele Fornea. *Mock plectic points.* 2023. arXiv: 2310.16758 [math.NT].

[Dia+23] Hansheng Diao et al. "Logarithmic Riemann-Hilbert correspondences for rigid varieties". In: *J. Amer. Math. Soc.* 36.2 (2023), pp. 483–562.

[DR14] Henri Darmon and Victor Rotger. "Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula". In: *Ann. Sci. Éc. Norm. Supér. (4)* 47.4 (2014), pp. 779–832. ISSN: 0012-9593. DOI: 10.24033/asens.2227. URL: https://doi.org/10.24033/asens.2227.

[DR17] Henri Darmon and Victor Rotger. "Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-functions". In: *J. Amer. Math. Soc.* 30.3 (2017), pp. 601–672. ISSN: 0894-0347,1088-6834. DOI: 10.1090/jams/861. URL: https://doi.org/10.1090/jams/861.

[DR22] Henri Darmon and Victor Rotger. "Stark-Heegner points and diagonal classes". In: 434. Heegner points, Stark-Heegner points, and diagonal classes. 2022, pp. 1–28. ISBN: 978-2-85629-959-3.

[EY21] Veronika Ertl and Kazuki Yamada. "Comparison between rigid and crystalline syntomic cohomology for strictly semistable log schemes with boundary". In: *Rend. Semin. Mat. Univ. Padova* 145 (2021), pp. 213–291. ISSN: 0041-8994,2240-2926. DOI: 10.4171/rsmup/81. URL: https://doi.org/10.4171/rsmup/81.

[EY22] Veronika Ertl and Kazuki Yamada. *Rigid analytic reconstruction of Hyodo–Kato theory.* 2022. arXiv: 1907.10964 [math.NT].

[EY23] Veronika Ertl and Kazuki Yamada. *Poincaré duality for rigid analytic Hyodo–Kato theory.* 2023. arXiv: 2009.09160 [math.AG].

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture.* Vol. 13. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné. Springer-Verlag, Berlin, 1988, pp. xviii+317. ISBN: 3-540-12175-7. DOI: 10.1007/978-3-662-02541-3. URL: https://doi.org/10.1007/978-3-662-02541-3.

[Fuk22] Kengo Fukunaga. "Triple product p-adic L-function attached to p-adic families of modular forms". In: *Tohoku Math. J. (2)* 74.3 (2022), pp. 441–488. ISSN: 0040-8735. DOI: 10.2748/tmj.20210501. URL: https://doi.org/10.2748/tmj.20210501.

[GD71] A. Grothendieck and J. A. Dieudonné. *Éléments de géométrie algébrique. I.* Vol. 166. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1971, pp. ix+466. ISBN: 3-540-05113-9; 0-387-05113-9.

[GJ78] Stephen Gelbart and Hervé Jacquet. "A relation between automorphic representations of GL(2) and GL(3)". In: *Ann. Sci. École Norm. Sup. (4)* 11.4 (1978), pp. 471–542. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1978_4_11_4_471_0.

[GM09] Eknath Ghate and Ariane Mézard. "Filtered modules with coefficients". In: *Trans. Amer. Math. Soc.* 361.5 (2009), pp. 2243–2261. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-08-04829-0. URL: https://doi.org/10.1090/S0002-9947-08-04829-0.

[GPJ23] Andrew Graham, Vincent Pilloni, and Joaquín Rodrigues Jacinto. *p-adic interpolation of Gauss–Manin connections on nearly overconvergent modular forms and p-adic L-functions.* 2023. arXiv: 2311.14438 [math.NT].

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes—with examples and exercises.* Second. Springer Studium Mathematik—Master. Springer

Spektrum, Wiesbaden, 2020, pp. vii+625. ISBN: 978-3-658-30732-5; 978-3-658-30733-2. DOI: 10.1007/978-3-658-30733-2. URL: https://doi.org/10.1007/978-3-658-30733-2.

[Har01] Urs T. Hartl. "Semi-stability and base change". In: *Arch. Math. (Basel)* 77.3 (2001), pp. 215–221. ISSN: 0003-889X,1420-8938. DOI: 10.1007/PL00000484. URL: https://doi.org/10.1007/PL00000484.

[Hid12] Haruzo Hida. *Geometric modular forms and elliptic curves*. Second. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012, pp. xiv+454. ISBN: 978-981-4368-64-3; 981-4368-64-4.

[Hid85] Haruzo Hida. "A p-adic measure attached to the zeta functions associated with two elliptic modular forms. I". In: *Invent. Math.* 79.1 (1985), pp. 159–195. ISSN: 0020-9910. DOI: 10.1007/BF01388661. URL: https://doi.org/10.1007/BF01388661.

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*. Vol. 26. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1993, pp. xii+386. ISBN: 0-521-43411-4; 0-521-43569-2. DOI: 10.1017/CBO9780511623691. URL: https://doi.org/10.1017/CBO9780511623691.

[HK91] Michael Harris and Stephen S. Kudla. "The central critical value of a triple product L-function". In: *Ann. of Math. (2)* 133.3 (1991), pp. 605–672. ISSN: 0003-486X. DOI: 10.2307/2944321. URL: https://doi.org/10.2307/2944321.

[Hsi21] Ming-Lun Hsieh. "Hida families and p-adic triple product L-functions". In: *Amer. J. Math.* 143.2 (2021), pp. 411–532. ISSN: 0002-9327. DOI: 10.1353/ajm.2021.0011. URL: https://doi.org/10.1353/ajm.2021.0011.

[Hun17] Pin-Chi Hung. "On the non-vanishing mod ℓ of central L-values with anticyclotomic twists for Hilbert modular forms". In: *J. Number Theory* 173 (2017), pp. 170–209. ISSN: 0022-314X. DOI: 10.1016/j.jnt.2016.09.013. URL: https://doi.org/10.1016/j.jnt.2016.09.013.

[HW22] Ting-Han Huang and Ju-Feng Wu. *Finite polynomial cohomology with coefficients*. 2022. arXiv: 2209.10289 [math.NT].

[Ich08] Atsushi Ichino. "Trilinear forms and the central values of triple product L-functions". In: *Duke Math. J.* 145.2 (2008), pp. 281–307. ISSN: 0012-7094. DOI: 10.1215/00127094-2008-052. URL: https://doi.org/10.1215/00127094-2008-052.

[Ike92] Tamotsu Ikeda. "On the location of poles of the triple L-functions". In: *Compositio Math.* 83.2 (1992), pp. 187–237. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1992__83_2_187_0.

[Ike98] Tamotsu Ikeda. "On the gamma factor of the triple L-function. II". In: *J. Reine Angew. Math.* 499 (1998), pp. 199–223. ISSN: 0075-4102. DOI: 10.1515/crll.1998.058. URL: https://doi.org/10.1515/crll.1998.058.

[Jac72] Hervé Jacquet. *Automorphic forms on* $\mathrm{GL}(2)$. *Part II.* Lecture Notes in Mathematics, Vol. 278. Springer-Verlag, Berlin-New York, 1972, pp. xiii+142.

[JL70] H. Jacquet and R. P. Langlands. *Automorphic forms on* $\mathrm{GL}(2)$. Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970, pp. vii+548.

[Kat04] Kazuya Kato. "p-adic Hodge theory and values of zeta functions of modular forms". In: 295. Cohomologies p-adiques et applications arithmétiques. III. 2004, pp. ix, 117–290.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*. Vol. 108. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1985, pp. xiv+514. ISBN: 0-691-08349-5; 0-691-08352-5. DOI: 10.1515/9781400881710. URL: https://doi.org/10.1515/9781400881710.

[LLZ23] Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu. "de Rham comparison and Poincaré duality for rigid varieties". In: *Peking Math. J.* 6.1 (2023), pp. 143–216. ISSN: 2096-6075,2524-7182. DOI: 10.1007/s42543-020-00031-5. URL: https://doi.org/10.1007/s42543-020-00031-5.

[LZ17] Ruochuan Liu and Xinwen Zhu. "Rigidity and a Riemann-Hilbert correspondence for p-adic local systems". In: *Invent. Math.* 207.1 (2017), pp. 291–343. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-016-0671-7. URL: https://doi.org/10.1007/s00222-016-0671-7.

[Maz94] B. Mazur. "On monodromy invariants occurring in global arithmetic, and Fontaine's theory". In: *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*. Vol. 165. Contemp. Math. Amer. Math. Soc., Providence, RI, 1994, pp. 1–20. ISBN: 0-8218-5180-2. DOI: 10.1090/conm/165/01599. URL: https://doi.org/10.1090/conm/165/01599.

[Miy06] Toshitsune Miyake. *Modular forms*. English. Springer Monographs in Mathematics. Translated from the 1976 Japanese original by Yoshitaka Maeda. Springer-Verlag, Berlin, 2006, pp. x+335. ISBN: 978-3-540-29592-1; 3-540-29592-5.

[MS00] Atsushi Murase and Takashi Sugano. "Local theory of primitive theta functions". In: *Compositio Math.* 123.3 (2000), pp. 273–302. ISSN: 0010-437X. DOI: 10.1023/A:1002051017269. URL: https://doi.org/10.1023/A:1002051017269.

[NN16] Jan Nekovář and Wiesława Nizioł. "Syntomic cohomology and p-adic regulators for varieties over p-adic fields". In: *Algebra Number Theory* 10.8 (2016). With appendices by Laurent Berger and Frédéric Déglise, pp. 1695–1790. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2016.10.1695. URL: https://doi.org/10.2140/ant.2016.10.1695.

[Pil20] V. Pilloni. "Higher coherent cohomology and p-adic modular forms of singular weights". In: *Duke Math. J.* 169.9 (2020), pp. 1647–1807. ISSN: 0012-7094. DOI: 10.1215/00127094-2019-0075. URL: https://doi.org/10.1215/00127094-2019-0075.

[PR87] I. Piatetski-Shapiro and Stephen Rallis. "Rankin triple L functions". In: *Compositio Math.* 64.1 (1987), pp. 31–115. ISSN: 0010-437X. URL: http://www.numdam.org/item?id=CM_1987__64_1_31_0.

[Sch13] Peter Scholze. "p-adic Hodge theory for rigid-analytic varieties". In: *Forum Math. Pi* 1 (2013), e1, 77. ISSN: 2050-5086. DOI: 10.1017/fmp.2013.1. URL: https://doi.org/10.1017/fmp.2013.1.

[Vat02] V. Vatsal. "Uniform distribution of Heegner points". In: *Invent. Math.* 148.1 (2002), pp. 1–46. ISSN: 0020-9910. DOI: 10.1007/s002220100183. URL: https://doi.org/10.1007/s002220100183.

[Yam22] Kazuki Yamada. *Hyodo-Kato theory with syntomic coefficients*. 2022. arXiv: 2005.05694 [math.AG].