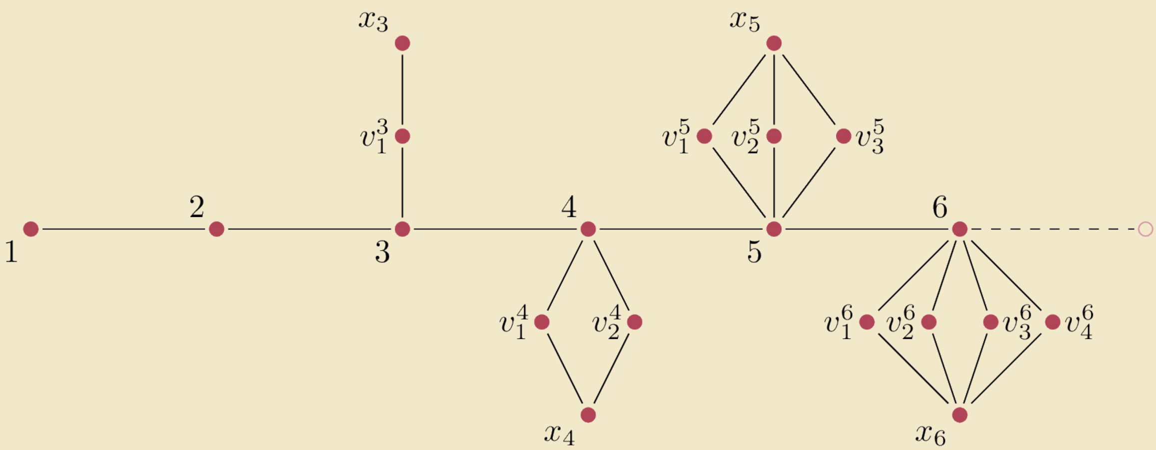
Fabian Gerle

Symmetric Feller Processes on Uniform State Spaces

Construction and Convergence



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Fabian Gerle

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Fabian Gerle

Symmetric Feller Processes on Uniform State Spaces:
Construction and Convergence

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For Marianne

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Bochum, February 2024

Fabian Gerle

Contents

Introduction

1

> *The story so far: In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.*
>
> — **Douglas Adams**
> The Restaurant at the End of the Universe

1.1 Introduction and main results

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. At the center of this thesis are symmetric Feller processes. That is, strong Markov processes defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in some topological space S that possess additional regularity properties. Many stochastic processes fall into this class, for example Brownian motion, Lévy processes or random walks. We generalize the state spaces from metric measure spaces to uniform measure spaces and show how hitting times play an important role in the analysis of such processes.

This research was initially motivated by the question under which conditions a sequence $X^{(n)}$ of symmetric Feller processes converges to a limiting process $X^{(\infty)}$.

1.1.1 Motivation

One of the earliest results of such a convergence is *Donskers invariance theorem*. It was obtained by Monroe D. Donsker as a result of his doctoral dissertation and published in [Don51]. Loosely speaking, Donsker showed that a simple symmetric random walk (linearly interpolated) converges in distribution to the Brownian motion as random variables on the space of continuous functions on the unit interval, $C([0, 1])$. More precisely, suppose $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed real-valued random variables with $\mathbb{E}[\xi] = 0$ and $\mathbb{E}\left[\xi^2\right] = \sigma^2$. Define $S_0 = 0$ and for each $n \in \mathbb{N}$ set $S_n := \sum_{k=1}^{n} \xi_k$. Moreover, for $n \in \mathbb{N},\ t \in [0, 1],\ \omega \in \Omega$ set

$$X_t^{(n)}(\omega) := \frac{1}{\sqrt{n}\sigma} S_{\lfloor nt \rfloor}(\omega) + \frac{nt - \lfloor nt \rfloor}{\sqrt{n}\sigma} \xi_{\lfloor nt \rfloor + 1}(\omega). \tag{1.1}$$

Then $X^{(n)} \colon \Omega \to C([0, 1])$ and

$$\mathbb{P}^{(n)} \Rightarrow \mathbb{W}, \tag{1.2}$$

weakly as probability measures on $C([0,1])$, where $\mathbb{W}$ denotes the *Wiener measure*. Sometimes this result is phrased as "the simple symmetric random walk converges to the Brownian motion in the *scaling limit*".

In [Sko56], Anatolii Skorokhod laid the groundwork for the analysis of processes that are not necessarily continuous but may contain jumps by introducing a topology (actually four different topologies) on the space of function $f\colon [0,\infty) \to S$ that are continuous from the right and possess limits from the left. Here S denotes a complete and separable metric space. We call such functions càdlàg[1] functions and denote the space of such functions by $D_S([0,\infty))$. We refer to the space $D_S([0,\infty))$ equipped with the Skorokhod topology as the *Skorokhod space* or *pathspace*.

Charles Stone considered in [Sto63] Markov processes on subsets of the real line such that "the random trajectories do not jump over points in the state space"[2] and depend continuously on a speed measure ν when considered on their "*natural scale*". Stone was able to show that under certain conditions on the convergence of the state spaces as well as the speed measure, such processes converge in the Skorokhod topology to a limiting process. Donsker's functional limit theorem can be considered an example of Stone's result.

More than 50 years later, Siva Athreya, Wolfgang Löhr and Anita Winter extended Stone's result to an invariance principle for random walks on metric measure trees in [ALW17]. Here, metric measure trees are metric spaces (T, r), that have a tree-like structure and are equipped with a measure ν. The speed-ν motion on a metric measure tree (T, r, ν) is a ν-symmetric Feller process which is determined by the structure of the tree, encoded in the metric r and the measure ν. The speed-ν motion was constructed earlier in [AEW13] using Dirichlet forms. It is worth pointing out that this construction makes use of the geometry of the tree through its metric. The speed-ν motion can therefore be considered to be on its *natural scale*.

In [ALW17], the authors were able to show the very elegant result that the speed-$\nu^{(n)}$ motions $X^{(n)}$ started in $\rho^{(n)}$ on a sequence of rooted metric measure trees $\left(\left(T^{(n)}, r^{(n)}, \rho^{(n)}, \nu^{(n)}\right)\right)_{n\in\mathbb{N}}$ converges weakly in path space to the speed-$\nu^{(\infty)}$ motion $X^{(\infty)}$ started in $\rho^{(\infty)}$ on a rooted metric measure tree $\left(T^{(\infty)}, r^{(\infty)}, \rho^{(\infty)}, \nu^{(\infty)}\right)$ whenever

$$\left(T^{(n)}, r^{(n)}, \rho^{(n)}, \nu^{(n)}\right) \longrightarrow \left(T^{(\infty)}, r^{(\infty)}, \rho^{(\infty)}, \nu^{(\infty)}\right), \tag{1.3}$$

as $n \to \infty$ in pointed Gromov-Hausdorff vague topology and a uniform bound on the lengths of edges emanating from a ball around the root holds. Pointed Gromov-Hausdorff vague convergence takes place when the rooted metric trees $\left(T^{(n)}, r^{(n)}, \rho^{(n)}\right)$ can be isometrically embedded into a common metric space (S, d) so

[1] from French: *continue à droite, limite à gauche*
[2] [Sto63, p. 638]

that $\left(T^{(n)}, r^{(n)}, \rho^{(n)}\right)$ converge in the pointed Hausdorff sense as subsets of the metric space (S, d) and the push-forwards of the measures $\nu^{(n)}$ under this embedding converge vaguely.

To illustrate this result consider again the simple symmetric random walk on $\mathbb{Z}$. We can consider $\mathbb{Z}$ as a (graph theoretic, discrete) tree where $x, y \in \mathbb{Z}$ are connected by an edge ($x \sim y$) if and only if $|x - y| = 1$. We let $r(x, y) = |x - y|$ be the Euclidean metric and $\nu(A) = \#(A \cap \mathbb{Z})$ the counting measure. Then, $\mathcal{X} = (\mathbb{Z}, r, 0, \nu)$ is a rooted metric measure tree and the speed ν-motion X on $\mathcal{X}$ is the ν-symmetric Feller process that jumps from $x \in \mathbb{Z}$ to $y \in \mathbb{Z}$ with $x \sim y$ at rate

$$\gamma_{xy} = \frac{1}{2\nu(\{x\})r(x, y)} = \frac{1}{2}. \tag{1.4}$$

The total jumprate at $x \in \mathbb{Z}$ is then $\gamma_x := \sum_{y:y\sim x} \gamma_{xy} = 1$. Hence, X is the continuous-time version of the simple symmetric random walk. Now define for each $n \in \mathbb{N}$ a rooted metric measure tree $\mathcal{X}^{(n)} = \left(T^{(n)}, r^{(n)}, 0, \nu^{(n)}\right)$ by setting

$$T^{(n)} := \mathbb{Z}, \quad r^{(n)}(x, y) := |x - y| / \sqrt{n} \quad \text{and} \quad \nu^{(n)}(A) := \nu(A) / \sqrt{n}. \tag{1.5}$$

The metric spaces $\left(T^{(n)}, r^{(n)}\right)$ are all naturally embedded into $\mathbb{R}$ and they converge to $(\mathbb{R}, d)$ in the Hausdorff topology where d denotes the Euclidean metric on $\mathbb{R}$. Moreover, for real numbers $a < b$, the set $[a, b] \cap T^{(n)}$ contains of the order of $\sqrt{n}(b - a)$ many points. More precisely,

$$\left(\sqrt{n}(a - b) - 1\right) / \sqrt{n} \le \nu^{(n)}([a, b]) \le \left(\sqrt{n}(a - b) + 1\right) / \sqrt{n}. \tag{1.6}$$

Consequently, $\nu^{(n)} \Rightarrow \lambda$ weakly as $n \to \infty$, where λ denotes the Lebesgue measure on $\mathbb{R}$. Moreover, the jump rates of the speed-$\nu^{(n)}$ motion $X^{(n)}$ is $\gamma_x^{(n)} = n$. The spaces $\mathcal{X}^{(n)}$ converge pointed Gromov-Hausdorff weakly to $\mathcal{X}^{(\infty)} = (\mathbb{R}, d, 0, \lambda)$ and the speed-$\nu^{(n)}$ motions converge to the speed-λ motion on $\mathbb{R}$ which is simply the standard Brownian motion. Note that the same result remains true when we rescale the metrics $r^{(n)}$ by a constant factor $c > 0$, as long as we make up for this rescaling by also rescaling the measures $\nu^{(n)}$ by c^{-1}. In this sense, constant factors can be shifted between the measure and the metric.

The result of Athreya, Löhr and Winter was extended by David Croydon in [Cro18] to so-called *resistance forms*. Resistance forms are a tool that was developed by Jun Kigami and others (cf. [Kig01]) to describe and analyze random walks on fractals and fractal-like graphs like the Sierpiński Gasket (see Figure 5.2). Technically, a resistance form is a symmetric bilinear form $\mathcal{E}$ on a subspace $\mathcal{F}$ of the real-valued functions on some set S satisfying certain conditions to ensure that $\mathcal{E}$ induces a metric

$\mathcal{R}$, called the *resistance metric*, on S by virtue of the following variational principle

$$\mathcal{R}(x,y) := \sup\left\{ \frac{|f(x)-f(y)|^2}{\mathcal{E}(f,f)} \,\middle|\, f \in \mathcal{F},\ \mathcal{E}(f,f) > 0 \right\}, \quad x,y \in S. \tag{1.7}$$

We will discuss the concepts related to resistance forms in more depth in Section 5.6. On the other hand, a resistance form together with a Radon measure ν on the metric space $(S,\mathcal{R})$ gives rise to a regular Dirichlet form on $L^2(S,\nu)$ which in turn uniquely defines a ν-symmetric Feller process with values in S. Again these processes can be considered to be on their *natural scale* as processes on the metric measure space $(S,\mathcal{R},\nu)$. Croydon showed that under an additional uniform recurrence condition an analogue of the invariance principle of [ALW17] holds. That is, the processes $X^{(n)}$ associated to a sequence of resistance forms $(\mathcal{E}^{(n)},\mathcal{F}^{(n)})$ on a sequence of sets $S^{(n)}$ started in $\rho^{(n)}$ converges weakly in path-space to a $\nu^{(\infty)}$-symmetric Feller process started in $\rho^{(\infty)}$, whenever

$$\left(S^{(n)},\mathcal{R}^{(n)},\rho^{(n)},\nu^{(n)}\right) \to \left(S^{(\infty)},\mathcal{R}^{(\infty)},\rho^{(\infty)},\nu^{(\infty)}\right) \tag{1.8}$$

pointed Gromov-Hausdorff weakly as $n \to \infty$.

All these results have in common that they are basically low dimensional in the sense that the processes hit points with positive probability, i.e. $\mathbb{P}_x(\{\exists t > 0 : X_t = y\}) > 0$ for all x, y in the state space. In other words, singletons have positive capacity (for the definition of capacities and other potential theoretic notions see Section 5.4). However this property fails in higher dimensions, for example for the Brownian motion in $\mathbb{R}^d$ for $d \geq 2$.

A complementary result that closes this gap was shown by Kohei Suzuki for Brownian motions on Riemannian Manifolds. The Brownian motion on a Riemannian manifold M equipped with the volume measure $\mathrm{d}V$ is again constructed by means of its Dirichlet form which is given in terms of the Laplace-Beltrami operator on M. The Laplace-Beltrami operator, on the other hand, is again related to the metric d on M through the Riemannian metric (see Section 5.7). The Brownian motion on a manifold M can therefore again be considered to be on its *natural scale*. In [Suz19a], Suzuki showed that under a uniform bound on the Ricci curvature of a sequence of Riemannian manifolds, the convergence of these manifolds in the Gromov-Hausdorff weak topology[3] implies pathwise convergence of the Brownian motions on said manifolds.

The common denominator of these complementary results is that the geometry of the state space and the probabilistic behavior of the processes defined on these state

[3]Suzuki actually uses *pointed measured Gromov convergence (pmG)* that was introduced in [GMS15]. However, this topology is weaker than the topology of Gromov-Hausdorff weak convergence (cf. [Suz19a, Remark 2.2. b)])

spaces are linked. This is what is meant by the expression that the processes are on their *natural scale*. This connection is maybe best illustrated by the *occupation time formula* for the speed-ν motion X on a metric measure tree (T, r, ν),

$$G_y f(x) := \mathbb{E}_x\left[\int_0^{\tau_y} f(X_t)\,\mathrm{d}t\right] = 2\int_T r(y, c(x,y,z)) f(z)\,\nu(\mathrm{d}z), \tag{1.9}$$

where $\tau_z := \inf\{\, t > 0 \mid X_t = y \,\}$ is the first hitting time of $y \in T$ and $c(x,y,z) \in T$ denotes the unique branchpoint of the three points $x, y, z \in T$. The occupation time formula relates the *Green operator* G_y on the left to the geometric structure and the speed measure through the *Green kernel* $g_y(x,z) = r(y, c(x,y,z))$ on the right.

If we now consider a ν-symmetric Feller process X on a "nice" metric measure space (S, d, ν) the question arises whether this process is on its natural scale and what is actually the natural scale for X?

Consider for example the random walk on a finite weighted graph $G = (V, \mu)$ (see Section 4.5.1). Here $V \neq \emptyset$ denotes the set of vertices and $\mu\colon V \times V \to [0,\infty)$ is a symmetric map that represents the weights (or inverse lengths) of edges between vertices. That is, two vertices $x, y \in V$ are connected by an edge of length μ_{xy}^{-1} if $\mu_{xy} > 0$ and there exists no edge between x and y if $\mu_{xy} = 0$. Such a graph comes with at least two natural metrics, the *simple graph distance* $d(x,y)$ which is the minimal number of vertices on a path from x to y (minus 1) and the *weighted graph distance* $d_\mu(x,y)$ which is simply the length of a shortest path between x and y. Neither of these metrics represents the natural scale for the random walk on G. Instead the natural scale is given by the resistance metric which can heuristically be understood as the electrical resistance between two vertices when we think of the graph as an electrical network where the vertices are connected by resistors with a resistance given by μ_{xy}^{-1}.

Moreover, if we have a sequence $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ of symmetric Feller processes living on a sequence of metric measure spaces $\left\{ \left(S^{(n)}, d^{(n)}, \nu^{(n)}\right) \,\middle|\, n \in \mathbb{N} \right\}$, under which conditions does this sequence converge to a limiting process? Since the processes have a priori no relation to the metrics $d^{(n)}$, we remove the metric from the state spaces and consider *uniform spaces* instead of metric spaces as state spaces for symmetric Feller processes. A Uniform space $(S, \mathcal{U})$ is a topological space with an additional structure that is just enough to define uniform continuity. In this sense uniform spaces are intermediates between topological spaces and metric spaces.

The idea to consider uniform spaces as state spaces is not new and goes back to Adam Jakubowski who introduced the Skorokhod topology on uniform spaces in [Jak86]. However, this idea had only little resonance.

1.1.2 Main results

One of the central results of this thesis is the introduction of uniform spaces as state spaces for stochastic processes. This entails a careful study of the space of càdlàg functions with values in a uniform space $(S, \mathcal{U})$. We show that the Skorokhod topology is uniformizable and describe the Skorokhod uniformity in terms of a family of pseudometrics (Proposition 3.14 and Theorem 3.16). Moreover, we show how many known quantitative results, in terms of the Skorokhod metric, can be reformulated as qualitative statements which hold true for the Skorokhod uniformity. For example, we prove a result about relative compactness in $D_S([0, \infty))$ by replacing the convergence of the modified modulus of continuity with a quantitative statement in Theorem 3.21.

Of particular interest is Theorem 3.27 where we characterize the convergence in $D_S([0, \infty))$ by the convergence of hitting times of certain sets. This result is new even in the context of metric spaces. The theorem was proven jointly with Gerónimo Rojas and it will appear in his dissertation for the metric case. For $A \subset S$ we define the first contact time of A by

$$\gamma_A(\omega) := \inf\left\{ t > 0 \,\middle|\, \{\omega(t), \omega(t-)\} \cap \overline{A} \neq \emptyset \right\}. \tag{1.10}$$

Moreover, for each $t > 0$ let $\theta_t\colon D_S([0, \infty)) \to D_S([0, \infty))$ with $\theta_t(\omega(\,\cdot\,)) := \omega(\,\cdot\, + t)$ denotes the *shift operator* on $D_S([0, \infty))$. Then Theorem 3.27 states the following.

Theorem. *Let $(S, \mathcal{U})$ be a uniform Hausdorff space, $\omega \in D_S([0, \infty))$ and $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0, \infty))$ be relatively compact. Then the following are equivalent.*

(i) $\lim_{n\to\infty} \omega_n = \omega$ *in the Skorokhod topology.*

(ii) For all $x \in S$, $U \in \mathcal{U}$, all continuity points $s \geq 0$ of ω, and all $D \in \mathcal{U}$, there exists a $E \in \mathcal{U}$ with $E \subset D$ open, such that

$$\tau_{(U\circ E)[x]}(\omega_n \circ \theta_s) \to \tau_{(U\circ E)[x]}(\omega \circ \theta_s), \quad \text{as } n \to \infty. \tag{1.11}$$

(iii) For all $x \in S$, all continuity points $s \geq 0$ of ω and all $U \in \mathcal{U}$ open such that $\tau_{U[x]}(\omega \circ \theta_s) = \gamma_{U[x]}(\omega \circ \theta_s)$ it holds that

$$\tau_{U[x]}(\omega_n \circ \theta_s) \to \tau_{U[x]}(\omega \circ \theta_s), \quad \text{as } n \to \infty. \tag{1.12}$$

In Theorem 3.48 we lift this statement to the space of probability measures on $D_S([0, \infty))$ and show how the convergence of hitting times can be used to show weak convergence of probability measures on $D_S([0, \infty))$. The theorem reads as follows.

Theorem. *Let* $(S, \mathcal{U})$ *be a separable uniform Hausdorff space with a countable base. Assume that* X, $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ *are* $D_S([0,\infty))$*-valued random variables with distribution* $\mathbb{P}^{X^{(n)}}$ *and* $\mathbb{P}^X$ *respectively. Then,* $\mathbb{P}^{X^{(n)}} \underset{n\to\infty}{\Longrightarrow} \mathbb{P}^X$ *if and only if the following conditions are satisfied.*

(i) The sequence $\left\{ \mathbb{P}^{X^{(n)}} \,\middle|\, n \in \mathbb{N} \right\}$ *is tight.*

(ii) There exists a countable dense set $T \subset \{ t > 0 \mid X_t = X_{t-} \text{ a.s.} \}$, *a countable dense subset* $D \subset S$ *and a countable base* $\mathcal{V} \subset \mathcal{U}$ *of* $\mathcal{U}$ *consisting of open entourages such that for all* $x \in D$, *all* $V \in \mathcal{V}$ *open with* $\tau_{V[x]}(X) = \gamma_{V[x]}(X)$ *a.s. and all* $s \in T$ *it holds that*

$$\tau_{V[x]}\left(X^{(n)} \circ \theta_s\right) \xrightarrow{d} \tau_{V[x]}(X \circ \theta_s). \tag{1.13}$$

We apply our analysis of the path-space to obtain a tightness criterion in Theorem 4.75 for Feller processes with values in uniform state spaces. The criterion states that a sequence of Feller processes $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ is tight when the probability that the processes move far from their starting point in a short time t goes uniformly to 0 in the starting point and n as $t \to 0$. Such a criterion was already shown in [ALW17, Corollary 4.3] as a corollary to Aldous' tightness criterion. Instead of using Aldous' criterion to proof Theorem 4.75 we present a direct proof using the Feller property.

Theorem. *For each* $n \in \mathbb{N}$ *let* $X^{(n)}$ *be a Feller process with values in a subset* S_n *of a locally compact Polish uniform space* $(S, \mathcal{U})$. *Assume that for every open entourage* $U \in \mathcal{U}$ *it holds that*

$$\lim_{t\to 0} \lim_{n\to\infty} \inf_{x\in S_n} \mathbb{P}_x((x, X_t^{(n)}) \in U) = 1. \tag{1.14}$$

Then for every sequence of initial distributions $\mu_n \in \mathcal{M}_1(S_n)$ *the family* $\left\{ X^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ *is tight in the one-point compactification* $(S_\vartheta, \mathcal{U}_\vartheta)$.

We follow up on the idea to analyze Feller processes by hitting times. For a symmetric Feller process X and a closed set $A \subset S$ we introduce the *killed process* X^A which is the same as X up to the first hitting time τ_A of A and is then moved to a cemetery state ϑ. In Theorem 4.65 and Theorem 4.66, we prove that the killed process is again a symmetric (strong) Feller process with state space $D_\vartheta := D \cup \{\vartheta\}$, where $D = S \setminus A$.

Theorem. *Let* X *be a* ν*-symmetric (strong) Feller process with values in* S_ϑ *and* $A \in \mathcal{B}_\vartheta$ *closed. Then the killed process* X^A *is again a* $\nu|_D$*-symmetric (strong) Feller process with values in* D_ϑ, *where* $D = S \setminus A$.

We apply this result in Theorem 4.72 to show that a symmetric doubly Feller process is already uniquely determined by its family of Green operators

$$G_A\colon \mathcal{B}_b \to \mathcal{B}_b, \quad G_A f(x) := \mathbb{E}_x\left[\int_0^{\tau_A} f(X_t)\,\mathrm{d}t\right]. \tag{1.15}$$

The theorem is stated as follows.

Theorem. *Let $(S, \mathcal{U})$ be compact and X be a ν-symmetric doubly Feller process with values in S_ϑ. Then X is uniquely determined by the family of Green operators*

$$\{\, G_A\colon \mathcal{B}_b \to \mathcal{B}_b \mid A \in \mathcal{B} \text{ closed} \,\}. \tag{1.16}$$

Other than in the situation of metric measure trees and resistance forms, points do generally not have positive capacity in our setup. That means we cannot define a resistance metric to introduce a *natural scale* for symmetric Feller processes on uniform spaces. We can however define a resistance between closed subsets of the state space in a very similar manner as in (1.7) using the Dirichlet form of the process.

Our final result is a convergence theorem for symmetric doubly Feller processes on compact uniform spaces. Theorem 6.1 can be formulated as follows.

Theorem. *Suppose $(S, \mathcal{U})$ is a compact uniform space and for each $n \in \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, $\nu^{(n)}$ is a Radon measure on $(S, \mathcal{B})$ with support $S^{(n)}$. Let further $X^{(n)}$ be a $\nu^{(n)}$-symmetric conservative doubly Feller process with values in S_n. Denote by $\mathbb{P}^{(n)} = \mathbb{P}^{X^{(n)}}$ the distribution of $X^{(n)}$ and assume that the following conditions hold.*

(C1) *$\nu^{(n)}$ converges Hausdorff weakly to $\nu^{(\infty)}$.*

(C2) *The family $\left\{ Q^{(n)} \;\middle|\; n \in \mathbb{N} \right\}$ of maps given by*

$$Q^{(n)}\colon S^{(n)} \times [0, \infty) \to \mathcal{M}_1(S), \quad (x, t) \mapsto Q^{(n)}_{x,t}(\,\cdot\,) := \mathbb{P}_x\left(X^{(n)}_t \in \,\cdot\,\right) \tag{1.17}$$

is uniformly equicontinuous.

(C3) *For every sequence $(x_n)_{n\in\mathbb{N}} \subset S$ with $x_n \in S^{(n)}$ and $\lim_{n\to\infty} x_n = x_\infty \in S^{(\infty)}$, the sequence $\left\{ \mathbb{P}^{(n)}_{x_n} \;\middle|\; n \in \mathbb{N} \right\}$ is tight as probability measures on $D_S([0, \infty))$.*

(C4) *The Green's functionals $G^{(n)}_A$ converge to $G^{(\infty)}_A$ in the following sense. For all bounded measurable functions $f \in \mathcal{B}_b(S)$ and all $A \in \mathcal{B}(S)$ with $\tau_A < \infty$, $\mathbb{P}^{(\infty)}_{x_\infty}$-a.s.,*

$$\lim_{n\to\infty} G^{(n)}_A f(x_n) = G^{(\infty)}_A f(x_\infty), \tag{1.18}$$

for all sequences $(x_n)_{n\in\mathbb{N}} \subset S$ with $x_n \in S^{(n)}$ and $\lim_{n\to\infty} x_n = x_\infty \in S^{(\infty)}$.

Then $X^{(n)}$ converges in distribution to $X^{(\infty)}$ for all sequences of initial distributions $\left(\mu^{(n)}\right)_{n\in\mathbb{N}} \subset \mathcal{M}_1(S)$ with $\mu^{(n)} \in \mathcal{M}_1(S^{(n)})$ and $\mu^{(n)} \Rightarrow \mu^{(\infty)} \in \mathcal{M}_1(S^{(\infty)})$. In other words,

$$\mathbb{P}^{(n)}_{\mu^{(n)}} \Rightarrow \mathbb{P}^{(\infty)}_{\mu^{(\infty)}} \tag{1.19}$$

weakly as probability measures on $D_S([0,\infty))$ as $n \to \infty$.

1.2 Outline

This thesis is structured as follows.

In Chapter 2 we introduce the notion of uniformities and uniform spaces. We present several different ways to define a uniform structure on a set S. Moreover, we explain how uniform spaces are related to topological spaces (uniformities induce topologies) and to metric spaces (metrics induce uniformities). In Section 2.5 we show that uniform spaces admit Cauchy sequences and therefore a notion of completeness. We introduce the notion of a *Polish uniform space* that is a separable and complete uniform space. This allows us to define *uniform measure spaces*. We also introduce a notion of uniform equicontinuity and prove a variant of the Arzelà-Ascoli theorem for uniform spaces in Theorem 2.46 and Lemma 2.47. We close this foundational chapter with a discussion of Hausdorff and Hausdorff weak convergence of subspaces of uniform spaces.

Chapter 3 is dedicated to the pathspace $D_S([0,\infty))$ of càdlàg functions with values in a uniform space $(S,\mathcal{U})$. We pick up an idea of Jakubowski [Jak86] and define a uniform structure on $D_S([0,\infty))$ that is compatible with the Skorokhod topology using a family of pseudometrics. We use the Skorokhod uniformity to reformulate many important results that are usually stated in terms of the Skorokhod metric in a more qualitative way. In Proposition 3.19 we give a useful criterion for convergence in the Skorokhod topology. In Section 3.3 we discuss the relatively compact subsets of $D_S([0,\infty))$ and give conditions for relative compactness in the Skorokhod topology. Section 3.4 is centered around Theorem 3.27 where we characterize the Skorokhod convergence by the convergence of hitting times of certain sets. We continue with a short discussion of the space of probability measures on a uniform space and show that the concept of the Prokhorov metric can be extended to define a uniform structure, the Prokhorov uniformity, on the space of probability measures on a uniform space. In Theorem 3.43 we give a characterization of tightness of a family of probability measures on $D_S([0,\infty))$ which will come in handy when we proof our tightness criterion Theorem 4.75 in Chapter 4. We conclude this chapter with our result on weak convergence of a sequence of probability measures on $D_S([0,\infty))$, Theorem 3.48.

We continue to introduce symmetric Feller processes with values in uniform spaces in Chapter 4. We first introduce Markov processes to fix some notations. In particular, we introduce filtrations, stopping times, semigroups, resolvents and ν-symmetry of semigroups. In Section 4.2 we introduce the normal and the strong Feller property and introduce the generator. We also state the Hille-Yosida theorem Proposition 4.40 to characterize Feller semigroups in terms of the generator. We then continue to show that Feller processes possess càdlàg modifications. Next, we discuss hitting times and give some bounds on hitting times. In Section 4.3 we introduce the killed process X^A that is killed upon hitting a closed set $A \subset S$. We show that the Markov property, the strong Markov property, symmetry as well as the normal and the strong Feller property carry over from X to the killed process X^A. We then briefly discuss recurrence and transience of Feller processes and continue to show one of our main results, Theorem 4.72, where we state that a symmetric doubly Feller process is already determined by its family of Green operators. Before we conclude this chapter with the discussion of two important examples, the random walk on graphs and Brownian motion on $\mathbb{R}^d$, we proof our tightness criterion Theorem 4.75 in Section 4.4.

In Chapter 5 we introduce Dirichlet forms. We start with the definition of a closed symmetric form, then define Dirichlet forms as closed symmetric forms that possess the Markov property. We begin Section 5.2 with a brief discussion of operators on Hilbert spaces and then illustrate the relationship between strongly continuous contraction semigroups, strongly continuous resolvents, generators and closed forms on $L^2(S,\nu)$. In the next subsection, we introduce the Markov property of the semigroup and show that a Markovian semigroup gives rise to a Dirichlet form. We conclude this section by explicitly extending a Feller semigroup $(P_t)_{t\geq 0}$ on $C_\infty(S)$ to a strongly continuous Markovian semigroup on $L^2(S,\nu)$. Thereby showing how a Feller process induces a Dirichlet form. We go on to define the extended Dirichlet space and discuss the implications of transience and recurrence for the extended Dirichlet space. Namely, the extended Dirichlet space is a Hilbert space if and only if the Dirichlet form is transient. In Section 5.4 we introduce important potential theoretic notions like the capacity. We begin with a general definition of Choquet capacities and then move on to define α-capacities with respect to a Dirichlet form $(\mathcal{E},\mathcal{D})$. The α-capacity is given by the following variational principle.

$$\operatorname{Cap}_\alpha(A) := \inf\left\{ \mathcal{E}_\alpha(f,f) \;\middle|\; f \in \mathcal{L}^A \right\}, \tag{1.20}$$

where $\mathcal{L}^A := \{ f \in \mathcal{D} \mid f \geq 1 \ \nu\text{-a.e. on } A \}$. Moreover, we characterize the minimizer of (1.20) in Theorem 5.51 and identify the minimizer with $p_A^\alpha(x) = \mathbb{E}_x\left[e^{-\alpha\tau_A}\right]$. For transient Dirichlet forms, we define the 0-capacity in Section 5.4.3 and proceed similarly as for the α-capacity. In Section 5.5 we define the resistance $\mathcal{R}(A,B)$ between two closed subsets of S as the inverse of the 0-capacity of the killed Dirichlet form. We conclude this chapter again with two examples. First, we formally introduce resistance forms and discuss some of their properties. Finally, we define the Brownian motion on Riemannian manifolds in a rather condensed form.

We proof the convergence theorem Theorem 6.1 in Chapter 6. We proceed thereby as follows. We first show that under the Hausdorff weak convergence of the state spaces and the uniform equicontinuity of the semigroup, there exist subsequential limits of the semigroup that again possess both the normal and the strong Feller property in Theorem 6.2. We then show in Theorem 6.4 that this already implies that the sequence of processes $X^{(n)}$ has subsequential limits in finite dimensional distributions. Together with the assumption that the sequence $X^{(n)}$ is tight, we obtain the existence of subsequential limits in path-space, Theorem 6.5. Finally, the convergence of the Green operators implies by Theorem 4.72 that all subsequential limits mus coincide, which proves the theorem. The last part of this chapter, Section 6.4, is dedicated to a discussion of the assumptions (C1) to (C4).

The last chapter, Chapter 7 contains remarks and conjectures that are potentially of interest for further research on the topic of convergence of symmetric Feller processes and uniform measure spaces as state spaces.

The appendices contain some important facts that are good to have at an arm's length but which have not found their way into the main text.

Uniform spaces 2

> *Gedanken ohne Inhalt sind leer, Anschauungen ohne Begriffe sind blind.*
>
> — **Immanuel Kant**
> Kritik der reinen Vernunft (B 75)

In this chapter, we introduce the notion of *uniformities* or *uniform spaces*. We will show that a metric induces a uniformity which in turn induces a topology; but not the other way round. In this sense uniform spaces are an intermediary between metric spaces and topological spaces.

Uniform spaces will serve as state spaces for our stochastic processes throughout this thesis. Although in many cases the uniform spaces under consideration will be metrizable we want, on the one hand, emphasize the sufficiency of the uniform structure for many results. On the other hand, we want to equip the state spaces with a structure that is related to the processes under consideration (think of resistance metrics) and the "correct" metric can be quite inaccessible.

Historically, the concept of uniform continuity for real-valued functions was introduced by Eduard Heine [Hei70] in 1870. Heine attributes the insight that a stronger notion of continuity is needed to Karl Weierstrass. The concept of uniform continuity was further extended to uniform continuity of functions on metric spaces by Maurice Fréchet [Fré06] in 1906 and Felix Hausdorff [Hau14] in 1914. It took until 1937 that André Weil formally introduced uniform spaces in [Wei37]. Weil used families of pseudometrics to define the uniform structure and we will present this approach in Section 2.3. A different approach was put forward in 1939 by John W. Tukey in his dissertation which has been recompiled and published as the monograph [Tuk40]. Tukey relied in his work on uniform coverings to define a uniform structure. In the 1950s and 1960s, there were further contributions to the theory of uniform spaces by Vadim A. Efremovič and Yuri M. Smirnov who constructed uniform spaces from the proximity relation we will introduce in Section 2.6. In this thesis, we will mainly rely on so-called diagonal uniformities which were used in [Bou66a] by the famous author's collective Nicolas Bourbaki which Weil was a founding member of. More on the history of uniform spaces can be found in the preface to [Isb64] and in the historical appendix in [Wil70].

2.1 Diagonal uniformities

We begin with a bit of motivation. Let (S, d) and (T, r) be metric spaces. A function $f : S \to T$ is continuous, if and only if the preimage $f^{-1}A$ of every open set $A \subset S$ is open in T. Furthermore, f is uniformly continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $r(f(x), f(y)) < \varepsilon$ for all $x, y \in S$ with $d(x, y) < \delta$. We can even measure the degree of continuity with the modulus of continuity or Lipschitz constants. In fact, the metric structure is not necessary to define uniform continuity. Write

$$B^d_\varepsilon := \left\{ (x, y) \in S^2 \,\middle|\, d(x, y) < \varepsilon \right\} \tag{2.1}$$

for the tube around the diagonal in S^2 with radius $\varepsilon > 0$ and analogously $B^r_\varepsilon \subset T^2$ for the ε-tube around the diagonal of T^2. Then the condition for uniform continuity can be reformulated as: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\{ (f(x), f(y)) \in T^2 \,\middle|\, (x, y) \in B^d_\delta \right\} \subset B^r_\varepsilon. \tag{2.2}$$

These tubes or *entourages* allow us to compare neighborhoods of different points across the whole space to each other. This idea is generalized by uniformities, in particular by diagonal uniformities.

Let S be a nonempty set. There are different ways to introduce a uniform structure on S. One way is via coverings and their refinements and another way is by families of subsets of $S \times S$. In the literature (cf. [Wil70]) the uniformities obtained from coverings are called *covering uniformities* and the latter are called *diagonal uniformities*. Both definitions are of course equivalent. We will mainly focus on diagonal uniformities.

We denote by

$$\Delta = \Delta(S) := \{ (x, x) \mid x \in S \} \subset S \times S \tag{2.3}$$

the diagonal of the space $S \times S$. Furthermore, we write

$$U^* := \{ (x, y) \in S \times S \mid (y, x) \in U \} \tag{2.4}$$

and say that U is symmetric if $U^* = U$. For two subsets U, V of $S \times S$ we define the *concatenation* of U and V as

$$U \circ V := \{ (x, y) \in S \times S \mid \exists\, z \in S : (x, z) \in V \text{ and } (z, y) \in U \}. \tag{2.5}$$

We can now define the main object of this section. The definition formalizes the intuition gained from the motivation above.

Definition 2.1 (Uniformities)**.** Let S be a nonempty set. A (diagonal) *uniformity* on S is a family $\mathcal{U} = \mathcal{U}(S)$ of subsets of $S \times S$ satisfying

(U1) if $U \in \mathcal{U}$ then $\Delta \subset U$,

(U2) if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$,

(U3) if $U \in \mathcal{U}$ then there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$,

(U4) if $U \in \mathcal{U}$ then $V^* \subset U$ for some $V \in \mathcal{U}$,

(U5) if $U \in \mathcal{U}$ and $U \subset V$ then $V \in \mathcal{U}$.

The elements U of a uniformity $\mathcal{U}$ are called *entourages* or *surroundings*. A pair $(S, \mathcal{U}(S))$ is called a *uniform space*. ◇

Most of the time is it enough to work with bases or even subbases of uniformities.

Definition 2.2 (Bases and subbases)**.** Let $(S, \mathcal{U})$ be a uniform space. A family $\mathcal{V} \subset \mathcal{U}$ of subsets of S is called a base of the uniformity $\mathcal{U}$ if

$$\mathcal{U} = \{\, U \subset S \times S \mid \exists\, V \in \mathcal{V} : U \supset V \,\}. \tag{2.6}$$

The elements of a base $\mathcal{V} \subset \mathcal{U}$ are called *basic entourages*.

A family $\mathcal{V} \subset \mathcal{U}$ is called a subbase of $\mathcal{U}$ if all finite intersections of elements of $\mathcal{V}$ form a base of $\mathcal{U}$. ◇

We have indicated in the introduction that metric spaces carry a uniform structure. It is instructive to have the following example in mind.

Example 2.3 (Metric spaces have uniform structure)**.** Let (S, d) be a metric space. Consider the family $\mathcal{V}$ of sets of the form

$$B_\varepsilon := \{\, (x, y) \in S \times S \mid d(x, y) < \varepsilon \,\}, \qquad \varepsilon > 0. \tag{2.7}$$

Clearly, $\mathcal{V}$ satisfies (U1) and (U4). For $0 < \varepsilon < \delta$ we have $B_\varepsilon \subset B_\delta$ and hence $B_\varepsilon \cap B_\delta = B_\varepsilon \in \mathcal{V}$ and $\mathcal{V}$ satisfies (U2). Finally, we have for every $\varepsilon > 0$ that $B_{\varepsilon/3} \circ B_{\varepsilon/3} = B_{2\varepsilon/3} \subset B_\varepsilon$, verifying (U3). Thus the family $\mathcal{V}$ is the base of a uniformity on S. We refer to this uniformity simply as the metric uniformity when there can be no confusion about the metric involved. ■

Given a uniformity $\mathcal{U}$ on S we can define neighborhoods of points by setting

$$U[x] := \{\, y \in S \mid (x, y) \in U \,\} \tag{2.8}$$

for some entourage $U \in \mathcal{U}$. This definition can be extended to neighborhoods of sets in a natural way by setting

$$U[A] := \bigcup_{x \in A} U[x] \tag{2.9}$$

for an entourage $U \in \mathcal{U}$ and a subset A of S.

We will show that these neighborhoods in fact give rise to a topology on S.

Recall from Definition A.4 and Proposition A.6 the properties of neighborhood bases.

Proposition & Definition 2.4 (Topologies and uniformities)**.** *Let $(S, \mathcal{U})$ be a uniform space and $\mathcal{V} \subset \mathcal{U}$ a base of $\mathcal{U}$. The family $\mathcal{N}_x = \{ V[x] \mid V \in \mathcal{V} \}$ forms a neighborhood base at $x \in S$ and thus $\mathcal{U}$ induces a topology on S which we call the* uniform topology *(associated with the uniformity $\mathcal{U}$) on S. Furthermore, any base $\mathcal{V}$ of $\mathcal{U}$ induces the same topology on S. We call any topology that can be obtained in this way from some uniformity* uniformizable.

Proof. We show that the family of subsets given by $\mathcal{N}_x := \{ V[x] \mid V \in \mathcal{V} \}$ at each $x \in S$ satisfies (i)–(iii) of Proposition A.6. By definition, $(x, x) \in V$ for all $V \in \mathcal{V}$ and consequently $x \in V[x]$. Assume that $N_1, N_2 \in \mathcal{N}_x$, then there exist $V_1, V_2 \in \mathcal{V}$ such that $N_j = V_j[x]$, $j = 1, 2$. By property (U2) of Definition 2.1 we have $V := V_1 \cap V_2 \in \mathcal{V}$ and hence

$$N_1 \cap N_2 = \{ y \in S \mid (x, y) \in V_1 \cap V_2 \} = V_1[x] \cap V_2[x] = V[x] \in \mathcal{N}_x \tag{2.10}$$

which implies (ii) of Proposition A.6. Consider $N \in \mathcal{N}_x$ with $N = V[x]$ for some $V \in \mathcal{V}$. By Definition 2.1 (U3) there exists a $U \in \mathcal{V}$ such that $U \circ U \subset V$ and consequently, for all $y \in U[x]$ and $z \in U[y]$ we have $(x, z) \in U \circ U \subset V$ and hence $U[y] \subset N$, verifying the final property of neighborhood bases.

Let $\mathcal{V}' \subset \mathcal{U}$ be another base of $\mathcal{U}$ then for each $x \in S$ the family $\mathcal{N}'_x := \{ V[x] \mid V \in \mathcal{V}' \}$ is a neighborhood base by the same arguments as above. Now, $\mathcal{N}_x$ and $\mathcal{N}'_x$ are bases for same neighborhood system at x and hence induce the same topologies on S, by Proposition A.3. ■

It follows immediately from the arguments laid out above that the uniform topology is first countable if the uniformity possesses a countable base. We will say that the uniform space $(S, \mathcal{U})$ has a countable base if the uniformity $\mathcal{U}$ has a countable base. This does not mean, and should not be confused with, that the topology induced by $\mathcal{U}$ has a countable base (second countable) but rather a countable local base (first countable).

By taking the product of the uniform topology, any uniformity on S induces a topology on $S \times S$. We say that an entourage is open, closed, compact etc. if it is open, closed, compact etc. with respect to the product of the uniform topology on $S \times S$. In the same way we define the interior and the closure and related notions of an entourage.

As is customary, we denote by A° the *interior* of $A \subset S$, that is the largest open set contained in A (cf. Definition A.2). We make the following simple observation.

Lemma 2.5 (interiors of entourages are again entourages)**.** *Let $(S, \mathcal{U})$ be a uniform space. Assume $U \in \mathcal{U}$ then $U^\circ \in \mathcal{U}$.*

Proof. Let $U \in \mathcal{U}$. In order to show the claim we show that there exists a $V \in \mathcal{U}$ such that $V \subset U^\circ$. The claim then follows from Definition 2.1 (U5). By definition of an entourage (U4), there exists a $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$. In order to show $V \in U^\circ$ we must show that every element $(x, y) \in V$ has a neighborhood that is contained in U. By construction,

$$(x, y) \in V[x] \times V[y] \subset V \circ V \circ V \subset U, \tag{2.11}$$

hence $V \circ V \circ V$ is the desired neighborhood. ■

The next result can be found in [Wil70, Theorem 35.6]. The proof is straightforward but we present it here for completeness sake.

Proposition & Definition 2.6 (Separating uniformities)**.** *Let $(S, \mathcal{U})$ be a uniform space. The uniformity $\mathcal{U}$ is called* separating *if*

$$\bigcap_{U \in \mathcal{U}} U = \Delta. \tag{2.12}$$

Furthermore, the uniformity $\mathcal{U}$ is separating if and only if (2.12) holds for some and hence for any base $\mathcal{V}$ of $\mathcal{U}$. The uniform topology is Hausdorff if and only if the uniformity $\mathcal{U}$ is separating

Proof. By definition of a base, it follows immediately that (2.12) holds for every base of $\mathcal{U}$ if it holds for $\mathcal{U}$. On the other hand, since every base is a subset of $\mathcal{U}$, (2.12) holds if it holds for some base $\mathcal{V}$ of $\mathcal{U}$.

Now assume that $\mathcal{U}$ is separating and let $x, y \in S$ be distinct. Then there exists a $U \in \mathcal{U}$ such that $(x, y) \notin U$. By Definition 2.1 (U3) and Lemma 2.5 there exists a $V \in \mathcal{U}$ open such that $V \circ V \subset U$. We claim that $V[x]$ and $V[y]$ are disjoint neighborhoods of x and y, respectively. If there exists a $z \in V[x] \cap V[y]$ then, by definition, $(x, y) \in V \circ V \subset U$ which was ruled out by assumption.

Now assume that $(S, \mathcal{T})$ is Hausdorff, where $\mathcal{T}$ is the uniform topology induced by $\mathcal{U}$. Let $x, y \in S$ be distinct. By definition of the uniform topology, there exist $V, W \in \mathcal{U}$ open such that $V[x] \cap W[y] = \emptyset$. Then $V \cap W \in \mathcal{U}$ is an (open) entourage that does not contain (x, y). ■

Different authors use slightly different definitions of uniformities. Isbell [Isb64] for example, includes the Hausdorff property in the definition of a uniformity.

The next lemma provides a convenient base for proofs involving uniformities. We say that $U \subset S \times S$ is symmetric, if $S = S^*$.

Lemma 2.7 ([Wil70, Theorem 35.9]). *The open, symmetric elements of $\mathcal{U}$ form a base of $\mathcal{U}$.*

Proof. Let $U \in \mathcal{U}$, then $U \cap U^* \in \mathcal{U}$, by Definition 2.1 (U4) and furthermore $U \cap U^* \subset U$. It remains to show that the open sets form a base. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be symmetric with the property that $V \circ V \circ V \subset U$. By Lemma 2.5 we have $U^\circ \in \mathcal{U}$, completing the proof. ■

It turns out that uniformities are the structure that is needed to define uniformly continuous functions – hence the name.

Definition 2.8 (Uniform continuity). Let $(S, \mathcal{U})$ and $(T, \mathcal{V})$ be two uniform spaces. A function $f: S \to T$ is *uniformly continuous*, if for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $\{ (f(x), f(y)) \mid (x, y) \in U \} \subset V$. ◇

If either S, T or both are metric spaces, the function $f: S \to T$ is uniformly continuous if and only if it is uniformly continuous with respect to the uniformities generated by metrics on S and/or T respectively.

It follows immediately from the definition of the metric uniformity that a function $f: S \to T$, where $(S, \mathcal{U})$ is a uniform space and (T, d) is a metric space, is uniformly continuous if and only if for every $\varepsilon > 0$ there exists a $U \in \mathcal{U}$ such that $d(f(x), f(y)) < \varepsilon$ whenever $(x, y) \in U$.

The trinity of topological, uniform and metric spaces becomes apparent when considering continuous functions: on topological spaces, we can only discern continuous from non-continuous functions. On uniform spaces, we can compare the degree of continuity at different points of a function, which leads to the notion of uniform continuity. In metric spaces, however, we can even measure the degree of continuity via the modulus of continuity and compare the degree of continuity across functions.

We conclude this section with a couple of examples.

Examples 2.9 (cf. [Wil70, Examples 35.3])**.** Let S be a non empty set.

(i) The uniformity $\mathcal{U} = \{ U \subset S \times S \mid \Delta \subset U \}$ is called the discrete uniformity. The discrete uniformity generates the discrete topology.

(ii) The uniformity $\mathcal{U} = \{S \times S\}$ is called the trivial uniformity. The trivial uniformity generates the trivial topology. ■

The next example illustrates that there may exist multiple uniformities that induce the same topology.

Example 2.10. Let $S = \mathbb{R}$. For any $r \in \mathbb{R}$ the sets of the form

$$U_r := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x > r \text{ and } y > r \right\} \cup \Delta \tag{2.13}$$

form a base for a uniformity $\mathcal{U}$ on $\mathbb{R}$ which is not the discrete uniformity (e.g. the unit ball is not contained in $\mathcal{U}$). On the other hand, for every x and every $r < x$ we have $U_r[x] = \{x\}$ and hence $\mathcal{U}$ generates the discrete topology on $\mathbb{R}$. ■

On the other hand, different metrics may induce the same uniformity.

Example 2.11. Let (S, d) be a metric space. Assume $\alpha > 0$, then the metrics $d, \alpha d$ and $\sqrt{d}$ all induce the same uniformity on S. ■

2.2 Weak uniformities

Similar to the weak topology induced by a family of functions one can define the weak uniformity.

Definition 2.12 (Weak uniformities)**.** Let S be a set and $(T, \mathcal{V})$ a uniform space. Further, let $\mathcal{F} := \{f : S \to T\}$ be a family of maps from S to T. The *weak uniformity* $\mathcal{U}_\mathcal{F}$ generated by $\mathcal{F}$ is the coarsest uniformity on S such that all $f \in \mathcal{F}$ are uniformly continuous. ◇

Proposition 2.13 (A base for weak uniformities)**.** *Let S be a non-empty set and $(T, \mathcal{V})$ a uniform space. Let further $\mathcal{F} \subset \{f : S \to T\}$ be a non empty family of maps from S to T and define for each $f \in \mathcal{F}$ the map $F_f : S \times S \to T \times T$ by*

$$F_f(x, y) = (f(x), f(y)). \tag{2.14}$$

Then the collection of sets

$$\mathcal{W} := \left\{ \bigcap_{i=1}^{n} F_{f_i}^{-1} V_i \;\middle|\; n \in \mathbb{N},\ f_i \in \mathcal{F},\ V_i \in \mathcal{V} \text{ for all } i \in \mathbb{N} \right\} \tag{2.15}$$

forms a base for the weak uniformity $\mathcal{U}_{\mathcal{F}}$ on S.

Proof. We first show that $\mathcal{W}$ is a base, i.e. satisfies properties (U1) to (U4). By definition, $\Delta(T) \subset V$ for all $V \in \mathcal{V}$ and clearly, $F_f^{-1}(\Delta(T)) = \Delta(S)$. Hence, $\Delta(S) \subset W$ for all $W \in \mathcal{W}$ showing (U1). Property (U2) follows immediately from the definition of $\mathcal{W}$. Properties (U3) and (U4) are consequences of the corresponding properties of $\mathcal{V}$. We only show (U3) as (U4) can be shown by a similar argument. Assume $W = F_f^{-1}V$ for some $f \in \mathcal{F}$ and $V \in \mathcal{V}$. Then there exists a $V' \in \mathcal{V}$ such that $V' \circ V' \subset V$ and set $W' := F_f^{-1}V'$. For $i = 1,2,3$ let $x_i \in S$ be such that $(x_1, x_2), (x_2, x_3) \in W'$. Then there exist $y_i \in T$, $i = 1,2,3$ such that $x_i \in f^{-1}\{y_i\}$ for $i = 1,2,3$ and $(y_1, y_2), (y_2, y_3) \in V'$. Thus, $(y_1, y_3) \in V$ and hence $(x_1, x_3) \in W$ and consequently $W' \circ W' \subset W$. The same conclusion follows for general $W \in \mathcal{W}$ from the observation that for two subsets $A, B \subset S \times S$ and $A', B' \subset S \times S$ such that $A' \circ A' \subset A$ and $B' \circ B' \subset B$ it holds that

$$(A' \cap B') \circ (A' \cap B') \subset (A' \circ A') \cap (B' \circ B') \subset A \cap B. \tag{2.16}$$

We have shown that $\mathcal{W}$ is indeed the base for a uniformity $\mathcal{U}_{\mathcal{F}}$ on S. It remains to show that every uniformity on S with respect to which all $f \in \mathcal{F}$ are uniformly continuous, contains $\mathcal{W}$. But this follows immediately from the definition of $\mathcal{W}$. ■

Remarks 2.14. (i) In (2.15) it suffices to restrict choice of the V_i to a base of $\mathcal{V}$.

(ii) In the case where (T, d) is a metric space the weak uniformity associated with a family $\mathcal{F} = \{f : S \to T\}$ is generated by the sets of the form

$$\bigcap_{i=1}^{n} F_{f_i}^{-1} B_\delta, \tag{2.17}$$

where $n \in \mathbb{N}$, $\delta > 0$, $f_j \in \mathcal{F}$ for all $1 \le i \le n$ and

$$B_\delta := \{\, (u, v) \in T \times T \mid d(x, y) < \delta \,\}. \tag{2.18}$$

To see this recall that the sets $B_\delta, \delta > 0$ are a base for the metric uniformity. Furthermore, the inclusion $B_\varepsilon \subset B_\delta$ for $\varepsilon < \delta$ is preserved under the preimage operation and hence we can choose $\delta = \min_{i=1,\dots,n}\{\delta_i\}$. ◊

For a more detailed account of weak uniformities see [Wil70, Chap. 37]. For our purpose weak uniformities generated by real-valued functions will suffice. Observe that the topology generated by the weak uniformity generated by $\mathcal{F}$ coincides with the weak topology generated by $\mathcal{F}$ (cf. the remark after [Wil70, Definition 37.7]).

2.3 Uniformities and pseudometrics

Recall from Definition A.26 that a *pseudometric* on a set S is a distance function $\rho\colon S \times S \to \mathbb{R}$ that satisfies all the axioms of a metric except that $\rho(x, y) = 0$ does not necessarily imply $x = y$. That is, ρ is non-negative definite, symmetric, satisfies $\rho(x, x) = 0$ for all $x \in S$ and the triangle inequality holds.

Pseudometrics, or rather families of pseudometrics provide a different way to characterize uniform spaces. Given a non-empty index set $\mathbb{I} \neq \emptyset$ and a family $\{ \rho_i \mid i \in \mathbb{I} \}$ of pseudometrics on S we can define a uniformity $\mathcal{U}$ on S using the sets of the form

$$U_\varepsilon^{\rho_i} := \{ (x, y) \in S \times S \mid \rho_i(x, y) < \varepsilon \} \quad i \in \mathbb{I}, \varepsilon > 0$$

as a base of $\mathcal{U}$. An important question is when is a uniformity generated by a family of pseudometrics separating or, equivalently, when is the uniform space $(S, \mathcal{U})$ Hausdorff.

Lemma 2.15 (Pseudometrics and separating uniformities)**.** *Let $S \neq \emptyset$ and $\Gamma = \{ \rho_i \mid i \in \mathbb{I} \}$ a family of pseudometrics on S. Then the uniformity generated by Γ is separating if for each pair $(x, y) \in S^2 \setminus \Delta$ there exists a $\rho \in \Gamma$ such that $\rho_i(x, y) > 0$.*

Proof. Let $(x, y) \in S^2 \setminus \Delta$. By assumption, there exists a $\varepsilon > 0$ and a $\rho \in \Gamma$ such that $\rho(x, y) > \varepsilon$. Hence, $(x, y) \notin U_\varepsilon^\rho$ and thus $\bigcap_{\rho\in\Gamma} \bigcap_{\varepsilon>0} U_\varepsilon^\rho = \Delta$. ■

More interestingly, every uniformity can be obtained from a family of pseudometrics (cf. [Bou66b, IX Theorem 1.4.1]). To construct such a family of pseudometrics on a uniform space $(S, \mathcal{U})$, consider the space $S \times S$ endowed with the product uniformity $\mathcal{U}^2$. That is the coarsest uniformity that makes the projections uniformly continuous. Then $\mathcal{U}$ is generated by the family of all pseudometrics that are uniformly continuous on $S \times S$ (see [Bou66b, IX §1.5]).

These observations lead to the following result. (see e.g. [Jak86])

Proposition 2.16 (Consistent families of pseudometrics)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space. Then there exists an index set $\mathbb{I}$ and a family $\{ \rho_i \mid i \in \mathbb{I} \}$ of pseudometrics on S generating $\mathcal{U}$ with the properties* *(i) for all $x, y \in S$ with $x \neq y$ there exists an $i \in \mathbb{I}$ such that $\rho_i(x, y) > 0$*

(ii) for all $i, j \in \mathbb{I}$ there exists an index $k \in \mathbb{I}$ such that $\max\{\rho_i, \rho_j\} \leq \rho_k$.

Proof. By Lemma 2.7 we can choose a base $\mathcal{V}$ of $\mathcal{U}$ consisting of the open and symmetric entourages $U \in \mathcal{U}$. For $U \in \mathcal{V}$ set $\rho_U(x, y) = \mathbb{1}_U((x, y))$. It is easy to check that ρ_U is a pseudometric. Furthermore, it is evident from the construction that the family $\{ \rho_U \mid U \in \mathcal{V} \}$ generates $\mathcal{U}$. By the Hausdorff property, for each pair $(x, y) \in S^2 \setminus \Delta$ there exists a basic entourage $U \in \mathcal{V}$ such that $(x, y) \notin U$ and hence $\rho_U(x, y) > 0$, showing (i). Now let $U, V \in \mathcal{V}$, by definition of a uniformity, $U \cap V \in \mathcal{U}$ and hence there exists a basic entourage $W \in \mathcal{V}$ such that $W \subset U \cap V$. Assume $\rho_U(x, y) > 0$ then, $(x, y) \notin U$ and hence $(x, y) \notin W$ and we have $\rho_U(x, y) = \rho_W(x, y) = 1$. The same holds for ρ_V, establishing (ii) ■

We use the common short-hand $a \vee b := \max\{a, b\}$ and $a \vee b := \min\{a, b\}$ for $a, b \in \mathbb{R}$. Analogously we set for real-valued functions $f, g \colon \Omega \to \mathbb{R}$

$$(f \vee g)(\omega) := \max\{f(\omega), g(\omega)\}, \quad (f \wedge g)(\omega) := \min\{f(\omega), g(\omega)\}, \quad \omega \in \Omega. \quad (2.19)$$

Without loss of generality we can always take the family $\{ \rho_i \mid i \in \mathbb{I} \}$ to be bounded by 1.

Lemma 2.17 (Truncated pseudometrics generate the same uniformity)**.** *Let $\{ \rho_i \mid i \in \mathbb{I} \}$ be a family of pseudometrics on S and $\mathcal{U}$ the uniformity generated by this family. Then $\mathcal{U}$ is also generated by the family $\{ \rho_i \wedge 1 \mid i \in \mathbb{I} \}$.*

Proof. The claim follows immediately from the observation that for all $0 < \varepsilon < 1$ and $i \in \mathbb{I}$

$$\left\{ (x, y) \in S^2 \,\middle|\, \rho_i(x, y) < \varepsilon \right\} = \left\{ (x, y) \in S^2 \,\middle|\, \rho_i(x, y) \wedge 1 < \varepsilon \right\}. \quad (2.20)$$

■

Apparently, there is a close connection between uniformities and (families of) pseudometrics. So it comes as no surprise that André Weil used families of pseudometrics, so called *gage structures*, to originally define uniformities in [Wei37].

Definition 2.18 (Gage structures)**.** Let S be a nonempty set and $I \neq \emptyset$ some set of indices. A family $\mathcal{G} = \{ \rho_i \mid i \in \mathbb{I} \}$ of pseudometrics on S is called a *gage structure* if it satisfies

(i) whenever $\rho_i, \rho_j \in \mathcal{G}$ then $\max \rho_i, \rho_j \in \mathcal{G}$

(ii) if ρ is a pseudometric on S and for every $\varepsilon > 0$ there exists a $\delta > 0$ and a $\rho' \in \mathcal{G}$ such that $\rho(x, y) < \varepsilon$ whenever $\rho'(x.y) < \delta$, then $\rho \in \mathcal{G}$. ◇

It can be shown (cf. [Kel75, Theorem 6.18]) that gage structures are in a one-to-one correspondence with uniformities.

The term "gage" does not appear in Weil's work in 1937 or in Doss' article [Dos49] in 1949. But it appears in the first edition of Kelley's [Kel75] in 1955. I have not been able to find out who first coined the term. In more recent publications about uniform spaces, one can also find the term "gauge" (see for example [HNV04]) which appears to be a copying error.

We have already seen, that every metric induces a uniform structure which in turn induces a topology. We are now interested in conditions under which these implications can be reversed. In other words, we seek conditions for a topological space to be uniformizable and for uniform spaces to be metrizable. We cite the following results from [Wil70] and omit the proofs.

Recall from Definition A.14 that a completely regular topological space is a topological space where points can be separated by continuous functions. As it turns out, the uniformizable topological spaces are exactly those that are completely regular.

Proposition 2.19 (Completely regular spaces are uniformizable)**.** *Let $(S, \mathcal{T})$ be a topological space. The topology $\mathcal{T}$ is uniformizable if and only if $(S, \mathcal{T})$ is completely regular.*

Proof. See [Wil70, Theorem 38.2]. ■

Next, we turn to the question which uniformities can be derived from a metric.

Proposition 2.20 (Metrizable uniform spaces)**.** *Let $(S, \mathcal{U})$ be a uniform space. Then the uniformity is pseudometrizable if and only if $\mathcal{U}$ has a countable base. Furthermore, $\mathcal{U}$ is metrizable if and only if $\mathcal{U}$ has a countable base and is Hausdorff.*

Proof. See [Wil70, Theorem 38.3 & Corollary 38.4]. ■

We will say that a uniform space $(S, \mathcal{U})$ is metrizable if the uniformity $\mathcal{U}$ is metrizable. It is important to observe that metrizability of the topology induced by $\mathcal{U}$ does not imply metrizability of $\mathcal{U}$ itself. For a pathological counterexample refer to [Wil70, Example 38.5].

2.4 Covering uniformities

Next, we give a brief introduction to covering uniformities. This is another, and equivalent, way to define a uniform structure and we will need this construction in the proof of Theorem 3.16. Some authors, notably John W. Tukey [Tuk40] and John R. Isbell [Isb64] advocate this approach. Isbell summarizes his opinion

> However, Weil's original axiomatization [via pseudometrics] is not at all convenient and was soon succeeded by two other versions: the orthodox (Bourbaki) [via diagonal uniformities] and the heretical (Tukey) [via uniform coverings]. The present author is a notorious heretic, and here advances the claim that in this book each system is used where it is most convenient, with the result that Tukey's system of uniform coverings is used nine-tenths of the time.[1]

Let $S \neq \emptyset$ be a non-empty set. Recall that a cover of S is a family $\mathcal{A} = \{A \subset S\}$ such that $S = \bigcup_{A \in \mathcal{A}} A$. Given a cover $\mathcal{A}$ of S and some $C \subset S$, the *star* of C with respect to $\mathcal{A}$ is the family

$$\operatorname{St}(C, \mathcal{A}) = \bigcup_{A \in \mathcal{A}\colon A \cap C \neq \emptyset} A. \tag{2.21}$$

Before we can define what a uniform covers or a covering uniformity is, we need a bit of vocabulary.

Definition 2.21 (Refinements)**.** Let $\mathcal{A}, \mathcal{B}$ be two covers of S. We say that

(i) $\mathcal{A}$ *refines* $\mathcal{B}$, $\mathcal{A} < \mathcal{B}$, if for each $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $A \subset B$.

(ii) $\mathcal{A}$ *star-refines* $\mathcal{B}$, $\mathcal{A} <\!\!* \mathcal{B}$, if for each $A \in \mathcal{A}$ there exists some $B \in \mathcal{B}$ such that $\operatorname{St}(A, \mathcal{A}) \subset B$.

(iii) $\mathcal{A}$ is a *barycentric refinements* of $\mathcal{B}$, $\mathcal{A} \sqsubset \mathcal{B}$, if the family of sets of the form $\{ \operatorname{St}(\{x\}, \mathcal{A}) \mid x \in S \}$ refines $\mathcal{B}$. ◊

Lemma 2.22 (Barycentric refinements of barycentric refinements are star refinements)**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be covers of S and assume that $\mathcal{A} \sqsubset \mathcal{B} \sqsubset \mathcal{C}$. Then $\mathcal{A} <\!\!* \mathcal{C}$.*

Proof. Let $A \in \mathcal{A}$. Since $\mathcal{A} \sqsubset \mathcal{B}$, there exists for each $x \in A$ a $B_x \in \mathcal{B}$ such that $\operatorname{St}(\{x\}, \mathcal{A}) \subset B_x$. By construction, we have $\operatorname{St}(A, \mathcal{A}) \subset \bigcup_{x \in A} B_x$ and $A \subset \bigcap_{x \in A} B_x$ which implies $\operatorname{St}(A, \mathcal{A}) \subset \operatorname{St}(\{x\}, \mathcal{B})$ for each $x \in A$. By assumption $\mathcal{B} \sqsubset \mathcal{C}$ and consequently there exists a $x \in A$ and $C \in \mathcal{C}$ such that $\operatorname{St}(A, \mathcal{A}) \subset \operatorname{St}(\{x\}, \mathcal{B}) \subset C$ and hence $\mathcal{A} <\!\!* \mathcal{C}$. ■

[1] [Isb64, p. v]

In uniform spaces, certain coverings play a special role.

Definition 2.23 (Uniform covers)**.** Let $(S, \mathcal{U})$ be a uniform space. A covering $\mathcal{A}$ of S is called a *uniform cover* if it is refined by a cover of the form

$$\mathcal{A}_U = \{ U[x] \mid x \in S \} \tag{2.22}$$

for some $U \in \mathcal{U}$. ◇

Proposition 2.24 (Properties of the family of uniform covers)**.** *Let $(S, \mathcal{U})$ be a uniform space and denote by μ the family of all uniform covers of S. Then the following hold*

(C1) If $\mathcal{A}_1, \mathcal{A}_2 \in \mu$ are uniform coverings then there exists another uniform covering $\mathcal{A}_3 \in \mu$ such that $\mathcal{A}_3 \lessdot\!\ast\, \mathcal{A}_1$ and $\mathcal{A}_3 \lessdot\!\ast\, \mathcal{A}_2$.

(C2) If $\mathcal{A} \in \mu$ and $\mathcal{A} < \mathcal{A}'$ for some covering $\mathcal{A}'$ of S, then $\mathcal{A}' \in \mu$.

Proof. Let $\mathcal{A} \in \mu$ be a uniform cover of S. Then there exists a $U \in \mathcal{U}$ such that $\mathcal{A}$ is refined by $\mathcal{A}_U := \{ U[x] \mid x \in S \}$. Choose $V \in \mathcal{U}$ such that $V \circ V \subset U$ and let $\mathcal{B} = \{ V[x] \mid x \in S \}$. For each $x \in S$ we have $\mathrm{St}(\{x\}, \mathcal{B}) \subset U[x]$ because each $V[y]$ for which $x \in V[y]$ is contained in $(V \circ V)[x] \subset U[x]$. Hence $\mathcal{B} \sqsubset \mathcal{A}$ and by Lemma 2.22 there exists another uniform cover $\mathcal{C}$ such that $\mathcal{C} \lessdot\!\ast\, \mathcal{A}$. What is left to show is that $\mathcal{A}_1, \mathcal{A}_2 \in \mu$ possess a common barycentric refinement. Without loss of generality, assume that let $U_1, U_2 \in \mathcal{U}$ are entourages that induce $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Now choose an open symmetric entourage $U \in \mathcal{U}$ such that $U \circ U \subset U_1 \cap U_2$ and denote the uniform cover induced by U by $\mathcal{A}$, then $\mathrm{St}(\{x\}, \mathcal{A}) \subset U_1[x] \cap U_2[x]$ and thus $\mathcal{A}$ is a barycentric refinement of both $\mathcal{A}_1$ and $\mathcal{A}_2$, which proves (C1).

The second assertion follows immediately from the definition of uniform covers. ■

The converse of Proposition 2.24 holds true, too.

Theorem 2.25 (Uniform covers induce uniformity)**.** *Let μ be a family of covers of the set S satisfying (C1) and (C2) of Proposition 2.24. Then the family*

$$\mathcal{V} := \left\{ \bigcup_{A \in \mathcal{A}} A \times A \;\middle|\; \mathcal{A} \in \mu \right\} \tag{2.23}$$

forms a base of a diagonal uniformity $\mathcal{U}$ and the collection of all uniform covers induced by $\mathcal{U}$ is μ.

Proof. Let $U \in \mathcal{V}$, i.e. there exists a cover $\mathcal{A} \in \mu$ such that $U = \bigcup_{A \in \mathcal{A}} A \times A$. We check the axioms (U1) to (U4) of Definition 2.1 one by one. Since $\mathcal{A}$ is a cover of S we readily get $\Delta \subset U$ and thus (U1). By construction, the elements of $\mathcal{V}$ are symmetric, implying (U4). Now assume $V \in \mathcal{V}$ is another element and $\mathcal{B} \in \mu$ is such that $V = \bigcup_{B \in \mathcal{B}} B \times B$. Then,

$$U \cap V = \bigcup_{A \in \mathcal{A}} A \times A \cap \bigcup_{B \in \mathcal{B}} B \times B = \bigcup_{A \in \mathcal{A}} \bigcup_{B \in \mathcal{B}} (A \cap B) \times (A \cap B) \tag{2.24}$$

and the family $C := \{ A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B} \}$ is a cover of S. By (C1) there exists a star-refinement $C' \in \mu$ of both $\mathcal{A}$ and $\mathcal{B}$. By construction, C' refines the cover C and thus $C \in \mu$ by virtue of (C2), which in turn implies $U \cap V \in \mathcal{V}$ and hence (U2). Finally, (U3) follows immediately when we choose $\mathcal{B}$ to be a star-refinement of $\mathcal{A}$ and U, V defined as before.

Next, we need to show that the uniform covers with respect to the uniformity $\mathcal{U}$ generated by $\mathcal{V}$ is just μ. It suffices to show that each $\mathcal{A} \in \mu$ is a uniform cover with respect to $\mathcal{U}$.

Let $\mathcal{A} \in \mu$ and $U \in \mathcal{U}$, as before, the entourage generated by $\mathcal{A}$. Choose an entourage $V \in \mathcal{U}$ such that $V \circ V \subset U$ then the cover $\{ V[x] \mid x \in S \}$ refines $\mathcal{A}$ and by definition $\mathcal{A}$ is a uniform cover with respect to $\mathcal{U}$. ∎

We call a family μ of covers of S satisfying (C1) and (C2) a *covering uniformity*. If μ satisfies only (C1), we say that μ is a base for a uniform covering.

The connection between covering uniformities and families of pseudometrics on S is straight-forward.

Lemma 2.26 (Pseudometrics and covering uniformities)**.** *Let $\mathbb{I}$ be a non empty index set and $(\rho_i)_{i \in \mathbb{I}}$ a family of pseudometrics on S satisfying (i) of Proposition 2.16. Then the family μ consisting of all covers of S of the form*

$$\mathcal{A}_{i,\varepsilon} := \{ B_i(x, \varepsilon) = \{ y \in S \mid \rho(x, y) < \varepsilon \} \mid x \in S \} \quad i \in \mathbb{I},\ \varepsilon > 0 \tag{2.25}$$

is a base for a covering uniformity of S.

Proof. We only need to show that μ satisfies condition (C1). Let $\varepsilon, \delta > 0$ and $i, j \in \mathbb{I}$. We need to find a star-refinement of both $\mathcal{A}_{i,\varepsilon}$ and $\mathcal{A}_{j,\delta}$. It follows from condition (i) of Proposition 2.16 that there exists an index $k \in \mathbb{I}$ such that $\rho_k \geq \max\{\rho_i, \rho_j\}$ which means $B_k(x, \varepsilon) \subset B_i(x, \varepsilon)$ and $B_k(x, \delta) \subset B_j(x, \delta)$ for all $x \in S$. That implies that $\mathcal{A}_{k,(\varepsilon \wedge \delta)/4}$ star-refines both $\mathcal{A}_{i,\varepsilon}$ and $\mathcal{A}_{j,\delta}$. ∎

Remark 2.27. Let $D \subset S$ be a dense subset of S. Then the conclusion of the last lemma still holds if we replace μ by the family of covers consisting of the sets

$$\mathcal{A}'_{i,\varepsilon} := \{ B_i(x,\varepsilon) = \{ y \in S \mid \rho(x,y) < \varepsilon \} \mid x \in D \} \quad i \in \mathbb{I},\ \varepsilon > 0, \tag{2.26}$$

as these are clearly covers of S and the same proof as before applies. ◊

It comes as no surprise, that uniform continuity can be defined in terms of covering uniformities equally well.

Proposition 2.28 (Uniform continuity [Wil70, Theorem 36.8])**.** *Let $(S, \mathcal{U})$ and $(T, \mathcal{V})$ be uniform Hausdorff spaces and denote by μ and ν the families of uniform covers of S and T, respectively. A function $f : S \to T$ is uniformly continuous if and only if any of the following two equivalent conditions is satisfied.*

(i) For each uniform cover $\mathcal{B} \in \nu$ of T there exists a uniform cover $\mathcal{A} \in \mu$ of S such that $f(\mathcal{A}) < \mathcal{B}$, where $f(\mathcal{A}) = \{ f(A) \mid A \in \mathcal{A} \}$.

(ii) For each uniform cover $\mathcal{B} \in \nu$ of T, the family $f^{-1}\mathcal{B} := \left\{ f^{-1}B \,\middle|\, B \in \mathcal{B} \right\}$ is a uniform cover of S.

Proof. First observe that the equivalence of the two conditions is an immediate consequence of Proposition 2.24. Now assume that $f : S \to T$ is uniformly continuous and let $\mathcal{B} \in \nu$ be a uniform cover of T. Then there exists an entourage $V \in \mathcal{V}$ such that $\mathcal{B}_V := \{ V[y] \mid y \in T \}$ is a refinement of $\mathcal{B}$. By uniform continuity, there exists a $U \in \mathcal{U}$ such that $(f(x), f(x')) \in V$ whenever $(x, x') \in U$ and hence, $f(\mathcal{A}_U) < \mathcal{B}_V < \mathcal{B}$.

Conversely, suppose that conditions (i) and (ii) hold and fix $V \in \mathcal{V}$. For $\mathcal{B} \in \nu$, write

$$V_{\mathcal{B}} := \bigcup_{B \in \mathcal{B}} B \times B, \tag{2.27}$$

and compare this to (2.23) to deduce that $V_{\mathcal{B}} \in \mathcal{V}$ for all $B \in \nu$. Then there exists a uniform cover $\mathcal{B} \in \nu$ such that $V_{\mathcal{B}} \subset V$ and by assumption a uniform cover $\mathcal{A} \in \mu$ of S such that $f(\mathcal{A}) < \mathcal{B}$. Hence, $(x, x') \in U_{\mathcal{A}}$ implies that $(f(x), f(x')) \in V_{\mathcal{B}} \subset V$. And since $V \in \mathcal{V}$ was arbitrary this proves uniform continuity of f, as claimed. ∎

The next result is well known for metric metric spaces from any introductory calculus course. We nevertheless prove it here for uniform spaces as the proof is rather instructive.

Lemma 2.29 (Continuous functions on compacta are uniformly continuous)**.** *Let $(S, \mathcal{U})$ and $(T, \mathcal{V})$ be uniform Hausdorff spaces and assume that S is compact. Then, every continuous function $f : S \to T$ is already uniformly continuous.*

Proof. Let $V \in \mathcal{V}$ and choose $V' \in \mathcal{V}$ open and symmetric such that $V' \circ V' \subset V$. Consider the open (uniform) cover $\mathcal{B}^f_{V'} \{ V'[f(x)] \mid x \in S \}$ of $f(S)$. By continuity of f, the family

$$\mathcal{A} = \left\{ f^{-1}V'[f(x)] \,\middle|\, x \in S \right\} \tag{2.28}$$

is an open cover of S. By definition of the uniform topology, there exist open entourages U_x such that

$$(U_x \circ U_x)[x] \subset f^{-1}V'[f(x)] \tag{2.29}$$

for all $x \in S$. By compactness, there exist finitely many $x_1, \ldots, x_N \in S$ such that the family $\left\{ U_{x_j}[x_j] \,\middle|\, j = 1, \ldots, N \right\}$ is an open cover of S. By definition of a uniformity, we obtain

$$U := \bigcap_{j=1}^{N} U_{x_j} \in \mathcal{U} \tag{2.30}$$

and furthermore, U is open.

Assume that $(y, z) \in U$. By construction, there exists a $j \in \{1, \ldots, N\}$ such that $(x_j, y) \in U_{x_j}$ and hence

$$\{y, z\} \subset (U_{x_j} \circ U)[x_j] \subset f^{-1}V'[f(x_j)]. \tag{2.31}$$

As a consequence, $\{f(y), f(z)\} \in V'[f(x_j)]$ and by a similar argument as before we finally obtain $(f(x), f(y)) \in V$ for all $(x, y) \in U$. ■

Much more can be said about covering uniformities and we refer the interested reader to Isbell's book [Isb64] for an in depth treatment of covering uniformities. The take-away from this section is that covering uniformities offer a different view of the uniform structure of a space.

2.5 Further properties of uniform spaces

We described in the first section of this chapter how uniform spaces are halfway between topological spaces and metric spaces with respect to their structure. Many structural properties known in metric spaces can be generalized to uniform spaces by exchanging quantitative statements for qualitative statements.

In this section, we explain how the notion of metric measure spaces can be extended to uniform measure spaces which will serve as the state spaces for the processes that are the focus of this research.

Definition 2.30 (totally bounded sets)**.** Let $(S, \mathcal{U})$ be a uniform space. A subset $A \subset S$ is *totally bounded* if for every open entourage $U \in \mathcal{U}$ there exists a finite collection of points $\{ x_i \in A \mid 1 \leq i \leq n \}$ in A such that

$$A \subset \bigcup_{i=1}^{n} U[x_i]. \qquad \diamond$$

Lemma 2.31 (A condition for totally boundedness)**.** *Let $(S, \mathcal{U})$ be a uniform space and $D \subset S$ a subset. Assume that for each $U \in \mathcal{U}$ there exists a totally bounded set $A \subset S$ such that*

$$D \subset \bigcup_{x \in A} U[x]. \tag{2.32}$$

Then D is totally bounded.

Proof. Fix $U \in \mathcal{U}$ open and choose $V, W \in \mathcal{U}$ open such that that $W \circ W \subset V$ and $V \circ V \subset U$. By assumption, there exists a totally bounded $A \subset S$ such that $D \subset \bigcup_{x \in A} W[x]$. Let $x_1, \ldots, x_n \in A$ be such that

$$A \subset \bigcup_{k=1}^{n} W[x_k]. \tag{2.33}$$

Since $\bigcup_{x \in W[x_k]} W[x] \subset V[x_k]$ for all $k = 1, \ldots, n$ it follows that

$$D \subset \bigcup_{k=1}^{n} V[x_k]. \tag{2.34}$$

Without loss of generality assume that for some $N \leqq n$ the points $x_1, \ldots, x_N$ are exactly those x_k for which $V[x_k] \cap D \neq \emptyset$. Now let $y_k \in V[x_k] \cap D$ for $k = 1, \ldots, N$. Then, $V[x_k] \subset U[y_k]$ and consequently

$$D \subset \bigcup_{k=1}^{N} U[y_k]. \tag{2.35}$$

■

Naturally, any uniform spaces $(S, \mathcal{U})$ induces a measurable space $(S, \mathcal{B}(S))$, where $\mathcal{B}(S)$ denotes the Borel σ-algebra generated by the open sets of $(S, \mathcal{U})$.

Recall from Definition A.35 the definition of a Radon measure.

Definition 2.32 (Boundedly finite measures)**.** Let $(S, \mathcal{U})$ be a locally compact uniform Hausdorff space. A Radon measure ν on $(S, \mathcal{B}(S))$ is *boundedly finite*, if $\nu(A) < \infty$ for every totally bounded set $A \subset S$. $\diamond$

Recall the definition of a *net* from Definition A.42: A net is a generalization of a sequence in the sense that we allow arbitrary directed sets $(\mathbb{I}, \succeq)$ as index sets.

The uniform structure allows us to define Cauchy sequences and nets as follows.

Definition 2.33 (Cauchy nets)**.** Let $(S, \mathcal{U})$ be a uniform space. A net $(x_\alpha)_{\alpha\in\mathbb{I}}$ is called a *Cauchy net* if for every open entourage $U \in \mathcal{U}$ there exists a $\alpha_0 \in \mathbb{I}$ such that

$$(x_\beta, x_\gamma) \in U \tag{2.36}$$

whenever $\beta, \gamma \succeq \alpha_0$. ♢

With the definition of Cauchy nets at hand we can introduce the notion of completeness for uniform spaces and define uniform measure spaces.

Definition 2.34 (Complete uniform spaces)**.** A uniform space $(S, \mathcal{U})$ is called complete if every Cauchy net converges. ♢

Next, we make two useful observations about compact sets.

Lemma 2.35 (Heine-Borel)**.** *Let $(S, \mathcal{U})$ be a complete uniform space. Then $A \subset S$ is compact if and only if A is closed and totally bounded.*

Proof. Assume $A \subset S$ is compact. Then A is closed and furthermore for every open entourage $U \in \mathcal{U}$, the covering $\{ U[x] \mid x \in A \}$ of A has a finite subcover, i.e. there exists a collection of points $\{x_1, \ldots, x_n\} \subset A$ such that $A \subset \bigcup_{i=1}^n U[x_i]$. The converse implication follows from [Wil70, Theorem 39.13]. ■

Lemma 2.36 (Totally bounded uniform neighborhoods)**.** *Let $(S, \mathcal{U})$ be a locally compact uniform Hausdorff space. Then there exists for every $x \in S$ an open entourage $U \in \mathcal{U}$ such that $\overline{U[x]}$ is compact.*

Proof. Fix $x \in S$. By local compactness, there exists a compact set $K_x \subset S$ and an open entourage $U \in \mathcal{U}$ such that $U[x] \subset K$. Suppose $\{ B_n \mid n \in \mathbb{N} \}$ is an open cover of the closure $\overline{U[x]}$. Then,

$$\{ B_n \mid n \in \mathbb{N} \} \cup \left\{ \complement \overline{U[x]} \right\} \tag{2.37}$$

is an open cover of K_x. By compactness of K_x, there exists a finite open subcover,

$$\{ B_1, \ldots, B_n \mid n \in \mathbb{N} \} \cup \left\{ \complement \overline{U[x]} \right\} \tag{2.38}$$

of K_x. Since $\complement\overline{U[x]} \cap \overline{U[x]} = \emptyset$, we have found with $\{ B_1, \ldots, B_n \mid n \in \mathbb{N} \}$ an open subcover of $\overline{U[x]}$ which is therefore compact. ■

A similar result holds true for compact subsets of S.

Lemma 2.37. *Let $(S, \mathcal{U})$ be a locally compact uniform Hausdorff space. For each $K \subset S$ compact there exists an open set $A \subset S$ such that $K \subset A \subset S$ and the closure $\overline{A}$ is compact.*

Proof. For each $x \in K$ choose by Lemma 2.36 an open entourage $U^x \in \mathcal{U}$ such that $U^x[x]$ is relatively compact. Take a finite subcover consisting of $x_1, \ldots, x_n \in K$ and $U_1, \ldots, U_n \in \mathcal{U}$ open such that

$$K \subset \bigcup_{i=1}^{n} U_n[x_n] =: A. \tag{2.39}$$

Then A is open and contained in the compact set

$$A \subset \bigcup_{i=1}^{n} \overline{U_n[x_n]} \subset S, \tag{2.40}$$

as claimed. ■

In order to use the classical results from probability theory, we need to make sure that our spaces are separable and complete. We adapt the terminology that is known from the theory of metric spaces and call such spaces Polish.

These assumptions can certainly be weakened to some degree, but it is not within the scope of this thesis to do so.

Definition 2.38 (Polish uniform space)**.** A metrizable uniform space $(S, \mathcal{U})$ is called *Polish uniform space* if is separable and complete. ◇

Some remarks about this definition are in order. First, observe that by Proposition 2.20 Polish uniform spaces are Hausdorff and possess a countable base for the uniformity $\mathcal{U}$. Furthermore, we note the following result for further reference.

Lemma 2.39 (Completely metrizable uniform spaces)**.** *[Wil70, Theorem 39.4] Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base (sc. $\mathcal{U}$ is metrizable). Assume further that $\mathcal{U}$ is complete. Then every metric on S that induces $\mathcal{U}$ is complete.*

This gives rise to the question why to consider Polish uniform spaces at all instead of relying on the well developed theory of metric measure spaces. The main reason is that we want to emphasize that the structural properties of the spaces that are important are those expressed by the uniformity and do not depend on the concrete metric that generates the uniform structure.

Lemma 2.40 (Lindelöf property)**.** *Let* $(S, \mathcal{U})$ *be a Polish uniform space. Then* $(S, \mathcal{U})$ *is Lindelöf.*

Proof. Metrizability and separability together imply the existence of a countable base of the uniform topology. Hence, every Polish uniform space is second countable and therefore, by Lemma A.19, Lindelöf. ■

For further reference, we introduce the analog of metric measure spaces for our uniform setup.

Definition 2.41 (Uniform measure spaces)**.** A *uniform measure space* is a triple $(S, \mathcal{U}, \nu)$, where $(S, \mathcal{U})$ is a Polish uniform space and ν is a σ-finite Radon measure on $(S, \mathcal{B}(S))$, where $\mathcal{B}(S)$ is the Borel σ-algebra, as usual. We write $\mathcal{B}_\nu(S)$ for the completion of $\mathcal{B}(S)$ with respect to ν, i.e.

$$\mathcal{B}_\nu(S) := \sigma\left(\mathcal{B}(S) \cup \{ A \subset N \in \mathcal{B}(S) \mid \nu(N) = 0 \}\right). \tag{2.41}$$

◇

Remark 2.42 (Properties of Radon measures on Polish spaces)**.** Let $A \subset S$ be totally bounded. By Lemma 2.35, $\overline{A}$ is compact and therefore, by Definition A.35,

$$\mu(A) \leq \mu(\overline{A}) < \infty. \tag{2.42}$$

Hence, every Radon measure on a Polish uniform space is boundedly finite. ◇

In the next chapter we will develop the theory of the Skorokhod space of càdlàg functions on a uniform space. We will be as general as the scope of this thesis permits in order to show that the assumptions on uniform measure spaces can be relaxed while still retaining a meaningful theory.

2.5.1 Uniform equicontinuity

In this section, we introduce the notion of equicontinuity of a family of real-valued functions on a uniform Hausdorff space and present a version of the celebrated Arzelà-Ascoli theorem that will be central to the proof of Theorem 6.2. The proof presented here is based on the proof of [DS58, Theorem IV.6.5.7].

Let S, T be two non empty sets, we denote by $\mathcal{F}(S;T) = \{f\colon S \to T\}$ the family of all maps f from S to T.

Lemma 2.43 (Uniformity of uniform convergence)**.** *Let $S \neq \emptyset$ be a set and $(T, \mathcal{V})$ a uniform Hausdorff space. Assume that $\mathcal{V}' \subset \mathcal{V}$ is a base of $\mathcal{V}$. Then the family of subsets of $\mathcal{F}(S;T)^2$ of the form*

$$\left\{ (f,g) \in \mathcal{F}(S;T)^2 \;\middle|\; (f(x), g(x)) \in V,\ \forall x \in S \right\}, \quad V \in \mathcal{V}' \tag{2.43}$$

is a base of a uniformity on $\mathcal{F}(S;T)$ which does not depend on the choice of the base $\mathcal{V}'$.

Proof. Let $\mathcal{V}' \subset \mathcal{V}$ be a base of $\mathcal{V}$ and assume that $\mathcal{W}'$ is the system of subsets of $\mathcal{F}(S;T)^2$ induced by $\mathcal{V}'$ as described in (2.43). We first show that $\mathcal{W}'$ is indeed a base for a uniformity on $\mathcal{F}(S;T)$. Clearly, $\Delta \subset W$ for all $W \in \mathcal{W}'$. The remaining properties of a base (U2) to (U4) follow readily from the analogous properties of $\mathcal{V}'$. Now Let $\mathcal{V}', \mathcal{V}'' \subset \mathcal{V}$ be two bases of the uniformity $\mathcal{V}$ and $\mathcal{W}', \mathcal{W}''$ the families of entourages defined as in (2.43). Let $W' \in \mathcal{W}'$, then there exists a $V' \in \mathcal{V}'$ such that $(f(x), g(x)) \in V'$ for all $(f,g) \in W'$ and $x \in S$. Since $\mathcal{V}'$ and $\mathcal{V}''$ were assumed to be bases of the same uniformity, there exists a $V'' \in \mathcal{V}''$ such that $V'' \subset V'$. Let $W'' = \left\{ (f,g) \in \mathcal{F}(S;T)^2 \;\middle|\; (f(x), g(x)) \in V'',\ \forall x \in S \right\}$, then $W'' \subset W'$ and we can deduce that $\mathcal{W}'$ is contained in the uniformity generated by $\mathcal{W}''$. By symmetry, we obtain the converse inclusion and ultimately the identity of the uniformities generated by $\mathcal{W}'$ and $\mathcal{W}''$, respectively. ■

Let $(T, \mathcal{V})$ be a uniform Hausdorff space and recall that a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(S;T)$ converges uniformly to a limit $f \in \mathcal{F}(S;T)$ if and only if for all $V \in \mathcal{V}$ there exists a $n_0 \in \mathbb{N}$ such that

$$\{ (f_n(x), f(x)) \mid x \in S \} \subset V \tag{2.44}$$

for all $n \geq n_0$.

We call the uniformity $\mathcal{W}$ on $C(S;T)$ as described in Lemma 2.43 the *uniformity of uniform convergence* for it induces the usual *topology of uniform convergence* .

Observe that both the uniformity of uniform convergences and the topology of uniform convergence fundamentally depend on the uniform structure of T.

More details on the uniformity of uniform convergence can be found in [Bou66a, Chapter X.1]. We collect some of the results in the following remarks.

Remarks 2.44. Let S be some set and $(T, \mathcal{V})$ a uniform Hausdorff space. We equip the space $\mathcal{F}(S;T)$ with the uniformity of uniform convergence which we denote by $\mathcal{W}$.

(i) Let $\mathcal{A}$ be some family of subsets of S. We can equip $\mathcal{F}(S;T)$ with the coarsest uniformity that makes the restrictions maps $f \mapsto f|_A$, $A \in \mathcal{A}$ uniformly continuous with respect to the uniformity of uniform convergence on $\mathcal{F}(A;T)$. This uniformity is called *uniformity of uniform convergence on the sets of* $\mathcal{A}$. One example is the uniformity of uniform convergence on compacta which is obtained by taking S to be a topological space and $\mathcal{A}$ to be the family of compact sets (cf. [Bou66a, Definition 2 X.1.2]).

(ii) The space $C(S;T)$ is a closed subset of the space $\mathcal{F}(S;T)$ equipped with the topology of uniform convergence (cf. [Bou66a, Theorem 2 X.1.6]). In particular, uniform limits of continuous functions are again continuous.

(iii) If T is complete, then so is $\mathcal{F}(S;T)$ equipped with the uniformity of uniform convergence (cf. [Bou66a, Theorem 1 X.1.5]). ♢

Next, we introduce the notion of uniform equicontinuity for a family of continuous functions and extend this definition to continuous functions that are defined on (possibly different) subsets of a common uniform space $(S, \mathcal{U})$.

Definition 2.45 (Uniform equicontinuity)**.** Let $(S, \mathcal{U})$ and $(T, \mathcal{V})$ be uniform Hausdorff spaces and $F \subset C(S;T)$ a family of continuous functions. We say that F is uniformly equicontinuous if for all $V \in \mathcal{V}$ open there exists a $U \in \mathcal{U}$ open such that

$$\bigcup_{f \in F} \{ (f(x), f(y)) \mid (x, y) \in U \} \subset V. \tag{2.45}$$

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions with each f_n defined on a subset $S_n \subset S$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous if for all open $V \in \mathcal{V}$ there exists an open $U \in \mathcal{U}$ such that

$$\bigcup_{n \in \mathbb{N}} \{ (f_n(x), f_n(y)) \mid (x, y) \in U \cap S_n \times S_n \} \subset V. \tag{2.46}$$

♢

We can now state and proof a version of the Arzelà-Ascoli theorem for uniform spaces.

Theorem 2.46 (Arzelà-Ascoli)**.** *Let $(S, \mathcal{U})$ and $(T, \mathcal{V})$ be uniform Hausdorff spaces and assume that $(S, \mathcal{U})$ is compact and $(T, \mathcal{V})$ is complete. A family $K \subset C(S;T)$ is relatively compact if and only if K is uniformly equicontinuous and the set*

$$\bigcup_{f \in K} f(S) \subset T \tag{2.47}$$

is relatively compact.

Proof. First assume that $K \subset C(S;T)$ is relatively compact and recall the definition of the uniformity $\mathcal{W}$ from Lemma 2.43. Let $V \in \mathcal{V}$ and choose $V' \in \mathcal{V}$ open and symmetric such that $V' \circ V' \subset V$, as usual. Fix $f \in K$, by compactness of S and continuity of f, we deduce that $f(S) \subset T$ is compact and hence there exist $x_1, \ldots, x_M \in S$ for some $M \in \mathbb{N}$ such that

$$f(S) \subset \bigcup_{i=1}^{M} V'[f(x_i)]. \tag{2.48}$$

Now choose $W \in \mathcal{W}$ open such that

$$W \subset \left\{ (f,g) \in \mathcal{F}(S;T)^2 \,\middle|\, (f(x), g(x)) \in V,\ \forall x \in S \right\}, \tag{2.49}$$

where $\mathcal{W}$ denotes the uniformity on $\mathcal{F}(S;T)$ as defined in Lemma 2.43. By relative compactness and Lemma 2.35, K is totally bounded and we can find $f_1, \ldots, f_N \in K$ for some $N \in \mathbb{N}$ such that

$$K \subset \bigcup_{j=1}^{N} W[f_j]. \tag{2.50}$$

For each $j = 1, \ldots, N$ let $x_1^{(j)}, \ldots, s_{M_j}^{(j)} \in S$, $M_j \in \mathbb{N}$ be a finite family of points such that (2.48) is satisfied for f_j. By construction we have

$$\bigcup_{f \in K} f(S) \subset \bigcup_{j=1}^{N} \bigcup_{i=1}^{M_j} (V' \circ V')[f_j(x_i^{(j)})] \subset \bigcup_{j=1}^{N} \bigcup_{i=1}^{M_j} V[f_j(x_i^{(j)})] \tag{2.51}$$

and since $V \in \mathcal{V}$ open was arbitrary, we have that $\bigcup_{f \in K} f(S)$ is totally bounded and hence relatively compact by Lemma 2.35.

Observe that by Lemma 2.29 each $f \in K$ is uniformly continuous. We continue to show that K is actually uniformly equicontinuous. Fix $V \in \mathcal{V}$ and choose $V' \in \mathcal{V}$, $W \in \mathcal{W}$ and $f_1, \ldots, f_N \in K$ as before with the only difference that we assume that $V' \circ V' \circ V' \circ V' \subset V$. By uniform continuity, there exist open entourages $U_1, \ldots, U_N \in \mathcal{U}$ such that

$$(f_j(x), f_j(y)) \in V' \tag{2.52}$$

for all $(x, y) \in U_j$ and $j = 1, \ldots, N$. We can take the intersection $U := \bigcap_{j=1}^{N} U_j$ of these entourages to obtain another open entourage $U \in \mathcal{U}$. By construction, we find for each $f \in K$ a $j \in \{1, \ldots, N\}$ such that $(f(x), f_j(x)) \in V'$ for all $x \in S$. In combination with (2.52) we obtain

$$(f(x), f(y)) \in V' \circ V' \circ V' \subset V \tag{2.53}$$

for all $(x, y) \in U$ and $f \in K$.

For the converse implication recall that by completeness of T, $C(S;T)$ is complete, too. It therefore suffices by Lemma 2.35 to show that K is totally bounded. To that end fix some $W \in \mathcal{W}$ open. By definition of $\mathcal{W}$ there exists a $V \in \mathcal{V}$ open such that

$$W' := \{ (f,g) \in \mathcal{F}(S;T) \mid (f(x), g(x)) \in V, \ \forall x \in S \} \subset W. \tag{2.54}$$

As before, we choose $V' \in \mathcal{V}$ open with the property that $V' \circ V' \circ V' \circ V' \subset V$. By uniform equicontinuity, there exists an open $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V'$ for all $(x,y) \in U$ and $f \in K$. Since S and $\bigcup_{f \in K} f(S)$ are totally bounded by assumption, we find finitely many $x_1, \dots, x_N \in S$ and $f_1, \dots, f_M \in K$ such that $\left\{ U[x_j] \,\middle|\, j = 1, \dots, N \right\}$ is an open cover of S and $\left\{ V'[f_i(x_j)] \,\middle|\, i = 1, \dots M, \ j = 1, \dots, N \right\}$ is an open cover of $\bigcup_{f \in K} f(S)$, i.e.

$$\bigcup_{f \in K} f(S) \subset \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} V'[f_i(x_j)]. \tag{2.55}$$

We claim that the family $\{ W[f_i] \mid i = 1, \dots, M \}$ is a finite open cover of K. Suppose this was not the case, then there exists a $f \in K$ such that $f \notin W[f_i]$ for all $i = 1, \dots, M$. This means, for each $i \in \{1, \dots, M\}$ there exists some $x \in S$ such that $(f(x), f_i(x)) \notin V'$. By (2.55) we can choose $i \in \{1, \dots, M\}$ such that $(f(y), f_i(y)) \in V$ for some $y \in S$. For convenience write $g := f_i$. By compactness of S we can find $(x, y) \in U$ such that $(f(x), g(x)) \notin V'$ but $(f(y), g(y)) \in V'$. By construction of U and uniform equicontinuity we obtain $(f(x), f(y)), (g(x), g(y)) \in V'$ and hence

$$(f(x), g(x)) \in V' \circ V' \circ V' \subset V, \tag{2.56}$$

in contradiction to the assumption. ∎

We present another formulation of the Arzelà-Ascoli theorem that is specifically tailored to our needs in Chapter 6. This formulation is due to [ALW17, Lemma 5.4] and a similar version can be found in [Cro18, Lemma 5.3]. Although the proof is very similar to the proof of Theorem 2.46, we give a detailed proof as both papers omit a proof and there are a few subtleties that require careful treatment.

Lemma 2.47 (Arzelà-Ascoli)**.** *Let $(S, \mathcal{U})$ and $(T, \mathcal{V})$ be uniform Hausdorff spaces with countable bases and assume that $(S, \mathcal{U})$ is compact and that $(T, \mathcal{V})$ is complete. Assume further that there are non-empty closed subsets $S_n \subset S$ for each $n \in \mathbb{N} \cup \{\infty\}$ and a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions such that $f_n \in C(S_n; T)$ and the sequence $(f_n)_{n \in \mathbb{N}}$ is uniformly equicontinuous. Suppose for each $x \in S_\infty$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset S$ with $x_n \in S_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$ with the property that $\{ f_n(x_n) \mid n \in \mathbb{N} \}$ is relatively compact in T.*

Then there exists a continuous function $f \in C(S_\infty; T)$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for all $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ with the property

$$\{ (f(x), f_{n_k}(y)) \,|\, (x, y) \in U \cap S_\infty \times S_{n_k} \} \subset V, \quad \forall k \in \mathbb{N}. \tag{2.57}$$

Proof. Let $V \in \mathcal{V}$ be open. By uniform equicontinuity, there exists an open entourage $U \in \mathcal{U}$ such that for all $n \in \mathbb{N}$

$$f_n(U) := \left\{ (f_n(x), f_n(y)) \,\middle|\, (x, y) \in U \cap S_n^2 \right\} \subset V. \tag{2.58}$$

Choose $U' \in \mathcal{U}$ open such that $U' \circ U' \subset U$. As S_∞ is a closed subset of a compact space, it is itself totally bounded and we can find finitely many $x^1, \ldots, x^N \in S_\infty$ such that

$$S_\infty \subset \bigcup_{j=1}^{N} U'[x^j]. \tag{2.59}$$

Furthermore, the x^j can be chosen in a way that

$$x^k \in U'[x^j] \;\Leftrightarrow\; k = j. \tag{2.60}$$

For each $j \in \{1, \ldots, N\}$ let $\left(x_n^j\right)_{n\in\mathbb{N}} \subset S$ be a sequence with $x_n^j \in S_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n^j = x^j$. By assumption such sequences exist and furthermore we can choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f_{n_k}(x_{n_k}^j) = z^j \in T, \quad \forall j \in \{1, \ldots, N\}. \tag{2.61}$$

For $x \in S_\infty$ set $\alpha(x) := \min\left\{ j = 1, \ldots, N \,\middle|\, x \in U'[x^j] \right\}$ and define $h_V : S_\infty \to T$ by setting

$$h_V(x) := z^{\alpha(x)}. \tag{2.62}$$

Observe that $h_V(x^j) = z^j$ for all $j \in \{1, \ldots, N\}$. It is worth noting that if T is path connected, h_V can be chosen to be continuous.

Let $(V_l)_{l\in\mathbb{N}} \subset \mathcal{V}$ be a sequence of open entourages such that

$$V_{l+1} \circ V_{l+1} \subset V_l. \tag{2.63}$$

For each $l \in \mathbb{N}$ define $h_l = h_{V_l} : S_\infty \to T$ as above but choose the sequence $(f_n)_{n\in\mathbb{N}}$ in the definition of h_{l+1} as a subsequence of that in the definition of h_l. We claim that the sequence $(h_l)_{l\in\mathbb{N}}$ is Cauchy with respect to the uniformity of uniform convergence $\mathcal{W}$ on $\mathcal{F}(S_\infty; T)$. To see this, take any $W \in \mathcal{W}$ open. By definition, there exists a $l_0 \in \mathbb{N}$ such that for $f, g \in \mathcal{F}(S_\infty; T)$,

$$\{ (f(x), g(x)) \mid x \in S_\infty \} \subset V_{l_0} \tag{2.64}$$

implies that $(f, g) \in W$. Now take $l_0 \leq k < l$ and fix $x \in S_\infty$. Choose $U_k, U_l \in \mathcal{U}$ open such that $f_n(U_k) \subset V_k$ and $f_n(U_l) \subset V_l$. As before take $U'_k, U'_l \in \mathcal{U}$ open such that $U'_k \circ U'_k \subset U_k$ and $U'_l \circ U'_l \subset U_l$, respectively. Denote by $y^k := x^{\alpha(x)}$ the x^j from the definition of h_k which determines the value of h_k at x, i.e. $h_k(x) = h_k(y^k)$ and in the same manner define $y^l \in S_\infty$ for h_l. By construction, we have $x \in U'_k[y^k] \cap U'_l[y^l]$ and thus $(y^k, y^l) \in U_k$, as $U_l \subset U_k$. Furthermore denote the sequences from the definition of h_k and h_l converging to y^k and y^l by $\left(y^k_n\right)_{n\in\mathbb{N}}$ and $\left(y^l_n\right)_{n\in\mathbb{N}}$ respectively. Observe that we can indeed take the same subsequences by construction. As U_k is open, we deduce that $(y^k_n, y^l_n) \in U_k$, eventually. Hence,

$$\left(f_n\left(y^k_n\right), f_n\left(y^l_n\right)\right) \in V_k \subset V_{l_0} \tag{2.65}$$

for all $n \in \mathbb{N}$ sufficiently large and consequently $(h_k(x), h_l(x)) \in V_{l_0}$. Since $x \in S_\infty$ was arbitrary we conclude that $(h_l, h_k) \in W$ which proves the claim that $(h_l)_{l\in\mathbb{N}}$ is Cauchy. By completeness of T together with Remarks 2.44 (iii) we have convergence of the sequence $(h_l)_{l\in\mathbb{N}}$ and we denote the limit by f.

It remains to show that f is continuous and satisfies (2.57). To that end take $V \in \mathcal{V}$ open and choose $V' \in \mathcal{V}$ open with $V' \circ V' \subset V$. Then there exists a $l_0 \in \mathbb{N}$ such that $(f(x), h_l(x)) \in V'$ for all $l \geq l_0$. By construction, there exists a $l_1 \in \mathbb{N}$ and a $U \in \mathcal{U}$ open such that $h_l(U) \subset V'$ for all $l \geq l_1$. Consequently, $f(U) \subset V$ and we have that f is even uniformly continuous. Finally, (2.57) holds by construction of f. ■

2.6 Proximity spaces

For further reference, we introduce the notion of proximities and show that proximity spaces are in a one-to-one relation with uniform spaces. The main source for this section is [Wil70, Chapter 40] where further details can be found. Willard traces the notion of proximities back to Frigyes Riesz (1908) [Rie08] and mentions works on proximity spaces by Alexander D. Wallace [Wal41], Vadim A. Efremovič [Efr52] and Yuri M. Smirnov [Smi52].

Definition 2.48 (Proximity spaces)**.** Let $S \neq \emptyset$ be a set. We call a binary relation $\bowtie$ on $\mathcal{P}(S)$ a *proximity (relation)* if for all subsets $A, B, C \subset X$ it holds

(P1) $\emptyset \not\bowtie A$ for all $A \subset S$

(P2) $\{x\} \bowtie \{x\}$ for all $x \in S$,

(P3) $A \bowtie B$ implies $B \bowtie A$,

(P4) $A \bowtie (B \cup C)$ if and only if $A \bowtie B$ or $A \bowtie C$,

(P5) if $A \not\bowtie B$ then there exist $E, F \subset S$ such that $E \cap F = \emptyset$ and $A \not\bowtie E^c$ and $B \not\bowtie F^c$.

If $\bowtie$ is a proximity relation on $\mathcal{P}(S)$, we call the pair $(S, \bowtie)$ a *proximity space* and we say that $A, B \subset S$ are close (or $\bowtie$-close) if $A \bowtie B$. If in addition

(P6) $\{x\} \bowtie \{y\}$ implies $x = y$,

we say that the proximity space $(S, \bowtie)$ is *separated* or that the proximity $\bowtie$ is *separating*. ◇

Examples 2.49 (Proximities). (i) For any set S and subsets $A, B \subset S$ we can define a proximity by $A \bowtie B$ if and only if $A \cap B \neq \emptyset$. It is easy to check, that this indeed defines a proximity and this proximity is called the discrete proximity and it is separating.

(ii) If (S, d) is a metric space we set $A \bowtie B$ if and only if

$$d(A, B) = \inf \{ d(x, y) \mid x \in A, y \in B \} = 0 \tag{2.66}$$

for $A, B \subset S$. Again, it is straightforward to check that this defines a separating proximity. ■

In the sequel, we omit the braces around singletons and simply write $x \bowtie y$ or $x \bowtie A$.

First, lets observe some simple facts.

Lemma 2.50 (Properties of proximities). *Let $(S, \bowtie)$ be a proximity space and $A, B, C \subset S$. Then the following hold.*

(i) $x \bowtie A$ for all $x \in A$,

(ii) if $A \cap B \neq \emptyset$ then $A \bowtie B$,

(iii) if $A \not\bowtie B$ and $C \subset B$ then $A \not\bowtie C$.

Proof. By writing $A = (A \setminus \{x\} \cup \{x\})$ we immediately obtain (i) from (P2) and (P4) of Definition 2.48. Using (i), we obtain (ii) by writing $A = (A \setminus \{x\} \cup \{x\})$ for some $x \in A \cap B$. Claim (iii) is another direct consequence of (P4). ■

The reason for the introduction of proximities is that they provide a further way to define uniformities on a set S. First, we observe that proximity spaces are topological spaces. To that end we introduce the notion of proximity neighborhoods.

Definition 2.51 (Proximity neighborhood). Let $(S, \bowtie)$ be a proximity space. For subsets $A, B \subset S$, we write $A \Subset B$ if $A \not\bowtie (S \setminus B)$. We call B a *proximity neighborhood* (*p-neighborhood*, or $\bowtie$*-neighborhood*) of A, if $A \Subset B$. ◇

Recall from Definition A.8 the definition of a closure operator and that we can associate a topology with a closure operator by virtue of Proposition A.9.

Proposition 2.52 (Topology induced by proximity)**.** *Let* $(S, \bowtie)$ *be proximity space. The operator* $\Gamma : \mathcal{P}(S) \to \mathcal{P}(S)$ *given by*

$$\Gamma(A) := \overline{A} := \{ x \in S \mid \{x\} \bowtie A \} \tag{2.67}$$

is a closure operator. Furthermore, the topology induced by Γ *is Hausdorff if and only if* $(S, \bowtie)$ *is separated.*

Proof. By Lemma 2.50 (i) we readily get $A \subset \Gamma(A)$. We proceed to show that $\Gamma(\Gamma(A)) = \Gamma(A)$. If $\Gamma(A) = S$, there is nothing to show. Assume instead that there exists a $x \notin \Gamma(A)$, i.e. $x \not\bowtie A$. By property (P5), there exist sets $E, F \subset S$ with $E \cap F = \emptyset$ such that $x \not\bowtie E^c$ and $A \not\bowtie F^c$. From $A \not\bowtie F^c$ we can deduce that $\Gamma(A) \subset F$. Now, since sets E and F are disjoint, we also have $F \subset E^c$ and hence $\Gamma(A) \subset E^c$ which implies $x \not\bowtie \Gamma(A)$ by Lemma 2.50 (iii) because $x \not\bowtie E^c$. From (P4) we can easily conclude that $\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B)$. Finally, $\Gamma(\emptyset) = \emptyset$ follows from (P1).

Now assume that $(S, \bowtie)$ is separated and take $x, y \in S$ with $x \neq y$. By (P6) we have $x \not\bowtie y$ and by (P5) we can find $E, F \subset S$ with $E \cap F = \emptyset$ such that $x \not\bowtie E^c$ and $y \not\bowtie F^c$. Then, because $\Gamma(A) = \Gamma(\Gamma(A))$, we have $x \not\bowtie \Gamma(E^c)$ and $y \not\bowtie \Gamma(F^c)$. Furthermore, $\Gamma(E^c)^c$ and $\Gamma(F^c)^c$ are disjoint open neighborhoods of y and x, respectively.

Conversely, assume that the topology $\mathcal{T}$ induced by Γ is Hausdorff and let $x, y \in S$ with $x \neq y$. Let $A, B \in \mathcal{T}$ be disjoint open neighborhoods of x and y, respectively. Then, A^c, B^c are closed and hence $x \not\bowtie A^c$ and $y \not\bowtie B^c$ which in turn yields $x \not\bowtie y$ by (P5), thus concluding the proof. ∎

2.7 Hausdorff and Hausdorff-weak convergence

We have observed throughout this chapter that known concepts from metric spaces can be generalized to uniform spaces by replacing quantitative statements by qualitative ones. We conclude this chapter with a generalization of the Hausdorff distance on the space of subsets of a metric space.

Let $A \neq \emptyset$ be some set. Recall (cf. Definition A.42) that a *net* in A, $\{ x_\alpha \in A \mid \alpha \in \mathbb{I} \}$ is a set where the index set $\mathbb{I}$ is a directed set (see Definition A.41) that is not necessarily countable.

Definition 2.53 (Hausdorff convergence)**.** Let $(S, \mathcal{U})$ be a uniform Hausdorff space. Assume that $(A_\alpha)_{\alpha\in\mathbb{I}} \subset \mathcal{P}(S)$ is a net in the family of subsets of S. We say that $(A_\alpha)_{\alpha\in\mathbb{I}}$ converges to some $A \in \mathcal{P}(S)$ in the Hausdorff sense, if and only if for all $U \in \mathcal{U}$ open there exists a $\alpha_0 \in \mathbb{I}$ such that

$$A_\alpha \subset U[A] \quad \text{and} \quad A \subset U[A_\alpha], \tag{2.68}$$

for all $\alpha \geq \alpha_0$. ◇

Proposition 2.54 (Hausdorff topology)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space. Then Hausdorff convergence uniquely determines a topology on the space $\mathcal{P}(S)$ of subsets of S and we call this topology the* Hausdorff topology*. Furthermore, the Hausdorff topology restricted to the family $\mathcal{K}(S) \subset \mathcal{P}(S)$ of closed subsets of S is itself Hausdorff.*

Proof. By Theorem A.44 it suffices to show that Hausdorff convergence determines a convergence class in the sense of Definition A.43 in order to show that Hausdorff convergence uniquely determines a topology on $\mathcal{P}(S)$. But this is trivial.

On the other hand, it is clear that the Hausdorff topology on $\mathcal{P}(S)$ cannot be Hausdorff as $A^\circ \subset U[\overline{A}]$ and $\overline{A} \subset U[A^\circ]$ for all open $U \in \mathcal{U}$ and $A \subset S$, where A° denotes the inner and $\overline{A}$ denotes the closure of A.

Now assume $A, B \in \mathcal{K}(S)$ are distinct closed subsets of S. Without loss of generality assume that there exists a $x \in A \setminus B$. As x is contained in the open set $\complement B$, there exists an open symmetric entourage $U \in \mathcal{U}$ such that $U[x] \subset \complement B$ which implies that $x \notin U[B]$. Hence every net $(A_\alpha)_{\alpha\in\mathbb{I}} \subset \mathcal{K}(S)$ that converges to A cannot converge to B and vice versa. ■

The next lemma shows that a Hausdorff convergent sequence of closed subsets of a uniform Hausdorff space satisfies the conditions on the domains in Lemma 2.47.

Lemma 2.55 (Approximating points in the Hausdorff limit)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base and $(S_n)_{n\in\mathbb{N}} \subset \mathcal{K}(S)$ a sequence of closed subsets of S. Assume that $(S_n)_{n\in\mathbb{N}}$ converges in the Hausdorff topology to some closed subset $S_\infty \in \mathcal{K}(S)$ of S. Then there exists for each $x \in S_\infty$ a sequence $(x_n)_{n\in\mathbb{N}} \subset S$ such that $x_n \in S_n$ and*

$$\lim_{n\to\infty} x_n = x. \tag{2.69}$$

Proof. Assume that $\lim_{n\to\infty} S_n = S_\infty \in \mathcal{K}(S)$ with respect to the Hausdorff topology and let $x \in S_\infty$. Fix $U \in \mathcal{U}$ open. By Hausdorff convergence, we have that $U[x] \cap S_n \neq \emptyset$, eventually. Now take a sequence $(U_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ of open entourages with

$U_{m+1} \subset U_m$ and $\bigcap_{m \in \mathbb{N}} U_m = \Delta$. By a diagonal argument we can find a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in S_n$ for all $n \in \mathbb{N}$ and $x_n \in U_{m_n}[x]$ for all $n \geq n_0 \in \mathbb{N}$ and a (not necessarily strictly) increasing sequence $(m_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{n \to \infty} m_n = \infty$. Since $x \in S_\infty$ was arbitrary this concludes the proof. ■

We now generalize the concept of a *correspondence* (cf. [BBI01, Definition 7.3.17]) to uniform spaces.

Lemma 2.56. *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base and $A, (A_n)_{n \in \mathbb{N}}$ subsets of S. Then, $A_n \to A$ in the Hausdorff topology if and only if there exist sets $(T_n)_{n \in \mathbb{N}}$ and for each $n \in \mathbb{N}$ surjective maps $\varphi_n \colon T_n \to A_n$, $\psi_n \colon T_n \to A$ with the property that for every open entourage $U \in \mathcal{U}$ there exists a natural number $N \in \mathbb{N}$ such that*

$$\{ (\varphi_n(z), \psi_n(z)) \mid z \in T_n \} \subset U \tag{2.70}$$

for all $n > N$.

Proof. We start with the necessity. Suppose $A_n \to A$ in the Hausdorff sense. Take a sequence $(U_m)_{m \in \mathbb{N}} \subset \mathcal{U}$ of open entourages with $U_{m+1} \subset U_m$ and $\bigcap_{m \in \mathbb{N}} U_m = \Delta$. We can choose $(U_m)_{m \in \mathbb{N}}$ so that for each $n \in \mathbb{N}$ there exists a minimal $m(n) \in \mathbb{N}$ such that $A_k \subset U_{m(n)}[A]$ and $A \subset U_{m(n)}[A_k]$ for all $k \geq n$. For each $n \in \mathbb{N}$ let T_n be defined as follows

$$T_n := \left\{ (x, y) \in A_n \times A \;\middle|\; (x, y) \in U_{m(n)} \right\}. \tag{2.71}$$

Let and φ_n, ψ_n the projections on the first and second component, respectively. By construction, φ_n and ψ_n are surjective for every $n \in \mathbb{N}$. Moreover, there exists for each $U \in \mathcal{U}$ open a $N \in \mathbb{N}$ such that $U_n \subset U$ for all $n \geq N$ and consequently, (2.70) holds.

Conversely, fix $U \in \mathcal{U}$ open. Then there exist sets T_n and surjective maps $\varphi_n \colon T_n \to A_n$, $\psi_n \colon T_n \to A$ satisfying (2.70) for all $n \geq N$ for some $N \in \mathbb{N}$. Hence,

$$A \subset \bigcup_{x \in A_n} U[x] = U[A_n] \quad \text{and} \quad A_n \subset \bigcup_{x \in A} U[x] = U[A] \tag{2.72}$$

proving sufficiency. ■

Recall the notion of uniform measure spaces from Definition 2.41 and the definition of the *support* $\mathrm{supp}(\nu)$ of a Radon measure ν from Definition A.36. Observe that the support of a Radon measure is always closed by definition.

Definition 2.57 (Hausdorff-weak convergence)**.** Let $(S, \mathcal{U})$ be a uniform Hausdorff space. For each $n \in \mathbb{N} \cup \{\infty\}$ let $\nu^{(n)}$ be a Radon measure on S with support $S_n \subset S$. We say that $(\nu^{(n)})_{n \in \mathbb{N}}$ converges *Hausdorff-weakly* (*Hausdorff-vaguely*) to $\nu^{(\infty)}$ if and only if $\nu^{(n)} \underset{n \to \infty}{\Longrightarrow} \nu^{(\infty)}$ weakly (vaguely) and $S_n \underset{n \to \infty}{\longrightarrow} S_\infty$ in the Hausdorff topology. ◊

Hausdorff-weak convergence is indeed stronger than weak convergence alone, as the following simple example demonstrates.

Example 2.58 (Hausdorff-weak vs. weak convergence)**.** Let $S = \mathbb{R}$ be equipped with the uniformity generated by the Euclidean metric. Assume that for each $n \in \mathbb{N}$,

$$\nu^{(n)} := \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_1, \tag{2.73}$$

where δ_x denotes the Dirac measure at $x \in S$. Then $\nu^{(n)} \underset{n\to\infty}{\Longrightarrow} \delta_0$ but $\operatorname{supp} \nu^{(n)} = \{0, 1\}$ for all $n \in \mathbb{N}$ whereas $\operatorname{supp} \delta_0 = \{0\}$. ■

Morally, the Hausdorff convergence of the supports ensures that no points disappear from the supports of the approximating sequence. This becomes crucial when we consider $\nu^{(n)}$ to be the speed measure of random processes. If points vanish from the support of the speed measure, the limiting process would be essentially *tunneling* through these points without visiting them. This breaks the path-wise convergence that we will introduce in the next chapter.

3 The path space

> *Not all those who wander are lost.*
>
> — **J.R.R. Tolkien**
> Lord of the Rings

It is well known that when (S, d) is a metric space, there exists a metric on the space $D_S([0,\infty))$ or right continuous functions with left limits $\omega\colon [0,\infty) \to S$ that metrizes the Skorokhod topology on $D_S([0,\infty))$.

We begin this chapter with a couple of observations about the space $D_S([0,\infty))$ when $(S, \mathcal{U})$ is a uniform Hausdorff space. Most importantly, we show that the knowledge of a large class of hitting times already determines a path (i.e. an element $\omega \in D_S([0,\infty))$) uniquely. We then proceed to show that in the situation where $(S, \mathcal{U})$ is a uniform Hausdorff space, there exists a uniformity on $D_S([0,\infty))$ that is compatible with the Skorokhod topology. We call this uniformity the *Skorokhod uniformity*. This idea goes back Itaru Mitoma [Mit83] and Adam Jakubowski [Jak86] who considered completely regular topological spaces which are just uniform spaces by Proposition 2.19. We then translate some known results for the Skorokhod topology in terms of the Skorokhod metric to the language of uniform spaces. Often this involves restating *quantitative* convergence statements (i.e. some distance goes to 0), as *qualitative* statements (i.e. for all $U \in \mathcal{U}$ open, there exists...).

The main result of this chapter is Theorem 3.27, which was proven in a joint effort with Gerónimo Rojas, characterizes the Skorokhod convergence in terms of the convergence of hitting times.

In the last section, we discuss random paths and give a criterion for the tightness of a family of probability measures on $D_S([0,\infty))$. This result will be crucial for the proof of our tightness criterion in Theorem 4.75. Finally, we show that Theorem 3.27 can be lifted to probability measures to obtain a characterization of weak convergence of probability measures on $D_S([0,\infty))$ in terms of the weak convergence of hitting times.

3.1 The space of càdlàg paths

Let $(S, \mathcal{U})$ be a uniform Hausdorff space. We denote by $D_S([0,\infty))$ the space of functions $\omega\colon [0,\infty) \to S$ that are continuous from the right and possess left limits at each $t > 0$. As is customary, we refer to such functions with the adjective càdlàg.

We introduce a family of homomorphisms $\{\, \theta_t \mid t \geq 0 \,\}$ on $D_S([0,\infty))$ by

$$\theta_t(\omega)(\,\cdot\,) = \omega \circ \theta_t(\,\cdot\,) = \omega(\,\cdot\, + t), \qquad t \geq 0. \tag{3.1}$$

For obvious reasons we call the family $\{\, \theta_t \mid t \geq 0 \,\}$ the family of *(time) shift operators*.

Denote by $\omega(t-) = \lim_{s \uparrow t} \omega(s)$ the left limit point of ω at $t > 0$. The points of discontinuity of ω are called *jumps* and we write

$$J(\omega) := \{\, t > 0 \mid \omega(t-) \neq \omega(t) \,\} \tag{3.2}$$

for the set of jump points of ω.

When S is a metric or metrizable space, càdlàg functions can only have countably many jumps (see e.g. [EK86, Lemma 4.5.1]). This is not true for general uniform spaces. This is illustrated by the following example which is due to Adam Jakubowski [Jak86, Example 1.2].

Example 3.1. Consider the space $S = [0,1]^{[0,1]} \cong \{f\colon [0,1] \to [0,1]\}$ equipped with the product topology, i.e. the topology of pointwise convergence. Analogously to the product topology, we can define the product uniformity on S as the weak uniformity generated by the projections $\{\, \pi_i \mid i \in [0,1] \,\}$ and observe that the product uniformity generates the product topology on S. Furthermore, S is Hausdorff as a product of Hausdorff spaces but has no countable base (compare [SS78, #105]).

Let $\omega\colon [0,1] \to S$ be defined as $\omega(t) = \mathbb{1}_{[0,t)}(x)$. Then $(\omega(t_n))_{n\in\mathbb{N}}$ converges pointwise to $\mathbb{1}_{[0,t)}(x)$ for every sequence $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ with $t_n \downarrow t \in [0,1]$. On the other hand, $(\omega(t_n))_{n\in\mathbb{N}}$ converges pointwise to $\mathbb{1}_{[0,t]}(x)$ for every such sequence with $t_n \uparrow t \in [0,1]$. Hence, $\omega \in D_S([0,1])$ but ω is discontinuous at every $t \in (0,1]$. ■

Instead of countably many jumps we have for càdlàg functions on general uniform Hausdorff spaces that there can only be countably many jumps exceeding a certain "size", in the following sense.

Lemma 3.2 (Discontinuity points of càdlàg paths)**.** *Let $(S,\mathcal{U})$ be a uniform Hausdorff space and $\omega \in D_S([0,\infty))$. Then the following hold*

(i) For every $U \in \mathcal{U}$ and $T > 0$ the set

$$J_U^T(\omega) := \{\, t \in [0,T] \mid (\omega(t), \omega(t-)) \notin U \,\} \tag{3.3}$$

is finite.

(ii) for every $U \in \mathcal{U}$ the set

$$J_U(\omega) := \{\, t > 0 \mid (\omega(t-), \omega(t)) \notin U \,\} \tag{3.4}$$

is at most countable.

(iii) If $\mathcal{U}$ has a countable base, then the set $J(\omega) = \{ t \geq 0 \mid \omega(t) \neq \omega(t-) \}$ of discontinuity points of ω is at most countable.

Proof. The second claim follows readily from the first by taking the limit $T \to \infty$ and the third claim follows from the second with the observation that the points of discontinuity of ω are

$$J(\omega) = \bigcup_{V \in \mathcal{V}} J_V(\omega) \tag{3.5}$$

for some base $\mathcal{V}$ of $\mathcal{U}$. By assumption $\mathcal{V}$ can be chosen to be countable and the countable union of countable sets is again countable.

To show the first claim, fix $U \in \mathcal{U}$ and $T > 0$ and assume that there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset J_U^T(\omega)$ with $t_n \uparrow t \in J_U^T(\omega)$. Now choose $V \in \mathcal{U}$ open and symmetric such that $V \circ V \subset U$ and observe that $V[\omega(t)] \cap V[\omega(t-)] = \emptyset$. Because ω is càdlàg, we have $\lim_{n \to \infty} \omega(t_n) = \omega(t-)$ and hence there exists some $n_0 \in \mathbb{N}$ such that

$$\omega(t_n) \in V[\omega(t-)] \quad \forall n > n_0. \tag{3.6}$$

From the existence of left limits, we deduce that there exists another entourage $W \in \mathcal{U}$, open and symmetric, with $W \circ W \subset V$ and a sequence $(s_n)_{n \in \mathbb{N}} \subset [0, T]$, not necessarily contained in $J_U^T(\omega)$, with $s_n \leq t_n$ and $s_n \uparrow t$ such that $\omega(s_n) \in W[\omega(t_n-)]$ for all $n \in \mathbb{N}$. By assumption $s_n \uparrow t$, there exists a $n_1 \in \mathbb{N}$ such that $\omega(s_n) \in W[\omega(t-)]$ for all $n > n_1$. Since $(\omega(s_n), \omega(t_n-)) \in W$ and $(\omega(s_n), \omega(t-)) \in W$ for all $n > n_1$, it follows that

$$(\omega(t_n-), \omega(t-)) \in W \circ W \subset V \quad \forall n > n_1. \tag{3.7}$$

By (3.6) we have $(\omega(t_n), \omega(t-)) \in V$ for all $n > n_0$ and together with (3.7) we deduce

$$(\omega(t_n), \omega(t_n-)) \in V \circ V \subset U \quad \forall n > n_0 \vee n_1, \tag{3.8}$$

in contradiction to the assumption $(t_n)_{n \in \mathbb{N}} \subset J_U^T(\omega)$.

By the same logic, there can not exist a decreasing sequence $(t_n)_{n \in \mathbb{N}} \subset J_U^T(\omega)$ with $t_n \downarrow t \in \mathbb{J}_U^T(\omega)$. Hence $J_U^T(\omega)$ has no cluster points and is thus finite. ■

Lemma 3.3 (càdlàg functions are measurable)**.** *Let S be a uniform Hausdorff space. Then every $\omega \in D_S([0, \infty))$ is Borel measurable.*

Proof. Fix $\omega \in D_S([0, \infty))$ and let $A \subset S$ be open. Assume that there exists a $t \geq 0$ such that $\omega(t) \in A$ and let $I \subset [0, \infty)$ be the largest interval such that $t \in I$ and $\omega(I) \subset A$. Then I is nonempty, open to the right by right continuity and either open or closed to the left depending on whether ω enters A continuously or by a jump. Either way, I has positive length and is Borel measurable. Furthermore, $\omega^{-1}A$ is at most a countable union of such intervals and hence measurable. ■

Recall that (cf. [EK86]), the modified modulus of continuity for càdlàg functions $\omega \in D_E([0,\infty))$ where (E,d) is a (Polish) metric space is defined as

$$w'(\omega,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} d(\omega(s),\omega(t)), \quad \delta > 0 \text{ and } T > 0, \tag{3.9}$$

where the infimum is taken over all partitions of $[0,T]$ of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \delta$ and $n \geq 1$.

Càdlàg functions $\omega \in D_E([0,\infty))$ are "almost continuous" in the sense that $\lim_{\delta\to 0} w'(\omega,\delta,T) = 0$ for all $T > 0$ (see [EK86, Lemma 3.6.2 (a)]). In the uniform setting we cannot measure the modulus of continuity but we can substitute the quantitative statement for a qualitative one.

We introduce the following notations for partitions of the time axis

$$\Pi := \left\{ \pi = (\pi_n)_{n\in\mathbb{N}_0} \;\middle|\; 0 = \pi_0 < \pi_1 < \pi_2 < \dots \right\} \tag{3.10}$$

and for $T > 0,\ N \in \mathbb{N}$,

$$\Pi_T^N := \left\{ \pi = (\pi_0,\pi_1,\pi_2,\dots,\pi_N) \;\middle|\; \pi_0 = 0 < \pi_1 < \cdots < \pi_{N-1} < T \leq \pi_N \right\}. \tag{3.11}$$

Every $\pi \in \Pi$ induces a unique partitions of $[0,\infty)$ via the map

$$\iota\colon \pi \mapsto \iota(\pi) := \{ [\pi_{i-1},\pi_i) \mid i \in \mathbb{N} \}, \quad \pi \in \Pi. \tag{3.12}$$

Since no confusion can arise, we use the same notation for the map that maps a $\pi \in \Pi_T^N$ to a partition of $[0,T]$, i.e.

$$\iota\colon \pi \mapsto \iota(\pi) := \{ [\pi_{i-1},\pi_i) \mid i = 1,\dots,N \}, \quad \pi \in \Pi_T^N. \tag{3.13}$$

It is often required to have some control over the length of the intervals of a partition. We write

$$L(\pi) := \sup_{I\in\iota(\pi)} \lambda(I) \quad \text{and} \quad l(\pi) := \inf_{I\in\iota(\pi)} \lambda(I) \tag{3.14}$$

for $\pi \in \Pi$ or $\pi \in \Pi_T^N$, where λ denotes the Lebesgue measure on $\mathbb{R}$.

In the following, we suppress the dependence on N from the notation and write $\Pi_T := \Pi_T^N$ if N is not explicitly needed

Lemma 3.4 (Modulus of continuity)**.** *Let* $(S, \mathcal{U})$ *be a uniform Hausdorff space and* $\omega \in D_S([0,\infty))$*. For all* $U \in \mathcal{U}$ *and* $T > 0$ *there exist a* $\delta > 0$ *and a partition* $\pi \in \Pi_T$ *of* $[0,T]$ *with* $l(\pi) > \delta$ *such that*

$$\{ (\omega(s), \omega(t)) \mid s,t \in I \} \subset U \quad \forall I \in \iota(\pi). \tag{3.15}$$

Proof. Fix $U \in \mathcal{U}$ and $T > 0$. Let $V \in \mathcal{U}$ be open and symmetric such that $V \circ V \subset U$. Further, let $\sigma_0 := 0$ and for $k \in \mathbb{N}$ set

$$\sigma_k := \inf \{ t > \sigma_{k-1} \mid (\omega(t), \omega(\sigma_{k-1})) \notin V \}, \tag{3.16}$$

where we set $\inf \emptyset = \infty$, as usual. If $\sigma_k = \infty$ for some $k \in \mathbb{N}$ we set $\sigma_l = \infty$ for all $l \geq k$. Observe that by existence of left limits the family $\{ \sigma_k \mid k \in \mathbb{N} \}$ contains no finite limit points and by right continuity we have $\sigma_{k+1} - \sigma_k > 0$ for all $k \in \mathbb{N}$. Hence $N := \inf \{ n \in \mathbb{N} \mid \sigma_N \geq T \}$ is finite. Now, let

$$\delta := \min_{k \geq 0} \{ \sigma_{k+1} - \sigma_k \mid \sigma_k \leq T \}. \tag{3.17}$$

Then, $\{ \sigma_k \mid k = 1, \ldots, N \}$ gives rise to a partition $\pi \in \Pi_T$ of $[0,T]$ with the desired properties. ■

3.1.1 Separation of paths by hitting times

In this section, we show that a path is uniquely determined by its hitting times of uniform neighborhoods. In the case where $(S, \mathcal{U})$ has a countable base we improve this result by reducing the number of neighborhoods.

First, recall the definition of a hitting time.

Definition 3.5 (Hitting times)**.** For $A \subset S$ we introduce the *(first) hitting time* operator $\tau_A \colon D_S([0,\infty)) \to [0,\infty]$ as

$$\tau_A(\omega) := \inf \{ t > 0 \mid \omega(t) \in A \}, \tag{3.18}$$

where we set $\inf \emptyset = \infty$, as usual. ◇

First, we show that the hitting times of a subset of *all* neighborhoods is separating on $D_S([0,\infty))$.

Proposition 3.6 (Separation by hitting times I)**.** *Let* $(S, \mathcal{U})$ *be a uniform Hausdorff space and* $\omega_1, \omega_2 \in D_S([0,\infty))$*. Then the following are equivalent*

(i) $\omega_1 = \omega_2$*,*

(ii) for all $x \in S$ there exists a $V \in \mathcal{U}$ such that for all $U \in \mathcal{U}$ with $U \subset V$ and all $t \geq 0$

$$\tau_{U[x]}(\omega_1 \circ \theta_t) = \tau_{U[x]}(\omega_2 \circ \theta_t). \tag{3.19}$$

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. For the reverse implication we proceed by contraposition. Assume $\omega_1 \neq \omega_2$. Then there exists a $t > 0$ such that $\omega_1(t) \neq \omega_2(t)$. By virtue of the Hausdorff property there exists a $V \in \mathcal{U}$ open such that $V[\omega_1(t)] \cap V[\omega_2(t)] = \emptyset$. By right continuity of the elements of $D_S([0,\infty))$ there exists a $\varepsilon > 0$ with $\omega_i(s) \in V[\omega_i(t)]$ for $i = 1, 2$ and all $s \in [t, t+\varepsilon)$. Hence

$$0 = \tau_{U[\omega(t)]}(\omega_1 \circ \theta_t) \neq \tau_{U[\omega(t)]}(\omega_2 \circ \theta_t) \geq \varepsilon > 0, \tag{3.20}$$

for all $U \in \mathcal{U}$ with $U \subset V$, concluding the proof. ■

Next, we introduce the notion of first contact times.

Definition 3.7 (Contact times)**.** Let $\omega \in D_S([0,\infty))$ and $A \subset S$. The *first contact time* of A by ω is defined as

$$\gamma_A(\omega) := \inf\left\{ t > 0 \,\middle|\, \{\omega(t), \omega(t-)\} \cap \overline{A} \neq \emptyset \right\}. \tag{3.21}$$

A set $A \subset S$ is called regular (for ω), if $\gamma_A(\omega) = \tau_A(\omega)$. ◇

As an immediate consequence of the definition observe that $\gamma_A(\omega) \leq \tau_A(\omega)$ for all $A \subset S$ and $\gamma_A(\omega) = \tau_A(\omega)$ if A is closed and ω is continuous.

Lemma 3.8 (Approximation of contact times by hitting times)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base, $A \subset S$ and $\omega \in D_S([0,\infty))$. Assume further that $\gamma_A(\omega) > 0$.*

(i) For any sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ of open entourages with $E_{n+1} \subset E_n$ and $\bigcap_{n\geq 1} E_n = \Delta$,

$$\lim_{n\to\infty} \tau_{E_n[A]}(\omega) = \gamma_A(\omega). \tag{3.22}$$

(ii) For each $s < \gamma_A(\omega)$ and $D \in \mathcal{U}$ there exists an open entourage $E \in \mathcal{U}$ with $E \subset D$ such that

$$\gamma_{E[A]}(\omega) \geq s. \tag{3.23}$$

Proof. We begin in the beginning and show (i) first. Observe that the Hausdorff property guarantees the existence of such a sequence. Furthermore, if $\Delta \in \mathcal{U}$, the topology generated by $\mathcal{U}$ is discrete and the statement becomes trivial. Clearly,

$\tau_{E_n[A]}(\omega)$ is an increasing sequence in n and by definition of $\gamma_A(\omega)$ and $\omega(t-)$, we find $\{\omega(\gamma_A(\omega)-), \omega(\gamma_A(\omega))\} \cap E_n[A] \neq \emptyset$ for all $n \geq 1$. Hence $\tau_{E_n[A]}(\omega) \leq \gamma_A(\omega)$ and the limit $\lim_{n\to\infty} \tau_{E_n[A]}(\omega)$ exists and

$$\lim_{n\to\infty} \tau_{E_n[A]}(\omega) \leq \gamma_A(\omega). \tag{3.24}$$

Denote $t_0 := \lim_{n\to\infty} \tau_{E_n[A]}(\omega)$. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that $\omega(\tau_{E_n[A]}(\omega) + \varepsilon_n) \in E_n[A]$ for each $n \in \mathbb{N}$. We have by construction $\lim_{n\to\infty} \omega(\tau_{E_n[A]}(\omega) + \varepsilon_n) \in \{\omega(t_0-), \omega(t_0)\}$ and $\lim_{n\to\infty} \omega(\tau_{E_n[A]}(\omega) + \varepsilon_n) \in \bigcap_{n\geq 1} E_n[A] = \overline{A}$ and hence

$$\gamma_A(\omega) \leq t_0 = \lim_{n\to\infty} \tau_{E_n[A]}(\omega). \tag{3.25}$$

For (ii) suppose the statement does not hold. Then there exist $0 < s < \gamma_A(\omega)$, $D \in \mathcal{U}$ and a sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ of open entourages with $E_{n+1} \subset E_n \subset D$ and $\bigcap_{n\geq 1} E_n = \Delta$ such that for all $n \in \mathbb{N}$,

$$\gamma_{E_n[A]}(\omega) < s. \tag{3.26}$$

Since $\gamma_{E_n[A]}(\omega)$ is an increasing sequence, the limit $\lim_{n\to\infty} \gamma_{E_n[A]}(\omega) =: t_1$ exists and $t_1 \leq s$. Furthermore, $\{\omega(t_1-), \omega(t_1)\} \cap \overline{E_n[A]} \neq \emptyset$ for all $n \in \mathbb{N}$. Because $\overline{E_{n+1}[A]} \subset \overline{E_n[A]}$, it follows that

$$\begin{aligned} \emptyset \neq \bigcap_{n\geq 1} (\{\omega(t_1-), \omega(t_1)\} \cap \overline{E_n[A]}) &= \{\omega(t_1-), \omega(t_1)\} \cap \bigcap_{n\geq 1} \overline{E_n[A]} \\ &= \{\omega(t_1-), \omega(t_1)\} \cap \overline{A}. \end{aligned} \tag{3.27}$$

Hence $\gamma_A(\omega) \leq s$, in contradiction to the assumption. ∎

In fact, $\gamma_A(\omega) \neq \tau_A(\omega)$ can only happen for exceptional sets and the regular sets are dense in the following sense.

Lemma 3.9 (Most neighborhoods are regular). *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base, $A \subset S$ and $\omega \in D_S([0,\infty))$. Then there exists for each $D \in \mathcal{U}$ an open entourage $E \in \mathcal{U}$ with $E \subset D$ such that*

$$\gamma_{E[A]}(\omega) = \tau_{E[A]}(\omega). \tag{3.28}$$

Proof. Fix $D \in \mathcal{U}$ and choose some $U \in \mathcal{U}$ open with $U \circ U \subset D$. If $U[A]$ is regular, we are done. Thus assume that

$$\tau_{U[A]}(\omega) > \gamma_{U[A]}(\omega) =: t_1. \tag{3.29}$$

We distinguish two cases of how ω behaves before time t_1. First consider the case where ω jumps to $\overline{U[A]}$ or, more precisely, to $\partial U[A]$ at time t_1. In that case we have $\omega(t_1) \in \overline{U[A]}$ and $\omega(t_1-) \neq \omega(t_1)$. Hence, we find for each $E \in \mathcal{U}$ with $E \subset U$ and $E[\omega(t_1-)] \cap (E \circ U)[A] = \emptyset$ some $\varepsilon = \varepsilon(E) > 0$ such that $\omega([t_1 - \varepsilon, t_1)) \subset E[\omega(t_1-)]$. Now take $s \in [t_1 - \varepsilon, t_1)$. By Lemma 3.8 there exists a $E' \in \mathcal{U}$ open with $E' \subset E$ such that $\gamma_{(E' \circ U)[A]}(\omega) > s$. By construction, we find $\gamma_{(E' \circ U)[A]}(\omega) = \tau_{(E' \circ U)[A]}(\omega) = \gamma_{U[A]}(\omega)$, as desired.

For the contrary case assume $\omega(t_1-) \in \overline{U[A]}$. Let $s_1 = t_1/2$, by Lemma 3.8 there exists an $E_1 \in \mathcal{U}$ open such that $E_1 \subset U$ and $\gamma_{(E_1 \circ U)[A]}(\omega) > s_1$. By definition of $\omega(t-)$, we find an $\varepsilon_1 > 0$ such that $\omega([t_1 - \varepsilon_1, t_1)) \subset (E_1 \circ U)[A]$. Hence,

$$s_1 < \gamma_{(E_1 \circ U)[A]}(\omega) \leq \tau_{(E_1 \circ U)[A]}(\omega) \leq t_1 - \varepsilon_1. \tag{3.30}$$

If we have equality of the contact time $\gamma_{(E_1 \circ U)[A]}(\omega)$ and the hitting time $\tau_{(E_1 \circ U)[A]}(\omega)$, we are done. If not set $t_2 := \gamma_{(E_1 \circ U)[A]}(\omega)$. If we find $\omega(t_2-) \neq \omega(t_2)$ and $\omega(t_2) \in \overline{(E_1 \circ U)[A]}$, we can use the same arguments as before to construct $E \in \mathcal{U}$, $E \subset U$ such that $\lim_{n\to\infty} \tau_{(U \circ E)[A]}(\omega_n) = \tau_{(U \circ E)[A]}(\omega)$. In the case where $\omega(t_2-) \in \overline{(E_1 \circ U)[A]}$ we proceed as before and take $s_2 = (s_1 + t_2)/2$ and $E_2 \in \mathcal{U}$ open with $E_1 \subset E_2 \subset U$ such that $t_3 := \gamma_{(E_2 \circ U)[A]}(\omega) > s_2$. Then we find some $\varepsilon_2 > 0$ such that $\omega([t_2 - \varepsilon_2, t_2)) \subset (E_2 \circ U)[A]$. We can repeat this construction inductively until we find some $E_n \in \mathcal{U}$ open with $E_n \subset U$ such that either $\gamma_{(E_n \circ U)[A]}(\omega) = \tau_{(E_n \circ U)[A]}(\omega)$ or $\omega(t_{n+1}-) \neq \omega(t_{n+1})$ and $\omega(t_{n+1}) \in \overline{(E_n \circ U)[A]}$. In both cases, we find an $E \in \mathcal{U}$ open with $E \subset U$ such that the hitting time of $(U \circ E)[A]$ and the contact time of $(U \circ E)[A]$ coincide. If this procedure does not terminate we end up with a strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$, a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ and a family of open entourages $(E_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ with $E_n \subset E_{n+1} \subset U$ such that

$$s_n < \gamma_{(E_n \circ U)[A]}(\omega) \leq \tau_{(E_n \circ U)[A]}(\omega) < t_n. \tag{3.31}$$

Since $E := \bigcup_{n\geq 1} E_n \in \mathcal{U}$ is open and $E \circ U \subset D$, by construction, we conclude $\gamma_{(E \circ U)[A]}(\omega) = \tau_{(E \circ U)[A]}(\omega)$. ■

The second condition in Proposition 3.6 can be sharpened significantly if we assume that ω has only countably many points of discontinuity.

Theorem 3.10 (Separation by hitting times II). *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base and $\omega_1, \omega_2 \in D_S([0,\infty))$. Then the following are equivalent*

(i) $\omega_1 = \omega_2$,

(ii) There exists dense subsets $D \subset S$ and $T \subset \{ t \geq 0 \mid \omega_i(s) = \omega_i(s-),\ i = 1,2 \} \subset [0,\infty)$ such that for all $x \in D$ and $s \in T$ and all $U \in \mathcal{U}$ open with $\tau_{U[x]}(\omega_i \circ \theta_s) = \gamma_{U[x]}(\omega_i \circ \theta_s)$, $i = 1,2$,

$$\tau_{U[x]}(\omega_1 \circ \theta_s) = \tau_{U[x]}(\omega_2 \circ \theta_s). \tag{3.32}$$

Proof. The first implication is again trivial. Conversely, assume there exists a $t > 0$ such that $\omega_1(t) \neq \omega_2(t)$. We argue along the same lines as in the proof of Proposition 3.6. There exist $W \in \mathcal{U}$ such that $W[\omega_1(t)] \cap W[\omega_2(t)] = \emptyset$. By right continuity, there exists a $V \in \mathcal{U}$ open with $V \circ V \subset W$ and $\varepsilon > 0$ such that $\omega_i(s) \in V[\omega_i(t)]$ for all $s \in [t, t+\varepsilon)$ and $i = 1,2$. By assumption, $\mathcal{U}$ has a countable base and we can apply Lemma 3.2 to deduce that there exists a continuity point $s \in [t, t+\varepsilon/2) \cap T$ of both ω_1 and ω_2. Now let $U \in \mathcal{U}$ open be such that $U \circ U \subset V$. Then there exists a $x \in D$ such that $\omega_1(s) \in U[x]$. We have constructed $U[x]$ and s in such a way that $\tau_{E \circ U[x]}(\omega_1 \circ \theta_s) = 0$ and $\tau_{E \circ U[x]}(\omega_2 \circ \theta_s) > \varepsilon/2$ for all $E \in \mathcal{U}$ open with $E \subset U$. By Lemma 3.9 there exists an open entourage $E \in \mathcal{U}$ with $E \subset U$ such that $\tau_{E \circ U[x]}(\omega_2 \circ \theta_s) = \gamma_{E \circ U[x]}(\omega_2 \circ \theta_s)$. Furthermore we have $\gamma_{E \circ U[x]}(\omega_1 \circ \theta_s) \leq \tau_{E \circ U[x]}(\omega_1 \circ \theta_s) = 0$ and the proof is complete. ■

3.2 Path space

Next, we want to obtain conditions for the convergence of paths. To do so, we first need to introduce a topology on $D_S([0,\infty))$. One can consider various topologies on $D_S([0,\infty))$. One possible choice being the topology of uniform convergence where we set $\lim_{n\to\infty} \omega_n = \omega$ if and only if for all open entourages $U \in \mathcal{U}$ there exists a $n_0 \in \mathbb{N}$ such that

$$(\omega_n(t), \omega(t)) \in U \quad \forall t \in [0,T] \text{ and } \forall n \geq n_0. \tag{3.33}$$

This topology is induced by the *uniformity of uniform convergence* (cf. Lemma 2.43), which has as a base the family of sets

$$\{ (\omega, \omega') \,\big|\, (\omega(t), \omega'(t)) \in U \quad \forall t \geq 0 \} \quad U \in \mathcal{U}. \tag{3.34}$$

Observe that the uniformity of uniform convergence has a countable base if the original uniformity $\mathcal{U}$ on S has a countable base. Thus, the topology of uniform convergence is first countable and hence determined by the converging sequences (cf. Proposition A.46) if $\mathcal{U}$ possesses a countable base.

Anatoliy Skorokhod observed in his seminal paper [Sko56]

> [T]he uniform topology in [$D_S([0,T])$] requires that the convergence of [ω_n] to [ω] imply that there exists a number such that for all n greater

than or equal to this number the points of discontinuity of $[\omega_n]$ coincide with the points of discontinuity of $[\omega]$. This means that if t is considered to be the time, we must assume the existence of an instrument which will measure time exactly, and physically this is an impossibility. It is much more natural to suppose that the functions we can obtain from each other by small deformations of the times scale lie close to each other.[1]

He introduced four topologies on $D_S([0,T])$ that take this observation into account. The strongest of the four, the J_1 topology is now commonly called *the* Skorokhod topology.

Definition 3.11 ([Sko56])**.** Let (S,d) be a metric space. The sequence $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,T])$ is called Skorokhod convergent (J_1-convergent) to $\omega \in D_S([0,T])$ if there exists a sequence of continuous bijections $\lambda_n\colon [0,T] \to [0,T]$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} d(\omega_n(t), \omega(\lambda_n(t))) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| = 0. \tag{3.35}$$

◊

It is important to note that this definition indeed well defines a topology on $D_S([0,T])$: Clearly, the topology described in Definition 3.11 is coarser than the topology of uniform convergence and hence is first countable if the topology of uniform convergence is first countable but this is the case because every metric uniformity possesses a countable base.

As indicated in the quote above, it is natural to think of $[0,T]$ or $[0,\infty)$ as a time interval and $\omega(t)$ describing the position of a particle in the space S at time t. With this interpretation in mind we call the elements of $D_S([0,\infty))$ *paths* and $D_S([0,\infty))$ itself the *pathspace*.

It is well known that the Skorokhod topology on $D_S([0,\infty))$ is metrizable when (S,d) itself is a metric space. For a detailed account of the Skorokhod metric we refer the reader to the classical book [EK86] by Stewart N. Ethier and Thomas G. Kurtz.

We will show in the following that a similar approach can applied to show that in the case where $(S,\mathcal{U})$ is a uniform Hausdorff space, the Skorokhod topology on $D_S([0,\infty))$ is uniformizable. Furthermore, we will explicitly construct the Skorokhod uniformity using the approach via families of pseudometrics. This Idea goes back to Itaru Mitoma [Mit83] and Adam Jakubowski [Jak86].

3.2.1 The Skorokhod uniformity

Let $(S,\mathcal{U})$ be a uniform space. Denote by $D_S := D_S([0,\infty))$ the space of càdlàg functions $\omega : [0,\infty) \to S$.

[1] [Sko56, p. 264f] with notation adapted to our notation.

We can define a uniformity on D_S that generates the Skorokhod topology on D_S using a similar approach as in the metric case. More precisely, we use the family of pseudometrics associated with the uniformity $\mathcal{U}$ to define a family of pseudometrics on D_S. This idea was introduced by Mitoma in [Mit83] and further developed by Jakubowski in [Jak86]. For reference, we present the construction suggested by Jakubowski.

We mimic the construction of the Skorokhod metric in the case of a metric space S (see e.g. [EK86]). For $s > 0$ let Λ_s denote the family of continuous and strictly increasing functions $\lambda\colon [0, s] \to [0, s]$ such that $\lambda(0) = 0$ and $\lambda(s) = s$.

Given any pseudometric ρ on S we can define a pseudometric $\tilde{\rho}^s$ on $D_S([0, s])$ via

$$\tilde{\rho}^s(\omega, \omega') = \inf_{\lambda \in \Lambda_s} \left(\sup_{t\in[0,s]} |\lambda(t) - t| \vee \sup_{t\in[0,s]} \rho\Big(\omega(t), \omega'(\lambda(t))\Big) \right). \tag{3.36}$$

For $s > 0$ consider the maps $q_s\colon D_S([0, \infty)) \to D_S([0, s + 1])$ given by

$$q_s(\omega)(t) = \begin{cases} \omega(t), & \text{if } t \in [0, s) \\ \omega(s), & \text{if } t \in [s, s + 1]. \end{cases} \tag{3.37}$$

For two paths $\omega, \omega' \in D_S([0, \infty))$ let

$$\zeta^\rho_s(\omega, \omega') := \tilde{\rho}^{s+1}(q_s(\omega), q_s(\omega')). \tag{3.38}$$

As a function in $s \in [0, \infty)$ this is an element of $D_{\mathbb{R}^+}([0, \infty))$ and we can define

$$\zeta^\rho(\omega, \omega') = \int_0^\infty e^{-s}(1 \wedge \zeta^\rho_s(\omega, \omega'))\mathrm{d}s. \tag{3.39}$$

It is straight-forward to show that this construction indeed yields a pseudometric on $D_S([0, \infty))$.

Lemma 3.12 (Pseudometrics on the pathspace)**.** *Let ρ be a pseudometric on S. Then ζ^ρ as defined above is a pseudometric on $D_S([0, \infty))$.*

Proof. Clearly, $\zeta^\rho(\omega, \omega) = 0$ for all $\omega \in D_S([0, \infty))$ and ζ^ρ is non-negative definite. The triangle inequality for ζ^ρ follows immediately if we can show that for every $s > 0$, the triangle inequality holds for $\tilde{\rho}^s$ as defined in (3.36). To show this, fix $s > 0$ and let

$\omega_a, \omega_b, \omega_c \in D_S([0,\infty))$. Then, by the triangle inequality for ρ,

$$\begin{aligned}
\tilde{\rho}^s(\omega_a,\omega_c) &= \inf_{\lambda\in\Lambda_s}\left(\sup_{t\in[0,s]}|\lambda(t)-t| \vee \sup_{t\in[0,s]}\rho\Big(\omega_a(t),\omega_c(\lambda(t))\Big)\right) \\
&\leq \inf_{\lambda\in\Lambda_s}\inf_{\lambda'\in\Lambda_s}\left(\sup_{t\in[0,s]}(|\lambda(t)-\lambda'(t)|+|\lambda'(t)-t|)\right. \\
&\qquad\qquad \left.\vee \sup_{t\in[0,s]}\Big(\rho\Big(\omega_a(t),\omega_b(\lambda'(t))\Big)+\rho\Big(\omega_b(\lambda'(t)),\omega_c(\lambda(t))\Big)\Big)\right) \\
&\leq \inf_{\lambda\in\Lambda_s}\inf_{\lambda'\in\Lambda_s}\left(\left(\sup_{t\in[0,s]}|\lambda'(t)-t| \vee \sup_{t\in[0,s]}\rho\Big(\omega_a(t),\omega_b(\lambda'(t))\Big)\right)\right. \\
&\qquad\qquad \left.+\left(\sup_{t\in[0,s]}|\lambda(t)-\lambda'(t)| \vee \sup_{t\in[0,s]}\rho\Big(\omega_b(\lambda'(t)),\omega_c(\lambda(t))\Big)\right)\right) \\
&= \tilde{\rho}^s(\omega_a,\omega_b)+\tilde{\rho}^s(\omega_b,\omega_c).
\end{aligned} \tag{3.40}$$

In the last equation, we have used the fact that with $\lambda \in \Lambda_s$ it follows that $\lambda^{-1} \in \Lambda_s$ and furthermore $\lambda^{-1} \circ \lambda' \in \Lambda_s$. ■

Denote by Λ the family of increasing continuous functions $\lambda\colon [0,\infty) \to [0,\infty)$ such that $\lambda(0) = 0$ and $\lambda(t) \to \infty$ as $t \to \infty$. Jakubowski has shown the following.

Proposition 3.13 (Convergence in the Skorokhod topology [Jak86, Proposition 4.1])**.** *Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of elements of $D_S([0,\infty))$ and $\omega \in D_S([0,\infty))$. Assume that ρ is a pseudometric on S and let ζ^ρ be defined as in (3.39). Then $\lim_{n\to\infty}\zeta^\rho(\omega_n,\omega) = 0$ if and only if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \Lambda$ such that for each $T \geq 0$*

$$\lim_{n\to\infty}\sup_{s\in[0,T]}|\lambda_n(s)-s| = 0 \tag{3.41}$$

$$\lim_{n\to\infty}\sup_{s\in[0,T]}\rho(\omega_n(\lambda_n(s)),\omega(s)) = 0. \tag{3.42}$$

Let $\mathbb{I}$ denote some index set and let $(\rho_i)_{i\in\mathbb{I}}$ be a family of pseudometrics on S that generates $\mathcal{U}$ with the properties described in Proposition 2.16. Analogously to the classical case where (S,d) is a metric space, we aim to define a uniform structure on $D_S([0,\infty))$ through the family $(\zeta^{\rho_i})_{i\in\mathbb{I}}$ of pseudometrics on $D_S([0,\infty))$.

Proposition 3.14 (Skorokhod uniformity generated by pseudometrics)**.** *Let $(S,\mathcal{U})$ be a uniform Hausdorff space and $(\rho_i)_{i\in\mathbb{I}}$ a family of pseudometrics that generates $\mathcal{U}$ and satisfies the conditions of Proposition 2.16. Then the family of pseudometrics $(\zeta^{\rho_i})_{i\in\mathbb{I}}$ on $D_S([0,\infty))$ as defined above, satisfies the conditions in Proposition 2.16 and induces a uniformity $\mathcal{D}$ on $D_S([0,\infty))$ which is Hausdorff.*

Proof. We first show that the family $(\zeta_i)_{i\in\mathbb{I}}$ satisfies condition (i) of Proposition 2.16. Assume $\omega, \omega' \in D_S([0,\infty))$ and $\omega \neq \omega'$. Then there exists a $t \geq 0$ such that $\omega(t) \neq \omega'(t)$. By the Hausdorff property there exists a basic entourage $U \in \mathcal{U}$ such that $U[\omega(t)] \cap U[\omega'(t)] = \emptyset$. By definition of uniformities and their bases, there exists another basic entourage $V \in \mathcal{U}$ such that $V \circ V \subset U$ and by right continuity we find an $\varepsilon > 0$ such that $V[\omega(s)] \cap V[\omega'(s')] = \emptyset$ for all $s, s' \in [t, t+\varepsilon)$. Since V is a basic entourage, there exists an $i \in \mathbb{I}$ and a $\delta > 0$ such that $V = \left\{ (x,y) \in S^2 \,\middle|\, \rho_i(x,y) < \delta \right\}$. As a consequence, we have for any $s > t + \varepsilon$ that

$$\tilde{\rho}^s(\omega, \omega') \geq \min\{\varepsilon/2, \delta\} > 0 \tag{3.43}$$

and hence $\zeta^{\rho_i}(\omega, \omega') > 0$. As an immediate consequence, we obtain that the uniformity $\mathcal{D}$ induced by $(\zeta_i)_{i\in\mathbb{I}}$ on $D_S([0,\infty))$ is Hausdorff.

The second property of Proposition 2.16 follows directly from the corresponding property of the family $(\rho_i)_{i\in\mathbb{I}}$. ■

It is important to know whether this construction of $\mathcal{D}$ depends on the choice of the family $(\rho_i)_{i\in\mathbb{I}}$ – it does not. But before we can show this fact we need to show that we can approximate elements of $D_S([0,\infty))$ by piecewise constant functions with countably many jumps.

We denote the family of piecewise constant functions by

$$\begin{aligned}\mathcal{E}_S([0,\infty)) := \{\omega \in D_S([0,\infty)) \,\big|\, \exists\, (x_n)_{n\in\mathbb{N}} \subset S,\ \pi \in \Pi :& \\ \omega|_I = x_i, \forall i \in \mathbb{N},\ I \in \iota(\pi)\}&.\end{aligned} \tag{3.44}$$

We refer to the elements of $\mathcal{E}_S([0,\infty))$ as *simple paths*.

Lemma 3.15 (Simple paths are dense in $D_S([0,\infty))$)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space and $(\rho_i)_{i\in\mathbb{I}}$ a family of pseudometrics on S, as before. Denote by $\mathcal{D}$ the uniformity on $D_S([0,\infty))$ generated by the family $(\zeta^{\rho_i})_{i\in\mathbb{I}}$. Then the family of piecewise constant functions $\mathcal{E}_S([0,\infty))$ is sequentially dense in $D_S([0,\infty))$. That is, for every $\omega \in D_S([0,\infty))$ there exists a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathcal{E}_S([0,\infty))$ such that $\lim_{n\to\infty} \omega_n = \omega$.*

It is worth pointing out that in general topological spaces, a subset $D \subset S$ is dense if and only for each $x \in S$ there exists a *net* $(x_\alpha)_\alpha$ such that $\lim_\alpha x_\alpha = x$. Thus, sequential denseness is stronger than denseness in general.

Proof. Let $\omega \in D_S([0,\infty))$. It is sufficient to show that for fixed $T > 0$ there exists a sequence $(\overline{\omega}_n)_{n\in\mathbb{N}} \subset \mathcal{E}_S([0,\infty))$ such that

$$\lim_{n\to\infty} \tilde{\rho}_i^T(\omega, \overline{\omega}_n) = 0 \quad \forall i \in \mathbb{I}. \tag{3.45}$$

To construct such a sequence, take a sequence of partitions $\left(\pi^{(n)}\right)_{n\in\mathbb{N}}$ of the interval $[0,T]$ such that $\pi^{(n)} \in \Pi^n_T$ and $\lim_{n\to\infty} L\left(\pi^{(n)}\right) = 0$ and define

$$\overline{\omega}_n(t) := \begin{cases} \omega\left(\pi^{(n)}_{i-1}\right), & \text{if } \pi^{(n)}_{i-1} \le t < \pi^{(n)}_i \wedge T, \ i = 1,\dots,n \\ \omega(T), & \text{if } t \ge T. \end{cases} \tag{3.46}$$

Now fix $\varepsilon > 0$ and $i \in \mathbb{I}$. By Lemma 3.4 there exists a partition $\pi \in \Pi_T$ of $[0,T]$ such that

$$\sup_{s,t\in I} \rho_i(\omega(s),\omega(t)) < \varepsilon \quad \forall I \in \iota(\pi). \tag{3.47}$$

Clearly, we can choose π such that $L(\pi) < \varepsilon$. Then there exists a $\lambda \in \Lambda_T$ and a $n \in \mathbb{N}$ such that $\sup_{t\in[0,T]} |\lambda(t)-t| < \varepsilon$ and the partition $\lambda\left(\pi^{(n)}\right) = \left(\lambda(\pi^{(n)}_0), \lambda(\pi^{(n)}_0), \dots, \lambda(\pi^{(n)}_n)\right)$ refines π, i.e. for all $J \in \iota\left(\lambda\left(\pi^{(n)}\right)\right)$ there exists an $I \in \iota(\pi)$ such that $J \subset I$. It follows that

$$\sup_{t\in[0,T]} \rho_i\left(\omega(t), \overline{\omega}_n(\lambda(t))\right) < \varepsilon \tag{3.48}$$

and consequently $\tilde{\rho}^T_i(\omega, \overline{\omega}_n) < \varepsilon$, which concludes the proof. ■

Theorem 3.16. *Let $(S,\mathcal{U})$ be a uniform Hausdorff space and $(\rho_i)_{i\in\mathbb{I}}$ and $(\sigma_j)_{j\in\mathbb{J}}$ two families of pseudometrics that generate $\mathcal{U}$ and satisfy the conditions of Proposition 2.16. Then the uniformities on $D_S([0,\infty))$ generated by $(\zeta^{\rho_i})_{i\in\mathbb{I}}$ and $(\zeta^{\sigma_j})_{j\in\mathbb{J}}$ coincide.*

Proof. We will make use of the covering uniformities introduced in Section 2.4 to prove this theorem. Observe that by Lemma 2.26 the families of covers of S of the form

$$\mathcal{A}_{i,\varepsilon} := \left\{ \{\, \omega \in D_S([0,\infty)) \mid \zeta^{\rho_i}(\omega,\omega_0) < \varepsilon \,\} \,\middle|\, \omega_0 \in D_S([0,\infty)) \right\} \quad i \in \mathbb{I},\ \varepsilon > 0 \tag{3.49}$$

and

$$\mathcal{B}_{j,\varepsilon} := \left\{ \{\, \omega \in D_S([0,\infty)) \mid \zeta^{\sigma_j}(\omega,\omega_0) < \delta \,\} \,\middle|\, \omega_0 \in D_S([0,\infty)) \right\} \quad j \in \mathbb{J},\ \delta > 0 \tag{3.50}$$

form bases of the covering uniformities μ and ν of $D_S([0,\infty))$, respectively. Once we can show that each $\mathcal{A}_{i,\varepsilon}$ is refined by some $\mathcal{B}_{j,\delta}$ we obtain $\mu \subset \nu$ by the definition of a base and the conclusion follows by symmetry. As before, it suffices to show that for fixed $T > 0$ and every pair $(i,\varepsilon) \in \mathbb{I}\times(0,\infty)$ there exist a pair $(j,\delta) \in \mathbb{J}\times(0,\infty)$ such that for every $\omega_0 \in D_S([0,\infty))$

$$\left\{ \omega \in D_S([0,\infty)) \,\middle|\, \tilde{\sigma}^T_j(\omega,\omega_0) < \delta \right\} \subset \left\{ \omega \in D_S([0,\infty)) \,\middle|\, \tilde{\rho}^T_i(\omega,\omega_0) < \varepsilon \right\}. \tag{3.51}$$

Both $(\rho_i)_{i\in\mathbb{I}}$ and $(\sigma_j)_{j\in\mathbb{J}}$ are bases for the same uniformity on S. By the definition of covering uniformities this implies that for each pair $(i,\varepsilon)\in\mathbb{I}\times(0,\infty)$ there exists a pair $(j,\delta)\in\mathbb{J}\times(0,\infty)$ such that the cover $\left\{ B_{\rho_i}(x,\varepsilon) \,\middle|\, x\in S \right\}$ is refined by the cover $\left\{ B_{\sigma_j}(x,\delta) \,\middle|\, x\in S \right\}$. Without loss of generality, we can assume that $\delta<\varepsilon$. Now fix $\omega_0\in D_S([0,\infty))$. For every $\omega\in\left\{ \omega\in D_S([0,\infty)) \,\middle|\, \tilde{\sigma}_j^T(\omega,\omega_0)<\delta \right\}$ there exists a $\lambda\in\Lambda_T$ such that

$$\sup_{t\in[0,T]} |\lambda(t)-t|<\delta \quad\text{and}\quad \sup_{t\in[0,T]} \sigma_j(\omega(\lambda(t)),\omega_0(t))<\delta \tag{3.52}$$

By choice of (j,δ), this immediately implies $\sup_{t\in[0,T]}\rho_i(\omega(\lambda(t)),\omega_0(t))<\varepsilon$. Together with $\sup_{t\in[0,T]}|\lambda(t)-t|<\delta<\varepsilon$ we readily obtain $\tilde{\rho}_i^T(\omega,\omega_0)<\varepsilon$, concluding the proof. ■

There is a plethora of results on the Skorokhod topology for metric spaces and even more for $\mathbb{R}$. The next result is due to Jakubowki and gives a handy way to translate these results to the uniform setting.

Proposition 3.17 ([Jak86, Theorem 4.3])**.** *Let $\mathcal{F}\subset C(S)$ be a family of continuous real functions on S that is closed under addition and generates the topology on S. For $f\in\mathcal{F}$ denote by $\hat{f}: D_S([0,\infty))\to D_{\mathbb{R}}([0,\infty))$ the map defined by $\hat{f}(\omega)(\,\cdot\,):=f(\omega(\,\cdot\,))$. Then the Skorokhod topology on $D_S([0,\infty))$ is generated by the family $\widehat{\mathcal{F}}:=\left\{ \hat{f} \,\middle|\, f\in\mathcal{F} \right\}$.*

Using the fact that the topology induced by the weak uniformity generated by a family of functions $\mathcal{F}$ coincides with the weak topology generated by these functions, we readily get the following.

Corollary 3.18. *Let $\mathcal{F}$ be a family of uniformly continuous functions mapping S to $\mathbb{R}$ that is closed under addition and generates the uniformity on S. Then the Skorokhod topology on $D_S([0,\infty))$ is uniformizable and generated by the weak uniformity generated by the family $\widehat{\mathcal{F}}$. Moreover, this uniformity does not depend on the choice of $\mathcal{F}$. We call this uniformity* Skorokhod uniformity *and denote it with $\mathcal{D}=\mathcal{D}(S)$.*

We can use this definition of the Skorokhod topology together with [EK86, Proposition 3.6.5] to obtain a useful characterization of Skorokhod convergence.

Proposition 3.19 (Characterization of Skorokhod convergence)**.** *Let S be a uniform Hausdorff space. Assume $(\omega_n)_{n\geq 1}\subset D_S$ and $\omega\in D_S$. Then $\omega_n\to\omega$ in the Skorokhod topology if and only if for all $t\geq 0$ and sequences $(t_n)_{n\geq 1}\subset(0,\infty)$ with $\lim_{n\to\infty}t_n=t$ the following three conditions are satisfied* *(i)* *For every* $U \in \mathcal{U}$ *open,*

$$\omega_n(t_n) \in U[\omega(t)] \cup U[\omega(t-)] \quad \textit{eventually}. \tag{3.53}$$

(ii) *If there exists a subsequence* $(t_k)_{k\in\mathbb{N}}$ *of* $(t_n)_{n\in\mathbb{N}}$ *such that* $\lim_{k\to\infty} \omega_k(t_k) = \omega(t)$*, then for all sequences* $(s_k)_{k\geq 1}$ *with* $s_k \geq t_k$ *for each* $k \geq 1$ *and* $\lim_{k\to\infty} s_k = t$ *it holds that* $\lim_{k\to\infty} \omega_k(s_k) = \omega(t)$*.*

(iii) *If there exists a subsequence* $(t_k)_{k\in\mathbb{N}}$ *of* $(t_n)_{n\in\mathbb{N}}$ *such that* $\lim_{k\to\infty} \omega_k(t_k) = \omega(t-)$*, then for all sequences* $(s_k)_{k\geq 1}$ *with* $0 \leq s_k \leq t_k$ *for each* $k \geq 1$ *and* $\lim_{k\to\infty} s_k = t$ *it holds that* $\lim_{k\to\infty} \omega_k(s_k) = \omega(t-)$*.*

Proof. In the case where S is a metric space, the statement is just a reformulation of [EK86, Proposition 3.6.5]. Without repeating the proof we assume that the statement holds for $S = \mathbb{R}$.

Throughout this proof let $\mathcal{F} \subseteq \{ f : S \to \mathbb{R} \mid f \text{ unif. cont.} \}$ denote a family of uniformly continuous functions that generates $\mathcal{U}$. Recall from Proposition 2.13 and Remarks 2.14 that this means that for every $U \in \mathcal{U}$ there exists a $m \in \mathbb{N}$, functions $f_1, \dots, f_m \in \mathcal{F}$ and $\delta > 0$ such that

$$\bigcap_{j=1}^{m} F_j^{-1} B_\delta \subset U, \tag{3.54}$$

where $B_\delta = \left\{ (u, v) \in \mathbb{R}^2 \,\middle\|\, u - v| < \delta \right\}$ and $F_j : S^2 \to \mathbb{R}^2$ is given by $F_j(x, y) = (f_j(x), f_j(y))$, as usual.

As in Proposition 3.17 define $\hat{f} : D_S([0, \infty)) \to D_\mathbb{R}([0, \infty))$ by $\hat{f}(\omega)(t) = f(\omega(t))$ for each $f \in \mathcal{F}$ and denote $\hat{\mathcal{F}} = \left\{ \hat{f} \,\middle|\, f \in \mathcal{F} \right\}$.

We start with necessity. Let $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0, \infty))$ and assume that $\omega_n \to \omega \in D_S([0, \infty))$ with respect to the Skorokhod topology on $D_S([0, \infty))$. Let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$ be such that $t_n \to t \geq 0$. By continuity, $\hat{f}(\omega_n) \to \hat{f}(\omega)$ for each $\hat{f} \in \hat{\mathcal{F}}$. Applying (i) to the $D_\mathbb{R}([0, \infty))$ valued sequence $\left(\hat{f}(\omega_n)\right)_{n\in\mathbb{N}}$, we find for each $\varepsilon > 0$ and $\hat{f} \in \hat{\mathcal{F}}$ a $n_0 = n_0(f) \in \mathbb{N}$ such that

$$\begin{aligned} &|\hat{f}(\omega_n)(t_n) - \hat{f}(\omega)(t)| \wedge |\hat{f}(\omega_n)(t_n) - \hat{f}(\omega)(t-)| \\ &= |f(\omega_n(t_n)) - f(\omega(t))| \wedge |f(\omega_n(t_n)) - f(\omega(t-))| < \varepsilon \end{aligned} \tag{3.55}$$

for all $n \geq n_0$.

Now let $U \in \mathcal{U}$ be open and $m \in \mathbb{N}$, $f_1, \dots, f_m \in \mathcal{F}$ and $\delta > 0$ be such that (3.54) is satisfied. Taking the maximum over all $n_0(f_j)$ as above we find a $N_0 \in \mathbb{N}$ such that

$$f_j(\omega_n(t_n)) \in B_\delta[f_j(\omega(t))] \cup B_\delta[f_j(\omega(t-))] \tag{3.56}$$

for all $n > N_0$ and all $j = 1, \dots, m$. By continuity of the f_j and after choosing a smaller $\delta > 0$ or a bigger $N_0 \in \mathbb{N}$, if necessary, we can assume that if $f_j(\omega_n(t_n)) \in B_\delta[f_j(\omega(t))]$ ($f_j(\omega_n(t_n)) \in B_\delta[f_j(\omega(t-))]$) for some $j = 1, \dots, m$ then the same holds for all $j = 1, \dots, m$. Thus, for every $n > N_0$ we obtain

$$\omega_n(t_n) \in (F_j^{-1} B_\delta)[\omega(t)] \quad \text{for all } j = 1, \dots, n \tag{3.57}$$

or

$$\omega_n(t_n) \in (F_j^{-1} B_\delta)[\omega(t-)] \quad \text{for all } j = 1, \dots, n \tag{3.58}$$

and hence, by construction,

$$\omega_n(t_n) \in U[\omega(t)] \cup U[\omega(t-)] \tag{3.59}$$

for all $n > N_0$. The same argument can be applied to show (ii) and (iii).

We now show sufficiency. Assume $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ and $\omega \in D_S([0,\infty))$ are such that for all $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $\lim_{n\to\infty} t_n = t \geq 0$ (i)–(iii) hold. If we can show that $\hat{f}(\omega_n) \to \hat{f}(\omega)$ in $D_\mathbb{R}([0,\infty))$, we are done by Lemma A.34. Let $f \in \mathcal{F}$, by continuity we can deduce that (i)–(iii) hold for the sequence $\left(\hat{f}(\omega_n)\right)_{n\in\mathbb{N}}$ and hence, by assumption, $\hat{f}(\omega_n) \to \hat{f}(\omega)$, concluding the proof. ■

Next, observe that the space of càdlàg paths over a Polish uniform space is itself again a Polish uniform space. We phrase this result as a corollary to the known result for Polish metric spaces but it is possible to prove this fact directly.

Lemma 3.20 (Completeness and separability of the Skorokhod uniformity)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base. Then the Skorokhod uniformity on $D_S([0,\infty))$ is separable if $(S, \mathcal{U})$ is separable and complete if $(S, \mathcal{U})$ is complete.*

Proof. By assumption, $(S, \mathcal{U})$ is metrizable and by Lemma 2.39 every metric that induces $\mathcal{U}$ is complete and the claim follows directly from the corresponding theorem for metric spaces (cf. [EK86, Theorem 3.5.6]). ■

Observe that we can drop the assumption that $\mathcal{U}$ has a countable base in the statement of Lemma 3.20 and still show that separability of S implies separability of $D_S([0,\infty))$ by the same proof as in the metric case. Furthermore, we strongly believe that also completeness can be shown without the assumption of first countability. But as we will only use the statement only for metrizable spaces, we do not intend to prove it here.

3.3 Relative compactness in $D_S([0,\infty))$

Recall the notation we introduced for partitions in (3.12) and (3.13) and (3.14).

We seek to prove the following theorem which is a qualitative restatement of the quantitative statement of [EK86, Theorem 3.6.3].

Theorem 3.21 (Relative compactness in $D_S([0,\infty))$)**.** *Let $(S,\mathcal{U})$ be a complete uniform Hausdorff space with a countable base and $A \subset D_S([0,\infty))$. Then A is relatively compact if and only if the following conditions are satisfied for every $T > 0$.*

(i) There exists a compact $\Gamma_T \subset S$ such that for all $\omega \in A$ and $t \in [0,T]$, $\omega(t) \in \Gamma_T$.

(ii) For all $U \in \mathcal{U}$ and $\omega \in A$ there exists a $\delta = \delta(U) > 0$ depending only on U and a partition $\pi^\omega \in \Pi_T$ of $[0,T]$ with $l(\pi^\omega) > \delta$ such that

$$\{ (\omega(s), \omega(t)) \mid s,t \in I \} \subset U \quad \forall I \in \iota(\pi^\omega). \tag{3.60}$$

As before, this theorem can be regarded as a corollary to the corresponding statement for metric spaces. We will nevertheless include a proof that relies on the uniform structure to highlight its significance.

The proof relies on a Lemma from [EK86] which holds verbatim for uniform Hausdorff spaces. We present it here together with a proof for sake of completeness. First, we introduce the following notation for an $\omega \in D_S([0,\infty))$ with countably many jumps. Given such an ω we define the jump-times $s_j(\omega)$ of ω as follows. Let $s_0(\omega) := 0$ and for $k = 1,2,\dots$ let

$$s_k(\omega) := \inf\{ t > s_{k-1}(\omega) \mid \omega(t) \neq \omega(t-) \}, \tag{3.61}$$

if $s_{k-1}(\omega) < \infty$ and $s_k(\omega) = \infty$ if $s_{k-1}(\omega) = \infty$. Here we use the convention that $\inf \emptyset = \infty$.

Lemma 3.22. *[EK86, Lemma 3.6.1] Let $(S,\mathcal{U})$ be a uniform Hausdorff space and $\Gamma \subset S$ a compact subset. Fix $\delta > 0$ and define $A(\Gamma,\delta)$ to be the set of piecewise constant paths $\omega \in \mathcal{E}_S([0,\infty))$ such that $\omega(t) \in \Gamma$ for all $t \geq 0$ and $s_k(\omega) - s_{k-1}(\omega) > \delta$ for all $k \in \mathbb{N}$ with $s_k(\omega) < \infty$. Then $A(\Gamma,\delta)$ is relatively compact (in $D_S([0,\infty))$).*

Proof. Let $(\omega_n)_{n\in\mathbb{N}} \subset A(\Gamma,\delta)$. We need to show that there exists a convergent subsequence $(\omega_{n_m})_{m\in\mathbb{N}}$ of $(\omega_n)_{n\in\mathbb{N}}$. For $k \in \mathbb{N}$ denote by $M_k := \{ n \in \mathbb{N} \mid s_k(\omega_n) < \infty \}$ the set of indices for which $(\omega_n)_{n\in\mathbb{N}}$ has at least k jumps and observe that $M_{k+1} \subset M_k$. In the case where $|M_1| < \infty$ there exists a subsequence $(\omega_{n_m})_{m\in\mathbb{N}}$ such that $\omega_{n_m}(t) = x_m \in \Gamma$ for all $t \geq 0$. Because Γ is compact there exists another subsequence that converges.

Now assume $M := \sup\{\, k \in \mathbb{N} \mid |M_k| = \infty \,\} \geq 1$. Observe that M may be infinite. Then there exists a subsequence $(\omega_{n_m})_{m\in\mathbb{N}}$ such that $s_k(\omega_{n_m}) < \infty$ for all $k \leq M$ and all $m \in \mathbb{N}$. Choosing an adequate subsequence if necessary, we can assume without loss of generality that $\lim_{m\to\infty} s_k(\omega_{n_m}) = t_k$ exists ($t_M = \infty$ is possible) and $\lim_{m\to\infty} \omega_{n_m}(s_k(\omega_{n_m})) = x_k \in \Gamma$ for all $k \leq M$. By assumption we have $t_k - t_{k-1} \geq \delta > 0$ for all $k \leq M$ and hence $\lim_{m\to\infty} \omega_{n_m} = \omega$, where

$$\omega(t) = \begin{cases} x_k, & t \in [t_{k-1}, t_k),\ k = 1, \ldots, M \\ x_M, & t \geq t_M. \end{cases} \tag{3.62}$$

■

Proof of Theorem 3.21. Let $A \subset D_S([0,\infty))$ and assume that the conditions (i) and (ii) of the theorem hold. By Lemma 3.20, $D_S([0,\infty))$ is complete and by Lemma 2.35 it suffices to show that A is totally bounded.

Let $(\rho_i)_{i\in\mathbb{I}}$ be a family of pseudometrics on S that generates $\mathcal{U}$ and write $(\zeta_i)_{i\in\mathbb{I}}$ for the family of pseudometrics on $D_S([0,\infty))$ induced by $(\rho_i)_{i\in\mathbb{I}}$, as before. Fix a pair $(i,\varepsilon) \in \mathbb{I} \times (0,\infty)$ and choose $T > 0$ large enough such that $\int_T^\infty e^{-t}\,\mathrm{d}t < \varepsilon/2$. By (ii) there exists a $\delta > 0$ and for each $\omega \in A$ a partition $\pi^\omega \in \Pi_T$ with $l(\pi^\omega) > \delta$ such that

$$\sup_{s,t\in I} \rho_i(\omega(s), \omega(t)) < \varepsilon/2 \quad \forall I \in \iota(\pi^\omega). \tag{3.63}$$

For $\omega \in A$ define $\overline{\omega}$ as

$$\overline{\omega}(t) := \begin{cases} \omega\left(\pi^\omega_{k-1}\right), & t \in [\pi^\omega_{k-1}, \pi^\omega_k),\ k = 1, \ldots, N_\omega \\ \omega\left(\pi^\omega_{N_\omega}\right) & t \geq \pi^\omega_{N_\omega}. \end{cases} \tag{3.64}$$

Then $\overline{\omega} \in A(\Gamma_T, \delta)$ and $\zeta_i(\omega, \overline{\omega}) < \varepsilon$.

As $(\zeta_i)_{i\in\mathbb{I}}$ generates the uniformity $\mathcal{D}$ on $D_S([0,\infty))$, we have shown that for every entourage $D \in \mathcal{D}$ there exists a compact set Γ_D and a $\delta_D > 0$ such that

$$A \subset \bigcup_{\omega\in A(\Gamma_D,\delta_D)} D[\omega]. \tag{3.65}$$

Since $A(\Gamma_D, \delta_D)$ is totally bounded by Lemma 3.22 it follows from Lemma 2.31 that A itself is totally bounded.

Now assume that A is relatively compact. Then every sequence $(\omega_n)_{n\in\mathbb{N}} \subset A$ contains a converging subsequence. Furthermore, every sequence $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ contains a converging subsequence and thus, by Proposition 3.19, every sequence $(\omega_n(t_n))_{n\in\mathbb{N}}$ of paths contains a converging subsequence. Therefore, the set $\{\, \omega(t) \mid \omega \in A,\ t \in [0,T] \,\}$ is contained in a compact set $\Gamma_T \subset S$.

Now assume that (ii) does not hold. Then there exist $T > 0$, $U \in \mathcal{U}$ open and a sequence $(\omega_n)_{n\in\mathbb{N}} \subset A$ such that for all partitions $\pi^{(n)} \in \Pi_T$ of $[0, T]$ with $l(\pi^n) \geq 1/n$ there exists an interval $I_n \in \iota(\pi^{(n)})$ and $s, t \in I_n$ such that

$$(\omega_n(s), \omega_n(t)) \notin U. \tag{3.66}$$

Choosing an adequate subsequence if necessary, we can assume without loss of generality that there exists a $\omega \in D_S([0, \infty))$, not necessarily in A, such that $\lim_{n\to\infty} \omega_n = \omega$. By Lemma 3.4 there exists a $\delta > 0$ and a partition $\pi \in \Pi_T$ with $l(\pi) > \delta$ such that

$$\{ (\omega(s), \omega(t)) \mid s, t \in I \} \subset U \quad \forall I \in \iota(\pi). \tag{3.67}$$

For large enough $n \in \mathbb{N}$ we can choose the partitions $\pi^{(n)}$ such that $\pi^{(n)}$ refines π and $L(\pi) < 2/n$. Now let $(s_n)_{n\in\mathbb{N}} \subset [0, T]$ and $(t_n)_{n\in\mathbb{N}} \subset [0, T]$ be such that $s_n < t_n$ and $s_n, t_n \in I_n$ for some $I_n \in \iota(\pi^{(n)})$. By compactness, there exist converging subsequences $(s_{n_k})_{k\in\mathbb{N}}$ and $(t_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} s_{n_k} = s$ and $\lim_{k\to\infty} t_{n_k} = t$. By construction we have $t_{n_k} - s_{n_k} \to 0$ and hence $s = t$. On the other hand we have $(\omega_{n_k}(s_{n_k}), \omega_{n_k}(t_{n_k})) \notin U$ for all $k \in \mathbb{N}$. Using the fact that the sequence $(\omega_{n_k})_{k\in\mathbb{N}}$ converges and Proposition 3.19 we deduce that $\lim_{k\to\infty} \omega_{n_k}(s_{n_k}) = \omega(t-)$ and $\lim_{k\to\infty} \omega_{n_k}(t_{n_k}) = \omega(t)$ and $(\omega(t-), \omega(t)) \notin U$. But this contradicts (3.67) thus concluding the proof. ■

Observe that in the proof we have used the existence of a countable base only in the first paragraph to justify that $D_S([0, \infty))$ is complete. If we can show the Conjecture 7.1 the assumption of first countability and hence also the implicit assumption of metrizability can be dropped from the statement of the theorem.

Lemma 3.23 (Skorokhod convergence)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space with a countable base. Assume that $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0, \infty))$ and $\omega \in D_S([0, \infty))$. Then* $\lim_{n\to\infty} \omega_n = \omega$ *in the Skorokhod topology if and only if the following two conditions are satisfied.*

(i) For all sequences $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ with $\lim_{n\to\infty} t_n = t < \infty$ *and all open entourages $U \in \mathcal{U}$ there exists a $n_0 \in \mathbb{N}$ such that*

$$\omega_n(t_n) \in U[\omega(t-)] \cup U[\omega(t)] \quad \forall n > n_0. \tag{3.68}$$

(ii) The sequence $(\omega_n)_{n\in\mathbb{N}}$ is relatively compact.

Proof. We have shown necessity of (i) already in the proof of Proposition 3.19 and (ii) follows directly from the convergence of the sequence $(\omega_n)_{n\in\mathbb{N}}$.

For sufficiency, we show that together with (i), relative compactness implies that (ii) and (iii) of Proposition 3.19 hold. To that end assume that Proposition 3.19

(ii) fails. We want to show that the sequence $(\omega_n)_{n\in\mathbb{N}}$ cannot be relatively compact. Assume that there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ such that $\lim_{n\to\infty} t_n = t < \infty$ and $\lim_{n\to\infty}(\omega_n(t_n)) = \omega(t)$. Assume further that there exists another sequence $(s_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $\lim_{n\to\infty} s_n = t$, $\lim_{n\to\infty} \omega_n(s_n) = \omega(t-)$ and $s_n \geq t_n$ for all $n \in \mathbb{N}$, where $\omega(t-) \neq \omega(t)$. By the Hausdorff property, there exists an open entourage $U \subset \mathcal{U}$ such that $U[\omega(t)] \cap U[\omega(t-)] = \emptyset$ and we can choose another open entourage $V \in \mathcal{U}$ with the property $V \circ V \circ V \circ V \subset U$. By right continuity and definition of the left limit point there exists $\varepsilon > 0$ such that

$$\omega(r) \in V[\omega(t-)],\ \forall r \in [t-\varepsilon, t) \quad \text{and} \quad \omega(r) \in V[\omega(t)],\ \forall r \in [t, t+\varepsilon). \tag{3.69}$$

Choose $\delta \in (0,\varepsilon)$ fixed but arbitrary. By Lemma 3.2 there exist continuity points $u \in [t-\delta/2, t)$ and $v \in [t, t+\delta/2)$ of ω. By (i) we have

$$\lim_{n\to\infty} \omega_n(u) = \omega(u) \in V[\omega(t-)] \quad \text{and} \quad \lim_{n\to\infty} \omega_n(v) = \omega(v) \in V[\omega(t)]. \tag{3.70}$$

Observe that there exists an open entourage $W \in \mathcal{U}$ such that $W[\omega(u)] \subset V[\omega(t-)]$ and $W[\omega(v)] \subset V[\omega(t)]$. After choosing an adequate subsequence we can assume without loss of generality that

$$\omega_n(u) \in V[\omega(t-)], \quad \omega_n(v) \in V[\omega(t)], \tag{3.71}$$

$s_n, t_n \in (u, v)$ and

$$\omega_n(s_n) \in V[\omega(t-)], \quad \omega_n(t_n) \in V[\omega(t)], \tag{3.72}$$

for all $n \in \mathbb{N}$. If we now collect all the pieces, observe that we constructed for each $n \in \mathbb{N}$

$$t - \delta/2 < u < t_n \leq s_n < v < t + \delta/2 \tag{3.73}$$

with the property

$$\begin{aligned} &V[\omega_n(u)] \cap V[\omega_n(t_n)] = \emptyset, \quad V[\omega_n(t_n)] \cap V[\omega_n(s_n)] = \emptyset \\ \text{and} \quad &V[\omega_n(s_n)] \cap V[\omega_n(v)] = \emptyset. \end{aligned} \tag{3.74}$$

Since $v - u \leq \delta$ and $\delta \in (0,\varepsilon)$ was arbitrary this is a contradiction to the relative compactness condition in Theorem 3.21 and hence $(\omega_n)_{n\in\mathbb{N}}$ is not relatively compact.

If instead condition (iii) of Proposition 3.19 fails, we can exchange the roles of s_n and t_n in the previous argument to deduce that $(\omega_n)_{n\in\mathbb{N}}$ is not relatively compact. ■

Note that we have shown in the proof of Lemma 3.23 that relative compactness of a sequence $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ implies that $(\omega_n)_{n\in\mathbb{N}}$ satisfies both conditions (iii) of Proposition 3.19.

3.4 Convergence of paths via hitting times

In this section, we show that the convergence of a sequence $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ of paths is equivalent to the convergence of hitting times.

Throughout this section let $(S, \mathcal{U})$ denote a uniform Hausdorff space and equip $D_S([0,\infty))$ with the Skorokhod uniformity. The first important observation is that the hitting time operator is upper semi-continuous for open sets.

Lemma 3.24 (Semi-continuity of hitting times)**.** *For $A \subset S$ be consider the hitting time operator*

$$\tau_A : D_S([0,\infty)) \to [0,\infty], \quad \tau_A(\omega) := \inf\{\, t \geq 0 \mid \omega(t) \in A \,\}. \tag{3.75}$$

If A is open, the hitting time operator τ_A is upper semi-continuous, i.e. for all $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ such that $\lim_{n\to\infty} \omega_n = \omega \in D_S([0,\infty))$,

$$\limsup_{n\to\infty} \tau_A(\omega_n) \leq \tau_A(\omega). \tag{3.76}$$

Proof. We proceed by contradiction. Let $A \subset S$ be open and assume there exists a sequence $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ such that $\omega_n \to \omega \in D_S([0,\infty))$ and $\limsup_{n\to\infty} \tau_A(\omega_n) > \tau_A(\omega)$. Passing over to subsequences, we can assume without loss of generality that $t_n := \tau_A(\omega_n) > \tau_A(\omega)$ and $\lim_{n\to\infty} t_n = t > \tau_A(\omega)$. By definition of the hitting time and Lemma 3.2 there exists a continuity point $s \in (\tau_A, t)$ of ω such that $\omega(s) \in A$. Hence, there exists an open entourage $U \in \mathcal{U}$ such that $U[\omega(s)] \subset A$ and. By construction, $\omega_n(s) \notin U[\omega(s)]$ for all $n \in \mathbb{N}$ in contradiction to Proposition 3.19 (i). ■

The convergence of *all* hitting times of neighborhoods is certainly sufficient for the convergence of a sequence of paths $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ but it is not a necessary condition.

Example 3.25. Let $S = \mathbb{R}$ and consider the sequence of constant paths $\omega_n \equiv 1/n$ and $\omega \equiv 0$. Clearly, $\omega_n \to \omega$, as $n \to \infty$. But for the hitting times of $B = B(1,1) = \{\, x \in \mathbb{R} \mid |x-1| < 1 \,\}$ we have $\tau_B(\omega_n) = 0$ and $\tau_B(\omega) = \infty$. ■

The example shows that we have to allow the convergence of hitting times to fail for a few exceptional sets. We will show that the Skorokhod convergence implies the convergence of the hitting times of all slightly enlarged neighborhoods. A similar result in the metric case was proved by Rojas [Roj24].

Recall from Definition 3.7 that the first contact time of ω to a set $A \subset S$ is defined as

$$\gamma_A(\omega) := \inf\left\{ t > 0 \,\middle|\, \{\omega(t), \omega(t-)\} \cap \overline{A} \neq \emptyset \right\}, \tag{3.77}$$

Lemma 3.26. *Let $(S, \mathcal{U})$ be a uniform Hausdorff space, $\omega \in D_S([0,\infty))$ and $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ such that $\omega_n \to \omega$ in the Skorokhod topology as $n \to \infty$. Assume $A \subset S$ is such that $\gamma_A(\omega) = \tau_A(\omega)$. Then $\lim_{n\to\infty} \tau_A(\omega_n) = \tau_A(\omega)$.*

Proof. Let $A \subset S$ be such that the first contact time is the first hitting time, i.e. $\gamma_A(\omega) = \tau_A(\omega)$.

Fix $t \in [0, \tau_A(\omega)) \cap \mathbb{Q}$. By assumption, $\omega(t)$ and $\omega(t-)$ are contained in the open set $\complement\overline{A}$ hence there exists a $V \in \mathcal{U}$ open such that $V[\omega(t)] \cap \overline{A} = \emptyset$ and $V[\omega(t-)] \cap \overline{A} = \emptyset$. By Proposition 3.19 we have that $\omega_n(t) \in V[\omega(t-)] \cup V[\omega(t)]$ for all $n \in \mathbb{N}$ large enough. By right continuity, we even find for every $n \in \mathbb{N}$ large enough some $\varepsilon_n > 0$ such that $\omega_n(s) \in V[\omega(t-)] \cup V[\omega(t)]$ for all $s \in [t, t+\varepsilon_n)$. Hence there cannot exist a subsequence $(\omega_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \tau_A(\omega_{n_k}) = t$ and consequently $\limsup_{n\to\infty} \tau_A(\omega_n) \geq t$ for all $t < \tau_A(\omega)$. On the other hand, we have by semi-continuity of the hitting times that $\limsup_{n\to\infty} \tau_A(\omega_n) \leq \tau_A(\omega)$. ∎

Theorem 3.27 (Convergence via hitting times)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space, $\omega \in D_S([0,\infty))$ and $(\omega_n)_{n\in\mathbb{N}} \subset D_S([0,\infty))$ be relatively compact. Then the following are equivalent.*

(i) $\lim_{n\to\infty} \omega_n = \omega$ *in the Skorokhod topology.*

(ii) For all $x \in S$, $U \in \mathcal{U}$, all continuity points $s \geq 0$ of ω, and all $D \in \mathcal{U}$, there exists a $E \in \mathcal{U}$ with $E \subset D$ open, such that

$$\tau_{(U\circ E)[x]}(\omega_n \circ \theta_s) \to \tau_{(U\circ E)[x]}(\omega \circ \theta_s), \quad \text{as } n \to \infty. \tag{3.78}$$

(iii) For all $x \in S$, all continuity points $s \geq 0$ of ω and all $U \in \mathcal{U}$ open such that $\tau_{U[x]}(\omega \circ \theta_s) = \gamma_{U[x]}(\omega \circ \theta_s)$ it holds that

$$\tau_{U[x]}(\omega_n \circ \theta_s) \to \tau_{U[x]}(\omega \circ \theta_s), \quad \text{as } n \to \infty. \tag{3.79}$$

Proof. We start with the implication (i) ⇒ (iii). Assume that $\omega_n \to \omega$ in the Skorokhod topology. Then also $\omega_n \circ \theta_s \to \omega \circ \theta_s$ for every continuity point $s \geq 0$ of ω. We can thus assume without loss of generality $s = 0$. Now assume that (iii) does not hold, i.e. there exists an $x \in S$ and a $U \in \mathcal{U}$ open such that $\tau_{U[x]}(\omega) = \gamma_{U[x]}(\omega)$ fails. By upper semi-continuity Lemma 3.24 this implies

$$t := \limsup_{n\to\infty} \tau_{U[x]}(\omega_n) < \tau_{U[x]}(\omega) = \gamma_{U[x]}(\omega) \tag{3.80}$$

and furthermore $\{\omega(t), \omega(t-)\} \subset \complement\overline{U[x]}$. We can thus find a $V \in \mathcal{U}$ open such that $V[\omega(t)] \cup V[\omega(t-)] \subset \complement\overline{U[x]}$. We can then choose a subsequence $(\omega_k)_{k\in\mathbb{N}}$ such that $t_k := \tau_{U[x]}(\omega_k) \to t$. By Proposition 3.19 we have $\omega_k(t_k) \in V[\omega(t)] \cup V[\omega(t-)]$ for all $k \in \mathbb{N}$ large enough and by right continuity there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0, \infty)$ such that $\omega_k([t_k, t_k + \varepsilon_k)) \subset V[\omega(t)] \cup V[\omega(t-)]$ for all $k \in \mathbb{N}$ large enough. In contradiction to the definition of t_k.

The implication (iii) $\Rightarrow$ (ii) is a direct consequence of Lemma 3.9.

Finally, we turn to the implication (ii) $\Rightarrow$ (i). Assume $(\omega_n)_{n\in\mathbb{N}}$ does not converge to $\omega \in D_S([0, \infty))$. We want to apply Lemma 3.23 and Lemma 3.2 to find a continuity point $s \geq 0$ of ω and a subsequence $(\omega_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} \omega_m(s) = x$ exists and $\omega(s) \neq x$. Then the claim follows because the hitting times of either $U[x]$ or $U[\omega(t)]$ do not converge for all $U \in \mathcal{U}$ open and sufficiently small.

Since $(\omega_n)_{n\in\mathbb{N}}$ is relatively compact by assumption, assume that (i) fails. Namely, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ such that $\lim_{n\to\infty} t_n = t \geq 0$ and a $U \in \mathcal{U}$ open such that

$$\omega_n(t_n) \notin U[\omega(t-)] \cup U[\omega(t)] \quad \text{for infinitely many } n \in \mathbb{N}. \tag{3.81}$$

Passing over to a subsequence $(t_k)_{k\in\mathbb{N}}$ we can assume that (3.81) holds for all $k \in \mathbb{N}$. Furthermore, $\sup_{k\in\mathbb{N}} t_k =: T < \infty$ and by assumption, $(\omega_k(t_k))_{k\in\mathbb{N}} \subset \Gamma_T$ for some $\Gamma_T \subset S$ compact. Hence, there exists a further subsequence $(t_l)_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \omega_l(t_l) = x \in \Gamma_T$. Let $V \in \mathcal{U}$ be open with $V \circ V \subset U$. We can choose yet another subsequence $(t_m)_{m\in\mathbb{N}}$ such that

$$\omega_m(t_m) \in V[x] \quad \text{for all } m \in \mathbb{N} \tag{3.82}$$

and observe that

$$V[x] \cap (V[\omega(t-)] \cup V[\omega(t)]) = \emptyset. \tag{3.83}$$

Furthermore, by right continuity of ω and by definition of $\omega(t-)$ there exists a $\varepsilon > 0$ such that $\omega([t-\varepsilon, t+\varepsilon]) \subset V[\omega(t-)] \cup V[\omega(t)]$. Observe that $(t_m)_{m\in\mathbb{N}}$ contains either an increasing or a decreasing subsequence – or both. Without loss of generality, we assume that $(t_m)_{m\in\mathbb{N}}$ is either increasing or decreasing itself and treat both cases separately.

Assume that $(t_m)_{m\in\mathbb{N}}$ is increasing. By Lemma 3.2 there exists a continuity point $s \in (t-\varepsilon, t)$ of ω. By construction we have $t_m \geq s$ eventually and thus

$$\lim_{m\to\infty} \tau_{V[x]}(\omega_m \circ \theta_s) = 0. \tag{3.84}$$

on the other hand, $\tau_{V[x]}(\omega \circ \theta_s) \geq \varepsilon$. Observe that this holds for all open entourages $W \in \mathcal{U}$ with $W \subset V$.

Now assume that $(t_m)_{m\in\mathbb{N}}$ is decreasing. Again by Lemma 3.2 there exists a continuity point $s \in (t - \varepsilon/4, t)$ of ω and we have

$$\limsup_{m\to\infty} \tau_{V[x]}(\omega_m \circ \theta_s) \leq \varepsilon/2, \tag{3.85}$$

while $\tau_{V[x]}(\omega \circ \theta_s) \geq \varepsilon$. ∎

We conclude this section with a few simple examples to highlight the importance of the assumptions in Theorem 3.27.

Example 3.28 (shifts to discontinuity points)**.** The restriction of the shifts in the statement of Theorem 3.27 (ii) is necessary because the starting point plays a special role in the Skorokhod topology and shifting the starting point to a point of discontinuity may break convergence in the Skorokhod topology. To illustrate this, consider the path

$$\omega(t) = \mathbb{1}_{[1,\infty)}(t) \in D_{\mathbb{R}}([0,\infty)) \tag{3.86}$$

and the sequence $(\omega_n)_{n\in\mathbb{N}} \subset D_{\mathbb{R}}([0,\infty))$ given by

$$\omega_n(t) = \mathbb{1}_{[1+1/n,\infty)}(t), \tag{3.87}$$

for every $n \in \mathbb{N}$. ∎

Example 3.29 (relative compactness is necessary)**.** Let $\omega \in D_{\mathbb{R}}([0,\infty))$ be defined as

$$\omega(t) := \mathbb{1}_{[1,\infty)}(t). \tag{3.88}$$

For each $n \in \mathbb{N}$ let $\omega_n \in D_{\mathbb{R}}([0,\infty))$ be defined as

$$\omega_n(t) := \mathbb{1}_{[1-1/n,1)}(t) + \mathbb{1}_{[1+1/n,\infty)}(t). \tag{3.89}$$

Clearly, ω is discontinuous at $t = 1$ and continuous on $[0,\infty) \setminus \{1\}$. For every sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $\lim_{n\to\infty} t_n = t \in [0,\infty) \setminus \{1\}$ we have $\omega_n(t_n) \to \omega(t)$. For every sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $\lim_{n\to\infty} t_n = 1$ we have $\omega_n(t_n) \in \{\omega(1), \omega(1-)\} = \{1, 0\}$. Hence, the condition (i) of Proposition 3.19 is satisfied. On the other hand, (ii) of Proposition 3.19 fails. To see that, consider the sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ defined as

$$t_n := 1 - \frac{1}{2n} \quad \text{and} \quad s_n := 1 + \frac{1}{2n}. \tag{3.90}$$

Then, $\lim_{n\to\infty} t_n = 1 = \lim_{n\to\infty} s_n$ and $t_n \leq s_n$ for every $n \in \mathbb{N}$. Furthermore, it holds for every $n \in \mathbb{N}$

$$\omega_n(t_n) = 1 = \omega(1) \quad \text{and} \quad \omega_n(s_n) = 0 = \omega(1-). \tag{3.91}$$

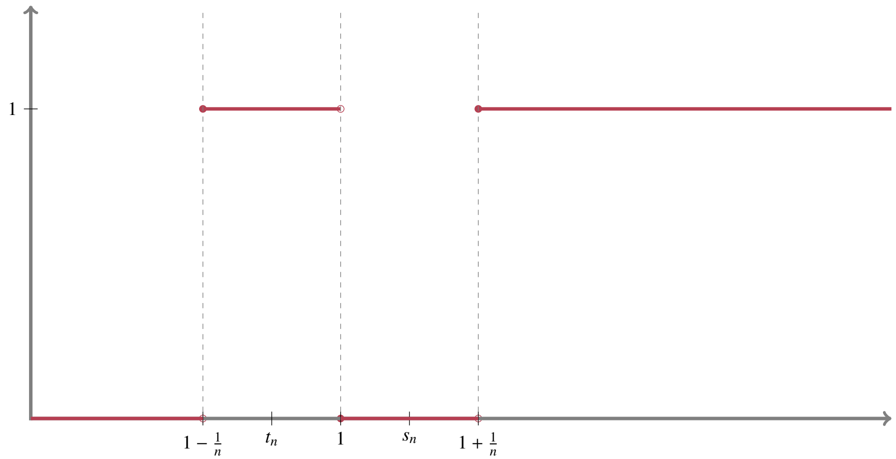

Fig. 3.1.: The process ω_n

Hence, we have found a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\omega_n(t_n) = \omega(t)$ and a sequence $(s_n)_{n\in\mathbb{N}}$ with the same limit, that dominates $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\omega_n(s_n) = \omega(t-) \neq \omega(t)$. Thus, ω_n does not converge to ω in the Skorokhod topology by Proposition 3.19.

On the other hand, we have

$$\tau_{B(x,\delta)}(\omega_n \circ \theta_s) \to \tau_{B(x,\delta)}(\omega \circ \theta_s) \tag{3.92}$$

for every $x \in \mathbb{R}$, $\delta > 0$ and every continuity point $s \geq 0$ of ω. ■

The following example was brought to our attention by Wolfgang Löhr [Löh21] and illustrates that the convergence of the hitting times can fail for infinitely many balls although the processes converge.

Example 3.30. Let $S = \mathbb{R}$ and $\omega \in D_S([0,\infty))$ be the path that starts in 1, waits for one unit of time and jumps by $1/2$; waits again $1/2$ unit of time and jumps by $1/4$ and so on in a geometric fashion. In other words

$$\omega(t) = \sum_{k=0}^{\infty} 2^{-k}\mathbb{1}_{[0,\sum_{j=0}^{k}2^{-j})}(t) = \sum_{k=0}^{\infty}(2-2^{-k})\mathbb{1}_{[2-2^{-k+1},2-2^{-k})}(t). \tag{3.93}$$

For $n \in \mathbb{N}$ and $t \geq 0$ let $\omega_n(t) := \omega(t) + 1/n$. Then $\omega_n \to \omega$ in the Skorokhod topology. Consider for example the ball $B(3,1)$ around 3 with radius 1. Then $\tau_{B(3,1)}(\omega_n) \leq 2$

and $\tau_{B(3,1)}(\omega) = \infty$. Furthermore, let $\delta > 0$ then there exists a $k \in \mathbb{N}$ such that $\varepsilon := (2 - 2^k) > 2 - \delta$ and for every $n \in \mathbb{N}$, we have

$$\tau_{B(3,1+\varepsilon)}(\omega_n) \leq 2 - 2^{-k+1} \tag{3.94}$$

but

$$\tau_{B(3,1+\varepsilon)}(\omega) = 2 - 2^{-k}. \tag{3.95}$$

hence, $\tau_{B(3,1+\varepsilon)}(\omega) - \tau_{B(3,1+\varepsilon)}(\omega_n) \geq 2^{-k}$, independently of n. ■

3.5 Random càdlàg paths

In this section, we introduce probability measures on the space of càdlàg functions $D_S([0,\infty))$ equipped with the Skorokhod uniformity. We will lift some of the statements we have proved in the last sections to the random elements. Namely, we will shortly introduce tightness for a family of probability measures and give criteria when a family of probability measures on $D_S([0,\infty))$ is tight. Finally, we will give a criterion for the convergence of a sequence of probability measures based on the hitting times of certain (uniform) neighborhoods.

We will consider mainly Polish uniform spaces. Recall from the discussion in Section 2.5 that if $(S, \mathcal{U})$ is a uniform Polish space, $D_S([0,\infty))$ also becomes a uniform Polish space when equipped with the Skorokhod uniformity $\mathcal{D}$. By Definition 2.38 and Lemma 2.39, $(S, \mathcal{U})$ and $(D_S([0,\infty)), \mathcal{D})$ are then completely metrizable. Instead of choosing one specific metric we continue as before and use the uniform structure thereby showing that many classical results can be translated by using uniformities instead of metrics.

We begin with some general remarks about probability measures on uniform spaces.

3.5.1 Probability measures on uniform spaces

Let $(S, \mathcal{U})$ be a uniform Hausdorff space. We equip S with the Borel-σ-field $\mathcal{B}$ and denote by $\mathcal{M}_1 = \mathcal{M}_1(S)$ the family of probability measures on $(S, \mathcal{B})$. In a similar fashion as we generalized the Skorokhod metric we can generalize the Prokhorov metric on the space of probability measures over a uniform space. Recall that there exists a family of pseudometrics $(\rho_i)_{i\in\mathbb{I}}$ for some $\mathbb{I} \neq \emptyset$ on S that generates $\mathcal{U}$ and satisfies (i) and (ii) of Proposition 2.16. For every $i \in \mathbb{I}$ we introduce the maps $\zeta_i : \mathcal{M}_1 \times \mathcal{M}_1 \to [0,1]$ as

$$\zeta_i(\mu, \nu) := \inf \{ \varepsilon > 0 \mid \mu(A) \leq \nu(B_i(A,\varepsilon)) + \varepsilon \text{ and } \nu(A) \leq \mu(B_i(A,\varepsilon)) + \varepsilon, \ \forall A \in \mathcal{B} \}, \tag{3.96}$$

where $B_i(A, \varepsilon) = \{ x \in S \mid \exists x \in A : \rho_i(x, y) < \varepsilon \}$ denotes the (open) ε-blowup of A with respect to ρ_i.

Proposition 3.31. *Let $(S, \mathcal{U})$ be a uniform space and $(\zeta_i)_{i \in \mathbb{I}}$ defined as above. Then the family $(\zeta_i)_{i \in \mathbb{I}}$ satisfies the conditions of Proposition 2.16 and the uniformity generated by $(\zeta_i)_{i \in \mathbb{I}}$ does not depend on the choice of the family $(\rho_i)_{i \in \mathbb{I}}$.*

Proof. First, let $\mu, \nu \in \mathcal{M}_1(S)$ with $\mu \neq \nu$. Without loss of generality there exists an $A \in \mathcal{B}$ such that $\mu(A) < \nu(A)$. Suppose

$$\inf \{ \varepsilon > 0 \mid \nu(A) \leq \mu(B_i(A, \varepsilon)) + \varepsilon \} = 0 \tag{3.97}$$

for all $i \in \mathbb{I}$. Then,

$$\nu(A) \leq \inf \{ \nu(U) \bigm| U \in \mathcal{B} \text{ open}, \ A \subset U \} + c, \tag{3.98}$$

where $c = \nu(A) - \mu(A) > 0$, in contradiction to the fact that ν is a probability measure and therefore (outer) regular. Hence we have verified (i) of Proposition 2.16.

Now let $i, j \in \mathbb{I}$. By (ii) of Proposition 2.16 there exists a $k \in \mathbb{I}$ such that $\rho_i \vee \rho_j \leq \rho_k$. Suppose again that $\mu, \nu \in \mathcal{M}_1(S)$ and $A \in \mathcal{B}$ such that $\mu(A) < \nu(A)$. Then, $B_i(A, \varepsilon) \cup B_j(A, \varepsilon) \subset B_k(A, \varepsilon)$ for all $\varepsilon > 0$. Consequently, $\mu(B_i(A, \varepsilon)) \vee \mu(B_j(A, \varepsilon)) \leq \mu(B_k(A, \varepsilon))$ and hence,

$$\inf \{ \varepsilon > 0 \mid \nu(A) \leq \mu(B_\alpha(A, \varepsilon)) + \varepsilon \} \geq \inf \{ \varepsilon > 0 \mid \nu(A) \leq \mu(B_k(A, \varepsilon)) + \varepsilon \} \tag{3.99}$$

for $\alpha = i, j$. Since $A \in \mathcal{B}$ was arbitrary we obtain $\zeta_i \vee \zeta_j \leq \zeta_k$, and this confirms Proposition 2.16 (ii).

The final part of the statement will follow from the next result, Proposition 3.32. □

We call the uniformity generated by $(\zeta_i)_{i \in \mathbb{I}}$ the *Prokhorov uniformity* and denote it by $\mathcal{D}_{\mathcal{M}}$. A different construction of the Prokhorov uniformity can be given in terms of the diagonal uniformity on S.

Proposition 3.32. *Let $(S, \mathcal{U})$ be a uniform Hausdorff space. Then the sets of the form*

$$D_{U,\varepsilon} := \left\{ (\mu, \nu) \in \mathcal{M}_1^2 \;\middle|\; \mu(A) \leq \nu(U[A]) + \varepsilon \text{ and } \nu(A) \leq \mu(U[A]) + \varepsilon, \ \forall A \in \mathcal{B} \right\}, \tag{3.100}$$

where $\varepsilon \in (0, 1)$ and $U \in \mathcal{U}$ form a base of the Prokhorov uniformity on $\mathcal{M}_1(S)$. The same holds true if we let ε range over a dense subset of $(0, 1)$ and U over some base $\mathcal{V}$ of $\mathcal{U}$.

Proof. We only show the second part of the statement. Let $D \in \mathcal{D}_{\mathcal{M}}$ without loss of generality we can assume that D is a basic entourage with respect to a family of pseudometric $(\zeta_i)_{i\in\mathbb{I}}$ as in Proposition 3.31. In other words, there exists a $i \in \mathbb{I}$ and a $0 < \delta < 1$ such that

$$D = \left\{ (\mu, \nu) \in \mathcal{M}_1^2 \,\middle|\, \zeta_i(\mu, \nu) < \delta \right\}. \tag{3.101}$$

By definition of ζ_i we can immediately conclude that D is of the form (3.100) with $U = \left\{ (x, y) \in S^2 \,\middle|\, \rho_i(x, y) < \delta \right\}$ and U an element of the basis $\mathcal{V}$ generated by the family of pseudometrics $(\rho_i)_{i\in\mathbb{I}}$. In every dense subset of $(0, 1)$ there exists an $\varepsilon > 0$ such that $\varepsilon < \delta$ and we find that

$$D_{U,\varepsilon} \subset D, \tag{3.102}$$

which yields the claim. Note that the uniformity $\mathcal{D}_{\mathcal{M}}$ does not depend on the choice of the base $\mathcal{V}$ or of the family $(\rho_i)_{i\in\mathbb{I}}$. ■

The following results are completely analogous of the results for the Prokhorov metric, as are their proofs. We leave the proofs to the reader as an exercise in handling uniformities.

Proposition 3.33. *Let $(S, \mathcal{U})$ be a uniform Hausdorff space and $\mathcal{D}_{\mathcal{M}}$ the Prokhorov uniformity on $\mathcal{M}_1(S)$. Then*

- *$(\mathcal{M}_1(S), \mathcal{D}_{\mathcal{M}})$ is separable if and only if $(S, \mathcal{U})$ is separable.*
- *$(\mathcal{M}_1(S), \mathcal{D}_{\mathcal{M}})$ has a countable base if and only if $(S, \mathcal{U})$ has a countable base.*
- *$(\mathcal{M}_1(S), \mathcal{D}_{\mathcal{M}})$ is complete if and only if $(S, \mathcal{U})$ is complete.*

The main feature of the Prokhorov metric is that it metricizes the weak convergence of (probability) measures. The analogue holds true for the Prokhorov uniformity.

Proposition 3.34. *Let $(S, \mathcal{U})$ be a uniform Hausdorff space and $\mathcal{D}_{\mathcal{M}}$ the Prokhorov uniformity on $\mathcal{M}_1(S)$. A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_1(S)$ converges weakly if and only if it converges with respect to the Prokhorov uniformity.*

As usual, we want to know more about the (relatively) compact subsets of $\mathcal{M}_1(S)$. Recall the following definition.

Definition 3.35 (Tightness)**.** Let $(S, \mathcal{U})$ be a uniform Hausdorff space. A family $A \subset \mathcal{M}_1(S)$ of probability measures is called *tight* if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that

$$\sup_{\mu\in A} \mu\left(\complement K_\varepsilon\right) < \varepsilon. \tag{3.103}$$

If $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $\Xi \subset \{X : \Omega \to S\}$ is a family of S-valued random variables, we say that Ξ is tight if the family $\left\{ \mathbb{P}^X = \mathbb{P} \circ X^{-1} \,\middle|\, X \in \Xi \right\} \subset \mathcal{M}_1(S)$ is tight. ◊

The next result is a simple extension of the well-known result for metric spaces.

Lemma 3.36. *Let $(S, \mathcal{U})$ be a uniform Polish space. Then every probability measure $\mu \in \mathcal{M}_1(S)$ is tight.*

For uniform spaces, we have the following version of Prokhorov's Theorem.

Proposition 3.37 (Prokhorov)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space and $A \subset \mathcal{M}_1(S)$ a family of probability measures on S.*

(i) If A is tight then A is relatively compact.

(ii) If in addition $(S, \mathcal{U})$ is separable, complete and possesses a countable base, then relative compactness of A implies tightness of A.

Proof. See [Kle14, Theorem 13.29]. The proof is easily adapted to the uniform setting. ■

3.5.2 Random paths

Let $(S, \mathcal{U})$ be a uniform Hausdorff space and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. We equip the space $(D_S([0, \infty)), \mathcal{D})$ with the Borel σ-field $\mathcal{B}_{\mathcal{D}}$ generated by the open sets and denote by $\pi_t : D_S([0, \infty)) \to S$ the projections $\omega \mapsto \pi_t(\omega) := \omega(t)$. The next result can be found in [EK86, Proposition 3.7.1] and the same proof can be applied almost verbatim for uniform spaces and we will omit it here.

Proposition 3.38. *Let $D \subset [0, \infty)$ be a dense subset. Then,*

$$\mathcal{B}_{\mathcal{D}} \supset \mathcal{B}'_{\mathcal{D}} := \sigma(\{ \pi_t \mid t \in [0, \infty) \}) = \sigma(\{ \pi_t \mid t \in D \}) . \tag{3.104}$$

If S is separable, we have $\mathcal{B}_{\mathcal{D}} = \mathcal{B}'_{\mathcal{D}}$.

For further reference, we cite the following result from Dudley's book [Dud02].

Lemma 3.39 ([Dud02, Proposition 4.1.7])**.** *Let S, T be two topological spaces. Then $\mathcal{B}(S) \otimes \mathcal{B}(T) \subset \mathcal{B}(S \times T)$. If both S and T are second countable, we have equality of the σ-fields.*

We continue with a simple observation.

Lemma 3.40. *Let* $(S, \mathcal{U})$ *be a separable uniform Hausdorff space with a countable base. Assume that* $U \in \mathcal{U}$ *is open. Then the sets*

$$\{\, \omega \in D_S([0,\infty)) \,\big|\, (\omega(s), \omega(t)) \in U \,\} \tag{3.105}$$

and

$$\{\, \omega \in D_S([0,\infty)) \,\big|\, (\omega(s-), \omega(t)) \in U \,\} \tag{3.106}$$

are $\mathcal{B}_{\mathcal{D}}$*-measurable for each* $0 \leq s < t$*.*

Proof. Let $0 \leq s < t$ and write $\pi_{s,t} \colon D_S([0,\infty)) \to S \times S$ for the map defined as $\pi_{s,t}(\omega) := (\pi_s(\omega), \pi_t(\omega)) = (\omega(s), \omega(t))$. By Proposition 3.38, the map $\pi_{s,t}$ is $\mathcal{B}_{\mathcal{D}} - \mathcal{B}^2$-measurable, where $\mathcal{B}^2$ denotes the product σ-field on $S \times S$. By assumption S is second countable and hence the product σ-field $\mathcal{B}^2$ and the Borel σ-field $\mathcal{B}(S \times S)$ on $S \times S$ generated by the product topology coincide by Lemma 3.39. Consequently, $U \in \mathcal{B}^2$ and the set in (3.105) is $\mathcal{B}_{\mathcal{D}}$-measurable as the preimage of a measurable set under a measurable map.

The measurability of the set in (3.106) follows by the same arguments if we can show that the map $\pi_{s-} \colon D_S([0,\infty)) \to S$ with $\pi_{s-}(\omega) := \omega(s-)$ is measurable. By definition of left limits, π_{s-} is the pointwise limit of $(\pi_{s_n})_{n\in\mathbb{N}}$ for every increasing sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \uparrow s$. Under the assumptions of the lemma S is metrizable and we can apply [Dud02, Theorem 4.2.2] to conclude that π_{s-} is, indeed, measurable. ∎

Assume that $X \colon \Omega \to D_S([0,\infty))$ is a $D_S([0,\infty))$-valued random variable that is a $\mathcal{A}$-$\mathcal{B}_{\mathcal{D}}$-measurable map. As before, we denote the Borel-σ-field on S by $\mathcal{B}$. By virtue of Proposition 3.38, the concatenations $X_t := \pi_t \circ X \colon \Omega \to D_S([0,\infty)) \to S$ are $\mathcal{A} - \mathcal{B}$-measurable for each $t \geq 0$ and hence S-valued random variables.

Proposition 3.41 (fdd's determine distribution of càdlàg random variables)**.** *Let* $(S, \mathcal{U})$ *be a separable uniform Hausdorff space and* X, Y *two* $D_S([0,\infty))$*-valued random variables. Assume that* X *and* Y *agree in their finite-dimensional distributions. Then,* X *and* Y *have the same distribution,* $\mathbb{P}^X = \mathbb{P}^Y$*.*

Proof. We use a classic Dynkin system argument. Consider the family

$$\mathcal{D} := \{\, A \in \mathcal{B}_D \mid \mathbb{P}(X \in A) = \mathbb{P}(Y \in A) \,\}. \tag{3.107}$$

Naturally, $\mathcal{D}$ is a Dynkin system. Let $T \subset [0, \infty)$ be a countable dense subset and define the family

$$\mathcal{E} := \left\{ \bigcap_{i=1}^{n} \pi_{t_i}^{-1} B_i \;\middle|\; n \in \mathbb{N};\; t_i \in T;\; B_i \in \mathcal{B}(S),\; \forall i = 1, \dots, n \right\}. \tag{3.108}$$

By measurability of the projections π_t we have $\mathcal{E} \subset \mathcal{B}_D$ and as a direct consequence of the definition of $\mathcal{E}$, the family is a π-system, i.e. $\mathcal{E}$ is closed under intersections. Furthermore, by Proposition 3.38, $\mathcal{E}$ generates $\mathcal{B}_D$.

By assumption, we have for all $A \in \mathcal{E}$,

$$\begin{aligned} \mathbb{P}(X \in A) &= \mathbb{P}\left((X_{t_1}, \dots, X_{t_n}) \in B_1 \times \cdots \times B_n\right) \\ &= \mathbb{P}\left((Y_{t_1}, \dots, Y_{t_n}) \in B_1 \times \cdots \times B_n\right) = \mathbb{P}(Y \in A) \end{aligned} \tag{3.109}$$

and hence $\mathcal{E} \subset \mathcal{D}$. By the Dynkin system argument (or Dynkin's π-λ-Theorem cf. e.g. [Kle14, Theorem 1.19]) we conclude that $\mathcal{D} = \mathcal{B}_D$ and hence $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all $A \in \mathcal{B}_D$. ■

Definition 3.42 (One-dimensional and finite-dimensional distributions)**.** Let $(S, \mathcal{U})$ be a uniform Hausdorff space and X a $D_S([0, \infty))$-valued random variable. The family of probability measures on S, $\mathbb{P}^{X_t} = \mathbb{P} \circ X_t^{-1}$ for $t > 0$ are called the *one dimensional distributions* of X. For any finite set $0 \leq t_1 < t_2 < \cdots < t_n$ of points we refer to the probability measure

$$\mathbb{P}^{(X_{t_1}, \dots, X_{t_n})} = \mathbb{P} \circ (X_{t_1}, \dots, X_{t_n})^{-1} \in \mathcal{M}_1(S^n) \tag{3.110}$$

as a *finite dimensional distribution* of X. ◇

It is useful to introduce the *canonical version* of X by identifying the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $(D_S([0, \infty)), \mathcal{B}_D, \mathbb{P}^X)$ where $\mathbb{P}^X := \mathbb{P} \circ X^{-1}$ denotes the push-forward of $\mathbb{P}$ under X. In this case, X is just the identity map and we can define the shift operator θ_t for $t > 0$ as

$$X \circ \theta_t = \omega(\,\cdot\, + t). \tag{3.111}$$

From now on we always implicitly assume that we are working with the canonical versions of the involved processes.

3.5.3 Tightness of random càdlàg paths

Let $(S, \mathcal{U})$ be a uniform Hausdorff space and X a $D_S([0, \infty))$-valued random variable. Recall the notations for partitions of the time axis we introduced in (3.10) and (3.12) to (3.14) of Section 3.1. For $T > 0$, $U \in \mathcal{U}$ and $\delta > 0$ we consider the event

$$W^T_{U,\delta}(X) := \{\exists \pi \in \Pi_T \text{ with } l(\pi) > \delta : (X_s, X_t) \in U \ \forall s,t \in I, \ I \in \iota(\pi)\} \subset \Omega. \quad (3.112)$$

For a family $\Xi \subset \{X : \Omega \to D_S([0,\infty))\}$ of $D_S([0,\infty))$-valued random variables defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $T > 0$, $U \in \mathcal{U}$, $\delta > 0$ we set

$$W^T_{U,\delta}(\Xi) := \bigcap_{X \in \Xi} W^T_{U,\delta}(X). \quad (3.113)$$

Theorem 3.43 (Tightness)**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space and Ξ a family of $D_S([0,\infty))$-valued random variables on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then Ξ is tight if and only if the following two conditions are satisfied.*

(i) For every $\varepsilon > 0$ and $T > 0$ there exists a compact set $\Gamma \subset S$ such that

$$\inf_{X \in \Xi} \mathbb{P}(\{ X_t \mid 0 \le t \le T \} \subset \Gamma) \ge 1 - \varepsilon. \quad (3.114)$$

(ii) For every $\varepsilon > 0$, $U \in \mathcal{U}$ and $T > 0$ there exists a $\delta > 0$ such that

$$\mathbb{P}\left(W^T_{U,\delta}(\Xi)\right) \ge 1 - \varepsilon. \quad (3.115)$$

Proof. Fix $\varepsilon > 0$. By tightness of Ξ there exists a $K \subset D_S([0,\infty))$ compact such that $\inf_{X \in \Xi} \mathbb{P}(X \in K) \ge 1 - \varepsilon$. As K is in particular relatively compact, we obtain (i) and (ii) from Theorem 3.21.

Now assume (i) and (ii) hold and fix $\varepsilon > 0$. We will construct a compact set $K \subset D_S([0,\infty))$ such that $\inf_{X \in \Xi} \mathbb{P}(X \in K) \ge 1 - \varepsilon$ establishing tightness of Ξ. Let $\Gamma \subset S$ be such that (3.114) holds for $\varepsilon/2$. Furthermore, let $U \in \mathcal{U}$ be open and choose $\delta > 0$ such that (3.115) holds for U and $\varepsilon/2$. Define $A = A(\Gamma, \delta)$ as in Lemma 3.22. Now let $\omega \in D_S([0,\infty))$ be such that there exists a partition $\pi \in \Pi_T$ of $[0,T]$ with $l(\pi) > \delta$ and

$$\{ (\omega(s), \omega(t)) \mid s,t \in I \} \subset U \quad \forall I \in \iota(\pi). \quad (3.116)$$

Furthermore, assume that $\omega([0,T]) \subset \Gamma$. By the same argument as in the proof of Theorem 3.21 we can find some $\overline{\omega} \in A$ and $D = D(U,\delta) \in \mathcal{D}$ open depending only on U and δ such that $(\omega, \overline{\omega}) \in D$. Since A is relatively compact by Lemma 3.22 we obtain from Lemma 2.31 that $D[A]$ is totally bounded and hence relatively compact by completeness. Taking $K = \overline{D[A]}$ we have found the desired compact set. ■

Next we want to derive equivalent conditions for the two conditions for tightness in Theorem 3.43. We start with condition (ii) and construct a partition explicitly.

Fix $\omega \in D_S([0,\infty))$, $U \in \mathcal{U}$ open and let $V \in \mathcal{U}$ open be such that $V \circ V \subset U$. For each $k \in \mathbb{N}_0$ we define τ_k, σ_k as follows. Let $\tau_0 = \sigma_0 \equiv 0$ and inductively define

$$\tau_k := \inf\{\, t > \tau_{k-1} \mid (\omega(t), \omega(\tau_{k-1})) \notin V \,\} \tag{3.117}$$

if $\tau_{k-1} < \infty$ and $\tau_k = \infty$ if $\tau_{k-1} = \infty$. And

$$\sigma_k := \sup\{\, t \le \tau_k \mid (\omega(t), \omega(\tau_k)) \notin V \text{ or } (\omega(t-), \omega(\tau_k)) \notin V \,\}, \tag{3.118}$$

if $\tau_k < \infty$ and $\sigma_k = \infty$, if $\tau_k = \infty$.

Furthermore, we write

$$\xi_k := \frac{\sigma_k + \tau_k}{2}, \quad k \in \mathbb{N}_0 \tag{3.119}$$

and observe that, by definition, $\lim_{k\to\infty} \xi_k = \infty$. Suppose now $\tau_{k+1} < \infty$. Then we have $(\omega(\tau_k), \omega(\tau_{k+1})) \notin V$ and hence $\sigma_{k+1} \ge \tau_k$. Thus,

$$\sigma_k \le \xi_k \le \tau_k \le \sigma_{k+1} \le \xi_{k+1} \le \tau_{k+1}, \tag{3.120}$$

for all $k \in \mathbb{N}_0$. If $\xi_k < \infty$ we obtain the following lower bound for the difference $\xi_{k+1} - \xi_k$ from (3.120)

$$\xi_{k+1} - \xi_k = \frac{\sigma_{k+1} + \tau_{k+1}}{2} - \frac{\sigma_k + \tau_k}{2} \ge \frac{\tau_k + \tau_{k+1}}{2} - \frac{\sigma_k + \tau_k}{2} = \frac{\tau_{k+1} - \sigma_k}{2}. \tag{3.121}$$

For the sake of readability we do not indicate the dependence of σ_k, τ_k and ξ_k on $V \in \mathcal{U}$ by notation at this point but note that the definitions very much depend on V (and U).

Now take $T > 0$ and $\delta > 0$. Assume that there exists a partition $\pi \in \Pi_T$ of $[0,T]$ with $l(\pi) > \delta$ such that $\{\, (\omega(s), \omega(t)) \mid s,t \in I \,\} \subset V$ for all $I \in \iota(\pi)$. From (3.120) we deduce that

$$\min\{\, \tau_{k+1} - \sigma_k \mid \tau_k < T \,\} > \delta, \tag{3.122}$$

as $\tau_{k+1} - \sigma_k \le \delta$ for some $k \ge 0$ would imply that any interval I of length at least δ with $\tau_k \in I$ also contains either σ_k or τ_{k+1} or both in its interior. That means there exist $s,t \in I$ such that $(\omega(s), \omega(t)) \notin V$, in contradiction to the assumption. Consequently, (3.121) implies that

$$\min\{\, \xi_{k+1} - \xi_k \mid \xi_k < T \,\} > \frac{\delta}{2}. \tag{3.123}$$

On the other hand, if we have (3.123) we can take

$$\pi' := (\xi_0, \xi_1, \ldots, \xi_{k+1}) \in \Pi_T \tag{3.124}$$

as a partition of $[0,T]$ with the properties $l(\pi') > \delta/2$ and $\{\, (\omega(s), \omega(t)) \mid s,t \in I \,\} \subset$

$V \circ V \subset U$ for all $I \in \iota(\pi')$.

It is now straightforward to extend our definitions of σ_k, τ_k and ξ_k to maps $D_S([0,\infty)) \to [0,\infty]$ for $k \in \mathbb{N}_0$. In fact, these maps are measurable.

Lemma 3.44 (Measurability of σ_k, τ_k, ξ_k)**.** *Let $(S, \mathcal{U})$ be a separable uniform Hausdorff space with a countable base. Then the maps $\sigma_k, \tau_k, \xi_k\colon D_S([0,\infty)) \to [0,\infty]$, $k \in \mathbb{N}_0$ as defined above are Borel measurable.*

Proof. By Lemma 3.20 and the subsequent remarks, $D_S([0,\infty))$ equipped with the Skorokhod uniformity is a separable uniform Hausdorff space. We proceed by induction. Clearly σ_0 and τ_0 are measurable as constant functions. Let $k \geq 1$ and consider the preimage $\tau_k^{-1}[0,s)$ of $[0,s)$ under τ_k for some $s > 0$. We use the event notation and write

$$\{\tau_k < s\} = \{\, \omega \in D_S([0,\infty)) \mid \tau_k(\omega) < s \,\} = \tau_k^{-1}[0,s). \tag{3.125}$$

We can decompose the set $\{\tau_k < s\}$ as

$$\{\tau_k < s\} = \{\tau_{k-1} < \infty\} \cap \bigcup_{t \in [0,s) \cap \mathbb{Q}} \Big(\{(\omega(\tau_{k-1}), \omega(t)) \notin V\} \cap \{\tau_{k-1} < t\} \Big). \tag{3.126}$$

By induction hypothesis, the sets $\{\tau_{k-1} < \infty\}$ and $\{\tau_{k-1} > t\}$ are measurable and it remains to show that the sets $\{(\omega(\tau_{k-1}), \omega(t)) \notin V\}$ are measurable. Again, by hypothesis τ_{k-1} is measurable and so is the map $\varphi_k\colon D_S([0,\infty)) \to S$, defined as $\varphi_k(\omega) = \omega(\tau_{k-1}(\omega))$, as composition of measurable maps by Lemma 3.3.We can thus follow the lines of the proof of Lemma 3.40 to conclude that $\{(\omega(\tau_{k-1}), \omega(t)) \in V\}$ is measurable, which implies the measurability of τ_k. Then, measurability of σ_k follows from the definition and the measurability of τ_k together with Lemma 3.40 and Lemma 3.3. Finally, measurability of ξ_k is a direct consequence of the measurability of σ_k and τ_k. ■

Given a random variable $X\colon \Omega \to D_S([0,\infty))$, the preceding lemma shows that the concatenations $\sigma_k^X := \sigma_k \circ X, \tau_k^X := \tau_k \circ X$ and $\xi_k^X := \xi_k \circ X$ are $[0,\infty]$-valued random variables. In order to make the dependence on the entourage V in the definition explicit, we add it to the superscript.

Lemma 3.45. *Let $(S, \mathcal{U})$ be a separable uniform Hausdorff space with a countable base and Ξ a family of $D_S([0,\infty))$-valued random variables on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then condition (ii) of Theorem 3.43 is equivalent to each of the following.*

(i) For all $U \in \mathcal{U}$ open and $T > 0$ it holds that

$$\liminf_{\delta \to 0} \inf_{X \in \Xi} \mathbb{P}\Big(W_{U,\delta}^T(\Xi) \Big) = 1. \tag{3.127}$$

(ii) *For all $U \in \mathcal{U}$ open and $T > 0$ it holds that*

$$\lim_{\delta\to 0} \inf_{X\in\Xi} \mathbb{P}\Big(\min\Big\{ \tau_{k+1}^{X,U} - \sigma_k^{X,U} \;\Big|\; k \in \mathbb{N} : \; \tau_k^{X,U} < T \Big\} \geq \delta\Big) = 1. \tag{3.128}$$

(iii) *For all $U \in \mathcal{U}$ open and $T > 0$ it holds that*

$$\lim_{\delta\to 0} \inf_{X\in\Xi} \mathbb{P}\Big(\min\Big\{ \xi_{k+1}^{X,U} - \xi_k^{X,U} \;\Big|\; k \in \mathbb{N} : \; \xi_k^{X,U} < T \Big\} \geq \delta\Big) = 1. \tag{3.129}$$

Proof. Clearly, (i) is equivalent to (ii) of Theorem 3.43. In the discussion above we have shown that (i) implies (ii) as well as (iii). Furthermore, we have shown that (iii) implies (i) hence the claim is established. ∎

3.5.4 Convergence of random càdlàg paths

For $A \subset S$ open recall the first hitting time operator $\tau_A : D_S([0,\infty)) \to \mathbb{R}_+$ from Definition 3.5 and the first contact time operator $\gamma_A : D_S([0,\infty)) \to \mathbb{R}_+$ from Definition 3.7, respectively.

Lemma 3.46 (Measurability of hitting times and contact times)**.** *Let $(S, \mathcal{U})$ be a separable uniform Hausdorff space with a countable base and $A \in \mathcal{B}$ a Borel subset of S. Then the maps $\tau_A, \gamma_A : D_S([0,\infty)) \to [0,\infty]$ are Borel measurable.*

Proof. Fix $s \geq 0$ and assume for now that $A \subset S$ is open. By right continuity of ω, the set $\tau_A^{-1}[0,s)$ can be written as

$$\{\tau_A < s\} = \bigcup_{t\in[0,s]\cap\mathbb{Q}} \{\, \omega \in D_S([0,\infty)) \mid \omega(t) \in A \,\} = \bigcup_{t\in[0,s]\cap\mathbb{Q}} \pi_t^{-1} A. \tag{3.130}$$

And the measurability of τ_A follows from the measurability of π_t for all $t \geq 0$. On the other hand, if $A \subset S$ is closed, we have by the same argument

$$\{\tau_A > s\} = \bigcap_{t\in[0,s]\cap\mathbb{Q}} \pi_t^{-1}(A \setminus S) \in \mathcal{B}(\mathbb{R}_{\geq 0}). \tag{3.131}$$

Now let $A \in \mathcal{B}(S)$ be arbitrary. Then A is the countable union of open or closed sets $(A_n)_{n\in\mathbb{N}}$, hence

$$\{\tau_A < s\} = \bigcup_{n\in\mathbb{N}} \{\tau_{A_n} < s\} \in \mathcal{B}(\mathbb{R}_{\geq 0}). \tag{3.132}$$

For the first contact time of an arbitrary Borel set $A \subset S$ we obtain

$$\begin{aligned}\{\gamma_A < s\} &= \bigcup_{t\in[0,s]\cap\mathbb{Q}} \left\{ \omega \in D_S([0,\infty)) \;\middle|\; \omega(t) \in \overline{A} \right\} \cup \left\{ \omega \in D_S([0,\infty)) \;\middle|\; \omega(t-) \in \overline{A} \right\} \\ &= \bigcup_{t\in[0,s]\cap\mathbb{Q}} \pi_t^{-1}\overline{A} \cup \pi_{t-}^{-1}\overline{A},\end{aligned} \tag{3.133}$$

which readily implies measurability of γ_A as π_{t-} is measurable for all $t \geq 0$ (see the proof of Lemma 3.40) ■

In metric spaces and in metrizable uniform spaces we have the following probabilistic version of Lemma 3.2.

Lemma 3.47 (cf. [EK86, Lemma 3.7.7]). *Let* $(S, \mathcal{U})$ *be a separable uniform Hausdorff space with a countable base. Assume that* X *is a* $D_S([0,\infty))$*-valued random variable. Then the set*

$$J(X) := \{\, t > 0 \mid \mathbb{P}(X_t \neq X_{t-}) > 0 \,\} \tag{3.134}$$

is at most countable.

Proof. Let $\delta > 0$ and $U \in \mathcal{U}$ open. For $T > 0$ fixed but arbitrary consider the set

$$J^T_{U,\delta}(X) := \{\, t \in [0,T] \mid \mathbb{P}((X_t, X_{t-}) \notin U) \geq \delta \,\}\,. \tag{3.135}$$

Assume the set $J^T_{U,\delta}(X)$ contains a sequence $(t_n)_{n\in\mathbb{N}}$ of distinct points and denote the jumps exceeding U by $A_n := \{(X_{t_n}, X_{t_n-}) \notin U\}$. By Fatou's lemma, we obtain

$$\mathbb{P}\left(\{(X_{t_n}, X_{t_n-}) \notin U \text{ infinitely often}\}\right) = \mathbb{P}(\liminf_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mathbb{P}(A_n) \geq \delta > 0, \tag{3.136}$$

in contradiction to Lemma 3.2 and hence $J^T_{U,\delta}(X)$ is finite. Letting $T \to \infty$ and $\delta \to 0$ we find that the set

$$J_U(X) := \{\, t > 0 \mid \mathbb{P}((X_t, X_{t-}) \notin U) > 0 \,\} \tag{3.137}$$

is at most countable. Finally, taking a sequence of open entourages $U_n \in \mathcal{U}$ with $U_n \supset U_{n+1}$ and $\bigcap_{n\geq 1} U_n = \Delta$, we conclude that

$$J(X) = \bigcup_{n\geq 1} J_{U_n}(X) \tag{3.138}$$

is at most countable. ■

Theorem 3.48 (Weak convergence of paths by weak convergence of hitting times)**.** *Let $(S, \mathcal{U})$ be a separable uniform Hausdorff space with a countable base. Assume that X, $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ are $D_S([0,\infty))$-valued random variables with distribution $\mathbb{P}^{X^{(n)}}$ and $\mathbb{P}^X$ respectively. Then, $\mathbb{P}^{X^{(n)}} \underset{n\to\infty}{\Longrightarrow} \mathbb{P}^X$ if and only if the following conditions are satisfied.*

(i) The sequence $\left\{ \mathbb{P}^{X^{(n)}} \,\middle|\, n \in \mathbb{N} \right\}$ is tight.

(ii) There exists a countable dense set $T \subset \{ t > 0 \mid X_t = X_{t-} \text{ a.s.} \}$, a countable dense subset $D \subset S$ and a countable base $\mathcal{V} \subset \mathcal{U}$ of $\mathcal{U}$ consisting of open entourages such that for all $x \in D$, all $V \in \mathcal{V}$ open with $\tau_{V[x]}(X) = \gamma_{V[x]}(X)$ a.s. and all $s \in T$ it holds that

$$\tau_{V[x]}\left(X^{(n)} \circ \theta_s\right) \xrightarrow{d} \tau_{V[x]}(X \circ \theta_s). \tag{3.139}$$

Proof. To keep the proof more readable we set $\mathbb{P}_n := \mathbb{P}^{X^{(n)}}$ for $n \in \mathbb{N}$. We begin with the implication "⇒". Assume $\mathbb{P}_n$ converges weakly to $\mathbb{P}^X$. Then (i) is obvious (cf. [Kal21, Theorem 23.2]) and it remains to show that (ii) holds. By Skorokhod's coupling theorem Theorem C.6 there exist $D_S([0,\infty))$-valued random variables $\xi, \left(\xi^{(n)}\right)_{n\in\mathbb{N}}$ on some probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ such that $\mathbb{P}^\xi = \mathbb{P}^X$ and $\mathbb{P}^{\xi^{(n)}} = \mathbb{P}_n$ for all $n \in \mathbb{N}$ and $\xi^{(n)} \to \xi$ $\mathbb{P}'$-a.s. Using Theorem 3.27 we conclude that there exists a $\mathbb{P}'$-nullset $N' \subset \Omega'$ such that for all $x \in S$, all $U \in \mathcal{U}$ open with $\tau_{U[x]}(\xi(\omega')) = \gamma_{U[x]}(\xi(\omega'))$ and all $s > 0$ with $\xi_s(\omega') = \xi_{s-}(\omega')$ it holds that

$$\lim_{n\to\infty} \tau_{U[x]}\left(\xi^{(n)}(\omega') \circ \theta_s\right) = \tau_{U[x]}\left(\xi(\omega') \circ \theta_s\right) \tag{3.140}$$

for all $\omega' \in \Omega' \setminus N'$. That implies (ii), as Lemma 3.47 ensures the existence of a countable dense set $T \subset \{ t > 0 \mid X_t = X_{t-} \text{ a.s.} \}$. Observe that we have actually shown the stronger conclusion that (3.139) holds for all $x \in S$ and $V \in \mathcal{U}$ open with $\tau_{V[x]}(X) = \gamma_{V[x]}(X)$.

For the reverse implication "⇐" assume that $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight. In order to show that $X^{(n)} \underset{n\to\infty}{\Longrightarrow} X$ we need to show that all subsequential limits of $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ have the same distribution. To that end assume there exists a random variable $Y : \Omega \to D_S([0,\infty))$ with distribution $\mathbb{P}^Y$ such that

$$\mathbb{P}_{n_k} \underset{k\to\infty}{\Longrightarrow} \mathbb{P}^Y \tag{3.141}$$

along a subsequence. Assume furthermore that (ii) holds, i.e. there exist countable dense subsets $D \subset S$ and $T \subset \{ t > 0 \mid X_t = X_{t-} \text{ a.s.} \}$ as well as a countable base $\mathcal{V} \subset \mathcal{U}$ of $\mathcal{U}$ such that (3.139) holds for all $x \in D$, $s \in T$ and all $V \in \mathcal{V}$ with $\tau_{V[x]}(X) = \gamma_{V[x]}(X)$. By (3.141) we have $X_{n_k} \underset{k\to\infty}{\Longrightarrow} Y$ and we can use what we have shown in the first part of the proof to deduce that for all $x \in D$ and $s \in T'$, where

$$T' \subset \{ t \geq 0 \mid X_t = X_{t-} \text{ and } Y_t = Y_{t-} \text{ a.s.} \} \tag{3.142}$$

is a countable dense subset by Lemma 3.47 and all $V \in \mathcal{V}$ with

$$\tau_{V[x]}(X) = \gamma_{V[x]}(X) \quad \text{and} \quad \tau_{V[x]}(Y) = \gamma_{V[x]}(Y) \tag{3.143}$$

it holds that

$$\tau_{V[x]}\left(Y^{(k)} \circ \theta_s\right) \xrightarrow{\text{d}} \tau_{V[x]}(Y \circ \theta_s) \stackrel{\text{d}}{=} \tau_{V[x]}(X \circ \theta_s). \tag{3.144}$$

Again, we can conclude from Skorokhod's coupling theorem, Theorem C.6 that there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $\zeta, \left(\xi^{(n)}\right)_{n\in\mathbb{N}}$ with $\mathbb{P}^{\zeta} = \mathbb{P}^{Y}$ and $\mathbb{P}^{\xi^{(k)}} = \mathbb{P}_k$ on Ω' such that $\xi^{(k)} \to \zeta$ almost surely. Furthermore, there exist a random variable ξ on Ω' with $\mathbb{P}^{\xi} = \mathbb{P}^{X}$ such that for all $x \in D$, $s \in T'$ and $V \in \mathcal{V}$ satisfying (3.143) there exists a nullset $N'(x, s, V) \subset \Omega'$ such that

$$\lim_{k\to\infty} \tau_{V[x]}\left(Y^{(k)}(\omega') \circ \theta_s\right) = \tau_{V[x]}(Y(\omega') \circ \theta_s) = \tau_{V[x]}(X(\omega') \circ \theta_s) \tag{3.145}$$

for all $\omega' \in \Omega' \setminus N'(x, s, V)$. As D, T' and $\mathcal{V}$ were assumed to be countable, the set

$$N' := \bigcup_{x\in D} \bigcup_{s\in T'} \bigcup_{V\in\mathcal{V}} N'(x, s, V) \tag{3.146}$$

is still a nullset and (3.145) holds for all $x \in D$, $s \in T'$ and all $V \in \mathcal{V}$ satisfying (3.143) outside the common nullset N'. From Theorem 3.10 we conclude that $\zeta(\omega') = \xi(\omega')$ for all $\omega' \in \Omega \setminus N'$, which implies that the laws of X and Y agree. Hence every subsequential limit of $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ has the same distribution as X and thus $\mathbb{P}_n \underset{n\to\infty}{\Longrightarrow} \mathbb{P}^X$, as claimed. ■

Symmetric Feller processes 4

> *At a purely formal level, one could call probability theory the study of measure spaces with total measure one, but that would be like calling number theory the study of strings of digits which terminate.*
>
> — **Terence Tao**
> Topics in random matrix theory

In this chapter, we introduce the main objects of this thesis, namely symmetric Feller processes. In this chapter we achieve two main results. First, we state with Theorem 4.72 that a Feller process X is uniquely determined by a family of Green operators associated with X. Finally, we give in Theorem 4.75 a tightness criterion for Feller processes.

The chapter is structured as follows: We first introduce time-homogeneous Markov processes and their semigroups. We then define Borel right processes and the associated resolvent or potential operators and Green operators. We further introduce the notions of symmetry and strong symmetry of Borel right processes. Next, we introduce the Feller property for semigroups. This leads to the notion of a Feller process and we will show that each Feller process possesses a modification with càdlàg sample paths. Furthermore, we will show that to each Feller semigroup, there exists a unique Feller process with càdlàg paths. This leads to the observation that a Feller process with càdlàg paths is uniquely determined by its family of resolvent operators. In preparation of the next chapter, we introduce Hunt processes, that is Feller processes with quasi-left continuous paths.

Up to this point, everything is standard and can be found in most textbooks on stochastic processes. Our main references are the books [Kal21] by Olav Kallenberg, [Kle14] by Achim Klenke and [CW05] by Kai Lai Chung and John B. Walsh. Although we state all our results in the framework of Polish uniform spaces, the classical results for Polish metric spaces apply as every Polish uniform space is completely metrizable by Definition 2.38 and Lemma 2.39. Nevertheless, we repeat the proofs of the most important results in order to remain as self-contained as possible and to highlight the fact that the actual *choice* of a metric does not matter and that all important properties are already captured by the uniform structure.

We then proceed to show that a Feller process is not only uniquely determined by its resolvent family but also by its family of Green operators. This result will play an important role in the proof of our convergence theorem Theorem 6.1.

Finally, we prove a tightness criterion for Feller processes that is closely related to Aldous's tightness criterion (cf. [Ald78, Theorem 1]) and conclude the chapter with a couple of examples.

Throughout this chapter let $(S, \mathcal{U}, \nu)$ denote a locally compact uniform measure space. As usual, we denote by $\mathcal{B}(S)$ the Borel σ-field on S. Furthermore, we write $\mathcal{B}(S;\mathbb{R})$ for the Borel measurable function $f\colon S \to \mathbb{R}$. When no confusion can occur, we drop the braces and write for both sets simply $\mathcal{B}$. We denote the set of bounded and Borel measurable functions by

$$\mathcal{B}_b = \mathcal{B}_b(S;\mathbb{R}) := \{\, f \in \mathcal{B}(S;\mathbb{R}) \mid \|f\|_\infty < a \in \mathbb{R} \,\} \tag{4.1}$$

and the family of non-negative Borel measurable functions by $\mathcal{B}^+ = \mathcal{B}^+(S,\mathbb{R}) \subset \mathcal{B}(S;\mathbb{R})$ with the obvious meaning of the combination $\mathcal{B}_b^+ = \mathcal{B}_b \cap \mathcal{B}^+$.

Finally we write $C = C(S;\mathbb{R})$ for the continuous real valued functions and introduce the following notations

$$C_b = C_b(S;\mathbb{R}) := \{\, f \in C \mid \|f\|_\infty < a \in \mathbb{R} \,\} \tag{4.2}$$

$$C_+ = C_+(S;\mathbb{R}) := \{\, f \in C \mid f(x) \geq 0\, \forall x \in S \,\} \tag{4.3}$$

$$C_\infty = C_\infty(S;\mathbb{R}) := \left\{\, f \in C_b \;\middle|\; \forall \varepsilon > 0\, \exists K \subset S \text{ compact s.t. } |f(x)| < \varepsilon,\ \forall x \in K^c \,\right\} \tag{4.4}$$

$$C_0 = C_0(S;\mathbb{R}) := \left\{\, f \in C_\infty \;\middle|\; \exists K \subset S \text{ compact s.t. } f(x) = 0\, \forall x \in K^c \,\right\} \tag{4.5}$$

for the bounded, the non-negative, the vanishing at infinity and the compactly supported continuous functions, respectively.

4.1 Markov processes

We begin with a fairly general introduction to stochastic processes. This section is kept as short as possible while trying to make this thesis as self-contained as possible. All the concepts put forward here are mathematical folklore and can be found in any textbook on Markov processes, for example [EK86], [MR06], [Lig10], [RY99, Chapter 3], [FOT11, Appendix 2], [Kal21], [Kle14], [KS98] or [CW05].

4.1.1 Stochastic processes

Definition 4.1 (Stochastic process)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $T \neq \emptyset$ some set of indices. A *stochastic process* on $(\Omega, \mathcal{A}, \mathbb{P})$ with index set T and values in the measurable space $(S, \mathcal{B})$ is a collection of mappings $X = \{ X_t \colon \Omega \to S \mid t \in T \}$ such that for each $t \in T$ the mapping $X_t \colon \Omega \to S$ is $\mathcal{A}/\mathcal{B}$-measurable and as such a S-valued random variable. ◇

Throughout this chapter, we assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and that $(S, \mathcal{U})$ is a Polish uniform space, that is, a separable uniform Hausdorff space with a countable base that is completely metrizable. It is useful to recall that in this situation the Borel σ-fields $\mathcal{B}(S \times S)$ and $\mathcal{B}(S) \otimes \mathcal{B}(S)$ coincide by Lemma 3.39.

Evaluating the process X at $k \in \mathbb{N}$ many points in T yields a probability measure on the product space S^k. This leads us to the following Definition (cf. Definition 3.42).

Definition 4.2 (Finite dimensional distributions & versions)**.** Let X be a stochastic process with values in $(S, \mathcal{B})$. For $k \in \mathbb{N}$ and $\{t_1, \dots, t_k\} \subset T$ define the *finite dimensional distribution* $\mathbb{P}^X_{t_1,\dots,t_k}$ of X as the push forward of the probability measure $\mathbb{P}$ on the product space $(S^k, \mathcal{B}(S^k))$ under the map $(X_{t_1}, \dots, X_{t_k}) \colon \Omega \to S^k$, i.e.

$$\mathbb{P}^X_{t_1,\dots,t_k}(A) := \mathbb{P}\left(\{(X_{t_1}, \dots, X_{t_k}) \in A\}\right), \quad A \in \mathcal{B}(S^k). \tag{4.6}$$

Furthermore, we say that two stochastic processes X and Y with values in $(S, \mathcal{B})$ are *versions* of each other, if they have the same finite-dimensional distributions, i.e. for all $k \in \mathbb{N}$ and $\{t_1, \dots, t_k\} \subset T$ it holds that $\mathbb{P}^X_{t_1,\dots,t_k} = \mathbb{P}^Y_{t_1,\dots,t_k}$. ◇

Note that in the above definition, the processes X and Y are not necessarily defined on the same probability space. In the case when X and Y are defined on the same probability space and their finite-dimensional distributions coincide, we say that X and Y are *modifications* of each other. In that case, we have $X_t = Y_t$ a.s. for all $t \in T$.

For the remainder of this chapter let X be a stochastic process indexed by time, i.e. we choose $T = [0, \infty)$ or, occasionally, $T = \mathbb{N}$ when we consider processes at discrete points in time.

For fixed $\omega \in \Omega$ we call the mapping $t \mapsto X_t(\omega)$ a *sample path*, or simply a *path*, of the process X. We say that the paths of X have a certain property (almost surely) if the mappings $t \mapsto X_t(\omega)$ have this property for ($\mathbb{P}$-almost) all $\omega \in \Omega$. By a slight abuse of terminology, we sometimes say that the process X has a property when the paths of X have that property.

We will only consider such processes which have almost surely right continuous paths with left limits. That is, we consider stochastic processes as random variables $X \colon \Omega \to D_S([0, \infty))$. This restriction will be justified later in this chapter when we show in Proposition 4.44 that Feller processes always admit a modification which has almost surely càdlàg sample paths.

If the state space S is not compact and X has càdlàg paths, the process X might leave the state space S in finite time with positive probability. We provide the following (non-rigorous) example as an illustration.

Example 4.3. Consider the process $X = (X_t)_{t\geq 0}$ with $X_t = (1 - W_t)^{-1}$, where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}$ started in 0. Then the process X explodes when the Brownian motion hits $\{1\}$ which happens with positive probability (even a.s.) in finite time as W has continuous paths. ■

Compactification

If the state space $(S, \mathcal{U})$ is locally compact, we can avoid the pitfalls that come with explosion by adjoining a point ϑ to S and consider the one-point compactification $(S_\vartheta, \mathcal{B}_\vartheta)$ of $(S, \mathcal{B})$ (see Definition A.22). Observe that every $f \in C_\infty$ can be canonically extended to a function $\hat{f} \in \hat{C} = C(S_\vartheta; \mathbb{R})$ by setting $\hat{f}(\vartheta) = 0$. Under this extension, every $f \in C_\infty$ is extended to a function $\hat{f} \in \hat{C}_\infty = C_\infty(S_\vartheta; \mathbb{R})$ and every $f \in C_0$ is extended to a function $\hat{f} \in \hat{C}_0 = C_0(S_\vartheta; \mathbb{R})$

The point ϑ serves as a *cemetery point* for the process, meaning that

$$\mathbb{P}(X_{t+s} \in \{\vartheta\} \,|\, X_t = \vartheta) = 1 \quad \text{for all } s, t \geq 0. \tag{4.7}$$

Extending this metaphor we define the *lifetime* ζ of X as

$$\zeta := \inf \{\, t \geq 0 \mid X_t = \vartheta \,\}, \tag{4.8}$$

where we define $\inf \emptyset = \infty$, as usual, and hence $\mathbb{P}(X_\infty = \vartheta) = 1$.

Generally, we call any state $x \in S$ that satisfies (4.7) *absorbing*.

Filtrations

Next, we introduce *filtrations*. Heuristically, filtrations capture the (incomplete) information available up to time $t \geq 0$.

Set $\mathcal{F}^0_\infty := \sigma(\{\, X_s \mid s \in [0, \infty) \,\})$ and $\mathcal{F}^0_t := \sigma(\{\, X_s \mid s \leq t \,\})$ for all $t \in [0, \infty)$, where $\sigma(\,\cdot\,)$ denotes the smallest σ-field that makes the content of the braces measurable.

For our purposes it is necessary to go a bit further into detail. We therefore collect some measure-theoretic notions in this paragraph for further reference.

Definition 4.4 (Admissible filtrations)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A family $(\mathcal{A}_t)_{t\geq 0}$ of sub σ-fields of $\mathcal{A}$ is called a *filtration* (of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$), if it is increasing, i.e. $\mathcal{A}_s \subset \mathcal{A}_t$ for all $s < t$. In that case, we say that $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ is a *filtered probability space*.

Suppose $X = (X_t)_{t\geq 0}$ is a stochastic process defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a measurable space $(S, \mathcal{B})$. A filtration $(\mathcal{A}_t)_{t\geq 0}$ of $(\Omega, \mathcal{A}, \mathbb{P})$ is called *admissible* (for X) if X_t is $\mathcal{A}_t/\mathcal{B}$-measurable for each $t \geq 0$. The process X is said to be *adapted* to $(\mathcal{A}_t)_{t\geq 0}$ if $(\mathcal{A}_t)_{t\geq 0}$ is admissible for X. ◊

Coming back to the heuristics for filtrations above, let X be adapted to the filtration $(\mathcal{A}_t)_{t\geq 0}$, and $A \in \mathcal{A}_t$ for some $t > 0$. Then we can decide whether the event A has occurred if we know the process X up to time t.

It is clear that $\left(\mathcal{F}_t^0\right)_{t\geq 0}$, where $\mathcal{F}_t^0 = \sigma(\{ X_s \mid s \leq t \})$ is an admissible filtration and we refer to it as the *minimal admissible filtration* or the *canonical filtration*.

A filtration $(\mathcal{A}_t)_{t\geq 0}$ is said to be *right continuous* if

$$\mathcal{A}_t = \mathcal{A}_{t+} := \bigcap_{s>t} \mathcal{A}_s \tag{4.9}$$

for all $t \geq 0$.

Remark 4.5 (Right continuous filtrations)**.** Every filtration $(\mathcal{A}_t)_{t\geq 0}$ can be turned into a right continuous filtration simply by setting $\mathcal{A}_t^+ := \mathcal{A}_{t+}$ for every $t \geq 0$. Clearly, $(\mathcal{A}_t^+)_{t\geq 0}$ is coarser that $\mathcal{A}$ in the sense that $\mathcal{A}_t \subset \mathcal{A}_t^+$ for every $t \geq 0$. If $\left(\mathcal{F}_t^0\right)_{t\geq 0}$ is the minimal admissible filtration then the right continuous filtration $(\mathcal{F}_t^+)_{t\geq 0}$ is an admissible filtration for X. ◊

Let $(S, \mathcal{B})$ be a measurable space and $\mu \in \mathcal{M}_1(S)$ a probability measure. We write $\mathcal{N}_\mu := \{ A \in \mathcal{B} \mid \mu(A) = 0 \}$ for the family of μ*-nullsets* or μ*-negligible* sets. For any sub σ-field $\mathcal{A}$ of $\mathcal{B}$ define the family of μ-negligible $\mathcal{A}$-sets as

$$\mathcal{N}_\mu(\mathcal{A}) \{ A \in \mathcal{A} \mid \mu(A) = 0 \} . \tag{4.10}$$

Recall that the *powerset* of a set A is the family of all subsets of A and denote the powerset of A by $\mathcal{P}(A)$. For any family of sets $\mathcal{A}$ we define

$$\mathcal{P}(\mathcal{A}) := \bigcup_{A\in\mathcal{A}} \mathcal{P}(A), \tag{4.11}$$

the union of all powersets of sets in $\mathcal{A}$. Now recall that the *completion* of a σ-field $\mathcal{A} \subset \mathcal{B}$ with respect to a measure $\mu \in \mathcal{M}_1(S)$ is defined as

$$\mathcal{A}^\mu := \sigma\left(\mathcal{A} \cup \mathcal{P}(\mathcal{N}_\mu(\mathcal{A}))\right) . \tag{4.12}$$

We define the universal completion of $\mathcal{A}$ with respect to a family $M \subset \mathcal{M}_1(S)$ of probability measures as the intersection of the completion of $\mathcal{A}^\mu$ over all probability measures $\mu \in M$:

$$\mathcal{A}^M := \bigcap_{\mu \in M} \mathcal{A}^\mu. \tag{4.13}$$

Definition 4.6 (Complete and augmented filtrations)**.** Let $(\mathcal{A}_t)_{t\geq 0}$ be a filtration of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

(i) The *completion* of the filtration $(\mathcal{A}_t)_{t\geq 0}$ with respect to $\mathbb{P}$ is the filtration $\left(\overline{\mathcal{A}}_t\right)_{t\geq 0}$ defined by

$$\overline{\mathcal{A}}_t := \sigma\left(\mathcal{A}_t \cup \mathcal{P}(\mathcal{N}_{\mathbb{P}}(\mathcal{A}))\right). \tag{4.14}$$

(ii) The *augmentation* of $(\mathcal{A}_t)_{t\geq 0}$ with respect to $\mathbb{P}$ is the filtration $(\mathcal{A}_t^*)_{t\geq 0}$ defined by

$$\mathcal{A}_t^* := \sigma\left(\mathcal{A}_t \cup \mathcal{P}(\mathcal{N}_{\mathbb{P}}(\mathcal{A}_t))\right). \tag{4.15}$$

◇

Some authors say that a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ satisfies the *usual conditions* if $(\mathcal{A}_t)_{t\geq 0}$ is right continuous and augmented with respect to $\mathbb{P}$ (cf. [Kle14, Definition 21.22]).

Again, it is an immediate consequence of the definition that if $(\mathcal{F}_t)_{t\geq 0}$ is admissible for X, then its completion $\left(\overline{\mathcal{F}}_t\right)_{t\geq 0}$ and augmentation $(\mathcal{F}_t^*)_{t\geq 0}$ are admissible for X as well.

Stopping times

Let $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ be a probability space and $\tau\colon \Omega \to [0, \infty]$ a random variable. We continue to interpret the positive real axis as time, in the same spirit we call such a random variable τ a *random time*. If X is a stochastic process on the same probability space, we write X_τ as a shorthand for the random variable $\omega \mapsto X_\tau(\omega) = X_{\tau(\omega)}(\omega)$. Using the established terminology, we refer to the process $(X_{t\wedge\tau})_{t\geq 0}$ as the process *killed at time* τ or, simply, the *killed process*. If the state space is locally compact, it is sometimes convenient to introduce a process $\hat{X}$ where we set $\hat{X}_\tau = \vartheta$ for the cemetery point of the one-point compactification S_ϑ of S, that is, we move the process to the cemetery right when it is killed.[1]

We are mostly interested in random times τ that are related to a stochastic process X in a way that we can determine whether $\tau \leq t$ if we know the process X up to time t. That leads to the following definition.

[1] Kallenberg [Kal21, p. 378] calls this terminology *morbid* but concedes that it is well established.

Definition 4.7 (Optional times and stopping times)**.** Let $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $\tau\colon \Omega \to [0,\infty]$ a random time.

(i) τ is called a $(\mathcal{A}_t)_{t\geq 0}$-*optional time* if

$$\{\tau < t\} \in \mathcal{A}_t \quad \forall t \geq 0. \tag{4.16}$$

(ii) τ is called a $(\mathcal{A}_t)_{t\geq 0}$-*stopping time* if

$$\{\tau \leq t\} \in \mathcal{A}_t \quad \forall t \geq 0. \tag{4.17}$$

◇

It is worth pointing out that many authors do not distinguish between stopping times and optional times. In [Kle14], Achim Klenke uses stopping times in the sense of our Definition 4.7. On the other hand, Kallenberg in [Kal21] or Chung and Walsh in [CW05] use the terms synonymously and use the term *optional time* for stopping times in the sense of Definition 4.7. The disambiguation we use here is due to Ioannis Karatzas and Steven E. Shreve as found in [KS98, Defintion 1.2.1]. The two definitions are quite similar. Indeed they coincide for right-continuous filtrations.

Lemma 4.8 (Stopping and optional times)**.** *Let $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Then, every stopping time is optional. If, in addition, the filtration $(\mathcal{A}_t)_{t\geq 0}$ is right continuous, then every optional time is a stopping time.*

Proof. Assume τ is a stopping time. Observe that

$$\left\{\tau \leq t - \frac{1}{n}\right\} \in \mathcal{A}_{t-\frac{1}{n}} \subset \mathcal{A}_t, \tag{4.18}$$

for every $t \geq 0$ and $n \in \mathbb{N}$ with $t \geq \frac{1}{n}$. Hence,

$$\{\tau < t\} = \bigcup_{n\in\mathbb{N}} \left\{\tau \leq t - \frac{1}{n}\right\} \in \mathcal{A}_t, \tag{4.19}$$

and we have shown the first assertion. Now let $(\mathcal{A}_t)_{t\geq 0}$ be right continuous and assume that τ is an optional time. Analogously to (4.19) we can write

$$\{\tau \leq t\} = \bigcap_{n\in\mathbb{N}} \left\{\tau < t + \frac{1}{n}\right\}. \tag{4.20}$$

Now, $\{\tau < t + 1/n\} \in \mathcal{A}_{t+\frac{1}{n}}$ for each $n \in \mathbb{N}$ and thus,

$$\{\tau \leq t\} \in \bigcap_{n\in\mathbb{N}} \mathcal{A}_{t+\frac{1}{n}} = \mathcal{A}_{t+} = \mathcal{A}_t, \tag{4.21}$$

which is what we wanted to show. ■

The simplest yet important example of stopping times are constant times. Let $s > 0$ as $s \le t$ is either the empty set or the whole of Ω and thus contained in $\mathcal{A}_t$ for every $t \ge 0$.

From now on we will always implicitly assume that $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\ge 0}, \mathbb{P})$ is a filtered probability space and omit the reference to the filtration when no confusion can occur. For a stopping time τ we introduce the *σ-field of the τ-past*,

$$\mathcal{A}_\tau := \{ A \in \mathcal{A} \mid A \cap \{\tau \le t\} \in \mathcal{A}_t \quad \forall t \ge 0 \} . \tag{4.22}$$

It is straightforward to check that $\mathcal{A}_\tau$ is indeed a σ-field.

Lemma 4.9. *Let τ, σ be stopping times with $\sigma(\omega) \le \tau(\omega)$ for all $\omega \in \Omega$. Then, $\mathcal{A}_\sigma \subset \mathcal{A}_\tau$.*

Proof. Let $A \in \mathcal{A}_\sigma$ and $t \ge 0$. By definition of $\mathcal{A}_\sigma$, we have $A \cap \{\sigma \le t\} \in \mathcal{A}_t$. Since τ is a stopping time we also have $\{\tau \le t\} \in \mathcal{A}_t$. By assumption, $\sigma \le \tau$ and thus $\{\tau \le t\} \subset \{\sigma \le t\}$. Hence

$$A \cap \{\tau \le t\} = (A \cap \{\sigma \le t\}) \cap \{\tau \le t\} \in \mathcal{A}_t \tag{4.23}$$

and therefore $A \in \mathcal{A}_\tau$. ■

We present a well-known lemma that shows that certain operations on optional times yield new optional times.

Lemma 4.10 (optional times)**.** *Let $(S, \mathcal{A}, (\mathcal{A}_t)_{t\ge 0}, \mathbb{P})$ be a filtered probability space. Assume $(\tau_n)_{n\in\mathbb{N}}$, τ and σ are optional times. Then the following random times are also optional:*

(i) $\sigma \vee \tau$ *and* $\sigma \wedge \tau$,

(ii) $\tau + \sigma$,

(iii) $\sup_{n\in\mathbb{N}} \tau_n$ *and* $\inf_{n\in\mathbb{N}} \tau_n$,

(iv) $\limsup_{n\to\infty} \tau_n$ *and* $\liminf_{n\to\infty} \tau_n$.

Proof. We obtain (i) from the simple observation

$$\{\sigma \vee \tau < t\} = \{\sigma < t\} \cap \{\tau < t\} \quad \text{and} \quad \{\sigma \wedge \tau \ge t\} = \{\sigma \ge t\} \cap \{\tau \ge t\}. \tag{4.24}$$

Now fix $t_0 > 0$. By (i) we have that $\sigma \wedge t_0$ and $\tau \wedge t_0$ are optional times. We first show that both random times are $\mathcal{A}_{t_0}$-measurable. Suppose $t \leq t_0$, then

$$\{\sigma \wedge t_0 < t\}, \{\tau \wedge t_0 < t\} \in \mathcal{A}_t \subset \mathcal{A}_{t_0}. \tag{4.25}$$

Now suppose $t > t_0$, then $(\sigma \wedge t_0)$ and $(\tau \wedge t_0)$ are both bounded by t and hence $\{\sigma \wedge t_0 < t\} = \{\tau \wedge t_0 < t\} = \Omega \in \mathcal{A}_{t_0}$. Now define $\hat{\sigma} := (\sigma \wedge t_0) + \mathbb{1}_{\sigma \geq t_0}$ and $\hat{\tau} := (\tau \wedge t_0) + \mathbb{1}_{\tau \geq t_0}$ and observe that both $\hat{\sigma}$ and $\hat{\tau}$ as well as the sum $\hat{\sigma} + \hat{\tau}$ are $\mathcal{A}_{t_0}$-measurable, by construction. It is now straight forward to check the equality

$$\{\sigma + \tau < t_0\} = \{\hat{\sigma} + \hat{\tau} < t_0\} \in \mathcal{F}_{t_0}. \tag{4.26}$$

Since $t_0 > 0$ was arbitrary, this proves (ii).

For (iii) we set $\hat{\sigma} := \inf_{n \in \mathbb{N}} \tau_n$ and $\hat{\tau} := \sup_{n \in \mathbb{N}} \tau_n$. Then, for all $t \geq 0$,

$$\{\hat{\sigma} < t\} = \bigcup_{n \in \mathbb{N}} \{\tau_n < t\} \in \mathcal{A}_t. \tag{4.27}$$

On the other hand, by Lemma 4.8, every optional time is a stopping time of the right continuous filtration $(\mathcal{A}_t^+)_{t \geq 0}$. Hence,

$$\{\hat{\tau} \leq t\} = \bigcap_{n \in \mathbb{N}} \{\tau_n \leq t\} \in \mathcal{A}_t^+. \tag{4.28}$$

Thus, $\hat{\tau}$ is again a stopping time with respect to the right continuous filtration $(\mathcal{A}_t^+)_{t \geq 0}$. Applying again Lemma 4.8, shows that $\hat{\tau}$ is indeed optional with respect to $(\mathcal{A}_t)_{t \geq 0}$.

Finally, (iv) follows from (iii) by the fact that

$$\limsup_{n \to \infty} \tau_n = \inf_{m \in \mathbb{N}} \sup_{n \geq m} \tau_n \tag{4.29}$$

and

$$\liminf_{n \to \infty} \tau_n = \sup_{m \in \mathbb{N}} \inf_{n \geq m} \tau_n, \tag{4.30}$$

thus completing the proof. ∎

Remark 4.11. Observe that (i), (ii) and the first part of (iii) of Lemma 4.10 hold also for stopping times. But in general, $\inf_{n \in \mathbb{N}} \tau_n$ is not again a stopping time for any sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times. ◇

4.1.2 Markov processes

We now introduce the Markov property and define Markov processes. Loosely speaking, the Markov property means that the past and the future of a stochastic process are independent of each other given the present. Despite the general definition, we will restrict ourselves to time-homogeneous Markov processes, as they come with useful analytic features like the transition semigroups and resolvent families. Furthermore, we will define stopping times and introduce the Green operators. Finally, we introduce a further subclass of Markov processes namely Borel right processes and we define what it means for such processes to be symmetric or strongly symmetric.

Definition 4.12 (Markov kernel)**.** Let $(\Omega, \mathcal{A})$ and $(S, \mathcal{B})$ be two measurable spaces. A *Markov kernel* (or *stochastic kernel*) is a map $\kappa\colon \Omega \times \mathcal{B} \to [0,1]$ with the following properties

(i) for each $B \in \mathcal{B}$, the map $\kappa(\,\cdot\,, B)\colon \Omega \to [0,1]$ is measurable,

(ii) $\kappa(\omega, \,\cdot\,)$ is a probability measure on $(S, \mathcal{B})$ for each $\omega \in \Omega$.

If instead of (ii), for all $\omega \in \Omega$, $\kappa(\omega, \,\cdot\,)$ is a finite Borel measure on $(S, \mathcal{B})$ with $\kappa(\omega, S) \leq 1$ for all $\omega \in \Omega$ then κ is called a *sub Markov* (or *substochastic*) kernel. ◊

Definition 4.13 (Markov process)**.** Let $X = \{ X_t \mid t \in [0, \infty] \}$ be a stochastic process on the filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \geq 0}, \mathbb{P})$. We say that X hast the *Markov property* if for each $t > 0$

$$\mathbb{E}[Y \mid \mathcal{A}_t] = \mathbb{E}[Y \mid X_t] \tag{4.31}$$

almost surely for all $\sigma(\{ X_s \mid s \geq t \})$ measurable Y.

The stochastic process X with state space S is called a *Markov process* if it possesses the Markov property and if there exists a family of probability measures $\{ \mathbb{P}_x \mid x \in S_\vartheta \}$ on $(\Omega, \mathcal{A})$ such that

(i) the map $x \mapsto \mathbb{P}_x(X_t \in B) \in [0,1]$ is Borel measurable for each $t \geq 0$ and $B \in \mathcal{B}$ and

(ii) for all $x \in S_\vartheta$ it holds that $\mathbb{P}_x(X_0 = x) = 1$. ◊

Denote by $\mathcal{M}_1(S) \subset \mathcal{M}_f(S) \subset \mathcal{M}(S)$ the set of probability measures on S, the set of finite measures on S and the set of all measures on $(S, \mathcal{B}(S))$, respectively. For a measure $\mu \in \mathcal{M}(S)$ we define

$$\mathbb{P}_\mu(\,\cdot\,) := \int_S \mathbb{P}_x(\,\cdot\,)\mu(\,\mathrm{d}x). \tag{4.32}$$

Taking $\mathbb{P}_\mu(\Omega)$ it is easy to see that $\mathbb{P}_\mu \in \mathcal{M}_1$ $(\in \mathcal{M}_f)$ if and only if $\mu \in \mathcal{M}_1$ $(\in \mathcal{M}_f)$. We denote the expectations with respect to $\mathbb{P}_x$ and $\mathbb{P}_\mu$ by $\mathbb{E}_x$ and $\mathbb{E}_\mu$, respectively.

For $x \in S$, $A \in \mathcal{B}$ and $t > s \geq 0$ define

$$p_{s,t}(x, A) := \mathbb{P}(X_t \in A \mid X_s = x). \tag{4.33}$$

Then $\{ p_{s,t} \mid t > s \geq 0 \}$ is a family of (sub) Markovian transition functions in the sense that $p_{s,t} \colon S \times \mathcal{B} \to [0,1]$ is a is a (sub) Markov kernel for each $t > s \geq 0$ and that for all $u > s > t \geq 0$, $x \in S$ and $A \in \mathcal{B}$ it holds that

$$p_{s,u}(x, A) = \int_S p_{s,t}(x, \, \mathrm{d}y) p_{t,u}(y, A). \tag{4.34}$$

The equation (4.34) is called *Chapman-Kolmogorov equation* and it is a consequence of the Markov property of X.

We call a Markov process *(time) homogeneous* when the associated transitions functions depend only on the difference $|t - s|$. In that case $p_{s,t}(x, A) = p_{t-s}(x, A) = \mathbb{P}_x(X_{t-s} \in A)$ and the Chapman-Kolmogorov equation (4.34) reads as

$$p_t(x, A) = \int_S p_{t-s}(y, A) p_s(x, \, \mathrm{d}y) \tag{4.35}$$

for $t > s \geq 0$.

Definition 4.14 (transition functions)**.** Let $(S, \mathcal{B})$ be a measurable space. A family $(p_t)_{t \geq 0}$ of (sub) Markov kernels $p_t \colon S \times \mathcal{B} \to [0,1]$ is called a family of *(sub) Markovian transition functions* if it satisfies the Chapman-Kolmogorov equation(4.35) for all $0 \leq s < t < \infty$ and $p_0(x, A) = \mathbb{1}_A(x)$. ◊

It is worth noting that the transition functions p_t are Markov kernels if and only if the associated process is non-explosive.

Suppose X is a Markov process, then for each $\omega \in \Omega$ the map $X(\omega) \colon [0, \infty) \to S$ is called a path. It is often useful to impose further regularity assumptions on these paths. For example, one can consider only Markov processes with continuous paths. This turns out to be rather restrictive as we want to allow the processes to have jumps. One possible choice is to consider processes that have càdlàg paths (more precisely, processes for which there exists a modification with càdlàg paths). As a first observation, homogeneous Markov processes cannot jump at fixed time-points, since that would break the homogeneity. As an illustration, consider the following example.

Example 4.15 (Processes with fixed jump times are non homogeneous)**.** Let $S = \{a, b, c\}$ and consider the process X that jumps at integer times from one point to one of the others with equal probability. It is easy to check that X is a Markov process. However, the transition probabilities are not homogeneous:

$$0 = \mathbb{P}\left(X_{3/4} \in \{b, c\} \mid X_{1/4} = a\right) \neq \mathbb{P}\left(X_{5/4} \in \{b, c\} \mid X_{3/4} = a\right) = 1. \tag{4.36}$$

■

It turns out that the holding times of a homogeneous Markov process X, i.e. the times that the process X spends in a point $x \in S$ before it jumps are exponentially distributed.

Lemma 4.16 (Holding times are exponentially distributed)**.** *Let X be a homogeneous Markov process with state space $(S, \mathcal{U})$. Assume that there exists a $x \in S$ such that*

$$T := \inf\{ t \geq 0 \mid X_t \neq x \} = \inf\{ t \geq 0 \mid X_t \neq X_{t-} \} \quad \mathbb{P}_x\text{-a.s.} \tag{4.37}$$

Then T is exponentially distributed under $\mathbb{P}_x$.

Proof. Observe that by (ii) of Definition 4.13 we have $\mathbb{P}_x(T > 0) = 1$. By time homogeneity and the Markov property we have for all $s, t > 0$,

$$\mathbb{P}_x(T > s + t \,|\, T > s) = \mathbb{P}_x(T > s + t \,|\, X_s = x) = \mathbb{P}_x(T > t). \tag{4.38}$$

This is the so-called *loss of memory property* that characterizes the exponential distribution. ■

4.1.3 The transition semigroup

For the remainder of this thesis, we will only be concerned with time homogeneous Markov processes.

Let $(S, \mathcal{B})$ be a measurable space and X a Markov process with values in S. Using the transition functions $(p_t)_{t\geq 0}$ of X, we can define for each $t \geq 0$ a linear operator on $B_b(S)$ by

$$P_t f = P_t f(\,\cdot\,) := \int_S f(y) p_t(\,\cdot\,, \mathrm{d}y) = \mathbb{E}_{\bullet}[f(X_t)], \quad t \geq 0. \tag{4.39}$$

The family $P = (P_t)_{t\geq 0}$ has some nice properties.

Proposition 4.17. *Let $(S, \mathcal{B})$ be a measurable space and X be a Markov process with values in S. Then the family $(P_t)_{t\geq 0}$ of operators on $B_b(S)$ defined above has the following properties for all $f, g \in B_b(S)$:*

(i) $P_0 f = f$,

(ii) $P_t P_s f = P_{s+t} f$ *for all* $s, t \geq 0$,

(iii) $P_t(\alpha f + \beta g) = \alpha P_t f + \beta P_t g$ *for all* $t \geq 0$,

(iv) if $f \geq 0$, then $P_t f \geq 0$ for all $t \geq 0$,

(v) $\|P_t f\|_\infty \leq \|f\|_\infty$.

Proposition 4.17 (i) and (ii) together imply that $P = (P_t)_{t\geq 0}$ is a semigroup on $B_b(S)$. i.e. P is equipped with a (commutative) binary operation $\circ$, there exists a neutral element but in general P_t has no inverse in P. Furthermore, (v) means that the semigroup P is *contractive* and (iv) says that P is positive. We say that P is the semigroup *determined* by the process X.

Definition 4.18 (positive contraction semigroups)**.** A family $T = (T_t)_{t\geq 0}$ of operators on a linear subspace of $\mathcal{F} \subset \mathcal{B}(S)$ containing constant functions that satisfies (i) to (v) of Proposition 4.17 is called a *semigroup of positive contraction operators* on $\mathcal{F}$. If, in addition, $T_t 1 = 1$ for all $t \geq 0$, then T is said to be *conservative*. ◇

We continue with the proof of the proposition.

Proof of Proposition 4.17. The property (i) follows from the definition of P_0 and (ii) is a consequence of the Chapman-Kolmogorov equation (4.39):

$$P_t P_s f(x) = \int_S \int_S f(z) p_s(y, \mathrm{d}z) p_t(x, \mathrm{d}y) = \int_S f(z) p_{s+t}(x, \mathrm{d}z) = P_{s+t}. \tag{4.40}$$

The positivity (iv) and linearity (iii) of P_t follow immediately from the definition (4.39) of P_t.

By (4.39) we have for all $x \in S$ and $t \geq 0$

$$P_t f(x) = \int_S f(y) p_t(x, dy) \leq \|f\|_\infty p_t(x, S) \leq \|f\|_\infty \tag{4.41}$$

and thus $\sup_{x\in S} P_t f(x) \leq \|f\|_\infty$, proving (v). ■

Now assume that ν is a finite Borel measure on $(S, \mathcal{B})$. For $p \in [1, \infty)$ we obtain by application of Jensen's inequality

$$\begin{aligned}\|P_t f\|_p^p &= \int_S |P_t f(x)|^p \nu(\mathrm{d}x) = \int_S |\mathbb{E}_x [f(X_t)]|^p \nu(\mathrm{d}x) \leq \int_S \mathbb{E}_x [|f(x)|^p] \nu(\mathrm{d}x) \\ &= \int_S P_t |f(x)|^p \nu(\mathrm{d}x) = \int_S |f(x)|^p P_t 1(x) \nu(\mathrm{d}x) \leq \int_S |f(x)|^p \nu(\mathrm{d}x) \\ &= \|f\|_p^p.\end{aligned} \tag{4.42}$$

Hence, (v) of Proposition 4.17 can be strengthened to (iii)* $\|P_t f\|_p \le \|f\|_p$ for all $t \ge 0$ and $p \in [1, \infty]$.

We have shown that every Markov process is associated with a family $(p_t)_{t\ge0}$ of (sub) Markov kernels satisfying the Chapman-Kolmogorov equation (4.35). Kolmogorov's celebrated extension theorem shows that the converse also holds. We present the theorem for further reference without proof as the proof can be found in any standard textbook on probability theory e.g. [Kal21, Theorem 11.4] or [Kle14, Theorem 14.36].

Theorem 4.19 (Kolmogorov's extension theorem)**.** *Let $(S, \mathcal{B})$ be a measurable space, $(p_t)_{t\ge0}$ a family of (sub) Markovian transition functions on S. Assume that $\mu \in \mathcal{M}_1(S)$. Then there exists a Markov process with state space S, initial distribution μ and transition functions $(p_t)_{t\ge0}$.*

Proof. See[Kal21, Theorem 11.4]. ∎

Assume that $(P_t)_{t\ge0}$ is a contraction semigroup on $\mathcal{B}_b(S)$. For $x \in S$, $A \in \mathcal{B}$ and $t \ge 0$ let

$$P_t \mathbb{1}_A(x) =: p_t(x, A). \tag{4.43}$$

Then the map $x \mapsto p_t(x, A)$ is Borel measurable because P_t is a linear operator on $\mathcal{B}_b(S)$. Furthermore, we have $p_0(x, A) = \mathbb{1}_A(x)$ and $(p_t)_{t\ge0}$ satisfies the Chapman-Kolmogorov equation:

$$p_t(x, A) = P_t \mathbb{1}_A(x) = P_s P_{t-s} \mathbb{1}_A(x) = P_s p_{t-s}(x, A) = \int_S p_{t-s}(y, A) p_s(x, \mathrm{d}y), \tag{4.44}$$

for $0 \le s < t < \infty$, where we used (4.39) in the last equality. Furthermore, we deduce from the contraction property, Proposition 4.17 (v) that

$$\sup_{x\in S} p_t(x, S) = \|P_t \mathbb{1}_S\|_\infty \le \|\mathbb{1}_S\|_\infty = 1, \tag{4.45}$$

thereby showing that P induces a family of Markovian transition function $(p_t)_{t\ge0}$ via (4.43). Approximating measurable functions by simple functions, it is straightforward to show that P is the semigroup induced by $(p_t)_{t\ge0}$. We have thus proved the following.

Corollary 4.20. *Let $(S, \mathcal{B})$ be a measurable space and $P = (P_t)_{t\ge0}$ a contraction semigroup on $\mathcal{B}_b(S)$. Assume that $\mu \in \mathcal{M}_1(S)$, then there exists a Markov process with state space S, initial distribution μ and transition semigroup P.*

While there always exists a Markov process with a given transition function (or semigroup), this process is generally not unique. Instead, we have that all processes with the same transition function are versions of each other.

Lemma 4.21 (Semigroup determines the finite-dimensional distributions)**.** *Let X, Y be two Markov processes on the measurable space $(S, \mathcal{B})$ with the same initial distribution $\mu \in \mathcal{M}_1(S)$. Assume that both processes have the same semigroup $P = (P_t)_{t\geq 0}$. Then, the finite-dimensional distributions of X and Y coincide.*

Proof. It is evident from the discussion above that both X and Y have the same transition functions $(p_t)_{t\geq 0}$. Let $A_1, \ldots, A_n \in \mathcal{B}$ and $0 \leq t_1 < \cdots < t_n$, then it follows by the Chapman-Kolmogorov equation,

$$\begin{aligned}
&\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) \\
&\quad = \int_S \mu(\,\mathrm{d}x_0) \int_{A_1} p_{t_1}(x_0,\,\mathrm{d}x_1)\cdots \int_{A_n} p_{t_n - t_{n-1}}(x_{n-1},\,\mathrm{d}x_n) \\
&\quad = \mathbb{P}(Y_{t_1} \in A_1, \ldots, Y_{t_n} \in A_n),
\end{aligned} \tag{4.46}$$

completing the proof. ■

Remark 4.22. By definition, the semigroup and hence the finite-dimensional distributions of a Markov process are already determined by their one-dimensional distributions. ◇

Now let $\nu \in \mathcal{M}(S)$ be a Radon measure on $(S, \mathcal{B})$. This measure will later serve as the *speed measure* for our processes in the sense that the time the process spends in a set $A \in \mathcal{B}$ will be roughly proportional to its measure $\nu(A)$.But first we explain what it means for a Markov process to be symmetric with respect to ν.

Definition 4.23 ((strong) symmetry)**.** Let X be a Markov process with values in the measure space $(S, \mathcal{B}, \nu)$. Then X is said to be ν-*symmetric* if the semigroup $P = (P_t)_{t\geq 0}$ determined by X satisfies

$$\int_S f(x) P_t g(x)\, \nu(\mathrm{d}x) = \int_S P_t f(x) g(x)\, \nu(\mathrm{d}x) \tag{4.47}$$

for all non-negative $f, g \in \mathcal{B}_b(S)$ and $t \geq 0$. In that case, we also refer to the family P as symmetric (with respect to ν).

If, in addition, the (sub) probability measures $p_t(x, \mathrm{d}y)$ are absolutely continuous with respect to ν for all $t \geq 0$ and $x \in S$, we say that the process X is *strongly symmetric* (with respect to ν). In that case, we denote the density of $p_t(x, \mathrm{d}y)$, with a slight abuse of notation, by $p_t(x, y)$. In that case, $P_t f(x)$ can be written as

$$P_t f(x) = \int_S f(y) p_t(x, y)\, \nu(\mathrm{d}y). \tag{4.48}$$

◇

Let $(S, \mathcal{B}, \nu)$ be a compact measure space. Suppose that X is a ν-symmetric Markov process with values in S. Then ν is a reversible measure for X in the sense that for all $A, B \in \mathcal{B}$ with $\nu(A), \nu(B) > 0$ and $t \geq 0$,

$$\begin{aligned} \nu(A)\mathbb{P}_{\nu_A}(X_t \in B) &= \int_A \mathbb{P}_x(X_t \in B)\,\nu(\mathrm{d}x) = \int_S P_t \mathbb{1}_B(x)\mathbb{1}_A(x)\,\nu(\mathrm{d}x) \\ &= \int_B \mathbb{P}_x(X_t \in A)\,\nu(\mathrm{d}x) = \nu(B)\mathbb{P}_{\nu_B}(X_t \in A), \end{aligned} \tag{4.49}$$

where $\nu_A = \left(\nu(A)^{-1}\nu\right)\big|_A$ and $\nu_B = \left(\nu(B)^{-1}\nu\right)\big|_B$ denote the renormalized restrictions of ν to A and B, respectively.

We take note of the following useful property of ν-symmetric Markov processes.

Lemma 4.24. *Let X be a ν-symmetric Markov process with values in S. Suppose that for $n \in \mathbb{N}$, $f_0, f_1, \ldots, f_n \in \mathcal{B}_b^+$ and $0 = t_0 < t_1 < \ldots t_n < \infty$. Then,*

$$\mathbb{E}_\nu\left[f_0(X_0)f_1(X_{t_1})\cdots f_n(X_{t_n})\right] = \mathbb{E}_\nu\left[f_0(X_{t_n})f_1(X_{t_n-t_1})\cdots f_{n-1}(X_{t_n-t_{n-1}})f_n(X_0)\right]. \tag{4.50}$$

Proof. We proceed by induction and start with the case $n = 1$. For $f, g \in \mathcal{B}_b^+(S)$ and $0 < t < \infty$, we have by symmetry

$$\begin{aligned} \mathbb{E}_\nu\left[f(X_0)g(X_t)\right] &= \int_S \mathbb{E}_x\left[f(X_0)g(X_t)\right]\,\nu(\mathrm{d}x) = \int_S f(x)P_t g(x)\,\nu(\mathrm{d}x) \\ &= \int_S \mathbb{E}_x\left[f(X_t)g(X_0)\right]\,\nu(\mathrm{d}x) = \mathbb{E}_\nu\left[f(X_t)g(X_0)\right]. \end{aligned} \tag{4.51}$$

Now suppose that the statement holds for some $n \in \mathbb{N}$ and let $f_0, \ldots, f_{n+1} \in \mathcal{B}_b^+$ and $0 = t_0 < \cdots < t_{n+1} < \infty$. Then,

$$\begin{aligned} \mathbb{E}_\nu\left[f_0(X_0)\cdots f_{n+1}(X_{t_{n+1}})\right] &= \mathbb{E}_\nu\left[f_0(X_0)\cdots(f_n \cdot P_{t_{n+1}-t_n}f_{n+1})(X_{t_n})\right] \\ &= \mathbb{E}_\nu\left[f_0(X_{t_n})\cdots(f_n \cdot P_{t_{n+1}-t_n}f_{n+1})(X_0)\right] \\ &= \int_S P_{t_{n+1}-t_n}f_{n+1}(x)\mathbb{E}_x\left[f_0(X_{t_n})\cdots f_n(X_0)\right]\,\nu(\mathrm{d}x) \\ &= \int_S f_{n+1}(x)P_{t_{n+1}-t_n}\mathbb{E}_x\left[f_0(X_{t_n})\cdots f_n(X_0)\right]\,\nu(\mathrm{d}x) \\ &= \int_S f_{n+1}(x)\mathbb{E}_x\left[\mathbb{E}_{X_{t_{n+1}-t_n}}\left[f_0(X_{t_n})\cdots f_n(X_0)\right]\right]\,\nu(\mathrm{d}x) \\ &= \int_S \mathbb{E}_x\left[f_0(X_{t_{n+1}})f_1(X_{t_{n+1}-t_1})\cdots f_{n+1}(X_0)\right]\,\nu(\mathrm{d}x) \\ &= \mathbb{E}_\nu\left[f_0(X_{t_{n+1}})f_1(X_{t_{n+1}-t_1})\cdots f_n(X_{t_{n+1}-t_n})f_{n+1}(X_0)\right], \end{aligned} \tag{4.52}$$

completing the proof. ∎

Next, we introduce the resolvent or potential operator (cf. [MR06]) of a Markov process as the Laplace transform of the semigroup.

Definition 4.25. Let X be a Markov process with values in S and $f \in B_b(S)$. Then, for each $\alpha > 0$ we set

$$R_\alpha f(x) := \mathbb{E}_x\left[\int_0^\infty f(X_t)e^{-\alpha t}\,\mathrm{d}t\right]. \tag{4.53}$$

The family $(R_\alpha)_{\alpha>0}$ of operators is called the *resolvent* associated with the process X. For $\alpha > 0$, the operator R_α is called the α-resolvent of X. ◊

Clearly, R_α is a linear operator mapping $B_b(S)$ to $B_b(S)$. By Fubini's Theorem, we can write

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\,\mathrm{d}t. \tag{4.54}$$

Applying Fubini's Theorem again and using the ν-symmetry of $(P_t)_{t\geq 0}$, we find that R_α is ν-symmetric as well. Observe that for $\alpha, \beta > 0$

$$\begin{aligned}
\left(R_\alpha - R_\beta\right) f &= \int_0^\infty e^{-\alpha t} P_t f\,\mathrm{d}t - \int_0^\infty e^{-\beta t} P_t f\,\mathrm{d}t \\
&= \int_0^\infty e^{-\beta t}\left(e^{-(\alpha-\beta)t} - 1\right) P_t f\,\mathrm{d}t \\
&= -(\alpha-\beta)\int_0^\infty \int_0^t e^{-\beta(t-s)-\alpha s} P_{(t-s)+s} f\,\mathrm{d}s\,\mathrm{d}t \\
&= -(\alpha-\beta)\int_0^\infty \int_0^\infty e^{-\alpha s}e^{-\beta t} P_s P_t f\,\mathrm{d}t\,\mathrm{d}s = -(\alpha-\beta)R_\alpha R_\beta f.
\end{aligned} \tag{4.55}$$

We have thus shown that the resolvent satisfies the *resolvent equation*

$$R_\alpha - R_\beta + (\alpha-\beta)R_\alpha R_\beta = 0 \quad \forall \alpha, \beta > 0. \tag{R1}$$

Remark 4.26. The resolvent has another, probabilistic, interpretation. Let $\alpha > 0$ and consider an exponential random variable ζ with expectation $1/\alpha$, independent of the process X. Let

$$\hat{X}_t =: \begin{cases} X_t, & t < \zeta \\ \vartheta, & t \geq \zeta \end{cases} \tag{4.56}$$

be the exponentially killed process. Denote by $\hat{\mathbb{E}}$ and $\hat{P}$ the expectation and the semigroup of $\hat{X}$, respectively. Then

$$\hat{P}_t f(x) = \hat{\mathbb{E}}\left[f(X_t); \zeta > t\right] = \mathbb{P}(\zeta > t) P_t f(x) = e^{-\alpha t} P_t f(x). \tag{4.57}$$

Thus, the α-resolvent can be regarded as the integrated semigroup of the process that is killed at an independent Exp(α)-time. When we consider $f = \mathbb{1}_A$ for some set $A \in \mathcal{B}$, the quantity $R_\alpha \mathbb{1}_A(x)$ is the expected time the process $\hat{X}$ spends in A, or *occupation time*, before it is killed at time ζ. ◊

Definition 4.27 (α-excessive functions)**.** Let $(P_t)_{t\geq 0}$ be the transition semigroup of a Markov process with values in $(S, \mathcal{B})$. Furthermore, let $\alpha > 0$. A non-negative measurable function $h \in \mathcal{B}^+(S)$ is called *α-excessive* with respect to $(P_t)_{t\geq 0}$ if

$$e^{-\alpha t} P_t h(x) \leq h(x) \tag{4.58}$$

and

$$\lim_{t\to 0} e^{-\alpha t} P_t h(x) = h(x), \tag{4.59}$$

for each $x \in S$. ◊

As an immediate consequence of Definition 4.27 we find that for every α-excessive function $h \in \mathcal{B}^+(S)$ and $s, t > 0$,

$$e^{-\alpha(s+t)} P_{s+t} h(x) = e^{-\alpha s} P_s e^{-\alpha t} P_t h(x) \leq e^{-\alpha t} P_t h(x), \tag{4.60}$$

and consequently the function $t \mapsto e^{-\alpha t} P_t h(x)$ is increasing as $t \to 0$. Furthermore, constant functions are α-excessive for every $\alpha > 0$ and $h \wedge g$ is α-excessive whenever f and g are α-excessive.

Lemma 4.28. *(i) Let $\alpha > 0$ and $(h_n)_{n\in\mathbb{N}} \subset \mathcal{B}^+(S)$ be an increasing sequence of α-excessive functions such that $\lim_{n\to\infty} h_n = h \in \mathcal{B}^+(S)$. Then, h is α-excessive, too.*

(ii) Let $f \in \mathcal{B}_b^+(S)$ be non-negative, bounded and measurable. Then the function $h := R_\alpha f$ is α-excessive for all $\alpha > 0$.

(iii) Let $h \in \mathcal{B}^+(S)$ be α-excessive for $\alpha > 0$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{B}(S)$ such that $R_\alpha f_n(x)$ is increasing as $n \to \infty$ and

$$\lim_{n\to\infty} R_\alpha f_n(x) = h(x), \tag{4.61}$$

for all $x \in S$.

Proof. Fix $\alpha > 0$ and let $(\subset_n)_{n\in\mathbb{N}} \mathcal{B}^+(S)$ be an increasing sequence of α-excessive functions with $\lim_{n\to\infty} h_n = h$. Then,

$$e^{-\alpha t} P_t h(x) = \lim_{n\to\infty} e^{-\alpha t} P_t h_n(x) \leq \lim_{n\to\infty} h_n = h. \tag{4.62}$$

Taking the limit $t \to 0$ we can interchange limits because of monotonicity and obtain

$$\lim_{t\to 0} e^{-\alpha t} P_t h(x) = \lim_{n\to\infty} \lim_{t\to 0} e^{-\alpha t} P_t h_n(x) = h(x) \tag{4.63}$$

and the first assertion (i) is established.

For the second claim (ii) assume that $f \in \mathcal{B}_b^+(S)$ is non-negative and bounded and let $\alpha > 0$. We start by showing that $h = R_\alpha f$ satisfies (4.58):

$$\begin{aligned} e^{-\alpha t} P_t h = e^{-\alpha t} P_t R_\alpha f = e^{-\alpha t} P_t \int_0^\infty e^{-\alpha s} P_s f \,\mathrm{d}s = \int_0^\infty e^{-\alpha(t+s)} P_{t+s} f \,\mathrm{d}s \\ = \int_t^\infty e^{-\alpha s} P_s f \,\mathrm{d}s \leq R_\alpha f = h. \end{aligned} \tag{4.64}$$

When we apply the limit for $t \to 0$ at (4.64) we obtain the equality (4.59), thus proving the claim.

For the last claim (iii) we construct the approximating sequence explicitly. Let $h \in \mathcal{B}^+(S)$ be α-excessive for some $\alpha > 0$. For $n \in \mathbb{N}$ set $h_n := h \wedge n$. By definition of the resolvent and substituting $t = ns$, we obtain

$$nR_{\alpha+n} h_n = \int_0^\infty n e^{-(\alpha+n)s} P_s h_n \,\mathrm{d}s = \int_0^\infty e^{-t} e^{-\alpha t/n} P_{t/n} h_n \,\mathrm{d}t. \tag{4.65}$$

Now, h_n is α-excessive since it is the minimum of two α-excessive functions. Therefore, the function

$$g_n := e^{-\alpha t/n} P_{t/n} h_n \tag{4.66}$$

is increasing for fixed $\alpha, t > 0$ as $n \to \infty$ and $\lim_{n\to\infty} g_n = h$. Hence, $nR_{\alpha+n} h_n$ is increasing in n and converges to h. Now observe that by the resolvent equation we have

$$nR_{\alpha+n} h_n = nR_\alpha \left(h_n - nR_{\alpha+n} h_n\right), \tag{4.67}$$

and therefore, the functions

$$f_n := n\left(h_n - nR_{\alpha+n} h_n\right) \tag{4.68}$$

are the desired sequence with $R_\alpha f_n \uparrow h$ as $n \to \infty$. ∎

The importance of α-excessive functions stems from the following fact.

Proposition 4.29. *Let $h \in \mathcal{B}^+(S)$ be α-excessive for some $\alpha > 0$. Then, the real valued stochastic process $(Y_t)_{t\geq 0} := (e^{-\alpha t} h(X_t))_{t\geq 0}$ is a supermartingale with respect to the canonical filtration $(\mathcal{A}_t)_{t\geq 0}$ for every initial distribution $\mu \in \mathcal{M}_1(S)$.*

Proof. The proof is straightforward. Fix $t > 0$, by α-excessivity of h we have $\mathbb{P}_\mu$-almost surely

$$\begin{aligned}\mathbb{E}\left[Y_{t+h} \,|\, \mathcal{A}_t\right] &= e^{-\alpha(t+h)}\mathbb{E}\left[h(X_{t+h}) \,|\, \mathcal{A}_t\right] = e^{-\alpha(t+h)}\mathbb{E}\left[P_h h(X_t) \,|\, \mathcal{A}_t\right] \\ &= e^{-\alpha t}e^{-\alpha h}P_h h(X_t) \leq e^{-\alpha t}h(X_t).\end{aligned} \tag{4.69}$$

■

4.2 Feller processes

In the previous chapter Chapter 3 we have examined the space of càdlàg functions on a uniform space $(S, \mathcal{U})$ with great care and in the previous section we have seen that the finite dimensional distributions of a Markov process are determined by its semigroup. We are interested in a stronger statement. Namely,we want to consider a class of Markov processes that are already uniquely determined by their semigroups. This is where the càdlàg paths come into play.

Proposition 4.30. *Let $(S, \mathcal{U})$ be a separable uniform Hausdorff space and X, Y two Markov processes with càdlàg paths, i.e. $X, Y\colon \Omega \to D_S([0, \infty))$. Assume further that X and Y have the same transition semigroup $P = (P_t)_{t\geq 0}$ and the same initial distribution $\mu \in \mathcal{M}_1(S)$. Then, X and Y have the same law.*

Proof. By Lemma 4.21, X and Y have the same finite-dimensional distributions and the result follows from Proposition 3.41. ■

As a consequence, in the case of Markov processes with càdlàg paths, the semigroup is a very powerful tool in the analysis of the process. Yet, this result is not satisfying as it is *a priori* not clear that a given Markov process even has a modification with càdlàg paths. To make sure that this is the case we need stronger assumptions on the semigroups.

For the remainder of this chapter assume that $(S, \mathcal{U})$ is a Polish uniform space, i.e. a separable and complete locally compact uniform Hausdorff space. Recall that under this assumption, $(S, \mathcal{U})$ is completely metrizable, yet we want to avoid fixing a *specific* metric. While some of the concepts can be further generalized, we refrain from doing so as this would go beyond the scope of this thesis.

Definition 4.31 (Feller semigroups)**.** A semigroup $(T_t)_{t\geq 0}$ of positive contraction operators is called a *Feller semigroup* if it has the following properties

(F1) $T_t f \in C_\infty(S)$ for all $f \in C_\infty(S)$ and $t \geq 0$,

(F2) $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$ for all $f \in C_\infty(S)$. ♢

Lemma 4.32. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup on $C_\infty(S)$. Then,*

$$(t, f, x) \mapsto T_t f(x) \tag{4.70}$$

is continuous as a function $[0,\infty) \times C_\infty(S) \times S \to \mathbb{R}$.

Proof. Let $(t, f, x), (s, g, y) \in [0,\infty) \times C_\infty(S) \times S$, then

$$\begin{aligned} |T_t f(x) - T_s g(y)| &= |T_t f(x) - T_t f(y) + T_t f(y) - T_s f(y) + T_s f(y) - T_s g(y)| \\ &\leq |T_t f(x) - T_t f(y)| + |T_t f(y) - T_s f(y)| + |T_s f(y) - T_s g(y)| \,. \end{aligned} \tag{4.71}$$

By **(F1)**, the first term vanishes as $y \to x$. For every $t > 0$, T_t is a contraction by assumption and the second term can be bounded by

$$\|T_t(T_{|s-t|} f - f)\|_\infty \leq \|T_{|s-t|} f - f\|_\infty . \tag{4.72}$$

By **(F2)** this bound converges to 0 as $s \to t$. Similarly, the last term is bounded by $\|f - g\|_\infty$ which also tends to 0 as $g \to f$. ■

Definition 4.33 (Feller processes)**.** Let X be a Markov process with values in S. We call X a *Feller process*, if the semigroup $(P_t)_{t\geq 0}$ associated with X satisfies **(F1)** and **(F2)** of Definition 4.31. ♢

Some remarks about the above definition are in order. As the name indicates, the definition goes back to a series of papers that William Feller wrote in the 1950s, e.g. [Fel52; Fel54]. Feller introduced the conditions above in the context of his analysis of diffusion processes [2]. However, Feller's original work is rarely cited today[3]. Instead, classical textbooks like [Mey66], [Dyn65] or [BG68] give a thorough account of the theory of Feller processes.

Some authors use slightly different definitions of the Feller property and require **(F1)** to hold for all bounded $f \in C_b$, instead.[4]

[2] [Fel54, Theorem 1]
[3] [CW05, Notes on Chapter §2.2, p. 73]
[4] See Martin Hairer's comment in [Hai]

Assume that X is a Markov process with values in a compact space S and assume that its associated semigroup $(P_t)_{t\geq 0}$ has the Feller property. Then, **(F1)** implies that for all $t \geq 0$ and $f \in C$ the maps $P_t f$ are continuous and thus

$$\int_S f(z)p_t(x, \mathrm{d}z) = \mathbb{E}_x\left[f(X_t)\right] = P_t f(x) \underset{x\to y}{\longrightarrow} P_t f(y) = \int_S f(z)p_t(x, \mathrm{d}y). \tag{4.73}$$

Hence, laws $\mathcal{L}_x$ and $\mathcal{L}_y$ of X started in x and y, respectively, converge as $x \to y$. In fact, the reverse implication is also true. We say that **(F1)** means that X *depends continuously on the starting point*. On the other hand, **(F2)** means that in probability (under $\mathbb{P}_x$),

$$\lim_{t\to 0} X_t = x. \tag{4.74}$$

Moreover, observe that under **(F1)** condition **(F2)** is equivalent to the seemingly weaker condition

$$\lim_{t\to 0} |P_t f(x) - f(x)| = 0, \tag{4.75}$$

for all $f \in C_\infty(S)$ and $x \in S$.[5]

We want to prove a slightly stronger result than (4.74).

Proposition 4.34 ([CW05, Proposition 2.2.2]). *Let X be a Feller process with values in S. Then X is* stochastically continuous, *i.e. for all $t > 0$, every initial distribution $\mu \in \mathcal{M}_1(S)$ and every open entourage $U \in \mathcal{U}$,*

$$\lim_{s\to t} \mathbb{P}_\mu\left((X_t, X_s) \in U\right) = 1. \tag{4.76}$$

Proof. It suffices to show the claim for $\mu = \delta_x$ for some $x \in S$. Let $U \in \mathcal{U}$ be open and $V \in \mathcal{U}$ such that $V \circ V \subset U$. Now choose a continuous function $h\colon S \times S \to [0, 1]$ such that $h(x, y) = 0$ if $(x, y) \notin U$ and $h(x, y) = 1$ if $(x, y) \in V$. Then $h \leq \mathbb{1}_U$ and we obtain

$$\mathbb{P}_x\left((X_t, X_s) \in U\right) \geq \mathbb{E}_x\left[h(X_t, X_s)\right], \tag{4.77}$$

for all $s, t > 0$. Consider the functions of the form

$$p(x, y) = \sum_{j=1}^{k} f_j(x)g_j(y), \tag{4.78}$$

where $f_j, g_j \in C_0(S)$ for $j \in \{1, \ldots, k\}$. These functions form a sub-algebra of

[5] See [RY99, Proposition III.2.4]

$C_\infty(S \times S)$ and separate points. We can therefore apply the Stone-Weierstrass Theorem (cf. [Con07, Corollary V.8.3]) to obtain a sequence

$$h_n(x, y) = \sum_{j=1}^{k_n} f_j^{(n)}(x) g_j^{(n)} \tag{4.79}$$

of functions the form (4.78) that converges uniformly to h.

Now observe that for all $f, g \in C_\infty(S)$ and $t > 0$ we have by (F2),

$$\begin{aligned} \mathbb{E}_x [f(X_t)g(X_{t+\delta})] &= \mathbb{E}_x [f(X_t)\mathbb{E}_{X_t} [g(X_\delta)]] = \mathbb{E}_x [f(X_t)P_\delta g(X_t)] \\ &\xrightarrow[\delta\to 0]{} \mathbb{E}_x [f(X_t)g(X_t)] . \end{aligned} \tag{4.80}$$

Consequently,

$$\lim_{\delta\to 0} \mathbb{P}_x((X_t, X_{t+\delta}) \in U) \geq \lim_{\delta\to 0} \mathbb{E}_x [h(X_t, X_{t+\delta})] = \mathbb{E}_x [h(X_t, X_t)] = 1. \tag{4.81}$$

Next, we have to consider the left limit $s \uparrow t$. To that end fix $t > 0$ and $0 < \delta < t$. Then,

$$\begin{aligned} \mathbb{E}_x [f(X_{t-\delta})g(X_t)] &= \mathbb{E}_x [f(X_{t-\delta})\mathbb{E}_{X_{t-\delta}} [g(X_\delta)]] = \mathbb{E}_x [f(X_{t-\delta})P_\delta g(X_{t-\delta})] \\ &= P_{t-\delta} (f P_\delta g) (x). \end{aligned} \tag{4.82}$$

As $\delta \to 0$, the right hand side converges to

$$P_t(fg)(x) = \mathbb{E}_x [f(X_t)g(X_t)] , \tag{4.83}$$

by Lemma 4.32. By the same argument as before we conclude that

$$\lim_{\delta\to 0} \mathbb{P}_x((X_t, X_{t-\delta}) \in U) \geq \lim_{\delta\to 0} \mathbb{E}_x [h(X_t, X_{t-\delta})] = \mathbb{E}_x [h(X_t, X_t)] = 1. \tag{4.84}$$

■

Condition (F1) in Definition 4.31 can be exchanged for another condition sometimes called the *strong Feller property*.

Definition 4.35 (Strong Feller property)**.** Let $(T_t)_{t\geq 0}$ be as in Definition 4.31 but assume that instead of (F1), $(T_t)_{t\geq 0}$ satisfies the *strong Feller property*

(F3) $T_t f \in C_b$ for all $f \in B_b(S)$ and $t > 0$.

Analogously to Definition 4.33 we call a Markov process X whose transition semigroup satisfies (F2) and (F3) a *strong Feller process* or simply *strongly Feller*. ◇

While the Feller property makes sure that the distribution of X at time t depends on the initial conditions continuously, the strong Feller property ensures that the process X behaves diffusively in the sense that point masses in the initial distribution are smoothed out by the semigroup.

As an example of a Markov process that is Feller but not strongly Feller consider the following simple process.

Example 4.36. Let X be the process on $\mathbb{R}$ that remains in its initial distribution forever, i.e. we have $\mathbb{P}_x(X_t = x) = 1$. Thus, the semigroup $(P_t)_{t\geq 0}$ of X is given by

$$P_t f(x) = \mathbb{E}_x\left[f(X_t)\right] = \mathbb{E}_x\left[f(x)\right] = f(x) \tag{4.85}$$

and P_t is the identity operator for all $t > 0$. Hence, **(F1)** and **(F2)** from Definition 4.31 hold but **(F3)** does not. ■

Note that, despite the name, the strong Feller property does not imply the normal Feller property. Instead, we have the following definition.

Definition 4.37 (Doubly Feller)**.** Let $(T_t)_{t\geq 0}$ be a semigroup of strongly continuous contraction operators. If $(T_t)_{t\geq 0}$ is both Feller and strongly Feller, i.e. $(T_t)_{t\geq 0}$ satisfies **(F1)** to **(F3)**, we say that $(T_t)_{t\geq 0}$ is *doubly Feller*. ◇

4.2.1 Resolvents and generators

Given a Feller semigroup $(T_t)_{t\geq 0}$ we can define the family of resolvent operators $(R_\alpha)_{\alpha>0}$ associated with $(T_t)_{t\geq 0}$ using (4.54), i.e.

$$R_\alpha f := \int e^{-\alpha t} T_t f \,\mathrm{d}t, \quad f \in C_\infty. \tag{4.86}$$

Observe that this definition of the resolvent coincides for Feller processes with the definition of resolvents given before, apart from the domain. This justifies using the same letter to designate both.

The resolvent has further remarkable properties. We write $C_\infty^+ := C_\infty \cap C^+$ for the non-negative continuous functions that vanish at infinity.

Lemma 4.38 (Resolvents and supermartingales)**.** *Let X be a Markov process with values in a uniform Hausdorff space $(S, \mathcal{U})$. Assume that $f \in C_\infty^+$, then for each $\alpha > 0$, the process $Y = (Y_t)_{t\geq 0}$ with*

$$Y_t := e^{-\alpha t} R_\alpha f(X_t), \quad t \geq 0, \tag{4.87}$$

is a supermartingale under $\mathbb{P}_\mu$ *for every initial distribution* $\mu \in \mathcal{M}_1(S)$.

Proof. The claim is a direct consequence of Lemma 4.28 and Proposition 4.29. ∎

Next, we introduce the *generator* of a Feller semigroup and present briefly the interrelationship between semigroups, resolvents and generators. We will keep this exposition as short as possible as we will go more into detail when discussing Dirichlet forms associated with Feller groups in the next chapter. Again, all of the following can be found in most standard textbooks covering Feller processes and we will refer to Kallenberg's book [Kal21] for most of the proofs.

Let $\mathcal{D} \subset C_\infty$ be the family of functions for which the limits

$$\Delta f := \lim_{t \downarrow 0} \frac{T_t f - f}{t} \tag{4.88}$$

exist in C_∞. By [Kal21, Theorem 17.6], Δ is a linear operator on C_∞ with domain $\mathcal{D}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_t f) = T_t \Delta f = \Delta T_t f, \quad t \geq 0. \tag{4.89}$$

We say that $(\Delta, \mathcal{D})$ is the *generator* of the Feller semigroup $(T_t)_{t\geq 0}$. The term generator stems from the fact that $(\Delta, \mathcal{D})$ determines the semigroup $(T_t)_{t\geq 0}$ uniquely (cf. [Kal21, Lemma 17.5]).

Furthermore, Δ satisfies

$$T_t f - f = \int_0^t T_s \Delta f \,\mathrm{d}s, \tag{4.90}$$

for all $f \in \mathcal{D}$ and $t \geq 0$, by [Kal21, Theorem 17.6].

On the other hand, the following relationship exists between the generator and the resolvent of a Feller semigroup.

Proposition 4.39. *Let* $(T_t)_{t\geq 0}$ *be a Feller semigroup on* C_∞ *with resolvents* $(R_\alpha)_{\alpha>0}$ *and generator* $(\Delta, \mathcal{D})$*. Then*

(i) for each $\alpha > 0$, αR_α *is an injective contraction operator on* C_∞ *and*

$$\lim_{\alpha\to\infty} \alpha R_\alpha = \mathrm{id} \tag{4.91}$$

in the strong operator topology,

(ii) $\mathcal{D} = R_\alpha C_\infty$ *independently of* $\alpha > 0$ *and* $\mathcal{D}$ *is dense in* C_∞,

(iii) for all $f \in \mathcal{D}$ and $\alpha > 0$, the relation

$$R_\alpha^{-1} f = (\alpha - \Delta) f \tag{4.92}$$

holds.

Proof. See [Kal21, Theorem 17.4]. ■

Here, (4.92) might be more familiar to the reader who knows the *resolvent* from a functional analytic context.

Recall that a linear operator Δ with domain $\mathcal{D} \subset B$, where B is some Banach space, is called *closed* if the *graph* of Δ, $G(\Delta) := \{ (\Delta f, f) \mid f \in \mathcal{D} \} \subset B^2$ is closed (cf. [Yos78, Definition II.6.2]). A linear operator $(\Delta, \mathcal{D})$ is called *closable*, if for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ with $\lim_{n\to\infty} f_n = 0$ it holds that $\lim_{n\to\infty} \Delta f_n = 0$ (cf. [Yos78, Proposition II.6.2]). In that case, Δ can be uniquely extended to an operator $\overline{\Delta}$ on $\overline{\mathcal{D}}$ by taking $G(\overline{\Delta}) := \overline{G(\Delta)}$.

Finally, assume that $(\Delta, \mathcal{D})$ is a closed operator. We call a linear subspace $D \subset \mathcal{D}$ a *core* for Δ if and only if the operator $(\Delta|_D, D)$ is closable and its closure is $(\Delta, \mathcal{D})$. It can be shown (cf. [Kal21, Lemma 17.8]) that the generator $(\Delta, \mathcal{D})$ of a Feller semigroup $(T_t)_{t\geq 0}$ is closed and that a linear subspace $D \subset \mathcal{D}$ is a core for $(\Delta, \mathcal{D})$ if and only if the range $(\alpha - \Delta)D$ is a dense subset of C_∞ for one and hence for every $\alpha > 0$.

The celebrated Hille-Yosida Theorem was proven independently by Einar Hille and Kōsaku Yosida in the middle of the last century. It characterizes those operators that uniquely determine a strongly continuous contraction semigroup. In the formulation we present here, it characterizes the generators of Feller semigroups.

Proposition 4.40 (Hille-Yosida)**.** *Let Δ be a linear operator on C_∞ with domain $\mathcal{D}$. Then Δ is closable and its closure $\overline{\Delta}$ is the generator of a Feller semigroup $(T_t)_{t\geq 0}$ on C_∞ if and only if the following conditions hold*

(i) $\mathcal{D}$ is dense in C_∞,

(ii) the range of $\alpha - \Delta$ is dense in C_∞ for some $\alpha > 0$,

(iii) $\Delta f(x) \leq 0$, for any $f \in \mathcal{D}$ and $x \in S$ such that $\|f \vee 0\|_\infty \leq f(x)$.

Proof. See [Kal21, Theorem 17.11]. ■

We conclude this brief discussion of the generator of a Feller process with the following useful result.

Proposition 4.41 (Dynkin's formula [Kal21, Lemma 17.21]). *Let X be a Feller process with values in S. Denote by $(P_t)_{t\geq 0}$ and $(\Delta, \mathcal{D})$ the semigroup and the generator associated with X, respectively. For $f \in \mathcal{D}$ define the process $\left(M_t^f\right)_{t\geq 0}$ by*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \Delta f(X_s)\,\mathrm{d}s. \tag{4.93}$$

Then

(i) M^f is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ under every initial distribution $\mu \in \mathcal{M}_1(S)$.

(ii) For every bounded optional time τ,

$$\mathbb{E}_x\left[f(X_\tau)\right] = f(x) + \mathbb{E}_x\left[\int_0^\tau \Delta f(X_s)\,\mathrm{d}s\right]. \tag{4.94}$$

Proof. We only show (i). The second assertion then follows by the optional sampling theorem. Fix $\mu \in \mathcal{M}_1$ and let $t, \delta > 0$ and $f \in \mathcal{D}$. Then,

$$M_{t+\delta}^f - M_t^f = f(X_{t+\delta}) - f(X_t) - \int_t^{t+\delta} \Delta f(X_s)\,\mathrm{d}s = M_\delta^f \circ \theta_t. \tag{4.95}$$

We can therefore apply the Markov property to obtain

$$\mathbb{E}_\mu\left[M_{t+\delta}^f \middle| \mathcal{F}_t\right] - M_t^f = \mathbb{E}_\mu\left[M_\delta^f \circ \theta_t \middle| \mathcal{F}_t\right] = \mathbb{E}_{X_t}\left[M_\delta^f\right]. \tag{4.96}$$

Now,

$$\mathbb{E}_{X_t}\left[\int_0^\delta \Delta f(X_s)\,\mathrm{d}s\right] = \int_0^\delta P_s \Delta f(X_t)\,\mathrm{d}s \tag{4.97}$$

and we can apply (4.90) to deduce that $\mathbb{E}_{X_t}\left[M_\delta^f\right] = 0$, $\mathbb{P}_\mu$-a.s. It then follows readily from (4.96) that M^f is a martingale. ■

4.2.2 Existence of Feller processes with càdlàg paths

Recall the one-point compactification $(S_\vartheta, \mathcal{U}_\vartheta)$ of the locally compact space $(S, \mathcal{U})$ and that every $f \in C_\infty(S)$ can be extended to function $\hat{f} \in C(S_\vartheta)$ by setting $\hat{f}(\vartheta) = 0$.

The following results are mathematical folklore but of fundamental importance for our treatment of Feller processes. For that reason, we choose to present them here along with their proofs. We follow again very closely Kallenberg's exposition in [Kal21, Chapter 17], where further background material on Feller processes can be found.

Lemma 4.42 (Extension of Feller semigroups [Kal21, Lemma 17.13]). *Every Feller semigroup* $(T_t)_{t\geq 0}$ *on* $C_\infty(S)$ *can be extended to a conservative Feller semigroup* $\left(\hat{T}_t\right)_{t\geq 0}$ *on the space* $C(S_\vartheta)$ *by setting*

$$\hat{T}_t f := f(\vartheta) + T_t\left(f - f(\vartheta)\right), \quad t \geq 0,\ f \in C(S_\vartheta). \tag{4.98}$$

Proof. First observe that for $f \in C(S_\vartheta)$, we have $(f - f(\vartheta)) \in C_0(S)$ and hence, $\hat{T}_t f \in C(S_\vartheta)$ for all $f \in C(S_\vartheta)$ and $t \geq 0$. The strong continuity and semigroup property then carry over from $(T_t)_{t\geq 0}$ to $(\hat{T})_{t\geq 0}$ by linearity.

Now let $f \in C(S_\vartheta)$ be non-negative and set $g := f(\vartheta) - f \in C_0(S)$. Then, $g \leq f(\vartheta)$ and we obtain

$$T_t g \leq T_t g^+ \leq \|T_t g^+\|_\infty \leq \|g^+\|_\infty \leq f(\vartheta), \tag{4.99}$$

where $g^+ := g \vee 0$ denotes the positive part of g, as usual. Thus,

$$\hat{T}_t f = f(\vartheta) - T_t g \geq 0. \tag{4.100}$$

Finally, we have $\hat{T}_t 1 = 1 + T_t 0 = 1$ and we can deduce the conservativeness and the contraction property of $(\hat{T}_t)_{t\geq 0}$. ■

Recall that a state $x \in S_\vartheta$ is called *absorbing* for a Markov process X if $p_t(x, \{x\}) = \mathbb{P}_x(X_t \in \{x\}) = 1$ for all $t \geq 0$.

First we show, that there is a Markov process associated with every Feller semigroup.

Proposition 4.43 (Existence [Kal21, Proposition 17.14]). *For every Feller semigroup* $(T_t)_{t\geq 0}$ *on* C_0 *there exists a unique family of Markovian transition functions* $(p_t)_{t\geq 0}$ *on* S_ϑ *satisfying*

$$T_t f(x) = \int f(y) p_t(x, \mathrm{d}y), \tag{4.101}$$

for each $f \in C_0$ *and* $t \geq 0$ *such that* ϑ *is absorbing for* $(p_t)_{t\geq 0}$.

Proof. By Lemma 4.42 the maps $f \mapsto \hat{T}_t f$ are positive linear functionals on $C(S_\vartheta)$ with norm 1 for each $t \geq 0$. Applying Riesz' representation Theorem (cf. [Rud87, Theorem 6.19]) we deduce that for each $x \in S_\vartheta$ and $t \geq 0$ there exists a unique probability measure $p_t(x, \cdot)$ on S_ϑ such that

$$\hat{T}_t f(x) = \int f(y) p_t(x, \mathrm{d}y) \tag{4.102}$$

for all $f \in C(S_\vartheta)$. By continuity, the right-hand side is a measurable function of x. Measurability of the maps $x \mapsto p_t(x, A)$ for all $A \in \mathcal{B}(S_\vartheta)$ and $t \geq 0$ is then obtained by an approximation argument and an application of the monotone class theorem. In the same fashion we can show that $p_0(x, A) = \mathbb{1}_A(x)$. From the semigroup property of $(\hat{T}_t)_{t\geq 0}$ we have

$$\hat{T}_t f(x) = \hat{T}_s \hat{T}_{t-s} f(x) = \int \int f(y) p_{t-s}(z, \mathrm{d}y) p_s(x, \mathrm{d}z) \tag{4.103}$$

for all $0 \leq s < t$ and by the same argument as before, we conclude that $(p_t)_{t\geq 0}$ satisfies the Chapman-Kolmogorov equation (4.35). Finally, (4.101) follows from (4.102) as well as

$$\int f(y) p_t(\vartheta, \mathrm{d}y) = \hat{T}_t f(\vartheta) = f(\vartheta) = 0, \tag{4.104}$$

for all $f \in C_0$. Hence, ϑ is indeed absorbing for $(p_t)_{t\geq 0}$. ■

We say that ϑ is absorbing for $X^{\pm}$ if $\mathbb{P}(X_t \in \{\vartheta\}) = 1$ for all $t \geq \zeta$, where

$$\zeta := \inf \{ t \geq 0 \mid \vartheta \in \{X_t, X_{t-}\} \}. \tag{4.105}$$

Proposition 4.44 (Feller processes admit càdlàg modifications [Kal21, Theorem 17.15])**.** *Let X be a Feller process with semigroup $(T_t)_{t\geq 0}$ and values in S. For every initial distribution $\mu \in \mathcal{M}_1(S)$, there exists a modification $\hat{X}$ of X with values in S_ϑ such that $\hat{X}$ has càdlàg paths and ϑ is absorbing for $\hat{X}^{\pm}$. If, in addition, $(T_t)_{t\geq 0}$ is conservative, then there exists a càdlàg modification $\hat{X}$ with values in S.*

From now on we will always assume that a Feller process has càdlàg paths and we will include this in our definition of a Feller process.

Definition 4.45 (Feller processes II)**.** A Markov process X with values in a Polish uniform space $(S, \mathcal{U})$ is a *Feller process* if it satisfies the following conditions

(i) for each $\omega \in \Omega$, $X(\omega)$ is càdlàg,

(ii) the semigroup $(P_t)_{t\geq 0}$ associated with X has the Feller property. ◇

Remark 4.46. Combining Propositions 4.30, 4.43 and 4.44 we can deduce that every Feller semigroup uniquely determines a Feller process. Consequently, every Feller process is uniquely determined by its family of resolvent operators. This is an immediate consequence of the definition (4.86) since the resolvent is the Laplace transform of the semigroup. ◇

Proposition 4.44 allows us to view a Feller process as a random variable $X : \Omega \to D_S([0, \infty))$, thus building the bridge to Section 3.5. As before, we introduce the *canonical version* of X by identifying Ω with $D_S([0,\infty))$ and setting $X_t(\omega) = \omega(t)$, where $\omega \in D_S([0,\infty))$. Recall further the *translation operators* $(\theta_t)_{t\geq 0}$, where $\theta_t\colon D_S([0,\infty)) \to D_S([0,\infty))$ is defined as $\theta_t(\omega) := \omega(t + \cdot\,)$ for each $t \geq 0$.

4.2.3 Feller processes and stopping times

Let X be a canonical Feller process on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in the Polish uniform space $(S, \mathcal{U})$. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the augmented filtration generated by X.

Recall that a random variable τ on $(\Omega, \mathcal{A}, P)$ with values in $[0,\infty)$ is called a $(\mathcal{F}_t)$-*stopping time* if for each $t \geq 0$ the event $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable. Whenever it is clear from the context which filtration we are using, we just say that τ is a stopping time. Further, recall the definition of the σ-field $\mathcal{F}_\tau$ from (4.22).

It is worth noting that the augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by a Feller process is always right continuous (cf. [CW05, Theorem 2.3.4]). From Lemma 4.8 we then deduce that for a Feller process every optional time is a stopping time (and vice versa).

Feller processes exhibit nice properties with respect to stopping times. First and foremost, Feller processes are strongly Markovian.

Proposition 4.47 (Feller processes have the strong Markov property)**.** *Let X be a Feller process with initial distribution $\mu \in \mathcal{M}_1(S)$. For every stopping time τ with $\mathbb{P}_x(\tau < \infty) = 1$ for all $x \in S$ and non-negative random variable $Y\colon \Omega \to \mathbb{R}$, we have*

$$\mathbb{E}_\mu\left[Y \circ \theta_\tau \,|\, \mathcal{F}_\tau\right] = \mathbb{E}_{X_\tau}\left[Y\right], \quad \mathbb{P}_\mu\text{-a.s.} \tag{4.106}$$

Proof. See [Kal21, Theorem 17.17]. ∎

On the other hand, Feller processes are *quasi left-continuous*.

Proposition 4.48 (Quasi left-continuity)**.** *Every Feller process X is quasi* left-continuous. *That is, for every stopping time τ and every sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \leq \tau_{n+1}$ and $\lim_{n\to\infty} \tau_n = \tau$ almost surely,*

$$\lim_{n\to\infty} X_{\tau_n} = X_\tau \quad \text{a.s. on } \{\tau < \infty\}. \tag{4.107}$$

Proof. See [CW05, Theorem 2.4.4]. ∎

While Feller's approach is more analytic, another approach to the same objects was developed by the mathematician (and *tennis ace* [Hol08]) Gilbert A. Hunt. Hunt's definition starts with Markov processes and their path properties. The following definition coincides roughly with Hunt's hypothesis A in [Hun56][6].

[6]Compare Chung's remarks in [Chu82, p. 135].

Definition 4.49 (Hunt processes)**.** Let X be a Markov process with values in S_ϑ. We call X a *Hunt process* if it satisfies the following conditions

(i) X is right continuous,

(ii) X has the strong Markov property,

(iii) X is quasi-left-continuous. ◊

Basically, Hunt processes are Feller processes but the approach is somewhat reversed.[7]

For some applications, it is useful to extend the Borel σ-algebra $\mathcal{B}_\vartheta$ on S_ϑ to include those sets that are "not seen" by a given Feller process X. This leads to the following

Definition 4.50 (nearly Borel measurable sets)**.** Let X be a Feller process with values in S_ϑ. A set $A \subset S_\vartheta$ is said to be *nearly Borel measurable* if there exist Borel measurable sets $A_1, A_2 \in \mathcal{B}_\vartheta$ such that $A_1 \subset A \subset A_2$ with

$$\mathbb{P}_\mu(X_t \in A_2 \setminus A_1 \text{ for some } t \geq 0) = 0 \tag{4.108}$$

for all initial distributions $\mu \in \mathcal{M}_1(S)$. We write $\mathcal{B}^n = \mathcal{B}^n(S_\vartheta)$ for the totality of all nearly Borel measurable sets. ◊

4.2.4 Hitting times

We have observed in Remark 4.46 that the family of resolvent operators $(R_\alpha)_{\alpha>0}$ of a Feller process X uniquely determines said process. On the other hand, we have seen in Remark 4.26 that for $\alpha > 0$ the resolvent R_α applied to $\mathbb{1}_A$ for some $A \in \mathcal{B}$ can be interpreted as the (expected) occupation time of the set A by the process X up to an $\text{Exp}(\alpha)$ distributed random time ζ. Using the usual approximations by simple functions and the linearity of the resolvent operator, we can deduce that the resolvent operators $(R_\alpha)_{\alpha>0}$ and *a fortiori* the Feller process X is uniquely determined by the occupation times of all Borel sets $A \in \mathcal{B}$ of the killed process $\hat{X}$, killed at an $\text{Exp}(\alpha)$ random time for all $\alpha > 0$.

Following up on this idea, one might suspect that the same holds true for another class of random times. Indeed, we will show that we can use hitting times of open sets instead of independent exponentially distributed random variables.

To that end, we start with a brief treatment of hitting times and stopping times in general.

[7] See [CW05, §3].

Let X be a Feller process with values in $(S, \mathcal{U})$ and denote by $(\mathcal{F}_t)_{t\geq 0}$ the augmented filtration generated by X. Recall that $(\mathcal{F}_t)_{t\geq 0}$ is right continuous (cf. [CW05, Theorem 2.3.4]).

Clearly, constant times $a \geq 0$ are stopping times since the events $\{a \leq t\}$ are either the empty set or the whole space Ω and thus $\mathcal{F}_t$-measurable for every $t \geq 0$. We write

$$\tau_A = \tau_A(X) := \inf \{ t > 0 \mid X_t \in A \} \tag{4.109}$$

for the first hitting time of the set $A \in \mathcal{B}$. Note that we use the same notation for the random hitting times as we did in Chapter 3 for the deterministic hitting times. This ambiguity should lead to no confusion as it is always clear from the context if the involved hitting times are random or not.

Recall the definition of the first contact time γ_A from Section 3.4 (3.21):

$$\gamma_A(\omega) := \inf \left\{ t > 0 \,\middle|\, \{\omega(t), \omega(t-)\} \cap \overline{A} \neq \emptyset \right\}. \tag{4.110}$$

Analogously we define the first contact time of the set $A \in \mathcal{B}$ by the process X as

$$\gamma_A = \gamma_A(X) := \inf \left\{ t \geq 0 \,\middle|\, \{X_t, X_{t-}\} \cap \overline{A} \neq \emptyset \right\}. \tag{4.111}$$

Proposition 4.51 (Hitting times of open and closed sets are stopping times)**.** *Let X be a Feller process with values in $(S, \mathcal{U})$. For each $A \in \mathcal{B}$, open or closed, the random times τ_A and γ_A are $(\mathcal{F}_t)_{t\geq 0}$-stopping times.*

Proof. Let $A \in \mathcal{B}$ be open. In the case where $\mathbb{P}(\{\tau_A < \infty\}) = 0$, the first hitting time τ_A is clearly a stopping time as $\{\tau_A < t\}$ is a $\mathbb{P}$-nullset for each $t > 0$ and $(\mathcal{F}_t)_{t\geq 0}$ was assumed to be augmented. Fix $t > 0$ and choose $\omega \in \{\tau_A < t\}$. Then there exists a $s > 0$ such that $\tau_A \leq s < t$ with $X_s(\omega) \in A$. Now, since A is open, there exists an open entourage $U \in \mathcal{U}$ such that $U[X_s(\omega)] \subset A$. By right continuity of the map $t \mapsto X_t(\omega)$, there exists a $\varepsilon > 0$ such that $X_r(\omega) \in U[X_s(\omega)] \subset A$ for all $r \in [s, s + \varepsilon)$. That means there exists a $q \in \mathbb{Q} \cap [0, t)$ such that $X_q(\omega) \in A$ and hence,

$$\{\tau < t\} = \bigcup_{q\in[0,t)\cap\mathbb{Q}} \{X_q \in A\} \in \mathcal{F}_t, \tag{4.112}$$

proving that τ_A is a $(\mathcal{F}_t)_{t\geq 0}$ stopping time.

Now suppose $A \in \mathcal{B}$ is closed. Let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ be a sequence of open sets such that $\overline{B}_{n+1} \subset B_n$ and $A \subset B_n$ for each $n \in \mathbb{N}$. Since $\mathcal{U}$ has a countable base, we can choose for example $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ to be a sequence of open entourages with $U_{n+1} \circ U_{n+1} \subset U_n$,

$\bigcap_{n\in\mathbb{N}} U_n = \Delta$ and set $B_n := U_n[A]$. Then the sequence $(\tau_{B_n})_{n\in\mathbb{N}}$ is increasing and bounded by τ_A. By right continuity, we have for each $B \in \mathcal{B}$

$$X_{\tau_B} \in \overline{B} \quad \text{on } \{\tau_B < \infty\}. \tag{4.113}$$

By construction, we find that $A = \bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \overline{B}_n$ and

$$T := \lim_{n\to\infty} \tau_{B_n} \le \tau_A. \tag{4.114}$$

Consequently, by quasi left continuity of X (Proposition 4.48),

$$X_T = \lim_{n\to\infty} X_{\tau_{B_n}} \in \bigcap_{n\in\mathbb{N}} \overline{B}_n = A \quad \text{on } \{T < \infty\}. \tag{4.115}$$

Hence, $\tau_A \le T$ and consequently $\tau_A = T$ almost surely on $\{T < \infty\}$. On the other hand, on $\{T = \infty\}$, we have $\tau_A \ge T = \infty$, by construction. It follows that

$$\tau_A = \lim_{n\to\infty} \tau_{B_n}, \tag{4.116}$$

and we conclude from Lemma 4.10, that τ_A is indeed a stopping time. It remains to show that γ_A is a stopping time. Let $A \in \mathcal{B}$ be open or closed and let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ denote a sequence of open sets, as before. From the definition of γ_A we obtain for each $t \ge 0$,

$$\{\gamma_A \le t\} = \bigcap_{n\in\mathbb{N}} \{\tau_{B_n} \le t\} \in \mathcal{F}_t, \tag{4.117}$$

proving that γ_A is indeed a stopping time. ■

Remark 4.52. In the proof of Proposition 4.51 we have actually shown the stronger statement that γ_A is a stopping time for every Borel set $A \in \mathcal{B}$. In fact, one can show that every hitting time of a Borel set is a stopping time. This is sometimes called the *Debut Theorem*. Usually, the proof involves Choquet's capacibility theorem (cf. Proposition 5.52). A proof using only elementary methods was given by Richard F. Bass in [Bas10]. ◇

Definition 4.53. A random time τ is called a *terminal time* if

$$\tau \circ \theta_t + t = \tau \tag{4.118}$$

$\mathbb{P}_x$-almost surely on $\{t \le \tau\}$ for any starting point $x \in S$. ◇

Lemma 4.54. *For all $A \in \mathcal{B}$ open or closed, the first hitting time τ_A is a terminal time.*

Proof. Let $\omega \in \{t \le \tau\}$. Then, $\tau_A \circ \theta_t = \tau_A - t$, almost surely. ■

Clearly, the event $\{\tau_A = 0\}$ is $\mathcal{F}_0$-measurable for every $A \in \mathcal{B}$ open or closed. With our remark above, the same holds for every Borel measurable set $A \in \mathcal{B}$ and by definition even for every nearly Borel measurable set $A \in \mathcal{B}^n$. By Blumenthal's $0-1$-law (cf. [Kal21, Corollary 17.18]) we therefore conclude

$$\mathbb{P}_x(\tau_A = 0) \in \{0, 1\} \tag{4.119}$$

for all $x \in S$ and $A \in \mathcal{B}^n$.

This leads to the following definition.

Definition 4.55 (regular points)**.** Let X be a Feller process with values in S_ϑ and $A \in \mathcal{B}^n$ a nearly Borel measurable set. We say that a point $x \in S$ is *regular for* A if

$$\mathbb{P}_x(\tau_A = 0) = 1. \tag{4.120}$$

We denote by $A^r \subset S$ the set of regular points for A, i.e.

$$A^r := \{ x \in S \mid \mathbb{P}_x(\tau_A = 0) = 1 \}. \tag{4.121}$$

Conversely, a point $x \in S$ is said to be *irregular* for A if it is not regular for A. In that case, $\mathbb{P}_x(\tau_A = 0) = 0$. We say that a set $A \in \mathcal{B}^n$ is *thin* if $A^r = \emptyset$. ◊

By right-continuity of X, we immediately obtain the relation,

$$A^\circ \subset A^r \subset \overline{A}. \tag{4.122}$$

Lemma 4.56. *Let X be a (strong) Feller process with values in S_ϑ and $A \subset S$ closed. Then,*

$$\lim_{t\to 0} \sup_{x\in K} \mathbb{P}_x(\tau_A \le t) = 0 \tag{4.123}$$

for all compact subsets $K \subset D = S \setminus A$.

Proof. Let $K \subset D$ be a compact subset of $D := S \setminus A$. Let $\varphi \in C_\infty(S)$ non negative be such that $\varphi(x) = 1$ for all $x \in K$ and $\varphi(x) = 0$ for all $x \in A$. By either the Feller property **(F1)** or the strong Feller property **(F3)** we have that $P_t\varphi \in C_b(S)$ for all $t > 0$. Furthermore, we have by **(F2)** that $\lim_{t\to 0} \|P_t\varphi - \varphi\| = 0$. For every $\varepsilon > 0$ we can thus find a $T > 0$ such that

$$\sup_{x\in A} P_t\varphi(x) < \varepsilon/2 \quad \text{and} \quad \inf_{x\in K} P_t\varphi(x) > 1 - \varepsilon/2, \tag{4.124}$$

for all $t \leq T$. Hence, for all $x \in K$,

$$\begin{aligned} 1-\varepsilon/2 < \mathbb{E}_x\left[\varphi(X_T)\right] &= \mathbb{E}_x\left[\varphi(X_T);\ T<\tau_A\right] + \mathbb{E}_x\left[\varphi(X_T);\ T\geq\tau_A\right] \\ &\leq \mathbb{P}_x(T<\tau_A) + \mathbb{E}_x\left[\varphi(X_T);\ T\geq\tau_A\right]. \end{aligned} \tag{4.125}$$

On the other hand, by the strong Markov property,

$$\mathbb{E}_x\left[\varphi(X_T);\ T\geq\tau_A\right] = \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_A}}\left[\varphi(X_{T-\tau_A})\right];\ T\geq\tau_A\right]. \tag{4.126}$$

Because $X_{\tau_A} \in A$, the inner expectation on the right can be bounded by $\varepsilon/2$, by virtue of (4.124). Consequently,

$$\mathbb{P}_x(T<\tau_A) > 1-\varepsilon \tag{4.127}$$

for all $x \in K$. We conclude the proof by letting $\varepsilon \to 0$. ■

We go even further and show that the probability $\mathbb{P}_x(\tau_A \leq t)$ decays at least linearly in t. This result seems to have escaped notice in the literature in the general form we present it here. A similar result for Feller processes on $\mathbb{R}^d$ was given in [BSW13, Theorem 5.1] and our proof is inspired by their proof.

Theorem 4.57. *Let X be a Feller process with values in S_ϑ and $A \subset S$ closed. For all compact subsets $K \subset D = S \setminus A$ there exists a constant $C > 0$ such that*

$$\mathbb{P}_x(\tau_A \leq t) \leq Ct \tag{4.128}$$

for all $x \in K$ and $t > 0$.

Proof. Let $K \subset D$. By Lemma 2.37 we can choose $\varphi \in C_\infty(S)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for all $x \in K$ and $\varphi(x) = 0$ for all $x \in A$. Denote by $(\Delta, \mathcal{D})$ the generator of the Feller semigroup $(P_t)_{t\geq 0}$ associated with X. Since $\mathcal{D} \subset C_\infty(S)$ is dense, we can assume without loss of generality that $\varphi \in \mathcal{D}$. Fix $t > 0$ and note that $t \wedge \tau_A$ is a bounded stopping time. We can therefore apply the Dynkin formula Proposition 4.41 (ii) to obtain

$$\mathbb{E}_x\left[1-\varphi\left(X_{t\wedge\tau_A}\right)\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_A} -\Delta\varphi(X_s)\,\mathrm{d}s\right]. \tag{4.129}$$

Observe that $\varphi\left(X_{t\wedge\tau_A}\right) = 0$ on $\{\tau_A \leq t\}$ and therefore $1-\varphi\left(X_{t\wedge\tau_A}\right) \geq \mathbb{1}_{\{\tau_A\leq t\}}$. Consequently,

$$\mathbb{P}_x(\tau_A \leq t) \leq \mathbb{E}_x\left[\int_0^{t\wedge\tau_A} -\Delta\varphi(X_s)\,\mathrm{d}s\right] \leq \mathbb{E}_x\left[t\wedge\tau_A\right]\|\Delta\varphi\|_\infty \leq Ct, \tag{4.130}$$

where $C = \|\Delta\varphi\|_\infty$. ■

Corollary 4.58 (exit times)**.** *Let X be a Feller process with values in S_ϑ. For each $x \in S$ and $U \in \mathcal{U}$ open, there exists a constant $C > 0$ such that for all $t > 0$,*

$$\mathbb{P}_x(\sigma_{U[x]} \leq t) \leq Ct. \tag{4.131}$$

Here, $\sigma_A = \tau_{\complement A}$ denotes the first exit time *from A.*

Recall from Definition 4.27 that for $\alpha > 0$ we call a measurable function $f \in \mathcal{B}(S)$ *α-excessive* with respect to the semigroup $(P_t)_{t\geq 0}$ if

$$P_t e^{-\alpha t} f(x) \leq f(x) \tag{4.132}$$

for all $t > 0$ and $\lim_{t\to 0} P_t e^{-\alpha t} f(x) = f(x)$ for all $x \in S$. For later reference, we note the following fact.

Proposition 4.59. *Let $A \in \mathcal{B}^n$ be a nearly Borel measurable set and τ_A the first hitting time of A. For each $\alpha > 0$ the function*

$$x \mapsto \mathbb{E}_x\left[e^{-\alpha\tau_A}\right], \quad x \in S \tag{4.133}$$

is α-excessive.

Proof. Let $\alpha, t > 0$. Then,

$$\begin{aligned} e^{-\alpha t} P_t \mathbb{E}_x\left[e^{-\alpha\tau_A}\right] &= P_t \mathbb{E}_x\left[e^{-\alpha t}\alpha^{-1}\int_{\tau_A}^{\infty} e^{-\alpha s}\,\mathrm{d}s\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_{X_t}\left[e^{-\alpha t}\alpha^{-1}\int_{\tau_A}^{\infty} e^{-\alpha s}\,\mathrm{d}s\right]\right] \\ &= \mathbb{E}_x\left[\int_{\tau_A\circ\theta_t}^{\infty} \alpha^{-1} e^{-\alpha(s+t)}\,\mathrm{d}s\right] = \mathbb{E}_x\left[\int_{t+\tau_A\circ\theta_t}^{\infty} \alpha^{-1} e^{-\alpha s}\,\mathrm{d}s\right]. \end{aligned} \tag{4.134}$$

Since τ_A is a terminal time, we have $\tau_A = t + \tau_A \circ \theta_t$ for all $t \leq \tau_A$. Hence, $\lim_{t\to 0} t + \tau_A \circ \theta_t = \tau_A$ and therefore,

$$\lim_{t\to 0} e^{-\alpha t} P_t \mathbb{E}_x\left[e^{-\alpha\tau_A}\right] = \mathbb{E}_x\left[e^{-\alpha\tau_A}\right]. \tag{4.135}$$

On the other hand we have $t + \tau_A \circ \theta_t \geq \tau_A$ if $t > \tau_A$, hence

$$e^{-\alpha t} P_t \mathbb{E}_x\left[e^{-\alpha\tau_A}\right] \leq \mathbb{E}_x\left[e^{-\alpha\tau_A}\right]. \tag{4.136}$$

∎

When we consider hitting times of a ν-symmetric Feller process X, the question *where* the process first hits a set $A \in \mathcal{B}^n$, naturally arises. This leads to the following definition of the *α-hitting distribution*. For each $\alpha > 0$ and nearly Borel set $A \in \mathcal{B}^n$ we set

$$H_A^\alpha(x, B) := \mathbb{E}_x\left[e^{-\alpha\tau_A};\ X_{\tau_A} \in B\right], \tag{4.137}$$

if $\alpha = 0$ we write

$$H_A^0(x, B) := \mathbb{P}_x(\tau_A < \infty;\ X_{\tau_A} \in B), \tag{4.138}$$

where $x \in S$ and $B \in \mathcal{B}$. Using Definition 4.12 one easily verifies the following.

Lemma 4.60. *For each $\alpha \geq 0$ and every nearly Borel set $A \in \mathcal{B}^n$, the α-hitting distribution*

$$H_A^\alpha : S \times \mathcal{B} \to [0, 1] \tag{4.139}$$

is a sub-Markov kernel.

As usual, we write $H_A^\alpha f(x)$ for the integral of a bounded Borel measurable function $f \in \mathcal{B}_b(S)$ with respect to the measure $H_A^\alpha(x, \mathrm{d}y)$. In other words,

$$H_A^\alpha f(x) = \mathbb{E}_x\left[f\left(X_{\tau_A}\right) e^{-\alpha\tau_A}\right] \quad \text{for } \alpha > 0 \text{ and} \quad H_A^0 f(x) = \mathbb{E}_x\left[f(X_{\tau_A}); \tau_A < \infty\right]. \tag{4.140}$$

In particular, we have

$$H_A^\alpha 1(x) = \mathbb{E}_x\left[e^{-\alpha\tau_A}\right] \quad \text{for } \alpha > 0 \text{ and} \quad H_A^0 1(x) = \mathbb{P}_x(\tau_A < \infty). \tag{4.141}$$

4.3 Symmetric Feller processes

As before let $(S, \mathcal{U})$ be a locally compact uniform Polish space and $\nu \in \mathcal{M}(S)$ a boundedly finite Radon measure on $(S, \mathcal{B})$. Recall the definition of a (strongly) ν-symmetric Markov process from Definition 4.23. Naturally, we say that a Feller process X with values in S is (strongly) ν-symmetric if it is (strongly) ν-symmetric in the sense of Definition 4.23. Observe that the same definition holds true when we consider the extension of X to the one-point compactification S_ϑ.

Suppose that X is a strongly ν-symmetric Feller process with semigroup $(P_t)_{t\geq 0}$ and resolvent $(R_\alpha)_{\alpha>0}$. Then for all $f \in C_\infty(S)$,

$$\begin{aligned} R_\alpha f(x) &= \int_0^\infty e^{-\alpha t} P_t f(x)\,\mathrm{d}t = \int_0^\infty e^{-\alpha t} \int_S p_t(x, y) f(y)\,\nu(\mathrm{d}y)\,\mathrm{d}t \\ &= \int_0^\infty \int_S e^{-\alpha t} p_t(x, y)\,\mathrm{d}t f(y)\,\nu(\mathrm{d}y) =: \int_S u_\alpha(x, y) f(y)\,\nu(\mathrm{d}y). \end{aligned} \tag{4.142}$$

The functions $u_\alpha(x, y)$ are called *α-resolvent kernels* or *α-potential densities* for their potential theoretic origin and the fact that the α-resolvent is sometimes referred to as the α-potential.

Furthermore, it follows from the ν-symmetry that p_t and u_α are symmetric for every $t \geq 0$ and $\alpha > 0$ (cf. [MR06, Chapter 3.3]). Observe that in the book [MR06] by Michael B. Marcus and Jay Rosen, the authors start with (4.142) as the definition of strong ν-symmetry and deduce the existence of a symmetric family p_t. Both definitions are equivalent as was shown by Rainer Wittmann in [Wit86] (cf. [MR06, Remark 3.3.5]).

By virtue of the Feller property and the strong continuity of the semigroup, we can choose the functions p_t such that the map $(t, x, y) \mapsto p_t(x, y)$ is continuous as a function on $[0, \infty) \times S \times S$. As a consequence, every strongly symmetric Feller process is *regular* in the sense of [Kal21, Chapter 26].

4.3.1 The killed process

Let $(S, \mathcal{U}, \nu)$ be a locally compact uniform measure space. Suppose that $A \in \mathcal{B}$ is closed. Given a ν-symmetric Feller process X with values in S_ϑ. Recall the definition of the lifetime $\xi = \inf\{\, t > 0 \mid X_t \in \{\vartheta\}\,\}$ of X. We introduce the process X^A which is the same as the process X but killed upon hitting the set A, i.e. for each $t \geq 0$ and $\omega \in \Omega$ we set

$$X_t^A(\omega) := \begin{cases} X_t(\omega), & t < \min\{\tau_A(\omega), \xi\} \\ \vartheta_A, & t \geq \min\{\tau_A(\omega), \xi\} \end{cases} \tag{4.143}$$

and

$$D := S \setminus A. \tag{4.144}$$

Note that D is again locally compact by Lemma A.18 and denote by $D_{\vartheta_A} = D \cup \{\vartheta_A\}$ its one-point compactification. Observe that the cemetery point ϑ_A does not necessarily coincide with the cemetery point ϑ of the original process X. Keeping this in mind, we just write ϑ for the cemetery point of X^A with an abuse of notation. As usual, we extend functions in $\mathcal{B}(D)$ to D_ϑ by setting $f(\vartheta) = 0$.

Further observe that every $f \in \mathcal{B}_b(D)$ and $f \in C_\infty(D)$ can be extended to $\mathcal{B}_b(S)$ and $C_\infty(S)$, respectively, by setting $f = 0$ on A. Moreover, we can identify

$$C_\infty(D) = \{\, f \in C_\infty(S) \mid f(x) = 0 \ \forall x \in A\,\}. \tag{4.145}$$

We can therefore consider the killed process as a stochastic process with state space D_{ϑ_A}. The question remains which properties of X the killed process X^A inherits. As a first property observe that since $A \subset S$ was assumed to be closed, every point $x \in D$ is regular for D, i.e.

$$\mathbb{P}_x(X_0^A = x) = 1 \quad \forall x \in D. \tag{4.146}$$

In the following, we will restrict ourselves to the case where we kill in a closed set $A \in \mathcal{B}$. Most of the results can be significantly generalized (see for example [CF11, Sections 3.2 ff.] or [FOT11, Sections 4.1 ff.]) to sufficiently regular sets.

Despite being of natural interest, the killed process has gotten little attention in the literature, as far as we can tell. Some results about the properties of the killed process were obtained by Kai Lai Chung in [Chu86] and more recently in [BLM18].

We start with the observation that the killed process is again Markov.

Lemma 4.61. *Let X be a Markov process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and values in S_ϑ. Suppose $A \in \mathcal{B}$ is closed, then the killed process X^A is also a Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$ with values in D_ϑ.*

Proof. The proof is straightforward. Let $s, t > 0$ and $B \in \mathcal{B}^D$ a Borel subset of $D = S \setminus A$. Then, for each $x \in D$,

$$\begin{aligned} \mathbb{P}_x\left(X^A_{t+s} \in B \mid \mathcal{F}_t\right) &= \mathbb{P}_x\left(X_s \circ \theta_t \in B,\ t < \tau_A,\ s < \tau_A \circ \theta_t \mid \mathcal{F}_t\right) \\ &= \mathbb{1}_{\{t<\tau_A\}} \mathbb{P}_{X_t}(X_s \in B,\ s < \tau_A) = \mathbb{P}_{X^A_t}\left(X^A_s \in B\right), \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.147}$$

■

Now, the semigroup associated with X^A, denoted by $\left(P^A_t\right)_{t\geq 0}$, is given by

$$P^A_t f(x) := \mathbb{E}_x\left[f\left(X^A_t\right)\right] = \mathbb{E}_x\left[f(X_t);\ t < \tau_A\right], \tag{4.148}$$

for $t \geq 0$, $f \in \mathcal{B}_b(D)$ and $x \in D$. Similarly, the resolvent $\left(R^A_\alpha\right)_{\alpha>0}$ associated with X^A can be written as

$$R^A_\alpha f(x) = \int_0^\infty e^{-\alpha t} P^A_t f(x)\,\mathrm{d}t = \mathbb{E}_x\left[\int_0^{\tau_A} e^{-\alpha t} f(X_t)\,\mathrm{d}t\right]. \tag{4.149}$$

Suppose X is a strong Markov process, then we have for all $\alpha > 0$ and $f \in \mathcal{B}^+_b(S)$ non negative,

$$\begin{aligned} H^\alpha_A\left(R_\alpha f(x)\right) &= \int_S R_\alpha f(y)\,\mathbb{E}_x\left[e^{-\alpha\tau_A};\ X_{\tau_A} \in \mathrm{d}y\right] \\ &= \int_S \mathbb{E}_y\left[\int_0^\infty e^{-\alpha t} f(X_t)\,\mathrm{d}t\right] \mathbb{E}_x\left[e^{-\alpha\tau_A};\ X_{\tau_A} \in \mathrm{d}y\right] \\ &= \mathbb{E}_x\left[\int_0^\infty e^{-\alpha(t+\tau_A)} \mathbb{E}_{X_{\tau_A}}\left[f(X_t)\right]\,\mathrm{d}t\right] \\ &= \mathbb{E}_x\left[\int_{\tau_A}^\infty e^{-\alpha t} f(X_t)\,\mathrm{d}t\right]. \end{aligned} \tag{4.150}$$

By combining (4.149) and (4.150) we obtain for all $\alpha > 0$, $f \in \mathcal{B}_b^+(S)$ and $x \in S$,

$$R_\alpha^A f(x) = R_\alpha f(x) - H_A^\alpha R_\alpha f(x), \tag{4.151}$$

which is a special case of *Dynkin's formula*[8] (cf. [Dyn65, §1 Theorem 5.1]) due to the prolific Eugene B. Dynkin.

Lemma 4.62. *Let X be a strong Markov process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and values in S_ϑ. Suppose $A \in \mathcal{B}$ is closed, then the killed process X^A is also a strong Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$ with values in D_ϑ.*

Proof. Let τ be a $(\mathcal{F}_t)_{t\geq 0}$ stopping time and $\Theta \in \mathcal{F}_\tau$. Then,

$$\begin{aligned}\mathbb{E}_x\left[P_t^A f\left(X_\tau^A\right);\ \tau < \infty;\ \Theta\right] &= \mathbb{E}_x\left[\mathbb{E}_{X_\tau}\left[f(X_t);\ t < \tau_A\right];\ \tau < \tau_A;\ \Theta\right] \\ &= \mathbb{E}_x\left[f\left(X_{\tau+t}\right);\ \tau + t < \tau_A;\ \Theta\right] \\ &= \mathbb{E}_x\left[f\left(X_{\tau+t}^A\right);\ \tau < \infty;\ \Theta\right].\end{aligned} \tag{4.152}$$

Because $\Theta \in \mathcal{F}_\tau$ was arbitrary, we conclude that for all $f \in \mathcal{B}(D_\vartheta)$ and $x \in D_\vartheta$,

$$\mathbb{E}_x\left[f\left(X_{\tau+t}^A\right)\middle|\mathcal{F}_\tau\right] = \mathbb{E}_{X_\tau}\left[f\left(X_t^A\right)\right] \quad \mathbb{P}_x - a.s. \tag{4.153}$$

Therefore, X^A is again strong Markov. ■

Clearly, X^A has càdlàg paths if X has càdlàg paths. We now show that the ν-symmetry of X is preserved under killing.

Lemma 4.63. *Let X be a ν-symmetric Feller process with values in S_ϑ. Suppose that $A \in \mathcal{B}_\vartheta$ is closed and denote $D := S \setminus A$. Then the killed process X^A is a ν-symmetric Markov process with values in D_ϑ.*

Proof. Let $t > 0$. Suppose $f, g \in C_\infty^+$ and fix $n \in \mathbb{N}$. Then, by Lemma 4.24,

$$\begin{aligned}&\int_S g(x)\mathbb{E}_x\left[f(X_t);\ X_{tk/2^n} \in D,\ k = 1, \ldots, 2^n\right] \nu(\mathrm{d}x) \\ &\quad = \int_S \mathbb{E}_x\left[g(X_0)\prod_{k=1}^{2^n} \mathbb{1}_D(X_{tk/2^n})f(X_t)\right] \nu(\mathrm{d}x) \\ &\quad = \int_S f(x)\mathbb{E}_x\left[g(X_t);\ X_{tk/2^n} \in D,\ k = 1, \ldots, 2^n\right] \nu(\mathrm{d}x)\end{aligned} \tag{4.154}$$

[8] see [FOT11, p. 154]. Compare also with the Dynkin Formula from Proposition 4.41.

By right continuity of X and because $D \subset S$ is open, we obtain

$$\int_S g(x) P_t^A f(x)\,\nu(\mathrm{d}x) = \int_S g(x)\mathbb{E}_x\left[f(X_t);\ t \le \tau_A\right]\,\nu(\mathrm{d}x)$$
$$= \int_S f(x)\mathbb{E}_x\left[g(X_t);\ t \le \tau_A\right]\,\nu(\mathrm{d}x) = \int_S f(x) P_t^A g(x)\,\nu(\mathrm{d}x) \tag{4.155}$$

as $n \to \infty$. Splitting f and g into positive and negative part and using the linearity of the integral completes the proof. ■

Next, we want to identify the generator of the killed process X^A.

Proposition 4.64. *Let X be a Feller process with values in S_ϑ and denote by $(\Delta, \mathcal{D})$ its generator. Let $A \in \mathcal{B}_\vartheta$ be closed and $D = S \setminus A$. Then the generator of the killed process X^A is given by $(\Delta^A, \mathcal{D}^A)$, where*

$$\mathcal{D}^A = \{\, f \in \mathcal{D} \mid f|_A = 0 \,\} = \mathcal{D} \cap C_\infty(D), \tag{4.156}$$

and $\Delta^A f = \Delta f$ for all $f \in \mathcal{D}^A$.

Proof. We first show that for all $f \in \mathcal{D}^A$ and $x \in D$,

$$\lim_{t \to 0} \left| \frac{P_t^A f(x) - f(x)}{t} - \frac{P_t f(x) - f(x)}{t} \right| = \lim_{t \to 0} \frac{\left|P_t^A f(x) - P_t f(x)\right|}{t} = 0. \tag{4.157}$$

It suffices to show (4.157) for non negative $f \in C_\infty^+(D) \cap \mathcal{D}$. Furthermore, we can extend $P_t^A f$ to $C_\infty(S)$ by setting $P_t^A f(x) = 0$ on A. Fix $f \in C_\infty^+(D) \cap \mathcal{D}$, we want to show (4.157) for all $x \in S$. By application of the strong Markov property, we obtain

$$P_t f(x) - P_t^A f(x) = \mathbb{E}_x\left[f(X_t);\ t \ge \tau_A\right] = \mathbb{E}_x\left[P_{t-\tau_A} f(X_{\tau_A});\ t \ge \tau_A\right]. \tag{4.158}$$

Now, by strong continuity **(F2)**, there exists a $\delta > 0$ such that

$$\|P_t f - f\|_\infty < \varepsilon, \tag{4.159}$$

for all $t \in [0, \delta)$. Suppose $t \in [0, \delta)$, note that this immediately implies $0 \le t - \tau_A < \delta$ on $\{t \ge \tau_A\}$ and hence,

$$\mathbb{E}_x\left[P_{t-\tau_A} f(X_{\tau_A});\ t \ge \tau_A\right] \le \mathbb{E}_x\left[f(X_{\tau_A}) + \varepsilon;\ t \ge \tau_A\right] \le \varepsilon \mathbb{P}_x(t \ge \tau_A), \tag{4.160}$$

because $X_{\tau_A} \in A$ by closedness of A and right continuity of the paths of X. Consequently, $f(X_{\tau_A}) = 0$. Clearly, the right-hand side is equal to 0 for all $x \in S$. By Theorem 4.57 we find for every $K \subset D$ compact a constant $C > 0$ such that

$$\sup_{x \in K} \mathbb{P}_x(t \geq \tau_A) \leq Ct. \tag{4.161}$$

Cleaning up now leads to

$$\lim_{t \to 0} \sup_{x \in K} \left| \frac{P_t^A f(x) - f(x)}{t} - \frac{P_t f(x) - f(x)}{t} \right| \leq \varepsilon C. \tag{4.162}$$

Since $\varepsilon > 0$ was arbitrary and $C > 0$ depends on K but not on ε, we find that

$$\frac{P_t^A f - f}{t} \longrightarrow \Delta f \in C_\infty(S) \tag{4.163}$$

uniformly on compacta as $t \to 0$ and therefore uniformly on D. Furthermore, because the left-hand side of (4.163) is equal to 0 for all $x \in A$, the right hand side is actually in $C_\infty(D)$. We have therefore shown that $\Delta^A f = \Delta f$ for all $f \in C_\infty(D) \cap \mathcal{D}$ and therefore $C_\infty(D) \cap \mathcal{D} \subset \mathcal{D}^A$. The converse relation "$\supset$" follows from the fact that $C_\infty(D) \subset C_\infty(S)$. ∎

A similar argument can be found in [BLM18, Theorem 2.3].

We are now in a position to show the main result about killed Feller processes.

Theorem 4.65. *Let X be a ν-symmetric Feller process with values in S_ϑ and $A \in \mathcal{B}_\vartheta$ closed. Then the killed process X^A is again a $\nu|_D$-symmetric Feller process with values in D_ϑ, where $D = S \setminus A$.*

Proof. We have already shown the symmetry of X^A in Lemma 4.63. It only remains to show that X^A is a Feller process. In Proposition 4.64 we have identified the generator $(\Delta^A, \mathcal{D}^A)$ of P_t^A and we want to apply the Hille-Yosida Theorem, Proposition 4.40, to conclude that P_t^A is Feller. Since $\mathcal{D}$ is dense in $C_\infty(S)$, it follows immediately that $\mathcal{D}^A = \mathcal{D} \cap C_\infty(D)$ is dense in $C_\infty(D)$ and (i) of Proposition 4.40 holds. Similarly, property (iii) follows from the corresponding property of Δ. In order to verify (ii), we need to show that the range of $(\alpha - \Delta^A)$ is dense in $C_\infty(D)$. By Proposition 4.39 (ii) we know that $\mathcal{D} = R_\alpha C_\infty(S)$ for all $\alpha > 0$. Fix some $\alpha > 0$, it suffices to show that

$$R_\alpha f(x) > 0 \tag{4.164}$$

for all $f \in C_\infty(S)$ with $f(x) > 0$. Because then we can argue that

$$R_\alpha \left(C_\infty(S) \setminus C_\infty(D)\right) \subset C_\infty(S) \setminus C_\infty(D) \tag{4.165}$$

and therefore the preimage of $f \in \mathcal{D}^A = \mathcal{D} \cap C_\infty(D)$ under R_α must be an element of $C_\infty(D)$. Consequently, because $(\Delta, \mathcal{D})$ itself satisfies property (ii) of Proposition 4.40 i.e. $(\alpha - \Delta)\mathcal{D}$ is dense in $C_\infty(S)$, we can conclude that

$$(\alpha - \Delta^A)\mathcal{D}^A = (\alpha - \Delta)\mathcal{D}^A = R_\alpha^{-1}\mathcal{D}^A \tag{4.166}$$

is dense in $C_\infty(D)$.

We show (4.164). Suppose $f \in C_\infty(S)$ and $x_0 \in S$ such that $f(x_0) = c > 0$. By continuity, there exists for each $\varepsilon > 0$ an open entourage $U \in \mathcal{U}$ such that $f(x) \geq c - \varepsilon$ for all $x \in U[x_0]$. Denote by $\sigma = \tau_{\complement U[x_0]}$ the first exit time from $U[x_0]$. Then,

$$R_\alpha f(x_0) = \int_0^\infty e^{-\alpha t} P_t f(x_0)\,\mathrm{d}t \geq \mathbb{E}_{x_0}\left[e^{-\alpha\sigma}\int_0^\sigma f(X_t)\,\mathrm{d}t\right] \geq \mathbb{E}_{x_0}\left[\sigma e^{-\alpha\sigma}\right](c - \varepsilon). \tag{4.167}$$

By Corollary 4.58 there exists a $t > 0$ such that $\mathbb{P}_{x_0}(\sigma > t) \geq \varepsilon$. That allows us to bound the expectation on the right of (4.167) from below by $e^{-\alpha t} t\varepsilon > 0$. Consequently, (4.164) is verified and the proof is finished. ∎

The final result of this section is basically due to [Chu86]. In the Theorem on p. 68 of [Chu86], Kai Lai Chung shows that X^A is doubly Feller whenever X is doubly Feller. We have decoupled the Feller property from the strong Feller property in Theorem 4.65. The proof that the strong Feller property is retained under killing is now rather simple.

Theorem 4.66 (the killed process is again strongly Feller)**.** *Let X be a strong Feller process with values in S and $A \in \mathcal{B}_\vartheta$ closed. Then the killed process is a strong Feller process with values in D_ϑ.*

Proof. First, assume that $(P_t)_{t\geq 0}$ has the property **(F3)**. Fix $f \in \mathcal{B}_b(D)$. For $x \in D$ and $t > s > 0$ let

$$\psi_s(x) := \mathbb{E}_x\left[f(X_{t-s});\ t - s < \tau_A\right]. \tag{4.168}$$

Clearly, $\psi_s \in \mathcal{B}_b(D)$. Hence,

$$\begin{aligned} P_s\psi_s(x) &= \mathbb{E}_x\left[\mathbb{E}_{X_s}\left[f(X_{t-s});\ t-s<\tau_A\right]\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_{X_s}\left[f(X_{t-s});\ t-s<\tau_A\right];\ s<\tau_A\right] \\ &\quad + \mathbb{E}_x\left[\mathbb{E}_{X_s}\left[f(X_{t-s});\ t-s<\tau_A\right];\ s\geq\tau_A\right] \\ &= \mathbb{E}_x\left[f(X_t);\ t<\tau_A\right] + \mathbb{E}_x\left[\mathbb{E}_{X_s}\left[f(X_{t-s});\ t-s<\tau_A\right];\ s\geq\tau_A\right] \end{aligned} \tag{4.169}$$

is continuous and bounded by **(F3)**. Bounding the last summand on the right, we obtain

$$\left|P_t^A f(x) - P_s\psi_s(x)\right| \leq \|\psi_s\|\,\mathbb{P}_x(\tau_A \leq s). \tag{4.170}$$

By virtue of Lemma 4.56 the right hand side converges to 0 uniformly on compact sets as $s \to 0$. Hence, $P_t^A f \in C_b(S)$ (see for example [Fol99, Proposition 4.38]). Moreover, $P_t^A f(x) = 0$ for all $x \in A$, by definition, and therefore $P_t^A f \in C_b(D)$. In conclusion, $\left(P_t^A\right)_{t\geq 0}$ satisfies the strong Feller property (**F3**).

It remains to show the strong continuity of P_t^A. Fix $f \in C_\infty(D)$, then

$$\left|P_t^A f(x) - P_t f(x)\right| = \left|\mathbb{E}_x\left[f(X_t);\ t \geq \tau_A\right]\right| \leq \|f\|_\infty \mathbb{P}_x(t \geq \tau_A). \tag{4.171}$$

By Lemma 4.56, the upper bound goes to 0 uniformly on compacta as $t \to 0$. Because $P_t f \to f$ uniformly as $t \to 0$ and $f \in C_\infty(D)$, we can conclude from (4.171) that $P_t^A f \to f$ uniformly as $t \to 0$.

■

4.3.2 Recurrence and transience

In this section, we introduce the notions of recurrence and transience for strongly symmetric Feller processes.

Definition 4.67 (Recurrence and transience)**.** A ν-symmetric Feller process X with values in $(S_\vartheta, \mathcal{U}_\vartheta)$ is *recurrent*, if

$$\int_0^\infty \mathbb{1}_A(X_t)\,\mathrm{d}t = \infty, \quad \mathbb{P}_x\text{-a.s.} \tag{4.172}$$

for all $x \in S$ and $A \in \mathcal{B}$ with $\nu(A) > 0$.

The process X is called *(uniformly) transient* if

$$\sup_{x\in S} \mathbb{E}_x\left[\int_0^\infty \mathbb{1}_K(X_t)\,\mathrm{d}t\right] < \infty, \tag{4.173}$$

for all $K \in \mathcal{B}$, compact. ◇

Again, there are various definitions of recurrence of a stochastic process. The definition we use here is sometimes called *Harris recurrence* (cf. [Kal21, Chapter 26]). Clearly, (4.172) implies that the first hitting time τ_A of every $A \in \mathcal{B}$ with $\nu(A) > 0$ is $\mathbb{P}_x$-a.s. finite for every starting point $x \in S$. Observe that in Definition 4.67 we consider the extension of X to the one-point compactification $(S_\vartheta, \mathcal{U}_\vartheta)$. Yet the equations (4.172) and (4.173) take only $x \in S$ and $B, K \in \mathcal{B}$ into account.

The next result is important but we refer for a proof to the literature as the proof requires some potential theoretic tools that we have not developed yet.

Proposition 4.68 (Recurrence dichotomy)**.** *Let X be a ν-symmetric Feller process. Then X is either recurrent or transient.*

Proof. See [Kal21, Theorem 26.17]. ■

As a consequence, we get the following result.

Lemma 4.69 (The killed process is transient)**.** *Let X be a ν-symmetric Feller process and $A \in \mathcal{B}$ closed with $\nu(A) > 0$. Then the killed process X^A, where $X^A_t = X_t$ on $\{t < \tau_A\}$ and $X^A_t = \vartheta$ on $\{t \geq \tau_A\}$ is transient.*

Proof. By Theorem 4.65, the process X^A is again a ν-symmetric (at least up to time $t = \tau_A = \zeta$) Feller process. If X was already transient, there is nothing to show. If X is recurrent, then $\tau_A < \infty$ $\mathbb{P}_x$-a.s. for every $x \in S$. Hence, (4.173) holds because $\vartheta \notin K$ for all $K \in \mathcal{B}$ compact. ■

4.3.3 Uniqueness by hitting times

For the remainder of this section we assume that $(S, \mathcal{U})$ is a compact uniform space and that ν is a Radon measure on S with full support. As before, let X denote a ν-symmetric Feller process with values in S_ϑ.

Note that in the situation of compact state spaces, transience of X is equivalent to the lifetime ζ of X being almost surely finite for every starting point $x \in S$.

We will apply the next lemma for killed processes but the result is in itself interesting. It shows that the whole resolvent family is already determined by the 0-resolvent if X is transient.

Lemma 4.70. *Let $(S, \mathcal{U})$ be compact and X a ν-symmetric and transient Feller process with values in S_ϑ. Define the* 0*-resolvent of X as*

$$Rf(x) := \mathbb{E}_x\left[\int_0^\infty f(X_s)\,\mathrm{d}s\right] = \mathbb{E}_x\left[\int_0^\zeta f(X_s)\,\mathrm{d}s\right] \tag{4.174}$$

for $f \in C_\infty(S)$ and $x \in S$. Then, $Rf \in C_\infty(S)$ and X is uniquely determined by the 0*-resolvent R.*

Proof. Let $f \in C_\infty(S)$. By definition of transience we have

$$\|Rf\|_\infty \leq \sup_{x\in S}\int_0^\infty |P_t f(x)|\,\mathrm{d}t \leq \|f\|_\infty \sup_{x\in S}\mathbb{E}_x\left[\int_0^\infty \mathbb{1}_S(X_t)\,\mathrm{d}t\right] < \infty. \tag{4.175}$$

Moreover, for every $n \in \mathbb{N}$ the function $G_n f$ defined by

$$G_n f(x) := \mathbb{E}_x \left[\int_0^n f(X_t)\, \mathrm{d}t \right] \tag{4.176}$$

is in $C_\infty(S)$ and $G_n f \to Rf$ uniformly and consequently, $Rf \in C_\infty$. We have therefore shown that $R\colon C_\infty(S) \to C_\infty(S)$ is a linear operator.

Now suppose that $f \in C_\infty^+(S)$ is non negative and choose $M < \infty$ such that $\|f\|_\infty \leq M$. Assume further that $\hat{X}$ is another Feller process with the same 0-resolvent R. Write $(R_\alpha)_{\alpha>0}$ and $\left(\hat{R}_\alpha\right)_{\alpha>0}$ for the resolvents of X and $\hat{X}$, respectively. By the resolvent equation (R1) we obtain for all $\alpha > 0$

$$R_\alpha f(x) = Rf(x) - \alpha R R_\alpha f(x). \tag{4.177}$$

Iterating this argument we get for all $0 < \alpha < M^{-1}$,

$$R_\alpha f(x) = \sum_{k=1}^{\infty} (-\alpha)^{k-1} R^k f(x). \tag{4.178}$$

An application of the same argument to $\hat{R}_\alpha$ yields $R_\alpha f = \hat{R}_\alpha f$ for all $\alpha \in (0, M^{-1})$ and hence for all $\alpha > 0$ by uniqueness of the Laplace transform. The extension of this equality to all $f \in C_\infty(S)$ is easily obtained by splitting f into positive and negative parts and applying monotone convergence to $f_n^\pm = f^\pm \mathbb{1}_{f^\pm \leq n}$. Finally, the claim follows from the fact that X is uniquely determined by its family of resolvent operators (see Remark 4.46). ■

For every $A \in \mathcal{B}$ with $\nu(A) > 0$ and $f \in \mathcal{B}_b(S)$ we introduce the *Green operator* G_A as follows

$$G_A f(x) := \mathbb{E}_x \left[\int_0^{\tau_A} f(X_s)\, \mathrm{d}s \right], \quad x \in S. \tag{4.179}$$

Lemma 4.71 (Green operators are bounded)**.** *Let $(S, \mathcal{U})$ be compact and X a ν-symmetric Feller process with values in S_ϑ. For each $A \in \mathcal{B}$ closed with $\nu(A) > 0$, the map $G_A : \mathcal{B}_b(S) \to \mathcal{B}_b(S)$ is a bounded linear operator.*

Proof. By Theorem 4.65 the killed process X^A is again Feller. Moreover, by Lemma 4.69, X^A is transient. Recall that $P_t^A f(x) := \mathbb{E}_x [f(X_t);\ t < \tau_A]$. Although the domain of P_t^A contains by definition only functions in $\mathcal{B}_b(D)$, where $D = S \setminus A$, P_t^A can easily extended to $\mathcal{B}_b(S)$. Suppose $f \in \mathcal{B}_b(S)$, then

$$\|G_A f\|_\infty \leq \sup_{x \in S} \int_0^\infty \left| P_t^A f(x) \right| \mathrm{d}t < \infty, \tag{4.180}$$

as before. ■

Theorem 4.72. *Let* $(S, \mathcal{U})$ *be locally compact and* X *be a* ν*-symmetric doubly Feller process with values in* S_ϑ*. Then* X *is uniquely determined by the family of Green operators*

$$\{ G_A : \mathcal{B}_b \to \mathcal{B}_b \mid A \in \mathcal{B} \text{ closed} \} . \tag{4.181}$$

Proof. Consider the process killed upon hitting the closure $A = \overline{U}$ of some open set $U \in \mathcal{B}$, i.e. $X^A = (X_{t \wedge \tau_A})_{t \geq 0}$. By assumption, ν has full support, hence every open $A \in \mathcal{B}$ has positive measure $\nu(A) > 0$. Consequently, by Lemma 4.69, we have that the killed process X^A is transient. Then the resolvent associated to X^A can be written as

$$R^A_\alpha f(x) := \mathbb{E}_x \left[\int_0^{\tau_A} e^{-\alpha s} f(X_s) \, \mathrm{d}s \right], \quad x \in S, \tag{4.182}$$

which we can extend to $\mathcal{B}_b(S)$, as before. The 0-resolvent R^A associated with X^A then coincides with the Green operator G_A associated with X. By Lemma 4.70, the killed process X^A is then uniquely determined by G_A. It therefore suffices to show that the resolvent $(R_\alpha)_{\alpha>0}$ of X is determined by the resolvents of X^A for a suitable collection of $A \in \mathcal{B}$. To that end let $a, b \in S$ and choose $U \in \mathcal{U}$ open such that $\overline{U[a]}, \overline{U[b]}$ are compact and $\overline{U[a]} \cap \overline{U[b]} = \emptyset$. Such a $U \in \mathcal{U}$ exists because of the Hausdorff property and the local compactness of S, as shown in Lemma 2.36. In order to save some ink we write $A = \overline{U[a]}$ and $B = \overline{U[b]}$. Now, define $\tau_0 := \tau_A$ and for $n \geq 0$ set

$$\tau_{n+1} := \inf \{ t > \tau_n \mid X_t \in A, \exists s \in [\tau_n, t] : X_s \in B \} . \tag{4.183}$$

Suppose that X is transient, then we have $\lim_{n\to\infty} \tau_n = \infty$, $\mathbb{P}_x$-almost surely for all $x \in S$. If, on the other hand, X is recurrent, we get $\tau_n < \infty$, $\mathbb{P}_x$-almost surely for all $x \in S$. By right continuity of X and the strong Markov property, we conclude that

$$\inf_{n\in\mathbb{N}} \inf_{x\in S} \mathbb{E}_x [\tau_n - \tau_{n-1}] := T > 0. \tag{4.184}$$

Hence,

$$\tau_n = \tau_0 + \sum_{j=1}^{n} \tau_j - \tau_{j-1}. \tag{4.185}$$

Consequently, we have by the strong law of large numbers $\lim_{n\to\infty} \tau_n = \infty$, $\mathbb{P}_x$-almost surely for all $x \in S$. Hence,

$$\begin{aligned} R_\alpha f(x) &= \mathbb{E}_x\left[\int_0^{\tau_0} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] + \sum_{n=0}^{\infty} \mathbb{E}_x\left[\int_{\tau_n}^{\tau_{n+1}} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] \\ &= R_\alpha^A f(x) + \sum_{n=0}^{\infty} \mathbb{E}_x\left[\int_{\tau_n}^{\tau_{n+1}} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] \end{aligned} \tag{4.186}$$

Since $A = \overline{U[a]}$ is compact there exist (not necessarily unique) minimizers $\underline{x}, \underline{x}_0 \in A$ of the variational problems

$$\mathbb{E}_{\underline{x}}\,[e^{-\alpha\tau_1}] = \inf\left\{\,\mathbb{E}_x\,[e^{-\alpha\tau_1}]\;\middle|\; x \in A\,\right\} \tag{4.187}$$

and

$$\mathbb{E}_{\underline{x}_0}\left[\int_0^{\tau_1} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] = \inf\left\{\,\mathbb{E}_x\left[\int_0^{\tau_1} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right]\;\middle|\; x \in A\,\right\}. \tag{4.188}$$

Applying the strong Markov property at the stopping times τ_n we obtain

$$\begin{aligned} \mathbb{E}_x\left[\int_{\tau_n}^{\tau_{n+1}} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] &= \mathbb{E}_x\left[\mathbb{E}\left[\int_{\tau_n}^{\tau_{n+1}} e^{-\alpha s} f(X_s)\,\mathrm{d}s\,\middle|\,\mathcal{F}_{\tau_n}\right]\right] \\ &= \mathbb{E}_x\left[e^{-\alpha\tau_n}\mathbb{E}_{X_{\tau_n}}\left[\int_0^{\tau_1} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right]\right] \\ &\geq \mathbb{E}_x\,[e^{-\alpha\tau_n}]\,\mathbb{E}_{\underline{x}_0}\left[\int_0^{\tau_1} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] \\ &\geq \mathbb{E}_x\,[e^{-\alpha\tau_0}]\,\mathbb{E}_{\underline{x}}\,[e^{-\alpha\tau_n}]\,\mathbb{E}_{\underline{x}_0}\left[\int_0^{\tau_1} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right]. \end{aligned} \tag{4.189}$$

Using the fact that

$$R_\alpha^A \mathbb{1}_S(x) = \mathbb{E}_x\left[\int_0^{\tau_0} e^{-\alpha s}\,\mathrm{d}s\right] = \alpha^{-1}\left(\mathbb{E}_x\,[e^{-\alpha\tau_0}] - 1\right) \tag{4.190}$$

we can write

$$\mathbb{E}_x\,[e^{-\alpha\tau_0}] = 1 - \alpha R_\alpha^A \mathbb{1}_S(x). \tag{4.191}$$

Now let $\underline{y}, \underline{y}_0 \in B$ be minimizers of

$$\mathbb{E}_{\underline{y}}\,[e^{-\alpha\tau_0}] = \inf\left\{\,\mathbb{E}_y\,[e^{-\alpha\tau_0}]\;\middle|\; y \in B\,\right\} \tag{4.192}$$

and

$$\mathbb{E}_{\underline{y}_0}\left[\int_0^{\tau_0} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] = \inf\left\{ \mathbb{E}_y\left[\int_0^{\tau_0} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] \,\middle|\, y \in B \right\}, \tag{4.193}$$

respectively. Using these minimizers and (4.190), we can estimate

$$\begin{aligned}
\mathbb{E}_{\underline{x}}\left[e^{-\alpha\tau_1}\right] &= \mathbb{E}_{\underline{x}}\left[e^{-\alpha\tau_B} e^{-\alpha(\tau_1-\tau_B)}\right] = \mathbb{E}_{\underline{x}}\left[\mathbb{E}\left[e^{-\alpha\tau_B} e^{-\alpha(\tau_1-\tau_B)} \,\middle|\, \mathcal{F}_{\tau_B}\right]\right] \\
&\geq \mathbb{E}_{\underline{x}}\left[e^{-\alpha\tau_B}\right] \mathbb{E}_{\underline{y}}\left[e^{-\alpha\tau_1}\right] \\
&= \left(1 - \alpha R_\alpha^B \mathbb{1}_S\left(\underline{x}\right)\right)\left(1 - \alpha R_\alpha^A \mathbb{1}_S\left(\underline{y}\right)\right)
\end{aligned} \tag{4.194}$$

and inductively

$$\mathbb{E}_{\underline{x}}\left[e^{-\alpha\tau_n}\right] \geq \left(\left(1 - \alpha R_\alpha^B \mathbb{1}_S\left(\underline{x}\right)\right)\left(1 - \alpha R_\alpha^A \mathbb{1}_S\left(\underline{y}\right)\right)\right)^n. \tag{4.195}$$

Similarly, we obtain

$$\begin{aligned}
\mathbb{E}_{\underline{x}_0}\left[\int_0^{\tau_1} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] &= \mathbb{E}_{\underline{x}_0}\left[\mathbb{E}\left[\int_0^{\tau_B} e^{-\alpha s} f(X_s)\,\mathrm{d}s + \int_{\tau_B}^{\tau_A} e^{-\alpha s} f(X_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_B}\right]\right] \\
&\geq R_\alpha^B f\left(\underline{x}_0\right) + \mathbb{E}_{\underline{x}_0}\left[e^{-\alpha\tau_B}\right] \mathbb{E}_{\underline{y}_0}\left[\int_0^{\tau_A} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] \\
&= R_\alpha^B f\left(\underline{x}_0\right) + \left(1 - \alpha R_\alpha^B \mathbb{1}_S\left(\underline{x}_0\right)\right) R_\alpha^A f\left(\underline{y}_0\right)
\end{aligned} \tag{4.196}$$

Plugging these estimates into (4.186), we obtain

$$\begin{aligned}
R_\alpha f(x) &\geq R_\alpha^A f(x) + \left(1 - \alpha R_\alpha^A \mathbb{1}_S(x)\right)\left(R_\alpha^B f\left(\underline{x}_0\right) + \left(1 - \alpha R_\alpha^B \mathbb{1}_S\left(\underline{x}_0\right)\right)\right) R_\alpha^A f\left(\underline{y}_0\right) \\
&\quad \times \sum_{n=0}^{\infty}\left(\left(1 - \alpha R_\alpha^B \mathbb{1}_S\left(\underline{x}\right)\right)\left(1 - \alpha R_\alpha^A \mathbb{1}_S\left(\underline{y}\right)\right)\right)^n \\
&=: R_\alpha^A f(x) + \left(1 - \alpha R_\alpha^A \mathbb{1}_S(x)\right) \underline{H}_\alpha(A, B, f)
\end{aligned} \tag{4.197}$$

By replacing the infima in (4.187), (4.188), (4.192) and (4.193) with suprema and writing $\overline{x}, \overline{x}_0, \overline{y}, \overline{y}_0$ for their respective maximizers, we obtain a similar upper bound

$$\begin{aligned} R_\alpha f(x) &\leq q R^A_\alpha f(x) + \left(1 - \alpha R^A_\alpha \mathbb{1}_S(x)\right)\left(R^B_\alpha f(\overline{x}_0) + \left(1 - \alpha R^B_\alpha \mathbb{1}_S(\overline{x}_0)\right)\right) R^A_\alpha f(\overline{y}_0) \\ &\quad \times \sum_{n=0}^{\infty} \left(\left(1 - \alpha R^B_\alpha \mathbb{1}_S(\overline{x})\right)\left(1 - \alpha R^A_\alpha \mathbb{1}_S(\overline{y})\right)\right)^n \\ &=: R^A_\alpha f(x) + \left(1 - \alpha R^A_\alpha \mathbb{1}_S(x)\right) \overline{H}_\alpha(A, B, f) \end{aligned} \tag{4.198}$$

Now let $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ be a family of open entourages such that $U \supset U_1 \supset U_2 \supset \dots$ and $\bigcap_{n\geq 1} U_n = \Delta$. Observe that the killed processes are again strongly Feller by Theorem 4.66. Therefore, $\underline{H}_\alpha(A, B, f)$ and $\overline{H}_\alpha(A, B, f)$ are continuous functions of $\underline{x}, \underline{x}_0, \underline{y}, \underline{y}_0$ and $\overline{x}, \overline{x}_0, \overline{y}, \overline{y}_0$, respectively. Therefore,

$$\limsup_{n\to\infty} \left| \overline{H}_\alpha(\overline{U_n[a]}, \overline{U_n[b]}, f) - \underline{H}_\alpha(\overline{U_n[a]}, \overline{U_n[b]}, f) \right| = 0, \tag{4.199}$$

and hence the upper and lower bounds in (4.197) and (4.198) converge to the same limit as we let $U \to \Delta$. Finally, we can write

$$R_\alpha f(x) = R^A_\alpha f(x) + (1 - \alpha R^A_\alpha \mathbb{1}_S(x)) \lim_{n\to\infty} \underline{H}_\alpha(U_n[a], U_n[b], f), \tag{4.200}$$

which concludes the proof. ■

By assumption, $(S, \mathcal{U})$ is a Polish uniform space. In particular, that means that there exists a sequence of open entourages $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $U_{n+1} \subset U_n$ and $\bigcap_{n\in\mathbb{N}} U_n = \Delta$. Upon closer inspection of the proof of Theorem 4.72 it turns out that the assumptions can be relaxed and we obtain the following.

Corollary 4.73. *Let X be a ν-symmetric doubly Feller process with values in S_ϑ. Suppose $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ is a decreasing sequence of open entourages with $\bigcap_{n\in\mathbb{N}} U_n = \Delta$. Then X is uniquely determined by the family of Green operators*

$$\left\{ G_{\overline{U_n[x]}} : \mathcal{B}_b \to \mathcal{B}_b \;\middle|\; n \in \mathbb{N},\ x \in \{a, b\} \subset S \right\}. \tag{4.201}$$

4.4 Tightness

Ultimately we are interested in the convergence of a sequence of Feller processes which may live on different subsets of a common state space. We have already developed some conditions for the convergence of random paths in Chapter 3. In order to apply Theorem 3.48 we need a good criterion for the tightness of a sequence

$(X_n)_{n\in\mathbb{N}}$ of processes. In the case of metric state spaces and strong Markov processes one has Aldous' tightness criterion that was developed by David Aldous in his dissertation and can be found for example in [Ald78], [Bil99, Theorem 16.10] or [Kal21, Theorem 23.11], to name a few. Aldous' criterion can be formulated as follows.

Proposition 4.74 (Aldous' tightness criterion)**.** *Let $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ be a sequence of stochastic processes with càdlàg paths with values in a metric space (S,d). Suppose that $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ is compactly contained, i.e. for every $T,\varepsilon>0$ there exists a compact set $K\subset S$ such that*

$$\liminf_{n\to\infty}\mathbb{P}\left(\left\{ X_t^{(n)} \;\middle|\; t\leq T \right\}\subset K\right)\geq 1--\varepsilon. \tag{4.202}$$

If, in addition, for every family $(\tau_n)_{n\in\mathbb{N}}$ of bounded optional times (with respect to $\sigma(X^{(n)})$) and every sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n>0$ and $\lim_{n\to\infty}\delta_n=0$,

$$\lim_{n\to\infty}\mathbb{P}\left(d\left(X^{(n)}_{\tau_n},X^{(n)}_{\tau_n+\delta_n}\right)>\varepsilon\right)=0,\quad\forall\varepsilon>0, \tag{4.203}$$

then the family $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ is tight.

Siva Athreya, Wolfgang Löhr and Anita Winter showed in [ALW17, Corollary 4.3] that a family of Feller processes satisfies Aldous' tightness criterion when the probability that the processes reach a given distance from the starting point before time t goes to zero uniformly in the starting point as t tends to zero.

We show that a similar result holds for uniform state spaces. But instead of applying Aldous' criterion we show the statement directly as we have the luxury of working with Feller processes which possess the strong Markov property and we don't need the full power of Aldous' theorem. The proof is inspired by the proof of [EK86, Lemma 3.8.1].

Theorem 4.75 (Tightness for Feller processes on uniform state spaces)**.** *For each $n\in\mathbb{N}$ let $X^{(n)}$ be a Feller process with values in a subset S_n of a locally compact Polish uniform space $(S,\mathcal{U})$. Assume that for every open entourage $U\in\mathcal{U}$ it holds that*

$$\lim_{t\to0}\lim_{n\to\infty}\inf_{x\in S_n}\mathbb{P}_x((x,X^{(n)}_t)\in U)=1. \tag{4.204}$$

Then for every sequence of initial distributions $\mu_n\in\mathcal{M}_1(S_n)$ the family $\left\{ X^{(n)} \;\middle|\; n\in\mathbb{N}\right\}$ is tight in the one-point compactification $(S_\vartheta,\mathcal{U}_\vartheta)$.

Proof. Write $\Xi:=\left\{ X^{(n)} \;\middle|\; n\in\mathbb{N}\right\}$. We want to apply Theorem 3.43 to prove the claim. That means that we have to show that the family Ξ is compactly contained (Theorem 3.43 (i)) and that jumps above a fixed threshold do not accumulate (Theorem 3.43

(ii)). As processes in the compact space $(S_\vartheta, \mathcal{U}_\vartheta)$, the family clearly satisfies (i) of Theorem 3.43, i.e. the family is compactly contained. It remains to show that Ξ also satisfies (ii) of Theorem 3.43. We want to apply one of the equivalent conditions from Lemma 3.45.

To that end fix $U \in \mathcal{U}$ open. For each $n \in \mathbb{N}$ define the random time $\tau^{(n)}$ as

$$\tau^{(n)} := \inf\left\{ t > 0 \,\middle|\, X_t^{(n)} \notin U\left[X_0^{(n)}\right] \right\}. \tag{4.205}$$

Observe that $\tau^{(n)}$ is the first hitting time of the closed set $\complement U\left[X_0^{(n)}\right]$ and therefore a stopping time by Proposition 4.51.

From the assumption, in particular (4.204), it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\inf_{n\in\mathbb{N}} \mathbb{P}_{\mu_n}\left(\tau^{(n)} \geq \delta\right) \geq 1 - \varepsilon \tag{4.206}$$

for all sequences of initial distributions $\mu_n \in \mathcal{M}_1(S_n)$.

Now set $\tau_0^{(n)} = 0$ and inductively define

$$\tau_k^{(n)} := \inf\left\{ t > \tau_{k-1}^{(n)} \,\middle|\, \left(X_{\tau_{k-1}^{(n)}}^{(n)}, X_t^{(n)}\right) \notin U \right\} \tag{4.207}$$

for each $k \in \mathbb{N}$ if $\tau_{k-1}^{(n)} < \infty$ and $\tau_k^{(n)} = \infty$, otherwise.

By the strong Markov property, we have for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $\tau_k^{(n)} < \infty$,

$$\mathbb{P}_{\mu_n}\left(\tau_{k+1}^{(n)} - \tau_k^{(n)} \geq \delta\right) = \mathbb{P}_{\hat{\mu}_{n,k}}\left(\tau^{(n)} \geq \delta\right), \tag{4.208}$$

where $\hat{\mu}_{n,k}$ denotes the distribution of $X^{(n)}$ at time $\tau_k^{(n)}$ when started in the initial distribution μ_n, i.e.

$$\hat{\mu}_{n,k}(A) = \mathbb{P}_{\mu_n}\left(X_{\tau_k^{(n)}}^{(n)} \in A\right), \quad A \in \mathcal{B}(S_n). \tag{4.209}$$

As $\hat{\mu}_{n,k} \in \mathcal{M}_1(S_n)$ for each $(n,k) \in \mathbb{N}^2$ with $\tau_k^{(n)} < \infty$, we can apply (4.206) to deduce that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\inf\left\{ \mathbb{P}_{\mu_n}\left(\tau_{k+1}^{(n)} - \tau_k^{(n)} \geq \delta\right) \,\middle|\, n,k \in \mathbb{N} :\ \tau_k^{(n)} < \infty \right\} \geq 1 - \varepsilon \tag{4.210}$$

for all sequences $(\mu_n)_{n\in\mathbb{N}}$ of initial distributions with $\mu_n \in \mathcal{M}_1(S_n)$. For convenience, we write for $k, n \in \mathbb{N}$

$$\xi_k^{(n)} := \tau_{k+1}^{(n)} - \tau_k^{(n)}, \tag{4.211}$$

if $\tau_{k-1}^{(n)} < \infty$ and $\xi_k^{(n)} = \infty$, otherwise.

For a fixed sequence $(\mu_n)_{n\in\mathbb{N}}$ of initial distributions an application of the strong Markov property at times $\tau_k^{(n)}$ together with (4.210) guarantees the existence of a sequence $\left(\zeta_k^{(n)}\right)_{k,n\in\mathbb{N}}$ of independent random variables satisfying the following conditions,

$$\xi_k^{(n)} \geq \zeta_k^{(n)} \quad \mathbb{P}_{\mu_n}\text{-a.s.} \tag{4.212}$$

and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathbb{P}_{\mu_n}\left(\zeta_k^{(n)} < \delta\right) < \varepsilon, \tag{4.213}$$

for all $k, n \in \mathbb{N}$.

Taking $\zeta_k^{(n)} \wedge 1$, if necessary, we can ensure that $\mathrm{Var}(\zeta_k^{(n)}) \leq 1$ for all $n, k \in \mathbb{N}$ while maintaining (4.212). By construction we have

$$\tau_k^{(n)} \geq \sum_{j=1}^{k} \zeta_j^{(n)} \quad \mathbb{P}_{\mu_n}\text{-a.s.} \tag{4.214}$$

We can therefore apply Kolmogorov's law of large numbers (cf. [Fel68, Section X.7]) to deduce that for each $T > 0$ and $n \in \mathbb{N}$ the number of $k \in \mathbb{N}$ with $\tau_k^{(n)} \leq T$ is $\mathbb{P}_{\mu_n}$-almost surely finite.

Moreover, by (4.213), there exists a $K > 0$ such that

$$\mathbb{E}_{\mu_n}\left[\zeta_k^{(n)}\right] \geq K \tag{4.215}$$

for all $k, n \in \mathbb{N}$. Consequently, there exists for each $\varepsilon > 0$ and $T > 0$ a number $M_\varepsilon(T) \in \mathbb{N}$ independent of $n \in \mathbb{N}$ such that

$$\mathbb{P}_{\mu_n}\left(\tau_{M_\varepsilon(T)}^{(n)} < T\right) < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{4.216}$$

Now fix $T, \varepsilon > 0$ and choose $\delta > 0$ such that

$$\inf\left\{ \mathbb{P}_{\mu_n}\left(\tau_{k+1}^{(n)} - \tau_k^{(n)} \geq \delta\right) \,\middle|\, k, n \in \mathbb{N} : \ \tau_k^{(n)} < T \right\} \geq \left(1 - \frac{\varepsilon}{2}\right)^{M_{\varepsilon/2}(T)^{-1}}. \tag{4.217}$$

To save some ink, we write $M_{\varepsilon/2} = M_{\varepsilon/2}(T)$ and obtain

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \mathbb{P}_{\mu_n}\left(\inf\left\{\, \xi_k^{(n)} \,\middle|\, k \in \mathbb{N} : \ \tau_k^{(n)} < T \,\right\} < \delta\right) \\
&\quad \leq \sup_{n\in\mathbb{N}} \Bigg[\mathbb{P}_{\mu_n}\left(\inf\left\{\, \xi_k^{(n)} \,\middle|\, k \leq M_{\varepsilon/2} \,\right\} < \delta\right) \\
&\qquad\qquad + \mathbb{P}_{\mu_n}\left(\inf\left\{\, \xi_k^{(n)} \,\middle|\, k \in \mathbb{N} : \ \tau_k^{(n)} < T \,\right\} < \delta, \ T > M_{\varepsilon/2}\right)\Bigg] \\
&\quad \leq \sup_{n\in\mathbb{N}} \mathbb{P}_{\mu_n}\left(\inf\left\{\, \xi_k^{(n)} \,\middle|\, k \leq M_{\varepsilon/2} \,\right\} < \delta\right) + \frac{\varepsilon}{2} \\
&\quad = 1 - \inf_{n\in\mathbb{N}} \mathbb{P}_{\mu_n}\left(\bigcap_{k=1}^{M_{\varepsilon/2}} \left\{\xi_k^{(n)} \geq \delta\right\}\right) + \frac{\varepsilon}{2} \leq 1 - \inf_{n\in\mathbb{N}}\left(\inf_{k\leq M_{\varepsilon/2}0} \mathbb{P}_{\mu_n}\left(\xi_k^{(n)} \geq \delta\right)\right)^{M_{\varepsilon/2}} + \frac{\varepsilon}{2} \\
&\quad \leq 1 - \left(1 - \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{4.218}$$

We can conclude that for each $T > 0$,

$$\lim_{\delta\to 0} \inf_{n\in\mathbb{N}} \mathbb{P}_{\mu_n}\left(\inf\left\{\, \tau_{k+1}^{(n)} - \tau_k^{(n)} \,\middle|\, k \in \mathbb{N} : \ \tau_k^{(n)} < T \,\right\} \geq \delta\right) = 1. \tag{4.219}$$

Now recall the definition of the random times σ_k from (3.118). Analogously, set $\sigma_0^{(n)} = 0$ and inductively define for $k \in \mathbb{N}$,

$$\sigma_k^{(n)} := \sup\left\{\, t \leq \tau_k^{(n)} \,\middle|\, \left\{X_t^{(n)}, X_{t-}^{(n)}\right\} \not\subset U\left[X_{\tau_k^{(n)}}^{(n)}\right] \right\}, \tag{4.220}$$

if $\tau_k^{(n)} < \infty$ and $\sigma_k^{(n)} = \infty$, otherwise.

Applying again the strong Feller property together with the strong continuity of the semigroups, we can deduce that $\sigma_k^{(n)} = \tau_k^{(n)}$ almost surely under $\mathbb{P}_{\mu_n}$ for every initial distribution $\mu_n \in \mathcal{M}_1(S_n)$. We have thus shown (3.128) and therefore established tightness of the family Ξ. ■

4.5 Examples

We conclude this chapter with two examples of strongly symmetric (doubly) Feller processes. These examples can be considered as our base examples. While both are strongly symmetric Feller processes, they differ in a fundamental way. Furthermore, both examples can be extended in different directions. The first example are random walks on graphs. This is a discrete example in the sense that the state space is countable and carries the discrete topology. The second example is Brownian motion where the state space is $\mathbb{R}^d$. While our first example can be extended to non-discrete examples (speed-v motion on the $\mathbb{R}$ trees [AEW13] and more generally to *resistance forms* [Cro18]), those examples remain basically *low dimensional* in the sense that these processes always hit points with a positive probability. Brownian motion has this property only in the case of $d = 1$ and can be further extended to other (high dimensional) state spaces like Riemannian manifolds (cf. [Suz19a]).

4.5.1 Random walks on graphs

We first introduce *weighted graphs* and collect some basic facts about them.

Weighted graphs

Definition 4.76 (Graphs)**.** Let $V \neq \emptyset$ be at most countable and $E \subset \{ e \subset V \mid \#e = 2 \}$, a subset of the family of two-element subsets of V. The pair (V, E) is called a *(undirected) graph* and the set V is the set of *vertices* whereas E is the set of *edges* of the graph. ◇

A *directed graph* is a generalization of an undirected graph that is obtained by taking $E \subset V \times V$. We will only be considering *undirected* graphs in this thesis and therefore drop the attribute *undirected*.

We say that two vertices $x, y \in V$ are connected by an edge if $\{x, y\} \in E$. In that case, we write $x \sim y$. All vertices $y \in V$ with $x \sim y$ are called *neighbors* of x. A *path* (of length $n \in \mathbb{N}$) is a $n+1$-tuple $(x_0, x_1, \ldots, x_n) \subset V^{n+1}$ with the property that $x_{k-1} \sim x_k$ for all $k = 1, \ldots, n$. We say that a path $(x_0, \ldots, x_n)$ is *simple* if $x_j \neq x_k$ for all $j, k = 0, \ldots, n$ with $j \neq k$. For two vertices $x, y \in V$ we denote the set of simple paths of length n connecting x and y by

$$\Gamma^n_{xy} := \left\{ (x_0, x_1, \ldots, x_n) \in V^{n+1} \,\middle|\, x = x_0 \sim x_1 \sim \cdots \sim x_n = y,\ x_j \neq x_k \text{ if } j \neq k \right\}, \tag{4.221}$$

and we write

$$\Gamma_{xy} := \bigcup_{n \in \mathbb{N}} \Gamma^n_{xy} \tag{4.222}$$

for the set of simple paths from x to y. Given a simple path $\gamma_{xy} = (x_0, x_1, \ldots, x_n) \in \Gamma_{xy}$ from x to y, we write

$$l(\gamma_{xy}) = n \tag{4.223}$$

for its *length*. A graph (V, E) is said to be *connected* if $\Gamma_{xy} \neq \emptyset$ for all pairs of vertices $(x, y) \in V^2$. We will assume from now on that the graphs under consideration are connected if not explicitly stated otherwise.

Furthermore, we introduce the *degree* of a vertex $x \in V$, as the number of neighbors of x, i.e.

$$\deg(x) := \#\{\, y \in V \mid x \sim y \,\}. \tag{4.224}$$

The *graph distance* d between two vertices $x, y \in V$ is defined as the length of the shortest path connecting x and y:

$$d(x, y) := \inf \left\{ l(\gamma_{xy}) \,\middle|\, \gamma_{xy} \in \Gamma_{xy} \right\}. \tag{4.225}$$

It is straightforward to check that d is indeed a metric.

We introduce an important generalization of graphs by assigning *weights* to the edges. One common interpretation of these weights are *conductances* in an electrical network. Here the conductance is the reciprocal of the *resistance* which can be seen as proportional to the length of an edge.

Definition 4.77 (Weighted graphs)**.** A *weighted graph* is a triple (V, E, μ), where (V, E) is a graph and $\mu \colon V \times V \to [0, \infty)$ is a symmetric map with $\mu(x, y) = \mu(y, x) > 0$ if and only if $\{x, y\} \in E$. ◇

We usually write $\mu_{xy} := \mu(x, y)$ and with a slight abuse of notation we write for an edge $e = \{x, y\} \in E$,

$$\mu_e = \mu(e) = \mu(x, y) = \mu(y, x) = \mu_{xy} = \mu_{yx}. \tag{4.226}$$

For a weighted graph, we can introduce weighted versions of the degree and the graph distance by

$$\deg_\mu(x) = \mu_x := \sum_{y : x \sim y} \mu_{xy} \tag{4.227}$$

and

$$d_\mu(x, y) := \inf \left\{ l_\mu(\gamma_{xy}) \,\middle|\, \gamma_{xy} \in \Gamma_{xy} \right\}, \tag{4.228}$$

where

$$l_\mu(\gamma_{xy}) := \sum_{k=1}^{l(\gamma_{xy})} \mu(x_{k-1}, x_k)^{-1}. \tag{4.229}$$

It is again a standard calculation to show that d_μ defines a metric on V.

Observe that the edges E of a weighted graph is determined by the weights $\left\{ \mu_{xy} \,\middle|\, x, y \in V \right\}$ because $\{x, y\} \in E \Leftrightarrow \mu_{xy} > 0$. For that reason, we often write (V, μ) for the weighted graph (V, E, μ).

Definition 4.78 (Degree conditions)**.** Let $G = (V, \mu)$ be a weighted graph.

(i) We say that G is of *finite local degree* if

$$\mu_x < \infty, \quad \forall x \in V. \tag{4.230}$$

(ii) The graph G satisfies the *controlled weights condition* if there exists a $\delta > 0$ such that

$$\frac{\mu_{xy}}{\mu_x} > \delta, \quad \forall x \in V \text{ and } y \in V \text{ with } x \sim y. \tag{4.231}$$

◇

Effective resistance

Let (V, μ) be a weighted graph. We introduce a bilinear form $\mathcal{E}$ on the space of real-valued maps $V \to \mathbb{R}$ by

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x,y \in V} \mu_{xy} \left(f(x) - f(y)\right)\left(g(x) - g(y)\right) \tag{4.232}$$

for $f, g \in \mathcal{D}(\mathcal{E})$, where

$$\mathcal{D}(\mathcal{E}) := \{ f \colon V \to \mathbb{R} \mid \mathcal{E}(f, f) < \infty \}. \tag{4.233}$$

We refer to the quantity $\mathcal{E}(f, f)$ as the *energy* of f and to $\mathcal{E}$ as the *energy form* associated with (V, μ). We will examine such bilinear forms in more depth in Chapter 5 where we also explore the deep connection between Dirichlet forms and symmetric Feller processes. Using the energy functional $\mathcal{E}$ we can define another metric on V by

$$\mathcal{R}(x, y) := \inf \{ \mathcal{E}(f, f) \mid f \in \mathcal{D}(\mathcal{E}), f(x) = 0, f(y) = 1 \}^{-1}. \tag{4.234}$$

We introduce the following shorthand notation

$$\mathcal{F}_x^y := \{ f \in \mathcal{D}(\mathcal{E}) \mid f(x) = 0, f(y) = 1 \}. \tag{4.235}$$

First, we check that $\mathcal{R}$ indeed defines a metric.

Lemma 4.79. *Let (V, μ) be a weighted graph of finite local degree and $\mathcal{E}$ the associated energy functional. Then $\mathcal{R}$ as defined in (4.234) is a metric on V.*

Proof. By definition of $\mathcal{E}$, we have $\mathcal{R}(x, y) \geq 0$ and $\mathcal{R}(x, x) = 0$ for all $x, y \in V$. On the other hand, if $x \neq y$ we can set $f(y) = 1$ and $f(z) = 0$ for all $z \in V \setminus \{y\}$ and obtain

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{u,v \in V} \mu_{uv}(f(u) - f(v))^2 = \frac{1}{2} \sum_{u \in V \colon u \sim y} \mu_{uy} = \mu_y < \infty. \tag{4.236}$$

Hence, $\mathcal{R}(x,y) \geq \mathcal{E}(f,f)^{-1} = \mu_y^{-1} > 0$. The symmetry of $\mathcal{R}$ follows from the fact that

$$\mathcal{E}(1-f, 1-f) = \mathcal{E}(1,1) - 2\mathcal{E}(1,f) + \mathcal{E}(f,f) = \mathcal{E}(f,f) \tag{4.237}$$

together with the fact that $(1-f) \in \mathcal{F}_y^x$ for all $f \in \mathcal{F}_x^y$.

It remains to show that $\mathcal{R}$ satisfies the triangle inequality. To that end fix $x, z \in V$ and observe that for all $f \in \mathcal{D}(\mathcal{E})$ with $f(x) \neq f(z)$ we have

$$h := \frac{f - f(x)}{f(z) - f(x)} \in \mathcal{F}_x^z, \tag{4.238}$$

and we can write the energy of h as

$$\mathcal{E}(h,h) = \frac{\mathcal{E}(f,f)}{(f(z)-f(x))^2}. \tag{4.239}$$

Consequently, we can rewrite (4.234) as

$$\mathcal{R}(x,z) = \sup\left\{ \frac{(f(z)-f(x))^2}{\mathcal{E}(f,f)} \;\middle|\; f \in \mathcal{D}(\mathcal{E}),\ \mathcal{E}(f,f) > 0 \right\}. \tag{4.240}$$

Now let $y \in V \setminus \{x,z\}$ be arbitrary. Applying (4.240), we arrive at

$$\begin{aligned} \mathcal{R}(x,z) &\leq \sup\left\{ \frac{(f(z)-f(y))^2}{\mathcal{E}(f,f)} + \frac{(f(y)-f(x))^2}{\mathcal{E}(f,f)} \;\middle|\; f \in \mathcal{D}(\mathcal{E}),\ \mathcal{E}(f,f) > 0 \right\} \\ &\leq \sup\left\{ \frac{(f(z)-f(y))^2}{\mathcal{E}(f,f)} \;\middle|\; f \in \mathcal{D}(\mathcal{E}),\ \mathcal{E}(f,f) > 0 \right\} \\ &\qquad + \sup\left\{ \frac{(f(y)-f(x))^2}{\mathcal{E}(f,f)} \;\middle|\; f \in \mathcal{D}(\mathcal{E}),\ \mathcal{E}(f,f) > 0 \right\} \\ &= \mathcal{R}(y,z) + \mathcal{R}(x,y). \end{aligned} \tag{4.241}$$

■

We call the metric $\mathcal{R}$ the *effective resistance metric* or simply *resistance metric*. The name stems from the interpretation of (V,μ) as an electrical network and it can be shown that $\mathcal{R}$ satisfies the usual rules for parallel resistors and resistors in series.

A good survey of the resistance metric on (finite) graphs can be found in the article [Wei18] by Tobias Weihrauch. Another rich source is the book [AF02] by David Aldous and James Allen Fill.

Remark 4.80. Clearly, the graph metric induces the discrete topology on V since

$$d(x,y) < 1 \;\Leftrightarrow\; x = y. \tag{4.242}$$

Let (V, μ) be of finite local degree. Fix $x \in V$ and suppose $c > 0$ is such that $\mu_x < c$. Then,

$$\left\{ y \in V \,\middle|\, d_\mu(x, y) < 1/c \right\} = \{x\} \tag{4.243}$$

and hence d_μ induces the discrete topology, too. The same holds for the resistance metric as is easy to check. For each $x, y \in V$, the function $\mathbb{1}_{V \setminus \{x\}} \in \mathcal{F}_x^y$. Hence,

$$\mathcal{R}(x, y) \geq \mathcal{E}(\mathbb{1}_{V\setminus\{x\}}, \mathbb{1}_{V\setminus\{x\}})^{-1} = \mu_x^{-1}, \tag{4.244}$$

and therefore $\mathcal{R}$ introduces the discrete topology on V by the same argument as before.

Now denote by $\mathcal{U}^d, \mathcal{U}^\mu$ and $\mathcal{U}^\mathcal{R}$ the uniformities generated by d, d_μ and $\mathcal{R}$, respectively. Recall from Example 2.3 that the sets of the form

$$U_\varepsilon^r := \left\{ (x, y) \in V^2 \,\middle|\, r(x, y) < \varepsilon \right\}, \quad \varepsilon > 0 \tag{4.245}$$

form a base of the uniformity $\mathcal{U}^r$, where r is one of the metrics d, d_μ and $\mathcal{R}$. Now, $U_1^r = \Delta$ and therefore d induces the discrete uniformity. This is not necessarily the case for the other metrics d_μ and $\mathcal{R}$. ◇

To illustrate the last remark consider the following two examples.

Example 4.81 (Line graph)**.** Let $G = (V, \mu)$ be the graph with $V = \mathbb{N}$ and

$$\mu_{xy} = \min\{x, y\}, \tag{4.246}$$

if $|x - y| = 1$ and $\mu_{xy} = 0$ otherwise. Then, $\mu_n = 2n - 1$ hence (V, μ) is of finite local degree. For every $\varepsilon > 0$ we have that

$$\left(\lceil \varepsilon^{-1} \rceil + 1, \lceil \varepsilon^{-1} \rceil + 2 \right) \in U_\varepsilon^\mu. \tag{4.247}$$

Therefore $\mathcal{U}^d \neq \mathcal{U}^\mu$. In this example, the metrics μ and $\mathcal{R}$ coincide because (V, μ) is a *tree*. ■

Example 4.82 (Line with attachements)**.** Now, let $G = (V, \mu)$ be the graph depicted in Figure 4.1. That is, G is the graph with vertices

$$V = \mathbb{N} \cup \left\{ v_i^j \,\middle|\, 1 \leq i \leq j,\ j \geq 3 \right\} \cup \left\{ x_j \,\middle|\, j \geq 3 \right\} \tag{4.248}$$

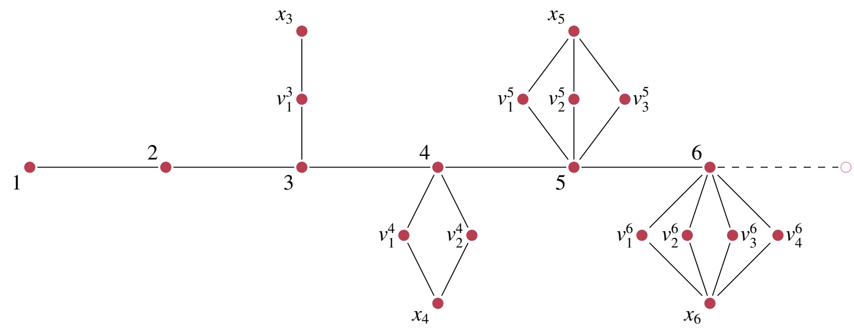

Fig. 4.1.: The graph $G = (V, \mu)$ from Example 4.82

and conductances

$$\begin{cases} \mu(n, n+1) = \mu(n+1, n) = 1, & n \in \mathbb{N}, \\ \mu(n, v_i^n) = \mu(v_i^n, n) = 1, & n \geq 3,\ 1 \leq i \leq n, \\ \mu(x_n, v_i^n) = \mu(v_i^n, x_n) = 1, & n \geq 3,\ 1 \leq i \leq n, \\ \mu(x, y) = 0, & \text{otherwise}. \end{cases} \tag{4.249}$$

First observe that $\mu_n = n$ for all $n \in \mathbb{N}$, $\mu_{v_i^j} = 2$ and $\mu_{x_n} = n - 2$. Therefore, G is of finite local degree. Furthermore, the metrics d and d_μ coincide and generate the discrete uniformity. We also have $\mathcal{R}(x, y) = d(x, y) = d_\mu(x, y) = |x - y|$ for all $x, y \in \mathbb{N}$. Whereas

$$\mathcal{R}(n, x_n) = \frac{2}{(n-2)}, \tag{4.250}$$

as can be easily checked using the series and parallel laws for resistors (cf. Appendix B.2). As a consequence, $\mathcal{R}$ does not generate the discrete uniformity since for each $\varepsilon > 0$ there exists a $n \in \mathbb{N}$ such that $(n, x_n) \in U_n^{\mathcal{R}}$. Recall that by Remark 4.80 all three metrics on V induce the discrete topology. So this is a further example of the case where different uniformities may induce the same topology (c.f. Example 2.10). ■

The speed-ν random walk

Let (V, μ) be a weighted graph and ν be a boundedly finite measure on V with full support. In other words, a map $\nu \colon V \to (0, \infty)$. Again, we use $\nu_x := \nu(x)$ as a shorthand. We refer to the triple (V, μ, ν) as a *weighted measure graph*.

Consider a continuous time Markov chain X with values in V, that is a Markov process with the countable and discrete state space V. Assume further that X jumps from $x \in V$ to a neighbor $y \sim x$ at rate

$$\eta_{xy} := \frac{\mu_{xy}}{2\nu_x}. \tag{4.251}$$

To put it differently, X is a random walk[9] on the graph (V, μ) that stays in a vertex $x \in V$ for an exponentially distributed random time $\xi := \inf\{ t > 0 \mid X_t \neq X_0 \}$ with expectation

$$\mathbb{E}_x[\xi] = \frac{\nu_x}{\mu_x} \tag{4.252}$$

and then jumps to one of the neighboring vertices $y \in \{ y \in V \mid y \sim x \}$ with probability

$$\mathbb{P}_x(X_\xi = y) = \frac{\mu_{xy}}{\mu_x}. \tag{4.253}$$

In the case where $\nu_x = c\mu_x$ for all $x \in V$ and some $c > 0$, we call X the *fixed speed random walk* and otherwise the *variable speed random walk* on V.

We will assume from now on that (V, μ) has finite local degree so that the holding times ξ are non-degenerate (i.e. $0 < \mathbb{E}_x[\xi] < \infty$ for all $x \in V$).

For a (continuous time) random walk X we introduce its (discrete time) *skeleton*[10] of X, usually denoted by $Z = (Z_n)_{n \geq 0}$, defined as

$$Z_n := X_{\tau_n}, \tag{4.254}$$

where $(\tau_n)_{n \geq 0}$ are defined inductively via $\tau_0 = 0$ and

$$\tau_{n+1} := \inf\left\{ t \geq \tau_n \,\middle|\, X_t \neq X_{\tau_n} \right\} \quad n \in \mathbb{N}. \tag{4.255}$$

Definition 4.83 (Speed-ν random walk)**.** Let (V, μ, ν) be a weighted measure graph with finite local degree. The Markov process X described above is called the *speed-ν random walk* (or *speed-ν motion*) on the graph (V, μ). We refer to the measure ν as the *speed measure* (of X). ♢

It is straightforward to check that the speed-ν random walk is both Feller and strongly Feller.

Conditions **(F1)** and **(F3)** follow trivially from the fact that every real-valued function $f : V \to \mathbb{R}$ on the discrete space V is continuous. On the other hand, we obtain **(F2)** from the fact that the holding time at x before the next jump is exponentially distributed with a finite parameter for all $x \in V$.

In order to show the symmetry of the speed-ν random walk we need to examine the semigroup further.

[9] We use the term *random walk* generally for a Markov process or a Markov chain on a discrete state space.

[10] It may be more common to call this object *the embedded discrete-time Markov chain*, but we prefer the term *skeleton* as it is much shorter. It is also not unprecedented (cf. [Szn11]).

Proposition 4.84. *Let (V, μ, ν) be a weighted measure graph and X the speed-ν random walk on (V, μ). Then for each $f \in \mathcal{B}_b(V)$, the generator of X is given by*

$$\mathcal{L}f(x) := \frac{1}{2\nu_x} \sum_{y \in V} \mu_{x,y} \left(f(y) - f(x) \right). \tag{4.256}$$

Thus, the semigroup $(P_t)_{t \geq 0}$ associated with X can be written as

$$P_t f(x) = \mathbb{E}_x \left[f(X_t) \right] = \sum_{n \geq 0} \frac{t^n}{n!} \mathcal{L}^n f(x) = e^{t\mathcal{L}} f(x), \quad t \geq 0, f \in \mathcal{B}_b(V). \tag{4.257}$$

Proof. Let $f \in \mathcal{B}_b(V)$ and fix $x \in V$ and consider the difference quotient

$$\frac{P_t f(x) - f(x)}{t} = t^{-1} \mathbb{E}_x \left[f(X_t) - f(X_0) \right]. \tag{4.258}$$

Write $J(t)$ for the number of jumps of X in the interval $[0, t]$. We can split the expectation on the right and obtain

$$\begin{aligned} \frac{P_t f(x) - f(x)}{t} &= t^{-1} \mathbb{E}_x \left[\left(f(X_t) - f(X_0) \right) \mathbb{1}_{\{J(t)=1\}} \right] \\ &\quad + t^{-1} \mathbb{E}_x \left[\left(f(X_t) - f(X_0) \right) \mathbb{1}_{\{J(t) \geq 2\}} \right]. \end{aligned} \tag{4.259}$$

For $z \in V$ denote by $\lambda_z := \mu_z / \nu_z$ the jump rate at z. Further, let

$$\lambda := \max \{ \lambda_z \mid z \sim x \}. \tag{4.260}$$

Then we can bound the probability that X has two or more jumps in $[0, t]$ by an Erlang$(2, \lambda)$-distribution, i.e.

$$\mathbb{P}_x(J(t) \geq 2) \leq 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}. \tag{4.261}$$

Observe that the right-hand side is in $o(t)$ as $t \to 0$. Denote by ξ the holding time of X before the first jump and recall that $\xi \sim \operatorname{Exp}(\lambda_x)$ under $\mathbb{P}_x$. Furthermore, let Z denote the discrete skeleton of X as defined in (4.254). Conditioning on the event $J(t) = 1$, we obtain

$$\begin{aligned} \mathbb{E}_x \left[\left(f(X_t) - f(X_0) \right) \mathbb{1}_{\{J(t)=1\}} \right] &= \mathbb{E}_x \left[f(Z_1) - f(Z_0) \right] \mathbb{P}_x(J(t) = 1) \\ &= \mathbb{E}_x \left[f(Z_1) - f(Z_0) \right] \left(\mathbb{P}_x(J(t) \geq 1) - \mathbb{P}_x(J(t) \geq 2) \right) \\ &= \mathbb{E}_x \left[f(Z_1) - f(Z_0) \right] \mathbb{P}_x(\xi < t) + o(t), \end{aligned} \tag{4.262}$$

as $t \to 0$. A similar argument applied to the second summand in (4.259) yields

$$\begin{aligned}\frac{P_t f(x) - f(x)}{t} &= t^{-1}\mathbb{E}_x\left[f(Z_1) - f(Z_0)\right]\mathbb{P}_x(\xi < t) + o(1) \\ &= \frac{1 - e^{-\lambda_x t}}{t}\sum_{y\in V}\frac{\mu_{xy}}{\mu_x}(f(y) - f(x)) + o(1) \\ &\xrightarrow[t\to 0]{} \lambda_x/\mu_x \sum_{y\in V}\mu_{xy}(f(y) - f(x)) = \mathcal{L}f(x).\end{aligned} \tag{4.263}$$

Thus, $\mathcal{L}$ is indeed the generator of the Feller process X and (4.257) follows with standard arguments. ∎

The operator $\mathcal{L}$ is sometimes referred to as the (discrete) *Laplacian* or the (discrete) *Laplace operator*.

The next result gives a hint at the intrinsic relationship between the resistance metric and the speed-ν random walk on (V, μ).

Lemma 4.85. *Let X be the speed-ν random walk on the weighted measure graph (V, μ, ν). Recall the energy form $\mathcal{E}$ from (4.232). Then, for all $f, g \in \mathcal{D}(\mathcal{E})$,*

$$\int_V -\mathcal{L}f g \,\mathrm{d}\nu = \mathcal{E}(f, g). \tag{4.264}$$

Proof. A straightforward calculation yields

$$\begin{aligned}\int_V -\mathcal{L}f g\,\mathrm{d}\nu &= \sum_{x\in V} -\mathcal{L}f(x)g(x)\nu_x = \sum_{x,y\in V}\mu_{xy}(f(x) - f(y))g(x) \\ &= \frac{1}{2}\sum_{x,y\in V}\mu_{xy}\left(f(x)g(x) - f(y)g(x)\right) + \frac{1}{2}\sum_{y,x\in V}\mu_{yx}\left(f(y)g(y) - f(x)g(y)\right) \\ &= \frac{1}{2}\sum_{x,y\in V}\mu_{xy}(f(y) - f(x))(g(y) - g(x)) = \mathcal{E}(f, g).\end{aligned} \tag{4.265}$$

∎

Proposition 4.86. *Let (V, μ, ν) be a weighted measure graph with locally finite degree. Then the speed-ν random walk on (V, μ) is both doubly Feller and strongly ν-symmetric.*

Proof. Let X be the speed-ν random walk on (V, μ). We have already shown that X is doubly Feller. By Lemma 4.85, the generator $\mathcal{L}$ is ν-symmetric, as

$$\int_V \mathcal{L}f g \,\mathrm{d}\nu = -\mathcal{E}(f, g) = -\mathcal{E}(g, f) = \int_V f \mathcal{L}g \,\mathrm{d}\nu. \tag{4.266}$$

This symmetry carries over to the semigroup $(P_t)_{t\geq 0}$ and the resolvent $(R_\alpha)_{\alpha>0}$ by Proposition 4.84.

Now define for $\alpha > 0$ and $x, y \in V$,

$$u_\alpha(x, y) := \nu_y^{-1} R_\alpha \mathbb{1}_y(x) = \nu_y^{-1} \int_0^\infty P_t \mathbb{1}_y(x) e^{\alpha t} \,\mathrm{d}t. \tag{4.267}$$

Again, checking the symmetry of u_α is a straight forward calculation

$$\begin{aligned} u_\alpha(x, y) &= \nu_y^{-1} R_\alpha \mathbb{1}_y(x) = (\nu_y \nu_x)^{-1} \int_V R_\alpha \mathbb{1}_y(z) \mathbb{1}_x(z) \,\mathrm{d}\nu \\ &= (\nu_y \nu_x)^{-1} \int_V \mathbb{1}_y(z) R_\alpha \mathbb{1}_x(z) \,\mathrm{d}\nu = \nu_x^{-1} R_\alpha \mathbb{1}_x(y) = u_\alpha(y, x). \end{aligned} \tag{4.268}$$

Finally, we have

$$\begin{aligned} \int_V u_\alpha(x, y) f(y)\, \nu(\mathrm{d}y) &= \int_V \int_0^\infty \nu_y^{-1} P_t \mathbb{1}_y(x) f(y) e^{-\alpha t} \,\mathrm{d}t\, \nu(\mathrm{d}y) \\ &= \int_0^\infty \int_V \nu_y^{-1} \mathbb{1}_y(x) P_t f(y) e^{-\alpha t}\, \nu(\mathrm{d}y) \,\mathrm{d}t \\ &= \int_0^\infty P_t f(x) e^{-\alpha t} \,\mathrm{d}t = R_\alpha f(x). \end{aligned} \tag{4.269}$$

Thus, the strong ν-symmetry of X is established (see the discussion at the beginning of Section 4.3). ∎

4.5.2 Brownian motion

Unsurprisingly, one of our base examples is *Brownian motion* which Kallenberg called "arguably the single most important object of modern probability"[11]. We assume that the reader is familiar with the basic properties of Brownian motion.

Let $d \in \mathbb{N}$ and consider the metric measure space $(\mathbb{R}^d, r, \lambda)$, where $r(x, y) = \|x - y\|$ is the Euclidean metric on $\mathbb{R}^d$ and λ denotes the Lebesgue measure.

[11] [Kal21, p.297]

Let $B = (B_t)_{t\geq 0}$ be the Brownian motion with values in $\mathbb{R}^d$. Recall (cf. [Sch21, Lemma 7.1]) that the semigroup $(P_t)_{t\geq 0}$ of B is given by

$$P_t f(x) := \int_{R^d} p_t(x,y) f(y)\, \lambda(\mathrm{d}y), \quad f \in \mathcal{B}_b(\mathbb{R}^d), x \in \mathbb{R}^d, t > 0, \tag{4.270}$$

where

$$p_t(x,y) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{\|y-x\|^2}{2t}}, \quad x, y \in \mathbb{R}^d, t > 0. \tag{4.271}$$

Proposition 4.87. *The semigroup $(P_t)_{t\geq 0}$ of the Brownian motion is both Feller and strongly Feller.*

Proof. We first show that P_t is strongly continuous, i.e. satisfies (**F2**). Let $f \in C_\infty(\mathbb{R}^d)$, then $\|f\|_\infty < \infty$ and f is uniformly continuous. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \forall x, y \in \mathbb{R}^d : \|x - y\| < \delta. \tag{4.272}$$

Then,

$$\begin{aligned}
\|P_t f - f\|_\infty &= \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_t(x,y) f(y)\, \lambda(\mathrm{d}y) - f(x) \right| \\
&\leq \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(x,y) |f(y) - f(x)|\, \lambda(\mathrm{d}y) \\
&= \sup_{x\in\mathbb{R}^d} \Bigg(\int_{\|x-y\|<\delta} p_t(x,y) |f(y) - f(x)|\, \lambda(\mathrm{d}y) \\
&\qquad + \int_{\|x-y\|\geq\delta} p_t(x,y) |f(y) - f(x)|\, \lambda(\mathrm{d}y) \Bigg) \\
&\leq \varepsilon + \sup_{x\in\mathbb{R}^d} \frac{2\|f\|_\infty}{(2\pi t)^{d/2}} \int_{\|x-y\|\geq\delta} e^{-\|x-y\|^2/(2t)}\, \lambda(\mathrm{d}y) \\
&= \varepsilon + 2\|f\|_\infty \mathbb{P}_0\left(\|B_t\| \geq \delta\right) \underset{t\to 0}{\longrightarrow} \varepsilon.
\end{aligned} \tag{4.273}$$

The claim then follows since $\varepsilon > 0$ was arbitrary.

The Feller property (**F1**) follows from the translation invariance of Brownian motion. Let again $f \in C_\infty(\mathbb{R}^d)$. Since $\|f\|_\infty < \infty$ we can apply dominated convergence to obtain

$$\lim_{x\to y} P_t f(x) = \lim_{x\to y} \mathbb{E}_x\left[f(B_t)\right] = \lim_{x\to y} \mathbb{E}_0\left[f(B_t + x)\right] = \mathbb{E}_0\left[f(B_t + y)\right] = P_t f(y). \tag{4.274}$$

Analogously, we obtain $\lim_{x\to\infty} P_t f(x) = 0$ and hence $P_t f \in C_\infty(\mathbb{R}^d)$.

In order to show the strong Feller property, (F3), fix $f \in \mathcal{B}_b(\mathbb{R}^d)$. Clearly, $P_t f$ is again bounded. We want to show that $P_t f$ is continuous at $x \in \mathbb{R}^d$. To that end let $R := 2\|x\|$ and write $U_R[0]$ for the ball around the origin with radius R. It suffices to show that $\lim_{n\to\infty} P_t f(x_n) = P_t f(x)$ for all sequences $(x_n)_{n\in\mathbb{N}} \subset U_R[0]$. Recall that

$$P_t f(x_n) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-\frac{\|x_n - y\|}{2t}} \lambda(\mathrm{d}y). \tag{4.275}$$

Thus, the claim follows by dominated convergence once we can show that the integrand is bounded by an integrable function. Suppose $\|y\| \geq 2R$, then

$$\|x_n - y\|^2 \geq (\|y\| - \|x_n\|)^2 \geq \frac{1}{4}\|y\|^2. \tag{4.276}$$

Consequently,

$$f(y) e^{-\frac{\|x_n - y\|}{2t}} \leq \|f\|_\infty \left(\mathbb{1}_{U_{2R}[0]} + \mathbb{1}_{\complement U_{2R}[0]} e^{-\frac{\|y\|}{8t}} \right), \tag{4.277}$$

which is the integrable bound we were seeking. Hence, $P_t f \in C_b(\mathbb{R}^d)$, as $x \in \mathbb{R}^d$ was arbitrary. ■

By definition, p_t is a symmetric function for all $t > 0$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^d} P_t f g \,\mathrm{d}\lambda &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(x,y) f(y)\, \lambda(\mathrm{d}y) g(x)\, \lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(y,x) g(x)\, \lambda(\mathrm{d}x) g(y)\, \lambda(\mathrm{d}y) = \int_{\mathbb{R}^d} f P_t g \,\mathrm{d}\lambda. \end{aligned} \tag{4.278}$$

In other words, Brownian motion is λ-symmetric. From (4.270) it is immediate that $p_t(x, \cdot)$ is the density of the probability measure $\mathbb{P}_x(B_t \in A) = P_t \mathbb{1}_A(x)$ with respect to λ. We have therefore shown the following.

Proposition 4.88. *For every $d \in \mathbb{N}$, d-dimensional Brownian motion is strongly λ-symmetric.* ■

In fact, it can be shown (cf. [Sch21, Example 7.14]) that the resolvent kernel of Brownian motion is given by

$$u_\alpha^d(x,y) = \frac{1}{\pi^{d/2}} \left(\frac{\sqrt{2\alpha}}{2|x-y|} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\sqrt{2\alpha}|x-y| \right), \tag{4.279}$$

where

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right) t^{-(\nu+1)}\,\mathrm{d}t, \quad z > 0 \tag{4.280}$$

denotes a modified Bessel function of the second[12] kind (see [Olv+10, §10.25] and [Olv+10, eq. 10.32.10]).

[12] In [Sch21], K_ν is identified as a Bessel function of the third kind, which seems to be a mistake.

Dirichlet Forms and symmetric Feller Processes

5

> *He stared at his feet. "I'm still very ignorant", he said, "but at least I'm ignorant about really important things."*
>
> — **Terry Pratchett**
> Diggers: The Second Book of the Nomes

5.1 Dirichlet Forms

Dirichlet forms are a rich analytical tool for the study of symmetric Feller processes. Informally speaking there is a one-to-one correspondence between a class of bilinear forms on $L^2(S, \nu)$ and a class of ν-symmetric Feller processes on the uniform measure space $(S, \mathcal{U}, \nu)$. In Section 4.5.1 we have already encountered an example of a Dirichlet form when we introduced the energy functional in (4.232). We give an introduction to Dirichlet forms and shine a light into the "black box" that is the theory of Dirichlet forms. We show in some detail how a symmetric Feller process gives rise to a Dirichlet form. Subsequently, we introduce some important potential theoretic notions and show how they relate to the Dirichlet form on the one hand and to the process on the other hand. Finally, we use Dirichlet forms to extend the examples of the previous section.

In order to keep this work reasonably bounded we refer the reader to the literature for deeper results. An extensive treatment of the theory of Dirichlet forms can be found in the monographs [FOT11] by Masatoshi Fukushima, Yoichi Oshima and Masayoshi Takeda and [CF11] by Zhen-Qing Chen and Masatoshi Fukushima. We focus exclusively on *symmetric* Dirichlet forms and include the symmetry in the definition of Dirichlet forms. A thorough treatment of the theory of not necessarily symmetric Dirichlet forms can be found in the book [MR92] by Zhi-Ming Ma and Michael Röckner. An extension of the hereinafter developed concepts to the not necessarily symmetric case is interesting but beyond the scope of this thesis and must remain a subject for further research. We begin with a brief discussion of symmetric forms on real Hilbert spaces.

Definition 5.1. Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{D} \subset \mathcal{H}$ a linear subspace. A *quadratic form* is a map $q\colon \mathcal{D} \to \mathbb{R}_{\geq 0}$ satisfying $q(\alpha f) = \alpha^2 q(f)$ for all $\alpha \in \mathbb{R}$. A map $\mathcal{E}\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ is a *symmetric bilinear form*, if the following hold

(i) $\mathcal{E}$ is symmetric, i.e.

$$\mathcal{E}(f, g) = \mathcal{E}(g, f) \tag{5.1}$$

for all $f, g \in \mathcal{D}$.

(ii) $\mathcal{E}$ is linear in each component, i.e.

$$\mathcal{E}(\alpha(f + g), h) = \alpha(\mathcal{E}(f, h) + \mathcal{E}(g, h)) \tag{5.2}$$

for all $f, g, h \in \mathcal{D}$ and $\alpha \in \mathbb{R}$.

$\mathcal{D}$ is called the *domain* of $\mathcal{E}$. To emphasize this, we sometimes write $\mathcal{D}(\mathcal{E})$. ◇

A quadratic form q uniquely determines a symmetric bilinear form $\mathcal{E}$ via *polarization*: $\mathcal{E}(f, g) := \frac{1}{2}(q(f + g) - q(f) - q(g))$ and *vice versa* every symmetric bilinear form uniquely determines a quadratic form via $q(f) := \mathcal{E}(f, f)$. To save some ink, we sometimes write $\mathcal{E}(f) := \mathcal{E}(f, f)$ for the quadratic form determined by the bilinear form $\mathcal{E}$. Furthermore, we drop the adjective bilinear from the notation for convenience.

A quadratic form q is called *positive (semi-) definite*[1] if $q(f)\,(\geq) > 0$ for all $f \in \mathcal{D}(\mathcal{E}) \setminus \{0\}$. A symmetric form $\mathcal{E}$ is called *positive (semi-) definite*, if the associated quadratic form is positive (semi-) definite.

We begin with some important observations.

Lemma 5.2 (Cauchy Schwarz)**.** *Let $\mathcal{E}$ be a positive semidefinite symmetric form with domain $\mathcal{D} \subset \mathcal{H}$. Let $f, g \in \mathcal{D}$ such that at least one of quantities $\mathcal{E}(f, f), \mathcal{E}(g, g)$ is non-zero. Then,*

$$\mathcal{E}(f, g)^2 \leq \mathcal{E}(f, f)\mathcal{E}(g, g). \tag{5.3}$$

Proof. Let $f, g \in \mathcal{D}$. Without loss of generality suppose that $\mathcal{E}(f, f) > 0$. For every $\lambda \in \mathbb{R}$ we have

$$Then, 0 \leq \mathcal{E}(g - \lambda f, g - \lambda f) = \mathcal{E}(g, g) - 2\lambda\mathcal{E}(f, g) + \lambda^2\mathcal{E}(f, f). \tag{5.4}$$

Now choose $\lambda = \mathcal{E}(f, g)/\mathcal{E}(f, f)$. Then,

$$0 \leq \mathcal{E}(g, g) - \mathcal{E}(f, g)^2/\mathcal{E}(f, f). \tag{5.5}$$

Rearranging (5.5) yields the desired inequality. ■

[1] Some authors (e.g. [FOT11]) use the term non-negative definite instead of positive semi-definite.

This is the classical Cauchy-Schwarz inequality with the only exception that we have to be mindful of the case where $\mathcal{E}(f, f) + \mathcal{E}(g, g) = 0$ while $f, g \neq 0$.

Lemma 5.3 (Triangle inequality)**.** *Let $\mathcal{E}$ be a positive semidefinite symmetric form with domain $\mathcal{D} \subset \mathcal{H}$. Suppose $f, g \in \mathcal{D}$ with $\mathcal{E}(f, f) + \mathcal{E}(g, g) > 0$. Then,*

$$\mathcal{E}(f + g, f + g)^{1/2} \leq \mathcal{E}(f, f)^{1/2} + \mathcal{E}(g, g)^{1/2}. \tag{5.6}$$

Proof. Let $f, g \in \mathcal{D}(\mathcal{E})$. Applying the Cauchy-Schwarz inequality from Lemma 5.2 we obtain

$$\begin{aligned} \mathcal{E}(f + g, f + g) &= \mathcal{E}(f, f) + 2\mathcal{E}(f, g) + \mathcal{E}(g, g) \\ &\leq \mathcal{E}(f, f) + 2\mathcal{E}(f, f)^{1/2}\mathcal{E}(g, g)^{1/2} + \mathcal{E}(g, g) \\ &\leq \left(\mathcal{E}(f, f)^{1/2} + \mathcal{E}(g, g)^{1/2}\right)^2, \end{aligned} \tag{5.7}$$

completing the proof. ■

Lemma 5.4. *Let $\mathcal{E}$ be a positive semidefinite symmetric form and denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathcal{H}$. Then, for each $\alpha > 0$, the form*

$$\mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g) + \alpha \langle f, g \rangle \tag{5.8}$$

is a positive definite symmetric bilinear form with domain $\mathcal{D}(\mathcal{E})$.

Proof. Symmetry and bilinearity are immediate consequences of the fact that $\mathcal{E}_\alpha$ is the sum of two symmetric bilinear forms. Similarly, positive definiteness follows from the fact that $\mathcal{E}_\alpha$ is the sum of a positive definite and a positive semidefinite form. ■

Recall that a *pre-Hilbert space* is a vector space equipped with a scalar product that is not necessarily complete.

Lemma 5.5. *The form $\mathcal{E}_\alpha$ is a scalar product on $\mathcal{D}(\mathcal{E})$ and $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ is a pre-Hilbert space for each $\alpha > 0$. Moreover, $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ determine equivalent metrics on $\mathcal{D}(\mathcal{E})$ for all $\alpha, \beta > 0$.*

Proof. By Lemma 5.4 it is clear that $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ is a real pre-Hilbert space. We need to show that

$$r_\alpha(f, g) := \sqrt{\mathcal{E}_\alpha(f - g, f - g)} \tag{5.9}$$

are equivalent metrics on $\mathcal{D}(\mathcal{E})$ for all $\alpha > 0$. Assume $0 < \alpha < \beta$, then

$$\frac{\alpha}{\beta}\mathcal{E}_\beta \leq \mathcal{E}_\alpha(f - g) \leq \frac{\beta}{\alpha}\mathcal{E}_\beta(f - g) \tag{5.10}$$

and r_α and r_β are even bi-Lipschitz equivalent. ■

We call a symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *closed* if $\mathcal{D}(\mathcal{E})$ is complete with respect to $\mathcal{E}_1$ (or, equivalently, with respect to $\mathcal{E}_\alpha$ for all $\alpha > 0$), i.e. every $\mathcal{E}_1$-Cauchy sequence is $\mathcal{E}_1$-convergent to an element of $\mathcal{D}(\mathcal{E})$. We say that the symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *closable* if for every $\mathcal{E}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{E})$ with $\lim_{n\to\infty} \langle f_n, f_n\rangle = 0$ it holds that $\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0$. As implied by the terminology, a closable symmetric form can be extended to a closed symmetric form in the following sense. We say that a symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an extension of the symmetric form $(\mathcal{E}', \mathcal{D}(\mathcal{E}'))$ if $\mathcal{D}(\mathcal{E}') \subset \mathcal{D}(\mathcal{E})$ and $\mathcal{E}|_{\mathcal{D}(\mathcal{E}')\times\mathcal{D}(\mathcal{E}')} = \mathcal{E}'$.

Proposition 5.6. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a closable symmetric form. Suppose $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{E})$ is an $\mathcal{E}_1$-Cauchy sequence. Then there exists a $f \in \mathcal{H}$ such that $\lim_{n\to\infty} f_n = f$ (in $\mathcal{H}$) and $\lim_{n\to\infty} \mathcal{E}(f_n) < \infty$ exists. Furthermore, let $\mathcal{D}(\overline{\mathcal{E}})$ denote the set of all $f \in \mathcal{H}$ such that there exists an $\mathcal{E}_1$-Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{E})$ with $\lim_{n\to\infty} f_n = f$ (in $\mathcal{H}$) and set*

$$\overline{\mathcal{E}}(f) := \lim_{n\to\infty} \mathcal{E}(f_n) \tag{5.11}$$

Then the value of $\overline{\mathcal{E}}(f)$ does not depend on the choice of $(f_n)_{n\in\mathbb{N}}$ and $(\overline{\mathcal{E}}, \mathcal{D}(\overline{\mathcal{E}}))$ is the smallest closed extension of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in the sense that every closed extension of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ also extends $(\overline{\mathcal{E}}, \mathcal{D}(\overline{\mathcal{E}}))$.

Proof. See [Kat95, Theorem VI.1.17]. ■

Let again denote $(S, \mathcal{U}, \nu)$ denote a uniform measure space. We will turn our focus to symmetric forms on the particular Hilbert space $L^2(S, \nu)$ equipped with the scalar product

$$\langle f, g\rangle := \int_S fg \,\mathrm{d}\nu. \tag{5.12}$$

We introduce the Markov property for symmetric forms.

Definition 5.7. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric form on $L^2(S, \nu)$. We say that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *Markovian* (has the Markov property) if for each $\varepsilon > 0$ there exists a function $\varphi_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ with the following properties

(i) $\varphi_\varepsilon(t) = t$ for all $t \in [0, 1]$

(ii) $-\varepsilon \le \varphi_\varepsilon(t) \le 1 + \varepsilon$ for all $t \in \mathbb{R}$

(iii) $0 \le \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \le t - s$ for all $s < t$

such that for all $f \in \mathcal{D}(\mathcal{E})$ we have $\varphi_\varepsilon \circ f \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\varphi_\varepsilon \circ f) \le \mathcal{E}(f)$. ◇

Next, we introduce the central object of this chapter.

Definition 5.8 (Dirichlet form)**.** Let $\mathcal{E}$ be a positive semi-definite symmetric bilinear form on $L^2(S, \nu)$ with domain $\mathcal{D}(\mathcal{E}) \subset L^2(S, \nu)$. Then, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a *Dirichlet form* if

(D1) $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $L^2(S, \nu)$,

(D2) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closed and

(D3) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is Markovian. ♢

Recall from (4.5) that we use $C_0 = C_0(S, \mathbb{R})$ to denote the compactly supported continuous real-valued functions on S. With a slight abuse of notation we write $C_0 \cap \mathcal{D}(\mathcal{E})$ for the elements of C_0 that are representatives of an element in $\mathcal{D}(\mathcal{E})$ as well as for those elements of $\mathcal{D}(\mathcal{E})$ that have a representative in C_0, depending on the context. For a $f \in \mathcal{D}(\mathcal{E})$ we denote by $\operatorname{supp}(f) = \operatorname{supp}(f \cdot \nu)$ the support of the measure $f \cdot \mathrm{d}\nu$. For the domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}$ we simply write $\mathcal{D}$ when no confusion can arise.

Definition 5.9. A Dirichlet form $(\mathcal{E}, \mathcal{D})$ is called *regular*, if

(D4) $C_0(S) \cap \mathcal{D}$ is both dense in $\mathcal{D}$ with respect to $\mathcal{E}_1$ and dense in $C_0(S)$ with respect to the uniform norm.

Furthermore, the Dirichlet form is called *local* if

(D5) for all $f, g \in \mathcal{D}$ such that $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are disjoint compact sets it holds that $\mathcal{E}(f, g) = 0$. ♢

We conclude this section with two examples of Dirichlet forms

Example 5.10 (Random walks on graphs)**.** Let $G = (V, \mu)$ be a finite weighted graph as defined in Definition 4.77 and $\nu \colon V \to \mathbb{R}_+$ a measure on V with $\nu_x = \nu(x) > 0$ for all $x \in V$. Clearly, $L^2(V, \nu)$ is just the space of all real functions $f \colon V \to \mathbb{R}$. Recall from Section 4.5.1 the definition of the energy form

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x,y \in V \colon x \sim y} \mu_{xy}(f(y) - f(x))(g(y) - g(x)), \tag{5.13}$$

with $\mathcal{D} = \{f \colon V \to \mathbb{R}\} = L^2(V, \nu)$. By definition, $\mathcal{E}$ is a symmetric bilinear form. We show that $(\mathcal{E}, \mathcal{D})$ is, indeed, a Dirichlet form. Because the weights μ_{xy} are all non-negative, the form $\mathcal{E}$ is positive semi definite. Properties (D1) and (D2) are satisfied, because $\mathcal{D} = L^2(V, \nu)$. In order to verify (D3), observe that it is enough to show that for $f \in L^2(V, \nu)$ it holds that $g := (f \wedge 1) \vee 0 \in L^2(V, \nu)$ and $\mathcal{E}(g, g) \le \mathcal{E}(f, f)$. The first part is obvious and since

$$\begin{aligned} (((f \wedge 1) \vee 0)(y) - ((f \wedge 1) \vee 0)(x))^2 &\le ((f \vee 0)(y) - (f \wedge 1)(x))^2 \\ &= ((f \wedge 1)(x) - (f \vee 0)(y))^2 \\ &\le (f(x) - f(y))^2 \end{aligned} \tag{5.14}$$

it follows that $\mathcal{E}(g, g) \le \mathcal{E}(f, f)$ and thus we have shown that $(\mathcal{E}, \mathcal{D})$ has the Markov property (D3) and hence is a Dirichlet form.

Furthermore, because G is a finite graph, we have $C_0(V) = L^2(V, \nu) = \mathcal{D}$ and thus, $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form.

On the other hand, observe that $\mathcal{E}(\mathbb{1}_{\{x\}}, \mathbb{1}_{\{y\}}) = \mu_{xy}$ whenever $x \neq y$ and thus $(\mathcal{E}, \mathcal{D})$ is not local. ∎

Example 5.11 (Brownian motion on $\mathbb{R}^d$)**.** Let $d \ge 1$ and set $(S, \nu) = (\mathbb{R}, \mathrm{d}x)$ the d-dimensional Euclidean space equipped with the Lebesgue measure. Denote by

$$H^1(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^d),\ 1 \le i \le d \right\} \tag{5.15}$$

the real Sobolev space of order 1. Here the derivative $\frac{\partial}{\partial x_i}$ are taken in the weak sense. Then a symmetric, positive semidefinite bilinear form is given by

$$\mathcal{E}(f, g) = \sum_{i=1}^{d} \frac{1}{2} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \,\mathrm{d}x. \tag{5.16}$$

We show that $(\mathcal{E}, H^1(\mathbb{R}^d))$ is a Dirichlet form.

For the sake of readability, we drop the space $\mathbb{R}^d$ from the notation of the various function spaces. To show (D1) observe that H^1 is a Banach space and that C_0^∞, the set of compactly supported, infinitely often continuously differentiable functions is a subset of H^1. Furthermore, C_0^∞ is a dense subset of L^2, which proves that H^1 is a dense linear subset of L^2. To check property (D2) consider a $\mathcal{E}_1$-Cauchy sequence $(f_n)_{n\ge1}$. Then (f_n) and $\left(\frac{\partial f_n}{\partial x_i}\right)$ are L^2-Cauchy sequences for all $1 \le i \le d$ and since L^2 is complete, the closedness of $(\mathcal{E}, H^1)$ follows. To show the Markov property, consider the following function (compare [FOT11, Exercise 1.2.1]). For $\varepsilon > 0$ let $\psi_\varepsilon(t) := (-\varepsilon \vee t) \wedge (1 + \varepsilon)$ and denote by $j(x) := \gamma^{-1} e^{-1/(1-x^2)}$ for $|x| < 1$ and $j(x) := 0$ for $|x| \ge 1$ a mollifier, where $\gamma = \int_{\mathbb{R}} j(x)\,\mathrm{d}x$. For $0 < \delta < \varepsilon$ set $j_\delta = \delta^{-1} j(x/\delta)$ and define

$$\varphi_\varepsilon(t) := (j_\delta * \psi_\varepsilon)(t) = \int_{\mathbb{R}} j_\delta(t - s)\psi_\varepsilon(s)\,\mathrm{d}s. \tag{5.17}$$

Then, φ_ε satisfies the properties (i)-(iii) of Definition 5.7 and additionally $\varphi_\varepsilon \in C_0^\infty$ and $|\varphi_\varepsilon'(t)| \le 1$ for all $t \in \mathbb{R}$ and $\varepsilon > 0$. Thus

$$\mathcal{E}(\varphi_\varepsilon(f), \varphi_\varepsilon(f)) = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \varphi_\varepsilon'(f)^2 \,\mathrm{d}x \le \mathcal{E}(f, f), \tag{5.18}$$

for all $f \in H^1$ and $\varepsilon > 0$, which proves (D3).

Next, we show that $(\mathcal{E}, H^1)$ is a regular Dirichlet form. Because C_0^∞ can be considered a subset of both H^1 and C_0 and because C_0^∞ is a dense subset of C_0, it is clear that $C_0 \cap H^1$ is uniformly dense in C_0. It remains to show that C_0^∞ is also $\mathcal{E}_1$-dense in H^1. To this end consider $f \in C^\infty \cap H^1$ and let $\delta \in C_0^\infty$ with $0 \le f \le 1$ and $\delta_{B(1,0)} \equiv 1$ be a smooth version of the indicator function on the unit ball at the origin. For $R > 0$ set $f_R(x) := f(x)\delta(x/R)$, then $f_R \in C_0^\infty$ for all $R > 0$ and by dominated convergence, $f_R \to f$ in L^2 as $R \to \infty$. Furthermore, we have

$$\frac{\partial f_R}{\partial x_i} = \frac{\partial f}{\partial x_i}\delta(x/R) + \frac{1}{R} f(x) \frac{\partial \delta}{\partial x_i}(x/R) \tag{5.19}$$

and with the same reasoning, $\frac{\partial f}{\partial x_i}\delta(x/R)$ converges in L^2 to $\frac{\partial f}{\partial x_i}$ whereas the second summand goes to 0 for all $1 \le i \le d$ as $R \to \infty$. We have shown that C_0^∞ is dense in $C^\infty \cap H^1$. Next, we show that $C^\infty \cap H^1$ is dense in H^1. Let $(\varphi_n)_{n\ge1} \subset C_0^\infty$ be a sequence of compactly supported smooth approximations of the identity and let $f \in H^1$. Then the convolutions $f_n := f * \varphi_n$ are in $C^\infty \cap H^1$ for each $n \in \mathbb{N}$. Because $f \in L^2$, it holds that f_n converges in L^2 to f as $n \to \infty$. Furthermore, $\frac{\partial f_n}{\partial x_i} = \frac{\partial f}{\partial x_i} * \varphi_n$ is in L^2 and hence $\frac{\partial f_n}{\partial x_i} \to \frac{\partial f}{\partial x_i}$ as $n \to \infty$ which proves the claim.

Finally, we show that the Dirichlet form $(\mathcal{E}, H^1)$ is even local. Let $f, g \in H^1$ be compactly supported with disjoint supports. Then $fg = 0$ almost everywhere. Moreover,

$$\operatorname{supp}\left(\frac{\partial f}{\partial x_i}\right) \subset \operatorname{supp}(f) \quad \text{and} \quad \operatorname{supp}\left(\frac{\partial g}{\partial x_i}\right) \subset \operatorname{supp}(g), \tag{5.20}$$

for all $1 \le i \le d$ which implies $\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} = 0$ almost everywhere which yields (D5). ■

5.2 Feller processes and Dirichlet forms

We have already seen in the previous chapter that there is a fundamental connection between semigroups of operators on $\mathcal{B}(S)$ and Markov processes. A similar connection exists between Dirichlet forms and semigroups of operators. With the subtle but important difference that the latter correspondence holds for operators on the L^2-space. Therefore the Dirichlet form theory is a *weak* theory whereas the theory presented in the last chapter is a *strong* theory.

5.2.1 Operators on Hilbert spaces and closed forms

Following the lines of [FOT11] we first describe how strongly continuous contraction semigroups, strongly continuous resolvents, non-positive definite self-adjoint operators and closed forms on Hilbert spaces are related to each other. Thereby establishing a one to one relation between these objects.

We then introduce the Markovian property of operators on $L^2(S, \nu)$ and show that the correspondence from the previous section can be extended to Markovian semigroups and resolvents and Dirichlet forms.

It then is just a small step to show that a symmetric Feller process induces a Markovian semigroup and therefore a Dirichlet form.

We begin with some basic functional analytic definitions to fix some notation. Henceforth, let $\mathcal{H}$ denote a non-empty, real or complex Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$.

Definition 5.12 (Linear operators)**.** A linear map $T\colon \mathcal{D}(T) \to \mathcal{H}$ is called a *linear operator* on $\mathcal{H}$ with *domain* $\mathcal{D}(T)$, if $\mathcal{D}(T) \subset \mathcal{H}$. We say that a linear operator T is *densely defined* if $\mathcal{D}(T)$ is a dense subset of $\mathcal{H}$.

For a linear operator T on $\mathcal{H}$ we introduce the *graph* $\mathcal{G}(T)$ of T as the subspace

$$\mathcal{G}(T) := \{\, (f, Tf) \mid f \in \mathcal{D}(T) \,\} \subset \mathcal{H}^2. \tag{5.21}$$

We call an operator T on $\mathcal{H}$ *closed* if $\mathcal{G}(T)$ is a closed subspace of $\mathcal{H}^2$. Given two linear operators T, V on $\mathcal{H}$ we write $T \subset V$ as a shorthand for $\mathcal{G}(T) \subset \mathcal{G}(V)$.

Finally, we define the *operator norm* of a linear operator T on $\mathcal{H}$ as

$$\|T\|_{\mathrm{op}} := \sup\{\, \|Tf\| \mid f \in \mathcal{D}(T),\ \|f\| \le 1 \,\}, \tag{5.22}$$

where $\|\cdot\|$ denotes the norm on $\mathcal{H}$ induced by the scalar product by $\|f\| := \sqrt{\langle f, f\rangle}$, $f \in \mathcal{H}$. ◊

It is easy to check that the space of bounded linear operators on a real or complex Hilbert space $\mathcal{H}$ with domain $\mathcal{H}$ form an algebra, denoted by $\mathfrak{B}(\mathcal{H})$, where

$$(T + V)f := Tf + Vf, \quad (TV)f := T(Vf), \quad (\alpha T)f := \alpha(Tf), \tag{5.23}$$

for all $T, V \in \mathfrak{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$ $(\in \mathbb{R})$ and $f \in \mathcal{H}$. On the other hand, $\mathfrak{B}(\mathcal{H})$ equipped with the norm $\|\cdot\|_{\mathrm{op}}$ becomes a Banach space or, more specifically, a *Banach algebra*. More details can be found in [Rud91, Chapters 10 & 12].

To a densely defined linear operator T on $\mathcal{H}$ we associate the *adjoint* operator T^* via

$$\langle Tf, g\rangle = \langle f, T^*g\rangle, \quad f \in \mathcal{D}(T). \tag{5.24}$$

The domain $\mathcal{D}(T^*)$ of T^* consists of all $g \in \mathcal{H}$ such that the mapping

$$f \mapsto (Tf, g) \in \mathcal{H}^2 \tag{5.25}$$

is continuous on $\mathcal{D}(T)$. Since T is densely defined, the adjoint T^* is unique and linear.

Definition 5.13. A linear operator T on $\mathcal{H}$ is called *symmetric* if

$$\langle Tf, g\rangle = \langle f, Tg\rangle, \tag{5.26}$$

for all $f, g \in \mathcal{D}(T)$.

A densely defined linear operator T on $\mathcal{H}$ satisfying

$$\mathcal{G}(T) = \mathcal{G}(T^*) \tag{5.27}$$

is called *self adjoint*. ◇

It is worth noting that the densely defined symmetric operators are those for which $T \subset T^*$. Hence, self-adjoint operators are symmetric and the two concepts coincide on $\mathfrak{B}(\mathcal{H})$.

Definition 5.14. A linear operator T on $\mathcal{H}$ is called *non-negative definite* or simply *non-negative* if

$$\langle Tf, f\rangle \geq 0, \quad \forall f \in \mathcal{D}(T). \tag{5.28}$$

Analogously, T is *non-positive (definite)* if $-T$ is non-negative. ◇

Definition 5.15 (Strongly continuous semigroup)**.** Let $(T_t)_{t\geq 0}$ be a semigroup of symmetric linear operators on $\mathcal{H}$ with domain $\mathcal{D}(T_t) = \mathcal{H}$ for each $t \geq 0$ and $T_0 = \mathrm{id}$. We say that $(T_t)_{t\geq 0}$ is

(i) *contractive* if $\|T_t f\|_2 \leq \|f\|_2$ for all $f \in \mathcal{H}$ or, equivalently, $\|T_t\|_{op} \leq 1$ for all $t \geq 0$.

(ii) *strongly continuous* if $\lim_{t\to 0}\|T_t f - f\| = 0$ for all $f \in \mathcal{H}$. ◇

In the same vein, we define resolvents.

Definition 5.16 (Resolvent)**.** A family $(G_\alpha)_{\alpha>0}$ of linear symmetric operators on $\mathcal{H}$ with domain $\mathcal{D}(G_\alpha) = \mathcal{H}$ for each $\alpha > 0$ is called a *resolvent* on $\mathcal{H}$ if

(i) $(G_\alpha)_{\alpha>0}$ satisfies the resolvent equation (R1)

$$G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0, \quad \forall \alpha, \beta > 0, \tag{R1}$$

(ii) αG_α is contractive for each $\alpha > 0$, i.e.

$$\|G_\alpha\|_{\mathrm{op}} \leq \alpha^{-1}. \tag{5.29}$$

If, in addition

(iii) $\lim_{\alpha\to\infty}\|\alpha G_\alpha f - f\| = 0$ for all $f \in L^2(S, \nu)$,

we say that $(G_\alpha)_{\alpha>0}$ is a strongly continuous resolvent. ◇

Lemma 5.17. *Let $(G_\alpha)_{\alpha>0}$ be a strongly continuous resolvent on $\mathcal{H}$. For each $\alpha > 0$, G_α is invertible.*

Proof. We need to show that

$$\ker(G_\alpha) = \{ f \in \mathcal{H} \mid G_\alpha f = 0 \} \tag{5.30}$$

is trivial. Suppose $f \in \mathcal{H}$ is such that $G_\alpha f = 0$. Using the resolvent equation (R1) we obtain $G_\beta f = 0$ for all $\beta > 0$. By strong continuity this implies

$$0 = \lim_{\alpha\to\infty} \|\alpha G_\alpha f - f\| = \|f\| \tag{5.31}$$

and therefore $f = 0$, completing the proof. ■

For a strongly continuous resolvent $(G_\alpha)_{\alpha>0}$ on $\mathcal{H}$ we introduce its *generator* as follows

$$\begin{cases} \Delta f & = \alpha f - G_\alpha^{-1} f \\ \mathcal{D}(\Delta) & = G_\alpha(H). \end{cases} \tag{5.32}$$

It is not clear *a priori* that this definition of Δ does not depend on our choice of $\alpha > 0$. To see this, let $\alpha, \beta > 0$. Then,

$$G_\alpha G_\beta \Big(\big(\alpha f - G_\alpha^{-1} f\big) - \big(\beta f - G_\beta^{-1} f\big) \Big) = (\alpha - \beta)G_\alpha G_\beta f + (G_\alpha - G_\beta) f = 0. \tag{5.33}$$

Because the kernels of G_α and G_β are trivial, we can conclude that the definition of the generator in (5.32) is independent of the choice of $\alpha > 0$.

We have the following important property of the generator which we will state without proof.

Proposition 5.18 ([FOT11, Lemma 1.3.1 (i)]). *The generator* $(\Delta, \mathcal{D}(\Delta))$ *of a strongly continuous resolvent* $(G_\alpha)_{\alpha>0}$ *on* $\mathcal{H}$ *is a non-positive definite self adjoint operator.*

We seek to explore the connection between closed symmetric forms, strongly continuous contraction semigroups and strongly continuous resolvents. As a first observation, we can easily get from a strongly continuous contraction semigroup to a strongly continuous resolvent.

Lemma 5.19. *Let* $(T_t)_{t\geq 0}$ *be a strongly continuous contraction semigroup on* $\mathcal{H}$*. For* $\alpha > 0$ *and* $f \in \mathcal{H}$ *write*

$$G_\alpha f := \int_0^\infty e^{-\alpha t} T_t f \,\mathrm{d}t, \tag{5.34}$$

where the integral is a Bochner integral (see Appendix C.1). Then the family $(G_\alpha)_{\alpha>0}$ *is a strongly continuous resolvent on* $\mathcal{H}$*.*

Proof. Let $(G_\alpha)_{\alpha>0}$ be defined as in (5.34) and fix $\alpha > 0$. First observe that $e^{-\alpha t} T_t f$ is indeed Bochner integrable as a function $[0,\infty) \to \mathcal{H}$. Then an application of Lemma C.5 shows that

$$\begin{aligned} \langle G_\alpha f, g \rangle &= \int_0^t e^{-\alpha t} \langle T_t f, g \rangle \\ &= \int_0^t e^{-\alpha t} \langle f, T_t g \rangle = \langle f, G_\alpha g \rangle, \end{aligned} \tag{5.35}$$

where we have used the symmetry of T_t. Now, linearity of T_t and linearity of the Bochner integral imply that G_α is in fact a symmetric linear operator with domain $\mathcal{D}(G_\alpha) = \mathcal{D}(T_t) = \mathcal{H}$.

For $\alpha, \beta > 0$ a straight forward calculation yields the resolvent equation (R1),

$$\begin{aligned} \left(G_\alpha - G_\beta\right) f &= \int_0^\infty e^{-\alpha t} T_t f \,\mathrm{d}t - \int_0^\infty e^{-\beta t} T_t f \,\mathrm{d}t \\ &= \int_0^\infty e^{-\beta t} \left(e^{-(\alpha-\beta)t} - 1\right) T_t f \,\mathrm{d}t \\ &= -(\alpha-\beta) \int_0^\infty \int_0^t e^{-\beta(t-s)-\alpha s} T_{(t-s)+s} f \,\mathrm{d}s \,\mathrm{d}t \\ &= -(\alpha-\beta) \int_0^\infty \int_0^\infty e^{-\alpha s} e^{-\beta t} T_s T_t f \,\mathrm{d}t \,\mathrm{d}s = -(\alpha-\beta) G_\alpha G_\beta f. \end{aligned} \tag{5.36}$$

Finally we obtain (5.29) from another application of Lemma C.5. Let $g \in \mathcal{H}$ with $\|g\| \le 1$, then

$$\begin{aligned} \|G_\alpha g\|^2 &= \left\langle \int_0^\infty e^{-\alpha t} T_t \,\mathrm{d}t, \int_0^\infty e^{-\alpha t} T_t \,\mathrm{d}t \right\rangle \\ &= \int_0^\infty \int_0^\infty e^{-\alpha t} e^{-\alpha s} \langle T_t g, T_s g\rangle \,\mathrm{d}t \,\mathrm{d}s \le \alpha^{-2}. \end{aligned} \tag{5.37}$$

■

The converse of the last result is the following.

Lemma 5.20. *Let $(G_\alpha)_{\alpha>0}$ be a strongly continuous resolvent on $\mathcal{H}$. For each $t \ge 0$ and $f \in \mathcal{H}$ set*

$$T_t f = \lim_{\alpha\to 0} e^{-\alpha t} \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!} (\alpha G_\alpha)^n f. \tag{5.38}$$

Then $(T_t)_{t\ge0}$ is a strongly continuous contraction semigroup and the resolvent induced by $(T_t)_{t\ge0}$ via (5.34) coincides with $(G_\alpha)_{\alpha>0}$.

Proof. Fix $t \ge 0$. We begin by showing that the limit in (5.38) exists. By strong continuity, we have that for each $f \in \mathcal{H}$ the map $\alpha \mapsto \alpha G_\alpha f$ is continuous. Therefore, for each $t \ge 0$ and $f \in \mathcal{H}$ the map

$$\alpha \mapsto e^{-\alpha t} \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!} (\alpha G_\alpha)^n f \tag{5.39}$$

is continuous. Furthermore, because the strong limit $\lim_{\alpha\to0} \alpha G_\alpha f$ exists the limit in (5.38) exists in the strong sense, too.

Applying the contractivity of αG_α we find that

$$\|T_t f\| \le \lim_{\alpha\to 0} e^{-\alpha t} \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!} \|(\alpha G_\alpha)^n f\| \le \|f\| \tag{5.40}$$

for all $f \in \mathcal{H}$. Next, we show the semigroup property of $(T_t)_{t\ge0}$. A straightforward calculation yields for all $s, t \geq 0$ and $f \in \mathcal{H}$,

$$\begin{aligned} T_{t+s}f &= \lim_{\alpha\to 0} e^{-\alpha(t+s)} \sum_{n=0}^{\infty} \frac{(\alpha(s+t))^n}{n!} (\alpha G_\alpha)^n f \\ &= \lim_{\alpha\to 0} e^{-\alpha(t+s)} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} s^j t^{n-j} \frac{\alpha^n}{n!} (\alpha G_\alpha)^n f \\ &= \lim_{\alpha\to 0} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(\alpha s)^j}{j!} \frac{(\alpha t)^{n-j}}{(n-j)!} (\alpha G_\alpha)^j (\alpha G_\alpha)^{n-j} f \\ &= \lim_{\alpha\to 0} \sum_{m,n=0}^{\infty} \frac{(\alpha s)^m}{m!} \frac{(\alpha t)^n}{n!} (\alpha G_\alpha)^m (\alpha G_\alpha)^n f = T_s T_t f \end{aligned} \tag{5.41}$$

Finally, the fact that the resolvent of $(T_t)_{t\geq 0}$ coincides with $(G_\alpha)_{\alpha>0}$ follows from the spectral theorem [Rud91, Theorem 12.23] (cf. [BGL14, p. 127]). ∎

Note that a Hilbert space always comes with a weak and a strong notion of convergence. We say that a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ converges weakly to a limit $f \in \mathcal{H}$, if for all $g \in \mathcal{H}$,

$$\lim_{n\to\infty} \langle f_n, g \rangle = \langle f, g \rangle . \tag{5.42}$$

In contrast, we say that $(f_n)_{n\in\mathbb{N}}$ converges to f in the strong limit if

$$\lim_{n\to\infty} \| f_n - f \| = 0. \tag{5.43}$$

Similar to our analysis of the Feller semigroup in the previous chapter we can introduce the *generator* of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $\mathcal{H}$ as

$$\begin{cases} \mathcal{D}(\Delta) &= \left\{ f \in \mathcal{H} \,\middle|\, \lim_{t\to 0} \frac{T_t f - f}{t} \text{ exists in the strong sense} \right\} \\ \Delta f &= \lim_{t\to 0} \frac{T_t f - f}{t}, \quad f \in \mathcal{D}(\Delta). \end{cases} \tag{5.44}$$

Indeed, the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ and the generator of the strongly continuous resolvent $(G_\alpha)_{\alpha>0}$ induced by $(T_t)_{t\geq 0}$ via (5.34) coincide [FOT11, Lemma 1.3.1 (ii)].

The next result shows how to obtain a strongly continuous resolvent and a strongly continuous contraction semigroup from a non-positive self-adjoint operator on $\mathcal{H}$. Again, we refer the reader to [FOT11] for a proof.

Proposition 5.21 ([FOT11, Lemma 1.3.2]). *Let Δ be a non-positive self-adjoint operator on $\mathcal{H}$.*

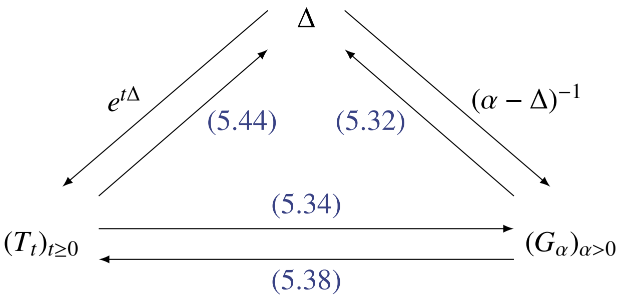

Fig. 5.1.: The relation between $(T_t)_{t\geq 0}$, $(G_\alpha)_{\alpha>0}$ and the generator Δ (cf. [FOT11, Diagram 1])

(i) $\{ T_t = \exp(t\Delta) \mid t \geq 0 \}$ *and* $\left\{ G_\alpha = (\alpha - \Delta)^{-1} \,\middle|\, \alpha > 0 \right\}$ *are a strongly continuous contraction semigroup and a strongly continuous resolvent on* $\mathcal{H}$ *respectively.*

(ii) *The generator of* $(T_t)_{t\geq 0}$ *as in (i) coincides with* Δ*. Furthermore, the strongly continuous contraction semigroup possessing* Δ *as its generator is unique and the same holds for the resolvent* $(G_\alpha)_{\alpha>0}$*.*

We have so far established a one-to-one correspondence between non-positive self-adjoint operators, strongly continuous contraction semigroups and strongly continuous resolvents. The next step is to show that there is a further one-to-one correspondence with closed symmetric forms. For a proof of this important fact, we refer the reader to the book [FOT11].

Proposition 5.22 ([FOT11, Theorem 1.3.1])**.** *There is a one-to-one correspondence between the family of closed symmetric forms* $\mathcal{E}$ *on* $\mathcal{H}$ *and the family of non-positive self adjoint operators* Δ *on* $\mathcal{H}$*. This correspondence is given by*

$$\begin{cases} \mathcal{D}(\mathcal{E}) &= \mathcal{D}\left(\sqrt{-\Delta}\right) \\ \mathcal{E}(f,g) &= \left\langle \sqrt{-\Delta} f, \sqrt{-\Delta} g \right\rangle, \quad f, g \in \mathcal{D}(\mathcal{E}). \end{cases} \tag{5.45}$$

Remark 5.23. The proof of Proposition 5.22 is fairly technical and will be omitted here as it can be found in [FOT11]. We still make some remarks about the proof.

(i) Another application of the spectral theorem shows that for the resolvent $(G_\alpha)_{\alpha>0}$ generated by Δ we have for all $\alpha > 0$ that $G_\alpha(\mathcal{H}) \subset \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}_\alpha (G_\alpha f, g) = \langle f, g \rangle, \quad f \in \mathcal{H}, \ g \in \mathcal{E}(\mathcal{D}). \tag{5.46}$$

(ii) The correspondence (5.45) can be restated as

$$\begin{cases} \mathcal{D}(\Delta) \subset \mathcal{D}(\mathcal{E}) \\ \mathcal{E}(f,g) = \langle -\Delta f, g\rangle, \quad f \in \mathcal{D}(\Delta),\ g \in \mathcal{D}(\mathcal{E}). \end{cases} \tag{5.47}$$

◇

For completeness sake, note that the symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ associated with Δ can be approximated using the semigroup $(T_t)_{t\geq 0}$ and the resolvent $(G_\alpha)_{\alpha>0}$ associated with Δ. To that end define for $f, g \in \mathcal{H}$,

$$\mathcal{E}^{(t)}(f,g) := t^{-1}\langle f - T_t f, g\rangle, \quad t > 0 \tag{5.48}$$

$$\mathcal{E}^{(\alpha)}(f,g) := \alpha\langle f - \alpha G_\alpha f, g\rangle, \quad \alpha > 0. \tag{5.49}$$

Then,

$$\begin{cases} \mathcal{D}(\mathcal{E}) &= \left\{ f \in \mathcal{H} \,\middle|\, \lim_{t\to 0} \mathcal{E}^{(t)}(f,f) < \infty \right\} \\ \mathcal{E}(f,g) &= \lim_{t\to 0} \mathcal{E}^{(t)}(f,f), \quad f,g \in \mathcal{D}(\mathcal{E}) \end{cases} \tag{5.50}$$

and

$$\begin{cases} \mathcal{D}(\mathcal{E}) &= \left\{ f \in \mathcal{H} \,\middle|\, \lim_{\alpha\to \infty} \mathcal{E}^{(\alpha)}(f,f) < \infty \right\} \\ \mathcal{E}(f,g) &= \lim_{\alpha\to \infty} \mathcal{E}^{(\alpha)}(f,g), \quad f,g \in \mathcal{D}(\mathcal{E}). \end{cases} \tag{5.51}$$

Furthermore, $\mathcal{E}^{(t)}$ and $\mathcal{E}^{(\alpha)}$ are increasing as $t \to 0$ and $\alpha \to \infty$, respectively. This handy approximation result is proven in [FOT11, Lemma 1.3.4].

5.2.2 Markovian operators and Dirichlet forms

From now on we consider the particular Hilbert space $L^2(S,\nu)$, where (S,ν) denotes a uniform measure space. As usual, we write

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = \left(\int_S f^2 \,\mathrm{d}\nu\right)^{1/2} \tag{5.52}$$

for the L^2-norm on $L^2(S,\nu)$.

Recall the shorthand $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$ for real numbers $a, b \in \mathbb{R}$. For real-valued functions $f, g\colon S \to \mathbb{R}$ we have set $f \vee g$ and $f \wedge g$ pointwise. Furthermore, we write

$$f^+ := f \vee 0 \quad \text{and} \quad f^- := -(f \wedge 0) \tag{5.53}$$

for the positive and the negative part of f, respectively. Clearly, these notations can be extended to the elements of $L^2(S,\nu)$ by applying them to a Borel-measurable representative.

The next result provides us with equivalent conditions to the Markov property of a closed symmetric form.

Lemma 5.24. *A closed symmetric form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *on* $L^2(S, \nu)$ *has the Markov property if and only if for all* $f \in \mathcal{D}(\mathcal{E})$ *it holds that* $g := 0 \vee (f \wedge 1) \in \mathcal{D}(\mathcal{E})$ *and*

$$\mathcal{E}(g, g) \leq \mathcal{E}(f, f). \tag{5.54}$$

Proof. Fix $f \in \mathcal{D}(\mathcal{E})$ and let $g = f^+ \wedge 1$, as above. The first implication is trivial, because $\varphi(t) := 0 \vee (t \wedge 1)$ satisfies (i) to (iii) of Definition 5.7 for all $\varepsilon > 0$ and $g = \varphi \circ f$. The converse implication follows readily from the observation that φ can be approximated by functions φ_ε satisfying (i) to (iii) of Definition 5.7. Letting $\varepsilon \to 0$ we obtain $g = \lim_{\varepsilon \to 0} \varphi_\varepsilon \circ f \in \mathcal{D}(\mathcal{E})$ by closedness of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Finally, (5.54) follows from the closedness of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ together with the Markov property of $\mathcal{E}$. ■

We call a real valued function $\gamma \colon \mathbb{R} \to \mathbb{R}$ a *normal contraction* if $\gamma(0) = 0$ and for all $s, t \in \mathbb{R}$,

$$|\gamma(s) - \gamma(t)| \leq |s - t|. \tag{5.55}$$

Lemma 5.25. *A closed symmetric form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *on* $L^2(S, \nu)$ *has the Markov property if and only if for all* $f \in \mathcal{D}(\mathcal{E})$ *and all normal contractions* $\gamma \colon \mathbb{R} \to \mathbb{R}$ *it holds that* $\gamma \circ f \in \mathcal{D}(\mathcal{E})$ *and*

$$\mathcal{E}(\gamma \circ f, \gamma \circ f) \leq \mathcal{E}(f, f). \tag{5.56}$$

Proof. See [CF11, Theorem 1.1.3] ■

The previous lemma has a very useful consequence.

Lemma 5.26. *Let* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *be a Dirichlet form. For every* $f \in \mathcal{D}(\mathcal{E})$ *it holds that*

$$\mathcal{E}(f^+, f^-) \leq 0. \tag{5.57}$$

Proof. For each $\varepsilon \in (0, 1)$ let $\gamma_\varepsilon(t) := t^+ - \varepsilon t^-$ and $\gamma(t) := t^+$. Then it is easy to check that $\gamma, \gamma_\varepsilon$ are normal contractions and $\gamma \circ \gamma_\varepsilon = \gamma$ for each $0 < \varepsilon < 1$. Hence, $\gamma \circ f, \gamma_\varepsilon \circ f \in \mathcal{D}(\mathcal{E})$ for any $f \in \mathcal{D}(\mathcal{E})$ and $0 < \varepsilon < 1$. Moreover,

$$\begin{aligned}\mathcal{E}(f^+, f^+) &= \mathcal{E}(\gamma \circ (\gamma_\varepsilon \circ f), \gamma \circ (\gamma_\varepsilon \circ f)) \\ &\leq \mathcal{E}(\gamma_\varepsilon \circ f, \gamma_\varepsilon \circ f) = \mathcal{E}(f^+ - \varepsilon f^-, f^+ - \varepsilon f^-).\end{aligned} \tag{5.58}$$

Hence,

$$\mathcal{E}(f^+, f^-) \leq \frac{\varepsilon}{2} \mathcal{E}(f^-, f^-) \tag{5.59}$$

and we conclude the proof by letting $\varepsilon \to 0$. ■

Definition 5.27 (Markovian and Dirichlet operators). (i) A linear operator V on $L^2(S, \nu)$ is called *positivity preserving* if

$$Vf \geq 0 \quad \nu\text{-a.e.} \tag{5.60}$$

for all $f \in L^2(S, \nu)$ with $f \geq 0$ ν-almost everywhere. We say that V is *Markovian* if V is bounded and

$$0 \leq Vf \leq 1 \quad \nu\text{-a.e.} \tag{5.61}$$

for all $f \in L^2(S, \nu)$ with $0 \leq f \leq 1$ ν-almost everywhere. Furthermore, a semigroup $(V_t)_{t\geq 0}$ of operators is said to be *Markovian* if V_t is Markovian for every $t \geq 0$. A strongly continuous resolvent $(G_\alpha)_{\alpha>0}$ is Markovian if for each $\alpha > 0$, αG_α is Markovian.

(ii) A closed and densely defined operator V on $L^2(S, \nu)$ is called a *Dirichlet operator* if

$$\langle Vf, (f-1)^+ \rangle \leq 0 \tag{5.62}$$

for all $f \in \mathcal{D}(V)$. ◊

Proposition 5.28. *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup, $(G_\alpha)_{\alpha>0}$ a strongly continuous resolvent, Δ a non-positive definite densely defined self adjoint operator and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a closed symmetric form on $L^2(S, \nu)$. Suppose that they are related to each other as described in the last section. Then the following are equivalent.*

(i) $(T_t)_{t\geq 0}$ is Markovian,

(ii) $(G_\alpha)_{\alpha>0}$ is Markovian,

(iii) Δ is a Dirichlet operator.

(iv) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form.

Proof. The equivalence of (i) to (iii) is [MR92, Proposition I.4.3] and the equivalence of (ii) and (iv) is due to [MR92, Theorem I.4.4]. ■

5.2.3 Symmetric Feller processes and Dirichlet forms

Dirichlet forms are a rich analytic tool for the analysis of symmetric Feller processes. To that end, we need to show that we can associate a Dirichlet form to a symmetric Feller process. Indeed, we can show that every Feller semigroup $(P_t)_{t\geq 0}$ of a ν-symmetric Feller process with values in $(S, \mathcal{U}, \nu)$ can be extended to a Markovian semigroup $(T_t)_{t\geq 0}$ on $L^2(S, \nu)$.

The following Lemma is essentially [CF11, Lemma 1.1.14. (ii)]. We want to provide a proof anyway because the result is central.

Proposition 5.29. *Let X be a ν-symmetric Feller process with values in a uniform measure space $(S, \mathcal{U}, \nu)$ with semigroup $(P_t)_{t\geq 0}$. Then there exists a unique extension of $(P_t)_{t\geq 0}$ to a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^2(S,\nu)$. Moreover, $(T_t)_{t\geq 0}$ is Markovian.*

Proof. First we show that for any given $t \geq 0$ the operator P_t on $B_b(S)$ can be uniquely extended to a linear contractive and symmetric operator on $L^2(S,\nu)$. Observe that each $f \in L^\infty(S,\nu)$ has a representative that is in $B_b(S)$ and that for two such representatives $f, g \in B_b(S)$ with $\|f-g\|_\infty = 0$ we have

$$\int |P_t f - P_t g| \,\mathrm{d}\nu \leq \int P_t |f-g| \,\mathrm{d}\nu \leq \int |f-g| \,\mathrm{d}\nu = 0 \tag{5.63}$$

and hence $\|P_t f - P_t g\|_\infty = 0$. Thus we can regard P_t as an operator on $L^\infty(S,\nu)$. By the contraction property, Proposition 4.17 (v), P_t can also be regarded as a bounded operator on $L^2(S,\nu) \cap L^\infty(S,\nu)$. It is easy to see that $L^2(S,\nu) \cap L^\infty(S,\nu)$ is dense in $L^2(S,\nu)$. Now let $f \in L^2(S,\nu)$ and $(f_n)_{n\in\mathbb{N}} \subset L^2(S,\nu) \cap L^\infty(S,\nu)$ such that $\lim_{n\to\infty}\|f-f_n\|_2 = 0$. By the contraction property of P_t we have

$$\lim_{n\to\infty} \|P_t f_n\|_2 \leq \lim_{n\to\infty} \|f_n\|_2 < \infty \tag{5.64}$$

and we can define $T_t f$ to be the $L^2(S,\nu)$ limit of $P_t f_n$.

To show uniqueness assume that $(g_n)_{n\in\mathbb{N}} \subset L^2(S,\nu) \cap L^\infty(S,\nu)$ is another sequence with $\lim_{n\to\infty}\|f-g_n\|_2 = 0$. Then, by linearity of P_t and the contraction property we have

$$\|P_t f_n - P_t g_n\|_2 = \|P_t(f_n - g_n)\|_2 \leq \|f_n - g_n\|_2 \to 0 \qquad \text{as } n\to\infty. \tag{5.65}$$

By assumption, P_t is positivity preserving and ν-symmetric and these properties as well as the contraction property carry over to T_t by approximation arguments. The semigroup property of $(T_t)_{t\geq 0}$ follows immediately from the semigroup property of $(P_t)_{t\geq 0}$. It remains to show that $(T_t)_{t\geq 0}$ is strongly continuous. We have that $C_\infty(S) \cap L^2(S,\nu)$ is dense in $L^2(S,\nu)$ (cf. [Rud87, Theorem 3.14]). For $\varepsilon > 0$ and $f \in L^2(S,\nu)$ let $g \in C_\infty(S) \cap L^2(S,\nu)$ such that $\|f-g\|_2 \leq \varepsilon$. Then, by the triangle inequality and the contraction property of T_t, we have

$$\|T_t f - g\|_2 \leq \|P_t g - g\|_2 + \|T_t f - T_t g\|_2 + \|f-g\|_2 \leq \|P_t g - g\|_2 + 2\varepsilon. \tag{5.66}$$

and

$$\|P_t g - g\|_2^2 \leq 2\|P_t g\|_2^2 - 2\langle P_t g, g\rangle. \tag{5.67}$$

Since P was assumed to be Feller, we have by property **(F2)** that $\lim_{t\to 0} P_t g(x) = g(x)$ for all $x \in S$ and by dominated convergence it follows that the right hand side of (5.67) goes to 0 as $t \to 0$, which concludes the proof because ε was arbitrary. ■

In the same manner, one can show that for each $\alpha > 0$ there exists a unique extension of the α-resolvent operator to a L^2-pendant.

Lemma 5.30. *Let $(R_\alpha)_{\alpha>0}$ be the resolvent of a ν-symmetric Feller process X. Then there exists for each $\alpha > 0$ a unique extension of of R_α to a ν-symmetric operator G_α on $L^2(S,\nu)$ such that αG_α is contractive and strongly continuous as $\alpha \to \infty$. Furthermore, the family $(G_\alpha)_{\alpha>0}$ satisfies the resolvent equation (R1).*

Proof. We could copy the proof of Proposition 5.29, instead we use the fact that $(P_t)_{t\geq 0}$ can be uniquely extended to a symmetric and strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^2(S,\nu)$. For $\alpha > 0$ define

$$G_\alpha f := \int_0^\infty e^{-\alpha t} T_t f \,\mathrm{d}t, \tag{5.68}$$

where the integral is defined in the Bochner sense (see Appendix C.1). By contractivity of $(T_t)_{t\geq 0}$ it follows that G_α is well defined and that αG_α is itself contractive and it agrees with R_α on $L^2(S,\nu) \cap B_b(S)$ by construction. To show that G_α is well-defined, we need to show that $G_\alpha f \in L^2(S,\nu)$ for $f \in L^2(S,\nu)$. Applying first Jensen's inequality then Fubini's Theorem and finally using the contractivity of $(T_t)_{t\geq 0}$ we get

$$\begin{aligned} \|G_\alpha f\|_2^2 &= \int_S \left(\int_0^\infty e^{-\alpha t} T_t f \,\mathrm{d}t \right)^2 \mathrm{d}\nu \leq \int_S \int_0^\infty e^{-2\alpha t} (T_t f)^2 \,\mathrm{d}t \,\mathrm{d}\nu \\ &= \int_0^\infty e^{-2\alpha t} \|T_t f\|_2^2 \,\mathrm{d}t \leq \frac{1}{2\alpha} \|f\|_2^2 < \infty \quad \forall \alpha > 0. \end{aligned} \tag{5.69}$$

Contractivity follows by the same arguments when we substitute $r = \alpha t$ in the inner integral:

$$\begin{aligned} \|\alpha G_\alpha f\|_2^2 &= \int_S \left(\int_0^\infty \alpha e^{-\alpha t} T_t f \,\mathrm{d}t \right)^2 \mathrm{d}\nu = \int_S \left(\int_0^\infty e^{-r} T_{r/\alpha} \,\mathrm{d}r \right)^2 \mathrm{d}\nu \\ &\leq \int_S \int_0^\infty e^{-2r} (T_{r/\alpha} f)^2 \,\mathrm{d}r \,\mathrm{d}\nu = \int_0^\infty e^{-2r} \|T_{r/\alpha} f\|_2^2 \,\mathrm{d}r \leq \frac{1}{2} \|f\|_2^2 \end{aligned} \tag{5.70}$$

In the same way as in (5.36) it is shown that $(G_\alpha)_{\alpha>0}$ satisfies the resolvent equation (R1). The uniqueness of G_α follows by approximation as in the proof of Proposition 5.29.

Next, we show strong continuity of αG_α as $\alpha \to \infty$. By substituting $r = \alpha t$, applying Jensen's inequality and Fubini's Theorem – in that order – we obtain

$$\begin{aligned}\|\alpha G_\alpha f - f\|_2^2 &= \int_S \left(\int_0^\infty \alpha e^{-\alpha t} T_t f \,\mathrm{d}t - f \right)^2 \mathrm{d}\nu = \int_S \left(\int_0^\infty \alpha e^{-\alpha t} (T_t f - f) \,\mathrm{d}t \right)^2 \mathrm{d}\nu \\ &= \int_S \left(\int_0^\infty e^{-r} (T_{r/\alpha} f - f) \,\mathrm{d}r \right)^2 \mathrm{d}\nu \le \int_0^\infty e^{-2r} \|T_{r/\alpha} f - f\|_2^2 \,\mathrm{d}r.\end{aligned} \tag{5.71}$$

By contractivity of T_t, the integrand on the right hand is dominated by $2e^{-2r}\|f\|_2^2$ and hence $\lim_{\alpha\to\infty}\|\alpha G_\alpha f - f\|_2^2 = 0$ by dominated convergence and the strong continuity of T_t. ■

5.3 Extension and transience of Dirichlet forms

Let $\mathcal{E}$ be a Dirichlet form with domain $\mathcal{D} := \mathcal{D}(\mathcal{E})$ on $L^2(S, \nu)$ where $(S, \mathcal{U}, \nu)$ denotes a locally compact uniform measure space, as usual. We adopt the terminology from the literature ([FOT11; CF11]) and refer to $\mathcal{D}$ as a *Dirichlet space*, where we implicitly equip $\mathcal{D}$ with the form $\mathcal{E}$. Recall that $L^\infty(S, \nu)$ is the family of ν-equivalence classes of ν-almost everywhere bounded, measurable functions. We want to extend the Dirichlet form $\mathcal{E}$ to functions in $L^\infty(S, \nu)$. We will show that this extension forms a Hilbert space if and only if the Dirichlet form is transient.

5.3.1 The extended Dirichlet space

We begin with the following observations.

Lemma 5.31. *Let V be a Markovian operator on $L^2(S, \nu)$. Then V can be uniquely extended to a Markovian operator on $L^\infty(S, \nu)$.*

Proof. We give an explicit construction of the extension. By Definition 2.41 we have that ν is σ-finite. Then there exists a strictly positive function $\varphi \in L^1(S, \nu)$, take for example the function

$$\varphi := \sum_{n=1}^\infty \alpha_n \nu(A_n)^{-1} \mathbb{1}_{A_n}, \tag{5.72}$$

where $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ is countable family of Borel subsets of S with $0 < \nu(A_n) < \infty$ for all $n \in \mathbb{N}$ and $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ a summable sequence of strictly positive real numbers. Now define for each $n \in \mathbb{N}$,

$$\varphi_n := (n\varphi) \wedge 1. \tag{5.73}$$

Then, $0 < \varphi_n \leq 1$ and the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is increasing with $\lim_{n\to\infty} \varphi_n = 1$, ν-a.e. Let $f \in L^\infty(S,\nu)$ be non-negative, then the product $\varphi_n f$ is in $L^2(S,\nu) \cap L^\infty(S,\nu)$ and we can set

$$Vf := \lim_{n\to\infty} V(\varphi_n f), \tag{5.74}$$

where the limit is taken in $L^\infty(S,\nu)$ and exists by the Markov property of V. We sometimes refer to V as the *potential operator*. Furthermore, we set

$$Vf := Vf^+ - Vf^- \tag{5.75}$$

for arbitrary $f \in L^\infty(S,\nu)$. ■

Lemma 5.32. *Let $(\mathcal{E},\mathcal{D})$ be a Dirichlet form on $L^2(S,\nu)$. Further, let $f \in L^\infty(S,\nu)$ and assume that there exists an $\mathcal{E}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $\lim_{n\to\infty} f_n = f$ in $L^\infty(S,\nu)$. Then the limit*

$$\mathcal{E}(f,f) := \lim_{n\to\infty} \mathcal{E}(f_n,f_n) \tag{5.76}$$

exists and is independent of the choice of $(f_n)_{n\in\mathbb{N}}$.

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ be an $\mathcal{E}$-Cauchy sequence and $f \in L^\infty(S,\nu)$ as above. The existence of the limit follows from the fact that $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{E}$-Cauchy. We want to prove this fact in detail. Fix $n,k \in \mathbb{N}$ and suppose without loss of generality that $\mathcal{E}(f_n,f_n) + \mathcal{E}(f_k,f_k) > 0$. Then,

$$\begin{aligned} |\mathcal{E}(f_n,f_n) - \mathcal{E}(f_k,f_k)| &= |\mathcal{E}(f_n - f_k, f_n + f_k)| \\ &\leq \mathcal{E}(f_n - f_k, f_n - f_k)^{1/2}\mathcal{E}(f_n + f_k, f_n + f_k)^{1/2} \\ &\leq \mathcal{E}(f_n - f_k, f_n - f_k)^{1/2}\left(\mathcal{E}(f_n,f_n)^{1/2} + \mathcal{E}(f_k,f_k)^{1/2}\right), \end{aligned} \tag{5.77}$$

where we used the Cauchy-Schwarz inequality from Lemma 5.2 in the first inequality and the triangle inequality from Lemma 5.3 in the second inequality. Rearranging now yields

$$\left|\mathcal{E}(f_n,f_n)^{1/2} - \mathcal{E}(f_k,f_k)^{1/2}\right| \leq \mathcal{E}(f_n - f_k, f_n - f_k)^{1/2}. \tag{5.78}$$

Therefore, $\sqrt{\mathcal{E}(f_n,f_n)}$ is a real-valued Cauchy sequence which converges. Therefore we immediately obtain the existence of $\lim_{n\to\infty} \mathcal{E}(f_n,f_n)$.

Recall the definition of $\mathcal{E}^{(t)}$ from (5.48). By Lemma 5.31 we can extend $\mathcal{E}^{(t)}$ to $L^\infty(S,\nu)$. It now suffices to show that

$$\lim_{t\to 0} \mathcal{E}^{(t)}(f,f) = \mathcal{E}(f,f). \tag{5.79}$$

For each $k \in \mathbb{N}$ we have that $f - f_k \in L^\infty(S, \nu)$ and $(f_n - f_k)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-Cauchy sequence that converges to $f - f_k$ ν-a.e. Therefore, by Fatou's property [CF11, Lemma 1.1.7] and the fact that $\mathcal{E}^{(t)}(f, f)$ is increasing as $t \to 0$ for all $f \in L^2(S, \nu)$, we obtain

$$\mathcal{E}^{(t)}(f - f_k, f - f_k) \leq \liminf_{n\to\infty} \mathcal{E}^{(t)}(f_n - f_k, f_n - f_k) \leq \lim_{n\to\infty} \mathcal{E}(f_n - f_k, f_n - f_k). \quad (5.80)$$

Taking the limit for $k \to \infty$ on both sides shows that $\lim_{k\to\infty} \mathcal{E}^{(t)}(f - f_k, f - f_k) = 0$. Using a similar argument as in (5.77) and (5.78), we obtain

$$\lim_{k\to\infty} \mathcal{E}^{(t)}(f_k, f_k) = \mathcal{E}^{(t)}(f, f). \quad (5.81)$$

Since $\mathcal{E}^{(t)}(f_k, f_k) \uparrow \mathcal{E}(f_k, f_k)$ as $t \to 0$ (see our remarks at the end of Section 5.2.1 or [FOT11, Lemma 1.3.4]), we conclude

$$\begin{aligned} \left| \lim_{t\to 0} \mathcal{E}^{(t)}(f, f)^{1/2} - \mathcal{E}(f_k, f_k)^{1/2} \right| &\leq \lim_{t\to 0} \mathcal{E}^{(t)}(f - f_k, f - f_k)^{1/2} \\ &\leq \lim_{n\to\infty} \mathcal{E}(f_n - f_k, f_n - f_k)^{1/2}, \end{aligned} \quad (5.82)$$

where we have applied the equivalent of (5.78) for $\mathcal{E}^{(t)}$ in the first inequality and (5.80) in the second inequality. Now, the right hand side of (5.82) goes to 0 as $k \to \infty$. Which completes the proof. ■

We have now justified the following definition.

Definition 5.33 (Extended Dirichlet space)**.** Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form on $L^2(S, \nu)$. Let $\mathcal{D}_e$ denote the collection of $f \in L^\infty(S, \nu)$ such that there exists a $\mathcal{E}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ with $\lim_{n\to\infty} f_n = f$ in $L^\infty(S, \nu)$. We call $\mathcal{D}_e$ the *extended Dirichlet space* and $(\mathcal{E}, \mathcal{D}_e)$ the *extended Dirichlet form* . ◇

It is worth noting that $\mathcal{D} = \mathcal{D}_e \cap L^2(S, \nu)$ (cf. [FOT11, Theorem 1.5.2 (iii)]).

5.3.2 Transient Dirichlet forms

Recall the definition of transience of a ν-symmetric Feller process from Definition 4.67. We introduce a closely related notion of transience of a Dirichlet form and show that the extended Dirichlet space becomes a Hilbert space whenever the Dirichlet form is transient.

We begin with the definition of a transient Dirichlet form. While quite abstract at first glance, we will fill this definition with a bit of life in the remainder of this section.

Definition 5.34 (Transient Dirichlet forms)**.** Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form on $L^2(S, \nu)$. We say that $(\mathcal{E}, \mathcal{D})$ is *transient* if there exists a $\psi \in L^1(S, \nu)$ with $\psi > 0$ ν-almost everywhere on S such that

$$\int_S |f|\psi \, \mathrm{d}\nu \leq \mathcal{E}(f, f)^{1/2}, \tag{5.83}$$

for all $f \in \mathcal{D}$. In that case we call ψ the *reference function* of $(\mathcal{E}, \mathcal{D})$. ◊

Let $(T_t)_{t\geq 0}$ be a Markovian semigroup on $L^2(S, \nu)$. For $t \geq 0$ and $f \in L^2(S, \nu)$ define

$$V_t f := \int_0^t T_s f \, \mathrm{d}s. \tag{5.84}$$

By the contraction property of $(T_t)_{t\geq 0}$, we can apply Fubini's theorem and with Jensen's inequality, we get

$$\begin{aligned} \|V_t f\|_2^2 &= \int_S (V_t f)^2 \, \mathrm{d}\nu \leq \int_S \int_0^t (T_s f)^2 \, \mathrm{d}s \, \mathrm{d}\nu = \int_0^t \int_S (T_s f)^2 \, \mathrm{d}\nu \, \mathrm{d}s \\ &\leq \int_0^t \|f\|_2^2 \, \mathrm{d}s = t\|f\|_2^2. \end{aligned} \tag{5.85}$$

Thus, V_t is a bounded symmetric operator on $L^2(S, \nu)$ for every $t > 0$.

Again, we want to extend $(T_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ to a different domain.

Lemma 5.35. *Each of the families of operators $(T_t)_{t\geq 0}, (V_t)_{t\geq 0}$ as above can be uniquely extended to $L^1(S, \nu)$ in a way such that for all $f \in L^1(S, \nu)$ and $s, t > 0$,*

$$T_s T_t f = T_{s+t} f, \quad \|T_t f\|_1 \leq \|f\|_1, \quad \|V_t f\|_1 \leq t\|f\|_1. \tag{5.86}$$

Moreover, T_t and $\frac{1}{t} V_t$ are Markovian for each $t > 0$.

Proof. First, let $f \in L^2(S, \nu) \cap L^1(S, \nu)$. By σ-finiteness, we can choose a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ of Borel subsets with $\nu(A_n) < \infty$, $A_n \subset A_{n+1}$ and $S = \bigcup_{n\geq 1} A_n$. Then for all $t > 0$ and $f \in L^2(S, \nu) \cap L^1(S, \nu)$,

$$\int_{A_n} |T_t f| \, \mathrm{d}\nu \leq \langle T_t |f|, \mathbb{1}_{A_n} \rangle = \langle |f|, T_t \mathbb{1}_{A_n} \rangle \leq \int_S |f| \, \mathrm{d}\nu, \tag{5.87}$$

where we have used the contraction property of T_t in the second inequality. Letting $n \to \infty$, we obtain $\|T_t f\|_1 \leq \|f\|_1$ and analogously $\|V_t f\|_1 \leq t\|f\|_1$ for all $t > 0$ and $f \in L^2(S, \nu) \cap L^1(S, \nu)$.

Now set $\varphi_n(t) := (-n \vee t) \wedge n$, $t \in \mathbb{R}$. Then, $\varphi_n \circ f \in L^2(S,\nu) \cap L^1(S,\nu)$ for all $f \in L^1(S,\nu)$:

$$\begin{aligned}\int_S (\Phi_n \circ f)\,\mathrm{d}\nu &= n^2 \nu(\{|f| > n\}) + n^2 \int_{\{|f|\le n\}} n^{-2} (\varphi_n \circ f)^2\,\mathrm{d}\nu \\ &\le n^2 \left(\nu(\{|f| > n\}) + \|f\|_1\right) < \infty.\end{aligned} \tag{5.88}$$

By the contractivity property of T_t, we immediately obtain the existence of the limit $T_t f := \lim_{n\to\infty} T_t(\varphi_n \circ f)$ in $L^1(S,\nu)$ for each $f \in L^1(S,\nu)$ and $t > 0$. The operators V_t can be extended in the same manner and the properties (5.86) as well as the Markov property of T_t and $t^{-1}V_t$ are an immediate consequence of this definition. ■

As an immediate consequence of the preceding lemma and the relation between $(T_t)_{t\ge0}$ and the resolvent $(G_\alpha)_{\alpha>0}$ given by (5.34) we obtain the existence of a unique extension of $(G_\alpha)_{\alpha>0}$ to a Markovian resolvent on $L^1(S,\nu)$. Moreover, we have for each $0 < s < t$ and $0 < \alpha < \beta$ and $f \in L^1_+(S,\nu) := \left\{ f \in L^1(S,\nu) \,\middle|\, f \ge 0 \ \nu\text{-a.e.} \right\}$ that

$$0 \le V_s f \le V_t f \quad \text{and} \quad 0 \le G_\beta f \le G_\alpha f, \tag{5.89}$$

ν-almost everywhere. Therefore, we can define for each $f \in L^1_+(S,\nu)$ a function $Vf\colon S \to [0,\infty]$ satisfying

$$\lim_{t\to\infty} V_t f = \lim_{n\to\infty} G_{1/n} f = Vf, \tag{5.90}$$

ν-almost everywhere. By (5.90) the function Vf is unique up to ν-equivalence. Observe that Vf can take the value $+\infty$ on a set of positive measure.

This leads us to the following definition (cf. [CF11, Definition 2.1.1]). Recall that we denote the completion of $\mathcal{B}$ with respect to ν by $\mathcal{B}_\nu$ (cf. Definition 2.41).

Definition 5.36. Let $(T_t)_{t\ge0}$ be Markovian semigroup on $L^2(S,\nu)$.

(i) $(T_t)_{t\ge0}$ is called *transient* if $Vf < \infty$ ν-a.e. for some $f \in L^1_+(S,\nu)$ with $f > 0$ ν-a.e.

(ii) $(T_t)_{t\ge0}$ is called *recurrent* if

$$\nu(\{\, x \in S \mid Vf(x) \in (0,\infty) \,\}) = 0 \tag{5.91}$$

for all $f \in L^1_+(S,\nu)$.

(iii) A set $A \in \mathcal{B}_\nu$ is called T_t-invariant if for every $t > 0$ and $f \in L^2(S,\nu)$,

$$T_t\left(\mathbb{1}_{\complement A} f\right) = 0 \quad \nu\text{-a.e. on } A. \tag{5.92}$$

(iv) $(T_t)_{t\geq 0}$ is called *irreducible* if any T_t-invariant set $A \in \mathcal{B}_\nu$ is ν-trivial, i.e. $\nu(A) = 0$ or $\nu(\complement A) = 0$. ◇

The next result gives some equivalent formulations for the transience and recurrence of $(T_t)_{t\geq 0}$ and formulates a recurrence-transience dichotomy. We will not prove this result here but instead refer the reader to the literature.

Proposition 5.37. *Let $(T_t)_{t\geq 0}$ be a Markovian semigroup and V as defined in (5.90).*

(i) $(T_t)_{t\geq 0}$ is transient if and only if for every $f \in L^1_+(S,\nu)$,

$$Vf < \infty \quad \nu\text{-a.e.} \tag{5.93}$$

(ii) The following three statements are equivalent

a) $Vf = \infty$ ν-a.e. for every $f \in L^1_+(S,\nu)$ with $f > 0$ ν-a.e.

b) There exists a $f \in L^1_+(S,\nu)$ such that $Vf = \infty$ ν-a.e.

c) $(T_t)_{t\geq 0}$ is recurrent.

(iii) Suppose that $(T_t)_{t\geq 0}$ is irreducible. Then $(T_t)_{t\geq 0}$ is either transient or recurrent.

Proof. See [CF11, Proposition 2.1.3]. ■

We can now show that there exists a one-to-one correspondence between transient semigroups and transient Dirichlet forms and that in that case the extended Dirichlet space becomes a real Hilbert space.

Theorem 5.38 ([CF11, Theorem 2.1.5])**.** *Let $(T_t)_{t\geq 0}$ be Markovian semigroup on $L^2(S,\nu)$ and $(\mathcal{E},\mathcal{D})$ the Dirichlet form associated with $(T_t)_{t\geq 0}$.*

(i) $(\mathcal{E},\mathcal{D})$ is transient if and only if $(T_t)_{t\geq 0}$ is transient.

(ii) Suppose that $(\mathcal{E},\mathcal{D})$ is transient with reference function $\psi \in L^1(S,\nu)$. Then,

$$\int_S |f|\psi \,\mathrm{d}\nu \leq \mathcal{E}(f,f)^{1/2} \tag{5.94}$$

for all $f \in \mathcal{D}_\mathrm{e}$ and the extended Dirichlet space $\mathcal{D}_\mathrm{e}$ is a real Hilbert space with inner product $\mathcal{E}$.

Proof. Fix $f \in L^2(S,\nu)$. For each $0 < s < t$ we have

$$V_t f - T_s V_t f = \int_0^t T_r f \,\mathrm{d}r - \int_0^t T_{r+s} f \,\mathrm{d}r = \int_0^s T_r f \,\mathrm{d}r - \int_t^{t+s} T_r f \,\mathrm{d}r. \tag{5.95}$$

Therefore, we obtain

$$\lim_{s\to 0} s^{-1}\langle V_t f - T_s V_t f, V_t f\rangle = \langle f, V_t f\rangle - \langle T_t f, V_t f\rangle < \infty. \tag{5.96}$$

Hence, $V_t f \in \mathcal{D}$ and by the same argument as before, we arrive at

$$\mathcal{E}(V_t f, g) = \langle f - T_t f, g\rangle \quad \forall g \in \mathcal{D}. \tag{5.97}$$

Now let $f \in L^1_+(S,\nu) \cap L^2(S,\nu)$. We claim that

$$\sup_{g\in\mathcal{D}} \frac{\langle |g|, f\rangle^2}{\mathcal{E}(g,g)} = \int_S f V f \,\mathrm{d}\nu. \tag{5.98}$$

Denote the left hand side of (5.98) by c and suppose that $c < \infty$. Using (5.97) and the fact that for all $t > 0$, $f, T_t f, V_t f \geq 0$ ν-a.e. we obtain for each $t > 0$,

$$\langle V_t f, f\rangle^2 \leq c\mathcal{E}(V_t f, V_t f) = c\left(\langle f, V_t f\rangle - \langle T_t f, V_t f\rangle\right) \leq c\,\langle f, V_t f\rangle \tag{5.99}$$

and consequently $\langle V_t f, f\rangle \leq c$. If we let $t \to \infty$, the inequality remains true and we obtain

$$\langle V f, f\rangle \leq c. \tag{5.100}$$

Now assume that the right-hand side of (5.98) is finite. We can apply Fubini's theorem and obtain

$$\int_S f V f \,\mathrm{d}\nu = \int_0^\infty \langle T_s f, f\rangle \,\mathrm{d}s. \tag{5.101}$$

By the contraction property of $(T_t)_{t\geq 0}$, we conclude that

$$\lim_{s\to\infty} \langle T_s f, f\rangle = \lim_{s\to\infty} \langle T_{s/2} f, T_{s/2}\rangle = 0. \tag{5.102}$$

Again, by (5.97) and the Cauchy-Schwarz inequalities for $\mathcal{E}$ and $\|\cdot\|_2$ we now obtain for all $g \in \mathcal{D}$ and $t > 0$,

$$\begin{aligned}
\langle |g|, f\rangle &= \mathcal{E}(V_t f, |g|) - \langle T_t f, |g|\rangle \leq \mathcal{E}(V_t f, V_t f)^{1/2}\mathcal{E}(g,g)^{1/2} + \|T_t f\|_2 \|u\|_2 \\
&= \sqrt{\langle f, V_t f\rangle - \langle T_t f, V_t f\rangle}\,\mathcal{E}(g,g)^{1/2} + \langle T_{2t} f, f\rangle^{1/2}\,\|u\|_2 \\
&\leq \langle f, V_t f\rangle^{1/2}\,\mathcal{E}(g,g)^{1/2} + \langle T_{2t} f, f\rangle^{1/2}\,\|u\|_2.
\end{aligned} \tag{5.103}$$

By (5.102) the right hand side of (5.103) converges to $\langle f, V f\rangle\,\mathcal{E}(g,g)^{1/2}$ when we let $t \to \infty$. Therefore,

$$c \leq \langle V f, f\rangle\,, \tag{5.104}$$

proving our claim.

Now suppose that $(\mathcal{E}, \mathcal{D})$ is transient with reference function $\psi \in L^1(S,\nu)$. Combining the definition of transience (5.83) with (5.98), we can deduce that

$$\int_S \psi V\psi \, \mathrm{d}\nu \leq 1. \tag{5.105}$$

Since ψ is strictly positive ν-a.e. we obtain that $V\psi < \infty$ ν-a.e. Hence, $(T_t)_{t\geq 0}$ is transient by Definition 5.36.

Now suppose that $(T_t)_{t\geq 0}$ is transient. By Proposition 5.37 (i) we have that $Vf < \infty$ $\nu - a.e.$ for every $f \in L^1_+(S,\nu)$. We can therefore choose $\varphi \in L^1_+(S,\nu)$ such that $\varphi > 0$, $V\varphi < \infty$ ν-a.e. and $\int_S \varphi \, \mathrm{d}\nu = 1$. Let

$$\psi := \varphi (V\varphi \vee 1)^{-1}, \tag{5.106}$$

by definition we have $0 < \psi \leq \varphi$ ν-a.e. Moreover,

$$\int_S \psi V\psi \, \mathrm{d}\nu \leq \int_S \varphi V\psi \, \mathrm{d}\nu = \int_S \psi V\varphi \, \mathrm{d}\nu \leq \int_S (\varphi/V\varphi)\, V\varphi \, \mathrm{d}\nu = \int_S \varphi \, \mathrm{d}\nu = 1. \tag{5.107}$$

When we plug this estimate in (5.98), we get that $(\mathcal{E}, \mathcal{D})$ is transient with reference function ψ, proving (i).

We turn to (ii). Suppose that $(\mathcal{E}, \mathcal{D})$ is transient. Fix $f \in \mathcal{D}_\mathrm{e}$, by definition of the extended Dirichlet space, there exists a $\mathcal{E}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $\lim_{n\to\infty} f_n = f$ in $L^\infty(S,\nu)$. By transience, (5.94) holds for all f_n, $n \in \mathbb{N}$. By definition of the reference function ψ we know that $f, f_n \in L^\infty(S, \psi\cdot\nu)$. Consequently, (5.94) also holds in the limit $n \to \infty$. Equation (5.94) also implies that $\mathcal{E}(f,f) = 0$ if and only if $f = 0$ ν-a.e. for all $f \in \mathcal{D}_\mathrm{e}$. It therefore remains to show that $\mathcal{D}_\mathrm{e}$ equipped with the scalar product $\mathcal{E}$ is complete. To that end, let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\mathrm{e}$ be a $\mathcal{E}$-Cauchy sequence. For each $n \in \mathbb{N}$ choose a $\mathcal{E}$-Cauchy sequence $(f_{n,m})_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} f_{n,m} = f_n$ ν-a.e. Note that

$$\mathcal{E}(f_n - f_{n,m}, f_n - f_{n,m}) = \lim_{k\to\infty} \mathcal{E}(f_{n,k} - f_{n,m}, f_{n,k} - f_{n,m}). \tag{5.108}$$

Therefore, $(f_{n,m})_{m\in\mathbb{N}}$ converges to f_n with respect to the scalar product $\mathcal{E}$. By a diagonal argument, we can assume without loss of generality that $(f_{n,m})_{m\in\mathbb{N}}$ is chosen so that

$$\lim_{n\to\infty} \mathcal{E}(f_n - f_{n,n}, f_n - f_{n,n}) = 0. \tag{5.109}$$

It is clear that $(f_{n,n})_{n\in\mathbb{N}} \subset \mathcal{D}$ is again a Cauchy sequence with respect to $\mathcal{E}$. Moreover, by virtue of (5.94) we can conclude that $(f_{n,n})_{n\in\mathbb{N}}$ is also Cauchy in $L^1(S, \psi\cdot\nu)$. Consequently, there exists a $f \in L^1(S, \psi\cdot\nu)$ such that $\lim_{n\to\infty} f_{n,n} = f$ ν-a.e. By definition, $f \in \mathcal{D}_\mathrm{e}$ and

$$\mathcal{E}(f_n - f, f_n - f) \leq \mathcal{E}(f_n - f_{n,n}, f_n - f_{n,n}) + \mathcal{E}(f_{n,n} - f, f_{n,n} - f). \tag{5.110}$$

The first summand on the right-hand side goes to 0 as $n \to \infty$ by (5.109). As for the second summand, recall that $(f_{n,n})_{n\in\mathbb{N}}$ is a $\mathcal{E}$-Cauchy sequence that converges ν-a.e. to $f \in \mathcal{D}_e$. By (5.108) we find that the second summand also tends to 0 as $n \to \infty$. We have therefore shown that $f_n \to f$ with respect to $\mathcal{E}$ as $n \to \infty$, thereby completing the proof. ∎

Using the recurrence-transience dichotomy from Proposition 5.37 (iii), Theorem 5.38 also characterizes a recurrent Dirichlet form in terms of its extended Dirichlet space. Here we call a Dirichlet form $(\mathcal{E}, \mathcal{D})$ *recurrent* if its associated Markovian semigroup $(T_t)_{t\geq 0}$ is recurrent. A more direct characterization with further useful implications is the following.

Theorem 5.39. *Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form on $L^2(S, \nu)$. Then $(\mathcal{E}, \mathcal{D})$ is recurrent if and only if $1 \in \mathcal{D}_e$ and*

$$\mathcal{E}(1, 1) = 0. \tag{5.111}$$

Proof. We only show necessity because we will only make use of this direction. For sufficiency see the proof of [CF11, Theorem 2.1.8].

Before we start with the actual proof we make the following observation. Let $\eta \in L^1(S, \nu) \cap L^\infty(S, \nu)$ with $\eta > 0$ ν-a.e. For each $f, g \in \mathcal{D}$ define,

$$\mathcal{E}^\eta(f, g) := \mathcal{E}(f, g) + \langle f, g \rangle_{\eta\cdot\nu}\,. \tag{5.112}$$

Here, $\langle \cdot\,, \cdot \rangle_{\eta\cdot\nu}$ denotes the inner product on $L^2(S, \eta \cdot \nu)$, i.e.

$$\langle f, g \rangle_{\eta\cdot\nu} = \int_S f(x) g(x) \eta(x)\, \nu(\mathrm{d}x). \tag{5.113}$$

Then, $(\mathcal{E}^\eta, \mathcal{D})$ is again a Dirichlet form on $L^2(S, \nu)$[2], because for all $f \in \mathcal{D}$ we have the inequality,

$$\mathcal{E}_1(f, f) \leq \mathcal{E}^\eta(f, f) + \langle f, f \rangle \leq \mathcal{E}(f, f) + (1 + \|\eta\|_\infty)\langle f, f \rangle\,. \tag{5.114}$$

We denote by $\left(T_t^\eta\right)_{t\geq 0}$ and $\left(G_\alpha^\eta\right)_{\alpha>0}$ the semigroup and the resolvent associated with the *perturbed Dirichlet form* $(\mathcal{E}^\eta, \mathcal{D})$. Observe that for every $f \in L^2(S, \nu)$, $g \in \mathcal{D}$ and $\alpha > 0$ it holds that

$$\mathcal{E}_\alpha(G_\alpha^\eta f, g) = \mathcal{E}_\alpha^\eta(G_\alpha^\eta f, g) - \left\langle G_\alpha^\eta f, g \right\rangle_{\eta\cdot\nu} = \left\langle f - \eta G_\alpha^\eta f, g \right\rangle_\nu \tag{5.115}$$

and consequently,

$$G_\alpha^\eta f = G_\alpha(f - \eta G_\alpha^\eta f), \tag{5.116}$$

[2] note that the *sets* $L^2(S, \nu)$ and $L^2(S, \eta \cdot \nu)$ are equal.

by virtue of (5.46).

To show that $1 \in \mathcal{D}_e$ we need to find a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $0 \leq f \leq 1$ and $\lim_{n\to\infty} f_n = 1$ ν-a.e. such that

$$\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0. \tag{5.117}$$

We claim that for η as above with the additional assumption $\|\eta\|_\infty \leq 1$, the sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ where

$$f_n = G^\eta_{1/n}\eta, \tag{5.118}$$

is such a sequence.

To prove the claim fix $\varepsilon > 0$. Note that $(\mathcal{E}, \mathcal{D})$ can be considered to be a Dirichlet form on the perturbed space $L^2(S, (\varepsilon + \eta) \cdot \nu)$. In view of this interpretation we obtain,

$$\begin{aligned} \mathcal{E}_1\left(G^\eta_\varepsilon(\varepsilon f + \eta f), g\right) &= \mathcal{E}\left(G^\eta_\varepsilon(\varepsilon f + \eta f), g\right) + \left\langle G^\eta_\varepsilon(\varepsilon f + \eta f), g\right\rangle_{(\eta+\varepsilon)\cdot\nu} \\ &= \mathcal{E}^\eta_\varepsilon\left(G^\eta_\varepsilon(\varepsilon f + \eta f), g\right) = \langle \varepsilon f + \eta f, g\rangle_\nu = \langle f, g\rangle_{(\varepsilon+\eta)\cdot\nu}\,. \end{aligned} \tag{5.119}$$

Hence, $G^\eta_\varepsilon(\varepsilon f + \eta f)$ is the 1-order resolvent of f with respect to the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(S, (\varepsilon + \eta) \cdot \nu)$. By the properties of the resolvent we immediately obtain

$$0 \leq G^\eta_\varepsilon(\varepsilon f + \eta f) \leq 1 \tag{5.120}$$

for all $f \in \mathcal{D}$ with $0 \leq f \leq 1$. If we now let first $\varepsilon \to 0$ and then $f \to 1$, we find that ν-a.e.,

$$0 \leq V^\eta\eta \leq 1, \tag{5.121}$$

where $V^\eta\eta = \lim_{\varepsilon\to0} G^\eta_\varepsilon\eta$, as in (5.89). Now take $f = \eta$ in (5.116), then

$$G^\eta_\varepsilon\eta = G_\varepsilon(\eta(1 - G^\eta_\varepsilon\eta)). \tag{5.122}$$

If we now let $\varepsilon \to 0$, we obtain together with (5.121),

$$0 \leq V(\eta(1 - V^\eta\eta)) \leq \lim_{\varepsilon\to0} G_\varepsilon(\eta(1 - G^\eta_\varepsilon\eta)) = V^\eta\eta \leq 1. \tag{5.123}$$

By definition of recurrence, Definition 5.36, we conclude that $V(\eta(1 - V^\eta\eta)) = 0$ ν-a.e. and therefore,

$$V^\eta\eta = 1 \quad \nu\text{-a.e.} \tag{5.124}$$

We have shown $f_n \uparrow 1$ ν-a.e. as $n \to \infty$ and it remains to show (5.117). Applying

(5.115), we obtain

$$\begin{aligned} 0 \le \mathcal{E}(f_n, f_n) \le \mathcal{E}_{1/n}(f_n, f_n) &= \langle \eta - \eta f_n, f_n \rangle_\nu \\ &= \int_S \eta(1 - f_n) f_n \, \mathrm{d}\nu \to 0, \end{aligned} \tag{5.125}$$

as $n \to \infty$, which proves the claim. ■

Corollary 5.40. *Let $(\mathcal{E}, \mathcal{D})$ be a recurrent Dirichlet form on $L^2(S, \nu)$. Then $1 \in \mathcal{D}_e$ and*

$$\mathcal{E}(1, f) = 0 \tag{5.126}$$

for all $f \in \mathcal{D}_e$.

Proof. Fix $f \in \mathcal{D}_e$ and let $(f_n)_{n\in\mathbb{N}} \subset D$ be defined as in the proof of Theorem 5.39. Instead of (5.125) we can write

$$\left|\mathcal{E}_{1/n}(f_n, f)\right| = \left|\int_S \eta(1 - f_n) f \, \mathrm{d}\nu\right| \le \|f\|_\infty \int_S |1 - f_n| \, \mathrm{d}\nu \to 0, \tag{5.127}$$

as $n \to \infty$, which implies the assertion. ■

Finally, we relate the transience of a ν-symmetric Feller process as defined in the previous chapter to the transience of its associated Dirichlet form.

Proposition 5.41. *Let $(S, \mathcal{U}, \nu)$ be a locally compact uniform measure space and suppose X is a transient ν-symmetric Feller process with values in $(S_\vartheta, \mathcal{U}_\vartheta)$. Then the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(S, \nu)$ associated with X is transient.*

Proof. Let $(P_t)_{t\ge0}$ denote the semigroup associated with X. Recall Lemma 2.40 and observe that $(S, \mathcal{U})$ is Lindelöf and consequently, by Lemma A.20, σ-compact. Therefore, we can write S as the union of countably many compact subsets $(K_n)_{n\in\mathbb{N}} \subset S$. By Definition 4.67, we have

$$c_n := \sup_{x\in S} \mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{K_n}(X_t) \, \mathrm{d}t\right] = \sup_{x\in S} \int_0^\infty P_t \mathbb{1}_{K_n}(x) \, \mathrm{d}t < \infty. \tag{5.128}$$

Now choose a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $\alpha_n > 0$ for all $n \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} \alpha_n < \infty$. Set

$$\varphi := \sum_{n=1}^\infty \frac{\alpha_n}{(c_n + \nu(K_n)) \vee 1} \mathbb{1}_{K_n}. \tag{5.129}$$

Then, $\varphi \in B_b(S)$, $\varphi \in L^1(S,\nu) \cap L^2(S,\nu)$, $\varphi > 0$ ν-a.e. and

$$\sup_{x \in S} \int_0^\infty P_t\varphi(x)\,\mathrm{d}t < \infty. \tag{5.130}$$

By definition, $T_t\varphi$ is a representative of $P_t\varphi$ and we can conclude that

$$V\varphi = \int_0^\infty T_t\varphi\,\mathrm{d}t < \infty \quad \nu\text{-a.e.} \tag{5.131}$$

Consequently, by Definition 5.36 we have that $(T_t)_{t\geq 0}$ is transient which implies the assertion by Theorem 5.38. ■

5.4 Potential theory

Potential theoretic concepts are an important tool in the analysis of Markov processes. In this section, we introduce some potential theoretic notions with the help of Dirichlet forms and show how they relate to the dynamics of the processes they are associated with. For more details see [FOT11], [CF11] and the classical books on potential theory [BG68] by Robert M. Blumenthal and Ronald Getoor or [DM79] by Claude Dellacherie and Paul-André Meyer.

5.4.1 Choquet capacities

The notion of capacity is at the very core of classical potential theory. We begin with the definition of *Choquet capacity* named after the French mathematician Gustave Choquet (1915–2006). We will use the following definition because it is tailored to our needs. For a more general definition see [DM79, Definition III.27].

Definition 5.42 (Choquet capacity)**.** Let $(S,\mathcal{U})$ be a uniform Hausdorff space. Denote by $\mathcal{K}$ the class of all compact subsets of S. An extended real valued set function φ that is defined on all the subsets of S is called a *Choquet capacity* on S if the following hold

(i) φ is increasing, i.e. $A \subset B$ implies that $\varphi(A) \leq \varphi(B)$.

(ii) For every increasing sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of S it holds that

$$\varphi\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup_{n\in\mathbb{N}} \varphi(A_n). \tag{5.132}$$

(iii) For every decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{K}$ it holds that

$$\varphi\left(\bigcap_{n\in\mathbb{N}} K_n\right) = \inf_{n\in\mathbb{N}} \varphi(K_n). \tag{5.133}$$

Given a Choquet capacity φ on S we say an arbitrary set $A \subset S$ is *φ-capacitable* or just *capacitable* if

$$\varphi(A) = \sup_{K\in\mathcal{K}, K\subset A} \varphi(K). \tag{5.134}$$

◊

The following result is a simplified version of the celebrated result of Choquet. A proof can be found for example in [DM79].

Proposition 5.43 (Choquet's capacibility theorem)**.** *Every Borel set is capacitable.*

Proof. See [DM79, Theorem III.28]. ■

We have the following useful characterization of Choquet capacities.

Proposition 5.44 (Theorem A.1.2 in [FOT11])**.** *Let $(S, \mathcal{U})$ be a uniform Hausdorff space and denote by $\mathcal{T}$ the uniform topology. Suppose $\varphi : \mathcal{T} \to \mathbb{R}^+ \cup \{\infty\}$ satisfies*

(i) for all $A, B \in \mathcal{T}$, $A \subset B \Rightarrow \varphi(A) \le \varphi(B)$,

(ii) for all $A, B \in \mathcal{T}$, $\varphi(A \cup B) + \varphi(A \cap B) \le \varphi(A) + \varphi(B)$,

(iii) for every increasing sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{T}$,

$$\varphi\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup_{n\in\mathbb{N}} \varphi(A_n). \tag{5.135}$$

For an arbitrary $A \subset S$ set

$$\varphi^*(A) := \inf_{B\in\mathcal{T},\, A\subset B} \varphi(B). \tag{5.136}$$

Then φ^* is a Choquet capacity. Moreover φ^* extends φ and is σ-subadditive.

Proof. By definition, we immediately obtain that φ^* is monotone and extends φ, i.e. $\varphi^*|_{\mathcal{T}} = \varphi$. Furthermore, we have by (ii) that for $A_1, A_2 \subset S$,

$$\begin{aligned}\varphi^*(A_1 \cup A_2) &= \inf\{\varphi(B) \mid B \in \mathcal{T},\ A_1 \cup A_2 \subset B\} \\ &= \inf\{\varphi(B_1 \cup B_2) \mid B_i \in \mathcal{T},\ A_i \subset B_i,\ i = 1, 2\} \\ &\leq \inf_{B_1 \in \mathcal{T},\ A_1 \subset B_1} \varphi(B_1) + \inf_{B_2 \in \mathcal{T},\ A_2 \subset B_2} \varphi(B_2) = \varphi^*(A_1) + \varphi^*(A_2).\end{aligned} \tag{5.137}$$

Hence, φ^* is subadditive. The claimed σ-subadditivity of φ^* follows immediately if we can show that φ^* satisfy property (ii) of Definition 5.42. To see that, let $(A_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of subsets of S and write $B_n := \bigcup_{i=1}^n A_i$. Then $(B_n)_{n\in\mathbb{N}}$ is an increasing sequence of subsets of S and by combining (5.132) and (5.137) we obtain

$$\varphi^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \varphi^*\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sup_{n\in\mathbb{N}} \varphi^*(B_n) \leq \sup_{n\in\mathbb{N}} \sum_{i=1}^n \varphi^*(A_i) = \sum_{n\in\mathbb{N}} \varphi^*(A_n). \tag{5.138}$$

Let's show that φ^* satisfies (ii) of Definition 5.42 first. Suppose $A_1, A_2, B_1, B_2 \in \mathcal{T}$ with $A_i \subset B_i$ and $\varphi(A_i), \varphi(B_i) < \infty$ for $i = 1, 2$. Using properties (i) and (ii) of φ, we have

$$\begin{aligned}\varphi(B_1 \cup B_2) + \varphi(A_1) &\leq \varphi(B_1 \cup (B_2 \cup A_1)) + \varphi(B_1 \cap (B_2 \cup A_1)) \\ &\leq \varphi(B_1) + \varphi(B_2 \cup A_1)\end{aligned} \tag{5.139}$$

and similarly,

$$\begin{aligned}\varphi(B_2 \cup A_1) + \varphi(A_2) &\leq \varphi(B_2 \cup (A_1 \cup A_2)) + \varphi(B_2 \cap (A_1 \cup A_2)) \\ &\leq \varphi(B_2) + \varphi(A_1 \cup A_2).\end{aligned} \tag{5.140}$$

Adding (5.139) and (5.140) and rearranging yields

$$\varphi(B_1 \cup B_2) - \varphi(A_1 \cup A_2) \leq \varphi(B_1) - \varphi(A_1) + \varphi(B_2) - \varphi(A_2). \tag{5.141}$$

Now let $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ with $A_n \subset B_n$ and $\varphi(A_n), \varphi(B_n) < \infty$ for all $n \in \mathbb{N}$. Suppose that for some $n \in \mathbb{N}$,

$$\varphi\left(\bigcup_{i=1}^n B_i\right) - \varphi\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \varphi(B_i) - \varphi(A_i). \tag{5.142}$$

Then, by (5.141),

$$\varphi\left(\bigcup_{i=1}^{n+1} B_i\right) - \varphi\left(\bigcup_{i=1}^{n+1} A_i\right) \leq \varphi\left(\bigcup_{i=1}^{n} B_i\right) - \varphi\left(\bigcup_{i=1}^{n} A_i\right) + \varphi(B_{n+1}) - \varphi(A_{n+1}) \\ \leq \sum_{i=1}^{n+1} \varphi(B_i) - \varphi(A_i). \tag{5.143}$$

Therefore, by induction, (5.142) holds in fact for all $n \in \mathbb{N}$.

Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of arbitrary subsets of S and set $A := \bigcup_{n\in\mathbb{N}} A_n$. Since $A_n \subset A$ for all $n \in \mathbb{N}$ we have $\varphi^*(A_n) \leq \varphi^*(A)$ and taking the supremum on the right hand side we obtain $\varphi^*(A) \geq \sup_{n\in\mathbb{N}} \varphi^*(A_n)$. It therefore remains to show that

$$\varphi^*(A) \leq \sup_{n\in\mathbb{N}} \varphi^*(A_n). \tag{5.144}$$

Without loss of generality, we can assume that the right-hand side of (5.144) is finite. By definition of φ^* we can find for each $\varepsilon > 0$ and $n \in \mathbb{N}$ a $B_n \in \mathcal{T}$ such that $A_n \subset B_n$ and

$$\varphi^*(A_n) \leq \varphi(B_n) \leq \varphi^*(A_n) + \frac{\varepsilon}{2^n}. \tag{5.145}$$

By assumption, the limit $\lim_{n\to\infty} \varphi^*(A_n)$ exists and by (5.145) we obtain

$$\lim_{n\to\infty} \varphi^*(A_n) = \lim_{n\to\infty} \varphi(B_n). \tag{5.146}$$

Now choose $k, n \in \mathbb{N}$ with $k < n$. Then, $A_k \subset B_k \cap B_n$ and we can extend the inequality (5.145) to

$$\varphi^*(A_k) \leq \varphi(B_k \cap B_n) \leq \varphi(B_k) \leq \varphi^*(A_k) + \frac{\varepsilon}{2^k}. \tag{5.147}$$

Therefore, $\varphi(B_k) - \varphi(B_k \cap B_n) \leq \varphi^*(A_k) + \varepsilon 2^{-k} - \varphi^*(A_k) = \varepsilon 2^{-k}$ and consequently

$$\sum_{k=1}^{n} \varphi(B_k) - \Phi(B_k \cap B_n) \leq \varepsilon \sum_{k=1}^{n} 2^{-k} < \varepsilon. \tag{5.148}$$

We can now apply (5.142) to obtain

$$\varphi\left(\bigcup_{k=1}^{n} B_k\right) - \varphi(B_n) = \varphi\left(\bigcup_{k=1}^{n} B_k\right) - \varphi\left(\bigcup_{k=1}^{n} B_n \cap B_k\right) \\ \leq \sum_{k=1}^{n} \varphi(B_k) - \varphi(B_k \cap B_n) < \varepsilon. \tag{5.149}$$

Now set $B := \bigcup_{n\in\mathbb{N}} B_n$. Then B is open and $A \subset B$ and we can conclude with (iii),

$$\varphi^*(A) \le \varphi(B) = \lim_{n\to\infty} \varphi\left(\bigcup_{k=1}^{n} B_n\right). \tag{5.150}$$

Taking the limit in (5.149) and applying (5.146) we find that

$$\lim_{n\to\infty} \varphi\left(\bigcup_{k=1}^{n} B_k\right) \le \lim_{n\to\infty} \varphi(B_n) + \varepsilon \le \lim_{n\to\infty} \varphi^*(A_n) + \varepsilon. \tag{5.151}$$

Finally, plugging this estimate into (5.150), we arrive at the desired inequality as $\varepsilon \to 0$.

It remains to show that φ^* satisfies property (iii) of Definition 5.42. Denote by $(K_n)_{n\in\mathbb{N}}$ a decreasing sequence of compact subsets of S. Since $\bigcap_{n\in\mathbb{N}} K_n \subset K_n$, we have $\varphi^*\left(\bigcap_{n\in\mathbb{N}} K_n\right) \le \varphi^*(K_n)$ for each $n \in \mathbb{N}$ and it remains to show that

$$\inf_{n\in\mathbb{N}} \varphi^*(K_n) \le \varphi^*\left(\bigcap_{n\in\mathbb{N}} K_n\right). \tag{5.152}$$

Again, we assume without loss of generality that the right-hand side is finite. By definition of φ^* we can find for each $\varepsilon > 0$ an open $B \in \mathcal{T}$ with $\varphi(B) < \infty$ such that $\bigcap_{n\in\mathbb{N}} K_n \subset B$ and

$$\varphi(B) \le \varphi^*\left(\bigcap_{n\in\mathbb{N}} K_n\right) + \varepsilon. \tag{5.153}$$

That means that for some $n \in \mathbb{N}$ it must hold that $\bigcap_{k=1}^{n} K_n \subset B$ and therefore $\varphi^*(K_l) \le \varphi(B)$ for all $l \ge n$. Now, since $\varepsilon > 0$ was arbitrary, this concludes the proof. ■

5.4.2 α-capacities

We now aim to define a capacity that is related to the ν-symmetric Feller process X through its Dirichlet form. We will also introduce a class of inverse capacities that we call *resistances* and show how these objects relate to the dynamics of the underlying process.

As usual, let $(S, \mathcal{U}, \nu)$ be a uniform measure space and denote by $\mathcal{T}$ the uniform topology, i.e. the open subsets of S with respect to the topology induced by the uniformity $\mathcal{U}$.

Furthermore, let X be a ν-symmetric Feller process with values in $(S, \mathcal{U})$ and denote by $(\mathcal{E}, \mathcal{D})$ the Dirichlet form associated with X. Recall the definition of the symmetric form $\mathcal{E}_\alpha(f,g) = \mathcal{E}(f,g) + \alpha \langle f,g \rangle$ for $f,g \in \mathcal{D}$ and $\alpha > 0$ and recall that $\mathcal{D}$ equipped with $\mathcal{E}_\alpha$ becomes a Hilbert space. Sometimes we write

$$\|f\|_{\mathcal{E}_\alpha} := \mathcal{E}_\alpha(f,f)^{1/2} \quad \text{and} \quad \|f\|_{\mathcal{E}} := \mathcal{E}(f,f)^{1/2} \tag{5.154}$$

for the (pseudo-)norms on $\mathcal{D}$ induced by $\mathcal{E}_\alpha$ and $\mathcal{E}$, respectively.

As before, we denote by $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha>0}$ the Markovian semigroup and the Markovian resolvent associated with $(\mathcal{E},\mathcal{D})$. The following definition is an analogue to Definition 4.27 for elements of $L^2(S,\nu)$.

Definition 5.45. Let $\alpha > 0$. An element $f \in L^2(S,\nu)$ is *α-excessive* (with respect to $(T_t)_{t\geq 0}$) if $f \geq 0$ ν-a.e. and for all $t > 0$,

$$e^{-\alpha t} T_t f \leq f \quad \nu\text{-a.e.} \tag{5.155}$$

We say that $f \in L^2(S,\nu)$ is α-excessive when the respective semigroup is evident from the context. ♢

Observe that α-excessive functions can be characterized via $\mathcal{E}_\alpha$, too.

Lemma 5.46 ([CF11, Lemma 1.2.4]). *Let $\alpha > 0$ and $f \in \mathcal{D}$. Then, f is α-excessive if and only if*

$$\mathcal{E}_\alpha(f,g) \geq 0 \tag{5.156}$$

for every $g \in \mathcal{D}$ with $g \geq 0$ ν-a.e.

Proof. Suppose $f \in \mathcal{D}$ is α-excessive and $g \geq 0$ ν-a.e. Then, $f - e^{-\alpha t}T_t f \geq 0$ ν-a.e. for all $t > 0$, by definition. Hence,

$$0 \leq \frac{1}{t}\left\langle f - e^{-\alpha t} T_t f, g \right\rangle = \frac{1}{t}\langle f - T_t f, g\rangle + \frac{1 - e^{-\alpha t}}{t}\langle T_t f, g\rangle \tag{5.157}$$

for all $t > 0$. Using the approximation of $\mathcal{E}$ by $\mathcal{E}^{(t)}$ given in (5.50) and the strong continuity of $(T_t)_{t\geq 0}$ we get that the right hand side of (5.157) converges to $\mathcal{E}_\alpha(f,g)$ as $t \to 0$.

For the converse implication suppose $f \in \mathcal{D}$ such that (5.156) holds for all non-negative $g \in \mathcal{D}$. For each $t > 0$ and $\alpha > 0$ we have

$$\begin{aligned} G_\alpha g - e^{-\alpha t} T_t G_\alpha g &= \int_0^\infty e^{-\alpha s} T_s g \,\mathrm{d}s - \int_0^\infty e^{-\alpha(s+t)} T_{s+t} g \,\mathrm{d}s \\ &= \int_0^t e^{-\alpha s} T_s g \,\mathrm{d}s \geq 0 \quad \nu\text{-a.e.} \end{aligned} \tag{5.158}$$

Therefore, by symmetry of T_t,

$$\left\langle f - e^{-\alpha t} T_t f, g \right\rangle = \langle f, g \rangle - \left\langle f, e^{-\alpha t} T_t g \right\rangle = \left\langle f, g - e^{-\alpha t} T_t g \right\rangle. \tag{5.159}$$

Recall that by Remark 5.23 (i) we have $\mathcal{E}_\alpha(f, G_\alpha g) = \langle f, g \rangle$. Hence,

$$\left\langle f - e^{-\alpha t} T_t f, g \right\rangle = \mathcal{E}_\alpha \left(f, G_\alpha g - e^{-\alpha t} T_t G_\alpha g \right) \geq 0, \tag{5.160}$$

by (5.159) and assumption. Consequently, $f - e^{-\alpha t} T_t f \geq 0$ ν-a.e. and f is α-excessive. ■

For each $A \in \mathcal{T}$ we introduce the family

$$\mathcal{L}^A := \{ f \in \mathcal{D} \mid f \geq 1 \ \nu\text{-a.e. on } A \} \tag{5.161}$$

and note that $\mathcal{L}^A$ is a convex and closed subset of the real Hilbert space $(\mathcal{D}, \mathcal{E}_\alpha)$ for each $\alpha > 0$.

Definition 5.47 (α-capacity)**.** For $A \in \mathcal{T}$ and $\alpha > 0$ the *α-capacity* of A is given as

$$\operatorname{Cap}_\alpha(A) := \inf \left\{ \mathcal{E}_\alpha(f, f) \;\middle|\; f \in \mathcal{L}^A \right\}, \tag{5.162}$$

where $\inf \emptyset := \infty$, by convention. For arbitrary subsets $B \subset S$ we define

$$\operatorname{Cap}_\alpha(B) := \inf \left\{ \operatorname{Cap}_\alpha(A) \;\middle|\; A \in \mathcal{T}, A \supset B \right\}. \tag{5.163}$$

For convenience, we write

$$\mathfrak{C}_\alpha := \left\{ A \subset S \;\middle|\; \operatorname{Cap}_\alpha(A) < \infty \right\} \tag{5.164}$$

for the family of subsets of S with finite α-capacity. ◇

Proposition 5.48. *Let $\alpha > 0$ and $A \in \mathfrak{C}_\alpha \cap \mathcal{T}$ be an open set with finite α-capacity. Then there exists a unique element $h_A^\alpha \in \mathcal{L}^A$ such that*

$$\mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha) = \operatorname{Cap}_\alpha(A). \tag{5.165}$$

Furthermore, h_A^α has the following properties.

(i) $0 \leq h_A^\alpha \leq 1$ *ν-almost everywhere on S and $h_A^\alpha = 1$ ν-almost everywhere on A.*

(ii) *h_A^α is α-excessive.*

(iii) *For every $f \in \mathcal{D}$ with $f = 0$ ν-a.e. on A it holds that*

$$\mathcal{E}_\alpha(h_A^\alpha, f) = 0. \tag{5.166}$$

(iv) For each $f \in \mathcal{D}$ with $f = 1$ ν-a.e. on A it holds that

$$\mathcal{E}_\alpha(h_A^\alpha, f) = \mathrm{Cap}_\alpha(A). \tag{5.167}$$

(v) If $B \in \mathfrak{C}_\alpha \cap \mathcal{T}$ is another open set with finite α-capacity such that $A \subset B$ then, $h_A^\alpha \leq h_B^\alpha$ ν-a.e. and

$$\mathrm{Cap}_\alpha(A) = \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha) \leq \mathcal{E}_\alpha(h_B^\alpha, h_B^\alpha) = \mathrm{Cap}_\alpha(B). \tag{5.168}$$

Proof. For each $A \in \mathfrak{C}_\alpha \cap \mathcal{T}$, the set $\mathcal{L}^A$ is a convex and closed subset of the Hilbert space $(\mathcal{D}, \mathcal{E}_\alpha)$. Therefore, the variational problem in (5.162) has a unique solution.

The first assertion (i) follows directly from the Markov property of $\mathcal{E}$, the fact that $f^+ \wedge 1 \in \mathcal{L}^A$ for all $f \in \mathcal{L}^A$ and $\|f^+ \wedge 1\|_2 \leq \|f\|_2$.

We use Lemma 5.46 to show (ii). Suppose $f \in \mathcal{D}$ is non-negative. Then for all $\varepsilon > 0$, $h_A^\alpha + \varepsilon f \in \mathcal{L}^A$. Without loss of generality we can assume that $\mathcal{E}_\alpha(f, f) > 0$ or, equivalently, $f \neq 0$. Hence,

$$0 \leq \mathcal{E}_\alpha(h_A^\alpha + \varepsilon f, h_A^\alpha + \varepsilon f) - \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha) = \varepsilon\left(2\mathcal{E}_\alpha(h_A^\alpha, f) + \varepsilon\mathcal{E}_\alpha(f, f)\right) \tag{5.169}$$

and therefore

$$\frac{\varepsilon}{2}\mathcal{E}_\alpha(f, f) \geq -\mathcal{E}_\alpha(h_A^\alpha, f) \tag{5.170}$$

which yields the desired inequality $\mathcal{E}(h_A^\alpha, f) \geq 0$ since $\varepsilon > 0$ was arbitrary.

Let $f \in \mathcal{D}$ be such that $f \geq 0$ ν-a.e. on S and $f = 0$ ν-a.e. on A. Then, for each $\varepsilon > 0$, we have $h_A^\alpha - \varepsilon f \in \mathcal{L}^A$. with the same argument as above we have

$$\frac{\varepsilon}{2}\mathcal{E}_\alpha(f, f) \geq \mathcal{E}_\alpha(h_A^\alpha, f). \tag{5.171}$$

Since $\varepsilon > 0$ was arbitrary and h_A^α is α-excessive by (ii) we obtain $\mathcal{E}_\alpha(h_A^\alpha, f) = 0$. The statement (iii) then follows if we consider the positive and negative part of f separately.

Property (iv) is a direct consequence of (iii) and the last property (v) follows from the fact that $\mathcal{L}^B \subset \mathcal{L}^A$ whenever $A \subset B$. ∎

The minimizer h_A^α is sometimes referred to as the *α-order equilibrium potential of A* (cf. [CF11, p.78]).

Let $A \subset S$ and note that $\mathrm{Cap}_\beta(A) > 0$ for some $\beta > 0$ implies $\mathrm{Cap}_\alpha(A) > 0$ for all $\alpha > 0$. It therefore suffices to consider only a particular $\alpha > 0$ e.g. $\alpha = 1$ when we talk about sets with capacity 0. We introduce some potential theoretic notions.

Definition 5.49. Let X be a ν-symmetric Feller process with values in S_ϑ and $(\mathcal{E}, \mathcal{D})$ the Dirichlet form associated with X. A set $A \subset S$ is said to be *$\mathcal{E}$-polar* or *polar for X*, if $\mathrm{Cap}_1(A) = 0$. A property that holds everywhere outside a polar set is said to hold *$\mathcal{E}$-quasi everywhere* and a increasing sequence $\{F_n\}_{n\geq 1}$ of closed subsets of S is called a *$\mathcal{E}$-nest* if $F_n \uparrow S$ and $\mathrm{Cap}_1(S \setminus F_k) \to 0$ as $n \to \infty$. For the sake of readability, we drop the $\mathcal{E}$ from the terminology when it is clear from the context which Dirichlet form we are referring to. ◇

A set $A \subset S$ which is polar in the sense of the above definition can be considered *small*. There are various other notions of smallness of sets which are intrinsically related. For example, from a measure-theoretic viewpoint we consider (Borel) sets as small if they have measure 0. It is an immediate consequence of the definition of the α-capacity that every open set $A \in \mathcal{B}$ with $\nu(A)$ has capacity zero. This follows simply from the fact that the function $0 \in L^2(S, \nu)$ can take any value on the nullset A. Sometimes, a set $A \subset S$ is also called X-polar (with respect to a process X) if it is contained a nearly Borel measurable set $B \in \mathcal{B}^n$ (see Definition 4.50) with $\mathbb{P}_x(\tau_B < \infty) = 0$ for all $x \in S$ (cf. [CF11, Definition A.2.6]). On the other hand, a set $A \subset S$ is called *thin* if it has no regular points (see Definition 4.55), i.e. if $\mathbb{P}_x(\tau_A = 0) = 0$ for all $x \in A$. A set that is contained in a countable union of thin sets is called *semipolar* (cf. [CF11, Definition A.2.6]). Some relations between these and further notions of smallness of sets are presented in the diagram in [FOT11, p. 158]. One important equivalence is the following.

Proposition 5.50. *Let X be a ν-symmetric Feller process with values in S_ϑ and $(\mathcal{E}, \mathcal{D})$ the Dirichlet form associated with X. A set $A \subset S$ is $\mathcal{E}$-polar if and only if it is ν-polar, i.e. A is contained in a nearly Borel measurable set $B \subset \mathcal{B}^n$ such that*

$$\mathbb{P}_\nu(\tau_B < \infty) = 0. \tag{5.172}$$

Proof. See [CF11, Theorem 3.1.3]. ■

Before we can show that α-capacities are capacities in the sense of Choquet we need an important characterization of the minimizer h_A^α of the variational problem in (5.162).

Theorem 5.51 (Characterization of minimizers)**.** *Let $\alpha > 0$ and $A \in \mathfrak{C}_\alpha$ be an open set with finite α-capacity and $h_A^\alpha \in \mathcal{L}^A$ the α-equilibrium potential of A.*

(i) $h_{A,\alpha}$ is the unique α-excessive element of $\mathcal{L}^A$ such that $h_A^\alpha = 1$ ν-almost everywhere on A.

(ii) h_A^α *is the unique element of* $\mathcal{L}^A$ *with*

$$\mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha) \le \mathcal{E}_\alpha(f, h_A^\alpha) \tag{5.173}$$

for all $f \in \mathcal{L}^A$.

(iii) h_A^α *is the minimal* α*-excessive element of* $\mathcal{L}^A$ *in the sense that for all* α*-excessive* $g \in \mathcal{L}^A$ *it holds that* $h_A^\alpha \le g$ ν*-a.e.*

Proof. By Proposition 5.48 we know that h_A^α is α-excessive and $= 1$ ν-a.e. on A. Let $h \in \mathcal{L}^A$ be another α-excessive function with $h = 1$ ν-a.e. on A. Then, $h_A^\alpha - h = 0$ ν-a.e. on A. Inspecting the proof of (iii) of Proposition 5.48 we realize that we have only used the α-excessivity of h_A^α. Therefore, the same property holds for h. Hence,

$$\mathcal{E}_\alpha(h_A^\alpha - h, h_A^\alpha - h) = 0, \tag{5.174}$$

which implies that $h = h_A^\alpha$ ν-a.e.

We turn to the proof of (iii). Suppose $h \in \mathcal{L}^A$ is α-excessive. As an immediate consequence of the definition of α-excessivity we obtain that $h_A^\alpha \wedge h$ is also α-excessive. Furthermore, $h_A^\alpha \wedge h = 1$ ν-a.e. on A and we can conclude from (i) that $h_A^\alpha \wedge h = h_A^\alpha$ which means $h_A^\alpha \le h$ ν-a.e.

It remains to proof the characterization (iii). Note that for every $f \in \mathcal{L}^A$ and $0 < \varepsilon < 1$ we have that $h_A^\alpha + \varepsilon(f - h_A^\alpha) \in \mathcal{L}^A$. Hence,

$$\begin{aligned} 0 &\le \mathcal{E}_\alpha\left(h_A^\alpha + \varepsilon(f - h_A^\alpha), h_A^\alpha + \varepsilon(f - h_A^\alpha)\right) - \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha) \\ &= \varepsilon\left(2\mathcal{E}_\alpha(h_A^\alpha, f - h_A^\alpha) + \varepsilon\mathcal{E}_\alpha(f - h_A^\alpha, f - h_A^\alpha)\right). \end{aligned} \tag{5.175}$$

And consequently

$$\mathcal{E}_\alpha(h_A^\alpha, f - h_A^\alpha) \ge -\frac{\varepsilon}{2}\mathcal{E}_\alpha(f - h_A^\alpha, f - h_A^\alpha), \tag{5.176}$$

letting $\varepsilon \to 0$ we have $\mathcal{E}_\alpha(h_A^\alpha, f - h_A^\alpha) \ge 0$. In other words, h_A^α satisfies (5.173). Now assume that $h \in \mathcal{L}^A$ is another function that satisfies (5.173). Then we can plug h_A^α into the inequality and obtain

$$\mathcal{E}_\alpha(h, h_A^\alpha - h) \ge 0. \tag{5.177}$$

Finally, we can conclude

$$\mathcal{E}_\alpha(h - h_A^\alpha, h - h_A^\alpha) = -\left(\mathcal{E}_\alpha(h, h_A^\alpha - h) + \mathcal{E}_\alpha(h_A^\alpha, h - h_A^\alpha)\right) \le 0 \tag{5.178}$$

and therefore $h = h_A^\alpha$ ν-a.e.

■

Proposition 5.52. *For each $\alpha \geq 0$ the α-capacity is a Choquet capacity.*

Proof. Fix $\alpha > 0$. It suffices to show that Cap_α satisfies the properties (i) to (iii) of Proposition 5.44 for open sets. We have already shown that Cap_α satisfies (i) in Proposition 5.48 (v).

We start by showing (ii). To that end let $A, B \in \mathfrak{C}_\alpha \cap \mathcal{T}$ be open sets with finite α-capacity. Recall that by Lemma 5.26, $\mathcal{E}(f^+, f^-) \leq 0$ for all $f \in \mathcal{D}$. This directly implies that $\mathcal{E}_\alpha(|f|, |f|) \leq \mathcal{E}_\alpha(f, f)$ for all $\alpha > 0$ and $f \in \mathcal{D}$. Therefore,

$$\begin{aligned} \mathrm{Cap}_\alpha(A \cup B) + \mathrm{Cap}_\alpha(A \cap B) &\leq \mathcal{E}_\alpha(h_A^\alpha \vee h_B^\alpha, h_A^\alpha \vee h_B^\alpha) + \mathcal{E}_\alpha(h_A^\alpha \wedge h_B^\alpha, h_A^\alpha \wedge h_B^\alpha) \\ &= \frac{1}{2}\Big(\mathcal{E}_\alpha(h_A^\alpha + h_B^\alpha, h_A^\alpha + h_B^\alpha) + \mathcal{E}_\alpha(|h_A^\alpha - h_B^\alpha|, |h_A^\alpha - h_B^\alpha|)\Big) \\ &\leq \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha) + \mathcal{E}_\alpha(h_B^\alpha, h_B^\alpha) = \mathrm{Cap}_\alpha(A) + \mathrm{Cap}_\alpha(B) \end{aligned} \tag{5.179}$$

Now let $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{C}_\alpha \cap \mathcal{T}$ be an increasing sequence of open sets with finite α-capacity. In order to show (iii) we can assume without loss of generality that $\sup_{n\in\mathbb{N}} \mathrm{Cap}_\alpha(A_n) < \infty$. Let $k, n \in \mathbb{N}$ with $k < n$. Then,

$$\mathcal{E}_\alpha(h_{A_n}^\alpha - h_{A_k}^\alpha, h_{A_n}^\alpha - h_{A_k}^\alpha) = \mathrm{Cap}_\alpha(A_n) + \mathrm{Cap}_\alpha(A_k) - 2\mathcal{E}_\alpha(h_{A_n}^\alpha, h_{A_k}^\alpha). \tag{5.180}$$

Note that $h_{A_n}^\alpha = 1$ ν-a.e. on $A_k \subset A_k$. Therefore, by (iv) of Proposition 5.48,

$$\mathcal{E}_\alpha(h_{A_n}^\alpha - h_{A_k}^\alpha, h_{A_n}^\alpha - h_{A_k}^\alpha) = \mathrm{Cap}_\alpha(A_n) - \mathrm{Cap}_\alpha(A_k). \tag{5.181}$$

Hence, $\left(h_{A_n}^\alpha\right)_{n\in\mathbb{N}}$ is a $\mathcal{E}_\alpha$-Cauchy sequence. By completeness of the Hilbert space $(\mathcal{D}, \mathcal{E}_\alpha)$, there exists a $h \in \mathcal{D}$ such that $\lim_{n\to\infty} h_{A_n}^\alpha = \alpha$ with respect to $\mathcal{E}_\alpha$. Evidently, $h = 1$ ν-a.e. on $A := \bigcup_{n\in\mathbb{N}} A_n$. Then we have for all $f \in \mathcal{D}$ with $f \geq 0$ ν-a.e.,

$$\mathcal{E}_\alpha(h, f) = \lim_{n\to\infty} \mathcal{E}_\alpha(h_{A_n}^\alpha, f) \geq 0, \tag{5.182}$$

by Proposition 5.48 (ii). Therefore, h is α-excessive by Lemma 5.46 and we can apply Theorem 5.51 (i) to conclude that

$$\sup_{n\in\mathbb{N}} \mathrm{Cap}_\alpha(A_n) = \lim_{n\to\infty} \mathcal{E}_\alpha(h_{A_n}^\alpha, h_{A_n}^\alpha) = \mathcal{E}_\alpha(h, h) = \mathrm{Cap}_\alpha(A). \tag{5.183}$$

■

We make the following observation about continuous functions.

Lemma 5.53. *Let $\alpha > 0$ and $f \in \mathcal{D} \cap C(S)$ be a continuous representative of an element of $\mathcal{D}$. Then,*

$$\mathrm{Cap}_\alpha(\{\, x \in S \mid |f(x)| > \lambda \,\}) \leq \lambda^{-2}\mathcal{E}_\alpha(f, f), \tag{5.184}$$

for all $\lambda > 0$.

Proof. By continuity of f, the set $A := (\{ x \in S \mid |f(x)| > \lambda \})$ is open. Furthermore, $\lambda^{-1}|f| \in \mathcal{L}^A$ and hence

$$\mathrm{Cap}_\alpha(A) \le \mathcal{E}_\alpha(\lambda^{-1}|f|, \lambda^{-1}|f|) \le \lambda^{-2}\mathcal{E}_\alpha(f, f), \tag{5.185}$$

where the last inequality is due to Lemma 5.26. ■

Let X be a ν-symmetric Feller process with values in $(S, \mathcal{U})$ and Dirichlet form $(\mathcal{E}, \mathcal{D})$. Recall (e.g. from (4.109)) that we write

$$\tau_A := \inf\{ t > 0 \mid X_t \in A \}. \tag{5.186}$$

for the first hitting time of a Borel set $A \in \mathcal{B}$. As a first result that relates the potential theoretic concepts we have developed so far to the process X we have the following.

Proposition 5.54 ([CF11, Lemma 3.1.1])**.** *Let* $\alpha > 0$ *and* $A \in \mathfrak{C}_\alpha \cap \mathcal{T}$ *be an open set with finite* α*-capacity. Define the function* $p_A^\alpha : S \to \mathbb{R}^+$ *by*

$$p_A^\alpha(x) = \mathbb{E}_x\left[e^{-\alpha\tau_A}\right] \quad x \in S. \tag{5.187}$$

Then, $p_A^\alpha = h_A^\alpha$ ν*-a.e.*

Proof. As usual we denote the L^2-semigroup associated with $(\mathcal{E}, \mathcal{D})$ by $(T_t)_{t\ge 0}$ and the Feller semigroup associated with the process X by $(P_t)_{t\ge 0}$. By Proposition 5.29, we have that for all $f \in \mathcal{B}_b(S) \cap L^2(S, \nu)$ and $t > 0$, $T_t f = P_t f$ ν-a.e. Moreover we have that p_A^α is α-excessive with respect to $(P_t)_{t\ge 0}$ by Proposition 4.59 and, as discussed above in Proposition 5.48, h_A^α is α-excessive with respect to $(T_t)_{t\ge 0}$. By definition of p_A^α, we have $p_A^\alpha(x) = 1$ for all $x \in A$. We can therefore apply (i) of Theorem 5.51 to prove the claim once we have shown that $p_A^\alpha \in \mathcal{D}$ because then $p_A^\alpha \in L^2(S, \nu)$ and therefore, p_A^α is also α-excessive with respect to $(T_t)_{t\ge 0}$.

By application of the Cauchy-Schwarz inequality to the scalar product $\mathcal{E}^{(t)}$ we can show that an α-excessive function $f \in L^2(S, \nu)$ is an element of $\mathcal{D}$ if there exists a $g \in \mathcal{D}$ with $f \le g$ ν-a.e. (cf. [CF11, Lemma 1.2.3]). It therefore suffices to show that

$$p_A^\alpha \le h_A^\alpha, \qquad \nu\text{-a.e.} \tag{5.188}$$

because then $p_A^\alpha \in L^2(S, \nu)$ and consequently in the domain of $\mathcal{E}$ due to the aforementioned.

In order to show (5.188) fix a Borel measurable representative $\tilde{h} \in \mathcal{B}_b^+(S)$ of the equivalence class $h_A^\alpha \in L^2(S, \nu)$ such that $\tilde{h}(x) = 1$ for all $x \in A$. Clearly, $\tilde{h}$ is α-

excessive with respect to $(P_t)_{t\geq 0}$ and by Proposition 4.29 the real-valued stochastic process

$$(Y_t)_{t\geq 0} := \left(e^{-\alpha t}\tilde{h}(X_t)\right)_{t\geq 0} \tag{5.189}$$

is a $\mathbb{P}_\mu$-supermartingale with respect to the canonical filtration $(\mathcal{A}_t)_{t\geq 0}$ for any initial distribution $\mu \in \mathcal{M}_1(S)$. Furthermore, Y_t is bounded by 1 for each $t \geq 0$ which implies uniform integrability of Y.

Consider the following construction. Let $\Gamma \subset (0, \infty)$ be a finite set with $a := \min \Gamma$ and $b := \max \Gamma$ and write

$$\tau(\Gamma, A) := \min\{\, t \in \Gamma \mid X_t \in A \,\}, \tag{5.190}$$

and set $\tau(\Gamma, A) = b$ if $\{\, t \in \Gamma \mid X_t \in A \,\} = \emptyset$. Clearly, $\tau(\Gamma, A)$ is a stopping time and $Y_{\tau(\Gamma,A)} = e^{-\alpha\tau(\Gamma,A)}$ on the event $\{\tau(\Gamma, A) < b\}$.

Let $g \in \mathcal{B}^+(S)$ be a non negative Borel measurable function with $\int_S g \,\mathrm{d}\nu = 1$ and set $\mu := g \cdot \nu$. Then,

$$\mathbb{E}_\mu\left[e^{-\alpha\tau(\Gamma,A)} \,\middle|\, \tau(\Gamma, A) < b\right] \leq \mathbb{E}_\mu\left[Y_{\tau(\Gamma,A)}\right]. \tag{5.191}$$

By uniform integrability we can apply the optional sampling theorem (cf. [Kle14, Theorem 10.21]) to obtain

$$\mathbb{E}_\mu\left[e^{-\alpha\tau(\Gamma,A)} \,\middle|\, \tau(\Gamma, A) < b\right] \leq \mathbb{E}_\mu\left[Y_a\right]. \tag{5.192}$$

Now fix $b > 0$ and choose for each $n \in \mathbb{N}$ a finite set $\Gamma_n \subset (0, b) \cap \mathbb{Q}$ such that $\Gamma_n \subset \Gamma_{n+1}$ and $\bigcup_{n\in\mathbb{N}} \Gamma_n = (0, b) \cap \mathbb{Q}$. As $n \to \infty$ the right hand side of (5.192) converges to

$$\mathbb{E}_\mu\left[Y_0\right] = \int_S \tilde{h} \,\mathrm{d}\mu = \int_S g\tilde{h} \,\mathrm{d}\nu = \left\langle g, \tilde{h}\right\rangle. \tag{5.193}$$

When we now let $b \to \infty$, the left hand side of (5.192) converges to

$$\mathbb{E}_\mu\left[e^{-\alpha\tau_A}\right] = \int_S p_A^\alpha \,\mathrm{d}\mu = \int_S g p_A^\alpha \,\mathrm{d}\nu = \left\langle g, p_A^\alpha\right\rangle. \tag{5.194}$$

We have thus shown $\left\langle g, p_A^\alpha\right\rangle \leq \left\langle g, \tilde{h}\right\rangle$ for all $g \in \mathcal{B}^+(S)$ with $\int_S g \,\mathrm{d}\nu = 1$ and hence $p_A^\alpha \leq \tilde{h}$ ν-a.e. which concludes the proof. ■

5.4.3 0-capacities

In the last section we have made use of the fact that $\mathcal{E}_\alpha$ turns $\mathcal{D}$ into a Hilbert space which ensured the existence of a unique minimizer for the variational problem that defines the α-capacity (5.162).

For the remainder of this section we assume that $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form.

Now suppose $(\mathcal{E}, \mathcal{D})$ is a transient Dirichlet form on $L^2(S, \nu)$ and recall that the extended Dirichlet space as defined in Definition 5.33 $\mathcal{D}_e$ then becomes a Hilbert space when equipped with the inner product $\mathcal{E}$ by Theorem 5.38. Analogously to the definition of the α-capacity in the previous section we can therefore introduce the notion of (0-)capacities for transient Dirichlet forms. To that end we write

$$\mathcal{L}_e^A := \{ f \in \mathcal{D}_e \mid f \geq 1 \ \nu\text{-a.e. on } A \}. \tag{5.195}$$

Definition 5.55 (0-Capacity)**.** Let $(\mathcal{E}, \mathcal{D})$ be a transient Dirichlet form. For open $A \in \mathcal{T}$ we define the (0-)capacity[3] as

$$\mathrm{Cap}(A) := \inf \left\{ \mathcal{E}(f, f) \,\middle|\, f \in \mathcal{L}_e^A \right\} \tag{5.196}$$

if $\mathcal{F}^A \neq \emptyset$ and $\mathrm{Cap}(A) := \infty$ otherwise. For arbitrary $B \subset S$ we define

$$\mathrm{Cap}(B) := \inf \left\{ \mathrm{Cap}(A) \,\middle|\, A \in \mathcal{T}, B \subset A \right\} \tag{5.197}$$

and denote by $\mathfrak{C} := \left\{ A \subset S \,\middle|\, \mathrm{Cap}(A) < \infty \right\}$ the subsets of S with finite (0-)capacity. ♢

By the same arguments laid out in the proof of Proposition 5.44, we can argue that the capacity defined in Definition 5.55 is a Choquet capacity. Furthermore, it is easy to check that the analogue of Lemma 5.53 remains true for the 0-capacity. In particular, there exists a unique minimizer $h_A \in \mathcal{L}_e^A$ to the variational problem in (5.196) such that

$$\mathrm{Cap}(A) = \mathcal{E}(h_A, h_A). \tag{5.198}$$

Lemma 5.56. *Let $f \in \mathcal{D}_e \cap C(S)$ be a continuous representative of an element of $\mathcal{D}_e$. Then,*

$$\mathrm{Cap}\left(\{ x \in S \mid |f(x)| > \lambda \}\right) \leq \lambda^{-2} \mathcal{E}(f, f), \tag{5.199}$$

for all $\lambda > 0$.

Proof. See the proof of Lemma 5.56. ■

Recall that a set $A \subset S$ has zero α-capacity for some $\alpha > 0$ then $\mathrm{Cap}_\alpha(A) = 0$ for all $\alpha > 0$. Also, note that sets of zero capacity give us a finer notion than sets

[3] We will use the notation Cap without index for the 0-capacity. Note that both our reference texts [CF11; FOT11] use the notation without index to denote the 1-capacity.

of measure zero since $\nu(A)$ implies $\mathrm{Cap}_\alpha(A) = 0$ but not the other way around. In Proposition 5.61 we will see that every set with zero α-capacity also has 0-capacity zero and vice versa. We could hence equivalently reformulate the following definition in terms of the α-capacity.

Definition 5.57 (Quasi continuous functions)**.** We call an extended real-valued function $f\colon S \to \mathbb{R} \cup \{-\infty, \infty\}$ *quasi continuous* if for each $\varepsilon > 0$ there exists an open set $A \in \mathcal{T}$ such that $\mathrm{Cap}(A) < \varepsilon$ and the restriction $f|_{\complement A}$ of f to the closed set $S \setminus A$ is finite and continuous. If $f|_{\complement A}$ is even finite and continuous on the complement of A with respect to the one-point compactification $S_\vartheta \setminus A$, we say that f is *quasi continuous in the restricted sense*. ◇

Proposition 5.58 ([FOT11, Theorem 2.1.3])**.** *Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form, then each $f \in \mathcal{D}_\mathrm{e}$ admits a quasi-continuous modification in the restricted sense which we will denote by $\tilde{f}$.*

Proof. By definition of the extended Dirichlet space $\mathcal{D}_\mathrm{e}$, we have that $\mathcal{D}$ is a dense subset of $\mathcal{D}_\mathrm{e}$. On the other hand, by definition of regularity, Definition 5.9, for each $f \in \mathcal{D}_\mathrm{e}$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\mathrm{e} \cap C_0(S)$ such that $\mathcal{E}_1(f_n - f, f_n - f) \to 0$ as $n \to \infty$. Then, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and we can assume without loss of generality that

$$\mathcal{E}_1(f_{n+1} - f_n, f_{n+1} - f) \leq 2^{-3n}. \tag{5.200}$$

Furthermore, $f_{n+1} - f_n$ is continuous. If we set

$$A_n := \{\, x \in S \;\big|\; |f_{n+1} - f_n| > 2^{-n} \,\}, \tag{5.201}$$

we can apply Lemma 5.56 to obtain

$$\mathrm{Cap}(A_n) \leq 2^{-n}. \tag{5.202}$$

Observe that $A_n \subset A_{n+1}$ and set

$$B_n := \bigcap_{k=n}^{\infty} \complement A_k. \tag{5.203}$$

Then, B_n is closed, $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$ and $\mathrm{Cap}(S \setminus B_n) \to 0$ as $n \to \infty$.[4] Now fix $N \in \mathbb{N}$, then we have for all $k, l > m \geq N$ and all $x \in B_N$ that

$$|f_k(x) - f_l(x)| \leq \sum_{i=N+1}^{\infty} |f_{i+1}(x) - f_i(x)| \leq \sum_{i=N+1}^{\infty} 2^{-i} = 2^{-N}. \tag{5.204}$$

[4] Recall that this means that $(B_n)_{n\in\mathbb{N}}$ is a $\mathcal{E}$-nest from Definition 5.49.

Consequently, for each $k \in \mathbb{N}$, the sequence of functions given by $f_n|_{B_k \cup \{\vartheta\}}$ (where we set $f_n(\vartheta) = 0$) converges uniformly as $n \to \infty$ and we can define

$$\tilde{f}(x) := \lim_{n\to\infty} f_n(x), \quad x \in \bigcup_{n=1}^{\infty} B_n. \tag{5.205}$$

Then, by uniform convergence, $\tilde{f}$ is the desired quasi-continuous representative of f since $\tilde{f} \in C_\infty(B_n)$ for each $n \in \mathbb{N}$ and $f = \tilde{f}$ ν-a.e. ■

Recall that we say that a property holds quasi everywhere (q.e.) if it holds outside a set of capacity zero. For now, we will use the term with respect to the 0-capacity. This ambiguity will be resolved once we prove Proposition 5.61.

Note that the elements of $\mathcal{D}_\mathrm{e}$ are equivalence classes of ν-a.e. identical functions and while $\tilde{f} = f$ ν-a.e. for all $f \in \mathcal{D}_\mathrm{e}$, $\tilde{f}$ itself describes an equivalence class of q.e. identical functions.

We make the following observations about quasi-continuous functions.

Lemma 5.59 ([FOT11, Lemma 2.1.6 & Theorem 2.1.4]). *Let $(\mathcal{E}, \mathcal{D})$ be a transient regular Dirichlet form on $L^2(S, \nu)$.*

(i) For each $f \in \mathcal{D}_\mathrm{e}$ and $\lambda > 0$,

$$\operatorname{Cap}\left(\left\{ x \in S \;\middle|\; \left|\tilde{f}(x)\right| > \lambda \right\}\right) \le \lambda^{-2}\mathcal{E}(f, f). \tag{5.206}$$

(ii) Suppose $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\mathrm{e}$ is a $\mathcal{E}$-Cauchy sequence. Then there exists a $f \in \mathcal{D}_\mathrm{e}$ such that $\lim_{n\to\infty} \tilde{f}_n = \tilde{f}$ q.e. and $f_n \to f$ with respect to $\mathcal{E}$ as $n \to \infty$.

Proof. We start with the first claim (i). Fix $f \in \mathcal{D}_\mathrm{e}$. Similarly to the proof of Proposition 5.58 there exists a sequence $(f_n)_{n\in\mathbb{N}} \in C_0(S) \cap \mathcal{D}_\mathrm{e}$ such that $\lim_{n\to\infty} f_n = f$ with respect to $\mathcal{E}$, by regularity. By assumption, $f = \tilde{f}$ q.e. Therefore, for each $\varepsilon > 0$ there exists an open set $A \in \mathcal{T}$ such that $\operatorname{Cap}(A) < \varepsilon$ and $f_n \to \tilde{f}$ uniformly on $S \setminus A$ as $n \to \infty$. Now let $\lambda > 0$, then we find for each $\delta > 0$ with $\delta < \lambda$ a $n_0 \in \mathbb{N}$ such that

$$\left\{ x \in S \;\middle|\; \left|\tilde{f}(x)\right| > \lambda \right\} \subset \{ x \in S \mid |f_n(x)| > \lambda - \delta \} \cup A \tag{5.207}$$

for all $n > n_0$. Consequently, by Lemma 5.56,

$$\operatorname{Cap}\left(\left\{ x \in S \;\middle|\; \left|\tilde{f}(x)\right| > \lambda \right\}\right) \le \mathcal{E}(f_n, f_n)(\lambda - \delta)^{-2} + \varepsilon. \tag{5.208}$$

The claim then follows when we let $n \to \infty$, $\delta \to 0$ and then $\varepsilon \to 0$.

For the second assertion (ii) let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\mathrm{e}$ be a $\mathcal{E}$-Cauchy sequence. By assumption and Theorem 5.38, $(\mathcal{D}_\mathrm{e}, \mathcal{E})$ is a real Hilbert space and therefore complete.

Consequently, there exists a $f \in \mathcal{D}_e$ with $\lim_{n\to\infty} \mathcal{E}(f_n - f, f_n - f) = 0$ and it remains to show that $\lim_{n\to\infty} \tilde{f}_n = f$ q.e.

Similarly as in the proof of Proposition 5.58 we set

$$A_n := \left\{ x \in S \;\middle|\; \left|\tilde{f}_{n+1} - \tilde{f}_n\right| > 2^{-n} \right\}. \tag{5.209}$$

By passing over to a subsequence, if necessary, we can assume by (i) that

$$\mathrm{Cap}(A_n) \le 2^{-n}. \tag{5.210}$$

By quasi continuity of $\tilde{f}_n$ we can choose for every $n \in \mathbb{N}$ a family $\left(E_k^{(n)}\right)_{k\in\mathbb{N}}$ of closed subsets of S with $E_k^{(n)} \subset E_{k+1}^{(n)}$ and

$$\mathrm{Cap}\left(S \setminus E_k^{(n)}\right) \le \frac{1}{2^n k} \tag{5.211}$$

such that $\tilde{f}_n$ is continuous on $E_k^{(n)}$ for all $k \in \mathbb{N}$. We can therefore set $E_k := \bigcap_{n\in\mathbb{N}} E_k^{(n)}$ to obtain a family of sets with $E_k \subset E_{k+1}$ and

$$\mathrm{Cap}(S \setminus E_k) \le 1/k \tag{5.212}$$

such that $\tilde{f}_n$ is continuous on E_k for each $k, n \in \mathbb{N}$. For $\varepsilon > 0$ we can therefore find open sets $B_1, B_2 \in \mathcal{T}$ with $\mathrm{Cap}(B_1), \mathrm{Cap}(B_2) \le \varepsilon/2$ and a $n_0 \in \mathbb{N}$ such that $A_n \subset B_1$ and $\tilde{f}_n$ is continuous on $S \setminus B_2$ for all $n > n_0$. Then, $\tilde{f}_n$ converges uniformly to $\tilde{f}$ on $B := B_1 \cup B_2$ and we can conclude the proof by letting $\varepsilon \to 0$.

■

For arbitrary $A \subset S$ consider the following family of functions

$$\mathcal{F}^A := \left\{ f \in \mathcal{D}_e \;\middle|\; \tilde{f} \ge 1 \text{ q.e. on } A \right\}. \tag{5.213}$$

Theorem 5.60. *Let $(\mathcal{E}, \mathcal{D})$ be a transient regular Dirichlet form on $L^2(S, \nu)$. Further, let $\emptyset \ne A \subset S$ be an arbitrary subset. Then the following hold.*

(i) The 0-capacity of A is given by the following variational problem

$$\mathrm{Cap}(A) = \inf\left\{ \mathcal{E}(f, f) \;\middle|\; f \in \mathcal{F}^A \right\} \tag{5.214}$$

(ii) Suppose $\mathcal{F}^A$ is non empty. Then there exists a unique minimizer $h_A \in \mathcal{F}^A$ to the variational problem (5.214) and

$$\mathrm{Cap}(A) = \mathcal{E}(h_A, h_A). \tag{5.215}$$

(iii) *The minimizer* h_A *from* (ii) *satisfies* $0 \le h_A \le 1$ ν*-a.e. and* $\tilde{h}_A = 1$ *q.e. on* A.

We call the minimizer h_A of the variational problem in (5.214) the 0-order equilibrium potential of A or simply the *equilibrium potential* of A.

Proof. We start with assertion (ii). Suppose $\mathcal{F}^A \neq \emptyset$. Then, clearly, the set $\mathcal{F}^A$ is convex and by Lemma 5.59 (ii) it is closed. Therefore, there exists a unique element $h_A \in \mathcal{F}^A$ such that

$$\mathcal{E}(h_A, h_A) \le \mathcal{E}(f, f) \tag{5.216}$$

for all $f \in \mathcal{F}^A$. By definition of capacity, there exists for each $\varepsilon > 0$ a $B \in \mathcal{T}$ open such that $A \subset B$ and

$$\mathrm{Cap}(A) > \mathrm{Cap}(B) - \varepsilon. \tag{5.217}$$

Then $\mathcal{L}_e^B \subset \mathcal{F}^A$ since $f \ge 1$ ν-a.e. on B implies $\tilde{f} \ge 1$ q.e. on B (cf. [FOT11, Lemma 2.1.4]). Consequently,

$$\mathrm{Cap}(B) \ge \mathcal{E}(h_A, h_A) \tag{5.218}$$

and hence $\mathrm{Cap}(A) \ge \mathcal{E}(h_A, h_A)$ since $\varepsilon > 0$ was arbitrary. For the reversed inequality fix a quasi-continuous modification $\tilde{h}_A$ of h_A. For each $\varepsilon > 0$ we can choose an open set B_ε such that $\mathrm{Cap}(B_\varepsilon) < \varepsilon$ and $\tilde{h}_A$ is continuous on $S \setminus B_\varepsilon$ and $\tilde{h}_A \ge 1$ for all $x \in A \setminus B_\varepsilon$. For convenience we write h_ε for the minimizer of the variational problem for the capacity of B_ε, i.e. $\mathrm{Cap}(B_\varepsilon) = \mathcal{E}(h_\varepsilon, h_\varepsilon)$. Observe that the set

$$E_\varepsilon := \left\{ x \in S \setminus B_\varepsilon \,\middle|\, \tilde{h}_A(x) > 1 - \varepsilon \right\} \cup B_\varepsilon \tag{5.219}$$

is open and $A \subset E_\varepsilon$. On the other hand, $h_A + h_\varepsilon \ge 1 - \varepsilon$ ν-a.e. on E_ε. Hence,

$$\begin{aligned} \mathrm{Cap}(A) \le \mathrm{Cap}(E_\varepsilon) &\le (1-\varepsilon)^{-1} \mathcal{E}(h_A + h_\varepsilon, h_A + h_\varepsilon) \\ &\le (1-\varepsilon)^{-2} \left(\mathcal{E}(h_A, h_A)^{1/2} + \mathcal{E}(h_\varepsilon, h_\varepsilon)^{1/2} \right)^2 \\ &\le (1-\varepsilon)^{-2} \left(\mathcal{E}(h_A, h_A)^{1/2} + \varepsilon^{1/2} \right)^2, \end{aligned} \tag{5.220}$$

where we have used the triangle inequality for $\mathcal{E}$ from Lemma 5.3 in the second line. Letting $\varepsilon \to 0$, we obtain

$$\mathrm{Cap}(A) \le \mathcal{E}(h_A, h_A), \tag{5.221}$$

therefore verifying (5.215). Furthermore, we obtain (i) as a direct consequence. Assertion (iii) follows from the observation that for each $f \in \mathcal{F}^A$ we have $g := (0 \vee f) \wedge 1 \in \mathcal{F}^A$ and $\tilde{g} = 1$ q.e. on A. ∎

We are now in a position to show the equivalence of the quasi notions with respect to the α-capacity and with respect to the 0-capacity.

Proposition 5.61 ([FOT11, Theorem 2.1.6]). *Let $(\mathcal{E}, \mathcal{D})$ be a transient regular Dirichlet form on $L^2(S, \nu)$. For all $A \subset S$ we have $\mathrm{Cap}(A) = 0$ if and only if $\mathrm{Cap}_1(A) = 0$. Furthermore, a function f is quasi-continuous with respect to the 0-capacity if and only if f is quasi-continuous with respect to the 1-capacity.*

Recall from Definition 5.49 that a set $A \subset S$ is said to be *$\mathcal{E}$-polar*, if $\mathrm{Cap}_1(A) = 0$.

Proof of Proposition 5.61. Recall from Theorem 5.38 that if $(\mathcal{E}, \mathcal{D})$ is transient there exists a function $\psi \in L^1(S, \nu)$ with $\psi > 0$ ν-a.e. called the *reference function* such that

$$\int_S |f|\psi \,\mathrm{d}\nu \leq \mathcal{E}(f, f)^{1/2} \tag{5.222}$$

for all $f \in \mathcal{D}_\mathrm{e}$. Consequently, $\mathrm{Cap}(A) = 0$ implies $\nu(A) = 0$. On the other hand, it is clear from the definition that

$$\mathrm{Cap}(A) \leq \mathrm{Cap}_1(A) \tag{5.223}$$

for all $A \subset S$. Now suppose $(A_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ is a decreasing sequence of relatively compact open subsets of S. We first show that $\lim_{n\to\infty} \mathrm{Cap}(A_n) = 0$ if and only if $\lim_{n\to\infty} \mathrm{Cap}_1(A_n) = 0$. By (5.223) we only need to show the implication

$$\lim_{n\to\infty} \mathrm{Cap}(A_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \mathrm{Cap}_1(A_n) = 0. \tag{5.224}$$

Using (5.222), we obtain that

$$h_{A_n} \to 0 \quad \nu\text{-a.e} \tag{5.225}$$

as $n \to \infty$, where h_{A_n} denotes the minimizer for the variational problem for $\mathrm{Cap}(A_n)$, as usual. By assumption the closure $\overline{A}_1$ is compact and $\mathcal{E}$ is regular. Consequently, there exists a continuous function $h \in \mathcal{D} \cap C_0(S)$ such that $h(x) \geq 1$ for all $x \in A_1$. Define

$$h_n := h_{A_n} \wedge h, \tag{5.226}$$

then $h_n \in L^2(S, \nu) \cap \mathcal{D}_\mathrm{e} = \mathcal{D}$. Hence,

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mathcal{E}_1(h_n, h_n) &\leq \sup_{n\in\mathbb{N}} \mathcal{E}(h_n, h_n) + \langle h, h\rangle \leq \sup_{n\in\mathbb{N}} \mathcal{E}(h, h) + \mathcal{E}(h_{A_n}, h_{A_n}) + \langle h, h\rangle \\ &\leq \sup_{n\in\mathbb{N}} \mathrm{Cap}(A_n) + \mathcal{E}_1(h, h) \leq \mathrm{Cap}(A_1) + \mathcal{E}_1(h, h) < \infty.\end{aligned} \tag{5.227}$$

We can now apply the Banach-Saks Theorem (cf. [CF11, Theorem A.4.1]) to argue that there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that the Cesàro means

$$g_k := \frac{1}{k}\sum_{j=1}^{k} h_{n_j} \tag{5.228}$$

converge with respect to $\mathcal{E}_1$. By (5.225) we can conclude that $h_n \to 0$ ν-a.e. as $n \to \infty$ and consequently $g_k \to 0$ ν-a.e. as $k \to \infty$. Observe that $g_k \in \mathcal{L}^{A_k}$ as $(A_n)_{n\in\mathbb{N}}$ was taken to be decreasing. We arrive therefore at

$$\mathrm{Cap}_1(A_k) \leq \mathcal{E}_1(g_k, g_k) \to 0, \tag{5.229}$$

as $n \to \infty$, showing (5.224).

Now let $A \subset S$ be arbitrary. From (5.223) we immediately obtain $\mathrm{Cap}(A) = 0$ if $\mathrm{Cap}_1(A) = 0$. Now observe that by σ-compactness there exists a decreasing sequence of relatively compact open sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $A \subset \bigcap_{n\in\mathbb{N}}$ and $\mathrm{Cap}_1(A_n) \to 0$ .Then the reverse implication follows from the statement above.

Now let $f\colon S \to \mathbb{R}$ be quasi-continuous with respect to the 1-capacity. Then, for every $\varepsilon > 0$ there exists an open set A such that $\mathrm{Cap}_1(A) < \varepsilon$ and $f|_{\complement A}$ is continuous. Then (5.223) implies $\mathrm{Cap}(A) < \varepsilon$, too. Now suppose f is quasi- continuous with respect to the 0-capacity. Then, by σ-compactness, there exists a increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of S such that $S = \bigcup_{n\in\mathbb{N}} K_n$. For every $n \in \mathbb{N}$ we can therefore find a decreasing sequence $\left(A_k^{(n)}\right)_{k\in\mathbb{N}} \subset \mathcal{T}$ such that $f|_{K_n\setminus A_k^{(n)}}$ is continuous and $\mathrm{Cap}(A_k^{(n)}) \to 0$ as $k \to \infty$. Then we can find for every $\varepsilon > 0$ and $n \in \mathbb{N}$ a $k \in \mathbb{N}$ such that

$$\mathrm{Cap}_1(A_k^{(n)}) \leq \varepsilon 2^{-n}. \tag{5.230}$$

Then, f is continuous on the complement of $A := \bigcup_{n\in\mathbb{N}} A_n$ and $\mathrm{Cap}_1(A) \leq \varepsilon$ and f is quasi-continuous with respect to the 1-capacity, as claimed. ■

Next, we want to characterize the minimizer h_A in a similar manner as in Theorem 5.51. We first need the following definition.

Definition 5.62. An element $f \in L^\infty(S, \nu)$ is called *excessive* (with respect to $(T_t)_{t\geq 0}$) if $f \geq 0$ ν-a.e. and for all $t \geq 0$,

$$T_t f \leq f \quad \nu\text{-a.e.} \tag{5.231}$$

◇

Again, we can characterize excessive functions via the Dirichlet form.

Lemma 5.63. *An element $f \in \mathcal{D}_e$ is excessive if and only if*

$$\mathcal{E}(f, g) \geq 0 \tag{5.232}$$

for every $g \in \mathcal{D}_e$ with $g \geq 0$ ν-a.e.

Proof. The claim follows from Lemma 5.46 by letting $\alpha \to 0$. ∎

Theorem 5.64. *Let $(\mathcal{E}, \mathcal{D})$ be a transient regular Dirichlet form on $L^2(S, \nu)$ and $\emptyset \neq A \subset S$ a non-empty subset of S. Denote by $h_A \in \mathcal{F}^A$ the equilibrium potential of A. Then h_A is the unique element of $\mathcal{D}_e$ such that $\tilde{h}_A = 1$ q.e. on A and*

$$\mathcal{E}(h_A, f) \geq 0, \tag{5.233}$$

for all $f \in \mathcal{D}_e$ with $\tilde{f} \geq 0$ q.e. on A.

Proof. Take $f \in \mathcal{D}_e$ with $\tilde{f} \geq 0$ q.e. on A. Then, for each $\varepsilon > 0$, $\varepsilon f + h_A \in \mathcal{F}^A$ and

$$\mathcal{E}(h_A, h_A) \leq \mathcal{E}(\varepsilon f + h_A, \varepsilon f + h_A). \tag{5.234}$$

Rearranging yields

$$\frac{\varepsilon}{2}\mathcal{E}(f, f) \geq -\mathcal{E}(h_A, f) \tag{5.235}$$

from which we can conclude that $\mathcal{E}(h_A, f) \geq 0$. Now suppose that $g \in \mathcal{F}^A$ is another element of $\mathcal{F}^A$ with $\tilde{g} = 1$ q.e. on A and $\mathcal{E}(f, g) \geq 0$ for all $f \in \mathcal{D}_e$ with $\tilde{f} \geq 0$ q.e. Suppose $h \in \mathcal{F}^A$, then $\widetilde{h - g} \geq 0$ q.e. on A and we obtain

$$\begin{aligned}\mathcal{E}(h, h) &= \mathcal{E}(g + (h - g), g + (h - g)) \\ &= \mathcal{E}(g, g) + 2\mathcal{E}(g, h - g) + \mathcal{E}(h - g, h - g) \geq \mathcal{E}(g, g).\end{aligned} \tag{5.236}$$

If we take $h = h_A$, we can conclude that $g = h_A$ from Theorem 5.60 (ii), which completes the proof.

∎

Corollary 5.65. *In the situation of Theorem 5.64, h_A is also the unique element of $\mathcal{D}_e$ such that $\tilde{h}_A = 1$ q.e. on A and*

$$\mathcal{E}(h_A, f) = 0 \tag{5.237}$$

for all $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on A.

Proof. Suppose $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on A, then $-f$ has the same property and both $\tilde{f}, \widetilde{-f} \geq 0$ q.e. on A. By Theorem 5.64 we obtain

$$\mathcal{E}(h_A, f) \geq 0 \quad \text{and} \quad \mathcal{E}(h_A, -f) = -\mathcal{E}(h_A, f) \geq 0. \tag{5.238}$$

Consequently, $\mathcal{E}(h_A, f) = 0$. For the converse implication suppose $g \in \mathcal{D}_e$ is another element with $\tilde{g} = 1$ q.e. on A satisfying (5.237) for all $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on A. Then, $h_A - g \in \mathcal{D}_e$ and $\widetilde{h_A - g} = 0$ q.e. on A. Therefore,

$$0 = \mathcal{E}(h_A, h_A - g) + \mathcal{E}(g, h_A - g) = \mathcal{E}(h_A, h_A) - \mathcal{E}(g, g) \tag{5.239}$$

which implies $\mathcal{E}(g, g) = \mathrm{Cap}(A)$ and therefore $g = h_A$ by uniqueness, Theorem 5.60 (ii). ■

Remark 5.66. For open sets $A \in \mathcal{T} \cap \mathfrak{C}$ with finite capacity, we have the analogue of Theorem 5.51 also for the 0-capacity where we replace $\mathcal{L}^A$ by $\mathcal{L}_e^A$ and drop the α. The proof is verbatim. ◊

We conclude this section by relating the analytic results developed above to probabilistic properties of a Feller process X which is associated with a transient regular Dirichlet form. The result is an analogue of Proposition 5.54.

Proposition 5.67. *Let X be a ν-symmetric Feller associated with a transient regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(S, \nu)$. For every Borel set $A \in \mathcal{B} \cap \mathfrak{C}$ with finite capacity, the function $x \mapsto p_A(x)$ given by*

$$p_A(x) := \mathbb{P}_x(\tau_A < \infty) \tag{5.240}$$

is a quasi-continuous version of the equilibrium potential h_A of A.

Proof. See [CF11, Corollary 3.4.3]. ■

Remark 5.68. From the start, the theory of Dirichlet forms is a theory of equivalence classes of L^2 functions. As such one would expect that we can only arrive at statements that are true outside a set of measure zero. The potential theory we have condensed on the last pages, however, enables us to make finer-grained statements, in the sense that they are true outside a set of zero capacity.[5] In order to obtain results that hold for *every* starting point of the process X we must therefore assume that every point $x \in S$ has positive capacity. This assumption leads to the *resistance forms* discussed under Section 5.6 ◊

5.5 Resistances

In this section, we introduce the notion of *resistance* between two sets of positive capacity. The resistance $\mathcal{R}(A, B)$ will be defined as the inverse of the 0-capacity of the

[5] see also the remarks in Section 7.3

set $B \subset S$ with respect to the Dirichlet form of the process X^A which is the process X but killed upon hitting the set $A \subset S$. Recall from (4.143) in Section 4.3.1 that X^A is given by

$$X_t^A = \begin{cases} X_t, & t < \tau_A \\ \vartheta, & t \geq \tau_A. \end{cases} \tag{5.241}$$

As usual assume that $(S, \mathcal{U}, \nu)$ is a locally compact uniform measure space and denote by $(S_\vartheta, \mathcal{U}_\vartheta)$ its one-point compactification. Furthermore, let X denote a ν-symmetric Feller process with values in S. We will assume throughout this section that X is associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(S, \nu)$.

5.5.1 The Dirichlet form of the killed process

Let $A \subset \mathcal{B}$ be a nonempty Borel measurable subset of S. in the following we will denote the complement of A in S by D, i.e.

$$D := S \setminus A. \tag{5.242}$$

We can identify the space $L^2(D, \nu)$ with the subspace

$$L^2(D, \nu) = \left\{ f \in L^2(S, \nu) \,\middle|\, f = 0 \ \nu\text{-a.e. on } A \right\} \subset L^2(S, \nu). \tag{5.243}$$

Let X be a ν-symmetric Feller process with values in S_ϑ. Recall from Theorem 4.65 that the killed process X^A is again a $\nu|_D$ symmetric Feller process with values in D_ϑ, where $D = S \setminus A$. Moreover, recall from (4.140) that for nearly Borel measurable $A \in \mathcal{B}^n$ and $\alpha > 0$ the α-hitting distribution H_A^α is given by

$$H_A^\alpha f(x) = \mathbb{E}_x \left[f\left(X_{\tau_A} e^{-\alpha \tau_A} \right) \right] \tag{5.244}$$

for $f \in \mathcal{B} \cap L^2(S, \nu)$, that is for a Borel measurable representative f of an element of $L^2(S, \nu)$. Let

$$\mathcal{F}_A := \left\{ f \in \mathcal{D} \,\middle|\, \tilde{f} = 0 \text{ q.e. on } A \right\}, \tag{5.245}$$

where $\tilde{f}$ denotes the quasi-continuous version of f. Then $\mathcal{F}_A$ is a closed linear subset of the Hilbert space $(\mathcal{D}, \mathcal{E}_\alpha)$ for every $\alpha > 0$. We denote its orthogonal complement (with respect to $\mathcal{E}_\alpha$) by $\mathcal{H}_A^\alpha$, i.e.

$$\mathcal{H}_A^\alpha := \{ g \in \mathcal{D} \mid \mathcal{E}_\alpha(f, g) = 0, \ \forall f \in \mathcal{F}_A \}. \tag{5.246}$$

Finally, denote by $\pi_A^\alpha : \mathcal{D} \to \mathcal{H}_A^\alpha$ the projection onto $\mathcal{H}_A^\alpha$.

Lemma 5.69. *Let $\alpha > 0$ and $f \in \mathcal{D}$ be α-excessive with respect to the process X. For every nearly Borel measurable set $A \subset \mathcal{B}^n$,*

$$\pi_A^\alpha f = H_A^\alpha f. \tag{5.247}$$

Furthermore, for every $f \in \mathcal{D}$, $H_A^\alpha |\tilde{f}| < \infty$ q.e. and $H_A^\alpha \tilde{f}$ is a quasi continuous version of $\pi_A^\alpha f$.

Proof. See [CF11, Lemma 3.2.1 & Theorem 3.2.2]. ■

Recall from (4.149) that the resolvent of the killed process X^A is given by

$$R_\alpha^A f(x) = \mathbb{E}_x \left[\int_0^{\tau_A} e^{-\alpha t} f(X_t)\, \mathrm{d}t \right] \tag{5.248}$$

for $f \in \mathcal{B}_b^+(D)$. Naturally, (5.248) can be extended to $\mathcal{B}_b^+(S)$. From (4.151) recall the Dynkin formula,

$$R_\alpha^A f(x) = R_\alpha f(x) - H_A^\alpha R_\alpha f(x) \tag{5.249}$$

holds for all $f \in \mathcal{B}_b^+(S)$, $x \in S$ and $\alpha > 0$. This equation can also be extended to hold for $f \in \mathcal{B}(S) \cap L^2(S, \nu)$ and q.e. $x \in S$. Clearly, $R_\alpha^A f(x) = 0$ for all $x \in A$ since every point of A is regular for A if A is closed. It can be shown (for a rigorous argument see [CF11, p. 105]) that $R_\alpha^A f \in \mathcal{F}_A$ for all $f \in \mathcal{B}(S) \cap L^2(S, \nu)$ and $\alpha > 0$. From Lemma 5.69 we can then deduce that (5.249) represents the orthogonal decomposition of f into the sum of elements of $\mathcal{F}_A$ and $\mathcal{H}_A^\alpha$ with respect to the scalar product $\mathcal{E}_\alpha$ on $\mathcal{D}$. Furthermore, $R_\alpha^A f$ is quasi continuous and

$$\mathcal{E}_\alpha \left(R_\alpha^A f, g \right) = \int_D f(x) g(x)\, \nu(\mathrm{d}x) \tag{5.250}$$

for every $g \in \mathcal{F}_A$. In the same way, it holds that

$$\int_D f(x) R_\alpha^A g(x)\, \nu(\mathrm{d}x) = \int_D R_\alpha^A f(x) g(x)\, \nu(\mathrm{d}x) \tag{5.251}$$

for all $f, g \in \mathcal{F}_A$ and $\alpha > 0$.

We can now identify the Dirichlet form associated with the killed process X^A.

Theorem 5.70. *Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form on $L^2(S, \nu)$ and $A \subset S$ closed with $\nu(A) > 0$. Then the bilinear form $(\mathcal{E}^D, \mathcal{D}^D)$, where*

$$\mathcal{D}^D = \mathcal{F}_A = \left\{ f \in \mathcal{D} \,\middle|\, \tilde{f} = 0 \text{ q.e. on } A \right\} \tag{5.252}$$

and $\mathcal{E}^D(f,f) = \mathcal{E}(f,f)$ for all $f \in \mathcal{D}^D$ is a regular Dirichlet form on $L^2(D,\nu)$. Furthermore, $(\mathcal{E}^D, \mathcal{D}^D)$ is associated with the ν-symmetric Feller process X^A.

Proof. We begin with the second claim. Recall from Theorem 4.65 that the killed process X^A is again a ν-symmetric Feller process with resolvent $\left(R_\alpha^A\right)_{\alpha>0}$ given by (5.248) Using Lemma 5.30, we can extend $\left(R_\alpha^A\right)_{\alpha>0}$ to a family of operators $\left(G_\alpha^A\right)_{\alpha>0}$ on $L^2(D,\nu)$ which gives rise to a Dirichlet form $(\mathcal{E}^D, \mathcal{D}^D)$ on $L^2(D,\nu)$ by (5.51). It follows from the discussion above that for each $\alpha > 0$,

$$\mathcal{F}_A = \left\{ G_\alpha^A f \;\middle|\; f \in L^2(D,\nu) \right\} \tag{5.253}$$

and (5.250) implies that $(\mathcal{E}^D, \mathcal{D}^D)$ is in fact the Dirichlet form associated with the killed process by virtue of (5.46).

The regularity of $(\mathcal{E}^D, \mathcal{D}^D)$ is due to [CF11, Theorem 3.3.9 (ii)]. ■

Corollary 5.71. *Let X be a ν-symmetric Feller process with values in S_ϑ and $A \in \mathcal{B}$ closed with* $\mathrm{Cap}_1(A) > 0$. *Then the Dirichlet form $(\mathcal{E}^D, \mathcal{D}^D)$ is transient.*

Proof. By Theorem 5.70, $(\mathcal{E}^D, \mathcal{D}^D)$ is associated with the killed process X^A. With Proposition 5.50 we can conclude analogously to Lemma 4.69, that X^A is transient and the claim follows from Proposition 5.41. ■

By virtue of Theorem 5.70 and Corollary 5.71 we can transfer the potential theoretic notions developed in Section 5.4 to the Dirichlet form $(\mathcal{E}^A, \mathcal{D}^A)$. Most notably, we can define the α-capacity Cap_α^A with respect to $(\mathcal{E}^A, \mathcal{D}^A)$ and, in the case where $\mathrm{Cap}_1(A) > 0$, the 0-capacity Cap^A in the same way as before. By Definition 5.33 the extended Dirichlet space with respect to $(\mathcal{E}^A, \mathcal{D}^A)$ is given by

$$\mathcal{D}_e^A = \left\{ f \in L^\infty(D,\nu) \;\middle|\; \exists (f_n)_{n\in\mathbb{N}} \subset \mathcal{D}^A \text{ Cauchy s.t. } \lim_{n\to\infty} f_n = f \text{ in } L^\infty(D,\nu) \right\}. \tag{5.254}$$

Note that, analogously to (5.243), we can identify $L^\infty(D,\nu)$ with the subspace

$$L^\infty(D,\nu) = \{ f \in L^\infty(S,\nu) \mid f = 0 \;\nu\text{-a.e. on } A \} \subset L^\infty(S,\nu). \tag{5.255}$$

Consequently, we can identify $\mathcal{D}_e^A$ with

$$\mathcal{D}_e^A = \mathcal{D}_e \cap L^\infty(D,\nu). \tag{5.256}$$

Recall from Theorem 5.38, that $\mathcal{D}_e^A$ becomes a real Hilbert space equipped with the inner product $\mathcal{E}^A$ if $(\mathcal{E}^A, \mathcal{D}^A)$ is transient.

We conclude this section with some potential theoretic properties of the killed process.

Proposition 5.72. *Let X be a ν-symmetric Feller process with values in S_ϑ and associated Dirichlet form $(\mathcal{E}, \mathcal{D})$. Moreover let $A \subset S$ be closed and denote by $(\mathcal{E}^A, \mathcal{D}^A)$ the Dirichlet form of the killed process X^A.*

(i) If an increasing sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}(S)$ of closed subsets of S is a $\mathcal{E}$-nest, then $(B_k \cap D)$ is a $\mathcal{E}^A$-nest.

(ii) For all $B \subset D$, $\mathrm{Cap}_1^A(B) \geq \mathrm{Cap}_1(B)$.

(iii) Suppose that $(\mathcal{E}, \mathcal{D})$ is transient, then $\mathrm{Cap}^A(B) \geq \mathrm{Cap}(B)$ for all $B \subset D$.

(iv) If $f \in \mathcal{F}_A$, then f is quasi-continuous with respect to $\mathcal{E}^A$ if and only if f is the restriction to D of a quasi-continuous function (with respect to $\mathcal{E}$) on S.

Proof. See [CF11, Theorem 3.3.8] ■

5.5.2 Effective resistance

We now introduce the *effective resistance* as a further potential theoretic notion. The effective resistance has long been recognized as an important tool in the analysis of Markov processes on graphs (see our example in Section 4.5.1). Peter G. Doyle and J. Laurie Snell in [DS84] trace some of the ideas regarding the electrical network interpretation of graphs back to the first half of the last century, in particular to [Kak45] by Shizuo Kakutani. The first application of the effective resistance seems to be found in the work [Nas59] by Crispin Nash-Williams. Despite its potential theoretic nature, the effective resistance is usually not treated in potential analytic texts on Markov processes e.g. [FOT11; CF11; BG68; RY99].

As usual, let $(S, \mathcal{U}, \nu)$ denote a uniform measure space and X a ν-symmetric Feller process. We denote the Dirichlet form associated with X by $(\mathcal{E}, \mathcal{D})$ and assume that it is regular.

Recall the definitions of $\mathcal{F}^A$ and $\mathcal{F}_A$ from (5.213) and (5.245), respectively. For $A \in \mathcal{B}$ closed with $\mathrm{Cap}_1(A) > 0$ and $B \subset D = S \setminus A$ set

$$\mathcal{F}_A^B := \left\{ f \in \mathcal{D}_e^A \;\middle|\; \tilde{f} \geq 1 \text{ q.e. on } B \right\}, \tag{5.257}$$

where $\mathcal{D}_e^A$ denotes the extended Dirichlet space (see Definition 5.33 and (5.254)) associated with the transient Dirichlet form $(\mathcal{E}^A, \mathcal{D}^A)$ (see Corollary 5.71).

Definition 5.73. Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form on $L^2(S, \nu)$. For two closed subsets $A, B \subset S$ with $\mathrm{Cap}_1(A), \mathrm{Cap}_1(B) > 0$ the *(effective) resistance* between A and B is defined as

$$\mathcal{R}(A, B) := \sup \left\{ \mathcal{E}(f, f)^{-1} \,\middle|\, f \in \mathcal{F}_A^B \right\}, \tag{5.258}$$

where we set $\sup \emptyset = 0$, as usual. ◊

We make note of the following properties of the effective resistance.

Proposition 5.74. *Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form on $L^2(S, \nu)$. Suppose $A, B \subset S$ are closed and have positive capacity. Then the effective resistance has the following properties.*

(i) $\mathcal{R}(A, B) \geq 0$.

(ii) *If* $\mathrm{Cap}^A(B) > 0$ *then,*

$$\mathcal{R}(A, B) = \mathrm{Cap}^A(B)^{-1} \tag{5.259}$$

and $\mathcal{R}(A, B) = \infty$ *else.*

(iii) $B \subset E \subset S$ *implies that* $\mathcal{R}(A, E) \leq \mathcal{R}(A, B)$ *and* $A \subset F \subset S$ *implies* $\mathcal{R}(F, B) \leq \mathcal{R}(A, B)$.

(iv) $\mathcal{R}(A, B) < \infty$.

(v) $\mathcal{R}(A, B) > 0$ *if and only if* $\mathrm{Cap}_1(A \cap B) = 0$ *and* $\mathrm{Cap}_1(B) < \infty$.

Proof. The first and the second assertion, (i) and (ii), follow directly from the definition.

We show (iii). Without loss of generality assume $\mathcal{R}(A, E) > 0$. For $B \subset E$ we have $\mathcal{F}_A^E \subset \mathcal{F}_A^B$ and consequently $\mathcal{R}(A, E) \leq \mathcal{R}(A, B)$. The second part follows analogously.

For (iv) suppose that $\mathcal{R}(A, B) > 0$. Then set $\mathcal{F}_A^B$ is non empty and therefore the 0-capacity $\mathrm{Cap}^A(B)$ of B with respect to $(\mathcal{E}^A, \mathcal{D}^A)$ is well defined. Now, $\mathrm{Cap}_1(B) > 0$ implies $\mathrm{Cap}_1^A(B) > 0$, by (ii), and consequently $\mathrm{Cap}^A(B) > 0$ by Proposition 5.61. The claim then follows from (ii).

To verify (v) note that $\mathrm{Cap}_1(A \cap B) > 0$ immediately implies that $\mathcal{F}_A^B = \emptyset$ and therefore $\mathcal{R}(A, B) = 0$. On the other hand, if $\mathrm{Cap}_1(B) = \infty$ that implies that $\mathcal{L}^C = \emptyset$ for all $C \in \mathcal{T}$ open with $B \subset C$. Since $\mathcal{F}_A = \mathcal{D}^A \subset \mathcal{D}$ this implies that there exists no $f \in \mathcal{F}_A$ with $f \geq 1$ ν-a.e. on B and therefore $\mathcal{F}_A^B = \emptyset$.[6] Now suppose $\mathrm{Cap}_1(A \cap B) = 0$ and $\mathrm{Cap}_1(B) < \infty$. First, observe that

$$\mathcal{R}(A, B) = \mathcal{R}(A, B \setminus N), \tag{5.260}$$

[6] This argument can be made more direct by applying, for example, [FOT11, Theorem 2.1.5].

where $N \subset S$ with $\mathrm{Cap}_1(N) = 0$. This is just a simple consequence of the definition of quasi continuity and quasi everywhere. We can therefore assume that $A \cap B = \emptyset$ or, equivalently, that B is a proper subset of $D = S \setminus A$. This, together with $\mathrm{Cap}_1(B) < \infty$, implies that $\mathcal{F}_A^B \neq \emptyset$ and therefore $\mathcal{R}(A, B) > 0$. ■

Remark 5.75. Note that the definition of the effective resistance can easily be extended to include arbitrary sets $B \subset S \setminus A$ since the 0-capacity is defined for such sets by Definition 5.55. Since we want the resistance to be symmetric (see Theorem 5.76) the question arises whether we can also extend the definition to allow arbitrary sets $A \subset S$ in the first argument. Some preliminary results in that direction are collected in Section 7.3.1. ◇

Theorem 5.76. *Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form. Suppose $A, B \subset S$ are closed and $\mathcal{R}(A, B) > 0$. Then the following hold.*

(i) There exists a unique maximizer $g_A^B \in \mathcal{F}_A^B$ to the variational problem (5.258) and

$$\mathcal{R}(A, B) = \mathcal{E}^{-1}(g_A^B, g_A^B). \tag{5.261}$$

(ii) The maximizer g_A^B from (i) satisfies $0 \leq g_A^B \leq 1$ ν-a.e. and $\tilde{g}_A^B = 1$ q.e. on B (and $\tilde{g}_A^B = 0$ q.e. on A, by definition).

(iii) g_A^B is the unique element of $\mathcal{F}_A^B$ with the property

$$\mathcal{E}(g_A^B, f) = 0 \tag{5.262}$$

for all $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on $A \cup B$.

(iv) Suppose that $(\mathcal{E}, \mathcal{D})$ is recurrent and $\mathrm{Cap}_1(A) < \infty$, then the effective resistance is symmetric,

$$\mathcal{R}(A, B) = \mathcal{R}(B, A). \tag{5.263}$$

Proof. We begin in the beginning and start with (i). By assumption $\mathcal{R}(A, B) > 0$ and therefore $\mathcal{F}_A^B \neq \emptyset$. The space $\mathcal{F}_A^B$ is a closed and convex subset of the real Hilbert space $(\mathcal{D}_e^A, \mathcal{E}^A)$. Therefore, there exists a unique minimizer of (5.258) and (5.261) follows from the fact that $\mathcal{E}^A(f, f) = \mathcal{E}(f, f)$ for all $f \in \mathcal{D}_e^A$.

The second claim (ii) follows immediately from the fact that g_A^B is the minimizer for the 0-capacity with respect to the Dirichlet form $(\mathcal{E}^A, \mathcal{D}^A)$ and Theorem 5.60 (ii).

Similarly, Corollary 5.65 states that g_A^B is the unique element of $\mathcal{F}_A^B$ with the property

$$\mathcal{E}^A(g_A^B, f) = 0 \tag{5.264}$$

for all $f \in \mathcal{D}_e^A$ with $\tilde{f} = 0$ q.e. on B. By definition of $\mathcal{D}_e^A$ those $f \in \mathcal{D}_e^A$ are exactly the $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on $A \cup B$. In the same way, g_A^B can be considered an element of $\mathcal{D}_e$ and $\mathcal{E}^A(g_A^B, f) = \mathcal{E}(g_A^B, f)$, again by definition, which implies (iii).

For the last assertion (iv) recall that by Corollary 5.40 recurrence of implies $1 \in \mathcal{D}_e$ and $\mathcal{E}(1, f) = 0$ for all $f \in \mathcal{D}$. Let

$$g := 1 - g_A^B. \tag{5.265}$$

Then, $g \in \mathcal{D}_e$ with $\tilde{g} = 1$ q.e. on A and $\tilde{g} = 0$ q.e. on B. Moreover,

$$\mathcal{E}(g, f) = \mathcal{E}(1 - g_A^B, f) = \mathcal{E}(1, f) - \mathcal{E}(g_A^B, f) = 0 \tag{5.266}$$

for all $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on $A \cup B$. Consequently, by (iii),

$$g = g_B^A \tag{5.267}$$

and therefore

$$\begin{aligned} \mathcal{R}(B, A)^{-1} &= \mathcal{E}(g_B^A, g_B^A) = \mathcal{E}(1 - g_A^B, 1 - g_A^B) \\ &= \mathcal{E}(1, 1) - 2\mathcal{E}(1, g_A^B) + \mathcal{E}(g_A^B, g_A^B) = \mathcal{E}(g_A^B, g_A^B) = \mathcal{R}(A, B)^{-1}, \end{aligned} \tag{5.268}$$

completing the proof ■

In the case where $(\mathcal{E}, \mathcal{D})$ is transient, we generally do not have symmetry of the effective resistance. We can, however, say the following. Suppose $(\mathcal{E}, \mathcal{D})$ is transient and $A, B \subset S$ are closed and $0 < \mathrm{Cap}_1(A)\,\mathrm{Cap}_1(B) < \infty$. Let $h := g_A^B + g_B^A$ where g_A^B and g_B^A are the maximizers from Theorem 5.76. Then, $h \in \mathcal{D}_e$ and $\tilde{h} = 1$ q.e. on $A \cup B$, by Theorem 5.76 (ii). Fix some $f \in \mathcal{D}_e$ with $\tilde{f} = 0$ q.e. on $A \cup B$ and observe that by Theorem 5.76 (iii),

$$\mathcal{E}(h, f) = \mathcal{E}(g_A^B, f) + \mathcal{E}(g_B^A, f) = 0. \tag{5.269}$$

By Theorem 5.60, this implies that $h = h_{A\cup B}$ where $h_{A\cup B}$ is the minimizer of the variational problem for $\mathrm{Cap}(A \cup B)$.

In the same manner as in Proposition 5.54 and Proposition 5.67 we can describe the maximizer of the variational problem for the resistance probabilistically.

Proposition 5.77. *Let X be a ν-symmetric Feller process with values in S_ϑ associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(S, \nu)$. Suppose $A, B \subset S$ are closed and* $0 < \mathrm{Cap}_1(A)\,\mathrm{Cap}_1(B) < \infty$ *and define the function $p_A^B \colon S \to [0, 1]$ as*

$$p_A^B(x) := \mathbb{P}_x(\tau_B < \tau_A,\ \tau_{A\cup B} < \infty). \tag{5.270}$$

Then p_A^B is a quasi-continuous version of g_A^B.

Proof. The claim follows directly from Proposition 5.67. If $(\mathcal{E}, \mathcal{D})$ is recurrent, we have $\tau_{A\cup B} < \infty$ almost surely and $\mathbb{P}_x^A(\tau_B < \infty) = \mathbb{P}_x(\tau_B < \tau_A)$ is a quasi continuous version of g_A^B. Here $\mathbb{P}^A$ denotes the probability with respect to the killed process X^A. If, on the other hand, $(\mathcal{E}, \mathcal{D})$ is transient we have that $\mathbb{P}_x^A(\tau_B < \infty) = \mathbb{P}_x(\tau_B < \tau_A,\ \tau_{A\cup B} < \infty)$, which is a quasi continuous version of g_A^B, again by Proposition 5.67. ■

5.6 Resistance forms

Resistance forms are closely related to Dirichlet forms. Informally speaking, resistance forms are Dirichlet forms for which the effective resistance, as defined in the previous section, between points is finite and therefore induces a metric on S. In particular, these are regular recurrent Dirichlet forms for which singletons have positive capacity. One example of a process associated with a resistance form is the random walk on a graph described in Section 4.5.1.

The concept of resistance forms is deeply rooted in the analysis of stochastic processes on fractals like the Sierpiński Gasket or the Sierpiński Carpet, named after Wracław Sierpiński [Sie16]. In the late 80s Martin T. Barlow, Richard F. Bass and Edwin Perkins described and constructed the Brownian motion on the Sierpiński Gasket in [BP88; BB89]. This research was continued for example by Shigeo Kusuoka and Zhou Yin in [KY92], Jun Kigami in [Kig95] and Volker Metz in [Met97]. The notion of *resistance forms* seems to first occur in [Kig01] and has since gained a lot of attention. Notable works include [Kig03; Kig12], [Kum04] by Takashi Kumagai, [KS05] by Kumagai and Karl-Theodor Sturm and [GT12] by Alexander Grigor'yan and Andras Telcs. More recently, David Croydon obtained results for the convergence of Feller processes associated with resistance forms in [Cro18] and further details were developed by Croydon together with Kumagai and Ben Hambly in [CHK17]. A good introduction to the topic of resistance forms in the context of random walks on graphs can be found in [Kum14]. Most of the results and their proofs presented here about resistance forms can be found in [Kig12]

Definition 5.78 (resistance forms)**.** Let S be a non-empty set. A quadratic form $(\mathcal{E}, \mathcal{F})$ on $\mathbb{R}^S$ is a called a *resistance form* if the following conditions are satisfied.

(i) The domain $\mathcal{F}$ of $\mathcal{E}$ is a linear subspace of $\mathbb{R}^S$ and contains the constant functions $f(x) = c \in \mathbb{R}$. Furthermore, $\mathcal{E}(f, f) = 0$ if and only if $f\colon S \to \mathbb{R}$ is constant.

(ii) Define an equivalence relation $\sim$ on $\mathcal{F}$ by $f \sim g$ if and only $f - g = c$ is constant. Then, the quotient space $\mathcal{F}/\sim$ equipped with the inner product $\mathcal{E}$ is a real Hilbert space.

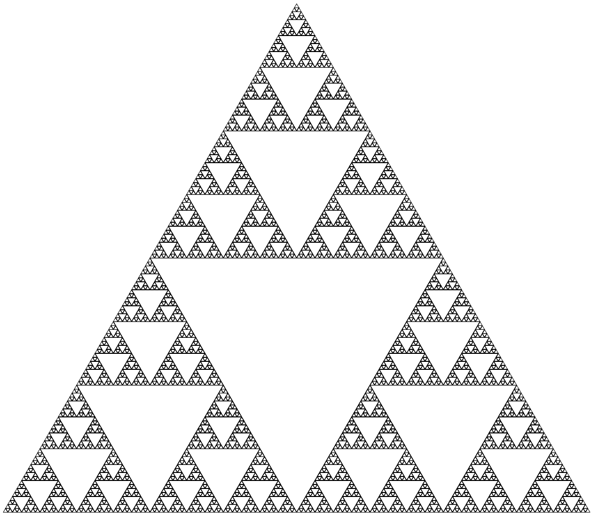

Fig. 5.2.: The Sierpiński Gasket with 5 iterations

(iii) $\mathcal{F}$ separates points in S, i.e. for $x, y \in S$ with $x \neq y$ there exists a $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

(iv) For all $x, y \in S$ it holds that

$$\mathcal{R}(x, y) := \sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} \;\middle|\; f \in \mathcal{F},\ \mathcal{E}(f, f) > 0 \right\} < \infty, \tag{5.271}$$

where $\sup \emptyset = 0$, as usual.

(v) If $f \in \mathcal{F}$ and $g := f^+ \wedge 1$, then $g \in \mathcal{F}$ and $\mathcal{E}(g, g) \leq \mathcal{E}(f, f)$. ◇

In the following, we will indicate the underlying set S by saying that $(\mathcal{E}, \mathcal{F})$ is a resistance form on S.

Note that in the definition of resistance forms, we do not assume any a priori structure on the set S. Instead, the resistance form itself induces a metric on S via (5.271).

Proposition 5.79. *Let $S \neq \emptyset$ and $(\mathcal{E}, \mathcal{F})$ be a resistance form on S. Then the resistance $\mathcal{R}\colon S \to S \to \mathbb{R}$ is a metric on S.*

Proof. By definition, $\mathcal{R}(x, y)$ is non negative and we immediately obtain $\mathcal{R}(x, x) = 0$. Suppose $x \neq y$. Then there exists a $f \in \mathcal{F}$ with $f(x) \neq f(y)$ by Definition 5.78 (iii) and by (i), we have $\mathcal{E}(f, f) > 0$. Consequently, $\mathcal{R}(x, y) > 0$. It remains to show that $\mathcal{R}$ satisfies the triangle inequality. Let $x, y, z \in S$, then

$$\begin{aligned}\mathcal{R}(x, z) &= \sup\left\{ \frac{|f(x) - f(z)|^2}{\mathcal{E}(f, f)} \,\middle|\, f \in \mathcal{F},\ \mathcal{E}(f, f) > 0 \right\} \\ &\leq \sup\left\{ \frac{|f(x) - f(y)|^2 + |f(y) - f(z)|^2}{\mathcal{E}(f, f)} \,\middle|\, f \in \mathcal{F},\ \mathcal{E}(f, f) > 0 \right\} \\ &\leq \mathcal{R}(x, y) + \mathcal{R}(y, z),\end{aligned} \tag{5.272}$$

therefore completing the proof. ■

Fix $x, y \in S$ with $x \neq y$ and choose $f \in \mathcal{F}$ with $f(y) \neq f(x)$. Then

$$\hat{f} := \frac{(f - f(x))}{f(y) - f(x)} \in \mathcal{F} \tag{5.273}$$

with $\hat{f}(x) = 0$ and $\hat{f}(y) = 1$. Moreover,

$$\begin{aligned}\frac{|f(x) - f(y)|}{\mathcal{E}(f, f)} &= \frac{(f(y) - f(x))^2 \left|\hat{f}(x) - f(x) - \hat{f}(y) + f(x)\right|}{(f(y) - f(x))^2 \mathcal{E}(\hat{f} - f(x), \hat{f} - f(x))} \\ &= \frac{\hat{f}(y)}{\mathcal{E}(\hat{f}, \hat{f})} = \mathcal{E}(\hat{f}, \hat{f})^{-1}.\end{aligned} \tag{5.274}$$

We can therefore rewrite the variational principle for the resistance in (5.271) as follows

$$\mathcal{R}(x, y) = (\inf \{ \mathcal{E}(f, f) \mid f \in \mathcal{F},\ \mathcal{E}(f, f) > 0,\ f(x) = 0,\ f(y) = 1 \})^{-1}. \tag{5.275}$$

In the following, we will tacitly assume that the space S is equipped with the resistance metric $\mathcal{R}$ when making topological statements like the next.

Lemma 5.80. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S. Then each $f \in \mathcal{F}$ is uniformly continuous.*

Proof. Let $f \in \mathcal{F}$. By definition of the resistance metric we have for all $x, y \in S$,

$$(f(x) - f(y))^2 \leq \mathcal{R}(x, y)\mathcal{E}(f, f), \tag{5.276}$$

which yields the claim. ■

Next, we want to extend the definition of the resistance to measure the resistance between a point and a set. In the same spirit as before we set for $A \subset S$

$$\mathcal{F}_A := \{ f \in \mathcal{F} \mid f|_A = 0 \}. \tag{5.277}$$

Definition 5.81. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S and $A \subset S$ non empty. For $x \in S$ we define the resistance between x and A as

$$\mathcal{R}(x, A) := \sup \left\{ \mathcal{E}(f, f)^{-1} \,\middle|\, f \in \mathcal{F}_A,\ f(x) \geq 1 \right\}, \tag{5.278}$$

where we set $\sup \emptyset = 0$, as usual. ◇

We will only focus on closed sets $A \subset S$ in the following. The results can, however, be extended to sufficiently regular sets (cf. [Kig12, Chapter 4]).

Theorem 5.82 (Green function)**.** *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S and $A \subset S$ non-empty and closed. Then $(\mathcal{F}_A, \mathcal{E})$ is a Hilbert space and there exists a unique map $g_A \colon S \times S \to \mathbb{R}$ with $g_A^x = g_A(x, \cdot\,) \in \mathcal{F}_A$ for all $x \in S$ and*

$$\mathcal{E}\left(g_A^x, f\right) = f(x) \tag{5.279}$$

for all $f \in \mathcal{F}_A$. Furthermore, $g_A(x, x) \geq 0$ for all $x \in S$ and $g_A(x, x) = 0$ if and only if $x \in B$.

Proof. First we show that $\mathcal{F}_A$ equipped with $\mathcal{E}$ is indeed a Hilbert space. By definition, the only constant function in $\mathcal{F}_A$ is the zero function and completeness follows from the fact that $(\mathcal{F}, \mathcal{E})$ is complete and A is closed. Now let $y \in A$ and $f \in \mathcal{F}_B$. Then,

$$|f(x)|^2 = |f(y) - f(x)|^2 \leq \mathcal{R}(x, y)\mathcal{E}(f, f) \tag{5.280}$$

for every $x \in S$. Consequently, the evaluation map $f \mapsto f(x)$ is a continuous linear functional $\mathcal{F}_A \to \mathbb{R}$. By Riesz' representation theorem [Yos78, Theorem II.6], there exists a unique $g_A^x \in \mathcal{F}_A$ such that

$$\mathcal{E}\left(g_A^x, f\right) = f(x) \tag{5.281}$$

for all $f \in \mathcal{F}_A$. Consequently, $g_A(x, x) = \mathcal{E}\left(g_A^x, g_A^x\right) \geq 0$. If $x \in B$, then $g_A(x, x) = 0$ because $g_A^x \in \mathcal{F}_A$. Conversely, suppose $g_A(x, x) = 0$. Then, $\mathcal{E}\left(g_A^x, g_A^x\right) = 0$ which means, $g_A^x = 0$. As a consequence, we obtain for every $f \in \mathcal{F}_A$,

$$f(x) = \mathcal{E}\left(g_A^x, f\right) = 0. \tag{5.282}$$

Therefore, $x \in A$ and the proof is finished. ∎

We call the map g_A the *Green function* or *Green kernel* associated with the resistance form $(\mathcal{E}, \mathcal{F})$. Equation (5.279) means that g_A is a *reproducing kernel* for the Hilbert space $(\mathcal{F}, \mathcal{E})$. The space $(\mathcal{F}, \mathcal{E})$ is therefore called a *reproducing kernel Hilbert space*.

Proposition 5.83. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S and $A \subset S$ non-empty and closed. For $x \in S$ the unique maximizer of (5.278) is given by $g_A^x / g_A(x,x)$. In particular,*

$$\mathcal{R}(x, A) = \mathcal{E}\left(g_A^x / g_A(x,x), g_A^x / g_A(x,x)\right) = g_A(x,x)^{-1}. \tag{5.283}$$

Proof. The case $x \in A$ is trivial. Suppose now that $x \in S \setminus A$. Clearly, the set $\{ f \in \mathcal{F}_A \mid f(x) \geq 1 \}$ is a closed convex subset of the Hilbert space $(\mathcal{F}_A, \mathcal{E})$ which implies the existence and uniqueness of a maximizer h of the variational problem (5.278). Note that by Definition 5.78 (v) we can assume without loss of generality that $0 \leq h \leq 1$ and $h(x) = 1$. By virtue of Theorem 5.82 we have that $g_A(x,x) > 0$. Set $\psi_A^x := g_A^x / g_A(x,x)$, then

$$\mathcal{E}\left(f - \psi_A^x, \psi_A^x\right) = \frac{\mathcal{E}\left(f - \psi_A^x, g_A^x\right)}{g_A(x,x)} = \frac{f(x) - 1}{g_A(x,x)} = 0 \tag{5.284}$$

for all $f \in \mathcal{F}_A$ with $f(x) = 1$. Consequently,

$$\begin{aligned} \mathcal{E}(h,h) &= \mathcal{E}\left(h - \psi_A^x, h - \psi_A^x\right) + \mathcal{E}\left(h - \psi_A^x, \psi_A^x\right) + \mathcal{E}\left(\psi_A^x, h\right) \\ &\leq \mathcal{E}\left(\psi_A^x, h\right) = g_A(x,x)^{-1} = \mathcal{E}\left(\psi_A^x, \psi_A^x\right). \end{aligned} \tag{5.285}$$

Since h was assumed to be the maximizer of (5.278), we have $\mathcal{E}(h,h) = \mathcal{E}\left(\psi_A^x, \psi_A^x\right)$ and therefore $h = \psi_A^x$. Finally,

$$\mathcal{E}\left(\psi_A^x, \psi_A^x\right) = \frac{\mathcal{E}\left(g_A^x, g_A^x\right)}{g_A(x,x)^2} = g_A(x,x)^{-1}, \tag{5.286}$$

thus completing the proof. ∎

For a subset A of S we write $S_A := (S \setminus A) \cup \{a\}$ to describe the set where A is replaced with a single point a. Moreover, we write

$$\mathcal{F}^A := \{ f \in \mathcal{F} \mid f|_A = c \in \mathbb{R} \} \tag{5.287}$$

for the subspace of $\mathcal{F}$ consisting only of functions which are constant on A. It is straightforward to verify the following fact.

Lemma 5.84. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S and $A \subset S$ non-empty and closed. Then $(\mathcal{E}, \mathcal{F}^A)$ is a resistance form on S_A.*

Note that we have to make some restrictions on the set A for Lemma 5.84 to hold. If A is open, for example, there might be no function $f \in \mathcal{F}^A$ that separates a from some $x \in \partial A$. Note that our assumption that A is closed is sufficient but not necessary for Lemma 5.84 to hold (cf. [Kig12, Chapter 4]).

If we think of the resistance form $(\mathcal{E}, \mathcal{F})$ representing some kind of electrical network, like in Section 4.5.1, the resistance form $(\mathcal{E}, \mathcal{F}^A)$ represents a transformation of the original electrical network where the whole set has been *shortened* or *fused* to a single point. The resistance form $(\mathcal{E}, \mathcal{F}^A)$ is therefore sometimes referred to as the *shortened* or *fused resistance form*. We denote the resistance associated with the fused resistance form $(\mathcal{E}, \mathcal{F}^A)$ by

$$\mathcal{R}_A(x, y) = \sup\left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} \;\middle|\; f \in \mathcal{F}^A,\ \mathcal{E}(f, f) > 0 \right\}. \tag{5.288}$$

Note that $\mathcal{R}_A(A, x) := \mathcal{R}_A(a, x) = \mathcal{R}(x, A)$.

Proposition 5.85. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S and $A \subset S$ non empty and closed. Then,*

$$g_A(x, y) = \frac{\mathcal{R}(A, x) + \mathcal{R}(A, y) - \mathcal{R}_A(x, y)}{2}, \tag{5.289}$$

for all $x, y \in S$.

Proof. See [Kig12, Theorem 4.3]. ∎

5.6.1 Resistance forms and Dirichlet forms

The close relationship between resistance forms and Dirichlet forms is salient. Condition (v) of Definition 5.78 is analogous to the Markov property of Dirichlet forms. On the other hand, (i) can be understood as a recurrence property in light of Theorem 5.39.

Similarly to the regularity of Dirichlet forms, we define regularity of resistance forms as follows.

Definition 5.86. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on S, where $S \neq \emptyset$. Denote by $C_0(S)$ the compactly supported functions $f : S \to \mathbb{R}$ that are continuous with respect to the metric $\mathcal{R}$ on S. We say that $(\mathcal{E}, \mathcal{F})$ is *regular* if and only if $\mathcal{F} \cap C_0(S)$ is dense in $C_0(S)$ with respect to the uniform norm $\|\cdot\|_\infty$. ◊

Indeed, when we equip the metric space $(S, \mathcal{R})$ associated with the resistance form $(\mathcal{E}, \mathcal{F})$ with a Borel regular measure, the resistance form gives rise to a Dirichlet form. We will assume in the following that ν is a Borel regular measure on $(S, \mathcal{R})$ with

$$0 < \nu(B_{\mathcal{R}}(x, r)) < \infty, \tag{5.290}$$

for all $x \in S$ and $r > 0$, where $B_{\mathcal{R}}(x, r)$ denotes the ball with respect to the resistance metric with radius $r > 0$ and center $x \in S$.

Proposition 5.87. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on S. Denote by $\mathcal{D}$ the closure of $\mathcal{F} \cap C_0(S)$ with respect to the inner product $\mathcal{E}_1$ on $\mathcal{F} \cap L^2(S, \nu)$ given by*

$$\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \langle f, g\rangle_\nu . \tag{5.291}$$

Then $\mathcal{E}$ can be uniquely extended to $\mathcal{D}$ and $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(S, \nu)$.

Proof. The extension of $\mathcal{E}$ to $\mathcal{D}$ is, of course, given by

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}(f_n, f_n), \tag{5.292}$$

where $f \in \mathcal{D}$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ is such that $\mathcal{E}_1(f - f_n, f - f_n) \to 0$ as $n \to \infty$. Clearly, this extension is unique and in particular $\mathcal{E}(f, f)$ does not depend on the choice of $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$.

By definition, $(\mathcal{E}, \mathcal{D})$ is a closed form on $L^2(S, \nu)$. The Markov property, Definition 5.78 (v), of $(\mathcal{E}, \mathcal{F})$ is preserved under the closure operation: Fix $f \in \mathcal{D}$ and set $g := f^+ \wedge 1$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\lim_{n\to\infty} f_n = f$ with respect to $\mathcal{E}_1$. The sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{F}$ with $g_n = f_n^+ \wedge 1$ is again a Cauchy sequence and converges to some $g \in \mathcal{D}$ with respect to $\mathcal{E}_1$. Clearly, $g = f^+ \wedge 1$ and

$$\mathcal{E}(g, g) = \lim_{n\to\infty} \mathcal{E}(g_n, g_n) \le \lim_{n\to\infty} \mathcal{E}(f_n, f_n) = \mathcal{E}(f, f). \tag{5.293}$$

It remains to show that $(\mathcal{E}, \mathcal{D})$ is regular. We have $\mathcal{F} \subset \mathcal{D}$ and consequently, $C_0(S) \cap \mathcal{D}$ is dense in $C_0(S)$ with respect to the uniform norm. On the other hand, $C_0(S) \cap \mathcal{D}$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}_1$, by definition of $\mathcal{D}$. Hence, $(\mathcal{E}, \mathcal{D})$ satisfies (D4) of Definition 5.9 and is therefore regular. ■

We will from now on assume that we are dealing with regular resistance forms. Then Proposition 5.87 allows us to apply the potential theory developed in the previous sections for the resistance form $(\mathcal{E}, \mathcal{F})$. Note that we will not always make the dependence on the Dirichlet form $(\mathcal{E}, \mathcal{D})$ associated with $(\mathcal{E}, \mathcal{F})$ explicit. Instead, we will simply speak about the 1-order capacity associated with the resistance form $(\mathcal{E}, \mathcal{F})$, for example.

If not explicitly stated otherwise, we will always assume that the resistance forms are defined on S, where $S \neq \emptyset$.

Lemma 5.88 (reproducing kernel Hilbert space). *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form. Denote by $(\mathcal{E}, \mathcal{D})$ the regular Dirichlet form associated with $(\mathcal{E}, \mathcal{F})$. Then there exists a* reproducing kernel *for the Hilbert space $(\mathcal{D}, \mathcal{E}_1)$. That is, for each $x \in S$ there exists a unique $\varphi_x \in \mathcal{D}$ such that*

$$\mathcal{E}_1(f, \varphi_x) = f(x), \tag{5.294}$$

for all $f \in \mathcal{D}$.

Note that in (5.294) we evaluate $f(x)$ for a continuous representative of $f \in \mathcal{D}$. By construction, such a continuous representative is unique and the expression makes sense.

Proof of Lemma 5.88. Fix $x \in S$. It suffices to show that the evaluation map $e_x : \mathcal{D} \to \mathbb{R}$ given by $e_x f = f(x)$ is a bounded operator. Then the existence of φ_x follows from Riesz' representation theorem (cf. [Yos78, Theorem II.6]). To that end let $f \in \mathcal{D}$ and assume that $f(x) \neq 0$. Without loss of generality, we can assume that $f(x) = 1$. Now let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ be a sequence with $f_n \to f$ with respect to $\mathcal{E}_1$. We can choose f_n such that $f_n(x) = 1$. Suppose we have

$$\mathcal{E}_1(f_n, f_n) = 1/n. \tag{5.295}$$

By definition of the resistance metric we get

$$|f_n(x) - f_n(y)| \leq \frac{\sqrt{\mathcal{R}(x, y)}}{\sqrt{n}} \leq \sqrt{\mathcal{R}(x, y)}, \tag{5.296}$$

for every $y \in S$. Consequently, $f_n(y) \geq 1/2$ for all $y \in B_{\mathcal{R}}(x, 1/4)$. Therefore,

$$\|f_n\|_2^2 = \int_S f^2 \, \mathrm{d}\nu \geq \int_{B_{\mathcal{R}}(x,1/4)} f(y)^2 \, \nu(\mathrm{d}y) \geq \frac{\nu(B_{\mathcal{R}}(x, 1/4))}{4} > 0, \tag{5.297}$$

by (5.290). This is clearly a contradiction to (5.295) and we have shown that there exists a constant $c_x > 0$ such that

$$1 = f(x) = e_x f \leq c_x \sqrt{\mathcal{E}_1(f, f)}. \tag{5.298}$$

Note that the bound in (5.297) does not depend on f which implies that for all $f \in \mathcal{D}$,

$$f(x) \le c_x \sqrt{\mathcal{E}_1(f, f)}. \tag{5.299}$$

Uniqueness of φ_x is clear. Suppose g had the same property, then

$$\mathcal{E}_1(\varphi_x - g, f) = f(x) - f(x) = 0, \tag{5.300}$$

for all $f \in \mathcal{D}$, which implies $g = \varphi_x$. ■

One of the fundamental properties of resistance forms is that singletons have positive capacity. This property makes it possible to associate a symmetric Feller process with a resistance form that is unique in distribution for every initial condition.

Proposition 5.89. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form. Each $x \in S$ has positive 1-order capacity,*

$$\mathrm{Cap}_1(\{x\}) > 0. \tag{5.301}$$

Proof. Fix $x \in S$ and denote by φ_x the unique element of $\mathcal{D}$ with $\mathcal{E}_1(\varphi_x, f) = f(x)$ for all $f \in \mathcal{D}$, as in Lemma 5.88. Note that $\varphi_x(x) = \mathcal{E}_1(\varphi_x, \varphi_x) > 0$ and therefore, $\varphi_x/\varphi_x(x)$ is well defined. Fix any $f \in \mathcal{D}$ with $f(x) \ge 1$ and set $a := f(x)$. Write $h_x^1 = \varphi_x/\varphi_x(x)$, then

$$\mathcal{E}_1(f, f) = \mathcal{E}_1\left(f - ah_x^1, f - ah_x^1\right) + \mathcal{E}_1\left(f - ah_x^1, ah_x^1\right) + \mathcal{E}\left(f, ah_x^1\right). \tag{5.302}$$

By the reproducing property of φ_x, we obtain

$$\mathcal{E}_1\left(f - ah_x^1, ah_x^1\right) = \frac{a}{\varphi_x(x)}\left(f(x) - ah_x^1(x)\right) = 0 \tag{5.303}$$

and similarly,

$$\mathcal{E}_1\left(f, ah_x^1\right) = \frac{a}{\varphi_x(x)} f(x) = \frac{a^2}{\varphi_x(x)}\mathcal{E}_1\left(h_x^1, \varphi_x\right) = \mathcal{E}_1\left(ah_x^1, ah_x^1\right). \tag{5.304}$$

Combining those with (5.302) and using the fact that $a \ge 1$, we arrive at

$$\mathcal{E}_1(f, f) = \mathcal{E}_1\left(f - ah_x^1, f - ah_x^1\right) + \mathcal{E}_1\left(ah_x^1, ah_x^1\right) \le \mathcal{E}_1\left(h_x^1, h_x^1\right). \tag{5.305}$$

Consequently, h_x^1 minimizes $\mathcal{E}_1(f, f)$ over the set $\{ f \in \mathcal{D} \mid f(x) \ge 1 \}$.

Now let $U \in \mathcal{U}$ be an open entourage. By definition of the 1-Capacity, we have

$$\begin{aligned} \mathrm{Cap}_1(U[x]) &= \inf\left\{ \mathcal{E}_1(f,f) \;\middle|\; f \in \mathcal{D},\ \mathcal{E}(f,f) > 0,\ f|_{U[x]} \geq 1 \right\} \\ &\geq \inf\{ \mathcal{E}_1(f,f) \mid f \in \mathcal{D},\ \mathcal{E}(f,f) > 0,\ f(x) \geq 1 \} \\ &= \mathcal{E}_1\left(h_x^1, h_x^1\right) = \varphi_x(x)^{-1}. \end{aligned} \tag{5.306}$$

Taking the limit inferior over a sequence $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ of open entourages with $\bigcap_{n\geq 1} U_n = \Delta$, we obtain $\mathrm{Cap}_1(\{x\}) = \varphi_x(x)^{-1} > 0$. ■

We have not only shown that singletons have positive capacity, we also have identified the minimizer for the 1-capacity of a point $x \in S$ to be h_x^1 and have given an expression for the 1-capacity in terms of the reproducing kernel φ_x.

Lemma 5.90. *The regular Dirichlet form* $(\mathcal{E}, \mathcal{D})$ *associated with a regular resistance form is transient.*

Proof. Since $\mathcal{D}$ is defined as the $\mathcal{E}_1$-closure of $\mathcal{F} \cap C_0(S)$, we immediately obtain $1 \in \mathcal{D}_e$ from $1 \in \mathcal{F}$. The claim then follows by Theorem 5.39. ■

5.6.2 Feller processes associated with resistance forms

Suppose $(\mathcal{E}, \mathcal{F})$ is a regular resistance form on S and $(\mathcal{E}, \mathcal{D})$ the regular Dirichlet form on $L^2(S, \nu)$ associated with $(\mathcal{E}, \mathcal{F})$. It follows from the standard theory of Dirichlet forms, a part of which that we have not presented in this chapter, that there exists a ν-symmetric Feller process X with values in S_ϑ that is associated with $(\mathcal{E}, \mathcal{D})$. See for example [FOT11, Theorem 7.2.1] or our remarks in Chapter 7. Moreover, as singletons have positive capacity by Proposition 5.89, it follows from [CF11, Theorem 3.1.12] that X is unique in distribution for every initial condition $\mu \in \mathcal{M}_1(S)$.

We can therefore give probabilistic interpretations of the Green function and the resistance. We will restrict to the case where $(S, \mathcal{R})$ is compact. Recall the definition of the Green operator $G_A \colon \mathcal{B}_b(S) \to \mathcal{B}_b(S)$,

$$G_A f(x) := \mathbb{E}_x\left[\int_0^{\tau_A} f(X_s)\,\mathrm{d}s\right], \quad x \in S,\ f \in \mathcal{B}_b(S) \tag{5.307}$$

from (4.179).

The next result shows that the Green function is, in fact, the integral kernel for the Green operator.

Proposition 5.91. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on S and X the ν-symmetric Feller process associated with $(\mathcal{E}, \mathcal{D})$. Suppose that the metric space $(S, \mathcal{R})$ is compact. Then,*

$$G_A f(x) = \int_S g_A(x, y) f(y)\, \nu(\mathrm{d}y), \tag{5.308}$$

for every $f \in \mathcal{B}_b(S)$ and $x \in S$.

Proof. See [Kig12, Theorem 10.10] and [Cro18, Lemma 3.1]. ∎

Note that by Proposition 5.77, we can identify

$$g_A(x, y) = \mathbb{P}_x(\tau_y < \tau_A) g_A(y, y). \tag{5.309}$$

Moreover, from Theorem 4.72, we know that the process X is uniquely determined by the family of Green operators $G_{\overline{U[x]}}$ for $x \in S$ and $U \in \mathcal{U}$ open. By Proposition 5.85, on the other hand, the Green kernel $g_A(x, y)$ can be expressed in terms of the resistance metric. In some sense the Green kernel $g_x(y, z)$ measures how much the triangle inequality for the triple (x, y, z) deviates from the identity.

Although random walks on graphs are the prime example of processes associated with resistance forms, the class of such examples is larger. We have already mentioned random walks on fractals like the Sierpiński gasket which can be constructed as the limit of discrete, self-similar graphs. The question that naturally arises is whether the random walks on such a sequence of graphs also converge to a limit. Further examples are continuous limits of discrete trees like the *continuum random tree* constructed by David Aldous in a series of papers [Ald91a; Ald91b; Ald93]. In [AEW13], Siva Athreya, Michael Eckhoff and Anita Winter constructed the Brownian motion on so-called $\mathbb{R}$-trees and in [ALW17] Athreya and Winter together with Wolfgang Löhr showed that the random walks on discrete trees converge to the Brownian motion on the $\mathbb{R}$-tree when the trees converge to an $\mathbb{R}$-tree. The following result is due to David Croydon [Cro18].

Theorem 5.92. *For each $n \in \mathbb{N} \cup \{\infty\}$ let $S^{(n)} \subset S$ be non empty and $\rho^{(n)} \in S^{(n)}$. Moreover, let $\left(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}\right)$ be a regular resistance form on $S^{(n)}$ and $\nu^{(n)}$ be a Borel regular measure on $\left(S^{(n)}, \mathcal{R}^{(n)}\right)$ with full support. Denote by $X^{(n)}$ the ν-symmetric Feller process associated with $(\mathcal{E}^{(n)}, \mathcal{D}^{(n)})$. Suppose $(S^{(n)}, \mathcal{R}^{(n)})$ is compact and*

$$\left(S^{(n)}, \mathcal{R}^{(n)}, \rho^{(n)}, \nu^{(n)}\right) \longrightarrow \left(S^{(\infty)}, \mathcal{R}^{(\infty)}, \rho^{(\infty)}, \nu^{(\infty)}\right) \tag{5.310}$$

with respect to the pointed Gromov-Hausdorff-weak topology. Then there exists a common metric space (S, d) *and for each* $n \in \mathbb{N} \cup \{\infty\}$*,* $(S^{(n)}, \mathcal{R}^{(n)})$ *can be embedded isometrically in* (S, d) *such that*

$$\mathbb{P}_{\rho^{(n)}}\left(X^{(n)} \in \cdot\right) \to \mathbb{P}_{\rho^{(\infty)}}\left(X^{(\infty)} \in \cdot\right) \tag{5.311}$$

weakly as processes with values in (S, d)*.*

Proof. This is a simplified version of [Cro18, Theorem 1.2]. ■

Note that despite many important examples of symmetric Feller processes are associated with resistance forms, such examples are basically low dimensional in the sense that the processes hit points almost surely. This fails to hold, for example, for Brownian motion already in dimension $d = 2$ (see Example 5.11).

5.7 Brownian Motion on Riemannian manifolds

Another class of examples of symmetric Feller processes are Brownian motions on Riemannian manifolds. These processes are in some sense complementary to those associated with resistance forms. We will keep this section almost painfully short because we only want to highlight the connection between the Riemannian metric and the Brownian motion itself. For the details of Riemannian manifolds, we rely on the book [Jos11] by Jürgen Jost. A (very) short construction of the Brownian motion on Riemannian manifolds can be found in [CF11, Section 2.2.5] and [FOT11, Example 5.7.2]. More results on Brownian motions on Riemannian manifolds under certain curvature conditions as well as a convergence result can be found in the paper [Suz19a] by Kohei Suzuki or [GL17] by Maria Gordina and Thomas Laetsch.

5.7.1 Riemannian Manifolds

We recall the basic concepts of Riemannian manifolds. For further details see [Jos11]. A different and less analytical approach to Riemannian manifolds can be found in [BBI01, Chapter 5.1].

Definition 5.93 (Differentiable manifold)**.** A connected and paracompact Hausdorff space M is called a *manifold* of dimension $d \in \mathbb{N}$ if every point $p \in M$ has a neighborhood U that is homeomorphic to an open subset O of $\mathbb{R}^d$. The homeomorphism

$$x\colon U \to O \tag{5.312}$$

is called a *(coordinate) chart*. A family of charts $\{ \{x_\alpha, U_\alpha\} \mid \alpha \in \mathbb{I} \}$ is called an *atlas* if $\{ U_\alpha \mid \alpha \in \mathbb{I} \}$ is an open cover of M. A manifold M is called *differentiable* if all chart transitions

$$x_\beta \circ x_\alpha^{-1} \colon x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta) \tag{5.313}$$

are infinitely often continuously differentiable, i.e. in C^∞, whenever $U_\alpha \cap U_\beta \neq \emptyset$. ◊

Let $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ and $O \subset \mathbb{R}^d$ open. The tangent space of O at the point $z \in O$,

$$T_z O \tag{5.314}$$

is the space $\{z\} \times E$, where E is the d-dimensional vector space spanned by the partial derivatives

$$\left.\frac{\partial}{\partial x^1}\right|_z, \dots, \left.\frac{\partial}{\partial x^d}\right|_z \tag{5.315}$$

at z. Suppose $O \subset \mathbb{R}^d$ and $O' \subset \mathbb{R}^c$ are open and $f \colon O \to O'$ is differentiable. For $z \in O$ we define the *derivative* $df(z) \colon T_z O \to T_{f(z)} O'$ by

$$v = \sum_{i=1}^d \frac{\partial}{\partial x^i} \mapsto \sum_{i=1}^d \sum_{j=1}^c v^i \frac{\partial f^j}{\partial x^i}(z) \frac{\partial}{\partial f^j}. \tag{5.316}$$

Definition 5.94 (Tangent space)**.** Let M be a differentiable manifold and $p \in M$. Define an equivalence relation on the set

$$\left\{ (x, v) \,\middle|\, x \colon U \to O \text{ is a chart with } p \in U \text{ and } v \in T_{x(p)} O \right\} \tag{5.317}$$

by setting

$$(x, v) \sim (y, w) \quad \iff \quad w = d(y \circ x^{-1}) v. \tag{5.318}$$

We denote the quotient space by $T_p M$ and say that $T_p M$ is the *tangent space* to M at p. ◊

The *tangent bundle* is the disjoint union of the tangent spaces $T_p M$, $p \in M$ and can itself be again equipped with a differentiable structure.

Definition 5.95 (Riemannian manifold)**.** A *Riemannian metric* on a differentiable manifold M is given by a scalar product on each tangent space $T_p M$ which depends smoothly on p. A *Riemannian manifold* is a differentiable manifold equipped with a Riemannian metric. ◊

The Riemannian metric can be represented as a positive definite symmetric matrix. Let $x = (x^1, \dots, x^d)$ be local coordinates, then a Riemannian metric can be written as

$$g = \left(g_{ij}(x)\right)_{i,j=1,\dots,d}. \tag{5.319}$$

Then, for $v, w \in T_pM$ we have

$$\langle v, w \rangle_p := \sum_{i,j} g_{ij}(x(p)) v^i w^j. \tag{5.320}$$

The representation (g_{ij}) is also called a *metric tensor*

The *volume element* $V(\mathrm{d}p)$ of (M, g) is given by

$$V(\mathrm{d}p) = \sqrt{g}\,\mathrm{d}p = \sqrt{\det(g_{ij})}\,\mathrm{d}p, \tag{5.321}$$

in local coordinates. Note that $V(\mathrm{d}p)$ constitutes a Radon measure on $(M, \mathcal{B}(M))$.

The Riemannian metric also gives rise to a metric on M. Let $\gamma\colon [a, b] \to M$ be a smooth curve, i.e. $\gamma \in C^\infty$. We write Γ for set of all such curves. The length of γ is defined as

$$L(\gamma) := \int_a^b \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \right\| \mathrm{d}t, \tag{5.322}$$

where $\|\cdot\|$ denotes the norm with respect to the Riemannian metric. For two points $p, q \in M$ we define

$$d(p, q) := \inf \{ L(\gamma) \mid \gamma \in \Gamma,\ \gamma(a) = p,\ \gamma(b) = q \} \tag{5.323}$$

and observe that $(M, d, V(\mathrm{d}x))$ is a metric measure space.

5.7.2 Brownian motion

There are different ways to define the Brownian motion on a Riemannian manifold. One way is through its Dirichlet form.

Let M be a Riemannian manifold with dimension $d \in \mathbb{N}$ and metric tensor (g_{ij}) in some local coordinates $x^1, \dots, x^d$. We want to construct a Dirichlet form on $L^2(M, \mathrm{d}V)$.

For a smooth function $f\colon M \to \mathbb{R}$ we define the *gradient* of f as the vector field given by

$$\nabla f := \operatorname{grad} f := \sum_{i,j=1}^{d} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \tag{5.324}$$

where g^{ij} is the ij-th entry of the inverse $(g_{ij})^{-1}_{ij=1,\dots,d}$ of the metric tensor. For compactly supported smooth functions $f, g \in C_0^\infty(M)$ we define a bilinear form $\mathcal{E}$ by

$$\mathcal{E}(f,g) := \frac{1}{2}\int_M \langle \nabla f, \nabla g\rangle_p \, V(\mathrm{d}p). \tag{5.325}$$

We can further define the divergence of a vector field $Z = \sum_{i=1}^d Z^i \frac{\partial}{\partial x^i}$ by

$$\operatorname{div} Z := \frac{1}{\sqrt{g}} \sum_{i=1}^{d} \frac{\partial}{\partial x^j}\left(\sqrt{g} Z^j\right) = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(\sqrt{g} g^{ij} \left\langle Z, \frac{\partial}{\partial x^i}\right\rangle\right). \tag{5.326}$$

Moreover the *Laplace-Beltrami* operator is defined by

$$\Delta f := -\operatorname{div}\operatorname{grad} f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i}\right). \tag{5.327}$$

For more details see the [Jos11, Chapter 3].

Then the quadratic form $\mathcal{E}$ can be written for $f, g \in C_0^\infty(M)$ as

$$\mathcal{E}(f,g) = -\frac{1}{2}\langle \Delta f, g\rangle_V = -\frac{1}{2}\langle f, \Delta g\rangle_V, \tag{5.328}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_V$ denotes the scalar product on $L^2(M, \mathrm{d}V)$. It can be shown that the quadratic form $(\mathcal{E}, C_0^\infty)$ is closable and that the closed form $(\mathcal{E}, \mathcal{D})$ is indeed a regular Dirichlet form. The $\mathrm{d}V$-symmetric Feller process X with values in M and Dirichlet form $(\mathcal{E}, \mathcal{D})$ is then called the *Brownian motion on M*. Compare this also to our Example 5.11.

Remark 5.96 (Convergence of Brownian motions on Riemannian manifolds)**.** In [Suz19a] and [Suz19b], Kohei Suzuki developed conditions under which the convergence of a sequence of Riemannian manifolds implies the (pathwise) convergence of the Brownian motions living on these manifolds. These conditions are basically Gromov-Hausdorff weak convergence of the manifolds as metric measure spaces, similar to the last example, and a lower bound on the so-called Ricci curvature to ensure that the limit is again a Riemannian manifold with a Brownian motion. ◊

It is important to point out that points on Riemannian manifolds do not have positive capacity, in general. Therefore, the Brownian motion on manifolds is generally not uniquely determined by its Dirichlet form for every starting *point* (see our remarks in Section 7.3). Instead, the results in [Suz19a] require that the Brownian motion is started in an initial distribution that is absolutely continuous with respect to the volume measure dV.

Remark 5.97. As noted at the beginning of this section, this result complements the result for resistance forms in different ways. On the one hand, it covers examples where points have capacity zero like the Brownian motion on $\mathbb{R}^d$ in dimensions $d \geq 2$ or the Ornstein-Uhlenbeck process (cf. [Suz19a, Remark 3.1]). On the other hand, the starting point here is a geometric structure, the Riemannian structure on the manifold. We use this structure, in particular the metric tensor, to construct the processes via their Dirichlet forms. Whereas in the case of resistance forms, the starting point is a bilinear form, which then induces the geometric structure on the state space and at the same time defines the processes. Another difference is that the speed measure and the geometry of the Riemannian manifold are related to each other through the Riemannian metric. In contrast to the resistance forms where the geometric structure, given by the resistance metric, and the speed measure are separated. ◊

Convergence of symmetric Feller processes

6

Invention, it must be humbly admitted, does not consist in creating out of void, but out of chaos.

— **Mary Wollstonecraft Shelley**
Frankenstein

In this chapter we formulate our main convergence result. We introduce four conditions and show in three steps which role these conditions play for the convergence of a sequence of symmetric doubly Feller processes. We first show that the sequence of semigroups has subsequential limits which are again doubly Feller. Then we show that the processes along such converging subsequence converge already weakly in the path space. Finally, we apply Theorem 4.72 to conclude that all subsequential limits must coincide.

In Section 6.3 we will discuss each of the conditions individually.

6.1 Statement of the theorem

For the remainder of this chapter let $(S, \mathcal{U})$ be a locally compact uniform Polish space. For each $n \in \mathbb{N}^\infty$ let $\nu^{(n)}$ denote a boundedly finite measure on $(S, \mathcal{B})$ with support $S^{(n)}$. Assume further that for each $n \in \mathbb{N}^\infty$ a $\nu^{(n)}$-symmetric doubly Feller process is given by $X^{(n)}$. We write

$$\mathbb{P}^{(n)} := \mathbb{P}^{X^{(n)}} \tag{6.1}$$

for the distribution of $X^{(n)}$. Generally, we will indicate all entities related to $X^{(n)}$ by a superscript (n).

Recall from Definition 2.57 that the sequence $\left(\nu^{(n)}\right)_{n\in\mathbb{N}}$ converges Hausdorff weakly if and only if the measures $\nu^{(n)}$ converge weakly and their supports converge in the Hausdorff topology.

Consider the following conditions.

(C1) $\nu^{(n)}$ converges Hausdorff weakly to $\nu^{(\infty)}$.

(C2) The family $\left\{ Q^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ of maps given by

$$Q^{(n)} \colon S^{(n)} \times [0, \infty) \to \mathcal{M}_1(S), \quad (x, t) \mapsto Q^{(n)}_{x,t}(\,\cdot\,) := \mathbb{P}_x\left(X^{(n)}_t \in \cdot\right) \tag{6.2}$$

is uniformly equicontinuous.

(C3) For every sequence $(x_n)_{n\in\mathbb{N}} \subset S$ with $x_n \in S^{(n)}$ and $\lim_{n\to\infty} x_n = x_\infty \in S^{(\infty)}$, the sequence $\left\{ \mathbb{P}^{(n)}_{x_n} \,\middle|\, n \in \mathbb{N} \right\}$ is tight as probability measures on $D_S([0, \infty))$.

(C4) The Green's functionals $G^{(n)}_A$ converge to $G^{(\infty)}_A$ in the following sense. For all bounded measurable functions $f \in \mathcal{B}_b(S)$ and all $A \in \mathcal{B}(S)$ with $\tau_A < \infty$, $\mathbb{P}^{(\infty)}_{x_\infty}$-a.s.,

$$\lim_{n\to\infty} G^{(n)}_A f(x_n) = G^{(\infty)}_A f(x_\infty), \tag{6.3}$$

for all sequences $(x_n)_{n\in\mathbb{N}} \subset S$ with $x_n \in S^{(n)}$ and $\lim_{n\to\infty} x_n = x_\infty \in S^{(\infty)}$.

We will exclusively consider the case where the space $(S, \mathcal{U})$ is compact which implies that the closed subsets $S^{(n)}$ are compact for each $n \in \mathbb{N}^\infty$. Moreover, we will assume that the processes $X^{(n)}$ are conservative for each $n \in \mathbb{N}^\infty$. We are confident that an extension to general state spaces $S^{(n)}$ is possible by approximation similar to [ALW17; Cro18]. Such an extension, however, remains subject to further research.

Theorem 6.1. *Assume that $(S, \mathcal{U})$ is compact and that $X^{(n)}$ is conservative for each $n \in \mathbb{N}^\infty$. Under conditions (C1), (C2), (C3) and (C4) $X^{(n)}$ converges in distribution to $X^{(\infty)}$ for all sequences of initial distributions $\left(\mu^{(n)}\right)_{n\in\mathbb{N}} \subset \mathcal{M}_1(S)$ with $\mu^{(n)} \in \mathcal{M}_1(S^{(n)})$ and $\mu^{(n)} \Rightarrow \mu^{(\infty)} \in \mathcal{M}_1(S^{(\infty)})$. In other words,*

$$\mathbb{P}^{(n)}_{\mu^{(n)}} \Rightarrow \mathbb{P}^{(\infty)}_{\mu^{(\infty)}} \tag{6.4}$$

weakly as probability measures on $D_S([0, \infty))$ as $n \to \infty$.

In order to proof Theorem 6.1 we will first show that under (C1) and (C2), the sequence $\left\{ X^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ has subsequential limits in the f.d.d. sense which are again doubly Feller. Next, we will show that the sequence $\left\{ X^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ has subsequential limits in pathspace if we additionally impose condition (C3). The final step is then to show that these subsequential limits coincide. We follow roughly the path that was laid out by [ALW17] and refined by [Cro18].

6.2 Existence of subsequential limits

For each $n \in \mathbb{N}^\infty$ denote the Feller semigroups associated with $X^{(n)}$ by $P^{(n)} = \left\{ P_t^{(n)} \,\middle|\, t \geq 0 \right\}$ and observe that for each $f \in C(S)$ and $x \in S^{(n)}$,

$$P_t^{(n)} f(x) = \int_{S^{(n)}} f \, \mathrm{d}Q^{(n)}(x,t), \tag{6.5}$$

where $Q^{(n)}$ is given by (6.2).

We begin by showing that the uniform continuity of the family $\left\{ Q^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ (condition (C2)) together with the Hausdorff convergence of the spaces $S^{(n)}$ (condition (C1)) implies that every subsequential limit of the semigroups $P^{(n)} = \left\{ P_t^{(n)} \,\middle|\, t \geq 0 \right\}$ is again a conservative Feller semigroup. Furthermore, we prove that such subsequential limits exist. The proof is based on the proofs of [ALW17, Proposition 5.2] and [Cro18, Lemma 5.4].

Theorem 6.2 (Convergence of semigroups)**.** *Let $(S, \mathcal{U})$ be compact and assume that conditions (C1) and (C2) hold. Then for every subsequence of $\left(P^{(n)}\right)_{n\in\mathbb{N}}$ there exists a further subsequence $\left(P^{(n_k)}\right)_{k\in\mathbb{N}}$ and a conservative doubly Feller semigroup $P = (P_t)_{t\geq 0}$ with the property that for every $\varepsilon > 0$, $f \in C(S)$ there exists a $U \in \mathcal{U}$ open and a $\delta > 0$ such that for every $k \in \mathbb{N}$ large enough,*

$$\left\{ \left(P_s f(x), P_t^{(n_k)} f(y)\right) \,\middle|\, (x,y) \in U \cap S^{(\infty)} \times S^{(n_k)},\ s,t > 0 : |t-s| < \delta \right\} \subset B_\varepsilon, \tag{6.6}$$

where $B_\varepsilon := \left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, |\alpha - \beta| < \varepsilon \right\}$.

Proof. Recall the definition of the Prokhorov uniformity from Section 3.5.1 and denote the Prokhorov uniformities on $\mathcal{M}_1(S)$ and $\mathcal{M}_1(S^{(n)})$ by $\mathcal{D}_\mathcal{M}$ and $\mathcal{D}_\mathcal{M}^{(n)}$, respectively. By assumption, $(S, \mathcal{U})$ is a Polish space and by virtue of Proposition 3.33 so are $(\mathcal{M}_1(S), \mathcal{D}_\mathcal{M})$ and $\left(\mathcal{M}_1(S_n), \mathcal{D}_\mathcal{M}^{(n)}\right)$, $n \in \mathbb{N}$. Consequently, the Prokhorov uniformities are completely metrizable by Proposition 2.20 and Lemma 2.39.

Fix $T > 0$ and write

$$Q_T^{(n)} : S^{(n)} \times [0,T] \to \mathcal{M}_1(S) \tag{6.7}$$

for the restriction of $Q^{(n)}$ to $S^{(n)} \times [0, \infty)$. Now take any subsequence $(n_k)_{k\in\mathbb{N}}$. For ease of notation, we simply use the index k to indicate elements from this sequence. Clearly, the family $\left\{ Q_T^{(k)} \,\middle|\, k \in \mathbb{N} \right\}$ is uniformly equicontinuous by assumption (C2). Furthermore, we have by assumption (C1) and Lemma 2.55 that for each $x \in S_\infty$ there exists a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in S_k$ and $\lim_{k\to\infty} x_k = x$. We can therefore apply the Arzelà-Ascoli theorem as formulated in Lemma 2.47 to obtain a continuous map $Q_T\colon S^{(\infty)} \times [0,T] \to \mathcal{M}_1(S)$ with the property that for all $V \in \mathcal{D}_{\mathcal{M}}$ open there exists a $U \in \mathcal{U}$ open and a $\delta > 0$ such that for all $k \in \mathbb{N}$ large enough,

$$\left\{ \left(Q_T^{(k)}(x,s), Q_T(y,t)\right) \,\middle|\, (x,y) \in U \cap S^{(k)} \times S^{(\infty)},\ s,t \in [0,T] : |s-t| < \delta \right\} \subset V. \quad (6.8)$$

By letting $T \to \infty$ we obtain a continuous map $Q\colon S^{(\infty)} \times [0,\infty) \to \mathcal{M}_1(S)$ with the same property (6.8) for all $s,t \in [0,\infty)$ with $|s-t| < \delta$. Note that because the spaces $S^{(n)}$, $n \in \mathbb{N} \cup \{\infty\}$ are all closed we can trivially extend the measures $Q(x,t)$ and $Q^{(n)}(x_n,t)$, $t \geq 0$, $x \in S^{(\infty)}$, $x_n \in S^{(n)}$ to probability measures on the whole of S. We will do so implicitly in the following.

Let $P^{(n)} := \left(P_t^{(n)}\right)_{t\geq 0}$ be defined as in (6.5) and analogously define $P = (P_t)_{t\geq 0}$ as the family of operators on $\mathcal{B}_b(S)$ given by

$$P_t f(x) := \int_S f \,\mathrm{d}Q(x,t). \quad (6.9)$$

Now take $f \in C(S^{(\infty)})$. Since Q is a continuous map, we have that $Q(x_n,t) \underset{n\to\infty}{\Longrightarrow} Q(x,t)$ weakly in $\mathcal{M}_1(S)$ for every sequence $(x_n)_{n\in\mathbb{N}} \subset S^{(\infty)}$ with $\lim_{n\to\infty} x_n = x$ which readily implies $P_t f(x_n) \underset{n\to\infty}{\longrightarrow} P_t f(x)$. Hence, $P_t f \in C(S^{(\infty)})$. By the same argument, we also obtain $P_t f \in C_b(S^{(\infty)})$ for all $f \in \mathcal{B}_b(S)$. Furthermore, P_t is a positive contraction operator on $C(S^{(\infty)})$, since $Q(x,t)$ is a probability measure for each $t \geq 0$ and $x \in S^{(\infty)}$. In order to show that P is a Feller semigroup it therefore remains to show that P is a strongly continuous semigroup, i.e.

$$P_s P_t f = P_{s+t} f \quad \forall s,t > 0 \quad (6.10)$$

and

$$\lim_{t\to 0} P_t f(x) = f(x) \quad (6.11)$$

for all $f \in C(S^{(\infty)})$ and $x \in S^{(\infty)}$.

We first show that $(P_t)_{t\geq 0}$ indeed satisfies (6.6). To that end fix a metric d on S that generates $\mathcal{U}$. We denote the Prokhorov metric (cf. Definition B.1) on $\mathcal{M}_1(S)$ induced by d by d_{Pr} and the Kantorovich-Rubinshtein metric (cf. Definition B.2) by d_{KR} and recall that both metrics are uniformly equivalent (cf. [Bog07, Theorem 8.10.43]). Hence, both metrics induce the Prokhorov uniformity $\mathcal{D}_{\mathcal{M}}$ on $\mathcal{M}_1(S)$. Furthermore, denote by

$$\mathrm{Lip}_1(S) := \mathrm{Lip}_1^d(S) := \{ f \in C(S) \mid |f(x) - f(y)| \leq d(x,y) \} \quad (6.12)$$

the family of real-valued Lipschitz continuous functions with Lipschitz constant at most 1 (with respect to the metric d). We first show (6.6) for $f \in \mathrm{Lip}_1(S)$. To that end observe that by definition of d_{KR} we have for $(x, y) \in S^{(\infty)} \times S^{(n)}$,

$$\begin{aligned} \left| P_s f(x) - P_t^{(k)} f(y) \right| &= \left| \int_S f \, \mathrm{d}Q(x, s) - \int_S f \, \mathrm{d}Q^{(k)}(y, t) \right| \\ &\leq d_{\mathrm{KR}}\left(Q(x, s), Q^{(k)}(y, t) \right), \end{aligned} \tag{6.13}$$

which implies the claim by (6.8). As any continuous function on a compact metric space can be approximated uniformly by Lip_1-functions (cf. [Mic00; Geo67]), we obtain (6.6) by an approximation argument.

By the same reasoning it suffices to show that both (6.10) and (6.11) hold for $f \in \mathrm{Lip}_1(S^{(\infty)})$. We first show the semigroup property (6.10). Fix $f \in \mathrm{Lip}_1(S^{(\infty)})$ and note that f can be extended to a function $\tilde{f} \in \mathrm{Lip}_1(S)$. In fact, any continuous function $f \in C(S^{(\infty)})$ can be extended to a continuous function $\tilde{f} \in C(S)$. Take $x \in S^{(\infty)}$ and denote by $(x_k)_{k \in \mathbb{N}} \subset S$ a sequence with $x_k \in S^{(k)}$ and $\lim_{k \to \infty} x_k = x$. Such a sequence exists by assumption (C1). By (6.6), we have for $t > 0$,

$$P_t f(x) = \lim_{k \to \infty} P_t^{(k)} \tilde{f}(x_k). \tag{6.14}$$

In particular we have for $s, t > 0$,

$$P_s P_t f(x) = \lim_{k \to \infty} P_s^{(k)} (\widetilde{P_t f})(x_k), \tag{6.15}$$

where $(\widetilde{P_t f}) \in C(S)$ again denotes the continuous extension of $P_t f \in C(S^{(\infty)})$. Applying the semigroup property of $P^{(k)}$ for $s, t > 0$ on the right-hand side we obtain

$$P_{s+t} f(x) = \lim_{k \to \infty} P_{s+t}^{(k)} \tilde{f}(x_k) = \lim_{k \to \infty} P_s^{(k)} P_t^{(k)} \tilde{f}(x_k). \tag{6.16}$$

With (6.15) in mind, it suffices to show that $P_s^{(k)} (\widetilde{P_t f})(x_k)$ and $P_s^{(k)} P_t^{(k)} \tilde{f}(x_k)$ have the same limit as $k \to \infty$. From assumption (C1) we know that the supports $S^{(n)}$ converge in the Hausdorff topology to $S^{(\infty)}$. By Lemma 2.56, there exist sets T_n and surjective maps $\varphi_n : T_n \to S_n$, $\psi_n : T_n \to S^{(\infty)}$ for all $n \in \mathbb{N}$ such that for all $U \in \mathcal{U}$ open,

$$\{ (\varphi_n(y), \psi_n(y)) \mid y \in T_n \} \subset U, \tag{6.17}$$

eventually. Hence,

$$\begin{aligned}\left|P_s^{(k)}P_t^{(k)}\tilde{f}(x_k) - P_s^{(k)}\widetilde{(P_tf)}(x_k)\right| &\leq \left|P_t^{(k)}\tilde{f}(x_k) - \widetilde{(P_tf)}(x_k)\right| \\ &\leq \sup_{y\in T_k}\left|P_t^{(k)}\tilde{f}\left(\varphi_k(y)\right) - \widetilde{(P_tf)}\left(\varphi_k(y)\right)\right| \\ &\leq \sup_{y\in T_k}\left|P_t^{(k)}\tilde{f}(\varphi_k(y)) - P_tf(\psi_k(y))\right| \\ &\quad + \sup_{y\in T_k}\left|P_tf(\psi_k(y)) - \widetilde{(P_tf)}\left(\varphi_k(y)\right)\right|\end{aligned} \tag{6.18}$$

where we applied the contraction property of $P^{(n)}$ in the first inequality and the triangle inequality in the second. Now it is easy to see that the right hand side tends to 0 as $k \to \infty$, as the first summand goes to 0 by construction of φ_k, ψ_k and their property (6.17) together with (6.6). Whereas the second summand goes to 0 by continuity of P_tf. We have thus shown that P is indeed a semigroup and it remains to show the strong continuity of P (6.11). Using (6.14) we obtain for $f \in C(S^{(\infty)})$ and $x \in S^{(\infty)}$,

$$P_0f(x) = \lim_{k\to\infty} P_0^{(k)}\tilde{f}(x_k) = \lim_{k\to\infty}\tilde{f}(x_k) = f(x), \tag{6.19}$$

where $(x_k)_{k\in\mathbb{N}} \subset S$ is chosen as before.

Finally, we deduce (6.11) from $\lim_{t\to 0} P_tf(x) = P_0f(x)$ which is a direct consequence of the continuity of Q. ∎

We have shown that there exists a subsequence $\left\{ X^{(n_k)} \;\middle|\; n_k \in \mathbb{N} \right\}$ of $\left\{ X^{(n)} \;\middle|\; n \in \mathbb{N} \right\}$ so that the semigroups $\left\{ P^{(n_k)} \;\middle|\; n_k \in \mathbb{N} \right\}$ converge in the sense of (6.6) to a limiting semigroup $(P_t)_{t\geq 0}$, which is again Feller. By Remark 4.46 there exists a unique Feller process X associated to $(P_t)_{t\geq 0}$. We need to examine if and in which sense the sequence of processes $\left\{ X^{(n_k)} \;\middle|\; n_k \in \mathbb{N} \right\}$ converges to X. But first, we verify that the limit is again symmetric with respect to the limit $\nu^{(\infty)}$ of the sequence $\left\{ \nu^{(n)} \;\middle|\; n \in \mathbb{N} \right\}$.

Lemma 6.3. *Under the assumptions of Theorem 6.2 suppose that the $P = (P_t)_{t\geq 0}$ is the limit of a subsequence $\left\{ P^{(n_k)} \;\middle|\; n_k \in \mathbb{N} \right\}$ of $\left\{ P^{(n)} \;\middle|\; n \in \mathbb{N} \right\}$ in the sense of (6.6). Then P is $\nu^{(\infty)}$-symmetric Feller semigroup.*

Proof. It only remains to show that P is $\nu^{(\infty)}$-symmetric by Theorem 6.2. For readability we again write just k instead of n_k for the index of the subsequence. We first show that for every $f \in C(S)$,

$$\lim_{k\to\infty}\int_S P_t^{(k)}f\,\mathrm{d}\nu^{(k)} = \int_S P_tf\,\mathrm{d}\nu^{(\infty)}. \tag{6.20}$$

Note that by (6.6) there exists a continuous extensions $\widetilde{P_t f}$ of $P_t f$ to $C(S)$ such that,

$$M_{t,f}^{(k)} := \sup_{x \in S^{(k)}} \left| P_t^{(k)} f(x) - \widetilde{P_t f}(x) \right| \to 0, \tag{6.21}$$

as $k \to \infty$. Consequently,

$$\int_S P_t^{(k)} f \, \mathrm{d}\nu^{(k)} \leq \int_S \widetilde{P_t f} \, \mathrm{d}\nu^{(k)} + M_{t,f}^{(k)} \nu^{(k)}(S). \tag{6.22}$$

Because S is compact we have $\nu^{(k)}(S) < \infty$ for all $k \in \mathbb{N}$.

Moreover, as $\lim_{n\to\infty} \nu^{(k)}(S) = \nu^{(\infty)}(S) < \infty$, the sequence $\nu^{(k)}(S)$ is uniformly bounded. Therefore we obtain (6.20) from (6.22) by weak convergence of $\nu^{(k)}$ and (6.21).

As an immediate consequence, we obtain the conclusion

$$\int_S f P_t g \, \mathrm{d}\nu^{(\infty)} = \lim_{k\to\infty} \int_S f P_t^{(k)} g \, \mathrm{d}\nu^{(k)} = \lim_{k\to\infty} \int_S P_t^{(k)} f g \, \mathrm{d}\nu^{(k)} = \int_S P_t f g \, \mathrm{d}\nu^{(\infty)}, \tag{6.23}$$

for all $f, g \in C(S)$. ■

Theorem 6.4 (Subsequential limits in f.d.d.)**.** *Under the assumptions of Theorem 6.2 suppose that the $P = (P_t)_{t\geq 0}$ is the limit of a subsequence $\left\{ P^{(n_k)} \,\middle|\, n_k \in \mathbb{N} \right\}$ of $\left\{ P^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ in the sense of (6.6) and that X is the $\nu^{(\infty)}$-symmetric Feller process associated with the semigroup $(P_t)_{t\geq 0}$. Then, for every sequence of starting points $(x_{n_k})_{k\in\mathbb{N}} \subset S$ with $x_{n_k} \in S^{(n_k)}$ and $\lim_{k\to\infty} x_{n_k} = x_\infty \in S^{(\infty)}$, $X^{(n_k)}$ converges to X in finite dimensional distributions. In other words, for every $N \in \mathbb{N}$, $f_1, \ldots, f_N \in C(S)$ and $0 \leq t_1 \leq \cdots \leq t_N$,*

$$\lim_{k\to\infty} \mathbb{E}_{x_{n_k}}^{(n_k)} \left[\prod_{j=1}^N f_j\left(X_{t_j}^{(n_k)}\right) \right] = \mathbb{E}_{x_\infty} \left[\prod_{j=1}^N f_j\left(X_{t_j}\right) \right], \tag{6.24}$$

where $\mathbb{E}^{(n)}$ denotes the expectation with respect to $\mathbb{P}^{(n)}$ and $\mathbb{E}$ the expectation with respect to $\mathbb{P}^X$.

Proof. Again we use just the index k instead of n_k for the subsequence. We proceed by induction. In the case $N = 1$, we have

$$\lim_{k\to\infty} P_{t_1}^{(k)} f(x_k) = P_{t_1} f(x_\infty), \tag{6.25}$$

which is true by Theorem 6.2. Suppose now that (6.24) holds for $N \in \mathbb{N}$. Applying the Markov property at t_N yields

$$\mathbb{E}_{x_k}^{(n_k)}\left[\prod_{j=1}^{N+1} f_j\left(X_{t_j}^{(k)}\right)\right] = \mathbb{E}_{x_k}\left[P_{t_{N+1}-t_N}^{(k)} f_N\left(X_{t_N}^{(k)}\right)\prod_{j=1}^{N} f_j\left(X_{t_j}^{(k)}\right)\right]. \tag{6.26}$$

As before, $P_{t_{N+1}-t_n} f_N$ can be extended to a continuous function $\widetilde{P_{t_{N+1}-t_n}} f_N$ on S and it holds that

$$\lim_{k\to\infty} \sup_{x\in S^{(k)}} \left|P_{t_{N+1}-t_N}^{(k)} f_N(x) - \widetilde{P_{t_{N+1}-t_n}} f_N(x)\right| = 0. \tag{6.27}$$

Combining this with (6.26), we arrive at

$$\lim_{k\to\infty} \mathbb{E}_{x_k}^{(n_k)}\left[\prod_{j=1}^{N+1} f_j\left(X_{t_j}^{(k)}\right)\right] = \lim_{k\to\infty} \mathbb{E}_{x_k}\left[\widetilde{P_{t_{N+1}-t_N}} f_N\left(X_{t_N}^{(k)}\right)\prod_{j=1}^{N} f_j\left(X_{t_j}^{(k)}\right)\right]. \tag{6.28}$$

Note that the factor $\widetilde{P_{t_{N+1}-t_N}} f_N\left(X_{t_N}^{(k)}\right) f_N\left(X_{t_N}^{(k)}\right)$ in the expectation on the right is a function of $X_{t_N}^{(k)}$ and we can apply the inductive hypothesis to obtain

$$\lim_{k\to\infty} \mathbb{E}_{x_k}^{(n_k)}\left[\prod_{j=1}^{N+1} f_j\left(X_{t_j}^{(k)}\right)\right] = \mathbb{E}_{x_\infty}\left[P_{t_{N+1}-t_N} f_N\left(X_{t_N}\right)\prod_{j=1}^{N} f_j\left(X_{t_j}\right)\right] = \mathbb{E}_{x_\infty}\left[\prod_{j=1}^{N+1} f_j\left(X_{t_j}\right)\right], \tag{6.29}$$

as claimed. ■

If we now add the tightness (C3) to our set of assumptions, we immediately obtain from Prokhorov's theorem Proposition 3.37 that for every subsequence of $\left\{\mathbb{P}^{(n)} \mid n \in \mathbb{N}\right\}$ and every sequence $(x_n)_{n\in\mathbb{N}} \subset S$ with $x_n \in S^{(n_n)}$ and $\lim_{n\to\infty} x_n = x_\infty \in S^{(\infty)}$, there exists a further subsequence $\left\{\mathbb{P}^{(n_k)} \mid n_k \in \mathbb{N}\right\}$ and a probability measure $\mathbb{P}_{x_\infty}$ on $D_S([0,\infty))$ such that $\mathbb{P}_{x_k}^{(n_k)} \Rightarrow \mathbb{P}_{x_\infty}$ weakly as $k \to \infty$. In conjunction with the previous result Theorem 6.4 we directly obtain the following result.

Theorem 6.5 (Subsequential limits in $D_S([0,\infty))$). *Let $(S,\mathcal{U})$ be compact and assume that conditions (C1), (C2) and (C3) hold. Then for every subsequence of $\left\{X^{(n)} \mid n \in \mathbb{N}\right\}$ there exists a further subsequence $\left\{X^{(n_k)} \mid n_k \in \mathbb{N}\right\}$ and a $\nu^{(\infty)}$-symmetric Feller process X such that*

$$\mathbb{P}_{\mu^{(k)}}^{(n_k)} \Rightarrow \mathbb{P}_{\mu^{(\infty)}}^{X}, \tag{6.30}$$

weakly as measures on $D_S([0,\infty))$ for every sequence $\left(\mu^{(k)}\right)_{k\in\mathbb{N}}$ of initial distributions with $\mu^{(k)} \in \mathcal{M}_1(S^{(k)})$ and $\mu^{(k)} \Rightarrow \mu^{(\infty)} \in \mathcal{M}_1(S^{(\infty)})$ weakly as $k \to \infty$.

6.3 Identification of subsequential limits

So far we have established that under conditions (C1), (C2) and (C3) the sequence $\left(X^{(n)}\right)_{n\in\mathbb{N}}$ possesses subsequential limits not only in a f.d.d. sense but also in a pathwise sense. In order to establish convergence of the sequence $\left\{ X^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$ we must therefore show that all subsequential limits coincide. Here comes our assumption (C4) into play.

Proof of Theorem 6.1. Suppose $\left\{ X^{(n_k)} \,\middle|\, n_k \in \mathbb{N} \right\}$ and $\left\{ X^{(n_l)} \,\middle|\, n_l \in \mathbb{N} \right\}$ are two subsequences along which the convergence in Theorem 6.5 holds. For ease of notation, we write again k instead of n_k and l instead of n_l for the indices. Denote the respective limits by X and $\hat{X}$. By Theorem 6.2 we have that both X and $\hat{X}$ are $\nu^{(\infty)}$-symmetric Feller processes on S_∞. Moreover, from (C4) we can conclude that for each $x \in S^{(\infty)}$ and $A \in \mathcal{B}(S)$ with $\tau_A < \infty$, $\mathbb{P}_x^{(\infty)}$-a.s. and every $f \in \mathcal{B}_b(S)$,

$$G_A f(x) = \hat{G}_A f(x), \tag{6.31}$$

where G_A and $\hat{G}_A$ denote the Green operators associated with X and $\hat{X}$, respectively. We can therefore apply Theorem 4.72 to obtain $X \overset{d}{=} \hat{X}$, concluding the proof. ■

By the same argument we obtain convergence in f.d.d. if we don't assume the tightness (C3) of the sequence $\left\{ X^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$.

Corollary 6.6. *Assume that $(S, \mathcal{U})$ is compact and that $X^{(n)}$ is conservative for each $n \in \mathbb{N}^\infty$. Under conditions (C1), (C2) and (C4) $X^{(n)}$ converges in f.d.d. to $X^{(\infty)}$ for all sequences of initial distributions $\left(\mu^{(n)}\right)_{n\in\mathbb{N}} \subset \mathcal{M}_1(S)$ with $\mu^{(n)} \in \mathcal{M}_1(S^{(n)})$ and $\mu^{(n)} \Rightarrow \mu^{(\infty)} \in \mathcal{M}_1(S^{(\infty)})$.*

6.4 Discussing the assumptions

Recall our discussion of the two examples at the end of the last chapter in Remark 5.97. In both examples the behavior of the process X is linked to the geometric structure of the state space. For the Brownian motion on a Riemannian manifold M through the Riemannian metric tensor g which induces both the metric d on the manifold and the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(M, \mathrm{d}V)$ that defines the Brownian motion. A different point of view is that the generator of the Brownian motion on M is given by the Laplace-Beltrami operator Δ, which is defined in terms of the metric tensor g and therefore related to the geometry of M.

A similar connection exists for the resistance form. Here the resistance form $(\mathcal{E}, \mathcal{F})$ itself is the link between the resistance metric $\mathcal{R}$ and the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(S, \nu)$ and consequently the process X.

This connection between the geometry of the state space and the process is sometimes (cf. [Sto63; AEW13; ALW17]) expressed by saying that the process is on its "*natural scale*".

6.4.1 Convergence of Green operators (C4)

For the resistance forms, the relation between the process and the metric of the state place is explicit in the following way. By Proposition 5.85 together with (5.309), we have that

$$\mathbb{P}_x(\tau_y < \tau_A) = \frac{g_A(x,y)}{g_A(y,y)} = \frac{\mathcal{R}(A,x) + \mathcal{R}(A,y) - \mathcal{R}(x,y)}{2\mathcal{R}(A,y)}. \tag{6.32}$$

Consequently, the probabilities of hitting one point before the other can be expressed in terms of their mutual distances and their distance to the starting point. Similarly, the Green kernel $g_A(x,y)$ and consequently the Green operator $G_A f(x) = \int_S g_A(x,y) f(y)\, \nu(\mathrm{d}y)$ is determined by the resistance metric. As a consequence, it is plausible that for resistance forms the Hausdorff-weak convergence (C1) already implies the convergence of the Green operators, (C4). Indeed, this is the case as shown in [Cro18, Lemma 5.5]. Note that the Green kernel can also be expressed in terms of the probabilities to hit one point before another, by (6.32).

Under our assumptions the processes are not on their "natural scale" as we do not have a *scale* (i.e. a metric) on the state spaces, to begin with. We have, however, defined a notion of resistance between sets in Section 5.5. We have shown in Proposition 5.77 that the minimizer of the variational problem for $\mathcal{R}(A,B)$ is given by $\mathbb{P}_x(\tau_B < \tau_A)$.

This leads us to believe that the following conjecture holds true.

Conjecture 6.7. *Suppose that $(S, \mathcal{U})$ is compact and that (C1) holds. Then (C4) is equivalent to the following condition.*

(C4*) *For all $A, B \in \mathcal{B}(S)$ closed with $\tau_A < \infty$ $\mathbb{P}_x^{(\infty)}$-a.s. for all $x \in S^{(\infty)}$,*

$$\lim_{n\to\infty} \mathbb{P}_{x_n}^{(n)}(\tau_A < \tau_B) = \mathbb{P}_{x_\infty}^{(\infty)}(\tau_A < \tau_B) \tag{6.33}$$

for every sequence $(x_n)_{n\in\mathbb{N}} \subset S$ with $x_n \in S^{(n)}$ and $\lim_{n\to\infty} x_n = x_\infty$.

Using the relation between the probabilities $\mathbb{P}_x(\tau_A < \tau_B)$ and the resistance $\mathcal{R}(A,B)$, we can rephrase Conjecture 6.7 as follows.

Conjecture 6.8. *Suppose that $(S, \mathcal{U})$ is compact and that (C1) holds. Then (C4) is equivalent to the following condition.*

(C4**) *For all $A, B \in \mathcal{B}(S)$ closed with* $\mathrm{Cap}_1(A) > 0$,

$$\lim_{n\to\infty} \mathcal{R}^{(n)}(A, B) = \mathcal{R}^{(\infty)}(A, B), \tag{6.34}$$

where $\mathcal{R}^{(n)}$ denotes the effective resistance associated with the process $X^{(n)}$, $n \in \mathbb{N}^\infty$, as defined in Definition 5.73.

6.4.2 Tightness (C3)

By Theorem 4.75, condition (C3) holds whenever the probability that $X^{(n)}$ moves far from its starting point in a short period of time is uniformly bounded in the starting point and $n \in \mathbb{N}$. Hausdorff weak convergence of the state spaces is one important property to ensure the tightness. Because it ensures that there are no areas of the state space, where the measure $\nu^{(\infty)}$ vanishes in the limit while having positive measure for all $n \in \mathbb{N}$. This would mean that the limiting process moves increasingly faster through these areas, causing (4.204) of Theorem 4.75 to fail.

For both special cases, for resistance forms and the Brownian motion on manifolds the tightness follows from Theorem 4.75 by the Gromov-Hausdorff weak convergence of the state spaces together with an additional condition. In [Cro18, Assumption 1.1 b)], this additional condition for resistance forms is a uniform recurrence condition given by

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathcal{R}^{(n)}\left(x_n, \complement B^{(n)}(x_n, r)\right) = \infty. \tag{6.35}$$

In [CHK17, Assumption 1.2], the authors instead assumed the stronger[1] *uniform volume doubling condition*, which claims that there exists a non-decreasing function $v\colon (0,\infty) \to (0,\infty)$ and constants $c_1, c_2, c_3 > 0$ such that $v(2r) \le c_1 v(r)$ for every $r > 0$ and

$$c_2 v(r) \le \nu^{(n)}\left(B^{(n)}(x, r_n)\right) \le c_3 v(r), \quad \forall x \in S^{(n)},\ r_n \in [R_0(n), R_\infty(n) + 1], \tag{6.36}$$

where

$$R_0(n) := \inf_{x,y\in S^{(n)},\ x\neq y} \left\{\mathcal{R}^{(n)}(x,y)\right\} \quad \text{and} \quad R_\infty(n) := \sup_{x,y\in S^{(n)}} \left\{\mathcal{R}^{(n)}(x,y)\right\}. \tag{6.37}$$

For Brownian motions on Riemannian manifolds, Suzuki imposes the following condition in [Suz19a, Lemma 5.6 (ii)] on the sequence of initial distribution. First,

[1] See Croydon's discussion of these assumptions in [Cro18, Remark 1.3 b)].

the sequence $\mu^{(n)}$ of initial distributions must be absolutely continuous with respect to the volume measure $\mathrm{d}V_n$. Moreover the Radon-Nikodym derivatives $\varphi_n = \frac{d\mu^{(n)}}{\mathrm{d}V_n}$ are uniformly bounded in the following sense. For every $r > 0$ there exists a $M_r > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup \left\{ |\varphi_n(z)| \,\middle|\, z \in B^{(n)}(x_n, r) \right\} < M_r < \infty. \tag{6.38}$$

It is far from obvious what additional assumption would be needed in our situation to apply Theorem 4.75.

6.4.3 Uniform equicontinuity of the semigroups (C2)

In [ALW17, Lemma 5.3] and in [Cro18, Lemma 5.2], the authors use a coupling argument to show the uniform equicontinuity of the semigroups (C2). Here the connection between the behavior of the process and the geometric structure of the state space given by the resistance metric plays a fundamental role.

6.4.4 Hausdorff-weak convergence of the state spaces (C1)

The results in [ALW17], [Cro18] and [Suz19a] all allow the processes to live on different state spaces. A part of the results, or more precisely, part of the assumption, is that the state spaces $S^{(n)}$ can be isometrically embedded into a common ambient space.

We also allow the processes to have different state spaces we do, however, assume that these state spaces are already subsets of an ambient space. This is due to the lack of a metric structure on the state spaces $S^{(n)}$. Without a metric structure we cannot define isometric embeddings and hence no Gromov-Hausdorff convergence.

A possible extension of our result would rely on the definition of convergence of uniform measure spaces.

Remarks and Outlook

7

It is possible to commit no mistakes and still lose. That is not a weakness; that is life.

— **Jean Luc Picard**
Star Trek: The Next Generation

In this chapter, we collect various remarks and present an outlook of further research questions that could not be answered in this thesis.

7.1 Chapter 3: The path space

In the discussion following Lemma 3.20 we already mentioned that it would make the following results a bit stronger if it could be shown that completeness of the uniform space $(S, \mathcal{U})$ implies completeness of the path space $D_S([0, \infty))$ *without* the assumption of metrizability. Indeed we conjecture that the following is true.

Conjecture 7.1. *Assume $(S, \mathcal{U})$ is a uniform Hausdorff space. If $(S, \mathcal{U})$ is complete then so is $D_S([0, \infty))$.*

7.2 Chapter 4: Symmetric Feller processes

In Example 4.15 we have shown that fixed jump times break the homogeneity property of a Markov process. The following conjecture seems like it should be well-known. Nevertheless, I have not been able to find a reference and will leave the proof as an open problem.[1]

Conjecture 7.2. *Let X be a homogeneous Markov process with state space $(S, \mathcal{U})$. Then for all $s \geq 0$ and $x \in S$*

$$\mathbb{P}_x(X_s \neq X_{s-}) = 0. \tag{7.1}$$

In Proposition 4.34 we have shown that this conjecture is indeed true if we assume that the semigroup of X has the Feller property.

[1] The author has posted this conjecture online: https://math.stackexchange.com/q/4443534/1054746

7.2.1 The killed process

In Section 4.3.1 we show that the Feller property, the strong Feller property and the ν-symmetry of the original process X carry over to the killed process X^A. It remains an open question whether also the strong ν-symmetry is preserved under killing.

7.2.2 Uniqueness by hitting times

In order to establish the uniqueness of a Feller process by its hitting times through the Green operator in Section 4.3.3, we have to make some quite restrictive assumptions. On the one hand, we assume that the state space is compact and on the other hand, we need the strong Feller property in the proof of Theorem 4.72. The compactness is needed to make sure that the 0-resolvent of a transient process and the Green operator are bounded operators. This assumption can certainly be relaxed as can be seen in our discussion of transient Dirichlet forms in Section 5.3.2. For more on this topic see [CF11, Section 2.1].

We have the following conjecture which might be too strong.

Conjecture 7.3. *Let $(S, \mathcal{U})$ be a locally compact uniform Hausdorff space and ν a Radon measure on $(S, \mathcal{B})$ with full support. Suppose X is a transient ν-symmetric Feller process with values in S_ϑ. Then the 0-resolvent given by*

$$Rf(x) := \int_0^\infty P_t f(x)\,\mathrm{d}t \tag{7.2}$$

for $f \in C_\infty(S)$ and $x \in S$ is a bounded operator mapping $C_\infty(S)$ to $C_\infty(S)$.

It is worth pointing out that the proof of Theorem 4.72 is the only point where we need the strong Feller property. If we could show that the same conclusion holds under the normal Feller property, we could significantly strengthen our results in Chapter 6.

7.3 Chapter 5: Dirichlet Forms and symmetric Feller Processes

We have tried to extract the most important parts from the very rich analytic theory of Dirichlet forms. This leads necessarily to some gaps. For example, we completely ignore the question if there is a ν-symmetric Feller associated with every Dirichlet form. The answer to this question is basically "yes" but a precise statement needs more potential theory than we can develop in this short summary. A partial answer is given by the following theorem.

Theorem 7.4 ([CF11, Theorem 1.5.1]). *Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form on $L^2(S, \nu)$, where S is a locally compact separable metric space and ν is a Radon measure on S with full support. Then there exists a Hunt process X with values in S and a ν-symmetric transition function such that $(\mathcal{E}, \mathcal{D})$ is the Dirichlet form associated with X.*

For a more general result see [CF11, Theorem 1.5.2]. Even uniqueness in distribution of the associated processes can be shown [CF11, Theorems 3.1.12 & 3.1.13]. However, uniqueness only holds for quasi-all starting points, in other words, outside a set of zero capacity.

7.3.1 Resistance

In Remark 5.75 we have hinted at a possible extension of the definition of the effective resistance to arbitrary subsets of S.

One possible extension is given by the following.

Definition 7.5. Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form on $L^2(S, \nu)$. For two subsets $A, B \subset S$ with $\mathrm{Cap}_1(A), \mathrm{Cap}_1(B) > 0$ and A closed the *(effective) resistance* between A and B is defined as

$$\mathcal{R}(A, B) := \sup \left\{ \mathcal{E}(f, f)^{-1} \;\middle|\; f \in \mathcal{F}_A^B \right\}, \tag{7.3}$$

where we set $\sup \emptyset = 0$. For arbitrary sets $A, B \subset S$ we define

$$\mathcal{R}(A, B) := \sup \{ \mathcal{R}(K, B) \mid K \supset A \text{ and } K \subset S \text{ closed} \} \tag{7.4}$$

whenever $\mathrm{Cap}_1(A) > 0$ and $\mathcal{R}(A, B) = 0$ otherwise. ◇

This definition would lead to the following consequence.

Proposition 7.6. *Suppose $A, B \subset S$ with positive capacity and $\mathcal{R}(A, B) > 0$, then*

$$\mathcal{R}(A, B) = \mathcal{R}(\overline{A}, B). \tag{7.5}$$

Proof. First observe that for $F \subset S$ with $A \subset F$, it holds hat $\mathcal{R}(F, B) \leq \mathcal{R}(A, B)$. To see that take any closed set $K \subset S$ with $F \subset K$, then K also contains A. Hence,

$$\begin{aligned} \mathcal{R}(F, B) &:= \sup \{ \mathcal{R}(K, B) \mid K \supset F \text{ and } K \subset S \text{ closed} \} \\ &\leq \sup \{ \mathcal{R}(K, B) \mid K \supset A \text{ and } K \subset S \text{ closed} \} = \mathcal{R}(A, B). \end{aligned} \tag{7.6}$$

Now the claim follows simply from the fact that $\overline{A} \subset K$ for all closed $K \subset S$ with $A \subset K$. ■

Furthermore, we conjecture that the same variational problem, (7.3) defines the effective resistance for arbitrary sets A, B.

Conjecture 7.7. *Let $(\mathcal{E}, \mathcal{D})$ be a regular Dirichlet form. Suppose $A, B \subset S$ are such that $\mathcal{R}(A, B) > 0$, then the resistance between A and B is given by the following variational problem*

$$\mathcal{R}(A, B) := \sup\left\{ \mathcal{E}(f, f)^{-1} \;\middle|\; f \in \mathcal{F}_A^B \right\}. \tag{7.7}$$

Sketch of proof. If A is closed (7.7) is just the definition of the resistance. Suppose A is not closed. Then $\mathcal{R}(A, B) = \mathcal{R}(\overline{A}, B)$ by Proposition 7.6. Now let $f \in \mathcal{F}_A$, i.e. $f \in \mathcal{D}$ is such that its quasi continuous version $\tilde{f}$ is 0 q.e. on A. By quasi continuity, $\tilde{f} = 0$ q.e. on $\overline{A}$. Hence, $\mathcal{F}_{\overline{A}} \subset \mathcal{F}_A$. The reverse inclusion is trivial and consequently $\mathcal{F}_{\overline{A}} = \mathcal{F}_A$. Recall the definition of the boundary of A, $\partial A := \overline{A} \setminus A^\circ$ from Definition A.2. Since for every $f \in \mathcal{F}_A$, $\tilde{f} = 0$ q.e. on ∂A, we have that $f = 0$ ν-a.e. on ∂A. Consequently, every limit in $L^\infty(S \setminus A, \nu)$ of an $\mathcal{E}$-Cauchy sequence in $\mathcal{F}_A$ is 0 ν-a.e. on $\overline{A} \setminus A$. That means we can identify $\mathcal{D}_e^A = \mathcal{D}_e^{\overline{A}}$ and consequently $\mathcal{F}_A^B = \mathcal{F}_{\overline{A}}^B$ which implies the statement. □

7.4 Chapter 6: Convergence of symmetric Feller processes

In the proof of Theorem 6.2 we stray from our path to completely avoid metrics. However, observe that the proof does not depend on the choice of the metric. We strongly believe that the statement is provable without using metrics. That would require an in-depth analysis of the convergence of (probability) measures on uniform spaces which would go beyond the scope of this thesis. It could be subject of further research in order to establish a full theory of stochastic processes on uniform spaces.

Topology

A

We formally introduce the basic topological concepts used throughout this thesis.

Throughout this chapter let $S \neq \emptyset$ be a non-empty set and denote by $\mathcal{P}(S) = \{A \subset S\}$ the powerset of S.

A.1 Fundamentals of topology

Definition A.1. Let S be a non-empty set. A family $\mathcal{T}$ of subsets of S is called a *topology* (on S), if it satisfies

(i) $S, \emptyset \in \mathcal{T}$,

(ii) any union of elements in $\mathcal{T}$ belongs again to $\mathcal{T}$,

(iii) finite intersections of elements in $\mathcal{T}$ belong again to $\mathcal{T}$.

The pair $(S, \mathcal{T})$ is called a *topological space*. When there can be no confusion about the topology $\mathcal{T}$ we sometimes refer to S as a topological space.

The elements of $\mathcal{T}$ are called *open sets*. A set $A \subset S$ is called *closed* if its *complement* $A^c := S \setminus A$ is open. ◇

Observe that openness and closedness are not complementary: A set can be both open and closed – or neither.

Definition A.2 (Interior and closure of sets)**.** Let $A \subset S$ be a set. The *interior* of A is defined as

$$A^\circ := \bigcup_{U \subset A \,:\, U \text{ open}} U. \tag{A.1}$$

Conversely, we define the *closure* of A as

$$\overline{A} := \bigcap_{K \supset A \,:\, K \text{ closed}} K. \tag{A.2}$$

We call the set theoretic difference $\partial A := \overline{A} \setminus A^\circ$ the *boundary* of the set A. ◇

It is easy to check that the interior of a set A is open and the closure is closed. Furthermore, a set $A \subset S$ is open if and only if $A = A^\circ$ and it is closed if and only if $A = \overline{A}$.

A neighborhood of $x \in S$ is a set V such that there exists an open set $U_x \in \mathcal{T}$ with $U_x \subset V$ and $x \in U_x$. We sometimes write

$$\mathcal{U}_x := \{ U \subset S \mid \exists U \in \mathcal{T} : U \subset V \text{ and } x \in U \} \tag{A.3}$$

for the *neighborhood system at* x.

The family of all neighborhood systems $\{ \mathcal{U}_x \mid x \in S \}$ is clearly determined by the topology on S. But the converse is also true.

Proposition A.3. *Let S be a topological space and $\mathcal{U}_x$ the neighborhood system of $x \in S$. Then $\mathcal{U}_x$ satisfies*

(i) $x \in U$ *for all* $U \in \mathcal{U}_x$,

(ii) $U \cap V \in \mathcal{U}_x$ *for all* $U, V \in \mathcal{U}_x$,

(iii) for all $U \in \mathcal{U}_x$ *there exists a* $V \in \mathcal{U}_x$, *such that* $U \in \mathcal{U}_y$ *for each* $y \in V$,

(iv) if $U \in \mathcal{U}_x$ *and* $U \subset V$, *then* $V \in \mathcal{U}_x$.

Furthermore

(v) $U \subset S$ *is open if and only if* U *contains a neighborhood of all its elements.*

Conversely, if for each $x \in S$ *there is a family* $\mathcal{U}_x$ *satisfying (i)–(iv), then the family of* open *sets in the sense of (v) is a topology and the neighborhood system of* x *in this topology is* $\mathcal{U}_x$.

We shall not provide a proof of this statement. Instead, we focus on a similar statement for bases of neighborhood systems.

Definition A.4. Let S be a topological space. A base of the neighborhood system at $x \in S$ or a *neighborhood base at* x is a family $\mathcal{N}_x \subset \mathcal{U}_x$ such that

$$\forall U \in \mathcal{U}_x \; \exists V \in \mathcal{N}_x : V \subset U. \tag{A.4}$$

The elements of the neighborhood base $\mathcal{N}_x$ are called *basic neighborhoods of* x. ♢

We categorize topological spaces by the size of their bases, i.e. whether they have a countable base or at least a countable neighborhood base at every point $x \in S$.

Definition A.5. A topological space $(S, \mathcal{T})$ is said to be *first countable*, if $\mathcal{T}$ possesses a countable neighborhood base $\mathcal{N}_x$ at every $x \in S$. We call a topological space *second countable* if the topology $\mathcal{T}$ has a countable base. ♢

Proposition A.6. *Let S be a topological space and $\mathcal{N}_x$ a neighborhood base at $x \in S$. Then $\mathcal{N}_x$ satisfies*

(i) $x \in U$ for all $U \in \mathcal{N}_x$,

(ii) for any $U_1, U_2 \in \mathcal{N}_x$ there exists $V \in \mathcal{N}_x$ such that $V \subset U_1 \cap U_2$,

(iii) for any $U \in \mathcal{N}_x$ there exists a $V \in \mathcal{N}_x$ such that for all $y \in V$ there exists a $W \in \mathcal{N}_y$ such that $W \subset U$.

Furthermore,

(iv) $U \subset S$ is open if and only if U contains a basic neighborhood of each of its elements.

Conversely, if we assign to each $x \in S$ a family $\mathcal{N}_x$ satisfying (I)–(III) and use (IV) to define the open subsets of S, then the result is a topology in which a neighborhood base for each $x \in S$ is given by $\mathcal{N}_x$.

Proof. Assertion (i) is evident from the definition of neighborhoods and the fact that $\mathcal{N}_x \subset \mathcal{U}_x$. To show (ii) let $U_1, U_2 \in \mathcal{N}_x$ and observe that $U := U_1^\circ \cap U_2^\circ$ is open and contains x, hence $U \in \mathcal{U}_x$. By definition Definition A.4, there exists a $V \in \mathcal{N}_x$ such that $V \subset U \subset U_1 \cap U_2$. Now, let $U \in \mathcal{N}_x$ then $U^\circ \in \mathcal{U}_x$ and by definition of $\mathcal{N}_x$ there exists a $V \in \mathcal{N}_x$ with $V \subset U^\circ$. Clearly, $U \in \mathcal{U}_y$ for all $y \in V$ and by definition of $\mathcal{N}_y$ there exists a $W \in \mathcal{N}_y$ with $W \subset U$, establishing (iii). For (iv) assume that $U \subset S$ is open, then U is a neighborhood for each of its elements and a fortiori contains a basic neighborhood of all its elements. Now assume that $U \subset S$ contains a basic neighborhood V_x of all its elements $x \in U$. Then $x \in V_x^\circ \subset U$ and by taking the union over all $x \in U$ we get $U \subset \bigcup_{x \in U} V_x \subset U$ and hence U is the union of open sets and thus open.

For the converse assertion assume that we are given a family $\mathcal{N}_x$ at each $x \in S$ satisfying (I)–(III). Let

$$\mathcal{T} := \{\, U \subset S \mid \forall x \in U \; \exists V \in \mathcal{N}_x \colon V \subset U \,\}. \tag{A.5}$$

We first show that $\mathcal{T}$ is a topology. Clearly $S, \emptyset \in \mathcal{T}$, the former because it contains all subsets and the latter because it has no elements. Let $\mathcal{U} \subset \mathcal{T}$ be any family of open sets. For each $x \in A := \bigcup_{U \in \mathcal{U}} U$ there exists a $U \in \mathcal{U}$ and a $V \in \mathcal{N}_x$ such that $V \subset U$. Thus $V \in A$ and A contains a basic neighborhood of each of its elements and thus $A \in \mathcal{T}$. Assume $n \in \mathbb{N}$ and $U_1, \dots, U_n \in \mathcal{T}$ and let $x \in U := \bigcap_{k=1}^n U_k$. Then there is a basic neighborhood $U_x^k \in \mathcal{N}_x$ with $U_x^k \subset U_k$ for all $k = 1, \dots, n$ and by (II) and induction there exists $V \in \mathcal{N}_x$ such that $V \subset \bigcap_{k=1}^n U_x^k \subset U$ and hence $U \in \mathcal{T}$ because $x \in U$ was arbitrary.

Now, for each $x \in S$ let $\mathcal{U}_x := \{\, U \subset S \mid \exists V \in \mathcal{N}_x \colon V \subset U \,\}$. It remains to be shown that $\mathcal{U}_x$ defines a neighborhood system at each $x \in S$. Let $x \in S$ and $U \in \mathcal{U}_x$, by (I) $x \in U$ and it suffices to show that $x \in U^\circ$. The assertion follows immediately from (III) if we can show that $U^\circ = \left\{ y \in U \,\middle|\, U \in \mathcal{U}_y \right\} =: \iota(U)$. The inclusion $U^\circ \subseteq \iota(U)$ is clear by (IV), the fact that U° is open and the definition of $\mathcal{U}_x$. For the converse inclusion, it is enough to show that $\iota(U) \in \mathcal{T}$. Let $y \in \iota(U)$, then there exists $V \in \mathcal{N}_y$ such that $V \subset U$. By (III) there exists a $B \subset \mathcal{N}_y$ such that for all $z \in B$ there exists a $W \in \mathcal{N}_z$ such that $W \subset V \subset U$. And by construction, all these W are subsets of U and hence $U \in \mathcal{U}_z$ for all $z \in B$ which means $B \subset \iota(U)$. Since y was arbitrary, $\iota(U)$ contains a basic neighborhood of all its elements and is thus open. ■

Remark A.7. In the proof of Proposition A.6 we have actually shown more. If at each $x \in S$ we have families $\mathcal{N}_x$ and $\mathcal{N}'_x$ satisfying (i)–(iii) then the topologies induced by $\{ \mathcal{N}_x \mid x \in S \}$ and $\{ \mathcal{N}'_x \,\middle|\, x \in S \}$ coincide. ◇

There are various ways to define a topology on a set S. In Section 2.6 we use the *closure operator* to define the topology induced by a proximity.

Definition A.8 (Closure operator)**.** Let $S \neq \emptyset$ be a set. A map $\Gamma \colon \mathcal{P}(S) \to \mathcal{P}(S)$ is called *closure operator* if it satisfies the following conditions for all $A, B \in \mathcal{P}(S)$

(i) $A \subset \Gamma(A)$,

(ii) $\Gamma(\Gamma(A)) = \Gamma(A)$,

(iii) $\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B)$,

(iv) $\Gamma(\emptyset) = \emptyset$.

If Γ is a closure operator, we write $\overline{A} := \Gamma(A)$. ◇

Clearly, the operation $\overline{A}$ defined by (A.2) defines a closure operator.

Proposition A.9. *Let Γ be a closure operator on $\mathcal{P}(S)$. The family*

$$\mathcal{T} := \left\{ U \in \mathcal{P}(S) \,\middle|\, \Gamma\left(\complement U\right) = \complement U \right\} \tag{A.6}$$

is a topology on S. In other words, the closed sets are those sets $A \in \mathcal{P}(S)$ for which $\Gamma(A) = A$ holds.

Proof. By (i) of Definition A.8, S is closed and hence, $\emptyset \in \mathcal{T}$. On the other hand, by (iv), $\emptyset$ is closed and hence S is open. Now let $U_1, \ldots, U_n \in \mathcal{T}$. Using (iii) and induction we get

$$\Gamma\left(\complement \bigcap_{j=1}^{n} U_j\right) = \Gamma\left(\bigcup_{j=1}^{n} \complement U_j\right) = \bigcup_{j=1}^{n} \Gamma\left(\complement U_j\right) = \bigcup_{j=1}^{n} \complement U_j = \complement \bigcap_{j=1}^{n} U_j \tag{A.7}$$

and hence $\bigcap_{j=1}^{n} U_j \in \mathcal{T}$.

It remains to show that if I is any index set and $\{ U_i \mid i \in I \} \subset \mathcal{T}$ we have that $\bigcup_{i \in I} U_i \in \mathcal{T}$. First observe that if $A, B \subset S$ with $A \subset B$, we have $\Gamma(B) = \Gamma(A) \cup \Gamma(B \setminus A)$ by (iii) and hence $\Gamma(A) \subset \Gamma(B)$. As $\bigcap_{i \in I} \complement U_i \subset \complement U_j$ for every $j \in I$, we conclude

$$\Gamma\left(\bigcap_{i \in I} \complement U_i\right) \subset \bigcap_{i \in I} \Gamma\left(\complement U_i\right) = \bigcap_{i \in I} \complement U_i. \tag{A.8}$$

Using (i) we obtain the reverse inclusion and hence

$$\Gamma\left(\bigcap_{i \in I} \complement U_i\right) = \bigcap_{i \in I} \complement U_i, \tag{A.9}$$

which implies, by definition of $\mathcal{T}$, $\bigcup_{i \in I} U_i \in \mathcal{T}$. ■

Recall the following fundamental concepts of topology.

Definition A.10. Let $(S, \mathcal{T})$ be a topological space.

(i) A point $x \in S$ is called a *cluster point* (or *point of accumulation*) of a set $A \subset S$ if every basic neighborhood $V \in \mathcal{N}_x$ of x contains some point $y \in A \setminus \{x\}$.

(ii) A set $K \subset S$ is called *compact* if and only if for every open covering of K, i.e. a family $\mathcal{U} \subset \mathcal{T}$ with $K \subset \bigcup_{U \in \mathcal{U}} U$, there exists a finite open subcover of K, i.e. there exists a family $\mathcal{V} \subset \mathcal{U}$ such that $K \subset \bigcup_{V \in \mathcal{V}} V$ and $|\mathcal{V}| < \infty$. ◇

Lemma A.11. *A set $A \subset S$ is closed if and only if all cluster points of A are contained in A. Furthermore $\overline{A} = A \cup \{ x \in S \mid x \text{ is a cluster point of } A \}$.*

Proof. Assume that $A \subset S$ is closed. Then $\complement A$ is open and for all $x \in \complement A$ there exists a basic neighborhood $V \in \mathcal{N}_x$ such that $V \cap A = \emptyset$. Hence all cluster points of A are already contained in A. Assume the converse is true, i.e. all cluster points of A are contained in A. Then, for each $y \in \complement A$ there exists a basic neighborhood $V \in \mathcal{N}_y$ such that $V \subset \complement A$ and hence $\complement A$ is open and A is closed. For the last part of the claim observe that $A \subset \overline{A}$ and $\overline{A}$ is closed and consequently contains all its cluster points and a fortiori all cluster points of A. To show the converse inclusion "$\supseteq$" observe that any closed subset $K \subset S$ that contains A has to contain all cluster points of A and so does the intersection of all such sets which is, by definition, $\overline{A}$. ■

Definition A.12 (relatively compact subsets)**.** Let $(S, \mathcal{T})$ be a topological Hausdorff space. A subset $A \subset S$ is *relatively compact* if its closure $\overline{A}$ is compact in $(S, \mathcal{T})$. ◇

Sometimes, especially in probability theory, it is useful to work with topological spaces that are "almost countable" in the following sense.

Definition A.13 (Separability)**.** Let $(S, \mathcal{T})$ be a topological space.

(i) A subset $A \subset S$ is said to be *dense* in S, if $\overline{A} = S$.

(ii) The topological space $(S, \mathcal{T})$ is called *separable*, if there exists a countable dense subset $D \subset S$. ◇

Definition A.14 (Complete regularity)**.** A topological space $(S, \mathcal{T})$ is called *completely regular* if for every closed set $A \subset S$ and every $x \in S \setminus A$ there exists a real valued continuous function $f : S \to \mathbb{R}$ with $f|_A = 0$ and $f(x) = 1$. ◇

Definition A.15 (Hausdorff property)**.** A topological space $(S, \mathcal{T})$ has the *Hausdorff property* if for every $x, y \in S$ with $x \neq y$ there exist neighborhoods $U_x \in \mathcal{U}_x$ and $U_y \in \mathcal{U}_y$ of x and y respectively such that $U_x \cap U_y = \emptyset$. ◇

If $(S, \mathcal{T})$ has the Hausdorff property we refer to $(S, \mathcal{T})$ as a Hausdorff space.

Hausdorff spaces have the following nice property.

Lemma A.16. *Let $(S, \mathcal{T})$ be a Hausdorff topological space. Then every compact subset of S is closed.*

Proof. Assume $K \subset S$ is compact and let $x \in S \setminus K$ be arbitrary. By the Hausdorff property, for each $y \in K$ there exist open neighborhoods $U_x(y)$ of x and V_y of y such that $U_x(y) \cap V_y = \emptyset$. Clearly, $\left\{ V_y \,\middle|\, y \in K \right\}$ is an open covering of K, hence, by compactness, there exists a finite subset $F \subset K$ such that $\left\{ V_y \,\middle|\, y \in F \right\}$ is already an open covering of K. Then

$$U_x := \bigcap_{y \in F} U_x(y) \tag{A.10}$$

is an open neighborhood of x, disjoint from K. Since $x \in S \setminus K$ was arbitrary this implies that $S \setminus K$ can be written as the union of open sets and is therefore open, which proves the claim. ∎

We introduce some concepts that are closely related to compactness.

Definition A.17. Let $(S, \mathcal{T})$ be a topological Hausdorff space.

(i) $(S, \mathcal{T})$ is said to be *locally compact* if each point has a compact neighborhood.

(ii) $(S, \mathcal{T})$ is called *σ-compact* if S is the union of countably many compact sets.

(iii) $(S, \mathcal{T})$ is said to be *Lindelöf* if every open cover of S has a countable subcover. ◊

Note that some authors refer to the property in Definition A.17 as *weakly locally compact* and require the existence of a local base of compact neighborhoods for each $x \in S$ for local compactness. However, the two definitions are equivalent for topological Hausdorff spaces by [Wil70, Theorem 18.2].

Clearly, if $(S, \mathcal{T})$ itself is compact, then S is a compact neighborhood of each $x \in S$ and $(S, \mathcal{T})$ is locally compact.

Lemma A.18. *Let $(S, \mathcal{T})$ be a locally compact Hausdorff space. Suppose $A \subset S$ is open or closed, then A is locally compact with respect to the subspace topology.*

Proof. First, let $A \subset S$ be closed. Fix $x \in A$ and take a compact neighborhood $K_x \subset S$ of x in S. Clearly, $K_x \cap A$ is closed in S and therefore compact. Since every open subset of A is of the form $U \cap A$ for some $U \in \mathcal{T}$, every open cover of $K_x \cap A$ in A gives rise to an open cover of K_x in S which possesses an open subcover $\{ U_k \in \mathcal{T} \mid k = 1, \ldots, n \}$ of K_x, by compactness. Then, $\{ U_k \cap A \mid k = 1, \ldots, n \}$ is a finite open subcover of $K_x \cap A$.

If A is open, on the other hand, the case is simple. Let $x \in A$, then there exists an open neighborhood $U_x \subset A$ of x which contains a compact neighborhood K_x (in S). But then K_x is also a compact neighborhood of x in A. ■

Lemma A.19. *Every second countable topological space is Lindelöf.*

Proof. Let $(S, \mathcal{T})$ be a second countable topological space. Suppose that an open cover of S is given by $\{ A_\alpha \in \mathcal{T} \mid \alpha \in \mathbb{I} \}$. Since $\mathcal{T}$ possesses a countable base $\{ U_n \in \mathcal{T} \mid n \in \mathbb{N} \}$, we can deduce that for each $\alpha \in \mathbb{I}$ there exists a $n \in \mathbb{N}$ such that

$$U_n \subset A_\alpha. \tag{A.11}$$

Now choose a subset $\mathbb{J} \subset \mathbb{I}$ such that for each $n \in \mathbb{N}$, $U_n \subset A_\beta$ for at most one $\beta \in \mathbb{J}$. Then, $\mathbb{J}$ is at most countable and $\left\{ A_\beta \,\middle|\, \beta \in \mathbb{J} \right\}$ is a countable open cover of S. ■

We can relate the different notions in Definition A.17 to each other in the following way.

Lemma A.20. *Let $(S, \mathcal{T})$ be a locally compact Lindelöf space. Then $(S, \mathcal{T})$ is σ-compact.*

Proof. By local compactness, there exists for each $x \in S$ a compact neighborhood K_x of x. By definition, there exists an open neighborhood U_x such that $U_x \subset K_x$ for each $x \in S$. Now, $\{ U_x \mid x \in S \}$ is an open cover of S and by the Lindelöf property there exists a countable collection $\{ x_n \in S \mid n \in \mathbb{N} \}$ such that

$$S \subset \bigcup_{n \in \mathbb{N}} U_{x_n} \subset \bigcup_{n \in \mathbb{N}} K_{x_n}. \tag{A.12}$$

Hence, $(S, \mathcal{T})$ is σ-compact. ■

Another notion related to compactness is that of a *paracompact* space.

Definition A.21 (Paracompactness)**.** Let $(S, \mathcal{T})$ be a topological Hausdorff space. A cover $\{ A_\alpha \in \mathcal{T} \mid \alpha \in \mathbb{I} \}$ is said to be *locally finite* if for every $x \in S$ there exists an open neighborhood U_x such that $U_x \cap A_\alpha \neq \emptyset$ for only finitely many $\alpha \in \mathbb{I}$. The space $(S, \mathcal{T})$ is called *paracompact* if every open cover possesses a locally finite refinement. In other words, for every open cover $\{ A_\alpha \in \mathcal{T} \mid \alpha \in \mathbb{I} \}$ of S there exists a locally finite open cover $\left\{ B_\beta \in \mathcal{T} \,\middle|\, \beta \in \mathbb{J} \right\}$ such that for all $\beta \in \mathbb{J}$ there exists an $\alpha \in \mathbb{I}$ such that $B_\beta \subset A_\alpha$. ◊

A.1.1 Compactification

Compactification is the process of embedding a topological space $(S, \mathcal{T})$ into a compact topological space $(\tilde{S}, \tilde{\mathcal{T}})$.

In the main text we use the *one-point compactification* or *Alexandrov compactification* of locally compact topological spaces.

Definition A.22 (One-point compactification)**.** Let $(S, \mathcal{T})$ be a locally compact Hausdorff topological space. The *one-point compactification* is the topological space $(S \cup \{\vartheta\}, \mathcal{T}_\vartheta)$, where $\vartheta \notin S$ is a single point and $\mathcal{T}_\vartheta := \mathcal{T} \cup \mathcal{T}'$, where

$$\mathcal{T}' := \left\{ (S \setminus C) \cup \{\vartheta\} \,\middle|\, C \subset S \text{ compact} \right\}. \tag{A.13}$$

◊

We also use the notation $S_\vartheta := S \cup \{\vartheta\}$ in the main text. Sometimes, the point ϑ in Definition A.22 is referred to as "point at infinity" and denoted by ∞. In the main text, we use the uncommon[1] ϑ instead of ∞, because the point in the one-point

[1] but not unprecedented (see e.g. [CW05])

compactification represents rather a cemetery state[2] of the state space of a Markov process than a point at infinity of some metric space.

We present the following useful properties of the one-point compactification.

Proposition A.23. *Let $(S, \mathcal{T})$ be a locally compact Hausdorff space and $(S_\vartheta, \mathcal{T}_\vartheta)$ its one-point compactification. Then the following assertions hold.*

(i) $(S_\vartheta, \mathcal{T}_\vartheta)$ *is compact and Hausdorff.*

(ii) The embedding map $\iota\colon S \hookrightarrow S_\vartheta$ *is continuous and open.*

(iii) When $(S, \mathcal{T})$ *is not already compact, then the image* $\iota(S)$ *of the embedding is dense in* S_ϑ.

(iv) When $(S, \mathcal{T})$ *is separable, then so is* $(S_\vartheta, \mathcal{T}_\vartheta)$.

Proof. Assume that $\mathcal{U} \subset \mathcal{T}_\vartheta$ is an open covering of S_ϑ. Then there exists a $V \in \mathcal{T}' \cap \mathcal{U}$, by assumption. I.e. $V = (S \setminus C) \cup \{\vartheta\}$ for some $C \subset S$ compact. Then $\mathcal{U} \setminus \{V\}$ is an open covering of C and by compactness of C, there is a finite open subcover $\mathcal{U}' \subset \mathcal{U}$ of C and furthermore $(\mathcal{U}' \cap \{V\}) \subset \mathcal{U}$ is a finite open cover of S_ϑ. To show the Hausdorff property it suffices to find disjoint open neighborhoods of ϑ and x for an arbitrary $x \in S$. By assumption, there exists a compact neighborhood C_x of x that contains an open neighborhood U_x of x. Then $C^c \cup \{\vartheta\}$ is an open neighborhood of ϑ and disjoint from U_x, which proves (i).

It is obvious that the embedding is an open map, i.e. it maps open sets to open sets. To show continuity, we need to show that the preimage of every set of the form $(S \setminus C) \cup \{\infty\}$ with $C \subset S$ compact is open. Hence, (ii) follows from Lemma A.16 and the fact that $(S, \mathcal{T})$ is Hausdorff by assumption.

Finally, (iii) follows because every compact set $C \subset S$ has non-empty complement in S and thus, ϑ is a cluster point of S in S_ϑ.

Assertion (iv) is trivial because we can amend the countable dense subset of $(S, \mathcal{T})$ by ϑ to get a countable dense subset of $(S_\vartheta, \mathcal{T}_\vartheta)$. ■

A.1.2 sequences

Given a topology $\mathcal{T}$ on a set S, it is possible to speak of converging sequences. A sequence is a subset of the ambient space S with countably many elements, thus the elements of a sequence can be indexed by the natural numbers and we write

$$(x_n)_{n\in\mathbb{N}} := \{\, x_n \in S \mid n \in \mathbb{N} \,\} \subset S. \tag{A.14}$$

[2]hence ϑ from greek: $\theta\alpha\nu\alpha\tau o\varsigma$, “Death”

Of course, we can index sets by arbitrary index sets I and we write $(x_i)_{i\in I}$ in that case. As a shorthand, we sometimes use the notation $(x_i)_i$, if the index set is clear.

Being a subset of a topological space $(S, \mathcal{T})$ we have a notion of cluster points for sequences from Definition A.10 (i). Recall the following definition of limit points of sequences.

Definition A.24. Let $(S, \mathcal{T})$ be a topological space and $(x_n)_{n\in\mathbb{N}} \subset S$ a sequence in S. Then $(x_n)_n$ converges to a *limit point* $x \in S$ if and only if for every open neighborhood U_x of x there exists an $n_0 \in \mathbb{N}$ such that $x_n \in U_x$ for all $n \geq n_0$. In that case, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. ♢

Observe that limit points are not well defined in general. Assume for example that S is equipped with the lump topology $\mathcal{T} = \{S, \emptyset\}$. Then S is the only open neighborhood for every $x \in S$ and by definition, every $x \in S$ is a limit point for every sequence in S. Clearly, the lump topology is not Hausdorff and it turns out that the Hausdorff property ensures uniqueness of limit points of sequences.

Lemma A.25. *Let $(S, \mathcal{T})$ be a Hausdorff topological space and $(x_n)_{n\in\mathbb{N}}$ a sequence in S. Assume that there exist $x, y \in S$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Then $x = y$.*

Proof. Assume $x \neq y$. Then there exist neighborhoods U_x and U_y of x and y, respectively, with $U_x \cap U_y = \emptyset$ by the Hausdorff property. Thus $x_n \notin U_x \cap U_y$ for any $n \in \mathbb{N}$, a contradiction to the assumption that both x and y are limit points of $(x_n)_n$. ■

A.1.3 Metrizable spaces

Recall the following definition.

Definition A.26 ((Pseudo) metric)**.** Let S be a non-empty set. A map $d\colon S \times S \to \mathbb{R}^+$ is a *pseudo metric* if it satisfies

(i) $d(x, y) = d(y, x)$ for all $x, y \in S$ (symmetry),

(ii) $d(x, y) \geq 0$ for all $x, y \in S$, (non negative definiteness),

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$ (triangle inequality).

If in addition $d(x, y) = 0$ if and only if $x = y$, we call d a *metric*.

We call a pair (S, d), where S is a set and d a (pseudo) metric on S a *(pseudo) metric space*. ♢

Given a metric d on a set S we write

$$B_d(x, \varepsilon) := \{ y \in S \mid d(x, y) < \varepsilon \} \tag{A.15}$$

for the *open ball* around $x \in S$ with radius $\varepsilon > 0$. When it is clear from the context which metric we are using, we sometimes drop the d from the notation of balls and write $B(\varepsilon, x) = B_\varepsilon(x)$. Using open balls we can always define a topology $\mathcal{T}$ on a metric space by taking $\mathcal{T}$ to be the collection of arbitrary unions and finite intersections of open balls. In that case, we say that d *generates* the topology $\mathcal{T}$. If not explicitly stated otherwise we always assume a metric space to be equipped with the topology generated by the given metric.

Definition A.27. A topological space $(S, \mathcal{T})$ is called *metrizable* if there exists a metric d on S that generates $\mathcal{T}$. ◊

Metrizable spaces have some nice features. For example, they are Hausdorff.

Proposition A.28. *Metrizable topological spaces are Hausdorff.*

Proof. Assume $(S, \mathcal{T})$ is metrizable and d is a metric that generates $\mathcal{T}$. Assume $x, y \in S$ with $x \neq y$, then there exists an $\varepsilon > 0$ such that $d(x, y) \geq 2\varepsilon$. Hence $B_d(x, \varepsilon)$ and $B_d(y, \varepsilon)$ are disjoint open neighborhoods of x and y, respectively. ■

Recall that a *Cauchy sequence* in a metric space (S, d) is a sequence $(x_n)_{n \in \mathbb{N}}$ such that for each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. This leads to the following definition.

Definition A.29. A metric d on a set S is called *complete* if every Cauchy sequence converges in the topology generated by d. Furthermore, we call the metric space (S, d) *complete* when d is a complete metric on S and we say that a topological space $(S, \mathcal{T})$ is completely metrizable when it is metrizable and there exists a complete metric that generates $\mathcal{T}$. ◊

Definition A.30. A metric space (S, d) is said to be *Polish* if it is complete and separable. ◊

In the main text, we are concerned with metrizable spaces without a specific metric. For a sensible treatment of these spaces, we need a slight weakening of the Polish property in the following sense.

Definition A.31 (Souslin and Lusin spaces)**.** A topological Hausdorff space $(S, \mathcal{T})$ is called a *Souslin space* if it is the image of Polish metric space under a continuous map Φ. If the map Φ is additionally bijective, we call $(S, \mathcal{T})$ a *Lusin space*. ◊

A.1.4 Weak topology

In this section, we collect some useful facts about weak or initial topologies.

Definition A.32. Let S be a nonempty set and (T, τ) a topological space. Assume $\mathcal{F} \subset \{f : S \to T\}$ is a non empty family of maps. The *initial* or *weak topology* generated by this family is the coarsest topology that makes every $f \in \mathcal{F}$ measurable. ◊

Let $\tau(\mathcal{F})$ be the weak topology on S as in the definition. Then it is easy to check that the family of sets

$$\left\{ f^{-1}A \,\middle|\, f \in \mathcal{F},\ A \in \tau \right\} \tag{A.16}$$

forms a subbase of the topology $\tau(\mathcal{F})$.

If the space (T, τ) is a metric space, we have the following useful characterization of $\tau(\mathcal{F})$.

Lemma A.33. *Let S be a non empty set, (T, d) a metric space and $\mathcal{F} \subset \{f : S \to T\}$ a non empty family of functions. Then the sets of the form*

$$\left\{ f^{-1}B(x, \varepsilon) \,\middle|\, f \in \mathcal{F},\ x \in T,\ \varepsilon > 0 \right\} \tag{A.17}$$

form a subbase of $\tau(\mathcal{F})$.

Lemma A.34. *Let S be a nonempty set and (T, τ) a topological Hausdorff space. Assume $\mathcal{F} \subset \{f : S \to T\}$ is a family of functions and equip S with the weak topology $\tau(\mathcal{F})$. A sequence $(x_n)_{n\in\mathbb{N}} \subset S$ in S converges to a limit $x \in S$ if and only if for all $f \in \mathcal{F}$, $f(x_n) \to f(x)$.*

Proof. The direction "⇒" is obvious by continuity. For the converse implication assume $(x_n)_{n\in\mathbb{N}} \subset S$ is such that there exists a $x \in S$ with

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in \mathcal{F}. \tag{A.18}$$

By definition of the weak topology $\tau(\mathcal{F})$, for every open neighborhood $U_x \in \tau(\mathcal{F})$ of x there exist finitely many $f_1, \ldots, f_m \in \mathcal{F}$ such that

$$x \in \bigcap_{j=1}^{m} f_j^{-1} V_j \subset U_x, \tag{A.19}$$

where $V_j \in \tau$ is an open neighborhood of $f_j(x)$ for every $j = 1, \ldots, m$. By assumption (A.18) we have $f_j(x_n) \in V_j$ eventually for every $j = 1, \ldots, m$. Hence, $x_n \in \bigcap_{j=1}^{m} f_j^{-1} V_j$ eventually, which means $\lim_{n\to\infty} x_n = x$, concluding the proof. ∎

A.2 Measures on topological spaces

Let $(S, \mathcal{T})$ be a topological space, recall that the *Borel σ-algebra* $\mathcal{B}(S)$ on S is the σ-algebra generated by the open sets. Another σ-algebra that can be defined on every topological space is the Baire σ-algebra $\mathrm{Ba}(S)$ which is generated by the continuous real-valued functions $C(S)$, i.e. $\mathrm{Ba}(S)$ is the smallest σ-algebra that makes all $f \in C(S)$ measurable. It is conceivable that the Borel σ-field and the Baire σ-field are not very different. Indeed the two σ-algebras coincide when S is a completely regular Souslin space (cf. [Bog07, Theorem 6.7.7]).

Definition A.35. Let $(S, \mathcal{T})$ be a locally compact topological Hausdorff space and $\mathcal{B}(S)$ the Borel σ-field on S. A *Radon measure* on $(S, \mathcal{B}(S))$ is a measure ν such that

(i) ν is finite on compact sets,

(ii) ν is *outer regular* on all Borel sets, i.e. for all $A \in \mathcal{B}(S)$

$$\nu(A) = \inf\{\, \nu(U) \mid U \in \mathcal{T}, U \supset A \,\}, \tag{A.20}$$

(iii) ν is *inner regular* on all open sets, i.e. for all $U \in \mathcal{T}$

$$\nu(U) = \sup\left\{\, \nu(K) \,\middle|\, K \subset S,\ \mathrm{K\ compact}, U \subset K \,\right\}. \tag{A.21}$$

◇

Throughout this thesis, we use the following notation for families of Radon measures, finite measures and probability measures on a topological Hausdorff space $(S, \mathcal{T})$ equipped with the Borel σ-algebra $\mathcal{B}(S)$.

$$\mathcal{M}(S) := \{\, \nu\colon \mathcal{B}(S) \to [0, \infty] \mid \nu \text{ Radon measure} \,\} \tag{A.22}$$

$$\mathcal{M}_f(S) := \{\, \nu \in \mathcal{M}(S) \mid \nu(S) < \infty \,\} \tag{A.23}$$

$$\mathcal{M}_1(S) := \left\{\, \nu \in \mathcal{M}_f(S) \,\middle|\, \nu(S) = 1 \,\right\} \tag{A.24}$$

Recall the following definition of the support of a Radon measure.

Definition A.36. Let $(S, \mathcal{T})$ be a topological Hausdorff space and $\nu \in \mathcal{M}(S)$ a Radon measure on $(S, \mathcal{B}(S))$. Let further $\mathcal{N} := \{ N \in \mathcal{T} \mid \nu(N) = 0 \}$ be the collection of open ν-nullsets. The *support* of ν is defined as

$$\operatorname{supp}(\nu) := \left(\bigcup_{N \in \mathcal{N}} N \right)^c , \tag{A.25}$$

i.e. the complement of the union of all open ν-nullsets. ◇

Remarks A.37. (i) By definition, the union of open sets is open and hence the support is always closed as the complement of an open set.

(ii) We frequently assume that the measures we are working with have *full support*, that is, $\operatorname{supp}(\nu) = S$. An important consequence of this assumption is that open sets $A \subset S$ always have strictly positive measure.

◇

Definition A.38. Let $(S, \mathcal{T})$ be a topological Hausdorff space, $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_f(S)$ and $\nu \in \mathcal{M}_f(S)$.

(i) We say that the sequence $(\nu_n)_{n\in\mathbb{N}}$ converges *weakly* to ν, if

$$\lim_{n\to\infty} \int_S f \,\mathrm{d}\nu_n = \int_S f \,\mathrm{d}\nu \quad \forall f \in C_b(S). \tag{A.26}$$

(ii) We say that the sequence $(\nu_n)_{n\in\mathbb{N}}$ converges *vaguely* to ν, if

$$\lim_{n\to\infty} \int_S f \,\mathrm{d}\nu_n = \int_S f \,\mathrm{d}\nu \quad \forall f \in C_0(S). \tag{A.27}$$

◇

Remark A.39 (Weak convergence of measures)**.** We can equip the space $\mathcal{M}_f(S)$ with the weak topology generated by the functions

$$\mathcal{F} = \left\{ \nu \mapsto \int_S f \,\mathrm{d}\nu \;\middle|\; f \in C_b(S) \right\}. \tag{A.28}$$

By Lemma A.34 the weak convergence of a sequence of measure $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_f(S)$ is equivalent to the convergence of $(\nu_n)_{n\in\mathbb{N}}$ in the weak topology generated by $\mathcal{F}$. ◇

A.3 Topologies and Nets

By definition, a topology always determines the convergent sequences. The converse is not true in general as there may be different topologies with the same convergent sequences.

Example A.40 (different topologies may have the same class of convergent sequences)**.** Let S be an uncountable set. Consider the discrete topology $\mathcal{T}_1 = \{ \{x\} \subset S \mid x \in S \}$ and the topology $\mathcal{T}_2 := \{ A \subset S \mid A^c \text{ is countable} \}$ consisting of complements of countable sets. Clearly, a sequence $(x_n)_{n\in\mathbb{N}} \subset S$ converges to $x \in S$ in the $\mathcal{T}_1$ topology if and only if there exists an $k \in \mathbb{N}$ such that $x_n = x$ for all $n \geq k$, that is, when $(x_n)_{n\in\mathbb{N}}$ is eventually constant. On the other hand, assume $\lim_{n\to\infty} x_n = x$ in the $\mathcal{T}_2$ topology but $(x_n)_{n\in\mathbb{N}}$ is not eventually constant. Then, for every $k \in \mathbb{N}$ the set $N_k := \{ x_n \mid n \geq k \} \setminus \{x\}$ is nonempty and countable. By definition of $\mathcal{T}_2$, N_1^c is an open neighborhood of x. Further, $N_1 \cap N_k \neq \emptyset$ for all $k \in \mathbb{N}$, thus $\{ x_n \mid n \geq k \}$ is not contained in N_1^c for any $k \in \mathbb{N}$, a contradiction. Hence, every converging sequence in $\mathcal{T}_2$ must also be eventually constant. ■

It turns out that there is a generalized notion of sequences, so-called *nets*, not to be confused with ε-nets, whose convergence already uniquely determines the topology. Before we introduce nets we need the following definition.

Definition A.41. A *directed set* is a set $\mathbb{I}$ equipped with a binary relation $\geq$ such that

(i) (transitivity) for all $\alpha, \beta, \gamma \in \mathbb{I}$ with $\alpha \geq \beta$ and $\beta \geq \gamma$ it holds that $\alpha \geq \gamma$,

(ii) (reflexivity) for all $\alpha \in \mathbb{I}$ it holds that $\alpha \geq \alpha$ and

(iii) (Archimedean property) for all $\alpha, \beta \in \mathbb{I}$ there exists a $\gamma \in \mathbb{I}$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. ◇

We can now introduce nets.

Definition A.42. Let S be any set. A *net* in S is a set $\{ x_\alpha \in S \mid \alpha \in \mathbb{I} \}$ where the index set $(\mathbb{I}, \geq)$ is a directed set. ◇

We adopt the following notions from Kelley's book [Kel75]. We say that a property holds for a net $(x_\alpha)_\alpha$ *eventually* if and only if there exists an $\alpha_0 \in \mathbb{I}$ such that x_α has that property for all $\alpha \geq \alpha_0$. We say that a property holds for a net $(x_\alpha)_\alpha$ frequently, if for every α such that x_α has the property, there exists a $\beta \geq \alpha$, such that the property also holds for x_β. A subnet of $(x_\alpha)_{\alpha\in\mathbb{I}}$ is a net $(x_\beta)_{\beta\in\mathbb{J}}$ with index set $\mathbb{J} \subset \mathbb{I}$ such that $(\mathbb{J}, \geq|_{\mathbb{J}\times\mathbb{J}})$ is itself a directed set.

We say that a net $(x_\alpha)_\alpha$ *converges* to a point $x \in S$ if and only if $(x_\alpha)_\alpha$ is eventually in every neighborhood $U \in \mathcal{U}_x$. In that case, we write $\lim_{\alpha\in\mathbb{I}} x_\alpha = x$, or, if there can be no confusion about the index set, $\lim_\alpha x_\alpha = x$.

We will show that the knowledge of convergent nets uniquely determines a topology if the class of convergent nets is sufficiently rich.

Definition A.43 (Convergence class)**.** Let $\mathcal{N}$ be a family of elements of the form $((x_\alpha)_{\alpha\in\mathbb{I}}, x)$ where $(x_\alpha)_{\alpha\in\mathbb{I}} \subset S$ is some net in S and $x \in S$. We say that a net $(x_\alpha)_{\alpha\in\mathbb{I}}$ converges to x with respect to $\mathcal{N}$ if and only if $((x_\alpha)_{\alpha\in\mathbb{I}}, x) \in \mathcal{N}$. In that case we write $\lim^{\mathcal{N}}_{\alpha\in\mathbb{I}} x_\alpha = x$. We call the family $\mathcal{N}$ a *convergence class* if and only if

(i) For every net $(x_\alpha)_{\alpha\in\mathbb{I}}$ with $x_\alpha = x$ for all $\alpha \in \mathbb{I}$ it holds that $\lim^{\mathcal{N}}_{\alpha\in\mathbb{I}} x_\alpha = x$.

(ii) If $(x_\alpha)_{\alpha\in\mathbb{I}}$ converges to x with respect to $\mathcal{N}$, then so does every subnet of $(x_\alpha)_{\alpha\in\mathbb{I}}$.

(iii) If $(x_\alpha)_{\alpha\in\mathbb{I}}$ does not converge to x with respect to $\mathcal{N}$ then there exists a subnet $(x_\beta)_{\beta\in\mathbb{J}}$ of $(x_\alpha)_{\alpha\in\mathbb{I}}$ such that no subset of $(x_\beta)_{\beta\in\mathbb{J}}$ converges to x.

(iv) For each $\alpha \in \mathbb{I}$ let $\mathbb{J}_\alpha$ be another directed set and denote the product of a directed set by
$$\mathbb{K} := \mathbb{I} \times \bigtimes_{\alpha\in\mathbb{I}} \mathbb{J}_\alpha. \tag{A.29}$$
Assume that $(x_\kappa)_{\kappa\in\mathbb{K}}$ converges to x with respect to $\mathcal{N}$. Let $f(\kappa) = f(\alpha, j) := (\alpha, j(\alpha))$, where $\kappa \in \mathbb{K}$, $\alpha \in \mathbb{I}$ and $j \in \bigtimes_{\alpha\in\mathbb{I}} \mathbb{J}_\alpha$. Then $(x_{f(\kappa)})_{\kappa\in\mathbb{K}}$ converges to x with respect to $\mathcal{N}$. ♢

It is easy to check that the convergent nets in a given topological space satisfy (i) to (iv) (cf. [Kel75, p. 74]). But also the converse holds true.

Theorem A.44 ([Kel75, Theorem 2.9])**.** *Let $\mathcal{N}$ be a convergence class for a set S and for each subset $A \subset S$ let $\Gamma(A)$ be the set of all points $x \in S$ such that there exists a net $(x_\alpha)_{\alpha\in\mathbb{I}} \subset A$ with $\lim^{\mathcal{N}}_{\alpha\in\mathbb{I}} x_\alpha = x$. Then Γ is a closure operator and $((x_\alpha)_{\alpha\in\mathbb{I}}, x) \in \mathcal{N}$ if and only if $(x_\alpha)_{\alpha\in\mathbb{I}}$ converges to x in the topology generated by Γ.*

Definition A.45. A topological space $(S, \mathcal{T})$ is called *sequential* if the topology $\mathcal{T}$ is determined by the convergent sequences. ♢

Proposition A.46 ([Wil70, Theorem 10.4])**.** *Let $(S, \mathcal{T})$ be a first-countable topological space. Then $\mathcal{T}$ is sequential.*

Metric spaces

B

B.1 Metrics on the space of probability measures

Let (S, d) be a metric space.

Definition B.1 (Prokhorov metric)**.** For two probability measures $\mu, \nu \in \mathcal{M}_1(S)$, define

$$d^*(\mu, \nu) := \inf \{ \varepsilon > 0 \mid \forall B \in \mathcal{B} : \ \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \}, \tag{B.1}$$

where $B^\varepsilon := \{ x \in S \mid \exists y \in B : \ d(x, y) < \varepsilon \}$ denotes the ε-blowup of B. Then the Prokhorov metric on $\mathcal{M}_1(S)$ is defined as

$$d_{\mathrm{Pr}}(\mu, \nu) := \max \{d^*(\mu, \nu), d^*(\nu, \mu)\}. \tag{B.2}$$

◇

Definition B.2 (Kantorovich-Rubinshtein metric)**.** Denote by $\mathrm{Lip}_1(S)$ the Lipschitz continuous functions $f : S \to \mathbb{R}$ with Lipschitz constant at most 1. On the space of probability measures $\mathcal{M}_1(S)$ we introduce the *Kantorovich-Rubinshtein norm* as

$$\|\mu\|_{\mathrm{KR}} := \sup \left\{ \int_S f \, \mathrm{d}\mu \,\middle|\, f \in \mathrm{Lip}_1(S), \ \|f\|_\infty \leq 1 \right\}, \quad \mu \in \mathcal{M}_1(S). \tag{B.3}$$

Then,

$$d_{\mathrm{KR}}(\mu, \nu) := \|\mu - \nu\|_{\mathrm{KR}} \tag{B.4}$$

is called the *Kantorovich-Rubinshtein metric* on $\mathcal{M}_1(S)$. ◇

B.2 The resistance metric

In Section 4.5.1 we have introduced the effective resistance metric on weighted graphs. In this appendix, we collect some useful properties of the effective resistance metric on finite graphs.

Let V be a finite set and recall from Definition 4.77 the definition of a weighted graph (V, μ). Furthermore, recall from (4.232) the definition of the energy form $\mathcal{E}$ given by

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x,y \in V} \mu_{xy} \left(f(x) - f(y)\right) \left(g(x) - g(y)\right) \tag{B.5}$$

for each $f\colon V \to \mathbb{R}$. Observe that in the case of finite V, the domain of $\mathcal{E}$ is unrestricted. In general, we denote by

$$\mathcal{F} := \{ f\colon V \to \mathbb{R} \mid \mathcal{E}(f, f) < \infty \} \tag{B.6}$$

the domain of $\mathcal{E}$.

Let $f\colon V \to \mathbb{R}$ and denote by $\tilde{f} := (f \vee 0) \wedge 1$ the unit truncation of f. Since $(\tilde{f}(x) - \tilde{f}(y))^2 \leq (f(x) - f(y))^2$ for all $x, y \in V$ we immediately obtain

$$\mathcal{E}(\tilde{f}, \tilde{f}) \leq \mathcal{E}(f, f). \tag{B.7}$$

Let $\mathcal{F}_x^y := \{ f\colon V \to \mathbb{R} \mid f(x) = 0, f(y) = 1 \}$. The effective resistance metric on V is given by[1]

$$\begin{aligned} \mathcal{R}(x, y) &= \inf \left\{ \mathcal{E}(f, f) \,\middle|\, \mathcal{F}_x^y \right\}^{-1} = \inf \left\{ \mathcal{E}(f, f) \,\middle|\, f \in \mathcal{F}_x^y,\ 0 \leq f \leq 1 \right\}^{-1} \\ &= \sup \left\{ \frac{(f(y) - f(x))^2}{\mathcal{E}(f, f)} \,\middle|\, f\colon V \to \mathbb{R},\ \mathcal{E}(f, f) > 0 \right\}. \end{aligned} \tag{B.8}$$

Define

$$\mu_x := \sum_{y \in V\colon y \sim x} \mu_{xy}. \tag{B.9}$$

Then, μ_x induces a measure on V and we can define the space $L^2(V, \mu)$ with the inner product

$$\langle f, g \rangle := \sum_{x \in V} \mu_x f(x) g(x) \tag{B.10}$$

Next, we introduce the discrete Laplace operator on $L^2(V, \mu)$ as follows

$$\mathcal{L} f(x) := \frac{1}{\mu_x} \sum_{y \in V\colon y \sim x} \mu_{xy}(f(y) - f(x)), \quad f \in L^2(V, \mu),\ x \in V. \tag{B.11}$$

Note that there is a close relationship between the Energy form $\mathcal{E}$ and the Laplace

[1] See (4.234), (4.240) and (B.7).

operator $\mathcal{L}$.[2] Let $f \in \mathcal{F} \cap L^2(V,\mu)$ and $g \in \mathcal{F}$. Then,

$$\begin{aligned}\langle -\mathcal{L}f, g\rangle &= -\sum_{x\in V}\mu_x \mathcal{L}f(x)g(x) = \sum_{x\in V}\sum_{y\in V:\ y\sim x}\mu_{xy}(f(x)-f(y))g(x)\\ &= \frac{1}{2}\left(\sum_{x,y\in V:\ x\sim y}\mu_{xy}(f(x)-f(y))g(x) + \sum_{x,y\in V:\ x\sim y}\mu_{yx}(f(y)-f(x))g(y)\right)\\ &= \frac{1}{2}\left(\sum_{x,y\in V:\ x\sim y}\mu_{xy}(f(x)-f(y))g(x) - \sum_{x,y\in V:\ x\sim y}\mu_{xy}(f(x)-f(y))g(y)\right)\\ &= \frac{1}{2}\sum_{x,y\in V:\ x\sim y}\mu_{xy}(f(x)-f(y))(g(x)-g(y)) = \mathcal{E}(f,g).\end{aligned} \tag{B.12}$$

With the Laplace operator we can define harmonic functions.

Definition B.3 (Harmonic functions)**.** Let (V,μ) be a weighted graph and $\mathcal{L}$ the discrete Laplace operator on $\mathcal{F}$. A function $f \in \mathcal{F}$ is *harmonic* in $x \in V$ if

$$\mathcal{L}f(x) = 0. \tag{B.13}$$

Moreover, we say that f is sub-harmonic in $x \in V$ if $\mathcal{L}f(x) \geq 0$ and f is super-harmonic in $x \in V$ if $\mathcal{L}f(x) \leq 0$. ◊

Suppose that V is finite. Then we can show that the minimizer of the resistance problem in (B.8) is a harmonic function.

Proposition B.4. *Let (V,μ) be a finite weighted graph. For any two vertices $x, y \in V$ there exists a unique minimizer h_x^y for the effective resistance problem (B.8). Moreover, h_x^y is the unique element of $\mathcal{F}_x^y$ that is harmonic on $V \setminus \{x,y\}$.*

Proof. Note that by finiteness of V the space $\mathcal{F}_x^y$ equipped with $\mathcal{E}$ is a convex and closed Hilbert space. Therefore, there exists a unique element $h_x^y \in \mathcal{F}_x^y$ with

$$\mathcal{E}(h_x^y, h_x^y) = \mathcal{R}(x,y)^{-1}. \tag{B.14}$$

Now suppose that $g\colon V \to \mathbb{R}$ is such that $g(x) = g(y) = 0$. For every $\lambda > 0$ we obtain $h_x^y - \lambda g \in \mathcal{F}_x^y$. Hence,

$$\mathcal{E}(h_x^y - \lambda g, h_x^y - \lambda g) \geq \mathcal{E}(h_x^y, h_x^y) = \mathcal{R}(x,y)^{-1} > 0 \tag{B.15}$$

[2]Compare this to the relationship between the generator and the Dirichlet form in (5.47).

and consequently,

$$\frac{\lambda}{2}\mathcal{E}(g,g) \geq \mathcal{E}(h_x^y, g). \tag{B.16}$$

Therefore, we can conclude that $\mathcal{E}(h_x^y, g) = 0$. Now, by (B.12), we also know that $\left\langle \mathcal{L}h_x^y, g\right\rangle = 0$. Choosing $g = \mathbb{1}_z$, we find that $\mathcal{L}f(z) = 0$ for all $z \in V \setminus \{x, y\}$. ∎

B.2.1 Network reduction rules

In order to calculate the effective resistance between two vertices of a finite graph the definition in (B.8) is quite unhandy. Instead, it is often more convenient to replace the graph with an equivalent graph. Recall that we can interpret a weighted graph (V, μ) as an electrical network with nodes V where two nodes $x, y \in V$ with $x \sim y$ are connected by a resistor with resistance given by μ_{xy}^{-1}. Analogously, we call the weights μ_{xy} *conductances*.

With this metaphor of an electrical network, we can interpret Proposition B.4 in this context. First, note that by Ohm's law the current between two nodes is given by the potential difference between them divided by the resistance. Therefore, $\mathcal{L}f(x)$ is the net current at $x \in V$ when a potential of f is applied to the network and $\mathcal{L}f(x) = 0$ means that there is the same amount of current flowing into x as is flowing out of x. The minimizer h_x^y then is the potential on the graph when we apply a source of one Volt to y and ground the network at x. Finally, the effective resistance between x and y is the reciprocal energy of this potential.

Now pick any two nodes $x, y \in V$ and connect them to an Ohm-meter to measure the resistance between them. We can replace the whole network $V \setminus \{x, y\}$ by a single resistor with resistance $\mathcal{R}(x, y)$ and the reading or our Ohm-meter would not change. Now, this leaves us with the same problem as before, calculating $\mathcal{R}(x, y)$. Instead, we can simplify the network successively by simple network reduction rules without changing the effective resistance between fixed nodes. The first rule says that two resistors in series can be replaced by a single resistor with the sum of the resistances of the replaced resistors.

Proposition B.5 (Serial rule)**.** *Let (V, μ) be a weighted graph and suppose that there are $u, v, w \in V$ such that the set of neighbors $\{ x \in V \mid x \sim v \}$ of v equals $\{u, w\}$. Define $(\hat{V}, \hat{\mu})$ where $\hat{V} = V \setminus \{v\}$ and $\hat{\mu}_{xy} = \mu_{xy}$ for all $x, y \in V \setminus \{u, v, w\}$ and*

$$\hat{\mu}_{u,w} = \frac{1}{\mu_{uv}^{-1} + \mu_{vw}^{-1}} = \frac{\mu_{uv}\mu_{vw}}{\mu_v}. \tag{B.17}$$

Then, $\mathcal{R}(x, y) = \hat{\mathcal{R}}(x, y)$ for all $x, y \in V \setminus \{v\}$.

Proof. Fix $x, y \in V \setminus \{v\}$ and denote the unique minimizer of the effective resistance problem by h. Then, h is harmonic in v and we can write

$$h(v) = \frac{\mu_{uv}h(u) + h(w)\mu_{wv}}{\mu_v}. \tag{B.18}$$

We claim that $\hat{h} = h|_{\hat{V}}$ is the unique minimizer of $\hat{\mathcal{E}}(f, f)$. Then,

$$\mu_{uv}(h(u) - h(v))^2 = \mu_{uv}\left(h(u) - \frac{\mu_{uv}h(u) + h(w)\mu_{wv}}{\mu_v}\right)^2 = \frac{\mu_{vw}^2}{\mu_v^2}(h(u) - h(w))^2 \tag{B.19}$$

and similarly

$$\mu_{vw}(h(v) - h(w))^2 = \frac{\mu_{uv}^2}{\mu_v^2}(h(u) - h(w))^2. \tag{B.20}$$

Thus,

$$\begin{aligned}\hat{\mathcal{E}}(\hat{h}, \hat{h}) - \mathcal{E}(h, h) &= \hat{\mu}_{uw}(\hat{h}(u) - \hat{h}(w))^2 - \mu_{uv}(h(u) - h(v))^2 - \mu_{vw}(h(v) - h(w))^2 \\ &= \left(\frac{\mu_{uv}\mu_{vw}}{\mu_v} - \frac{\mu_{uv}\mu_{vw}^2}{\mu_v^2} - \frac{\mu_{vw}\mu_{uv}^2}{\mu_v^2}\right)(h(u) - h(w))^2 \\ &= \left(\frac{\mu_{uv}\mu_{vw}}{\mu_v} - \frac{\mu_{uv}\mu_{vw}}{\mu_v}\left(\frac{\mu_{vw} + \mu_{uv}}{\mu_v}\right)\right)(h(u) - h(w))^2 = 0.\end{aligned} \tag{B.21}$$

It remains to show that $\hat{h}$ is indeed the unique minimizer of $\hat{\mathcal{E}}$. It suffices to show that $\hat{h}$ is harmonic in u and w, by Proposition B.4. A similar calculation as above for $\hat{\mathcal{L}}\hat{h}(u) - \mathcal{L}h(u)$ yields the claim. ■

Note that replacing two adjoining edges by a single edge can lead to a graph that is no longer a simple graph in the sense that two vertices can be joined by two edges with different weights. In this case, we can apply the following rule for parallel resistors and replace them by a single resistor with the sum of the conductances as conductance.

We state this rule rather informally and without proof because a formal statement would create too much notational overhead.

rule B.6 (Parallel rule)**.** Let (V, μ) be a finite weighted graph and suppose that there is an additional edge between x and y with weight μ'_{xy}. Then the edges μ_{xy} and μ'_{xy} can replaced by a single edge with weight $\hat{\mu}_{xy} = \mu_{xy} + \mu'_{xy}$. ■

Some loose ends

C

C.1 The Bochner Integral

Let $\mathfrak{B} \neq \emptyset$ denote a Banach space with norm $\|\cdot\|$ and let $(\Omega, \mathcal{A}, \mu)$ be a complete and σ-finite measure space. We want to define the integral of a map $F\colon \Omega \to \mathfrak{B}$.

The Bochner integral is defined similar to the Lebesgue integral and its construction is due to Salomon Bochner [Boc33].

Let $\mathfrak{E} = \mathfrak{E}(\Omega; \mathfrak{B})$ be the space of elementary functions i.e.

$$\mathfrak{E} := \left\{ F = \sum_{i=1}^{n} f_i \, \mathbb{1}_{A_i} \,\middle|\, n \in \mathbb{N},\ f_i \in \mathfrak{B},\ A_i \in \mathcal{A} : \mu(A_i) < \infty,\ A_i \cap A_j = \emptyset,\ i \neq j \in \{1, \dots, n\} \right\}. \tag{C.1}$$

Definition C.1 (Bochner integral)**.** Let $F \in \mathfrak{E}$. The *Bochner integral* of F with respect to μ is defined as

$$\int_B F \, \mathrm{d}\mu := \sum_{i=1}^{n} f_i \, \mu(A_i \cap B), \qquad B \in \mathcal{A}. \tag{C.2}$$

We say that a function $F\colon \Omega \to \mathfrak{B}$ is *Bochner integrable* if there exists an sequence $(F_n)_{n\in\mathbb{N}} \subset \mathfrak{E}$ of elementary functions satisfying

(i) $\lim_{n\to\infty} F_n = F$, μ-a.e. and

(ii) $\lim_{n\to\infty} \int_\Omega \|F_n - F\| \, \mathrm{d}\mu = 0$.

In that case, the Bochner integral of F (with respect to μ) is defined as

$$\int_B F \, \mathrm{d}\mu := \lim_{n\to\infty} \int_B F_n \, \mathrm{d}\mu, \qquad B \in \mathcal{A}. \tag{C.3}$$

◊

Not only is the Bochner integral defined similar to the Lebesgue integral, it also exhibits most of the properties of the Lebesgue integral. The following properties are shown in the same way as for the Lebesgue integral.

Proposition C.2. *(i) The Bochner integral $\int_B F \, \mathrm{d}\mu$ does not depend on the choice of the approximating sequence $(F_n)_{n\in\mathbb{N}} \subset \mathfrak{E}$.*

(ii) The Bochner integral is linear.

(iii) The Bochner integral is monotone,

$$F \le G \ \mu\text{-a.e.} \ \Rightarrow \ \int_B F \,\mathrm{d}\mu \le \int_B G \,\mathrm{d}\mu. \tag{C.4}$$

Definition C.3 (Strong measurability)**.** Let $F\colon \Omega \to \mathfrak{B}$. We say that F is *strongly measurable* (or *Bochner measurable*) if there exists a sequence $(F_n)_{n\in\mathbb{N}} \subset \mathfrak{E}$ of elementary functions with

$$\lim_{n\to\infty} \|F_n(\omega) - F(\omega)\| = 0 \tag{C.5}$$

for almost all $\omega \in \Omega$. ◊

Strong measurability yields a handy criterion for Bochner integrability.

Theorem C.4 ([Rao04, Theorem VII.5])**.** *A function $F\colon \Omega \to \mathfrak{B}$ is Bochner integrable if and only if F is strongly measurable and $\|F\|$ is Lebesgue integrable.*

Lemma C.5. *Let $\mathcal{H}$ be a real or complex Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and $F\colon \Omega \to \mathcal{H}$ Bochner integrable. Then for each $g \in \mathcal{H}$, the function $\omega \mapsto \langle F(\omega), g\rangle$ is Lebesgue integrable and*

$$\left\langle \int_B F \,\mathrm{d}\mu, g \right\rangle = \int_B \langle F, g\rangle \,\mathrm{d}\mu. \tag{C.6}$$

Proof. Fix $g \in \mathcal{H}$. First assume that F is elementary, i.e.

$$F = \sum_{i=1}^{n} f_i \mathbb{1}_{A_i} \tag{C.7}$$

for some $n \in \mathbb{N}$ and $f_i \in \mathcal{H}$, $A_i \in \mathcal{A}$. Then,

$$\left\langle \int_B F \,\mathrm{d}\mu, g \right\rangle = \sum_{i=1}^{n} \mu(A_i \cap B)\,\langle f_i, g\rangle = \int_B \sum_{i=1}^{n} \langle f_i, g\rangle \,\mathbb{1}_{A_i} \,\mathrm{d}\mu = \int_B \langle F, g\rangle \,\mathrm{d}\mu \tag{C.8}$$

and clearly the Lebesgue integral on the right-hand side exists.

Now let $F\colon \Omega \to \mathfrak{B}$ be Bochner integrable. By Theorem C.4, there exists a sequence $(F_n)_{n\in\mathbb{N}} \subset \mathfrak{E}$ such that $\|F_n - F\| \to 0$ outside a set of measure 0 as $n \to \infty$. Therefore, by application of the definition of the Bochner integral and (C.8),

$$\left\langle \int_B F \,\mathrm{d}\mu, g \right\rangle = \lim_{n\to\infty} \left\langle \int_B F_n \,\mathrm{d}\mu, g \right\rangle = \lim_{n\to\infty} \int_B \langle F_n, g\rangle \,\mathrm{d}\mu. \tag{C.9}$$

The Cauchy-Schwarz inequality now yields

$$|\langle F_n, g\rangle - \langle F, g\rangle| = |\langle F_n - F, g\rangle| \leq \|F_n - F\| \, \|g\| \underset{n\to\infty}{\longrightarrow} 0, \tag{C.10}$$

for μ-almost all $\omega \in \Omega$.

Without loss of generality, we can choose the sequence $(F_n)_{n\in\mathbb{N}}$ in such a way that $(\|F_n\|)_{n\in\mathbb{N}}$ is increasing. Then the functions $\langle F_n, g\rangle$ are bounded μ-almost everywhere by the integrable function $\|F\| \, \|g\|$ and an application of the dominated convergence theorem yields the claim. ■

C.2 Probabilistic results in the uniform setting

Here we collect some important results from probability theory that can be found in most textbooks but are usually formulated for (Polish) metric spaces. We show how the proofs can easily be generalized to the uniform setting.

As always, let $(\Omega, \mathcal{A}, \mathbb{P})$ denote some probability space.

Theorem C.6 (Skorokhod coupling [Kal21, Theorem 5.31])**.** *Let $(S, \mathcal{U})$ be a separable uniform Hausdorff space and $X, X_1, X_2, \ldots : \Omega \to S$ random variables. Assume that $X_n \xrightarrow{d} X$, then there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, Y_1, Y_2, \ldots : \Omega' \to S$ such that $\mathbb{P}_X = \mathbb{P}'_Y$ and $\mathbb{P}_{X_n} = \mathbb{P}'_{Y_n}$ for all $n \in \mathbb{N}$ and furthermore*

$$Y_n \to Y \quad \mathbb{P}'\text{-a.s.} \tag{C.11}$$

The proof relies on the following general result.

Lemma C.7. *Let $\mathbb{I}$ be some index set and $(\Omega_\alpha, \mathcal{A}_\alpha, \mathbb{P}_\alpha)$ a probability space for each $\alpha \in \mathbb{I}$. Then there exist independent random variables X_α on Ω_α with*

$$\mathcal{L}(X_\alpha) = \mathbb{P}_\alpha, \qquad \alpha \in \mathbb{I}. \tag{C.12}$$

Proof. See [Kal21, Corollary 8.25]. ■

Proof of Theorem C.6. We start with the case where $S = \{1, \ldots, m\}$ is finite. Set

$$p_k := \mathbb{P}(X = k) \quad \text{and} \quad p_k^n := \mathbb{P}(X_n = k). \tag{C.13}$$

Let η be defined on a probability space $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and uniformly distributed on $[0, 1]$. We can construct random variables $(Y_n)_{n\in\mathbb{N}}$ on the product space $\Omega' = \Omega_1$

with $\mathcal{L}(Y_n) = X_n$ by setting $Y_n = k$ whenever $X = k$ and $\eta \le p_k^n/p_k$. By assumption, $p_k^n/p_k \to 1$ as $n \to \infty$ for each $1 \le k \le m$. Hence, $Y_n \to Y$ $\mathbb{P}'$-a.s.

Now let $(S, \mathcal{U})$ be an arbitrary separable uniform Hausdorff space and $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ a sequence of open entourages with $U_{n+1} \subset U_n$ and $\bigcap_{n\in\mathbb{N}} U_n = \Delta$. Fix $p \in \mathbb{N}$ and suppose further that $\left(B_j\right)_{j\in\mathbb{N}} \subset \mathcal{B}$ is a partition of S into subsets with $\mathbb{P}(X \in \partial B_j) = 0$ such that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset S$ with $B_j \subset U_p[x_j]$ for all $j \in \mathbb{N}$. We can choose $m \in \mathbb{N}$ large enough so that

$$\mathbb{P}\left(X \notin \bigcup_{k\le m} B_k\right) < 2^{-p}. \tag{C.14}$$

Moreover, write $B_0 := \bigcap_{k\le m} \complement B_k$. Let $\kappa, (\kappa_n)_{n\in\mathbb{N}}$ be random variables with $\kappa = k$ when $X \in B_k$ and $\kappa_n = k$ when $X_n \in B_k$ for $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m\}$. By assumption, $\kappa_n \xrightarrow{\mathrm{d}} \kappa$ as $n \to \infty$. We can therefore apply the result for finite S and conclude that there exist random variables $\tilde{\kappa}, (\tilde{\kappa}_n)_{n\in\mathbb{N}}$ on some probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ such that $\tilde{\kappa} \overset{d}{=} \kappa$, $\tilde{\kappa}_n \overset{d}{=} \kappa_n$ and $\tilde{\kappa}_n \to \tilde{\kappa}$, $\mathbb{P}'$-a.s. as $n \to \infty$. Define now further random variables ξ_n^k on Ω' with values in S and distributions

$$\mathcal{L}(\xi_n^k) = \mathcal{L}(X_n \mid X_n \in B_k). \tag{C.15}$$

Moreover, define

$$Y_n^p := \sum_{k\in\mathbb{N}} \xi_n^k \, \mathbb{1}_{\tilde{\kappa}_n = k}. \tag{C.16}$$

Then, $Y_n^p \overset{d}{=} X_n$ and, by construction,

$$\left\{\left(Y_n^p, X\right) \notin U_p\right\} \subset \{\tilde{\kappa}_n \neq \kappa\} \cup \{X \in B_0\}, \tag{C.17}$$

for all $n, p \in \mathbb{N}$. Let Y be defined on Ω' with $Y \overset{d}{=} X$. Because of the almost sure convergence $\tilde{\kappa}_n \to \tilde{\kappa}$ and (C.14), we conclude that for every $p \in \mathbb{N}$ there exists a $n_p \in \mathbb{N}$ such that

$$\mathbb{P}'\left(\bigcup_{n\ge n_p} \left\{\left(Y_n^p, Y\right)\right\}\right) < 2^{-p}. \tag{C.18}$$

Without loss of generality, we can assume that the sequence $\left(n_p\right)_{p\in\mathbb{N}}$ is increasing. Applying the Borel-Cantelli Theorem, find that

$$\left\{ (Y_n^p, Y) \,\middle|\, n > n_p \right\} \subset U \tag{C.19}$$

for all but finitely many $p \in \mathbb{N}$. We can therefore apply a diagonal argument and set

$Y_n := Y_n^p$ for some $n \in \{n_p, \ldots, n_{p+1} - 1\}$ to finally obtain

$$X_n \overset{d}{=} Y_n \to Y, \tag{C.20}$$

$\mathbb{P}'$-a.s. as $n \to \infty$. ■

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List of Symbols

Notation	Description	Page(s)
$\overline{A}$	Closure of the set A	245
A^c, $\complement A$	$:= S \setminus A$ for $A \subset S$, complement of the set A in S	245
A°	Interior of the set A	17, 245
A^r	$:= \{ x \in S \mid \mathbb{P}_x(\tau_A = 0) = 1 \}$, set of regular points for A	118
$x \vee y$	$:= \max\{x, y\}$	22, 167
$x \wedge y$	$:= \min\{x, y\}$	22, 167
f^-	$:= -(f \wedge 0)$, negative part of f	167
f^+	$:= f \vee 0$, positive part of f	112, 167
$\varphi_* \nu$	$:= \nu \circ \varphi^{-1}$, the push forward of ν under φ	
$\mathcal{A}_{t+}$	$:= \bigcap_{s>t} \mathcal{A}_s$, right limit of the filtration $(\mathcal{A}_t)_{t \geq 0}$	89
$\mathcal{B}_x$	Neighborhood base at x	246
$\mathcal{B}^n$	The family of nearly Borel measurable sets	115
$\mathcal{B}(S)$	Borel σ-algebra generated by the open sets of S	
$B(S)$	Borel measurable real valued functions on S	
$B_b(S)$	Banach space of bounded measurable functions on S	
$C_b(S)$	Continuous bounded functions	
$C^\infty(S)$	Infinitely often continously differentiable functions on S	224
$C_\infty(S)$	The space of continuous functions on S vanishing at infinity	
$C_0(S)$	Continuous and compactly supported functions on S	
$\Delta(X)$	$:= \{ (x, x) \in X \times X \mid x \in X \}$, the diagonal of the set $X \times X$	14
$d_{\mathrm{H}}(A, B)$	Hausdorff distance between $A, B \subset S$	
d_{KR}	Kantorovich-Rubinshtein metric on the space of (probability) measures	
d_{Pr}	Prokhorov metric on the space of (probability) measures	
$D_S([0, \infty))$	Space of càdlàg functions $f : [0, \infty) \to S$	

Notation	**Description**	**Page(s)**
$\emptyset$	The empty set	
$\mathcal{F}_\infty^0$	Smallest σ-Algebra that makes all $\{ X_t \mid t \geq 0 \}$ measurable	88
$\mathcal{F}_t^0$	Smallest σ-Algebra that makes all $\{ X_s \mid s \in [0, t] \}$ measurable	88
$\mathcal{L}(X)$	The distribution (law) of X	
Lip_α	Lipschitz continuous functions with Lipschitz constant at most $\alpha > 0$	232
$L^p(S, \nu)$	Lebesgue's p-space on (S, ν), $p \geq 1$	
$\mathcal{M}(S)$	The set of measures on $(S, \mathcal{B}(S))$	94
$\mathcal{M}_1(S)$	The set of probability measures on $(S, \mathcal{B}(S))$	94
$\mathcal{M}_f(S)$	The set of finite measures on $(S, \mathcal{B}(S))$	94
$\mathbb{N}^\infty$	$:= \mathbb{N} \cup \{\infty\}$, the set of extended natural numbers	
$\mathcal{P}(A)$	Powerset of the set A, i.e. the family of all subsets of A	89
$\bowtie$	Proximity relation	38
$\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$)	(strictly) positive reals	154
$\sigma(\quad)$	The smallest σ-algebra that makes the content of the braces measurable	88
$(S, \mathcal{T})$	Topological space	
ϑ	Cemetery state / point at infinity in the Alexandrov compactification	88
$\{ \theta_t \mid t \geq 0 \}$	The family of shift operators on $D_S([0, \infty))$	114
$\mathcal{U}_x$	Neighborhood system at x	246
$U \circ V$	$:= \left\{ (x, y) \in S^2 \,\middle|\, \exists\, z \in S : (x, z) \in V \text{ and } (z, y) \in U \right\}$	14
$(S, \mathcal{U})$	Uniform space	15

Index

Symmetric Feller Processes on Uniform State Spaces

This thesis investigates symmetric Feller processes on uniform state spaces equipped with a measure, aiming to extend and unify diverse results on the convergence of such processes on metric measure spaces. It has been shown in previous works that processes that are related to both the metric and the measure of their state spaces through their Dirichlet forms converge whenever the state spaces converge. However, it is not always clear which metric is the right one to consider and there might even be many such metrics. The main focus lies therefore on abstracting from the metric structure of the spaces and instead considering their uniform structure. This approach warrants an in-depth analysis of uniform spaces as state spaces for Feller processes including an analysis of the Skorokhod topology on the the space of paths on such spaces. One of the main results is that the convergence of a family of hitting times implies the convergence of paths. Moreover, symmetric Feller processes on uniform state spaces and their Dirichlet forms are introduced and studied. As a result of a detailed study of killed processes, it is demonstrated that symmetric Feller processes are uniquely determined by their Green operators. Finally, five conditions for the convergence of symmetric Feller processes on uniform state spaces are identified and discussed.

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