On the Derived Category of the Fargues-Fontaine Curve

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Abstract

Let X be the schematic Fargues-Fontaine curve as defined in [FF]. Following arguments of Bondal and van den Bergh we show that $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ generates the derived category of quasi-coherent $\mathcal{O}_X$-modules. By a theorem of Keller the latter is equivalent to the derived category of an associated differential graded algebra. We give an explicit description of this algebra in terms of rings of adèles on X and determine the dg modules corresponding to all coherent $\mathcal{O}_X$-modules. We also take first steps in determining the multiplicative structure of a huge skew field first discovered by Colmez. This involves explicit computations in the heart of a t-structure constructed by Le Bras.

Zusammenfassung

Sei X die schematische Fargues-Fontaine-Kurve wie in [FF] eingeführt. Argumenten von Bondal und van den Bergh folgend zeigen wir, dass $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ die derivierte Kategorie der quasi-kohärenten $\mathcal{O}_X$-Modulen erzeugt. Letztere ist nach einem Satz von Keller äquivalent zur derivierten Kategorie einer assoziierten differentiell graduierten Algebra. Wir geben eine explizite Beschreibung dieser Algebra mittels Adèlen auf X und bestimmen die zu allen kohärenten $\mathcal{O}_X$-Moduln korrespondierenden DG-Moduln. Darüber hinaus unternehmen wir erste Schritte in der Bestimmung der multiplikativen Struktur eines riesigen Schiefkörpers, der zuerst von Colmez entdeckt wurde. Dies beinhaltet explizite Berechnungen im Herzen einer von Le Bras konstruierten t-Struktur.

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Contents

0 Introduction

Fix a prime p. Let E be a nonarchimedean local field of residue characteristic p and let F be a perfectoid field of characteristic p. Let $X = X_{E,F}$ be the Fargues-Fontaine curve associated with E, F. This is a connected separated Noetherian regular scheme of dimension 1 which was discovered and studied extensively by Laurent Fargues and Jean-Marc Fontaine in [FF]. Using this curve Fargues and Fontaine were able to give geometric proofs of fundamental results in p-adic Hodge theory. More recently ([FS]), Fargues and Scholze used families of Fargues-Fontaine curves in order to develop a geometric version of the local Langlands program.

The motivation for this project comes from the question of Fargues ([LeB], Question 7.18) whether the Fargues-Fontaine curve is in some sense the Brauer-Severi variety associated with a particular skew field $\mathscr{C}$ that was originally studied by Colmez in his work on Banach-Colmez spaces ([Col]). In his dissertation [LeB] Le Bras established a connection between Colmez' skew field and the Fargues-Fontaine curve by showing that the category of Banach-Colmez spaces is equivalent to the heart of a certain nontrivial t-structure on the bounded derived category of coherent sheaves on the curve.

The aim of this thesis is to make use of this connection and apply derived tilting theorems to give an algebraic description of the derived category of X and to gain a better understanding of the skew field $\mathscr{C}$.

To put this into perspective, recall the following classical example. Let K be a field and let A be a k-linear Grothendieck category. An object T of A is called a tilting object if the functor

$$\mathrm{RHom}(T, -) : \mathsf{D}(\mathsf{A}) \to \mathsf{D}(\mathsf{Mod}_{\mathrm{End}(T)})$$

is an equivalence of categories. There are explicit criteria characterizing tilting objects (see e.g. [Kel], §4.6). One can use these criteria to show that if $\mathbb{P}^1_k$ denotes the projective line over k then $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)$ is a tilting object for the category of quasi-coherent sheaves on $\mathbb{P}^1_k$, i.e. the functor $\mathrm{RHom}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1), -)$ induces an equivalence

$$\mathsf{D}(\mathbb{P}^1_k) \xrightarrow{\cong} \mathsf{D}(R).$$

Here $R = \mathrm{End}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)) \cong \begin{pmatrix} k & 0 \\ k[x,y]_1 & k \end{pmatrix}$ where $k[x,y]_1$ denotes homogeneous elements of degree 1.

Although it is far from being a classical variety, the Fargues-Fontaine curve X shares a lot of features with the projective line over the field $k = E$. In particular, this concerns the classification of vector bundles (cf. Theorem 1.29 below). However, there are significant differences leading to the fact that the sheaf $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ is not a tilting object in the classical sense above. The issue is that $\mathrm{Ext}^1(\mathcal{O}_X(1), \mathcal{O}_X) = H^1(X, \mathcal{O}_X(-1)) \neq 0$ (in fact, it is infinite dimensional over the base field). However, following the strategy of [BvdB], Theorem 3.1.1, we give a self-contained and streamlined proof of the fact that $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ generates $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ in the sense of triangulated categories (cf. Theorem 3.3). This allows us to make use of a more general tilting construction due to Keller.

Namely, let A be a Grothendieck abelian category such that $\mathsf{D}(\mathsf{A})$ is generated by a compact object P. Then Keller showed that $\mathsf{D}(\mathsf{A})$ is equivalent to $\mathsf{D}(\mathscr{A})$ where $\mathscr{A}$ is a differential graded algebra (dg algebra, for short) obtained by replacing P by a K-injective complex and whose cohomology is given by $\mathrm{Ext}^*(P, P)$.

By applying Keller's result to $\mathsf{A} = \mathsf{QCoh}_X$, the category of quasi-coherent sheaves on X, we obtain an equivalence between $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ and the derived category of the dg algebra

$$\mathscr{A} = \mathrm{RHom}(\mathcal{O}_X \oplus \mathcal{O}_X(1), \mathcal{O}_X \oplus \mathcal{O}_X(1))$$

(cf. Theorem 2.20). The dg algebra $\mathscr{A}$ may be computed by using an injective resolution $\mathcal{I}$ of $\mathcal{O}_X \oplus \mathcal{O}_X(1)$. Then $\mathscr{A} = \mathrm{Hom}^\bullet(\mathcal{I}, \mathcal{I})$, where $\mathrm{Hom}^\bullet$ denotes the Hom complex, see §2.1.

As one of our main results we show that the dg algebra $\mathscr{A}$ admits an astonishingly explicit algebraic description. To explain this, let $E(X)$ be the function field of X and denote by $\mathcal{E}_X$ the constant $\mathcal{O}_X$-module with value $E(X)$. Denote by $|X|$ the set of closed points of X, and let

$$\mathbf{A}_X^0 := \prod_{x\in|X|} \widehat{\mathcal{O}}_{X,x} \subset \prod_{x\in|X|}{}' \operatorname{Frac}(\widehat{\mathcal{O}}_{X,x}) =: \mathbf{A}_X$$

be the rings of (integral) adèles on X. For a ring R denote by $M_2(R)$ the ring of 2×2-matrices over R. In §3.2 we describe $\operatorname{Hom}^\bullet(\mathcal{I},\mathcal{I})$ in terms of $E(X), \mathbf{A}_X^0$ and $\mathbf{A}_X$. We use this in order to obtain a dg algebra $\mathscr{A}^{ad}$ which is isomorphic to $\mathscr{A}$. Our main result is the following (cf. Theorem 3.24):

Theorem 0.1. *Let X be the Fargues-Fontaine curve and let $\mathscr{A}$ be the dg algebra described above. Then $\mathscr{A}$ is isomorphic to a differential graded algebra $\mathscr{A}^{ad}$ with underlying complex*

$$M_2(E(X)) \times M_2(\mathbf{A}_X^0) \to M_2(\mathbf{A}_X)$$

with an explicit an easy to write down differential.

If F is algebraically closed then there is a particularly simple classification of vector bundles on X. For coherent sheaves on X there is a well-defined notion of degree. Putting this together with the rank function on vector bundles, Fargues and Fontaine introduced a general formalism of slopes leading to a simple classification result. Namely, for each $\lambda \in \mathbf{Q}$ there is a unique stable vector bundle $\mathcal{O}_X(\lambda)$ of Harder-Narasimhan slope λ, and every vector bundle decomposes as a direct sum of stable bundles (cf. Theorem 1.29). The bundle $\mathcal{O}_X(\lambda)$ is the pushforward of a line bundle along a finite étale morphism of curves. In Proposition 3.20 we establish a connection between the differential graded algebras for such finite étale coverings of curves. Moreover, we show that the derived pushforward and pullback along the covering map translate into derived scalar restriction and extension between differential graded modules. This allows us to determine explicitly the differential graded $\mathscr{A}$-modules corresponding to the vector bundles $\mathcal{O}_X(\lambda)$ under the equivalence of categories $\mathsf{D}_{qcoh}(\mathcal{O}_X) \cong \mathsf{D}(\mathscr{A})$. In fact, we determine what happens for coherent $\mathcal{O}_X$-modules in general (cf. §3.4).

In principle, our results allow to translate any problem about $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ into a corresponding problem about differential graded $\mathscr{A}$-modules. An interesting question that we do not discuss is how to make explicit Le Bras' t-structure in terms of $\mathsf{D}(\mathscr{A})$.

We note in passing that many of our results and arguments work for abstract curves in general and also in the setting of generalized Riemann spheres as introduced by Fargues and Fontaine (cf. §1.3).

In the last part we take a first step in investigating the multiplicative structure of Colmez' skew field $\mathscr{C}$ which is not well understood. Let us mention that $\mathscr{C}$ is a rather mysterious object containing the central E-division algebras of invariant $\frac{1}{h}$ for all $h > 0$, as well as all untilts of F in the sense of Scholze. Note that the latter are in correspondence with the closed points of X.

Let us sketch the construction of $\mathscr{C}$ following Le Bras. Let C be the heart of Le Bras' t-structure on $\mathsf{D}^b(X)$. Let $\mathsf{C}^0 \subseteq \mathsf{C}$ be the Serre subcategory consisting of objects that are finite direct sums of copies of the structure sheaf $\mathcal{O}_X$. The localization $\mathsf{Q} := \mathsf{C}/\mathsf{C}^0$ is a semisimple abelian category with a unique simple object, and Colmez' skew field $\mathscr{C}$ is the endomorphism algebra of this simple object. We point out that the aforementioned properties of the category Q are stated in [LeB] without proof. We give a detailed account of the arguments in §4.2. Since $\mathcal{O}_X(1)$ represents the simple object of Q, the general theory of localization shows that $\mathscr{C}$ is a filtered colimit

$$\mathscr{C} = \operatorname{End}_{\mathsf{Q}}(\mathcal{O}_X(1)) = \varinjlim_{s:\mathcal{O}_X(1)\to\mathcal{F}} \operatorname{Hom}_{\mathsf{C}}(\mathcal{O}_X(1), \mathcal{F})$$

where the index set runs over all morphisms $s : \mathcal{O}_X(1) \to \mathcal{F}$ which become isomorphisms in Q and is filtered by using pushouts. Therefore, the elements of $\mathscr{C}$ are represented by roofs of morphisms in C of the form

$$\begin{tikzcd} & \mathcal{F} & \\ \mathcal{O}_X(1) \arrow[ur, "f"] & & \mathcal{O}_X(1) \arrow[ul, "s"'] \end{tikzcd}$$

where $\ker(s), \operatorname{coker}(s) \in \mathsf{C}^0$ and for which we write $s^{-1}f$. Here $\mathcal{F}$ is an object of C which becomes isomorphic to $\mathcal{O}_X(1)$ in Q. A first but important step in determining $\mathscr{C}$ consists of giving a complete list of the possible objects $\mathcal{F}$, i.e. in determining the indexing set of the above colimit. We do so in Proposition 4.9:

Proposition 0.2. *(i) If $f : \mathcal{O}_X(1) \to \mathcal{F}$ is a morphism which becomes an isomorphism in Q then $\mathcal{F} \cong \mathcal{O}_X^n \oplus \mathcal{G}$ for some $n \geq 1$ and where $\mathcal{G}$ is either $\mathcal{O}_X(1)$, a skyscraper sheaf C_x for some closed point $x \in |X|$, or $\mathcal{O}_X(-\frac{1}{h})[1]$ for some $h \geq 1$.*

(ii) If $\mathcal{F}$ is of the form in (i) and if $f : \mathcal{O}_X(1) \to \mathcal{F}$ is a nonzero morphism in C then f becomes an isomorphism in Q.

The appearance of the objects $\mathcal{O}_X(-\frac{1}{h})[1]$ shows that $\mathscr{C}$ also contains all central E-division algebras of invariant $-\frac{1}{h}$. Although this has not been observed before, we do not know if $\mathscr{C}$ contains still other central E-division algebras.

The multiplication in $\mathscr{C}$ is given by the pushout of roofs: Given two roofs

$$\begin{tikzcd} & \mathcal{F} & \\ \mathcal{O}_X(1) \arrow[ur, "f"] & & \mathcal{O}_X(1) \arrow[ul, "s"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathcal{G} & \\ \mathcal{O}_X(1) \arrow[ur, "g"] & & \mathcal{O}_X(1) \arrow[ul, "t"'] \end{tikzcd}$$

the product $t^{-1}g \cdot s^{-1}f$ is given by the outer roof of the diagram

$$\begin{tikzcd} & & \mathcal{F} \coprod_{\mathcal{O}_X(1)} \mathcal{G} & & \\ & \mathcal{F} \arrow[ur] & & \mathcal{G} \arrow[ul] & \\ \mathcal{O}_X(1) \arrow[ur, "f"] & & \mathcal{O}_X(1) \arrow[ul, "s"'] \arrow[ur, "g"] & & \mathcal{O}_X(1) \arrow[ul, "t"'] \end{tikzcd}$$

where the upper square is the pushout of the morphisms s and g in the abelian category C.

We note that the degree function on coherent sheaves is additive in short exact sequences and can therefore be extended to objects of $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ and C. An important step in making the multiplication of the skew field $\mathscr{C}$ explicit is to compute the above pushouts in C. In Proposition 4.10 we solve this problem completely:

Proposition 0.3. *Let $\mathcal{F}, \mathcal{G} \in \mathsf{C}$ be of degree 1 and let $f : \mathcal{O}_X(1) \to \mathcal{F}$, $g : \mathcal{O}_X(1) \to \mathcal{G}$ be two nonzero morphisms.*

(i) If $\mathcal{F} = \mathcal{O}_X(1)$ then their pushout in C is isomorphic to $\mathcal{G}$.

(ii) If $\mathcal{F} = \mathcal{G} = C_x$ for some $x \in |X|$ then their pushout in C is isomorphic to C_x.

(iii) If $\mathcal{F} = C_x$ and $\mathcal{G} = C_y$ for some $x \neq y$ then their pushout in C is isomorphic to $\mathcal{O}_X(-1)[1]$.

(iv) *Let* $\mathcal{F} = \mathcal{O}_X(-\frac{1}{h})[1]$ *for some* $h \geq 1$ *and* $\mathcal{G} = C_x$ *for some* $x \in |X|$*. Let* $t \in H^0(X, \mathcal{O}_X(1))$ *be such that* $x = \infty_t$*. Moreover, suppose that the map* f *is the class of the extension*

$$0 \to \mathcal{O}_X(-\frac{1}{h}) \to \mathcal{O}_X^{h+1} \xrightarrow{p} \mathcal{O}_X(1) \to 0$$

where $p = \begin{pmatrix} s_1 & \dots & s_{h+1} \end{pmatrix}$ *with* $s_1, \dots, s_{h+1} \in H^0(X, \mathcal{O}_X(1))$*. Then*

$$\mathcal{O}_X(-\frac{1}{h})[1] \coprod_{\mathcal{O}_X(1), \mathsf{C}} C_x \cong \begin{cases} \mathcal{O}_X(-\frac{1}{h})[1] & \text{if } t \in \langle s_1, \dots, s_{h+1} \rangle_E, \\ \mathcal{O}_X(-\frac{1}{h+1})[1] & \text{otherwise.} \end{cases}$$

Here $\langle s_1, \dots, s_{h+1} \rangle_E$ *denotes the* E*-subspace of* $H^0(X, \mathcal{O}_X(1))$ *generated by* $s_1, \dots, s_{h+1}$*.*

(v) *Let* $\mathcal{F} = \mathcal{O}_X(-\frac{1}{h})[1]$ *and* $\mathcal{G} = \mathcal{O}_X(-\frac{1}{h'})[1]$ *for some* $h, h' \geq 1$*. Let* f *and* g *be represented by extensions*

$$0 \to \mathcal{O}_X(-\frac{1}{h}) \to \mathcal{O}_X^{h+1} \xrightarrow{p} \mathcal{O}_X(1) \to 0$$

and

$$0 \to \mathcal{O}_X(-\frac{1}{h'}) \to \mathcal{O}_X^{h'+1} \xrightarrow{q} \mathcal{O}_X(1) \to 0,$$

respectively. Write $p = \begin{pmatrix} s_1 & \dots & s_{h+1} \end{pmatrix}$ *and* $q = \begin{pmatrix} s_{h+2} & \dots & s_{h+h'+2} \end{pmatrix}$ *for some* $s_i \in H^0(X, \mathcal{O}_X(1))$ *and set* $n := \dim_E \langle s_1, \dots, s_{h+h'+2} \rangle_E$*. Then the pushout of* f *and* g *in* C *is isomorphic to* $\mathcal{O}_X(-\frac{1}{n-1})[1]$*.*

In order to make explicit the skew field $\mathscr{C}$ in terms of the above colimit formula, one needs to compute $\mathrm{Hom}_{\mathsf{C}}(\mathcal{O}_X(1), \mathcal{F})$ for the various possibilities of $\mathcal{F}$. This is a simple task which is already implicit in the work of Fargues, Fontaine and Le Bras.

The final step in determining $\mathscr{C}$ is understanding the transition maps in the above colimit. This is a question we hope to come back to in the future. A second follow-up project is concerned with the connection between the skew field $\mathscr{C}$ and the dg algebra $\mathscr{A}^{ad}$. As explained earlier, for this one needs to transfer Le Bras' t-structure to the derived category of $\mathscr{A}^{ad}$ and mimic the construction of $\mathscr{C}$.

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1 Curves

In §1.1 we recollect the basic definitions about abstract curves as introduced in [FF], §5. In §1.2 we explain how to define a degree function on coherent sheaves on a curve, and we recall the Harder-Narasimhan formalism introduced in [FF], §5.5. In §1.3 we recall the notion of a generalized Riemann sphere as in [FF], §5.6.4. Finally, in §1.4 we briefly introduce the Fargues-Fontaine curve and state some of its properties as well as the classification of vector bundles.

1.1 Generalities

Definition 1.1 ([Ked], Definition 3.3.4)**.** An *(abstract) curve* is a connected separated Noetherian scheme which is regular of dimension 1.

Lemma 1.2 ([Det], Lemma 1.2)**.** *A connected separated scheme is an abstract curve if and only if it admits a finite affine open covering by spectra of Dedekind domains one of which is not a field. In particular, any curve is normal and integral.*

Let us fix some notation: If X is a curve then we denote by $|X|$ its set of closed points. We let $\eta \in X$ be the generic point and denote by $E(X) := \mathcal{O}_{X,\eta}$ the function field of X. For each $x \in |X|$ the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring which we may view as a subring of $E(X)$. We have $\mathrm{Frac}(\mathcal{O}_{X,x}) = E(X)$. If $U \subseteq X$ is non-empty open then we also view $\Gamma(U, \mathcal{O}_X)$ as a subring of $E(X)$. If U is affine, $E(X)$ is the field of fractions of $\Gamma(U, \mathcal{O}_X)$.

For $x \in |X|$ we denote by
$$v_x : E(X)^\times \to \mathbf{Z}$$
the normalized valuation and by ϖ_x a uniformizer of the discrete valuation ring $\mathcal{O}_{X,x}$. We set $v_x(0) := \infty$. Then $\mathcal{O}_{X,x} = \{a \in K \mid v_x(a) \geq 0\}$, and by [GW], Proposition 3.29, if $U \subseteq X$ is a nonempty open subset then
$$\Gamma(U, \mathcal{O}_X) = \{f \in E(X) \mid \forall x \in |U| : v_x(f) \geq 0\}.$$

Definition 1.3. If X is a curve then $\mathrm{Div}(X)$ denotes the free abelian group on the closed points of X. An element of $\mathrm{Div}(X)$ is called a *divisor on* X. A divisor $D = \sum_{x\in|X|} m_x \cdot [x]$ is called *effective* if $m_x \geq 0$ for all $x \in |X|$. Any divisor $D = \sum_{x\in|X|} m_x \cdot [x]$ gives rise to an $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ as follows: For any open set $U \subseteq X$ we set
$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in E(X) \mid \forall x \in |U| : v_x(f) + m_x \geq 0\}.$$
If $V \subseteq U$ then the restriction from U to V is the inclusion $\Gamma(U, \mathcal{O}_X(D)) \hookrightarrow \Gamma(V, \mathcal{O}_X(D))$. The scalar multiplication $\Gamma(U, \mathcal{O}_X) \times \Gamma(U, \mathcal{O}_X(D)) \to \Gamma(U, \mathcal{O}_X(D))$ is given by multiplication within $E(X)$. If $f \in E(X)^\times$ then we define the *divisor of* f by
$$\mathrm{div}(f) := \sum_{x\in|X|} v_x(f) \cdot [x] \in \mathrm{Div}(X).$$
The divisors of the form $\mathrm{div}(f)$ for some $f \in E(X)^\times$ are called *principal divisors*.

Remark 1.4. (i) If $D = \sum_{x\in|X|} m_x[x]$ and $D' = \sum_{x\in|X|} m'_x[x]$ are divisors on X with $m_x \leq m'_x$ for all $x \in |X|$ then we have a natural inclusion
$$\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(D').$$

(ii) For every $D \in \mathrm{Div}(X)$ the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ is invertible with inverse $\mathcal{O}_X(-D)$. The sequence of abelian groups
$$\begin{aligned} 1 \to \Gamma(X, \mathcal{O}_X)^\times \to E(X)^\times \overset{\mathrm{div}}{\to} \mathrm{Div}(X) &\to \mathrm{Pic}(X) \to 1 \\ D &\mapsto [\mathcal{O}_X(D)] \end{aligned}$$
is exact (cf. [GW], equation (11.12.6) and Propositions 11.27/11.38).

(iii) A divisor D on X is effective if and only if $\mathcal{O}_X(-D)$ is an ideal of $\mathcal{O}_X$. The corresponding closed subscheme of X is denoted by D as well. By definition there is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

(cf. [GW], Remark 11.25). Twisting with $\mathcal{O}_X(D)$ yields an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D \to 0.$$

Indeed, write $D = \sum_x n_x[x]$. There is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \xrightarrow{\cong} \mathcal{O}_D$ whose germ at each point $x \in \operatorname{supp}(D)$ is given by the $\mathcal{O}_{X,x}$-linear isomorphism

$$\mathcal{O}_{X,x}/\varpi_x^{n_x} \otimes_{\mathcal{O}_{X,x}} \varpi_x^{-n_x}\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\varpi_x^{n_x},$$

sending $1 \otimes f$ to $\varpi_x^{n_x} f$.

Definition 1.5 ([Ked], Definition 3.3.4)**.** A *complete curve* is an abstract curve X together with a nonzero additive map $\deg : \operatorname{Div}(X) \to \mathbf{Z}$ which is nonnegative on effective divisors and zero on principal divisors.

Example 1.6. Let k be a field and X a connected smooth projective curve over k. Set $\deg(x) := [k(x) : k]$ for $x \in |X|$. Then $(X, \deg)$ is a complete curve (cf. [Har1], Corollary 6.10).

Remark 1.7. If X is a complete curve then the degree map on $\operatorname{Div}(X)$ induces a map

$$\deg : \operatorname{Pic}(X) \to \mathbf{Z}$$

(cf. Remark 1.4 (ii)).

Lemma 1.8 ([FF], Lemme 5.1.5)**.** *If X is a complete curve then $\Gamma(X, \mathcal{O}_X)$ is a subfield of $E(X)$ which is algebraically closed within $E(X)$.*

Proof. Let $f \in E(X)^\times$. By [GW], Proposition 3.29 (3), $\Gamma(X, \mathcal{O}_X) = \bigcap_{x \in |X|} \mathcal{O}_{X,x}$. Hence, $f \in \Gamma(X, \mathcal{O}_X)$ if and only if $\operatorname{div}(f) \geq 0$. Since $\deg(\operatorname{div}(f)) = 0$, the latter holds if and only if $\operatorname{div}(f) = 0$. Since the valuations are multiplicative we have $\operatorname{div}(f^{-1}) = -\operatorname{div}(f)$. It follows that $\Gamma(X, \mathcal{O}_X)$ is a subring of $E(X)$ which is closed under taking inverses, hence it is a field which we call E.

It remains to show that an element of $E(X)^\times$ that is algebraic over E lies in E. Let $f \in E(X)^\times$ and suppose that there are $n \in \mathbf{N}_{\geq 1}$ and $a_0, \ldots, a_{n-1} \in E$ such that $f^n + \sum_{i=0}^{n-1} a_i f^i = 0$. For $x \in |X|$ and $0 \leq i \leq n-1$ we have

$$v_x(a_i f^i) = v_x(a_i) + v_x(f^i) = i \cdot v_x(f).$$

In particular, the elements $v_x(a_i f^i), 1 \leq i \leq n-1$, are pairwise distinct integers. By the strict triangle inequality,

$$v_x(\sum_{i=0}^{n-1} a_i f^i) = \min\{i \cdot v_x(f) \mid 0 \leq i \leq n-1\}.$$

Assume that there exists $x \in |X|$ such that $v_x(f) \neq 0$. Then we obtain

$$\begin{aligned} n \cdot v_x(f) = v_x(f^n) &= v_x(\sum_{i=0}^{n-1} a_i f^i) \\ &= \min\{i \cdot v_x(f) \mid 0 \leq i \leq n-1\} \\ &= \begin{cases} 0, & v_x(f) > 0 \\ (n-1) \cdot v_x(f), & v_x(f) < 0. \end{cases} \end{aligned}$$

In the first case, $n = 0$, a contradiction. In the second case, $v_x(f) = 0$, a contradiction as well. It follows that $v_x(f) = 0$ for all $x \in |X|$, whence $f \in E$. ☐

Definition 1.9. If X is a complete curve then the field $\Gamma(X, \mathcal{O}_X)$ is called the **field of definition** of X.

Lemma 1.10 ([FF], Lemme 5.4.1)**.** *Let X be a complete curve admitting a closed point $\infty \in |X|$ such that $\deg([\infty]) = 1$ and such that $X \setminus \{\infty\}$ is affine. The following are equivalent:*

(i) $\mathrm{Pic}(X \setminus \{\infty\}) = 0$.

(ii) The map $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$ *is an isomorphism.*

Definition 1.11. Let X be a complete curve satisfying the equivalent conditions of Lemma 1.10. For $k \in \mathbf{Z}$ we define

$$\mathcal{O}_X(k) := \mathcal{O}_X(k[\infty]).$$

Note that this definition is independent of the choice of a closed point ∞.

1.2 Slope theory for coherent sheaves on complete curves

Throughout this section let X be a complete curve. In particular, it comes with a degree map $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$. We will extend this to coherent sheaves on X. An inportant result for us will be that it is additive in short exact sequences.

Definition 1.12. Let $\mathcal{F}$ be a nonzero vector bundle on X. The *degree* of $\mathcal{F}$ is

$$\deg \mathcal{F} := \deg(\bigwedge^{\mathrm{rk}\,\mathcal{F}} \mathcal{F}).$$

Remark 1.13 ([Ked], Example 3.3.2)**.** If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of vector bundles on X then $\deg \mathcal{F} = \deg \mathcal{F}' + \deg \mathcal{F}''$, i.e. deg is additive in short exact sequences of vector bundles. This relies on the fact that there is a natural isomorphism

$$\bigwedge^{\mathrm{rk}\,\mathcal{F}} \mathcal{F} \cong (\bigwedge^{\mathrm{rk}\,\mathcal{F}'} \mathcal{F}') \otimes_{\mathcal{O}_X} (\bigwedge^{\mathrm{rk}\,\mathcal{F}''} \mathcal{F}'')$$

of $\mathcal{O}_X$-modules.

Definition 1.14. (i) A coherent $\mathcal{O}_X$-module $\mathcal{G}$ is a *torsion sheaf* if its stalk at the generic point of X is 0.

(ii) Let $\mathcal{G}$ be a torsion sheaf on X. We define its *degree* to be

$$\deg \mathcal{G} := \sum_{x \in X} \deg(x) \cdot \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{G}_x).$$

Remark 1.15. Let $\mathcal{G} \neq 0$ be a torsion sheaf on X. Then $\mathrm{supp}(\mathcal{G}) := \{x \in X \mid \mathcal{G}_x \neq 0\}$ is a proper closed subset of X, hence is finite.

If $\mathcal{E}$ is a coherent sheaf on X then $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ is a vector bundle and $\mathcal{G}$ is a torsion sheaf on X. We define $\deg \mathcal{E} := \deg \mathcal{F} + \deg \mathcal{G}$. One can now show that deg is additive in short exact sequences of coherent sheaves on X.

Let us introduce the Harder-Narasimhan formalism for vector bundles on X.

Definition 1.16. Let $\mathcal{F}$ be a vector bundle on X.

(i) The *(Harder-Narasimhan) slope* of $\mathcal{F}$ is the rational number

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\mathrm{rk}\,\mathcal{F}}.$$

(ii) $\mathcal{F}$ is called *semistable* if for every nonzero subbundle $\mathcal{F}' \subset \mathcal{F}$ (the quotient need not be locally free)

$$\mu(\mathcal{F}') \leq \mu(\mathcal{F}).$$

Theorem 1.17 ([FF], Théorème 5.5.2). *Every vector bundle $\mathcal{F}$ on X admits a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$$

such that

(i) $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable for each $1 \leq i \leq r$, and

(ii) the sequence of slopes $(\mu(\mathcal{F}_i/\mathcal{F}_{i-1}))_{1\leq i\leq r}$ is strictly decreasing.

Definition 1.18. Let $\mathcal{F}$ be a vector bundle on X. Let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$ be its Harder-Narasimhan filtration as in Theorem 1.17. The *slope multiset* of $\mathcal{F}$ is the multiset of rational numbers of cardinality $\operatorname{rk}\mathcal{F}$ containing $\mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ with multiplicity $\operatorname{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})$ and which is ordered decreasingly. If $\mu = (\mu_0, \ldots, \mu_r)$ denotes the multiset of slopes of $\mathcal{F}$ then the *HN polygon* $\mathrm{HN}(\mathcal{F})$ is the polygon with slope μ_i on the interval $(i, i+1)$.

We define a partial order on HN polygons as follows: if HN_1 and HN_2 are two polygons, then we say that $\mathrm{HN}_1 \leq \mathrm{HN}_2$ if HN_1 lies on or below HN_2 and the polygons have the same endpoints (cf. [BFHHLWY], Definition 2.2.12).

The following result gives an obstruction on the HN polygon of an extension of vector bundles.

Lemma 1.19 ([Ked], Lemma 3.4.17). *If $0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{F}_2 \to 0$ is a short exact sequence of vector bundles on X then $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{F}_1 \oplus \mathcal{F}_2)$.*

1.3 Generalized Riemann spheres

Definition 1.20 ([FF], Définition 5.6.21). A generalized Riemann sphere is a pair (X, E_∞) where

- X is a complete curve with field of definition E, and
- $E_\infty|E$ is an algebraic field extension which is Galois with Galois group $\widehat{\mathbf{Z}}$

such that

(i) for each closed point x of X, $E_\infty|E$ embeds into $\kappa(x)|E$,

(ii) there is a closed point $\infty \in |X|$ of degree 1 such that for all intermediate extensions $E \subseteq E' \subseteq E_\infty$ with $E'|E$ finite there is a closed point of $X \times_E E'$ in the fiber above ∞ whose complement is the spectrum of a principal ideal domain, and

(iii) $H^1(X, \mathcal{O}_X) = 0$.

Let X be a generalized Riemann sphere. For $h \in \mathbf{Z}$ let $E_h := E_\infty^{h\widehat{\mathbf{Z}}}$. This is a cyclic extension of degree h of E, and $E_\infty = \bigcup_h E_h$. We set

$$X_h := X \times_E E_h$$

and let

$$\pi_h : X_h \to X$$

be the projection. If $y \in |X_h|$ is a closed point then we set $\deg([y]) := \deg([\pi_h(y)])$ and extend this linearly to a function $\deg : \mathrm{Div}(X_h) \to \mathbf{Z}$.

Lemma 1.21. *For any $h \geq 1$ the scheme X_h is a complete curve with field of definition E_h.*

Proof. First of all, $\Gamma(X_h, \mathcal{O}_{X_h}) = \Gamma(X, \mathcal{O}_X) \otimes_E E_h \cong E_h$. If $U = \mathrm{Spec}(R) \subseteq X$ is non-empty affine open then $\pi_h^{-1}(U) = \mathrm{Spec}(R) \times_E E_h \cong \mathrm{Spec}(R \otimes_E E_h)$. As $E_h|E$ is separable and E is algebraically closed inside $E(X) = \mathrm{Frac}(R)$, the ring $R \otimes_E E_h$ is an integral domain (cf. [Jac], Cor. 1 on page 203), showing that X_h is an integral scheme.

Set $S := R \otimes_E E_h$. Then $S|R$ is a finite ring extension, hence $\dim(S) = \dim(R) = 1$. Fix a closed point ∞_h above ∞ such that $X_h \setminus \{\infty_h\}$ is the spectrum of a principal ideal domain. Since E_h embeds into the residue fields of the closed points of X, the fiber of π_h over each closed point consists of h distinct points, and the group $\mathrm{Gal}(E_h|E)$ acts simply transitively on each fiber. In particular, $\mathrm{Gal}(E_h|E)$ does not fix the affine open subscheme $X_h \setminus \{\infty_h\}$ (if $h \neq 1$). It follows that X_h admits a covering by two spectra of principal ideal domains, so that X_h is indeed a curve. Let us write $E(X)$ and $E_h(X_h)$ for the function fields of X and X_h, respectively. We claim that $E_h(X_h) = E(X) \otimes_E E_h$.

As before, using Jacobson's result, the right hand side is an integral domain. Since it is an integral ring extension of the field $E(X)$, it is a field itself. The inclusions $E(X) \subseteq E_h(X_h)$ and $E_h \subseteq E_h(X_h)$ induce a canonical field homomorphism $E(X) \otimes_E E_h \hookrightarrow E_h(X_h)$. On the other hand, the inclusion $R \subseteq E(X)$ induces a ring homomorphism $R \otimes_E E_h \hookrightarrow E(X) \otimes_E E_h$ which extends to a field homomorphism $E_h(X_h) = \mathrm{Frac}(R \otimes_E E_h) \hookrightarrow E(X) \otimes_E E_h$. This gives the desired isomorphism $E_h(X_h) \cong E(X) \otimes_E E_h$.

It remains to prove that X_h is complete. Note that $\mathrm{Gal}(E_h|E)$ also acts on $E_h(X_h) = E(X) \otimes_E E_h$ via the second factor, making $E_h(X_h)|E(X)$ a Galois extension with Galois group isomorphic to $\mathrm{Gal}(E_h|E)$. If $y \in |X_h|$ corresponds to the nonzero prime ideal $\mathfrak{p}$ in the coordinate ring S of some affine open subset of X_h and if $f \in E_h(X_h) = \mathrm{Frac}(S)$ and $\sigma \in \mathrm{Gal}(E_h|E)$ then

$$\begin{aligned} v_y(\sigma(f)) &= \max\{n \in \mathbf{N} \mid \sigma(f) \in \mathfrak{p}^n\} \\ &= \max\{n \in \mathbf{N} \mid f \in \sigma^{-1}(\mathfrak{p})^n\} \\ &= v_{\sigma^{-1}(y)}(f). \end{aligned}$$

This implies

$$\begin{aligned} \deg(\mathrm{div}(\sigma(f))) &= \sum_{y \in |X_h|} v_y(\sigma(f)) \cdot \deg(y) \\ &= \sum_{y \in |X_h|} v_{\sigma^{-1}(y)}(f) \cdot \deg(\sigma^{-1}(y)) \\ &= \sum_{y \in |X_h|} v_y(f) \cdot \deg(y) \\ &= \deg(\mathrm{div}(f)). \end{aligned}$$

Let $g \in E(X)^\times$. We may also view g as an element of $E_h(X_h) = E(X) \otimes_E E_h$. We will write $\mathrm{div}_X(g)$ (resp. $\mathrm{div}_{X_h}(g)$) for the principal divisor on X (resp. on X_h) associated with g. Let $x \in |X|$ and $y \in \pi_h^{-1}(x)$. Since the map π_h is totally decomposed at the closed points we have $v_y(g) = v_x(g)$. This shows

$$\begin{aligned} \deg(\mathrm{div}_{X_h}(g)) &= \sum_{y \in |X_h|} v_y(g) \deg(y) \\ &= \sum_{x \in |X|} \sum_{\pi_h(y) = x} v_x(g) \deg(x) \\ &= h \cdot \deg(\mathrm{div}_X(g)). \end{aligned}$$

Now let $f \in E_h(X_h)$. Using that X is complete we obtain

$$\begin{aligned} 0 &= h \cdot \deg(\mathrm{div}_X(N(f))) \\ &= \deg(\mathrm{div}_{X_h}(\prod_\sigma \sigma(f)) \\ &= \sum_\sigma \deg(\mathrm{div}_{X_h}(\sigma(f))) \\ &= h \cdot \deg(\mathrm{div}_{X_h}(f)), \end{aligned}$$

showing that also X_h is complete. $\square$

Definition 1.22. Let X be a generalized Riemann sphere. Fix a compatible system $(\infty_h)_h \in \varprojlim_{h\geq 1} |X_h|$ of closed points of degree 1 and a generator $\sigma \in \mathrm{Aut}(X_h/X) \cong \mathbf{Z}/h\mathbf{Z}$. For $k \geq 1$ we define

$$\mathcal{O}_{X_h}(k) := \mathcal{O}_{X_h}(\sum_{i=0}^{k-1}[\sigma^i(\infty_h)]).$$

Remark 1.23.
- If X is a generalized Riemann sphere then for each $h \geq 1$ the map $\deg : \mathrm{Pic}(X_h) \to \mathbf{Z}$ is an isomorphism (cf. Lemma 1.10). Recall that by the way the degree map on X_h is obtained from the one on X we have $\deg(\alpha(\infty_h)) = \deg(\infty_h) = 1$ for all $\alpha \in \mathrm{Aut}(X_h/X)$. Therefore, up to isomorphism $\mathcal{O}_{X_h}(k)$ is the unique line bundle of degree k on X_h. In particular, this shows that it does not depend on the choice of generator σ.
- If we write $\infty := \infty_1$ then we have $\mathcal{O}_{X_h}(h) = \mathcal{O}_{X_h}(\pi_h^{-1}(\infty))$.

Definition 1.24 ([FF], Définition 5.6.22)**.** For $d \in \mathbf{Z}$ and $h \in \mathbf{N}_{\geq 1}$ we define

$$\mathcal{O}_X(d,h) := \pi_{h,*}\mathcal{O}_{X_h}(d).$$

If $(d,h) = 1$ and $\lambda = \frac{d}{h} \in \mathbf{Q}$ then we set

$$\mathcal{O}_X(\lambda) := \pi_{h,*}\mathcal{O}_{X_h}(d).$$

One has the following classification of vector bundles, assuming that certain extensions of vector bundles always admit global sections:

Theorem 1.25 ([FF], Théorème 5.6.26)**.** *Let X be a generalized Riemann sphere. Suppose that for all h and all $n \geq 1$, if*

$$0 \to \mathcal{O}_{X_h}(-\frac{1}{n}) \to \mathcal{F} \to \mathcal{O}_{X_h}(1) \to 0$$

is a short exact sequence of vector bundles then $H^0(X_h, \mathcal{F}) \neq 0$. Then

(i) the HN filtration of a vector bundle on X splits, and

(ii) the assignment

$$\begin{aligned} \{(\lambda_i)_{1\leq i\leq n} \in \mathbf{Q}^n \mid n \in \mathbf{N}, \lambda_1 \geq \cdots \geq \lambda_n\} &\longrightarrow \{\textit{vector bundles on } X\}/\sim \\ (\lambda_1, \ldots, \lambda_n) &\longmapsto [\bigoplus_{i=1}^n \mathcal{O}_X(\lambda_i)] \end{aligned}$$

is a bijection.

1.4 The Fargues-Fontaine curve

Fix a prime p and let E be a nonarchimedean local field of residue characteristic p. Let F be a complete algebraically closed nontrivially valued nonarchimedean field of characteristic p. The (schematic) Fargues-Fontaine curve $X = X_{E,F}$ is a complete curve with field of definition E whose closed points are all of degree 1. It was first defined by Fargues and Fontaine (cf. [FF], Définition 6.5.1). We have $X = Proj(\bigoplus_{n\geq 0} B^{\varphi=\pi^n})$ where B is a Fréchet algebra depending on E and F and which admits a Frobenius endomorphism φ. If $x \in |X|$ is a closed point then we let $\iota_x : \{x\} \to X$ be the inclusion and write $C_x = \kappa(x)$ for its residue field. Abusing notation we will also write $C_x = \iota_{x,*}\kappa(x)$ for the skyscraper sheaf on X with value C_x at x. We list some important properties of the curve:

Theorem 1.26 ([FF], Théorème 6.5.2)**.** *(i) If $E'|E$ is finite then there is a canoncical isomorphism*

$$X_{E'} \xrightarrow{\cong} X_E \otimes_E E'.$$

(ii) The complement of each closed point x of X is the spectrum of a principal ideal domain. Its residue field C_x is a complete algebraically closed field whose tilt is isomorphic to F. All untilts of F arise in this way.

(iii) There is a bijection

$$(B^{\varphi=\pi} \setminus \{0\})/E^{\times} \xrightarrow{\cong} |X|, \quad t \mapsto \infty_t.$$

(iv) If $E'|E$ is finite then via the étale covering $X_{E'} \to X_E$ the fiber over a closed point $x \in |X_E|$ consists of $[E' : E]$ points of the same residue field as the one of x.

(v) The degree map induces an isomorphism $\mathrm{Pic}(X) \xrightarrow{\cong} \mathbf{Z}$.

(vi) One has $H^1(X, \mathcal{O}_X) = 0$.

Corollary 1.27. *The Fargues-Fontaine curve is a generalized Riemann sphere.*

Remark 1.28. There is also a version of the Fargues-Fontaine curve as an adic space X^{ad} over $Spa(E)$ and a morphism of locally ringed spaces $X^{ad} \to X$ pullback along which induces an equivalence of categories between vector bundles on X and X^{ad} (cf. [KL], Theorem 8.7.7).

One has a classification of vector bundles:

Theorem 1.29 ([FF], Théorème 8.2.10)**.** *(i) For each $\lambda \in \mathbb{Q}$ the semistable vector bundles of slope λ on X are isomorphic to finite direct sums of $\mathcal{O}_X(\lambda)$.*

(ii) The Harder-Narasimhan filtration of any vector bundles splits.

(iii) Every vector bundle $\mathcal{F}$ on X admits a direct sum decomposition

$$\mathcal{F} \cong \bigoplus_{\lambda} \mathcal{O}_X(\lambda)$$

where the λ run over the slope multiset of $\mathcal{F}$ (cf. Definition 1.18).

2 Derived categories and t-structures

In §2.1 we construct the Hom complex. In §2.2 we recall the basic definition of a generator of a triangulated category. In §2.3 we recall the definition of a t-structure and explain how to construct kernels and cokernels in the heart in the case where the t-structure comes from a torsion pair on an abelian category. This will be helpful when we compute pushouts in §4. Finally, in §2.4 we introduce basic constructions of differential graded algebras, show how Hom complexes give rise to differential graded algebras, and sketch a proof of the tilting theorem of Keller that we will use later.

2.1 The complex $\mathrm{Hom}^\bullet$

Let A be an additive category admitting countable direct products. If X, Y are two complexes of A then we let $\mathrm{Hom}^\bullet_{\mathsf{A}}(X,Y)$ be the total (product) complex associated to the double complex $(\mathrm{Hom}_{\mathsf{A}}(X^n, Y^m))_{n,m}$. This defines an additive bifunctor

$$\mathrm{Hom}^\bullet_{\mathsf{A}}(-,-) : \mathsf{CoCh}_{\mathsf{A}}^{op} \times \mathsf{CoCh}_{\mathsf{A}} \longrightarrow \mathsf{CoCh}_{\mathbf{Z}}$$

(cf. [KS1], Lemma 11.6.1 and Example 11.6.2 (i)). It induces an additive bifunctor

$$\mathrm{Hom}^\bullet_{\mathsf{A}}(-,-) : \mathsf{K}(\mathsf{A})^{op} \times \mathsf{K}(\mathsf{A}) \longrightarrow \mathsf{K}(\mathbf{Z}).$$

In the following we will write $\mathrm{Hom}_{\mathsf{A}}(X,Y)^n$ instead of $\mathrm{Hom}^\bullet_{\mathsf{A}}(X,Y)^n$ for the nth graded piece. Explicitly,

$$\mathrm{Hom}_{\mathsf{A}}(X,Y)^n := \prod_{k\in\mathbf{Z}} \mathrm{Hom}_{\mathsf{A}}(X^k, Y^{n+k}),$$

and if $f \in \mathrm{Hom}^\bullet_{\mathsf{A}}(X,Y)^n$ then $d^n f \in \mathrm{Hom}^\bullet_{\mathsf{A}}(X,Y)^{n+1}$ has kth entry

$$(d^n f)^k = d_Y^{k+n} \circ f^k + (-1)^{n+1} f^{k+1} \circ d_X^k \in \mathrm{Hom}_{\mathsf{A}}(X^k, Y^{n+1+k}).$$

By [KS1], Proposition 11.7.3,

$$H^0(\mathrm{Hom}^\bullet_{\mathsf{A}}(X,Y)) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{A})}(X,Y).$$

Proposition 2.1 ([KS1], Corollary 14.3.2 and equation (13.4.2)). *Let k be a field and let A be a k-linear Grothendieck abelian category. Then $\mathrm{Hom}^\bullet_{\mathsf{A}}$ admits a right derived functor*

$$\mathrm{RHom}_{\mathsf{A}} : \mathsf{D}(\mathsf{A})^{op} \times \mathsf{D}(\mathsf{A}) \to \mathsf{D}(k).$$

Moreover, $H^k(\mathrm{RHom}_{\mathsf{A}}(X,Y)) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(X, Y[k]) =: \mathrm{Ext}^k_{\mathsf{A}}(X,Y)$ for all complexes X, Y.

As usual the functor $\mathrm{RHom}_{\mathsf{A}}$ may be computed by using resolutions by K-injective objects (cf. [Sta], Lemma 070K). Specifically, if X, Y are two complexes and if we choose a quasi-isomorphism $Y \to I$ where I is K-injective then

$$\mathrm{RHom}_{\mathsf{A}}(X,Y) \cong \mathrm{Hom}^\bullet_{\mathsf{A}}(X, I).$$

2.2 Generators of triangulated categories

Definition 2.2. Let D be a triangulated category. An object $\mathcal{P}$ of D is called a *(weak) generator* of D if for any object $\mathcal{F}$ of D, $\mathrm{Hom}_{\mathsf{D}}(\mathcal{P}[n], \mathcal{F}) = 0$ for all $n \in \mathbb{Z}$ implies $\mathcal{F} = 0$.

Remark 2.3. If $\mathsf{D} = \mathsf{D}(\mathsf{A})$ is the derived category of an abelian category A then a complex K is a generator of $\mathsf{D}(\mathsf{A})$ if and only if $\mathrm{RHom}(K, -)$ detects the zero object (cf. Proposition 2.1).

Definition 2.4. Let X be a scheme. We denote by $\mathsf{D}_{qcoh}(\mathcal{O}_X) \subseteq \mathsf{D}(\mathcal{O}_X)$ the full subcategory of complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Moreover, we say that a complex $\mathcal{F}$ of $\mathcal{O}_X$-modules is *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles.

Bondal and van den Bergh proved the following:

Theorem 2.5 ([BvdB], Theorem 3.1.1 (2)). *Assume that X is a quasicompact quasiseparated scheme. Then $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ is generated by a single perfect complex.*

2.3 Torsion pairs and t-structures

Definition 2.6. Let A be an abelian category. A *torsion pair* in A is a pair of classes of objects $(\mathcal{T}, \mathcal{F})$ of A such that

(i) $\mathrm{Hom}_{\mathsf{A}}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$,

(ii) if $T \in \mathsf{A}$ such that $\mathrm{Hom}_{\mathsf{A}}(T, F) = 0$ for all $F \in \mathcal{F}$ then $T \in \mathcal{T}$,

(iii) if $F \in \mathsf{A}$ such that $\mathrm{Hom}_{\mathsf{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ then $F \in \mathcal{F}$, and

(iv) for every $X \in \mathsf{A}$ there is a short exact sequence

$$0 \to T \to X \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair then $\mathcal{T}$ is called the *torsion class* and $\mathcal{F}$ is called the *torsionfree class*.

Definition 2.7. Let D be a triangulated category. A *t-structure* on D is a pair of strictly full subcategories $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ of D satisfying the condition below. Write $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

(i) $\mathsf{D}^{\leq 0} \subset \mathsf{D}^{\leq 1}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$,

(ii) $\mathrm{Hom}_{\mathsf{D}}(X, Y) = 0$ for $X \in \mathsf{D}^{\leq 0}$ and $Y \in \mathsf{D}^{\geq 1}$, and

(iii) for each $X \in \mathsf{D}$ there is a distinguished triangle

$$A \to X \to B \to A[1]$$

with $A \in \mathsf{D}^{\leq 0}$ and $B \in \mathsf{D}^{\geq 1}$.

If $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a t-structure then the full subcategory $\mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$ is called its *heart*.

Example 2.8 ([KS2], Example 10.1.3 (i))**.** Let A be an abelian category. Let $\mathsf{D}^{\leq 0}$ be the full subcategory of $\mathsf{D}^b(\mathsf{A})$ consisting of the complexes $K^\bullet$ such that $H^i(K^\bullet) = 0$ for all $i > 0$, and $\mathsf{D}^{\geq 0}$ the one consisting of complexes $K^\bullet$ satisfying $H^i(K^\bullet) = 0$ for all $i < 0$. The pair $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a t-structure on $\mathsf{D}^b(\mathsf{A})$ called the *trivial t-structure*. The functor $H^0 : \mathsf{D}^b(\mathsf{A}) \to \mathsf{A}$ induces an equivalence $\mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0} \cong \mathsf{A}$.

Proposition 2.9 ([KS2], Proposition 10.1.4)**.** *Let D be a triangulated category. Let $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ be a t-structure on D. The inclusion $\mathsf{D}^{\leq n} \to \mathsf{D}$ (resp. $\mathsf{D}^{\geq 0} \to \mathsf{D}$) has a right adjoint functor $\tau^{\leq n} : \mathsf{D} \to \mathsf{D}^{\leq n}$ (resp. a left adjoint functor $\tau^{\geq n} : \mathsf{D} \to \mathsf{D}^{\geq n}$).*

Proof. We may assume $n = 0$. Let $X \in \mathsf{D}$. By the third axiom of a t-structure we may embed X into a distinguished triangle $X_0 \to X \to X_1 \xrightarrow{[1]}$ with $X_0 \in \mathsf{D}^{\leq 0}$ and $X_1 \in \mathsf{D}^{\geq 1}$. One can show that the assignment $X \mapsto X_0$ (resp. $X \mapsto X_1$) yields a functor $\tau^{\leq 0}$ (resp. $\tau^{\geq 1}$) as desired. $\square$

Remark 2.10 ([KS2], eq. (10.1.1))**.** We have

$$\begin{cases} \tau^{\leq n}(X[m]) \cong \tau^{\leq n+m}(X)[m], \\ \tau^{\geq n}(X[m]) \cong \tau^{\geq n+m}(X)[m]. \end{cases}$$

In particular, $\tau^{\geq 0}(X) \cong \tau^{\geq 1}(X[-1])[1]$.

The functors $\tau^{\leq n}$ and $\tau^{\geq n}$ are called the *truncation functors* with respect to the t-structure.

Proposition 2.11 ([KS2], Proposition 10.1.11 (i))**.** *The heart of a t-structure is an abelian category.*

Proof. We sketch the construction of the kernel and the cokernel of a morphism. Let $f : X \to Y$ be a morphism in the heart. We may embed f into a distinguished triangle $X \xrightarrow{f} Y \to Z \xrightarrow{[1]}$. Then

$$\begin{cases} \ker(f) \cong \tau^{\leq 0}(Z[-1]), \\ \operatorname{coker}(f) \cong \tau^{\geq 0}(Z). \end{cases}$$

□

There is a relation between torsion pairs in an abelian category and t-structures on its bounded derived category:

Proposition 2.12 ([Mat], Proposition 2.6). *Let* A *be an abelian category and let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* A. *Let*

$$\mathsf{D}^{\leq 0} := \{X^\bullet \in \mathsf{D}^b(\mathsf{A}) \mid H^i(X^\bullet) = 0 \text{ for } i > 0, H^0(X^\bullet) \in \mathcal{T}\},$$

and

$$\mathsf{D}^{\geq 0} := \{X^\bullet \in \mathsf{D}^b(\mathsf{A}) \mid H^i(X^\bullet) = 0 \text{ for } i < -1, H^{-1}(X^\bullet) \in \mathcal{F}\}.$$

Then $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ *is a* t*-structure on* $\mathsf{D}^b(\mathsf{A})$.

Proof. We verify the third axiom: Let $X^\bullet$ be a complex of A with bounded cohomology. Since $(\mathcal{T}, \mathcal{F})$ is a torsion pair in A there is a short exact sequence

$$0 \to T \xrightarrow{\mu} H^0(X^\bullet) \xrightarrow{\pi} F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Consider the following commutative diagram of exact sequences in A obtained by pullback along μ from the lower horizontal sequence:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{im}(d^{-1}) \arrow[r, "\mu''"] \arrow[d, equal] & E \arrow[r] \arrow[d, "\mu'"] & T \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \operatorname{im}(d^{-1}) \arrow[r] & \ker(d^0) \arrow[r] \arrow[d] & H^0(X^\bullet) \arrow[r] \arrow[d, "\pi"] & 0 \\
 & & F \arrow[r, equal] \arrow[d] & F \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Note that pullback preserves monomorphisms, epimorphisms, and kernels. Moreover, since the map $\ker(d^0) \to H^0(X^\bullet)$ is an epimorphism, the pullback is also a pushout and hence it also preserves cokernels.

Let $d^{-1} = i \circ \rho$ be the canonical factorization of d^{-1} through $\operatorname{im}(d^{-1})$ and let $\tilde{d}^{-1} := \mu'' \circ \rho$. Let $X'^\bullet$ be the following subcomplex of $X^\bullet$:

$$X'^\bullet : \cdots \to X^{-2} \to X^{-1} \xrightarrow{\tilde{d}^{-1}} E \to 0 \to \cdots$$

By construction, $X'^\bullet \in \mathsf{D}^{\leq 0}$. Let $X''^\bullet$ be the quotient complex $X^\bullet / X'^\bullet$. We obtain a distinguished triangle

$$X'^\bullet \to X^\bullet \to X''^\bullet \to X'^\bullet[1]$$

in $\mathsf{D}^b(\mathsf{A})$. It remains to show that $X''^\bullet \in \mathsf{D}^{\geq 1}$. By construction, $H^i(X''^\bullet) = 0$ for $i < 0$. Now $X''^0 = X^0/E$, $X''^1 = X^1$, and we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d, "d^0"] & X^0/E \arrow[r] \arrow[d, "\tilde{d}^0"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & X^1 \arrow[r, equal] & X^1 \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{d}^0$ denotes the 0-th differential of $X''^\bullet$. Hence, $H^0(X''^\bullet) = \ker(\tilde{d}^0) \cong \ker(d^0)/E \cong H^0(X^\bullet)/T \cong F \in \mathcal{F}$, so that $X''^\bullet \in \mathsf{D}^{\geq 1}$. □

The following result gives a recipe to construct kernels and cokernels of morphisms in the heart of a t-structure coming from a torsion pair.

Corollary 2.13. *Let A be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$. Let $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ be the corresponding t-structure of $\mathsf{D}^b(\mathsf{A})$ as in Proposition 2.12 with associated heart $\mathsf{C} := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$. Let $f : X^\bullet \to Y^\bullet$ be a morphism in C. Embed f into a distinguished triangle*

$$X^\bullet \xrightarrow{f} Y^\bullet \to Z^\bullet \to X^\bullet[1].$$

Embed $H^{-1}(Z^\bullet)$ into a short exact sequence in A

$$0 \to T \to H^{-1}(Z^\bullet) \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Let

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & T \arrow[d] \\
\ker(d_Z^{-1}) \arrow[r] & H^{-1}(Z^\bullet).
\end{tikzcd}$$

be a pullback diagram in A. In other words, $E \subseteq \ker(d_Z^{-1})$ is such that $E/\operatorname{im}(d_Z^{-2}) \cong T$ is the torsionfree part of $H^{-1}(Z^\bullet)$. Then

$$\ker_{\mathsf{C}}(f) \cong (\cdots \to \overset{-1}{Z^{-2}} \to \overset{0}{E} \to \overset{1}{0} \to \cdots)$$

and

$$\operatorname{coker}_{\mathsf{C}}(f) \cong (\cdots \to 0 \to \overset{-1}{Z^{-1}/E} \to \overset{0}{Z^0} \to \overset{1}{Z^1} \to \cdots).$$

Proof. This follows by combining Proposition 2.11 with the proof of Proposition 2.12. □

2.4 Differential graded algebras and derived tilting

Throughout this section let R be a commutative unital ring.

Definition 2.14. A *differential graded R-algebra* (*dg R-algebra*, for short) is a $\mathbf{Z}$-graded R-algebra $\mathscr{A} = \bigoplus_{n \in \mathbf{Z}} \mathscr{A}^n$ endowed with a differential d of degree 1 such that

$$d(ab) = d(a)b + (-1)^n a d(b)$$

for all $a \in \mathscr{A}^n, b \in \mathscr{A}$. A *homomorphism of differential graded R-algebras* is a homogeneous morphism of degree 0 of the underlying $\mathbf{Z}$-graded R-algebras commuting with the differentials.

Remark 2.15. We may view the ring R as a differential graded R-algebra concentrated in degree 0.

Example 2.16. Let A be an abelian category with enough injectives. Let X be a complex of A. Then $\mathscr{A} := \mathrm{RHom}_{\mathsf{A}}(X, X)$ (cf. §2.1) can be given the structure of a dg algebra. Choose a quasi-isomorphism $X \to I$ where I is K-injective. Then we have $\mathscr{A} = \mathrm{Hom}^\bullet_{\mathsf{A}}(I, I)$. Note that we replace both entries with

I because we will define a multiplication on $\mathscr{A}$ which comes from composition of morphisms $I \to I$. Let $f \in \mathscr{A}^n$ and $g \in \mathscr{A}^m$. We define their product $f * g \in \mathscr{A}^{n+m}$ by

$$(f * g)^k = f^{k+m} \circ g^k \qquad \text{for all } k \in \mathbf{Z}.$$

This makes $\mathscr{A}$ into a $\mathbf{Z}$-graded algebra with unit element $1_{\mathscr{A}} = (id_{I^k})^k \in \mathscr{A}^0$. It remains to verify the Leibniz rule:

$$\begin{aligned}
&(d(f) * g)^k + (-1)^n (f * d(g))^k \\
=&[d_I^{k+m+n} \circ f^{k+m} + (-1)^{n+1} f^{k+m+1} \circ d_I^{k+m}] \circ g^k + (-1)^n f^{k+m+1} \circ [d_I^{k+m} \circ g^k + (-1)^{m+1} g^{k+1} \circ d_I^k] \\
=&d_I^{k+n+m} \circ f^{k+m} \circ g^k + (-1)^{n+m+1} f^{k+m+1} \circ g^{k+1} \circ d_I^k \\
=&d_I^{k+n+m} \circ (f * g)^k + (-1)^{n+m+1} (f * g)^{k+1} \circ d_I^k \\
=&d(f * g)^k.
\end{aligned}$$

Definition 2.17. Let $\mathscr{A}$ be a dg R-algebra. A *differential graded (right) $\mathscr{A}$-module* (*dg $\mathscr{A}$-module*, for short) is a $\mathbf{Z}$-graded (right) $\mathscr{A}$-module $\mathscr{M} = \bigoplus_{n \in \mathbf{Z}} \mathscr{M}^n$ endowed with a differential d of degree 1 such that

$$d(ma) = d(m)a + (-1)^n m d(a)$$

for all $a \in \mathscr{A}$ and $m \in \mathscr{M}^n$. A *homomorphism of differential graded $\mathscr{A}$-modules* is a homogeneous morphism of degree 0 of the underlying $\mathbf{Z}$-graded $\mathscr{A}$-modules commuting with the differentials. The category of dg $\mathscr{A}$-modules is denoted $\mathsf{Mod}_{\mathscr{A}}$. There is a forgetful functor

$$\mathsf{Mod}_{\mathscr{A}} \longrightarrow \mathsf{CoCh}_R$$

of abelian categories. Here CoCh_R denotes the category of cochain complexes of R-modules. For $n \in \mathbf{Z}$ the *n-th cohomology group* $H^n(\mathscr{M})$ of a dg $\mathscr{A}$-module $\mathscr{M}$ is defined as the n-th cohomology group of the underlying cochain complex of R-modules.

In a similar way one defines the notion of a dg *left* $\mathscr{A}$-module. However, if we say dg $\mathscr{A}$-module we will always mean a dg right $\mathscr{A}$-module.

Example 2.18. Fix the setting of Example 2.16. For any complex Y of A the complex $\mathscr{M} = \mathrm{RHom}_{\mathsf{A}}(X, Y)$ can be given the structure of a dg $\mathscr{A}$-module as follows: Choose a quasi-isomorphism $Y \to J$ with J K-injective, so that $\mathscr{M} = \mathrm{Hom}^\bullet_{\mathsf{A}}(I, J)$. If $f \in \mathscr{M}^n$ and $g \in \mathscr{A}^m$ then we define

$$(f * g)^k = f^{k+m} \circ g^k \qquad \text{for all } k \in \mathbf{Z}.$$

This makes $\mathscr{M}$ into a $\mathbf{Z}$-graded $\mathscr{A}$-module, and the Leibniz rule holds by the same computation as in Example 2.16.

Lemma 2.19 ([Sta], Lemma 09JJ)**.** *The category $\mathsf{Mod}_{\mathscr{A}}$ is abelian and has arbitrary limits and colimits.*

If $\mathscr{A}$ is a dg algebra then we denote by $\mathsf{D}(\mathscr{A})$ the derived category of the category of dg $\mathscr{A}$-modules.

Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a homomorphism of dg R-algebras. If $\mathscr{N}$ is a dg $\mathscr{B}$-module then we can view it as a dg $\mathscr{A}$-module via φ. This yields a functor $\varphi_* : \mathsf{Mod}_{\mathscr{B}} \to \mathsf{Mod}_{\mathscr{A}}$ called restriction of scalars slong φ. It admits a right derived version $\mathrm{R}\,\varphi_* : \mathsf{D}(\mathscr{B}) \to \mathsf{D}(\mathscr{A})$ (cf. [Sta], Lemma 09LI).

Let $\mathscr{A}$ be a dg R-algebra. Let $\mathscr{M}$ be a dg $\mathscr{A}$-module and $\mathscr{N}$ a dg left $\mathscr{A}$-module. Then we can consider the tensor product $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$ of the underlying graded objects. Recall that it is defined by

$$(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})^l = \mathrm{coker}\Big(\bigoplus_{r+t+s=l} \mathscr{M}^r \otimes_R \mathscr{A}^t \otimes_R \mathscr{N}^s \to \bigoplus_{p+q=l} \mathscr{M}^p \otimes_R \mathscr{N}^q \Big)$$

$$x \otimes a \otimes y \mapsto x \otimes ay - xa \otimes y$$

for $l \in \mathbf{Z}$. Abusing notation we write $x \otimes y$ for the class of $x \otimes y$ in $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$. We define a differential d on $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$ by

$$d(x \otimes y) = d(x) \otimes y + (-1)^n x \otimes d(y)$$

for all $x \in \mathscr{M}^n$ and $y \in \mathscr{N}$ homogeneous. This makes $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$ into a dg $\mathscr{A}$-module.

Now let again $\varphi : \mathscr{A} \to \mathscr{B}$ be a homomorphism of dg R-algebras and let $\mathscr{M}$ be a dg $\mathscr{A}$-module. We can view $\mathscr{B}$ as a dg left $\mathscr{A}$-module via φ. Abusing notation we will write $\mathscr{B}$ instead of $\varphi_* \mathscr{B}$. Consider the dg $\mathscr{A}$-module $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}$. Via multiplication on the right factor it becomes a dg $\mathscr{B}$-module which we call the extension of scalars of $\mathscr{M}$ along φ and denote it by $\varphi^* \mathscr{M}$. This defines a functor $\varphi^* : \mathsf{Mod}_{\mathscr{A}} \to \mathsf{Mod}_{\mathscr{B}}$ which admits a left derived version $\mathrm{L}\,\varphi^* : \mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ (cf. [Sta], Lemma 09LS).

We will make use of the following tilting theorem due to Bernhard Keller in the case where X is a curve.

Theorem 2.20 ([Sta], Theorem 09M5)**.** *Let X be a quasicompact quasiseparated scheme. There exists a differential graded algebra $\mathscr{A}$ such that $\mathsf{D}_{qcoh}(\mathcal{O}_X) \cong \mathsf{D}(\mathscr{A})$.*

Proof. By Theorem 2.5 there exists a perfect complex $\mathcal{P}$ of $\mathcal{O}_X$-modules which generates $\mathsf{D}_{qcoh}(\mathcal{O}_X)$. Set

$$\mathscr{A} := \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}).$$

For $n \in \mathbf{Z}$ its n-th cohomology group is

$$\begin{aligned} H^n(\mathscr{A}) &\overset{2.1}{=} \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{P}, \mathcal{P}[n]) \\ &= \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}). \end{aligned}$$

In other words, the cohomology of $\mathscr{A}$ computes the Ext-groups of $\mathcal{P}$. By [Sta], Lemma 09LZ, this implies that the functor

$$- \otimes^{\mathbf{L}}_{\mathscr{A}} \mathcal{K} : \mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathcal{O}_X)$$

(cf. [Sta], Lemma 09LX) is fully faithful. Hence, its right adjoint

$$\mathrm{RHom}(\mathcal{K}, -) : \mathsf{D}(\mathcal{O}_X) \to \mathsf{D}(\mathscr{A})$$

is a left quasi-inverse. On the other hand, the essential image of $- \otimes^{\mathbf{L}}_{\mathscr{A}} \mathcal{P}$ is contained in $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ (cf. [Sta], Lemma 09M3). Since $\mathcal{P}$ generates $\mathsf{D}_{qcoh}(\mathcal{O}_X)$, the kernel of the restriction of $\mathrm{RHom}(\mathcal{P}, -)$ to $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ is zero. The statement now follows from a formal result (see [Sta], Lemma 09J1). $\square$

3 A derived equivalence

Using the construction of Bondal and van den Bergh in the proof of Theorem 2.5, in §3.1 we will compute a particularly simple generator of $\mathsf{D}_{qcoh}(\mathcal{O}_X)$ if X is a curve admitting a closed point whose complement is affine. In §3.2 we compute certain sets of morphisms of injective $\mathcal{O}_X$-modules in terms of adèles on X. This finally leads to an explicit description of a dg algebra $\mathscr{A}$ such that $\mathsf{D}_{qcoh}(\mathcal{O}_X) \cong \mathsf{D}(\mathscr{A})$ which we will establish in §3.3. Finally, in §3.4 we compute explicit dg modules corresponding to coherent sheaves on the curve along the equivalence $\mathsf{D}_{qcoh}(\mathcal{O}_X) \cong \mathsf{D}(\mathscr{A}^{ad})$.

3.1 A simple generator of the derived category of a curve

Lemma 3.1. *Let X be a ringed space, $U \subseteq X$ an open subspace and denote by $j : U \to X$ the inclusion. Let $\mathcal{G} \in \mathsf{Mod}(\mathcal{O}_U)$ be a skycraper sheaf at a closed point of U. Then $j_!\mathcal{G} = j_*\mathcal{G}$.*

Proof. Let $x \in U$ be a closed point and M an $\mathcal{O}_{X,x}$-module such that $\mathcal{G} \cong \iota_{x,*}M$. Since $j_!$ is left adjoint to restriction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{G}, j_*\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, (j_*\mathcal{G})_{|U}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{G}).$$

By taking the preimage of the identity on the right, we obtain a canonical morphism $j_!\mathcal{G} \to j_*\mathcal{G}$. Over U this map is the identity of $\mathcal{G}$ and hence an isomorphism. If $y \notin U$ then both $(j_!\mathcal{G})_y$ and $(j_*\mathcal{G})_y$ are equal to 0. For $j_!$ this is clear from the definition. For j_* we use the fact that x is a closed point of U, hence of X, and for $y \notin U$ we therefore find an open subset $V_y \subset X$ containing y but not x. Then $\mathcal{G}(V_y \cap U) = 0$ and hence $(j_*\mathcal{G})_y = \varinjlim_{y \in V \subset X} \mathcal{G}(V \cap U) = 0$. ☐

Lemma 3.2 ([Sta], special case of Lemma 09IR)**.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme, let $t \in A$ be a regular element and denote by $j : U = D(t) \hookrightarrow X$ the corresponding open embedding. Let $\mathcal{G} \in \mathsf{Mod}_{\mathcal{O}_X}$ be the sheaf corresponding to the A-module A/tA. If $\mathcal{F} \in \mathsf{D}_{qcoh}(\mathcal{O}_X)$ with $\mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{G}[n], \mathcal{F}) = 0$ for all $n \in \mathbb{Z}$ then the canonical map $\mathcal{F} \to \mathrm{R}\, j_*(\mathcal{F}_{|U})$ is an isomorphism.*

Proof. The unit of the adjunction between the direct and inverse image gives a canonical comparison morphism $\mathcal{F} \to Rj_*(\mathcal{F}_{|U})$. Since this morphism is functorial in $\mathcal{F}$ and since $\mathcal{F}$ is quasi-isomorphic to a K-injective complex, in order to prove the statement we may assume that $\mathcal{F}$ itself is K-injective. Since X is affine, we have an equivalence of categories $\mathsf{D}_{qcoh}(\mathcal{O}_X) \cong \mathsf{D}(A)$ (see [Sta], Lemma 06Z0). Using this equivalence we will view $\mathcal{F}$ as a complex $M^\bullet$ of A-modules. Applying the functor $\mathrm{Hom}_{\mathsf{D}(A)}(\cdot, M^\bullet)$ to the short exact sequence of A-modules

$$0 \to A \xrightarrow{t} A \to A/tA \to 0$$

and looking at the long exact cohomology sequence, we get that for all $n \in \mathbb{Z}$ the map

$$\mathrm{Hom}_{\mathsf{D}(A)}(A, M^\bullet[n]) = \mathrm{Ext}^n_A(A, M^\bullet) \to \mathrm{Ext}^n_A(A, M^\bullet) = \mathrm{Hom}_{\mathsf{D}(A)}(A, M^\bullet[n])$$

is an isomorphism. Since $M^\bullet[n]$ is a K-injective complex, by [Sta], Lemma 070I, we have

$$\mathrm{Hom}_{\mathsf{D}(A)}(A, M^\bullet[n]) = \mathrm{Hom}_{\mathsf{K}(A)}(A, M^\bullet[n]) = \mathrm{Hom}_{\mathsf{CoCh}(A)}(A, M^\bullet[n])/\sim,$$

where $\sim$ denotes the homotopy relation. Now $\mathrm{Hom}_{\mathsf{CoCh}(A)}(A, M^\bullet[n])$ is isomorphic to $\ker(d^n : M^n \to M^{n+1})$ by sending a morphism of complexes $f : A \to M^\bullet[n]$ to the image of 1_A under f. Moreover, such a morphism is homotopic to zero if and only if it factors through $\mathrm{im}(d^{n-1})$. Hence,

$$\mathrm{Hom}_{\mathsf{CoCh}(A)}(A, M^\bullet[n])/\sim \,\cong \ker(d^n)/\,\mathrm{im}(d^{n-1}) = H^n(M^\bullet).$$

It follows that for all $n \in \mathbb{Z}$, multiplication by t on $H^n(M^\bullet)$ is bijective.

Using that localization at t is exact (and by definition of j_*), the cohomology groups of the complex $j_*(M^\bullet_{|U})$ are the localizations at t of the cohomology groups of the complex $M^\bullet$. Thus, the canonical map $H^n(M^\bullet) \to$

$H^n(j_*(M^\bullet_{|U}))$ is an isomorphism for all n by the bijectivity of multiplication by t on $H^n(M^\bullet)$. This means that $M^\bullet \to j_*(M^\bullet_{|U})$ is a quasi-isomorphism of complexes, hence is an isomorphism in $\mathsf{D}(A)$. Note that since $M^\bullet$ is K-injective the functor $\mathrm{R}\, j_*$ is computed by applying j_* in each degree. □

Theorem 3.3. *Let X be a curve admitting a closed point ∞ such that $X \setminus \{\infty\}$ is affine. Let k be a nonzero integer. Then $\mathcal{O}_X \oplus \mathcal{O}_X(k[\infty])$ generates $\mathsf{D}_{qcoh}(\mathcal{O}_X)$.*

Proof. We follow the strategy of [Sta], Theorem 09IS, which is due to Bondal and van den Bergh (cf. [BvdB], Theorem 3.1.1).

Write $U := X \setminus \{\infty\}$. By [Det], Lemma 1.4, we may choose an affine open neighborhood $V = \mathrm{Spec}(A)$ of ∞ such that the prime ideal $\mathfrak{p} \subset A$ corresponding to ∞ is principal. We choose a generator $t \in A$ of $\mathfrak{p}$. In other words, $\{\infty\} = \mathrm{V}(t)$ as subsets of $V = \mathrm{Spec}(A)$. We have the following diagram of open immersions:

$$\begin{array}{ccc} U \cap V & \xrightarrow{h} & U \\ \big\downarrow g & & \big\downarrow f \\ V & \xrightarrow{j} & X. \end{array}$$

Note that $U \cap V = \mathrm{D}(t)$ as a subscheme of $V = \mathrm{Spec}(A)$. Let d be a positive integer. Denote by $\mathcal{G}$ the $\mathcal{O}_V$-module associated to the A-module A/t^dA. Over $\mathrm{D}(t) = V \setminus \{\infty\}$, t is a unit, so that $\mathcal{G}_{|\mathrm{D}(t)} = 0$. Moreover, $\mathcal{G}_\mathfrak{p} \cong A_\mathfrak{p}/\mathfrak{p}^d A_\mathfrak{p} \cong A/\mathfrak{p}^d = A/t^dA$. Hence, $\mathcal{G}$ is the skyscraper sheaf at ∞ on V with value A/t^dA. By Lemma 3.1 we have $j_*\mathcal{G} = j_!\mathcal{G}$, which is the skyscraper sheaf at ∞ on X with value A/t^dA.

We claim that $\mathcal{P} := \mathcal{O}_X \oplus j_*\mathcal{G}$ generates $\mathsf{D}(\mathcal{O}_X)$. Suppose we have $\mathcal{F} \in \mathsf{D}(\mathcal{O}_X)$ such that $\mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{P}[n], \mathcal{F}) = 0$ for all $n \in \mathbf{Z}$. We may assume that $\mathcal{F}$ is a K-injective complex (cf. [Sta], Proposition 077P and Theorem 079P). Using the adjunction between extension by zero and pullback along j we obtain that for all $n \in \mathbf{Z}$,

$$0 = \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(j_!\mathcal{G}[n], \mathcal{F}) = \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_V)}(\mathcal{G}[n], \mathcal{F}_{|V}).$$

By Lemma 3.2 this implies that the canonical map

$$\mathcal{F}_{|V} \longrightarrow \mathrm{R}\, g_*(\mathcal{F}_{|U\cap V})$$

is an isomorphism. By [Sta], Lemma 08FE, the right hand side is canonically isomorphic to $\mathrm{R}\, f_*(\mathcal{F}_{|U})_{|V}$, showing that the canonical map

$$\mathcal{F} \longrightarrow \mathrm{R}\, f_*(\mathcal{F}_{|U})$$

is an isomorphism over V. That it is an isomorphism over U follows immediately from the definition of the pushforward. Using this together with the adjunction between pullback and pushforward along f we obtain that for all $n \in \mathbf{Z}$

$$\begin{aligned} 0 = \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{O}_X[n], \mathcal{F}) &= \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{O}_X[n], \mathrm{R}\, f_*(\mathcal{F}_{|U})) \\ &= \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_U)}(\mathcal{O}_U[n], \mathcal{F}_{|U}). \end{aligned}$$

Since U is affine, $\mathcal{O}_U$ generates $\mathsf{D}(\mathcal{O}_U)$ (cf. [Sta], Theorem 06Z0), showing that $\mathcal{F}_{|U} = 0$, whence $\mathcal{F} \cong \mathrm{R}\, f_*(\mathcal{F}_{|U}) = 0$, concluding the proof of the fact that $\mathcal{O}_X \oplus j_*\mathcal{G}$ generates $\mathsf{D}_{qcoh}(\mathcal{O}_X)$.

By Remark 1.4 there are short exact sequences of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(d[\infty]) \to j_*\mathcal{G} \to 0$$

and

$$0 \to \mathcal{O}_X(-d[\infty]) \to \mathcal{O}_X \to j_*\mathcal{G} \to 0.$$

If $\mathcal{F} \in \mathsf{D}(\mathcal{O}_X)$ then the functor $\mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(-, \mathcal{F})$ is cohomological (see [Sta], Lemma 0149). Therefore, the distinguished triangles induced by the previous exact sequences induce long exact sequences of abelian groups

$$\begin{aligned} \ldots \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(j_*\mathcal{G}[n], \mathcal{F}) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{O}_X(d[\infty])[n], \mathcal{F}) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{O}_X[n], \mathcal{F}) \\ \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(j_*\mathcal{G}[n-1], \mathcal{F}) \to \ldots \end{aligned}$$

and

$$\begin{aligned} \ldots \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(j_*\mathcal{G}[n], \mathcal{F}) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{O}_X[n], \mathcal{F}) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(\mathcal{O}_X(-d[\infty])[n], \mathcal{F}) \\ \to \mathrm{Hom}_{\mathsf{D}(\mathcal{O}_X)}(j_*\mathcal{G}[n-1], \mathcal{F}) \to \ldots \end{aligned}$$

Now if k is positive then the first long exact sequence shows that together with $\mathcal{O}_X \oplus j_*\mathcal{G}$ also $\mathcal{O}_X \oplus \mathcal{O}_X(k[\infty])$ is a generator.

Similarly, for negative k the second long exact sequence shows the desired result. □

3.2 Morphisms of sheaves and adèles on curves

Recall the following elementary fact:

Lemma 3.4. *Let A be an integral domain with field of fractions K. The multiplication map*

$$m : K \to \mathrm{End}_A(K), x \mapsto (y \mapsto xy),$$

is a K-linear isomorphism.

Proof. It is enough to observe that any A-linear endomorphism of K is automatically K-linear because $K = \mathrm{Frac}(A)$. □

Lemma 3.5. *Let A be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $a \in \mathbf{Z}$. Then the map*

$$\begin{aligned} K &\to \mathrm{Hom}_A(K, K/\mathfrak{m}^a), \\ x &\mapsto m_x := (y \mapsto xy + \mathfrak{m}^a) \end{aligned}$$

is a K-linear isomorphism.

Proof. The map is clearly K-linear. If $x \in K$ with $m_x = 0$ then $xy \in \mathfrak{m}^a$ for all $y \in K$, which implies $x = 0$. For the surjectivity let $g : K \to K/\mathfrak{m}^a$ be given. Let π be a uniformizer of A. For $n \geq 0$ write $g(\pi^{-n}) = x_n + \mathfrak{m}^a$ for $n \geq 0$. Since g is A-linear for every $n \geq 0$ we have

$$x_n + \mathfrak{m}^a = g(\pi \cdot \pi^{-n-1}) = \pi g(\pi^{-n-1}) = \pi x_{n+1} + \mathfrak{m}^a,$$

so that $x_n - \pi x_{n+1} \in \mathfrak{m}^a$ and hence $\pi^n x_n - \pi^{n+1} x_{n+1} \in \mathfrak{m}^{a+n}$ for all n. This implies that $x := \lim_{n\to\infty} \pi^n x_n$ exists in K with $\pi^n x_n - x \in \mathfrak{m}^{a+n}$ for all n. Consequently,

$$m_x(\pi^{-n}) = \pi^{-n} x + \mathfrak{m}^a = \pi^{-n}(x - \pi^n x_n + \pi^n x_n) + \mathfrak{m}^a = x_n + \mathfrak{m}^a = g(\pi^{-n}).$$

The A-linearity of g and m_x then imply that $g = m_x$. □

Proposition 3.6. *Suppose that A is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Denote by $\widehat{A}$ the $\mathfrak{m}$-adic completion of A and by $\widehat{K}$ the fraction field of $\widehat{A}$. Let $a, b \in \mathbf{Z}$. There are natural A-linear isomorphisms $\mathrm{Hom}_A(K, K/\mathfrak{m}^a) \xrightarrow{\cong} \widehat{K}$ and $\mathrm{Hom}_A(K/\mathfrak{m}^b, K/\mathfrak{m}^a) \xrightarrow{\cong} \mathfrak{m}^{a-b}\widehat{A}$ such that the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_A(K,K) \arrow[r, "can"] & \mathrm{Hom}_A(K,K/\mathfrak{m}^a) \arrow[d, "\cong"] \\
K \arrow[u, "\cong"', "3.4"] \arrow[r, "\subseteq"] & \widehat{K}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_A(K/\mathfrak{m}^b,K/\mathfrak{m}^a) \arrow[r, "can"] \arrow[d, "\cong"'] & \mathrm{Hom}_A(K,K/\mathfrak{m}^a) \arrow[d, "\cong"] \\
\mathfrak{m}^{a-b}\widehat{A} \arrow[r, "\subseteq"] & \widehat{K}
\end{tikzcd}$$

commute.

Proof. First note that for any $n \in \mathbf{Z}$ we have $K \cap \mathfrak{m}^n\widehat{A} = \mathfrak{m}^n$ as $\widehat{A}$-submodules of $\widehat{K}$. Since K is dense in $\widehat{K}$ and $\mathfrak{m}^n\widehat{A}$ is open, this implies that the map

$$K/\mathfrak{m}^n \to \widehat{K}/\mathfrak{m}^n\widehat{A}$$

is an A-linear isomorphism.

Now let $f \in \mathrm{Hom}_A(K,K/\mathfrak{m}^a)$. Base change to $\widehat{A}$ gives us an $\widehat{A}$-linear map $f \otimes id_{\widehat{A}} : K \otimes_A \widehat{A} \to K/\mathfrak{m}^a \otimes_A \widehat{A}$. Under the identifications $K \otimes_A \widehat{A} \cong \widehat{K}$ and $K/\mathfrak{m}^a \otimes_A \widehat{A} \cong \widehat{K}/\mathfrak{m}^a\widehat{A}$ the latter map corresponds to an $\widehat{A}$-linear map $\widehat{f} : \widehat{K} \to \widehat{K}/\mathfrak{m}^a\widehat{A}$. Say f is given by $f(\pi^{-n}) = x_n + \mathfrak{m}^a$ for $n \geq 1$ and some $x_n \in K$, where π is a fixed uniformizer of A. Then $\widehat{f}(\pi^{-n}) = x_n + \mathfrak{m}^a\widehat{A}$ for all $n \geq 1$.

We claim that the A-linear map

$$\mathrm{Hom}_A(K,K/\mathfrak{m}^a) \to \mathrm{Hom}_{\widehat{A}}(\widehat{K},\widehat{K}/\mathfrak{m}^a\widehat{A}), f \mapsto \widehat{f},$$

is an isomorphism.

For the injectivity, let $f : K \to K/\mathfrak{m}^a$ be A-linear with $\widehat{f} = 0$. Write $f(\pi^{-n}) = x_n + \mathfrak{m}^a$ for $n \geq 1$. We obtain $x_n \in \mathfrak{m}^a\widehat{A} \cap K = \mathfrak{m}^a$ for all $n \geq 1$ and hence $f = 0$.

Now let $g \in \mathrm{Hom}_{\widehat{A}}(\widehat{K},\widehat{K}/\mathfrak{m}^a\widehat{A})$. Write $g(\pi^{-n}) = x_n + \mathfrak{m}^a\widehat{A}$ for some $x_n \in \widehat{K}$ and all $n \geq 1$. Using that the canonical map $K/\mathfrak{m}^a \to \widehat{K}/\mathfrak{m}^a\widehat{A}$ is an isomorphism we may assume that the x_n lie in K. Now define $f : K \to K/\mathfrak{m}^a$ by $f(\pi^{-n}) := x_n + \mathfrak{m}^a$ and extend A-linearly (note that $K = A[\pi^{-1}]$). This is well-defined: Since g is $\widehat{A}$-linear we have

$$\pi x_{n+1} + \mathfrak{m}^a\widehat{A} = \pi g(\pi^{-n-1}) = g(\pi^{-n}) = x_n + \mathfrak{m}^a\widehat{A}$$

and hence $\pi x_{n+1} - x_n \in \mathfrak{m}^a\widehat{A} \cap K = \mathfrak{m}^a$. Finally, $\widehat{f} = g$.

By Lemma 3.5 the map

$$\widehat{K} \to \mathrm{Hom}_{\widehat{A}}(\widehat{K},\widehat{K}/\mathfrak{m}^a\widehat{A}),$$
$$x \mapsto m_x := (y \mapsto xy + \mathfrak{m}^a\widehat{A})$$

is an $\widehat{A}$-linear isomorphism. Moreover, from our explicit construction it follows that the first diagram commutes. To conclude the proof it remains to observe that an element $x \in \widehat{K}$ lies in $\mathfrak{m}^{a-b}\widehat{A}$ if and only if $\mathfrak{m}^b\widehat{A} \subseteq \ker(m_x)$. $\square$

From now on we fix a Dedekind domain A with field of fractions K. If $\mathfrak{p} \subset A$ is a nonzero prime ideal then we write $\widehat{A}_{\mathfrak{p}}$ for its $\mathfrak{p}$-adic completion and $\widehat{K}_{\mathfrak{p}}$ for the field of fractions of $\widehat{A}_{\mathfrak{p}}$.

Definition 3.7. We call

$$\mathbf{A}_K := \prod_{\substack{\mathfrak{p} \subset A \\ \text{maximal}}}' \widehat{K}_{\mathfrak{p}}$$

the *ring of adèles of* K, where the restricted direct product is with respect to the subrings $\widehat{A}_{\mathfrak{p}} \subset \widehat{K}_{\mathfrak{p}}$, and

$$\mathbf{A}_K^0 := \prod_{\substack{\mathfrak{p} \subset A \\ \text{maximal}}} \widehat{A}_{\mathfrak{p}}$$

the *ring of integral adèles of* K.

Remark 3.8. Via the diagonal embeddings $A \overset{\Delta}{\hookrightarrow} \mathbf{A}_K^0$ and $K \overset{\Delta}{\hookrightarrow} \mathbf{A}_K$, $\mathbf{A}_K^0$ is an A-algebra and $\mathbf{A}_K$ a K-algebra.

Lemma 3.9. *The multiplication map*

$$K \otimes_A \mathbf{A}_K^0 \longrightarrow \mathbf{A}_K$$

is an isomorphism of K*-algebras.*

Proof. If A is a field then $A = K = \mathbf{A}_K^0 = \mathbf{A}_K$ and the statement is trivially true. Let us now assume that A is not a field.

Since K is a localization of A, it is a flat A-module. The map of interest is thus the base change to K of the inclusion $\mathbf{A}_K^0 \subset \mathbf{A}_K$ followed by the multiplication map $K \otimes_A \mathbf{A}_K \to \mathbf{A}_K$ which is an isomorphism because $\mathbf{A}_K$ is a K-algebra. It remains to show the surjectivity.

For this, let $f = (f_{\mathfrak{p}})_{\mathfrak{p}} \in \mathbf{A}_K$ be an adèle. For each maximal ideal $\mathfrak{p}$ of A we denote by $v_{\mathfrak{p}} : K \to \mathbf{Z}$ the $\mathfrak{p}$-adic valuation on K. It extends uniquely to a valuation $\widehat{K}_{\mathfrak{p}} \to \mathbf{Z}$ which we call $v_{\mathfrak{p}}$ as well. Let Q denote the finite set of maximal ideals $\mathfrak{q}$ of A such that $f_{\mathfrak{q}} \notin \widehat{A}_{\mathfrak{q}}$. For each maximal ideal $\mathfrak{p}$ of A we may choose $\pi_{\mathfrak{p}} \in A$ such that its image in $\widehat{A}_{\mathfrak{p}}$ is a uniformizer. Then $\alpha := \prod_{\mathfrak{q} \in Q} \pi_{\mathfrak{q}}^{-v_{\mathfrak{q}}(f_{\mathfrak{q}})} \in A$ because $v_{\mathfrak{q}}(f_{\mathfrak{q}}) < 0$ for all $\mathfrak{q} \in Q$. Define $g := \alpha \cdot f \in \mathbf{A}_K$. If $\mathfrak{p} \notin Q$ then $g_{\mathfrak{p}} = \alpha f_{\mathfrak{p}} \in \widehat{A}_{\mathfrak{p}}$ because $\alpha \in A$ and $f_{\mathfrak{p}} \in \widehat{A}_{\mathfrak{p}}$. If $\mathfrak{q} \in Q$ then

$$v_{\mathfrak{q}}(g_{\mathfrak{q}}) = v_{\mathfrak{q}}(f_{\mathfrak{q}}) - \sum_{\mathfrak{q}' \in Q} v_{\mathfrak{q}'}(f_{\mathfrak{q}'}) v_{\mathfrak{q}}(\pi_{\mathfrak{q}'}) = -\sum_{\mathfrak{q}' \neq \mathfrak{q}} \underbrace{v_{\mathfrak{q}'}(f_{\mathfrak{q}'})}_{<0} \underbrace{v_{\mathfrak{q}}(\pi_{\mathfrak{q}'})}_{\geq 0} \geq 0.$$

Hence $g_{\mathfrak{p}} \in \widehat{A}_{\mathfrak{p}}$ for all $\mathfrak{p}$, whence $g \in \mathbf{A}_K^0$, and $\alpha^{-1} \otimes g \in K \otimes_A \mathbf{A}_K^0$ maps to f under the multiplication map. $\square$

Remark 3.10. Let $\mathfrak{a}$ be a fractional ideal of A. Since A is noetherian, the direct product of any family of flat A-modules is flat (cf. [Cha], Theorem 2.1). Hence, $\mathbf{A}_K^0$ is a flat A-module and therefore the map $\mathfrak{a} \otimes_A \mathbf{A}_K^0 \to K \otimes_A \mathbf{A}_K^0$ is injective. The isomorphism $K \otimes_A \mathbf{A}_K^0 \cong \mathbf{A}_K$ then identifies $\mathfrak{a}\mathbf{A}_K^0$ with $\mathfrak{a} \otimes_A \mathbf{A}_K^0$ as A-submodules of $\mathbf{A}_K$.

We define the following partial ordering on the set of nonzero ideals of A: If I, J are two fractional ideals of A then $I \leq J$ whenever $I \supseteq J$. In this case there is a natural projection $A/J \to A/I$. This makes the family $(A/I)_I$ into a projective system of A-modules.

Lemma 3.11. *There is an* A*-linear isomorphism* $\mathbf{A}_K^0 \cong \varprojlim_I A/I$*.*

Proof. Since A is Dedekind every nonzero ideal I of A admits a unique decomposition $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ with $v_{\mathfrak{p}}(I) \in \mathbf{Z}$. By the Chinese Remainder Theorem,

$$A/I \cong \prod_{\mathfrak{p}} A/\mathfrak{p}^{v_{\mathfrak{p}}(I)}.$$

Using that projective limits commute with arbitrary products, we have the following series of ring isomorphisms:

$$\varprojlim_I A/I \cong \prod_{\mathfrak{p}} \varprojlim_I A/\mathfrak{p}^{v_{\mathfrak{p}}(I)} \cong \prod_{\mathfrak{p}} \varprojlim_{n\geq 1} A/\mathfrak{p}^n \cong \prod_{\mathfrak{p}} \varprojlim_{n\geq 1} A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} = \prod_{\mathfrak{p}} \widehat{A}_{\mathfrak{p}} = \mathbf{A}_K^0.$$

□

Let $\mathfrak{a}$ be a fractional ideal of A and let $\mathfrak{p}$ be a maximal ideal. We denote by $\widehat{\mathfrak{a}}_{\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of $\mathfrak{a}$, i.e. $\widehat{\mathfrak{a}}_{\mathfrak{p}} = \varprojlim_n \mathfrak{a}/\mathfrak{p}^n\mathfrak{a}$ where the index set runs over all n such that $\mathfrak{p}^n \subset \mathfrak{a}$. Recall that we have $\widehat{\mathfrak{a}}_{\mathfrak{p}} \cong \mathfrak{a} \otimes_A \widehat{A}_{\mathfrak{p}}$. If $\mathfrak{a}\widehat{A}_{\mathfrak{p}}$ denotes the $\widehat{A}_{\mathfrak{p}}$-submodule of $\widehat{K}_{\mathfrak{p}}$ generated by $\mathfrak{a}$ then by flatness of $A \to \widehat{A}_{\mathfrak{p}}$ we have $\mathfrak{a} \otimes_A \widehat{A}_{\mathfrak{p}} \cong \mathfrak{a}\widehat{A}_{\mathfrak{p}}$.

Proposition 3.12. *Let $\mathfrak{a} \subset K$ be a fractional ideal of A. There is a unique K-linear isomorphism*

$$\Phi : \mathrm{Hom}_A(K, K/\mathfrak{a}) \xrightarrow{\cong} \mathbf{A}_K$$

such that for each maximal ideal $\mathfrak{p}$ of A the diagram

$$\begin{tikzcd}
\mathrm{End}_A(K) \arrow[r, "can"] & \mathrm{Hom}_A(K, K/\mathfrak{a}) \arrow[r, "-\otimes id_{\widehat{A}_{\mathfrak{p}}}"] \arrow[d, "\cong"', "\Phi"] & \mathrm{Hom}_{\widehat{A}_{\mathfrak{p}}}(\widehat{K}_{\mathfrak{p}}, \widehat{K}_{\mathfrak{p}}/\widehat{\mathfrak{a}}_{\mathfrak{p}}) \arrow[d, "\cong"', "3.5"] \\
K \arrow[u, "\cong", "3.4"'] \arrow[r, "\Delta"] & \mathbf{A}_K \arrow[r, "proj_{\mathfrak{p}}"] & \widehat{K}_{\mathfrak{p}}
\end{tikzcd}$$

commutes. If $\mathfrak{b} \subset K$ is another fractional ideal then there is a unique A-linear isomorphism

$$\Psi : \mathrm{Hom}_A(K/\mathfrak{b}, K/\mathfrak{a}) \xrightarrow{\cong} \mathfrak{b}^{-1}\mathfrak{a}\mathbf{A}_K^0$$

making the diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(K/\mathfrak{b}, K/\mathfrak{a}) \arrow[r, "can"] \arrow[d, "\cong"', "\Psi"] & \mathrm{Hom}_A(K, K/\mathfrak{a}) \arrow[r, "-\otimes id_{\widehat{A}_{\mathfrak{p}}}"] \arrow[d, "\cong"', "\Phi"] & \mathrm{Hom}_{\widehat{A}_{\mathfrak{p}}}(\widehat{K}_{\mathfrak{p}}/\widehat{\mathfrak{b}}_{\mathfrak{p}}, \widehat{K}_{\mathfrak{p}}/\widehat{\mathfrak{a}}_{\mathfrak{p}}) \arrow[d, "\cong"', "3.5"] \\
\mathfrak{b}^{-1}\mathfrak{a}\mathbf{A}_K^0 \arrow[r, "\subseteq"] & \mathbf{A}_K \arrow[r, "proj_{\mathfrak{p}}"] & \widehat{\mathfrak{b}}_{\mathfrak{p}}^{-1}\widehat{\mathfrak{a}}_{\mathfrak{p}}
\end{tikzcd}$$

commutative for any nonzero prime ideal $\mathfrak{p}$.

Proof. **Step 1**: Consider the A-submodule $\mathrm{Hom}_A(K/A, K/\mathfrak{a})$ of $\mathrm{Hom}_A(K, K/\mathfrak{a})$. We claim that it generates $\mathrm{Hom}_A(K, K/\mathfrak{a})$ as a K-vector space.

To see this, let $f : K \to K/\mathfrak{a}$ be A-linear. Its restriction to A is not injective because $K/\mathfrak{a}$ is a torsion A-module. Therefore, there is a nonzero element $x \in A$ such that $f(x) = 0$. This means $x \cdot f$ contains A in its kernel. Thus, $x \cdot f \in \mathrm{Hom}(K/A, K/\mathfrak{a})$, proving the claim.

Taking the base change of the inclusion $\mathrm{Hom}_A(K/A, K/\mathfrak{a}) \hookrightarrow \mathrm{Hom}_A(K, K/\mathfrak{a})$ along $A \hookrightarrow K$, we get an isomorphism

$$K \otimes_A \mathrm{Hom}_A(K/A, K/\mathfrak{a}) \xrightarrow{\cong} \mathrm{Hom}_A(K, K/\mathfrak{a}).$$

Indeed, the injectivity of this map follows from the flatness of $A \hookrightarrow K$, and the surjectivity is the claim above.

Step 2: Therefore, it suffices to construct an A-linear isomorphism

$$\mathrm{Hom}_A(K/A, K/\mathfrak{a}) \xrightarrow{\cong} \mathfrak{a}\mathbf{A}_K^0$$

such that for all $\mathfrak{p}$ the diagram

$$\begin{tikzcd} \mathrm{Hom}_A(K/A, K/\mathfrak{a}) \arrow[r, "-\otimes id_{\widehat{A}_\mathfrak{p}}"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\widehat{A}_\mathfrak{p}}(\widehat{K}_\mathfrak{p}/\widehat{A}_\mathfrak{p}, \widehat{K}_\mathfrak{p}/\widehat{\mathfrak{a}}_\mathfrak{p}) \arrow[d, "\cong"', "3.5"] \\ \mathfrak{a}\mathbf{A}_K^0 \arrow[r, "proj_\mathfrak{p}"] & \widehat{a}_\mathfrak{p} \end{tikzcd}$$

commutes. Indeed, the base change to K then gives a unique K-linear isomorphism between $\mathrm{Hom}_A(K, K/\mathfrak{a})$ and

$$K \otimes_A \mathfrak{a}\mathbf{A}_K^0 \cong K \otimes_A \mathfrak{a} \otimes_A \mathbf{A}_K^0 \cong K \otimes_A \mathbf{A}_K^0 \cong \mathbf{A}_K$$

making the diagrams of the proposition commute if we identify $\widehat{\mathfrak{a}}_\mathfrak{p} \otimes_A K$ with $\widehat{K}_\mathfrak{p}$.

Step 3: For any nonzero element $x \in A$ evaluation at $x^{-1} + A$ gives an isomorphism between $\mathrm{Hom}_A(x^{-1}A/A, K/\mathfrak{a})$ and $x^{-1}\mathfrak{a}/\mathfrak{a}$. The latter is isomorphic to $\mathfrak{a}/x\mathfrak{a} \cong A/xA \otimes_A \mathfrak{a}$ via multiplication by x. Now K/A is the colimit of the A-modules $x^{-1}A/A$. By the universal property of colimits $\mathrm{Hom}_A(K/A, K/\mathfrak{a})$ is isomorphic to the limit of the A-modules $A/xA \otimes_A \mathfrak{a}$ where x runs over the nonzero elements of A and the transition maps are the natural ones. Here the indexing set is ordered by divisibility in A. In this special situation the base change commutes with the limit because $\mathfrak{a}$ is finitely presented and the A-modules A/xA are of finite length (allowing for a Mittag-Leffler argument; cf. the proof of [Jen], Théorème 7.7). Therefore, the limit is isomorphic to $\mathbf{A}_K^0 \otimes_A \mathfrak{a} \cong \mathfrak{a}\mathbf{A}_K^0$ (by Lemma 3.11, using that the set of nonzero principal ideals is cofinal in the set of all nonzero ideals of A).

We now show that the first diagram commutes. Let $\alpha \in K$. The corresponding map $K \to K/\mathfrak{a}$ sends x to $\alpha x + \mathfrak{a}$. Choose $y \in A$ such that $\alpha y \in \mathfrak{a}$. Then under the isomorphism $K \otimes_A \mathrm{Hom}_A(K/A, K/\mathfrak{a}) \xrightarrow{\cong} \mathbf{A}_K$ the above map corresponds to $y^{-1} \otimes m_{\alpha y}$ on the left. Going through the various isomorphisms from step 3 we see that $m_{\alpha y}$ corresponds to the adèle $(\alpha y)_\mathfrak{p} \in \mathbf{A}_K^0$. Hence, we indeed obtain that the adèle corresponding to α is $(\alpha)_\mathfrak{p}$, showing the commutativity of the first diagram.

Generalizing the construction in step 3 we now show that $\mathrm{Hom}_A(K/\mathfrak{b}, K/\mathfrak{a})$ is isomorphic to $\mathfrak{b}^{-1}\mathfrak{a}\mathbf{A}_K^0$. If $x \in A$ is a nonzero element such that $\mathfrak{b} \subseteq x^{-1}A$ then evaluation at $x^{-1} + \mathfrak{b}$ gives an isomorphism between $\mathrm{Hom}_A(x^{-1}A/\mathfrak{b}, K/\mathfrak{a})$ and $x^{-1}\mathfrak{b}^{-1}\mathfrak{a}/\mathfrak{a}$. The latter is isomorphic to $\mathfrak{b}^{-1}\mathfrak{a}/x\mathfrak{a} \cong \mathfrak{b}^{-1}/xA \otimes_A \mathfrak{a} \cong A/xA \otimes \mathfrak{b}^{-1}\mathfrak{a}$ via multiplication by x. Now $K/\mathfrak{b}$ is the colimit of the A-modules $x^{-1}A/\mathfrak{b}$ such that $\mathfrak{b} \subseteq x^{-1}A$, because the set of such x is cofinal in the set of all nonzero elements of A. By the universal property of colimits $\mathrm{Hom}_A(K/\mathfrak{b}, K/\mathfrak{a})$ is isomorphic to the limit of the A-modules $A/xA \otimes_A \mathfrak{b}^{-1}\mathfrak{a}$. This limit is isomorphic to $\mathbf{A}_K^0 \otimes_A \mathfrak{b}^{-1}\mathfrak{a} \cong \mathfrak{b}^{-1}\mathfrak{a}\mathbf{A}_K^0$.

For the commutativity of

$$\begin{tikzcd} \mathrm{Hom}_A(K/A, K/\mathfrak{a}) \arrow[r, "-\otimes id_{\widehat{A}_\mathfrak{p}}"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\widehat{A}_\mathfrak{p}}(\widehat{K}_\mathfrak{p}/\widehat{A}_\mathfrak{p}, \widehat{K}_\mathfrak{p}/\widehat{\mathfrak{a}}_\mathfrak{p}) \arrow[d, "\cong"', "3.5"] \\ \mathfrak{a}\mathbf{A}_K^0 \arrow[r, "proj_\mathfrak{p}"] & \widehat{a}_\mathfrak{p} \end{tikzcd}$$

we note that the left vertical isomorphism is the projective limit of the isomorphisms $\mathrm{Hom}_A(x^{-1}A/A, K/\mathfrak{a}) \cong \mathfrak{a}/x\mathfrak{a}$ considered above. Likewise, the isomorphism from Lemma 3.5 can be constructed as the inverse limit of the analogous isomorphisms $\mathrm{Hom}_{\widehat{A}_\mathfrak{p}}(x^{-1}\widehat{A}_\mathfrak{p}/\widehat{A}_\mathfrak{p}, \widehat{K}_\mathfrak{p}/\widehat{\mathfrak{a}}_\mathfrak{p}) \cong \widehat{\mathfrak{a}}_\mathfrak{p}/x\widehat{\mathfrak{a}}_\mathfrak{p}$. In both cases we may let x run through the nonzero elements of A because $K/A = \bigcup_{x\in A\setminus\{0\}} x^{-1}A/A$ and $\widehat{K}_\mathfrak{p}/\widehat{A}_\mathfrak{p} = \bigcup_{x\in A\setminus\{0\}} x^{-1}\widehat{A}_\mathfrak{p}/\widehat{A}_\mathfrak{p}$, using that $\widehat{A}_\mathfrak{p} \subseteq \widehat{K}_\mathfrak{p}$ is open and $K \subset \widehat{K}_\mathfrak{p}$ is dense. Therefore, it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(x^{-1}A/A, K/\mathfrak{a}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\widehat{A}_\mathfrak{p}}(\widehat{K}_\mathfrak{p}/\widehat{A}_\mathfrak{p}, \widehat{K}_\mathfrak{p}/\widehat{\mathfrak{a}}_\mathfrak{p}) \arrow[d, "\cong"] \\
\mathfrak{a}/x\mathfrak{a} \arrow[r] & \widehat{\mathfrak{a}}_\mathfrak{p}/x\widehat{\mathfrak{a}}_\mathfrak{p}
\end{tikzcd}$$

commutes. Since the lower horizontal map is induced by the inclusion $\mathfrak{a} \to \widehat{\mathfrak{a}}_\mathfrak{p}$ this follows directly from the definition of the vertical isomorphisms. ☐

Remark 3.13. The isomorphism Φ is characterized by the following property: Given an A-linear map $f : K \to K/\mathfrak{a}$ let $x = (x_\mathfrak{p})_\mathfrak{p}$ denote the corresponding adèle. For any $\mathfrak{p}$ let $\widehat{f}_\mathfrak{p} : \widehat{K}_\mathfrak{p} \to \widehat{K}_\mathfrak{p}/\widehat{\mathfrak{a}}_\mathfrak{p}$ denote the base change of f along $A \to \widehat{A}_\mathfrak{p}$. Then the relation between f and x is characterized by the fact that for any $\mathfrak{p}$ the map $\widehat{f}_\mathfrak{p}$ is $x_\mathfrak{p}$ times the residue class map $\widehat{K}_\mathfrak{p} \to \widehat{K}_\mathfrak{p}/\widehat{\mathfrak{a}}_\mathfrak{p}$.

Let X be a curve with field of definition E. Let $\mathcal{E}_X$ denote the constant $\mathcal{O}_X$-module associated with the function field $E(X)$ of X. Similarly to the affine case let

$$\mathbf{A}_X^0 := \prod_{x\in|X|} \widehat{\mathcal{O}}_{X,x} \subset \prod_{x\in|X|}{}' \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x}) =: \mathbf{A}_X$$

denote the ring of *(integral) adèles over* X where the restricted direct product is taken with respect to the subrings $\widehat{\mathcal{O}}_{X,x} \subset \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})$. Recall that a *fractional ideal sheaf* on X is a subsheaf $\mathcal{I} \subset \mathcal{E}_X$ that is a coherent $\mathcal{O}_X$-module.

Lemma 3.14. *The map*

$$E(X) \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}_X), \alpha \mapsto \alpha \cdot id_{\mathcal{E}_X},$$

is an $E(X)$-linear isomorphism.

Proof. Since the restriction maps of $\mathcal{E}_X$ are the identity maps this follows from Lemma 3.4. ☐

If $\mathcal{I}$ is a fractional ideal sheaf on X then we write

$$\mathcal{I}\mathbf{A}_X^0 := \prod_{x\in|X|} \widehat{\mathcal{I}}_x \subseteq \mathbf{A}_X.$$

Note that the arguments in step 3 above also show that $\mathfrak{a}\mathbf{A}_K^0 \cong \prod_\mathfrak{p} \widehat{\mathfrak{a}}_\mathfrak{p}$. Therefore, the notation $\mathcal{I}\mathbf{A}_X^0$ is not an illegitimate abuse of notation if $X = \mathrm{Spec}(A)$ is affine.

Corollary 3.15. *Let X be a curve with field of definition E and let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{E}_X$ be fractional ideal sheaves. Denote by $\Delta : E(X) \to \mathbf{A}_X$ the diagonal embedding. The isomorphisms from Proposition 3.12 glue to isomorphisms*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_X, \mathcal{E}_X/\mathcal{I}) \xrightarrow{\cong} \mathbf{A}_X$$

and

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_X/\mathcal{J}, \mathcal{E}_X/\mathcal{I}) \xrightarrow{\cong} \mathcal{J}^{-1}\mathcal{I}\mathbf{A}_X^0$$

such that the diagrams

$$\begin{array}{ccc}
\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}_X) & \xrightarrow{can} & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_X, \mathcal{E}_X/\mathcal{I}) \\
\cong \uparrow 3.14 & & \downarrow \cong \\
E(X) & \xrightarrow{\quad\Delta\quad} & \mathbf{A}_X.
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_X/\mathcal{J}, \mathcal{E}_X/\mathcal{I}) & \xrightarrow{can} & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_X, \mathcal{E}_X/\mathcal{I}) \\
\cong \downarrow & & \downarrow \cong \\
\mathcal{J}^{-1}\mathcal{I}\mathbf{A}_X^0 & \xrightarrow{\quad\subseteq\quad} & \mathbf{A}_X.
\end{array}$$

commute.

Proof. This follows from Proposition 3.12 and uses the uniqueness statement therein. $\square$

Example 3.16. Let X be a complete curve with closed point ∞ whose complement is the spectrum of a principal ideal domain. For $k \in \mathbf{Z}$ we have the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(k)$. Choose a uniformizer ϖ of $\mathcal{O}_{X,\infty}$ and let $\varepsilon \in \mathbf{A}_X^0$ be the adèle defined by

$$\varepsilon_x := \begin{cases} \varpi, & x = \infty \\ 1 & \text{else.} \end{cases}$$

Then $\mathcal{O}_X(k)\mathbf{A}_X^0 = \varepsilon^{-k}\mathbf{A}_K^0$.

Lemma 3.17. *Let X be a curve and let $\mathcal{I}$ be a fractional ideal sheaf on X. Then $\mathcal{E}_X/\mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

Proof. By [Har2], Proposition 7.17, an $\mathcal{O}_X$-module $\mathcal{F}$ is injective if and only if for each $x \in X$ its stalk $\mathcal{F}_x$ at x is an injective $\mathcal{O}_{X,x}$-module. Since X is a curve, all its local rings are principal ideal domains. By [HS], Chapter I, Theorem 7.1, a module over a principal ideal domain is injective if and only if it is divisible. Now for every point $x \in X$, the stalk $\mathcal{E}_{X,x} = E(X) = \mathrm{Frac}(\mathcal{O}_{X,x})$ is a divisible $\mathcal{O}_{X,x}$-module, hence is injective. Therefore, $\mathcal{E}_X$ is an injective $\mathcal{O}_X$-module. Since any quotient of a divisible module is again divisible (cf. [HS], Chapter I, Proposition 7.2), it follows that $\mathcal{E}_X/\mathcal{I}$ is an injective $\mathcal{O}_X$-module. $\square$

Proposition 3.18. *Let $\mathcal{I}, \mathcal{J} \subset \mathcal{E}_X$ be fractional ideal sheaves. Then there is an exact sequence of abelian groups*

$$0 \to \mathrm{Hom}(\mathcal{J}, \mathcal{I}) \to \mathcal{J}^{-1}\mathcal{I}\mathbf{A}_X^0 \to \mathbf{A}_X/E(X) \to \mathrm{Ext}^1(\mathcal{J}, \mathcal{I}) \to 0.$$

In particular, we have E-linear isomorphisms

$$\mathcal{J}^{-1}\mathcal{I}\mathbf{A}_X^0 \cap E(X) \cong \mathrm{Hom}(\mathcal{J}, \mathcal{I})$$

and

$$\mathbf{A}_X/(E(X) + \mathcal{J}^{-1}\mathcal{I}\mathbf{A}_X^0) \cong \mathrm{Ext}^1(\mathcal{J}, \mathcal{I}).$$

Proof. The statement follows from Corollary 3.15 and the snake lemma applied to the following diagram all of whose rows and columns are exact by the injectivity of $\mathcal{E}_X$ and $\mathcal{E}_X/\mathcal{I}$:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & 0 \arrow[d] & \mathrm{Hom}(\mathcal{J},\mathcal{I}) \arrow[d] & \\
0 \arrow[r] & \underbrace{\mathrm{Hom}(\mathcal{E}_X/\mathcal{J},\mathcal{E}_X)}_{=0} \arrow[r] \arrow[d] & \underbrace{\mathrm{Hom}(\mathcal{E}_X,\mathcal{E}_X)}_{\cong E(X)} \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{J},\mathcal{E}_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underbrace{\mathrm{Hom}(\mathcal{E}_X/\mathcal{J},\mathcal{E}_X/\mathcal{I})}_{\cong \mathcal{J}^{-1}\mathcal{I}\mathbf{A}_X^0} \arrow[r] & \underbrace{\mathrm{Hom}(\mathcal{E}_X,\mathcal{E}_X/\mathcal{I})}_{\cong \mathbf{A}_X} \arrow[r] & \mathrm{Hom}(\mathcal{J},\mathcal{E}_X/\mathcal{I}) \arrow[r] \arrow[d] & 0 \\
 & & & \mathrm{Ext}^1(\mathcal{J},\mathcal{I}) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}
$$

□

3.3 An explicit differential graded algebra

Throughout this section let X be a generalized Riemann sphere over a field E (cf. §1.3). Fix $k \in \mathbf{Z}$. Recall that by Theorem 3.3 $\mathcal{P} := \mathcal{O}_X \oplus \mathcal{O}_X(k)$ is a perfect generator of $\mathsf{D}_{qcoh}(\mathcal{O}_X)$. Set $\mathcal{P}_h := \pi_h^* \mathcal{P} = \mathcal{O}_{X_h} \oplus \mathcal{O}_{X_h}(kh)$, a perfect generator of $\mathsf{D}_{qcoh}(\mathcal{O}_{X_h})$.

Definition 3.19. For $h \geq 1$ set $\mathscr{A}_h := \mathrm{RHom}_{\mathcal{O}_{X_h}}(\mathcal{P}_h, \mathcal{P}_h)$.

$\mathscr{A}_h$ is a dg E_h-algebra (cf. Example 2.16), and by Theorem 2.20, $\mathsf{D}_{qcoh}(\mathcal{O}_{X_h}) \cong \mathsf{D}(\mathscr{A}_h)$.

Proposition 3.20. *There is a homomorphism of dg E-algebras*

$$\varphi_h : \mathscr{A}_1 \longrightarrow \mathscr{A}_h$$

such that the diagrams

$$
\begin{tikzcd}
\mathsf{D}_{qcoh}(\mathcal{O}_{X_h}) \arrow[r, "\cong"] \arrow[d, "\pi_{h,*}"'] & \mathsf{D}(\mathscr{A}_h) \arrow[d, "\mathrm{R}\,\varphi_{h,*}"] \\
\mathsf{D}_{qcoh}(\mathcal{O}_X) \arrow[r, "\cong"] & \mathsf{D}(\mathscr{A}_1)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{D}_{qcoh}(\mathcal{O}_{X_h}) \arrow[r, "\cong"] & \mathsf{D}(\mathscr{A}_h) \\
\mathsf{D}_{qcoh}(\mathcal{O}_X) \arrow[r, "\cong"] \arrow[u, "\pi_h^*"] & \mathsf{D}(\mathscr{A}_1). \arrow[u, "\mathrm{L}\,\varphi_h^*"']
\end{tikzcd}
$$

commute.

Proof. The map φ_h is the canonical one mentioned in [Sta], section 0B6A. We need to see that it is a homomorphism of dg algebras. Explicitly, φ_h is given as follows: The complex of $\mathcal{O}_{X_h}$-modules

$$\mathcal{K}_h := \quad \mathcal{E}_{X_h} \oplus \mathcal{E}_{X_h} \longrightarrow \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)$$

is K-injective (see Lemma 3.17) and quasi-isomorphic to $\mathcal{P}_h = \mathcal{O}_{X_h} \oplus \mathcal{O}_{X_h}(kh)$. Note that we have $\mathcal{K}_h \cong \pi_h^* \mathcal{K}_1$.

Now if $f \in \mathscr{A}_1 = \mathrm{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{K}_1, \mathcal{K}_1)$ is homogeneous of degree n with $f = (f^k)_k$ then $\varphi_h(f) = (\pi_h^*(f^k))_k \in \mathscr{A}_h$ is again a homogeneous map $\mathcal{K}_h \to \mathcal{K}_h$ of degree n. One checks that this indeed gives a homomorphism of dg algebras.

Let $\mathcal{F} \in \mathsf{D}_{qcoh}(\mathcal{O}_{X_h})$. By [KS1], Theorem 14.4 (c) there is an isomorphism (in $\mathsf{D}(E)$)

$$\mathrm{RHom}_{\mathcal{O}_{X_h}}(\pi_h^*\mathcal{P}, \mathcal{F}) \cong \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{P}, \pi_{h,*}\mathcal{F})$$

which is functorial in $\mathcal{F}$. This is an isomorphism in $\mathsf{D}(\mathscr{A})$ if the left hand side is given the dg $\mathscr{A}$-module structure induced by scalar restriction along φ, showing the commutativity of the first diagram.

The second diagram commutes by uniqueness of adjoint functors since the horizontal functors are equivalences. $\square$

Proposition 3.21. *The canonical map*

$$\mathscr{A}_1 \otimes_E E_h \longrightarrow \mathscr{A}_h$$

is an isomorphism of dg E_h-algebras.

Proof. Consider the Cartesian square

$$\begin{tikzcd} X_h \arrow[r, "\pi_h"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(E_h) \arrow[r] & \mathrm{Spec}(E). \end{tikzcd}$$

Since $\mathcal{P} \in \mathsf{D}_{qcoh}(\mathcal{O}_X)$ is perfect, by [Sta], Lemma 0AA7, the canonical map

$$\mathscr{A}_1 \otimes_E E_h = \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}) \longrightarrow \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{P}_h, \mathcal{P}_h) = \mathscr{A}_h$$

is an isomorphism in $D(E_h)$. By construction it is comptatible with the algebra structures, hence is an isomorphism of dg E_h-algebras. $\square$

We will now use the results of §3.2 to make the dg algebras more explicit. Let us fix some notation.

Fix a uniformizer ϖ of $\mathcal{O}_{X,\infty}$. For $h \geq 1$ and $y \in |X_h|$ with $\pi_h(y) = \infty$ set $\varpi_y := \pi_h^\sharp(\varpi)$, a uniformizer of $\mathcal{O}_{X_h,y}$. Let $\varepsilon_h \in \mathbf{A}^0_{X_h}$ be the adèle defined by

$$(\varepsilon_h)_y = \begin{cases} \varpi_y, & y \in \pi_h^{-1}(\infty) \\ 1, & \text{else,} \end{cases}$$

and set $J_h := \begin{pmatrix} \varepsilon_h & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbf{A}^0_{X_h})$.

Definition 3.22. We denote by $\mathscr{A}_h^{ad}$ the differential graded E_h-algebra with graded pieces

$$(\mathscr{A}_h^{ad})^n := \begin{cases} M_2(E_h(X_h)) \times M_2(\mathbf{A}^0_{X_h}), & n = 0, \\ M_2(\mathbf{A}_{X_h}), & n = 1, \\ 0 & \text{else,} \end{cases}$$

differential

$$\begin{aligned} d := d^0 : (\mathscr{A}_h^{ad})^0 &\to (\mathscr{A}_h^{ad})^1 \\ (M, N) &\mapsto M - J_h^k N J_h^{-k}, \end{aligned}$$

and whose algebra structure is given by the formulae

$$\begin{aligned}(M,N)*(A,B) &= (M\cdot A, N\cdot B),\\ (M,N)*P &= J_h^k N J_h^{-k}\cdot P,\\ P*(M,N) &= P\cdot M,\\ P*Q &= 0\end{aligned}$$

for $(M,N),(A,B)\in(\mathscr{A}_h^{ad})^0$ and $P,Q\in(\mathscr{A}_h^{ad})^1$.

Proposition 3.23. *Pullback along π_h induces a homomorphism of dg E-algebras*

$$\varphi_h^{ad}:\mathscr{A}_1^{ad}\longrightarrow\mathscr{A}_h^{ad}.$$

Proof. For each $y\in X_h$ the local homomorphism of local rings $\pi_{h,y}^\sharp:\mathcal{O}_{X,\pi_h(y)}\to\mathcal{O}_{X_h,y}$ extends uniquely to a local ring homomorphism $\widehat{\mathcal{O}}_{X,\pi_h(y)}\to\widehat{\mathcal{O}}_{X_h,y}$ and further to a field homomorphism $\mathrm{Frac}(\widehat{\mathcal{O}}_{X,\pi_h(y)})\to\mathrm{Frac}(\widehat{\mathcal{O}}_{X_h,y})$ which we still denote by $\pi_{h,y}^\sharp$.

If $y=\eta_h$ is the generic point of X_h then $\pi_h(y)=\eta$ is the generic point of X, and we obtain a map $\pi_{h,\eta_h}:E(X)=\mathcal{O}_{X,\eta}\to\mathcal{O}_{X_h,\eta_h}=E_h(X_h)$ between the function fields. We extend it to a map $M_2(E(X))\to M_2(E_h(X_h))$ by applying it to each entry.

If we let y run over all the closed points of X_h then we obtain an E-algebra homomorphism

$$\mathbf{A}_X^0=\prod_{x\in|X|}\widehat{\mathcal{O}}_{X,x}\to\prod_{x\in|X|}\prod_{y\in\pi_h^{-1}(x)}\widehat{\mathcal{O}}_{X_h,y}=\mathbf{A}_{X_h}^0$$

and an $E(X)$-algebra homomorphism

$$\mathbf{A}_X=\prod_{x\in|X|}{}'\,\mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})\to\prod_{x\in|X|}{}'\prod_{y\in\pi_h^{-1}(x)}\mathrm{Frac}(\widehat{\mathcal{O}}_{X_h,y})=\mathbf{A}_{X_h}$$

given by sending an adèle $(f_x)_{x\in|X|}$ to $((\pi_{h,y}^\sharp(f_x))_{y\in\pi_h^{-1}(x)})_{x\in|X|}$. We extend this map to matrices by applying it componentwise.

Abusing notation we will denote all the maps

$$M_2(E(X))\to M_2(E_h(X_h)),\quad M_2(\mathbf{A}_X^0)\to M_2(\mathbf{A}_{X_h}^0),\quad\text{and}\quad M_2(\mathbf{A}_X)\to M_2(\mathbf{A}_{X_h})$$

simply by $\pi_h^\sharp$. This induces a map $\varphi_h^{ad}:\mathscr{A}_1^{ad}\to\mathscr{A}_h^{ad}$ on the level of the underlying complexes.

To show that φ_h^{ad} is a homomorphism of dg algebras amounts to the commutativity of the diagram

$$\begin{array}{ccc} M_2(E(X))\times M_2(\mathbf{A}_X^0) & \xrightarrow{d} & M_2(\mathbf{A}_X)\\ \downarrow{\scriptstyle\pi_h^\sharp} & & \downarrow{\scriptstyle\pi_h^\sharp}\\ M_2(E_h(X_h))\times M_2(\mathbf{A}_{X_h}^0) & \xrightarrow{d} & M_2(\mathbf{A}_{X_h}).\end{array}$$

Let $(M,N)\in(\mathscr{A}_1^{ad})^0$. If we first apply the differential of $\mathscr{A}_1^{ad}$ and then $\pi_h^\sharp$ then we obtain the matrix

$$\pi_h^\sharp(M)-\pi_h^\sharp(J_1)^k\pi_h^\sharp(N)\pi_h^\sharp(J_1)^{-k}.$$

On the other hand, if we first apply $\pi_h^\sharp$ and then the differential of $\mathscr{A}_h^{ad}$ then we get the matrix

$$\pi_h^\sharp(M) - J_h^k \pi_h^\sharp(N) J_h^{-k}.$$

Hence, it suffices to see that $\pi_h^\sharp(J_1) = J_h$ which in turn boils down to $\pi_h^\sharp(\varepsilon) = \varepsilon_h$, which is true by definition. $\square$

Theorem 3.24. *For each $h \geq 1$ there is an isomorphism of dg E_h-algebras*

$$\psi_h : \mathscr{A}_h \xrightarrow{\cong} \mathscr{A}_h^{ad}$$

such that the diagram

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "\varphi_h"] \arrow[d, "\cong"', "\psi_1"] & \mathscr{A}_h \arrow[d, "\cong"', "\psi_h"] \\
\mathscr{A}_1^{ad} \arrow[r, "\varphi_h^{ad}"] & \mathscr{A}_h^{ad}
\end{tikzcd}$$

commutes.

Proof. Recall that

$$\mathcal{K}_h := \quad \mathcal{E}_{X_h} \oplus \mathcal{E}_{X_h} \xrightarrow{p_h := \begin{pmatrix} can & 0 \\ 0 & can \end{pmatrix}} \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)$$

is a K-injective complex (see Lemma 3.17) which is quasi-isomorphic to $\mathcal{P}_h = \mathcal{O}_{X_h} \oplus \mathcal{O}_{X_h}(kh)$.

Also recall that for $n \in \mathbf{Z}$ the nth graded piece $\mathscr{A}^n$ is given by

$$\mathscr{A}_h^n = \mathrm{Hom}_{\mathcal{O}_{X_h}}(\mathcal{K}_h, \mathcal{K}_h)^n = \prod_{l \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{O}_{X_h}}(\mathcal{K}_h^l, \mathcal{K}_h^{l+n}).$$

This can be illustrated as follows, where the labels indicate the degrees of the morphisms:

$$\begin{tikzcd}
\mathcal{E}_{X_h}^2 \arrow[r] \arrow[d, "0"] \arrow[dr, "1"] & \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh) \arrow[d, "0"] \arrow[dl, "-1"'] \\
\mathcal{E}_{X_h}^2 \arrow[r] & \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh).
\end{tikzcd}$$

Since $\mathcal{E}_{X_h}/\mathcal{O}_{X_h}$ and $\mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)$ are torsion $\mathcal{O}_{X_h}$-modules and $\mathcal{E}_{X_h}$ is torsion-free, there are no nonzero morphisms of degree -1 from $\mathcal{K}_h$ into itself. Hence, $\mathscr{A}^n = 0$ for $n \neq 0, 1$, and

$$\begin{aligned}
\mathscr{A}_h^0 &= \mathrm{End}(\mathcal{E}_{X_h} \oplus \mathcal{E}_{X_h}) \times \mathrm{End}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) \\
&= M_2(\mathrm{End}(\mathcal{E}_{X_h})) \times \begin{pmatrix} \mathrm{End}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}) & \mathrm{Hom}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh), \mathcal{E}_{X_h}/\mathcal{O}_{X_h}) \\ \mathrm{Hom}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) & \mathrm{End}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{A}_h^1 &= \mathrm{Hom}(\mathcal{E}_{X_h} \oplus \mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) \\
&= \begin{pmatrix} \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}) & \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}) \\ \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) & \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) \end{pmatrix}.
\end{aligned}$$

Recall that we denote by $p_h : \mathcal{E}_{X_h} \oplus \mathcal{E}_{X_h} \to \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)$ the differential of $\mathcal{K}_h$. Let $f \in \mathrm{End}(\mathcal{E}_{X_h} \oplus \mathcal{E}_{X_h})$ and $g \in \mathrm{End}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh))$. The differential $d : \mathscr{A}_h^0 \longrightarrow \mathscr{A}_h^1$ of $\mathscr{A}$ post- (resp. pre-)composes f (resp. g) with p_h and takes the difference of the resulting morphisms of degree 1, i.e. it takes the pair (f, g) to $p_h \circ f - g \circ p_h$.

The algebra structure on $\mathscr{A}_h$ is given by the formulae

$$\begin{aligned}(a_1, b_1) * (a_2, b_2) &= (a_1 \circ a_2, b_1 \circ b_2),\\ (a, b) * c &= b \circ c,\\ c * (a, b) &= c \circ a,\\ c * d &= 0,\end{aligned}$$

for all $(a, b), (a_1, b_1), (a_2, b_2) \in \mathscr{A}_h^0$ and $c, d \in \mathscr{A}_h^1$.

Recall that

$$\mathcal{O}_{X_h}(kh)_y = \begin{cases} \mathfrak{m}_y^k = \varpi_y^k \mathcal{O}_{X_h,y}, & y \in \pi_h^{-1}(\infty) \\ \mathcal{O}_{X_h,y}, & \text{else,} \end{cases}$$

showing that $\mathcal{O}_{X_h}(kh)\mathbf{A}^0_{X_h} = \varepsilon_h^k \mathbf{A}^0_{X_h}$.

By Proposition 3.15 we have the following commutative diagrams:

$$\begin{tikzcd} \mathrm{End}(\mathcal{E}_{X_h}) \arrow[r, hook] & \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}) \arrow[d, "\cong"] & \mathrm{End}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}) \arrow[l, hook'] \arrow[d, "\cong"] \\ E_h(X_h) \arrow[u, "\cong"] \arrow[r, hook, "\Delta"] & \mathbf{A}_{X_h} & \mathbf{A}^0_{X_h} \arrow[l, hook', "\supseteq"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{End}(\mathcal{E}_{X_h}) \arrow[r, hook] & \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}) \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh), \mathcal{E}_{X_h}/\mathcal{O}_{X_h}) \arrow[l, hook'] \arrow[d, "\cong"] \\ E_h(X_h) \arrow[u, "\cong"] \arrow[r, hook, "\Delta"] & \mathbf{A}_{X_h} & \varepsilon_h^k \mathbf{A}^0_{X_h} \arrow[l, hook', "\supseteq"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{End}(\mathcal{E}_{X_h}) \arrow[r, hook] & \mathrm{Hom}(\mathcal{E}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{E}_{X_h}/\mathcal{O}_{X_h}, \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(kh)) \arrow[l, hook'] \arrow[d, "\cong"] \\ E_h(X_h) \arrow[u, "\cong"] \arrow[r, hook, "\Delta"] & \mathbf{A}_{X_h} & \varepsilon_h^{-k} \mathbf{A}^0_{X_h} \arrow[l, hook', "\supseteq"'] \end{tikzcd}$$

Abusing notation from now on we will not write Δ anymore and view it as an inclusion. Let $\widetilde{\mathscr{A}_h^{ad}}$ be the dg E_h-algebra with underlying complex

$$\begin{aligned} M_2(E_h(X_h)) \times \begin{pmatrix} \mathbf{A}^0_{X_h} & \varepsilon_h^k \mathbf{A}^0_{X_h} \\ \varepsilon_h^{-k} \mathbf{A}^0_{X_h} & \mathbf{A}^0_{X_h} \end{pmatrix} &\xrightarrow{\tilde{d}} M_2(\mathbf{A}_{X_h}) \\ (A, B) &\longmapsto A - B \end{aligned}$$

and whose algebra structure is given by the formulae

$$\begin{aligned}(A_1, B_1) * (A_2, B_2) &= (A_1 \cdot A_2, B_1 \cdot B_2),\\ (A, B) * C &= B \cdot C,\\ C * (A, B) &= C \cdot A,\\ C * D &= 0,\end{aligned}$$

for all $(A,B),(A_1,B_1),(A_2,B_2) \in \mathscr{A}_h^0$ and $C,D \in \mathscr{A}_h^1$.

The commutative diagrams above yield an isomorphism of differential graded E_h-algebras $\widetilde{\psi_h} : \mathscr{A}_h \xrightarrow{\cong} \widetilde{\mathscr{A}_h^{ad}}$.

We have an isomorphism of E_h-algebras

$$\chi_h : \begin{pmatrix} \mathbf{A}^0_{X_h} & \varepsilon_h^k \mathbf{A}^0_{X_h} \\ \varepsilon_h^{-k}\mathbf{A}^0_{X_h} & \mathbf{A}^0_{X_h} \end{pmatrix} \xrightarrow{\cong} M_2(\mathbf{A}^0_{X_h})$$

$$A \longmapsto J_h^{-k} A J_h^k.$$

Recall that we denote by $d : \mathscr{A}_h^{ad,0} \to \mathscr{A}_h^{ad,1}$ the differential of the dg E_h-algebra $\mathscr{A}_h^{ad}$ which is given by $d(A,B) = (A, J_h^k B J_h^{-k})$. The diagram

$$\begin{tikzcd} M_2(E_h(X_h)) \times \begin{pmatrix} \mathbf{A}^0_{X_h} & \varepsilon_h^k \mathbf{A}^0_{X_h} \\ \varepsilon_h^{-k}\mathbf{A}^0_{X_h} & \mathbf{A}^0_{X_h} \end{pmatrix} \arrow[r, "\tilde{d}"] \arrow[d, "id\times\chi_h"] & M_2(\mathbf{A}_{X_h}) \arrow[d, "id"] \\ M_2(E_h(X_h)) \times M_2(\mathbf{A}^0_{X_h}) \arrow[r, "d"] & M_2(\mathbf{A}_{X_h}) \end{tikzcd}$$

commutes. Therefore, we obtain an isomorphism of differential graded E_h-algebras $\widetilde{\mathscr{A}_h^{ad}} \xrightarrow{\cong} \mathscr{A}_h^{ad}$ which we still call χ_h. The composition $\psi_h := \chi_h \circ \widetilde{\psi_h}$ defines the desired isomorphism $\mathscr{A}_h \xrightarrow{\cong} \mathscr{A}_h^{ad}$.

The commutativity of

$$\begin{tikzcd} \mathscr{A}_1 \arrow[r, "\varphi_h"] \arrow[d, "\cong"', "\psi_1"] & \mathscr{A}_h \arrow[d, "\cong"', "\psi_h"] \\ \mathscr{A}_1^{ad} \arrow[r, "\varphi_h^{ad}"] & \mathscr{A}_h^{ad} \end{tikzcd}$$

follows immediately from the construction. □

Let us do a sanity check and compute the cohomology of $\mathscr{A}^{ad} := \mathscr{A}_1^{ad}$. By the proof of Theorem 2.20 it should be isomorphic to $\operatorname{Ext}^*(\mathcal{P},\mathcal{P})$. We have

$$H^*(\mathscr{A}^{ad}) = \begin{pmatrix} E(X)\cap \mathbf{A}^0_X & E(X)\cap\varepsilon^k\mathbf{A}^0_X \\ E(X)\cap\varepsilon^{-k}\mathbf{A}^0_X & E(X)\cap\mathbf{A}^0_X \end{pmatrix} \oplus \begin{pmatrix} \mathbf{A}_X/(E(X)+\mathbf{A}^0_X) & \mathbf{A}_X/(E(X)+\varepsilon^k\mathbf{A}^0_X) \\ \mathbf{A}_X/(E(X)+\varepsilon^{-k}\mathbf{A}^0_X) & \mathbf{A}_X/(E(X)+\mathbf{A}^0_X) \end{pmatrix}$$

on which the algebra structure is induced by the one on $\mathscr{A}^{ad}$.

By Proposition 3.18 applied to $\mathcal{I},\mathcal{J} \in \{\mathcal{O}_X, \mathcal{O}_X(k), \mathcal{O}_X(-k)\}$ we have short exact sequences

$$0 \to H^0(X,\mathcal{O}_X) \to \mathbf{A}^0_X \to \mathbf{A}_X/E(X) \to H^1(X,\mathcal{O}_X) \to 0,$$

$$0 \to H^0(X,\mathcal{O}_X(k)) \to \varepsilon^{-k}\mathbf{A}^0_X \to \mathbf{A}_X/E(X) \to H^1(X,\mathcal{O}_X(k)) \to 0,$$

$$0 \to H^0(X,\mathcal{O}_X(-k)) \to \varepsilon^{k}\mathbf{A}^0_X \to \mathbf{A}_X/E(X) \to H^1(X,\mathcal{O}_X(-k)) \to 0.$$

Using this we may identify

$$H^*(\mathscr{A}^{ad}) = \begin{pmatrix} H^0(X,\mathcal{O}_X) & H^0(X,\mathcal{O}_X(-k)) \\ H^0(X,\mathcal{O}_X(k)) & H^0(X,\mathcal{O}_X) \end{pmatrix} \oplus \begin{pmatrix} H^1(X,\mathcal{O}_X) & H^1(X,\mathcal{O}_X(-k)) \\ H^1(X,\mathcal{O}_X(k)) & H^1(X,\mathcal{O}_X) \end{pmatrix}$$

which is exactly $\mathrm{Ext}^*(\mathcal{O}_X \oplus \mathcal{O}_X(k), \mathcal{O}_X \oplus \mathcal{O}_X(k))$, as it should (cf. the proof of Theorem 2.20).

3.4 The dg modules associated with coherent sheaves

Example 3.25 (Line bundles)**.** Let d be an integer. Then

$$\mathcal{L}^{(d)} = (\mathcal{E}_X \longrightarrow \mathcal{E}_X/\mathcal{O}_X(d))$$

is a K-injective complex of quasi-coherent $\mathcal{O}_X$-modules which is quasi-isomrphic to $\mathcal{O}_X(d)$. Therefore, the differential graded $\mathscr{A}$-module corresponding to $\mathcal{O}_X(d)$ is $\mathrm{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{K}, \mathcal{L}^{(d)})$. The following diagram illustrates the situation.

$$\begin{tikzcd} \mathcal{E}_X \oplus \mathcal{E}_X \arrow[r, "p"] \arrow[d, "0"] \arrow[dr, "1"] & \mathcal{E}_X/\mathcal{O}_X \oplus \mathcal{E}_X/\mathcal{O}_X(k) \arrow[d, "0"] \\ \mathcal{E}_X \arrow[r, "q"] & \mathcal{E}_X/\mathcal{O}_X(d). \end{tikzcd}$$

We obtain the complex

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}_X^2, \mathcal{E}_X) \times \mathrm{Hom}(\mathcal{E}_X/\mathcal{O}_X \oplus \mathcal{E}_X/\mathcal{O}_X(k), \mathcal{E}_X/\mathcal{O}_X(d)) &\longrightarrow \mathrm{Hom}(\mathcal{E}_X^2, \mathcal{E}_X/\mathcal{O}_X(d)) \\ (f,g) &\longmapsto q \circ f - g \circ p. \end{aligned}$$

The $\mathscr{A}$-module structure is given by the formulae

$$\begin{aligned} (m,n) * (a,b) &= (m \circ a, n \circ b), \\ (m,n) * c &= n \circ c, \\ p * (a,b) &= p \circ a, \\ p * c &= 0 \end{aligned}$$

for all $(a,b) \in \mathscr{A}^0$, $c \in \mathscr{A}^1$, $(m,n) \in \mathrm{Hom}^\bullet(\mathcal{K}, \mathcal{L}^{(d)})^0$ and $p \in \mathrm{Hom}^\bullet(\mathcal{K}, \mathcal{L}^{(d)})^1$.

By Proposition 3.15 the dg $\mathscr{A}$-module $\mathrm{Hom}^\bullet(\mathcal{K}, \mathcal{L}^{(d)})$ corresponds via base change along the isomorphism $\mathscr{A} \xrightarrow{\cong} \widetilde{\mathscr{A}^{ad}}$ to the dg $\widetilde{\mathscr{A}^{ad}}$-module $\widetilde{\mathscr{M}(d)}$ with underlying complex

$$\begin{aligned} E(X)^2 \times (\varepsilon^{-d}\mathbf{A}_X^0 \oplus \varepsilon^{k-d}\mathbf{A}_X^0) &\longrightarrow \mathbf{A}_X^2 \\ (M,N) &\longmapsto M - N \end{aligned}$$

concentrated in degree 0 and 1. Its $\widetilde{\mathscr{A}^{ad}}$-module structure is given by the formulae

$$\begin{aligned} (M,N) * (A,B) &= (M \cdot A, N \cdot B), \\ (M,N) * C &= N \cdot C, \\ P * (A,B) &= P \cdot A, \\ P * C &= 0 \end{aligned}$$

for all $(A,B) \in (\widetilde{\mathscr{A}^{ad}})^0$, $C \in (\widetilde{\mathscr{A}^{ad}})^1$, $(M,N) \in \widetilde{\mathscr{M}(d)}^0$ and $P \in \widetilde{\mathscr{M}(d)}^1$.

Example 3.26 (Skyscraper sheaves)**.** Let $x \in X$ be a closed point and let $\iota_x : \{x\} \hookrightarrow X$ be the inclusion. Let $d \in \mathbf{N}_{\geqslant 1}$. Then $\mathcal{O}_{d[x]} = \iota_{x,*}\mathcal{O}_{X,x}/\mathfrak{m}_x^d$ is the skyscraper sheaf at x with value $\mathcal{O}_{X,x}/\mathfrak{m}_x^d$. There is a short exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{O}_{X,x}/\mathfrak{m}_x^d \longrightarrow E(X)/\mathfrak{m}_x^d \longrightarrow E(X)/\mathcal{O}_{X,x} \to 0$$

the two nonzero terms on the right are divisible $\mathcal{O}_{X,x}$-modules, hence are injective because $\mathcal{O}_{X,x}$ is a principal ideal domain (cf. [HS], Chapter I, Theorem 7.1). Applying $\iota_{x,*}$ yields a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \iota_{x,*}\mathcal{O}_{X,x}/\mathfrak{m}_x^d \longrightarrow \iota_{x,*}E(X)/\mathfrak{m}_x^d \longrightarrow \iota_{x,*}E(X)/\mathcal{O}_{X,x} \to 0$$

in which the two nonzero terms on the right are injective quasi-coherent $\mathcal{O}_X$-modules (cf. [Har2], Proposition 7.17). Hence, the complex of $\mathcal{O}_X$-modules

$$\mathcal{G}_x^{(d)} := \quad \iota_{x,*}E(X)/\mathfrak{m}_x^d \longrightarrow \iota_{x,*}E(X)/\mathcal{O}_{X,x}$$

is K-injective and quasi-isomorphic to $\iota_{x,*}\mathcal{O}_{X,x}/\mathfrak{m}_x^d$. From now on if M is an $\mathcal{O}_{X,x}$-module we also write M instead of $\iota_{x,*}M$ for the corresponding skyscraper sheaf on X.

The underlying complex of the corresponding differential graded $\mathscr{A}$-module $\operatorname{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{K}, \mathcal{G}_x^{(d)})$ is

$$\operatorname{Hom}(\mathcal{E}_X/\mathcal{O}_X \oplus \mathcal{E}_X/\mathcal{O}_X(k), E(X)/\mathfrak{m}_x^d) \to \operatorname{Hom}(\mathcal{E}_X^2, E(X)/\mathfrak{m}_x^d) \times \operatorname{Hom}(\mathcal{E}_X/\mathcal{O}_X \oplus \mathcal{E}_X/\mathcal{O}_X(k), E(X)/\mathcal{O}_{X,x})$$

$$\to \operatorname{Hom}(\mathcal{E}_X^2, E(X)/\mathcal{O}_{X,x})$$

which is concentrated in degrees $-1, 0$, and 1.

Note that if $\mathcal{F}$ is an $\mathcal{O}_X$-module and M and $\mathcal{O}_{X,x}$-module then by the adjunction between pullback and pushforward along ι_x there is an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \iota_{x,*}M) \cong \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M)$$

which is functorial in $\mathcal{F}$ and M.

Suppose now that $x \neq \infty$. Denote by $\widehat{\mathfrak{m}}_x = \mathfrak{m}_x\widehat{\mathcal{O}}_{X,x} \subset \widehat{\mathcal{O}}_{X,x}$ the maximal ideal. Then by Proposition 3.6 the above complex is isomorphic to

$$\widetilde{\mathscr{M}_x^{(d)}} \quad = \quad (\widehat{\mathfrak{m}}_x^d)^2 \to \operatorname{Frac}(\widehat{\mathcal{O}}_{X,x})^2 \times \widehat{\mathcal{O}}_{X,x}^2 \to \operatorname{Frac}(\widehat{\mathcal{O}}_{X,x})^2$$

using that $\mathcal{O}_X(k)_x = \mathcal{O}_{X,x}$. The first differential is given by $f \mapsto (f, f)$, the second one sends (g, h) to $g - h$. The scalar multiplication with elements of $\widetilde{\mathscr{A}^{ad}}$ works as follows: If $(A, B) \in (\widetilde{\mathscr{A}^{ad}})^0$, $C \in (\widetilde{\mathscr{A}^{ad}})^1$, $K \in (\widetilde{\mathscr{M}_x^{(d)}})^{-1}$, $(M, N) \in (\widetilde{\mathscr{M}_x^{(d)}})^0$ and $P \in (\widetilde{\mathscr{M}_x^{(d)}})^1$ then

$$\begin{aligned} K * C &= K \cdot C_x, \\ K * (A, B) &= K \cdot B_x, \\ (M, N) * (A, B) &= (M \cdot A_x, N \cdot B_x), \\ (M, N) * C &= N \cdot C_x, \\ P * (A, B) &= P \cdot A_x, \\ P * C &= 0. \end{aligned}$$

If $x = \infty$ then we obtain

$$\widetilde{\mathscr{M}_\infty^{(d)}} \quad = \quad (\widehat{\mathfrak{m}}_\infty^d \oplus \widehat{\mathfrak{m}}_\infty^{d+k}) \to \operatorname{Frac}(\widehat{\mathcal{O}}_{X,\infty})^2 \times (\widehat{\mathcal{O}}_{X,\infty} \oplus \widehat{\mathfrak{m}}_\infty^k) \to \operatorname{Frac}(\widehat{\mathcal{O}}_{X,\infty})^2$$

since $\mathcal{O}_X(k)_\infty = \mathfrak{m}_\infty^{-k}$. For the differential as well as for the scalar multiplication with elements of $\widetilde{\mathscr{A}^{ad}}$ we have the same formulae as in the case $x \neq \infty$.

Multiplication by π_∞^k yields $\widehat{\mathcal{O}}_{X,\infty}$-linear isomorphisms

$$\widehat{\mathcal{O}}_{X,\infty} \xrightarrow{\cong} \widehat{\mathfrak{m}}_\infty^k \quad \text{and} \quad \widehat{\mathfrak{m}}_\infty^d \xrightarrow{\cong} \widehat{\mathfrak{m}}_\infty^{d+k}.$$

Hence, $\widetilde{\mathscr{M}_\infty^{(d)}}$ is isomorphic to

$$(\widehat{\mathfrak{m}}_\infty^d)^2 \to \mathrm{Frac}(\widehat{\mathcal{O}}_{X,\infty})^2 \times \widehat{\mathcal{O}}_{X,\infty}^2 \to \mathrm{Frac}(\widehat{\mathcal{O}}_{X,\infty})^2.$$

To summarize, for any closed point x of X the differential graded $\widetilde{\mathscr{A}^{ad}}$-module associated to $\mathcal{O}_{X,x}/\mathfrak{m}_x^d$ viewed as a skyscraper sheaf at x is given by the complex

$$\widetilde{\mathscr{M}_x^{(d)}} \quad = \quad (\widehat{\mathfrak{m}}_x^d)^2 \to \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})^2 \times \widehat{\mathcal{O}}_{X,x}^2 \to \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})^2$$

with differentials $f \mapsto (f,f)$ and $(g,h) \mapsto g-h$. Note, however, that the scalar multiplication is different for $x=\infty$. Its cohomology is $(\widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x^d)^2$ in degree 0. The projection

$$\begin{tikzcd}
(\widehat{\mathfrak{m}}_x^d)^2 \arrow[r] \arrow[d] & \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})^2 \times \widehat{\mathcal{O}}_{X,x}^2 \arrow[r] \arrow[d] & \mathrm{Frac}(\widehat{\mathcal{O}}_{X,x})^2 \arrow[d] \\
0 \arrow[r] & (\widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x^d)^2 \arrow[r] & 0
\end{tikzcd}$$

induces isomorphisms in cohomology, hence is an isomorphism in $\mathsf{D}(\widetilde{\mathscr{A}^{ad}})$. Note that we have an $\mathcal{O}_{X,x}$-linear isomorphism

$$\mathcal{O}_{X,x}/\mathfrak{m}_x^d \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x^d.$$

Example 3.27 (Vector bundles)**.** Let $d \in \mathbf{Z}$ and $h \in \mathbf{N}_{\geq 1}$ such that $(d,h)=1$. Let $\mathcal{O}_X(\frac{d}{h}) := \pi_{h,*}\mathcal{O}_{X_h}(d)$, the unique stable vector bundle of slope $\lambda = \frac{d}{h}$.

We first compute $\mathrm{RHom}_{\mathcal{O}_{X_h}}(\mathcal{P}_h, \mathcal{O}_{X_h}(d))$. There is a short exact sequence of $\mathcal{O}_{X_h}$-modules

$$0 \to \mathcal{O}_{X_h}(d) \to \mathcal{E}_{X_h} \to \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(d) \to 0.$$

We obtain a K-injective complex of $\mathcal{O}_{X_h}$-modules

$$\mathcal{L}_h^{(d)} := \quad \mathcal{E}_{X_h} \to \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(d)$$

which is quasi-isomorphic to $\mathcal{O}_{X_h}(d)$. Hence, we need to compute $\mathrm{Hom}^\bullet_{\mathcal{O}_{X_h}}(\mathcal{K}_h, \mathcal{L}_h^{(d)})$, that is morphisms of different degrees $\mathcal{K}_h \to \mathcal{L}_h^{(d)}$ as illustrated by the following diagram:

$$\begin{tikzcd}
\mathcal{K}_h : & \mathcal{E}_{X_h}^2 \arrow[r] \arrow[d, "0"] \arrow[dr, "1"] & \mathcal{E}_{X_h}/\mathcal{O}_{X_h} \oplus \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(hk) \arrow[d, "0"] \\
\mathcal{L}_h^{(d)} : & \mathcal{E}_{X_h} \arrow[r] & \mathcal{E}_{X_h}/\mathcal{O}_{X_h}(d).
\end{tikzcd}$$

Fix a uniformizer ϖ of $\mathcal{O}_{X,\infty}$. Recall that $\varepsilon_h \in \mathbf{A}_{X_h}^0$ is the adèle with entry $\varpi_y := \pi_{h,y}^\sharp(\varpi)$ if $y \in \pi_h^{-1}(\infty)$ and 1 otherwise. More generally, let $\varepsilon_{h,d} \in \mathbf{A}_{X_h}^0$ be the adèle with entry ϖ_y if $y = \sigma^i(\infty_h)$ for some $0 \leq i \leq d-1$, and 1 otherwise. The latter notation $\sigma^i(\infty_h)$ was intoduced in Definition 1.22. With this notation, $\varepsilon_{h,h} = \varepsilon_h$.

It follows that the differential graded $\widetilde{\mathscr{A}_h^{ad}}$-module $\widetilde{\mathscr{M}_h(d)}$ corresponding to $\mathcal{O}_{X_h}(d)$ has underlying complex

$$(E_h(X_h) \oplus E_h(X_h)) \times (\varepsilon_{h,d}^{-1}\mathbf{A}_{X_h}^0 \oplus \varepsilon_{h,d}^{-1}\varepsilon_h^k \mathbf{A}_{X_h}^0) \to (\mathbf{A}_{X_h} \oplus \mathbf{A}_{X_h})$$

where the differential takes the difference of matrices. The $\widetilde{\mathscr{A}_h^{ad}}$-module structure is given by the formulae

$$\begin{aligned}(M, N) * (A, B) &= (M \cdot A, N \cdot B),\\ (M, N) * C &= N \cdot C,\\ P * (A, B) &= P \cdot A,\\ P * C &= 0\end{aligned}$$

for all $(A, B) \in (\widetilde{\mathscr{A}_h^{ad}})^0$, $C \in (\widetilde{\mathscr{A}_h^{ad}})^1$, $(M, N) \in \widetilde{\mathscr{M}_h(d)}^0$ and $P \in \widetilde{\mathscr{M}_h(d)}^1$.

Via base change along the isomorphism $\widetilde{\mathscr{A}_h^{ad}} \xrightarrow{\cong} \mathscr{A}_h^{ad}$ we obtain a dg $\mathscr{A}_h^{ad}$-module $\mathscr{M}_h(d)$ which has the same underlying complex as $\widetilde{\mathscr{M}_h(d)}$ and whose scalar multiplication works according to the formulae

$$\begin{aligned}(M, N) * (A, B) &= (M \cdot A, N \cdot J_h^k B J_h^{-k}),\\ (M, N) * C &= N \cdot C,\\ P * (A, B) &= P \cdot A,\end{aligned}$$

for all $(A, B) \in (\mathscr{A}_h^{ad})^0$, $C \in (\mathscr{A}_h^{ad})^1$, $(M, N) \in \mathscr{M}_h(d)^0$ and $P \in \mathscr{M}_h(d)^1$.

We may view the latter as a dg $\mathscr{A}_1^{ad}$-module $\mathscr{M}(\frac{d}{h})$ via scalar restriction along the homomorphism $\mathscr{A}_1^{ad} \xrightarrow{\varphi_h^{ad}} \mathscr{A}_h^{ad}$. By Proposition 3.20, $\mathscr{M}(\frac{d}{h})$ is the dg $\mathscr{A}_1^{ad}$-module corresponding to $\mathcal{O}_X(\frac{d}{h})$ along the equivalence $\mathsf{D}_{qcoh}(\mathcal{O}_X) \xrightarrow{\cong} \mathsf{D}(\mathscr{A}_1^{ad})$.

Let us remark that the above examples cover the dg modules associated with all coherent sheaves on X.

4 Le Bras' nontrivial t-structure on the derived category of the Fargues-Fontaine curve

Let X be the Fargues-Fontaine curve. We first recall the t-structure on the bounded derived category of coherent sheaves on X which was studied by Le Bras in [LeB], §5.2. In §4.2 we explain the construction of the skew field of Colmez. Finally, we investigate its multiplicative structure by computing pushouts in §4.3.

4.1 A nontrivial t-structure on $D^b(X)$

Let $\mathcal{F}$ be a nonzero coherent sheaf on X. Write it as $\mathcal{F} = \mathcal{E} \oplus \mathcal{T}$ where $\mathcal{E}$ is a vector bundle and $\mathcal{T}$ a torsion sheaf. We write $\mathcal{F} \geq 0$ if all of the slopes in the slope multiset of $\mathcal{F}$ are nonnegative. We write $\mathcal{F} < 0$ if $\mathcal{T} = 0$ and if all the slopes in the slope multiset of $\mathcal{F}$ are negative.

Proposition 4.1 ([LeB], Proposition 5.5)**.** *The full subcategories*

$$\begin{aligned} \mathsf{D}^{\geq 0} &= \{\mathcal{F} \in \mathsf{D}^b(X) \mid H^0(\mathcal{F}) \geq 0 \text{ and } H^i(\mathcal{F}) = 0 \text{ for all } i > 0\}, \\ \mathsf{D}^{\leq 0} &= \{\mathcal{F} \in \mathsf{D}^b(X) \mid H^{-1}(\mathcal{F}) < 0 \text{ and } H^i(\mathcal{F}) = 0 \text{ for all } i < -1\} \end{aligned}$$

define a t-structure on $\mathsf{D}^b(X)$. Its heart $\mathsf{D}^{\geq 0} \cap \mathsf{D}^{\leq 0}$ will be denoted by C.

The objects of C are isomorphic to direct sums $\mathcal{F}'[1] \oplus \mathcal{F}''$ with $\mathcal{F}' < 0$ and $\mathcal{F}'' \geq 0$ (see [LeB], computation after Proposition 5.5).

It will be useful to compute the truncations of particularly simple objects with respect to this t-structure. If $\mathcal{F} = \bigoplus_\lambda \mathcal{O}(\lambda) \oplus \mathcal{T}$ with $\mathcal{T}$ torsion then we write $\mathcal{F}^{\geq 0} = \bigoplus_{\lambda \geq 0} \mathcal{O}(\lambda) \oplus \mathcal{T}$ and $\mathcal{F}^{<0} = \bigoplus_{\lambda < 0} \mathcal{O}(\lambda)$.

Lemma 4.2. *Let $\mathcal{F}$ be a coherent sheaf on X, viewed as an object of $\mathsf{D}^b(X)$ concentrated in degree 0.*

(i) $\tau^{\leq 0}\mathcal{F} \cong \mathcal{F}^{\geq 0}$ and $\tau^{\geq 0}\mathcal{F} \cong \mathcal{F}$,

(ii) $\tau^{\leq 0}(\mathcal{F}[1]) \cong \mathcal{F}[1]$ and $\tau^{\geq 0}(\mathcal{F}[1]) \cong \mathcal{F}^{<0}[1]$.

Proof. Since $\mathcal{F} \in \mathsf{D}^{\geq 0}$ the canonical map $\mathcal{F} \to \tau^{\geq 0}\mathcal{F}$ is an isomorphism (see [KS2], Prop. 10.1.6) Similarly the map $\tau^{\leq 0}(\mathcal{F}[1]) \to \mathcal{F}[1]$ is an isomorphism.

Consider the short exact sequence of coherent sheaves

$$0 \to \underbrace{\mathcal{F}^{\geq 0}}_{\in \mathsf{D}^{\leq 0}} \to \mathcal{F} \to \underbrace{\mathcal{F}^{<0}}_{\in \mathsf{D}^{\geq 1}} \to 0.$$

Hence by [KS2], Prop. 10.1.4 and its proof, $\tau^{\leq 0}\mathcal{F} \cong \mathcal{F}^{\geq 0}$. Moreover, $\tau^{\geq 0}(\mathcal{F}[1]) \cong \tau^{\geq 1}(\mathcal{F})[1] \cong \mathcal{F}^{<0}[1]$ using [KS2], formula 10.1.1 on page 413. □

4.2 The skew field of Colmez

We may extend the degree function to bounded complexes of coherent sheaves by setting

$$\deg(\mathcal{F}^\bullet) := \sum_{i \in \mathbf{Z}} (-1)^i \deg \mathcal{F}^i.$$

Since it is additive in short exact sequences of coherent sheaves, it is invariant under quasi-isomorphisms. Therefore, it can be defined on $\mathsf{D}^b(X)$.

Lemma 4.3. *The degree function is additive in short exact sequences in C.*

Proof. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence in C. Then there exists a morphism $\mathcal{H} \to \mathcal{F}[1]$ in $\mathsf{D}^b(X)$ such that

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$$

is an exact triangle in $\mathsf{D}^b(X)$ (cf. [KS1], Proposition 13.1.13). Now by definition of the triangulated structure on $\mathsf{D}^b(X)$ an exact triangle is one that is isomorphic to the image of an exact triangle in $\mathsf{K}^b(X)$ under the localization functor. Since deg is invariant under quasi-isomorphisms we may reduce to the case where

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$$

is an exact triangle in $\mathsf{K}^b(X)$. Now an exact triangle in $\mathsf{K}^b(X)$ is one that is isomorphic to the triangle associated to a termwise split short exact sequence of complexes (cf. [Sta], Definition 014Q). Since a homotopy equivalence is a quasi-isomorphism we may thus further reduce to the case where

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$$

is the triangle associated to a termwise split short exact sequence of complexes, i.e.

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of complexes, and for each $i \in \mathbf{Z}$ the sequence of coherent sheaves

$$0 \to \mathcal{F}^i \to \mathcal{G}^i \to \mathcal{H}^i \to 0$$

is split exact. In this case deg is additive. $\square$

Definition 4.4 ([Sta], Definition 02MO)**.** Let A be an abelian category. A Serre subcategory of A is a nonempty full subcategory A^0 of A such that given an exact sequence

$$A \to B \to C$$

with $A, C \in \mathsf{A}^0$ then also $B \in \mathsf{A}^0$.

Lemma 4.5 ([Sta], Lemma 02MP)**.** *Let A be an abelian category. Let A^0 be a subcategory of A. Then A^0 is a Serre subcategory if and only if the following conditions are satisfied:*

(i) $0 \in \mathsf{A}^0$,

(ii) A^0 *is a strictly full subcategory of* A,

(iii) any subobject or quotient of an object of A^0 *is an object of* A^0*, and*

(iv) if $A \in \mathsf{A}$ *is an extension of objects of* A^0 *then also* $A \in \mathsf{A}^0$.

Moreover, a Serre subcategory is an abelian category and the inclusion functor is exact.

Proposition 4.6. *The full subcategory* C^0 *of objects of degree* 0 *is a Serre subcategory of* C.

Proof. Note that the degree function on C takes values only in $\mathbf{N}_0$. Indeed, recall that any object of C is isomorphic to a direct sum $\mathcal{F}'[1] \oplus \mathcal{F}''$ with $\mathcal{F}' < 0$ and $\mathcal{F}'' \geq 0$. Hence, its degree is $\underbrace{\deg \mathcal{F}''}_{\geq 0} - \underbrace{\deg \mathcal{F}'}_{\leq 0} \geq 0$. An object of degree 0 in C is thus a finite direct sum of copies of the structure sheaf $\mathcal{O}$. That this is a Serre subcategory of C follows from the the previous lemma together with the fact that $\deg : \mathsf{C} \to \mathbf{N}_0$ is additive in short exact sequences of C. $\square$

By [Sta], Lemma 02MS, we obtain an abelian category $\mathsf{Q} = \mathsf{C}/\mathsf{C}^0$ together with an exact functor $F : \mathsf{C} \to \mathsf{Q} = \mathsf{C}/\mathsf{C}^0$ which is essentially surjective and whose kernel is C^0. The objects of Q are the objects of C and a morphism from $x \to y$ is a fraction $s^{-1}f$ where $f : x \to y'$ is a morphism in C and $s : y \to y'$ is an element of the multiplicative system

$$S := \{f \in \mathrm{Mor}(\mathsf{C}) \mid \ker(f), \mathrm{coker}(f) \in \mathsf{C}^0\}.$$

The category Q is called the localization of C at the set S.

Remark 4.7 ([LeB], §7.3)**.** Note that the objects of degree 0 do not form a Serre subcategory of the category Coh_X of coherent sheaves on X. For example, $\mathcal{O}(-1)$ is a subobject of $\mathcal{O}$ but it has degree -1. Instead, the objects of rank 0 do so. These are precisely the torsion objects, and the localization $\mathsf{Coh}_X/\mathsf{Coh}_X^{tors}$ identifies via $\mathcal{F} \mapsto \mathcal{F}_\eta$ with the category of finite dimensional $E(X)$-vector spaces. In particular, this category is semisimple with a unique simple object.

Proposition 4.8 ([LeB], §7.3)**.** *The category* $\mathsf{Q} = \mathsf{C}/\mathsf{C}^0$ *is semisimple with a unique simple object.*

Proof. We first show that the degree function deg is additive in short exact sequences in Q. As we have seen before it is additive in short exact sequences in C. Now if $X \cong Y$ in Q then $\deg X = \deg Y$. Indeed, let $a : X \to Y$ be an isomorphism in Q. Then $a = s^{-1}f$ for some $s : Y' \to Y$ in S and $f : X \to Y'$ in C. Denote by $F : \mathsf{C} \to \mathsf{Q}$ the localization functor. Its kernel is C^0 (cf. [Sta], Lemma 02MS). By the proof of [Sta], Lemma 05QG, $F(\ker(f))$ is a kernel of $s^{-1}f$ and similarly for the cokernel. This means that both $\ker(f)$ and $\mathrm{coker}(f)$ lie in the kernel of the localization functor F, hence $\deg\ker(f) = \deg\mathrm{coker}(f) = 0$, i.e. $f \in S$. This implies

$$\begin{aligned}\deg Y = \deg Y' &= \deg\mathrm{coker}(f) + \deg\mathrm{im}(f)\\ &= \deg\mathrm{coker}(f) + \deg X - \deg\ker(f)\\ &= \deg X.\end{aligned}$$

Now let $s^{-1}f : X \to Y$ be a monomorphism in Q, where $f : X \to Y'$ and $s : Y \to Y'$. Then

$$\begin{aligned}\deg Y = \deg Y' &= \deg\mathrm{coker}(f) + \deg\mathrm{im}(f)\\ &= \deg\mathrm{coker}(f) + \deg X - \deg\ker(f)\\ &= \deg\mathrm{coker}(s^{-1}f) + \deg X,\end{aligned}$$

showing that the degree function is indeed additive in short exact sequences in Q.

Now we prove that any object of Q is isomorphic to a finite direct sum of copies of $\mathcal{O}(1)$. This follows from the following short exact sequences in C for $d \geq 2$ and $k \geq 1$ (cf. [LeB], Lemme 7.3):

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(d-1) \to \mathcal{O}(d) \to 0$$

$$0 \to \mathcal{O} \to \mathcal{O}(k) \to \iota_{\infty,*}\mathcal{O}_\infty/\mathfrak{m}_\infty^k \to 0$$

and

$$0 \to \mathcal{O} \to \iota_{\infty,*}\mathcal{O}_\infty/\mathfrak{m}_\infty^k \to \mathcal{O}(-k)[1] \to 0.$$

Indeed, by the classification of vector bundles on X every vector bundle is a finite direct sum of $\mathcal{O}(\lambda)$ for $\lambda \in \mathbb{Q}$, where $\mathcal{O}(\frac{d}{h}) = \pi_{h,*}\mathcal{O}_h(d)$ if $(d,h) = 1$. By looking at the first sequence on the degree h covering and applying the direct image functor, we obtain inductively that $\mathcal{O}(\frac{d}{h}) \cong \mathcal{O}(\frac{1}{h})^{\oplus d'}$ in Q for some $d' \geq 1$. By looking at the second sequence on the degree h covering and using that the extension of residue fields is trivial we obtain $\mathcal{O}(\frac{1}{h}) \cong \iota_{\infty,*}\kappa(\infty) \cong \mathcal{O}(1)$ for all $h \geq 1$. Now if $d \geq 1$ then the second and third sequences together imply $\mathcal{O}(-\frac{d}{h})[1] \cong \iota_{\infty,*}\mathcal{O}_{X,\infty}/\mathfrak{m}_\infty^d \cong \mathcal{O}(\frac{d}{h})$. Again using the first and second sequence we see that every torsion sheaf is isomorphic to a finite direct sum of copies of $\mathcal{O}(1)$.

Together with the additivity of the $\mathbf{N}_0$-valued degree function on C we can now conclude the proof of the proposition. It remains to see that $\mathcal{O}(1)$ is simple in Q. Given a subobject $\mathcal{F}$ of $\mathcal{O}(1)$ then $\mathcal{F} \cong \mathcal{O}(1)^{\oplus n}$ which by additivity of deg forces n to be either 0 or 1. $\square$

For example, for any $h \geqslant 1$ the vector bundle $\mathcal{O}(\frac{1}{h})$ represents the simple object of Q. By Schur's lemma, its endomorphism ring in Q is a skew field which we call $\mathscr{C}$. Le Bras constructed an equivalence of categories between C and BC, the category of Banach-Colmez spaces introduced by Colmez ([Col]). This equivalence identifies C/C^0 with a localization of BC, from which Le Bras deduced that $\mathscr{C}$ is the skew field first studied by Colmez in [Col], §5 and §9. We aim to give a more explicit description of the skew field $\mathscr{C}$.

4.3 Multiplicative structure of $\mathscr{C}$

By [Sta], 05Q1,

$$\mathscr{C} = \mathrm{End}_{\mathsf{Q}}(\mathcal{O}(1)) = \varinjlim_{s:\mathcal{O}(1)\to\mathcal{F}} \mathrm{Hom}_{\mathsf{C}}(\mathcal{O}(1),\mathcal{F})$$

where the index set runs over all morphisms $s : \mathcal{O}(1) \to \mathcal{F}$ which lie in S and is filtered by using pushouts. More explicitly, if $s : \mathcal{O}(1) \to \mathcal{F}$ and $t : \mathcal{O}(1) \to \mathcal{G}$ are morphisms in S then we form their pushout

$$\begin{tikzcd} \mathcal{O}(1) \arrow[r, "s"] \arrow[d, "t"'] & \mathcal{F} \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{F} \coprod_{\mathcal{O}(1)} \mathcal{G} \end{tikzcd}$$

and the composition is an element of S (see the proof of [Sta], Lemma 02MS).

An element of $\mathscr{C}$ can be illustrated as a roof of morphisms in C of the form

$$\begin{tikzcd} & \mathcal{F} & \\ \mathcal{O}(1) \arrow[ur, "f"] & & \mathcal{O}(1) \arrow[ul, "s"'] \end{tikzcd}$$

where $\ker(s), \mathrm{coker}(s) \in \mathsf{C}^0$ and for which we write $s^{-1}f$. Here $\mathcal{F}$ is an object of C which becomes isomorphic to $\mathcal{O}(1)$ in Q.

The multiplication in $\mathscr{C}$ is given by the pushout of roofs: Given two roofs

$$\begin{tikzcd} & \mathcal{F} & \\ \mathcal{O}(1) \arrow[ur, "f"] & & \mathcal{O}(1) \arrow[ul, "s"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathcal{G} & \\ \mathcal{O}(1) \arrow[ur, "g"] & & \mathcal{O}(1) \arrow[ul, "t"'] \end{tikzcd}$$

the product $t^{-1}g \cdot s^{-1}f$ is given by the outer roof of the diagram

$$\begin{tikzcd} & & \mathcal{F} \coprod_{\mathcal{O}(1)} \mathcal{G} & & \\ & \mathcal{F} \arrow[ur] & & \mathcal{G} \arrow[ul] & \\ \mathcal{O}(1) \arrow[ur, "f"] & & \mathcal{O}(1) \arrow[ul, "s"'] \arrow[ur, "g"] & & \mathcal{O}(1) \arrow[ul, "t"'] \end{tikzcd}$$

where the upper square is the pushout of the morphisms s and g in C.

Proposition 4.9. *(i) If $f : \mathcal{O}(1) \to \mathcal{F}$ is a morphism in S then $\mathcal{F} \cong \mathcal{O}^n \oplus \mathcal{G}$ for some $n \geq 1$ and where $\mathcal{G}$ is either $\mathcal{O}(1)$, a skyscraper sheaf C_x for some closed point $x \in |X|$, or $\mathcal{O}(-\frac{1}{h})[1]$ for some $h \geq 1$.*

(ii) If $\mathcal{F}$ is of the form in (i) and $f : \mathcal{O}(1) \to \mathcal{F}$ is a nonzero morphism then $f \in S$.

Proof. Recall that we can write $\mathcal{F} = \mathcal{F}'[1] \oplus \mathcal{F}''$ for some coherent sheaves $\mathcal{F}'$ and $\mathcal{F}''$ where $\mathcal{F}'$ only has negative slopes and $\mathcal{F}''$ only has nonnegative slopes. Moreover, we have the additive degree function $\deg : \mathsf{C} \to \mathbf{N}_0$. That s is an element of S therefore implies that $\deg \mathcal{F} = 1$. By [FF], Proposition 5.6.23 (5), $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(\frac{1}{h})) = 0$ for all $h \geq 2$. Hence, if $f : \mathcal{O}(1) \to \mathcal{F}$ is an element of S then $\mathcal{F}$ is one of the objects mentioned in the statement.

In order to prove the second part of the statement, let us investigate the cases separately.

Firstly, since $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)) = H^0(X, \mathcal{O}) = E$ (cf. [FF], Théorème 6.4.1), any nonzero morphism $\mathcal{O}(1) \to \mathcal{O}(1)$ is an automorphism, hence lies in S.

Secondly, any nonzero map $\mathcal{O}(1) \to C_x$ is automatically surjective and has kernel $\mathcal{O}$ for reasons of degree and rank, hence lies in S.

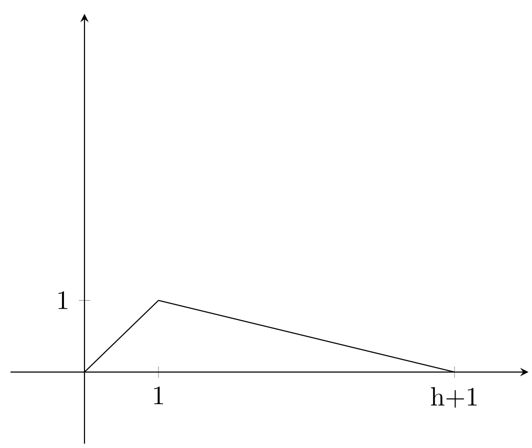
Now let $h \geq 1$. A morphism $f : \mathcal{O}(1) \to \mathcal{O}(-\frac{1}{h})[1]$ in C corresponds to a class of extensions

$$0 \to \mathcal{O}(-\frac{1}{h}) \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

Since $\mathcal{E}[1]$ is isomorphic to the mapping cone of f, we may read off the kernel and cokernel of f from the vector bundle $\mathcal{E}$ using Lemma 4.2:

$$\ker(f) \cong \tau^{\leq 0}\mathcal{E} \cong \mathcal{E}^{\geq 0},$$
$$\mathrm{coker}(f) \cong \tau^{\geq 0}(\mathcal{E}[1]) \cong \mathcal{E}^{<0}[1].$$

Let us use Lemma 1.19 in order to restrict the possibilties for $\mathcal{E}$. First let us draw the HN polygon of $\mathcal{O}(-\frac{1}{h}) \oplus \mathcal{O}(1)$:



It follows that the slopes of $\mathcal{E}$ are between 0 and 1. However, none of its slopes can be strictly between 0 and 1. Indeed, any such could be written as $\mu = \frac{r}{s}$ with $r, s \in \mathbf{Z}$, $r < s$ and $s > 1$, and the straight line from the origin to the point (s, r) crosses through $\mathrm{HN}(\mathcal{O}(-\frac{1}{h}) \oplus \mathcal{O}(1))$. Hence, there are only two possibilties: Either the largest slope of $\mathcal{E}$ is equal to 1. In this case, $\mathcal{E} \cong \mathcal{O}(-\frac{1}{h}) \oplus \mathcal{O}(1)$ is a split extension. Or $\mathcal{E}$ has only slope 0 and is hence isomorphic to $\mathcal{O}^{h+1}$.

In the case of the split extension we obtain $\mathrm{coker}_{\mathsf{C}}(f) \cong \mathcal{E}^{<0}[1] \cong \mathcal{O}(-\frac{1}{h})[1]$, so that f does not lie in S. If $\mathcal{E} \cong \mathcal{O}^{h+1}$ then $f \in S$.

Note that the morphisms $f : \mathcal{O}(1) \to \mathcal{O}(-\frac{1}{h})[1]$ corresponding to a nonsplit extension class are precisely the nonzero morphisms $\mathcal{O}(1) \to \mathcal{O}(-\frac{1}{h})[1]$, and that the space of such is nonempty because

$$\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(-\frac{1}{h})[1]) \cong H^1(X, \mathcal{O}(\frac{-1-h}{h})) \cong H^1(X_h, \mathcal{O}_{X_h}(-h-1)) \cong \mathcal{O}_{X_h,\infty_h}/(E_h + \mathfrak{m}_{\infty_h}^{h+1}).$$

To conclude the proof recall that $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}) = H^0(\mathcal{O}(-1)) = 0$. Hence, if $f : \mathcal{O}(1) \to \mathcal{G}$ is a morphism in S with $\mathcal{G}$ as in the statement then also the map $\mathcal{O}(1) \xrightarrow{(0,f)} \mathcal{O}^n \oplus \mathcal{G}$ lies in S because $\ker_{\mathsf{C}}(0, f) = \ker_{\mathsf{C}}(f)$ and $\mathrm{coker}_{\mathsf{C}}(0, f) \cong \mathcal{O}^n \oplus \mathrm{coker}_{\mathsf{C}}(f)$. $\square$

In order to understand the composition of morphisms in the filtered colimit above, we are led to compute pushouts of diagrams

$$\begin{tikzcd} \mathcal{O}(1) \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{F} \\ \mathcal{G} & \end{tikzcd}$$

in the category C, where $\mathcal{F}$ and $\mathcal{G}$ are objects as in the previous lemma.

Proposition 4.10. *Let $\mathcal{F}, \mathcal{G} \in \mathsf{C}$ be of degree 1 and let $f : \mathcal{O}(1) \to \mathcal{F}$, $g : \mathcal{O}(1) \to \mathcal{G}$ be two nonzero morphisms.*

(i) If $\mathcal{F} = \mathcal{O}(1)$ then their pushout in C is isomorphic to $\mathcal{G}$.

(ii) If $\mathcal{F} = \mathcal{G} = C_x$ for some $x \in |X|$ then their pushout in C is isomorphic to C_x.

(iii) If $\mathcal{F} = C_x$ and $\mathcal{G} = C_y$ for some $x \neq y$ then their pushout in C is isomorphic to $\mathcal{O}(-1)[1]$.

(iv) Let $\mathcal{F} = \mathcal{O}(-\frac{1}{h})[1]$ for some $h \geq 1$ and $\mathcal{G} = C_x$ for some $x \in |X|$. Let $t \in H^0(X, \mathcal{O}(1))$ be such that $x = \infty_t$. Moreover, suppose that the map f is the class of the extension

$$0 \to \mathcal{O}(-\frac{1}{h}) \to \mathcal{O}^{h+1} \xrightarrow{p} \mathcal{O}(1) \to 0$$

where $p = \begin{pmatrix} s_1 & \dots & s_{h+1} \end{pmatrix}$ with $s_1, \dots, s_{h+1} \in H^0(X, \mathcal{O}(1))$. Then

$$\mathcal{O}(-\frac{1}{h})[1] \coprod_{\mathcal{O}(1), \mathsf{C}} C_x \cong \begin{cases} \mathcal{O}(-\frac{1}{h})[1] & \text{if } t \in \langle s_1, \dots, s_{h+1} \rangle_E, \\ \mathcal{O}(-\frac{1}{h+1})[1] & \text{otherwise.} \end{cases}$$

Here $\langle s_1, \dots, s_{h+1} \rangle_E$ denotes the E-subspace of $H^0(X, \mathcal{O}(1))$ generated by $s_1, \dots, s_{h+1}$.

(v) Let $\mathcal{F} = \mathcal{O}(-\frac{1}{h})[1]$ and $\mathcal{G} = \mathcal{O}(-\frac{1}{h'})[1]$ for some $h, h' \geq 1$. Let f and g be represented by extensions

$$0 \to \mathcal{O}(-\frac{1}{h}) \to \mathcal{O}^{h+1} \xrightarrow{p} \mathcal{O}(1) \to 0$$

and

$$0 \to \mathcal{O}(-\frac{1}{h'}) \to \mathcal{O}^{h'+1} \xrightarrow{q} \mathcal{O}(1) \to 0,$$

respectively. Write $p = \begin{pmatrix} s_1 & \dots & s_{h+1} \end{pmatrix}$ and $q = \begin{pmatrix} s_{h+2} & \dots & s_{h+h'+2} \end{pmatrix}$ for some $s_i \in H^0(X, \mathcal{O}(1))$ and set $n := \dim_E \langle s_1, \dots, s_{h+h'+2} \rangle_E$. Then the pushout of f and g in C is isomorphic to $\mathcal{O}(-\frac{1}{n-1})[1]$.

Proof. (i) Since $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)) \cong E$, f is an isomorphism. Therefore,

$$\begin{tikzcd} \mathcal{O}(1) \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{O}(1) \arrow[d, "g\circ f^{-1}"] \\ \mathcal{G} \arrow[r, "id"] & \mathcal{G} \end{tikzcd}$$

is a pushout diagram in C.

(ii) Note that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(1), C_x) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}(1)_x, C_x) \cong C_x$ by the adjunction of pullback and pushforward along ι_x and because $\mathcal{O}(1)_x$ is a free $\mathcal{O}_{X,x}$-module of rank 1. Hence, a nonzero map $\mathcal{O}(1) \to C_x$ is automatically surjective, and we have $g = af$ for some $a \in C_x^\times$. Now

$$\begin{aligned} &\mathrm{coker}_{\mathsf{C}}(\begin{pmatrix} f \\ -af \end{pmatrix} : \mathcal{O}(1) \to C_x \oplus C_x) \\ \cong &\mathrm{coker}_{\mathsf{C}}(\begin{pmatrix} f \\ 0 \end{pmatrix} : \mathcal{O}(1) \to C_x \oplus C_x) \\ \cong &C_x \end{aligned}$$

by applying the automorphism given by the matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ on the target.

(iii) The map $\begin{pmatrix} f \\ -g \end{pmatrix} : \mathcal{O}(1) \to C_x \oplus C_y$ is surjective as a map of $\mathcal{O}$-modules. Its kernel (in Coh_X) is therefore a vector bundle of rank 1 and degree -1, hence is isomorpic to $\mathcal{O}(-1)$. Therefore, we have a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \to C_x \oplus C_y \to 0$$

which by rotation of the corresponding exact triangle induces a short exact sequence

$$0 \to \mathcal{O}(1) \to C_x \oplus C_y \to \mathcal{O}(-1)[1] \to 0$$

in the category C, showing that $\mathrm{coker}_{\mathsf{C}} \begin{pmatrix} f \\ -g \end{pmatrix} \cong \mathcal{O}(-1)[1]$.

(iv) The map g fits into the short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{O} \xrightarrow{t} \mathcal{O}(1) \xrightarrow{g} C_x \to 0.$$

The map of complexes

$$\begin{tikzcd} \mathcal{O}(-\frac{1}{h}) \arrow[d] & \\ \mathcal{O}^{h+1} \arrow[r, "p"] & \mathcal{O}(1) \end{tikzcd}$$

is a quasi-isomorphism, and up to post-composition with its inverse the map f is the morphism of complexes

$$\begin{tikzcd} & \mathcal{O}(1) \arrow[d, "id"] \\ \mathcal{O}^{h+1} \arrow[r, "p"] & \mathcal{O}(1). \end{tikzcd}$$

Therefore, the map $\begin{pmatrix} f \\ -g \end{pmatrix} : \mathcal{O}(1) \to \mathcal{O}(-\frac{1}{h})[1] \oplus C_x$ can be replaced by the map of complexes

$$\begin{tikzcd} & \mathcal{O}(1) \arrow[d, "\begin{pmatrix} id \\ -g \end{pmatrix}"] \\ \mathcal{O}^{h+1} \arrow[r, "\begin{pmatrix} p \\ 0 \end{pmatrix}"] & \mathcal{O}(1) \oplus C_x. \end{tikzcd}$$

The mapping cone of the latter is the complex

$$Z : \mathcal{O}(1) \oplus \mathcal{O}^{h+1} \xrightarrow{d} \mathcal{O}(1) \oplus C_x$$

with differential $d = \begin{pmatrix} id & p \\ -g & 0 \end{pmatrix}$. By the formulas for kernels and cokernels from Corollary 2.13 we obtain

$$\ker_{\mathsf{C}}(\begin{pmatrix} f \\ -g \end{pmatrix}) \cong H^{-1}(Z)^{\geq 0}[0] = \ker(d)^{\geq 0}[0]$$

and

$$\operatorname{coker}_{\mathsf{C}}(\begin{pmatrix} f \\ -g \end{pmatrix}) \cong (Z^{-1}/\ker(d)^{\geq 0} \xrightarrow{\bar{d}} Z^0) \cong \ker(d)^{<0}[1].$$

Hence, we are left with computing $\ker(d)$. The kernel of the composition of d with the projection onto C_x is $\mathcal{O} \oplus \mathcal{O}^{h+1}$. Restricting d to the latter and composing with the projection onto $\mathcal{O}(1)$ yields the map $\tilde{d} = \begin{pmatrix} t & s_1 & \dots & s_{h+1} \end{pmatrix} : \mathcal{O}^{h+2} \longrightarrow \mathcal{O}(1)$ with $\ker(d) = \ker(\tilde{d})$. By the above formula for the cokernel of $\begin{pmatrix} f \\ -g \end{pmatrix}$ it remains to see that $\ker(\tilde{d})$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-\frac{1}{h})$ if $t \in \langle s_1, \dots, s_{h+1} \rangle_E$ and to $\mathcal{O}(-\frac{1}{h+1})$ otherwise.

If $t = \sum_i \alpha_i s_i$ for $\alpha_i \in E$ then set

$$A := \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\alpha_1 & 1 & 0 & \dots & 0 \\ -\alpha_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -\alpha_{h+1} & 0 & \dots & 0 & 1 \end{pmatrix} \in GL_{h+2}(E).$$

Then

$$\begin{pmatrix} t & s_1 & \dots & s_{h+1} \end{pmatrix} \cdot A = \begin{pmatrix} 0 & s_1 & \dots & s_{h+1} \end{pmatrix}.$$

Hence, by precomposing $\tilde{d}$ with the automorphism of $\mathcal{O}^{h+2}$ given by A we obtain the morphism $\begin{pmatrix} 0 & s_1 & \dots & s_{h+1} \end{pmatrix} : \mathcal{O}^{h+2} \to \mathcal{O}(1))$. Since $s_1, \dots, s_{h+1}$ are linearly independent (cf. Lemma 4.11), its kernel is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-\frac{1}{h})$. If instead the elements $t, s_1, \dots, s_{h+1}$ are linearly independent then by Lemma 4.11 the kernel of $\tilde{d}$ is $\mathcal{O}(-\frac{1}{h+1})$.

(v) We need to compute $\operatorname{coker}_{\mathsf{C}}(\begin{pmatrix} f \\ -g \end{pmatrix} : \mathcal{O}(1) \to \mathcal{O}(-\frac{1}{h})[1] \oplus \mathcal{O}(-\frac{1}{h'})[1])$. Consider the quasi-isomorphisms

$$\begin{tikzcd} \mathcal{O}(-\frac{1}{h}) \arrow[d] & \\ \mathcal{O}^{h+1} \arrow[r, "p"] & \mathcal{O}(1) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{O}(-\frac{1}{h'}) \arrow[d] & \\ \mathcal{O}^{h'+1} \arrow[r, "q"] & \mathcal{O}(1) \end{tikzcd}$$

where $\mathcal{O}(-\frac{1}{h})$ and $\mathcal{O}(-\frac{1}{h'})$ are in degree -1 (so that we should rather write $\mathcal{O}(-\frac{1}{h})[1]$ and $\mathcal{O}(-\frac{1}{h'})[1]$). We may then replace the map $\begin{pmatrix} f \\ -g \end{pmatrix}$ by the morphism of complexes

$$\begin{tikzcd} & \mathcal{O}(1) \arrow[d, "\begin{pmatrix} id \\ -id \end{pmatrix}"] \\ \mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1} \arrow[r, "\begin{pmatrix} p & \\ & q \end{pmatrix}"] & \mathcal{O}(1)^2. \end{tikzcd}$$

Its mapping cone is the complex

$$Z : \mathcal{O}(1) \oplus \mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1} \xrightarrow{d} \mathcal{O}(1)^2$$

where $d = \begin{pmatrix} id & p & 0 \\ -id & 0 & q \end{pmatrix}$. By Corollary 2.13 we obtain

$$\ker_{\mathsf{C}}(\begin{pmatrix} f \\ -g \end{pmatrix}) \cong H^{-1}(Z)^{\geq 0}[0] = \ker(d)^{\geq 0}[0]$$

and

$$\operatorname{coker}_{\mathsf{C}}(\begin{pmatrix} f \\ -g \end{pmatrix}) \cong (Z^{-1}/\ker(d)^{\geq 0} \xrightarrow{\bar{d}} Z^0) \cong \ker(d)^{<0}[1].$$

Hence, we are left with computing $\ker(d)$. By applying the automorphism $\begin{pmatrix} id & 0 \\ id & id \end{pmatrix}$ of $\mathcal{O}(1)^2$ the map d corresponds to the map

$$\mathcal{O}(1) \oplus \mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1} \xrightarrow{\begin{pmatrix} id & p & 0 \\ 0 & p & q \end{pmatrix}} \mathcal{O}(1)^2.$$

By applying the automorphism given by the matrix $\begin{pmatrix} 1 & -p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left hand side, the latter map can be replaced by

$$\mathcal{O}(1) \oplus \mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1} \xrightarrow{\begin{pmatrix} id & 0 & 0 \\ 0 & p & q \end{pmatrix}} \mathcal{O}(1)^2.$$

The kernel of its composition with the projection to the first copy of $\mathcal{O}(1)$ is $\mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1}$. The kernel of its composition with the projection to the second copy of $\mathcal{O}(1)$ is

$$\mathcal{O}(1) \oplus \ker(\mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1} \xrightarrow{(p \quad q)} \mathcal{O}(1)).$$

To compute the second direct summand, we may assume that $h \geq h'$. By Lemma 4.11, the elements $s_1, \ldots, s_{h+1}$ are linearly independent. In particular, $n \geq h + 1$. Since we may choose a maximal linearly independent subset of $s_1, \dots, s_{h+h'+2}$, up to reordering the elements $s_{h+2}, \dots, s_{h+h'+2}$ we may assume that $s_1, \dots s_n$ are linearly independent. Then the elements $s_{n+1}, \dots, s_{h+h'+2}$ lie in the E-subspace of $H^0(X, \mathcal{O}(1))$ spanned by the $s_1, \dots, s_n$. Hence, we find a matrix $A \in M_{n,h+h'+2-n}(E)$ such that $\begin{pmatrix} s_1 & \dots & s_n \end{pmatrix} \cdot A = \begin{pmatrix} s_{n+1} & \dots & s_{h+h'+2} \end{pmatrix}$. Set $\widetilde{A} := \begin{pmatrix} I_n & A \\ 0 & -I_{h+h'+2-n} \end{pmatrix} \in GL_{h+h'+2}(E) = \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}^{h+h'+2})$. Then $\begin{pmatrix} s_1 & \dots & s_{h+h'+2} \end{pmatrix} \cdot \widetilde{A} = \begin{pmatrix} s_1 & \dots & s_n & 0 & \dots & 0 \end{pmatrix}$.

Hence, by precomposition with the automorphism of $\mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1}$ given by the matrix $\widetilde{A}$ we identify the map $\begin{pmatrix} p & q \end{pmatrix} = \begin{pmatrix} s_1 & \dots & s_{h+h'+2} \end{pmatrix}$ with the map

$$\begin{pmatrix} s_1 & \dots & s_n & 0 & \dots & 0 \end{pmatrix} : \mathcal{O}^{h+1} \oplus \mathcal{O}^{h'+1} \longrightarrow \mathcal{O}(1).$$

The kernel of this map is $\ker \begin{pmatrix} s_1 & \dots & s_n \end{pmatrix} \oplus \mathcal{O}^{h+h'+2-n}$. Since the elements $s_1, \dots, s_n$ are linearly independent, $\ker \begin{pmatrix} s_1 & \dots & s_n \end{pmatrix} \cong \mathcal{O}(-\frac{1}{n-1})$ by Lemma 4.11. Putting everything together we now know that $\ker(d) \cong \mathcal{O}(-\frac{1}{n-1}) \oplus \mathcal{O}^{h+h'+2-n}$. Therefore,

$$\mathcal{O}(-\frac{1}{h}) \coprod_{\mathcal{O}(1),\mathsf{C}} \mathcal{O}(-\frac{1}{h'}) \cong \ker(d)^{<0}[1] \cong \mathcal{O}(-\frac{1}{n-1})[1].$$

$\square$

Lemma 4.11. *Let $h \geq 1$. A set of $h+1$ global sections $s_0, \dots, s_h$ of $\mathcal{O}(1)$ is E-linearly independent if and only if the map $\mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1)$ has kernel $\mathcal{O}(-\frac{1}{h})$.*

Moreover, in this case the map $\mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1)$ is surjective, so that we obtain a short exact sequence

$$0 \to \mathcal{O}(-\frac{1}{h}) \to \mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1) \to 0.$$

Proof. Suppose that $s_0, \dots, s_h$ are linearly independent over E. For $0 \leq i \leq h$ the map $\mathcal{O} \xrightarrow{s_i} \mathcal{O}(1)$ is surjective (even an isomorphism) over $D_+(s_i)$. Hence, the direct sum map $\mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1)$ is surjective over $\bigcup_{i=0}^h D_+(s_i) = X$ using the linear independence (see [FF], Théorème 6.5.2 (3) and (4)). Its kernel $\mathcal{F}$ is a subobject of $\mathcal{O}^{h+1}$ not containing a copy of $\mathcal{O}$. Indeed, if there was $0 \neq \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_h \end{pmatrix} \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}^{h+1}) \cong E^{h+1}$ such that the composition with $\mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1)$ is zero then $\sum_{i=0}^h \alpha_i s_i = 0$ contradicting the linear independece of the s_i. Therefore, all of the slopes of $\mathcal{F}$ are negative. Its HN polygon has endpoint $(h, -1)$ by Lemma 1.19. By concavity of the HN polygon $\mathcal{F}$ is isomorphic to $\mathcal{O}(-\frac{1}{h})$, as desired. In particular, we obtain a short exact sequence $0 \to \mathcal{O}(-\frac{1}{h}) \to \mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1) \to 0$.

Conversely, suppose that the kernel of the map $\mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1)$ is $\mathcal{O}(-\frac{1}{h})$. If $\sum_i \alpha_i s_i = 0$ for some $\alpha_i \in E$ which are not all 0 then the map $\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_h \end{pmatrix} : \mathcal{O} \longrightarrow \mathcal{O}^{h+1}$ is injective and the composition with $\mathcal{O}^{h+1} \xrightarrow{(s_0,\dots,s_h)} \mathcal{O}(1)$ is zero, hence the kernel of the latter map contains a copy of $\mathcal{O}$. This is impossible since $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}(-\frac{1}{h})) = 0$. $\square$

References

[BFHHLWY] C. Birkbeck, T. Feng, D. Hansen, S. Hong, Q. Li, A. Wang, L. Ye, *Extensions of vector bundles on the Fargues-Fontaine curve*, `arXiv:1705.00710`

[BvdB] A. Bondal, M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Moscow Mathematical Journal, (3)1, 1–36, 2003. `arXiv:math/0204218`

[Cha] S. U. Chase, *Direct products of modules*, Transactions of the American Mathematical Society, 97(3), 457-473, 1960.

[Col] P. Colmez, *Espaces de Banach de Dimension Finie*, Journal of the Institute of Mathematics of Jussieu, 1(3), 331-439, 2002.

[Det] J. C. Dette, *Abstrakte Kurven und fast euklidische Ringe*, master thesis, University of Duisburg-Essen, 2020.

[FF] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque, 2018, ISBN: 978-2-85629-896-1.

[FS] L. Fargues, P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459, 2021.

[Gro] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Mathematical Journal, (2), 9 (2): 119-221, DOI: 10.2748/tmj/1178244839.

[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg + Teubner Verlag, first edition, 2010.

[Har1] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[Har2] R. Hartshorne, *Residues and Duality*, Springer, 1966.

[HS] P. J. Hilton, U. Stammbach, *A course in homological algebra*, Springer, 1997.

[Jac] N. Jacobson, *Lectures in Abstract Algebra, III. Theory of Fields and Galois Theory*, Springer, 2012.

[Jen] C. U. Jensen, *Les Foncteurs Derives de lim et leurs Applications en Theorie des Modules*, Springer, 1972.

[Ked] K. Kedlaya, *Sheaves, stacks, and shtukas*, preprint, `https://swc-math.github.io/aws/2017/2017KedlayaNotes.pdf`

[Kel] B. Keller, *Derived categories and tilting*, lecture notes, `https://webusers.imj-prg.fr/~bernhard.keller/ictp2006/lecturenotes/keller.pdf`

[KL] K. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, arXiv:1301.0792v5

[KS1] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Springer, 2006.

[KS2] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Springer, 1990.

[LeB] A.-C. Le Bras, *Espaces de Banach-Colmez et Faisceaux Cohérents sur la Courbe de Fargues-Fontaine*, Duke Mathematical Journal, 167(18) 3455-3532, 2018. `https://doi.org/10.1215%2F00127094-2018-0034`.

[Mat] F. Mattiello, *On the heart associated to a faithful torsion pair*, master thesis, `https://algant.eu/documents/theses/mattiello.pdf`, 2011.

[Sta] *Stacks Project.*