Uniformization of complex projective klt varieties by bounded symmetric domains

Dissertation

Eingereicht in teilweiser Erfüllung des Abschlusses
Dr. rer. nat.

von

Aryaman Patel

Betreuer: **Prof. Dr. Daniel Greb**

Gutachter: **Prof. Dr. Benoît Claudon (Rennes)** und **Prof. Dr. Daniel Greb (Essen)**.
Verteidigt am **18.09.2023**.

Fakultät für Mathematik

UNIVERSITÄT DUISBURG ESSEN

Offen im Denken

DOI: 10.17185/duepublico/81480
URN: urn:nbn:de:hbz:465-20240410-064926-3

Abstract

Using classical results from non Abelian Hodge theory and more contemporary ones developed for complex projective varieties with Kawamata log terminal (klt) singularities, we deduce necessary and sufficient conditions for such varieties to be uniformized by each of the four classical irreducible Hermitian symmetric spaces of non compact type. We also deduce necessary and sufficient conditions for uniformization by a polydisk in the klt setting, which generalizes a classical result of Simpson.

Zussamenfassung

Mithilfe von klassischen Resultaten aus der nicht-Abelschen Hodgetheorie und modernen Resultaten über komplexe projektive Varietäten mit Kawamata-log-terminalen (klt) Singularitäten, geben wir jeweils notwendige und hinreichende Bedingungen für die Uniformisierung solcher Varietäten durch jeden der vier klassischen irreduziblen Hermitesch-symmetrischen Räume von nicht kompaktem Typ. Wir geben in diesem klt-Setting auch notwendige und hinreichende Bedingungen für die Uniformisierung durch eine Polydisk und verallgemeinern damit ein klassisches Resultat von Simpson.

Contents

1 Introduction and main results **4**
1.1 The uniformization problem 4
1.2 The role of Hodge theory 5
1.3 Recent developments 6
1.4 Main results 6
1.5 Structure of the thesis 8

I Uniformization theorem **10**

2 Preliminaries **10**
2.1 Morphisms and sheaves 10
2.2 Higgs sheaves and stability 11
2.3 Uniformization 13
2.4 Klt spaces and $\mathbb{Q}$-Chern classes 15
2.5 Harmonic bundles 17
2.6 A weak characterization of klt varieties with split cotangent sheaf 19
2.7 Another remark on stability 20

3 Revisiting Simpson's results **21**
3.1 The tangent bundle of $\mathcal{D}$ 21
3.2 The tangent bundle of $X = \mathcal{D}/\Gamma$ 23
3.3 Classical results of Simpson 25

4 Extending Simpson's result to the klt case **26**
4.1 Auxiliary remarks 26
4.2 Proof of Theorem 1.1 30

II Applications **32**

5 Uniformization by a polydisk **32**
5.1 Sufficient conditions 32
5.2 Necessary conditions 38

6 Uniformization by Hermitian symmetric space of type ***CI*** **40**
6.1 Sufficient conditions 40
6.2 Necessary conditions 41

7 Uniformization by Hermitian symmetric space of type ***DIII*** **42**
7.1 Sufficient conditions 43
7.2 Necessary conditions 44

8 Uniformization by Hermitian symmetric space of type ***BDI*** **45**
8.1 Sufficient conditions 45
8.2 Necessary conditions 46

9 Uniformization by Hermitian symmetric space of type $AIII$ 47

9.1 Sufficient conditions 48

9.2 Necessary conditions when $p \neq q$ 49

9.3 Necessary conditions when $p = q$ 50

1 Introduction and main results

After reading the title of this thesis one might wonder what the words uniformization, klt, and bounded symmetric domains mean. We begin by addressing these curiosities, and explaining the goal of this work. The aim of this section is to introduce the uniformization problem, and to convince the reader that it is indeed an interesting one, and has applications in algebraic and complex geometry.
Throughout this thesis, we work with algebraic varieties over the field $\mathbb{C}$ of complex numbers.

1.1 The uniformization problem

A *Hermitian manifold* can be regarded as the complex analytic version of a Riemannian manifold. More precisely, it is a complex manifold with a Hermitian metric on its holomorphic tangent bundle. A *Hermitian symmetric space* is a a Hermitian manifold with the property that at each point, there is an inversion symmetry which preserves the Hermitian structure.
We are interested in the Hermitian symmetric spaces $\mathcal{D}$ of non-compact type. Each such $\mathcal{D}$ can be expressed as the quotient of a real algebraic semisimple Lie group G_0 by a maximal compact subgroup K_0, which is unique up to conjugation (see Section 2.3 for details). Irreducible Hermitian symmetric spaces of non-compact type were classified by Elie Cartan into six types- the four classical, and two exceptional ones. It was shown by Harish-Chandra that each Hermitan symmetric space of non-compact type can be realized as a *bounded symmetric domain* inside a complex vector space (see Section 3.1 for details).

Bounded symmetric domains and their quotients are of classical interest in complex geometry. Varieties which are quotients of bounded symmetric domains are known to admit special tensors and automorphic forms, so they are interesting from an arithmetic point of view as well. The characterization of such varieties is then a natural problem to consider. In view of the minimal model program, it is important to work with singular varieties. One of the most important classes of singularities occurring in the minimal model program are the Kawamata log terminal (klt for short) singularities (see Section 2.4). For details about singularities of the minimal model program, we refer the reader to [24].
The goal of this thesis is to answer the following question, which we call the uniformization problem.

Let X be a complex projective variety with Kawamata log terminal (klt) singularities. What are necessary and sufficient conditions so that the universal cover $\tilde{X}$ of X is a bounded symmetric domain?

The uniformization problem has been studied in various settings. For example, in [39], Yau showed that smooth, projective surfaces of general type which satisfy equality in the Miyaoka-Yau inequality are uniformized by the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. Simpson formulated necessary and sufficient conditions for a smooth complex projective variety of aribitrary dimension to have a bounded symmetric domain as its universal cover in [34], and as a consequence obtained explicit conditions for uniformization by the polydisk and by the ball. In the ball case, Simpson's result has been generalized to smooth quasi-projective varieties (see [8]), and projective varieties with Kawamata log terminal (klt) singularities (see [13, 15]). The main result of this thesis, Theorem 1.1, settles the uniformization problem for arbitrary bounded symmetric domains, in the projective klt setting.

1.2 The role of Hodge theory

In his classical work [34], Simpson developed a method to construct representations of the topological fundamental group of a complex algebraic variety, which are the same as vector bundles on the variety with vanishing curvature. To do this, he used techniques from harmonic analysis and partial differential equations. He solved the Yang-Mills equation on holomorphic vector bundles with interaction terms over complex Kähler varieties. The solutions to this equation yield flat connections on the vector bundles if certain Chern classes are zero. As an application in the smooth projective case, one arrives at necessary and sufficient conditions for a smooth projective variety to be uniformized by any Hermitian symmetric space of non-compact type.An important fact is that an irreducible holomorphic vector bundle $\mathcal{E}$ on a projective variety has a Hermitian-Yang-Mills metric if and only if it is stable. Moreover, if the Chern classes of $\mathcal{E}$ satisfy $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) \cdot [K_X]^{n-2} = 0$, then any Hermitian-Yang-Mills metric on $\mathcal{E}$ is flat.

One of the main goals of [34] was to parametrize complex variations of Hodge structures as defined by Griffiths. A *complex variation of Hodge structure* is a C^∞ vector bundle V together with a decomposition $V = \bigoplus_{p+q=w} V^{p,q}$, a flat connection D, satisfying the following *Griffiths transversality* condition

$$D : V^{p,q} \to \mathcal{A}^{0,1}(V^{p+1,q-1}) + \mathcal{A}^{1,0}(V^{p,q}) + \mathcal{A}^{0,1}(V^{p,q}) + \mathcal{A}^{1,0}(V^{p-1,q+1})$$

and with a polarization. In the above expression $\mathcal{A}^{i,j}(V^{p,q})$ denotes $V^{p,q}$-valued forms of type (i,j). A polarization is a Hermitian form which makes the Hodge decomposition orthogonal, and is positive definite on $V^{p,q}$ if p is even, and is negative definite if p is odd. For a VHS V coming from a family of manifolds, there is a lattice in V preserved by the connection D, which Griffiths included as part of his definition. Relaxing this condition gives more complex VHSs, so a VHS can be deformed in a continuous family.

Considering infinitesimal deformations of a VHS motivates the definition of a *system of Hodge bundles* (see Section 2, Definition 2.16). These are special instances of more general objects called *Higgs sheaves* (see Section 2, Definition 2.7). A system of Hodge bundles arises from a VHS in a natural way. A more interesting problem is to construct a VHS starting from a system of Hodge bundles. The upshot is that an irreducible complex VHS corresponds precisely to a system of Hodge bundles $\mathcal{E}$ which i stable and satisfies $c_1(\mathcal{E}) = 0$, and $c_2(\mathcal{E}) \cdot [K_X]^{n-2} = 0$. A VHS on a variety X gives a holomorphic map from the universal cover $\tilde{X}$ of X to the classifying space $\mathcal{D}$ of Hodge structures. Of prime interest to us is the case when $\mathcal{D}$ is a Hermitian symmetric space of non-compact type.

When X is a curve, one can construct a non-trivial VHS on X in the following way. Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L}^{\otimes 2} = K_X$, and set $\mathcal{E}^{1,0} = \mathcal{L}$, and $\mathcal{E}^{0,1} = \mathcal{L}^\vee$. Then $\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ together with Higgs field θ given by $\theta^{1,0} : \mathcal{L} \cong \mathcal{L}^\vee \otimes K_X$, $\theta^{0,1} : \mathcal{L}^\vee \to 0$ is a system of Hodge bundles. It is easily checked that $c_1(\mathcal{E}) = c_2(\mathcal{E}) = 0$. If the genus of X is ≥ 2, then the only saturated subsystem of Hodge sheaves of $\mathcal{E}$ is $\mathcal{L}$, which has negative degree, thus $\mathcal{E}$ is stable and gives a VHS. The classifying map is an isomorphism between the universal cover $\tilde{X}$ and the upper half plane $\mathbb{H}$.

Now suppose X is of any dimension n. There is a canonical system of Hodge bundles $\mathcal{E}$ on X given by $\mathcal{E}^{1,0} = \Omega^1_X$, $\mathcal{E}^{0,1} = \mathcal{O}_X$, and the Higgs field given by sending Ω^1_X to itself and $\mathcal{O}_X$ to zero. If $\mathcal{E}$ is stable then the Bogomolov-Gieseker inequality $(2(n+1)c_2(X) - nc_1(X)^2) \cdot [K_X]^{n-2} \geq 0$ holds. If equality holds, then $\tilde{X}$ is the unit ball $\mathbb{B}^n$. In the case that X is a surface of general type, one recovers the result of Miyaoka and Yau which says that $c_1(X)^2 - 3c_2(X) \leq 0$, and if equality holds then X is a ball quotient. Similarly, the tangent bundle of X splitting as a direct sum of line bundles of negative degree, together with the Chern class equality $(c_1(X)^2 - 2c_2(X)) \cdot [K_X]^{n-2}$ are sufficient conditions for X to be a quotient of the polydisk $\mathbb{H}^n$. This result was slightly improved by Beauville (see [3, Theorem B]), who showed that the assumption on the Chern classes of X is not necessary.

1.3 Recent developments

The results of this thesis owe a great deal to the works [13] and [15], which extend Simpson's non-abelian Hodge correspondence to complex projective varieties with klt singularities. The main result of [13] is a Miyaoka-Yau inequality for projective varieties of general type with klt singularities, formulated in terms of $\mathbb{Q}$-Chern classes (see Section 2.4), known as the *$\mathbb{Q}$-Miyaoka-Yau inequality.* The fact that klt singularities are quotient in codimension two allows one to define the first and second $\mathbb{Q}$-Chern classes of the tangent sheaf $\mathcal{T}_X$ of X, which is locally free over the big open subset of X where X has only quotient singularities. If X is smooth in codimension two, these agree with the usual Chern classes of $\mathcal{T}_X$. The authors also prove that equality in the $\mathbb{Q}$-Miyaoka-Yau inequality is a necessary and sufficient condition for a projective klt variety which is smooth in codimension two, to be uniformized by the ball.

The protagonist in the proof of the uniformization result of [13] is the system of Hodge sheaves $\mathcal{E} = (\Omega^1_X)^{**} \oplus \mathcal{O}_X$, which is the correct analog of the system of Hodge bundles $\Omega^1_X \oplus \mathcal{O}_X$, in the singular setting. Using the semistability of $\mathcal{T}_X$ and $(\Omega^1_X)^{*}*$ for X klt and of general type, it is shown that $\mathcal{E}$ is stable as a Higgs sheaf on X. A crucial result is that X admits a global Galois, quasi-étale cover $\gamma : Y \to X$, such that the étale fundamental groups of Y and of its smooth locus Y_{reg} are isomorphic. This is known as a *maximally quasi-étale* cover, and the aim is to show that it is smooth. Using the relationship between certain representations of the fundamental group and variations of Hodge structures in the singular setting, also dveloped in [13], it is shown that the system of Hodge sheaves $\gamma^{[**]}(\Omega^{[1]}_X \oplus \mathcal{O}_X) = \Omega^{[1]}_Y \oplus \mathcal{O}_Y$ is locally free. The solution to the Lipman-Zariski conjecture for klt spaces then implies that Y is smooth, as desired. Then Yau's uniformization result implies that Y is a ball quotient, and a further argument shows that the same is true for X.

In [15], the authors show that Higgs bundles living over the smooth locus X_{reg} of a projective klt variety admit harmonic metrics. Let $(\mathcal{E}, \theta)$ be a reflexive Higgs sheaf on the smooth locus X_{reg} of a klt space X of dimension n, let $\mathcal{E}'$ denote the reflexive extension of $\mathcal{E}$ to X, and let H be an ample divisor on X. Then the main result of [15] says that $(\mathcal{E}, \theta)$ being polystable with respect to H and satisfying the equalities $\widehat{ch}_1(\mathcal{E}') \cdot [H]^{n-1} = 0$ and $\widehat{ch}_2(\mathcal{E}') \cdot [H]^{n-2} = 0$ is equivalent to $\mathcal{E}$ being locally free, and $(\mathcal{E}, \theta)$ being induced by a tame, purely imaginary harmonic bundle whose associated flat bundle is semisimple (see Section 2.5). As a consequence one obtains necessary and sufficient conditions for a projective klt variety X with ample canonical divisor to be uniformized by the ball. The improvement from the uniformization result of [13] is that one does not need to assume X to be smooth in codimension two.

The proof of the uniformization result of [15] is the same spirit as that of [13], and we use it as a template to prove the uniformization results appearing in this thesis.

1.4 Main results

We work in the same setting as [15] and formulate the following analog of Simpson's result of uniformization by Hermitian symmetric spaces of non compact type, for a complex projective variety with klt singularities and ample canonical divisor.

Theorem 1.1. *Let X be a complex projective klt variety with ample canonical divisor K_X. Let $\mathcal{D}$ be a Hermitian symmetric space of non-compact type. Then $X \cong \mathcal{D}/\Gamma$, where Γ is a discrete cocompact subgroup of $Aut(\mathcal{D})$, whose action on $\mathcal{D}$ is fixed point free in codimension one, if and only if:*

1. *The smooth locus X_{reg} admits a uniformizing system of Hodge bundles (P, θ) for any Hodge group G_0 of which $Aut(\mathcal{D})$ is a quotient by a discrete central subgroup, and*
2. *X satisfies the $\mathbb{Q}$-Chern class equality $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$, where $\mathcal{E}'$ is the reflexive extension of the system of Hodge bundles $P \times_K \mathfrak{g}$ to X.*

The notation appearing in Theorem 1.1 will be made clear in the subsequent sections. As a consequence of Theorem 1.1, we derive necessary and sufficient conditions for a projective klt variety X with ample canonical divisor to be uniformized by each of the four classical Hermitian symmetric spaces of noncompact type, and by the polydisk. In the polydisk case, our result is a slight generalization of Beauville's, in that we only assume that the tangent bundle $\mathcal{T}_{X_{reg}}$ of the smooth locus X_{reg} of X splits as a direct sum of line bundles (see Theorem 5.1). Due to the semistability of the tangent sheaf $\mathcal{T}_X$ with respect to the canonical divisor K_X, which was shown in [13], we do not need to assume that the line bundles appearing in the decomposition of $\mathcal{T}_{X_{reg}}$ have negative degree.
In each case, the conditions consist of the tangent bundle of the smooth locus $\mathcal{T}_{X_{reg}}$ admitting a reduction in structure group, and X satisfying a $\mathbb{Q}$-Chern class equality. For example, the statement for uniformization by the Hermitian symmetric space $\mathcal{H}_n$ of type CI (also known as the Siegel upper half space), is as follows.

Theorem 1.2. *Let X be a projective klt variety of dimension $n(n+1)/2$ such that the canonical divisor K_X is ample. Then $X \cong \mathcal{H}_n/\Gamma$, where Γ is a discrete cocompact subgroup of $Aut(\mathcal{H}_n) = PSp(2n, \mathbb{R})$, whose action on $\mathcal{D}$ is fixed point free in codimension one, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong Sym^2(\mathcal{E})$
- $[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{n-2} = 0,$

where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}, and $\mathcal{E}'$ denotes the reflexive extension of $\mathcal{E}$ to X.

Theorem 1.2 is a consequence of combining Propositions 6.1 and 6.2 in Section 6. Analogous results to Theorem 1.2 for Hermitian symmetric spaces of types $DIII$, BDI, and $AIII$ for $p \neq q$ are Theorems 7.3, 8.3, and 9.3 in Sections 7, 8, and 9 respectively. Necessary and sufficient conditions for uniformization by the polydisk and by the Hermitian symmetric space of type $AIII$ for $p = q$ are formulated separately in Sections 5 and 9 respectively.

Another consequence of Theorem 1.1 is the following Kazhdan type result, which is a klt version of [34, Corollary 9.5].

Corollary 1.3. *Let $\mathcal{D}$ be a Hermitian symmetric space of non-compact type, and let X be a projective klt quotient of $\mathcal{D}$ by a Γ as in Theorem 1.1, with K_X ample. Then for any $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, the conjugate variety X^σ is also a quotient of $\mathcal{D}$.*

This statement appears again as Corollary in Section 4 and is proved there, following the proof of Theorem 1.1.

1.5 Structure of the thesis

In order to prove our main uniformization theorem and subsequently apply it to particular bounded symmetric domains, it is important to understand Simpson's uniformization result and its philosophy in the smooth projective setting. The thesis is divided in two parts. Part I consists of preliminaries, Simpson's classical results from [34] in detail, and the proof of Theorem 1.1. In Part II we apply Theorem 1.1 to the classical irreducible bounded symmetric domains, and the polydisk.

In Section 2 we introduce objects, morphisms, and conventions that will be used in the following sections of the thesis. We start by discussing nef and ample sheaves, and Galois and quasi-étale morphisms of normal varieties. We then talk about Higgs sheaves and their stability, which play a central role in the proof of the main theorem. We then introduce objects and notions specific to the uniformization problem, as they appear in [34]. Next, we move to the klt setting where we make definitions of klt spaces and pairs, and introduce $\mathbb{Q}$-Chern classes of reflexive sheaves living on these. It is convenient that $\mathbb{Q}$-Chern classes have the same numerical behaviour (e.g. with respect to direct sums and tensor products) as the usual Chern classes. Finally, we discuss harmonic bundles and related structures necessary to state the main results of [15] which will be used in the proof of the main uniformization result.

In Section 3 we discuss in detail the material appearing in Sections 8 and 9 of [34]. For the sake of thoroughness, we give proofs of some important facts stated therein. In particular, using the equivalence between isomorphism classes of flat G-bundles and conjugacy classes of representations $\rho : \pi_1(X) \to G$, we show that each Hermitian symmetric space $\mathcal{D}$ admits a uniformizing system of Hodge bundles. Moreover, we show that this descends to a uniformizing system of Hodge bundles on any smooth projective quotient X of $\mathcal{D}$. We conclude the section by recalling the main results of [34] on which the proofs of our uniformization results are based. We remark that one should be careful when working with bounded symmetric domains whose automorphism group is not connected, the reason for this should become apparent in sections 5-9.

Section 4 is dedicated to extending the uniformization results of [34] to the projective, klt setting, and is divided into two parts. In the first part we prove some auxiliary statements about the stability of certain Higgs sheaves that will be used in the proof of the main theorem. For example, we show that a torsion free system of Hodge sheaves being (poly/semi)stable as a system of Hodge sheaves is equivalent to it being (poly/semi)stable as a Higgs sheaf. Using this, together with the semistablity of the tangent sheaf of a klt variety of general type, we show that the system of Hodge bundles $P \times_K \mathfrak{g}$ (introduced in Section 2.3) is always polystable as a Higgs bundle on the smooth locus X_{reg} of X. In the second part of the section we prove Theorem 1.1. As a corollary, we show that X being a quotient of a bounded symmetric domain implies that any Galois conjugate X^σ is too.

In Sections 5-9, we apply Theorem 1.1 to the polydisk and each of the four classical irreducible bounded symmetric domains $\mathcal{D}$ and determine necessary and sufficient conditions for a projective klt variety X with ample canonical divisor to have $\mathcal{D}$ as it's universal cover. We also provide examples wherever possible. The necessary and sufficient conditions consist of (a) the tangent bundle of the smooth locus X_{reg} admitting a reduction in structure group to K, and (b) X satisfying the $\mathbb{Q}$-Chern class equality. To make these conditions explicit for a particular $\mathcal{D}$, the trick is to identify the vector bundles associated to the adjoint representations of K on the Lie algebras $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$. One can then write down the tangent bundle as $\mathcal{T}_{X_{reg}} = P \times_K \mathfrak{g}^{-1,1}$, and the $\mathbb{Q}$-Chern class equality follows from equating $c_2(P \times_K \mathfrak{g})$ to zero.

Acknowledgements

I would like to thank my advisor Prof. Daniel Greb firstly for introducing me to the problem and suggesting it as the topic of my thesis, and secondly for the many insightful discussions leading up to it. I would also like to thank my co-advisor Prof. Ulrich Görtz, and Prof. Jochen Heinloth for many discussions that were crucial to proving our results. I am grateful to Matteo Costantini for his detailed comments and feedback on the article version, which significantly improved it. I am also grateful to Adrian Langer for his helpful comments on the first version of this article, and for pointing me to his paper from which I learned a lot. I would also like to thank Prof. Philippe Eyssidieux for answering my questions about automorphism groups of bounded symmetric domains.

I am immensely grateful for all the friends I made over the course of my PhD. You made my time here beautiful and unforgettable.

Finally, I acknowledge the RTG 2553 of ESAGA at the University of Duisburg-Essen for financial support throughout my PhD.

Part I

Uniformization theorem

We remind the reader that throughout this thesis we work over the field $\mathbb{C}$ of complex numbers.

2 Preliminaries

In this section we introduce objects and notation that will be used to formulate our main results. We follow the conventions of [13, Section 2], [15, Sections 2 and 3] and of [34].

2.1 Morphisms and sheaves

We recall notions of positivity for vector bundles and coherent sheaves that will be used throughout.

Definition 2.1 (Nef and ample sheaves, [15, Definition 2.4])**.** Let X be a normal projective variety and let $\mathcal{F} \neq 0$ be a nontrivial coherent sheaf on X. Then $\mathcal{F}$ is called *ample* (resp. *nef*) if the locally free sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \in \mathrm{Pic}(\mathbb{P}(\mathcal{F}))$ is ample (resp. nef).

Following are some well known facts about ample and nef sheaves which we list without proof.

Lemma 2.2 ([15], Facts 2.5). *Let X be a normal projective variety. Then the following statements hold.*

- *Ample sheaves on X are nef.*
- *A direct sum of sheaves on X is nef if and only if every summand is nef.*
- *Pullbacks and quotient sheaves of nef sheaves are nef.*
- *A sheaf $\mathcal{E}$ on X is nef if and only if for every morphism $\gamma : C \to X$ from a smooth curve C, the pullback $\gamma^*\mathcal{E}$ is nef on C.*

A closed subset Z of a normal, quasi projective variety is called *small* if the codimension of Z in X is at least two. An open subset U of X is called *big* if the complement $X \setminus U$ is small.

Definition 2.3 (Covering maps and Galois morphisms)**.** A *covering map* is a finite, surjective map $\gamma : Y \to X$ of normal, quasi-projective varieties. The covering map γ is called *Galois* if there exists a finite group $G \subset \mathrm{Aut}(Y)$ such that γ is isomorphic to the quotient map $Y \to Y/G$.

Definition 2.4 (Quasi étale morphism)**.** A morphism $f : X \to Y$ between normal varieties is called *quasi étale* if f is of relative dimension zero, and étale in codimension one. In other words, f is quasi étale if $\dim(X) = \dim(Y)$ and if there exists a subset $Z \subset X$ of codimension at least two such that the restricted map $f|_{X\setminus Z} : X \setminus Z \to Y$ is étale.

Definition 2.5 (Saturated subsheaf)**.** Let $\mathcal{E}$ be a coherent sheaf on X. A coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ is said to be *saturated* in $\mathcal{E}$ if the quotient sheaf $\mathcal{E}/\mathcal{F}$ is torsion-free. Morever, the saturation of a subsheaf $\mathcal{F} \subset \mathcal{E}$ is the kernel of the map $\mathcal{E} \to (\mathcal{E}/\mathcal{F})/(\text{torsion})$.

2.2 Higgs sheaves and stability

Next, we discuss Higgs sheaves and their stability on smooth quasi-projective varieties. We first introduce more general objects, namely sheaves with operators.

Definition 2.6 (Sheaf with an operator, [13, Definition 4.1])**.** Let X be a smooth, quasi-projective variety, and $\mathcal{W}$ a coherent sheaf of $\mathcal{O}_X$-modules. A *sheaf with a $\mathcal{W}$-valued operator* is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a coherent sheaf on X and $\theta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{W}$ is an $\mathcal{O}_X$-linear sheaf morphism.

A Higgs sheaf on X is an example of a Ω^1_X-valued operator with some additional conditions on the morphism θ. More precisely,

Definition 2.7 (Higgs sheaves)**.** Let X be a smooth, quasi-projective variety. A *Higgs sheaf* on X is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a coherent sheaf of $\mathcal{O}_X$-modules, and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ is an Ω^1_X-valued operator called the *Higgs field*, such that the composed morphism

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\theta \otimes \mathrm{Id}} \mathcal{E} \otimes \Omega^1_X \otimes \Omega^1_X \xrightarrow{\mathrm{Id} \otimes \wedge} \mathcal{E} \otimes \Omega^2_X$$

is zero. The composed morphism is usually denoted $\theta \wedge \theta$.

One can also define Higgs sheaves on normal varieties X (see [13, Definition 5.1]) by replacing Ω^1_X in the above definition with the sheaf $\Omega^{[1]}_X$ of reflexive differentials. Higgs sheaves defined this way behave well under reflexive pullback, as is explained in more detail in [13, Section 4].

We remark that an alternative definition of Higgs sheaves on normal varieties was introduced in [28, Section 4]. The advantage of this definition is that one can define define duals of Higgs sheaves, and also extend reflexive Higgs sheaves on the smooth locus of a klt variety to reflexive Higgs sheaves on the whole variety. This does not seem immediately possible with Higgs sheaves in the sense of Definition 2.7. However, in this article we use Definition 2.7 because the proof of the main theorem relies on some results of [15], where this definition is used, and we work in the same generality as [15].
The constructions in [28, Section 4] are however of independent interest, and we point the interested reader to the article [28] for details.

Definition 2.8 (Morphism of Higgs Sheaves, [13, Definition 5.2])**.** In the setting of Definition 2.7, a *morphism of Higgs sheaves* $f : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ is a morphism $f : \mathcal{E}_1 \to \mathcal{E}_2$ of sheaves which is compatible with the Higgs fields, i.e., $(f \otimes \mathrm{Id}_{\Omega^1_X}) \circ \theta_1 = \theta_2 \circ f$.

An important notion that will be used later in this note is that of a Higgs subsheaf. To make a definition we must first define an invariant subsheaf of a sheaf with an operator.

Definition 2.9 (Invariant subsheaf, [13, Definition 4.8])**.** Let X be a smooth quasi-projective variety and $(\mathcal{E}, \theta)$ a sheaf with a $\mathcal{W}$-valued operator, as in Definition 2.6. A coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ is called *θ-invariant* if the map $\theta : \mathcal{F} \to \mathcal{E} \otimes \mathcal{W}$ factors through $\mathcal{F} \otimes \mathcal{W}$. We call $\mathcal{F}$ *generically θ-invariant* if the restriction $\mathcal{F}|_U$ is invariant with respect to $\theta|_U$, where $U \subset X$ is the maximal dense open subset of X where $\mathcal{W}$ is locally free.

Let $(\mathcal{E}, \theta)$ be a Higgs sheaf on X. A subsheaf $\mathcal{F} \subset \mathcal{E}$ is called a *sub-Higgs sheaf* if $\mathcal{F}$ is θ-invariant, and if $(\mathcal{F}, \theta|_{\mathcal{F}})$ is a Higgs sheaf.

Remark 2.10. For applications to uniformization (Sections 4-9), we work exclusively with Higgs sheaves over the smooth locus X_{reg} of a projective klt variety X (see Section 2.4). If $\mathcal{F} \subset \mathcal{E}$ is any subsheaf, then X_{reg} being quasi-projective and $\Omega^1_{X_{reg}}$ being locally free implies that the functor $- \otimes \Omega^1_{X_{reg}}$ is exact. In particular, $\mathcal{F} \otimes \Omega^1_{X_{reg}} \subset \mathcal{E} \otimes \Omega^1_{X_{reg}}$ holds.

Remark 2.11. Let $(\mathcal{E}, \theta)$ be a Higgs sheaf on a smooth, quasi-projective variety X. Then any generically θ-invariant subsheaf $\mathcal{F} \subset \mathcal{E}$ is actually θ-invariant, because Ω^1_X is locally free. Moreover, every θ-invariant subsheaf $\mathcal{F}$ is actually a Higgs subsheaf $(\mathcal{F}, \theta|_{\mathcal{F}})$, because $\mathcal{F} \otimes \Omega^1_X \subset \mathcal{E} \otimes \Omega^1_X$ by Remark 2.10, thus $\theta|_{\mathcal{F}} \wedge \theta|_{\mathcal{F}} = 0$.

Following [13], on a normal, quasi projective variety X, we denote by $N^1(X)_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space of numerical Cartier divisor classes. For any sheaf $\mathcal{F}$ on X whose determinant is $\mathbb{Q}$-Cartier, we denote the corresponding element of $N^1(X)_{\mathbb{Q}}$ by $[\mathcal{F}] = [\det \mathcal{F}]$. The following construction gives a way to compute intersection products between Weil and Cartier divisors.

Intersection products ([13, Construction 2.17]). Let X be a normal, projective variety of dimension n, and let $\mathcal{F}$ be a non-zero coherent sheaf on X. Then $\det \mathcal{F}$ is a Weil-divisorial sheaf, say $\det \mathcal{F} = \mathcal{O}_X(D)$, for some Weil divisor D on X. Then D defines a rational equivalence class δ of an $(n-1)$-dimensional cycle on X. For $(n-1)$ line bundles $\mathcal{L}_1$, $\mathcal{L}_2$,...,$\mathcal{L}_{n-1}$ on X, we can form the cap product, and consider the number

$$\deg(\delta \cap [L_1] \cap \cdots \cap [\mathcal{L}_{n-1}]) \in \mathbb{Z},$$

where $[\mathcal{L}_i]$ denotes the element of $N^1(X)_{\mathbb{Q}}$ corresponding to the line bundle $\mathcal{L}_i$. The above value depends only on the numerical classes $[\mathcal{L}_i]$, thus the sheaf $\mathcal{F}$ induces a well-defined $\mathbb{Q}$-multilinear form on $N^1(X)_{\mathbb{Q}}^{\times(n-1)}$. Following [13], we denote the multilinear form obtained as above also by $[\mathcal{F}]$. If $\det \mathcal{F}$ is $\mathbb{Q}$-Cartier, then there is a numerical class $[\mathcal{F}] \in N^1(X)_{\mathbb{Q}}$, and it agrees with the one obtained by applying the above construction to $\mathcal{F}$ (see [13, Remark 2.19]).
Henceforth, we denote the intersection product of a coherent sheaf $\mathcal{F}$ with numerical classes $\alpha_1, ..., \alpha_{n-1} \in N^1(X)_{\mathbb{Q}}$ by $[\mathcal{F}] \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-1} \in \mathbb{Q}$.

We can now define the slope of any coherent sheaf on a normal projective variety with respect to a nef, $\mathbb{Q}$-Cartier divisor as follows.

Definition 2.12 (Slope with respect to a nef divisor)**.** Let X be a normal projective variety of dimension n, and let H be a nef, $\mathbb{Q}$-Cartier divisor on X. If $\mathcal{F}$ is a torsion free, coherent sheaf on X, the *slope* of $\mathcal{F}$ with respect to H is given by

$$\mu_H(\mathcal{F}) = \frac{[\mathcal{F}] \cdot [H]^{n-1}}{\operatorname{rank}(\mathcal{F})}.$$

We call $\mathcal{F}$ *semistable* with respect to H if for any subsheaf $\mathcal{E} \subset \mathcal{F}$ with $0 < \operatorname{rank}(\mathcal{E}) < \operatorname{rank}(\mathcal{F})$, we have $\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F})$. We call $\mathcal{F}$ *stable* with respect to H if the strict inequality holds for all $\mathcal{E}$.

For situations that arise later in the thesis, we would like to use notions of slope and stability which work for sheaves defined over the smooth locus of a normal projective variety. This is developed in [14, Section 2]. For completeness, we state the relevant definitions which appear therein.
It should be pointed out that all sheaves we work with are assumed to algebraic. For coherent analytic sheaves $\mathcal{F}$ on the analytic space associated to X_{reg}, the pushforward $j_*\mathcal{F}$ is in general not analytic, as remarked in [14, Remark 2.24]. We will view X_{reg} as a quasi projective variety over $\mathbb{C}$, hence coherent sheaves on X_{reg} will always be algebraic.

Definition 2.13 (Slope for sheaves on the smooth locus)**.** Let X be a normal, projective variety of dimension n, and let $\mathcal{E}$ be a torsion free, coherent sheaf of rank r on the smooth locus X_{reg}. If H is a nef, $\mathbb{Q}$-Cartier divisor on X, define the slope of $\mathcal{E}$ with respect to H as

$$\mu_H(\mathcal{E}) = \frac{[j_*\mathcal{E}] \cdot [H]^{n-1}}{r}$$

where $j : X_{reg} \to X$ is the inclusion.

As earlier, we call $\mathcal{E}$ *semistable* with respect to H if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ on X_{reg} with $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$, we have $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$. Call $\mathcal{E}$ *stable* with respect to H if the strict equality holds.
These notions can be extended to sheaves with operator on X_{reg} as follows.

Definition 2.14 (Stability for sheaves on the smooth locus)**.** Let the setting be as in Definition 2.13. Let $\mathcal{W}$ be a coherent sheaf on X_{reg}, and let $\theta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{W}$ be a $\mathcal{W}$-valued operator. We say that $(\mathcal{E}, \theta)$ is *semistable* with respect to H if the inequality $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$ holds for all generically θ-invariant subsheaves $\mathcal{F}$ of $\mathcal{E}$ with $0 < rank(\mathcal{F}) < rank(\mathcal{E})$. Call $(\mathcal{E}, \theta)$ *stable* with respect to H if the strict inequality $\mu_H(\mathcal{F}) < \mu_H(\mathcal{E})$ holds. Direct sums of stable sheaves with operator of the same slope are called *polystable*.

A Higgs sheaf on the smooth locus X_{reg} of a normal projective variety X is called *stable* (resp. *semistable*, *polystable*) with respect to H if it is stable (resp. semistable, polystable) with respect to H as a sheaf with an $\Omega^1_{X_{reg}}$-valued operator.
Lemmas 2.26 and 2.27 in [14] give useful properties of the above notions of slope and stability.

Remark 2.15. In order to check the stability of a Higgs sheaf, it is sufficient to check only the saturated sub-Higgs sheaves, because passing to the saturation of a subsheaf does not decrease the slope.

2.3 Uniformization

The notions of a system of Hodge bundles, principle bundles, and uniformizing systems of Hodge bundles play a central role in the proof of the main theorem. We state the definitions here as they are in Simpson's paper [34]. The variety X is assumed to be smooth and quasi-projective through Definitions 2.16-2.24.

Definition 2.16 (System of Hodge sheaves)**.** A *system of Hodge sheaves* on X is a Higgs sheaf (E, θ) as in Definition 2.7, together with a splitting $E = \bigoplus_{p,q} E^{p,q}$ such that restricting the Higgs field θ to each $E^{p,q}$, we get $\mathcal{O}_X$-linear maps $\theta|_{E^{p,q}} : E^{p,q} \to E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X$. If E is locally free, we say it is a *system of Hodge bundles*.

Let E be a system of Hodge sheaves. A *subsystem of Hodge sheaves* F of E is a sub-Higgs sheaf of E which has the same decomposition $F = \bigoplus_{p,q} F^{p,q}$ as E, with Higgs field given by $\theta|_F : F^{p,q} \to F^{p-1,q+1} \otimes \Omega^1_X$. A system of Hodge sheaves E is *semistable* (resp. *stable*) with respect to a nef, $\mathbb{Q}$-Cartier divisor if for any proper subsystem of Hodge sheaves $F \subset E$, we have $\mu_H(F) \leq \mu_H(E)$ (resp. $\mu_H(F) < \mu_H(E)$). We call E *polystable* if it is a direct sum of stable subsystems of Hodge sheaves of the same slope.

Definition 2.17 (Complexification of a Lie group)**.** A *complexification* of a real Lie group G is a complex Lie group $G_{\mathbb{C}}$ containing G as a closed, real Lie subgroup such that the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ is a complexification of the Lie algebra $\mathfrak{g}$ of G i.e., $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. The group G is then called a *real form* of the group $G_{\mathbb{C}}$.

Not every real Lie group G admits a complexification in the above sense. For example, the universal cover of $SL(2, \mathbb{R})$ does not admit such a complexification. The following is a collection of facts from [4, Chapters 2 and 3].

Lemma 2.18. *A real Lie group G admits a complexification in the sense of Definition 2.17 if and only if G is a linear group. If a complexification exists, it is not necessarily unique. Compact Lie groups always admit complexifications.*

We are specifically interested in automorphism groups of bounded symmetric domains, which are real Lie groups with some additional properties.

Definition 2.19 (Hodge group)**.** A *Hodge group* is a semisimple real algebraic Lie group G_0, together with a Hodge decomposition of the complexified Lie algebra

$$\mathfrak{g} = \bigoplus_p \mathfrak{g}^{p,-p}$$

such that $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{r,-r}] \subset \mathfrak{g}^{p+r,-p-r}$ and such that $(-1)^{p+1}\operatorname{Tr}(\operatorname{ad}(U) \circ \operatorname{ad}(\bar{V})) > 0$ for $U, V \in \mathfrak{g}^{p,-p} \setminus \{0\}$. Here $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der}(\mathfrak{g})$ denotes the adjoint representation of the Lie algebra $\mathfrak{g}$.

Let G_0 be a Hodge group as in the above definition, and let $K_0 \subset G_0$ be the subgroup corresponding to the Lie algebra $\mathfrak{k} = \mathfrak{g}_0^{0,0}$. It is the subgroup of elements k such that $ad(k)$ preserves the Hodge decomposition of $\mathfrak{g}$. In particular $\operatorname{ad}(k)$ preserves the positive definite form $(-1)^{p+1}\operatorname{Tr}(\operatorname{ad}(U) \circ \operatorname{ad}(\bar{V}))$ so K_0 is compact.

Remark 2.20. Henceforth, we will fix the Hodge group G_0 to be linear, because this is the case for applications to uniformization (see Sections 4-9 and [34, Section 9]). Thus G_0 and K_0 will admit complexifications in the sense of Definition 2.17 (see [25, Chapter VII, Section 9]). We denote by G and K the complexifications of G_0 and K_0 respectively.

Definition 2.21 (Principal system of Hodge bundles)**.** A *principal system of Hodge bundles* on X for a Hodge group G_0 is a principal K-bundle P on X together with a morphism of vector bundles

$$\theta : \mathcal{T}_X \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_X$.

The morphism θ maps local sections of $\mathcal{T}_X$ to elements of $\mathfrak{g}^{-1,1}$, thus the Lie bracket appearing in the above definition is the usual Lie bracket of $\mathfrak{g}$ extended to the bundle $P \times_K \mathfrak{g}^{-1,1}$.
To any local section a of the vector bundle $P \times_K \mathfrak{g}$, we can associate the map $a \mapsto (v \mapsto [\theta(v), a])$, for all local sections v of $\mathcal{T}_X$. It is easy to check that this gives $P \times_K \mathfrak{g}$ the structure of a system of Hodge bundles, hence the name.

Definition 2.22 (Hodge group of Hermitian type)**.** A Hodge group G_0 of *Hermitian type* is a Hodge group such that the Hodge decomposition of $\mathfrak{g}$ has only types $(1,-1)$, $(0,0)$, and $(-1,1)$, and such that G_0 has no compact factors.

In the case of a Hodge group of Hermitian type, $K_0 \subset G_0$ is a maximal compact subgroup, and the quotient $\mathcal{D} = G_0/K_0$ is a *Hermitian symmetric space of non-compact type*, and it can also be realized as a *bounded symmetric domain.* Moreover, all bounded symmetric domains arise in this way.

We use the following result of Simpson about Lie algebras of Hodge groups of Hermitian type multiple times later in the thesis, so we state it here.

Lemma 2.23 ([34, Corollary 9.3])**.** *Let $\mathfrak{g}$ be the complexified Lie algebra of a Hodge group of Hermitian type. Let $W \subset \mathfrak{g}$ be a sub-Hodge structure such that $[\mathfrak{g}^{-1,1}, W] \subset W$, where $[\cdot,\cdot]$ denotes the Lie bracket of $\mathfrak{g}$. Then*

$$\dim(W^{-1,1}) \geq \dim(W^{1,-1})$$

and if equality holds then W is a direct sum of ideals of $\mathfrak{g}$.

Definition 2.24 (Uniformizing system of Hodge bundles)**.** Let G_0 be a Hodge group of Hermitian type. A *uniformizing system of Hodge bundles* on X for G_0 is a principal system of Hodge bundles (P, θ) on X such that map $\theta : \mathcal{T}_X \to P \times_K \mathfrak{g}^{-1,1}$ is an isomorphism of sheaves.

A uniformizing system of Hodge bundles on X corresponds to a reduction of structure group for $\mathcal{T}_X$ to $K \to GL(n, \mathbb{C})$, where the map $K \to GL(n, \mathbb{C})$ is given by the adjoint representation of K on $\mathfrak{g}^{-1,1}$. We refer the reader to Section 3 of this thesis for more details.
Note that if $W \subset P \times_K \mathfrak{g}$ is a subsystem of Hodge sheaves of a uniformizing system of Hodge bundles $P \times_K \mathfrak{g}$, then W is locally a sub-Hodge structure of $\mathfrak{g}$. In particular, Lemma 2.23 holds locally for W.

Definition 2.25 (Uniformizing variation of Hodge structure)**.** A *uniformizing variation of Hodge structure* on X is a uniformizing system of Hodge bundles (P, θ) together with a flat metric on the associated bundle $P \times_K \mathfrak{g}$.

The table below shows bounded symmetric domains $\mathcal{D} = G_0/K_0$, and the associated groups G_0, K_0, and their respective complexifications G, K, which we use to prove our uniformization statements (Sections 5-9).

Bounded symmetric domain	G_0	K_0	G	K
Polydisk ($\mathbb{H}^n$)	$SL(2,\mathbb{R})^n$	$U(1)^n$	$SL(2,\mathbb{C})^n$	$(\mathbb{C}^*)^n$
Type CI ($\mathcal{H}_n$)	$Sp(2n,\mathbb{R})$	$U(n)$	$Sp(2n,\mathbb{C})$	$GL(n,\mathbb{C})$
Type DIII ($\mathcal{D}_n$)	$SO^*(2n)$	$U(n)$	$SO^*(2n,\mathbb{C})$	$GL(n,\mathbb{C})$
Type BDI ($\mathcal{B}_n$)	$SO(2,n)$	$SO(2) \times SO(n)$	$SO(2+n,\mathbb{C})$	$SO(2,\mathbb{C}) \times SO(n,\mathbb{C})$
Type AIII ($\mathcal{A}_{pq}$)	$SU(p,q)$	$S(U(p) \times U(q))$	$SL(p+q,\mathbb{C})$	$S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$

2.4 Klt spaces and $\mathbb{Q}$-Chern classes

One of six main classes of singularities one encounters when running the minimal model program in dimension ≥ 3 are klt singularities, which we define next. We first introduce some basic terminology which can be found in any standard reference on birational geometry, such as [24].

A *contraction* is a projective morphism $f : X \to Z$ of normal varieties with connected fibres, i.e. such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. A birational contraction is called a *blowup* or *blowdown* depending on the variety we start with.
A *boundary* on a variety is a $\mathbb{Q}$-Weil divisor $D = \sum_i a_i D_i$ with coefficients $0 \leq a_i \leq 1$ for all i. If D is a boundary on X then (X, D) is called a *log variety* or *log pair.* For $f : X \to Y$ a birational morphism, the boundary D' of Y is given by the direct image of the boundary D of X, i.e., $D' = f_*D$. A *log resolution* $f : \tilde{X} \to X$ is a resolution of singularities such that the union $\bigcup_i \tilde{D}_i \cup \mathrm{Exc}(f)$ of proper transforms of the D_i and the exceptional locus of f is a simple normal crossing divisor.
Let $f : \tilde{X} \to X$ be a projective birational morphism between normal varieties, and let D be a $\mathbb{Q}$-Weil divisor on X. Suppose that $K_X + D$ is $\mathbb{Q}$-Cartier. Then we can write

$$K_{\tilde{X}} + \tilde{D} \equiv f^*(K_X + D) + \sum_E a(E, D)E,$$

where $\tilde{D}$ denotes the proper transform of D and $a(E,D) \in \mathbb{Q}$. Note that the numbers $a(E,D)$ depend only on X, D, and E and not on f. They are called the *discrepancies.* Define

$$\text{discrep}(X,D) = \inf_E\{a(E,D)|E \text{ is an exceptional divisor on X}\}.$$

We are now ready to make the definitions of klt pairs and spaces.

Definition 2.26 (klt pair)**.** A *klt pair* (X,Δ) consists of a normal variety X, and a $\mathbb{Q}$-Weil divisor $\Delta = \sum_i a_i D_i$, where $a_i \in \mathbb{Q} \cap (0,1)$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and such that the discrepancy satisfies $\text{discrep}(X,\Delta) > -1$.

Definition 2.27 (klt space)**.** A normal, quasi-projective variety X is called a *klt space* if there exisits an effective $\mathbb{Q}$-Weil divisor Δ on X such that (X,Δ) is a klt pair.

Klt spaces of dimension 2 are known to have only quotient singularities. In dimension ≥ 3, a klt space X does not in general have only quotient singularities (see Example 2.29). A standard example of a klt singularity that is not quotient is that of a cone over a Fano variety. We first give a definition of a Fano variety and then discuss an example.

Definition 2.28. A *Fano variety* is a normal projective variety with klt singularities such that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Example 2.29. Let Y be a smooth projective subvariety of $\mathbb{P}^n$ with hyperplane divisor H, such that $K_Y \sim aH$ for some $a \in \mathbb{Q}$.
Set $\pi : \tilde{X} = \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(H)) \to Y$, and let E be a section such that $E|_Y \equiv -H$. The cone X in $\mathbb{P}^{n+1}$ over Y is normal, it is the contraction $f : \tilde{X} \to X$ of E, and we have

$$\begin{aligned} K_{\tilde{X}} &\sim -2E + \pi^*(K_Y - H) \\ &\sim -2E + \pi^*((a-1)H). \end{aligned}$$

It follows that $K_X \sim (a-1)H$. Note that they are both Weil divisors which coincide outside the vertex of X. Thus K_X is ample if and only if $a < 1$.
Writing $K_{\tilde{X}} \sim f^*K_X + bE$ and restricting to E gives that $b = -1-a$, so it follows that X has klt singularities if and only if $a < 0$. Moreover, X is a Fano variety of and only if Y is a Fano variety.

One can study the geometry of klt spaces using a more general notion of Chern classes, namely the $\mathbb{Q}$*-Chern classes*, or *orbifold Chern classes.*
Let X be a klt space as in Definition 2.27. After excluding a suitable subset $Z \subset X$ of codimension ≥ 3, only quotient singularities remain, and $X \setminus Z$ can be equipped with the structure of a $\mathbb{Q}$-variety that admits a global, Cohen-Macaulay cover (see [13, Section 3.4]). Thus following Mumford's work [32], Chern classes can be defined on $X \setminus Z$. Since $\text{codim}_X(Z) \geq 3$, one can construct on any klt space of dimension $n \geq 3$ intersection products with first and second $\mathbb{Q}$-Chern classes of any reflexive sheaf $\mathcal{E}$ on X. The associated symmetric $\mathbb{Q}$-multilinear forms can be written as follows.

$$\begin{aligned} \widehat{c}_1(\mathcal{E}) : N^1(X)^{n-1}_{\mathbb{Q}} \to \mathbb{Q}, &\quad (\alpha_1, \ldots, \alpha_{n-1}) \mapsto \widehat{c}_1(\mathcal{E}) \cdot \alpha_1 \cdots \alpha_{n-1} \\ \widehat{c}_1(\mathcal{E})^2 : N^1(X)^{n-2}_{\mathbb{Q}} \to \mathbb{Q}, &\quad (\alpha_1, \ldots, \alpha_{n-2}) \mapsto \widehat{c}_1^2(\mathcal{E}) \cdot \alpha_1 \cdots \alpha_{n-2} \\ \widehat{c}_2(\mathcal{E}) : N^1(X)^{n-1}_{\mathbb{Q}} \to \mathbb{Q}, &\quad (\alpha_1, \ldots, \alpha_{n-2}) \mapsto \widehat{c}_2(\mathcal{E}) \cdot \alpha_1 \cdots \alpha_{n-2}. \end{aligned}$$

In later sections we will also consider $\mathbb{Q}$-Chern characters, which are defined as follows

$$\begin{aligned} \widehat{ch}_1(\mathcal{E}) &= \widehat{c}_1(\mathcal{E}) \\ \widehat{ch}_2(\mathcal{E}) &= \frac{1}{2}(\widehat{c}_1(\mathcal{E})^2 - 2\widehat{c}_2(\mathcal{E})). \end{aligned}$$

The subject of $\mathbb{Q}$-Chern classes has been covered in greater detail in [13, Section 3]. We will use the following important observation about the behaviour of $\mathbb{Q}$-Chern classes under quasi-etale covers.

Lemma 2.30 ([13, Lemma 3.16])**.** *Let $\gamma : Y \to X$ be a quasi-etale morphism between projective klt spaces. Then the following equalities hold for all reflexive sheaves $\mathcal{E}$ on X, and all numerical classes $\alpha_1, \ldots, \alpha_{n-1} \in N^1(X)_{\mathbb{Q}}$*

$$\widehat{c}_1(\gamma^{[*]}\mathcal{E}) \cdot \gamma^*\alpha_1 \cdot \cdots \cdot \gamma^*\alpha_{n-1} = (\deg \gamma)\widehat{c}_1(\mathcal{E}) \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-1},$$

where $\gamma^{[*]}\mathcal{E} = (\gamma^*\mathcal{E})^{**}$. Analogous statements hold for $\widehat{c}_1(\gamma^{[*]}\mathcal{E})^2$ and $\widehat{c}_2(\gamma^{[*]}\mathcal{E})$.

We will use the following formula for the $\mathbb{Q}$-Chern classes of the direct sum of two reflexive sheaves E and F of ranks r and r' respectively.

$$\widehat{c}_k(E \oplus F) = \sum_{i=0}^{k} \widehat{c}_i(E) \cdot \widehat{c}_{k-i}(F).$$

So in particular we have

$$\widehat{c}_1(E \oplus F) = \widehat{c}_1(E) + \widehat{c}_1(F) \tag{1}$$

$$\widehat{c}_2(E \oplus F) = \widehat{c}_2(E) + \widehat{c}_1(E) \cdot \widehat{c}_1(F) + \widehat{c}_2(F). \tag{2}$$

2.5 Harmonic bundles

Harmonic bundles play an important role in nonabelian Hodge theory. We give here some basic definitions, facts, and results established in [15, Section 3], which will be used in the proof of the main uniformization result, Theorem 1.1.

Definition 2.31 ([15, Fact and Definition 3.1])**.** Let M be a complex manifold, and let $\mathbb{E} = (E, \bar{\partial}, \theta, h)$ be tuple consisting of the following data

- A holomorphic vector bundle $(E, \bar{\partial})$ and a Hermitian metric h on E.
- A Higgs field $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$, where $\mathcal{E} = \ker \bar{\partial}$ is the sheaf of holomorphic sections.

Denote also by θ the induced $\mathcal{A}^0$-linear morphism $\theta : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$. Let $\theta^h : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$ be the adjoint of θ with respect to the metric h, and let ∂ be the $(1,0)$-part of the unique Chern connection compatible with both the metric h and the complex structure $\bar{\partial}$. Then

$$\nabla_{\mathbb{E}} = \partial + \bar{\partial} + \theta + \theta^h$$

is a connection. The tuple $\mathbb{E} = (E, \bar{\partial}, \theta, h)$ is called a *harmonic bundle* if $\nabla_{\mathbb{E}}$ is flat.

For a harmonic bundle $\mathbb{E} = (E, \bar{\partial}, \theta, h)$ as in the above definition, denote the associated flat bundle by $(E, \nabla_{\mathbb{E}})$.

Definition 2.32 ([15, Notation 3.3])**.** Let M be a complex manifold.

- Let $\mathcal{E}$ be a locally free sheaf on M with associated holomorphic bundle $(E, \bar{\nabla})$. Then we say $\mathcal{E}$ *admits a harmonic bundle structure* if there exists a harmonic bundle of the form $(E, \bar{\partial}, \theta, h)$.
- Let $(\mathcal{E}, \theta)$ be a locally free Higgs sheaf on M with associated holomorphic bundle $(E, \bar{\nabla})$. Then we say $(\mathcal{E}, \theta)$ *admits a harmonic bundle structure* if there exists a harmonic bundle of the form $(E, \bar{\partial}, \theta, h)$.

- A flat bundle (E, ∇) is said to *admit a harmonic bundle structure* if there exists a harmonic bundle $\mathbb{E} = (E, \bar{\partial}, \theta, h)$ such that $\nabla = \nabla_{\mathbb{E}}$.

If $\mathcal{E}$ is a locally free sheaf on a smooth, quasi-projective variety X, then $\mathcal{E}$ is said to admit a harmonic bundle structure if its analytification $\mathcal{E}^{an}$ on the complex manifold X^{an} admits a harmonic bundle structure. An analogous statement holds for Higgs bundles $(\mathcal{E}, \theta)$ on X.
In the situation of Definition 2.32, let N be a complex submanifold of M, and let $(\mathcal{E}, \theta)$ be a locally free Higgs sheaf on M. Then the locally free Higgs sheaf $(\mathcal{E}, \theta)|_N$ on N admits a harmonic bundle structure via restriction.

To study Higgs bundles on a quasi-projective varieties, the authors of [15] look at *tame harmonic bundles.*

Definition 2.33 (Tame harmonic bundle, [15, Definition 3.5])**.** Let M be a complex manifold, let $D \subset M$ be a simple normal crossing divisor, and let $\mathbb{E} = (E, \bar{\partial}, \theta, h)$ be a harmonic bundle on $M \setminus D$. Then $\mathbb{E}$ is called *tame with respect to* (M, D) if there exists a locally free sheaf $\mathcal{E}_M$ on M, a morphism of sheaves

$$\theta_M : \mathcal{E}_M \to \mathcal{E}_M \otimes \Omega^1_M(\log D)$$

and an isomorphism $\mathcal{E}_M|_{M\setminus D} \cong \mathcal{E}$ that identifies $\theta_M|_{M\setminus D}$ with θ. Call $(\mathcal{E}_M, \theta_M)$ an *extension* of $(\mathcal{E}, \theta)$.

Definition 2.34 (Purely imaginary bundles, [15, Fact and Definition 3.6])**.** In the setup of Definition 2.33, assume that $(E, \bar{\partial}, \theta, h)$ is tame, and let $(\mathcal{E}_M, \theta_M)$ be an extension of $(\mathcal{E}, \theta)$. If $D_i \subset D$ is any component, and if $x \in D_i$ is any point, consider the residue and its restriction to x

$$\mathrm{res}_{D_i}\theta_M \in \mathrm{End}(\mathcal{E}_M|_{D_i}), \quad \textit{and} \quad (\mathrm{res}_{D_i}\theta_M)|_x \in \mathrm{End}(\mathcal{E}_M|_x).$$

Then the set of eigenvalues of $(\mathrm{res}_{D_i}\theta_M)|_x$ is independent of the choice of $(\mathcal{E}_M, \theta_M)$.
The harmonic bundle $(E, \bar{\partial}, \theta, h)$ is called *purely imaginary with respect to* (M, D) if all eigenvalues of the residues of θ_M along the irreducible components of D are purely imaginary for any extension $(\mathcal{E}_M, \theta_M)$ of $(\mathcal{E}, \theta)$.

Proposition 2.35 (Tame and purely imaginary bundles on quasi-projective bundles, [15, Fact and Definition 3.7])**.** *Let X be a smooth quasi-projective variety, and let $\mathbb{E} = (E, \bar{\partial}, \theta, h)$ be a harmonic bundle on X^{an}. Let $\bar{X}_1$ and $\bar{X}_2$ be two smooth projective compactifications of X such that $D_i = \bar{X}_i \setminus X$, $i \in \{1, 2\}$ are simple normal crossing divisors. Then $\mathbb{E}$ is tame and purely imaginary with respect to $(\bar{X}_1, D_1)$ if and only if $\mathbb{E}$ is tame and purely imaginary with respect to $(\bar{X}_2, D_2)$.*

The above proposition shows that the notion of tame purely imaginary does not depend on the compactification. Thus one can talk about *tame and purely imaginary harmonic bundles* on the analytification X^{an} of a smooth quasi-projective variety X.
Denote by TPI-locFree_X the family of isomorphism classes of locally free sheaves on X that admit a tame, purely imaginary harmonic bundle structure. Let TPI-Higgs_X denote the family of isomorphism classes of locally free Higgs sheaves on X that admit a tame, purely imaginary harmonic bundle structure. Write $(\mathcal{E}, \theta) \in \text{TPI-Higgs}_X$ to denote that a Higgs sheaf $(\mathcal{E}, \theta)$ on X is locally free and admits a tame, purely imaginary harmonic bundle structure $\mathbb{E}$. If $(\mathcal{E}, \theta) \in \text{TPI-Higgs}_X$ and if X is a big open subset of a normal projective variety $\bar{X}$, we say that $(\mathcal{E}, \theta)$ is *induced* by $\mathbb{E}$.

In order to prove the sufficiency part of Theorem 1.1, we make use of the following deep existence result.

Theorem 2.36 ([12, Theorem 1.14]). *Let X be a projective klt variety. Then X admits a quasi-étale cover $\gamma : Y \to X$ such that the natural map of étale fundamental groups $\widehat{\pi}_1(Y_{reg}) \to \widehat{\pi}_1(Y)$ is an isomorphism. This is known as a* maximally quasi-étale cover

We also state the following results of [15], which are crucial to the proof of the main uniformization result.

Proposition 2.37 ([15, Proposition 3.17]). *Let X be a projective klt space. Let $\gamma : Y \to X$ be a maximally quasi-étale cover, let $Y^o = \gamma^{-1} X_{reg}$, and let $\gamma|_{Y^o} : Y^o \to X_{reg}$ be the restricted morphism, which is étale. Then for any $(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}}) \in$ TPI-Higgs$_{X_{reg}}$, there exists a locally free Higgs sheaf $(\mathcal{F}, \theta)$ on Y, and an isomorphism*

$$(\mathcal{F}, \theta)|_{Y^o} = (\gamma|_{Y^o})^*(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}}).$$

In particular, if $\mathcal{E}$ denotes the reflexive extension of $\mathcal{E}_{X_{reg}}$ to X, then $\gamma^{[*]}\mathcal{E} \cong \mathcal{F}$ is locally free, and all the Chern classes of $\mathcal{F}$ vanish.

The following theorem is one of the main results of the paper [15], and allows us to pass from the singular to the smooth setting in the proof of Theorem 1.1.

Theorem 2.38 ([15, Theorem 5.1]). *Let X be a projective klt space of dimension $n \geq 2$. Let H be an ample divisor on X and use H to equip X_{reg} with a Kähler metric. Let $(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}})$ be a reflexive Higgs sheaf on X_{reg} and let $\mathcal{E}$ denote the reflexive extension of $\mathcal{E}_{X_{reg}}$ to X. Then the following statements are equivalent.*

- *The Higgs sheaf $(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}})$ is (poly)stable with respect to H, and the $\mathbb{Q}$-Chern characters satisfy*

$$\widehat{ch}_1(\mathcal{E}) \cdot [H]^{n-1} = 0 \quad \text{and} \quad \widehat{ch}_2(\mathcal{E}) \cdot [H]^{n-2} = 0.$$

- *The sheaf $\mathcal{E}_{X_{reg}}$ is locally free and $(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}})$ is induced by a tame, purely imaginary harmonic bundle whose associated flat bundle is (semi)simple.*

2.6 A weak characterization of klt varieties with split cotangent sheaf

The following observation is a weaker version of [16, Proposition 4.1], in a more general setting.

Lemma 2.39. *Let X be normal, complex quasi-projective variety of dimension n with klt singularities. Assume that the sheaf of reflexive differentials $\Omega_X^{[1]}$ is of the form*

$$\Omega_X^{[1]} \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n. \tag{3}$$

where $\mathcal{L}_i$ is a rank one $\mathbb{Q}$-Cartier sheaf for all $1 \leq i \leq n$. Then X has only quotient singularities.

Proof. Since each $\mathcal{L}_i$ is assumed to be $\mathbb{Q}$-Cartier, there is a minimal number $N_i \in \mathbb{N}$ for each i, such that $\mathcal{L}_i^{[\otimes N_i]}$ is locally free. Note that for any point $x \in X$, for each $1 \leq i \leq n$, we can find an open neighbourhood $U_i = U_i(x)$ of x over which $\mathcal{L}_i^{[\otimes N_i]}$ is trivial, i.e. such that $(\mathcal{L}_i^{[\otimes N_i]})|_{U_i} \cong \mathcal{O}_{U_i}$. Let $V = \bigcap_i U_i$. Then V is a non-empty open neighbourhood of x as it is the intersection of finitely many non-empty open neighbourhoods of x.
Consider the restriction of $\mathcal{L}_1^{[\otimes N_1]}$ to V and observe that it is trivial. Let $\gamma_1 : V_1 \to V$ be the associated index one quasi-étale cover which is cyclic of order N_1. The variety V_1 again has klt singularities, and we have $\gamma_1^{[*]}\mathcal{L}_1 \cong \mathcal{O}_{V_1}$. In particular, it follows that $\Omega_{V_1}^{[1]} \cong \gamma_1^{[*]}\Omega_V^{[1]}$, i.e., $\Omega_{V_1}^{[1]} \cong \gamma_1^{[*]}(\mathcal{L}_1|_V) \oplus \cdots \oplus \gamma_1^{[*]}(\mathcal{L}_n|_V) \cong \mathcal{O}_{V_1} \oplus \cdots \oplus \gamma_1^{[*]}(\mathcal{L}_n|_V)$. Note that since $\mathcal{L}_2^{[\otimes N_2]}$ is trivial over V, $\gamma_1^{[*]}\mathcal{L}_2^{[\otimes N_2]}$ is trivial over V_1. Now let $\gamma_2 : V_2 \to V_1$ be the associated index one quasi-étale cover which is cyclic of order N_2. We again have that

V_2 is klt and $\gamma_2^{[*]}\gamma_1^{[*]}\mathcal{L}_2 \cong \mathcal{O}_{V_2}$. Thus it follows that $\Omega_{V_2}^{[1]} \cong \gamma_2^{[*]}\Omega_{V_1}^{[1]} \cong \gamma_2^{[*]}\gamma_1^{[*]}(\mathcal{L}_1|_V) \oplus \cdots \oplus \gamma_2^{[*]}\gamma_1^{[*]}(\mathcal{L}_n|_V) \cong \mathcal{O}_{V_2} \oplus \mathcal{O}_{V_2} \oplus \cdots \oplus \gamma_2^{[*]}\gamma_1^{[*]}(\mathcal{L}_n|_V)$.
Continuing in this way $n-2$ more times we arrive at an index one quasi-étale cover $\gamma_n : V_n \to V_{n-1}$ of order N_n, such that V_n is klt and $\gamma_n^{[*]} \ldots \gamma_1^{[*]}\mathcal{L}_n \cong \mathcal{O}_{V_n}$. This implies that $\Omega_{V_n}^{[1]} \cong \gamma_n^{[*]} \ldots \gamma_1^{[*]}\Omega_V^{[1]} \cong \mathcal{O}_{V_n}^{\oplus n}$ i.e., $\Omega_{V_n}^{[1]}$ and $\mathcal{T}_{V_n}$ are both free. The solution for the Lipman-Zariski conjecture for spaces with klt singularities ([11, Theorem 16.1]) then asserts that V_n is smooth.
The composed map $\gamma = \gamma_1 \circ \cdots \circ \gamma_n : V_n \to V$ is a covering map of quasi-projective varieties. Then it follows from [12, Theorem 3.7] that there exists a normal, quasi-projective variety $\tilde{V}_n$ and a finite surjective morphism $\tilde{\gamma} : \tilde{V}_n \to V_n$. such that the following holds. There exist finite groups $H \subset G$ such that the composed map $\gamma \circ \tilde{\gamma} : \tilde{V}_n \to V$ and $\tilde{\gamma}$ are Galois with groups G and H respectively. The map $\gamma \circ \tilde{\gamma}$ is called the *Galois closure* of γ. Moreover, the branch loci of γ and $\gamma \circ \tilde{\gamma}$ are equal. This implies that the map $\gamma \circ \tilde{\gamma} : \tilde{V}_n \to V$ is again quasi-étale, and $\tilde{V}_n$ has at worst klt singularities. We have $\Omega_{\tilde{V}_n}^{[1]} = \tilde{\gamma}^{[*]}\gamma^{[*]}\Omega_V^{[1]} = \tilde{\gamma}^{[*]}\mathcal{O}_{V_n}^{\oplus n} = \mathcal{O}_{\tilde{V}_n}^{\oplus n}$. It follows again by the solution to the Lipman-Zariski conjecture for klt spaces that $\tilde{V}_n$ is smooth. Thus we conclude that X has quotient singularities only. □

2.7 Another remark on stability

We now make an observation about the stability of certain vector bundles associated to a semistable bundle.

Lemma 2.40. *Let X be a n-dimensional algebraic variety over $\mathbb{C}$ and let H be an ample divisor on X. Let $\mathcal{E}$ be a rank r vector bundle on X such that $Sym^2(\mathcal{E})$ is semistable with respect to H. Then $\mathcal{E}$ and $\mathcal{E}nd(\mathcal{E})$ are semistable with respect to H.*

Proof. Note that $\mathrm{Sym}^2(\mathcal{E})$ is a vector bundle of rank $r(r+1)/2$. Let $c_t(\mathcal{E}) = \sum_{i=0}^r c_i(\mathcal{E})t^i$ be the Chern polynomial of $\mathcal{E}$, and let $\alpha_1, ..., \alpha_r$ be the Chern roots of $\mathcal{E}$. Then the Chern polynomial of the p-th symmetric power $\mathrm{Sym}^p(\mathcal{E})$ is given by

$$c_t(\mathrm{Sym}^p(\mathcal{E})) = \prod_{i_1 \leq \cdots \leq i_p} (1 + (\alpha_i + \alpha_j)t).$$

In particular, we have that $c_1(\mathrm{Sym}^2(\mathcal{E}))$ is the coefficient of t in the expression $\prod_{i \leq j}(1+(\alpha_i+\alpha_j)t)$. A quick computation shows that $c_1(\mathrm{Sym}^2(\mathcal{E})) = (r+1)\sum_{i=1}^r \alpha_i = (r+1)c_1(\mathcal{E})$. The slope of $\mathrm{Sym}^2(\mathcal{E})$ with respect to H is given by

$$\mu_H(\mathrm{Sym}^2(\mathcal{E})) = \frac{c_1(\mathrm{Sym}^2(\mathcal{E})) \cdot [H]^{n-1}}{r(r+1)/2} = \frac{2c_1(\mathcal{E}) \cdot [H]^{n-1}}{r} = 2\mu_H(\mathcal{E}).$$

Suppose $\mathcal{E}$ is not H-semistable. Let $\mathcal{F} \subset \mathcal{E}$ be a subsheaf of rank $r' < r$ such that $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$. Consider the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

where $\mathcal{G} = \mathcal{E}/\mathcal{F}$. Note that taking Sym^2 preserves surjective maps, so the map $\mathrm{Sym}^2(\mathcal{E}) \to \mathrm{Sym}^2(\mathcal{G})$ is surjective. This implies that $\mathrm{Sym}^2(\mathcal{G}^\vee) \subset \mathrm{Sym}^2(\mathcal{E}^\vee)$. Since $\mathrm{rank}(\mathcal{G}) = r - r'$ and $c_1(\mathcal{G}) = c_1(\mathcal{E}) - c_1(\mathcal{F})$, we have

$$\mu_H(\mathcal{G}) = \frac{(c_1(\mathcal{E}) - c_1(\mathcal{F})) \cdot H^{n-1}}{r - r'} = \frac{r}{r - r'}\mu_H(\mathcal{E}) - \frac{r'}{r - r'}\mu_H(\mathcal{F}).$$

Thus $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$ implies that $\mu_H(\mathcal{G}) < \mu_H(\mathcal{E})$, which further implies $\mu_H(\mathcal{G}^\vee) = -\mu_H(\mathcal{G}) > -\mu_H(\mathcal{E}) = \mu_H(\mathcal{E}^\vee)$. But this means that $\mu_H(\mathrm{Sym}^2(\mathcal{G}^\vee)) > \mu_H(\mathrm{Sym}^2(\mathcal{E}^\vee))$, which implies that $\mathrm{Sym}^2(\mathcal{E}^\vee)$ is not H-semistable, and hence neither is $\mathrm{Sym}^2(\mathcal{E})$, a contradiction.

Since $\mathcal{E}$ is semistable with respect to H, so is the dual bundle $\mathcal{E}^\vee$, and the endomorphism bundle $\mathcal{E}nd(\mathcal{E}) \cong \mathcal{E} \otimes \mathcal{E}^\vee$, because the tensor product of semistable bundles is semistable. □

An analogous statement to Lemma 2.40 holds for the second wedge power.

Lemma 2.41. *Let X be a n-dimensional algebraic variety and let H be an ample divisor on X. Let $\mathcal{E}$ be a vector bundle of rank $r \geq 3$ on X such that $\bigwedge^2(\mathcal{E})$ is semistable with respect to H. Then $\mathcal{E}$ and $\mathcal{E}nd(\mathcal{E})$ are semistable with respect to H.*

The proof of this Lemma is essentially the same as that of Lemma 2.40, and is therefore omitted. Applying $\bigwedge^2$ also preserves surjective maps, and a simple computation shows that $\mu_H(\bigwedge^2 \mathcal{E}) = 2\mu_H(\mathcal{E})$, just as in the Sym^2 case.
It was pointed out to us by A. Langer that this Lemma does not work when the rank of $\mathcal{E}$ is 2. This is because $\bigwedge^2(\mathcal{E})$ will be a line bundle in this case, which is always semistable, but this does not imply that $\mathcal{E}$ is semistable.

3 Revisiting Simpson's results

In this section, our goal is to study the structure of the tangent bundle of a Hermitian symmetric space $\mathcal{D}$ of non-compact type. This structure descends to the tangent bundle of a smooth projective quotient $X = \mathcal{D}/\Gamma$, where Γ is a discrete, cocompact group of automorphisms of $\mathcal{D}$ which acts freely. We use the notation of Sections 8 and 9 of [34], and we use the theory developed in Chapter 12 of [6].

3.1 The tangent bundle of $\mathcal{D}$

Let G_0 be a Hodge group, and K_0 the subgroup corresponding to the Lie algebra $\mathfrak{k}_0 = \mathfrak{g}_0^{0,0}$, as in the paragraph following Definition 2.19. Since G_0 and K_0 are linear groups (see Remark 2.20), they admit complexifications in the sense of Definition 2.17. Let G and K be complexifications of G_0 and K_0 respectively.

Let X be a smooth projective variety of dimension n over $\mathbb{C}$, and let (P, θ) be a principal system of Hodge bundles on X for G. Recall from Definition 2.21 that P is a principal K-bundle on X, and θ is a morphism of vector bundles

$$\theta : \mathcal{T}_X \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_X$.

Definition 3.1. A *metric* H for a principal system of Hodge bundles is a C^∞ reduction of structure group of P from K to K_0, i.e., a principal K_0-bundle $P_H \subset P$.

Let (P, θ) be a principal system of Hodge bundles with metric $P_H \subset P$. This reduction in structure group corresponds to a Hermitian metric H on the associated system of Hodge bundles $E = P \times_K \mathfrak{g} \cong P_H \times_{K_0} \mathfrak{g}$. Let $d'_H \in \mathcal{A}^1(\mathrm{End}(E))$ denote the Chern connection on E with respect to H, and let d_H be the K_0-connection on P_H from which d'_H is induced. Now we view E as a G_0-bundle $E' = R_H \times_{G_0} \mathfrak{g}^{-1,1}$, where $R_H = P_H \times_{K_0} G_0$, and G_0 acts on $\mathfrak{g}$ via the adjoint action. The map $\theta : \mathcal{T}_X \to P \times_K \mathfrak{g}^{-1,1}$ gives an $\mathrm{End}(E')$-valued one-form $\theta' \in \mathcal{A}^{1,0}(\mathrm{End}(E'))$, as discussed in the paragraph following Definition 2.21. Let $\bar{\theta}' \in \mathcal{A}^{0,1}(\mathrm{End}(E'))$ be the adjoint of θ' with respect to H, i.e., $\langle \theta' u, v \rangle_H = \langle u, \bar{\theta}' v \rangle_H$ for local sections u, v of E. Let σ and $\bar{\sigma}$ be the G_0-connections on R_H which induce θ' and $\bar{\theta}'$ respectively. Then $D'_H = d'_H + \theta' + \bar{\theta}' \in \mathcal{A}^1(\mathrm{End}(E'))$ is a connection on E. Let $D_H = d_H + \sigma + \bar{\sigma}$ denote the G_0-connection on R_H which induces D'_H.

Definition 3.2. A *principal variation of Hodge structure* for a Hodge group G_0 is a principal system of Hodge bundles (P, θ) together with a metric P_H such that the curvature of the associated connection D_H is zero.

Let (P, θ) together with metric (P_H, D_H) be a principal variation of Hodge structure on X, and let $\widetilde{X}$ be a universal cover of X. We denote by π the quotient map $\pi : \widetilde{X} \to X$. Let $\widetilde{R}_H$ be the pullback of the G_0-bundle $R_H = P_H \times_{K_0} G_0$ to $\widetilde{X}$, then $\widetilde{R}_H = \pi^*(P_H \times_{K_0} G_0) = \widetilde{P}_H \times_{K_0} G_0$, where $\widetilde{P}_H = \pi^* P_H$ is a principal K_0-bundle on $\widetilde{X}$. The flat connection D_H on R_H pulls back to a flat connection on $\widetilde{R}_H$. Since $\widetilde{R}_H$ is a principal bundle with a flat connection on a simply connected space $\widetilde{X}$, we have a trivialisation $\phi : \widetilde{R}_H = \widetilde{P}_H \times_{K_0} G_0 \cong \widetilde{X} \times G_0$, hence $\widetilde{R}_H$ admits a global section. A global section of $\widetilde{P}_H \times_{K_0} G_0$ corresponds to a K_0-equivariant map $\varphi : \widetilde{P}_H \to G_0$, which induces a map $\widetilde{P}_H/K_0 \to G_0/K_0$. Since $\widetilde{X}$ is diffeomorphic to $\widetilde{P}_H/K_0$, we get a map $\widetilde{X} \to G_0/K_0$, which sends a point $x \in \widetilde{X}$ to a right K_0-coset $\varphi \widetilde{P}_{H,x} \subset G_0$.

Let H be a Lie group. Then, corresponding to every flat principal H-bundle $P \to X$, there is a group homomorphism $\rho : \pi_1(X) \to H$, known as the *holonomy morphism* or the *monodromy morphism.* More precisely, the following is true.

Theorem 3.3 ([31], Theorem 2.9)**.** *The correspondence, which sends each flat principal H-bundle over X to its holonomy morphism, induces a bijection*

$$\{\textit{isomorphism classes of flat } H\textit{-bundles over } X\} \cong \{\textit{conjugacy classes of homomorphism } \rho : \pi_1(X) \to H\}.$$

In view of this, the flat G_0-bundle R_H on X corresponds to a homomorphism $\sigma : \pi_1(X) \to G_0$. Recall from the preceding paragraph that we have a trivialisation $\phi : \pi^* R_H = \widetilde{R}_H \cong \widetilde{X} \times G_0$. From the proof of [31, Theorem 2.9], it follows that the flat bundle R_H can be expressed as

$$R_H = \widetilde{R}_H / \pi_1(X) = (\widetilde{X} \times G_0)/\pi_1(X),$$

where $\pi_1(X)$ acts on $\widetilde{X} \times G_0$ diagonally- by deck transformations on $\widetilde{X}$, and via σ on G_0. This action is described explicitly in [31], p.58-59. Setting $\phi = (\phi_1, \phi_2)$, we have that $\phi(r \cdot \gamma) = (\phi_1(r)\gamma, \sigma(\gamma)^{-1}\phi_2(r))$ for all local sections $r \in \widetilde{R}_H$ and all $\gamma \in \pi_1(X)$. The map $\widetilde{X} \to G_0/K_0$, given by $x \mapsto \varphi \widetilde{P}_{H,x}$ is equivariant under the representation σ, i.e.,

$$\varphi \widetilde{P}_{H,\gamma x} = \varphi(\gamma \cdot \widetilde{P}_{H,x}) = \sigma(\gamma)(\varphi \widetilde{P}_{H,x})$$

for all $\gamma \in \pi_1(X)$, and $x \in \widetilde{X}$.

Note that $\mathcal{D} = G_0/K_0$ is a homogeneous space and its tangent bundle $T_{\mathcal{D}}$ is a homogeneous vector bundle on $\mathcal{D}$. Since G_0 is a Hodge group, G_0/K_0 is in fact a *reductive domain*, i.e., there is an $\mathrm{Ad}(K_0)$-invariant decomposition of the Lie algebra of G_0 as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{m}$. From the discussion in [6, Chapter 12.2], there is an isomorphism $G_0 \times_{K_0} \mathfrak{m} \cong T_{\mathcal{D}}$. To obtain the holomorphic tangent bundle $\mathcal{T}_{\mathcal{D}}$ of $\mathcal{D}$, consider the splitting $\mathfrak{m} \otimes \mathbb{C} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, where $\mathfrak{m}^+ = \bigoplus_{p>0} \mathfrak{g}^{p,-p}$, and $\mathfrak{m}^- = \bigoplus_{p<0} \mathfrak{g}^{p,-p}$. By [6, Lemma 12.2.2], this splitting defines a complex structure on $\mathfrak{m}$. By [6, Lemma-Definition 12.2.3], the holomorphic tangent bundle of $\mathcal{D}$ is given by $\mathcal{T}_{\mathcal{D}} \cong G_0 \times_{K_0} \mathfrak{m}^-$, where K_0 acts on $\mathfrak{m}^-$ via the adjoint action. Let H_0 be the maximal compact subgroup of G_0 containing K_0. Then G_0/H_0 is a *symmetric space.* The associated *Cartan decomposition* of $\mathfrak{g}_0$ is $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{p}_0$. The *Cartan involution* $\iota : \mathfrak{g}_0 \to \mathfrak{g}_0$ is defined such that $\iota|_{\mathfrak{h}_0} = \mathrm{id}$, and $\iota|_{\mathfrak{p}_0} = -\mathrm{id}$. The Cartan decomposition is reflected in the tangent bundle of $\mathcal{D}$ as follows. There is a canonical projection $\omega : \mathcal{D} = G_0/K_0 \to G_0/H_0$ with respect to which the tangent space at any point $x \in \mathcal{D}$ splits into vertical and horizontal tangent spaces given by

$$T^v_{\mathcal{D},x} = \text{fiber of } \omega \text{ through } x, \quad T^h_{\mathcal{D},x} = \text{orthogonal complement of } T^v_{\mathcal{D},x}.$$

From [6, Lemma 12.5.2], we know that the ± 1-eigenspaces of the Cartan involution on $\mathfrak{g}$ are given by $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} = \bigoplus_{j \text{ even}} \mathfrak{g}^{-j,j}$, and $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C} = \bigoplus_{j \text{ odd}} \mathfrak{g}^{-j,j}$. Moreover, there are canonical identifications $T^v_{\mathcal{D},x} = \mathfrak{h}_0/\mathfrak{k}_0$, and $T^h_{\mathcal{D},x} = \mathfrak{p}_0$. By [6, Proposition 12.5.3], the tangent bundle $T_{\mathcal{D}}$ canonically decomposes as $T_{\mathcal{D}} = T^v_{\mathcal{D}} \oplus T^h_{\mathcal{D}}$ into vertical and horizontal components. Both components are homogeneous vector bundles on $\mathcal{D}$ for the adjoint action of K_0 on $\mathfrak{g}_0$. They can be written as

$$T^v_{\mathcal{D}} = G_0 \times_{K_0} \mathfrak{h}_0/\mathfrak{k}_0, \quad T^h_{\mathcal{D}} = G_0 \times_{K_0} \mathfrak{p}_0.$$

The vertical tangent bundle $T^v_{\mathcal{D}}$ is holomorphic, because its fibers are complex submanifolds of $\mathcal{D}$ (see [6, Problem 4.4.2]). The horizontal tangent bundle is in general not holomorphic. The subbundle $G_0 \times_{K_0} \mathfrak{g}^{-1,1}$ of the complexification $T^h_{\mathcal{D}} \otimes \mathbb{C}$ has the structure of a holomorphic vector bundle, and is called the *holomorphic horizontal tangent bundle* of $\mathcal{D}$.
Now suppose that G_0 is a Hodge group of Hermitian type. Then K_0 is a maximal compact subgroup of G_0, and $\mathcal{D} = G_0/K_0$ is the associated Hermitian symmetric space of non-compact type. While there may be several G_0 corresponding to a fixed $\mathcal{D}$, their identity components are all isogenous, so the complexified Lie algebra $\mathfrak{g}$ and its Hodge decomposition are determined (see [29, Section II, Corollary 3.30]).

Lemma 3.4. *The holomorphic tangent bundle $\mathcal{T}_{\mathcal{D}}$ of the Hermitian symmetric space $\mathcal{D}$ can be written as*

$$\mathcal{T}_{\mathcal{D}} \cong P' \times_K \mathfrak{g}^{-1,1},$$

where P' is a principal K-bundle on $\mathcal{D}$.

Proof. Let Q^+ and Q^- denote the Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{g}^{-1,1}$ and $\mathfrak{g}^{1,-1}$ of $\mathfrak{g}$ respectively. Then Q^+ and Q^- are abelian unipotent subgroups of G stabilized by conjugation by K ([38, Section 2.4]). There is a Zariski open embedding

$$j : \mathcal{D} \cong G_0/K_0 \hookrightarrow G/(K \ltimes Q^-) \cong \mathcal{D}^*$$

called the Borel embedding (see [20, Chapter VIII, Section 7]), where $\mathcal{D}^*$ is a complex homogeneous projective variety known as the *compact dual* of the Hermitian symmetric space $\mathcal{D}$. By the discussion in the preceding paragraph, the holomorphic tangent bundle of $\mathcal{D}^*$ can be expressed as $\mathcal{T}_{\mathcal{D}^*} \cong G \times_{(K \ltimes Q^-)} \mathfrak{g}^{-1,1}$, where $K \ltimes Q^-$ acts on $\mathfrak{g}^{-1,1}$ via the adjoint action. The adjoint action of Q^- on $\mathfrak{g}^{-1,1}$ is trivial, i.e., $qXq^{-1} = 0$ for all $q \in Q^-$, $X \in \mathfrak{g}^{-1,1}$. Thus the adjoint action of $K \ltimes Q^-$ on $\mathfrak{g}^{-1,1}$ factors through K. Hence $\mathcal{T}_{\mathcal{D}^*}$ admits a reduction in structure group from $K \ltimes Q^-$ to K, and we can write $\mathcal{T}_{\mathcal{D}^*} \cong P \times_K \mathfrak{g}^{-1,1}$, where P is a principal K-bundle on $\mathcal{D}^*$ such that $P \times_K (K \ltimes Q^-) \cong G$. It follows that $P \cong G/Q^-$ as principal K-bundles over $\mathcal{D}^*$. Expressing the holomorphic tangent bundle $\mathcal{T}_{\mathcal{D}}$ of $\mathcal{D}$ as the restriction of $\mathcal{T}_{\mathcal{D}^*}$ to $\mathcal{D}$, we get $\mathcal{T}_{\mathcal{D}} \cong P' \times_K \mathfrak{g}^{-1,1}$, where $P' = P|_{\mathcal{D}}$ is principal K-bundle on $\mathcal{D}$. □

Hence, it is clear that the holomorphic tangent bundle of a Hermitian symmetric space of non-compact type is horizontal.
If X admits a uniformizing variation of Hodge structure for a Hodge group G_0 of Hermitian type, then we know from the proof of [34, Proposition 9.1] (Proposition 3.6 below), that the $\pi_1(X)$-equivariant map $\widetilde{X} \to G_0/K_0 = \mathcal{D}$ is an isomorphism. The differential of this map is an isomorphism $\theta : \mathcal{T}_{\widetilde{X}} \cong P' \times_K \mathfrak{g}^{-1,1}$.

3.2 The tangent bundle of $X = \mathcal{D}/\Gamma$

We would now like to invert the previous construction. Let X be a smooth projective variety with universal cover $\widetilde{X}$. Suppose there is a $\pi_1(X)$-equivariant holomorphic isomorphism $\phi : \widetilde{X} \to \mathcal{D}$, where $\mathcal{D}$ is a Hermitian symmetric space of noncompact type. Let $\pi : \widetilde{X} \to X$ denote the projection map. Let $G_0 = \mathrm{Aut}(\mathcal{D})$ be the full automorphism group of $\mathcal{D}$, then G_0 is a Hodge group of Hermitian type with maximal compact subgroup K_0, and we have $\mathcal{D} = G_0/K_0$.

Lemma 3.5. *In this situation, the variety X admits a uniformizing variation of Hodge structure (P, θ) for the Hodge group G_0.*

Proof. The differential of the holomorphic isomorphism $\phi : \widetilde{X} \cong \mathcal{D}$ is the isomorphism $d\phi : \mathcal{T}_X \cong \mathcal{T}_{\mathcal{D}}$ of holomorphic tangent bundles. From Lemma 3.4 it follows that $\mathcal{T}_{\widetilde{X}} \cong \widetilde{P} \times_K \mathfrak{g}^{-1,1}$, where $\widetilde{P}$ is a prinicipal K-bundle on $\widetilde{X}$. Recall that, by [6, Lemma-Definition 12.2.3], we have $\mathcal{T}_{\mathcal{D}} \cong G_0 \times_{K_0} \mathfrak{g}^{-1,1}$. Thus it follows that $\mathcal{T}_{\widetilde{X}} \cong \widetilde{P}' \times_{K_0} \mathfrak{g}^{-1,1}$, where $\widetilde{P}' = \phi^* G_0$ is a principal K_0-bundle on $\widetilde{X}$. Hence, we may view $\widetilde{P}'$ as a reduction in structure group of $\widetilde{P}$ from K to K_0.

There is natural principal G_0 bundle $\widetilde{R}$ on $\mathcal{D}$ called the *Higgs principal bundle* (see [6, Definition 12.4.1]). It is given by

$$\widetilde{R} = G_0 \times_{K_0} G_0 \cong \mathcal{D} \times G_0,$$

where the isomorphism is a G_0-equivariant map given by $[g, g'] \mapsto ([g], gg')$. The *Higgs connection* ω_H on $\widetilde{R}$ is obtained by pulling back the Maurer-Cartan form on G_0 to $\mathcal{D} \times G_0$, and ω_H is flat. Thus we get a principal G_0-bundle $\widetilde{R}'$ on $\widetilde{X}$ given by $\widetilde{R}' = \phi^* \widetilde{R} = \phi^*(G_0 \times_{K_0} G_0) \cong \widetilde{P}' \times_{K_0} G_0$. The connection $\omega'_H = \phi^* \omega_H$ on $\widetilde{R}'$ is flat, and there is a trivialisation $\widetilde{R}' \cong \widetilde{X} \times G_0$. The fundamental group $\pi_1(X)$ acts on $\widetilde{X} \cong \mathcal{D}$ by holomorphic automorphisms, so we have a representation $\sigma : \pi_1(X) \to G_0$. The isomorphism $\phi : \widetilde{X} \cong G_0/K_0$ is equivariant under σ i.e., we have $\phi(\gamma x) = \sigma(\gamma)\phi(x)$ for all $x \in \widetilde{X}$ and $\gamma \in \pi_1(X)$. It follows that the differential $d\phi$ is also equivariant under σ. The action of $\pi_1(X)$ on $\widetilde{X}$ lifts to a left action of $\pi_1(X)$ on the principal K-bundle $\widetilde{P}$. Thus there is a left action of $\pi_1(X)$ on the associated bundle $\mathcal{T}_{\widetilde{X}}$, and we have $\mathcal{T}_X \cong \mathcal{T}_{\widetilde{X}}/\pi_1(X)$. By the equivariance of $d\phi$ under the representation σ, we have an isomorphism $\theta : \mathcal{T}_X \cong P \times_K \mathfrak{g}^{-1,1}$, where $P = \widetilde{P}/\pi_1(X)$ is a principal K-bundle on X. The pair (P, θ) is a uniformizing system of Hodge bundles on X for the Hodge group G_0. Since we also have $\mathcal{T}_{\widetilde{X}} \cong \widetilde{P}' \times_{K_0} \mathfrak{g}^{-1,1}$, it follows that $\mathcal{T}_X \cong P' \times_{K_0} \mathfrak{g}^{-1,1}$, where $P' = \widetilde{P}'/\pi_1(X)$ is a principal K_0 bundle on X, and $P' \times_{K_0} K \cong P$. Note that P' is a metric for (P, θ).

Recall that $\widetilde{R}' = \widetilde{P}' \times_{K_0} G_0$ is a flat principal G_0-bundle on $\widetilde{X}$, and we can associate to it the system of Hodge bundles $\widetilde{E} = \widetilde{R}' \times_{G_0} \mathfrak{g}$ via the adjoint action of G_0 on $\mathfrak{g}$. We can view $\widetilde{E}$ as a K_0-bundle $\widetilde{E}' = \widetilde{P}' \times_{K_0} \mathfrak{g}$. As a K_0-bundle $\widetilde{E}'$ decomposes as a direct sum $\widetilde{E}' = \bigoplus_{i \in \{-1,0,1\}} \widetilde{P}' \times_{K_0} \mathfrak{g}^{i,-i}$. The flat connection ω_H on $\widetilde{R}'$ induces a flat connection ω'_H on the associated bundle $\widetilde{E}$. From [6, Proposition 13.1.1], there is a decomposition $\omega'_H = \widetilde{d}_H + \sigma + \bar{\sigma}$, where $\widetilde{d}_H$ is the Chern connection for the Hodge metric (see Definition 3.1) on $\widetilde{E}'$, $\sigma \in \mathcal{A}^{1,0}(\mathrm{End}(\widetilde{E}))$, and $\bar{\sigma}$ is the adjoint of σ with respect to the Hodge metric. By slight abuse of notation, let $\omega_H = \widetilde{d}_H + \sigma + \bar{\sigma}$ be the associated splitting as principal connections.

There is an action of $\pi_1(X)$ on $\widetilde{R}$ by automorphisms, which comes from the representation $\sigma : \pi_1(X) \to G_0$. This action is described explicitly in the proof of [31, Theorem 2.9]. The quotient $R' = (\widetilde{P}' \times_{K_0} G_0)/\pi_1(X) = P' \times_{K_0} G_0$ is a flat principal G_0-bundle on X by the correspondence of Theorem 3.3.

The system of Hodge bundles $E = P \times_K \mathfrak{g}$ on X admits a Hermitian metric H corresponding to the reduction of structure group P' of P from K to K_0. Let d'_H be the K_0-connection on P' which induces the Chern connection on E for H. Let θ' denote the connection on E corresponding to θ, $\bar{\theta}'$ the adjoint of θ' with respect to H, and σ', $\bar{\sigma}'$ the G_0 connections on R' which induce θ', $\bar{\theta}'$ respectively. Then $D_H = d'_H + \sigma' + \bar{\sigma}'$ is a G_0-connection on R', and the connections $D_H, d'_H, \sigma', \bar{\sigma}'$ pull back to $\widetilde{D}_H, \widetilde{d}_H, \sigma, \bar{\sigma}$ respectively. Recall that $\widetilde{D}_H$ is a flat connection on $\widetilde{R}'$. Since flatness can be checked locally, and X and $\widetilde{X}$ are locally diffeomorphic, it follows that D_H is a flat connection on R'.

Thus, (P, θ) together with the metric P' is a principal variation of Hodge structure on X for the Hodge group G_0. It is in fact a uniformizing variation because the differential θ is an isomorphism. □

3.3 Classical results of Simpson

In the case when X is a smooth compact complex manifold, Simpson derives the following necessary and sufficient conditions for X to be uniformized by a Hermitian symmetric space $\mathcal{D}$ of noncompact type.

Proposition 3.6 ([34, Proposition 9.1])**.** *Let X be a smooth compact complex manifold and let $\widetilde{X}$ be the universal cover of X. Then $\widetilde{X}$ is isomorphic to the bounded symmetric domain $\mathcal{D}$ if and only if X admits a uniformizing variation of Hodge structure for some Hodge group G_0 with $\mathcal{D} = G_0/K_0$.*

A more algebraic formulation of the above Proposition, which is more useful in practice, is the following result.

Theorem 3.7 ([34, Theorem 2])**.** *Let X be a smooth compact complex manifold. Then $\widetilde{X}$ is biholomorphic to $\mathcal{D}$ if and only if there is a uniformizing system of Hodge bundles (P, θ) for a Hodge group G_0 of Hermitian type corresponding to $\mathcal{D}$, such that $P \times_K \mathfrak{g}$ is polystable with respect to K_X as a Higgs bundle on X, and $c_2(P \times_K \mathfrak{g}) \cdot [K_X]^{n-2} = 0$.*

Theorem 3.7 turns out to be useful in formulating explicit necessary and sufficient conditions for a complex projective variety with klt singularities to be uniformized by each of the four classical Hermitian symmetric spaces $\mathcal{D}$ of non-compact type.

Remark 3.8. To determine necessary conditions for a projective variety X over $\mathbb{C}$ to be uniformized by a Hermitian symmetric space $\mathcal{D}$ of non-compact type, we must fix a Hodge group G_0 associated to $\mathcal{D}$. In general, choosing G_0 to be the connected component $\mathrm{Aut}^0(\mathcal{D})$ of the automorphism group of $\mathcal{D}$ will not give necessary conditions.

In general, choosing G_0 to be the full automorphism group of the boounded symmetric domain will always work, but we have a little more freedom. More precisely,

Lemma 3.9. *In order to determine necessary conditions for X to have a bounded symmetric domain $\mathcal{D}$ as its universal cover, we can choose the G_0 to be a cover of $Aut(\mathcal{D})$, such that the kernel of the covering map $\varphi : G_0 \to Aut(\mathcal{D})$ is a discrete central subgroup of G_0.*

Proof. Let K_0 and M_0 denote the maximal compact subgroups of G_0 and $Aut(\mathcal{D})$ respectively. Note that G_0 and $Aut(\mathcal{D})$ have the same Lie algebra, and the isomorphism $G_0/K_0 \cong Aut(\mathcal{D})/M_0$ is compatible with the map φ. This means that the image of K_0 via φ is M_0, and the kernel of $\varphi|_{K_0}$ is a discrete central subgroup of K_0.
Let K and M denote the complexifications of K_0 and M_0 respectively, which exist because K_0 and M_0 are compact. Then, M is a quotient of K by a discrete central subgroup. Note that K acts on the complexified Lie algebra $\mathfrak{g}$ of G_0 via the adjoint representation $Ad : K \to Aut(\mathfrak{g})$, $k \mapsto Ad_k = kXk^{-1}$, $X \in \mathfrak{g}$. The kernel of the adjoint representation Ad is the center of K. It follows that the action of K on $\mathfrak{g}$ factors through M. From the proof of Lemma 3.5, we know that the tangent bundle of X can be expressed as $\mathcal{T}_X \cong P \times_M \mathfrak{g}^{-1,1}$, where P is a principal M-bundle on X. Since the action of K on $\mathfrak{g}^{-1,1}$ factors through M, the tangent bundle of X can also be expressed as $\mathcal{T}_X \cong P' \times_K \mathfrak{g}^{-1,1}$, where $P' \cong P \times_M K$ is a principal K- bundle on X. This concludes the proof. □

4 Extending Simpson's result to the klt case

In this section, our goal is to prove Theorem 1.1. Henceforth, we work in the klt setting instead of the smooth setting. We first make some observations which will help us.

4.1 Auxiliary remarks

The following observation about the semistability of the tangent sheaf of a variety of general type is a slight generalization of a result of H.Guenancia ([18, Theorem A]). It allows us to do away with the assumption that the tangent sheaf has negative degree, which was made by Simpson in his uniformization result ([34, Corollary 9.7]).

Proposition 4.1 ([13], Theorem 7.1)**.** *Let X be a projective, klt variety of general type whose canonical divisor K_X is nef. Then the sheaves $\mathcal{T}_X$ and $\Omega_X^{[1]}$ are semistable with respect to K_X.*

In general, the (semi)stability of a sheaf on a normal, projective variety X in the sense of Definition 2.14 is equivalent to the (semi)stability of the sheaf restricted to the smooth locus X_{reg}. The following is a special case of [14, Lemma 2.26].

Lemma 4.2. *Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal, projective variety X and let H be a nef, $\mathbb{Q}$-Cartier divisor on X. Then $\mathcal{E}$ is semistable (resp. stable) with respect to H if and only if $\mathcal{E}|_{X_{reg}}$ is semistable (resp. stable) with respect to H.*

Proof. Let $j : X_{reg} \to X$ denote the inclusion. If for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$ we have $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$, then it follows that $\mu_H(\mathcal{F}|_{X_{reg}}) > \mu_H(\mathcal{E}|_{X_{reg}})$. Indeed, by Definition 2.13 we have $\mu_H(\mathcal{F}|_{X_{reg}}) = \mu_H(j_*\mathcal{F}|_{X_{reg}}) = \mu_H(\mathcal{F}^{**}) = \mu_H(\mathcal{F})$, and similarly $\mu_H(\mathcal{E}|_{X_{reg}}) = \mu_H(\mathcal{E})$.
Conversely, the same argument shows that if $\mu_H(\mathcal{F}|_{X_{reg}}) > \mu_H(\mathcal{E}|_{X_{reg}})$ then we have $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$. □

Remark 4.3. In particular, if X is a projective, klt variety of general type with K_X nef, the vector bundles $\mathcal{T}_{X_{reg}}$ and $\Omega^1_{X_{reg}}$ on X_{reg} are semistable with respect to K_X.

An important observation due to A. Langer is that the stability of a system of Hodge sheaves is equivalent to the stability of the underlying Higgs sheaf. This was first proved for systems of Hodge sheaves over a smooth projective variety X (see [26, Proposition 8.1]). The proof uses that systems of Hodge sheaves are fixed points of the $\mathbb{C}^*$-action on the moduli space of Higgs sheaves (see [35, Lemma 4.1]) on X. Moreover, [36, Lemma 6.8] says that a torsion free Higgs sheaf E on X is the same thing as a pure coherent sheaf $\mathcal{E}$ of dimension $n = \dim(X)$ on a projective completion Z of the cotangent bundle, such that the support of $\mathcal{E}$ does not meet the divisor at infinity. Then $\mathcal{E}$ being a $\mathbb{C}^*$-fixed point implies that the Quot scheme $Quot_Z(\mathcal{E})$ inherits a $\mathbb{C}^*$- action. Since $Quot_Z(\mathcal{E})$ is projective, the limit of the orbit of any point $\mathcal{F} \in Quot_Z(\mathcal{E})$ under the $\mathbb{C}^*$-action exists in $Quot_Z(\mathcal{E})$, and corresponds to a quotient system of Hodge sheaves of E on X that has the same numerical invariants as $\mathcal{F}$.
This result was generalized to the setting where X is normal and projective (see [27, Corollary 3.5]). It is shown that even in this more general setting, a torsion free Higgs sheaf is a fixed point of the $\mathbb{C}^*$-action if and only if it is a system of Hodge sheaves. To prove the result it is sufficient to show that the maximally destabilizing Higgs subsheaf of a system of Hodge sheaves is a system of Hodge sheaves. This follows from the fact that the maximally destabilizing Higgs subsheaf is unique, so it must be a fixed point of the $\mathbb{C}^*$-action. We give the following independent proof of this result for systems of Hodge sheaves on the smooth locus of a projective variety X with klt singularities.

Lemma 4.4. *Let X be a projective variety with klt singularities. Then the stability conditions for a torsion free system of Hodge sheaves on the smooth locus X_{reg} and the stability conditions of the underlying Higgs sheaf in the sense of Definition 2.14, are equivalent.*

Proof. Let $E = \bigoplus_{p=0}^{n} E^p$ be a torsion free system of Hodge sheaves on X_{reg} with Higgs field θ, i.e., we have $\theta : E^p \to E^{p-1} \otimes \Omega^1_{X_{reg}}$ for all $0 \le p \le n$. Let $F \subset E$ be a Higgs subsheaf. The idea is to construct a subsystem of Hodge sheaves $F' = \bigoplus_{p=0}^{n} F'^p$ of E such that F and F' have the same rank and first Chern class. For each i, define $G_i = F \cap \bigoplus_{p<i} E^p$, where $0 \le i \le n+1$. So for example $G_0 = 0$, $G_1 = F \cap E_0$, and $G_{n+1} = F \cap \bigoplus_{p<n+1} E^p = F$. Thus the $G_i's$ give a filtration of F

$$0 = G_0 \subset G_1 \subset \cdots \subset G_n \subset G_{n+1} = F.$$

Since $\theta(F) \subset F \otimes \Omega^1_{X_{reg}}$, and $\theta(\bigoplus_{p<i} E^p) \subset (\bigoplus_{p<i-1} E^p) \otimes \Omega^1_{X_{reg}}$, it follows that $\theta(G_i) \subset G_{i-1} \otimes \Omega^1_{X_{reg}}$ for all $0 \le i \le n+1$. Let $\{F'^i\}_{i=0}^{n}$ be the quotients of this filtration, i.e., $F'^i = G_{i+1}/G_i$ for all i. Consider the sequence of maps

$$G_{i+1} \hookrightarrow \bigoplus_{p<i+1} E^p \to E^i$$

where the first map is the inclusion and the second is projection. The kernel of the composed map is G_i, so the image of G_{i+1} in E^i is isomorphic to the quotient $F'^i = G_{i+1}/G_i$. Thus we have $F'^i \subset E^i$ for all $0 \le i \le n$. Moreover, since $\theta(G_i) \subset G_{i-1} \otimes \Omega^1_{X_{reg}}$, and tensoring with $\Omega^1_{X_{reg}}$ is exact, it follows that $\theta(F'^i) \subset F'^{i-1} \otimes \Omega^1_{X_{reg}}$ for all $0 \le i \le n$. Hence $F' = \bigoplus_{p=0}^{n} F'^p \subset \bigoplus_{p=0}^{n} E^p$ is a subsystem of Hodge sheaves. To see that F and F' have the same numerical invariants, we look at the series of short exact sequences

$$\begin{aligned}
&0 \to G_0 = 0 \to G_1 \to F'^0 \to 0\\
&0 \to G_1 \to G_2 \to F'^1 \to 0\\
&\vdots\\
&0 \to G_{n-1} \to G_n \to F'^{n-1} \to 0\\
&0 \to G_n \to G_{n+1} = F \to F'^n \to 0
\end{aligned}$$

From the first exact sequence it follows that $\operatorname{rank}(G_1) = \operatorname{rank}(F'^0)$. This implies that $\operatorname{rank}(G_2) = \operatorname{rank}(F'^0) + \operatorname{rank}(F'^1)$, and repeating this gives $\operatorname{rank}(G_i) = \operatorname{rank}(F'^0) + \cdots + \operatorname{rank}(F'^{i-1})$. From the last exact sequence we get $\operatorname{rank}(F) = \operatorname{rank}(F'^0) + \cdots + \operatorname{rank}(F'^n) = \operatorname{rank}(F')$. The same computation holds for first Chern classes, so we have $c_1(F) = c_1(F')$. □

The above result is true also for reflexive systems of Hodge sheaves on a normal variety in the sense of [28, Section 4]. More precisely, the stability conditions for a reflexive system of Hodge sheaves and the underlying reflexive Higgs sheaf in the sense of [28, Section 4] are equivalent. This is remarked in [28, Section 4.10].

Recall that any bounded symmetric domain $\mathcal{D}$ can be expressed as a quotient $\mathcal{D} = G_0/K_0$, where G_0 is a Hodge group of Hermitian type with maximal compact subgroup K_0. Their complexifications are denoted G and K respectively. The Lie algebra $\mathfrak{g}$ of G decomposes as $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$. We know from the discussion in Section 3.1 that any smooth, projective quotient X of $\mathcal{D}$ satisfies $\mathcal{T}_X \cong P \times_K \mathfrak{g}^{-1,1}$, where P is a principal K-bundle on X. Moreover, X admits a system of Hodge bundles $P \times_K \mathfrak{g}$ which has a Hodge decomposition induced from that of $\mathfrak{g}$.

We will now consider the system of Hodge bundles $P \times_K \mathfrak{g}$ over the smooth locus X_{reg} of a projective, klt variety X. The semistability of the bundle $P \times_K \mathfrak{g}^{0,0}$ is necessary to show that the system of Hodge bundles $P \times_K \mathfrak{g}$ is polystable as a Higgs bundle in the sense of Definition 2.14. This will play a central role in the proofs of statements that appear later. We observe the following.

Lemma 4.5. *Let X be a projective, klt variety of general type, and let (P, θ) define a uniformizing system of Hodge bundles for a Hodge group G_0 on X_{reg}. Then the vector bundle $P \times_K \mathfrak{g}^{0,0}$ is semistable with respect to K_X.*

Proof. Let $\mathcal{T}_{X_{reg}} = \bigoplus V_i$ be a decomposition corresponding to irreducible representations of $K \to \mathrm{Aut}(\mathfrak{g}^{-1,1})$. First suppose that the group G is connected. Let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ be the decomposition into simple ideals. Then $\mathfrak{g}^{-1,1} = \bigoplus \mathfrak{g}_i^{-1,1}$ is the decomposition into irreducible representations of K. Since $\mathcal{T}_{X_{reg}}$ is semistable by Proposition 4.1, it follows that the $V_i \cong P \times_K \mathfrak{g}_i^{-1,1}$ and their duals $P \times_K \mathfrak{g}_i^{1,-1}$ are also semistable. Since the Lie bracket $\mathfrak{g}_i^{-1,1} \otimes \mathfrak{g}_i^{1,-1} \to \mathfrak{g}_i^{0,0}$ is surjective (see [34, proof of Proposition 9.6]), we get a surjective map of vector bundles

$$(P \times_K \mathfrak{g}_i^{-1,1}) \otimes (P \times_K \mathfrak{g}_i^{1,-1}) \to P \times_K \mathfrak{g}_i^{0,0},$$

where the left hand side is a semistable vector bundle of degree zero and the right hand side is a vector bundle of degree zero. It follows that $P \times_K \mathfrak{g}_i^{0,0}$ is also semistable. Indeed, if it is not, then neither is the dual bundle $(P \times_K \mathfrak{g}_i^{0,0})^\vee$. Note that $(P \times_K \mathfrak{g}_i^{0,0})^\vee$ is also of degree zero, and is a subbundle of $(P \times_K \mathfrak{g}_i^{-1,1}) \otimes (P \times_K \mathfrak{g}_i^{1,-1})$. Thus, if some subsheaf of $(P \times_K \mathfrak{g}_i^{0,0})^\vee$ destabilizes it, then it also destabilizes $(P \times_K \mathfrak{g}_i^{-1,1}) \otimes (P \times_K \mathfrak{g}_i^{1,-1})$, which is a contradiction. Since $P \times_K \mathfrak{g}^{0,0} = \bigoplus_i P \times_K \mathfrak{g}_i^{0,0}$, it follows that $P \times_K \mathfrak{g}^{0,0}$ is also a semistable vector bundle of degree zero.

Now suppose that G is not connected. Let G' be the connected component of G, let $K' = K \cap G'$, and let $f' : Y' \to X_{reg}$ be a finite étale cover which the structure group of P can be reduced to K'. Note that Y' can be completed to a projective, klt variety Y such that K_Y is ample, and there is a finite quasi-étale map $f : Y \to X$, which restricts to f' on Y'. Since the V_i's are semistable with respect to K_X, the pullbacks f'^*V_i are semistable with respect to $f^*K_X = K_Y$. If we decompose $\mathcal{T}_Y = \bigoplus_k V'_k$ corresponding to irreducible components of the representation $K' \to \mathrm{Aut}(\mathfrak{g}^{-1,1})$, the V'_k's are direct summands of the f'^*V_i, so they are also semistable with respect to K_Y. Since $V'_k \cong f'^*(P \times_K \mathfrak{g}_k^{-1,1}) = f'^*P \times_{K'} \mathfrak{g}_k^{-1,1}$, and G' is connected, it follows that $f'^*(P \times_K \mathfrak{g}^{0,0})$ is semistable with respect to K_Y of degree zero. Since f' is finite and étale, it follows that $P \times_K \mathfrak{g}^{0,0}$ is also semistable with respect to K_X of degree zero. Indeed, if not, then the pullback along f' of any destabilizing subsheaf of $P \times_K \mathfrak{g}^{0,0}$ will destabilize $f'^*(P \times_K \mathfrak{g}^{0,0})$. □

The following observation is at the heart of the proof of Theorem 1.1, and we again split the proof into two cases, namely G connected and disconnected.

Proposition 4.6. *Let X be a projective, klt variety with ample canonical divisor K_X and let (P, θ) define a uniformizing system of Hodge bundles for a Hodge group G_0 on X_{reg}. Then the system of Hodge bundles $P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}. If $\mathfrak{g}$ is a simple Lie algebra, then it is K_X-stable.*

Remark 4.7. The system of Hodge bundles $P \times_K \mathfrak{g}$ on X_{reg} being K_X-polystable is equivalent to the restriction $(P \times_K \mathfrak{g})|_{X'}$ being K_X-polystable on X', for any big open subset $X' \subset X_{reg}$. Thus we may replace X_{reg} by any big open subset X' in Proposition 4.6.

Proof of Proposition 4.6. First, suppose that the complexification G of G_0 is connected. By assumption, there is an isomorphism of vector bundles $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$ on X_{reg}. Write $\mathcal{T}_{X_{reg}} = \bigoplus_i V_i$, where each V_i corresponds to an irreducible component of the representation $K \hookrightarrow \mathrm{Aut}(\mathfrak{g}^{-1,1})$. We know from Remark

4.3 that $\mathcal{T}_{X_{reg}}$ is semistable with respect to K_X, and since K_X is ample by assumption, $\mathcal{T}_{X_{reg}}$ has negative slope with respect to K_X. It follows that all direct summands V_i are semistable with respect to K_X and have the same negative slope as $\mathcal{T}_{X_{reg}}$.

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a decomposition of $\mathfrak{g}$ as a direct sum of simple ideals. Then each $P \times_K \mathfrak{g}_i$ is a subsystem of Hodge bundles of $P \times_K \mathfrak{g}$ (see [34, p.903]), and we have a decomposition $P \times_K \mathfrak{g} = \bigoplus_i P \times_K \mathfrak{g}_i$. Note that if G is not connected, we do not in general have a global decomposition of $P \times_K \mathfrak{g}$ into the bundles $P \times_K \mathfrak{g}_i$ associated to the simple ideals $\mathfrak{g}_i$ of $\mathfrak{g}$. Let $\mathfrak{g}^{-1,1} = \bigoplus_i \mathfrak{g}_i^{-1,1}$ be the decomposition into irreducible representations of $K \subset \mathrm{Aut}(\mathfrak{g}^{-1,1})$. Since $V_i \cong P \times_K \mathfrak{g}_i^{-1,1}$ for all i, the bundles $P \times_K \mathfrak{g}_i^{-1,1}$ and their duals $P \times_K \mathfrak{g}_i^{1,-1}$ are semistable with respect to K_X. From the proof of Lemma 4.5, we know that the bundle $P \times_K \mathfrak{g}_i^{0,0}$ is semistable with respect to K_X of degree zero. Thus each $P \times_K \mathfrak{g}_i$ is a system of Hodge bundles on X_{reg} of degree zero.

By Lemma 4.4, to show that $P \times_K \mathfrak{g}$ is polystable as a Higgs bundle, it is sufficient to show that it is polystable as a system of Hodge bundles on X_{reg}. Let $W \subset P \times_K \mathfrak{g}_i$ be a subsystem of Hodge sheaves. Then we know from Lemma 2.23 that $\mathrm{rank}(W^{-1,1}) \geq \mathrm{rank}(W^{1,-1})$, i.e., $\mu_{K_X}(W) \leq 0$, and if equality holds then $W = P \times_K \mathfrak{g}_i$, because $\mathfrak{g}_i$ is simple. Thus each $P \times_K \mathfrak{g}_i$ is K_X-stable as a Higgs bundle of degree zero, and since $P \times_K \mathfrak{g} = \bigoplus_i P \times_K \mathfrak{g}_i$, it follows that $P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}, of degree zero. Note that if $\mathfrak{g}$ is a simple Lie algebra, then the only non trivial ideal of $\mathfrak{g}$ is itself. So in this case, $P \times_K \mathfrak{g}$ is even K_X-stable as a Higgs bundle on X_{reg}.

Now suppose G is not connected. Let G' denote the connected component of G and let $K' = K \cap G'$. Let $f : Y' \to X_{reg}$ be a finite étale cover over which the structure group of P can be reduced from K to K', and let Y be a normal projective completion of Y' such that f can be extended to a quasi-étale map $\gamma : Y \to X$. Note that Y is again klt, $Y' \subset Y_{reg}$ is a big open subset, and $K_Y = \gamma^* K_X$ is still ample. Thus $\mathcal{T}_{Y'}$ is semistable with respect to K_Y of negative degree.

We can write $\mathcal{T}_{Y'} = f^*\mathcal{T}_{X_{reg}} = f^*(P \times_K \mathfrak{g}^{-1,1}) = f^*P \times_{K'} \mathfrak{g}^{-1,1}$. Then $\mathcal{T}_{Y_{reg}} = P' \times_{K'} \mathfrak{g}^{-1,1}$, where P' is a principal K'-bundle on Y_{reg} such that $P'|_{Y'} = f^*P$. Since G' is connected, we know that the system of Hodge bundles $P' \times_{K'} \mathfrak{g}$ is K_Y-polystable as a Higgs bundle on Y_{reg}. Therefore by Remark 4.7, $f^*(P \times_K \mathfrak{g})$ is K_Y-polystable as a Higgs bundle on Y'.

If $W \subset P \times_K \mathfrak{g}$ is any saturated subsystem of Hodge sheaves with $\deg(W) \geq 0$, then f^*W is a subsystem of Hodge sheaves of $f^*(P \times_K \mathfrak{g})$ of degree zero, and is therefore a direct summand of $f^*(P \times_K \mathfrak{g})$ by [34, Proposition 3.3]. This implies that locally on Y', f^*W is a sub-Hodge structure of $\mathfrak{g}$. Then by Lemma 2.23, f^*W is locally a direct sum of simple ideals of $\mathfrak{g}$, i.e., f^*W is locally a direct sum of the $P \times_{K'} \mathfrak{g}_i$. Hence the same holds for W on X_{reg}. From the discussion in [34, p.903], there is a unique finest global decomposition

$$P \times_K \mathfrak{g} = \bigoplus_j \mathcal{E}_j \tag{4}$$

on X_{reg} such that each $\mathcal{E}_j$ is a subsystem of Hodge bundles of $P \times_K \mathfrak{g}$, and is locally a direct sum of simple ideals of $\mathfrak{g}$. We want to show that each $\mathcal{E}_j$ in the decomposition 4 is K_X-stable as a system of Hodge bundles of degree zero on X_{reg}.

There is similarly a finest global decomposition $f^*P \times_{K'} \mathfrak{g} = \bigoplus_k \mathcal{E}'_k$ on Y', and since $f^*P \times_{K'} \mathfrak{g}$ is K_Y-polystable as a system of Hodge bundles of degree zero, each $\mathcal{E}'_k$ is K_Y-stable as a system of Hodge bundles of degree zero. The pullbacks $f^*\mathcal{E}_j$ are direct sums of the $\mathcal{E}'_k$, thus each $\mathcal{E}_j$ also has degree zero.

Now we argue as in the proof of [34, Corollary 9.4]. If $W \subset \mathcal{E}_j$ is any saturated subsystem of Hodge sheaves of degree ≥ 0, then W must have degree zero, and we know that W is a locally a direct sum of simple ideals of $\mathfrak{g}$. Therefore we must have $W = \mathcal{E}_j$ by minimality of the decomposition (4). Thus $\mathcal{E}_j$ is stable with respect to K_X as a system of Hodge bundles of degree zero. It follows that $P \times_K \mathfrak{g}$ is K_X-polystable as a system of

Hodge bundles, which is equivalent to $P \times_K \mathfrak{g}$ being K_X-polystable as a Higgs bundle on X_{reg}, by Lemma 4.4. $\square$

We are now ready to prove Theorem 1.1.

4.2 Proof of Theorem 1.1

We split the proof into two steps, in order to make it more readable. The first step is to show that if $X \cong \mathcal{D}/\Gamma$, then X satisfies the two conditions of Theorem 1.1. The second step is to prove the converse.

Proof. **Step I**. Suppose we can write $X \cong \mathcal{D}/\Gamma$, for a Hermitian symmetric space $\mathcal{D}$ of noncompact type. Let $G_0 = \mathrm{Aut}(\mathcal{D})$ be the full automorphism group of $\mathcal{D}$, then $\Gamma \subset G_0$. In this case G_0 is a Hodge group of Hermitian type, and K_0 is a maximal compact subgroup of G_0. Let G and K denote complexifications of G_0 and K_0 respectively. Since the smooth locus X_{reg} is (the analytic space associated to) a quasi-projective variety, its fundamental group $\pi_1(X_{reg})$ is finitely generated, and isomorphic to Γ. Then from Selberg's lemma (see [1]), it follows that Γ admits a normal torsion free subgroup $\widehat{\Gamma}$ of finite index. Thus the quotient map $\mathcal{D} \to \mathcal{D}/\Gamma = X$ factors as

$$\mathcal{D} \xrightarrow{\pi} \mathcal{D}/\widehat{\Gamma} \xrightarrow{\gamma} \mathcal{D}/\Gamma = X,$$

where $\mathcal{D}/\widehat{\Gamma} = Y$ is smooth and projective because $\widehat{\Gamma}$ acts freely and cocompactly on $\mathcal{D}$. The map $\gamma : Y \to X$ is quasi-étale and Galois with group $H = \Gamma/\widehat{\Gamma}$. Recall from Lemma 3.4 that the holomorphic tangent bundle of $\mathcal{D}$ satisfies $\mathcal{T}_{\mathcal{D}} \cong \tilde{P} \times_K \mathfrak{g}^{-1,1}$, where $\tilde{P}$ is a principal K-bundle on $\mathcal{D}$. Since Y is uniformized by $\mathcal{D}$, it follows from the classical Theorem 3.7 that Y admits a uniformizing system of Hodge bundles (P', θ) for the Hodge group G_0, such that $c_1(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-1} = c_2(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-2} = 0$. There is hence an isomorphism of vector bundles $\mathcal{T}_Y \cong P' \times_K \mathfrak{g}^{-1,1}$ on Y, where $P' = \tilde{P}/\widehat{\Gamma}$.
By the purity of branch locus, $\gamma : Y \to X$ branches only over the singular locus of X. Let $Y^o = \gamma^{-1}(X_{reg})$. Then $\gamma|_{Y^o} : Y^o \to X_{reg}$ is étale, and $Y^o \subset Y$ is a big open subset. We have $\mathcal{T}_{Y^o} = \mathcal{T}_Y|_{Y^o} = P'' \times_K \mathfrak{g}^{-1,1}$, where $P'' = P'|_{Y^o}$. The action of H on Y restricts to a free action on Y^o and lifts to a left action on $\mathcal{T}_{Y^o}$. On X_{reg}, we have $\mathcal{T}_{X_{reg}} \cong \mathcal{T}_{Y^o}/H$, and the isomorphism $\mathcal{T}_{Y^o} \cong P'' \times_K \mathfrak{g}^{-1,1}$ is H-equivariant because the isomorphism $\mathcal{T}_{\mathcal{D}} \cong \tilde{P} \times_K \mathfrak{g}^{-1,1}$ is Γ-equivariant. Thus there is an isomorphism $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$, where $P \cong P''/H$ is a principal K-bundle on X_{reg}. It follows that (P, θ) is a uniformizing system of Hodge bundles on X_{reg}.
Let $\mathcal{E}$ be the system of Hodge bundles $P \times_K \mathfrak{g}$ on X_{reg}, and let $\mathcal{E}'$ denote the unique extension of $\mathcal{E}$ to X as a reflexive sheaf. Then, $\gamma^{[*]}\mathcal{E}' \cong P' \times_K \mathfrak{g}$ on Y, because $\gamma^{[*]}\mathcal{E}$ and $P' \times_K \mathfrak{g}$ are both reflexive and agree on the big open subset Y^o of Y. Moreover, we have $K_Y = \gamma^* K_X$. From the behaviour of $\mathbb{Q}$-Chern classes under quasi-étale covers (Lemma 2.30), it follows that

$$(a)\ c_1(\mathcal{E}') \cdot [K_X]^{n-1} = (\deg(\gamma))^{-1} c_1(\gamma^{[*]}\mathcal{E}') \cdot [\gamma^* K_X]^{n-1} = (\deg(\gamma))^{-1} c_1(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-1}$$
$$(b)\ \widehat{c}_2(\mathcal{E}') \cdot [K_X]^{n-2} = (\deg(\gamma))^{-1} \widehat{c}_2(\gamma^{[*]}\mathcal{E}') \cdot [\gamma^* K_X]^{n-2} = (\deg(\gamma))^{-1} c_2(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-2}.$$

The right hand sides of equalities (a) and (b) are zero, again by Theorem 3.7. Thus it follows that $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$. This proves one implication of Theorem 1.1.

Step II. Conversely, suppose that X is a projective klt variety which satisfies the two conditions of Theorem 1.1. Let G_0, K_0, G, and K be as in Step I. By assumption, we have an isomorphism of vector bundles $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$ on X_{reg}.
Let $\gamma : Y \to X$ be a Galois, maximally quasi-étale cover with Galois group H. The existence of such a cover

is known by [12, Theorem 1.14]. Note that Y is again klt (see [24, Proposition 5.20]), and $K_Y = \gamma^* K_X$ is ample. Since γ branches only over the singular locus of X, the restricted map $\gamma|_{Y^o} : Y^o \to X_{reg}$ is étale. Here $Y^o = \gamma^{-1}(X_{reg}) \subset Y_{reg}$, and note that Y^o is a big open subset of Y. Then $\mathcal{T}_{Y^o} = (\gamma|_{Y^o})^* \mathcal{T}_{X_{reg}} \cong (\gamma|_{Y^o})^*(P \times_K \mathfrak{g}^{-1,1}) = P' \times_K \mathfrak{g}^{-1,1}$, where $P' = (\gamma|_{Y^o})^* P$ is a principal K-bundle on Y^o. Thus there is an isomorphism of vector bundles $\theta' : \mathcal{T}_{Y_{reg}} \cong P'' \times_K \mathfrak{g}^{-1,1}$ on Y_{reg}, where P'' is a principal K-bundle on Y_{reg} such that $P''|_{Y^o} = P'$. Note that $\mathcal{T}_{Y_{reg}}$ has negative degree, and is semistable with respect to K_Y, again by Remark 4.3.

Let $\mathcal{F} = P'' \times_K \mathfrak{g}$, then $\mathcal{F}$ is a system of Hodge bundles on Y_{reg} with Higgs field coming from θ'. By Proposition 4.6, we know that $\mathcal{F}$ is K_Y-polystable as a Higgs bundle on Y_{reg}, and of degree zero. Let $\mathcal{F}'$ and $\mathcal{E}'$ be the unique extensions of $\mathcal{F}$ to Y and of $P \times_K \mathfrak{g}$ to X respectively, as reflexive sheaves. Since $\gamma^{[*]}\mathcal{E}'$ and $\mathcal{F}'$ agree on the big open subset Y^o, we have $\mathcal{F}' \cong \gamma^{[*]}\mathcal{E}'$. By assumption, $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$ holds, and by the behaviour of $\mathbb{Q}$-Chern classes under quasi-étale covers (Lemma 2.30), we have $\widehat{ch}_2(\mathcal{F}') \cdot [K_Y]^{n-2} = 0$. From Theorem 2.38, it follows that $(\mathcal{F}, \theta')$ is induced by a purely imaginary harmonic bundle whose associated flat bundle is semisimple, i.e., $(\mathcal{F}, \theta') \in \text{TPI-Higgs}_{Y_{reg}}$. Since Y is already maximally quasi-étale, Proposition 2.37 implies that the reflexive extension $\mathcal{F}'$ of $\mathcal{F}$ is locally free. Since $\mathcal{T}_Y$ is a direct summand of $\mathcal{F}'$, it follows that $\mathcal{T}_Y$ is locally free. The solution to the Lipman-Zariski conjecture for klt spaces (see [11, Theorem 16.1]) asserts that Y is smooth. We may thus apply Simpson's classical Theorem 3.7 to conclude that Y is uniformized by $G_0/K_0 = \mathcal{D}$.

This means that $Y \cong \mathcal{D}/\Gamma'$, where Γ' is a discrete, cocompact, torsion free group of automorphisms of $\mathcal{D}$, and $X \cong Y/H$, where H acts fixed point freely in codimension one on Y by automorphisms. By arguments analogous to those in the proof of [13, Theorem 1.3], (specifically, step (1.3.3) $\Longrightarrow$ (1.3.1)), there is an exact sequence of groups $1 \to \Gamma' \to \Gamma \to H \to 1$, where Γ is a discrete, cocompact subgroup of $\mathrm{Aut}(\mathcal{D})$ acting properly discontinuously, and fixed point freely in codimension one, such that $X \cong \mathcal{D}/\Gamma$. This concludes the proof of Theorem 1.1. ☐

Corollary 4.8. *Let $\mathcal{D}$ be a Hermitian symmetric space of non-compact type, and let X be a projective klt quotient of $\mathcal{D}$ by a Γ as in Theorem 1.1, with K_X ample. Then for any $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, the conjugate variety X^σ is also a quotient of $\mathcal{D}$.*

Proof. The variety X^σ again has klt singularities and ample canonical divisor K_{X^σ}. From the proof of Theorem 1.1 we know that X admits a smooth, quasi-étale cover $\gamma : Y \to X$ such that Y is uniformized by $\mathcal{D}$. The conjugate variety $Y^\sigma = X^\sigma \times_X Y$ is again smooth, and by the result for smooth projective varieties ([34, Corollary 9.5]), it follows that Y^σ is also uniformized by $\mathcal{D}$. Thus Y^σ admits a uniformizing system of Hodge bundles $\mathcal{E}$ which is polystable with respect to K_Y, and has vanishing first and second Chern classes. The map $\gamma^\sigma : Y^\sigma \to X^\sigma$ is also quasi-étale and finite, since these properties are preserved under base change. Thus there is a uniformizing system of Hodge bundles $\mathcal{F}$ on X^σ_{reg} which is polystable with respect to K_{X^σ}, such that the reflexive extension $\mathcal{F}'$ of $\mathcal{F}$ to X^σ pulls back to $\mathcal{E}$, i.e., $(\gamma^\sigma)^{[*]}(\mathcal{F}') \cong \mathcal{E}$. By the behaviour of $\mathbb{Q}$-Chern classes under quasi-́etale covers (Lemma 2.30), the first and second $\mathbb{Q}$-Chern classes of $\mathcal{F}'$ vanish. Then we conclude by Theorem 1.1 that X^σ is uniformized by $\mathcal{D}$. ☐

Part II

Applications

We are now in good shape to apply Theorem 1.1 to the polydisk, and each of the four classical irreducible bounded symmetric domains. Before we begin, we remark that for each bounded symmetric domain $\mathcal{D}$, necessary and sufficient conditions for uniformization by $\mathcal{D}$ are formulated separately. Recall from Section 4 that to obtain such conditions, we must choose a Hodge group G_0 associated to $\mathcal{D}$.

Remark A. If we choose G_0 to be a cover of $\mathrm{Aut}(\mathcal{D})$ such that the kernel of the covering map is a discrete central subgroup of G_0, the sufficient conditions for uniformization by $\mathcal{D}$ are also necessary. However in some cases, such a choice of G_0 implies that the complexification G is disconnected, and this does not give meaningful (nice) conditions on the tangent bundle $\mathcal{T}_{X_{reg}}$. Thus in cases when G is disconnected, we formulate necessary conditions for uniformization by $\mathcal{D}$ up to a finite, quasi-étale cover.

5 Uniformization by a polydisk

We are now ready to state the first uniformisation result, which is an application of Theorem 1.1. This is an extension of [34, Corollary 9.7], to the klt setting. We make the steps of the proof of Theorem 1.1 explicit in this section and therefor do not use any of the results of Section 4.

5.1 Sufficient conditions

Theorem 5.1. *Let X be an n-dimensional projective klt variety of general type whose canonical divisor K_X is big and nef. Suppose that the tangent sheaf $\mathcal{T}_{X_{reg}}$ of the smooth locus X_{reg} splits as a direct sum*

$$\mathcal{T}_{X_{reg}} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \tag{5}$$

where each direct summand $\mathcal{L}_i$ is a line bundle. Then the canonical model X_{can} of X admits a Galois, quasi-étale cover $\gamma : Y \to X_{can}$, where Y is a smooth projective variety whose universal cover is the polydisk $\mathbb{H}^n$.

Let X be as in Theorem 5.1 and let E be a vector bundle on the smooth locus X_{reg} of X. Let $\mathcal{D}^1(E)$ denote the sheaf of differential operators $\Delta : E \to E$ of order ≤ 1 whose symbol $\sigma(\Delta)$ is scalar. The *Atiyah class* of E is the class $at(E)$ of the extension

$$0 \to \mathcal{E}nd(E) \to \mathcal{D}^1(E) \xrightarrow{\sigma} \mathcal{T}_X \to 0$$

in $H^1(X, \mathcal{E}nd(E) \otimes \Omega^1_X)$. We refer the reader to [22, p. 180] for further details.
In order to prove Theorem 5.1, we would like to define the Atiyah class of a reflexive sheaf $\mathcal{L}$ of rank one on X. This turns out to be the extension class of an exact sequence, whose restriction to X_{reg} is the exact sequence corresponding to the classical Atiyah class $at(\mathcal{L}|_{X_{reg}})$ of the line bundle $\mathcal{L}|_{X_{reg}}$. This construction is used to prove an analog of [3, Lemma 3.1], and may also be of independent interest.

Proposition 5.2. *Let X be a projective klt variety of dimension n and let $\mathcal{L}$ be a line bundle on the smooth locus X_{reg}. Consider the following short exact sequence of sheaves*

$$0 \to \mathcal{E}nd(\mathcal{L}) \to \mathcal{D}^1(\mathcal{L}) \xrightarrow{\sigma} \mathcal{T}_{X_{reg}} \to 0 \tag{6}$$

on X_{reg}, known as the Atiyah sequence of $\mathcal{L}$. Let $j : X_{reg} \to X$ be the natural inclusion map. Then, the sequence

$$0 \to j_*\mathcal{E}nd(\mathcal{L}) \to j_*\mathcal{D}^1(\mathcal{L}) \to j_*\mathcal{T}_{X_{reg}} \to 0 \tag{7}$$

is exact on X.

Proof. Note that since $\mathcal{L}$ is a line bundle on X_{reg}, we have $\mathcal{E}nd(\mathcal{L}) \cong \mathcal{O}_{X_{reg}}$. Moreover, since $\mathcal{O}_X$ and $j_*\mathcal{O}_{X_{reg}}$ are both reflexive and agree on X_{reg}, we have $j_*\mathcal{O}_{X_{reg}} \cong \mathcal{O}_X$. Let $Z \subset X$ be the subset of X consisting of points which are not quotient singularities. Since X is klt, it follows that Z has codimension ≥ 3 in X. Let $X' = X \setminus Z$ denote the complement of Z in X and let $i : X_{reg} \to X'$ denote the inclusion map. Note that X' is $\mathbb{Q}$-factorial and hence $i_*\mathcal{L}$ is a $\mathbb{Q}$-Cartier sheaf on X'.
The extension class $at(\mathcal{L})$ of the Atiyah sequence (6) on X_{reg} is an element of $\mathrm{Ext}^1(\mathcal{T}_{X_{reg}}, \mathcal{O}_{X_{reg}}) \cong H^1(X_{reg}, \Omega^1_{X_{reg}})$, and coincides with the first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$. Since $i_*\mathcal{L}$ is $\mathbb{Q}$-Cartier on X', we can associate to it a cohomology class $c_1(i_*\mathcal{L}) \in H^1(X', \Omega^1_{X'}) \cong \mathrm{Ext}^1(\mathcal{O}_{X'}, \Omega^1_{X'})$, which we call the first Chern class of $i_*\mathcal{L}$, (see [17, Section 4]). Moreover, $c_1(i_*\mathcal{L})$ corresponds to the extension class of a short exact sequence in $\mathrm{Ext}^1(\mathcal{O}_{X'}, \Omega^1_{X'})$ which is locally split, hence the dual sequence of this exact sequence is also exact and locally split (see [17, Section 4]). Since this dual sequence is precisely the pushforward along i_* of the Atiyah sequence (6), we have a short exact sequence

$$0 \to \mathcal{O}_{X'} \to i_*\mathcal{D}^1(\mathcal{L}) \to \mathcal{T}_{X'} \to 0 \tag{8}$$

on X'. Let $i' : X' \to X$ denote the inclusion map. To complete the proof we want the pushforward of the exact sequence (8) along i'_* to stay exact.
Since the direct image functor is left exact it follows that the sequence

$$0 \to \mathcal{O}_X \to j_*\mathcal{D}^1(\mathcal{L}) \to \mathcal{T}_X \to R^1 j_*\mathcal{O}_{X_{reg}}$$

is exact on X.
From [19, Corollary 1.9], we know that $\mathcal{H}^{p+1}_Z(\mathcal{O}_X) \cong R^p i'_*\mathcal{O}_{X'}$, for all $p > 0$, so in particular, we have $\mathcal{H}^2_Z(\mathcal{O}_X) \cong R^1 j_*\mathcal{O}_{X_{reg}}$. In order to show that the sequence (7) is exact it suffices to work affine locally, so we may assume $X = \mathrm{Spec}(A)$, for some Noetherian ring A. Let $\mathcal{I} \subset \mathcal{O}_X$ denote the ideal sheaf associated to the closed subset $Z \subset X$. Let $x \in Z$ be any point. Then we have by [10, Lemma 2.4] that

$$\mathrm{depth}_Z\mathcal{O}_X = \mathrm{inf}_{x\in Z}(\mathrm{depth}_{\mathcal{I}_x}\mathcal{O}_{X,x}) = \mathrm{inf}_{\mathcal{I}_x \supset \mathfrak{p}}(\mathrm{depth}(\mathcal{O}_{X,\mathfrak{p}}))$$

where $\mathcal{I}_x$ denotes the localization of $\mathcal{I}$ at x, $\mathfrak{p}$ is a prime ideal such that $\mathcal{I}_x \subset \mathfrak{p} \subset \mathcal{O}_{X,x}$, and $\mathrm{depth}(\mathcal{O}_{X,\mathfrak{p}})$ denotes the depth of the local ring $\mathcal{O}_{X,\mathfrak{p}}$ over its maximal ideal. By the assumptions on X, we know that X is a Cohen-Macaulay scheme, which means that every local ring $\mathcal{O}_{X,x}$ of X is Cohen-Macaulay, and the same holds for $\mathcal{O}_{X,\mathfrak{p}}$, where $\mathfrak{p} \subset \mathcal{O}_{X,x}$ is any prime ideal. The Cohen-Macaulay condition implies that $\mathrm{depth}(\mathcal{O}_{X,\mathfrak{p}}) = \dim(\mathcal{O}_{X,\mathfrak{p}}) = \mathrm{height}(\mathfrak{p})$. Since the codimension of Z in X is ≥ 3, we have $\dim(\mathcal{O}_{X,\mathfrak{p}}) = \mathrm{height}(\mathfrak{p}) \geq 3$, i.e., $\mathrm{depth}(\mathcal{O}_{X,\mathfrak{p}}) \geq 3$ for all primes $\mathfrak{p}$ such that $\mathcal{I}_x \subset \mathfrak{p} \subset \mathcal{O}_{X,x}$, $x \in Z$. It follows that $\mathrm{depth}_Z\mathcal{O}_X = \mathrm{inf}_{\mathcal{I}_x \supset \mathfrak{p}}(\mathrm{depth}(\mathcal{O}_{X,\mathfrak{p}})) \geq 3$. Thus by [19, Theorem 3.8], we have $\mathcal{H}^p_Z(\mathcal{O}_X) = 0$ for $p < 3$, i.e., in particular $\mathcal{H}^2_Z(\mathcal{O}_X) = 0$. This implies that $R^1 i'_*\mathcal{O}_{X'} = R^1 j_*\mathcal{O}_{X_{reg}} = 0$, and hence that the sequence (7) is exact on X. □

Lemma 5.3. *Suppose X satisfies the assumptions in Theorem 5.1, and let $\mathcal{L}_i$ be a direct summand of $\mathcal{T}_X$ for each $1 \leq i \leq n$. Then we have $\widehat{c}_1(\mathcal{L}_i)^2 = 0$ for each $1 \leq i \leq n$.*

Proof. Recall from Section 2.2, that for a reflexive sheaf $\mathcal{L}_i$ on X, $\widehat{c}_1(\mathcal{L}_i)^2 : N_1(X)_{\mathbb{Q}}^{\times(n-2)} \to \mathbb{Q}$ is a $\mathbb{Q}$-multilinear form which maps a tuple $(\alpha_1, ..., \alpha_{n-2}) \in N^1(X)_{\mathbb{Q}}^{\times(n-2)}$ to $\widehat{c}_1(\mathcal{L}_i)^2 \cdot \alpha_1 \dots \alpha_{n-2} \in \mathbb{Q}$, such that the properties listed in [13, Theorem 3.13], hold.
Recall also that K_X is big and nef, so that we can choose a sufficiently increasing and divisible sequence of numbers $0 << m_1 << \dots << m_{n-2}$ and a general tuple of elements $(H_1, ..., H_{n-2}) \in \prod_i |m_i K_X|$, and set $S = H_1 \cap \dots \cap H_{n-2}$. Note that S has only quotient singularities and hence is contained entirely in $X' = X \setminus Z$, where $Z \subset X$ is the subset of X containing points which are not quotient singularities.
In order to show $\widehat{c}_1(\mathcal{L}_i)^2 = 0$, it suffices to show $\widehat{c}_1(\mathcal{L}_i)^2 \cdot S = 0$ for every surface S of complete intersection constructed as above. Since $S \subset X'$, the latter equality is equivalent to $\widehat{c}_1(\mathcal{L}_i|_{X'})^2 \cdot S = 0$. Since X' is $\mathbb{Q}$-factorial, $\mathcal{L}_i|_{X'}$ is $\mathbb{Q}$-Cartier and hence has an associated first Chern class $c_1(\mathcal{L}_i|_{X'}) \in H^1(X', \Omega^1_{X'})$. This cohomology class can be identified with an element in $H^2(X', \mathbb{Q})$, which we also denote by $c_1(\mathcal{L}_i|_{X'})$. Thus we have $\widehat{c}_1(\mathcal{L}_i|_{X'})^2 \cdot S = c_1(\mathcal{L}_i|_{X'})^2 \cdot S$, where the latter intersection product is computed as the cup product of $c_1(\mathcal{L}_i|_{X'})^2 \in H^4(X', \mathbb{Q})$ and $[S] \in H^{2(n-2)}(X', \mathbb{Q})$. Choose an integer m large enough so that $(\mathcal{L}_i|_{X'})^{\otimes m}$ is locally free. Then $c_1(\mathcal{L}_i|_{X'})^2 \cdot S = 0$ implies that $c_1((\mathcal{L}_i|_{X'})^{\otimes m})^2 \cdot S = 0$ and conversely, hence we may assume for the remainder of the proof that $\mathcal{L}_i|_{X'}$ is locally free.
Recall that the first Chern class $c_1(\mathcal{L}_i|_{X'})$ corresponds to the extension class $\widehat{at}(\mathcal{L}_i|_{X'})$ of the exact sequence

$$0 \to \mathcal{O}_{X'} \to i_*\mathcal{D}^1(\mathcal{L}_i|_{X_{reg}}) \to \mathcal{T}_{X'} \to 0 \tag{9}$$

in $H^1(X', \Omega^1_{X'})$. We know from [3, Lemma 3.1], that the above exact sequence restricted to X_{reg} is exact and splits over the sub-bundle $\mathcal{F} = \bigoplus_{j \neq i} \mathcal{L}_j|_{X_{reg}} \subset \mathcal{T}_{X_{reg}}$, hence $at(\mathcal{L}_i|_{X_{reg}}) \in H^1(X_{reg}, \Omega^1_{X_{reg}})$ vanishes in $H^1(X_{reg}, \mathcal{F}^*)$ and comes from $H^1(X_{reg}, \mathcal{L}_i^*|_{X_{reg}})$.
The pushforward of the $\mathcal{O}_{X_{reg}}$-linear map $\mathcal{F} \to \mathcal{D}^1(\mathcal{L}_i|_{X_{reg}})$ along the inclusion $i : X_{reg} \to X$ gives an $\mathcal{O}_{X'}$-linear map $i_*\mathcal{F} \to i_*\mathcal{D}^1(\mathcal{L}_i|_{X_{reg}})$. Over an open subset $U \subset X'$, this map sends a local section $u \in H^0(U, i_*\mathcal{F}) \cong H^0(U_{reg}, \mathcal{F})$ to $D_u \in H^0(U, i_*\mathcal{D}^1(\mathcal{L}_i|_{X_{reg}})) \cong H^0(U_{reg}, \mathcal{D}^1(\mathcal{L}_i|_{X_{reg}}))$, where $U_{reg} = i^{-1}U = U \cap X_{reg}$. Moreover, the pushed forward symbol map $i_*\sigma : i_*\mathcal{D}^1(\mathcal{L}_i|_{X_{reg}}) \to \mathcal{T}_{X'}$ maps D_u to u, for every local section u of $i_*\mathcal{F}$. Hence we see that the exact sequence (9) splits over the subsheaf $i_*\mathcal{F}$ of $\mathcal{T}_{X'}$. Thus its extension class $\widehat{at}(\mathcal{L}_i|_{X'}) \in H^1(X', \Omega^1_{X'})$ vanishes in $H^1(X', (i_*\mathcal{F})^*)$, hence comes from $H^1(X', (\mathcal{L}_i|_{X'})^*)$. Since $(\mathcal{L}_i|_{X'})^*$ is a locally free sheaf of rank 1 on X', we have $\widehat{at}(\mathcal{L}_i|_{X'})^2 \in H^2(X', \bigwedge^2(\mathcal{L}_i|_{X'})^*) = 0$. This implies $c_1(\mathcal{L}_i|_{X'})^2 = 0$, and hence $\widehat{c}_1(\mathcal{L}_i)^2 \cdot S = 0$, for every surface $S \subset X$ of complete intersection constructed as earlier. ☐

The proof of Theorem 5.1 is based on the following key auxiliary Propositions.

Proposition 5.4. *Let X be an n-dimensional projective, klt variety of general type whose canonical divisor K_X is big and nef. Suppose that the tangent sheaf $\mathcal{T}_{X_{reg}}$ of the smooth locus X_{reg} splits as in (5). Then, the tangent sheaf of the smooth locus of the canonical model X_{can} of X also splits as in (5), where each direct summand is a line bundle of negative degree.*

Proof. Let $\pi : X \to X_{can}$ be the birational crepant morphism to the canonical model X_{can}. We know from [15] that X_{can} is also projective and klt and its canonical divisor $K_{X_{can}}$ is ample.
Let $U \subset X$ be the largest open subset of X restricted to which π is an isomorphism. Let $V \subset X_{can}$ be the open subset defined as the intersection $V = \pi(U) \cap X_{can,reg}$, where $X_{can,reg}$ denotes the smooth locus of X_{can}. Since $X_{can} \setminus \pi(U)$ is a codimension ≥ 2 subset of X_{can}, it follows that $X_{can,reg} \setminus V$ is a codimension ≥ 2 subset of $X_{can,reg}$. The subset $X_{reg} \setminus \pi^{-1}(V)$ is a codimension ≥ 1 subset of X_{reg}.
The restriction $\mathcal{T}_{X_{reg}}|_{\pi^{-1}(V)} = \mathcal{T}_{\pi^{-1}(V)}$ splits as a direct sum of line bundles because we have by assumption that $\mathcal{T}_{X_{reg}} \cong \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_n$, where each $\mathcal{L}_i$ is a line bundle. Since we have $\pi^{-1}(V) \cong V$, it follows that the

corresponding tangent bundle $\mathcal{T}_V$ of V also splits as a direct sum of line bundles, say $\mathcal{T}_V = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$. Let $\mathcal{M}'_i$ denote the unique extension of $\mathcal{M}_i$ as a reflexive sheaf of rank 1 over $X_{can,reg}$ for all $1 \leq i \leq n$. Then by uniqueness of the extensions, we have that $\mathcal{M}'_1 \oplus \cdots \oplus \mathcal{M}'_n \cong \mathcal{T}_{X_{can,reg}}$. Since $X_{can,reg}$ is smooth, $\mathcal{T}_{X_{can,reg}}$ must be locally free. This implies that the summand $\mathcal{M}'_i$ must be locally free, i.e. a line bundle, for all $1 \leq i \leq n$.

We know from Proposition 4.1 that the tangent sheaf $\mathcal{T}_{X_{can}}$ is semistable with respect to $K_{X_{can}}$. Since $K_{X_{can}}$ is ample, we have $[K_{X_{can}}]^n > 0$ and it follows that

$$\mu_{K_{X_{can}}}(\mathcal{T}_{X_{can}}) = \frac{[\det(\mathcal{T}_{X_{can}})] \cdot [K_{X_{can}}]^{n-1}}{\operatorname{rank}(\mathcal{T}_{X_{can}})} = -\frac{[K_{X_{can}}]^n}{n} < 0.$$

Since $\mathcal{T}_{X_{can,reg}}$ decomposes as a direct sum of line bundles, it follows that $\mathcal{T}_{X_{can}}$ decomposes as a direct sum of rank 1 reflexive sheaves on X_{can}. Namely, $\mathcal{T}_{X_{can}} \cong \bigoplus_{i=1}^n j_*\mathcal{M'}_i$, where $j : X_{can,reg} \to X_{can}$ denotes the inclusion. For each direct summand $j_*\mathcal{M}'_i$ of $\mathcal{T}_{X_{can}}$, we have $\mu_{K_{X_{can}}}(j_*\mathcal{M}'_i) \leq \mu_{K_{X_{can}}}(\mathcal{T}_{X_{can}}) < 0$ by the semistability of $\mathcal{T}_{X_{can}}$. This implies that $c_1(j_*\mathcal{M}'_i) \cdot [K_{X_{can}}]^{n-1} < 0$ for all $1 \leq i \leq n$. Note that $[K_{X_{can}}]^{n-1}$ corresponds to the class of a smooth curve in X_{can} because $K_{X_{can}}$ is ample. Hence $c_1(\mathcal{M}'_i) \cdot [K_{X_{can}}]^{n-1} = c_1(j_*\mathcal{M}'_i) \cdot [K_{X_{can}}]^{n-1} < 0$ for all $1 \leq i \leq n$. Thus the tangent bundle $\mathcal{T}_{X_{can,reg}}$ of $X_{can,reg}$ splits as a direct sum of line bundles of negative degree. □

Proposition 5.5. *Let X be an n-dimensional projective klt variety such that the tangent sheaf $\mathcal{T}_{X_{reg}}$ of the regular locus X_{reg} splits as a direct sum of line bundles as in (5). Let $\gamma : Y \to X$ be a Galois, quasi-étale cover. Then, the tangent sheaf $\mathcal{T}_{Y_{reg}}$ of the regular locus Y_{reg} also splits as a direct sum of line bundles.*

Proof. Note that since $\gamma : Y \to X$ is quasi-étale, Y is again projective and klt of dimension n. By purity of branch locus, we know that γ branches only over the singular locus of X. It follows that $\gamma^{-1}(X_{reg}) = Y^o$, where Y^o denotes the open subset of Y restricted to which the map γ is étale. Note that $Y^o \subset Y_{reg}$, because any singular point of Y necessarily belongs to the branching locus of γ. Since γ is finite, the codimension of the branching locus of γ in Y is equal to the codimension of the singular locus of X in X. This implies that Y^o is a big open subset of Y, and in particular of the smooth locus Y_{reg}.

Since the restricted map $\gamma : Y^0 \to X_{reg}$ is finite and étale, we have $K_{Y^o} \cong \gamma^* K_{X_{reg}}$, and $\mathcal{T}_{Y^o} \cong \gamma^*\mathcal{T}_{X_{reg}} \cong \gamma^*\mathcal{L}_1 \oplus \cdots \oplus \gamma^*\mathcal{L}_n$, where the $\mathcal{L}_i$'s are the line bundles appearing in the direct sum decomposition of $\mathcal{T}_{X_{reg}}$. Let $\mathcal{L}'_i$ denote the unique extension of $\gamma^*\mathcal{L}_i$ as a reflexive sheaf over Y_{reg}, for all $1 \leq i \leq n$. Then by uniqueness of the reflexive extension, we have $\mathcal{L}'_1 \oplus \cdots \oplus \mathcal{L}'_n \cong \mathcal{T}_{Y_{reg}}$. Since Y_{reg} is smooth, $\mathcal{T}_{Y_{reg}}$ is locally free, which implies that each $\mathcal{L}'_i$ is locally free, i.e. a line bundle for all $1 \leq i \leq n$. □

Proposition 5.6. *Let X be a projective, klt variety of dimension n, and let the canonical divisor K_X be ample. Suppose the tangent sheaf $\mathcal{T}_{X_{reg}}$ of the smooth locus of X splits as a direct sum of line bundles as in (5), and suppose that X is maximally quasi-étale, i.e., there is an isomorphism of étale fundamental groups $\widehat{\pi}_1(X_{reg}) \cong \widehat{\pi}_1(X)$. Then X is smooth.*

Proof. Since $\mathcal{T}_{X_{reg}} \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, and we have assumed K_X to be ample, the semistability of $\mathcal{T}_{X_{reg}}$ with respect to K_X (Remark 4.3) implies that each line bundle $\mathcal{L}_i$ appearing in the direct sum decomposition of $\mathcal{T}_{X_{reg}}$ has negative degree i.e., $c_1(\mathcal{L}_i) \cdot [K_X]^{n-1} < 0$ for all $1 \leq i \leq n$.

Let $G_0 = SL(2, \mathbb{R})^n$, then G_0 is a connected Hodge group corresponding to the polydisk $\mathbb{H}^n$, which is a Hermitian symmetric space of noncompact type. Let $G = SL(2, \mathbb{C})^n$, a complexification of G_0, and let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})^n$, the Lie algebra of G. Explicitly, $\mathfrak{g}$ consists of n-tuples of trace zero 2×2 complex matrices

$\left(\begin{bmatrix}\alpha_j & \beta_j \\ \gamma_j & -\alpha_j\end{bmatrix}\right)_{j=1}^n$. Since G_0 is a Hodge group of *Hermitian type*, $\mathfrak{g}$ has the following Hodge decomposition

$$\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1},$$

with $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{q,-q}] \subset \mathfrak{g}^{p+q,-p-q}$ for $p, q \in \{-1, 0, 1\}$, where $[\cdot,\cdot]$ denotes the Lie bracket of $\mathfrak{g}$. The summands $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$ consist of n-tuples of the form $\left(\begin{bmatrix}0 & \beta_j \\ 0 & 0\end{bmatrix}\right)_{j=1}^n$, $\left(\begin{bmatrix}\alpha_j & 0 \\ 0 & -\alpha_j\end{bmatrix}\right)_{j=1}^n$, and $\left(\begin{bmatrix}0 & 0 \\ \gamma_j & 0\end{bmatrix}\right)_{j=1}^n$ respectively. Note that $\mathbb{H}^n \cong G_0/K_0$, where $K_0 = U(1)^n$ is a maximal compact subgroup of G_0. Let K denote the complexification of K_0, then we have $K \cong (\mathbb{C}^*)^n$, and K sits inside $SL(2,\mathbb{C})^n$ as the set of all n-tuples of diagonal matrices.

Since $\mathcal{T}_{X_{reg}}$ splits as a direct sum of line bundles, it admits a reduction in structure group from $GL(n,\mathbb{C})$ to $K = (\mathbb{C}^*)^n$. Let P denote the principal $K = (\mathbb{C}^*)^n$-bundle over X_{reg} associated to $\mathcal{T}_{X_{reg}}$, i.e., P has the same locally trivializing cover and the same transition functions as $\bigoplus_{i=1}^n \mathcal{L}_i$. Note that K acts on $\mathfrak{g}^{-1,1}$ via the adjoint representation, which in the present case is simply by scaling. Thus $\mathbb{C}^{\oplus n}$ and $\mathfrak{g}^{-1,1}$ are isomorphic as K-representations, and it follows from the associated bundle construction that there is an isomorphism of vector bundles

$$\theta : \mathcal{T}_{X_{reg}} \longrightarrow P \times_K \mathfrak{g}^{-1,1} \tag{10}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. Thus (P, θ) is a uniformizing system of Hodge bundles on X_{reg}. In particular, $E = P \times_K \mathfrak{g}$ is a system of Hodge bundles on X_{reg}. It has a Hodge decomposition $E = E^{-1,1} \oplus E^{0,0} \oplus E^{1,-1}$ induced by the Hodge decomposition of $\mathfrak{g}$, i.e., $E^{i,-i} \cong P \times_K \mathfrak{g}^{i,-i}$ for $i \in \{-1, 0, 1\}$.

The isomorphism θ induces a map from $\mathcal{T}_{X_{reg}}$ to the endomorphism bundle $\mathcal{E}nd(E)$. This map is given locally by sending an element $a \in \mathcal{T}_{X_{reg},x} \cong \mathfrak{g}^{-1,1}$ to the endomorphism $b \mapsto [\theta(a), b]$ in $\mathcal{E}nd(E)_x$. Hence θ corresponds to an element in $H^0(X_{reg}, \Omega^1_{X_{reg}} \otimes \mathcal{E}nd(E))$ via the isomorphism $\mathcal{H}om(\mathcal{T}_{X_{reg}}, \mathcal{E}nd(E)) \cong \Omega^1_{X_{reg}} \otimes \mathcal{E}nd(E)$. Since $\Omega^1_{X_{reg}} \otimes \mathcal{E}nd(E) \cong \Omega^1_{X_{reg}} \otimes E \otimes E^* \cong \mathcal{H}om(E, E \otimes \Omega^1_{X_{reg}})$, we see that θ eventually corresponds to a morphism $\widehat{\theta} : E \to E \otimes \Omega^1_{X_{reg}}$. The map $\widehat{\theta}$ is given locally by sending an element $a \in E_x \cong \mathfrak{g}$ to the map $b \mapsto [\theta(b), a] \in \mathcal{H}om(\mathcal{T}_{X_{reg},x}, E_x) \cong \Omega^1_{X_{reg},x} \otimes E_x$. Thus on the direct summands $E^{p,q}$ appearing in the Hodge decomposition of E, we have $\widehat{\theta} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_{X_{reg}}$. Moreover, it is straightforward to see that $\widehat{\theta} \wedge \widehat{\theta} = 0$, where $\widehat{\theta} \wedge \widehat{\theta}$ denotes the composition

$$\widehat{\theta} \wedge \widehat{\theta} : E \xrightarrow{\widehat{\theta}} E \otimes \Omega^1_{X_{reg}} \xrightarrow{\widehat{\theta} \otimes Id} E \otimes \Omega^1_{X_{reg}} \otimes \Omega^1_{X_{reg}} \xrightarrow{Id \otimes [\wedge]} E \otimes \Omega^2_{X_{reg}}.$$

Thus $\widehat{\theta}$ is in fact a Higgs field on E. We can decompose $\mathfrak{g}$ as as direct sum of simple ideals $\mathfrak{g} \cong \bigoplus_i \mathfrak{g}_i$, where $\mathfrak{g}_i \cong \mathfrak{sl}(2,\mathbb{C})$ for all $1 \le i \le n$. Since $G = SL(2,\mathbb{C})^n$ is connected, we have a global decomposition $E = P \times_K \mathfrak{g} = \bigoplus_i P \times_K \mathfrak{g}_i$, Each $\mathfrak{g}_i$ is a sub-Hodge structure of $\mathfrak{g}$, i.e., there is a decomposition $\mathfrak{g} = \mathfrak{g}_i^{-1,1} \oplus \mathfrak{g}_i^{0,0} \oplus \mathfrak{g}_i^{1,-1}$ for each i. Moreover, $\mathfrak{g}_i^{-1,1}$ is irreducible as a K-representation, and from the decomposition of $\mathcal{T}_{X_{reg}}$ it follows that $\mathcal{L}_i \cong P \times_K \mathfrak{g}_i^{-1,1}$. Similarly, we have $P \times_K \mathfrak{g}_i^{0,0} \cong \mathcal{O}_{X_{reg}}$ for all i. Hence we have $E_i = P \times_K \mathfrak{g}_i \cong \mathcal{L}_i \oplus \mathcal{L}_i^* \oplus \mathcal{O}_{X_{reg}}$ for all $1 \le i \le n$, and each E_i is a subsystem of Hodge sheaves of E.

We want to show that E is K_X-polystable as a Higgs bundle on X_{reg}, and by Lemma 4.4 it is sufficient to show that E is K_X-polystable as a system of Hodge bundles. To do this, we will show that E_i is a K_X-stable as a system of Hodge bundles, and hence as a Higgs bundle on X_{reg}, for all $1 \le i \le n$.

Let $\mathcal{F} \subset E_i$ be a subsystem of Hodge sheaves with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(E_i)$. For each local section $a \in \mathcal{F}$, we can identify the image $\widehat{\theta}(a)$ with a local section $(b \mapsto [\theta(b), a]) \in \mathcal{H}om(\mathcal{T}_{X_{reg}}, E_i)$, with $b \in \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$.

For each non-zero local section $b \in \mathcal{T}_{X_{reg}}$, $\theta(b)$ can be expressed as an n-tuple of trace zero upper-diagonal 2×2 matrices of the form

$$\left(\begin{bmatrix} 0 & \beta_j \\ 0 & 0 \end{bmatrix}\right)_j,$$

$\beta_j \in \mathbb{C}$ for all $1 \leq j \leq n$. If $\mathcal{F}$ is a subsystem of Hodge sheaves of E_i of rank one, then $\mathcal{F}$ is a subsheaf of one of the direct summands of E_i. If $\mathcal{F} \subset \mathcal{L}_i$, then $c_1(\mathcal{F}) \cdot [K_X]^{n-1} < 0$ because $c_1(\mathcal{L}_i) \cdot [K_X]^{n-1} < 0$ and $\mathcal{L}_i$ is semistable. If $\mathcal{F} \subset \mathcal{L}_i^*$, then any non-zero local section $a \in \mathcal{F}$ can be represented by an n-tuple of the form

$$\left(0, \ldots, 0, \begin{bmatrix} 0 & 0 \\ \alpha_i & 0 \end{bmatrix}, 0, \ldots, 0\right),$$

where the matrix is in the i-th position. The bracket $[\theta(b), a]$ gives

$$\left(0, \ldots, 0, \begin{bmatrix} \alpha_i\beta_i & 0 \\ 0 & -\alpha_i\beta_i \end{bmatrix}, 0, \ldots, 0\right).$$

Thus the image $\widehat{\theta}(a)$ is a non-zero local section in $\mathcal{O}_{X_{reg}} \otimes \Omega^1_{X_{reg}}$, for all non-zero $a \in \mathcal{F}$, which implies that $\widehat{\theta}(\mathcal{F})$ is not contained in $\mathcal{F} \otimes \Omega^1_{X_{reg}}$. It follows that $\mathcal{F}$ is not $\widehat{\theta}$-invariant, hence not a Higgs subsheaf of E_i. Similarly, we see that if $\mathcal{F} \subset \mathcal{O}_{X_{reg}}$, then $\mathcal{F}$ is not $\widehat{\theta}$-invariant and hence not a Higgs subsheaf of E_i. If $\mathcal{F}$ is a rank two subsystem of Hodge sheaves of E_i, then $\mathcal{F}$ is a subsheaf of either $\mathcal{L}_i \oplus \mathcal{L}_i^*$, or $\mathcal{L}_i \oplus \mathcal{O}_{X_{reg}}$, or $\mathcal{L}_i^* \oplus \mathcal{O}_{X_{reg}}$. If $\mathcal{F} \subset \mathcal{L}_i \oplus \mathcal{O}_{X_{reg}}$, then it is straightforward to see that $c_1(\mathcal{F}) \cdot [K_{X_{reg}}]^{n-1} < 0$. If $\mathcal{F} \subset \mathcal{L}_i^* \oplus \mathcal{O}_{X_{reg}}$, then any non-zero local section $a \in \mathcal{F}$ can be expressed as a n-tuple of the form

$$\left(0, \ldots, 0, \begin{bmatrix} \gamma_i & 0 \\ \alpha_i & -\gamma_i \end{bmatrix}, 0, \ldots, 0\right)$$

with $\alpha_i, \gamma_i \in \mathbb{C}^*$. For a non-zero local section $b \in \mathcal{T}_{X_{reg}}$ as earlier, the bracket $[\theta(b), a]$ is then

$$\left(0, \ldots, 0, \begin{bmatrix} \alpha_i\beta_i & -2\gamma_i\beta_i \\ 0 & -\alpha_i\beta_i \end{bmatrix}, 0, \ldots, 0\right).$$

Thus the image $\widehat{\theta}(a)$ is a non-zero local section of $(\mathcal{L}_i \oplus \mathcal{O}_{X_{reg}}) \otimes \Omega^1_{X_{reg}}$, i.e., the image $\widehat{\theta}(\mathcal{F})$ is not contained in $\mathcal{F} \otimes \Omega^1_{X_{reg}}$. It follows that $\mathcal{F}$ is not $\widehat{\theta}$-invariant and hence not a Higgs subsheaf of E_i. It can similarly be verified that if $\mathcal{F} \subset \mathcal{L}_i \oplus \mathcal{L}_i^*$, then $\mathcal{F}$ is not $\widehat{\theta}$-invariant and hence not a Higgs subsheaf of E_i. These are all the cases, hence we conclude that any proper subsystem of Hodge sheaves of E_i has slope strictly less than zero, which implies that E_i is K_X-stable as a Higgs bundle for all $1 \leq i \leq n$. Since E is a direct sum of all the E_i's, it follows that E is K_X-polystable as a Higgs bundle on X_{reg}.
Note that $E \cong \mathcal{T}_{X_{reg}} \oplus \Omega^1_{X_{reg}} \oplus \mathcal{O}^{\oplus n}_{X_{reg}}$, and let $\mathcal{F}_X$ be the unique extension of E as a reflexive sheaf on X. Then we have $\mathcal{F}_X \cong \mathcal{T}_X \oplus \Omega^{[1]}_X \oplus \mathcal{O}^{\oplus n}_X$, and by construction it follows that $c_1(\mathcal{F}) = 0$. Note that $\mathcal{T}_X = \bigoplus_i j_*\mathcal{L}_i$, where $j : X_{reg} \to X$ denotes the natural inclusion. Therefore $\widehat{c}_2(\mathcal{F}_X) = \sum_i \widehat{c}_1(j_*\mathcal{L}_i)^2$ by the formula (2). It follows from Lemma 5.3 that $\widehat{c}_1(j_*\mathcal{L}_i)^2 = 0$ for all $1 \leq i \leq n$ and hence $\widehat{c}_2(\mathcal{F}_X) = 0$. In particular, we have $\widehat{ch}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = \widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = 0$. Theorem 2.38 thus implies that $(E, \widehat{\theta}) \in \text{TPI-Higgs}_{X_{reg}}$.
Then Proposition 2.37 together with the assumption that X is maximally quasi-étale implies that $\mathcal{F}_X$ is locally free. Since $\mathcal{T}_X$ is a direct summand of $\mathcal{F}_X = \mathcal{T}_X \oplus \Omega^{[1]}_X \oplus \mathcal{O}^{\oplus n}_X$, it follows that $\mathcal{T}_X$ must be locally free. The solution of the Lipman-Zariski conjecture for klt spaces ([11, Theorem 16.1]) thus asserts that X is smooth, and we are done. □

Proof of Theorem 5.1

Suppose X is an n-dimensional projective, klt variety of general type which satisfies the assumptions of Theorem 5.1. Recall from Proposition 5.4 that the canonical model X_{can} of X is projective, klt, and the tangent sheaf of its regular locus splits as a direct sum of line bundles of negative degree. Moreover, the canonical divisor $K_{X_{can}}$ of X_{can} is ample. Let $\gamma : Y \to X_{can}$ be a Galois, maximally quasi-étale cover (see Lemma 2.36). Y is again klt, andthe canonical divisor $K_Y = \gamma^* K_{X_{can}}$ is ample. Moreover, by Proposition 5.5 and the semistability of $\mathcal{T}_Y$ with respect to K_Y (Proposition 4.1), we have that $\mathcal{T}_{Y_{reg}}$ splits as a direct sum of line bundles of negative degree. Thus Y satisfies the assumptions of Proposition 5.6 and we conclude that Y is smooth. It follows that Y also satisfies the assumptions of [34, Corollary 9.7], hence Y is uniformized by the polydisk $\mathbb{H}^n$. □

Remark 5.7. While the splitting of the tangent bundle $\mathcal{T}_{X_{reg}}$ of the smooth locus X_{reg} of X as a direct sum of line bundles is a sufficient condition for (the canonical model of) X to be uniformized by the polydisk, it is not a necessary condition. The reason, as pointed out by Catanese and Di Scala in [7], is that the group $Aut(\mathbb{H}^n)$ of holomorphic automorphisms of the polydisk is not $PSL(2,\mathbb{R})^n$, but is the semidirect product of $PSL(2,\mathbb{R})^n$ and the symmetric group σ_n. Thus there is a split sequence

$$1 \to Aut(\mathbb{H})^n \hookrightarrow Aut(\mathbb{H}^n) \to \sigma_n \to 1$$

where $Aut(\mathbb{H})^n = Aut^0(\mathbb{H}^n) = PSL(2,\mathbb{R})^n$. The group $Aut(\mathbb{H}^n)$ is a disconnected Hodge group of Hermitian type. The splitting of the tangent bundle $\mathcal{T}_{\mathbb{H}^n} \cong \bigoplus_{i=1}^n \mathcal{T}_{\mathbb{H}}$ descends to a quotient $X \cong \mathbb{H}^n/\Gamma$, where $\Gamma \subset Aut(\mathbb{H}^n)$ acts discretely and cocompactly on $\mathbb{H}^n$, only if Γ acts diagonally on $\mathbb{H}^n$, i.e., only if $\Gamma \subset Aut(\mathbb{H})^n = PSL(2,\mathbb{R})^n$. However, since any subgroup $\Gamma \subset Aut(\mathbb{H}^n)$ can be expressed as an extension of a subgroup Γ' of $PSL(2,\mathbb{R})^n$ by a subgroup H of σ_n because of the above exact sequence, every projective, klt quotient X of a polydisk admits a Galois, quasi-étale cover $\gamma : Y \to X$ such that Y is also a polydisk quotient whose tangent sheaf $\mathcal{T}_Y$ splits as a direct sum of reflexive sheaves of rank one.

5.2 Necessary conditions

Next we would like to formulate a necessary condition for a projective klt variety of general type to be uniformized by the polydisk. This will be a more precise version of Remark 5.7.

Proposition 5.8. *Let X be a n-dimensional projective klt variety with K_X is ample, and such that $X \cong \mathbb{H}^n/\widehat{\Gamma}$, where $\widehat{\Gamma} \subset PSL(2,\mathbb{R})^n \rtimes \sigma_n = Aut(\mathbb{H}^n)$ is a discrete cocompact subgroup acting fixed point freely in codimension one. Then X admits a smooth quasi-étale cover $\gamma : Y \to X$ such that K_Y is ample, and the tangent bundle $\mathcal{T}_Y$ splits as a direct sum of line bundles.*

Proof. Due to the structure of the automorphism group $Aut(\mathbb{H}^n) = PSL(2,\mathbb{R})^n \rtimes \sigma_n$, we know that $\widehat{\Gamma}$ can be expressed as an extension

$$1 \to \Gamma' \to \widehat{\Gamma} \to H \to 1,$$

where $\Gamma' \subset PSL(2,\mathbb{R})^n$ is normal of finite index in $\widehat{\Gamma}$, and $H \subset \sigma_n$ is finite. The quotient map $\pi : \mathbb{H}^n \to X$ thus factors as

$$\mathbb{H}^n \to \mathbb{H}^n/\Gamma' \to \mathbb{H}^n/\widehat{\Gamma} = X,$$

where $Y' = \mathbb{H}^n/\Gamma'$ is also klt and projective, and the map $Y' \to X$ is Galois and quasi-étale. Using Selberg's lemma we know that Γ' has a torsion free normal subgroup Γ of finite index. The quotient map $\pi' : \mathbb{H}^n \to Y'$

thus factors as

$$\mathbb{H}^n \to \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Gamma' = Y'.$$

Since Γ acts freely and cocompactly on $\mathbb{H}^n$, the quotient $Y = \mathbb{H}^n/\Gamma$ is a smooth projective variety. The map $Y \to Y'$ is Galois and quasi-étale, and after composing with the map $Y' \to X$, we get a quasi-étale map $Y \to X$. Recall from Lemma 3.4 that the tangent bundle of $\mathbb{H}^n$ can be expressed as $\mathcal{T}_{\mathbb{H}^n} \cong P \times_K \mathfrak{g}^{-1,1}$, where $K = (\mathbb{C}^*)^n$ is maximal compact inside $SL(2,\mathbb{C})^n$, and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})^n$. The action of Γ on $\mathbb{H}^n$ lifts to a left action of Γ on $\mathcal{T}_{\mathbb{H}^n}$ via pushforward of tangent spaces, so it follows that $\mathcal{T}_Y \cong P' \times_K \mathfrak{g}^{-1,1}$, where $P' \cong P/\Gamma$ as principal K-bundles on Y. Since $\mathcal{T}_Y$ also has structure group $K = (\mathbb{C}^*)^n$, it splits as a direct sum of line bundles. Since the map $\gamma : Y \to X$ is finite, and K_X is ample by assumption, it follows that $K_Y = \gamma^* K_X$ is also ample. The semistability of $\mathcal{T}_Y$ with respect to K_Y then implies that the line bundles appearing in the direct sum decomposition of $\mathcal{T}_Y$ all have negative degree. ☐

Remark 5.9. If X is a smooth projective quotient of the polydisk, then we *do not* in general have a representation $\pi_1(X) \to PSL(2,\mathbb{R})^n$. Therefore the action of $\pi_1(X)$ on $\mathbb{H}^n$ is not in general compatible with the reduction in structure group of $\mathcal{T}_{\mathbb{H}^n}$ from $GL(n,\mathbb{C})$ to $(\mathbb{C}^*)^n = K$. Hence the splitting of the tangent bundle into a direct sum of line bundles does not in general descend to the quotient variety X.
There is however always a representation $\pi_1(X) \to PSL(2,\mathbb{R})^n \rtimes \sigma_n$, and the action of $\pi_1(X)$ on $\mathbb{H}^n$ is compatible with the reduction in structure group of $\mathcal{T}_{\mathbb{H}^n}$ from $GL(n,\mathbb{C})$ to $(\mathbb{C}^*)^n \rtimes \sigma_n = K'$. The tangent bundle of X thus always admits a reduction in structure group from $GL(n,\mathbb{C})$ to K' i.e., we can write $\mathcal{T}_X \cong P \times_{K'} \mathfrak{g}^{-1,1}$, for some principal $K' = (\mathbb{C}^*)^n \rtimes \sigma_n$-bundle P on X.

Example 5.10. Let C_1,...,C_n be smooth projective curves, each of genus $g_i \geq 2$, and let G be a finite group acting faithfully by automorphisms on each of the n curves. Necessary and sufficient conditions for a finite group G to act on a curve by automorphisms are given by [2, Theorem 2.1], which is a reformulation of the *Riemann existence theorem.* Consider the cartesian product $Y = C_1 \times \cdots \times C_n$ and let X be the quotient variety $X = Y/G$. If the action of G on Y is free, then X is said to be a *variety isogenous to a product of curves of unmixed type.* The universal cover of X (and of course of Y) is the polydisk $\mathbb{H}^n$.
We restrict our attention to product-quotient surfaces i.e., quotients $X = (C_1 \times C_2)/G$, where G acts diagonally on $C_1 \times C_2$. If the action of G is not free, then we know from [2, Remark 2.4] that G has a finite set of fixed points, and X has a finite number of cyclic quotient singularities, which are rational. In particular, X is klt, and the quotient map $C_1 \times C_2 \to X$ is quasi-étale. Since G acts diagonally, the tangent bundle of the smooth locus X_{reg} splits as a direct sum of line bundles, so it follows from Theorem 5.1 that X is uniformized by the bidisk $\mathbb{H}^2$.
If $\tilde{X}$ is any resolution of singularities of X then we have equality of topological fundamental groups $\pi_1(X) = \pi_1(\tilde{X})$. Let $p_g(\tilde{X})$ and $q(\tilde{X})$ denote the geometric genus and irregularity of $\tilde{X}$ respectively. Then all surfaces $X = (C_1 \times C_2)/G$ with $g_1, g_2 \geq 2$ where the action of G on $C_1 \times C_2$ is unmixed, and such that X has only rational double point singularities and $p_g(\tilde{X}) = q(\tilde{X}) = 0$ have been classified in [2, Theorem 0.20]. A list of all such surfaces is given in [2, Table 2], and each entry of the table gives an example of a singular polydisk quotient. For example, $g_1 = 3$, $g_2 = 22$, and $G = PSL(2,7)$ produces a surface with $\pi_1(X) = (\mathbb{Z}/2\mathbb{Z})^2$, which is finite, while $g_1 = 4$, $g_2 = 21$, and $G = \sigma_5$ produces a surface with $\pi_1(X) = \mathbb{Z}^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$, which is infinite.

6 Uniformization by Hermitian symmetric space of type CI

For any $n \in \mathbb{Z}_{\geq 1}$, the Siegel upper half space $\mathcal{H}_n$ is a Hermitian symmetric space of non-compact type, of dimension $n(n+1)/2$. The bounded symmetric domain realization of CI is as follows (see [30, Ch.4, Sec.2]).

$$CI = \{Z \in \mathcal{D}_{n,n} : Z^T = Z\},$$

where the domain $\mathcal{D}_{n,n}$ is given by

$$\mathcal{D}_{n,n} = \{Z \in M(n,n,\mathbb{C}) \cong \mathbb{C}^{n^2} : I_n - \bar{Z}^T Z > 0\}. \tag{11}$$

In the above definition, the condition in the bracket is that the matrix $I_n - \bar{Z}^T Z$ is positive definite. We know, for example from [37, p.108], that the automorphism group of $\mathcal{H}_n$ is $PSp(2n,\mathbb{R})$, which is a Hodge group of Hermitian type. Another Hodge group associated to $\mathcal{H}_n$ is $G_0 = Sp(2n,\mathbb{R})$, which is a cover of $PSp(2n,\mathbb{R})$ and has maximal compact subgroup $K_0 = U(n)$. Thus we have $\mathcal{H}_n \cong G_0/K_0 = Sp(2n,\mathbb{R})/U(n)$. The complexified Lie algebra $\mathfrak{g}$ of G_0 has complex dimension $n(2n+1)$, and it follows from the list on [20, p.341], that $\mathfrak{g}$ can be expressed as the set of complex $2n \times 2n$ trace zero matrices $\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$, where A is a complex $n \times n$ matrix, and B and C are complex symmetric $n \times n$ matrices. By definition, the Lie algebra $\mathfrak{g}$ admits a Hodge decomposition given by $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$, where $\mathfrak{g}^{-1,1}$ consists of matrices of the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, $\mathfrak{g}^{0,0}$ consists of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$, and $\mathfrak{g}^{1,-1}$ consists of matrices of the form $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. The Lie algebras $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$ have complex dimensions $n(n+1)/2$, n^2, and $n(n+1)/2$ respectively.

The compact dual of $\mathcal{H}_n$ is the Lagrangian Grassmannian Y_n, which is a homogeneous complex projective variety of dimension $n(n+1)/2$ for the action of $G = Sp(2n,\mathbb{C})$. It parametrizes all Lagrangian (i.e. maximally isotropic) subspaces of a complex symplectic vector space of dimension $2n$. We know from [9] that there is an open embedding $j : \mathcal{H}_n \hookrightarrow Y_n$, and the tangent bundle of Y_n is given by $\mathcal{T}_{Y_n} \cong \mathrm{Sym}^2(\mathcal{E})$, where $\mathcal{E}$ is the tautological vector bundle on Y_n, and corresponds to the standard representation of $K_0 = U(n)$. It follows that the tangent bundle of $\mathcal{H}_n$ can be expressed as $\mathcal{T}_{\mathcal{H}_n} \cong \mathrm{Sym}^2(\mathcal{E}')$, where $\mathcal{E}'$ denotes the restriction of $\mathcal{E}$ to $\mathcal{H}_n$.

We want to formulate necessary and sufficient conditions for a complex projective klt variety of dimension $n(n+1)/2$ with ample canonical divisor to be uniformized by the Siegel upper half space $\mathcal{H}_n$. To do this, we apply Theorem 1.1.

6.1 Sufficient conditions

Proposition 6.1. *Let X be a projective, klt variety of dimension $n(n+1)/2$ for some $n \in \mathbb{Z}_{\geq 1}$ with K_X ample. Suppose that the tangent bundle of the regular locus X_{reg} of X satisfies $\mathcal{T}_{X_{reg}} \cong Sym^2(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}. Let $\mathcal{E}'$ denote the reflexive extension of $\mathcal{E}$ to X, and suppose that the Chern class equality*

$$[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{n-2} = 0 \tag{12}$$

holds on X. Then $X \cong \mathcal{H}_n/\Gamma$, where $\Gamma \subset PSp(2n,\mathbb{R})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on the Siegel upper half space $\mathcal{H}_n$.

Proof. Let $j : X_{reg} \to X$ denote the natural inclusion. Then we have $\mathcal{T}_X \cong j_*\mathcal{T}_{X_{reg}} \cong \mathrm{Sym}^{[2]}(\mathcal{E}') = \mathrm{Sym}^2(\mathcal{E}')^{**}$ by the uniqueness of reflexive extension. Let $G_0 = Sp(2n, \mathbb{R})$, $K_0 = U(n)$ a maximal compact subgroup of G_0, Then complexifications G and K of G_0 and K_0 respectively, are $G = Sp(2n, \mathbb{C})$ and $K = GL(n, \mathbb{C})$.

Let P be the frame bundle of $\mathcal{E}$. Then P is a principal $K = GL(n, \mathbb{C})$-bundle on X_{reg} and we can write $\mathcal{E} \cong P \times_K V$, where V denotes the typical fiber of $\mathcal{E}$. It follows that $\mathcal{T}_{X_{reg}} \cong \mathrm{Sym}^2(P \times_K V) = P \times_K \mathrm{Sym}^2(V)$, so in particular $\mathcal{T}_{X_{reg}}$ admits a reduction in structure group from $GL(n(n+1)/2, \mathbb{C})$ to K. As mentioned earlier, every element of $\mathfrak{g}^{-1,1}$ can by expressed as an $n \times n$ complex symmetric matrix, and the adjoint action of K on such an element is given by conjugation by unitary matrices. Hence it follows that $\mathrm{Sym}^2(V)$ and $\mathfrak{g}^{-1,1}$ are isomorphic as K-representations. Thus there is an isomorphism

$$\theta : \mathcal{T}_{X_{reg}} \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. The latter statement is clear because G_0 is a Hodge group, so the Lie bracket of any two elements of $\mathfrak{g}^{-1,1}$ lands in $\mathfrak{g}^{-2,2}$, which is zero. It follows that (P, θ) is a uniformizing system of Hodge bundles on X_{reg}.

Hence we can form the system of Hodge bundles $E = P \times_K \mathfrak{g} = P \times_K \mathfrak{g}^{-1,1} \oplus P \times_K \mathfrak{g}^{0,0} \oplus P \times_K \mathfrak{g}^{1,-1}$, where we have $P \times_K \mathfrak{g}^{1,-1} \cong \mathrm{Sym}^2(\mathcal{E}^\vee) \cong \Omega^1_{X_{reg}}$, and $P \times_K \mathfrak{g}^{0,0} \cong \mathcal{E}nd(\mathcal{E})$. The last isomorphism follows from observing that $\mathfrak{g}^{0,0}$ and $\mathrm{End}(V)$ are isomorphic as K-representations. Indeed, they are isomorphic as $\mathbb{C}$-vector spaces, and the adjoint action of K on $\mathfrak{g}^{0,0}$ is compatible with the action of K on $\mathrm{End}(V)$ by conjugation.

Note that the system of Hodge bundles $E = P \times_K \mathfrak{g} \cong \mathcal{T}_{X_{reg}} \oplus \mathcal{E}nd(\mathcal{E}) \oplus \Omega^1_{X_{reg}}$ is in particular a Higgs bundle with Higgs field $\widehat{\theta} : E \to E \otimes \Omega^1_{X_{reg}}$ given by sending a local section u of E to the local section $v \mapsto [\theta(v), u]$ of $\mathcal{H}om(\mathcal{T}_{X_{reg}}, E) \cong E \otimes \Omega^1_{X_{reg}}$. It is clear that E has slope zero with respect to K_X. By Proposition 4.6, we know that E is K_X-polystable as a Higgs bundle on X_{reg}. In fact, E is K_X-stable as a Higgs bundle on X_{reg} because $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ is a simple Lie algebra.

Let $\mathcal{F}_X$ denote the reflexive extension of E to X, then $\mathcal{F}_X \cong \mathcal{T}_X \oplus \mathcal{E}nd(\mathcal{E}') \oplus \Omega^{[1]}_X$. It is clear that $\widehat{c}_1(\mathcal{F}_X) \cdot [K_X]^{n-1} = 0$. Moreover, we compute using the formulae (1) and (2)

$$\widehat{c}_2(\mathcal{F}_X) = 2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(X)^2,$$

so by the assumption of the Proposition we have $\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = \widehat{ch}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = 0$.

Therefore, X satisfies the two conditions of Theorem 1.1, and we conclude that $X \cong \mathcal{H}_n/\Gamma$, where $\Gamma \subset PSp(2n, \mathbb{R})$ is a discrete, cocompact subgroup acting fixed point freely in codimension one on $\mathcal{H}_n$. ☐

6.2 Necessary conditions

Since the automorphism group $\mathrm{Aut}(\mathcal{H}_n) = PSp(2n, \mathbb{R})$ is a connected Lie group, we choose the associated Hodge group to be $G_0 = Sp(2n, \mathbb{R})$, and its complexification to be $G = Sp(2n, \mathbb{C})$, which is also connected. Thus by Remark A the sufficient conditions of Proposition 6.1 are also necessary conditions for a projective klt variety with ample canonical divisor to be uniformized by $\mathcal{H}_n$.

Proposition 6.2. *Let X be a projective klt variety of dimension $n(n+1)/2$ with K_X ample, such that $X = \mathcal{H}_n/\widehat{\Gamma}$, where $\widehat{\Gamma} \subset Aut(\mathcal{H}_n)$ acts discretely, cocompactly, and fixed point freely in codimension one on $\mathcal{H}_n$. Then the tangent bundle of the regular locus of X satisfies $\mathcal{T}_{X_{reg}} \cong Sym^2(\mathcal{E})$, where $\mathcal{E}$ is a rank n vector bundle on X. Moreover, X satisfies the Chern class equality 12.*

Proof. We know from Theorem 1.1 that the smooth locus X_{reg} of X admits a uniformizing system of Hodge bundles (P, θ) corresponding to the Hodge group $G_0 = Sp(2n, \mathbb{R})$, such that $E = P \times_K \mathfrak{g}$ is K_X-polystable

as a Higgs bundle on X_{reg}, and $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$, where E' denotes the unique reflexive extension of E to X. Thus there is an isomorphism $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$. Recall from the proof of Proposition 6.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\mathrm{Sym}^2(V)$, where V is a complex n-dimensional vector space. So we can write $\mathcal{T}_{X_{reg}} \cong P \times_K \mathrm{Sym}^2(V) = \mathrm{Sym}^2(\mathcal{E})$, where $\mathcal{E} = P \times_K V$ is a rank n vector bundle on X_{reg}. We also have $\Omega^1_{X_{reg}} \cong \mathrm{Sym}^2(\mathcal{E}^\vee) \cong P \times_K \mathfrak{g}^{1,-1}$, and $\mathcal{E}nd(\mathcal{E}) \cong P \times_K \mathfrak{g}^{0,0}$, from which it follows that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{T}_{X_{reg}} \oplus \mathcal{E}nd(\mathcal{E}) \oplus \Omega^1_{X_{reg}}$.
Let $\mathcal{E}'$ denote the reflexive extension of $\mathcal{E}$ to X. Then $E' = \mathcal{T}_X \oplus \mathcal{E}nd(\mathcal{E}') \oplus \Omega^{[1]}_X$, and the equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$ is equivalent to

$$\widehat{c}_2(\mathcal{T}_X \oplus \mathcal{E}nd(\mathcal{E}') \oplus \Omega^{[1]}_X) \cdot [K_X]^{n-2} = 0.$$

Using the formula (2) for the second $\mathbb{Q}$-Chern class of a direct sum of reflexive sheaves, the above equality can be rephrased as

$$[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{n-2} = 0.$$

Thus it follows that X satisfies the required Chern class equality (12), this concludes the proof. □

Putting together Propositions 6.1 and 6.2, we arrive at Theorem 1.2.

7 Uniformization by Hermitian symmetric space of type $DIII$

This example is very similar to the Siegel upper half space. The Hermitian symmetric space of type $DIII$, which we denote by $\mathcal{D}_n$, can be expressed as the quotient G_0/K_0, where the associated Hodge group is $G_0 = SO^*(2n)$ defined as

$$SO^*(2n) = \{M \in SL(2n, \mathbb{C}) : M^T M = I_{2n}, \bar{M}^T J_n M = J_n\},$$

where J_n is the matrix given by $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We know from [30, Ch.4, Sec.2] that the maximal compact subgroup K_0 of G_0 is given by

$$K_0 = \left\{ M = \begin{bmatrix} U & 0 \\ 0 & \bar{U} \end{bmatrix} : M \in SU(n,n) \right\},$$

from which it is clear that $U \in U(n) \cong K_0$, for $n \in \mathbb{Z}_{\geq 1}$. The bounded symmetric domain realization of $DIII$ is as follows (see again [30, Ch. 4, Sec. 2]).

$$DIII = \{Z \in \mathcal{D}_{n,n} : Z^T = -Z\},$$

where the domain $\mathcal{D}_{n,n}$ is as defined in the expression (11). From [37, p. 111], the group of holomorphic automorphisms of $\mathcal{D}_n$ is $\mathrm{Aut}(\mathcal{D}_n) = PSO^*(2n)$, so $\mathrm{Aut}(\mathcal{D}_n)$ is a quotient of G_0 by a discrete central subgroup. From the list on [20, p.341], the complexified Lie algebra $\mathfrak{g}$ of G_0 has complex dimension $n(2n-1)$, and can be expressed as the set of complex $2n \times 2n$ trace zero matrices $\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$, where A is a complex $n \times n$ matrix, and B and C are complex skew-symmetric $n \times n$ matrices. Since G_0 is a Hodge group of Hermitian type, the Lie algebra $\mathfrak{g}$ has a Hodge decomposition given by $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$, where $\mathfrak{g}^{-1,1}$ consists of matrices of the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, $\mathfrak{g}^{0,0}$ consists of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$, and $\mathfrak{g}^{1,-1}$ consists

of matrices of the form $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. The Lie algebras $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$ have dimensions $n(n-1)/2$, n^2, and $n(n-1)/2$ respectively.
The compact dual of $\mathcal{D}_n$ is the Isotropic Grassmannian I_n, which is a homogeneous complex projective variety of dimension $n(n-1)/2$ for the action of $G = SO^*(2n, \mathbb{C})$. It parametrizes all isotropic subspaces of a complex vector space of dimension $2n$ equipped with a complex inner product. Just as in the case of the Lagrangian Grassmannian, there is an open embedding $j : \mathcal{D}_n \hookrightarrow I_n$. The tangent bundle of I_n is given by $\mathcal{T}_{I_n} \cong \bigwedge^2(\mathcal{E})$, where $\mathcal{E}$ is the tautological vector bundle on I_n, and corresponds to the standard representation of $K_0 = U(n)$. It follows that the tangent bundle of $\mathcal{D}_n$ can be expressed as $\mathcal{T}_{\mathcal{D}_n} \cong \bigwedge^2(\mathcal{E}')$, where $\mathcal{E}'$ denotes the restriction of $\mathcal{E}$ to $\mathcal{D}_n$.

We want to formulate necessary and sufficient conditions for a complex projective variety of dimension $n(n-1)/2$ with klt singularities and ample canonical divisor to be uniformized by the Hermitian symmetric space $\mathcal{D}_n$ of type DIII.

7.1 Sufficient conditions

A version of Proposition 6.1 also holds in this case, and the proof is essentially the same. For completeness, we repeat the proof.

Proposition 7.1. *Let X be a projective, klt variety of dimension $n(n-1)/2$ for some $n \in \mathbb{Z}_{\geq 1}$ with K_X ample. Suppose that the tangent bundle of the regular locus X_{reg} of X satisfies $\mathcal{T}_{X_{reg}} \cong \bigwedge^2(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}. Let $\mathcal{E}'$ denote the reflexive extension of $\mathcal{E}$ to X, and suppose that the Chern class equality*

$$[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{n-2} = 0 \tag{13}$$

holds on X. Then $X \cong \mathcal{D}_n/\Gamma$, where $\Gamma \subset PSO^*(2n, \mathbb{R})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on the domain $\mathcal{D}_n$ of type DIII.

Proof. Let $j : X_{reg} \to X$ denote the natural inclusion. Then we have $\mathcal{T}_X \cong j_*\mathcal{T}_{X_{reg}} \cong \bigwedge^{[2]}(\mathcal{E}') = (\bigwedge^2(\mathcal{E}'))^{**}$ by the uniqueness of reflexive extension. Let $G_0 = SO^*(2n, \mathbb{R})$, $K_0 = U(n)$ a maximal compact subgroup of G_0. Then we choose complexifications G and K of G_0 and K_0 respectively, to be $G = SO^*(2n, \mathbb{C})$ and $K = GL(n, \mathbb{C})$.
Let P be the frame bundle of $\mathcal{E}$. Then P is a principal $K = GL(n, \mathbb{C})$-bundle on X_{reg} and we can write $\mathcal{E} \cong P \times_K V$, where $V \cong \mathbb{C}^n$ denotes the typical fiber of $\mathcal{E}$. It follows that $\mathcal{T}_{X_{reg}} \cong \bigwedge^2(P \times_K V) = P \times_K \bigwedge^2(V)$, so in particular $\mathcal{T}_{X_{reg}}$ admits a reduction in structure group from $GL(n(n-1)/2, \mathbb{C})$ to K. Note that analogous to the previous case, each element of $\mathfrak{g}^{-1,1}$ can be expressed as an $n \times n$ anti-symmetric matrix, and the action of K is given again by conjugation by unitary matrices. Hence $\bigwedge^2(V)$ and $\mathfrak{g}^{-1,1}$ are isomorphic as K-representations. Thus there is an isomorphism

$$\theta : \mathcal{T}_{X_{reg}} \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. It follows that (P, θ) is a uniformizing system of Hodge bundles on X_{reg}.
Hence we can form the system of Hodge bundles $E = P \times_K \mathfrak{g} = P \times_K \mathfrak{g}^{-1,1} \oplus P \times_K \mathfrak{g}^{0,0} \oplus P \times_K \mathfrak{g}^{1,-1}$, where we have $P \times_K \mathfrak{g}^{1,-1} \cong \bigwedge^2(\mathcal{E}^\vee) \cong \Omega^1_{X_{reg}}$, and $P \times_K \mathfrak{g}^{0,0} \cong \mathcal{E}nd(\mathcal{E})$. The last isomorphism follows from observing, as in Proposition 6.1, that $\mathfrak{g}^{0,0}$ and $\mathrm{End}(V)$ are isomorphic as K-representations. Hence we have

$P \times_K \mathfrak{g}^{0,0} \cong P \times_K \mathrm{End}(V) = \mathcal{E}nd(P \times_K V)$.
Note that the system of Hodge bundles $E = P \times_K \mathfrak{g} \cong \mathcal{T}_{X_{reg}} \oplus \mathcal{E}nd(\mathcal{E}) \oplus \Omega^1_{X_{reg}}$ is in particular a Higgs bundle with Higgs field $\widehat{\theta} : E \to E \otimes \Omega^1_{X_{reg}}$ given by sending a local section u of E to the local section $v \mapsto [\theta(v), u]$ of $\mathcal{H}om(\mathcal{T}_{X_{reg}}, E) \cong E \otimes \Omega^1_{X_{reg}}$. It is clear that E has slope zero with respect to K_X. We know from Proposition 4.6 that E is K_X-polystable as a Higgs bundle on X_{reg}. In fact, E is K_X-stable as a Higgs bundle on X_{reg} because again $\mathfrak{g}$ is a simple Lie algebra.
Let $\mathcal{F}_X$ denote the reflexive extension of E to X, then $\mathcal{F}_X \cong \mathcal{T}_X \oplus \mathcal{E}nd(\mathcal{E}') \oplus \Omega^{[1]}_X$. It is clear that $\widehat{c}_1(\mathcal{F}_X) \cdot [K_X]^{n-1} = 0$. Moreover, we compute again using (2)

$$\widehat{c}_2(\mathcal{F}_X) = 2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(X)^2,$$

so by the assumption of the Proposition we have $\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = \widehat{ch}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = 0$.
Again we see that X satisfies the two conditions of Theorem 1.1, and we conclude that $X \cong \mathcal{D}_n/\Gamma$, where $\Gamma \subset PSO^*(2n, \mathbb{R})$ is a discrete, cocompact subgroup, acting fixed point freely in codimension one on $\mathcal{D}_n$. □

7.2 Necessary conditions

Since $\mathrm{Aut}(\mathcal{D}_n) = PSO^*(2n)$ is a connected Lie group, we take the associated Hodge group to be $G_0 = SO^*(2n)$, and its complexification to be $G = SO^*(2n, \mathbb{C})$, which is also connected. Then Remark A implies that the sufficient conditions of Proposition 7.1 are also necessary conditions for a projective klt variety with ample canonical divisor to be uniformized by $\mathcal{D}_n$. We have $K_0 = U(n)$ and its complexification $K = GL(n, \mathbb{C})$. The proof is essentially the same as that of Proposition 6.2.

Proposition 7.2. *Let X be a projective klt variety of dimension $n(n-1)/2$ with K_X ample, such that $X = \mathcal{D}_n/\widehat{\Gamma}$, where $\widehat{\Gamma} \subset Aut(\mathcal{D}_n)$ acts discretely, cocompactly, and fixed point freely in codimension one on $\mathcal{D}_n$. Then the tangent bundle of the regular locus of X satisfies $\mathcal{T}_{X_{reg}} \cong \bigwedge^2(\mathcal{E})$, where $\mathcal{E}$ is a rank n vector bundle on X_{reg}. Moreover, X satisfies the Chern class equality 13.*

Proof. From Theorem 1.1, it follows that X_{reg} admits a uniformizing system of Hodge bundles (P, θ) corresponding to the Hodge group $G_0 = SO^*(2n)$, such that the system of Hodge bundles $E = P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}. Moreover, the equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$. holds, where E' denotes the extension of E to X as a reflexive sheaf. Hence there is an isomorphism $\theta : \mathcal{T}_Y \cong P \times_K \mathfrak{g}^{-1,1}$. Recall from the proof of Proposition 7.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\bigwedge^2(V)$, where V is a complex n-dimensional vector space. So we can write $\mathcal{T}_{X_{reg}} \cong P \times_K \bigwedge^2(V) = \bigwedge^2(\mathcal{E})$, where $\mathcal{E} = P \times_K V$ is a rank n vector bundle on Y. We also have $\Omega^1_{X_{reg}} \cong \bigwedge^2(\mathcal{E}^\vee) \cong P \times_K \mathfrak{g}^{1,-1}$, and $\mathcal{E}nd(\mathcal{E}) \cong P \times_K \mathfrak{g}^{0,0}$, from which it follows that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{T}_{X_{reg}} \oplus \mathcal{E}nd(\mathcal{E}) \oplus \Omega^1_{X_{reg}}$.
Let $\mathcal{E}'$ denote the reflexive extension of $\mathcal{E}$ to X. Then $E' = \mathcal{T}_X \oplus \mathcal{E}nd(\mathcal{E}') \oplus \Omega^{[1]}_X$, and the Chern class equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$ is equivalent to

$$\widehat{c}_2(\mathcal{T}_X \oplus \mathcal{E}nd(\mathcal{E}') \oplus \Omega^{[1]}_X) \cdot [K_X]^{n-2} = 0.$$

Using the formula (2) for the second $\mathbb{Q}$-Chern class of a direct sum of vector bundles, the above equality can be rephrased as

$$[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}) - (n-1)\widehat{c}_1(\mathcal{E})^2] \cdot [K_X]^{n-2} = 0.$$

Thus it follows that X satisfies the Chern class equality (13), and we are done. □

Putting together Propositions 7.1 and 7.2, we arrive at the following necessary and sufficient condition for a projective klt variety X with ample canonical divisor to be uniformized by the Hermitian symmetric space $\mathcal{D}_n$ of type $DIII$.

Theorem 7.3. *Let X be a projective klt variety of dimension $n(n-1)/2$ such that the canonical divisor K_X is ample. Then $X \cong \mathcal{D}_n/\Gamma$, where $\Gamma \subset PSO^*(2n, \mathbb{R})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on $\mathcal{D}_n$, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \bigwedge^2(\mathcal{E})$
- $[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{n-2} = 0,$

where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}, and $\mathcal{E}'$ denotes the reflexive extension of $\mathcal{E}$ to X.

8 Uniformization by Hermitian symmetric space of type BDI

The Hermitian symmetric space of type BDI, which we denote by $\mathcal{B}_n$, can be expressed as the quotient G_0/K_0, where we may take the associated Hodge group to be $G_0 = SO(2, n)$, and $K_0 = SO(2) \times SO(n)$ a maximal compact subgroup of G_0, for $n \in \mathbb{Z}_{\geq 2}$. The bounded symmetric domain realization of BDI is more involved than the other examples, we refer the reader to [30, p. 75] for details. When $n = 2$, the domain $\mathcal{B}_2$ is isomorphic to the bidisk $\mathbb{H}^2$, which we have already studied. Thus we may assume $n \geq 3$. The Lie group G_0 has two connected components, and the automorphism group $PSO(2, n)$ of $\mathcal{B}_n$ (see again [37]) is a quotient of $SO(2, n)$ by a discrete central subgroup. The dimension of $\mathcal{B}_n$ as a complex manifold is n. The complexified Lie algebra of G_0 is $\mathfrak{g} = \mathfrak{so}(2+n, \mathbb{C})$. The Lie algebra $\mathfrak{g}$ has complex dimension $(2+n)(1+n)/2$, and using [20, p. 341], can be expressed as the set of complex $(n+2) \times (n+2)$ matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is a 2×2 orthogonal matrix, D is a $n \times n$ orthogonal matrix, and B and C are $2 \times n$ and $n \times 2$ matrices respectively. The group $G_0 = SO(2, n)$ is in fact a Hodge group of Hermitian type, therefore the Lie algebra $\mathfrak{g}$ has a Hodge decomposition given by $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$, where $\mathfrak{g}^{-1,1}$ consists of those matrices of $\mathfrak{g}$ of the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, $\mathfrak{g}^{0,0}$ consists of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, and $\mathfrak{g}^{1,-1}$ consists of matrices of the form $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. The Lie algebras $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$ are of dimensions n, $n(n-1)/2 + 1$, and n respectively. By Lemma 3.4, we know there is be a principal $K = SO(2, \mathbb{C}) \times SO(n, \mathbb{C})$-bundle P on $\mathcal{B}_n$ such that there is an isomorphism $\theta : \mathcal{T}_{\mathcal{B}_n} \to P \times_K \mathfrak{g}^{-1,1}$. Then there is an orthogonal vector bundle $\mathcal{W}$ of rank n, and a line bundle $\mathcal{L}$ such that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\bigwedge^2(\mathcal{W} \oplus \mathcal{L} \oplus \mathcal{L}^\vee)$. From the Hodge decomposition of the Lie algebra $\mathfrak{g}$, it follows that $P \times_K \mathfrak{g}^{-1,1} \cong \mathcal{T}_{\mathcal{B}_n} \cong \mathcal{H}om(\mathcal{W}, \mathcal{L})$, $P \times_K \mathfrak{g}^{0,0} \cong \bigwedge^2(\mathcal{W}) \oplus \mathcal{O}_{\mathcal{B}_n}$, and $P \times_K \mathfrak{g}^{1,-1} \cong \Omega^1_{\mathcal{B}_n} \cong \mathcal{H}om(\mathcal{W}, \mathcal{L}^\vee)$. Note that, since $\mathcal{W}$ is an orthogonal bundle, it is self-dual.

We want to determine necessary and sufficient conditions for a projective, klt variety of dimension n with ample canonical divisor to be uniformized by $\mathcal{B}_n$.

8.1 Sufficient conditions

We apply Theorem 1.1 to prove the following.

Proposition 8.1. *Let X be a projective, klt variety of dimension $n \geq 3$ with K_X ample. Suppose that the tangent bundle of the regular locus X_{reg} of X satisfies $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{W}, \mathcal{L})$, where $\mathcal{W}$ is an orthogonal vector* *bundle of rank n, and $\mathcal{L}$ is a line bundle on X_{reg}. Let $\mathcal{W}'$ and $\mathcal{L}'$ denote the reflexive extensions of $\mathcal{W}$ and $\mathcal{L}$ respectively to X, and suppose that the Chern class equality*

$$[\widehat{c}_2(\mathcal{W}' \wedge \mathcal{W}') + 2\widehat{c}_2(\mathcal{W}') - n\widehat{c}_1(\mathcal{L}')^2] \cdot [K_X]^{n-2} = 0 \tag{14}$$

holds on X. Then $X \cong \mathcal{B}_n/\Gamma$, where $\Gamma \subset Aut(\mathcal{B}_n)$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on $\mathcal{B}_n$.

Proof. Let $j : X_{reg} \to X$ denote the natural inclusion. Then we have $\mathcal{T}_X \cong j_*\mathcal{T}_{X_{reg}} \cong j_*\mathcal{H}om(\mathcal{W},\mathcal{L}) \cong \mathcal{H}om(\mathcal{W}',\mathcal{L}')$ by the uniqueness of reflexive extension. Let $G_0 = SO(2,n)$, $K_0 = SO(2)\times SO(n)$ the maximal compact subgroup of G_0, and let G and K denote complexifications of G_0 and K_0 respectively. We choose $G = SO(2+n,\mathbb{C})$, then $K = SO(2,\mathbb{C}) \times SO(n,\mathbb{C})$.
We can deduce from [5, Table 2] that the Lie algebra $\mathfrak{g} = \mathfrak{so}(2+n,\mathbb{C})$ and the typical fiber of the vector bundle $\bigwedge^2(\mathcal{W}\oplus\mathcal{L}\oplus\mathcal{L}^\vee)$ are isomorphic as K-representations, although this is also straightforward to check. Moreover, the vector bundle $\bigwedge^2(\mathcal{W}\oplus\mathcal{L}\oplus\mathcal{L}^\vee)$ admits a reduction in structure group from $GL((n+2)(n+1)/2,\mathbb{C})$ to $K = SO(2,\mathbb{C}) \times SO(n,\mathbb{C})$. Thus there is a principal K-bundle P on X_{reg} such that $P \times_K \mathfrak{g} \cong \bigwedge^2(\mathcal{W}\oplus\mathcal{L}\oplus\mathcal{L}^\vee)$. Recall from the earlier discussion that the Hodge decomposition of $\mathfrak{g}$ gives $P\times_K \mathfrak{g}^{-1,1} \cong \mathcal{H}om(\mathcal{W},\mathcal{L})$, $P \times_K \mathfrak{g}^{0,0} \cong \bigwedge^2\mathcal{W}\oplus\mathcal{O}_{X_{reg}}$, and $P\times_K\mathfrak{g}^{1,-1}\cong\mathcal{H}om(\mathcal{W},\mathcal{L}^\vee)$. Since $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{W},\mathcal{L})$ by assumption, we have the following isomorphism

$$\theta : \mathcal{T}_{X_{reg}} \to P\times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u),\theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. It follows that (P,θ) gives a uniformizing system of Hodge bundles on X_{reg}.
The system of Hodge bundles $E = P\times_K\mathfrak{g} \cong \mathcal{T}_{X_{reg}} \oplus \bigwedge^2\mathcal{W}\oplus\mathcal{O}_{X_{reg}}\oplus\Omega^1_{X_{reg}}$ is in particular a Higgs bundle with Higgs field $\widehat{\theta} : E \to E\otimes\Omega^1_{X_{reg}}$ given by sending a local section u of E to the local section $v \mapsto [\theta(v),u]$ of $\mathcal{H}om(\mathcal{T}_{X_{reg}},E) \cong E\otimes\Omega^1_{X_{reg}}$. Since $\mathcal{W}$ is self-dual, we have that $\det(\mathcal{W}) \cong \mathcal{O}_{X_{reg}}$, and hence $c_1(E) = c_1(P\times_K\mathfrak{g}) = 0$. We know again from Proposition 4.6 that the system of Hodge bundles E is K_X-polystable as a Higgs bundle on X_{reg}. In fact, E is K_X-stable as a Higgs bundle on X_{reg} because $\mathfrak{so}(2+n,\mathbb{C})$ is a simple Lie algebra for $n \geq 3$.
Let $\mathcal{F}_X$ denote the reflexive extension of E to X. It is clear that $\widehat{c}_1(\mathcal{F}_X)\cdot[K_X]^{n-1} = 0$. Moreover by (2), we have

$$\widehat{c}_2(\mathcal{F}_X)\cdot[K_X]^{n-2} = \widehat{c}_2(\bigwedge^2(\mathcal{W}'\oplus\mathcal{L}'\oplus\mathcal{L}'^\vee))\cdot[K_X]^{n-2}.$$

Expanding the right hand side of the above equality gives $[\widehat{c}_2(\mathcal{W}'\wedge\mathcal{W}') + 2\widehat{c}_2(\mathcal{W}') - n\widehat{c}_1(\mathcal{L}')^2]\cdot[K_X]^{n-2}$, which by assumption is zero. Therefore, we have $\widehat{ch}_2(\mathcal{F}_X)\cdot[K_X]^{n-2} = 0$.
It follows from Theorem 1.1 that $X \cong \mathcal{B}_n/\Gamma$, where $\Gamma\subset \mathrm{Aut}(\mathcal{B}_n)$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on $\mathcal{B}_n$. This concludes the proof. □

8.2 Necessary conditions

Note that the full automorphism group of $\mathcal{B}_n$ is a quotient of $G_0 = SO(2,n)$ by a discrete central subgroup, thus the complexification of G_0 is $G = SO(2+n,\mathbb{C})$. The complexification of the maximal compact subgroup of $\mathrm{Aut}(\mathcal{B}_n)$ is a quotient of $K = SO(2,\mathbb{C})\times SO(n,\mathbb{C})$ by a discrete central subgroup. Since G is connected, it folloes from Remark A that the sufficient criteria for uniformization by $\mathcal{B}_n$ are also necessary.

Proposition 8.2. *Let X be a projective klt variety of dimension $n \geq 3$ with K_X ample, such that $X = \mathcal{B}_n/\widehat{\Gamma}$, where $\widehat{\Gamma}\subset Aut(\mathcal{B}_n)$ is a discrete, cocompact subgroup which acts fixed point freely in codimension one on $\mathcal{B}_n$.*

Then the tangent bundle of the regular locus of X satisfies $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{V}, \mathcal{M})$, where $\mathcal{V}$ is an orthogonal vector bundle of rank n, and $\mathcal{M}$ is a line bundle on X_{reg}. Moreover, X satisfies the Chern class equality 14.

Proof. Again from Theorem 1.1, it follows that the smooth locus X_{reg} of X admits a uniformizing system of Hodge bundles (P, θ) corresponding to the Hodge group $G_0 = SO(2, n)$, such that $E = P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}. Furthermore, the equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$ is satisfied, where E' denotes the reflexive extension of E to X. Hence there is an isomorphism $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$. Recall that from the proof of Proposition 8.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\mathrm{Hom}(W, L)$, where W and L are complex n and 1-dimensional vector spaces respectively. So we can write $\mathcal{T}_{X_{reg}} \cong P \times_K \mathrm{Hom}(W, L) = \mathcal{H}om(\mathcal{W}, \mathcal{L})$, where $\mathcal{W}$ is an orthogonal rank n vector bundle, and $\mathcal{L}$ is a line bundle on X_{reg}. We also have $\Omega^1_{X_{reg}} \cong \mathcal{H}om(\mathcal{W}, \mathcal{L}^\vee) \cong P \times_K \mathfrak{g}^{1,-1}$, and $\bigwedge^2 \mathcal{W} \oplus \mathcal{O}_{X_{reg}} \cong P \times_K \mathfrak{g}^{0,0}$, from which it follows that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\bigwedge^2(\mathcal{W} \oplus \mathcal{L} \oplus \mathcal{L}^\vee)$.
Let E', $\mathcal{W}'$, and $\mathcal{L}'$ denote the reflexive extensions of E, $\mathcal{W}$, and $\mathcal{L}$ respectively to X. Then the equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$ is equivalent to

$$\widehat{c}_2(\bigwedge^2(\mathcal{W}' \oplus \mathcal{L}' \oplus \mathcal{L}'^\vee)) \cdot [K_X]^{n-2} = 0.$$

Expanding the above expression using the formulae (1) and (2) for $\mathbb{Q}$-Chern classes, we arrive at the following expression.

$$[\widehat{c}_2(\mathcal{W}' \wedge \mathcal{W}') + 2\widehat{c}_2(\mathcal{W}') - n\widehat{c}_1(\mathcal{L}')^2] \cdot [K_X]^{n-2} = 0.$$

Therefore, X satisfies the $\mathbb{Q}$-Chern class equality (14). □

Putting together Propositions 8.1 and 8.2, we arrive at the following necessary and sufficient condition for a projective klt variety X with ample canonical divisor to be uniformized by the Hermitian symmetric space $\mathcal{B}_n$ of type BDI.

Theorem 8.3. *Let X be a projective klt variety of dimension $n \geq 3$, such that the canonical divisor K_X is ample. Then $X \cong \mathcal{B}_n/\Gamma$, where $\Gamma \subset Aut(\mathcal{B}_n)$ is a discrete, cocompact subgroup acting fixed point freely in codimension one on $\mathcal{B}_n$, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{W}, \mathcal{L})$
- $[\widehat{c}_2(\mathcal{W}' \wedge \mathcal{W}') + 2\widehat{c}_2(\mathcal{W}') - n\widehat{c}_1(\mathcal{L}')^2] \cdot [K_X]^{n-2} = 0,$

where $\mathcal{W}$ is an orthogonal vector bundle of rank n, $\mathcal{L}$ is a line bundle on X_{reg}, and $\mathcal{W}'$ and $\mathcal{L}'$ denote the reflexive extensions of $\mathcal{W}$ and $\mathcal{L}$ to X.

9 Uniformization by Hermitian symmetric space of type $AIII$

The Hermitian symmetric space of type $AIII$, denoted $\mathcal{A}_{pq}$, can be expressed as the quotient G_0/K_0, where we may take $G_0 = SU(p, q)$, and $K_0 = S(U(p) \times U(q))$ its maximal compact subgroup, for $p, q \in \mathbb{Z}_{\geq 1}$, $pq > 1$. The domain $\mathcal{A}_{pq}$ can also be realized as follows (see [30])

$$\mathcal{A}_{pq} = \{Z \in M(p, q, \mathbb{C}) \cong \mathbb{C}^{pq} : I_q - \bar{Z}^T Z > 0\},$$

where the "> 0" again means positive definiteness. The complex dimension of $\mathcal{A}_{pq}$ is pq. The complexified Lie algebra of $SU(p, q)$ is $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$. The Lie algebra $\mathfrak{g}$ has complex dimension $(p+q)^2 - 1$, and again using

the list on [20, p.341], can expressed as the set of complex $(p+q)\times(p+q)$ matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is a $p\times p$ matrix, B is a $p\times q$ matrix, C is a $q\times p$ matrix, and D is a $q\times q$ matrix, and $Tr(A)+Tr(D)=0$. Note that $G_0 = SU(p,q)$ is a Hodge group of Hermitan type, and $\mathfrak{g}$ has a Hodge decomposition $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$ as in Definition 2.22, where $\mathfrak{g}^{-1,1}$ consists of matrices of the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, $\mathfrak{g}^{0,0}$ consists of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, and $\mathfrak{g}^{1,-1}$ consists of matrices of the form $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. The Lie algebras $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$ have complex dimensions pq, p^2+q^2-1, and pq respectively.
Let P be a principal $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$-bundle on $\mathcal{A}_{pq}$ such that $\theta : \mathcal{T}_{\mathcal{A}_{pq}} \cong P \times_K \mathfrak{g}^{-1,1}$, which we know exists by Lemma 3.4. Then there are vector bundles $\mathcal{V}$ and $\mathcal{W}$ of ranks p and q on $\mathcal{A}_{pq}$, such that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$, the bundle of trace zero endomorphisms of $\mathcal{V} \oplus \mathcal{W}$. From the Hodge decomposition of $\mathfrak{g}$, we get $P \times_K \mathfrak{g}^{-1,1} \cong \mathcal{H}om(\mathcal{V},\mathcal{W}) \cong \mathcal{T}_{\mathcal{A}_{pq}}$, $P \times_K \mathfrak{g}^{0,0} \cong (\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0$, and $P \times_K \mathfrak{g}^{1,-1} \cong \mathcal{H}om(\mathcal{W},\mathcal{V}) \cong \Omega^1_{\mathcal{A}_{pq}}$. The domain $\mathcal{A}_q$ corresponding to $p=1$ is the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$. In this case we have $\mathcal{V} \cong \Omega^1_{\mathbb{B}^q}$, $\mathcal{W} = \mathcal{O}_{\mathbb{B}^q}$, and the system of Hodge bundles is given by $P \times_K \mathfrak{g} \cong \mathcal{E}nd_0(\Omega^1_{\mathbb{B}^q} \oplus \mathcal{O}_{\mathbb{B}^q})$

As usual, the goal is to formulate necessary and sufficient conditions for a projective, klt variety of dimension pq with ample canonical divisor to be uniformized by $\mathcal{A}_{pq}$.

9.1 Sufficient conditions

We again apply Theorem 1.1 to make the following characterization.

Proposition 9.1. *Let X be a projective, klt variety of dimension pq for some $p, q \in \mathbb{Z}_{\geq 1}$, $pq > 1$, with K_X ample, such that the tangent bundle of the regular locus X_{reg} of X satisfies $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{V},\mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of ranks p and q on X_{reg}. Let $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X, and suppose that the Chern class equality*

$$[2(p+q)(\widehat{c}_2(\mathcal{V}') + \widehat{c}_2(\mathcal{W}')) - (p+q-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_X]^{n-2} = 0 \tag{15}$$

holds on X. Then $X \cong \mathcal{A}_{pq}/\Gamma$, where $\Gamma \subset Aut(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on the domain $\mathcal{A}_{pq}$ of type AIII.

Proof. Let $j : X_{reg} \to X$ denote the natural inclusion. Then we have $\mathcal{T}_X \cong j_*\mathcal{T}_{X_{reg}} \cong j_*\mathcal{H}om(\mathcal{V},\mathcal{W}) \cong \mathcal{H}om(\mathcal{V}',\mathcal{W}')$ by the uniqueness of reflexive extension. Let $G_0 = SU(p,q)$, $K_0 = S(U(p)\times U(q))$ the maximal compact subgroup of G_0, and let G and K denote complexifications of G_0 and K_0 respectively. We choose $G = SL(p+q,\mathbb{C})$ and $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$.
Let $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$ denote the bundle of trace zero endomorphisms of $\mathcal{V} \oplus \mathcal{W}$. We deduce again from [5, Table 2], that the Lie algebra $\mathfrak{g} = \mathfrak{sl}(p+q,\mathbb{C})$ and the typical fiber $\mathrm{End}_0(\mathbb{C}^p,\mathbb{C}^q)$ of $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$ are isomorphic as K-representations. Moreover, the vector bundle $\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$ admits a reduction in structure group from $GL((p+q)^2-1,\mathbb{C})$ to $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$. Thus there is a principal K-bundle P on X_{reg} associated to $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$ i.e., we have $P \times_K \mathfrak{g} \cong \mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$. The Hodge decomposition of $\mathfrak{g}$ together with identifying isomorphic K-representations using [5, Table 2] gives $P \times_K \mathfrak{g}^{-1,1} \cong \mathcal{H}om(\mathcal{V},\mathcal{W})$, $P\times_K \mathfrak{g}^{0,0} \cong (\mathcal{E}nd(\mathcal{V})\oplus\mathcal{E}nd(\mathcal{W}))_0$, and $P\times_K \mathfrak{g}^{1,-1} \cong \mathcal{H}om(\mathcal{W},\mathcal{V})$. Since $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{V},\mathcal{W})$ by assumption, we have the following isomorphism

$$\theta : \mathcal{T}_{X_{reg}} \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. It follows that (P, θ) gives a uniformizing system of Hodge bundles on X_{reg}.
The system of Hodge bundles $E = P \times_K \mathfrak{g} \cong \mathcal{T}_{X_{reg}} \oplus (\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0 \oplus \Omega^1_{X_{reg}}$ is in particular a Higgs bundle with Higgs field $\widehat{\theta} : E \to E \otimes \Omega^1_{X_{reg}}$ given by sending a local section u of E to the local section $v \mapsto [\theta(v), u]$ of $\mathcal{H}om(\mathcal{T}_{X_{reg}}, E) \cong E \otimes \Omega^1_{X_{reg}}$. Note that E fits into the following short exact sequence

$$0 \to E = \mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W}) \to \mathcal{E}nd(\mathcal{V} \oplus \mathcal{W}) \to \mathcal{L} \to 0 \tag{16}$$

where $\mathcal{L}$ is a line bundle of degree zero. Thus we have $c_1(E) = c_1(\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})) = 0$. From Proposition 4.6, it follows that E is K_X-polystable as a Higgs bundle on X_{reg}, and in fact E is K_X-stable as a Higgs bundle on X_{reg} because $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$ is a simple Lie algebra.
Let $\mathcal{F}_X$ denote the reflexive extension of E to X. Then $\mathcal{F}_X$ also fits into an exact sequence

$$0 \to \mathcal{F}_X \to \mathcal{E}nd(\mathcal{V}' \oplus \mathcal{W}') \to \mathcal{L}' \to 0$$

where $\mathcal{L}'$ is a coherent sheaf of degree 0. Thus $\widehat{c}_1(\mathcal{F}_X) \cdot [K_X]^{n-1} = \widehat{c}_1(\mathcal{E}nd(\mathcal{V}' \oplus \mathcal{W}')) \cdot [K_X]^{n-1} = 0$, and $\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = \widehat{c}_2(\mathcal{E}nd(\mathcal{V}' \oplus \mathcal{W}')) \cdot [K_X]^{n-2}$. Moreover, we compute using (2)

$$\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = [2(p+q)(\widehat{c}_2(\mathcal{V}') + \widehat{c}_2(\mathcal{W}')) - (p+q-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_X]^{n-2},$$

so by the assumption of the Proposition we have $\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = \widehat{ch}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = 0$.
Thus the conditions of Theorem 1.1 are satisfied, and it follows that $X \cong \mathcal{A}_{pq}/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup acting fixed point freely in codimension one on the domain $\mathcal{A}_{pq}$. This concludes the proof. ☐

In order to determine the necessary conditions for uniformization by $\mathcal{A}_{pq}$, we consider two separate cases, namely when (i) $p \neq q$, and (ii) $p = q$. This is because the group of automorphisms $\mathrm{Aut}(\mathcal{A}_{pq})$ of $\mathcal{A}_{pq}$ is connected when $p \neq q$, and is a quotient of $SU(p,q)$ by a discrete central subgroup. However when $p = q$, $\mathrm{Aut}(\mathcal{A}_{pp})$ has two connected components, and is a quotient of $SU(p,p) \rtimes \mathbb{Z}_2$ by a discrete central subgroup (see [37]).

9.2 Necessary conditions when $p \neq q$

From [37, p. 114], we know that the automorphism group of $\mathcal{A}_{pq}$ when $p \neq q$ is $\mathrm{Aut}(\mathcal{A}_{pq}) = PSU(p,q)$. This is a connected Lie group and is a quotient of $SU(p,q)$ by a discrete central subgroup. Thus we can choose the associated Hodge group to be $G_0 = SU(p,q)$, and its complexification as $G = SL(p+q, \mathbb{C})$, which is also connected. Thus the sufficient conditions of Proposition 9.1 are also necessary conditions for a projective klt variety with ample canonical divisor to be uniformized by $\mathcal{A}_{pq}$. The proof is essentially the same as that of Proposition 6.2.

Proposition 9.2. *Let X be a projective klt variety of dimension pq, $p \neq q$ with K_X ample, such that $X = \mathcal{A}_{pq}/\widehat{\Gamma}$, where $\widehat{\Gamma} \subset Aut(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup acting fixed point freely in codimension one on $\mathcal{A}_{pq}$. Then the tangent bundle of the regular locus of X satisfies $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of ranks p and q on X_{reg}. Moreover, X satisfies the Chern class equality 15.*

Proof. By Theorem 1.1, the smooth locus X_{reg} of X admits a uniformizing system of Hodge bundles (P, θ) corresponding to the Hodge group $G_0 = SU(p,q)$, such that the system of Hodge bundles $E = P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}. Moreover, X satisfies the $\mathbb{Q}$-Chern class equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$, where E' denotes the unique reflexive extension of E to X. Hence there is an isomorphism $\theta : \mathcal{T}_{X_{reg}} \cong$

$P \times_K \mathfrak{g}^{-1,1}$. Recall from the proof of Proposition 9.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\mathrm{Hom}(V, W)$, where V and W are complex p and q-dimensional vector spaces. So we can write $\mathcal{T}_{X_{reg}} \cong P \times_K \mathrm{Hom}(V, W) = \mathcal{H}om(\mathcal{V}, \mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of ranks p and q on X_{reg}. We also have $\Omega^1_{X_{reg}} \cong \mathcal{H}om(\mathcal{W}, \mathcal{V}) \cong P \times_K \mathfrak{g}^{1,-1}$, and $(\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0 \cong P \times_K \mathfrak{g}^{0,0}$, from which it follows that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$. Let $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X. Then the Chern class equality $\widehat{c}_2(E') \cdot [K_X]^{n-2} = 0$ is equivalent to

$$\widehat{c}_2(\mathcal{E}nd_0(\mathcal{V}' \oplus \mathcal{W}')) \cdot [K_X]^{n-2} = 0.$$

Using the short exact sequence (16), and the formula (2) for the second $\mathbb{Q}$-Chern class of a direct sum of reflexive sheaves, the above equality can be rephrased as

$$[2(p+q)(\widehat{c}_2(\mathcal{V}') + \widehat{c}_2(\mathcal{W}')) - (p+q-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_X]^{n-2} = 0.$$

Thus it follows that X satisfies the Chern class equality (15), which completes the proof. ☐

Putting together Propositions 9.1 and 9.2, we arrive at the following necessary and sufficient condition for a projective klt variety X with ample canonical divisor to be uniformized by the Hermitian symmetric space $\mathcal{A}_{pq}$ ($p \neq q$) of type $AIII$.

Theorem 9.3. *Let X be a projective klt variety of dimension $pq \geq 2$, $p, q \in \mathbb{Z}_{\geq 1}$, $p \neq q$, such that the canonical divisor K_X is ample. Then $X \cong \mathcal{A}_{pq}/\Gamma$, where $\Gamma \subset Aut(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on $\mathcal{A}_{pq}$, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{V}, \mathcal{W})$
- $[2(p+q)(\widehat{c}_2(\mathcal{V}') + \widehat{c}_2(\mathcal{W}')) - (p+q-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_X]^{n-2} = 0,$

where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of ranks p and q on X_{reg}, and $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X.

Note that when $p = 1$ and $q = n$, the associated domain is the unit ball $\mathbb{B}^n$. In this case we have $\mathcal{V} = \Omega^1_{X_{reg}}$, $\mathcal{W} = \mathcal{O}_{X_{reg}}$, and the system of Hodge bundles is given by $P \times_K \mathfrak{g} = \mathcal{E}nd_0(\Omega^1_{X_{reg}} \oplus \mathcal{O}_{X_{reg}})$. The condition on the structure of the tangent bundle is tautological, and the Chern class condition is the well known Bogomolov-Miyaoka-Yau equality satisfied by ball quotients.

9.3 Necessary conditions when $p = q$

In this case the group of holomorphic automorphisms has two connected components. Again from [37] we know that $Z \mapsto Z^T$ is an automorphism of $\mathcal{A}_{pp}$ of order 2 that is not contained in the connected component of $\mathrm{Aut}(\mathcal{A}_{pp})$. This defines a group homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{A}_{pp})$, i.e., there is a splitting $\mathrm{Aut}(\mathcal{A}_{pp}) = \mathrm{Aut}^0(\mathcal{A}_{pp}) \rtimes \mathbb{Z}_2$, where $\mathrm{Aut}^0(\mathcal{A}_{pp}) = PSU(p,p)$. It follows that $\mathrm{Aut}(\mathcal{A}_{pp})$ is a quotient of $SU(p,p) \rtimes \mathbb{Z}_2$ by a discrete central subgroup. Let G denote the chosen complexification of $SU(p,p) \rtimes \mathbb{Z}_2$, and K the complexification of $K_0 = S(U(p) \times U(p)) \rtimes \mathbb{Z}_2$. Then the tangent bundle of the regular locus of a projective, klt quotient of $\mathcal{A}_{pp}$ admits a reduction in structure group to K. However, as in the polydisc case, such a reduction in structure group does not give a meaningful condition on the tangent sheaf. Therefore, we state the result up to a 2:1 quasi-étale cover. The proof is essentially the same as that of Proposition 6.2.

Proposition 9.4. *Let X be a projective klt variety of dimension p^2 with K_X ample, such that X is a quotient of $\mathcal{A}_{pp}$ by a discrete, cocompact subgroup Γ of $Aut(\mathcal{A}_{pp})$ acting fixed point freely in codimension one. Then*

X admits a 2:1 quasi-étale cover X' such that the tangent bundle of the regular locus $X'_{reg} \subset X'$ satisfies $\mathcal{T}_{X'_{reg}} \cong \mathcal{H}om(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of rank p on X_{reg}. Moreover, X' satisfies the Chern class equality 15 for $p = q$.

Proof. Since the automorphism group $\mathrm{Aut}(\mathcal{A}_{pp})$ has two connected components, the group Γ can be expressed as an extension of a normal subgroup Γ' by a subgroup of $\mathbb{Z}_2$. Thus the quotient map $\mathcal{A}_{pp} \to X$ factors as

$$\mathcal{A}_{pp} \to \mathcal{A}_{pp}/\Gamma' \to \mathcal{A}_{pp}/\Gamma = X,$$

where $X' = \mathcal{A}_{pp}/\Gamma'$, and the map $X' \to X$ is a Galois, 2:1 quasi-étale cover. Note that X' is again projective, klt, and $K_{X'}$ is ample. Again from Theorem 1.1, it follows that the smooth locus X'_{reg} of X' admits a uniformizing system of Hodge bundles (P, θ) corresponding to the Hodge group $G_0 = SU(p, p)$, such that the system of Hodge bundles $E = P \times_K \mathfrak{g}$ is $K_{X'}$-polystable as a Higgs bundle on X'_{reg}. Moreover, X' satisfies the $\mathbb{Q}$-Chern class equality $\widehat{c}_2(E') \cdot [K_{X'}]^{n-2} = 0$, where E' is the reflexive extension of E to X. Therefore, there is an isomorphism $\theta : \mathcal{T}_{X'_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$ of vector bundles. Recall from the proof of Proposition 9.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\mathrm{Hom}(V, W)$, where V and W are p-dimensional complex vector spaces. So we can write the tangent bundle as $\mathcal{T}_{X'_{reg}} \cong P \times_K \mathrm{Hom}(V, W) = \mathcal{H}om(\mathcal{V}, \mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of rank p on X'_{reg}. Moreover, we have $\Omega^1_{X'_{reg}} \cong P \times_K \mathfrak{g}^{1,-1} \cong \mathcal{H}om(\mathcal{W}, \mathcal{V})$, and $P \times_K \mathfrak{g}^{0,0} \cong (\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0$. Thus the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$. Let $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X'. Then $E' \cong \mathcal{E}nd_0(\mathcal{V}' \oplus \mathcal{W}')$, and the $\mathbb{Q}$-Chern class equality $\widehat{c}_2(E') \cdot [K_{X'}]^{n-2} = 0$ is equivalent to

$$[\widehat{c}_2(\mathcal{E}nd_0(\mathcal{V}', \mathcal{W}'))] \cdot [K_{X'}]^{n-2} = 0.$$

Expanding this using Chern class formulae (1) and (2) we arrive at the following expression

$$[4p(\widehat{c}_2(\mathcal{V})' + \widehat{c}_2(\mathcal{W}')) - (2p-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_{X'}]^{n-2} = 0.$$

Thus X' satisfies the Chern class equality as claimed, and this completes the proof. □

Example 9.5. As mentioned earlier, when $p = 1$, the corresponding domain $\mathcal{A}_{1q}$ is the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$. It is well known that projective klt ball quotients satisfy the $\mathbb{Q}$-Bogomolov-Miyaoka-Yau equality.
In the two-dimensional case, there are many interesting examples of smooth ball quotient surfaces such as the *fake projective planes* considered by Keum in [23].
Hirzebruch described a method to construct smooth projective surfaces X satisfying $c_1(X)^2 = 3c_2(X)$ starting with line arrangements on the projective plane $\mathbb{P}^2$ in [21].
Another class of examples is given by algebraic surfaces X such that the intersection form on $H^2(X, \mathbb{Q})$ is positive definite. These are smooth surfaces of general type and satisfy $c_1(X)^2 = 9$ and $c_2(X) = 3$, so in particular they are ball quotients.
One can obtain examples of singular ball quotients by taking quotients of fake projective planes by finite automorphism groups. There are only four possibilities for non-trivial automorphism groups of fake projective planes, determined by Prasad and Yeung in [33]. Namely, the groups $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/7\mathbb{Z}$, $7:3$, and $(\mathbb{Z}/3\mathbb{Z})^2$. Quotients of fake projective planes and their minimal resolutions have been classified completely by Keum in [23]. When the group of automorphisms has prime order, he obtains the following.

Theorem 9.6 ([23, Theorem 1.1]). *Let G be a group of automorphisms of a fake projective plane X. Let $Z = X/G$ and $\nu : Y \to Z$ be a minimal resolution. Then the following statements hold.*

- *If the order of G is 3, then Z has 3 singular points of type $\frac{1}{3}(1, 2)$, and Y is a minimal surface of general type with $K_Y^2 = 0$, $p_g = 0$.*

- *If the order of Y is 7, then Z has 3 singular points of type $\frac{1}{7}(1,3)$, and Y is a minimal elliptic surface of Kodaira dimension 1 with 2 multiple fibres. The pair of the multiplicities is one of the following three cases:* $(2,3)$, $(2,4)$, $(3,3)$.

Note that the surface Z is a singular ball quotient with isolated quotient singularities.

References

[1] R.C. Alperin, "An elementary account of Selberg's lemma." *Enseign. Math(2)*, 33(3-4):269-273, (1987).

[2] Ingrid Bauer, Fabrizio Catanese, Fritz Grunewald, and Roberto Pignatelli, "Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups." *American Journal of Mathematics*, 134(4), 993-1049 (2012).

[3] Arnaud Beauville, "Complex manifolds with split tangent bundle." *Complex analysis and algebraic geometry*, de Gruyter, Berlin, 61–70 (2000).

[4] Nicolas Bourbaki, *Éléments de mathématique: Groupes et algèbres de lie, chapitres 2-3.* Paris: Hermann, (1972).

[5] Steven B. Bradlow, Oscar Garcia-Prada, and Peter B. Gothen, "Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces", *Geometriae Dedicata*, 122(1), 185-213 (2006).

[6] James Carlson, Stefan Müller-Stach, and Chris Peters, "Period mappings and period domains", Cambridge University Press, (2017).

[7] Fabrizio Catanese, Antonio J. Di Scala, "A characterization of varieties whose universal cover is the polydisk or a tube domain." *Mathematische Annalen* 356(2), 419-438 (2013).

[8] Ya Deng, Benoit Caldorel, "A characterization of complex quasi-projective manifolds uniformized by unit balls." *Mathematische Annalen*, 384, 1833-1881, (2022).

[9] Gerard van der Geer, "The cohomology of the moduli space of abelian varieties." *Handbook of Moduli Vol. I, Volume 24 of Adv. Lect. Math. (ALM).* Somerville, MA: International Press, 415-457, (2013).

[10] Patrick Graf, "The generalized Lipman-Zariski problem.", *Mathematische Annalen*, 362(1-2), 241-264 (2015).

[11] Daniel Greb, Stefan Kebekus, Sandor J. Kovacs, and Thomas Peternell, "Differential forms on log canonical spaces". *Publications mathematiques de l'IHES* 114(1), 87-169 (2011).

[12] Daniel Greb, Stefan Kebekus, and Thomas Peternell, "Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties" *Duke Mathematical Journal*, 165(10), 1965-2004 (2016).

[13] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji, "The Miyaoka-Yau inequality and uniformisation of canonical models." *Ann. Sci. Ec. Norm. Supér. (4)*, 52(6):1487–1535, (2019).

[14] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji, "Nonabelian Hodge theory for klt spaces and descent theorems for vector bundles" *Compositio Mathematica*, 155(2), 289-323 (2019).

[15] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji, "Harmonic metrics on Higgs sheaves and uniformization of varieties of general type." *Mathematische Annalen*, 378(3), 1061-1094 (2020).

[16] Daniel Greb, Stefan Kebekus, and Thomas Peternell, "Projectively flat KLT varieties" *Journal de l'École polytechnique — Mathématiques*, 8, (2021).

[17] Daniel Greb, Stefan Kebekus, and Thomas Peternell, "Projective flatness over klt spaces and uniformisation of varieties with nef anti-canonical divisor." *J. Algebraic Geom.*, 31, 467-496 (2022).

[18] Henri Guenancia, "Semistability of the tangent sheaf of singular varieties", *Algebraic Geometry*, 3(5), 508-542, (2016).

[19] Robin Hartshorne, "Local Cohomology: A seminar given by A. Grothendieck, Harvard University. Fall, 1961" *Springer*, Vol. 41 (2006).

[20] Sigurdur Helgason, "Differential geometry, Lie groups, and symmetric spaces", Academic Press, (1979).

[21] Friedrich Hirzebruch, "Arrangements of lines and algebraic surfaces", *Arithmetic and geometry. Birkhäuser, Boston, MA*, 113-140, (1983).

[22] Daniel Huybrechts, *Complex geometry: an introduction*, Vol. 78. Berlin: Springer (2005).

[23] JongHae Keum, "Quotients of fake projective planes", *Geometry and Topology*, 12(4), 2497-2515, (2008).

[24] János Kóllar, Shigefumi Mori, Charles Herbert Clemens, and Alessio Corti, "Birational geometry of algebraic varieties", Cambridge University Press, Vol. 134 (1998).

[25] Anthony W. Knapp and A. W. Knapp, "Lie groups beyond an introduction", Vol. 140, Boston: Birkhäuser, (1996).

[26] Adrian Langer, "A note on Bogomolov's instability and Higgs sheaves." *Algebraic Geometry: A Volume in Memory of Paolo Francia*, edited by Mauro C. Beltrametti, Fabrizio Catanese, Ciro Ciliberto, Antonio Lanteri and Claudio Pedrini, Berlin, New York: De Gruyter, 237-256 (2002).

[27] Adrian Langer, "Semistable modules over Lie algebroids in positive characteristic." *Documenta Mathematica*, 19, 509-540 (2014).

[28] Adrian Langer, "Bogomolov's inequality and Higgs sheaves on normal varieties in positive characteristic", arXiv preprint, *arXiv:2210.08767* (2022).

[29] James S. Milne, "Lie algebras, algebraic groups, and Lie groups, 2013" *Available on www.jmilne.org/math* (2013).

[30] Ngaiming Mok, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Vol. 6, World Scientific, (1989).

[31] Shigeyuki Morita, "Geometry of characteristic classes", No. 199, American Mathematical Society, (2001).

[32] David Mumford, "Towards an enumerative geometry of the moduli space of curves", *Arithmetic and Geometry, vol. II, volume 36 of Progr. Math.*, 271-328, Birkhäuser Boston (1983).

[33] Gopal Prasad and Sai-Kee Yeung, "Fake projective planes", *Inventiones Mathematicae*, 168(2), 321-370 (2007).

[34] Carlos T. Simpson, "Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization." *Journal of the American Mathematical Society*, 867-918, (1988).

[35] Carlos T.Simpson, "Higgs bundles and local systems." *Publications Mathematiques de l'IHES* 75, 5-95 (1992).

[36] Carlos T. Simpson, "Moduli of representations of the fundamental group of a smooth projective variety II." *Publications Mathematiques de l'IHES* 80, 5-79 (1995).

[37] Masaru Takeuchi, "On the fundamental group and the group of isometries of a symmetric space", *J. Fac. Sci. Univ. Tokyo Sect. I* 10, 88-123 (1964).

[38] Aldo Conca, Sandra Di Rocco, Jan Draisma, June Huh, Bernd Strumfels, Filippo Viviani, and Filippo Viviani, "A tour on Hermitian symmetric manifolds" *Combinatorial algebraic geometry: Levico Terme, Italy 2013, Editors: Sandra Di Rocco, Bernd Strumfels*, 149-239 (2014).

[39] S.T. Yau, "Calabi's conjecture and some new results in algebraic geometry.", *Proceedings of the National Academy of Sciences* 74(5), 1798-1799 (1977).