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THREE-FOLD PRODUCTS OF ELLIPTIC CURVES, p-ADIC L-FUNCTIONS AND RATIONAL POINTS

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Preface. This thesis is the result of the last ten years of me studying mathematics. It has been quite a journey and I would not be here without the various sorts of help from various people.

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Jonas Franzel

Abstract. The present dissertation consists of two main parts. Both parts arise from the desire of a systematic construction of rational points on elliptic curves. On the one hand, we study the existence of parametrizations of an elliptic curve over the rationals by considering merely its self-fold triple product. This is done via an analysis of Galois representations and its follow-up contextualization into the framework of the Tate Conjecture. On the other hand, we construct a new p-adic L-function for the symmetric cube of a Hida family. This p-adic L-function provides a factorization of a restricted balanced triple product p-adic L-function. Moreover, we prove an interpolation property of this novel p-adic L-function and formulate a conjecture on the leading term of its Taylor expansion at a point of vanishing.

Zusammenfassung. Die vorliegende Dissertation verfolgt zwei Hauptziele, die beide aus dem Wunsch nach einer systematischen Konstruktion rationaler Punkte auf elliptischen Kurven entstehen. Einerseits untersuchen wir die Existenz von Parametrisierungen einer elliptischen Kurve über den rationalen Zahlen, indem wir lediglich ihr selbstgefaltetes Tripelprodukt in Betracht ziehen. Dies geschieht durch eine Analyse von Galois-Darstellungen und derer anschließenden Einordnung in den Kontext der Tate-Vermutung. Andererseits konstruieren wir eine neue p-adische L-Funktion für den symmetrischen Kubus einer Hida-Familie. Diese p-adische L-Funktion liefert eine Faktorisierung einer eingeschränkten, ausgewogenen p-adischen Tripelprodukt L-Funktion. Wir beweisen darüber hinaus eine Interpolationseigenschaft dieser neuartigen p-adischen L-Funktion und formulieren eine Vermutung über den führenden Term ihrer Taylor-Entwicklung an einem Nullpunkt.

Contents

1 Introduction

Let $A/\mathbb{Q}$ be an elliptic curve of conductor N. The algebro-geometric version of the celebrated modularity theorem of Wiles [Wil95], Taylor–Wiles [TW95] and Breuil–Conrad–Diamond–Taylor [BCDT01] provides the existence of a so-called *modular parametrization*

$$\mathrm{CH}^1(X_0(N))_0 \longrightarrow A, \tag{1.1}$$

where $X_0(N)$ denotes the modular curve of level $\Gamma_0(N)$ and $\mathrm{CH}^1(\,\cdot\,)_0$ denotes the subgroup of *null-homologous* algebraic cycles in the Chow group $\mathrm{CH}^1(\,\cdot\,)$.

Among several equivalent formulations of the *modularity* of elliptic curves over the rationals, the above form is of particular interest in the study of points on the elliptic curve. In the language of L-functions, the modularity theorem states that there exists a newform $f_A \in \mathrm{S}_2(\Gamma_0(N))$ with rational Fourier coefficients such that

$$L(A/\mathbb{Q}, s) = L(f_A, s).$$

This is established by showing that the p-adic Tate module $V_p(A) = \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1)$ is a constituent of the p-adic étale cohomology of the modular curve $X_0(N)$. The parametrization (1.1) is then deduced from combining the work of Shimura [Shi71] and Carayol [Car86] with Faltings' proof [Fal83] of a certain case of the Tate Conjecture for abelian varieties over number fields. More precisely, the *Eichler–Shimura Construction* provides an elliptic curve E_{f_A} attached to f_A, constructed as a quotient of $\mathrm{CH}^1(X_0(N))_0$, whose Hasse–Weil L-series equals the complex L-function of f_A up to finitely many factors. With the complete equality being established by Carayol, the *Isogeny Theorem* of Faltings shows that A and E_{f_A} are isogenous as a consequence of having the same L-series, and we can form the composition

$$\mathrm{CH}^1(X_0(N))_0 \longrightarrow E_{f_A} \longrightarrow A.$$

A key feature of (1.1) stems from considering special collections of algebraic points on $X_0(N)$, arising from the moduli of elliptic curves with complex multiplication by a quadratic imaginary field K. The resulting images of certain degree-zero divisors supported on these points provide points on A which are defined over abelian extensions of K. These include the so-called *Heegner points*, which provide one of the most fruitful approaches to the Birch–Swinnerton-Dyer (BSD) Conjecture, such as the result

$$\mathrm{ord}_{s=1} L(A/\mathbb{Q}, s) \leq 1 \quad \Longrightarrow \quad \mathrm{rank}(A(\mathbb{Q})) = \mathrm{ord}_{s=1} L(A/\mathbb{Q}, s) \quad \text{and} \quad \#\text{Ш}(A/\mathbb{Q}) < \infty.$$

This statement follows from combining the *Gross–Zagier Formula* [GZ86], which relates canonical heights of Heegner points to the central critical derivatives of $L(A/K, s)$, with a method of Kolyvagin [Kol90]. We refer the interested reader to the subsequent work of Gross [Gro91] for a discussion of this.

Aiming to generalize (1.1), one option is to replace $X_0(N)$ by a variety X of dimension $d > 1$ and $\mathrm{CH}^1(X_0(N))_0$ by $\mathrm{CH}^j(X)_0$ for $0 \leq j \leq d$. Bertolini–Darmon–Prasanna [BDP14] discuss this approach for the product of a Kuga–Sato variety with a self-fold product of the considered elliptic curve in a complex multiplication situation and provide a novel construction of so-called *Chow–Heegner points* on the elliptic curve.

In contrast to the setting of [BDP14], the main idea of the present dissertation is to study the existence of such a parametrization considering merely the three-fold product A^3 and without assuming A to have complex multiplication. This is done via an analysis on the level of Galois representations and putting this into the context of the Tate Conjecture.

We give a brief overview of our approach. The Tate Conjecture claims the surjectivity of the associated (geometric) p-adic étale cycle class map, whose construction we will recall in Section 2.5, for smooth projective varieties Z defined over a number field K:

$$\mathrm{cl}^p_{\text{ét},\mathrm{g}} \otimes \mathbb{Q}_p \colon \mathrm{CH}^n(Z)(K) \otimes \mathbb{Q}_p \longrightarrow \mathrm{H}^{2n}_{\text{ét}}(\bar{Z}, \mathbb{Q}_p)(n)^{G_K}.$$

In our case, we study the p-adic Tate module $V_p(A)$ of the elliptic curve and its *appropriately twisted* triple tensor product $V_p(A)^{\otimes 3}(-1) = \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)^{\otimes 3}(2)$. As will be explained in greater detail in Section 3, this particular twist allows us to establish a projection of p-adic Galois representations

$$\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)^{\otimes 3}(2) \longrightarrow\!\!\!\!\!\rightarrow V_p(A)^{\oplus 2} \tag{1.2}$$

onto two copies of the p-adic Tate module of A. This is done by considerations on the level of Frobenius-traces. The left-hand side of (1.2) is a part of the *Künneth decomposition* for the triple product:

$$\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) = \bigoplus_{j_1+j_2+j_3=3} \mathrm{H}^{j_1}_{\text{ét}}(\bar{A}, \mathbb{Q}_p) \otimes \mathrm{H}^{j_2}_{\text{ét}}(\bar{A}, \mathbb{Q}_p) \otimes \mathrm{H}^{j_3}_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(2).$$

Therefore, (1.2) provides two $G_{\mathbb{Q}}$-equivariant projections

$$\pi_{\text{ét},i} \colon \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) \longrightarrow\!\!\!\!\!\rightarrow V_p(A). \tag{1.3}$$

As will be explained in greater detail in Section 4, these components $\pi_{\text{ét},i}$ may be viewed as non-trivial elements of $\mathrm{H}^4_{\text{ét}}(\bar{A}^4, \mathbb{Q}_p)(2)^{G_{\mathbb{Q}}}$ by using *Poincaré duality* along with a Künneth decomposition as above. Hence, by assuming the Tate Conjecture in this setting, we obtain elements $\Pi_i^{?,(p)} \in \mathrm{CH}^2(A^4)(\mathbb{Q}) \otimes \mathbb{Q}_p$. By basic notions from the theory of algebraic cycles and intersection theory, which we will recall in Section 2, such elements induce maps

$$\begin{aligned} \Pi^{?,(p)}_{i,*} \colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \otimes \mathbb{Q}_p &\longrightarrow A(\mathbb{Q}) \otimes \mathbb{Q}_p \\ \Gamma &\longmapsto \mathrm{pr}_{A,*}(\Pi_i^{?,(p)} \cdot \mathrm{pr}^*_{A^3}(\Gamma)). \end{aligned}$$

Here, $\mathrm{pr}_{A,*}$ denotes the pushforward and $\mathrm{pr}^*_{A^3}$ denotes the pullback on Chow groups with respect to the natural projections pr_A and pr_{A^3} from $A^4 = A^3 \times A$ onto A and A^3, and $\cdot$ denotes the intersection product. Note that, as a consequence of the formulation of the Tate Conjecture, we have to involve p-adic coefficients on the level of the algebraic cycles, so that we only obtain a parametrization with coefficients in $\mathbb{Q}_p$. Therefore, we are led to address a possible normalization procedure in order to obtain a parametrization defined over the rationals. Using methods from p-adic Hodge theory to be recalled in Section 2.3, including a *comparison isomorphism*, we pass to de Rham cohomology over $\mathbb{Q}_p$ and obtain analogues

$$\pi_{\mathrm{dR},i} \colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2] \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1] \tag{1.4}$$

of the projections $\pi_{\text{ét},i}$ of Galois representations. The advantage is that the de Rham cohomology has a natural rational structure which we can use for *normalizing* the maps.

The above mentioned article [BDP14] was a first inspiration for the following result.

Theorem A. *If the Tate Conjecture is true for A^4 in codimension two, then there exist algebraic cycles $\Pi_i^? \in \mathrm{CH}^2(A^4)(\mathbb{Q})$ inducing rational parametrizations*

$$\Pi_{i,*}^? \colon\ \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow A(\mathbb{Q})$$

along with projections

$$\Pi_{i,\text{ét},*}^{?,p} \colon\ \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) \twoheadrightarrow V_p(A) \quad \textit{and} \quad \Pi_{i,\mathrm{dR},*}^? \colon\ \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \twoheadrightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1],$$

which approximate the non-conjectural maps $\pi_{\text{ét},i}$ of (1.3) *and their analogues $\pi_{\mathrm{dR},i}$ of* (1.4) *in de Rham cohomology.*

We recall how to obtain the induced maps $\Pi_{i,\mathrm{dR},*}^?$ in Section 2.2, using a cycle class map, a Künneth decomposition and Poincaré duality in de Rham cohomology. The maps $\Pi_{i,\text{ét},*}^{?,p}$ are their p-adic analogues in étale cohomology and are described in Section 2.5.

In particular, Theorem A provides parametrizations of the elliptic curve which map algebraic cycles $\Gamma \in \mathrm{CH}^2(A^3)_0(L)$ defined over any number field L to points $P_i^?(\Gamma) \in A(L)$ on A defined over the same number field. This is explained in greater detail in Section 4, using the notions recalled in Section 2. Moreover, this study of parametrizations via threefold products leads us to a conjecture on the rank of its relevant Chow group in codimension two. As the parametrizations $\Pi_{i,*}^?$ conjecturally arise from two independent components of a projection, we pose the following implication:

$$\mathrm{rank}(A(\mathbb{Q})) \geq 1 \quad \overset{?}{\Longrightarrow} \quad \mathrm{rank}(\mathrm{CH}^2(A^3)_0(\mathbb{Q})) \geq 2.$$

We will consider this aspect towards the end of Section 4.

It turns out that the underlying idea of (1.2) can be carried out more generally, considering eigenforms $\theta \in \mathrm{S}_k(\Gamma_0(N))$ of arbitrary even weight $k \geq 2$. This yields the following decomposition result on the level of p-adic Galois representations, which is discussed in Section 3.3. Here, $\rho_{\theta,p}$ denotes the $G_{\mathbb{Q}}$-representation attached to θ as constructed by Deligne [Del71], whose properties are recalled in Section 3.1.

Proposition B. *There is an isomorphism of p-adic Galois representations*

$$\rho_{\theta,p}^{\otimes 3}(1-k) \cong \mathrm{Sym}^3(\rho_{\theta,p})(1-k) \oplus \rho_{\theta,p}^{\oplus 2}. \tag{1.5}$$

The map of (1.2) now arises from taking θ to be the eigenform $f_A \in \mathrm{S}_2(\Gamma_0(N))$ attached to A as a consequence of the modularity theorem.

This result naturally draws our attention to the symmetric cube that appears in the decomposition (1.5), which will be the main focus for the rest of the thesis. With Proposition B leading to a factorization of complex L-functions

$$L(\theta^{\otimes 3}, s+k-1) = L(\mathrm{Sym}^3\theta, s+k-1) \cdot L(\theta, s)^2, \tag{1.6}$$

we move on to the second main part of the thesis – the study of p-adic L-functions. When the forms θ vary in a p-adic Hida family $\boldsymbol{\theta}$ of square-free level, we have the following known p-adic L-functions at hand:

- The restriction of the Mazur–Kitagawa p-adic L-function (cf. [MSD74], [MTT86] [Kit91], [GS93]) to the central critical line:

$$L_p^{\mathrm{cc}}(\boldsymbol{\theta})(k) = L_p^{\mathrm{MK}}(\boldsymbol{\theta})(k, \tfrac{k}{2}),$$

- The restriction of the balanced triple product p-adic L-function (cf. [Hsi21]) to the diagonal:

$$L_p^{\Delta}(\boldsymbol{\theta})(k) = \mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{\theta}, \boldsymbol{\theta}, \boldsymbol{\theta})(k,k,k)^2.$$

These particular restrictions interpolate the central critical values appearing for $s = \frac{k}{2}$ in (1.6) for the complex L-functions attached to the forms $\theta_k^\sharp$ that are associated with the weight-k specializations of $\boldsymbol{\theta}$. Our goal is to establish a factorization of the above form on the level of p-adic L-functions. We approach this by introducing a p-adic L-function $L_p^{\mathrm{Sym}^3}(\boldsymbol{\theta})(k)$ which interpolates the central critical values for the complex symmetric cube L-functions. Under Assumptions 1, 2, 3, specified in Section 5.5, we are able to prove the following result on our proposed p-adic L-function.

Theorem C. *Let $\boldsymbol{f}$ be the p-adic Hida family passing through the newform attached to a semi-stable elliptic curve $A/\mathbb{Q}$ with split multiplicative reduction at p at its weight-two specialization. There exists a p-adic L-function $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$, providing a p-adic analytic function on a neighborhood of $k=2$, such that*

1. $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \mathscr{L}_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)^2$ *is a square,*
2. $L_p^{\Delta}(\boldsymbol{f})(k) = L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) \cdot L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2$,
3. $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ *interpolates the central critical values* $L(\mathrm{Sym}^3 f_k^\sharp, \frac{3k-2}{2})$,
4. $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ *has an exceptional zero at* $k=2$.

Note that the first point of the theorem is a formal consequence of the second one, as the triple product p-adic L-function is constructed as a square and the Mazur–Kitagawa p-adic L-function appears as a square in the factorization:

$$L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \frac{L_p^{\Delta}(\boldsymbol{f})(k)}{L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2} = \left(\frac{\mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{f}, \boldsymbol{f}, \boldsymbol{f})(k,k,k)}{L_p^{\mathrm{cc}}(\boldsymbol{f})(k)}\right)^2 = \mathscr{L}_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)^2.$$

In particular, this shows that the order of vanishing of the symmetric cube p-adic L-function at $k=2$ is even and at least two.

The last section of this thesis is dedicated to the study of a *BSD type* formula for the triple product p-adic L-function, introducing a conjectural *regulator* for the symmetric cube p-adic L-function. We briefly elaborate on what is done in this final part. The restriction of the Mazur–Kitagawa p-adic L-function is known to vanish to order at least two at $k=2$ and Bertolini–Darmon [BD07] provide a formula for its second derivative. More precisely, the authors construct a global point $\mathbf{P} \in A(\mathbb{Q}) \otimes \mathbb{Q}$, whose squared image under the formal group logarithm $\log_A\colon A(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ attached to A describes the derivative up to a non-zero rational scalar:

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{cc}}(\boldsymbol{f})(k)_{|k=2} = a \cdot \log_A(\mathbf{P})^2 \qquad \text{for some } a \in \mathbb{Q}^\times.$$

With both p-adic L-functions in the factorization of $L_p^\Delta(\boldsymbol{f})(k)$ vanishing to order at least two at $k=2$, one of them appearing as a square, the triple product p-adic L-function vanishes to even order at least six at $k=2$. We use the above result of Bertolini–Darmon and obtain the following formula by a direct computation.

Theorem D. *There exists $\mathbf{P} \in A(\mathbb{Q}) \otimes \mathbb{Q}$ and $a \in \mathbb{Q}^\times$, such that*

$$\frac{\mathrm{d}^6}{\mathrm{d}k^6} L_p^\Delta(\boldsymbol{f})(k)_{|k=2} = 90a^2 \cdot \frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)_{|k=2} \cdot \log_A(\mathbf{P})^4.$$

As a consequence of Theorem D, we are left to understand the contribution coming from the p-adic L-function for the symmetric cube, for which we follow the lines of [BSV21]. In this article, Bertolini–Seveso–Venerucci consider an elliptic curve twisted by two Artin representations and introduce a conjecture on the order of vanishing of the relevant p-adic L-function at the point of interest and the leading term of its Taylor expansion. This is done by introducing a *Garrett–Nekovář regulator*, using the work of Nekovář [Nek06]. We adapt the constructions for introducing a regulator for our scenario via extended Selmer groups and p-adic height pairings. This regulator for the symmetric cube aims to take care of the leading term appearing on the right-hand side of the equation in Theorem D. We close the thesis by formulating a conjecture that encompasses the aforementioned concepts and propose a BSD type formula for the triple product p-adic L-function in a situation of minimal rank.

Structure of the Thesis. The present work is structured as follows. In Section 2, we introduce and recall relevant notions on algebraic cycles and cohomology theories. We prove Proposition B in Section 3, which is where the fundamental discussion of the relevant Galois representations is carried out. Building on the discussion of Section 3, in Section 4 we place (1.2) in the framework of the Tate Conjecture and study how to normalize the so-obtained algebraic cycles in order to arrive at Theorem A. In Section 5, we construct the p-adic L-function for the symmetric cube and discuss the proof of Theorem C. Finally, in Section 6, we introduce the concepts needed for discussing the derivative on the right-hand side of Theorem D. This is where we formulate a conjecture on the precise order of vanishing, as well as a resulting BSD type formula in light of Theorem D in a situation of minimal rank.

Notational Conventions. If not explicitely stated otherwise, all appearing fields are of characteristic zero and cardinality $\leq 2^{\aleph_0}$, so that they may be viewed inside $\mathbb{C}$. Number fields are considered as embedded in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Moreover, we fix a complex embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, as well as p-adic embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$ for each rational prime p. In this way, all finite extensions of $\mathbb{Q}$ are simultaneously realized as a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$. This provides in particular embeddings $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ of absolute Galois groups, whose images G_p are the *decomposition groups* at p, and we denote the relevant *inertia subgroups* by I_p. When writing Frob_p we refer to an *arithmetic* Frobenius element at p. Finally, when considering p-adic étale cohomology groups or more generally p-adic Galois representations V, for any $n \in \mathbb{Z}$, we will write $V(n)$ for the n-th *Tate-twist*, meaning that we tensor with $\mathbb{Q}_p(n)$. This changes the action by a power of the p-adic cyclotomic character $\chi_p \colon G_{\mathbb{Q}} \to \mathbb{Z}_p^{\times}$.

2 Algebraic Cycles and Cohomology

This section is meant to collect some basic material and introduce certain theories that will be used throughout our work. It concerns in particular concepts related to algebraic cycles and those that are derived from algebraic cycles when considering certain cohomology theories. We provide references for further exposition of the matter below. Additional works that influenced the exposition of the material in the first subsections include [Dau13] and [Lil21]. We keep the proofs presented in this section at a minimum and point the reader to the references therein for further details.

2.1 **Chow Groups.** A comprehensive presentation of the notions to be introduced in this section can be found in [Sta23, Tag 0AZ6]. For any smooth projective variety X over a field K, let $\mathrm{C}^m(X)$ denote the group of algebraic cycles of codimension m. This is the free abelian group generated by codimension m subvarieties of $X_{\bar{K}}$. Considering rational equivalence $\sim_{\mathrm{rat}}$ on $\mathrm{C}^m(X)$, we set

$$\mathrm{CH}^m(X) = \mathrm{C}^m(X)/\sim_{\mathrm{rat}}$$

to be the *Chow group* of codimension m algebraic cycles on X. For any field extension L/K inside $\mathbb{C}$, we denote

$$\mathrm{CH}^m(X)(L) = \{[Z] \in \mathrm{CH}^m(X) : Z \sim_{\mathrm{rat}} Z^\sigma \text{ for all } \sigma \in \mathrm{Aut}_L(\mathbb{C})\}.$$

The elements of this group are said to be *defined over L*. For any $0 \leq m, n \leq \dim(X)$, there is an *intersection product*

$$\mathrm{CH}^m(X) \times \mathrm{CH}^n(X) \longrightarrow \mathrm{CH}^{m+n}(X),$$

which endows $\mathrm{CH}^*(X) = \bigoplus_{m\geq 0} \mathrm{CH}^m(X)$ with the structure of a graded ring. The last notion we want to recall concerns morphisms $f\colon X \to Y$ of varieties defined over K. If f is flat, respectively proper, then the pullback, respectively pushforward, of algebraic cycles preserves $\sim_{\mathrm{rat}}$, so that there are induced maps

$$f^*\colon \mathrm{CH}^m(Y) \longrightarrow \mathrm{CH}^m(X), \qquad \text{respectively} \quad f_*\colon \mathrm{CH}^m(X) \longrightarrow \mathrm{CH}^{\dim(Y)-\dim(X)+m}(Y).$$

In particular, combining the two previous notions, any element $\Pi \in \mathrm{CH}^m(X \times Y)$ induces

$$\begin{aligned} \Pi^*\colon \mathrm{CH}^j(Y) &\longrightarrow \mathrm{CH}^{m-\dim(Y)+j}(X), \quad \Gamma_Y \longmapsto \mathrm{pr}_{X,*}(\Pi \cdot \mathrm{pr}_Y^*(\Gamma_Y)), \\ \Pi_*\colon \mathrm{CH}^j(X) &\longrightarrow \mathrm{CH}^{m-\dim(X)+j}(Y), \quad \Gamma_X \longmapsto \mathrm{pr}_{Y,*}(\Pi \cdot \mathrm{pr}_X^*(\Gamma_X)), \end{aligned}$$

where for $Z = X, Y$, $\mathrm{pr}_Z\colon X \times Y \to Z$ is the projection map, and $\cdot$ denotes the intersection product. Note that the above notions of an intersection product and the induced maps on Chow groups restrict to classes defined over extensions of the base field inside $\mathbb{C}$.

2.2 **Cycle Classes and Induced Maps on de Rham Cohomology.** Let X be a smooth projective variety defined over K as before. For more details on its algebraic de Rham cohomology and the relevant notions that we will recall in this section, we refer the reader to [Sta23, Chapter 0FK4] or [Har75]. The algebraic de Rham cohomology groups are equipped with an alternating, non-degenerate pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_X\colon \mathrm{H}^j_{\mathrm{dR}}(X/K)\times \mathrm{H}^{2\dim(X)-j}_{\mathrm{dR}}(X/K)[\dim(X)]\longrightarrow K,\qquad 0\le j\le 2\dim(X),$$

which is called the *Poincaré pairing.* In particular, this pairing provides an isomorphism

$$\mathrm{H}^{2\dim(X)-j}_{\mathrm{dR}}(X/K)[\dim(X)]\cong \mathrm{H}^j_{\mathrm{dR}}(X/K)^\vee \tag{2.1}$$

of filtered K-vector spaces, which is referred to as *Poincaré duality.* Here, $[\,\cdot\,]$ refers to a shift in the Hodge filtration, i.e., as vector spaces one has $V[t]=V$, but $\mathrm{Fil}^i\,V[t]=\mathrm{Fil}^{i+t}\,V$. We will recall the notion of filtered vector spaces when it becomes relevant for a more detailed discussion.

For a morphism $f\colon X\to Y$ of varieties over K as above with $\dim(X)=d$ and $\dim(Y)=e$, one has maps

$$f^*_{\mathrm{dR}}\colon \mathrm{H}^{j'}_{\mathrm{dR}}(Y/K)\longrightarrow \mathrm{H}^{j'}_{\mathrm{dR}}(X/K),\qquad f_{\mathrm{dR},*}\colon \mathrm{H}^{j}_{\mathrm{dR}}(X/K)\longrightarrow \mathrm{H}^{j-2(d-e)}_{\mathrm{dR}}(Y/K)[e-d],$$

which are mutual adjoints (for the appropriate choice of $j'=2d-j$) with respect to the Poincaré pairing, i.e.

$$\langle f^*_{\mathrm{dR}}\alpha,\beta\rangle_X=\langle\alpha,f_{\mathrm{dR},*}\beta\rangle_Y\qquad\text{for }\alpha\in \mathrm{H}^{2d-j}_{\mathrm{dR}}(Y/K)\text{ and }\beta\in \mathrm{H}^{j}_{\mathrm{dR}}(X/K).$$

They further respect the Hodge filtration that the cohomology inherits. Moreover, any element $\Pi\in\mathrm{CH}^m(X\times Y)(K)$ induces maps

$$\begin{aligned}\Pi^*_{\mathrm{dR}}\colon \mathrm{H}^{j'}_{\mathrm{dR}}(Y/K)&\longrightarrow \mathrm{H}^{j'-2(e-m)}_{\mathrm{dR}}(X/K)[m-e],\\ \Pi_{\mathrm{dR},*}\colon \mathrm{H}^{j}_{\mathrm{dR}}(X/K)&\longrightarrow \mathrm{H}^{j-2(d-m)}_{\mathrm{dR}}(Y/K)[m-d],\end{aligned}$$

as follows. Let

$$\mathrm{cl}_{\mathrm{dR}}\colon\ \mathrm{CH}^m(X\times Y)(K)\longrightarrow \mathrm{H}^{2m}_{\mathrm{dR}}(X\times Y/K)[m]$$

denote the *cycle class map* for the variety $X\times Y$ over K. Its target cohomology group has a *Künneth decomposition* of the form

$$\mathrm{H}^{2m}_{\mathrm{dR}}(X\times Y/K)[m]=\bigoplus_{j+j'=2m}\mathrm{H}^{j}_{\mathrm{dR}}(X/K)\otimes\mathrm{H}^{j'}_{\mathrm{dR}}(Y/K)[m]$$

and we denote by

$$\mathrm{pr}_j\colon\ \mathrm{H}^{2m}_{\mathrm{dR}}(X\times Y/K)[m]\longrightarrow\!\!\!\!\rightarrow \mathrm{H}^{j}_{\mathrm{dR}}(X/K)\otimes\mathrm{H}^{2m-j}_{\mathrm{dR}}(Y/K)[m]$$

the projection onto its j-th component. Note that we have an isomorphism

$$\mathrm{Hom}(\mathrm{H}^{2d-j}_{\mathrm{dR}}(X/K),\mathrm{H}^{2m-j}_{\mathrm{dR}}(Y/K)[m-d])\cong \mathrm{H}^{j}_{\mathrm{dR}}(X/K)\otimes\mathrm{H}^{2m-j}_{\mathrm{dR}}(Y/K)[m]$$

arising from Poincaré duality (2.1) along with the isomorphism

$$A^\vee\otimes B\xrightarrow{\sim}\mathrm{Hom}(A,B),\quad \Psi\otimes b\longmapsto(a\mapsto\Psi(a)\cdot b).$$

We can now define a map

$$\begin{aligned}(\,\cdot\,)_{\mathrm{dR},*}\colon\ \mathrm{CH}^m(X\times Y)(K) &\longrightarrow \mathrm{Hom}(\mathrm{H}^{2d-j}_{\mathrm{dR}}(X/K),\mathrm{H}^{2m-j}_{\mathrm{dR}}(Y/K)[m-d])\\ \Pi &\longmapsto \Pi_{\mathrm{dR},*}\end{aligned}$$

by the diagram

$$\begin{tikzcd}
\mathrm{CH}^m(X\times Y)(K) \arrow[r, "\mathrm{cl}_{\mathrm{dR}}"] \arrow[d, "(\,\cdot\,)_{\mathrm{dR},*}"] & \mathrm{H}^{2m}_{\mathrm{dR}}(X\times Y/K)[m] \arrow[d, two heads, "\mathrm{pr}_j"] \\
\mathrm{Hom}(\mathrm{H}^{2d-j}_{\mathrm{dR}}(X/K),\mathrm{H}^{2m-j}_{\mathrm{dR}}(Y/K)[m-d]) & \mathrm{H}^{j}_{\mathrm{dR}}(X/K)\otimes\mathrm{H}^{2m-j}_{\mathrm{dR}}(Y/K)[m]. \arrow[l, "\sim"']
\end{tikzcd} \tag{2.2}$$

The map

$$\Pi^*_{\mathrm{dR}}\colon\ \mathrm{H}^{j'}_{\mathrm{dR}}(Y/K)\longrightarrow \mathrm{H}^{j'-2(e-m)}_{\mathrm{dR}}(X/K)[m-e]$$

is defined to be the adjoint (for $j'=j+2(e-m)$) with respect to the Poincaré pairing, so that one has

$$\langle \Pi^*_{\mathrm{dR}}\alpha,\beta\rangle_X=\langle \alpha,\Pi_{\mathrm{dR},*}\beta\rangle_Y \qquad \text{for } \alpha\in\mathrm{H}^{j+2(e-m)}_{\mathrm{dR}}(Y/K) \text{ and } \beta\in\mathrm{H}^{2d-j}_{\mathrm{dR}}(X/K).$$

The cycle class maps for X and Y link (cf. [Sta23, Tag 0FWC, Tag 0FFG]) the flat pullback and proper pushforward maps f^* and f_* on Chow groups to their de Rham avatars in the form of commutative diagrams

$$\begin{tikzcd}
\mathrm{CH}^m(Y)(K) \arrow[r, "\mathrm{cl}_{\mathrm{dR}}"] \arrow[d, "f^*"] & \mathrm{H}^{2m}_{\mathrm{dR}}(Y/K)[m] \arrow[d, "f^*_{\mathrm{dR}}"] \\
\mathrm{CH}^m(X)(K) \arrow[r, "\mathrm{cl}_{\mathrm{dR}}"] & \mathrm{H}^{2m}_{\mathrm{dR}}(X/K)[m]
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
\mathrm{CH}^m(X)(K) \arrow[r, "\mathrm{cl}_{\mathrm{dR}}"] \arrow[d, "f_*"] & \mathrm{H}^{2m}_{\mathrm{dR}}(X/K)[m] \arrow[d, "f_{\mathrm{dR},*}"] \\
\mathrm{CH}^{m-(d-e)}(Y)(K) \arrow[r, "\mathrm{cl}_{\mathrm{dR}}"] & \mathrm{H}^{2m-2(d-e)}_{\mathrm{dR}}(Y/K)[m-d+e].
\end{tikzcd}$$

Considerations of cycle class maps also lead to subgroups of the Chow groups that are of particular interest for us. We denote by

$$\mathrm{CH}^m(X)_0(K)=\ker(\mathrm{cl}_{\mathrm{dR}})$$

the subgroup of *null-homologous* cycles defined over K. For any $\Pi\in\mathrm{CH}^m(X\times Y)(K)$ and any morphism $f\colon X\to Y$ as above, assumed to be flat for the pullback and proper for the pushforward, the attached maps on Chow groups can then also be viewed as

$$\begin{aligned}\Pi_*\colon\ \mathrm{CH}^j(X)_0(K)&\longrightarrow\mathrm{CH}^{m-d+j}(Y)_0(K),\\ \Pi^*\colon\ \mathrm{CH}^j(Y)_0(K)&\longrightarrow\mathrm{CH}^{m-e+j}(X)_0(K),\\ f_*\colon\ \mathrm{CH}^j(X)_0(K)&\longrightarrow\mathrm{CH}^{e-d+j}(Y)_0(K),\\ f^*\colon\ \mathrm{CH}^j(Y)_0(K)&\longrightarrow\mathrm{CH}^{j}(X)_0(K).\end{aligned}$$

Most of our future study builds on an analysis of Galois representations. With its origin lying in studying certain p-adic étale cohomology groups instead of de Rham cohomology groups, we are interested in passing from one cohomology theory to the other. More precisely, we want to use the merit of de Rham cohomology, inheriting a natural rational structure, and are therefore interested in a *comparison theorem* in order to normalize objects in the étale context, which are a priori only defined up to p-adic coefficients.

The following two subsections will provide the tools for passing from considerations on étale cohomology to de Rham cohomology and therefore provide a bridge between the notions that we have just defined and their p-adic étale avatars that we will consider later. In fact, once we have introduced the p-adic étale versions, the upcoming material will enable us to view the concepts on de Rham cohomology over p-adic fields as a natural image of those p-adic étale concepts.

2.3 **Recalls of p-adic Hodge Theory.** This subsection is devoted to summarizing notions from p-adic Hodge theory that are needed to state the comparison result between p-adic étale cohomology and de Rham cohomology over p-adic fields.

This subject employs Fontaine's rings $B_{\mathrm{cris}}, B_{\mathrm{st}}, B_{\mathrm{dR}}$ of p-adic periods. As we will be mostly using B_{dR}, let us briefly recall its construction. For further details on the construction and basic properties, in particular of the period rings B_{st} and B_{cris}, we refer the reader to the comprehensive work of Fontaine–Ouyang [FO08]. Let $R = \lim_{n\in\mathbb{Z}_{\geq 0}} \mathcal{O}_{\mathbb{C}_p}/(p)$ be the ring obtained as an inverse limit along the p-power maps. Its elements $(x_n)_{n\in\mathbb{N}} \in R$ are identified with collections $(x^{(n)})_{n\in\mathbb{Z}_{\geq 0}}$ of elements $x^{(n)} \in \mathcal{O}_{\mathbb{C}_p}$ in the ring of integers with $(x^{(n+1)})^p = x^n$. Denoting by $W(R)$ the ring of Witt vectors over R, there is (cf. [FO08, Lemma 5.9, Proposition 5.11]) a surjective homomorphism

$$W(R) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}, \quad (a_n)_{n\in\mathbb{Z}_{\geq 0}} \longmapsto \textstyle\sum_{n\geq 0} p^n a_n^{(n)}.$$

Its kernel is a principal ideal, generated by an element $\xi \in W(R)$, such that $\xi - p = [\varpi]$ is the *Teichmüller representative* $[\varpi] = (\varpi, 0, 0, \dots)$ of an element $\varpi \in R$ with $\varpi^{(0)} = -p$. Fontaine's ring of *de Rham periods* is defined to be

$$B_{\mathrm{dR}} = \mathrm{Frac}\Big(\lim_{n\in\mathbb{Z}_{\geq 0}} W(R)[\tfrac{1}{p}]/(\xi)^n\Big).$$

Let F be a finite extension of $\mathbb{Q}_p$. We can now associate to any p-adic representation V of G_F the F-vector space

$$\mathbf{D}_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_F}.$$

This vector space is equipped with a decreasing, separated and exhaustive filtration, which we will introduce later on.

In the same style as above, there are vector spaces $\mathbf{D}_{\mathrm{st}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)$, whose construction employs Fontaine's rings B_{st} and B_{cris}. In order to recall their shape and basic properties, we fix the following notation. Let F_0 be the maximal unramified extension of $\mathbb{Q}_p$ in F and let σ be the absolute Frobenius acting on F_0, inducing the p-th power map on the residue field of F_0. To any p-adic representation V, there are the following associated vector spaces:

(i) $\mathbf{D}_{\mathrm{st}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_F}$ – a vector space over F_0, equipped with a σ-semilinear bijective map $\phi_{\mathrm{st}}\colon \mathbf{D}_{\mathrm{st}}(V) \to \mathbf{D}_{\mathrm{st}}(V)$ and a linear map $\mathrm{N}_{\mathrm{st}}\colon \mathbf{D}_{\mathrm{st}}(V) \to \mathbf{D}_{\mathrm{st}}(V)$, called *monodromy operator*, satisfying $\mathrm{N}_{\mathrm{st}} \circ \phi_{\mathrm{st}} = p \cdot \phi_{\mathrm{st}} \circ \mathrm{N}_{\mathrm{st}}$.

(ii) $\mathbf{D}_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_F}$ – a vector space over F_0, equipped with a σ-semilinear bijective map $\phi_{\mathrm{cris}}\colon \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$.

We further set $\phi_{\star,F} = \phi_{\star}^{[F_0:\mathbb{Q}_p]}$ for $\star \in \{\mathrm{st}, \mathrm{cris}\}$. The above vector spaces are related by:

- $\mathbf{D}_{\mathrm{st}}(V)^{\mathrm{N}_{\mathrm{st}}=0} = \mathbf{D}_{\mathrm{cris}}(V)$,
- injective maps
$$\jmath_{\mathrm{st}}\colon \mathbf{D}_{\mathrm{st}}(V) \otimes_{F_0} F \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V) \quad \text{and} \quad \jmath_{\mathrm{cris}}\colon \mathbf{D}_{\mathrm{cris}}(V) \otimes_{F_0} F \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V),$$
- $\dim_{F_0}(\mathbf{D}_{\mathrm{cris}}(V)) \le \dim_{F_0}(\mathbf{D}_{\mathrm{st}}(V)) \le \dim_F(\mathbf{D}_{\mathrm{dR}}(V)) \le \dim_{\mathbb{Q}_p}(V)$.

The representation V is said to be

$$\begin{cases} \textit{crystalline} & \text{if } \dim_{F_0}(\mathbf{D}_{\mathrm{cris}}(V)) = \dim_{\mathbb{Q}_p}(V), \\ \textit{semistable} & \text{if } \dim_{F_0}(\mathbf{D}_{\mathrm{st}}(V)) = \dim_{\mathbb{Q}_p}(V), \\ \textit{de Rham} & \text{if } \dim_F(\mathbf{D}_{\mathrm{dR}}(V)) = \dim_{\mathbb{Q}_p}(V). \end{cases}$$

In particular, crystalline representations are semistable, which in turn are de Rham. Furthermore, if V is semistable, then $\jmath_{\mathrm{st}}$ is an isomorphism, while if V is crystalline, then $\mathrm{N}_{\mathrm{st}} = 0$ on $\mathbf{D}_{\mathrm{st}}(V)$, $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{st}}(V)$ and both $\jmath_{\mathrm{cris}}$ and $\jmath_{\mathrm{st}}$ are isomorphisms.

More on Filtered Vector Spaces and $\mathbf{D}_{\mathrm{dR}}$. We will now concentrate on the assignment provided by $\mathbf{D}_{\mathrm{dR}}(\,\cdot\,)$ and consider it from a categorial viewpoint. Denote by $\mathbf{Fil}_F$ the category whose objects are finite dimensional F-vector spaces U equipped with a decreasing, exhausted and separated filtration $\{\mathrm{Fil}^j\, U\}_{j\in\mathbb{Z}}$, with morphisms $\eta\colon U \to U'$ given by F-linear maps respecting the filtration. This means:

- $\mathrm{Fil}^j\, U$ are sub F-vector spaces of U,
- $\mathrm{Fil}^{j+1}\, U \subseteq \mathrm{Fil}^j\, U$,
- $\mathrm{Fil}^j\, U = 0$ for $j \gg 0$ and $\mathrm{Fil}^j\, U = U$ for $j \ll 0$,
- $\eta(\mathrm{Fil}^j\, U) \subseteq \mathrm{Fil}^j\, U'$.

We provide some more notions in that regard which are of interest for our study. A morphism $\eta\colon U \to U'$ in $\mathbf{Fil}_F$ is said to be *strict* if for all $j \in \mathbb{Z}$ one has

$$\eta(\mathrm{Fil}^j\, U) = \mathrm{Fil}^j\, U' \cap \mathrm{im}(\eta).$$

Strict morphisms are of importance for the notion of *short exact* sequences in the category of filtered vector spaces. Indeed, a short exact sequence in $\mathbf{Fil}_F$ is of the form

$$U' \xhookrightarrow{\ \eta\ } U \xrightarrow{\ \mu\ } U'',$$

for strict morphisms η and μ such that $\mathrm{im}(\eta) = \ker(\mu)$. Given two objects $U, U' \in \mathbf{Fil}_F$, we define their tensor product as the F-vector space $U \otimes U'$ together with a filtration given by

$$\mathrm{Fil}^j(U \otimes U') = \sum_{j_1+j_2=j} \mathrm{Fil}^{j_1} U \otimes \mathrm{Fil}^{j_2} U'.$$

The unit object is given by the base field F with $\mathrm{Fil}^j F = 0$ for $j > 0$ and $\mathrm{Fil}^j F = F$ for $j \leq 0$. Lastly, any $U \in \mathbf{Fil}_F$ has a dual given by the F-linear dual $U^\vee$ as its underlying vector space together with a filtration defined by

$$\mathrm{Fil}^j U^\vee = (\mathrm{Fil}^{1-j} U)^\perp = \{\varphi \in \mathrm{Hom}_F(U, F) : \varphi(x) = 0 \text{ for all } x \in \mathrm{Fil}^{1-j} U\}.$$

For any p-adic representation V of G_F, the attached F-vector space $\mathbf{D}_{\mathrm{dR}}(V)$ has a filtration induced by the one on B_{dR}, which we will now describe. Let

$$B_{\mathrm{dR}}^+ = \lim_{n \in \mathbb{Z}_{\geq 0}} W(R)[\tfrac{1}{p}]/(\xi)^n,$$

so that $B_{\mathrm{dR}} = \mathrm{Frac}\, B_{\mathrm{dR}}^+$. Denoting by $\mathfrak{m}_{\mathrm{dR}}^+$ the maximal ideal of B_{dR}^+, the filtration on B_{dR} is given by

$$\mathrm{Fil}^j B_{\mathrm{dR}} = \mathfrak{m}_{\mathrm{dR}}^{+,i}.$$

Now, $\mathbf{D}_{\mathrm{dR}}(V)$ is an object of $\mathbf{Fil}_F$ by setting

$$\mathrm{Fil}^j \mathbf{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^j B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_F}.$$

We are now restricting our attention to the category $\mathbf{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_F)$ of p-adic de Rham representations of G_F. The proof of [FO08, Theorem 2.13] contains a particular observation concerning short exact sequences which we would like to point out. We recall the relevant argument for convenience.

Lemma 2.1. *If the middle term of a short exact sequence in* $\mathbf{Rep}_{\mathbb{Q}_p}(G_F)$ *is de Rham, then so are its surrounding terms.*

Proof. Let V be a p-adic de Rham representation of G_F and let

$$V' \xhookrightarrow{f} V \xrightarrow{g} V''$$

be an exact sequence in $\mathbf{Rep}_{\mathbb{Q}_p}(G_F)$. Then, we have

$$\mathbf{D}_{\mathrm{dR}}(V') \xhookrightarrow{\mathbf{D}_{\mathrm{dR}}(f)} \mathbf{D}_{\mathrm{dR}}(V) \xrightarrow{\mathbf{D}_{\mathrm{dR}}(g)} \mathbf{D}_{\mathrm{dR}}(V'') \tag{2.3}$$

with $\mathrm{im}(\mathbf{D}_{\mathrm{dR}}(f)) = \ker(\mathbf{D}_{\mathrm{dR}}(g))$ and we can compute:

$$\begin{aligned}
\dim \mathbf{D}_{\mathrm{dR}}(V) &= \dim V \\
&= \dim V' + \dim V'' \\
&\geq \dim \mathbf{D}_{\mathrm{dR}}(V') + \dim \mathbf{D}_{\mathrm{dR}}(V'') \\
&= \dim \mathrm{im}(\mathbf{D}_{\mathrm{dR}}(f)) + \dim \ker(\mathbf{D}_{\mathrm{dR}}(f)) + \dim \mathbf{D}_{\mathrm{dR}}(V'') \qquad (2.4) \\
&= \dim \ker(\mathbf{D}_{\mathrm{dR}}(g)) + \dim \mathbf{D}_{\mathrm{dR}}(V'') \\
&\geq \dim \ker(\mathbf{D}_{\mathrm{dR}}(g)) + \dim \mathrm{im}(\mathbf{D}_{\mathrm{dR}}(g)) \\
&= \dim \mathbf{D}_{\mathrm{dR}}(V).
\end{aligned}$$

From this it follows that $\dim V' + \dim V'' = \dim \mathbf{D}_{\mathrm{dR}}(V') + \dim \mathbf{D}_{\mathrm{dR}}(V'')$, concluding the proof. $\square$

We note that the computation (2.4) further shows that the morphism $\mathbf{D}_{\mathrm{dR}}(g)$ of (2.3) is surjective, so that $\mathbf{D}_{\mathrm{dR}}$ actually maps short exact sequences to short exact sequences. Indeed, one has the following result.

Theorem 2.2 ([FO08, Theorem 5.29])**.** *The assignment* $\mathbf{D}_{\mathrm{dR}}(\,\cdot\,)$ *provides an exact, faithful tensor functor*

$$\mathbf{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_F) \longrightarrow \mathbf{Fil}_F\,.$$

We will now state the aforementioned comparison result between étale cohomology and de Rham cohomology. This *comparison isomorphism* (cf. [Fal89], [Tsu99]) will be used for transferring considerations on the level of Galois representations arising from étale cohomology groups to vector spaces offering a rational structure that we employ for a certain normalization process. Let X be a smooth projective variety as before and let F be a finite extension of $\mathbb{Q}_p$.

Theorem 2.3 (Comparison Isomorphism)**.** *The étale cohomology group* $\mathrm{H}^m_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(j)$ *is a p-adic de Rham representation of G_F and there is a canonical isomorphism*

$$\mathrm{comp}_{\mathrm{dR}}\colon\ \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^m_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(j)) \xrightarrow{\ \sim\ } \mathrm{H}^m_{\mathrm{dR}}(X/F)[j] \tag{2.5}$$

in $\mathbf{Fil}_F$*.*

In particular, as $\mathrm{comp}_{\mathrm{dR}}$ identifies the filtrations, we get an isomorphism

$$\overline{\mathrm{comp}}_{\mathrm{dR}} = \overline{\mathrm{q}_{\mathrm{Fil}^0} \circ \mathrm{comp}_{\mathrm{dR}}}\colon\ \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^m_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(j))/\operatorname{Fil}^0 \xrightarrow{\ \sim\ } \mathrm{H}^m_{\mathrm{dR}}(X/F)[j]/\operatorname{Fil}^0 \tag{2.6}$$

by the universal property of quotients with respect to the projection

$$\mathrm{q}_{\mathrm{Fil}^0}\colon\ \mathrm{H}^m_{\mathrm{dR}}(X/F)[j] \twoheadrightarrow \mathrm{H}^m_{\mathrm{dR}}(X/F)[j]/\operatorname{Fil}^0\,.$$

2.4 **Selmer Groups and the Bloch–Kato Logarithm.** We retain the notation of the preceding subsection and let F be a finite extension of $\mathbb{Q}_p$ and let V be a p-adic de Rham representation of G_F.

For recalling the definition of the *Selmer groups* in the style of Bloch–Kato, in addition to [BK90] we refer to [Nek00] and [Bel09] for further treatment of the material. Let $\mathrm{H}^1_{\mathrm{cts}}(F, V)$ denote the continuous Galois cohomology group (cf. [Tat76]). One defines nested subspaces

$$\mathrm{H}^1_e(F, V) \subseteq \mathrm{H}^1_f(F, V) \subseteq \mathrm{H}^1_g(F, V) \subseteq \mathrm{H}^1_{\mathrm{cts}}(F, V)$$

by setting

$$\mathrm{H}^1_\star(F, V) = \begin{cases} \ker(\mathrm{H}^1_{\mathrm{cts}}(F, V) \longrightarrow \mathrm{H}^1_{\mathrm{cts}}(F, V \otimes B_{\mathrm{cris}}^{\phi_{\mathrm{cris}}=1})) & \text{if } \star = e \\ \ker(\mathrm{H}^1_{\mathrm{cts}}(F, V) \longrightarrow \mathrm{H}^1_{\mathrm{cts}}(F, V \otimes B_{\mathrm{cris}})) & \text{if } \star = f \\ \ker(\mathrm{H}^1_{\mathrm{cts}}(F, V) \longrightarrow \mathrm{H}^1_{\mathrm{cts}}(F, V \otimes B_{\mathrm{dR}})) & \text{if } \star = g. \end{cases}$$

It will be of later interest to also consider a number field K and a p-adic representation V of G_K. In that case, we define $\mathrm{H}^1_\star(K,V)$ as the subspace of $\mathrm{H}^1_{\mathrm{cts}}(K,V)$ consisting of elements whose restriction at any finite place ν of K belongs to

$$\begin{cases} \mathrm{H}^1_\star(K_\nu,V) & \text{if } \nu \mid p \\ \begin{cases} 0 & \text{if } \star = e \\ \mathrm{H}^1_{\mathrm{ur}}(K_\nu,V) & \text{if } \star = f \\ \mathrm{H}^1_{\mathrm{cts}}(K_\nu,V) & \text{if } \star = g \end{cases} & \text{if } \nu \nmid p. \end{cases}$$

Here, the *unramified* cohomology group $\mathrm{H}^1_{\mathrm{ur}}(K_\nu,V)$ is defined to be the kernel of the map $\mathrm{H}^1_{\mathrm{cts}}(K_\nu,V) \to \mathrm{H}^1_{\mathrm{cts}}(I_\nu,V)$, in which I_ν denotes the inertia subgroup.

Considering Fontaine's period rings, there is (cf. [BK90, (1.17.1)]) a short exact sequence

$$\mathbb{Q}_p \hookrightarrow B_{\mathrm{cris}}^{\phi_{\mathrm{cris}}=1} \oplus \mathrm{Fil}^0 B_{\mathrm{dR}} \twoheadrightarrow B_{\mathrm{dR}}. \tag{2.7}$$

Tensoring with V and taking Galois cohomology gives a connecting homomorphism

$$\delta^0_V\colon \mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^1_{\mathrm{cts}}(F,V),$$

which is essentially the *Bloch–Kato exponential* as introduced by Bloch–Kato in [BK90]. This assignment is natural in the representation, i.e. the collection of maps δ^0_V forms a natural transformation

$$\mathbf{D}_{\mathrm{dR}}(\,\cdot\,) \Longrightarrow \mathrm{H}^1_{\mathrm{cts}}(F,\,\cdot\,)$$

of functors $\mathbf{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_F) \longrightarrow \mathbf{Vect}_F$. In the next lemma, we modify this to suit our applications.

Lemma 2.4. *We can modify both the functors $\mathbf{D}_{\mathrm{dR}}(\,\cdot\,)$ and $\mathrm{H}^1_{\mathrm{cts}}(F,\,\cdot\,)$ as follows.*

(i) Cutting out the kernel of the connecting homomorphism gives a functor

$$\begin{aligned} \overline{\mathbf{D}}_{\mathrm{dR}}(\,\cdot\,)\colon \mathbf{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_F) &\longrightarrow \mathbf{Vect}_F \\ V &\longmapsto \mathbf{D}_{\mathrm{dR}}(V)/\ker(\delta^0_V) \\ (\varphi\colon V \to W) &\longmapsto (\overline{\mathbf{D}}_{\mathrm{dR}}(\varphi)\colon \mathbf{D}_{\mathrm{dR}}(V)/\ker(\delta^0_V) \to \mathbf{D}_{\mathrm{dR}}(W)/\ker(\delta^0_W)), \end{aligned}$$

where $\overline{\mathbf{D}}_{\mathrm{dR}}(\varphi)$ is the map induced by the universal property of quotients for the composition $\mathrm{q}_{\ker(\delta^0_W)} \circ \mathbf{D}_{\mathrm{dR}}(\varphi)$.

(ii) Restricting to the image of the connecting homomorphism gives a functor

$$\begin{aligned} \mathrm{im}(\delta^0_{\cdot})\colon \mathbf{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_F) &\longrightarrow \mathbf{Vect}_F \\ V &\longmapsto \mathrm{im}(\delta^0_V) \\ (\varphi\colon V \to W) &\longmapsto (\mathrm{H}^1_{\mathrm{cts}}(F,\varphi)_{|\mathrm{im}(\delta^0_V)}\colon \mathrm{im}(\delta^0_V) \to \mathrm{im}(\delta^0_W)). \end{aligned}$$

Proof. As we will see, (i) follows from the universal property of quotients, while for (ii) one has to observe that $\mathrm{H}^1_{\mathrm{cts}}(F,\varphi)$ respects the images of the connecting homomorphisms. This will be proven by considering the commutative diagram arising from applying cohomology to the relevant morphism of short exact sequences.

Ad (i): First of all note that for any morphism $\varphi\colon V \to W$ of de Rham representations, the assigned morphism $\overline{\mathbf{D}}_{\mathrm{dR}}(\varphi)$ is meaningful as $\mathbf{D}_{\mathrm{dR}}(\varphi)(\ker(\delta^0_V)) \subseteq \ker(\delta^0_W)$, so that the universal property of quotients provides the desired morphism. In order to prove that

$$\overline{\mathbf{D}}_{\mathrm{dR}}(\mathrm{id}_V) = \mathrm{id}_{\overline{\mathbf{D}}_{\mathrm{dR}}(V)} \qquad \text{and} \qquad \overline{\mathbf{D}}_{\mathrm{dR}}(\varphi \circ \varphi') = \overline{\mathbf{D}}_{\mathrm{dR}}(\varphi) \circ \overline{\mathbf{D}}_{\mathrm{dR}}(\varphi')$$

for any two morphisms $\varphi'\colon U \to V$ and $\varphi\colon V \to W$, we show that both sides of the respective desired equality satisfy the universal property for the considered quotient. The identity case being clear, let us have a look at the composition. We claim that both the maps $\overline{\mathbf{D}}_{\mathrm{dR}}(\varphi \circ \varphi')$ and $\overline{\mathbf{D}}_{\mathrm{dR}}(\varphi) \circ \overline{\mathbf{D}}_{\mathrm{dR}}(\varphi')$ fit in the diagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dR}}(U) \arrow[r, "\mathbf{D}_{\mathrm{dR}}(\varphi\circ\varphi')"] \arrow[d, two heads, "\mathrm{q}_{\ker(\delta^0_U)}"'] & \mathbf{D}_{\mathrm{dR}}(W) \arrow[r, two heads, "\mathrm{q}_{\ker(\delta^0_W)}"] & \overline{\mathbf{D}}_{\mathrm{dR}}(W) \\
\overline{\mathbf{D}}_{\mathrm{dR}}(U) \arrow[urr, dashed] & &
\end{tikzcd}$$

as the dashed arrow making it commutative. Indeed, we compute

$$\begin{aligned}
\overline{\mathbf{D}}_{\mathrm{dR}}(\varphi) \circ \overline{\mathbf{D}}_{\mathrm{dR}}(\varphi') \circ \mathrm{q}_{\ker(\delta^0_U)} &= \overline{\mathbf{D}}_{\mathrm{dR}}(\varphi) \circ \mathrm{q}_{\ker(\delta^0_V)} \circ \mathbf{D}_{\mathrm{dR}}(\varphi') \\
&= \mathrm{q}_{\ker(\delta^0_W)} \circ \mathbf{D}_{\mathrm{dR}}(\varphi) \circ \mathbf{D}_{\mathrm{dR}}(\varphi') \\
&= \mathrm{q}_{\ker(\delta^0_W)} \circ \mathbf{D}_{\mathrm{dR}}(\varphi \circ \varphi').
\end{aligned}$$

Ad (ii): With the compatibility properties given, we only have to make sure the images are respected, which is immediate from the construction. Indeed, tensoring the short exact sequence (2.7) with V and W and applying cohomology to

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d] & V \otimes (B_{\mathrm{cris}}^{\phi_{\mathrm{cris}}=1} \oplus \mathrm{Fil}^0 B_{\mathrm{dR}}) \arrow[r, two heads] \arrow[d] & V \otimes B_{\mathrm{dR}} \arrow[d] \\
W \arrow[r, hook] & W \otimes (B_{\mathrm{cris}}^{\phi_{\mathrm{cris}}=1} \oplus \mathrm{Fil}^0 B_{\mathrm{dR}}) \arrow[r, two heads] & W \otimes B_{\mathrm{dR}}
\end{tikzcd}$$

gives a commutative diagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dR}}(V) \arrow[r, "\delta^0_V"] \arrow[d, "\mathbf{D}_{\mathrm{dR}}(\varphi)"'] & \mathrm{H}^1_{\mathrm{cts}}(F, V) \arrow[d, "\mathrm{H}^1_{\mathrm{cts}}(F,\varphi)"] \\
\mathbf{D}_{\mathrm{dR}}(W) \arrow[r, "\delta^0_W"'] & \mathrm{H}^1_{\mathrm{cts}}(F, W).
\end{tikzcd}$$

This shows that $\mathrm{H}^1_{\mathrm{cts}}(F,\varphi)(\mathrm{im}(\delta^0_V)) \subseteq \mathrm{im}(\delta^0_W)$ as required. $\square$

We record the following result, which is immediate from the above.

Proposition 2.5. *There is a natural isomorphism*

$$\exp_{\mathrm{BK},\cdot}\colon \overline{\mathbf{D}}_{\mathrm{dR}}(\,\cdot\,) \overset{\sim}{\Longrightarrow} \mathrm{im}(\delta^0_{\cdot})$$

of functors $\mathbf{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_F) \to \mathbf{Vect}_F$.

We call $\exp_{\mathrm{BK},V}$ the *Bloch–Kato exponential* for V. Its inverse will be denoted $\log_{\mathrm{BK},V}$ and called the *Bloch–Kato logarithm* for V.

Going into the defining diagrams, [BK90, Corollary 3.8.4] provides an explicit description of the appearing kernels and images:

$$\begin{aligned}\ker(\delta^0_V) &= \mathrm{im}\Big(\mathbf{D}_{\mathrm{cris}}(V)^{\phi_{\mathrm{cris}}=1} \oplus \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \mathbf{D}_{\mathrm{dR}}(V), \quad (x,y) \longmapsto x-y\Big),\\ \mathrm{im}(\delta^0_V) &= \mathrm{H}^1_e(F,V).\end{aligned}$$

If we further assume our de Rham representation V to satisfy $\mathbf{D}_{\mathrm{cris}}(V)^{\phi_{\mathrm{cris},F}=1} = 0$, we have in particular $\mathbf{D}_{\mathrm{cris}}(V)^{\phi_{\mathrm{cris}}=1} = 0$ and thus

$$\ker(\delta^0_V) = \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V) \quad \text{and} \quad \mathrm{im}(\delta^0_V) = \mathrm{H}^1_f(F,V). \tag{2.8}$$

For the second equality, one simply notes that $\mathrm{H}^1_e(F,V) \subseteq \mathrm{H}^1_f(F,V)$ have the same dimension, which follows again by [BK90, Corollary 3.8.4] and the additional assumption. The Bloch–Kato logarithm for a p-adic de Rham representation V of G_K with $\mathbf{D}_{\mathrm{cris}}(V)^{\phi_{\mathrm{cris}}=1} = 0$ hence is of the form

$$\log_{\mathrm{BK},V}\colon \mathrm{H}^1_f(F,V) \overset{\sim}{\longrightarrow} \mathbf{D}_{\mathrm{dR}}(V)/\,\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V). \tag{2.9}$$

Remark 2.6. As noted in [BDP17], the above assumptions are satisfied in particular for representations arising as $V = \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j)$ for X a smooth projective variety over a number field K having good reduction at a prime ν of K lying above p, where now $F = K_\nu$.

Recalling the comparison isomorphism modulo filtrations (2.6), the following result provides another description of the image of the Bloch–Kato logarithm for representations $V = \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j)$ with X a smooth projective variety as above. It is a *twisted* version of the Poincaré duality isomorphism (2.1) with respect to filtrations.

Lemma 2.7. *There is a canonical isomorphism*

$$\mathrm{H}^{2j-1}_{\mathrm{dR}}(X/F)/\,\mathrm{Fil}^j \cong (\mathrm{Fil}^{j^*}\,\mathrm{H}^{2j^*-1}_{\mathrm{dR}}(X/F))^\vee \tag{2.10}$$

induced by the Poincaré pairing, where $j^* = \dim(X) - j + 1$.

Proof. We consider the Poincaré pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_X\colon \mathrm{H}^m_{\mathrm{dR}}(X/F) \times \mathrm{H}^{2\dim(X)-m}_{\mathrm{dR}}(X/F)[\dim(X)] \longrightarrow F$$

with $m = 2j-1$, providing an isomorphism

$$\mathrm{H}^{2j-1}_{\mathrm{dR}}(X/F) \cong \mathrm{H}^{2\dim(X)-2j+1}_{\mathrm{dR}}(X/F)[\dim(X)]^\vee$$

of filtered F-vector spaces. In particular, we find that

$$\mathrm{H}_{\mathrm{dR}}^{2j-1}(X/F)/\operatorname{Fil}^j \cong \mathrm{H}_{\mathrm{dR}}^{2\dim(X)-2j+1}(X/F)[\dim(X)]^\vee/\operatorname{Fil}^j. \tag{2.11}$$

It is left to have a closer look at the right-hand side, for which we consider the projection

$$\mathrm{H}_{\mathrm{dR}}^{2\dim(X)-2j+1}(X/F)[\dim(X)]^\vee \twoheadrightarrow (\operatorname{Fil}^{1-j}\mathrm{H}_{\mathrm{dR}}^{2\dim(X)-2j+1}(X/F)[\dim(X)])^\vee$$

given by the dual of the inclusion. Its target is qual to the right-hand side of (2.10) and its kernel equals

$$(\operatorname{Fil}^{1-j}\mathrm{H}_{\mathrm{dR}}^{2\dim(X)-2j+1}(X/F)[\dim(X)])^\perp = \operatorname{Fil}^j(\mathrm{H}_{\mathrm{dR}}^{2\dim(X)-2j+1}(X/F)[\dim(X)]^\vee),$$

which is precisely the j-th filtration piece on the right-hand side of (2.11). □

We call the isomorphism (2.10) *twisted Poincaré duality isomorphism* and henceforth denote it by PD_X.

2.5 **Étale Cohomology and p-adic Abel–Jacobi Maps.** Let now X be a smooth projective variety over a field K of characteristic zero. As mentioned before, the *geometric* étale cohomology groups $\mathrm{H}_{\text{ét}}^j(\bar{X},\mathbb{Q}_p)$ and their twists are of particular interest in our considerations of Galois representations. While for these we are extending scalars of the variety to an algebraic closure, one can also take *arithmetic* étale cohomology groups (*continuous* étale cohomology in the sense of [Jan88]) of X over K, without extending scalars. We denote these groups by $\mathrm{H}_{\text{ét}}^{j'}(X,\mathbb{Q}_p)$. Their relation to the Galois cohomology of $\mathrm{H}_{\text{ét}}^j(\bar{X},\mathbb{Q}_p)$ leads to the construction of the p-adic étale cycle class and Abel–Jacobi maps that we are interested in. Moreover, as remarked earlier, in the p-adic case, the material presented in the previous two sections enables one to view notions on de Rham cohomology as certain images of their étale versions. Therefore, by definition, there is a compatibility between considerations on de Rham cohomology over p-adic fields and p-adic étale cohomology. After briefly introducing some concepts on the level of étale cohomology, we explain how to pass to their de Rham versions.

Let us start by recalling the construction of the p-adic étale cycle class map, following Nekovář [Nek00]. For X as above, we consider the cycle class map

$$\mathrm{cl}_{\text{ét},\mathsf{a}}^p\colon\ \mathrm{CH}^n(X)(K) \longrightarrow \mathrm{H}_{\text{ét}}^{2n}(X,\mathbb{Q}_p(n))$$

with values in p-adic arithmetic étale cohomology. There is a *Hochschild–Serre spectral sequence*

$$\mathrm{H}_{\mathrm{cts}}^i(K,\mathrm{H}_{\text{ét}}^j(\bar{X},\mathbb{Q}_p)(n)) \Longrightarrow \mathrm{H}_{\text{ét}}^{i+j}(X,\mathbb{Q}_p(n))$$

linking arithmetic and geometric étale cohomology, which degenerates at the second page (cf. [Del68], [Del80]). More precisely, there exists a decreasing filtration

$$\cdots \subseteq \operatorname{Fil}^j\mathrm{H}_{\text{ét}}^m(X,\mathbb{Q}_p(n)) \subseteq \cdots \subseteq \operatorname{Fil}^0\mathrm{H}_{\text{ét}}^m(X,\mathbb{Q}_p(n)) = \mathrm{H}_{\text{ét}}^m(X,\mathbb{Q}_p(n))$$

in such a way that there are isomorphisms

$$\alpha_{m,j}^{(n)}\colon\ \operatorname{Fil}^j\mathrm{H}_{\text{ét}}^m(X,\mathbb{Q}_p(n))/\operatorname{Fil}^{j+1}\mathrm{H}_{\text{ét}}^m(X,\mathbb{Q}_p(n)) \xrightarrow{\sim} \mathrm{H}_{\mathrm{cts}}^j(K,\mathrm{H}_{\text{ét}}^{m-j}(\bar{X},\mathbb{Q}_p)(n)).$$

This provides various maps, including the *p-adic étale cycle class map*

$$\mathrm{cl}^p_{\text{ét},\mathbf{g}}\colon\ \mathrm{CH}^n(X)(K) \longrightarrow \mathrm{H}^{2n}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(n).$$

The image of this map actually consists of G_K-invariants, as it is constructed as the composition

$$\mathrm{CH}^n(X)(K) \longrightarrow \mathrm{H}^{2n}_{\text{ét}}(X,\mathbb{Q}_p(n)) \twoheadrightarrow \mathrm{H}^{2n}_{\text{ét}}(X,\mathbb{Q}_p(n))/\operatorname{Fil}^1 \xrightarrow{\sim} \mathrm{H}^{2n}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(n)^{G_K} \tag{2.12}$$

of the cycle class map for p-adic arithmetic étale cohomology with the natural projection and the isomorphism $\alpha^{(n)}_{2n,0}$. This map is object of the Tate Conjecture for smooth projective varieties X over number fields K, claiming surjectivity of the map

$$\mathrm{cl}^p_{\text{ét},\mathbf{g}} \otimes \mathbb{Q}_p\colon\ \mathrm{CH}^n(X)(K)\otimes\mathbb{Q}_p \longrightarrow \mathrm{H}^{2n}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(n)^{G_K}.$$

As this will be of interest at a later point, we will come back to this and discuss its relevance for us at the beginning of Section 4.

Remark 2.8. We note that $\ker(\mathrm{cl}^p_{\text{ét},\mathbf{g}})$ is independent of the choice of the prime p. In fact, the kernel $\ker(\mathrm{cl}^p_{\text{ét},\mathbf{g}})$ is precisely the group of null-homologous cycles $\mathrm{CH}^n(X)_0(K)$ as defined in Section 2.2.

Given $\Pi \in \mathrm{CH}^m(X\times Y)(K)$, with Y of dimension e, we use the cycle class map to define its induced maps on p-adic étale cohomology

$$\begin{aligned}\Pi^{*,p}_{\text{ét}}\colon \mathrm{H}^{2m-j'}_{\text{ét}}(\bar{Y},\mathbb{Q}_p) &\longrightarrow \mathrm{H}^{2d-j'}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(m-e),\\ \Pi^{p}_{\text{ét},*}\colon \mathrm{H}^{2d-j}_{\text{ét}}(\bar{X},\mathbb{Q}_p) &\longrightarrow \mathrm{H}^{2m-j}_{\text{ét}}(\bar{Y},\mathbb{Q}_p)(m-d),\end{aligned}$$

by considering the exact étale analogue of diagram (2.2). More precisely, we define $(\,\cdot\,)^p_{\text{ét},*}$ through the diagram

$$\begin{tikzcd} \mathrm{CH}^m(X\times Y)(K) \arrow[r, "\mathrm{cl}^p_{\text{ét},\mathbf{g}}"] \arrow[d, "(\,\cdot\,)^p_{\text{ét},*}"'] & \mathrm{H}^{2m}_{\text{ét}}(\overline{X\times Y},\mathbb{Q}_p)(m) \arrow[d, two heads, "\mathrm{pr}_j"'] \\ \operatorname{Hom}(\mathrm{H}^{2d-j}_{\text{ét}}(\bar{X},\mathbb{Q}_p),\mathrm{H}^{2m-j}_{\text{ét}}(\bar{Y},\mathbb{Q}_p)(m-d)) & \mathrm{H}^{j}_{\text{ét}}(\bar{X},\mathbb{Q}_p)\otimes\mathrm{H}^{2m-j}_{\text{ét}}(\bar{Y},\mathbb{Q}_p)(m). \arrow[l, "\sim"'] \end{tikzcd}$$

The assignment $(\,\cdot\,)^{*,p}_{\text{ét}}$ is again defined to be its adjoint with respect to the Poincaré pairing on étale cohomology. Note that as before one can make the relevant adjustments for extensions L/K inside $\mathbb{C}$ and that the resulting maps are actually G_L-equivariant.

When considering the local case of a finite extension F of $\mathbb{Q}_p$, the de Rham versions of the above maps arise through the commutative diagrams

$$\begin{tikzcd} \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^{j}_{\text{ét}}(\bar{Y},\mathbb{Q}_p)) \arrow[r, "\mathbf{D}_{\mathrm{dR}}(\Pi^{*,p}_{\text{ét}})"] \arrow[d, "\mathrm{comp}_{\mathrm{dR}}"', "\wr"] & \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^{j-2(e-m)}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(m-e)) \arrow[d, "\mathrm{comp}_{\mathrm{dR}}"', "\wr"] \\ \mathrm{H}^{j}_{\mathrm{dR}}(Y/F) \arrow[r, "\Pi^{*,p}_{\mathrm{dR}}"] & \mathrm{H}^{j-2(e-m)}_{\mathrm{dR}}(X/F)[m-e] \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dR}}(\mathrm{H}^j_{\text{ét}}(\bar{X},\mathbb{Q}_p)) \arrow[r, "\mathbf{D}_{\mathrm{dR}}(\Pi^p_{\text{ét},*})"] \arrow[d, "\mathrm{comp}_{\mathrm{dR}}"', "\wr"] & \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^{j-2(d-m)}_{\text{ét}}(\bar{Y},\mathbb{Q}_p)(m-d)) \arrow[d, "\mathrm{comp}_{\mathrm{dR}}"', "\wr"] \\
\mathrm{H}^j_{\mathrm{dR}}(X/F) \arrow[r, "\Pi^p_{\mathrm{dR},*}"] & \mathrm{H}^{j-2(d-m)}_{\mathrm{dR}}(Y/F)[m-d].
\end{tikzcd}$$

Let now K be a number field. We close this section by briefly discussing the p-adic étale Abel–Jacobi map. By using the material of the previous subsections, this can be used to introduce the p-adic Abel–Jacobi map on de Rham cohomology over p-adic fields K_ν. On étale cohomology, the map is of the form

$$\mathrm{AJ}^p_{\text{ét},X} \colon \mathrm{CH}^j(X)_0(K) \longrightarrow \mathrm{H}^1_{\mathrm{cts}}(K, \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j)),$$

induced by the following observation. Given $\Pi \in \mathrm{CH}^j(X)_0(K)$, the definition (2.12) of $\mathrm{cl}^p_{\text{ét},\mathsf{g}}$ reveals that $\mathrm{cl}^p_{\text{ét},\mathsf{a}}(\Pi) \in \mathrm{Fil}^1\,\mathrm{H}^{2j}_{\text{ét}}(X,\mathbb{Q}_p(j))$, so that we can set

$$\mathrm{AJ}^p_{\text{ét},X}(\Pi) = \alpha^{(j)}_{2j,1}([\mathrm{cl}^p_{\text{ét},\mathsf{a}}(\Pi)]).$$

We note (cf. [Nek00]) that the image of $\mathrm{AJ}^p_{\text{ét},X}$ lies in the Bloch–Kato Selmer group $\mathrm{H}^1_f(K, \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j))$. As before, a change in the field of definition on the left-hand side translates to an appropriate change on the right-hand side for the Galois cohomology.

We can now introduce the p-adic Abel–Jacobi map on de Rham cohomology. We assume the variety X to have good reduction at a prime ν lying above p. By restricting the target of $\mathrm{AJ}^p_{\text{ét},X}$ to K_ν, we can consider the Bloch–Kato logarithm (2.9) along with the comparison isomorphism modulo filtrations (2.6) and the twisted Poincaré duality isomorphism (2.10):

$$\begin{aligned}
\log_{\mathrm{BK},X} &\colon \mathrm{H}^1_f(K_\nu, \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j)) \xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j))/\,\mathrm{Fil}^0,\\
\overline{\mathrm{comp}}_{\mathrm{dR},X} &\colon \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j))/\,\mathrm{Fil}^0 \xrightarrow{\sim} \mathrm{H}^{2j-1}_{\mathrm{dR}}(X/K_\nu)[j]/\,\mathrm{Fil}^0,\\
\mathrm{PD}_X &\colon \mathrm{H}^{2j-1}_{\mathrm{dR}}(X/K_\nu)[j]/\,\mathrm{Fil}^0 \xrightarrow{\sim} (\mathrm{Fil}^{j^*}\,\mathrm{H}^{2j^*-1}_{\mathrm{dR}}(X/K_\nu))^\vee.
\end{aligned}$$

We denote their composition by

$$\log^\nu_X \colon \mathrm{H}^1_f(K_\nu, \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X},\mathbb{Q}_p)(j)) \xrightarrow{\sim} (\mathrm{Fil}^{j^*}\,\mathrm{H}^{2j^*-1}_{\mathrm{dR}}(X/K_\nu))^\vee. \tag{2.13}$$

This map will be referred to as the *ν-adic logarithm*. We denote its inverse by $\exp^\nu_X$ and call it the *ν-adic exponential*. Finally, we can define

$$\mathrm{AJ}^\nu_{\mathrm{dR},X} = \log^\nu_X \circ \mathrm{res}_\nu \circ \mathrm{AJ}^p_{\text{ét},X} \colon \mathrm{CH}^j(X)_0(K) \longrightarrow (\mathrm{Fil}^{j^*}\,\mathrm{H}^{2j^*-1}_{\mathrm{dR}}(X/K_\nu))^\vee \tag{2.14}$$

to be the *ν-adic Abel–Jacobi map* on de Rham cohomology.

The compatibility of ν-adic Abel–Jacobi maps for varities X and Y through the induced maps introduced before takes the form of the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^j(X)_0(K) \arrow[r, "\mathrm{AJ}^\nu_{\mathrm{dR},X}"] \arrow[d, "\Pi_*"'] & (\mathrm{Fil}^{d-j+1}\,\mathrm{H}^{2(d-j)+1}_{\mathrm{dR}}(X/K_\nu))^\vee \arrow[d, "\Pi^{*,p,\vee}_{\mathrm{dR}}"'] \\
\mathrm{CH}^{m-d+j}(Y)_0(K) \arrow[r, "\mathrm{AJ}^\nu_{\mathrm{dR},Y}"] & (\mathrm{Fil}^{e-m+d-j+1}\,\mathrm{H}^{2(e-m+d-j)+1}_{\mathrm{dR}}(Y/K_\nu))^\vee
\end{tikzcd} \tag{2.15}$$

for cycles $\Pi \in \mathrm{CH}^m(X \times Y)(K)$. This is referred to as *functoriality* of Abel–Jacobi maps with respect to correspondences.

Remark 2.9. To be precise, the map $\Pi_{\mathrm{dR}}^{*,p,\vee}$ on the right-hand side of diagram (2.15) is given by the dual of the filtration piece $\mathrm{Fil}^{e-m+d-j+1}\,\Pi_{\mathrm{dR}}^{*,p}$.

3 Self-Fold Triple Products

We will now turn our attention towards the particular situation that we are interested in – three-fold products of elliptic curves over the rational numbers. Let $A/\mathbb{Q}$ be an elliptic curve with corresponding modular form $f_A \in \mathrm{S}_2^{\mathrm{new}}(\Gamma_0(\mathrm{cond}(A)))$ and associated p-adic Galois representation

$$\rho_{f_A,p}\colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Q}_p),$$

whose properties we recall below. Considering its three-fold tensor product $\rho_{f_A,p}^{\otimes 3}$, we want to know about the existence of a projection of representations

$$\rho_{f_A,p}^{\otimes 3}(n) \longrightarrow\!\!\!\!\!\rightarrow \rho_{f_A,p}^{\oplus m} \tag{3.1}$$

for an appropriate twist $n \in \mathbb{Z}$ and multiplicity $m \geq 1$.

With regard to our subsequent considerations of Hida families and p-adic L-functions attached to those, we will address this question in a more general context. More precisely, we let $\theta \in \mathrm{S}_k(\Gamma_0(N))$ be an eigenform of even weight $k \geq 2$ and ask the very same question for the attached p-adic Galois representation $\rho_{\theta,p}$. We will answer this by decomposing the triple product, revealing the original representation as one of the components of the direct sum.

On the one hand, when specializing to the weight-two case of an elliptic curve, the projection (3.1) that this section provides will be of fundamental interest for our upcoming discussion of conjectural parametrizations via three-fold products. On the other hand, those eigenforms considered in the general situation show up when looking at the *weight-k specializations* of the Hida family that we will consider later. Again, the result will play an important role in our study of complex and p-adic L-functions, more precisely in the factorization of such.

3.1 **Galois Representations Attached to Modular Forms.** Let $\theta \in \mathrm{S}_k(\Gamma_0(N))$ be an eigenform of even weight $k \geq 2$ and denote by K_θ the number field generated by its Fourier coefficients $\{a_n(\theta)\}_n$. As constructed by Deligne (cf. [Del71], [Rib77, Theorem 2.1]), we consider the p-adic Galois representation

$$\rho_{\theta,p}\colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(K_\theta \otimes \mathbb{Q}_p)$$

with the following property. For any prime $q \nmid Np$, the representation $\rho_{\theta,p}$ is unramified at q with

$$\mathrm{tr}(\rho_{\theta,p}(\mathrm{Frob}_q)) = a_q(\theta), \qquad \det(\rho_{\theta,p}(\mathrm{Frob}_q)) = q^{k-1}.$$

Remark 3.1. Note that this is the contragredient representation of that originally constructed by Deligne. For example, for an elliptic curve $A/\mathbb{Q}$, the representation $\rho_{f_A,p}$ is realized by the p-adic Tate module $V_p(A)$, which is the dual of the first étale cohomology group $\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)$.

We will be looking at those representations through the following $\mathfrak{p}$-adic variants. Decomposing $K_\theta \otimes \mathbb{Q}_p$ as the product of the various completions $K_{\theta,\mathfrak{p}}$ for primes $\mathfrak{p}$ of $\mathcal{O}_{K_\theta}$ lying above p, the representation $\rho_{\theta,p}$ decomposes as the direct sum of representations

$$\rho_{\theta,\mathfrak{p}}\colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(K_{\theta,\mathfrak{p}})$$

with coefficients in a finite extension of $\mathbb{Q}_p$. This viewpoint is useful as one has the following result.

Proposition 3.2 ([Rib77, Theorem 2.3]). *The representations $\rho_{\theta,\mathfrak{p}}$ are simple.*

The importance of this observation lies in the fact of semi-simple representations being up to isomorphism uniquely determined by a particular *finite* amount of Frobenius-traces via a density argument. We will discuss this in greater detail in the next subsection.

3.2 **Characterization of Semi-Simple Galois Representations.** We will now have a look at how considerations on the level of Frobenius-traces characterize semi-simple Galois representations. To put ourselves in the setting in which we want to apply these methods, we consider representations of $G_{\mathbb{Q}}$ with coefficients in a finite extension $F/\mathbb{Q}_p$.

Let us fix the following simple observation.

Lemma 3.3. *Let $\rho \twoheadrightarrow \rho'$ be a projection of semi-simple representations. Then, given any $\sigma \in G_{\mathbb{Q}}$, each eigenvalue for $\rho'(\sigma)$ is also one for $\rho(\sigma)$.*

Proof. Let $\varphi \colon V \twoheadrightarrow W$ be the corresponding $G_{\mathbb{Q}}$-equivariant projection between the representation spaces of ρ and ρ'. As V is semi-simple, there exists a section, i.e. a $G_{\mathbb{Q}}$-equivariant map $\varphi' \colon W \hookrightarrow V$ such that $\varphi \circ \varphi' = \mathrm{id}_W$. Let $w \in W$ be an eigenvector of $\rho'(\sigma)$ with eigenvalue μ, so that in particular $w \neq 0$. Then $\varphi'(w) \neq 0$ and

$$\rho(\sigma)(\varphi'(w)) = \varphi'(\rho'(\sigma)(w)) = \varphi'(\mu w) = \mu\varphi'(w),$$

so μ is also an eigenvalue for $\rho(\sigma)$. ☐

Therefore, checking the eigenvalues for $\rho(\sigma)$ and $\rho'(\sigma)$ for an *accessible* amount of $\sigma \in G_{\mathbb{Q}}$ may be the first thing to consider when hoping for a projection $\rho \twoheadrightarrow \rho'$. In our case, these elements will be the Frobenius elements Frob_q for primes $q \nmid Np$.

Remark 3.4. To be precise, in general one has to replace the representations by a suitable scalar extension to make sense of talking about actual eigenvalues. However, we will not get into this as in the end we are working with Frob_q-traces instead of Frob_q-eigenvalues, which always belong to the field of definition.

Taking this observation a step further, we now introduce the relevant material for the characterization, as can be found in [FWG+92]. Let S be a finite set of primes including p and let $K/\mathbb{Q}$ be a finite Galois extension unramified outside of S. A particular version of the Chebotarev Density Theorem (cf. Corollary 2.4 of loc. cit.) provides a finite set $T_{K,S}$ of primes disjoint from S in such a way that the conjugacy classes of the Frobenius automorphisms at q for $q \in T_{K,S}$ cover all of $\mathrm{Gal}(K/\mathbb{Q})$.

Now, let $d \geq 1$ be an integer and take L to be the finite Galois extension of $\mathbb{Q}$ containing all the Galois extensions $K/\mathbb{Q}$ of degree $[K : \mathbb{Q}] < p^{2d^2}$ which are unramified outside of S. Considering the finite set of primes $T_{L,S}$ defined as above, semi-simple Galois representations of dimension d are characterized by the following result.

Proposition 3.5 ([FWG+92, Proposition 2.7]). *Let $\rho, \rho' \colon G_{\mathbb{Q}} \to \mathrm{GL}_d(F)$ be two semi-simple Galois representations which are unramified outside of S and satisfy*

$$\mathrm{tr}(\rho(\mathrm{Frob}_q)) = \mathrm{tr}(\rho'(\mathrm{Frob}_q)) \quad \textit{for all } q \in T_{L,S}.$$

Then ρ and ρ' are isomorphic.

Remark 3.6. We note that we replaced $\mathbb{Q}_p$ by F in the above result as its proof does not depend on that choice. In our situation of representations associated to an eigenform $\theta \in \mathrm{S}_k(\Gamma_0(N))$, these finite extensions will be the $\mathfrak{p}$-adic completions $K_{\theta,\mathfrak{p}}$, appearing in the decomposition $K_\theta \otimes \mathbb{Q}_p = \prod_{\mathfrak{p}|p} K_{\theta,\mathfrak{p}}$ mentioned before.

3.3 **The Appropriate Twist of the Triple Product and its Decomposition.** We will now compare Frobenius-eigenvalues and -traces for our Galois representations of interest in order to determine how to appropriately twist the triple product representation. Note that in the end, we have to consider the relevant $\mathfrak{p}$-adic components of the representations of interest, but as the Frobenius-eigenvalues of the original representation give those of the components, we have a look at these.

As the twists of the representation are described by powers of the p-adic cyclotomic character, let us briefly recall its construction. Let $n \geq 1$ be an integer and let μ_{p^n} denote the group of p^n-th roots of unity in $\bar{\mathbb{Q}}^\times$. Fixing a primitive p^n-th root of unity ζ_{p^n} generating μ_{p^n}, we can define the *mod p^n cyclotomic character* to be the map

$$\chi_{p,n}\colon G_\mathbb{Q} \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times, \quad \text{such that } \sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{p,n}(\sigma)} \text{ for } \sigma \in G_\mathbb{Q}.$$

When n varies, the elements $\chi_{p,n}(\sigma)$ form a compatible system and hence yield an element $\chi_p(\sigma) \in \mathbb{Z}_p^\times$. The so obtained map

$$\chi_p\colon G_\mathbb{Q} \longrightarrow \mathbb{Z}_p^\times$$

is referred to as the *p-adic cyclotomic character*. This map is unramified at primes $q \neq p$, and moreover satisfies $\chi_p(\mathrm{Frob}_q) = q$ for such primes. Twisting a p-adic Galois representation ρ by $n \in \mathbb{Z}$ thus affects the Frob_q-eigenvalues by multiplication by q^n.

We can now turn to the study of representations of interest. Let $\theta \in \mathrm{S}_k(\Gamma_0(N))$ be an eigenform. We fix a prime p and consider the associated p-adic Galois representation from Section 3.1. In the following, we refrain from carrying the prime p in the index of the representation and denote it by ρ_θ.

A direct computation shows how the triple product $\rho_\theta^{\otimes 3}$ should be twisted in order to have a chance for a projection onto ρ_θ.

Lemma 3.7. *For every prime $q \nmid Np$, each* Frob_q*-eigenvalue for ρ_θ appears as a* Frob_q*-eigenvalue for $\rho_\theta^{\otimes 3}(1-k)$.*

Proof. Let α_q and β_q be the roots of the characteristic polynomial of $\rho_\theta(\mathrm{Frob}_q)$. Since $\alpha_q\beta_q = q^{k-1}$, we immediately notice that for $\rho_\theta^{\otimes 3}(1-k)(\mathrm{Frob}_q)$ we find $\alpha_q^3 q^{1-k}$ and $\beta_q^3 q^{1-k}$, both with multiplicity one, as well as α_q and β_q, both with multiplicity three. ☐

Our goal is now to establish a projection

$$\rho_\theta^{\otimes 3}(1-k) \twoheadrightarrow \rho_\theta^{\oplus 2}. \tag{3.2}$$

This is done by decomposing the left-hand side, using considerations on the level of Frob_q-traces. More precisely, we will write down a representation, having the same Frobenius-traces as our triple product for the right amount of primes, so that, considering the $\mathfrak{p}$-adic components of both of these, we can apply Proposition 3.5 to those for each $\mathfrak{p} \mid p$. Here, the finite set S will be given by $S(N) = \{q \text{ prime} : q \mid N\} \cup \{p\}$, and we set $T(N)$ to be $T_{L,S(N)}$ as in the notation of Section 3.2.

Proposition 3.8. *There is an isomorphism of p-adic Galois representations*

$$\rho_\theta^{\otimes 3}(1-k) \cong \rho_\theta^{\oplus 2} \oplus \operatorname{Sym}^3(\rho_\theta)(1-k).$$

Proof. Let $q \in T(N)$, so that in particular $q \nmid Np$. We compute that

$$\begin{aligned}
\operatorname{tr}(\rho_\theta^{\otimes 3}(1-k)(\operatorname{Frob}_q)) &= q^{1-k}(\alpha_q^3 + \beta_q^3) + 3a_q(\theta) \\
&= \operatorname{tr}(\operatorname{Sym}^3(\rho_\theta)(1-k)(\operatorname{Frob}_q)) + \operatorname{tr}(\rho_\theta^{\oplus 2}(\operatorname{Frob}_q)) \\
&= \operatorname{tr}((\operatorname{Sym}^3(\rho_\theta)(1-k) \oplus \rho_\theta^{\oplus 2})(\operatorname{Frob}_q)).
\end{aligned}$$

Therefore, in particular the $\mathfrak{p}$-adic components on both sides share the same Frob_q-traces. Noting that tensor products of semi-simple representations are semi-simple (cf. [Hoc71, Theorem 12.2]), as are subrepresentations of semi-simple representations, the stability of semi-simplicity under taking direct sums ensures that all the considered representations are semi-simple. Now, Proposition 3.5 yields an isomorphism as desired on each $\mathfrak{p}$-component, assembling to an isomorphism as claimed. □

Remark 3.9. One might ask whether one can find a higher multiplicity of ρ_θ inside the triple product $\rho_\theta^{\otimes 3}(1-k)$. The summand that has to be studied further for such a higher multiplicity result is $\operatorname{Sym}^3(\rho_\theta)(1-k)$. As its Frob_q-traces are given by

$$\begin{aligned}
\operatorname{tr}(\operatorname{Sym}^3(\rho_\theta)(1-k)(\operatorname{Frob}_q)) &= q^{1-k}(\alpha_q^3 + \beta_q^3) + a_q(\theta) \\
&= q^{1-k}(\alpha_q^3 + \beta_q^3) + \operatorname{tr}(\rho_\theta(\operatorname{Frob}_q)),
\end{aligned} \tag{3.3}$$

in order to find another copy of ρ_θ, one has to realize the *error term* $q^{1-k}(\alpha_q^3 + \beta_q^3)$ as the Frob_q-trace of some p-adic representation. As will be presented in what follows, this is indeed possible in a *complex multiplication case*. However, let us point out that the multiplicity of ρ_θ can not be four. Indeed, if this were the case, then we would have $q^{1-k}a_q(\theta)^3 = 4a_q(\theta)$, so that by choosing a prime $q \nmid N$ with $a_q(\theta) \neq 0$, we arrive at $\sqrt{q^{k-1}} \in \mathbb{Q}$, which forms a contradiction as $k \geq 2$ is even.

Complex Multiplication. In this section, we want to improve the above discussion in a special case, whose origin lies in assuming our elliptic curve $A/\mathbb{Q}$ to have complex multiplication by a quadratic imaginary field K. Let ψ_A be the Hecke character of infinity type $(1,0)$ attached to A, inducing the weight-two newform $\theta_{\psi_A} = f_A$ that establishes the modularity of A. In this situation, we will prove that we can now obtain a projection of the form

$$\rho_{f_A}^{\otimes 3}(-1) \twoheadrightarrow \rho_{f_A}^{\oplus 3}. \tag{3.4}$$

More precisely, we will show in greater generality that the twisted triple tensor product Galois representations of cusp forms induced by odd powers of a Hecke character ψ of infinity type $(1,0)$ split further, now allowing to find three copies of the original representation.

Let $k \geq 2$ be an even integer and consider the Hecke character ψ^{k-1} of K of infinity type $(k-1,0)$. As $k-1$ is odd, its central character is given by the quadratic Dirichlet character attached to K, and so ψ^{k-1} gives rise to a cusp form

$$\theta_{\psi^{k-1}} \in \mathrm{S}_k(\Gamma_0(-\operatorname{disc}(K)\,\mathrm{N}\mathfrak{f}_{\psi^{k-1}})),$$

where $\mathfrak{f}_{\psi^{k-1}}$ denotes the conductor of ψ^{k-1}. For the *good* primes, the Frob_q-eigenvalues of the attached Galois representation can be described in terms of the Hecke character ψ^{k-1}, depending on the ramification behavior of q in the field K:

$$\begin{cases} \{\psi^{k-1}(\mathfrak{q}), \psi^{k-1}(\mathfrak{q}')\} & \text{if } q\,\mathcal{O}_K = \mathfrak{q}\,\mathfrak{q}' \\ \{\pm\psi^{k-1}(\mathfrak{q})^{1/2}\} & \text{if } q\,\mathcal{O}_K = \mathfrak{q}. \end{cases} \tag{3.5}$$

The next result is a follow-up version of Proposition 3.8 in this particular situation of Hecke characters.

Proposition 3.10. *There is an isomorphism of p-adic representations*

$$\rho_{\theta_{\psi^{k-1}}}^{\otimes 3}(1-k) \cong \rho_{\theta_{\psi^{k-1}}}^{\oplus 3} \oplus \rho_{\theta_{\psi^{3k-3}}}(1-k).$$

Proof. As remarked earlier, we will now analyze the component $\mathrm{Sym}^3(\rho_{\theta_{\psi^{k-1}}})(1-k)$. Set $N_{k-1} = -\operatorname{disc}(K)\,\mathrm{N}\mathfrak{f}_{\psi^{k-1}}$ and let, in the notation introduced above, q be a prime not in $S(N_{k-1})$. Denoting $\alpha_{q,k-1}$ and $\beta_{q,k-1}$ the Frob_q-eigenvalues of $\rho_{\theta_{\psi^{k-1}}}$, by using (3.3) and (3.5) we can compute

$$\begin{aligned} q^{1-k}(\alpha_{q,k-1}^3 + \beta_{q,k-1}^3) + a_q(\theta_{\psi^{k-1}}) &= q^{1-k}(\alpha_{q,k-1}^3 + \beta_{q,k-1}^3) + \mathrm{tr}(\rho_{\theta_{\psi^{k-1}}}(\mathrm{Frob}_q)) \\ &= q^{1-k}(\alpha_{q,3k-3} + \beta_{q,3k-3}) + \mathrm{tr}(\rho_{\theta_{\psi^{k-1}}}(\mathrm{Frob}_q)) \\ &= \mathrm{tr}(\rho_{\theta_{\psi^{3k-3}}}(1-k)(\mathrm{Frob}_q)) + \mathrm{tr}(\rho_{\theta_{\psi^{k-1}}}(\mathrm{Frob}_q)) \\ &= \mathrm{tr}((\rho_{\theta_{\psi^{3k-3}}}(1-k) \oplus \rho_{\theta_{\psi^{k-1}}})(\mathrm{Frob}_q)). \end{aligned}$$

It should be noted that $S(N_{3k-3}) \subseteq S(N_{k-1})$, so our choice of primes to consider makes sense. Again, executing the same argument as before concerning the $\mathfrak{p}$-adic components, Proposition 3.5 concludes the proof. ☐

Specializing Proposition 3.10 to the weight-two case of an elliptic curve $A/\mathbb{Q}$, we obtain the enhanced version (3.4) of (3.2) in the case of complex multiplication.

4 Conjectural Parametrizations via Three-Fold Products

In this section, we study the possibility of obtaining conjectural parametrizations of elliptic curves through the analysis of the three-fold Galois representation that we discussed in the previous section. The fundamental ingredient for these conjectural considerations is the following statement.

Conjecture 4.1 (Tate Conjecture)**.** *If Z is a smooth projective variety over a number field K, then*

$$\mathrm{cl}^p_{\text{ét,g}} \otimes \mathbb{Q}_p \colon \; \mathrm{CH}^n(Z)(K) \otimes \mathbb{Q}_p \longrightarrow \mathrm{H}^{2n}_{\text{ét}}(\bar{Z}, \mathbb{Q}_p)(n)^{G_K}$$

is surjective.

Before we explain how to put our analysis of Galois representations into the context of the Tate Conjecture, let us recall in which case elements of $\mathrm{CH}^n(Z)(K)$ are of interest for parametrizations. Let X and Y be smooth projective varieties over K and let $\Pi \in \mathrm{CH}^n(X \times Y)(K)$. Denoting by $\mathrm{pr}_X \colon X \times Y \to X$ and $\mathrm{pr}_Y \colon X \times Y \to Y$ the natural projection maps, recall (cf. Section 2.1) that one has an induced map

$$\begin{aligned} \Pi_* \colon \; \mathrm{CH}^{\dim(X)+j-n}(X)_0(K) &\longrightarrow \mathrm{CH}^j(Y)_0(K) \\ \Delta &\longmapsto \mathrm{pr}_{Y,*}(\Pi \cdot \mathrm{pr}_X^*(\Delta)). \end{aligned}$$

This is of particular interest for us when we take Y to be our elliptic curve $A/\mathbb{Q}$ and $j = 1$. More precisely, we will be considering this in the case of $X = A^3$, so that the sought for elements live in the Chow group of the four-fold A^4.

As a consequence of the formulation of the Tate Conjecture, we necessarily encounter p-adic coefficients and so we will also address the question of an appropriate normalization in order to get towards a global parametrization with rational coefficients.

4.1 **Projections onto the Tate Module and the Tate Conjecture.** We will now discuss how certain Galois equivariant maps with target the Tate module of A come into play for obtaining conjectural parametrizations of A. The Tate Conjecture will be used through the following simple observation.

Lemma 4.2. *Any G_K-equivariant projection $\varphi \colon \mathrm{H}^{2j-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(j) \twoheadrightarrow V_p(A)$ corresponds to a non-trivial element*

$$Z_\varphi \in \mathrm{H}^{2j^*}_{\text{ét}}(\overline{X \times A}, \mathbb{Q}_p)(j^*)^{G_K} \qquad \textit{for } j^* = \dim(X) + 1 - j.$$

Proof. Starting with the fact that any G-invariant map $V \to W$ corresponds to an element of $(V^\vee \otimes W)^G$, we see that φ corresponds to a non-zero element of

$$\begin{aligned} Z_\varphi &\in (\mathrm{H}^{2j-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(j)^\vee \otimes V_p(A))^{G_K} \\ &= (\mathrm{H}^{2j-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p)^\vee(-j) \otimes \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1))^{G_K} \\ &= (\mathrm{H}^{2\dim(X)-2j+1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(\dim(X) - j) \otimes \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1))^{G_K} \\ &\subseteq \mathrm{H}^{2j^*}_{\text{ét}}(\overline{X \times A}, \mathbb{Q}_p)(j^*)^{G_K}. \end{aligned}$$

Here, we have further used Poincaré duality and the Künneth decomposition. □

In the following, we will explain how to situate ourselves within the aforementioned scenario for our particular case of interest. Let $f_A \in \mathrm{S}_2(\Gamma_0(\mathrm{cond}(A)))$ be the cusp form attached to A by the modularity theorem. The decomposition of Proposition 3.8 provides a projection

$$\rho_{f_A}^{\otimes 3}(-1) \twoheadrightarrow \rho_{f_A}^{\oplus 2}$$

of p-adic Galois representations, which translates to a $G_{\mathbb{Q}}$-invariant projection

$$\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)^{\otimes 3}(2) \twoheadrightarrow V_p(A)^{\oplus 2}.$$

Combining this with the natural projection arising from the Künneth decomposition lets us end up with a $G_{\mathbb{Q}}$-equivariant projection

$$\pi_{\text{ét}} \colon \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) \twoheadrightarrow V_p(A)^{\oplus 2}.$$

We refer to its components by $\pi_{\text{ét},i}$ and record the following observation, performing the steps from the previous general considerations in this particular case.

Lemma 4.3. *If the Tate Conjecture is true for A^4 in codimension two, then the two components of the $G_{\mathbb{Q}}$-equivariant projection $\pi_{\text{ét}}$ give rise to elements $\Pi_i^{?,(p)} \in \mathrm{CH}^2(A^4)(\mathbb{Q}) \otimes \mathbb{Q}_p$, consequently inducing parametrizations*

$$\Pi_{i,*}^{?,(p)} \colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \otimes \mathbb{Q}_p \longrightarrow A(\mathbb{Q}) \otimes \mathbb{Q}_p.$$

Proof. By Lemma 4.2, we know that we can view the component $\pi_{\text{ét},i}$ as a non-trivial element $Z_{\pi_{\text{ét},i}} \in \mathrm{H}^4_{\text{ét}}(\bar{A}^4, \mathbb{Q}_p)(2)^{G_{\mathbb{Q}}}$. Assuming the validity of the Tate Conjecture thus gives rise to an element

$$\Pi_i^{?,(p)} \in \mathrm{CH}^2(A^4)(\mathbb{Q}) \otimes \mathbb{Q}_p \quad \text{such that} \quad (\mathrm{cl}^p_{\text{ét},\mathbf{g}} \otimes \mathbb{Q}_p)(\Pi_i^{?,(p)}) = Z_{\pi_{\text{ét},i}}.$$

As introduced in Section 2 and recalled at the beginning of this section, this yields

$$\Pi_{i,*}^{?,(p)} \colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \otimes \mathbb{Q}_p \longrightarrow A(\mathbb{Q}) \otimes \mathbb{Q}_p. \qquad \square$$

As we are ultimately interested in obtaining objects that are defined over $\mathbb{Q}$ without the need of extending coefficients to $\mathbb{Q}_p$, in the following we will study the process of *normalizing* the original projection $\pi_{\text{ét}}$ in an appropriate way in order to naturally get towards a *global* parametrization.

4.2 **Passing to de Rham Cohomology.** We will now describe how $\pi_{\text{ét}}$ induces a corresponding projection π_{dR} in de Rham cohomology, whose components we will again denote by $\pi_{\mathrm{dR},i}$. This morphism in de Rham cohomology has the advantage that its domain and target have natural $\mathbb{Q}$-structures, which we will use for normalizing our original projection on the level of étale cohomology.

Applying $\mathbf{D}_{\mathrm{dR}}$ to the projection $\pi_{\text{ét}} \colon \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) \twoheadrightarrow \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1)^{\oplus 2}$ and using the respective comparison isomorphism (2.5) on both sides induces a map

$$\pi_{\mathrm{dR}} \colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2] \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]^{\oplus 2}$$

defined by the diagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) \arrow[r, two heads, "\mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét}})"] \arrow[d, "\mathrm{comp}_{\mathrm{dR},A^3}"', "\wr"] & \mathbf{D}_{\mathrm{dR}}(V_p(A)^{\oplus 2}) \arrow[d, "\mathrm{comp}^{\oplus 2}_{\mathrm{dR},A}"', "\wr"] \\
\mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2] \arrow[r, "\pi_{\mathrm{dR}}"] & \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]^{\oplus 2}.
\end{tikzcd} \tag{4.1}$$

Remark 4.4. Note that $\mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét}})_i = \mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét},i})$, so it does not matter at which point we decide to consider the respective components separately for studying π_{dR}.

We record two useful properties of the induced map on de Rham cohomology.

Lemma 4.5. *The map π_{dR} vanishes on* $\mathrm{Sym}^3(\mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1])[-1]$.

Proof. As $\pi_{\text{ét}}$ vanishes on $\mathrm{Sym}^3(\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1))(-1)$, it follows by its very construction that π_{dR} vanishes on the image of $\mathbf{D}_{\mathrm{dR}}(\mathrm{Sym}^3(\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1))(-1))$ under the relevant comparison isomorphism. Note that $\mathrm{Sym}^3(\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1))(-1)$ is indeed a de Rham representation, being the kernel of the surjection

$$\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)^{\otimes 3}(2) \twoheadrightarrow V_p(A)^{\oplus 2}$$

of de Rham representations which arises from Proposition 3.8. This follows for example from Lemma 2.1. The assertion then follows from the fact that the comparison isomorphism on the triple product side restricts to an isomorphism

$$\mathbf{D}_{\mathrm{dR}}(\mathrm{Sym}^3(V_p(A))(-1)) \cong \mathrm{Sym}^3(\mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1])[-1]. \qquad \square$$

Lemma 4.6. *The map π_{dR} is surjective, and so is each of its filtration parts*

$$\mathrm{Fil}^j \pi_{\mathrm{dR}} \colon \mathrm{Fil}^{j+2} \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p) \twoheadrightarrow (\mathrm{Fil}^{j+1} \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p))^{\oplus 2}.$$

Proof. As $\mathbf{D}_{\mathrm{dR}}$ is exact by Theorem 2.2, it takes the short exact sequence

$$\ker(\pi_{\text{ét}}) \hookrightarrow \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) \xrightarrow{\pi_{\text{ét}}} V_p(A)^{\oplus 2}$$

in $\mathbf{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ to a short exact sequence

$$\mathbf{D}_{\mathrm{dR}}(\ker(\pi_{\text{ét}})) \hookrightarrow \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) \xrightarrow{\mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét}})} \mathbf{D}_{\mathrm{dR}}(V_p(A)^{\oplus 2})$$

in $\mathbf{Fil}_{\mathbb{Q}_p}$. Therefore, $\mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét}})$, and thus π_{dR} by its definition (4.1), are surjective. Recall that in the category of filtered vector spaces, short exact sequences are those having *strict* morphisms as the injection and surjection, i.e.

$$\begin{aligned}
\mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét}})(\mathrm{Fil}^j \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2))) &= \mathrm{Fil}^j \mathbf{D}_{\mathrm{dR}}(V_p(A)^{\oplus 2}) \cap \mathrm{im}(\mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét}})) \\
&= \mathrm{Fil}^j \mathbf{D}_{\mathrm{dR}}(V_p(A)^{\oplus 2}).
\end{aligned}$$

As the comparison isomorphisms respect the filtrations, being strict as well, this finishes the proof of surjectivity of $\mathrm{Fil}^j \pi_{\mathrm{dR}}$. $\square$

For a deeper examination of the maps $\pi_{\text{ét}}$ and π_{dR}, we position ourselves within a context enriched with additional knowledge about the Bloch–Kato logarithm, achieved through the imposition of the assumption of good reduction at p. Recalling the Bloch–Kato logarithm (2.9), the comparison isomorphism modulo filtrations (2.6) and the twisted Poincaré duality isomorphism (2.10), the p-adic logarithms that are of interest for us are of the following form.

Lemma 4.7. *The p-adic logarithms for A^3 and A give isomorphisms of the form*

$$\begin{aligned}\log^p_{A^3} \colon \mathrm{H}^1_f(\mathbb{Q}_p, \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) &\xrightarrow{\sim} (\mathrm{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3, \mathbb{Q}_p))^\vee, \\ \log^p_A \colon \mathrm{H}^1_f(\mathbb{Q}_p, V_p(A)) &\xrightarrow{\sim} \Omega^1(A/\mathbb{Q}_p)^\vee.\end{aligned}$$

Proof. As introduced in (2.13), these maps are given by the composition of the following isomorphisms:

$$\begin{aligned}\log_{\mathrm{BK},A^3} \colon \mathrm{H}^1_f(\mathbb{Q}_p, \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) &\xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2))/\,\mathrm{Fil}^0, \\ \overline{\mathrm{comp}}_{\mathrm{dR},A^3} \colon \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2))/\,\mathrm{Fil}^0 &\xrightarrow{\sim} \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2]/\,\mathrm{Fil}^0, \\ \mathrm{PD}_{A^3} \colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)/\,\mathrm{Fil}^0 &\xrightarrow{\sim} (\mathrm{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p))^\vee,\end{aligned}$$

respectively,

$$\begin{aligned}\log_{\mathrm{BK},A} \colon \mathrm{H}^1_f(\mathbb{Q}_p, V_p(A)) &\xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(V_p(A))/\,\mathrm{Fil}^0, \\ \overline{\mathrm{comp}}_{\mathrm{dR},A} \colon \mathbf{D}_{\mathrm{dR}}(V_p(A))/\,\mathrm{Fil}^0 &\xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]/\,\mathrm{Fil}^0, \\ \mathrm{PD}_A \colon \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)/\,\mathrm{Fil}^0 &\xrightarrow{\sim} (\mathrm{Fil}^1 \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p))^\vee.\end{aligned}$$ ☐

We further have the following compatibility result through the various ingredients of these p-adic logarithms.

Proposition 4.8. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{H}^1_f(\mathbb{Q}_p, \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) \arrow[r, "\mathrm{H}^1_f(\mathbb{Q}_p, \pi_{\text{ét},i})"] \arrow[d, "\log_{\mathrm{BK},A^3}"', "\wr"] & \mathrm{H}^1_f(\mathbb{Q}_p, V_p(A)) \arrow[d, "\log_{\mathrm{BK},A}"', "\wr"] \\
\mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2))/\,\mathrm{Fil}^0 \arrow[r, two heads, "\overline{\mathbf{D}}_{\mathrm{dR}}(\pi_{\text{ét},i})"] \arrow[d, "\overline{\mathrm{comp}}_{\mathrm{dR},A^3}"', "\wr"] & \mathbf{D}_{\mathrm{dR}}(V_p(A))/\,\mathrm{Fil}^0 \arrow[d, "\overline{\mathrm{comp}}_{\mathrm{dR},A}"', "\wr"] \\
\mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2]/\,\mathrm{Fil}^0 \arrow[r, two heads, "\overline{\pi_{\mathrm{dR},i}}"] \arrow[d, "\mathrm{PD}_{A^3}"', "\wr"] & \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]/\,\mathrm{Fil}^0 \arrow[d, "\mathrm{PD}_A"', "\wr"] \\
(\mathrm{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p))^\vee \arrow[r, "\pi^{\mathrm{ad},\vee}_{\mathrm{dR},i}"] & \Omega^1(A/\mathbb{Q}_p)^\vee.
\end{tikzcd}$$

Proof. The top square commutes by Proposition 2.5 and (2.8). For showing commutativity of the middle square one applies the universal property of quotients in various situations.

More precisely, one shows that both the maps $\overline{\pi_{\mathrm{dR},i}} \circ \overline{\mathrm{comp}}_{\mathrm{dR},A^3}$ and $\overline{\mathrm{comp}}_{\mathrm{dR},A} \circ \overline{\mathbf{D}}_{\mathrm{dR}}(\pi_{\text{ét},i})$ qualify for being the unique dashed arrow making the diagram

$$\begin{tikzcd} \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) \arrow[rr, two heads, "\mathrm{q}_{\mathrm{Fil}^0} \circ \mathrm{comp}_{\mathrm{dR},A} \circ \mathbf{D}_{\mathrm{dR}}(\pi_{\text{ét},i})"] \arrow[d, two heads, "\mathrm{q}_{\mathrm{Fil}^0}"'] & & \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]/\operatorname{Fil}^0 \\ \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2))/\operatorname{Fil}^0 \arrow[urr, dashed] & & \end{tikzcd}$$

commutative. Finally, the bottom square commutes by definition: $\pi^{\mathrm{ad}}_{\mathrm{dR},i}$ is the adjoint of $\pi_{\mathrm{dR},i}$ with respect to the Poincaré pairing and the twisted Poincaré duality isomorphisms PD_A and PD_{A^3} are given by that pairing. $\square$

4.3 **A More Detailed Look at the Relevant Filtration Step.** Let us have a look at a specific filtration piece of our projection π_{dR}. We take ω_A to be the canonical differential on A and write

$$\mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p) = \mathbb{Q}_p\omega_A \oplus \mathbb{Q}_p\eta_A$$

for another element $\eta_A \in \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)$, which may be chosen to satisfy $\langle \omega_A, \eta_A \rangle_A = 1$. We then have

$$\begin{aligned} \operatorname{Fil}^0 \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1] &= \operatorname{Fil}^1 \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p) = \Omega^1(A/\mathbb{Q}_p) = \mathbb{Q}_p\omega_A, \\ \operatorname{Fil}^0 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2] &= \operatorname{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p) = \mathbb{Q}_p\varpi_A^{123} \oplus \mathbb{Q}_p\varpi_A^{12} \oplus \mathbb{Q}\varpi_A^{13} \oplus \mathbb{Q}_p\varpi_A^{23}, \end{aligned}$$

where for J a set of indices,

$$\varpi_A^J = \mathrm{pr}^*_{A,1,\mathrm{dR}}\,\vartheta_1 \wedge \mathrm{pr}^*_{A,2,\mathrm{dR}}\,\vartheta_2 \wedge \mathrm{pr}^*_{A,3,\mathrm{dR}}\,\vartheta_3, \quad \text{with } \vartheta_j = \begin{cases} \omega_A & \text{if } j \in J \\ \eta_A & \text{if } j \notin J. \end{cases}$$

Here, $\mathrm{pr}_{A,j} \colon A^3 \to A$ is the projection onto the j-th component of the triple product. By Lemma 4.6, we now have a projection

$$\operatorname{Fil}^0 \pi_{\mathrm{dR}} \colon \operatorname{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p) \twoheadrightarrow \Omega^1(A/\mathbb{Q}_p)^{\oplus 2}.$$

As provided by Lemma 4.5, the vanishing of $\pi_{\text{ét}}$ on $\mathrm{Sym}^3(\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1))(-1)$ implies the vanishing of the induced map π_{dR} on the corresponding 4-dimensional $\mathbb{Q}_p$-vector space $\mathrm{Sym}^3(\mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1])[-1]$, which is generated by

$$\varpi_A^{\emptyset}, \qquad \varpi_A^{123}, \qquad \varpi_A^{12} + \varpi_A^{13} + \varpi_A^{23}, \qquad \varpi_A^{1} + \varpi_A^{2} + \varpi_A^{3}.$$

Out of these generators, only ϖ_A^{123} and $\varpi_A^{12} + \varpi_A^{13} + \varpi_A^{23}$ belong to the filtration step we are interested in, so we obtain a two-dimensional subspace $H \subseteq \operatorname{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)$ and an isomorphism

$$\overline{\operatorname{Fil}^0 \pi_{\mathrm{dR}}} \colon \operatorname{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)/H \xrightarrow{\sim} \Omega^1(A/\mathbb{Q}_p)^{\oplus 2}.$$

Complex Multiplication. By our analysis on the level of Galois representations (cf. Proposition 3.10), in the case of A having complex multiplication, we obtain

$$\mathrm{Fil}^0 \pi_{\mathrm{dR}} \colon \mathrm{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p) \twoheadrightarrow \Omega^1(A/\mathbb{Q}_p)^{\oplus 3}.$$

Hence, we only have to cut out a one-dimensional subspace $H \subseteq \mathrm{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)$ for getting an isomorphism onto $\Omega^1(A/\mathbb{Q}_p)^{\oplus 3}$. Further filtration considerations show that π_{dR} vanishes on ϖ_A^{123} and hence we get an isomorphism

$$\overline{\mathrm{Fil}^0 \pi_{\mathrm{dR}}} \colon \mathrm{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)/\mathbb{Q}_p \varpi_A^{123} \xrightarrow{\sim} \Omega^1(A/\mathbb{Q}_p)^{\oplus 3}.$$

Indeed, the differential ϖ_A^{123} generates $\mathrm{Fil}^3 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)$, which under π_{dR} maps onto

$$(\mathrm{Fil}^1 \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1])^{\oplus 3} = (\mathrm{Fil}^2 \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p))^{\oplus 3} = 0.$$

4.4 **Normalization and Conjectural Global Cycles.** The remainder of this section concerns two objectives, which we discuss in separate subsections. For these, we use the comparison between étale and de Rham cohomology and take advantage of the rational structure of the latter. More precisely, we normalize our projection $\pi_{\text{ét}}$ by requiring that the induced projection π_{dR} respects the $\mathbb{Q}$-structures of the relevant de Rham cohomologies. This results in having an induced projection of the form

$$\pi_{\mathrm{dR}} \colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \twoheadrightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1]^{\oplus 2}.$$

On the one hand, we study the conjectural existence of corresponding cycles with rational coefficients, inducing a map on de Rham cohomology as above. Consequently, our study aims to provide a conjectural parametrization

$$\mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow A(\mathbb{Q})^{\oplus 2}.$$

This gives a conjectural map which non-conjecturally produces rational points on the elliptic curve from rational cycles on its three-fold product. On the other hand, the subsequent subsection concerns a non-conjectural map

$$\mathrm{H}^1_f(\mathbb{Q}, \pi_{\text{ét},i}) \circ \mathrm{AJ}^p_{\text{ét},A^3} \colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}, V_p(A))$$

which conjecturally produces rational points on the elliptic curve from rational cycles on its three-fold product. Their relation will be displayed by a commutative diagram provided by Lemma 4.15.

Recall the cycle class map in de Rham cohomology

$$\mathrm{cl}_{\mathrm{dR}} \colon \mathrm{CH}^2(A^4)(\mathbb{Q}) \longrightarrow \mathrm{H}^4_{\mathrm{dR}}(A^4/\mathbb{Q})[2]$$

and the Künneth decomposition

$$\mathrm{H}^4_{\mathrm{dR}}(A^4/\mathbb{Q})[2] = \bigoplus_{j+j'=4} \mathrm{H}^j_{\mathrm{dR}}(A^3/\mathbb{Q}) \otimes \mathrm{H}^{j'}_{\mathrm{dR}}(A/\mathbb{Q})[2].$$

Both the cycle class map and the Künneth decomposition were used to define the maps on de Rham cohomology induced by cycles in Section 2.2. More precisely, recall the diagram

$$\begin{tikzcd}
\mathrm{CH}^2(A^4)(\mathbb{Q}) \arrow[rr, "\mathrm{cl}_{\mathrm{dR}}"] \arrow[d, "(\cdot)_{\mathrm{dR},*}"'] & & \mathrm{H}^4_{\mathrm{dR}}(A^4/\mathbb{Q})[2] \arrow[d, two heads, "\mathrm{pr}_3"'] \\
\mathrm{Hom}(\mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2], \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1]) & & \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[1] \otimes \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1], \arrow[ll, "\sim"']
\end{tikzcd} \tag{4.2}$$

where the map below arises from Poincaré duality along with the canonical isomorphism $A^\vee \otimes B \cong \mathrm{Hom}(A, B)$.

Remark 4.9. Note that we changed the left vertical map of the diagram by a twist of two. This only effects considerations on the level of filtrations.

As we have already explained in Lemma 4.2 for the étale case, we can also use the above for viewing any map $\tau\colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \to \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1]$ as an element

$$\begin{aligned}
Z_\tau &\in \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2]^\vee \otimes \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1] \\
&= \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[1] \otimes \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1] \\
&\subseteq \mathrm{H}^4_{\mathrm{dR}}(A^4/\mathbb{Q})[2].
\end{aligned}$$

The commutative diagram (4.2) implies that in the above notation,

$$\mathrm{cl}_{\mathrm{dR}}(\Pi) = Z_{\Pi_{\mathrm{dR},*}} \quad \text{for any } \Pi \in \mathrm{CH}^2(A^4)(\mathbb{Q}).$$

Therefore, if given a map $\tau\colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \to \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1]$, the corresponding element Z_τ is of the form $Z_\tau = \mathrm{cl}_{\mathrm{dR}}(\Pi_\tau)$ for some $\Pi_\tau \in \mathrm{CH}^2(A^4)(\mathbb{Q})$, then τ actually *comes from* a correspondence in the sense that

$$\tau = \Pi_{\tau,\mathrm{dR},*}.$$

This will be interesting in particular for the rational version of our induced map

$$\pi_{\mathrm{dR},i}\colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \twoheadrightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1].$$

The above discussion yields:

Proposition 4.10. *If there exists a cycle* $\Pi_i^? \in \mathrm{CH}^2(A^4)(\mathbb{Q})$ *such that* $\mathrm{cl}_{\mathrm{dR}}(\Pi_i^?) = Z_{\pi_{\mathrm{dR},i}}$*, then*

$$\Pi^?_{i,\mathrm{dR},*} = \pi_{\mathrm{dR},i}.$$

In light of the Tate Conjecture, such algebraic cycles conjecturally exist at the cost of extending coefficients to $\mathbb{Q}_p$. In the following, we therefore want to describe an *approximation process* that can be applied after using the Tate Conjecture, in order to conjecturally obtain algebraic cycles with rational coefficients that are of the above form.

Proposition 4.11. *If the Tate Conjecture is true for* A^4 *in codimension two, then there exist cycles* $\tilde{\Pi}_i^? \in \mathrm{CH}^2(A^4)(\mathbb{Q})$ *such that*

1. $\tilde{\Pi}^{?,p}_{i,\text{ét},*}\colon \mathrm{H}^3_{\text{ét}}(\bar{A}^3,\mathbb{Q}_p)(2) \twoheadrightarrow V_p(A)$ *is a surjection of* $\mathbb{Q}_p[G_{\mathbb{Q}}]$*-modules.*

2. $\tilde{\Pi}^{?}_{i,\mathrm{dR},*}\colon \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \twoheadrightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1]$ *is a surjection of* $\mathbb{Q}$*-vector spaces.*

Proof. Our approach follows an argument which is also used in [BDP14, Lemma 2.9]. If the Tate conjecture is true for A^4 in codimension two, then there exist elements $\alpha_{i,j} \in \mathbb{Q}_p$ and $\Pi^?_{i,j} \in \mathrm{CH}^2(A^4)(\mathbb{Q})$, such that

$$Z_{\pi_{\text{ét},i}} = \sum_{j=1}^{t_i} \alpha_{i,j}\mathrm{cl}^p_{\text{ét}}(\Pi^?_{i,j}).$$

By multiplying $Z_{\pi_{\text{ét}}}$ with a suitable power of p we may assume that $\alpha_{i,j} \in \mathbb{Z}_p$. We impose that $\pi_{\text{ét}}$ has been modified beforehand in such a way. We now choose a vector $(\beta_{i,1},\dots,\beta_{i,t_i}) \in \mathbb{Z}^{t_i}$ that is *sufficiently close* to $(\alpha_{i,1},\dots,\alpha_{i,t_i}) \in \mathbb{Z}_p^{t_i}$ and obtain a cycle

$$\tilde{\Pi}^?_i = \sum_{j=1}^{t_i} \beta_{i,j}\Pi^?_{i,j} \in \mathrm{CH}^2(A^4)(\mathbb{Q}),$$

which induces surjections

$$\tilde{\Pi}^{?,p}_{i,\text{ét},*}\colon\ \mathrm{H}^3_{\text{ét}}(\bar{A}^3,\mathbb{Q}_p)(2) \longrightarrow\!\!\!\!\!\rightarrow V_p(A), \qquad \tilde{\Pi}^{?}_{i,\mathrm{dR},*}\colon\ \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q})[2] \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q})[1],$$

of $\mathbb{Q}_p[G_{\mathbb{Q}}]$-modules, respectively $\mathbb{Q}$-vector spaces. □

Remark 4.12. By viewing $\tilde{\Pi}^?_i$ as being defined over $\mathbb{Q}_p$ via an embedding of its coefficients, we also obtain

$$\tilde{\Pi}^{?}_{i,\mathrm{dR},*}\colon\ \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2] \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]$$

as an induced map. This is nothing but $\tilde{\Pi}^{?,p}_{i,\mathrm{dR},*}$ as defined via $\tilde{\Pi}^{?,p}_{i,\text{ét},*}$ through the commutative diagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3,\mathbb{Q}_p)(2)) \arrow[r, "\mathbf{D}_{\mathrm{dR}}(\tilde{\Pi}^{?,p}_{i,\text{ét},*})"] \arrow[d, "\mathrm{comp}_{\mathrm{dR},A^3}"', "\wr"] & \mathbf{D}_{\mathrm{dR}}(V_p(A)) \arrow[d, "\mathrm{comp}_{\mathrm{dR},A}"', "\wr"] \\
\mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2] \arrow[r, "\tilde{\Pi}^{?,p}_{i,\mathrm{dR},*}"] & \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1].
\end{tikzcd}$$

By mimicking the approximation process on the level of the non-conjectural projections we therefore obtain projections which are p-adically close to the original ones and conjecturally induced by rational algebraic cycles.

Remark 4.13. This can be thought of as a linear algebra principle for $\mathbb{Q}_p$-vector spaces which admit a $\mathbb{Q}$-structure, the idea being that for a surjective map on $\mathbb{Q}_p$-vector spaces there is a rational approximation which is still surjective.

We replace $\pi_{\text{ét}}$ and π_{dR} by these new maps that are conjecturally induced by rational algebraic cycles and keep denoting them by the same symbols. With the replacements being made, Proposition 4.10 can be reformulated to the following.

Proposition 4.14. *If the Tate Conjecture is true for A^4 in codimension two, then there exist algebraic cycles $\Pi_i^? \in \mathrm{CH}^2(A^4)(\mathbb{Q})$ such that*

$$\Pi^?_{i,\mathrm{dR},*} = \pi_{\mathrm{dR},i}.$$

We therefore obtain conjectural parametrizations defined over the rationals

$$\Pi^?_{i,*}\colon\ \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow A(\mathbb{Q})$$

which are related to our non-conjectural maps by the following observation.

Lemma 4.15. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{CH}^2(A^3)_0(\mathbb{Q}) \arrow[r, "\mathrm{AJ}^p_{\text{ét},A^3}"] \arrow[d, hook] \arrow[rrr, bend left, "\Pi^?_{i,*}"] & \mathrm{H}^1_f(\mathbb{Q}, \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) \arrow[r, "\mathrm{H}^1_f(\mathbb{Q},\pi_{\text{ét},i})"] \arrow[d, "\mathrm{res}_p"'] & \mathrm{H}^1_f(\mathbb{Q}, V_p(A)) \arrow[d, "\mathrm{res}_p"'] & A(\mathbb{Q}) \arrow[l, "\mathrm{AJ}^p_{\text{ét},A}"'] \arrow[d, hook] \\
\mathrm{CH}^2(A^3)_0(\mathbb{Q}_p) \arrow[r, "\mathrm{AJ}^p_{\text{ét},A^3}"] \arrow[ddddr, bend right, "\mathrm{AJ}^p_{\mathrm{dR},A^3}"'] & \mathrm{H}^1_f(\mathbb{Q}_p, \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2)) \arrow[r, "\mathrm{H}^1_f(\mathbb{Q}_p,\pi_{\text{ét},i})"] \arrow[d, "\mathrm{log}_{\mathrm{BK},A^3}"', "\wr"] & \mathrm{H}^1_f(\mathbb{Q}_p, V_p(A)) \arrow[d, "\mathrm{log}_{\mathrm{BK},A}"', "\wr"] & A(\mathbb{Q}_p) \arrow[l, "\mathrm{AJ}^p_{\text{ét},A}"'] \arrow[ddddl, bend left, "\mathrm{AJ}^p_{\mathrm{dR},A}"] \\
& \mathbf{D}_{\mathrm{dR}}(\mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2))/\operatorname{Fil}^0 \arrow[r, two heads, "\overline{\mathbf{D}}_{\mathrm{dR}}(\pi_{\text{ét},i})"] \arrow[d, "\overline{\mathrm{comp}}_{\mathrm{dR},A^3}"', "\wr"] & \mathbf{D}_{\mathrm{dR}}(V_p(A))/\operatorname{Fil}^0 \arrow[d, "\overline{\mathrm{comp}}_{\mathrm{dR},A}"', "\wr"] & \\
& \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p)[2]/\operatorname{Fil}^0 \arrow[r, two heads, "\overline{\pi_{\mathrm{dR},i}}"] \arrow[d, "\mathrm{PD}_{A^3}"', "\wr"] & \mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)[1]/\operatorname{Fil}^0 \arrow[d, "\mathrm{PD}_{A}"', "\wr"] & \\
& (\operatorname{Fil}^2 \mathrm{H}^3_{\mathrm{dR}}(A^3/\mathbb{Q}_p))^\vee \arrow[r, "\Pi^{?,*,p,\vee}_{i,\mathrm{dR}}"] & \Omega^1(A/\mathbb{Q}_p)^\vee. &
\end{tikzcd} \tag{4.3}$$

Proof. The middle squares are commutative by Proposition 4.8 and the compatibility with the Abel–Jacobi maps is given by definition. The conjectural parametrization $\Pi^?_{i,*}$ fits on top of the diagram since the outer arrows form a commutative diagram by (2.15). □

Remark 4.16. The compatibility that this commutative diagram provides explains the choice of our normalization on the non-conjectural map. It contains the relation between the conjectural maps

$$\Pi^?_{i,*}\colon\ \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow A(\mathbb{Q})$$

and the non-conjectural map

$$\mathrm{H}^1_f(\mathbb{Q}, \pi_{\text{ét},i}) \circ \mathrm{AJ}^p_{\text{ét},A^3}\colon\ \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}, V_p(A))$$

that we want to consider in the upcoming subsection.

We can use the conjectural cycles to define conjectural rational points on A:

$$P_i^?(\Gamma) = \Pi_{i,*}^?(\Gamma) \in A(\mathbb{Q}),$$

related to non-conjectural objects arising from our projection on the level of Galois representations. Note that these conjectural rational points on A come in pairs, as $\pi_{\text{ét}}$ consists of two components, giving rise to a pair of cycles $(\Pi_1^?, \Pi_2^?) \in \mathrm{CH}^2(A^4)(\mathbb{Q})^{\oplus 2}$ inducing

$$\mathbf{\Pi}_*^? = (\Pi_{1,*}^?, \Pi_{2,*}^?)\colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow A(\mathbb{Q})^{\oplus 2}. \tag{4.4}$$

Although we are not going to dive into this topic in this thesis, this can become interesting once special collections of algebraic cycles $\Gamma \in \mathrm{CH}^2(A^3)_0(\mathbb{Q})$ are considered.

Remark 4.17. We would like to point out that a certain seemingly natural choice of a cycle on the three-fold product will not be of interest for these kinds of considerations. As is presented in [GS95], one can modify the diagonal cycle Δ on A^3 to obtain a homologically trivial cycle in the following way. Let e be a rational point of A. Then, besides Δ, there are the following obvious subvarieties of codimension two of A^3 defined over $\mathbb{Q}$:

$$\begin{aligned} \Delta_1 &= \{(x,e,e) : x \in A(\mathbb{Q})\}, \\ \Delta_2 &= \{(e,x,e) : x \in A(\mathbb{Q})\}, \\ \Delta_3 &= \{(e,e,x) : x \in A(\mathbb{Q})\}, \\ \Delta_{12} &= \{(x,x,e) : x \in A(\mathbb{Q})\}, \\ \Delta_{23} &= \{(e,x,x) : x \in A(\mathbb{Q})\}, \\ \Delta_{13} &= \{(x,e,x) : x \in A(\mathbb{Q})\}. \end{aligned}$$

It is shown in [GS95, Proposition 3.1] that the cycle

$$\Delta - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3 \in \mathrm{CH}^2(A^3)(\mathbb{Q})$$

is homologous to zero and therefore defines an element denoted $\Delta_e \in \mathrm{CH}^2(A^3)_0(\mathbb{Q})$. However, in our case of elliptic curves, [GS95, Proposition 4.1] shows that Δ_e is even rationally equivalent to zero, and so images under our parametrizations are trivial.

Shifting the point of view to studying the Chow group of the three-fold product *via* rational points on the elliptic curve, we propose the following conjecture.

Conjecture 4.18. *If the rank of $A(\mathbb{Q})$ is at least one, then the rank of $\mathrm{CH}^2(A^3)_0(\mathbb{Q})$ is at least two.*

This conjecture fits into the overall picture of our previous discussion by the fact that our conjectural parametrization (4.4) arises from two independent projections on the level of Galois representations as remarked before. Indeed, the origin being the projection

$$\pi_{\text{ét}}\colon \mathrm{H}^3_{\text{ét}}(\bar{A}^3, \mathbb{Q}_p)(2) \twoheadrightarrow V_p(A)^{\oplus 2},$$

its two components $\pi_{\text{ét},i}$ should produce two independent cycles $\Pi_i^?$ in the Chow group $\mathrm{CH}^2(A^4)(\mathbb{Q})$. The resulting components $\Pi_{i,*}^?$ of the conjectural parametrization (4.4) are further expected to provide independent rational points on the elliptic curve.

4.5 **A Note on Global Selmer Classes.** Let us take a final look at some of the non-conjectural parts of the above diagram, which involves global cohomology classes. More precisely, consider the map

$$\Phi_{\pi_{\text{ét},i}} = \mathrm{H}^1_f(\mathbb{Q}, \pi_{\text{ét},i}) \circ \mathrm{AJ}^p_{\text{ét},A^3} \colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}, V_p(A)).$$

As the right-hand side equals the Selmer group $\mathrm{Sel}_p(A/\mathbb{Q})$ (cf. [Bel09, Proposition 2.13]), we are actually considering classical Selmer classes

$$\Phi_{\pi_{\text{ét},i}}(\Gamma) \in \mathrm{Sel}_p(A/\mathbb{Q}) \quad \text{for } \Gamma \in \mathrm{CH}^2(A^3)(\mathbb{Q}).$$

Recall that the Selmer group consists of elements in the Galois cohomology of $V_p(A)$, whose restrictions at p are contained in the image of the local Kummer map

$$\kappa_A^{(p)} \colon A(\mathbb{Q}_p) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, V_p(A)).$$

Moreover, we remind (cf. [Bel09], [Sil09]) that we have an induced injection

$$\kappa_A \colon A(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Sel}_p(A/\mathbb{Q}),$$

which is an isomorphism if and only if the p-primary component $Ш(A/\mathbb{Q})[p^\infty]$ of the Tate–Shafarevich group is finite.

We are led to pose the following conjecture on the images of cycles under the non-conjectural map $\Phi_{\pi_{\text{ét},i}}$ of diagram (4.3).

Conjecture 4.19. *For any $\Gamma \in \mathrm{CH}^2(A^3)_0(\mathbb{Q})$, there exists a point $\tilde{P}_i^?(\Gamma) \in A(\mathbb{Q})$ such that $\Phi_{\pi_{\text{ét},i}}(\Gamma) = \kappa_A(\tilde{P}_i^?(\Gamma))$.*

Let us have a look at possible local analogues. By definition, considering restrictions of those classes, there are local points $P_i^{(p)}(\Gamma) \in A(\mathbb{Q}_p)$ such that

$$\kappa_A^{(p)}(P_i^{(p)}(\Gamma)) = \Phi_{\pi_{\text{ét},i}}(\Gamma)_{|G_{\mathbb{Q}_p}}.$$

We may thus define

$$\Phi^{(p)}_{\pi_{\text{ét},i}} = \mathrm{H}^1_f(\mathbb{Q}_p, \pi_{\text{ét},i}) \circ \mathrm{AJ}^p_{\text{ét},A^3} \colon \mathrm{CH}^2(A^3)_0(\mathbb{Q}_p) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, V_p(A))$$

to be the p-adic analogue of $\Phi_{\pi_{\text{ét},i}}$, so that $\Phi^{(p)}_{\pi_{\text{ét},i}}(\Gamma) = \kappa_A^{(p)}(P_i^{(p)}(\Gamma))$. This is precisely what the conjecture aims for, but over $\mathbb{Q}_p$ instead of $\mathbb{Q}$. The big commutative diagram (4.3) further gives that

$$\begin{aligned}
\kappa_A^{(p)}(P_i^{(p)}(\Gamma)) &= \Phi^{(p)}_{\pi_{\text{ét},i}}(\Gamma) \\
&= \exp^p_{A^3}(\Pi^{?,p,*,\vee}_{i,\mathrm{dR}}(\mathrm{AJ}^p_{\mathrm{dR},A^3}(\Gamma))) \\
&= \exp^p_{A^3}(\mathrm{AJ}^p_{\mathrm{dR},A}(\Pi^?_{i,*}(\Gamma))) \\
&= \mathrm{AJ}^p_{\text{ét},A}(P_i^?(\Gamma)).
\end{aligned}$$

This means that the local Kummer images of the local analogues coincide with the local Abel–Jacobi images of the conjectural global points when viewed inside $A(\mathbb{Q}_p)$.

5 A p-adic L-Function for the Symmetric Cube

In this section, we construct a p-adic L-function for the symmetric cube of a Hida family, guided by the factorization of complex L-functions

$$L(\theta^{\otimes 3}, s+k-1) = L(\operatorname{Sym}^3 \theta, s+k-1) \cdot L(\theta, s)^2 \tag{5.1}$$

for eigenforms $\theta \in \mathrm{S}_k(\Gamma_0(N))$ of even weight $k \geq 2$. This factorization is provided by our decomposition (cf. Proposition 3.8) of the Galois representation $\rho_\theta^{\otimes 3}(1-k)$. When θ varies in a p-adic Hida family $\boldsymbol{\theta}$, the central critical values $L(\theta^{\otimes 3}, \frac{3k-2}{2})$ and $L(\theta, \frac{k}{2})$ within the above factorization can be p-adically interpolated by the following known p-adic L-functions:

- The restriction of the Mazur–Kitagawa p-adic L-function (cf. [MSD74], [MTT86] [Kit91], [GS93]) to the central critical line:

$$L_p^{\mathrm{cc}}(\boldsymbol{\theta})(k) = L_p^{\mathrm{MK}}(\boldsymbol{\theta})(k, \tfrac{k}{2}),$$

- The restriction of the balanced triple product p-adic L-function (cf. [Hsi21]) to the diagonal:

$$L_p^{\Delta}(\boldsymbol{\theta})(k) = \mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{\theta}, \boldsymbol{\theta}, \boldsymbol{\theta})(k,k,k)^2.$$

Our goal is to obtain p-adic interpolation of the central critical values $L(\operatorname{Sym}^3 \theta, \frac{3k-2}{2})$ for the symmetric cube complex L-function in such a way that the above factorization (5.1) of complex L-functions is mirrored by a factorization of p-adic L-functions.

After agreeing on some standard notation concerning Hida theory, we recall the relevant information on the two p-adic L-functions that we want to use and formulate the general conjecture on the existence of the sought for p-adic L-function for the symmetric cube. A special emphasis will lie on a particular case of elliptic curves, in which we prove the conjecture under certain additional assumptions.

5.1 **Hida Families.** Let $f \in \mathrm{S}_2(\Gamma_0(N))$ be a p-ordinary newform of level $N = N'p^r$ with $r \in \{0,1\}$ and $p \geq 5$ not dividing N'. Write

$$X^2 - a_p(f)X + p = (X - \alpha_p) \cdot (X - \beta_p),$$

with $\alpha_p, \beta_p \in \bar{\mathbb{Q}}$. Under our fixed embedding, we assume $\alpha_p \in \mathbb{Z}_p^\times$ and $\beta_p \in p\,\mathbb{Z}_p$. If $r = 0$, the *p-stabilization* of f is the cusp form $f^o \in \mathrm{S}_2(\Gamma_0(N'p))$, defined by

$$f^o(z) = f(z) + \beta_p \cdot f(pz). \tag{5.2}$$

If $r = 1$, we put $f^o = f$. Hida's theory (cf. [Hid86b], [Hid86a]) associates to f a neighborhood $U_f \subseteq \mathscr{X} = \operatorname{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ of the weight $k = 2$ and a formal q-expansion

$$\boldsymbol{f} = \sum_{n \geq 1} \boldsymbol{a}_n\, q^n, \quad \boldsymbol{a}_n \in \mathscr{A}(U_f),$$

called *(p-adic) Hida family*, with $\mathscr{A}(U_f)$ denoting the ring of $\mathbb{C}_p$-valued p-adic analytic functions on U_f. The neighborhood U_f is understood by means of the embedding $\mathbb{Z} \hookrightarrow \mathscr{X}$ defined by $k \mapsto (\,\cdot\,)^{k-2}$, realizing $\mathbb{Z}$ as a dense subset, and is contained in the residue class of 2 modulo $p-1$. We fix the following notation for the *specializations* of $\boldsymbol{f}$ at *classical* weights $k \in U_f^{\mathrm{cl}} = U_f \cap \mathbb{Z}_{\geq 2}$:

(1) For any $k \in U_f^{\mathrm{cl}}$, the weight-k specialization $f_k = \sum_{n\geq 1} \boldsymbol{a}_n(k)q^n$ is the q-expansion of a normalized p-ordinary eigenform of weight k on $\Gamma_0(N'p)$ and $f_2 = f^o$. We put $a_n(f_k) = \boldsymbol{a}_n(k)$.

(2) These eigenforms are new at the primes dividing the *tame level* N', but not new at p for $k > 2$. In this case, f_k arises as the p-stabilization of a newform $f_k^\sharp$ of weight k on $\Gamma_0(N')$. In the terminology introduced in (5.2), one has $f_k = f_k^{\sharp,o}$, satisfying

$$f_k(z) = f_k^\sharp(z) + \beta_p(k) \cdot f_k^\sharp(pz).$$

For notational consistency, we put $f_2^\sharp = f_2$.

(3) We define punctured sets $U_f^\circ = U_f \smallsetminus \{2\}$ and $U_f^{\mathrm{cl},\circ} = U_f^{\mathrm{cl}} \smallsetminus \{2\}$.

Remark 5.1. Note that the classical weights are necessarily even as p is assumed to be odd and elements in the neighborhood U_f are congruent to 2 modulo $p-1$.

In the following two subsections, we will recall the relevant material on the two known p-adic L-functions that we want to consider. When it comes to discussing the respective assumptions for those functions, we will focus on those that are of particular impact on the elliptic curve scenario that we ultimately want to study. For a complete and general discussion of all the assumptions and their background, we refer to the works already mentioned in the introduction of this section.

5.2 **The Restricted Mazur–Kitagawa p-adic L-Function.** We consider a Hida family $\boldsymbol{f}$ as in the previous section. Its classical specializations f_k are ordinary eigenforms of even weight, so that each comes equipped with a p-adic L-function $L_p(f_k, s)$ once a complex period Ω_{f_k} is chosen (cf. [MSD74], [MTT86]). As discussed in [GS93], the so-called *Mazur–Kitagawa p-adic L-function* $L_p^{\mathrm{MK}}(\boldsymbol{f})(k,s)$ combines those p-adic L-functions of the weight-k specializations as a single two-variable p-adic L-function, which is analytic in a neighborhood of $(k,s) = (2,1)$. It interpolates the complex special values $L(f_k^\sharp, n)$ with $k \in U_f^{\mathrm{cl}}$ and $1 \leq n \leq k-1$ as we will now recall.

We pin down the algebraic parts of the L-functions for $f_k^\sharp$ for the critical values as follows, using the canonical periods $\Omega_k^\pm \in \mathbb{C}^\times$, chosen to satisfy

$$\Omega_k^\pm \Omega_k^\mp = \langle f_k^\sharp, f_k^\sharp \rangle$$

as in [BD07] (see also [Shi77]). Here, the Petersson scalar product is normalized so that

$$\langle g, h \rangle = 4\pi^2 \iint_{\Gamma_0(N)\backslash\,\mathfrak{h}} g(z)\bar{h}(z)y^{k-2}\mathrm{d}x\mathrm{d}y,$$

the integral taken over any fundamental region for the action of $\Gamma_0(N)$ on the upper half-plane $\mathfrak{h}$. With these periods, for any integer $1 \leq n \leq k-1$ determining the sign $\epsilon_{2n} = (-1)^{n-1}$, we fix the algebraic part as in [BD07]:

$$L^{\mathrm{alg}}(f_k^\sharp, n) = \frac{L(f_k^\sharp, n)}{(\pi\sqrt{-1})^{n-1} \cdot \Omega_k^{\epsilon_{2n}}}.$$

We are interested in the restriction of the Mazur–Kitagawa p-adic L-function to the central critical line $(k, \frac{k}{2})$ and denote the resulting p-adic L-function by

$$L_p^{\mathrm{cc}}(\boldsymbol{f})(k) = L_p^{\mathrm{MK}}(\boldsymbol{f})(k, \tfrac{k}{2}).$$

This *central critical* p-adic L-function satisfies the following interpolation property (cf. [BSV22a]):

$$L_p^{\mathrm{cc}}(\boldsymbol{f})(k) = \lambda_k^{\epsilon_k} \cdot \frac{\Gamma(\frac{k}{2})}{(-1)^{\frac{k}{2}-1} \cdot 2^{\frac{k}{2}-1}} \cdot \mathscr{E}_p^{\mathrm{cc}}(\boldsymbol{f}, k) \cdot \mathscr{E}_{p,\mathrm{old}}^{\mathrm{cc}}(\boldsymbol{f}, k) \cdot L^{\mathrm{alg}}(f_k^\sharp, \tfrac{k}{2}). \tag{5.3}$$

In this formula, $\lambda_k^\pm \in \mathbb{C}_p$ are p-adic numbers with $\lambda_2^\pm = 1$ as in [BD07, Theorem 1.5], which are non-vanishing after possibly shrinking U_f (cf. [BD07, Proposition 1.7]). The factors

$$\mathscr{E}_p^{\mathrm{cc}}(\boldsymbol{f}, k) = 1 - a_p(f_k)^{-1} p^{\frac{k}{2}-1} \quad \text{and} \quad \mathscr{E}_{p,\mathrm{old}}^{\mathrm{cc}}(\boldsymbol{f}, k) = 1 - \varepsilon_p(k) a_p(f_k)^{-1} p^{\frac{k}{2}-1}$$

are modified Euler factors at p, for which we put $\varepsilon_p(k) = 0$ if $f_k = f_k^\sharp$ and $\varepsilon_p(k) = 1$ otherwise.

Remark 5.2. Our labeling of the Euler factors stems from their appearance being dependent on whether the considered form is new or old at p. These Euler factors in fact only appear in the latter case.

5.3 **The Restricted Balanced Triple Product p-adic L-Function.** Let us now briefly discuss the balanced triple product p-adic L-function as constructed by Hsieh in [Hsi21] and recall the relevant interpolation formula that is of interest for us. We will address some of the imposed assumptions that are of particular impact for our considerations, but will not go into full detail with most of the technical assumptions and rather refer to loc. cit. for a complete discussion of the techniques involved.

Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triplet of Hida families as in the previous section, such that the least common multiple of their tame levels is square-free. Under further technical assumptions, the above mentioned article constructs a p-adic L-function $\mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(k, l, m)$, called the *balanced square-root triple product p-adic L-function.* Its square interpolates the complex values $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, \frac{k+l+m-2}{2})$ for weight triples (k, l, m) that lie in the *balanced* region, i.e. those such that $k+l+m$ is greater than $2k$, $2l$ and $2m$. We are interested in the restriction of this square-root p-adic L-function to the diagonal. More precisely, we consider the Hida family $\boldsymbol{f}$ of square-free tame level N' and put

$$L_p^{\Delta}(\boldsymbol{f})(k) = \mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{f}, \boldsymbol{f}, \boldsymbol{f})(k, k, k)^2.$$

For any $k \in U_f^{\mathrm{cl}}$, let us look at the complex triple product L-function $L(f_k^{\sharp,\otimes 3}, s)$, whose special values this p-adic L-function interpolates. The functional equation relating its values at s and $3k-2-s$ has a sign that can be written as a product of local root numbers

$$\epsilon(f_k^{\sharp,\otimes 3}) = -\prod_{q|N} \epsilon_q(f_k^{\sharp,\otimes 3}), \qquad \epsilon_q(f_k^{\sharp,\otimes 3}) \in \{\pm 1\}.$$

Observe the contribution at ∞ as the triple (k,k,k) is balanced, so that $\epsilon_\infty(f_k^{\sharp,\otimes 3}) = -1$. We define a subset of the primes dividing the tame level by

$$\Sigma^- = \{q \mid N' : \epsilon_q(f_k^{\sharp,\otimes 3}) = -1\}$$

and remark that this does not depend on the weight k that is chosen, as being pointed out in the introduction of [Hsi21]. We assume from now on that Σ^- has odd cardinality.

The algebraic parts that are going to be interpolated by the restricted balanced triple product p-adic L-function are as follows (cf. [GH93], [HK91]). For any integer $k \leq m \leq 2k-2$, we define

$$L^{\mathrm{alg}}(f_k^{\sharp,\otimes 3}, m) = \frac{L(f_k^{\sharp,\otimes 3}, m)}{(\pi\sqrt{-1})^{4m-3k+3} \cdot \langle f_k^\sharp, f_k^\sharp \rangle^3}.$$

The following interpolation property for the restricted balanced triple product p-adic L-function is proven in [Hsi21, Theorem B] (see also [BCS23] for an explicit description of the various ingredients):

$$L_p^\Delta(\boldsymbol{f})(k) = \mathscr{A}_{\boldsymbol{f}}^\Delta \cdot \frac{\Gamma(\frac{3k-2}{2}) \cdot \Gamma(\frac{k}{2})^3 \cdot c_{f_k}^3}{(-1)^{3k+3} \ \cdot 2^{3k-5}} \cdot (\sqrt{-1})^{1-3k} \cdot \frac{\mathscr{E}_p^\Delta(\boldsymbol{f},k)^2}{\mathscr{E}_{p,\mathrm{old}}^\Delta(\boldsymbol{f},k)^3} \cdot L^{\mathrm{alg}}(f_k^{\sharp,\otimes 3}, \tfrac{3k-2}{2}). \tag{5.4}$$

In this formula, $\mathscr{A}_{\boldsymbol{f}}^\Delta = \prod_{q|N'} \mathrm{Loc}_q \in \mathbb{Q}^\times$ is an explicit non-zero rational number, c_{f_k} is the *congruence number* of f_k as defined by Hida in [Hid81, (0.3)] and the modified Euler factors are defined as

$$\mathscr{E}_p^\Delta(\boldsymbol{f},k) = (1 - a_p(f_k)^{-1} p^{\frac{k}{2}-1})^3 \cdot (1 - a_p(f_k)^{-3} p^{\frac{3k-2}{2}-1}),$$
$$\mathscr{E}_{p,\mathrm{old}}^\Delta(\boldsymbol{f},k) = (1 - \varepsilon_p(k) a_p(f_k)^{-2} p^{k-1}) \cdot (1 - \varepsilon_p(k) a_p(f_k)^{-2} p^{k-2}).$$

We point out that c_{f_k} is a positive integer in our case by Theorem A (0.4_b) of loc. cit., noting that our level is divisible by the odd prime p.

5.4 **Construction of the Symmetric Cube p-adic L-Function.** Let again $\boldsymbol{f}$ be a Hida family of square-free tame level N' and consider the newforms $f_k^\sharp \in \mathrm{S}_k(\Gamma_0(N'))$ giving rise to the weight-k specializations $f_k = f_k^{\sharp,o}$ for $k \in U_f^{\mathrm{cl}}$. We start by pinning down the algebraic parts of the critical L-values for the symmetric cube, using the algebraic parts already mentioned in the previous two subsections:

$$L^{\mathrm{alg}}(f_k^\sharp, n) = \frac{L(f_k^\sharp, n)}{(\pi\sqrt{-1})^{n-1} \cdot \Omega_k^{\epsilon_{2n}}}, \qquad L^{\mathrm{alg}}(f_k^{\sharp,\otimes 3}, m) = \frac{L(f_k^{\sharp,\otimes 3}, m)}{(\pi\sqrt{-1})^{4m-3k+3} \cdot \langle f_k^\sharp, f_k^\sharp \rangle^3},$$

for any integer $1 \leq n \leq k-1$, determining the sign $\epsilon_{2n} = (-1)^{n-1}$, and any integer $k \leq m \leq 2k-2$. Recalling the factorization

$$L(f_k^{\sharp,\otimes 3}, m) = L(\mathrm{Sym}^3 f_k^\sharp, m) \cdot L(f_k^\sharp, m+1-k)^2, \tag{5.5}$$

this immediately gives the following result.

Proposition 5.3. *For any integer $k \leq m \leq 2k-2$ such that $L(f_k^\sharp, m+1-k) \neq 0$, choosing the sign $\epsilon_{2m-2k+2} = \epsilon_{2m+2} = (-1)^m$, one has*

$$L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, m) = \frac{L(\mathrm{Sym}^3 f_k^\sharp, m)}{(\pi\sqrt{-1})^{2m-k+3} \cdot \frac{\langle f_k^\sharp, f_k^\sharp\rangle^3}{\Omega_k^{\epsilon_{2m+2},2}}} \in K_{f_k^\sharp}.$$

Proof. If the symmetric cube L-function vanishes at m, the above definition of its algebraic part gives $L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, m) = 0$ and the statement trivially holds. In the non-vanishing situation we can compute that

$$\frac{L^{\mathrm{alg}}(f_k^{\sharp,\otimes 3}, m)}{L^{\mathrm{alg}}(f_k^\sharp, m+1-k)^2} = L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, m),$$

concluding the claim. $\square$

Remark 5.4. We give a few remarks on the statement of Proposition 5.3.

(i) The reader might want to compare the formula to the one in [KS00], addressing the conjectures of [Del79] and [Zag77]. Note that in our normalization and notation, the period for the symmetric cube is

$$\frac{\langle f_k^\sharp, f_k^\sharp\rangle^3}{\Omega_k^{\pm,2}} = \Omega_k^\pm \cdot \Omega_k^{\mp,3},$$

which appears to be of the same form as the period introduced right before [KS00, Proposition 4.1] with an opposite choice of sign. Moreover, note that the *extra* powers of $\pi\sqrt{-1}$ are a little different, so is their way of fixing the algebraic part of $L(f_k^\sharp, n)$ for critical n, possibly due to different normalization choices.

(ii) In addition to the formula for the algebraic part we remark that [KS00, Proposition 4.1] establishes $L(\mathrm{Sym}^3 f_k^\sharp, m)$ being holomorphic for any critical integer $k \leq m \leq 2k-2$, in particular at the central critical value $m = \frac{3k-2}{2}$. We refer the reader to [KS99] concerning holomorphicity of symmetric cube L-functions.

As mentioned before, we use the p-adic L-functions $L_p^{\mathrm{cc}}(\boldsymbol{f})(k) = L_p^{\mathrm{MK}}(\boldsymbol{f})(k, \frac{k}{2})$, as well as $L_p^{\Delta}(\boldsymbol{f})(k) = \mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{f}, \boldsymbol{f}, \boldsymbol{f})(k,k,k)^2$. Both of these p-adic L-functions are viewed as $\mathbb{C}_p$-valued p-adic analytic functions on U_f and we assume that $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ does not vanish identically.

We fix the following observation on the vanishing behavior:

Lemma 5.5. *If $L_p^{\Delta}(\boldsymbol{f})(k)$ does not vanish at $k \in U_f^{\mathrm{cl}}$, then $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ does not either.*

Proof. Assuming non-vanishing of the triple product p-adic L-function, its interpolation formula (5.4) provides non-vanishing of the Euler factors, which are given by

$$1 - a_p(f_k)^{-1} p^{\frac{k}{2}-1} \qquad \text{and} \qquad 1 - a_p(f_k)^{-3} p^{\frac{3k-2}{2}-1}.$$

The first factor is precisely the Euler factor appearing in the interpolation formula (5.3) for the Mazur–Kitagawa p-adic L-function, so we are not in the situation of an extra zero coming from the interpolation. Moreover, since by assumption the complex special value $L(f_k^{\sharp,\otimes 3}, \frac{3k-2}{2})$ is non-zero, so is $L(f_k^{\sharp}, \frac{k}{2})$ by the factorization (5.5) for $m = \frac{3k-2}{2}$ and the finiteness of the complex values for the symmetric cube. The interpolation formula (5.3) concludes the claim. $\square$

Definition 5.6. We define the *symmetric cube p-adic L-function* as the meromorphic function on U_f given by

$$L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = L_p^{\Delta}(\boldsymbol{f})(k) \cdot L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^{-2}.$$

The definition of this ratio formally provides the desired factorization on the level of p-adic L-functions, given that we are considering finite values of this a priori meromorphic function.

Definition 5.7. For any classical weight $k \in U_f^{\mathrm{cl}}$, we define the following terms:

1. $\mathscr{A}_{\boldsymbol{f}} = -\mathscr{A}_{\boldsymbol{f}}^{\Delta} \in \mathbb{Q}^{\times}$,
2. $\mathscr{B}_k = \Gamma(\frac{3k-2}{2}) \cdot \Gamma(\frac{k}{2}) \cdot 2^{3-2k} \in \mathbb{Q}^{\times}$,
3. $\mathscr{C}_{\boldsymbol{f},k} = c_{f_k}^3 \in \mathbb{Z}_{\geq 1}$,
4. $\mathscr{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f}, k) = \mathscr{E}_p^{\Delta}(\boldsymbol{f}, k)^2 \cdot \mathscr{E}_p^{\mathrm{cc}}(\boldsymbol{f}, k)^{-2}$,
5. $\mathscr{E}_{p,\mathrm{old}}^{\mathrm{Sym}^3}(\boldsymbol{f}, k) = \mathscr{E}_{p,\mathrm{old}}^{\Delta}(\boldsymbol{f}, k)^3 \cdot \mathscr{E}_{p,\mathrm{old}}^{\mathrm{cc}}(\boldsymbol{f}, k)^2$,
6. $\mathfrak{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f}, k) = \mathscr{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f}, k)^2 \cdot \mathscr{E}_{p,\mathrm{old}}^{\mathrm{Sym}^3}(\boldsymbol{f}, k)^{-1}$.

Remark 5.8. We would like to mention a couple of remarks on the modified Euler factors introduced in the above definition.

(i) Recalling the definition of the modified Euler factors for the Mazur–Kitagawa and balanced triple product p-adic L-function, we point out that $\mathscr{E}_{p,\mathrm{old}}^{\mathrm{Sym}^3}(\boldsymbol{f}, k)$ equals

$$(1 - \varepsilon_k(p) a_p(f_k)^{-2} p^{k-1})^3 \cdot (1 - \varepsilon_k(p) a_p(f_k)^{-2} p^{k-2})^3 \cdot (1 - \varepsilon_k(p) a_p(f_k)^{-1} p^{\frac{k}{2}-1})^2,$$

and $\mathscr{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f}, k)$ equals

$$(1 - a_p(f_k)^{-1} p^{\frac{k}{2}-1})^2 \cdot (1 - a_p(f_k)^{-3} p^{\frac{3k-2}{2}-1}).$$

(ii) For points $k \in U_f^{\mathrm{cl},\circ}$ such that $f_k \neq f_k^\sharp$, the ratio of modified Euler factors that we have introduced is finite by the p-ordinariness of each f_k. Indeed, the Euler factors of the form $\mathscr{E}_{p,\mathrm{old}}^{\mathrm{Sym}^3}(\boldsymbol{f},k)$ are non-vanishing as there does not exist a pair (n,m) of positive integers such that $a_p(f_k)^n = p^m$.

We propose the following conjecture on the symmetric cube p-adic L-function.

Conjecture 5.9. *The p-adic L-function $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ for the symmetric cube is a p-adic analytic function on U_f such that*

1. $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \mathscr{L}_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)^2$ *is a square,*
2. $L_p^{\Delta}(\boldsymbol{f})(k) = L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) \cdot L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2$,
3. $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ *satisfies the following interpolation property for any $k \in U_f^{\mathrm{cl}}$:*

$$L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \mathscr{A}_{\boldsymbol{f}} \cdot \mathscr{B}_k \cdot \mathscr{C}_{\boldsymbol{f},k} \cdot \lambda_k^{\epsilon_k} \cdot \sqrt{-1}^{1-3k} \cdot \mathfrak{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f},k) \cdot L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, \tfrac{3k-2}{2}),$$

in which $\epsilon_k = (-1)^{\frac{k}{2}-1}$.

Note that the first point is a formal consequence of the second point, as $L_p^{\Delta}(\boldsymbol{f})(k)$ is constructed as a square and $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ appears as a square in the factorization:

$$L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \frac{L_p^{\Delta}(\boldsymbol{f})(k)}{L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2} = \left(\frac{\mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{f},\boldsymbol{f},\boldsymbol{f})(k,k,k)}{L_p^{\mathrm{cc}}(\boldsymbol{f})(k)}\right)^2 = \mathscr{L}_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)^2.$$

Proposition 5.10. *If the central critical p-adic L-function $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ does not vanish for any $k \in U_f$, then Conjecture 5.9 holds true.*

Proof. With the definition of $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ as the ratio of $L_p^{\Delta}(\boldsymbol{f})(k)$ by $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2$ and the respective interpolation formulae (5.3) and (5.4) at hand, the interpolation property for the symmetric cube p-adic L-function is nothing but a formal computation, keeping track of the various factors of different nature and dependency. ☐

Remark 5.11. Concerning the assumption of a non-vanishing denominator, let us point out the following remarks.

(i) Outside of the points $k \in U_f^{\mathrm{cl},\circ}$ where the complex special value $L(f_k^\sharp, \frac{k}{2})$ vanishes, we are in the situation of a non-vanishing Mazur–Kitagawa p-adic L-function by the very same reason that we have pointed out concerning the finiteness of our ratio of modified Euler factors in (ii) of Remark 5.8: The Euler factors of (5.3) are non-vanishing as there does not exist a pair (n,m) of positive integers such that $a_p(f_k)^n = p^m$, since the forms f_k are p-ordinary. Therefore, the restriction of the Mazur–Kitagawa p-adic L-function to the central critical line is non-vanishing for those particular $k \in U_f^{\mathrm{cl},\circ}$.

(ii) We will take a closer look at the delicate point $k = 2$ in a special situation of an elliptic curve, which is what we are ultimately interested in. In this particular situation, as will be discussed below, we know that $L_p^{\mathrm{cc}}(\boldsymbol{f})(2) = 0$, and so we can not simply compute the ratio via the interpolation formulae, having a vanishing denominator. Outside of $k = 2$ we will gain control by an extra assumption in light of the previous remark, but the missing point has to be taken care of by a separate argument.

5.5 **The Elliptic Curve Case.** Let now $A/\mathbb{Q}$ be a semi-stable elliptic curve of conductor $N = N'p$ and $\operatorname{sign}(A/\mathbb{Q}) = -1$, having an odd prime $p \geq 5$ of split multiplicative reduction. Further, let $\boldsymbol{f}$ denote the Hida family specializing in weight-two to the newform $f_A \in \mathrm{S}_2(\Gamma_0(N))$ attached to A by the modularity theorem. The semi-stability assumption arises from [Hsi21] as we want the tame level to be square-free.

Remark 5.12. The functional equation of the two-variable p-adic L-function $L_p^{\mathrm{MK}}(\boldsymbol{f})(k,s)$ relating the values at (k,s) and $(k,k-s)$ has a sign that is independent of k and therefore denoted by $\operatorname{sign}(\boldsymbol{f})$. In our particular case of interest, in which $A/\mathbb{Q}$ has *split* multiplicative reduction at p, one has

$$\operatorname{sign}(\boldsymbol{f}) = -\operatorname{sign}(A/\mathbb{Q}),$$

which excludes the case of positive sign for the Hasse–Weil L-function $L(A/\mathbb{Q}, s)$ from our attention as in this case $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ vanishes identically.

After possibly shrinking U_f, we impose the following main assumption on the complex special values.

Assumption 1. *For every $k \in U_f^{\mathrm{cl},\circ}$, the central critical value $L(f_k^\sharp, \frac{k}{2})$ does not vanish.*

Recall the p-adic numbers $\lambda_k^\pm \in \mathbb{C}_p$ of (5.3), which satisfy $\lambda_2^\pm = 1$ and are non-zero after possibly shrinking U_f. We further put the following assumption on those discrepancy factors coming from the p-adic interpolation.

Assumption 2. *For every $k \in U_f^{\mathrm{cl},\circ}$, the element $\lambda_k^{\epsilon_k}$ is p-integral.*

Remark 5.13. The assumption on the $\lambda_k^{\epsilon_k}$ fulfilling an integrality property seems reasonable as they arise from comparing two integral structures. The one side considers the modular symbols associated with the forms $f_k^\sharp$, having values in the dual of the space of degree $k-2$ homogeneous polynomials with coefficients in the number field generated by the Fourier coefficients of $f_k^\sharp$. The other side deals with a certain image of a measure-valued modular symbol, provided by [BD07, Theorem 1.5]. We refer to [GS93], in particular Theorem 5.13 and Section 6.

Concerning the use of the triple product p-adic L-function, note that the condition on Σ^- is automatically satisfied by our assumption on $A/\mathbb{Q}$ being semi-stable. Indeed, as noted in [BD07], this results in the number of primes of split multiplicative reduction for $A/\mathbb{Q}$ being even. Since those are the primes giving local root number -1, excluding the split prime p which does not belong to Σ^-, there is an odd number left. In fact (cf. [GK92]), the local root numbers (considering the triple $(2,2,2)$) are given by $-a_q(f_A)^3$, resulting in the global sign being

$$\begin{aligned}\epsilon &= (-1)\cdot \prod_{\substack{q\mid N\\ \text{split mult.}}} -a_q(f_A)^3 \cdot \prod_{\substack{\ell\mid N\\ \text{non-split mult.}}} -a_\ell(f_A)^3\\ &= (-1)\cdot \prod_{\substack{q\mid N\\ \text{split mult.}}} (-1)\cdot \prod_{\substack{\ell\mid N\\ \text{non-split mult.}}} -(-1)^3\\ &= -1.\end{aligned}$$

Here, we used that $a_q(f_A) = -1$ for q of non-split multiplicative reduction for $A/\mathbb{Q}$ and $a_\ell(f_A) = 1$ for ℓ of split multiplicative reduction.

This sign analysis enables us to prove the following result on the vanishing of the complex symmetric cube L-function.

Proposition 5.14. *The symmetric cube L-function $L(\mathrm{Sym}^3 f_A, s+1)$ vanishes to positive odd order at the point $s = 1$.*

Proof. By the above discussion of signs, the complex triple product L-function $L(f_A^{\otimes 3}, s+1)$ vanishes to odd order at $s = 1$. Recalling the factorization (5.1) for $\theta = f_A$ with $k = 2$, having an odd order of vanishing at $s = 1$ on the left-hand side, the fact that the Hasse–Weil L-function appears as a square on the right-hand side shows the contribution of $L(\mathrm{Sym}^3 f_A, s+1)$ to this order of vanishing. We conclude that

$$\mathrm{ord}_{s=1} L(\mathrm{Sym}^3 f_A, s+1) \geq 1 \quad \text{is odd.} \qquad \square$$

Before we get to our main result in this elliptic curve situation, we make the following final assumption on the special values of interest.

Assumption 3. *For every $k \in U_f^{\mathrm{cl},\circ}$, the value $L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, \frac{3k-2}{2})$ is p-integral.*

We are now able to formulate our main result on the p-adic L-function for the symmetric cube associated with the elliptic curve $A/\mathbb{Q}$.

Theorem 5.15. *Let $\boldsymbol{f}$ be the Hida family passing through the newform attached to a semi-stable elliptic curve $A/\mathbb{Q}$ with split multiplicative reduction at p at the weight-2 specialization and assume that Assumption 1, 2 and 3 are satisfied. Then, Conjecture 5.9 holds true and the p-adic L-function $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ has an exceptional zero at $k = 2$, which is of even order at least two.*

Proof. With the quantities for the interpolation formula given as in Definition 5.7, under Assumption 1, we already have the desired interpolation formula for all $k \in U_f^{\mathrm{cl},\circ}$:

$$L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \mathscr{A}_{\boldsymbol{f}} \cdot \mathscr{B}_k \cdot \mathscr{C}_{\boldsymbol{f},k} \cdot \lambda_k^{\epsilon_k} \cdot \sqrt{-1}^{1-3k} \cdot \mathfrak{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f}, k) \cdot L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, \tfrac{3k-2}{2}). \tag{5.6}$$

By construction, we view $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ as a meromorphic function in the weight variable k, noting that it is continuous (in fact analytic) at $k = 2$ as a result of the boundedness in a neighborhood of $k = 2$ by Assumptions 2 and 3. Indeed, using the interpolation formula in the punctured neighborhood $U_f^{\mathrm{cl},\circ}$, the values $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ factor as a product of

- a p-adically bounded part given by $\mathscr{A}_{\boldsymbol{f}} \cdot \mathscr{B}_k \cdot \mathscr{C}_{\boldsymbol{f},k} \cdot \sqrt{-1}^{1-3k}$,
- a p-adically bounded part given by $\lambda_k^{\epsilon_k}$ by Assumption 2,
- $\mathfrak{E}_p^{\mathrm{Sym}^3}(\boldsymbol{f}, k)$ tending to 0 as $k \to 2$,
- a p-adically bounded part given by $L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, \frac{3k-2}{2})$ by Assumption 3.

In particular, the above factorization shows that

$$\lim_{k\to 2} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = 0,$$

which, by continuity, gives that $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ vanishes at $k = 2$ as claimed. The interpolation property now holds by our observations on the vanishing of $L(\mathrm{Sym}^3 f_A, s+1)$ at $s = 1$ and the vanishing of the Euler factor appearing in the formula for $k = 2$. Note that we are dealing with a scenario of an *exceptional zero* at $k = 2$, since the p-adic L-function vanishes at $k = 2$ independently of the vanishing of the complex symmetric cube L-function. This is due to the shape of the Euler factor appearing in the interpolation formula. Finally, the order of vanishing is even, as

$$L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) = \frac{L_p^{\Delta}(\boldsymbol{f})(k)}{L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2} = \left(\frac{\mathscr{L}_p^{\mathrm{bal}}(\boldsymbol{f},\boldsymbol{f},\boldsymbol{f})(k,k,k)}{L_p^{\mathrm{cc}}(\boldsymbol{f})(k)}\right)^2 = \mathscr{L}_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)^2$$

is a square by construction. $\square$

6 A BSD-Formula for the Triple Product p-adic L-Function

We remain in the setting of the previous discussion and let $A/\mathbb{Q}$ be a semi-stable elliptic curve with $\operatorname{sign}(A/\mathbb{Q}) = -1$, split multiplicative reduction at a prime $p \geq 5$ and associated p-adic Hida family $\boldsymbol{f}$ passing through (the modular form of) A at its weight-two specialization. As before, we work under the following list of assumptions from the previous section for every classical weight $k \in U_f^{\mathrm{cl},\circ}$ away from $k = 2$:

Assumption 1: *The central critical value $L(f_k^\sharp, \frac{k}{2})$ does not vanish.*

Assumption 2: *The element $\lambda_k^{\epsilon_k}$ is p-integral.*

Assumption 3: *The value $L^{\mathrm{alg}}(\mathrm{Sym}^3 f_k^\sharp, \frac{3k-2}{2})$ is p-integral.*

In the previous section, we introduced a p-adic L-function for the symmetric cube, providing a factorization of p-adic analytic functions

$$L_p^\Delta(\boldsymbol{f})(k) = L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k) \cdot L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2. \tag{6.1}$$

In the following, we want to study the orders of vanishing at $k = 2$ along with the relevant derivatives of the involved p-adic L-functions. More precisely, using a result of Bertolini–Darmon described below, we prove a formula for the sixth derivative of the triple product p-adic L-function, relating it to a global point on the elliptic curve. Afterwards, we address the meaning of the derivative of the symmetric cube p-adic L-function that appears as a factor in the so-obtained formula (6.2). Our goal is to obtain a description of this derivative in terms of a *regulator term* for the symmetric cube p-adic L-function.

6.1 **Derivatives and Rational Points.** In our particular situation of the elliptic curve having split multiplicative reduction at p and $\operatorname{sign}(A/\mathbb{Q}) = -1$, the restriction of the Mazur–Kitagawa p-adic L-function $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ vanishes to order at least two at $k = 2$. The following result was proven by Bertolini–Darmon [BD07], assuming a mild technical condition subsequently removed by Mok [Mok11, Section 6].

Theorem 6.1. *There exists a global point $\mathbf{P} \in A(\mathbb{Q}) \otimes \mathbb{Q}$ and a scalar $a \in \mathbb{Q}^\times$ such that*

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{cc}}(\boldsymbol{f})(k)_{|k=2} = a \cdot \log_A(\mathbf{P})^2,$$

where $\log_A \colon A(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is the formal group logarithm attached to A.

We now want to use this theorem to provide a formula for a derivative of the triple product p-adic L-function. Recall that, as a result of Theorem 5.15, the symmetric cube p-adic L-function vanishes to even order at least two at $k = 2$.

The aforementioned result on the triple product p-adic L-function is as follows.

Theorem 6.2. *The restricted triple product p-adic L-function vanishes to even order at least six at $k = 2$ and there exists $\mathbf{P} \in A(\mathbb{Q}) \otimes \mathbb{Q}$ and $a \in \mathbb{Q}^\times$ such that*

$$\frac{\mathrm{d}^6}{\mathrm{d}k^6} L_p^\Delta(\boldsymbol{f})(k)_{|k=2} = 90a^2 \cdot \frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)_{|k=2} \cdot \log_A(\mathbf{P})^4. \tag{6.2}$$

Proof. As both the symmetric cube p-adic L-function and the Mazur–Kitagawa p-adic L-function vanish to even order at least two at $k = 2$, the factorization (6.1) yields that $L_p^{\Delta}(\boldsymbol{f})(k)$ vanishes to even order at least six. Using the product rule, we obtain that

$$\frac{\mathrm{d}^6}{\mathrm{d}k^6} L_p^{\Delta}(\boldsymbol{f})(k)_{|k=2} = \sum_{n=0}^{6} \binom{6}{n} \cdot \frac{\mathrm{d}^{6-n}}{\mathrm{d}k^{6-n}} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)_{|k=2} \cdot \frac{\mathrm{d}^n}{\mathrm{d}k^n} L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2_{|k=2}. \tag{6.3}$$

By the vanishing of $L_p^{\mathrm{cc}}(\boldsymbol{f})(k)$ at $k = 2$ being of even order at least two, the first, second, and third derivative of its square vanish at $k = 2$. Therefore, only the summands for $n \in \{4, 5, 6\}$ in (6.3) are possibly non-zero. The positive even order of vanishing of the symmetric cube p-adic L-function implies that its first derivative at $k = 2$ vanishes as well. Therefore, the only remaining term of (6.3) that can be non-zero appears at $n = 4$ and is given by

$$\begin{aligned} &\binom{6}{4} \cdot \frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)_{|k=2} \cdot \frac{\mathrm{d}^4}{\mathrm{d}k^4} L_p^{\mathrm{cc}}(\boldsymbol{f})(k)^2_{|k=2} \\ &\qquad = \binom{6}{4} \cdot \binom{4}{2} \cdot \frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)_{|k=2} \cdot \left(\frac{\mathrm{d}^2}{\mathrm{d}k^2} L_p^{\mathrm{cc}}(\boldsymbol{f})(k)_{|k=2}\right)^2. \end{aligned}$$

Finally, using Theorem 6.1, we obtain $\mathbf{P} \in A(\mathbb{Q}) \otimes \mathbb{Q}$ and $a \in \mathbb{Q}^\times$ along with the desired formula. $\square$

6.2 **Ordinariness of the Symmetric Cube.** In this section, we check the *ordinariness* of the relevant Galois representations at p.

Let F be a finite extension of $\mathbb{Q}_p$ and V be an F-vector space giving a p-adic representation of $G_{\mathbb{Q}}$. Following Greenberg [Gre94], we call V *ordinary at* p if there exists a descending filtration $\{\mathrm{Fil}^j V\}_{j \in \mathbb{Z}}$ of $G_{\mathbb{Q}_p}$-stable F-subspaces of V which is exhaustive and separated such that the inertia subgroup I_p acts on the quotients $\mathrm{Fil}^j V / \mathrm{Fil}^{j+1} V$ via the j-th power χ_p^j of the cyclotomic character χ_p recalled in Section 3.3.

Let now $\theta \in \mathrm{S}_k(\Gamma_0(N))$ be a p-ordinary newform of even weight $k \geq 2$ and consider the p-adic representation $\rho_{\theta,\mathfrak{p}}$ for $\mathfrak{p} \mid p$ as in Section 3.1.

Lemma 6.3. *The p-adic representation* $(\mathrm{Sym}^3 \rho_{\theta,\mathfrak{p}})(1-k)$ *is ordinary at p, being of the form*

$$(\mathrm{Sym}^3 \rho_{\theta,\mathfrak{p}})(1-k)_{|G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_p^{2(k-1)}\delta^{-3} & * & * & * \\ & \chi_p^{k-1}\delta^{-1} & * & * \\ & & \delta & * \\ & & & \chi_p^{1-k}\delta^3 \end{pmatrix}.$$

Proof. By [Wil88, Theorem 2.1.4], we have

$$\rho_{\theta,\mathfrak{p}|G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_p^{k-1}\delta^{-1} & * \\ 0 & \delta \end{pmatrix}$$

with δ being the unramified character sending Frob_p to the unit root of the Hecke polynomial $X^2 - a_p(\theta)X + p^{k-1}$ of θ at p. Thus, its symmetric cube is equivalent to

$$\begin{pmatrix} \chi_p^{3(k-1)}\delta^{-3} & * & * & * \\ & \chi_p^{2(k-1)}\delta^{-1} & * & * \\ & & \chi_p^{k-1}\delta & * \\ & & & \delta^3 \end{pmatrix}$$

and so we observe that the relevant twist by $1-k$ is ordinary, being of the claimed form. $\square$

We will now begin with the final part of the thesis in which we adapt the discussion of [BSV21] for introducing a *regulator* for the symmetric cube. Building on the work of [Nek06], this is done by discussing an appropriate p-adic *height pairing* on an *extended Selmer group*. Our goal is to formulate a conjecture on this regulator, describing the derivative of the symmetric cube appearing in Theorem 6.2.

We refer to the aforementioned work of Nekovář for a detailed introduction of the underlying notions and to the work of Bertolini, Seveso and Venerucci for the ideas towards a regulator that we aim to mimic.

6.3 **Nekovář Selmer Complexes and the p-adic Height Pairing.** Let us start by giving an idea of the picture which the *extended Selmer group* fits in for the case of an elliptic curve. We refer the reader to [Nek06, Sections 0.10, 12.1.8] for further elaboration. This example, without going into detail, may be kept in mind as it displays the direct connection to the Bloch–Kato Selmer groups discussed in Section 2.4.

Consider the p-adic Tate module of an elliptic curve $A/\mathbb{Q}$ with ordinary reduction at p, sitting in the middle of an exact sequence of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$V_p(A)^+ \hookrightarrow V_p(A) \twoheadrightarrow V_p(A)^-.$$

Here, $V_p(A)^\pm$ is of dimension one over $\mathbb{Q}_p$ and the inertia subgroup I_p acts trivially on $V_p(A)^-$. The *extended Selmer group* for the elliptic curve sits in the middle of an exact sequence

$$\mathrm{H}^0(\mathbb{Q}_p, V_p(A)^-) \hookrightarrow \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(A)) \twoheadrightarrow \mathrm{Sel}(\mathbb{Q}, V_p(A)) \tag{6.4}$$

with

$$\dim_{\mathbb{Q}_p}(\mathrm{H}^0(\mathbb{Q}_p, V_p(A)^-)) = \begin{cases} 1 & \text{if } A \text{ has split multiplicative reduction at } p, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we think of $\mathrm{H}^0(\mathbb{Q}_p, V_p(A)^-)$ as a space of p-adic periods, detecting the presence of an extra zero for the p-adic L-function of A.

For the convenience of the reader, we give a brief introduction to the theory of Nekovář, as assembled in [Ven13, Appendix A]. Let $\mathtt{B}$ be a complete noetherian ring with maximal ideal $\mathfrak{m}_{\mathtt{B}}$ and finite residue field $\mathtt{B}/\mathfrak{m}_{\mathtt{B}}$ of characteristic p. Further, let $K/\mathbb{Q}$ be a number field and fix a finite set S_f of primes of K, containing every prime above p. The maximal algebraic extension of K which is unramified outside of $S = S_f \cup \{\nu \mid \infty\}$ is denoted by $K_S \subseteq \bar{K}$. For any prime $\nu \in S_f$, we consider the fixed embedding $\imath_\nu \colon \bar{K} \hookrightarrow \bar{K}_\nu$ with its induced map $\imath_\nu^*\colon G_{K_\nu} \hookrightarrow G_K$ of absolute Galois groups and denote the corresponding decomposition group by $G_\nu = \imath_\nu^*(G_{K_\nu})$.

For the following, we fix $G \in \{G_{K_S}, G_{K_\nu}, G_\nu\}$. Given any admissible $\mathtt{B}[G]$-module $\mathtt{M}$ in the sense of [Nek06, Section 3], we denote by $\mathrm{C}^\bullet_{\mathrm{cts}}(G, \mathtt{M})$ the complex of (non-homogeneous) continuous cochains. Its image in the derived category of complexes of $\mathtt{B}$-modules (cf. [Har66]) will be denoted $\mathbf{R}\Gamma_{\mathrm{cts}}(G, \mathtt{M})$ and its cohomology will be denoted $\mathrm{H}^\cdot(G, \mathtt{M})$. We further use the abbreviations

$$\begin{aligned} \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M}) &= \mathrm{C}^\bullet_{\mathrm{cts}}(G_{K_\nu}, \mathtt{M}), \\ \mathbf{R}\Gamma_{\mathrm{cts}}(K_\nu, \mathtt{M}) &= \mathbf{R}\Gamma_{\mathrm{cts}}(G_{K_\nu}, \mathtt{M}), \\ \mathrm{H}^\cdot(K_\nu, \mathtt{M}) &= \mathrm{H}^\cdot(G_{K_\nu}, \mathtt{M}). \end{aligned}$$

Remark 6.4. In the context of the above notions, it is worth to point out the following:

(i) If $\mathtt{M}$ is a $\mathtt{B}[G]$-module of finite type over $\mathtt{B}$, then $\mathtt{M}$ is admissible precisely when G acts continuously with respect to the $\mathfrak{m}_{\mathtt{B}}$-adic topology on $\mathtt{M}$, and $\mathrm{C}^\bullet_{\mathrm{cts}}(G, \mathtt{M})$ is the usual continuous cochain complex.

(i) Any admissible $\mathtt{B}[G_{K_S}]$-module can be viewed as an admissible $\mathtt{B}[G_\nu]$-module via $\imath_\nu^*$ for $\nu \in S_f$. Moreover, note that there is a natural isomorphism

$$\mathrm{C}^\bullet_{\mathrm{cts}}(G_\nu, \mathtt{M}) \cong \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M})$$

induced by $G_\nu \cong G_{K_\nu}$ via $\imath_\nu^*$ and the identity on $\mathtt{M}$.

The restriction from G_{K_S} to G_ν arising from $G_\nu \subseteq G_K \twoheadrightarrow G_{K_S}$ induces a restriction map

$$\mathrm{res}_\nu\colon \mathrm{C}^\bullet_{\mathrm{cts}}(G_{K_S}, \mathtt{M}) \longrightarrow \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M})$$

and we denote the induced map on cohomology by the same symbol. Furthermore, we set the sum of the restriction maps to be $\mathrm{res}_{S_f} = \bigoplus_{\nu \in S_f} \mathrm{res}_\nu$.

The definition of a Selmer complex à la Nekovář requires to introduce certain local conditions that we now want to address. Let $\mathtt{M}$ be an admissible $\mathtt{B}[G_{K_S}]$-module. A *local condition* $\Delta_\nu(\mathtt{M})$ for $\mathtt{M}$ at $\nu \in S_f$ is a complex of $\mathtt{B}$-modules $U_\nu^+(\mathtt{M})$ together with a morphism

$$i_\nu^+ = i_\nu^+(\mathtt{M})\colon U_\nu^+(\mathtt{M}) \longrightarrow \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M})$$

of complexes of $\mathtt{B}$-modules. Here, we write $\Delta_\nu(\mathtt{M}) = U_\nu^+(\mathtt{M})$ when the morphism i_ν^+ is clear from the context. Given local conditions $\Delta(\mathtt{M}) = \{\Delta_\nu(\mathtt{M})\}_{\nu \in S_f}$, the associated *Nekovář Selmer complex* is defined (cf. [Nek06, Section 6]) to be

$$\widetilde{\mathrm{C}}^\bullet_{\mathrm{cts}}(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M})) = \mathrm{cone}\Big(\mathrm{C}^\bullet_{\mathrm{cts}}(G_{K_S}, \mathtt{M}) \oplus \bigoplus_{\nu \in S_f} U_\nu^+(\mathtt{M}) \longrightarrow \bigoplus_{\nu \in S_f} \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M})\Big)[-1]. \quad (6.5)$$

In this definition, the map of complexes is given by $\mathrm{res}_{S_f} - i^+_{S_f}$, where $i^+_{S_f} = \bigoplus_{\nu \in S_f} i^+_\nu$.

We denote by $\mathbf{R}\widetilde{\Gamma}_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M}))$ the image of $\widetilde{\mathrm{C}}^\bullet_{\mathrm{cts}}(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M}))$ in the derived category and set its cohomology to be

$$\widetilde{\mathrm{H}}^\cdot_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M})) = \mathrm{H}^\cdot(\mathbf{R}\widetilde{\Gamma}_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M}))).$$

Remark 6.5. If $\mathtt{M}$ is of finite type over $\mathtt{B}$ and the local conditions $U_\nu^+(\mathtt{M})$ are complexes with cohomology of finite type over $\mathtt{B}$, then this translates to $\tilde{\mathrm{H}}^{\cdot}_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M}))$ being of finite type over $\mathtt{B}$.

We are particularly interested in the following type of local conditions. Let $\mathtt{M}$ be a free $\mathtt{B}$-module of finite type with a continuous $\mathtt{B}$-linear action of G_{K_S}. We assume that for every $\nu \in S_f$, there exists a short exact sequence of $\mathtt{B}[G_\nu]$-modules

$$\mathtt{M}_\nu^+ \xhookrightarrow{i_\nu^+} \mathtt{M} \xrightarrow{q_\nu^-} \mathtt{M}_\nu^-$$

with $\mathtt{M}_\nu^\pm$ free as $\mathtt{B}$-modules. The local conditions that we will consider are then given by $\Delta_\nu(\mathtt{M}) = \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M}_\nu^+)$ with morphisms $i_\nu^+ \colon \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M}_\nu^+) \to \mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M})$ induced by i_ν^+ for $\nu \in S_f$. Keeping the above choices of $\mathtt{B}[G_\nu]$-submodules $i_\nu^+ \colon \mathtt{M}_\nu^+ \hookrightarrow \mathtt{M}$ in mind, we set $\Delta(\mathtt{M}) = \{\mathrm{C}^\bullet_{\mathrm{cts}}(K_\nu, \mathtt{M}_\nu^+)\}_{\nu \in S_f}$ and denote

$$\begin{aligned}
\tilde{\mathrm{C}}^\bullet_f(G_{K_S}, \mathtt{M}) &= \tilde{\mathrm{C}}^\bullet_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M})),\\
\mathbf{R}\tilde{\Gamma}_f(G_{K_S}, \mathtt{M}) &= \mathbf{R}\tilde{\Gamma}_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M})),\\
\tilde{\mathrm{H}}^{\cdot}_f(G_{K_S}, \mathtt{M}) &= \tilde{\mathrm{H}}^{\cdot}_f(G_{K_S}, \mathtt{M}; \Delta(\mathtt{M})).
\end{aligned}$$

Here, the cohomology groups are $\mathtt{B}$-modules of finite type. One obtains an exact triangle in the derived category of $\mathtt{B}$-modules (cf. [Ven13, (157)])

$$\bigoplus_{\nu \in S_f} \mathbf{R}\Gamma_{\mathrm{cts}}(K_\nu, \mathtt{M}_\nu^-)[-1] \longrightarrow \mathbf{R}\tilde{\Gamma}_f(G_{K_S}, \mathtt{M}) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cts}}(G_{K_S}, \mathtt{M}) \longrightarrow \bigoplus_{\nu \in S_f} \mathbf{R}\Gamma_{\mathrm{cts}}(K_\nu, \mathtt{M}_\nu^-), \tag{6.6}$$

which in turn provides a long exact sequence of $\mathtt{B}$-modules

$$\cdots \to \bigoplus_{\nu \in S_f} \mathrm{H}^{j-1}(K_\nu, \mathtt{M}_\nu^-) \to \tilde{\mathrm{H}}^j_f(G_{K_S}, \mathtt{M}) \to \mathrm{H}^j(G_{K_S}, \mathtt{M}) \to \bigoplus_{\nu \in S_f} \mathrm{H}^j(K_\nu, \mathtt{M}_\nu^-) \to \cdots$$

by taking cohomology. Here, the last map is obtained by composing res_{S_f} with the sum of the maps induced by q_ν^-.

Finally, we give a brief overview of the ingredients for the relevant height pairing that we want to use for introducing a regulator for the symmetric cube p-adic L-function. Let T be a bounded complex of admissible $\mathtt{B}[G]$-modules of finite type over $\mathtt{B}$ and denote by $J = \ker(\bar{\mathtt{B}} \to \mathtt{B})$ the augmentation ideal of $\mathtt{B}$, where $\bar{\mathtt{B}} = \mathtt{B}[\![\Gamma]\!]$ is an Iwasawa algebra over $\mathtt{B}$. On the one hand, there is (cf. [Nek06, Section 0.16]) a *derived Bockstein map*

$$\tilde{\boldsymbol{\beta}} \colon \mathbf{R}\tilde{\Gamma}_f(T) \longrightarrow \mathbf{R}\tilde{\Gamma}_f(T)[1] \otimes J/J^2.$$

On the other hand, one has (cf. [Nek06, (0.9.3)]) a cup product of the form

$$\cup \colon \mathbf{R}\tilde{\Gamma}_f(T) \otimes \mathbf{R}\tilde{\Gamma}_f(T^\vee(1)) \longrightarrow \omega[-3],$$

where ω is the *dualizing complex* (cf. [Nek06, Section 0.4]). Based on the Bockstein map and the cup product, the *height pairing* is defined as the composition

$$(\cup[1] \otimes J/J^2) \circ (\tilde{\boldsymbol{\beta}} \otimes \mathbf{R}\tilde{\Gamma}_f(T^\vee(1))) \colon \mathbf{R}\tilde{\Gamma}_f(T) \otimes \mathbf{R}\tilde{\Gamma}_f(T^\vee(1)) \longrightarrow \omega \otimes J/J^2[-2].$$

Its component of the form

$$\tilde{\mathrm{H}}^1_f(T) \otimes \tilde{\mathrm{H}}^1_f(T^\vee(1)) \longrightarrow \mathrm{H}^0(\omega) \otimes J/J^2$$

is what will be of special interest for us and will be referred to as the *p-adic height pairing.*

6.4 **Extended Selmer Groups for the Symmetric Cube.** From here on, we focus on the symmetric cube and introduce variants of the theory presented in the previous subsection and the notions therein along the lines of [BSV21].

Let $\boldsymbol{f}$ be the p-adic Hida family as in the introduction of this section. It passes through the newform $f_A \in \mathrm{S}_2(\Gamma_0(N'p))$ of a semi-stable elliptic cuve $A/\mathbb{Q}$ with $\mathrm{sign}(A/\mathbb{Q}) = -1$ and split multiplicative reduction at the prime $p \geq 5$ at its weight-two specialization. As before, we assume it to satisfy Assumptions 1, 2 and 3. We denote by $V(\boldsymbol{f})$ the Galois representation (cf. [BSV22b]) that comes with the Hida family $\boldsymbol{f}$, which interpolates the Galois representations associated with the specializations. In particular, it specializes to the p-adic Tate module $V_p(f_A)$ attached to f_A at weight two. Further, we define $V(\mathrm{Sym}^3\,\boldsymbol{f})$ to be the representation associated with the symmetric cube, i.e.

$$V(\mathrm{Sym}^3\,\boldsymbol{f}) = \mathrm{Sym}^3\,V(\boldsymbol{f}) \otimes \Xi.$$

In this definition, $\Xi\colon G_{\mathbb{Q}} \to \mathscr{O}(U_{f_A})^\times$ is the character satisfying

$$\Xi(\sigma)(k) = \chi_p(\sigma)^{1-k} \quad \text{for every } \sigma \in G_{\mathbb{Q}} \text{ and } k \in U_{f_A}^{\mathrm{cl}},$$

and $\mathscr{O}(U_{f_A})$ is the ring of bounded analytic functions on U_{f_A}. This *big* representation thus gives rise to the appropriate twist of the symmetric cube that appears in the decomposition result of Proposition 3.8. In particular, we have as a weight-two specialization

$$V_p(\mathrm{Sym}^3\,f_A) = \mathrm{Sym}^3(V_p(f_A))(-1).$$

We let $\Lambda(U_{f_A})$ be the ring of analytic functions on U_{f_A} bounded by 1. The ideal of analytic functions in $\mathscr{O}(U_{f_A})$ vanishing at $k = 2$ will be denoted $\mathscr{J}$. We define $K_{N'p}$ to be the maximal algebraic extension of $\mathbb{Q}$ which is unramified at all the rational primes not dividing $N'p$ and denote its absolute Galois group by $G_{N'p}$. In the terminology used in Section 6.3, S is the set of primes dividing $N'p$.

In our case, $(\mathtt{B}, \mathtt{M})$ is either of the pairs $(\mathbb{Z}_p, \mathtt{V}_p(\mathrm{Sym}^3\,f_A))$ and $(\Lambda(U_{f_A}), \mathtt{V}(\mathrm{Sym}^3\,\boldsymbol{f}))$, where

$$\mathtt{V}_p(\mathrm{Sym}^3\,f_A) \subseteq V_p(\mathrm{Sym}^3\,f_A), \quad \text{resp.} \quad \mathtt{V}(\mathrm{Sym}^3\,\boldsymbol{f}) \subseteq V(\mathrm{Sym}^3\,\boldsymbol{f}),$$

is a $\mathbb{Z}_p$-lattice, resp. $\Lambda(U_{f_A})$-lattice, preserved by the action of $G_{N'p}$. We keep denoting the maximal ideal of $\mathtt{B}$ by $\mathfrak{m}_{\mathtt{B}}$ and equip $\mathtt{M}$ with the $\mathfrak{m}_{\mathtt{B}}$-adic topology. Furthermore, we let $(B, M) = (\mathtt{B}[\frac{1}{p}], \mathtt{M}[\frac{1}{p}])$ and equip the absolute Galois groups $G_{N'p}$ and $G_{\mathbb{Q}_\ell}$ for any prime $\ell \mid N'p$ with the profinite topology. The relevant complex of continuous (non-homogeneous) cochains of G_K with values in M is denoted by

$$\mathrm{C}^\bullet_{\mathrm{cts}}(K, M) = \mathrm{C}^\bullet_{\mathrm{cts}}(G_K, \mathtt{M}) \otimes_{\mathtt{B}} B,$$

where K is any of the fields considered before. The local conditions of interest for us arise from the inclusion within a short exact sequence

$$V(\boldsymbol{f})^+ \hookrightarrow V(\boldsymbol{f}) \twoheadrightarrow V(\boldsymbol{f})^-$$

of $\mathscr{O}(U_{f_A})[G_{\mathbb{Q}_p}]$-modules as in [BSV21, Section 2.1] or [BSV22b, (102)]. We put

$$V(\mathrm{Sym}^3\,\boldsymbol{f})^+ = \mathrm{Sym}^3\,V(\boldsymbol{f})^+ \otimes \Xi$$

and fix, for M being any of the two representations of interest, a $G_{\mathbb{Q}_p}$-stable $\mathtt{B}$-lattice $\mathtt{M}^+$ mapping into $\mathtt{M}$ under the inclusion $M^+ \hookrightarrow M$. In the notation used before, for any prime $\ell \mid N'p$, we consider the local conditions given by

$$U_\ell^+(M) = \begin{cases} \mathrm{C}^\bullet_{\mathrm{cts}}(\mathbb{Q}_p, M^+) = \mathrm{C}^\bullet_{\mathrm{cts}}(\mathbb{Q}_p, \mathtt{M}^+) \otimes B & \text{if } \ell = p \\ 0 & \text{if } \ell \neq p. \end{cases}$$

Then, as introduced in (6.5), the Nekovář Selmer complex is of the form

$$\tilde{\mathrm{C}}^\bullet_{\mathrm{cts}}(K_{N'p}, M) = \mathrm{cone}\Big(\mathrm{C}^\bullet_{\mathrm{cts}}(K_{N'p}, M) \oplus \mathrm{C}^\bullet_{\mathrm{cts}}(\mathbb{Q}_p, M^+) \longrightarrow \bigoplus_{\ell \mid N'p} \mathrm{C}^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, M)\Big)[-1].$$

Again, we set $\mathbf{R}\tilde{\Gamma}_f(\mathbb{Q}, M)$ to be the image of $\tilde{\mathrm{C}}^\bullet_{\mathrm{cts}}(K_{N'p}, M)$ in the derived category of bounded complexes of B-modules with cohomology of finite type over B and denote its cohomology

$$\tilde{\mathrm{H}}^\cdot_f(\mathbb{Q}, M) = \mathrm{H}^\cdot(\mathbf{R}\tilde{\Gamma}_f(\mathbb{Q}, M)).$$

The first cohomology group $\tilde{\mathrm{H}}^1_f(\mathbb{Q}, M)$ is called the *Nekovář extended Selmer group* of M. We refer the reader to [Nek06, 12.5.9.2 (ii), 12.7.13.3 (ii)] for a discussion of the independence of these groups on the fixed sets of primes.

Just as in [BSV21, Section 2.3], this group should again be identified with the *naive* extended Selmer group

$$\mathrm{Sel}^\dagger(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) = \mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \oplus \mathrm{H}^0(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-), \tag{6.7}$$

where the $G_{\mathbb{Q}_p}$-invariants on the right-hand side are thought of as the *p-adic periods* for the symmetric cube. Here,

$$V_p(\mathrm{Sym}^3 f_A)^- = \mathrm{Sym}^3(V_p(f_A)^-)(-1)$$

is a $\mathbb{Q}_p$-module of dimension one with trivial action of I_p. Note that $V_p(\mathrm{Sym}^3 f_A)^-$ is defined via the one-dimensional $\mathbb{Q}_p$-module $V_p(f_A)^-$, on which an arithmetic Frobenius acts as multiplication by the unit root α_p of the Hecke polynomial of f_A at p. Therefore, an arithmetic Frobenius of $G_{\mathbb{Q}_p}$ acts on $V_p(\mathrm{Sym}^3 f_A)^-$ with eigenvalue $p^{-1}\alpha_p^3$.

Remark 6.6. Recall the exact sequence (6.4) for the case of an elliptic curve, having the extended Selmer group as its middle term, providing an identification

$$\tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(A)) = \mathrm{Sel}(\mathbb{Q}, V_p(A)) \oplus \mathrm{H}^0(\mathbb{Q}_p, V_p(A)^-).$$

The right-hand side is exactly what we call the naive extended Selmer group for the appropriate representation.

The idea for the identification (6.7) is as follows. Recalling (6.6), we have an exact triangle

$$\mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_p, M^-)[-1] \longrightarrow \mathbf{R}\tilde{\Gamma}_f(\mathbb{Q}, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cts}}(K_{N'p}, M) \xrightarrow{q^- \circ \mathrm{res}_p} \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_p, M^-), \tag{6.8}$$

where q^- is the map on complexes induced by the projection $M \twoheadrightarrow M^-$. This triangle induces a long exact sequence in cohomology

$$\cdots \to \mathrm{H}^{j-1}(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-) \to \tilde{\mathrm{H}}^j_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \to \mathrm{H}^j(K_{N'p}, V_p(\mathrm{Sym}^3 f_A)) \to \cdots.$$

The main claim to be checked is that from this long exact sequence one can extract a short exact sequence (cf. [Nek06, 12.5.9.2 (iii)])

$$\mathrm{H}^0(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-) \hookrightarrow \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \twoheadrightarrow \mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)). \tag{6.9}$$

For this, it is essential to show that (cf. [Nek06, 6.1.4 & 11.3.2])

$$\mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) = \ker\Big(\mathrm{H}^1(K_{N'p}, V_p(\mathrm{Sym}^3 f_A)) \xrightarrow{q^- \circ \mathrm{res}_p} \mathrm{H}^1(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-)\Big).$$

The short exact sequence (6.9) would then provide an identification

$$\begin{aligned}\tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) &= \mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \oplus \mathrm{H}^0(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-)\\ &= \mathrm{Sel}^\dagger(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A))\end{aligned}$$

as desired.

Finally, the definition of the p-adic height pairing that we are interested in builds on a certain Bockstein map and a cup product pairing. More precisely, the Bockstein map appears as

$$\tilde{\beta}\colon\ \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \longrightarrow \tilde{\mathrm{H}}^2_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \otimes \mathscr{J}/\mathscr{J}^2$$

and the relevant cup product is of the form

$$\cup\colon\ \tilde{\mathrm{H}}^2_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \otimes \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \longrightarrow \mathbb{Q}_p.$$

As introduced at the end of Section 6.3, both of the above maps arise from *derived* versions when considering the appropriate component in cohomology. We tensor the Bockstein map $\tilde{\beta}$ with the identity on $\tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A))$, the cup product $\cup$ with the identity on $\mathscr{J}/\mathscr{J}^2$ and define the p-adic height pairing

$$\langle\!\langle \cdot\,, \cdot \rangle\!\rangle\colon\ \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \otimes \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) \longrightarrow \mathscr{J}/\mathscr{J}^2$$

to be their composition. We expect it to be skew-symmetric by analogous arguments along the lines of the proof of [BSV21, Proposition 2.1], using results from [Ven13, Appendix C].

6.5 **A Conjectural Regulator for the Symmetric Cube p-adic L-Function.** Concluding this thesis, we assemble the above to introduce a regulator and propose a conjecture along the lines of [BSV21] which puts that regulator into the context of Theorem 6.2. The Nekovář extended Selmer group will henceforth be identified with the naive extended Selmer group, and we set its dimension over the rationals to be

$$r^\dagger(\mathrm{Sym}^3 A) = \dim \tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)).$$

Definition 6.7. We define the *regulator* for the symmetric cube to be

$$R_p(\mathrm{Sym}^3 A) = \det\left(\langle\!\langle Q_i, Q_j\rangle\!\rangle\right)_{1\le i,j\le r^\dagger(\mathrm{Sym}^3 A)} \in \left(\mathscr{J}^{r^\dagger(\mathrm{Sym}^3 A)}/\mathscr{J}^{r^\dagger(\mathrm{Sym}^3 A)+1}\right)/\mathbb{Q}^{\times,2},$$

where $Q_1, \ldots, Q_{r^\dagger(\mathrm{Sym}^3 A)}$ form a $\mathbb{Q}$-basis of $\tilde{\mathrm{H}}^1_f(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A))$.

Note that the determinant is only defined up to multiplication by the square of a non-zero rational and we further use the natural multiplication $(J/J^2)^r \to J^r/J^{r+1}$ to consider the determinant as an element of the quotient on the right-hand side.

The following conjecture is the analogue of [BSV21, Conjecture 1.1] for the symmetric cube p-adic L-function constructed in Section 5.

Conjecture 6.8. *The p-adic L-function $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ has order of vanishing $r^\dagger(\mathrm{Sym}^3 A)$ at $k = 2$ and the following equality holds in the quotient of $\mathscr{J}^{r^\dagger(\mathrm{Sym}^3 A)}/\mathscr{J}^{r^\dagger(\mathrm{Sym}^3 A)+1}$ by the multiplicative action of $\mathbb{Q}^{\times,2}$:*

$$\frac{\mathrm{d}^{r^\dagger(\mathrm{Sym}^3 A)}}{\mathrm{d}k^{r^\dagger(\mathrm{Sym}^3 A)}} L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)_{|k=2} = R_p(\mathrm{Sym}^3 A).$$

Wrapping up the thesis, we provide a more specialized conjecture by merging Conjecture 6.8 and Theorem 6.2 in a situation of minimal analytic rank of the complex symmetric cube L-function.

This follow-up conjecture arises from various steps that we now want to address. The assumption on the minimal analytic rank for the complex symmetric cube L-function conjecturally gives $\dim \mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A)) = 1$. Pairing this with the expectation of $\mathrm{H}^0(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-)$ being one-dimensional by the presence of an extra zero (cf. Theorem 5.15), Conjecture 6.8 predicts $r^\dagger(\mathrm{Sym}^3 A) = 2$ to be the relevant order of vanishing by the identification of Selmer groups. The conjectured regulator for the symmetric cube is of the form

$$R_p(\mathrm{Sym}^3 A) = \langle\!\langle Q, \omega\rangle\!\rangle^2,$$

with generators $Q \in \mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A))$ and $\omega \in \mathrm{H}^0(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-)$. This would follow from the pairing being skew-symmetric, so that the diagonal in the defining matrix is zero, and the anti-diagonal entries differ only by a sign.

In conclusion, we propose the following BSD-formula for the triple product p-adic L-function.

Conjecture 6.9. *Assume that the order of vanishing of the complex symmetric cube L-function $L(\mathrm{Sym}^3 f_A, s+1)$ at $s = 1$ is equal to one. Then, the order of vanishing of the p-adic symmetric cube L-function $L_p^{\mathrm{Sym}^3}(\boldsymbol{f})(k)$ at $k = 2$ is equal to two. Further, there exist $\mathbf{P} \in A(\mathbb{Q}) \otimes \mathbb{Q}$, $b \in \mathbb{Q}^\times$, $Q \in \mathrm{Sel}(\mathbb{Q}, V_p(\mathrm{Sym}^3 f_A))$ and $\omega \in \mathrm{H}^0(\mathbb{Q}_p, V_p(\mathrm{Sym}^3 f_A)^-)$ such that*

$$\frac{\mathrm{d}^6}{\mathrm{d}k^6} L_p^{\Delta}(\boldsymbol{f})(k)_{|k=2} = 90 b^2 \cdot \langle\!\langle Q, \omega\rangle\!\rangle^2 \cdot \log_A(\mathbf{P})^4.$$

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