Evolutionary Analysis of a Singular Minimal Surface Equation

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Abstract

Let $D \subset \mathbb{R}^n$ be a domain, $\Omega = D \times (0, T)$ with $0 < T \leq \infty$ and $0 < \phi_0(x), u_0(x) \in C^0(\overline{D})$. Our interest lies in the solvability of the initial boundary value problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad & \text{on } \Omega, \\ u(x,t) = \phi_0(x) \quad & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) \quad & \text{on } D \times \{0\}, \end{aligned} \tag{$P^{\gamma}_{\phi_0,u_0}$}$$

for $\gamma \in \mathbb{R} \setminus \{0\}$. By application of a fixed point argument we prove, that ($P^{\pm}_{\phi_0,u_0>c}$) is uniquely solvable, as long as ϕ_0, u_0 are chosen „sufficently large“ and the domain D has nonnegative inward mean curvature $H_D(y) \geq 0$ for all $y \in \partial D$. Hence, we derive the usual *a priori* estimates, which are required for the applicability of a fixed point argument.

Furthermore, we prove that in the case $\gamma < 0$ there is a solution, even if the initial and boundary values are chosen to be 0. Moreover we show, that if the solution $u(x,t)$ for any $\gamma \in \mathbb{R}$ is smooth enough, there is a subsequence of times t_k, for which $u(x, t_k)$ converges to a solution of the stationary problem

$$\begin{aligned} \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad & \text{on } D, \\ u(x) = \phi_0(x) \quad & \text{on } \partial D, \end{aligned}$$

for $t_k \to \infty$. Afterwards, we derive an *a priori* interior gradient bound for the case $\gamma < 0$, unrelated to the existence theory.

Finally, we investigate the case of „low initial and boundary values“ for $\gamma > 0$. We prove, that a singularity must occur after finite time, which implies that there is no classical solution for such initial and boundary values that exists for all times. However, we are able to prove, that there is a maximum value $\hat{T}$, such that the problem has a unique solution for all times $0 < T < \hat{T}$. Motivated by the elliptic case, we can prove under additional assumptions about the regularity of the solution $u(x,t)$ at time $\hat{T}$, that $u(\cdot, \hat{T})$ remains $\frac{1}{2}$-Hölder continuous.

Zusammenfassung

Seien $D \subset \mathbb{R}^n$ ein Gebiet, $\Omega = D \times (0,T)$ mit $0 < T \leq \infty$ und $0 < \phi_0(x), u_0(x) \in C^0(\overline{D})$. Unser Interesse gilt dem Rand-Anfangswertproblem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \quad & \text{in } \Omega, \\ u(x,t) = \phi_0(x) \quad & \text{auf } \partial D \times (0,T), \\ u(x,0) = u_0(x) \quad & \text{auf } D \times \{0\}, \end{aligned} \tag{$P^{\gamma}_{\phi_0,u_0}$}$$

für $\gamma \in \mathbb{R} \setminus \{0\}$. Wir zeigen durch Anwendung eines Fixpunktarguments, dass das Problem ($P^{\pm}_{\phi_0,u_0>c}$) für geeignet gewählte Anfangs- und Randwerte und Gebiete D mit nichtnegativer, nach innen gerichteter mittlerer Krümmung $H_D(y) \geq 0$ für alle $y \in \partial D$ stets eine eindeutige klassische Lösung besitzt. Zur Anwendbarkeit des Fixpunktarguments werden die üblichen *a priori* Schranken hergeleitet.

Im weiteren Verlauf zeigen wir, dass für $\gamma < 0$ auch dann noch eine Lösung existiert, wenn die Anfangs- und Randwerte auf 0 abfallen. Zudem wird bewiesen, dass es zu einer genügend glatten Lösung für beliebiges $\gamma \in \mathbb{R}$ eine Teilfolge von Zeiten t_k gibt, für welche $u(x, t_k)$ bei $t_k \to \infty$ gegen eine Lösung des stationären Problems

$$\begin{aligned} \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \quad & \text{in } D, \\ u(x) = \phi_0(x) \quad & \text{auf } \partial D, \end{aligned}$$

konvergiert. Anschließend leiten wir im Kontext der *a priori* Schranken eine *a priori* innere Gradientenschranke für den Fall $\gamma < 0$ her.

Schlussendlich untersuchen wir, was im Fall $\gamma > 0$ für „zu niedrig liegende Anfang- und Randwerte" geschieht. Wir zeigen, dass nach endlicher Zeit eine Singularität auftreten muss, sodass für diese Rand- und Anfangswerte keine klassische Lösung für alle Zeiten existiert. Außerdem beweisen wir, dass es in diesem Fall eine maximale Zeit $\hat{T}$ gibt, sodass das Problem für alle Zeiten $0 < T < \hat{T}$ eine eindeutige Lösung besitzt. Motiviert durch den elliptischen Fall können wir unter zusätzlichen Annahmen an die Regularität der Lösung zum Zeitpunkt $\hat{T}$ zeigen, dass sie bezüglich der x-Variablen $\frac{1}{2}$-Hölderstetig bleibt.

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Contents

1 Introduction

In geometric analysis, equations containing the mean curvature H have been of particular interest for several decades. Given a domain $D \subset \mathbb{R}^n$ and a smooth function $u\colon D \to \mathbb{R}, u \in C^2(D)$, we can express the graph's mean curvature $H(u)$ by

$$H(u) = D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right) = \operatorname{div} \left(\frac{Du}{\sqrt{1+|Du|^2}} \right),$$

where the graph of u is given by $\operatorname{graph}(u(x)) = (x, u(x)), x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and we use the convention to sum over repeated indices. The best-known equation containing the mean curvature is the minimal surface equation

$$H(u) = 0,$$

which is usually paired with a boundary condition, forming the Dirichlet problem

$$\begin{aligned} H(u) &= 0 \quad \text{on } D, \\ u &= \phi_0 \ \ \text{on } \partial D. \end{aligned}$$

Other well known and vastly studied equations related to the mean curvature are the „*Hanging Drop Problem*“

$$\begin{aligned} H(u) + \kappa u &= 0 \quad \text{on } D, \\ u &= \phi_0 \ \ \text{on } \partial D, \end{aligned}$$

where $\kappa > 0$ is a constant determined by the density of the liquid and the „*Hanging Roof Problem*“

$$\begin{aligned} \sqrt{1+|Du|^2} H(u) - \frac{1}{u} &= 0 \quad \text{on } D, \\ u &= \phi_0 \ \ \text{on } \partial D. \end{aligned}$$

Our interest lies in studying a generalization of the Hanging Roof Problem from an evolutionary point of view. More precisely, let $D \subset \mathbb{R}^n$ be a domain, $\gamma \in \mathbb{R}$ and set $v := \sqrt{1+|Du|^2}$. Consider the equation

$$H(u) = \frac{\gamma}{uv} \quad \text{on } D, \tag{1.1}$$

which is the Euler-Lagrange equation to the energy

$$\int_D u^\gamma v \, \mathrm{d}x. \tag{1.2}$$

Calculating the divergence in the definition of $H(u)$ and multiplying by v yields

$$\Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u}. \tag{1.3}$$

Denoting $\dot{u} = \frac{\mathrm{d}}{\mathrm{d}t} u$, the corresponding flow is then given by

$$-\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u}. \tag{PDE}$$

Our first goal is to prove existence and uniqueness of a classical solution for any choice of $\gamma \in \mathbb{R} \setminus \{0\}$, that attains prescribed boundary and initial values $\phi_0(x), u_0(x) > 0$, whose regularity will be specified later. Defining

$$\Omega := D \times (0,T), \quad \mathcal{S}\Omega := \partial D \times (0,T), \quad \mathcal{B}\Omega := D \times \{0\}, \quad \mathcal{C}\Omega := \partial D \times \{0\},$$

where $0 < T \leq \infty$, this leads to the initial boundary value problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } \Omega, \\ u(x,t) &= \phi_0(x) && \text{on } \mathcal{S}\Omega, \\ u(x,0) &= u_0(x) && \text{on } \mathcal{B}\Omega. \end{aligned} \tag{$P^{\gamma}_{\phi_0,u_0}$}$$

By a *classical* solution we understand a function $u = u(x,t)\colon D \times (0,T) \to \mathbb{R}$, that is twice differentiable in x, once in t and continuous up to the boundary of the domain, in short $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$. Note, that continuity up to the boundary can only be achieved, if the two functions ϕ_0, u_0 coincide in the corner $\mathcal{C}\Omega$. This type of restriction is called *compatibility condition* and for higher regularity up to the boundary, say $u \in H_{2+\alpha}(\Omega)$ (for a definition of the Hölder spaces $H_{2+\alpha}(\Omega)$ see Appendix, Sec 5.4), additional compatibility conditions have to be presupposed [cf. Appendix, Sec. 5.5].

To prove existence and uniqueness of a classical solution for ($P^{\gamma}_{\phi_0,u_0}$) we apply a fixed point theorem. To elaborate this topic further, let us discuss the general setting we are working with. Therefore, let $X = (x,t) \in \Omega, z \in \mathbb{R}, p \in \mathbb{R}^n$. Let further $(a^{ij})_{i,j=1,\ldots,n} = (a^{ij}(X,z,p))$ be a matrix-valued function and $a = a(X,z,p) \in \mathbb{R}$. Denoting $u_{ij} := D_{ij}u$, we say that an operator P given by

$$Pu = -\dot{u} + Eu := -\dot{u} + a^{ij}(X,z,p)u_{ij} + a(X,z,p)$$

is parabolic in a subset $\mathcal{A} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, if

$$0 < \lambda(X,z,p)|\xi|^2 \leq a^{ij}(X,z,p)\xi_i\xi_j \leq \Lambda(X,z,p)|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n. \tag{1.4}$$

The values λ and Λ may be taken as the smallest and largest eigenvalue of the matrix

(a^{ij}) respectively and we call E the elliptic operator associated with P. Furthermore, we call P parabolic in u, if (1.4) holds for $z = u(X)$ and $p = Du(X)$.
Now assume the equation

$$Pu = -\dot{u} + a^{ij}(X, u(X), Du(X))u_{ij} + a(X, u(X), Du(X)) = 0.$$

To find a solution to this equation, we study the linear problem

$$-\dot{u} + a^{ij}(X, w(X), Dw(X))u_{ij} + a(X, w(X), Dw(X)) = 0 \tag{1.5}$$

for some given function $w(X)$ and the unknown $u(X)$. By defining an operator $Q(w) = u$ if and only if u is a solution to (1.5), we then have to prove existence of a fixed point $\hat{u}$ of the operator Q, that is the existence of a function that fulfills $Q(\hat{u}) = \hat{u}$. Approaching the problem with this idea has the advantage that we can use known results about linear parabolic equations.

One well-known fixed point theorem is that of Leray-Schauder, see for example the book by Ladyzenskaya, Solonnikov, Ural'ceva [LSU88, p. 450] or by Gilbarg-Trudinger [GT01, p. 286-288] for an application in the parabolic and elliptic case respectively. However, we will apply a different version, the Schauder fixed point theorem, that can be found in Lieberman [Lie96, p. 205-208]. It is used in combination with the a priori estimates to establish the existence of a solution on a domain $\Omega_\epsilon := D \times [0, \epsilon)$ that is possibly very small in time direction. If additionally the a priori estimates hold on all of $\Omega = D \times (0, T), 0 < T \leq \infty$, then Arzela-Ascoli's theorem can be used to establish long time existence.

Hence, the solvability of the initial boundary value problem $(P^{\gamma}_{\phi_0,u_0})$ is reduced to the derivation of a priori bounds for any classical solution $u \in C^{2,1}(\Omega)$, which is split into four parts.

i) Show that $u(x, t)$ is bounded.

ii) Establish a bound for $Du(x, t)$ on the boundary $\mathcal{P}\Omega := \mathcal{S}\Omega \cup \mathcal{B}\Omega \cup \mathcal{C}\Omega$.

iii) Establish a bound for $Du(x, t)$ on all of Ω.

iv) Derive Hölder-Norm bounds for $Du(x, t)$ on Ω.

As it turns out, the derivation of a priori estimates for the case $\gamma > 0$ is different from the case $\gamma < 0$. Therefore, we split $(P^{\gamma}_{\phi_0,u_0})$ into the categories $\gamma > 0$ and $\gamma < 0$ denoted by $(P^{+}_{\phi_0,u_0})$ and $(P^{-}_{\phi_0,u_0})$ respectively, neglecting the case $\gamma = 0$, that has already been studied extensively (see for example related papers from Huisken or the book by Ecker [Eck04]). We derive a priori estimates for $(P^{+}_{\phi_0,u_0}), (P^{-}_{\phi_0,u_0})$ in chapter 2.

If certain conditions for the domain Ω and the initial and boundary values $\phi_0(x), u_0(x)$ are met, we are able to derive an existence and uniqueness theory that is based on the aforementioned fixed point argument. For instance, it is crucial for long time existence that the solution starts at a sufficient height and it will be necessary for the derivation of a boundary gradient estimate, that the domain D in the definition of $\Omega = D \times (0,T)$ has non-negative inward mean curvature. Once existence is established, we prove, by using results from FRIEDMAN [Fri83, p. 71-75], that every classical solution $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ is already arbitrarily smooth in Ω with respect to all its variables, denoted by $u(x,t) \in C^\infty(\Omega) \cap C(\overline{\Omega})$. Existence, uniqueness and regularity will be proven in chapter 3.

Compared to previously achieved results (see for example STONE [Sto94]) our existence theory covers all cases $\gamma \in \mathbb{R} \setminus \{0\}$ as opposed to only $\gamma = 1$ and we are also able to relax the condition $H_D(y) \geq c$ for a positive constant $c > 0$ to $H_D(y) \geq 0$.

Once we have shown existence of a solution to the problem $(P^-_{\phi_0,u_0>0})$, we can employ an approximation device to solve the problem (P^-_0) given by

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} \quad \text{on } \Omega, \\ u(x,t) &= 0 \quad \text{on } \mathcal{P}\Omega. \end{aligned} \tag{P^-_0}$$

We consider for some (small) $\epsilon > 0$ the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} \quad \text{on } \Omega, \\ u(x,t) &= \epsilon \quad \text{on } \mathcal{P}\Omega, \end{aligned} \tag{$P^-_{0+\epsilon}$}$$

which has a unique solution due to the existence theory derived in chapters 2 and 3. We prove in chapter 4, that the solution $u^\epsilon(x,t)$ to problem $(P^-_{0+\epsilon})$ converges uniformly to a solution $u(x,t)$ for (P^-_0), which moreover attains the regularity $u(x,t) \in C^\infty(\Omega) \cap C(\overline{\Omega})$. Another interesting question that will be addressed in chapter 4 regards the convergence of a solution $u(x,t)$ of $(P^\gamma_{\phi_0,u_0})$ as $t \to \infty$ to a solution of the stationary boundary value problem

$$\begin{aligned} \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} \qquad \text{on } D, \\ u(x) &= \phi_0(x) \quad \text{on } \partial D. \end{aligned} \tag{$P^S_{\phi_0}$}$$

It will be proven, that whenever Ω, ϕ_0, u_0 are smooth enough for the solution $u(x,t)$ to be in $H_{2+\alpha}(\Omega)$, there is a subsequence t_k of t such that $u(x,t_k)$ converges uniformly to a classical solution $u(x)$ of the stationary boundary value problem $(P^S_{\phi_0})$.
Unrelated to the existence theory we prove in chapter 4 an a priori interior gradient estimate by applying a method that is due to KOREVAAR [Kor86].
Finally, we analyze the case of „low initial and boundary values“, when $\gamma > 0$. We prove, that there cannot be a solution that exists for all times $T > 0$ and moreover, that there is a „largest time“ $\hat{T}$, so that the problem $(P^+_{\phi_0,u_0<\hat{c}})$ has a solution for all

times $0 < t < \hat{T}$. Under additional assumptions about the regularity of the solution at time T_0 we can prove, motivated by the elliptic case, that $u(\cdot, \hat{T})$ remains $\frac{1}{2}$-Hölder continuous.

2 A priori Estimates

In this chapter we derive the a priori estimates necessary for the application of the fixed point theorem, starting with

2.1 Upper and lower bounds for $u(x,t)$

While estimates for the gradient $|Du|$ are independent of the choice of γ, the same is not true for $|u(x,t)|$. However, the lower bounds for $|u|$ are crucial for us to prove the existence of a gradient bound. Thus, we will investigate the two cases $(P^+_{\phi_0,u_0>c})$ and $(P^-_{\phi_0,u_0>c})$ separately. As it turns out, the constant c for negative γ can be taken as 0, whereas the constant c for positive γ has to be a larger value, depending on the diameter of D, γ and the dimension n.

We begin by analyzing the problem $(P^-_{\phi_0,u_0>0})$, that is

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } \Omega, \\ u(x,t) &= \phi_0(x) > 0 && \text{on } \mathcal{S}\Omega, \\ u(x,0) &= u_0(x) > 0 && \text{on } \mathcal{B}\Omega, \end{aligned} \tag{$P^-_{\phi_0,u_0>0}$}$$

with $\gamma < 0$.

2.1.1 Boundedness of $u(x,t)$ for $(P^-_{\phi_0,u_0>0})$

To prove that any solution $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ to $(P^-_{\phi_0,u_0>0})$ is uniformly bounded on Ω, we apply the comparison principle [cf. Appendix, Sec. 5.3]. The comparison functions are obtained from the corresponding elliptic boundary value problem on ball shaped domains, cf. Dierkes [Die19].

Lemma 2.1 (Foliation of $\mathbb{R}^n \times \mathbb{R}^+$) [Die19, Thm. 2.1] *Let $n \geq 2$ and $\gamma < 0$. There exists a foliation of $\mathbb{R}^n \times \mathbb{R}^+$ determined by concave rotational symmetric functions $v_\lambda = v_\lambda(x) \colon B_\lambda(0) \to \mathbb{R}^+, \lambda > 0$ arbitrary, $v_\lambda(0) = \lambda \cdot v_1(0)$ and $Dv_\lambda(0) = 0$. Furthermore,* *for each* $\lambda > 0$ *the functions* $v_\lambda \in C^\omega(B_\lambda(0)) \cap C^0(\overline{B_\lambda}(0))$, *solve the Dirichlet problem*

$$\begin{aligned} \operatorname{div}\left(\frac{Dv_\lambda}{\sqrt{1+|Dv_\lambda|^2}}\right) &= \frac{\gamma}{v_\lambda\sqrt{1+|Dv_\lambda|^2}} && \text{on } B_\lambda(0), \\ v_\lambda &= 0 && \text{on } \partial B_\lambda(0). \end{aligned} \tag{2.1}$$

□

These solutions possess two important properties that make them ideal comparison functions. On the one hand, every solution $v_\lambda(x)$ to equation (2.1) is also a time independent solution to ($P^-_{\phi_0,u_0>0}$) on ball-shaped domains, since $\frac{\partial v_\lambda}{\partial t} \equiv 0$. Hence, v_λ solves the initial boundary value problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } B_\lambda(0) \times (0,T), \\ u(x,t) &= 0 && \text{on } \partial B_\lambda(0) \times [0,T), \\ u(x,0) = u_0(x) &= u(x) && \text{on } B_\lambda(0) \times \{0\}. \end{aligned} \tag{P_B}$$

On the other hand, the symmetry and concavity imply the estimates

$$\begin{aligned} & v_\lambda(\theta x + (1-\theta)y) \geq \theta v_\lambda(x) + (1-\theta) v_\lambda(y), && \theta \in [0,1], \\ \overset{y=-x,|y|=|x|,\theta=\frac{1}{2}}{\Rightarrow} \quad & v_\lambda(0) \geq v_\lambda(x), && \theta \in [0,1] \end{aligned}$$

and

$$\begin{aligned} & v_\lambda(\theta x + (1-\theta)y) \geq \theta v_\lambda(x) + (1-\theta) v_\lambda(y), && \theta \in [0,1], \\ \overset{x=0,|y|=\lambda}{\Rightarrow} \quad & v_\lambda((1-\theta)y) \geq \theta v_\lambda(0), && \theta \in [0,1]. \end{aligned}$$

The first estimate implies $0 \leq v_\lambda(x) \leq v_\lambda(0)$ for every $x \in B_\lambda(0)$. If additionally $0 \leq |x| < \frac{\lambda}{2}$, it is $v_\lambda(x) \in \left[\frac{1}{2}v_\lambda(0), v_\lambda(0)\right]$. Hence, by using the property $v_\lambda(0) = \lambda v_1(0)$ we can achieve, that the solution $v_\lambda(x)$ of (2.1) becomes arbitrarily small on all of $B_\lambda(0)$ and arbitrarily large on $B_{\frac{\lambda}{2}}(0)$ by adjusting the value for λ accordingly. This observation leads to

Proposition 2.2 (A priori estimates for $u(x,t)$**)** *Let* $\Omega = D \times (0,T)$ *and let* $\phi_0(x), u_0(x) \in C^0(\overline{D})$ *with* $u_0(x) = \phi_0(x)$ *on* $\mathcal{C}\Omega$. *Let further* $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ *be a solution to* ($P^-_{\phi_0,u_0>0}$). *Then there exist positive constants* c_1^- *and* C_1^- *with*

$$0 < c_1^-(u_0,\phi_0,\Omega,\gamma,n) \leq u(x,t) \leq C_1^-(u_0,\phi_0,\Omega,\gamma,n) < \infty \tag{2.2}$$

for every $(x,t) \in \Omega$. □

Proof. We use the solutions $v_\lambda(x,t) \equiv v_\lambda(x)$ for all $t \in [0,T)$ as comparison function to deduce the a priori estimates via the comparison principle. Starting with the upper bound, we choose $\lambda > 0$ big enough and use the scaling property of $v_\lambda(x)$ to guarantee that

$$\begin{aligned} \overline{D} &\subset B_{\frac{\lambda}{2}}(0), \\ v_\lambda(x) &\geq \phi_0(x) \quad \text{ on } \mathcal{S}\Omega, \\ v_\lambda(x) &\geq u_0(x) \quad \text{ on } \mathcal{B}\Omega. \end{aligned}$$

This is always possible since $u_0, \phi_0 \in C^0(\overline{D})$ paired with $u_0, \phi_0 > 0$ implies the existence of constants $c_{\min}, c_{\max}$ with $0 < c_{\min} \leq u_0, \phi_0 \leq c_{\max} < \infty$ and $v_\lambda(x) \overset{\lambda\to\infty}{\to} \infty$ for every $x \in \overline{D} \subset B_{\frac{\lambda}{2}}(0)$. Let $\overline{\lambda}$ be a λ that fulfills these three properties. Let P be the parabolic operator given by

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u}.$$

The function $v_{\overline{\lambda}}$ then suffices

$$\begin{aligned} Pv_{\overline{\lambda}}(x,t) = 0 = Pu(x,t) \quad &\text{on } \Omega, \\ v_{\overline{\lambda}}(x,t) \geq u(x,t) \quad &\text{on } \mathcal{P}\Omega. \end{aligned}$$

Application of the comparison principle [cf. Appendix, Sec. 5.3] yields

$$u(x,t) \leq v_{\overline{\lambda}}(x) \leq v_{\overline{\lambda}}(0) = \overline{\lambda} v_1(0) = C_1^-(u_0, \phi_0, \Omega, \gamma, n).$$

To obtain the lower a priori estimate we argue in a similar fashion by covering the set $\overline{D}$ with balls $B_{\frac{\lambda_i}{2}}(x_i), \lambda_i \in \mathbb{R}^+, x_i \in \overline{D}$ and using the comparison principle on each ball. Since $\overline{D}$ is compact, we deduce the existence of a finite subcover. From this we can define a function $\eta(x)$ that is bounded below by a positive constant, which simultaneously is a lower bound for $u(x,t)$.

Therefore, let $x_0 \in \overline{D}$ be an arbitrary point. Observe that the equation

$$-\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \quad \text{on } \Omega \tag{2.3}$$

does not explicitly depend on $X = (x,t)$. Hence, we can translate the solution $v_\lambda(x)$, which solves (2.1) on $B_\lambda(0)$, by x_0 to obtain a solution on $B_\lambda(x_0)$. In other words the function $v_\lambda(x - x_0)$ is a solution to

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \quad &\text{on } B_\lambda(x_0) \times (0,T), \\ u(x,t) = 0 \quad &\text{on } \partial B_\lambda(x_0) \times [0,T), \\ u(x,0) = u_0(x) = u(x) \quad &\text{on } B_\lambda(x_0) \times \{0\}, \end{aligned} \tag{$P_{B_\lambda(x_0)}$}$$

with the same scaling properties as $v_\lambda(x)$. Now consider the set $D \cap B_\lambda(x_0)$ with boundary parts $\partial D \cap B_\lambda(x_0)$ and $D \cap \partial B_\lambda(x_0)$. The scaling properties of $v_\lambda(x - x_0)$ combined with the estimates

$$\begin{aligned} 0 < c_{\min} \leq \phi_0(x), u_0(x) \quad & \text{on } \overline{D}, \\ v_\lambda(x - x_0) = 0 < u(x,t) \quad & \text{on } (D \cap \partial B_\lambda(x_0)) \times [0,T). \end{aligned}$$

guarantee the existence of a (small) $\lambda > 0$ with

$$\begin{aligned} v_\lambda(x - x_0) &\leq u(x,t) \quad \text{on } (D \cap \partial B_\lambda(x_0)) \times [0,T), \\ v_\lambda(x - x_0) &\leq \phi_0(x) \quad \text{on } (\partial D \cap B_\lambda(x_0)) \times [0,T), \\ v_\lambda(x - x_0) &\leq u_0(x) \quad \text{on } (D \cap B_\lambda(x_0)) \times \{0\}. \end{aligned} \tag{2.4}$$

Let $\underline{\lambda}$ be a λ that fulfills these conditions. Combining the estimates in (2.4) leads to

$$\begin{aligned} Pv_{\underline{\lambda}}(x - x_0, t) = 0 = Pu(x,t) \quad & \text{on } (D \cap B_{\underline{\lambda}}(x_0)) \times [0,T), \\ v_{\underline{\lambda}}(x - x_0, t) \leq u(x,t) \quad & \text{on } \mathcal{P}((D \cap B_{\underline{\lambda}}(x_0)) \times [0,T)). \end{aligned}$$

Application of the comparison principle yields

$$v_{\underline{\lambda}}(x - x_0, t) \leq u(x,t) \quad \text{on } (\overline{D} \cap B_{\underline{\lambda}}(x_0)) \times [0,T), \tag{2.5}$$

which implies that the same estimate is true on the set $(\overline{D} \cap B_{\underline{\lambda}/2}(x_0)) \times [0,T)$. Note, that we restrict our attention to balls with radius $\underline{\lambda}/2$, since on this set the solution $v_{\underline{\lambda}}(x - x_0, t)$ fulfills the estimate

$$v_{\underline{\lambda}}(x - x_0, t) \geq \frac{1}{2} v_{\underline{\lambda}}(0,t) = c(\underline{\lambda}, v_1(0)) > 0$$

for a fixed choice of $\underline{\lambda}$. If we repeat this process for every point $x \in \overline{D}$, we obtain a cover of $\overline{D}$ with open balls, which, by compactness of $\overline{D}$, has a finite subcover $(B_{\frac{\lambda_i}{2}}(x_i))_{i \in \{1,\ldots,k\}}$. On each ball there exists a function $v_{\lambda_i}(x - x_i, t)$ with

$$\frac{1}{2} v_{\lambda_i}(0,t) \leq v_{\lambda_i}(x - x_i, t) \leq u(x,t) \quad \text{on } (\overline{D} \cap B_{\frac{\lambda_i}{2}}(x_i)) \times [0,T).$$

We continue each function $v_{\lambda_i}(x - x_i, t)$ by 0 on $\overline{\Omega} \setminus ((\overline{D} \cap B_{\lambda_i}(x_i)) \times [0,T))$ and denote this continuation by $\hat{v}_{\lambda_i}$. Now define

$$\eta(x) \equiv \eta(x,t) := \max_{i \in \{1,\ldots,k\}} \hat{v}_{\lambda_i}(x - x_i, t) \equiv \max_{i \in \{1,\ldots,k\}} \hat{v}_{\lambda_i}(x - x_i)$$

Then we set

$$\min_{x\in\overline{D}} \eta(x) =: c_1^-(u_0, \phi_0, \Omega, \gamma, n)$$

and by construction it is clear that

$$0 < c_1^-(u_0, \phi_0, \Omega, \gamma, n) \le \eta(x,t) \le u(x,t) \quad \text{on } \Omega,$$

which is the desired lower a priori bound. □

In the upcoming section we study the case $\gamma > 0$, which is given by

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } \Omega, \\ u(x,t) &= \phi_0(x) > c && \text{on } \mathcal{S}\Omega, \\ u(x,0) &= u_0(x) > c && \text{on } \mathcal{B}\Omega, \end{aligned} \tag{$P^+_{\phi_0,u_0>c}$}$$

with a constant $c \ge 0$ that has to be further specified.

2.1.2 Boundedness of $u(x,t)$ for ($P^+_{\phi_0,u_0>c}$)

If we compare the case ($P^+_{\phi_0,u_0>c}$) with ($P^-_{\phi_0,u_0>0}$) we neither can use the same comparison principle because $a(z) = -\frac{\gamma}{z}$ is no longer non-increasing in z, nor do we have the comparison functions from the foliation lemma. Nonetheless, it is possible to show the existence of upper and lower bounds for solutions to ($P^+_{\phi_0,u_0>c}$) by application of a different comparison principle, see also Stone [Sto94, p. 171, Lemma 4.1.].

Proposition 2.3 (A priori estimates for $u(x,t)$) *Let $\Omega = D \times (0,T)$, $d := \operatorname{diam}(D)$ and let $\phi_0(x), u_0(x) \in C^0(\overline{D})$ with $\phi_0(x) = u_0(x)$ on $\mathcal{C}\Omega$. Let further $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ be a solution to ($P^+_{\phi_0,u_0>c}$), where the constant c in ($P^+_{\phi_0,u_0>c}$) is now chosen so that*

$$\begin{aligned} c &> 2d\sqrt{\frac{\gamma}{2n-\gamma}}, && \text{if } \gamma \in (0,1], \\ c &> \frac{2d\gamma}{\sqrt{2n-1}}, && \text{if } \gamma > 1. \end{aligned} \tag{2.6}$$

Then there exist positive constants c_1^+ and C_1^+ with

$$0 < c_1^+(d,\gamma,n) \le u(x,t) \le C_1^+(u_0,\phi_0) < \infty \tag{2.7}$$

for every $(x,t) \in \Omega$. □

Proof. We begin with the upper bound for either choice of $\gamma > 0$. Let $\epsilon > 0$ be arbitrary and small, let P be the parabolic operator associated with ($P^+_{\phi_0,u_0>c}$), that is

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u} = -\dot{u} + vH(u) - \frac{\gamma}{u},$$

and let $v(x,t)$ be the constant function $v(x,t) = c_{\max} + \epsilon$, where as in the previous proof $c_{\max}$ is a constant which suffices $u_0, \phi_0 \leq c_{\max} < \infty$. Then for any $\gamma > 0$

$$\begin{aligned} Pv(x,t) &= -\frac{\gamma}{c_{\max}+\epsilon} < 0 = Pu(x,t) \quad &&\text{on } \Omega, \\ v(x,t) &> u(x,t) &&\text{on } \mathcal{P}\Omega. \end{aligned}$$

Hence, the requirements for the weak comparison principle [cf. Appendix, Sec. 5.3] are fulfilled and its application yields $u(x,t) < c_{\max} + \epsilon =: C_1^+$, which is the desired upper bound.

For the lower bound we first study the case $\gamma \in (0,1]$. Define

$$r_1 = \frac{1}{c}\left(d^2 + \left(\frac{c}{2}\right)^2\right) \geq d$$

as the radius of a spherical cap given below. Let $x_0 \in D$ be an arbitrary point and define the spherical cap $\theta_1(x,t) \equiv \theta_1(x)$ by

$$\theta_1(x) := r_1 + \frac{c}{2} - \sqrt{r_1^2 - |x-x_0|^2}.$$

For the application of the weak comparison principle we have to calculate $P\theta_1$. It is well-known that the mean curvature of an n-dimensional sphere with radius r is given by $\frac{n}{r}$. However, since we also need the expression $\sqrt{1+|D\theta_1|^2}$, we may as well calculate $H(\theta_1)$. Letting $x_0 = (x_1^0, \ldots, x_n^0)$ we have

$$D_i\theta_1 = \frac{x_i - x_i^0}{\sqrt{r_1^2 - |x-x_0|^2}}, \quad |D\theta_1|^2 = \frac{|x-x_0|^2}{r_1^2 - |x-x_0|^2}, \quad 1+|D\theta_1|^2 = \frac{r_1^2}{r_1^2 - |x-x_0|^2},$$

so that

$$H(\theta_1) = D_i\left(\frac{x_i - x_i^0}{\sqrt{r_1^2 - |x-x_0|^2}} \cdot \frac{\sqrt{r_1^2 - |x-x_0|}}{r_1}\right) = D_i\left(\frac{x_i - x_i^0}{r_1}\right) = \frac{n}{r_1}.$$

We must have

$$P\theta_1 > Pu = 0 \quad \text{on } \Omega.$$

Inserting the terms calculated above yields

$$\begin{aligned} P\theta_1 &= -\dot{\theta}_1 + \sqrt{1+|D\theta_1|^2} H(\theta_1) - \frac{\gamma}{\theta_1} \\ &= \frac{r_1}{\sqrt{r_1^2 - |x-x_0|^2}} \cdot \frac{n}{r_1} - \gamma \cdot \frac{1}{r_1 + \frac{c}{2} - \sqrt{r_1^2 - |x-x_0|^2}}, \end{aligned}$$

thus, the requirement $P\theta_1 > 0$ is equivalent to

$$\sqrt{r_1^2 - |x-x_0|^2} < \frac{n}{\gamma+n}\left(r_1 + \frac{c}{2}\right).$$

This in return is always fulfilled if

$$r_1 < \frac{n}{\gamma+n}\left(r_1 + \frac{c}{2}\right) \Leftrightarrow r_1 < \frac{n}{2\gamma} \cdot c. \tag{2.8}$$

At this point we use the definition of $r_1 = \frac{1}{c}\left(d^2 + \left(\frac{c}{2}\right)^2\right)$ to conclude

$$c > 2d\sqrt{\frac{\gamma}{2n-\gamma}},$$

which is the restriction we have chosen for c in (2.6) for the case $\gamma \in (0,1]$. Moreover, we must have on the boundary

$$\theta_1(x,t) < u(x,t) \quad \text{on } \mathcal{P}\Omega.$$

To achieve this we prove $\frac{c}{2} \leq \theta_1 \leq c$ so that on the one hand θ_1 is strictly positive everywhere and on the other hand $\theta_1(x,t) < u(x,t)$ on $\mathcal{P}\Omega$. The relation $\frac{c}{2} \leq \theta_1$ is obvious since $0 \leq \sqrt{r_1^2 - |x-x_0|^2} \leq r_1$. The relation $\theta_1 \leq c$ is equivalent to

$$r_1 - \frac{c}{2} - \sqrt{r_1^2 - |x-x_0|^2} \leq 0,$$

so that we either need to require

$$r_1 \leq \frac{c}{2} \Leftrightarrow d^2 + \frac{c^2}{4} \leq \frac{c^2}{2} \Leftrightarrow c \geq 2d$$

or, in case $2d\sqrt{\frac{\gamma}{2n-\gamma}} \leq c \leq 2d$

$$\left(r_1 - \frac{c}{2}\right)^2 \leq r_1^2 - |x-x_0|^2.$$

This inequality is always fulfilled if

$$r_1^2 - r_1 c + \frac{c^2}{4} \leq r_1^2 - d^2 \Leftrightarrow \frac{1}{c}\left(d^2 + \frac{c^2}{4}\right) \leq r_1,$$

which is obviously true from the definition of r_1. Hence, we may apply the weak comparison principle [cf. Appendix, Sec. 5.3], yielding the desired lower bound

$$c_1^+ := \frac{c}{2} \leq \theta_1 < u(x,t) \quad \text{on } \Omega.$$

We see from this choice for c that we cannot simply use the same argument for any $\gamma > 0$ because it does not yield any results for $\gamma \geq 2n$ and the results for γ close to $2n$ become increasingly worse. Instead, we modify the radius of the spherical cap defined above in a suitable way to obtain a lower bound for the case $\gamma > 1$. Therefore, let $\epsilon \geq 1$ be a constant at our disposal and set

$$r_\epsilon = \frac{1}{c}\left((d\epsilon)^2 + \left(\frac{c}{2\epsilon}\right)^2\right) \geq d.$$

We define the spherical cap θ_ϵ by

$$\theta_\epsilon := r_\epsilon + \frac{c}{2} - \sqrt{r_\epsilon^2 - |x - x_0|^2}$$

and observe that this time $\frac{c}{2} \leq \theta_\epsilon \leq \frac{c}{2} + \frac{c}{2\epsilon^2} \leq c$ for $\epsilon \geq 1$. Obviously, $\frac{c}{2} \leq \theta_\epsilon$ for the same reason as in the first case. The inequality $\theta_\epsilon \leq \frac{c}{2} + \frac{c}{2\epsilon^2}$ is equivalent to

$$r_\epsilon - \frac{c}{2\epsilon^2} - \sqrt{r_\epsilon^2 - |x - x_0|^2} \leq 0,$$

so that we either need to require

$$r_\epsilon \leq \frac{c}{2\epsilon^2} \Leftrightarrow d^2\epsilon^2 + \frac{c^2}{4\epsilon^2} \leq \frac{c^2}{2\epsilon^2} \Leftrightarrow c \geq 2d\epsilon^2,$$

or, in case $\frac{2d\gamma}{\sqrt{2n-1}} \leq c \leq 2d\epsilon^2$

$$\left(r_\epsilon - \frac{c}{2\epsilon^2}\right)^2 \leq r_\epsilon^2 - |x - x_0|^2.$$

This inequality is always fulfilled if

$$r_\epsilon^2 - \frac{c}{\epsilon^2} r_\epsilon + \frac{c^2}{4\epsilon^4} \leq r_\epsilon^2 - d^2 \Leftrightarrow \frac{1}{c}\left(d^2\epsilon^2 + \frac{c^2}{4\epsilon^2}\right) \leq r_\epsilon,$$

which is obviously true by the definition of r_ϵ. Thus, we have $\theta_\epsilon < u(x,t)$ on $\mathcal{P}\Omega$. To show $P\theta_\epsilon > Pu = 0$ on Ω we can copy the steps from above until (2.8), where we now have

$$r_\epsilon < \frac{n}{2\gamma} \cdot c.$$

Inserting the definition of $r_\epsilon = \frac{1}{c}\left((d\epsilon)^2 + \left(\frac{c}{2\epsilon}\right)^2\right)$ then yields

$$\left((d\epsilon)^2 + \left(\frac{c}{2\epsilon}\right)^2\right) < \frac{n}{2\gamma} \cdot c^2 \Leftrightarrow d^2\epsilon^2 < c^2 \cdot \left(\frac{n}{2\gamma} - \frac{1}{4\epsilon^2}\right)$$

so that for the choice $\epsilon^2 = \gamma > 1$ we obtain

$$c > \frac{2d\gamma}{\sqrt{2n-1}},$$

which is the restriction we have chosen for c in (2.6) for the case $\gamma > 1$. Hence, for this choice of c the weak comparison principle yields

$$c_1^+ = \frac{c}{2} \leq \theta_\epsilon < u(x,t) \quad \text{on } \Omega,$$

which completes the proof. □

Remark. Note, that the bounds found in this chapter are not extendable for arbitrary data ϕ_0, u_0. In fact, DIERKES and HUISKEN have shown [DH90] that in the elliptic case for $\gamma = 1$ there is no (classical) solution to

$$\begin{aligned} \Delta u - \frac{u_i u_j}{1+|Du|^2} u_{ij} &= \frac{1}{u} \quad \text{on } D, \\ u &= \phi_0 \quad \text{on } \partial D, \end{aligned}$$

if we assume that

$$\sup_{\partial D} |\phi_0| < \frac{|D|}{\mathcal{H}_{n-1}(\partial D)},$$

where $|D|$ denotes the Lebesgue measure of D and $\mathcal{H}_n$ denotes the n-dimensional Hausdorff measure. In chapter 4 we will prove, that a similar result is true in the parabolic case. Hence, we restrict our analysis to the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } \Omega, \\ u(x,t) &= \phi_0(x) > c && \text{on } \mathcal{S}\Omega, \\ u(x,0) &= u_0(x) > c && \text{on } \mathcal{B}\Omega, \end{aligned} \tag{$P^\pm_{\phi_0,u_0>c}$}$$

where c is chosen large enough to fulfill the conditions defined above.

2.2 Boundary gradient estimates

In this step we prove bounds for the gradient Du on the parabolic boundary $\mathcal{P}\Omega$ of the domain Ω. Since we have

$$u(x,0) = u_0(x) \quad \text{on } \mathcal{B}\Omega,$$

we immediately obtain $Du(x,0) = Du_0(x)$ so that $|Du|$ is bounded on $\mathcal{B}\Omega$ as long as $|Du_0(x)|$ is bounded. Hence, it remains to prove the boundedness of $|Du|$ on $\mathcal{S}\Omega$.

Since the operator P given by

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u}$$

is not necessarily uniformly parabolic, we need to state structure conditions for the domain Ω, namely that the spatial boundary $\mathcal{S}\Omega$ is (inward) mean-convex and exhibits at least C^2-regularity to assure that the spatial distance function $d(x)$ is twice differentiable [cf. Appendix, Sec. 5.6].

To achieve a bound for $|Du|$ on $\mathcal{S}\Omega$ we use methods from Lieberman [Lie96, chapter 10], which are tied closely to those in Gilbarg-Trudinger [GT01, chapter 14] for the elliptic case. The general idea is to show the boundedness of the expression

$$[u]'_1 := \sup_{X\in\mathcal{S}\Omega, Y\in\Omega, s\le t} \frac{|u(X)-u(Y)|}{|X-Y|},$$

where $X=(x,t), Y=(y,s)$, by using suitable comparison functions and applying the comparison principle. Note that if $Du \in C^0(\overline{\Omega})$ (which implies the total differentiability of Du), the relation $|Du| \le [u]'_1$ holds, for if $(Y_n)_{n\in\mathbb{N}}$ with $Y_n = (y_n, s_n)$ is an arbitrary sequence in Ω with $\lim_{n\to\infty, s_n\le t} Y_n = X = (x,t)$, we have

$$|Du(X)| = \lim_{n\to\infty} \frac{|u(X)-u(Y_n)|}{|X-Y_n|} \le \sup_{X\in\mathcal{S}\Omega, Y\in\Omega, s\le t} \frac{|u(X)-u(Y)|}{|X-Y|} = [u]'_1.$$

For the application of the comparison principle we define an auxiliary operator $\overline{P}$ by

$$\overline{P}v := -\dot{v} + a^{ij}(X, v(X), Dv(X))v_{ij} + a(X, u(X), Du(X)),$$

where $u \in C^{2,1}(\Omega)\cap C^0(\overline{\Omega})$ is a solution to $(P^{\pm}_{\phi_0, u_0>c})$ and v is an arbitrary function in $C^{2,1}(\Omega)\cap C^0(\overline{\Omega})$. Obviously $\overline{P}u \equiv Pu$ but this particular choice of $\overline{P}$ has an advantage when estimating a. Next we look at a parabolic neighborhood N of an arbitrary point $X_0 = (x_0, t_0) \in \mathcal{S}\Omega$. Assume that there is a positive constant R_0 for which

$$Q(X_0, R_0) := \{(x,t)\in\mathbb{R}^{n+1} \mid |x-x_0|^2 \le R_0^2, t_0 - R_0^2 \le t \le t_0\} \subset N.$$

Furthermore, set $M := \sup_{X \in \overline{\Omega}} |u - \phi_0|$ where $\phi_0(X) = \phi_0(x,t) \equiv \phi_0(x)$ $\forall t \in [0,T[$ is the prescribed boundary data on $\mathcal{S}\Omega$.
Then we search for functions $w^\pm \in C^0(\overline{N \cap \Omega}) \cap C^{2,1}(N \cap \Omega)$ that fulfill

$$\pm\overline{P}w^\pm \leq 0 \quad \text{on } N \cap \Omega, \tag{2.9a}$$

$$w^+ \geq \phi_0 = u \geq w^- \quad \text{on } N \cap \mathcal{P}\Omega, \tag{2.9b}$$

$$w^+ \geq u \geq w^- \quad \text{on } \mathcal{P}N \cap \Omega, \tag{2.9c}$$

$$w^\pm(X_0) = \phi_0(X_0). \tag{2.9d}$$

If such functions exist we can use the conditions a),b) and c) to apply the comparison principle and deduce

$$w^+ \geq u \geq w^- \quad \text{on } N \cap \Omega.$$

Making use of d) we obtain the inequalities

$$\frac{w^-(Y) - w^-(X_0)}{|Y - X_0|} \leq \frac{u(Y) - u(X_0)}{|Y - X_0|} \leq \frac{w^+(Y) - w^+(X_0)}{|Y - X_0|} \quad \text{on } N \cap \Omega.$$

If there are constants $L^\pm$ with

$$\frac{w^+(Y) - w^+(X_0)}{|Y - X_0|} \leq L^+, \frac{w^-(X_0) - w^-(Y)}{|Y - X_0|} \leq L^-, \tag{2.10}$$

for all $Y \in Q(X_0, R_0)$ then we can use the bounds $L^\pm$ inside $Q(X_0, R_0)$ and $\frac{M}{R_0}$ outside of $Q(X_0, R_0)$ respectively to deduce

$$[u]'_{1;X_0} := \sup_{Y \in \Omega, s \leq t_0} \frac{|u(X_0) - u(Y)|}{|X_0 - Y|} \leq \max\left\{L^+, L^-, \frac{M}{R_0}\right\}.$$

If $L^\pm, R_0$ can be chosen independently of X_0, this estimate yields an a priori bound for $[u]'_1$. With this method we are able to prove the following

Proposition 2.4 (A priori estimate for $|Du|$ on the boundary) *Let $\Omega = D \times (0,T)$ be a domain with C^2-boundary, where D has non-negative inward mean curvature $H_D(y) \geq 0$ for every $y \in \partial D$. Assume further that $u_0, \phi_0 \in C^2(\overline{D})$ with $u_0 = \phi_0$ on $\mathcal{C}\Omega$ and let $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ with $Du \in C^0(\overline{\Omega})$ be a solution to ($P^\pm_{\phi_0, u_0 > c}$). Then there exists a constant C_2, such that*

$$|Du| \leq C_2 \quad \text{on } \mathcal{P}\Omega, \tag{2.11}$$

where $C_2 = C_2(\gamma, u_0, \phi_0, n, c_1^\pm, H_D)$, with $c_1^\pm$ being the lower bounds from chapter 2.1. □

Remark. As explained before the proposition, we can still obtain a modulus of continuity estimate, if we merely assume $u \in C^{2,1} \cap C^0(\overline{\Omega})$. In this case we instead have

$$\sup_{X \in \mathcal{P}\Omega, Y \in \Omega, s \leq t} \frac{|u(X) - u(Y)|}{|X - Y|} \leq C_2,$$

which can then be used in the upcoming a priori estimates for the gradient on Ω (see also Gilbarg-Trudinger [GT01, p.353]).

Proof. Since the shape of D remains unchanged for any time $t \in [0, T)$, we can use time-independent barrier functions of the form

$$w^{\pm}(x) = \phi_0(x) + f(d(x)).$$

Here, $\phi_0(x)$ are the boundary values on $\mathcal{S}\Omega$, $d(x,t) \equiv d(x)$ is the distance to the spatial boundary ∂D and f is a C^2 function of one variable which is increasing ($f' > 0$) and concave ($f'' < 0$). Since $d(x)$ is independent of t and $\partial D \in C^2$, the distance function satisfies $d \in C^{2,1}(N \cap \Omega)$ with $\dot{d} \equiv 0$ [cf. Appendix, Sec. 5.6].

Moreover, we will make use of the Bernstein $\mathcal{E}$ function [cf. Appendix, Sec. 5.1], defined by

$$\mathcal{E}(X, z, p) := a^{ij}(X, z, p) p_i p_j.$$

Since a^{ij} is assumed to be the quadratic part of a parabolic differential operator, the Bernstein function satisfies $\lambda |p|^2 \leq \mathcal{E} \leq \Lambda |p|^2$, where λ and Λ can be taken as the smallest and largest eigenvalue of a^{ij} respectively. For the specific choice $a^{ij}(p) = \delta^{ij} - \frac{p_i p_j}{1+|p|^2}$ we obtain

$$\mathcal{E} = \frac{|p|^2}{1 + |p|^2} = \lambda |p|^2.$$

Let us begin with the construction of an upper barrier function $w^+(x)$ which, from now on, will be denoted by $w(x)$. Inserting the function $w(x) = \phi_0(x) + f(d(x))$ into the parabolic operator $\overline{P}$ yields

$$\overline{P}w = a^{ij}(Dw) D_{ij}\phi_0 - \dot{\phi}_0 + f'' a^{ij}(Dw) D_i d D_j d + f'[a^{ij}(Dw) D_{ij} d - \dot{d}] + a(u).$$

Making use of $\dot{\phi}_0(x), \dot{d}(x) \equiv 0$ and defining

$$\mathcal{E}_d := a^{ij}(Dw) D_i d D_j d,$$

we are left with

$$\overline{P}w = a^{ij}(Dw)D_{ij}\phi_0 + f''\mathcal{E}_d + f'[a^{ij}(Dw)D_{ij}d] + a(u). \tag{2.12}$$

Out of the four conditions that need to be fulfilled by the barrier function $w(x)$, the first one ($\overline{P}w \leq 0$) is the most difficult one to prove. The first major step is the estimation of (2.12) by an expression of the form

$$\overline{P}w \leq [f'' + c(f')^2]\mathcal{E}_d \tag{2.13}$$

with a positive constant c. Once obtained, we can choose the function f to fulfill all conditions that are required for $w = f(d) + \phi_0$ to be a barrier function.

All forthcoming calculations will be carried out in a principal coordinate system [cf. Appendix, Sec. 5.6]. Denoting the principal curvatures by $\kappa_i, i = 1, \ldots, n-1$, the distance function d suffices

$$Dd(x) = Dd(y) = (0, ..., 0, 1) \quad \text{and} \quad D^2d(x) = \operatorname{diag}\left[\frac{-\kappa_i(y)}{1 - \kappa_i(y)d(x)}, 0\right]$$

in these coordinates, where $y \in \partial D$ and $x \in D$ is close enough [in the sense of Appendix, Sec. 5.6] to ∂D. This implies

$$\sum_{i=1}^{n} D_{ii}d(x) = \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d(x)} \leq \sum_{i=1}^{n-1} -\kappa_i = \sum_{i=1}^{n} D_{ii}d(y) \tag{2.14}$$

for $x \in D$ close enough to $y \in \partial D$, which is easily seen by distinguishing the cases $\kappa_i < 0$ and $\kappa_i \geq 0$.

Next, we point out some estimates that will be needed later on. If we assume $f' \geq 2 \sup |D\phi_0|$, the gradient Dw satisfies

$$\begin{aligned} |Dw| &= |f'Dd + D\phi_0| \leq f' + |D\phi_0| \leq 2f', \\ |Dw| &\geq f' - |D\phi_0| \geq \frac{1}{2}f'. \end{aligned} \tag{2.15}$$

Moreover, the operator $\overline{P}$ has the largest eigenvalue $\Lambda = 1$. Hence, there exist a constant $c > 0$ and a number $p_0 > 0$ so that for any $p \geq p_0$

$$\Lambda = 1 \leq c\frac{|p|^2}{1 + |p|^2} = c\mathcal{E}(p). \tag{2.16}$$

Note that it suffices to discuss the case $|Du| \geq p_0$ because once we have found a bound $B > 0$ for this case we obtain a bound for the general case by $|Du| \leq \max\{p_0, B\}$.

Another important estimate is given by

$$\begin{aligned}\mathcal{E}_d(Dw) &= a^{ij}(Dw)D_i d D_j d = a^{nn}(Dw)\\ &= 1 - \frac{|D_n w|^2}{1+|Dw|^2} = \frac{1+|Dw|^2-|D_n w|^2}{1+|Dw|^2} \geq \frac{1}{1+|Dw|^2},\end{aligned}$$

where we made use of the fact that $Dd = (0, \ldots, 0, 1)$ in a principal coordinate system. Applying (2.15) for $f' \geq 2 \sup |D\phi_0|$ we obtain

$$\mathcal{E}(Dw) = \frac{|Dw|^2}{1+|Dw|^2} \leq 4(f')^2 \frac{1}{1+|Dw|^2} \leq 4(f')^2 \mathcal{E}_d(Dw). \tag{2.17}$$

Next we estimate the components of (2.12) to achieve the form (2.13). Let $\eta = (1, ..., 1)$ be the n-dimensional vector consisting of only ones. Then by using (2.16) and (2.17) we have

$$\begin{aligned}a^{ij}(Dw)D_{ij}\phi_0 &\leq a^{ij}(Dw)\eta_i\eta_j|D^2\phi_0| \leq \Lambda(Dw)|\eta|^2|D^2\phi_0|\\ &\leq cn|D^2\phi_0|\mathcal{E}(Dw) = c(n,\phi_0)\mathcal{E}(Dw) \leq c(n,\phi_0)(f')^2\mathcal{E}_d(Dw).\end{aligned}$$

For the last term of (2.12), which is $a(u)$, we make use of the boundedness of $u(x,t) \geq c_1^{\pm} > 0$ from the previous section and (2.17) to conclude that

$$a(u) = \frac{\gamma}{u} \leq \frac{|\gamma|}{u} \leq \frac{|\gamma|}{c_1^{\pm}} = c(\gamma, c_1^{\pm})\Lambda(Dw) \leq c(\gamma, c_1^{\pm})(f')^2\mathcal{E}_d(Dw).$$

What remains of (2.12) to be estimated is the expression $f'[a^{ij}(Dw)D_{ij}d]$. Here we use a decomposition from LIEBERMAN [Lie96, p. 244]. It is easily verified that we can write $a^{ij}(p) = a_{\infty}^{ij}\left(\frac{p}{|p|}\right) + a_0^{ij}(p)$, where

$$a_{\infty}^{ij}\left(\frac{p}{|p|}\right) = \delta^{ij} - \frac{p_i p_j}{|p|^2}, \quad a_0^{ij}(p) = \frac{p_i p_j}{|p|^2(1+|p|^2)}.$$

Now, if $\xi_1, \xi_2 \in S^{n-1}$, where S^{n-1} is the $(n-1)$-dimensional unit-sphere, we have Lipschitz continuity of a_{∞}^{ij}, hence there is a constant $L > 0$ with

$$|a_{\infty}^{ij}(\xi_1) - a_{\infty}^{ij}(\xi_2)| \leq L|\xi_1 - \xi_2|$$

for all $\xi_1, \xi_2 \in S^{n-1}$. Moreover, there is a value $p_0 > 0$ such that for every $p \geq p_0$ we have

$$|p|a_0^{ij}(p) = \frac{p_i p_j}{|p|(1+|p|^2)} \leq c\frac{|p|^2}{1+|p|^2} = c\mathcal{E}(p),$$

so that yet again we obtain by (2.17)

$$|Dw|a_0^{ij}(Dw) \leq c\mathcal{E}(Dw) \leq c(f')^2\mathcal{E}_d(Dw).$$

Proceeding with a_∞^{ij} we need additional estimates. First observe $Dd(x) = \nu(y)$, where $\nu(y)$ is the inward normal vector pointing from $y \in \partial D$ towards $x \in D$ close to ∂D. This relation can be seen easily by calculating

$$x = y + \nu(y)d \quad \Rightarrow \nu(y) = \frac{x-y}{d(x)} = \frac{x-y}{|x-y|}$$

and differentiating $d(x) = |x-y|$, which yields $Dd(x) = \frac{x-y}{|x-y|}$ as well. Therefore, we can estimate

$$\begin{aligned} |Dw - |Dw|\nu| &= |Dw - f'\nu + f'\nu - |Dw|\nu| \leq |Dw - f'\nu| + |\nu| \cdot |f' - |Dw|| \\ &= |f'Dd + D\phi_0 - f'\nu| + ||Dw| - f'| = |D\phi_0| + ||Dw| - f'|. \end{aligned} \tag{2.18}$$

Making use of the inverse triangular inequality

$$||a| - |b|| \leq |a - b|, \quad a, b \in \mathbb{R}^n,$$

we also have, with $a = Dw, b = f'Dd$,

$$||Dw| - f'| = ||f'Dd + D\phi_0| - |f'Dd|| \leq |f'Dd + D\phi_0 - f'Dd| = |D\phi_0|.$$

Inserting this estimate in (2.18) we obtain for $f' \geq 2\sup|D\phi_0|$

$$|Dw - |Dw|\nu| \leq 2|D\phi_0| \Rightarrow \left|\frac{Dw}{|Dw|} - \nu\right| \leq 2\frac{|D\phi_0|}{|Dw|} \leq 4\frac{|D\phi_0|}{f'}, \tag{2.19}$$

where we made use of (2.15) for the last inequality. The remaining condition that needs to be met involves the inward mean curvature $H_D(y)$ at a point $y \in \partial D$. If ν is the inward normal vector in a principal coordinate system, then $\nu = (0, ..., 0, 1)$ and we can calculate

$$a_\infty^{ii}(\nu) = \begin{cases} 1, & \text{if } i = 1, ..., n-1 \\ 0, & \text{if } i = n. \end{cases}$$

If we combine this with the properties of $D^2 d(x)$ in a principal coordinate system and (2.14), we obtain

$$a_\infty^{ij}(\nu)D_{ij}d(x) = a_\infty^{ii}(\nu)D_{ii}d(x) = \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d(x)} \leq \sum_{i=1}^{n-1} -\kappa_i = -H_D(y),$$

where $H_D(y) = \sum_{i=1}^{n-1} \kappa_i$ is the inward mean curvature of ∂D at $y \in \partial D$.

If we now assume that $H_D(y) \geq 0$, the last inequality implies that we also have $a^{ij}_\infty(\nu)D_{ij}d(x) \le 0$ and we can use the decomposition and the previous estimates (especially (2.19) in the third step) to conclude

$$\begin{aligned} f'[a^{ij}(Dw)D_{ij}d] &= f'\left[a^{ij}_\infty\left(\frac{Dw}{|Dw|}\right)D_{ij}d(x) + a^{ij}_0(Dw)D_{ij}d(x)\right] \\ &\le f'\left[a^{ij}_\infty\left(\frac{Dw}{|Dw|}\right) - a^{ij}_\infty(\nu)\right]D_{ij}d(x) + f'a^{ij}_0(Dw)D_{ij}d(x) \\ &\le 4|D\phi_0|\sup_{y\in\partial D} H_D(y) + 2|Dw|a^{ij}_0(Dw)D_{ij}d(x) \\ &\le c(\phi_0, H_D(y))\Lambda(Dw) + c(H_D(y)), n)\mathcal{E}(Dw) \\ &\le c(\phi_0, n, H_D(y))(f')^2\mathcal{E}_d(Dw), \end{aligned}$$

as long as $|Dw| \ge p_0$ and $f' \ge 2\sup|D\phi_0|$.

Combining this estimate with the ones above, we have shown that there exists a constant $c = c(\gamma, n, \phi_0, c_1^\pm, H_D(y))$ such that for $|Dw| \ge p_0$ and $f' \ge 2\sup|D\phi_0|$ the estimate

$$\begin{aligned} \overline{P}w &= a^{ij}(Dw)D_{ij}\phi_0 + f''\mathcal{E}_d + f'[a^{ij}(Dw)D_{ij}d] + a(u) \\ &\le [f'' + c(f')^2]\mathcal{E}_d(Dw) \end{aligned}$$

holds.

Now we can solve the ordinary differential equation

$$f''(d) + c(f')^2(d) = 0,$$

which yields

$$\begin{aligned} f'(d) &= \frac{1}{cd(x)+k}, \\ f(d) &= \frac{1}{c}\ln\left(\frac{c}{k}d(x) + 1\right), \end{aligned}$$

with a constant k at our disposal. The second constant we obtained from integration was chosen so that $f(0) = 0$. Note, that for small and positive k and x close enough to ∂D, the function $f'(d)$ becomes arbitrarily large.

We have proven that if we choose $f(d)$ as the function defined above and let k be positive and small enough to guarantee $f' \ge 2\sup|D\phi_0|$, then $w(x) = \phi_0(x) + f(d(x))$ suffices condition a) ($\overline{P}w \le 0$) from the beginning of this chapter. Having proven that condition a) is fulfilled, we will now show that the other conditions, b), c) and d) are also fulfilled, if we choose the set N in a specific way. Conditions b) and d) are immediately fulfilled from the definition of w because $f(d) \ge 0$ and $f(0) = 0$. To meet condition c) we set $N = B_\alpha(x_0) = \{x \mid |x - x_0| \le \alpha\}$, then $\mathcal{P}N = \{x \mid d(x) = \alpha\}$

so that c) becomes

$$w(x) = f(\alpha) + \phi_0(x) \geq u(x) \quad \text{on } \mathcal{P}N \cap \Omega.$$

With $M = \sup_{\Omega}|u - \phi_0|$ this inequality is always fulfilled if we choose α so that $f(\alpha) = M$. Since we had

$$f(d) = \frac{1}{c} \ln\left(\frac{c}{k} d(x) + 1\right),$$

this leads to the choice

$$\alpha = \frac{k}{c}(\exp(Mc) - 1).$$

Note, that the smaller the value for k, the smaller the value for α so that by choosing k sufficiently small we reduce the distance of x to ∂D, which increases the value for $f'(d)$. Hence, the conditions a) to d) are fulfilled and thus $w(x) = \phi_0(x) + f(d(x))$ is an upper barrier for $u(x,t)$.
Let us now briefly discuss why the function

$$w^-(x) := \phi_0(x) - f(d(x)),$$

for the same choice of $f(d)$, is a lower barrier for $u(x,t)$. First, note that upon inserting w^- into the parabolic operator $\overline{P}$ we obtain

$$\overline{P}w^- = a^{ij}(Dw)D_{ij}\phi_0 - f''\mathcal{E}_d - f'[a^{ij}(Dw)D_{ij}d] + a(u)$$

with the same notation as before. (2.15) still holds and in accordance with (2.16), (2.17) we obtain

$$-\Lambda \geq -c\mathcal{E}(p), \quad -\mathcal{E}(Dw) \geq -4(f')^2\mathcal{E}_d(Dw).$$

We may estimate the first expression in (2.5) by

$$a^{ij}(Dw)D_{ij}\phi_0 \geq -a^{ij}(Dw)\eta_i\eta_j|D^2\phi_0| \geq -\Lambda(Dw)|\eta|^2|D^2\phi_0| \geq -c(n,\phi_0)(f')^2\mathcal{E}_d(Dw)$$

and the last expression by

$$a(u) \geq -\frac{|\gamma|}{u} \geq -\frac{|\gamma|}{c_1^{\pm}} \geq -c(\gamma, c_1^{\pm})(f')^2\mathcal{E}_d(Dw).$$

In addition it is clear from our previous calculations for $f'[a^{ij}(Dw)D_{ij}d]$, that under the conditions $H_D(y) \geq 0$ and $f' \geq 2\sup|D\phi_0|$ we have

$$-f'[a^{ij}(Dw)D_{ij}d] \geq -c(\phi_0, n, H_D(y))(f')^2\mathcal{E}_d(Dw)$$

and thus we end up with the inequality

$$\overline{P}w^- \geq -[f'' + c(f')^2]\mathcal{E}_d$$

with a positive constant c. This leads to the differential equation

$$f''(d) + cf'(d) = 0$$

as in the case for the upper barrier. Hence, if we choose $f(d)$ the same way as before we obtain a). b) and d) immediately follow from the definition of $w^-(x)$ and taking $N = B_\alpha(x_0)$, condition c) becomes

$$u(x) \geq w(x) = \phi_0(x) - f(\alpha) \quad \text{on } \mathcal{P}N \cap \Omega,$$

so that we may take $f(\alpha) = M$ again to also fulfill c). This proves that $w^-(x) = \phi_0(x) - f(d(x))$ is a lower barrier for u. It is obvious that $f(d) \in C^\infty$ and from our assumptions we have $\phi_0(x), d(x) \in C^2(\overline{D})$, hence $w^\pm(x) \in C^2$. Additionally $f'(d)$ is bounded for $d \geq 0$, which implies Lipschitz continuity for $f(d)$ and hence there are constants $L^\pm$ so that (2.10) is fulfilled. Moreover, ∂D is a compact subset of $\mathbb{R}^n$ so there is a finite cover of ∂D consisting of balls $B_{\alpha_i}(x_i), x_i \in \partial D, i = 1, \ldots, k$, where α_i is chosen accordingly to fulfill condition c). Hence, we obtain an upper bound independent of the choice of $X_0 \in \mathcal{P}\Omega$, which completes the proof. □

2.3 Global gradient estimate

The classical approach for the derivation of global gradient estimates uses the fact that Du solves a similar partial differential equation as u itself. Usually one differentiates the equation with respect to x_k and multiplies by u_k afterwards. The resulting expression is estimated in a way that allows for the application of the weak maximum principle to show that $|Du|$ is bounded over all of $\mathcal{P}\Omega \cup \Omega$ by a constant that depends on the bound for $|Du|$ on $\mathcal{P}\Omega$. Lieberman [Lie96, p. 259-264] has done such a calculation that is applicable in our case for any choice of $\gamma \in \mathbb{R} \setminus \{0\}$. However, this approach has the disadvantage that the bound depends on the time T and becomes unbounded when $T \to \infty$. Thus, this method can only be used for the case $T < \infty$ and we have to make use of a different method deployed by Ecker and Huisken [EH89] as well as Stone [Sto94], to obtain a bound for $T = \infty$, which requires to work locally on the surfaces graph$(u(x,t))$.

We begin with the canonical approach from Lieberman [Lie96]. To state his results we need to define some new expressions first. Therefore, let

$$c_1^{\pm} \le m = \inf_{\Omega} u, \quad M = \sup_{\Omega} u \le C_1^{\pm}, \quad \xi = |Du|^2$$

and for $(X, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ define the operators

$$\delta := D_z + |p|^{-2} p \cdot D_x, \quad \overline{\delta} := p \cdot D_p,$$

where

$$\begin{aligned} D_x f(X, z, p) &= (D_{x_1} f(X, z, p), \dots, D_{x_n} f(X, z, p)), \\ D_p f(X, z, p) &= (D_{p_1} f(X, z, p), \dots, D_{p_n} f(X, z, p)). \end{aligned}$$

Furthermore, assume that there are a matrix valued function (a_*^{ij}) with smallest eigenvalue λ_* and largest eigenvalue Λ_* respectively and a vector valued function (f_j) such that (a^{ij}) can be decomposed as

$$a^{ij}(X, z, p) = a_*^{ij}(X, z, p) + \frac{1}{2}[p_i f_j(X, z, p) + p_j f_i(X, z, p)].$$

Finally, we define the quantities

$$\begin{aligned} A &= \frac{1}{\mathcal{E}} \left(\frac{\xi}{2\lambda_*} \sum_{i,j=1}^{n} (\overline{\delta} a_*^{ij})^2 + (\overline{\delta} - 1)\mathcal{E} \right), \\ B &= \frac{1}{\mathcal{E}} \left(\delta \mathcal{E} + (\overline{\delta} - 1) a \right), \\ C &= \frac{1}{\mathcal{E}} \left(\frac{\xi}{2\lambda_*} \sum_{i,j=1}^{n} (\delta a_*^{ij})^2 + \delta a \right) \end{aligned}$$

and

$$A_\infty, B_\infty, C_\infty = \lim_{|p| \to \infty} \sup_{\Omega \times [m,M]} A, B, C. \tag{2.20}$$

Then we may use the following theorem from Lieberman [Lie96, p.263, Thm. 11.1].

Proposition 2.5 (Gradient estimate on Ω for $T < \infty$) *Let $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ with $Du \in C^0(\overline{\Omega})$ and suppose that $Pu = 0$ on Ω. Suppose that P is parabolic at u and that the quantities A_∞, B_∞ and C_∞ are finite. If*

$$\min\{A_\infty, C_\infty, B_\infty + 2|A_\infty C_\infty|^{\frac{1}{2}}\} \le 0,$$

then there are positive constants k, k_1 with

$$\sup_{\Omega} |Du| \leq k_1 \mathrm{e}^{kT},$$

where k_1 is a constant determined only by $\sup\limits_{\mathcal{P}\Omega} |Du|, A_\infty, B_\infty, C_\infty, c_1^\pm, C_1^\pm$ and the limit behavior of (2.20). □

Thus, the derivation of a gradient bound for $T < \infty$ over all of Ω can be reduced to the calculation of the quantities A_∞, B_∞ and C_∞. The parabolic operator P was given by

$$Pu = -\dot{u} + a^{ij}(Du)u_{ij} + a(u) = -\dot{u} + \left(\delta^{ij} - \frac{u_i u_j}{1+|Du|^2}\right) u_{ij} - \frac{\gamma}{u},$$

so that for the choice

$$a_*^{ij} := \delta^{ij} \quad \text{and} \quad f_i := -\frac{p_i}{1+|p|^2}$$

we have

$$a_*^{ij} + \frac{1}{2}(p_i f_j + p_j f_i) = \delta^{ij} + \frac{1}{2}\left(-\frac{p_i p_j}{1+|p|^2} - \frac{p_j p_i}{1+|p|^2}\right) = \delta^{ij} - \frac{p_i p_j}{1+|p|^2} = a^{ij}(p).$$

Hence, we obtain

$$\mathcal{E}(p) = a^{ij}(p)p_i p_j = \frac{|p|^2}{1+|p|^2}, \quad \lambda_* = \Lambda_* = 1$$

and obviously any kind of differentiation of a_*^{ij} yields the value 0 since the matrix is constant, thus

$$\sum_{i,j=1}^{n} (\bar{\delta} a_*^{ij})^2 = \sum_{i,j=1}^{n} (\delta a_*^{ij})^2 = 0.$$

This simplifies the quantities A, B, C to

$$A = \frac{1}{\mathcal{E}}(\bar{\delta} - 1)\mathcal{E}, \quad B = \frac{1}{\mathcal{E}}(\delta\mathcal{E} + (\bar{\delta} - 1)a), \quad C = \frac{1}{\mathcal{E}}\delta a$$

and here we obtain

$$\begin{aligned}
\overline{\delta}\mathcal{E}(p) &= p \cdot D_p \left(\frac{|p|^2}{1+|p|^2} \right) = \frac{2|p|^2(1+|p|^2) - |p|^2 \cdot 2|p|^2}{(1+|p|^2)^2} = \frac{2|p|^2}{(1+|p|^2)^2}, \\
\delta\mathcal{E}(p) &= (D_z + |p|^{-2} p D_x) \left(\frac{|p|^2}{1+|p|^2} \right) = 0, \\
\overline{\delta}a(z) &= p \cdot D_p \left(-\frac{\gamma}{z} \right) = 0, \\
\delta a(z) &= (D_z + |p|^{-2} p D_x) \left(-\frac{\gamma}{z} \right) = \frac{\gamma}{z^2}.
\end{aligned}$$

Therefore, it is

$$\begin{aligned}
A &= \frac{1+|p|^2}{|p|^2} \cdot \left(\frac{2|p|^2}{(1+|p|^2)^2} - \frac{|p|^2}{1+|p|^2} \right) = \frac{2}{1+|p|^2} - 1, \\
B &= \frac{1+|p|^2}{|p|^2} \cdot \left(0 + 0 - \left(-\frac{\gamma}{z} \right) \right) = \frac{1+|p|^2}{|p|^2} \cdot \frac{\gamma}{z}, \\
C &= \frac{1+|p|^2}{|p|^2} \cdot \frac{\gamma}{z^2},
\end{aligned}$$

and letting $|p| \to \infty$ yields

$$A_\infty = -1, \quad B_\infty = \sup_{z \in [m,M]} \frac{\gamma}{z}, \quad C_\infty = \sup_{z \in [m,M]} \frac{\gamma}{z^2},$$

which are all finite since we have shown in the previous chapters, that there are constants $c_1^\pm, C_1^\pm$ with $0 < c_1^\pm \leq m \leq u \leq M \leq C_1^\pm < \infty$ for either choice of $\gamma \in \mathbb{R} \setminus \{0\}$. Moreover, it is $A_\infty = -1 < 0$, so by using proposition 2.5 we conclude

Proposition 2.6 (Global gradient bound for $T < \infty$) *Let $\Omega = D \times (0,T)$ with $T < \infty$. Let $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ with $Du \in C^0(\overline{\Omega})$ be a solution to $(P^\pm_{\phi_0, u_0 > c})$ and assume further that $u_0, \phi_0 \in C^2(\overline{D})$ with $u_0 = \phi_0$ on $\mathcal{C}\Omega$. Then there are positive constants $k, \tilde{C}_3$ such that*

$$|Du| \leq \tilde{C}_3 e^{kT} \quad \text{on } \Omega, \tag{2.21}$$

where $\tilde{C}_3 = \tilde{C}_3 \left(\sup_{\mathcal{P}\Omega} |Du|, c_1^\pm, C_1^\pm \right)$, with the constants $c_1^\pm, C_1^\pm$ from chapter 2.1 . □

For a gradient bound independent of T we proceed by working locally on the surfaces M_t generated by graph$(u(x,t))$, see also [EH89] and [Sto94]. For a detailed explanation of this procedure we refer to [Appendix, Sec. 5.7].

Proposition 2.7 (Global gradient bound) *Let $\Omega = D \times (0,T)$, where $T \leq \infty$. Let $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ with $Du \in C^0(\overline{\Omega})$ be a solution to $(P^\pm_{\phi_0, u_0 > c})$ and assume that*

$u_0, \phi_0 \in C^2(\overline{D})$ with $u_0 = \phi_0$ on $\mathcal{C}\Omega$. Then there is a positive constant C_3 such that

$$|Du| \leq C_3 \quad \text{on } \Omega, \tag{2.22}$$

where $C_3 = C_3(c_1^{\pm}, C_1^{\pm}, C_2)$, with $c_1^{\pm}, C_1^{\pm}$ being the constants from chapter 2.1 and C_2 the constant from chapter 2.2. □

Remark. If the solution is merely in $C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ we can still obtain a gradient estimate by using the modulus of continuity estimate from chapter 2.2. In this case it also suffices, that $u_0, \phi_0 \in C^0(\overline{\Omega})$ with $u_0 = \phi_0$ on $\mathcal{C}\Omega$.

Proof. Let us assume that the solution $u(x,t)$ defines a graph M_t and set

$$F(x,t) := (x, u(x,t))$$

for every $(x,t) \in \Omega = D \times (0,T)$. We call $F(x,t)$ the flow of the surfaces $\operatorname{graph}(u(x,t))$.

Let $g : \Omega \to M_0 \subset \mathbb{R}^n, g(x,t) = p$ be the diffeomorphism from Lemma 5.14 in [Appendix, Sec. 5.7], where M_0 is the initial surface given by $M_0 = \operatorname{graph}(u(x,0))$. Then there is another way to describe the flow $\tilde{F}(p,t) = F(g(x,t),t)$ with images $\tilde{M}_t$, that suffice $M_0 = \operatorname{graph}(u(x,0)) = \operatorname{graph}(\tilde{u}(p,0)) = \tilde{M}_0$. Additionally, the surfaces $\tilde{M}_t$ generated by the graph of $\tilde{u}(p,t)$ are equivalent to the surfaces M_t generated by the graph of $u(x,t)$ up to the tangential diffeomorphism $g(x,t)$. Hence, it suffices to bound the gradient of the alternative flow $\tilde{F}(p,t)$. $\tilde{F}(p,t)$ satisfies

$$\begin{aligned} \dot{\tilde{F}}(p,t) &= v^{-1}\left(vH(\tilde{u}) - \frac{\gamma}{\tilde{u}}\right)\nu(p,t) && \text{on } M_0 \times [0,\infty), \\ \tilde{F}(p,t) &= \tilde{F}(p,0) && \text{on } \partial M_0 \times [0,T), \\ \tilde{F}(p,0) &= p && \text{on } M_0, \end{aligned} \tag{PF}$$

where all quantities are now evaluated at the point (p,t) and understood as function of $\tilde{u}(p,t) = \tilde{F}(p,t) \cdot e_{n+1}$. For this flow the time derivative of the unit normal is given by (see Stone [Sto94, p. 173] and Ecker [Eck04, p. 121])

$$\partial_t \nu = -\nabla\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}t} \cdot \nu\right) = -\nabla\left(\left(H - \frac{\gamma}{\tilde{u}v}\right)\nu \cdot \nu\right) = -\nabla H - \frac{\gamma}{\tilde{u}^2 v^2}\nabla(\tilde{u}v).$$

Since we also have the relation $v = (\nu \cdot e_{n+1})^{-1}$, it follows for the time derivative of v

$$\dot{v} = \frac{\mathrm{d}}{\mathrm{d}t}((\nu \cdot e_{n+1})^{-1}) = v^2(\nabla H \cdot e_{n+1}) + \frac{\gamma}{\tilde{u}^2}(\nabla(\tilde{u}v) \cdot e_{n+1}). \tag{2.23}$$

Moreover, we obtain by the definition of $\Delta = \Delta_{M_t}$

$$\Delta v = g^{ij} D_i D_j v + H\nu^i D_i v \tag{2.24}$$

with $g^{ij} = \delta^{ij} - \nu^i \nu^j$. Abbreviating $\nu \cdot e_{n+1} =: \nu^{n+1}$, the partial derivatives of v are given by

$$\begin{aligned} D_i v &= D_i(\nu^{n+1})^{-1} = -v^2 D_i \nu^{n+1}, \\ D_j D_i v &= D_j(-v^2 D_i \nu^{n+1}) = -D_j(v^2) D_i \nu^{n+1} - v^2 D_i D_j \nu^{n+1}. \end{aligned}$$

Inserting this in (2.24) yields

$$\begin{aligned} \Delta v &= -v^2(g^{ij} D_i D_j \nu^{n+1} + H\nu^i D_i \nu^{n+1}) - g^{ij} D_j(v^2) D_i \nu^{n+1} \\ &= -v^2 \Delta \nu^{n+1} + \frac{2}{v} g^{ij} D_i v D_j v \\ &= -v^2 \Delta \nu^{n+1} + 2\frac{|\nabla v|^2}{v}, \end{aligned}$$

so that, after making use of the Jacobi field equation

$$\Delta \nu = -\nu |A|^2 - \nabla H,$$

we end up with

$$\Delta v = v|A|^2 + 2\frac{|\nabla v|^2}{v} + v^2(\nabla H \cdot e_{n+1}). \tag{2.25}$$

Now we can subtract the two equations (2.23) and (2.25) to obtain

$$\dot{v} = \Delta v - |A|^2 v - 2\frac{|\nabla v|^2}{v} + \frac{\gamma}{\tilde{u}^2}(\nabla(\tilde{u} v) \cdot e_{n+1}).$$

Dividing this equation by v and noting

$$\begin{aligned} \Delta(\ln(v)) &= g^{ij} D_i D_j (\ln(v)) + H\nu^i D_i(\ln(v)) = g^{ij} D_i \left(\frac{D_j v}{v}\right) + H\nu^i \frac{D_i v}{v} \\ &= \frac{1}{v}\Delta v - \frac{1}{v^2} g^{ij} D_i v D_j v = \frac{1}{v}\Delta v - \frac{|\nabla v|^2}{v^2} \end{aligned}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln(v)) = \Delta(\ln(v)) - |A|^2 - \frac{|\nabla v|^2}{v^2} + \frac{\gamma}{\tilde{u}^2}(\nabla \tilde{u} \cdot e_{n+1}) + \frac{\gamma}{\tilde{u} v}(\nabla v \cdot e_{n+1}). \tag{2.26}$$

Using the evolution equation in combination with the identity $H = v\Delta\tilde{u}$ gives us

$$\dot{\tilde{u}} = \frac{\mathrm{d}\tilde{F}}{\mathrm{d}t} \cdot e_{n+1} = \Delta \tilde{u} - \frac{\gamma}{\tilde{u} v^2}.$$

Hence, by introducing the auxiliary function

$$\chi := \sigma \ln(v) + \tilde{u}$$

with a positive constant $\sigma > 0$ to be chosen later, we may multiply (2.26) with σ to obtain the estimate

$$\dot{\chi} \leq \Delta\chi + \gamma\left(\frac{\sigma}{\tilde{u}^2}(\nabla\tilde{u}\cdot e_{n+1}) + \frac{\sigma}{\tilde{u}v}(\nabla v\cdot e_{n+1}) - \frac{1}{\tilde{u}v^2}\right). \tag{2.27}$$

Next, observe that

$$\nabla\chi = \frac{\sigma}{v}\nabla v + \nabla\tilde{u},$$

so inequality (2.27) may be expressed as

$$\dot{\chi} \leq \Delta\chi + \gamma\left(\frac{1}{\tilde{u}}(\nabla\chi\cdot e_{n+1}) - \frac{1}{\tilde{u}}(\nabla\tilde{u}\cdot e_{n+1}) + \frac{\sigma}{\tilde{u}^2}(\nabla\tilde{u}\cdot e_{n+1}) - \frac{1}{\tilde{u}v^2}\right).$$

Also, since v, ν are functions of $\tilde{u}$ evaluated at p, we have

$$\nabla\tilde{u}\cdot e_{n+1} = ((D\tilde{u}, 0) - D_i\tilde{u}\nu^i\nu)\cdot e_{n+1} = 0 - \frac{1}{v}D_i\tilde{u}\nu^i = \frac{|D\tilde{u}|^2}{v^2}$$

and thus

$$\begin{aligned}\dot{\chi} &\leq \Delta\chi + \gamma\left(\frac{1}{\tilde{u}}(\nabla\chi\cdot e_{n+1}) + \frac{|D\tilde{u}|^2}{\tilde{u}^2v^2}\sigma - \frac{|D\tilde{u}|^2}{\tilde{u}v^2} - \frac{1}{\tilde{u}v^2}\right)\\ &= \Delta\chi + \gamma\left(\frac{1}{\tilde{u}}(\nabla\chi\cdot e_{n+1}) + \frac{|D\tilde{u}|^2}{\tilde{u}^2v^2}\sigma - \frac{1}{\tilde{u}}\right)\\ &= \Delta\chi + \frac{\gamma}{\tilde{u}}(\nabla\chi\cdot e_{n+1}) + \frac{\gamma}{\tilde{u}^2}\left(\frac{|D\tilde{u}|^2}{v^2}\sigma - \tilde{u}\right).\end{aligned} \tag{2.28}$$

At this point, we have to distinguish the cases $\gamma > 0$ and $\gamma < 0$. If $\gamma > 0$ we may proceed as STONE [Sto94] by estimating $\frac{|D\tilde{u}|^2}{v^2} \leq 1$ and choosing $\sigma < c_1^+ \leq \tilde{u}$ to obtain

$$\dot{\chi} \leq \Delta\chi + \frac{\gamma}{\tilde{u}}(\nabla\chi\cdot e_{n+1}). \tag{2.29}$$

If $\gamma < 0$ we work on the set $M_t^1 := \{(p,t) \in M_0 \times [0,T) \mid |D\tilde{u}| > 1\}$. If this set is empty, then

$$\sup_{(p,t)\in M_0\times[0,T)} |D\tilde{u}| = \sup_{X\in\Omega}|Du| \leq 1$$

and the result follows. If M_t^1 is not empty, we have

$$\frac{|D\tilde{u}|^2}{v^2} \geq \frac{1}{2} \quad \text{on } M_t^1.$$

Hence, if we choose $\sigma > 2C_1^- \geq 2\tilde{u}$, we obtain

$$\frac{|D\tilde{u}|^2}{v^2}\sigma - \tilde{u} \geq 0 \quad \text{on } M_t^1$$

and thus,

$$\dot{\chi} \leq \Delta\chi + \frac{\gamma}{\tilde{u}}(\nabla\chi \cdot e_{n+1}) \quad \text{on } M_t^1. \tag{2.30}$$

For both equations (2.29), (2.30) the weak maximum principle on manifolds [cf. Appendix, Sec. 5.3] yields

$$\max_{M_0\times[0,T)} \chi \leq \max\left\{\max_{M_0} \chi(\cdot,0), \max_{\partial M_0\times[0,T)} \chi\right\}. \tag{2.31}$$

From the definition of χ we infer the existence of a constant $C_3 = C_3(c_1^\pm, C_1^\pm, C_2)$ with

$$\sup_{X\in\Omega} |Du| = \sup_{(p,t)\in M_0\times[0,T)} |D\tilde{u}| \leq C_3 \tag{2.32}$$

or in case $\gamma < 0$

$$\sup_{X\in\Omega} |Du| = \sup_{(p,t)\in M_0\times[0,T)} |D\tilde{u}| \leq \max\{1, C_3\}, \tag{2.33}$$

which concludes the proof. □

2.4 Hölder gradient estimate

The estimates from the previous chapters imply that the parabolic operator P given by

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u}$$

is uniformly parabolic since the ratio $\frac{\Lambda}{\lambda} = 1 + |Du|^2$ is bounded. Hence, we have access to results for uniformly parabolic operators which easily yield the remaining a priori Hölder gradient estimates. For completeness sake, we cite the relevant propositions from Lieberman [Lie96]. For the definition of time-dependent Hölder spaces H_a with $a \in \mathbb{R}^+$ see [Appendix, Sec. 5.4]. Additionally, define the function

$$\psi_0(x) := \begin{cases} u_0(x), & \text{if } x \in D, \\ \phi_0(x), & \text{if } x \in \partial D, \end{cases}$$

that combines the initial and boundary values ϕ_0, u_0 [cf. Appendix, Sec. 5.5].

Proposition 2.8 (Interior Hölder gradient estimate) [Lie96, p. 305, Thm.

12.3] *Let P be a parabolic operator of the form*

$$Pu = -\dot{u} + a^{ij}(X, u(X), Du(X))u_{ij} + a(X, u(X), Du(X))$$

and let $u \in C^{2,1}(\Omega)$ satisfy $Pu = 0$ and $|u| + |Du| \leq K$ in Ω for some positive constant $K \in \mathbb{R}$. Suppose further that the function $a^{ij}(X, z, p)$ is differentiable with respect to (X, z, p) and that a^{ij} and a are continuous. Let μ_K be a constant for which the inequality

$$\mu_K \geq K(|a^{ij}_x| + |a^{ij}_z||p|) + |a|$$

holds and let λ_K, Λ_K be the smallest and largest eigenvalue of a^{ij} respectively. Then there is a constant $\alpha > 0$ such that for any $\Omega' \subset\subset \Omega$ we have

$$[Du]_{\alpha;\Omega'} \leq C(n, K, \lambda_K, \Lambda_K, \mu_K, \mathrm{diam}\Omega)\delta^{-\alpha},$$

where $\delta = \mathrm{dist}(\Omega', \mathcal{P}\Omega)$. □

Proposition 2.9 (Global Hölder gradient estimate) [Lie96, p. 309, Thm. 12.10] *Suppose that P is the operator defined in the previous proposition, let $\beta \in (0, 1]$ and let $\Omega = D \times (0, T)$ suffice $\mathcal{P}\Omega \in H_{1+\beta}$. Suppose further that $a^{ij}(X, z, p)$ is continuously differentiable with respect to (X, z, p). If $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ with $Du \in L^\infty$ satisfies $Pu = 0$ on $\Omega, u = \psi_0$ on $\mathcal{P}\Omega$ for some $\psi_0 \in H_{1+\beta}$ fulfilling the compatibility condition $P\psi_0 = 0$, then there are positive constants α and C determined only by $n, \beta, \lambda_K, \Lambda_K, \mathrm{diam}\ \Omega$ and $\frac{\mu_K}{K}$ such that*

$$[Du]_\alpha \leq C[K + |\phi_0|_{1+\beta} + \mu_K],$$

where μ_K is defined as in the previous proposition. □

Both propositions are applicable to ($P^{\pm}_{\phi_0,u_0>c}$) for either choice of $\gamma \in \mathbb{R} \setminus \{0\}$, since $0 < c_1^{\pm} \leq |u| \leq C_1^{\pm}$ and $|Du| \leq C_3$ and thus $a^{ij}(Du)$ and $a(u)$ are continuously differentiable for every $(X, u(X), Du(X)) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Hence, the application of the last proposition yields the desired last a priori estimate, which concludes this chapter.

3 Existence, Uniqueness, Regularity

3.1 Existence and uniqueness

The a priori estimates allow for the application of a fixed point argument to prove the existence of a solution to ($P^{\pm}_{\phi_0,u_0>c}$). We have shown in the previous chapter, that there are constants $\delta \in (1, 2]$ and $M_\delta \in \mathbb{R}^+$ only depending on values determined by the operator P, the initial and boundary values $u_0(x), \phi_0(x)$ and the set Ω but not on u or its derivatives, such that every solution $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ fulfills the estimate

$$|u|_\delta \leq M_\delta,$$

as long as the initial and boundary data and the set Ω fulfill appropriate conditions. Therefore, we can make use of the following two propositions from LIEBERMAN [Lie96].

Proposition 3.1 (Short Time Existence) [Lie96, p. 206, Thm. 8.2.] *Assume that for any bounded subset $\mathcal{K}$ of $\Omega \times \mathbb{R} \times \mathbb{R}^n$ there is a positive constant $\lambda_{\mathcal{K}}$ such that*

$$\lambda_{\mathcal{K}} |\xi|^2 \leq a^{ij}(X, z, p)\xi_i \xi_j$$

for any $(X, z, p) \in \mathcal{K}$ and any $\xi \in \mathbb{R}^n$. Assume further that $X \to a^{ij}(X, u(X), Du(X))$ and $X \to a(X, u(X), Du(X))$ are Hölder continuous and define $\Omega_\epsilon := \{X \in \Omega \mid t < \epsilon\}$. Suppose $\mathcal{P}\Omega \in H_\delta$ and $\psi_0 \in H_\delta(\overline{\Omega})$ for some $\delta \in (1, 2)$, where ψ_0 are the combined initial and boundary values. Then there is a positive constant ϵ such that the problem

$$\begin{aligned} Pu &= 0 \quad \text{on } \Omega_\epsilon, \\ u &= \psi_0 \quad \text{on } \mathcal{P}\Omega_\epsilon, \end{aligned} \tag{3.1}$$

has a solution $u \in H_{2+\alpha}(\Omega') \cap C^0(\overline{\Omega}_\epsilon)$, where $\alpha \in (0, 1]$ and Ω' is any compact subset of Ω_ϵ. If $\mathcal{P}\Omega \in H_{2+\alpha}$ and $\psi_0 \in H_{2+\alpha}(\Omega)$ fulfills the compatibility condition $P\psi_0 = 0$ on $\mathcal{C}\Omega$, then $u \in H_{2+\alpha}(\Omega_\epsilon)$. □

Proposition 3.2 (Long Time Existence) [Lie96, p. 207, Thm. 8.3.] *Suppose that $\Omega = D \times [0, T), T \leq \infty, \psi_0$ and P are as in the proposition above. If there are constants $\delta \in (1, 2]$ and $M_\delta \in \mathbb{R}^+$ (independent of the ϵ from the previous proposition) such that any solution u of (3.1) satisfies the estimate*

$$|u|_\delta \leq M_\delta \quad \text{on } \Omega,$$

then there is a solution of

$$\begin{aligned} Pu &= 0 \quad \text{on } \Omega, \\ u &= \psi_0 \quad \text{on } \mathcal{P}\Omega. \end{aligned} \tag{3.2}$$

□

Combining these two propositions yields a solution u to problem ($P^{\pm}_{\phi_0, u_0 > c}$). For uniqueness we make use of results for the linear theory, especially for uniformly parabolic linear operators. Results are once again taken from LIEBERMAN [Lie96].

Proposition 3.3 (Uniqueness and regularity) [Lie96, p. 94, Thm. 5.14.] *Suppose we are given the initial boundary value problem*

$$\begin{aligned} Lu &= f \quad \text{on } \Omega, \\ u &= \psi_0 \quad \text{on } \mathcal{P}\Omega, \end{aligned} \tag{3.3}$$

where L is a parabolic operator given by

$$Lu = -\dot{u} + a^{ij}u_{ij} + b^i u_i + cu$$

and ψ_0 are the combined initial and boundary values. Suppose further that there are positive constants $\lambda, \Lambda, A, B, C$ such that for every $\xi \in \mathbb{R}^n$ and some $\alpha \in (0,1)$ we have

$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \quad |a^{ij}|_\alpha \le A, \quad |b^i|_\alpha \le B, \quad |c|_\alpha \le C.$$

If $\mathcal{P}\Omega \in H_{2+\alpha}, \psi_0 \in H_{2+\alpha}(\Omega)$ and $f \in H_\alpha(\Omega)$ for some $\alpha \in (0,1)$ then there is a unique solution $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ to (3.3). If also the compatibility condition $P\psi_0 = 0$ on $\mathcal{C}\Omega$ is fulfilled, then $u \in H_{2+\alpha}(\Omega)$ and

$$|u|_{2+\alpha} \le C(A, B, C, n, \alpha, \Omega)(|f|_\alpha + |\psi_0|_{2+\alpha}).$$

□

Remark ([Lie96, p. 93]). The condition $|c|_\alpha \le C$ instead of $|c|_\alpha \le 0$ is sufficient for the application of the maximum principle that is used to prove the previous proposition. To see this, define

$$L_k v := Lv - kv, f_k(X) := \mathrm{e}^{-kt} f(X), (\psi_0)_k(X) := \mathrm{e}^{-kt}\psi_0(X).$$

If $|c|_\alpha \le k$ for the operator L, then $|c_k|_\alpha \le 0$ for the operator L_k. Hence, there is a solution u_k to

$$\begin{aligned} L_k u_k &= f_k \quad \text{on } \Omega, \\ u_k &= (\psi_0)_k \quad \text{on } \mathcal{P}\Omega. \end{aligned}$$

Then, the function $u(X) := \mathrm{e}^{kt}u_k(X)$ solves

$$Lu = -k\mathrm{e}^{kt}u_k(X) + \mathrm{e}^{kt}Lu_k = \mathrm{e}^{kt}L_k u_k = \mathrm{e}^{kt} f_k = f$$

and thus $u(X)$ is a solution to

$$\begin{aligned} Lu &= f \quad \text{on } \Omega, \\ u &= \psi_0 \quad \text{on } \mathcal{P}\Omega. \end{aligned}$$

□

The application of these three propositions to ($P^{\pm}_{\phi_0,u_0>c}$) leads to

Theorem 3.4 (Existence and uniqueness for ($P^{\pm}_{\phi_0,u_0>c}$)) *Let $\Omega = D\times(0,T), T \leq \infty$ be a domain with $\mathcal{P}\Omega \in H_{2+\alpha}$, let $D \subset \mathbb{R}^n$ be a domain with non-negative inward mean curvature $H_D(y) \geq 0$ for every $y \in \partial D$ and let $\psi_0(x) \equiv \psi_0(x,t) \in H_{2+\alpha}(\Omega)$ be given by*

$$\psi_0(x) = \begin{cases} u_0(x), & \text{if } x \in D, \\ \phi_0(x), & \text{if } x \in \partial D. \end{cases}$$

Suppose for $\gamma \in \mathbb{R} \setminus \{0\}$ we are given the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \quad & \text{on } \Omega, \\ u(x,t) = \phi_0(x) \quad & \text{on } \mathcal{S}\Omega, \\ u(x,0) = u_0(x) \quad & \text{on } \mathcal{B}\Omega, \end{aligned} \tag{3.4}$$

where $\phi_0(x), u_0(x)$ are chosen so that

$$\begin{aligned} &\phi_0(x), u_0(x) > 0, && \text{if } \gamma < 0, \\ &\phi_0(x), u_0(x) > 2d\sqrt{\frac{\gamma}{2n-\gamma}}, && \text{if } \gamma \in (0,1], \\ &\phi_0(x), u_0(x) > \frac{2\gamma d}{\sqrt{2n-1}}, && \text{if } \gamma > 1, \end{aligned}$$

for $d := \operatorname{diam}(D)$. Then there is a unique solution $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ to problem (3.4), which is also in $H_{2+\alpha}(\Omega')$ for any $\Omega' \subset\subset \Omega$, with $\alpha \in (0,1)$.
If further $\psi_0(x)$ fulfills the compatibility condition $P\psi_0 = 0$ on $\mathcal{C}\Omega$ then $u \in H_{2+\alpha}(\Omega)$ and

$$|u|_{2+\alpha} \leq C(c_1^{\pm}, C_1^{\pm}, C_2, C_3, |\psi_0|_{2+\alpha}),$$

where $c_1^{\pm}, C_1^{\pm}, C_2, C_3$ are the constants from chapter 2. □

Proof. The a priori estimates from chapter 2 allow for the application of propositions 3.1 and 3.2, yielding a solution u to (3.4). Since the gradient is a priori bounded and $\lambda(Du) = \frac{1}{1+|Du|^2}, \Lambda = 1$ are the smallest and largest eigenvalues of a^{ij} respectively, there is a constant $\hat{\lambda} > 0$ such that $\hat{\lambda}|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$ for every $\xi \in \mathbb{R}^n$. Moreover, by setting $b^i, c = 0, f = \frac{\gamma}{u}$, the equation $Pu = 0$ with

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u}$$

is equivalent to $Lu = f$. Thus, to apply proposition 3.3, we have to prove that there is a constant $\alpha \in (0,1)$ so that $|a^{ij}|_\alpha \leq A$ and $f \in H_\alpha(\Omega)$. But the Hölder continuity of u and Du follows once again immediately from the a priori estimates and since

$$a^{ij}(p) = \delta^{ij} - \frac{p_i p_j}{1+|p|^2}, \quad a(z) = \frac{\gamma}{z}$$

are Hölder continuous functions on the set $\mathcal{S} := \{(z,p) \in \mathbb{R}\times\mathbb{R}^n \mid 0 < c_1^\pm \leq z, |p| \leq C_3\}$ with the constants $c_1^\pm, C_3$ from section 2, the compositions $a^{ij}(Du), a(u)$ are Hölder continuous for some constant $\alpha \in (0,1)$ as well. Hence, we may apply proposition 3.3, which concludes the proof. □

3.2 Higher Regularity

In this section we make use of regularity results for parabolic equations found in FRIEDMAN [Fri83].

Proposition 3.5 (Local spatial Regularity) [Fri83, p. 72, Thm. 10] *Let $\Omega = D \times (0,T) \subset \mathbb{R}^{n+1}$ be a domain and L be a parabolic operator given by*

$$Lu = -\dot{u} + a^{ij}(x,t)u_{ij} + b^i(x,t)u_i + c(x,t)u.$$

Assume that

$$D_x^m a^{ij}, D_x^m b^i, D_x^m c, D_x^m f \quad (0 \leq m \leq p)$$

are Hölder continuous with $\alpha \in (0,1)$ in Ω, where the operator D_x^m is to be understood as any combination of partial derivatives with respect to $x_i, i = 1,\ldots,n$, whose order adds up to m. If u is a solution to $Lu = f$ on Ω, then

$$D_x^m u, D_t D_x^k u \quad (0 \leq m \leq p+2; 0 \leq k \leq p)$$

exist on every cylinder $Q \subset\subset \Omega$ and are Hölder continuous with exponent α on Q. □

Proposition 3.6 (Local Regularity in Time) [Fri83, p. 74, Thm. 11] *Let L and Ω be given as in the proposition above. Assume now that also*

$$D_x^m D_t^k a^{ij}, D_x^m D_t^k b^i, D_x^m D_t^k c, D_x^m D_t^k f \quad (0 \leq m+2k \leq p; 0 \leq k \leq q)$$

are Hölder continuous with $\alpha \in (0,1)$ on Ω. If u is a solution to $Lu = f$ on Ω, then

$$D_x^m D_t^k u \quad (0 \leq m+2k \leq p+2; 0 \leq k \leq q+1)$$

exist on every cylinder $Q \subset\subset \Omega$ and are Hölder continuous with exponent α on Q. □

These properties can be extended to the whole domain, if Ω and the initial and boundary values ψ_0 are smooth enough and ψ_0 fulfills compatibility conditions.

Proposition 3.7 (Regularity on all of Ω) [Fri83, p.75, Thm. 12] *Let L be a parabolic operator in Ω. Assume that*

$$D_x^m D_t^k a_{ij}, D_x^m D_t^k b_i, D_x^m D_t^k c, D_x^m D_t^k f \quad (0 \leq m + k \leq p)$$

are uniformly Hölder continuous in Ω. Assume further that the functions ϕ from the local representation of ∂D [cf. Appendix, Sec. 5.6] are such that

$$D_x^{m+2} D_t^k \phi, D_x^m D_t^{k+1} \phi \quad (m \geq -2, k \geq -1, m + k \leq p)$$

are Hölder continuous, which implies $\mathcal{P}\Omega \in H_{p+\alpha}$ for some $\alpha \in (0, 1]$. Assume finally that $\psi_0 \in H_{2+\alpha}$, that $L\psi_0 = f$ on $\mathcal{C}\Omega$ and that, as a function of the local parameters of ∂D, ψ_0 is a function satisfying the condition that

$$D_x^{m+2} D_t^k \psi_0, D_x^m D_t^{k+1} \psi_0 \quad (m \geq -2, k \geq -1, m + k \leq p)$$

are Hölder continuous, wheras on $\mathcal{B}\Omega$

$$D_x^{m+2} \psi_0 \quad (-2 \leq m \leq p)$$

are Hölder continuous. If u is the solution to

$$\begin{aligned} Lu &= f \quad \text{on } \Omega, \\ u &= \psi_0 \quad \text{on } \mathcal{P}\Omega, \end{aligned}$$

then the functions

$$D_x^{m+2} D_t^k u, D_x^m D_t^{k+1} u \quad (m \geq -2, k \geq -1, m + 2k \leq p)$$

are uniformly Hölder continuous on Ω. □

Hence, by invoking these three propositions we obtain

Proposition 3.8 (Higher regularity) *Every classical solution $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ to problem ($P^\gamma_{\phi_0,u_0}$) is in $C^\infty(\Omega) \cap C(\overline{\Omega})$ with locally Hölder continuous derivatives of arbitrary order in space and time. If moreover the conditions from proposition 3.7 and the compatibility conditions of order $\lfloor \frac{p}{2} + 1 \rfloor$ for $p \in \mathbb{N}$ are fulfilled [cf. Appendix, Sec:*

5.5], the functions

$$D_x^{m+2}D_t^k u, D_x^m D_t^{k+1} u \quad (m \geq -2, k \geq -1, m+2k \leq p)$$

are uniformly Hölder continuous on Ω. □

Proof. Setting $b^i = c = 0$ and $f = \frac{\gamma}{u}$ the problem ($P^{\gamma}_{\phi_0,u_0}$) is equivalent to

$$\begin{aligned} Lu &= f \quad \text{on } \Omega, \\ u &= \psi_0 \quad \text{on } \mathcal{P}\Omega, \end{aligned}$$

where ψ_0 is the function that combines the initial and boundary values

$$\psi_0(x) = \begin{cases} u_0(x), & \text{if } x \in D, \\ \phi_0(x), & \text{if } x \in \partial D. \end{cases}$$

The functions

$$f(z) = \frac{\gamma}{z}, \quad a^{ij}(p) = \delta^{ij} - \frac{p_i p_j}{1+|p|^2}$$

have bounded derivatives of arbitrary order on the set

$$\mathcal{S} := \{(z,p) \in \mathbb{R} \times \mathbb{R}^n \mid 0 < c_1^{\pm} \leq z, |p| \leq C_3\}$$

with the constants $c_1^{\pm}, C_3$ from section 2. This implies Lipschitz and thus Hölder continuity for any derivative of $f(z), a^{ij}(p)$ on $\mathcal{S}$. Since the composition of two Hölder continuous functions with exponents α_1, α_2 is once again Hölder continuous with exponent $\alpha_1 \cdot \alpha_2$, the Hölder continuity of $D_x^m a^{ij}(Du)$ and $D_x^m f(u)$ merely depends on the continuity properties of Du and u. Thus, if we initially assume that $u \in C^{2,1}(\Omega)$, the a priori estimates imply that u and Du are locally Hölder continuous and hence $a^{ij}(Du)$ and $f(u)$ are locally Hölder continuous as well. This enables the application of proposition 3.5 for $p = 0$ from which follows that $D_x^2 u$ and $D_t u$ exist and are locally Hölder continuous.

This in return yields that not only $a^{ij}(Du)$ and $f(u)$ but also $D_x a^{ij}(Du)$ and $D_x f(u)$ are locally Hölder continuous. Using proposition 3.5 once again with $p = 1$ yields existence and local Hölder continuity of $D_x^3 u$ and $D_t D_x u$. Hence, by repeating this process arbitrarily often, we obtain existence and local Hölder continuity of $D_x^m u$ and $D_t D_x^k u$ for every $m, k \in \mathbb{N}$.

Proceeding with the differentiability in time we see that once again $D_x^m D_t^k a^{ij}(Du)$ and $D_x^m D_t^k f(u)$ are Hölder continuous if the respective functions u, Du and their derivatives are. Since from proposition 3.5 we already know that $D_t D_x^l u$ is locally Hölder continuous for arbitrary values of l, we can apply proposition 3.6 with $q = 1$, which yields local Hölder continuity of $D_x^m D_t^2 u$. This guarantees that $D_x^m D_t^2 a^{ij}(Du)$ and $D_x^m D_t^2 f(u)$ are locally Hölder continuous and another application of proposition 3.6 grants local Hölder continuity of $D_x^m D_t^3 u$. Repeating this process, we achieve local Hölder continuity of $D_x^m D_t^k u$ for arbitrary values of $m, k \in \mathbb{N}$.

Under the additional assumptions from proposition 3.7, we can prove global Hölder continuity with the same reasoning. □

4 Further properties of solutions

4.1 Solution to the Problem (P_0^-)

In this chapter we will construct a unique solution to the problem (P_0^-)

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u &= \frac{\gamma}{u} \quad \text{on } \Omega, \\ u(x,t) &= 0 \quad \text{on } \mathcal{P}\Omega. \end{aligned} \tag{P_0^-}$$

Since $\lim_{u \to 0} \left|\frac{\gamma}{u}\right| = \infty$ poses a problem when reaching the boundary of Ω in (P_0^-), we make use of the approximating problems $(P_{0+\epsilon}^-)$ which, according to the fixed point theory deduced above, have a unique solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$.

Proposition 4.1 (Unique solvability of (P_0^-)) *Let $\Omega = D \times (0,T), T \leq \infty$ be a domain with $\mathcal{P}\Omega \in H_{2+\alpha}$ and let $D \subset \mathbb{R}^n$ have non-negative inward mean curvature $H_D(y) \geq 0$ for every $y \in \partial D$. Then the problem*

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u &= \frac{\gamma}{u} \quad \text{on } \Omega, \\ u(x,t) &= 0 \quad \text{on } \mathcal{P}\Omega, \end{aligned} \tag{P_0^-}$$

has a unique solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$. □

Proof. Let $\epsilon > 0$ and consider the family of problems

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u &= \frac{\gamma}{u} \quad \text{on } \Omega, \\ u(x,t) &= \epsilon \quad \text{on } \mathcal{P}\Omega. \end{aligned} \tag{$P_{0+\epsilon}^-$}$$

According to our previous results these problems have a unique solution $u^\epsilon(x,t) \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ for every $\epsilon > 0$, which moreover suffice $u^\epsilon(x,t) \in H_{2+\alpha}(\Omega')$ for any $\Omega' \subset\subset \Omega$. We can use a method from Dierkes [Die19, p. 515, Proof of Thm. 1.6] to show that $u^\epsilon(x,t)$ converges uniformly to a solution $u(x,t)$ of (P_0^-).

We interpret $u^\epsilon = u^\epsilon(x,t)$ as sequence in ϵ which, by the a priori estimates obtained earlier, is bounded for every $\epsilon > 0$. Moreover, it is monotonically increasing in ϵ, which can be easily seen by application of the comparison principle [cf. Appendix, Sec. 5.3]: If $0 < \epsilon_1 \le \epsilon_2 \le 1$ we have

$$\begin{aligned} Pu^{\epsilon_1} = 0 = Pu^{\epsilon_2} \quad &\text{on } \Omega, \\ u^{\epsilon_1} = \epsilon_1 \le \epsilon_2 = u^{\epsilon_2} \quad &\text{on } \mathcal{P}\Omega, \end{aligned}$$

where

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u},$$

so that by application of the comparison principle we obtain $u^{\epsilon_1} \le u^{\epsilon_2}$ on $\overline{\Omega}$. Since u^ϵ is a bounded and monotone sequence, the pointwise limes

$$\lim_{\epsilon \downarrow 0} u^\epsilon(x,t) = u(x,t)$$

exists everywhere on $\overline{\Omega}$. Furthermore, since $u^\epsilon(x,t) \in H_{2+\alpha}(\Omega')$ for any $\Omega' \subset\subset \Omega$, the theorem of Arzela-Ascoli yields a subsequence ϵ_k for which all derivatives of $u^{\epsilon_k}(x,t)$ that appear in the operator P converge uniformly to a function $\tilde{u}(x,t)$ on any compact subset Ω' contained in Ω. Since $u^\epsilon(x,t)$ already has a pointwise limes, both limites have to agree so that by inserting $u^\epsilon(x,t)$ into the initial boundary value problem ($P^-_{0+\epsilon}$) and letting $\epsilon \downarrow 0$ we see, that $u(x,t)$ is a solution to (P^-_0) which also fulfills the a priori estimates. Hence, by the regularity theory we have $u(x,t) \in C^\infty(\Omega)$.

Next, we prove that $u(x,t)$ is also continuous up to the boundary by showing that the convergence is uniform on $\overline{\Omega}$. Observe, that the function $u^{\epsilon_1} + \epsilon_2 - \epsilon_1$ fulfills

$$\begin{aligned} P(u^{\epsilon_1} + \epsilon_2 - \epsilon_1) &= -\dot{u}^{\epsilon_1} + a^{ij}(Du^{\epsilon_1}) D_{ij} u^{\epsilon_1} - \frac{\gamma}{u^{\epsilon_1} + \epsilon_2 - \epsilon 1} \\ &= -\dot{u}^{\epsilon_1} + a^{ij}(Du^{\epsilon_1}) D_{ij} u^{\epsilon_1} - \frac{\gamma}{u^{\epsilon_1}} + \frac{\gamma}{u^{\epsilon_1}} - \frac{\gamma}{u^{\epsilon_1} + \epsilon_2 - \epsilon 1} \\ &= Pu_{\epsilon_1} + \frac{\gamma(\epsilon_2 - \epsilon_1)}{u^{\epsilon_1}(u^{\epsilon_1} + \epsilon_2 - \epsilon_1)} = \frac{\gamma(\epsilon_2 - \epsilon_1)}{u^{\epsilon_1}(u^{\epsilon_1} + \epsilon_2 - \epsilon_1)} \end{aligned}$$

so comparing $u^{\epsilon_1} + \epsilon_2 - \epsilon_1$ with u^{ϵ_2} yields for $\gamma < 0$

$$\begin{aligned} P(u^{\epsilon_1} + \epsilon_2 - \epsilon_1) = \frac{\gamma(\epsilon_2 - \epsilon_1)}{u^{\epsilon_1}(u^{\epsilon_1} + \epsilon_2 - \epsilon_1)} \le 0 = Pu^{\epsilon_2} \quad &\text{on } \Omega, \\ u^{\epsilon_1} + \epsilon_2 - \epsilon_1 = \epsilon_2 = u^{\epsilon_2} \quad &\text{on } \mathcal{P}\Omega, \end{aligned}$$

which, by application of the comparison principle, gives us

$$u^{\epsilon_2} \le u^{\epsilon_1} + (\epsilon_2 - \epsilon_1) \Leftrightarrow u^{\epsilon_2} - u^{\epsilon_1} \le \epsilon_2 - \epsilon_1 \quad \text{on } \overline{\Omega}.$$

Hence, the convergence is uniform in ϵ on $\overline{\Omega}$ and thus $u(x,t) \in C^\infty(\Omega) \cap C(\overline{\Omega})$.

Uniqueness also follows easily from the comparison principle: Assume $\hat{u}$ is another (different) solution to (P_0^-). Then we immediately have

$$\begin{aligned} P\hat{u} = Pu = 0 \quad &\text{on } \Omega, \\ \hat{u} = u = 0 \quad &\text{on } \mathcal{P}\Omega, \end{aligned}$$

so after application of the comparison principle in either direction we have $\hat{u} \geq u$ and $\hat{u} \leq u$ on $\overline{\Omega}$ which implies $\hat{u} = u$ on $\overline{\Omega}$ and concludes the proof. □

Remark. The method presented above can also be used in cases where $\psi_0 \not\equiv 0$ and $\psi_0 = 0$ on subsets $S \subset \mathcal{P}\Omega$. Thus, we can relax the restriction $u_0, \phi_0 > 0$ in ($P^-_{\phi_0, u_0 > 0}$) to $\phi_0, u_0 \geq 0$ instead.

4.2 Convergence to a stationary solution of ($P^\pm_{\phi_0, u_0 > c}$)

In this section we prove, that a solution $u(x,t)$ to ($P^\pm_{\phi_0, u_0 > c}$) has a subsequence of times $t_k \to \infty$, for which $u(x, t_k)$ converges to a solution of the stationary problem

$$\begin{aligned} \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u = \frac{\gamma}{u} \quad &\text{on } D, \\ u(x) = \phi_0(x) \quad &\text{on } \partial D. \end{aligned} \tag{$P^S_{\phi_0}$}$$

We achieve this by making use of the Arzela-Ascoli theorem as well as an idea from Huisken [Hui89, p.375], which can also be found in Stone [Sto94, p.175], that consists in differentiating the energy

$$\int_D u^\gamma v \, \mathrm{d}x$$

related to the (PDE).

Lemma 4.2 *Assume $\Omega = D \times (0, \infty)$ and $u(x,t) \in H_{2+\alpha}(\Omega)$ for some $\alpha \in (0,1]$. If*

$$\int_0^\infty \int_D \dot{u}^2(x,t) \, \mathrm{d}x \, \mathrm{d}t < \infty,$$

then

$$\lim_{t \to \infty} \dot{u}(x,t) = 0,$$

for every $x \in D$. □

Proof. We prove the claim by contradiction. Suppose it is not

$$\lim_{t\to\infty} \dot{u}(x,t) = 0 \quad \text{for every } x \in D.$$

Then there are numbers $\epsilon_0 > 0, N_0 \in \mathbb{N}$ and a sequence of points $X_k = (x_k, t_k) \in \Omega$ with $\lim\limits_{k\to\infty} t_k = \infty$, so that

$$|\dot{u}(X_k)| \geq \epsilon_0 \quad \text{for every } k \geq N_0.$$

Define

$$K_{\frac{\epsilon_0}{2}} := \left\{(x,t) \in \Omega \mid |\dot{u}(X)| \geq \frac{\epsilon_0}{2}\right\}$$

and observe that

$$\int_0^\infty \int_D \dot{u}^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \geq \frac{\epsilon_0^2}{4}|K_{\frac{\epsilon_0}{2}}|,$$

which, by the property

$$\int_0^\infty \int_D \dot{u}^2(x,t)\,\mathrm{d}x\,\mathrm{d}t < \infty,$$

implies that $|K_{\frac{\epsilon_0}{2}}| < \infty$. Since $\dot{u}$ is assumed to be uniformly Hölder continuous on Ω, there is a constant $K \in \mathbb{R}$, so that

$$|\dot{u}(X) - \dot{u}(X_k)| \leq K|X - X_k|^\alpha$$

for every $X \in \Omega$ and some $\alpha \in (0,1]$. If we choose $\delta > 0$ to suffice

$$\delta \leq \left(\frac{\epsilon_0}{2K}\right)^{\frac{1}{\alpha}}$$

and consider the sets

$$N_\delta(X_k) := \{X \in \Omega \mid |X - X_k| < \delta\},$$

then by Hölder continuity of $\dot{u}(X)$ it follows

$$|\dot{u}(X) - \dot{u}(X_k)| \leq K \cdot |X - X_k|^\alpha \leq K\delta^\alpha \leq \frac{\epsilon_0}{2},$$

for every $X \in N_\delta(X_k)$. Hence, for every $X \in N_\delta(X_k)$ with $k \geq N_0$ we have $|\dot{u}(X)| \geq \frac{\epsilon_0}{2}$, so that for every $k \geq N_0$

$$N_\delta(X_k) \subset K_{\frac{\epsilon_0}{2}}.$$

Furthermore, since $t_k \to \infty$, we can find a subsequence k_i with $k_1 \geq N_0$ and $k_i \to \infty$ as $i \to \infty$, for which $N_\delta(X_{k_i}) \cap N_\delta(X_{k_j}) = \emptyset$. Then we have

$$|K_{\frac{\epsilon_0}{2}}| \geq |\cup_{k \geq N_0} N_\delta(X_k)| \geq |\cup_{i \in \mathbb{N}} N_\delta(X_{k_i})| = \sum_{i=1}^{\infty} |N_\delta(X_{k_i})| = \infty,$$

since obviously $|N_\delta(X_{k_i})| \geq c(k_i) > 0$ for every $k_i \in \mathbb{N}$. This is a contradiction to $|K_{\frac{\epsilon_0}{2}}| < \infty$ and the claim follows. □

Proposition 4.3 (Convergence to a stationary solution) *Let $\Omega = D \times (0,T), T \leq \infty$ be a domain with $\mathcal{P}\Omega \in H_{2+\alpha}$. Let $D \subset \mathbb{R}^n$ be a domain with non-negative inward mean curvature $H_D(y) \geq 0$ for every $y \in \partial D$ and let $\psi_0(x) \equiv \psi_0(x,t) \in H_{2+\alpha}(\Omega)$ be given by*

$$\psi_0(x) = \begin{cases} u_0(x), & \text{if } x \in D \\ \phi_0(x), & \text{if } x \in \partial D, \end{cases}$$

fulfilling the compatibility condition $P\psi_0 = 0$ on $\mathcal{C}\Omega$. Then the unique solution $u \in H_{2+\alpha}(\Omega) \cap C^\infty(\Omega)$ to the problem

$$-\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad \text{on } \Omega,$$
$$u(x,t) = \psi_0(x) \quad \text{on } \mathcal{P}\Omega,$$

has a subsequence of times t_k, for which $u(x,t_k)$ converges to a solution $u_\infty(x)$ of the stationary problem

$$\Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad \text{on } D,$$
$$u(x) = \phi_0(x) \quad \text{on } \partial D. \tag{$P^S_{\phi_0}$}$$

If the solution to ($P^S_{\phi_0}$) is unique then not only a subsequence but the whole sequence $u(x,t)$ converges to $u_\infty(x)$ for $t \to \infty$. □

Proof. Since $u \in H_{2+\alpha}(\Omega)$, the theorem of Arzela-Ascoli yields a subsequence of times t_k, for which

$$u(x,t_k) \to u_\infty(x), Du(x,t_k) \to Du_\infty(x), D^2u(x,t_k) \to D^2u_\infty(x), \dot{u}(x,t_k) \to \dot{u}_\infty(x)$$

uniformly on Ω, as $t_k \to \infty$. Our goal is to prove, that the function $u_\infty(x)$ is a solution to the stationary problem. To achieve this, we show that the relation

$$\int_0^\infty \int_D \dot{u}^2(x,t)\,\mathrm{d}x\,\mathrm{d}t < \infty \tag{4.1}$$

holds. If this integral is finite, we must have

$$\lim_{t_k\to\infty} \dot{u}(x,t_k) = 0$$

by Lemma 4.2. We obtain estimate (4.1) by differentiating the energy

$$\int_D u^\gamma v \,\mathrm{d}x \tag{4.2}$$

in time direction, where $v = \sqrt{1+|Du|^2}$, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D u^\gamma v\,\mathrm{d}x = \int_D \gamma u^{\gamma-1}\dot{u}v\,\mathrm{d}x + \int_D u^\gamma \dot{v}\,\mathrm{d}x. \tag{4.3}$$

For the upcoming calculations we use the notation

$$\nu^i = \frac{u_i}{v} \quad \text{and} \quad g^{ij} = \delta^{ij} - \nu^i\nu^j.$$

This results in the relation

$$g^{ij}u_{ij} = \Delta u - \frac{u_i u_j}{1+|Du|^2}u_{ij} = v\cdot \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = vD_i\nu^i$$

and by the definition of v we also have

$$\dot{v} = \frac{\dot{u}_i u_i}{v} = \frac{u_i}{v}D_i(\dot{u}) = \nu^i D_i(\dot{u}).$$

If we now substitute $\nu^i D_i(\dot{u})$ for $\dot{v}$ in (4.3) and integrate by parts, where boundary terms are zero because $\dot{u}\equiv 0$ on $\mathcal{S}\Omega$, we can continue the previous calculations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_D u^\gamma v\,\mathrm{d}x &= \int_D \gamma u^{\gamma-1}\dot{u}v\,\mathrm{d}x + \int_D u^\gamma\dot{v}\,\mathrm{d}x \\
&= \int_D \gamma u^{\gamma-1}\dot{u}v\,\mathrm{d}x - \int_D D_i(u^\gamma\nu^i)\dot{u}\,\mathrm{d}x \\
&= \int_D \gamma u^{\gamma-1}\dot{u}v\,\mathrm{d}x - \int_D \gamma u^{\gamma-1}u_i\frac{u_i}{v}\dot{u} + u^\gamma D_i(\nu^i)\dot{u}\,\mathrm{d}x \\
&= \int_D \gamma u^{\gamma-1}\dot{u}v\,\mathrm{d}x - \int_D \gamma u^{\gamma-1}\frac{|Du|^2}{v}\dot{u} + \frac{1}{v}u^\gamma g^{ij}u_{ij}\dot{u}\,\mathrm{d}x \\
&= \int_D \gamma u^{\gamma-1}\dot{u}\left(\frac{1+|Du|^2-|Du|^2}{v}\right) - \frac{1}{v}u^\gamma g^{ij}u_{ij}\dot{u}\,\mathrm{d}x \\
&= \int_D \frac{1}{v}u^\gamma\dot{u}\left(\frac{\gamma}{u} - g^{ij}u_{ij}\right)\mathrm{d}x.
\end{aligned}$$

At this point we make use of the (PDE), that states

$$\dot{u} = g^{ij}u_{ij} - \frac{\gamma}{u},$$

so we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D u^\gamma v \,\mathrm{d}x = -\int_D \frac{u^\gamma}{v}\dot{u}^2 \,\mathrm{d}x.$$

This equation remains valid for $\gamma = 0$, since here we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_D v \,\mathrm{d}x &= \int_D \dot{v} \,\mathrm{d}x = -\int_D D_i\nu^i \dot{u} \,\mathrm{d}x \\ &= -\int_D \frac{1}{v} g^{ij}u_{ij}\dot{u} \,\mathrm{d}x = -\int_D \frac{1}{v}\dot{u}^2 \,\mathrm{d}x.\end{aligned}$$

Now we may apply the a priori estimates $|u| \geq c_1^\pm > 0$ and $|Du| \leq C_3$ from the previous chapters to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D u^\gamma v \,\mathrm{d}x = -\int_D \frac{u^\gamma}{v}\dot{u}^2 \,\mathrm{d}x \leq -C(c_1^\pm, C_3)\int_D \dot{u}^2 \,\mathrm{d}x,$$

so after integration from 0 to T it follows, that

$$\begin{aligned}\int_0^T \int_D \dot{u}^2 \,\mathrm{d}x \,\mathrm{d}t &\leq \frac{1}{C(c_1^\pm, C_3)}\left(\int_D u^\gamma(x,0)v(x,0) \,\mathrm{d}x - \int_D u^\gamma(x,T)v(x,T) \,\mathrm{d}x\right) \\ &\leq \frac{1}{C(c_1^\pm, C_3)}\int_D u_0^\gamma(x)\cdot\sqrt{1+|Du_0(x)|^2} \,\mathrm{d}x = C(c_1^\pm, C_3, D, u_0, Du_0) < \infty,\end{aligned}$$

where we used that $u^\gamma, v \geq 0$. Since the bound is independent of T, we can let $T \to \infty$ to deduce that

$$\int_0^\infty \int_D \dot{u}^2(x,t) \,\mathrm{d}x \,\mathrm{d}t < \infty. \tag{4.4}$$

Hence, if we insert $u(x,t_k)$ and its derivatives into the (PDE) and let $t_k \to \infty$, we see, that $u_\infty(x)$ is a solution to the stationary problem.

Now, consider the case of unique solvability of $(P^S_{\phi_0})$. Let us assume there is another subsequence $u(x,t_i)$ that converges to a different function $\hat{u}_\infty(x)$ for $i \to \infty$. Because of our foregoing calculations, $\hat{u}_\infty$ is another stationary solution to the problem $(P^S_{\phi_0})$ but the solution to this problem is unique so that we have $u_\infty(x) = \hat{u}_\infty(x)$. Hence, every subsequence of $u(x,t)$ converges to the same function, which implies that the whole sequence fulfills $\lim\limits_{t\to\infty} u(x,t) = u_\infty(x)$. □

Remark. With further restrictions on the initial and boundary data we can show as in Stone [Sto94], that the convergence is of exponential order and that the whole sequence $u(x,t)$ converges to $u_\infty(x)$ even if the solution to $(P^S_{\phi_0})$ is not unique. □

4.3 Interior gradient estimate for $(P^-_{\phi_0,u_0>0})$

We prove an interior gradient estimate for the case $\gamma < 0$. The proof is based on a variant of a method from KOREVAAR [Kor86]. We start by quoting the theorem from KOREVAAR [Kor86] which yields an interior gradient estimate for a similar problem in the elliptic case.

Lemma 4.4 (Interior Gradient Bound for Solutions to the Prescribed Mean Curvature Equation) *Let $x \in \mathbb{R}^n, B_1 = \{x \mid |x| < 1\}$ and let $u \in C^3(B_1)$ be a negative solution to the equation*

$$\frac{1}{v} g^{ij} u_{ij} - h(x,u(x)) = 0,\ \frac{\partial h}{\partial u} \geq 0,\ |h| + |D_x h| \leq M,$$

with $v = \sqrt{1+|Du|^2}, u_{ij} = D_{ij}u, g^{ij} = \delta^{ij} - u_i u_j (1+|Du|^2)^{-1}$. Then there is a constant c with

$$|Du(0)| \leq c,$$

where c only depends on M, n and $u(0)$. □

For the applicability of this lemma we study the *mirrored Problem to* $(P^-_{\phi_0,u_0>0})$ which is given by

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } \Omega, \\ u(x,t) &= -\phi_0(x) < 0 && \text{on } \mathcal{S}\Omega, \\ u(x,0) &= -u_0(x) < 0 && \text{on } \mathcal{B}\Omega, \end{aligned} \qquad (\overline{P^-_{\phi,u_0>0}})$$

where $\gamma < 0$ and ϕ_0, u_0 are the initial and boundary values of $(P^-_{\phi_0,u_0>0})$. Observe that $-u$ solves $(\overline{P^-_{\phi,u_0>0}})$ if and only if u solves $(P^-_{\phi_0,u_0>0})$. Hence, all bounds for $|u|$ and $|Du|$ are also valid for $-u$ and every bound for $-u$ and its gradient will be valid for u.

Proposition 4.5 (Interior gradient estimate for the mirrored problem) *Let $\phi_0, u_0 \in C^0(\overline{D})$ with $\phi_0 = u_0$ on $\mathcal{C}\Omega$, let $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ be a negative solution to $(\overline{P^-_{\phi,u_0>0}})$ and let $\delta = \delta(x,t) = \min\left\{\frac{1}{2}\mathrm{dist}(x,\partial D), \frac{1}{2}t, 1\right\}$. Then there is a constant $C = C(c_1^-, n, \delta)$, with*

$$|Du(X)| \leq C \quad \text{on } \Omega \tag{4.5}$$

for every $X = (x,t) \in \Omega$, where c_1^- is the constant from chapter 2.1. □

Corollary 4.6 (Interior gradient estimate for ($P^-_{\phi_0,u_0>0}$)) *Let ϕ_0, u_0 and δ be chosen as in the previous proposition. If $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ is a positive solution to ($P^-_{\phi_0,u_0>0}$) then there is a constant $C = C(c_1^-, n, \delta)$, with*

$$|Du(X)| \leq C \quad \text{on } \Omega \tag{4.6}$$

for every $X = (x,t) \in \Omega$, where c_1^- is the constant from chapter 2.1. □

The corollary is an immediate consequence of proposition 4.5.

Proof (of proposition 4.5). Once again set

$$\nu^i = \frac{u_i}{v} \quad \text{and} \quad g^{ij} = \delta^{ij} - \nu^i \nu^j.$$

Furthermore, we introduce the operator L given by

$$Lw := -\dot{w} + g^{ij} w_{ij},$$

so that we may express the parabolic operator P with

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u - \frac{\gamma}{u},$$

by

$$Pu = -\dot{u} + g^{ij} u_{ij} - \frac{\gamma}{u} = Lu - \frac{\gamma}{u}.$$

We will also make use of a cutoff function. Therefore, let $X_0 = (x_0, t_0)$ be an arbitrary point in Ω and observe, that we can assume $\Omega = D \times (0, \infty)$ since we have shown, that every solution to ($P^-_{\phi_0,u_0>0}$) exists for all times. Set $u_{X_0} := -u(X_0) > 0$ and for $0 < \delta \leq 1$ let $Q_\delta(X_0) := B_\delta(x_0) \times (0, 2t_0)$ be a cylinder centered around X_0, where δ is always chosen small enough to guarantee that $Q_\delta \subset \Omega$. Then we define $\eta(x,t,z) := f \circ \mu(x,t,z)$ with $f(\mu) = \mathrm{e}^{k\mu} - 1$ and the cutoff function

$$\mu \colon B_\delta(x_0) \times (0, 2t_0) \times \mathbb{R}^- \to \mathbb{R}^+, \mu(x,t,z) = \left(\frac{\delta^2}{2u_{X_0}} z + \frac{t(2t_0 - t)}{t_0^2} (\delta^2 - |x - x_0|^2) \right)_+,$$

where $(g)_+ := \max\{g, 0\}$ denotes the positive part of g and $k > 0$ is a constant at our disposal. Observe, that μ is zero on the boundary of $Q_\delta(X_0)$, non-negative in $Q_\delta(X_0)$ and differentiable, whenever it is positive. Define the function

$$h(x,t) := \eta(x,t,u(x,t)) \cdot v(x,t).$$

Since $h(x_0, t_0) = \mathrm{e}^{\frac{1}{2}k\delta^2} - 1 \neq 0$, the set on which $h(x,t)$ is positive is not empty. Moreover, $h(x,t) = 0$ on the parabolic boundary of $Q_\delta(X_0)$ as well as $h(x, 2t_0) = 0$ for every $x \in B_\delta(x_0)$. Hence, there must be a point P_0 in the interior of $Q_\delta(X_0)$, at which $h(x,t)$ attains a positive maximum. At this point we have

i) $h_i = 0$,

ii) $h_{ij} \leq 0$ (negative semidefinite Hessian),

iii) $\dot{h} = 0$.

Now, write $(\eta)_i$ for $\frac{\mathrm{d}\eta}{\mathrm{d}x_i}$ and compute

$$\begin{aligned} Lh &= g^{ij}(\eta v)_{ij} - \frac{\mathrm{d}}{\mathrm{d}t}(\eta v) \\ &= g^{ij}((\eta)_{ij} v + (\eta)_i v_j + (\eta)_j v_i + \eta v_{ij}) - \dot{\eta} v - \eta \dot{v} \\ &= vL\eta + \eta L v + 2g^{ij}(\eta)_i v_j. \end{aligned}$$

Since at P_0 we have

$$0 = h_i = (\eta)_i v + \eta v_i \Leftrightarrow (\eta)_i = -\frac{\eta v_i}{v},$$

we can insert this expression for $(\eta)_i$ to obtain

$$Lh = vL\eta + \eta\left(Lv - \frac{2}{v} g^{ij} v_j v_i\right). \tag{4.7}$$

In the next step we show that

$$Lv - \frac{2}{v} g^{ij} v_i v_j \geq 0. \tag{4.8}$$

Calculating the derivatives of v yields

$$\begin{aligned} v_i &= \frac{u_k u_{ki}}{v} = \nu^k u_{ki}, \\ v_{ij} &= \nu^k u_{kij} + u_{ki}\left(\frac{u_{kj} v - u_k v_j}{v^2}\right) = \nu^k u_{kij} + \frac{1}{v}(u_{ki} u_{kj} - \nu^l u_{lj} u_{ki} \nu^k). \end{aligned} \tag{4.9}$$

Note, that if we interpret the expression

$$u_{ki} u_{kj} - \nu^l u_{lj} u_{ki} \nu^k =: M_{ij}$$

as matrix $M = (M_{ij}) \in \mathbb{R}^{n \times n}$, then M is positive semi-definite as long as $|\nu| \leq 1$. To see this, let $U = (u_{ij}) \in \mathbb{R}^{n \times n}$ and $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$. Further let $\otimes$ be the dyadic product of two vectors so that for $x, y \in \mathbb{R}^n$ we have $x \otimes y = x \cdot y^T$ where the dot now denotes the usual matrix multiplication and the scalar product is denoted by $\langle \cdot, \cdot \rangle$.

Since U is symmetric, we may express M by

$$M = UU - ((U\nu) \otimes (U\nu)) = UU - U(\nu \cdot \nu^T)U = U(I - \nu\nu^T)U,$$

where I is the unit matrix. Hence, we obtain with $Ux = y \in \mathbb{R}^n$

$$x^T M x = x^T U^T (I - \nu\nu^T) U x = y^T (I - \nu\nu^T) y = |y|^2 - |\langle y, \nu\rangle|^2 \geq 0$$

if $|\nu| \leq 1$ and thus M is a positive semi-definite matrix. Multiplying (4.9) by g^{ij} yields

$$g^{ij} v_{ij} = g^{ij} u_{ijk} \nu^k + \frac{1}{v} g^{ij} (u_{ki} u_{kj} - \nu^k u_{ki} u_{lj} \nu^l) \tag{4.10}$$

and by interpreting the last term as trace of a product of the two positive semi-definite matrices $G := g^{ij}$ and $M = M_{ij}$ we obtain [cf. Appendix, Sec. 5.2]

$$\frac{1}{v} g^{ij} (u_{ki} u_{kj} - \nu^k u_{ki} u_{lj} \nu^l) = \frac{1}{v} tr(GM) \geq 0.$$

Thus, the only expression from (4.10) that remains to be estimated is $g^{ij} u_{ijk} \nu^k$. We use the (PDE) to continue

$$g^{ij} u_{ijk} \nu^k = (g^{ij} u_{ij})_k \nu^k - g^{ij}_k u_{ij} \nu^k = \left(\dot{u} + \frac{\gamma}{u}\right)_k \nu^k - g^{ij}_k u_{ij} \nu^k.$$

Furthermore, we have

$$\begin{aligned} -g^{ij}_k = (\nu^i \nu^j)_k &= \frac{1}{v}((u_{ik} - \nu^i \nu^l u_{lk})\nu^j + \nu^i (u_{jk} - \nu^j \nu^l u_{kl})) \\ &= \frac{2}{v}(u_{ik} - \nu^i \nu^l u_{lk})\nu^j, \end{aligned}$$

which leads to

$$\begin{aligned} -g^{ij}_k u_{ij} \nu^k &= \frac{2}{v}(u_{ik} - \nu^i \nu^l u_{lk})\nu^j u_{ij} \nu^k \\ &= \frac{2}{v}(v_i v_i - v_l \nu^l v_i \nu^i) = \frac{2}{v}(g^{ij} v_i v_j). \end{aligned}$$

Collecting terms, we have shown that

$$\begin{aligned} g^{ij} v_{ij} &= g^{ij} u_{ijk} \nu^k + \frac{1}{v} g^{ij} (u_{ki} u_{kj} - \nu^k u_{ki} u_{lj} \nu^l) \\ &\geq g^{ij} u_{ijk} \nu^k = \left(\dot{u} + \frac{\gamma}{u}\right)_k \nu^k + \frac{2}{v} g^{ij} v_i v_j. \end{aligned} \tag{4.11}$$

Returning to the original task of showing (4.8), we also need the time-derivative of v, which is given by

$$\dot{v} = \dot{u}_k \frac{u_k}{v} = \dot{u}_k \nu^k.$$

Thus, by using this relation and (4.11) we obtain

$$\begin{aligned} Lv - \frac{2}{v} g^{ij} v_i v_j &= -\dot{v} + g^{ij} v_{ij} - \frac{2}{v} g^{ij} v_i v_j \\ &\geq -\dot{v} + \left(\dot{u} + \frac{\gamma}{u}\right)_k \nu^k + \frac{2}{v} g^{ij} v_i v_j - \frac{2}{v} g^{ij} v_i v_j \\ &= -\dot{u}_k \nu^k + \dot{u}_k \nu^k - \frac{\gamma u_k}{u^2} \nu^k = -\gamma \frac{|Du|^2}{vu^2} > 0, \end{aligned}$$

since $-\gamma > 0$.
Now that (4.8) is proven, we obtain from (4.7) together with $\dot{h} = 0$ in P_0

$$Lh = g^{ij} h_{ij} - \dot{h} = g^{ij} h_{ij} \geq vL\eta. \tag{4.12}$$

Observe that, since h_{ij} is negative semi-definite at the point P_0, its product with the positive semi-definite matrix h^{ij} has a non-positive trace, that is

$$g^{ij} h_{ij} \leq 0.$$

Thus, if the assumption of a large (unbound) gradient leads to the estimate $L\eta < 0$, we obtain a contradiction and the gradient must be bounded. Therefore, we continue calculating the components of $L\eta = g^{ij}(\eta)_{ij} - \dot{\eta}$. Since $\eta = f \circ \mu(x, t, u(x, t))$ with f and μ defined as before, we have

$$\begin{aligned} \dot{\eta} &= f'(\partial_t \mu + \mu_z \dot{u}), \\ (\eta)_i &= f'(\mu_i + \mu_z u_i), \\ (\eta)_{ij} &= f''(\mu_i + \mu_z u_i)(\mu_j + \mu_z u_j) + f'(\mu_{ij} + \mu_{iz} u_j + \mu_{zj} u_i + \mu_{zz} u_i u_j + \mu_z u_{ij}), \end{aligned}$$

where f and its derivatives are evaluated at μ and $\partial_t \mu$ stands for the derivative of $\mu(x, t, z)$ in t-direction. We see from the definition of μ, that

$$0 \leq \mu \leq \delta^2 \leq 1, \quad \mu_z = \frac{\delta^2}{2u_{X_0}} \leq \frac{1}{2u_{X_0}}, \quad \mu_{zz} = \mu_{iz} = 0, \quad \mu_{ij} = -2\delta^{ij},$$

which simplifies the expression for $(\eta)_{ij}$ to

$$(\eta)_{ij} = f'' \left(\mu_i + \frac{\delta^2}{2u_{X_0}} u_i\right) \left(\mu_j + \frac{\delta^2}{2u_{X_0}} u_j\right) + f' \left(\mu_{ij} + \frac{\delta^2}{2u_{X_0}} u_{ij}\right).$$

Now, we calculate $L\eta$

$$\begin{aligned} L\eta &= -\dot{\eta} + g^{ij}(\eta)_{ij} \\ &= f'' g^{ij}\left(\mu_i\mu_j + \frac{\delta^2}{2u_{X_0}}\mu_i u_j + \frac{\delta^2}{2u_{X_0}}\mu_j u_i + \frac{\delta^4}{4u_{X_0}^2}u_i u_j\right) \\ &+ f'\left(g^{ij}\mu_{ij} - \partial_t\mu + \frac{\delta^2}{2u_{X_0}}(g^{ij}u_{ij} - \dot{u})\right). \end{aligned} \tag{4.13}$$

Here, we have

$$\begin{aligned} &g^{ij}\mu_i\mu_j \geq \lambda|\mu|^2 \geq 0, \\ &g^{ij}u_i u_j = \sum_i u_i^2 - \sum_{i,j}\frac{u_i^2 u_j^2}{1+|Du|^2} = \sum_i u_i^2\left(\frac{1+|Du|^2 - \sum_j u_j^2}{1+|Du|^2}\right) = \frac{|Du|^2}{1+|Du|^2}, \end{aligned}$$

with λ being the smallest eigenvalue of g^{ij} and

$$\begin{aligned} g^{ij}(\mu_i u_j + \mu_j u_i) &= 2\sum_i \mu_i u_i - \frac{1}{1+|Du|^2}\sum_{i,j}\mu_i u_i u_j^2 + \mu_j u_j u_i^2 \\ &= 2\sum_i \mu_i u_i \frac{1+|Du|^2 - \sum_j u_j^2}{1+|Du|^2} = \frac{2\mu_i u_i}{1+|Du|^2}. \end{aligned}$$

Using the (PDE), we may also deduce

$$\frac{\delta^2}{2u_{X_0}}(g^{ij}u_{ij} - \dot{u}) = \frac{\gamma\delta^2}{2uu_{X_0}} > 0,$$

because $\gamma, u < 0$ and $u_{X_0} > 0$. Inserting these results into expression (4.13) for $L\eta$ we are left with

$$\begin{aligned} L\eta &\geq f''\left(\frac{\delta^2}{2u_{X_0}}\frac{2\mu_i u_i}{1+|Du|^2} + \frac{\delta^4}{4u_{X_0}^2}\frac{|Du|^2}{1+|Du|^2}\right) + f'(g^{ij}\mu_{ij} - \partial_t\mu) \\ &= f''\frac{\delta^4|Du|^2 + 4\delta^2 u_{X_0}\mu_i u_i}{4u_{X_0}^2(1+|Du|^2)} + f'(g^{ij}\mu_{ij} - \partial_t\mu). \end{aligned}$$

From the properties of μ we infer

$$\mu_{ij} = -2\delta^{ij} \Rightarrow g^{ij}\mu_{ij} = -2g^{ii} = -2\left(\delta^{ii} - \frac{|Du|^2}{1+|Du|^2}\right) \geq -2n$$

as well as

$$-\partial_t\mu \geq -\frac{4\delta^2}{t_0},$$

so that the coefficient of f' is bounded below by $-(2n + \frac{4\delta^2}{t_0})$. By our choice of δ this expression is always bounded below, no matter what we choose for t_0. For instance if

t_0 is close to 0, then

$$\delta \leq \frac{t_0}{2} \Rightarrow -\frac{4\delta^2}{t_0} \geq -t_0 \overset{t_0 \to 0}{\to} 0$$

and if t_0 is large, then

$$\delta \leq 1 \Rightarrow -\frac{4\delta^2}{t_0} \geq -\frac{4}{t_0} \overset{t_0 \to \infty}{\to} 0.$$

For the coefficient of f'' we estimate the enumerator of the fraction first. Whenever μ is not zero it follows

$$|D\mu|^2 = \sum_{i=1}^{n} \mu_i^2 = \frac{t^2(2t_0 - t)^2}{t_0^4}(4|x - x_0|^2) \leq 4\delta^2$$

and thus we can estimate

$$\mu_i \leq |D\mu| \leq 2\delta \quad \text{and also} \quad u_i \leq |Du|.$$

Hence, if we assume $|Du| \geq \frac{16u_{X_0}}{\delta}$, we obtain for the enumerator

$$\delta^4|Du|^2 + 4\delta^2 u_{X_0} u_i \mu_i \geq \delta^4|Du|^2 - 8\delta^3|Du|u_{X_0} \geq \frac{\delta^4}{2}|Du|^2.$$

If we further assume that $|Du| > \max\left(3, \frac{16u_{X_0}}{\delta}\right)$, we can estimate the coefficient of f'' by

$$\frac{\delta^4|Du|^2 + 4\delta^2 u_{X_0} u_i \mu_i}{4u_{X_0}^2(1 + |Du|^2)} \geq \frac{\delta^4}{8u_{X_0}^2} \frac{|Du|^2}{1 + |Du|^2} > \frac{\delta^4}{10u_{X_0}^2}.$$

Summarizing the calculations up to this point we see that $L\eta$ is bounded below by an expression of the form $c_1 f'' + c_2 f'$, where $c_1 > 0, c_2 > -\infty$ and $f(\mu) = \mathrm{e}^{k\mu} - 1$. The derivatives of f are $f' = k\mathrm{e}^{k\mu}$, $f'' = k^2\mathrm{e}^{k\mu}$ so that by choosing k sufficiently large, we can guarantee that $L\eta \geq 0$. Hence, under the assumption that $|Du| > \max\left(3, \frac{16u_{X_0}}{\delta}\right)$, we have shown the relation (cf. (4.12))

$$g^{ij}h_{ij} \geq vL\eta > 0 \quad \text{in } P_0.$$

But since also $g^{ij}h_{ij} \leq 0$ in P_0 as we have shown earlier, the assumption $|Du| > \max\left(3, \frac{16u_{X_0}}{\delta}\right) := \tilde{C}$ at P_0 must be wrong. Hence,

$$|Du(P_0)| \leq \tilde{C} \Rightarrow v(P_0) \leq 1 + \tilde{C} =: \hat{C}.$$

Thus, for every point $X \in Q_\delta(X_0)$ we have

$$\eta(x, t, u(x)) \cdot v(x) \leq \eta(P_0) \cdot v(P_0) \leq \hat{C} \cdot \mathrm{e}^{k\mu(P_0)} \leq \hat{C} \cdot \mathrm{e}^{k\delta^2}. \tag{4.14}$$

Especially for $X = X_0$ we obtain

$$\left(\mathrm{e}^{\frac{1}{2}k\delta^2} - 1\right) v(X_0) \leq \hat{C} \cdot \mathrm{e}^{k\delta^2}, \tag{4.15}$$

which yields the desired interior gradient bound. □

4.4 Existence of a Singularity after Finite Time for $(P^+_{\phi_0,u_0})$

We have proven earlier, that for sufficiently large initial and boundary values u_0, ϕ_0 there is always a solution for the case $\gamma > 0$ which exists for all times $t > 0$. Hence, the question arises what happens if the boundary values are lower than the assumed threshold. From the structure of the parabolic operator P with

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u - \frac{\gamma}{u}$$

it is clear that if $u \to 0$, then $u \notin C^{2,1}$, since if we had $u \in C^{2,1}$ then

$$\left| -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u \right| < \infty,$$

but also

$$\lim_{u \to 0} \left| \frac{\gamma}{u} \right| = \infty.$$

Thus, we must have some sort of irregularity whenever $u = 0$ so we may say that u has a singularity if it reaches the value 0. By comparison with cone-shaped surfaces which are slowly moving towards the $\{u = 0\}$-plane we will be able to demonstrate that there cannot be a solution that exists for all times $t > 0$, if the initial and boundary values ϕ_0, u_0 are too small.

Proposition 4.7 *Let $\Omega = D \times (0, T)$ with $T \leq \infty$. Consider the problem*

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u &= \frac{\gamma}{u} && \text{on } \Omega, \\ u(x,t) &= \phi_0(x) && \text{on } \mathcal{S}\Omega, \\ u(x,0) &= u_0(x) && \text{on } \mathcal{B}\Omega. \end{aligned} \tag{$P^+_{\phi_0,u_0}$}$$

Let $x_0 \in D$ be a point that fulfills

$$\operatorname{dist}(x_0, \partial D) \geq \operatorname{dist}(\tilde{x}, \partial D), \quad \text{for every } \; \tilde{x} \in \overline{D},$$

which always exists since the distance function is continuous and $\overline{D}$ is a compact set. Set $r_0 := dist(x_0, \partial D)$ and let $\delta \in \mathbb{R}^+, 0 \leq t_0 < T$ be arbitrary values. If the initial and *boundary values suffice*

$$\sup_{x\in D} \phi_0 < r_0\sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}}, \quad \sup_{x\in D} u_0 < \delta\sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}} = \sqrt{\frac{\gamma}{\frac{n}{\delta^2}+\frac{1}{2t_0}}},$$

then there is no classical solution $u \in C^{2,1}(\Omega)\cap C(\overline{\Omega})$ *to problem* $(P^+_{\phi_0,u_0})$. □

Proof. We intend to apply the weak comparison principle [cf. Appendix, Sec. 5.3]. Recall that the parabolic operator P is given by

$$Pu = -\dot{u} + vH(u) - \frac{\gamma}{u} = 0,$$

with $v = \sqrt{1+|Du|^2}$ and

$$H(u) = D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right)$$

being the mean curvature of graph(u). Let x_0, t_0 be as in the proposition and define $\Omega_{t_0} := D\times(0,t_0)$. Now, consider the function

$$f(x,t) := a\sqrt{\frac{\delta^2}{t_0}(t_0-t)+|x-x_0|^2},$$

with $a,\delta\in\mathbb{R}^+$. Observe, that

$$f(x_0,t_0)=0 \quad \text{and} \quad f(x,t)>0 \quad \text{for} \quad (x,t)\in(\Omega_{t_0}\cap\mathcal{P}\Omega_{t_0}).$$

Let us assume, that $u(x,t)$ is a classical solution to $(P^+_{\phi_0,u_0})$ in every point $(x,t)\in\Omega_{t_0}$, which especially implies that there are no singularities up to that time. By using the comparison principle we show that u is bounded above by f. Then, for $(x,t)\to(x_0,t_0)$ we obtain $0\le u\le f=0$ and thus u must have a singularity at the time t_0.
We begin by calculating the derivatives of f

$$\dot{f} = -\frac{a\delta^2}{2t_0}\left(\sqrt{\frac{\delta^2}{t_0}(t_0-t)+|x-x_0|^2}\right)^{-1}$$

and also, by abbreviating $g(t) := \frac{\delta^2}{t_0}(t_0 - t)$ with $x_0 = (x_1^0, \ldots, x_n^0)$,

$$
\begin{aligned}
D_i f &= a \frac{x_i - x_i^0}{\sqrt{g(t) + |x - x_0|^2}}, \\
1 + |Df|^2 &= 1 + a^2 \frac{|x - x_0|^2}{g(t) + |x - x_0|^2} = \frac{g(t) + (1 + a^2)|x - x_0|^2}{g(t) + |x - x_0|^2}, \\
H(f) &= D_i \left(a \frac{x_i - x_i^0}{\sqrt{g(t) + |x - x_0|^2}} \cdot \frac{\sqrt{g(t) + |x - x_0|^2}}{\sqrt{g(t) + (1 + a^2)|x - x_0|^2}} \right) \\
&= a D_i \left(\frac{x_i - x_i^0}{\sqrt{g(t) + (1 + a^2)|x - x_0|^2}} \right) \\
&= a \sum_{i=1}^{n} \frac{\sqrt{g(t) + (1 + a^2)|x - x_0|^2} - (x_i - x_i^0) \cdot \frac{(1+a^2)(x_i - x_i^0)}{\sqrt{g(t) + (1+a^2)|x - x_0|^2}}}{g(t) + (1 + a^2)|x - x_0|^2} \\
&= a \cdot \frac{n[g(t) + (1 + a^2)|x - x_0|^2] - (1 + a^2)|x - x_0|^2}{[g(t) + (1 + a^2)|x - x_0|^2]^{\frac{3}{2}}}.
\end{aligned}
$$

Hence, we obtain for Pf

$$
\begin{aligned}
Pf &= -\dot{f} + \sqrt{1 + |Df|^2} H(f) - \frac{\gamma}{f} \\
&= \frac{a\delta^2}{2t_0} \cdot \frac{1}{\sqrt{g(t) + |x - x_0|^2}} \\
&+ \frac{\sqrt{g(t) + (1 + a^2)|x - x_0|^2}}{\sqrt{g(t) + |x - x_0|^2}} \cdot a \cdot \frac{n[g(t) + (1 + a^2)|x - x_0|^2] - (1 + a^2)|x - x_0|^2}{[g(t) + (1 + a^2)|x - x_0|^2]^{\frac{3}{2}}} \\
&- \frac{\gamma}{a\sqrt{g(t) + |x - x_0|^2}} \\
&= \frac{a}{\sqrt{g(t) + |x - x_0|^2}} \cdot \left(\frac{\delta^2}{2t_0} + n - \frac{(1 + a^2)|x - x_0|^2}{g(t) + (1 + a^2)|x - x_0|^2} - \frac{\gamma}{a^2} \right).
\end{aligned}
$$

To show that f lies above the solution u, we need to achieve

$$
\begin{aligned}
Pf < Pu = 0 &\quad \text{on } \Omega_{t_0}, \\
f > u &\quad \text{on } \mathcal{P}\Omega_{t_0}.
\end{aligned}
$$

Since

$$
\frac{a}{\sqrt{g(t) + |x - x_0|^2}} > 0 \quad \text{and} \quad \frac{(1 + a^2)|x - x_0|^2}{g(t) + (1 + a^2)|x - x_0|^2} \in [0, 1)
$$

for every $(x, t) \in \Omega_{t_0}$, the inequality $Pf < 0$ is fulfilled if

$$
\frac{\delta^2}{2t_0} + n - \frac{\gamma}{a^2} < 0 \Leftrightarrow a < \sqrt{\frac{\gamma}{n + \frac{\delta^2}{2t_0}}}.
$$

Set $r_0 := \operatorname{dist}(x_0, \partial D)$ and consider the cylinder $P_{r_0}(x_0,t_0) := B_{r_0} \times (0,t_0)$. Note, that B_{r_0} is a „largest ball" contained in D (there might be more than one of these balls). Then we obtain on $\mathcal{P}P_{r_0}$

$$\begin{aligned} f(x,t) &= a\sqrt{\frac{\delta^2}{t_0}(t_0-t)+r_0^2} \geq a r_0 \quad \text{on } \partial B_{r_0}(x_0)\times(0,t_0),\\ f(x,0) &= a\sqrt{\delta^2+|x-x_0|^2} \geq a\delta \qquad \text{on } B_{r_0}(x_0)\times\{0\}. \end{aligned}$$

Since $f(x,t)$ is increasing with greater distance to x_0, we can use the same estimates on $D\times\{0\}$ and $\partial D\times(0,t_0)$ respectively. Assume now, that

$$\sup_{x\in D}\phi_0 < r_0\sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}}, \quad \sup_{x\in D} u_0 < \delta\sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}}.$$

Then there is an $\epsilon > 0$, such that also

$$\sup_{x\in D}\phi_0 \leq r_0\left(\sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}}-\epsilon\right), \quad \sup_{x\in D} u_0 \leq \delta\left(\sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}}-\epsilon\right).$$

If we set

$$a := \sqrt{\frac{\gamma}{n+\frac{\delta^2}{2t_0}}}-\frac{\epsilon}{2}$$

the claim follows. □

Remark. Observe, that by increasing δ, the bound for ϕ_0 becomes smaller while the one for u_0 increases. Thus, there is no obvious choice for δ. However, we can choose $\delta = r_0$, which maximizes the value of the minimum of both data, implying that there is no solution, if

$$\max\left\{\sup_{x\in D}\phi_0, \sup_{x\in D} u_0\right\} \leq \sqrt{\frac{\gamma}{n+\frac{r_0^2}{2t_0}}}\,r_0 = \sqrt{\frac{\gamma}{\frac{n}{r_0^2}+\frac{1}{2t_0}}}.$$

□

Corollary. We can use the comparison principle the opposite way to improve the lower bounds from chapter 2 for large $\gamma > 1$. The function

$$f(x,t) = a\sqrt{\frac{\delta^2}{t_0}(t_0-t)+|x-x_0|^2}$$

lies beneath a solution u to $(P^+_{\phi_0,u_0>c})$ on $\Omega = D \times (0,T)$ with $0 < T \leq \infty$, if

$$\begin{aligned} Pf > Pu = 0 \quad &\text{on } \Omega, \\ f < u \quad &\text{on } \mathcal{P}\Omega. \end{aligned}$$

Let $x_0 \in D$ be an arbitrary point and set $t_0 = T, d := \operatorname{diam}(D)$. The condition $Pf > 0$ is fulfilled if

$$\frac{\delta^2}{2T} + n - 1 - \frac{\gamma}{a^2} > 0 \Leftrightarrow a > \sqrt{\frac{\gamma}{n + \frac{\delta^2}{2T} - 1}}.$$

Restricting the problem to the cylindrical domain $P_d(x_0) := B_d(x_0) \times (0,T)$, we obtain on its boundary

$$\begin{aligned} f(x,t) &= a\sqrt{\frac{\delta^2}{T}(T-t) + d^2} \leq a\sqrt{\delta^2 + d^2} \quad \text{on } \partial B_d(x_0) \times (0,T), \\ f(x,0) &= a\sqrt{\delta^2 + |x - x_0|^2} \leq a\sqrt{\delta^2 + d^2} \quad \text{on } B_d(x_0) \times \{0\}. \end{aligned}$$

Now, observe that $\overline{D} \subset\subset B_d$. Since $f(x,t)$ is monotonically increasing with greater distance to x_0, the above conditions are especially fulfilled on $\mathcal{P}\Omega$. Thus, there is a solution to $(P^+_{\phi_0,u_0>c})$, if

$$\min\left\{\inf_{x\in D} \phi_0, \inf_{x\in D} u_0\right\} > \sqrt{\frac{\gamma(\delta^2 + d^2)}{n + \frac{\delta^2}{2T} - 1}} \overset{\delta \to 0}{\to} \sqrt{\frac{\gamma}{n-1}}d.$$

□

4.5 Solutions to $(P^+_{\phi_0,u_0})$ for low Initial and Boundary Values

Throughout this chapter let $\Omega = D \times (0,T)$ with $0 < T \leq \infty$ and $H_D(y) \geq 0$ for all $y \in \partial D$, where $H_D(y)$ denotes the inward mean curvature of D at the point y. Consider the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u &= \frac{\gamma}{u} \quad \text{on } \Omega, \\ u(x,t) &= \psi_0(x) \quad \text{on } \mathcal{P}\Omega, \end{aligned} \tag{$P^+_{\psi_0}$}$$

where $\psi_0(x) \in H_{2+\alpha}(\Omega), \psi_0(x) > 0$ with

$$\psi_0(x) = \begin{cases} u_0(x), & \text{if } x \in D \\ \phi_0(x), & \text{if } x \in \partial D \end{cases}$$

are the initial and boundary values that can now be chosen arbitrarily small.

Proposition 4.8 (Lower bound for small times) *Set*

$$\underline{\psi_0} := \min_{x \in \overline{D}} \psi_0(x), \quad T_0 := \frac{\underline{\psi_0}^2}{2\gamma}, \quad \Omega_t := D \times (0,t).$$

For every $\tau \in (0, T_0)$ there is a value $\delta > 0$ independent of the solution $u(x,t)$, so that every classical solution $u(x,t)$ to the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} \qquad \text{on } \Omega_\tau, \\ u(x,t) &= \psi_0(x) \quad \text{on } \mathcal{P}\Omega_\tau, \end{aligned} \tag{P_τ}$$

is a priori bounded below by δ. □

Proof. We use the weak comparison principle to construct a lower bound [cf. Appendix, Sec. 5.3]. Let $\delta > 0$ be a constant at our disposal. For $t_0 > 0$ to be specified later consider the function

$$f\colon (0, t_0) \to \mathbb{R}, f(t) = \sqrt{2\gamma(t_0 - t)} + \delta,$$

with derivative

$$f'(t) = -\frac{\sqrt{\gamma}}{\sqrt{2(t_0 - t)}}.$$

Given the parabolic operator

$$Pu = -\dot{u} + vH(u) - \frac{\gamma}{u},$$

the function $f(t)$ satisfies

$$Pf = -f'(t) - \frac{\gamma}{f} = \frac{\sqrt{\gamma}}{\sqrt{2(t_0 - t)}} - \frac{\sqrt{\gamma}}{\sqrt{2(t_0 - t)} + \frac{\delta}{\sqrt{\gamma}}} > 0$$

for every $t \in (0, t_0), \delta > 0$ and thus $Pf > Pu$ on Ω_{t_0}. For the estimate on the boundary let t_0 fulfill

$$\sqrt{2\gamma t_0} + \delta < \underline{\psi_0}, \tag{4.16}$$

which means that for small $\epsilon > 0$ we can take t_0 as

$$\sqrt{2\gamma t_0} + \delta = \underline{\psi_0} - \epsilon \Leftrightarrow t_0(\epsilon, \delta) = \frac{(\underline{\psi_0} - \epsilon - \delta)^2}{2\gamma},$$

where now ϵ and δ are chosen to fulfill $\epsilon + \delta < \underline{\psi_0}$. Then (4.16) implies, that $f(0) <$

$u(x,0)$ and since $f(t)$ is monotonically decreasing in t we also have $f(t) < u(x,t)$ for all $(x,t) \in \partial D \times (0,t_0)$. Hence, it is

$$f(t) < u(x,t) \quad \text{on } \mathcal{P}\Omega_{t_0}.$$

Since P is a parabolic operator, we can apply the weak comparison principle to obtain

$$f(t) < u(x,t) \quad \text{on } \Omega_{t_0}.$$

By letting $\epsilon, \delta \to 0$ this estimate proves, that for every $\tau \in (0,T_0)$ there is a value $\delta > 0$, independent of the solution $u(x,t)$, such that

$$0 < \delta \leq f(t) < u(x,t) \quad \text{for all } (x,t) \in \Omega_\tau,$$

which concludes the proof. □

Corollary. The lower bound from proposition 4.8 together with the other a priori estimates developed in chapter 2 allow us to conclude that for any boundary and initial values $\psi_0(x) > 0$ there is a (small) time $T_0 > 0$ such that for every $\tau \in (0,T_0)$ the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} \qquad \text{on } \Omega_\tau, \\ u(x,t) &= \psi_0(x) \quad \text{on } \mathcal{P}\Omega_\tau, \end{aligned} \tag{P_τ}$$

has a unique classical solution, which is regular in the interior and continuous up to the boundary of Ω_τ. □

Next we prove, that the unique classical solution which was obtained for small times $0 < \tau < T_0$ exists up to a time $\hat{T} \geq T_0$, at which for the first time

$$\lim_{t \uparrow \hat{T}} \min_{x \in \overline{D}} u(x,t) = 0.$$

In other words, the solution exists as long as $\min\limits_{x \in \overline{D}} u(x,t) > 0$. For the proof we make use of the following lemma.

Lemma 4.9 *Let $\Omega = D \times (0,T)$ with $0 < T \leq \infty$ and $H_D(y) \geq 0$ for all $y \in \partial D$. Assume, that $u(x,t)$ is the unique classical solution to the problem*

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u &= \frac{\gamma}{u} \qquad \text{on } \Omega, \\ u(x,t) &= \psi_0(x) \quad \text{on } \mathcal{P}\Omega, \end{aligned} \tag{$P^+_{\psi_0}$}$$

for every $\tau \in (0,T_0)$ with $0 < T_0 \leq T$. Then exactly one of the two statements is *correct.*

i) $\lim_{t\uparrow T_0} \min_{x\in\overline{D}} u(x,t) = 0.$

ii) $u(x,T_0) > 0$ *for all* $x \in D$ *and there is a time* $\bar{T} > T_0$ *so that* $(P^+_{\psi_0})$ *has a unique solution for all* $\tau \in (0,\bar{T})$.

□

Proof. Denote by $u(x,t)$ the unique solution for every $\tau \in (0,T_0)$ and define

$$\underline{u}(t) := \min_{x\in\overline{D}} u(x,t).$$

For $\epsilon,\delta > 0$ with $\epsilon+\delta < \underline{u}(t)$ consider the function

$$g\colon (0,T_0) \to \mathbb{R}, g(t) = t + \frac{(\underline{u}(t)-\epsilon-\delta)^2}{2\gamma}.$$

Then exactly one of the following two statements is correct.

i) There are numbers $\epsilon,\delta > 0$ with $\epsilon+\delta < \underline{u}(t)$ and $\tau_1 \in (0,T_0)$, so that $g(\tau_1) > T_0$.

ii) For every $\epsilon,\delta > 0$ with $\epsilon+\delta < \underline{u}(t)$ and every $t \in (0,T_0)$ we have $g(t) \le T_0$.

The second case implies

$$t + \frac{(\underline{u}(t)-\epsilon-\delta)^2}{2\gamma} \le T_0 \Leftrightarrow \underline{u}(t) \le \sqrt{2\gamma(T_0-t)} + \epsilon + \delta$$

for every $\epsilon,\delta > 0$ with $\epsilon+\delta < \underline{u}(t)$ and $t \in (0,T_0)$ and hence, by letting $\epsilon,\delta \to 0$,

$$\lim_{t\uparrow T_0} \min_{x\in\overline{D}} u(x,t) = 0.$$

If instead the first case is true, define

$$t_1(\epsilon,\delta) := \frac{(\underline{u}(\tau_1)-\epsilon-\delta)^2}{2\gamma}, \quad \bar{T}(\epsilon,\delta) = \tau_1 + t_1(\epsilon,\delta) = g(\tau_1) > T_0,$$

as well as for some small $\bar{\delta} > 0$

$$f_1(t)\colon (\tau_1,\tau_1+t_1) \to \mathbb{R}, f_1(t) = \sqrt{2\gamma(t_1-(t-\tau_1))} + \bar{\delta}.$$

For $\tau \in (\tau_1,\bar{T})$ and $\Omega_{\tau_1,\tau} := D \times (\tau_1,\tau)$ consider the problem

$$\begin{aligned} -\dot{v} + \Delta v - \frac{D_i v D_j v}{1+|Dv|^2} D_{ij} v = \frac{\gamma}{v} \qquad & \text{on } \Omega_{\tau_1,\tau}, \\ v(x,t) = u(x,\tau_1) \quad & \text{on } \mathcal{P}\Omega_{\tau_1,\tau}. \end{aligned}$$

As in the proof of proposition 4.8, we can use the weak comparison principle to show that the function $f_1(t)$ lies below every solution $v(x,t)$ to the problem above for all times $t \in (\tau_1, \bar{T})$. Thus, by the a priori estimates of section 2 there is a unique solution $v(x,t)$ which solves the problem for $t \in (\tau_1, \bar{T})$ and agrees with the unique solution $u(x,t)$ for the problem with times $0 < t < T_0$ at time $t = \tau_1$. Hence, the function

$$U(x,t) = \begin{cases} u(x,t), & \text{if } t \in [0, \tau_1) \\ v(x,t), & \text{if } t \in [\tau_1, \bar{T}) \end{cases}$$

solves the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad & \text{on } \Omega_\tau, \\ u(x,t) = \psi_0(x) \quad & \text{on } \mathcal{P}\Omega_\tau, \end{aligned}$$

for every $\tau \in (0, \bar{T})$ with $\bar{T} > T_0$. □

By the application of Lemma 4.9 we obtain

Proposition 4.10 (Maximum Existence Time $\hat{T}$) *Let $\Omega = D \times (0,T)$ with $0 < T \leq \infty$ and $H_D(y) \geq 0$ for every $y \in \partial D$. The problem*

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad & \text{on } \Omega, \\ u(x,t) = \psi_0(x) \quad & \text{on } \mathcal{P}\Omega, \end{aligned} \tag{$P^+_{\psi_0}$}$$

with $\psi_0(x) \in H_{2+\alpha}(\Omega), \psi_0(x) > 0$ has a unique solution that exists up to a time $\hat{T}$, at which for the first time

$$\lim_{t \uparrow \hat{T}} \min_{x \in \overline{D}} u(x,t) = 0.$$

□

Proof. According to proposition 4.8 there is a time $T_0 > 0$, such that there is a unique solution for all $\tau \in (0, T_0)$. Now, distinguish the two cases from Lemma 4.9. If the first case is true we are done. If the second case is true, then there is a unique solution $u(x,t)$ for all times $\tau \in (0, \bar{T})$ with $\bar{T} > T_0$. Thus, we must have $\underline{u}(T_0) := \min_{x \in \overline{D}} u(x, T_0) > 0$. Now define for small $\epsilon_0, \delta_0 > 0$ with $\epsilon_0 + \delta_0 < \underline{u}(T_0)$

$$t_1(\epsilon_0, \delta_0) := \frac{(\underline{u}(T_0) - \epsilon_0 - \delta_0)^2}{2\gamma}, \quad T_1 = T_0 + \frac{\underline{u}^2(T_0)}{2\gamma},$$

as well as for small $\delta_1 > 0$

$$f_1(t)\colon (T_0, T_0 + t_1) \to \mathbb{R}, f_1(t) = \sqrt{2\gamma(t_1 - (t - T_0))} + \delta_1.$$

For $\tau \in (T_0, T_1)$ and $\Omega_{T_0,\tau} := D \times (T_0, \tau)$ consider the problem

$$\begin{aligned} -\dot{v} + \Delta v - \frac{D_i v D_j v}{1 + |Dv|^2} D_{ij} v = \frac{\gamma}{v} \qquad & \text{on } \Omega_{T_0,\tau}, \\ v(x,t) = u(x, T_0) \quad & \text{on } \mathcal{P}\Omega_{T_0,\tau}. \end{aligned}$$

By the same argument as in proposition 4.8, the function $f_1(t)$ can be used to show the existence of a $\delta > 0$, independent of the solution $u(x,t)$, such that

$$u(x,t) > \delta \quad \text{for all } t \in (T_0, T_0 + t_1), x \in \overline{D}.$$

This implies that there is a unique solution $v(x,t)$ for every $\tau \in (T_0, T_0 + t_1)$ which, for $\epsilon_0, \delta_0 \to 0$ can be extended to a unique solution for every $\tau \in (T_0, T_1)$. Moreover, it agrees at time $t = T_0$ with the solution $u(x,t)$ of the problem

$$\begin{aligned} -\dot{u} + \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u = \frac{\gamma}{u} \qquad & \text{on } \Omega_{\bar{T}}, \\ u(x,t) = \psi_0(x) \quad & \text{on } \mathcal{P}\Omega_{\bar{T}}, \end{aligned}$$

with $\bar{T} > T_0$. With the same reasoning as in Lemma 4.9, this implies the existence of a function

$$U(x,t) = \begin{cases} u(x,t), & \text{if } t \in (0, T_0) \\ v(x,t), & \text{if } t \in [T_0, T_1), \end{cases}$$

which is the unique solution to the problem for every $\tau \in (0, T_1)$. Repeating this process, we obtain a sequence of times, inductively defined by

$$T_{n+1} := \frac{\underline{u}^2(T_n)}{2\gamma} + T_n,$$

that can only converge to a time $\hat{T} < \infty$, if $\lim_{n\to\infty} \underline{u}(T_n) = 0$. Since we have shown in section 4.4, that for low initial and boundary values a singularity must occur after finite time, the claim follows. □

Now that we know that for low initial and boundary values there is a unique solution that remains smooth as long as it is positive, we would like to study its behavior at the time when the singularity occurs. Inspired by the elliptic case we have the following result, which holds under the assumption that the solution exists at time $\hat{T}$ when the singularity occurs and remains smooth in every point where it is positive. We then obtain the statement by working locally on the surface (see also the work from KORE-

VAAR and SIMON [KS87] as well as TENNSTÄDT [Ten17]).

Proposition 4.11 (A regularity result for solutions u to ($P^+_{\phi_0,u_0}$) with low boundary and initial values) *Let $\Omega = D \times (0,T)$ with $0 < T \leq \infty$. Assume that D has non-negative inward mean curvature $H_D(y) \geq 0$ for every $y \in \partial D$. Consider the problem*

$$-\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u = \frac{\gamma}{u} \quad \text{on } \Omega,$$
$$u(x,t) = \psi_0(x) \quad \text{on } \mathcal{P}\Omega,$$

where ϕ_0 and u_0 are chosen „small enough“ (cf. proposition 4.7) to force a singularity after finite time. Assume that $\hat{T}$ is the first time at which

$$\lim_{t\uparrow\hat{T}} \min_{x\in\overline{D}} u(x,t) = 0.$$

Define $\Omega_{\hat{T}} := D \times (0,\hat{T})$ and the sets of singular points

$$\mathcal{S} := \{(x,\hat{T}) \mid x \in D, u(x,\hat{T}) = 0\} \subset \mathbb{R}^{n+1}, \ \mathcal{S}_x := \{x \in D \mid u(x,\hat{T}) = 0\} \subset \mathbb{R}^n.$$

Assume that at least $u \in C^{3,1}(\Omega_{\hat{T}} \setminus \mathcal{S}) \cap C^0(\overline{\Omega_{\hat{T}}})$ as well as

$$\sup_{x\in\mathcal{K}} \dot{u}(x,\hat{T}) \leq C_0 < \infty$$

for every $\mathcal{K} \subset\subset (D \setminus \mathcal{S}_x)$. Then $u(\cdot,\hat{T})$ is $\frac{1}{2}$-Hölder continuous. □

Proof. We work locally on the surface graph(u) [cf. Appendix, Sec. 5.7]. First, note that since $u \in C^0(\overline{\Omega_{\hat{T}}})$ and $u(x,t) > 0$ for all $t \in [0,\hat{T})$, there is a number $\delta > 0$, which in general depends on the solution u, such that for every $T_0 > \hat{T}$ the set $\{u < \delta\} \subset\subset (D \times (0,T_0))$. Thus, there is a function $\phi \in C^2_c(\mathbb{R}^{n+1})$ with the properties

$$\phi = 0 \ \text{ in } \{u < \delta\}, \quad \phi = \psi_0 \ \text{ close to } \mathcal{P}\Omega_{\hat{T}}, \quad \|\phi\|_{C^2(\mathbb{R}^{n+1})} \leq C_\phi = C_\phi(\delta,\psi_0) < \infty.$$

Now define $\eta : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\eta(s) := (\mathrm{e}^{Ks} - 1)\mathrm{e}^{-2C_\phi K}$$

with a constant $K > 0$ to be chosen later. If we set $(u-\phi)^+ := \max\{u-\phi, 0\}$, we have $0 \leq \eta((u-\phi)^+) \leq 1$, since the weak comparison principle implies, that $u(x,t) < \phi(x) + \hat{\epsilon}$ for every $\hat{\epsilon} > 0$, which yields $u \leq C_\phi$. Let $\epsilon > 0$ and M be the maximum of

$$f(x,t) := \frac{\eta((u-\phi)^+)}{\nu^{n+1} + \epsilon}$$

on $\overline{\Omega}_{\hat{T}}$. $f(x,t)$ is continuous on $\overline{\Omega}_{\hat{T}}$, non-negative, $f = 0$ on $\{u = 0\} := \mathcal{S}$ and positive in $\{0 < u < \delta\}$, thus f attains its maximum M in a point $(x_M, t_M) \in \{u > 0\} := \Omega_{\hat{T}} \backslash \mathcal{S}$. At this point the function Ψ given by

$$\Psi(x,t) := \eta((u-\phi)^+) - M(\nu^{n+1} + \epsilon) \leq 0$$

fulfills

$$\Psi(x_M, t_M) = 0, \dot{\Psi}(x_M, t_M) \geq 0, \nabla\Psi(x_M, t_M) = 0, \Delta\Psi(x_M, t_M) \leq 0.$$

Calculate the gradient

$$\nabla\Psi = \eta'\nabla(u-\phi) - M\nabla\nu^{n+1}$$

and the Laplace operator

$$\Delta\Psi = \eta''|\nabla(u-\phi)|^2 + \eta'\Delta(u-\phi) - M\Delta\nu^{n+1}. \tag{4.17}$$

Since we are working locally on the surfaces, the identity $\Delta u = H\nu^{n+1}$ holds and from the Jacobi field equation we obtain

$$\Delta\nu^{n+1} = -\nu^{n+1}|A|^2 - e_{n+1} \cdot \nabla H.$$

Inserting both equations in (4.17) yields

$$\Delta\Psi = \eta''|\nabla(u-\phi)|^2 + \eta' H\nu^{n+1} - \eta'\Delta\phi + M\nu^{n+1}|A|^2 + M\nabla H \cdot e_{n+1}. \tag{4.18}$$

Making use of the equation $Pu = 0$ with

$$Pu = -\dot{u} + vH - \frac{\gamma}{u}$$

leads to

$$-\dot{u} + vH = \frac{\gamma}{u} \Leftrightarrow H = \nu^{n+1}\left(\frac{\gamma}{u} + \dot{u}\right), \tag{4.19}$$

which we can differentiate to obtain

$$\nabla H = \left(\nabla\nu^{n+1}\frac{\gamma}{u} - \nu^{n+1}\frac{\gamma}{u^2}\nabla u\right) + \left(\nabla\nu^{n+1}\dot{u} + \nu^{n+1}\nabla\dot{u}\right). \tag{4.20}$$

Inserting (4.19) and (4.20) for $H, \nabla H$ in (4.18) and sorting by terms with and without time derivative gives us

$$\begin{aligned}\Delta\Psi &= \left\{\eta''|\nabla(u-\phi)|^2 + \eta'\frac{\gamma}{u}(\nu^{n+1})^2 - \eta'\Delta\phi + M\nu^{n+1}|A|^2 + M\left(\nabla\nu^{n+1}\frac{\gamma}{u} - \nu^{n+1}\frac{\gamma}{u^2}\nabla u\right) \cdot e_{n+1}\right\} \\ &\quad + \left\{\eta'\dot{u}(\nu^{n+1})^2 + M\left(\nabla\nu^{n+1}\dot{u} + \nu^{n+1}\nabla\dot{u}\right) \cdot e_{n+1}\right\} \\ &\quad = \{A\} + \{B\}.\end{aligned}$$

When dealing with part $\{A\}$, we can proceed exactly as in [Ten17]. Noting the identities

$$M\nabla\nu^{n+1} = \eta'\nabla(u-\phi), \quad \nabla u\cdot e_{n+1} = \frac{|Du|^2}{1+|Du|^2} = |\nabla u|^2, \quad |\nabla u|^2 + (\nu^{n+1})^2 = 1,$$

we may rewrite $\{A\}$ to

$$\{A\} = \eta''|\nabla(u-\phi)|^2 + \eta'\frac{\gamma}{u} - \eta'\left(\Delta\phi + \frac{\gamma}{u}\nabla\phi\cdot e_{n+1}\right) + M\nu^{n+1}|A|^2 - M\nu^{n+1}\gamma\frac{|\nabla u|^2}{u^2}.$$

Also, since $\Psi(x_M, t_M) = 0$, it is

$$\nu^{n+1} = \frac{\eta - M\epsilon}{M} \leq \frac{1}{M} \Leftrightarrow \sqrt{1+|Du|^2} \geq M$$

and moreover, by definition of the tangential derivative

$$\begin{aligned}
|\nabla(u-\phi)|^2 &= |D(u-\phi)|^2 - \frac{|Du\cdot D(u-\phi)|^2}{1+|Du|^2} \\
&= |Du|^2 - 2Du\cdot D\phi + |D\phi|^2 - \frac{|Du|^4 - 2|Du|^2 Du\cdot D\phi + (Du\cdot D\phi)^2}{1+|Du|^2} \\
&= \frac{|Du|^2 - 2Du\cdot D\phi + |D\phi|^2 + |Du|^2|D\phi|^2 - (Du\cdot D\phi)^2}{1+|Du|^2} \\
&\geq \frac{|Du|^2 - 2C|Du|}{1+|Du|^2} \overset{|Du|\to\infty}{\to} 1.
\end{aligned}$$

Hence, there is a constant M_0, depending only on C_ϕ, such that $|\nabla(u-\phi)|^2 > \frac{1}{2}$, if $M > M_0$. Let us assume that $M > M_0$. Then, since in (x_M, t_M) the identity

$$-M\nu^{n+1} = -\eta + M\epsilon \geq -\eta$$

holds, we can estimate

$$\{A\} \geq \frac{1}{2}\eta'' + \eta'\frac{\gamma}{u} - \eta'\left(\Delta\phi + \frac{\gamma}{u}\nabla\phi\cdot e_{n+1}\right) - \eta\frac{\gamma}{u^2}. \tag{4.21}$$

When estimating $\{B\}$ observe the relation

$$\nabla\dot{u}\cdot e_{n+1} = \partial_t(\nabla u\cdot e_{n+1}) = \partial_t(|\nabla u|^2) = -\partial_t(\nu^{n+1})^2 = -2\nu^{n+1}\dot{\nu}^{n+1}$$

and moreover, since $\dot{\Psi}(x_M, t_M) \geq 0$, suppressing the dependence of (x_M, t_M)

$$\dot{\Psi} = \eta'(\dot{u} - \dot{\phi}) - M\dot{\nu}^{n+1} \geq 0 \Leftrightarrow -M\dot{\nu}^{n+1} \geq \eta'\dot{\phi} - \eta'\dot{u}.$$

Hence, at (x_M, t_M) it follows for $\{B\}$

$$\begin{aligned}\{B\} &= \eta'\dot{u}(\nu^{n+1})^2 + \dot{u}M\nabla\nu^{n+1}\cdot e_{n+1} + M\nu^{n+1}\nabla\dot{u}\cdot e_{n+1}\\ &= \eta'\dot{u}(\nu^{n+1})^2 + \dot{u}\eta'\nabla u\cdot e_{n+1} - \dot{u}\eta'\nabla\phi\cdot e_{n+1} - 2M\dot{\nu}^{n+1}(\nu^{n+1})^2\\ &\geq \eta'\dot{u}(\nu^{n+1})^2 + \eta'\dot{u}|\nabla u|^2 - 2\eta'\dot{u}(\nu^{n+1})^2 + 2\eta'\dot{\phi}(\nu^{n+1})^2 - \eta'\dot{u}\nabla\phi\cdot e_{n+1}\\ &= \eta'\dot{u}(1-2(\nu^{n+1})^2) + 2\eta'\dot{\phi}(\nu^{n+1})^2 - \eta'\dot{u}\nabla\phi\cdot e_{n+1}.\end{aligned}$$

Combining the estimates for $\{A\}$ and $\{B\}$ and making use of $\Delta\Psi(x_M, t_M) \leq 0$, we obtain

$$\begin{aligned}0 &\geq \Delta\Psi(x_M, t_M) = \{A\} + \{B\}\\ &\geq \frac{1}{2}\eta'' + \eta'\frac{\gamma}{u} - \eta'\left(\Delta\phi + \frac{\gamma}{u}\nabla\phi\cdot e_{n+1}\right) - \eta\frac{\gamma}{u^2}\\ &\quad + \eta'\dot{u}(1-2(\nu^{n+1})^2) + \eta'(2\dot{\phi}(\nu^{n+1})^2 - \dot{u}\nabla\phi\cdot e_{n+1}).\end{aligned}$$

Let us now, additionally to $M > M_0$, assume, that $(x_M, t_M) \in \{u < \delta\}$. In this set we have $\phi \equiv 0$ and thus

$$\eta'' + 2\eta'\frac{\gamma}{u} - 2\eta\frac{\gamma}{u^2} + 2\eta'\dot{u}\left(1 - \frac{2}{1+|Du|^2}\right) \leq 0.$$

Clearly $\left|1 - \frac{2}{1+|Du|^2}\right| \leq 1$. Additionally, there is a constant C_0, for which $\dot{u}(x_M, t_M) \leq C_0$. To see this, assume first, that $t_M < \hat{T}$. Then there is a $\tau \in (t_M, \hat{T})$ for which the problem is uniquely solvable on $\Omega_\tau = D \times [0, \tau)$ and the solution lies in $H_{2+\alpha}(\Omega')$ for every $\Omega' \subset\subset \Omega_\tau$. Since also $x_M \notin \partial D$ (else $f(x_M, t_M) = 0$, contradicting $M > 0$), we can find a set Ω', for which $(x_M, t_M) \in \Omega'$ and choose C_0 as the Hölder norm of $u(x,t)$ on Ω'. If instead $t_M = \hat{T}$ we make use of the assumption

$$\sup_{x\in\mathcal{K}} \dot{u}(x, \hat{T}) \leq C_0 < \infty$$

for every $\mathcal{K} \subset\subset (D \setminus \mathcal{S}_x)$. Since $x_M \notin \partial D$ and $u(x_M, t_M) \neq 0$, we can find a compact subset $\mathcal{K} \subset\subset (D \setminus \mathcal{S}_x)$ with $(x_M, t_M) \in \mathcal{K}$. Thus, we obtain

$$\eta'' + 2\eta'\frac{\gamma}{u} - 2\eta\frac{\gamma}{u^2} - 2\eta' C_0 \leq 0$$

at (x_M, t_M). Inserting the definition of $\eta(s) = (\mathrm{e}^{Ks} - 1)\mathrm{e}^{-2C_\phi K}$ this is equivalent to

$$\begin{aligned}&K^2u^2\mathrm{e}^{Ku} + 2Ku\gamma\mathrm{e}^{Ku} - 2(\mathrm{e}^{Ku}-1)\gamma - 2Ku^2C_0\mathrm{e}^{Ku} \leq 0\\ &\qquad\Leftrightarrow (2\gamma - 2Ku\gamma - K^2u^2 + 2Ku^2C_0)\mathrm{e}^{Ku} \geq 2\gamma.\end{aligned}$$

Setting $Ku =: s$, we obtain the inequailty

$$\left(2\gamma - 2s\gamma + s^2\left(\frac{2C_0}{K} - 1\right)\right) \geq 2\gamma,$$

which, by taking $K > 2C_0$, is only fulfilled for $s \leq 0$, a contradiction to $u > 0$. Hence, it is either $(x_M, t_M) \in \{u \geq \delta\}$ and $M > M_0$ or it is $M \leq M_0$. Therefore, assume now $(x_M, t_M) \in \{u \geq \delta\}$ and $M > M_0$. In this case we have the estimates

$$\left|\Delta\phi + \frac{\gamma}{u}\nabla\phi \cdot e_{n+1}\right| \leq C(\delta, \gamma, c_1^+, C_\phi)$$

and

$$\left|2\dot{\phi}(\nu^{n+1})^2 - \dot{u}\nabla\phi \cdot e_{n+1}\right| \leq C(C_0, C_3, C_\phi),$$

where we used the same argument as above to estimate $\dot{u}(x_M, t_M)$ and c_1^+, C_3 are the constants from section 2. Thus, every coefficient of η' is bounded and they can be combined to a new constant $C > 0$ to obtain

$$\eta'' - C\eta' - \eta\frac{\gamma}{u^2} \leq 0,$$

which in return leads to

$$\left(K^2 - KC - \frac{2\gamma}{\delta}\right)\mathrm{e}^{K(u-\phi)^+} \leq -\frac{2\gamma}{c_1^+}.$$

Choosing K big enough to suffice

$$K^2 - KC - \frac{2\gamma}{\delta} > 0$$

leads to a contradiction. Hence, it must be $M \leq M_0$.

Let us briefly discuss, why a similar procedure can be applied to

$$g(x,t) := \frac{\eta((\phi - u)^+)}{\nu^{n+1} + \epsilon},$$

resulting in the same bound M_0 as above. Note first, that $(\phi - u)^+ = 0$ on $\{u < \delta\}$, thus we only have to consider the case $\{u \geq \delta\}$. Defining

$$\Psi(x,t) := \eta((\phi - u)^+) - M(\nu^{n+1} + \epsilon) \leq 0$$

and making use if its properties at the interior maximum (x_M, t_M) yields

$$\begin{aligned}\Delta\Psi &= \left\{\eta''|\nabla(\phi - u)|^2 - \eta'\frac{\gamma}{u}(\nu^{n+1})^2 + \eta'\Delta\phi + M\nu^{n+1}|A|^2 + M\left(\nabla\nu^{n+1}\frac{\gamma}{u} - \nu^{n+1}\frac{\gamma}{u^2}\nabla u\right)\cdot e_{n+1}\right\} \\ &\quad + \left\{-\eta'\dot{u}(\nu^{n+1})^2 + M\left(\nabla\nu^{n+1}\dot{u} + \nu^{n+1}\nabla\dot{u}\right)\cdot e_{n+1}\right\} \\ &= \{A\} + \{B\}.\end{aligned}$$

Comparing this to the previous expression for $\Delta\Psi$, only a few signs have changed. We can simplify this further to the estimate

$$\begin{aligned}
0 &\geq \Delta\Psi(x_M, t_M) = \{A\} + \{B\} \\
&\geq \frac{1}{2}\eta'' - \eta'\frac{\gamma}{u} + \eta'\left(\Delta\phi + \frac{\gamma}{u}\nabla\phi \cdot e_{n+1}\right) - \eta\frac{\gamma}{u^2} \\
&+ \eta'\dot{u}(2(\nu^{n+1})^2 - 1) - \eta'(2\dot{\phi}(\nu^{n+1})^2 + \dot{u}\nabla\phi \cdot e_{n+1}),
\end{aligned}$$

where once again only a few signs have changed. At this point in the proof every estimate is made in regard to the absolute value of any of the coefficients. Hence, the exact same estimates as in the first case can be used here to obtain the value M_0 as the upper bound for $g(x,t)$. Thus, we have proven

$$\frac{\eta(|u-\phi|)}{\nu^{n+1} + \epsilon} \leq M_0$$

for every $\epsilon > 0$ and with $\epsilon \to 0$ it follows

$$\eta(|u-\phi|)\sqrt{1+|Du|^2} \leq M_0$$

and also

$$|u-\phi||D(u-\phi)| \leq \frac{1}{K}\mathrm{e}^{2C_\phi K}\eta(|u-\phi|)(\sqrt{1+|Du|^2} + |D\phi|) \leq \frac{1}{K}\mathrm{e}^{2C_\phi K}(M_0 + C_\phi).$$

As described in Tennstädt [Ten17] the function $|u-\phi|$ may be extended continuously by 0 outside of $\{u > 0\}$ and it follows, that $(u-\phi)^2$ is Lipschitz continuous, which implies $\frac{1}{2}$-Hölder continuity for $(u-\phi)$. Since $\phi \in C^2(\Omega)$, we thus conclude, that $u(\cdot, T_0)$ is $\frac{1}{2}$-Hölder continuous. □

5 Appendix

5.1 Eigenvalues of (a^{ij}) and Bernstein $\mathcal{E}$ function

The general form of a semilinear parabolic partial differential equation of second order is given by

$$Pu := -\dot{u} + a^{ij}(X, u(X), Du(X))u_{ij} + a(X, u(X), Du(X)) = 0,$$

with $u \in C^{2,1}(\Omega)$ and $X = (x,t) \in \Omega \subset \mathbb{R}^n \times (0,T)$, where $T \leq \infty$ and Ω is a domain. For the operator P given by

$$Pu = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u - \frac{\gamma}{u},$$

we obtain the values

$$\begin{aligned} a^{ij}(X, u(X), Du(X)) = a^{ij}(Du(X)) &= \delta^{ij} - \frac{u_i u_j}{1+|Du|^2}, \\ a(X, u(X), Du(X)) = a(u(X)) \qquad &= -\frac{\gamma}{u}, \end{aligned}$$

with δ^{ij} being the Kronecker-Delta which is equal to 1 if $i = j$ and 0 otherwise.

An important parameter in the analysis of semilinear partial differential equations are the smallest and largest eigenvalue of the matrix (a^{ij}). We denote these values by λ and Λ for the smallest and largest eigenvalue respectively. The matrix (a^{ij}) suffices

$$\lambda(X,z,p)|\xi|^2 \leq a^{ij}(X,z,p)\xi_i\xi_j \leq \Lambda(X,z,p)|\xi|^2$$

for every $(X,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ at which the operator P is parabolic. If, in addition, the ratio $\frac{\Lambda}{\lambda}$ is uniformly bounded we say that P is uniformly parabolic in (X,z,p).

For the operator P defined above it is $\lambda = \frac{1}{1+|p|^2}$ and $\Lambda = 1$, which results in $\frac{\Lambda}{\lambda} = 1 + |p|^2 \overset{|p|\to\infty}{\to} \infty$ [cf. Lieberman [Lie96, p.204, eq. (8.4)]]. Hence, this operator only becomes uniformly parabolic if we are able to derive a priori bounds for $|u|$ and $|Du|$.

A useful tool for discussing semilinear (parabolic) partial differential equations is the Bernstein $\mathcal{E}$ function, defined by

$$\mathcal{E}(X,z,p) := a^{ij}(X,z,p)p_i p_j.$$

$\mathcal{E}$ always fulfills the estimates $\lambda|p|^2 \leq \mathcal{E} \leq \Lambda|p|^2$ and for the choice of a^{ij} given above it is [cf. Lieberman [Lie96, p.204, eq. (8.4)]]

$$\mathcal{E} = \frac{|p|^2}{1+|p|^2} = \lambda|p|^2.$$

5.2 Some linear algebra

Definitions 5.1 (Notation, Definiteness, Trace)

i) *We denote by* $\mathbb{R}^{n\times n}$ *the space of all* $(n\times n)$*-matrices. If* A *is an* $(n\times n)$*-matrix with components* a^{ij} *we write* $A = (a^{ij}) \in \mathbb{R}^{n\times n}$*. Furthermore, the matrix* $A = (a^{ij})$ *is symmetric if* $a^{ij} = a^{ji}$ *for every* $i, j = 1, \dots, n$*.*

ii) *We say a matrix* $A \in \mathbb{R}^{n\times n}$ *is positive semi-definite if* $x^T Ax = a^{ij}x_i x_j \geq 0$ *for all* $x = (x_1, ..., x_n) \in \mathbb{R}^n$*. We say the matrix* $A \in \mathbb{R}^{n\times n}$ *is negative semi-definite if* $x^T Ax \leq 0$ *for all* $x \in \mathbb{R}^n$*. Note that a parabolic operator* P *suffices* $0 < \lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j$ *which especially means that* (a^{ij}) *is positive semi-definite.*

iii) *The trace of a matrix* $A \in \mathbb{R}^{n\times n}$ *is given by* $\text{trace}(A) = tr(A) = a^{ii}$*.* □

For the proof of the comparison principle, we need to estimate the expression $a^{ij}u_{ij}$. If we interpret a^{ij} and u_{ij} as the entries of two symmetric $(n \times n)$-matrices A and U then this sum can be expressed as $a^{ij}u_{ij} = tr(AU)$.

Proposition 5.2 (Rules for the trace operator) *Let* $A = (a^{ij}), B = (b^{ij}) \in \mathbb{R}^{n\times n}$ *be symmetric* $(n \times n)$*-matrices. Then the following holds*

i) *If both matrices are positive semi-definite then* $tr(AB) \geq 0$*.*

ii) *If one matrix is positive semi-definite and the other is negative semi-definite then* $tr(AB) \leq 0$*.* □

Proof. It is well known that for a symmetric matrix A there is an orthogonal matrix O with $O^T O = I_{R^{n\times n}}$ so that $A = ODO^T$, where D is a diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ being the eigenvalues of A. Furthermore, it is well known that A is positive semi-definite if and only if $\lambda_i \geq 0\ \forall i \in \{1, \dots, n\}$ and that the trace operator is invariant under cyclic permutations, which means that $tr(ABC) = tr(CAB) = tr(BCA)$ for $A, B, C \in \mathbb{R}^{n\times n}$. Inserting the i-th basis vector e_i in the expression $x^T Ax$ also yields $e_i^T Ae_i = a_{ii} \geq 0$ if A is positive semi-definite and $a_{ii} \leq 0$ if A is negative semi-definite respectively.
To proof either claim, let B be the positive semi-definite matrix without loss of generality. We can write

$$tr(AB) = tr(ODO^T B) = tr(DO^T BO),$$

where D is the diagonal matrix consisting of the eigenvalues of A which are all positive in the first case and all negative in the second case. Defining $y := Ox \in \mathbb{R}^n$, we see that the matrix $O^T BO$ is positive semi-definite because

$$x^T O^T BOx = (Ox)^T BOx = y^T By \geq 0.$$

This means that all diagonal entries of $O^T BO$ have to be non-negative so that both claims follow by the definition of the trace operator. □

5.3 Comparison and Maximum Principles

We begin with a slight modification of a comparison principle that can be found in LIEBERMAN [Lie96, p. 219, Thm 9.1].

Proposition 5.3 (Comparison Principle) *Let P be the quasilinear operator defined by*

$$Pu = -\dot{u} + a^{ij}(X, u, Du)D_{ij}u + a(X, u, Du).$$

Suppose that a^{ij} is independent of z and that $a(X, z, p)$ is non-increasing in z for every fixed $(X, p) \in \Omega \times \mathbb{R}^n$. If u and v are functions in $C^{2,1}(\Omega) \cap C(\overline{\Omega})$ such that $Pu \geq Pv$ on Ω and $u \leq v$ on $\mathcal{P}\Omega$ and if P is parabolic with respect to u or v, then $u \leq v$ on $\Omega \cup \mathcal{P}\Omega$. □

Proof. We argue very similarly to [Lie96]. Therefore, set $w = (u - v)\mathrm{e}^{\lambda t}$ for a constant λ at our disposal. Since $u \leq v$ on $\mathcal{P}\Omega$, we have $w \leq 0$ on $\mathcal{P}\Omega$. Let us assume there is a point $(x_0, t_0) \in \Omega$ at which $w(x, t)$ attains a positive maximum. At this point we have

$$w(x_0, t_0) > 0, \quad Dw(x_0, t_0) = 0, \quad w_t(x_0, t_0) = 0, \quad D^2 w(x_0, t_0) \leq 0.$$

Furthermore, calculating w_t yields

$$w_t = (u - v)_t \mathrm{e}^{\lambda t} + \lambda(u - v)\mathrm{e}^{\lambda t} = 0 \Leftrightarrow -(u - v)_t = \lambda(u - v).$$

Combining these properties we obtain

$$\begin{aligned}
0 &\leq Pu(X_0) - Pv(X_0) \\
&= -(u - v)_t + a^{ij}(X_0, Du(X_0))D_{ij}(u - v) + a(X_0, u(X_0), Du(X_0)) - a(X_0, v(X_0), Dv(X_0)) \\
&\leq -\lambda(u(X_0) - v(X_0)),
\end{aligned}$$

which cannot be true if we choose $\lambda > 0$. Hence, the assumption $w > 0$ was false and we have $w \leq 0$, implying $u \leq v$ on $\Omega \cup \mathcal{P}\Omega$. □

Remark. The comparison principle in LIEBERMAN [Lie96] appears to have a small mistake, where he requires „an increasing positive constant k such that $a(X, z, p) + k(M)z$ is a decreasing function of z on $\Omega \times [-M, M] \times \mathbb{R}^n$“. To the author's understanding it should instead be $a(X, z, p) - k(M)z$, which would then be in accordance with the modified version from above. □

If $\gamma > 0$, the requirements for the application of the comparison principle are no longer fulfilled. However, a weaker version still holds.

Proposition 5.4 (Weak Comparison Principle) [Lie96, p. 220, Lemma 9.4.] *Suppose that u and v are in $C^{2,1}(\Omega) \cap C(\overline{\Omega})$ and that P is parabolic at u or at v. If $Pu > Pv$ on Ω and if $u < v$ on $\mathcal{P}\Omega$, then $u < v$ on $\Omega \cup \mathcal{P}\Omega$.* □

Proof. Define $w = u - v$ and assume that $w \geq 0$ somewhere in Ω. Then there is a first time $t_0 = \inf\{t \mid w(x,t) \geq 0, x \in D\}$ at which the function $w(x,t)$ becomes non-negative. Since $w < 0$ on $\mathcal{P}\Omega$ and $w(x,t)$ is continuous, there must be a point $X_0 = (x_0, t_0)$ at which $w(X_0) = 0$. Since $w(\cdot, t_0)$ attains its maximum at X_0, we have $Dw(X_0) = 0$, thus $Du(X_0) = Dv(X_0)$, and $D^2 w(X_0) \leq 0$. Also $w_t(X_0) \geq 0$ from the choice of t_0. These observations lead to

$$0 < Pu(X_0) - Pv(X_0) = -w_t(X_0) + a^{ij}(X_0, u(X_0), Du(X_0))w_{ij}(X_0) \leq 0,$$

which is a contradiction. Hence, the assumption $w \geq 0$ on Ω was wrong and we obtain $u < v$ on Ω. □

When working locally on the surfaces graph$(u(x,t))$ we have the following weak maximum principle (see Ecker [Eck04, p. 24, Prop. 3.1], with proof [Eck04, p. 122]), which has to be slightly modified since the manifold we study has a boundary.

Proposition 5.5 (Weak Maximum Principle on Manifolds) *Let M^n be a compact n-dimensional manifold with boundary and $F(\cdot,t) = F_t \colon M^n \to \mathbb{R}^{n+1}$ with $M_t = F_t(M^n)$. Suppose $h \colon M^n \times [t_1, t_0) \to \mathbb{R}$ is sufficiently smooth for $t > t_1$, continuous on $M^n \times [t_1, t_0]$ and satisfies an inequality of the form*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) h \leq a \cdot \nabla^{M_t} h,$$

where Δ_{M_t} and ∇^{M_t} are the laplace and nabla operator on the surfaces M_t respectively. Then

$$\max_{M^n \times [t_1, t_0)} h \leq \max\left\{\max_{M^n} h(\cdot, t_1), \max_{\partial M^n \times [t_1, t_0)} h\right\}.$$

For the vector field $a \colon M^n \times [t_1, t_0) \to \mathbb{R}^{n+1}$ we only require that it is well-defined in a neigbourhood of all maximum points of h. □

Proof. We can imitate the proof from Ecker, in which he shows, that the function

$h(p,t)$ cannot have an *interior* maximum, if

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta_{M_t}\right) h \leq a \cdot \nabla^{M_t} h.$$

However, since the manifold has a boundary, we can only conclude from the absence of an interior maximum, that

$$\max_{M^n \times [t_1,t_0)} h \leq \max \left\{ \max_{M^n} h(\cdot, t_1), \max_{\partial M^n \times [t_1,t_0)} h \right\},$$

as opposed to

$$\max_{M^n \times [t_1,t_0)} h \leq \max_{M^n} h(\cdot, t_1),$$

for manifolds without boundary. □

5.4 Hölder continuity

Definitions are taken from Lieberman [Lie96, p. 46,47].

Definition 5.6 (Hölder continuity for $\alpha \in (0,1]$) *We say that a function f defined on $\Omega \subset \mathbb{R}^{n+1}$ is Hölder continuous at $X_0 = (x_0, t_0)$ with exponent $\alpha \in (0,1]$ if the quantity*

$$[f]_{\alpha;X_0} = \sup_{X \in \Omega \setminus \{X_0\}} \frac{|f(X) - f(X_0)|}{|X - X_0|^\alpha}$$

is finite. If the semi-norm

$$[f]_{\alpha;\Omega} = \sup_{X_0 \in \Omega} [f]_{\alpha;X_0}$$

is finite, we say that f is uniformly Hölder continuous in Ω. If f is uniformly Hölder continuous on any $\Omega' \subset\subset \Omega$ we say that f is locally Hölder continuous in Ω. □

For Hölder continuity of higher order let $\beta \in (0,2]$ and set

$$\langle f \rangle_{\beta,X_0} := \sup \left\{ \frac{f(x_0,t) - f(X_0)|}{|t - t_0|^{\beta/2}} \;\middle|\; (x_0,t) \in \Omega \setminus \{X_0\} \right\}, \quad \langle f \rangle_{\beta;\Omega} := \sup_{X_0 \in \Omega} \langle f \rangle_{\beta;X_0}.$$

Definition 5.7 (Hölder continuity for $a > 1$) *Let $a > 1$ with $a = k + \alpha, k \in \mathbb{N}, \alpha \in$*

$(0, 1]$. *Then we define*

$$\langle f \rangle_{a;\Omega} := \sum_{|\beta|+2j=k-1} \langle D_x^\beta D_t^j f \rangle_{\alpha+1},$$
$$[f]_{a;\Omega} := \sum_{|\beta|+2j=k} [D_x^\beta D_t^j f]_\alpha,$$
$$|f|_{a;\Omega} := \sum_{|\beta|+2j\le k} \sup |D_x^\beta D_t^j f| + [f]_a + \langle f \rangle_a.$$

We set $H_a(\Omega) := \{f \mid |f|_a < \infty\}$ *which is a Banach space with norm* $|\cdot|_a$. □

Remarks

i) It is generally true that $\langle f \rangle_a \le |f|_0 + [f]_a$ [cf. [Lie96, p. 46]].

ii) As we can see from the definition, the inclusion $C^{2,1}(\Omega) \subset H_{2+\alpha}(\Omega)$ holds.

iii) For the applicability of the Schauder fixed point theorem we are particularly interested in the Hölder space $H_a(\Omega)$ with $a \in (1, 2)$. Functions in this space fulfill

$$|f|_{a;\Omega} = \sup |f| + \sup |Df| + [Df]_\alpha + \langle f \rangle_{\alpha+1} < \infty.$$

Together with the first remark this leads to the objective of showing boundedness of $\sup |u|, \sup |Du|$ and $[Du]_\alpha$ for $\alpha > 0$. □

5.5 Compatibility Conditions

It is desirable that a solution $u(x,t)$ to $(P^\gamma_{\phi_0,u_0})$ is continuous up to the boundary. Since we prescribe data in two different ways

$$u(x, 0) = \phi_0(x) \quad \text{on } \mathcal{C}\Omega,$$
$$u(x, 0) = u_0(x) \quad \text{on } \mathcal{C}\Omega.$$

we may encounter a problem in the corner $\mathcal{C}\Omega$. We avoid this by imposing compatibility conditions. If we want to achieve higher regularity up to the boundary of the domain Ω, the (PDE) imposes additional restrictions that have to be fulfilled.

Definition 5.8 (Compatibility Conditions) [LSU88, p.318-320] *The compatibility conditions consist in the fact that the derivatives* $\frac{\mathrm{d}}{\mathrm{d}t}u(x, 0)$*, which can be determined for* $t = 0$ *by means of the equation and initial condition* $u_0(x,t) \equiv u_0(x)$*, must satisfy* *for $x \in \mathcal{S}\Omega$ the boundary conditions. Introducing the notation*

$$u^{(k)}(x) := \frac{\mathrm{d}}{\mathrm{d}t} u(x,t)\big|_{t=0},$$
$$Pu = -\dot{u} + E(u) = -\dot{u} + \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_{ij} u,$$

it is obvious, that on the set $\mathcal{C}\Omega$ the functions $u^{(k)}(x)$ $(k = 0, 1)$ are determined by

$$u^{(0)}(x) = u_0(x),$$
$$u^{(1)}(x) = E(u_0),$$

while higher order derivatives are given by

$$u^{(k+1)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} E(u(x,t))\big|_{t=0}.$$

We then say that the compatibility conditions of order $m \geq 0$ are fulfilled, if

$$u^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} \phi_0, \quad k = 0, \dots, m \text{ on } \mathcal{C}\Omega.$$

□

Remark. For most existence and regularity results we only need the compatibility condition of order 1 to be fulfilled, which allows for the solution $u(x,t)$ to be in $H_{2+\alpha}(\Omega)$. If we define

$$\psi_0(x) := \begin{cases} u_0(x), & \text{if } x \in D, \\ \phi_0(x), & \text{if } x \in \partial D, \end{cases}$$

these can be easily expressed by $\psi_0(x) \in H_{2+\alpha}$ with $P\psi_0(x) = 0$, since $\dot{u}_0 = \dot{\phi}_0 = \dot{\psi}_0 = 0$.

5.6 Boundary regularity types and the distance function

Boundary regularity is crucial for the existence of barrier functions on the spatial boundary $\mathcal{S}\Omega$ of the domain Ω. Furthermore, the regularity of the spatial distance function is related closely to the boundary's regularity.

Definition 5.9 **(C^k-boundary)** *We say a set $D \subset \mathbb{R}^n$ has a C^k-boundary, written $\partial D \in C^k$, if for every point $x \in \partial D$ there is a neighborhood N of x such that $\partial D \cap N$ can be represented in the form*

$$x_n = \phi(x_1, ..., x_{n-1}),$$

where ϕ is a function that is k times differentiable. We say that $\Omega = D \times (0,T)$ has *a C^k-boundary if D has a C^k boundary and we write $\mathcal{P}\Omega \in C^k$. Hölder continuous boundaries $\mathcal{P}\Omega \in H_a$ with $a = k + \alpha, k \in \mathbb{N}, \alpha \in (0,1]$ are defined in the same way, where now $\phi \in H_a$.* □

Additionally, we note some useful properties for the distance function $d(x)$ defined by

$$d(x) := \inf_{y \in \partial D} |x - y|$$

with $D \subset \mathbb{R}^n$. Proofs can be found in GILBARG-TRUDINGER [GT01, p. 354-355] and SERRIN [SH69, p. 420-422, chapter 1.3.].

Definition 5.10 (Principal coordinate system) *Let $D \subset \mathbb{R}^n$ be a bounded domain and let $y_0 \in \partial D$ be an arbitrary boundary point of D with $\partial D \in C^k, k \geq 2$. Then there is a neighborhood $N(y_0)$ of y_0 and a function $\phi = \phi(x') = \phi(x_1, ..., x_{n-1})$ such that $N \cap \partial D$ is given by the equation $x_n = \phi(x')$ with $D\phi(y_0') = 0$. In this case the curvature of ∂D is described by the orthogonal invariants of the Hessian matrix $[D^2\phi]$ evaluated at y_0'. The eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$ of $[D^2\phi(y_0')]$ are called the* ***principal curvatures*** *of ∂D at y_0 and the corresponding eigenvectors are called the principal directions of ∂D at y_0. We call a coordinate system* ***principal coordinate system*** *if the $x_1, ..., x_{n-1}$ axes lie along principal directions corresponding to $\kappa_1, ..., \kappa_{n-1}$ at y_0.* □

Proposition 5.11 (Differentiability of $d(x)$) *Let $D \subset \mathbb{R}^n$ be bounded and $\partial D \in C^k$ for $k \geq 2$. Define for $\mu > 0$*

$$\Gamma_\mu := \{x \in \overline{D} \mid d(x) < \mu\}.$$

Then there exists a positive constant μ depending on D such that $d \in C^k(\Gamma_\mu)$. □

Proposition 5.12 (Properties of $d_i(x), d_{ij}(x)$) *In a principal coordinate system with axes $x_i, i = 1, ..., n-1$ lying along the principal directions and axis x_n lying along the normal vector of the surface ∂D pointing from the point $y_0 \in \partial D$ towards $x_0 \in \Gamma_\mu$ we have at x_0*

$$Dd(x_0) = (0, ..., 0, 1)$$

and

$$D^2 d(x_0) = -\left(\frac{\kappa_1}{1 - \kappa_1 d}, ..., \frac{\kappa_{n-1}}{1 - \kappa_{n-1} d}, 0\right)_{\text{diag}},$$

where $\kappa_1, ..., \kappa_{n-1}$ are the principal curvatures of ∂D at y_0. □

5.7 Geometry on the surface graph$(u(x,t))$

Let us assume that the solution $u(x,t)$ defines a graph M_t and set

$$F(x,t) := (x, u(x,t))$$

for every $(x,t) \in \Omega = D \times [0,T)$. Since the solution $u(x,t)$ is supposed to fulfill the initial and boundary conditions

$$u(x,t) = \phi_0(x) \quad \text{on } \mathcal{S}\Omega,$$
$$u(x,0) = u_0(x) \quad \text{on } \mathcal{B}\Omega,$$

the corresponding graph at $t = 0$ has to be given by the points

$$F(x,0) = (x, u(x,0)) = (x, u_0(x)) \quad \text{on } D.$$

This graph (for the fixed time $t = 0$) defines a hypersurface M_0 of $\mathbb{R}^{n+1}$ of which the evolution in time is given by the hypersurfaces M_t. In this setting we may now understand the solution $u(x,t)$ as the last component

$$u(x,t) = F(x,t) \cdot e_{n+1}$$

of the function $F(x,t)$, which allows to perform calculations directly on the surfaces M_t.

Instead of discussing the hypersurfaces above a fixed point $x \in D \subset \mathbb{R}^n$, it is sometimes more beneficial to fix a point p in the initial surface M_0 and analyze its evolution in direction of the normal ν relative to the graph of $u(x,t)$. To make this more precise we use a paragraph from Ecker [Eck04, p. 7-8].

Definition 5.13 (Surfaces moving in normal direction) *Let M^n be an n-dimensional manifold. Consider the family of smooth embeddings $F_t = F(\cdot,t) : M^n \to \mathbb{R}^{n+1}$ with $M_t = F_t(M^n)$. We say the surfaces M_t move in direction ν with velocity $\mathcal{H}$, if*

$$\frac{\mathrm{d}F}{\mathrm{d}t}(p,t) = \vec{H}(F(p,t)) = \mathcal{H}(F(p,t))\nu \tag{5.1}$$

for $p \in M^n$ and $t \in [0,T)$.

Lemma 5.14 (Normal Motion and Tangential Diffeomorphisms) *Let $F_t = F(\cdot,t) : M^n \to \mathbb{R}^{n+1}$ with $M_t = F_t(M^n)$ be a family of embeddings satisfying the equation*

$$\left(\frac{\mathrm{d}F}{\mathrm{d}t}(x,t)\right)^{\perp} = \vec{H}(F(x,t))$$

for $x \in M^n$. Here $\perp$ denotes the projection onto the normal space of $F_t(M^n)$. Let $g_t(\cdot) = g(\cdot, t)$ be a family of diffeomorphisms on M^n satisfying

$$D_q F(g(x,t),t) \cdot \frac{\mathrm{d}g}{\mathrm{d}t}(x,t) = -\left(\frac{\mathrm{d}F}{\mathrm{d}t}(g(x,t),t)\right)^T,$$

where $D_q F$ denotes differentiation of F with respect to its first n components. If we set

$$\tilde{F}_t(p) = \tilde{F}(p,t) = F(g(x,t),t) = F(g_t(x),t),$$

then $M_t = \tilde{F}_t(M^n) = F_t(M^n)$ and

$$\frac{\mathrm{d}\tilde{F}}{\mathrm{d}t}(p,t) = H(\tilde{F}(p,t)).$$

□

Let us apply this lemma to the parabolic operator P given by

$$Pu = -\dot{u} + vH(u) - \frac{\gamma}{u}.$$

It is well known that the normal ν of a graph $(x, u(x,t))$ is given by

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}} = \frac{(-Du, 1)}{v}$$

with $v = \sqrt{1 + |Du|^2}$. To obtain the projection $\left(\frac{\mathrm{d}F}{\mathrm{d}t}(x,t)\right)^\perp$ of $\frac{\mathrm{d}F}{\mathrm{d}t}(x,t)$ onto the normal space of M_t we calculate

$$\frac{\mathrm{d}F}{\mathrm{d}t}(x,t) \cdot \nu = (0, \dot{u}) \cdot \frac{(-Du, 1)}{v} = \frac{\dot{u}}{v}.$$

Since $u(x,t)$ solves the equation

$$-\dot{u} + vH(u) = \frac{\gamma}{u} \Leftrightarrow \dot{u} = vH(u) - \frac{\gamma}{u},$$

we may set

$$\mathcal{H} := \frac{1}{v}\left(vH(u) - \frac{\gamma}{u}\right)$$

in (5.1) to obtain

$$\left(\frac{\mathrm{d}F}{\mathrm{d}t}(x,t)\right)^\perp = \frac{\dot{u}}{v}\nu = \mathcal{H}\nu = \vec{H}.$$

Now, according to lemma 5.14, there is a tangential diffeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n, g(x,t) = p$ satisfying

$$D_q F(g(x,t),t) \cdot \frac{\mathrm{d}g}{\mathrm{d}t}(x,t) = -\left(\frac{\mathrm{d}F}{\mathrm{d}t}(g(x,t),t)\right)^T,$$

so that the new flow

$$\tilde{F}_t(p) = \tilde{F}(p,t) = F(g(p,t),t)$$

satisfies

$$\frac{\mathrm{d}\tilde{F}}{\mathrm{d}t}(p,t) = \vec{H}(\tilde{F}(p,t)) = \frac{1}{v}\left(vH(u) - \frac{\gamma}{u}\right)\nu$$

and also $M_t = \tilde{M}_t$, where $\tilde{M}_t$ are the surfaces generated by the graph

$$\operatorname{graph}(\tilde{u}(x,t)) = \{(g(x,t), u(g(x,t),t)) \in \mathbb{R}^{n+1} \mid x \in D, t \in [0,T)\}$$

with $\tilde{u}(x,t) = \tilde{F}(p,t) \cdot e_{n+1}$. □

For either flow $F(x,t)$ and $\tilde{F}(p,t)$ there is a set of useful identities that are available to us when working locally on the surfaces. To introduce these identities we first need to define a notion of tangential gradient and laplace operator on graph(u). Given an arbitrary function $f \in C^2$ these quantities are given by

$$\nabla_{M_t} f := (Df, 0) - D_i f \nu^i \nu = \left(Df - \frac{Df \cdot Du}{1+|Du|^2}Du, \frac{Df \cdot Du}{1+|Du|^2}\right)$$

and

$$\Delta_{M_t} f = g^{ij} D_i D_j f + H\nu^i D_i f,$$

where the operator D denotes the usual differentiation in $\mathbb{R}^n$, $\nu = \frac{(-Du,1)}{\sqrt{1+|Du|^2}}$ is the unit normal with respect to graph(u), $g^{ij} = \delta^{ij} - \nu^i\nu^j$ is the inverse of the first fundamental form and $H = H(u(x,t))$ is the mean curvature of graph(u) at (x,t). Whenever it is clear that we work locally on the surface graph(u), we will simply write ∇ for ∇_{M_t} and Δ for Δ_{M_t} respectively. From the definition of ∇f we infer

$$\begin{aligned}|\nabla f|^2 &= |Df|^2 - 2\frac{(Df \cdot Du)^2}{1+|Du|^2} + \frac{(Df \cdot Du)^2}{(1+|Du|^2)^2}|Du|^2 + \frac{(Df \cdot Du)^2}{(1+|Du|^2)^2} \\ &= |Df|^2 - (Df \cdot \nu)^2 = |Df|^2 - (D_i f \nu^i)^2\end{aligned}$$

and for $f = u$ we also obtain the identities

$$\nabla u = \left(\frac{Du}{1+|Du|^2}, \frac{|Du|^2}{1+|Du|^2}\right), \quad |\nabla u|^2 = \frac{|Du|^2}{1+|Du|^2} = e_{n+1} \cdot \nabla u.$$

Since $H(u)$ is defined by

$$H(u) := \frac{1}{v} g^{ij} u_{ij},$$

we obtain from the definition of the laplace operator

$$\Delta u = vH - H \cdot \frac{|Du|^2}{v} = H\nu^{n+1}.$$

Moreover, we have the Jacobi field equation

$$\Delta\nu = -\nu|A|^2 - \nabla H,$$

where $|A|$ is the norm of the second fundamental form [cf. [DHT10, p. 163, Prop. 2]]. $\square$

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