A functorial approach to the stability of vector bundles

Dissertation

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von

Dario Weißmann
geboren in Emmendingen, Deutschland

vorgelegt beim Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen

Gutachter:
Prof. Dr. Georg Hein
Prof. Dr. Stefan Schröer

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Abstract

On a normal projective variety the locus of μ-stable vector bundles that remain μ-stable on *all* Galois covers prime to the characteristic $p \geq 0$ is open in the moduli space of Gieseker semistable sheaves. On a smooth projective curve of genus at least 2 this locus is big in the moduli space of stable vector bundles.

The moduli space of μ-stable vector bundles admits a canonical stratification defined via the decomposition type of a vector bundle. We give mostly sharp dimension estimates for these strata over a smooth projective curve of genus at least 2.

As an application we obtain a mostly sharp estimate of the dimension of the closure of prime to p trivializable stable vector bundles in the moduli space of stable vector bundles over a smooth projective curve of genus at least 2. In rank 2 we give a description of its irreducible components.

Zusammenfassung

Der Ort der μ-stabilen Vektorbündel auf einer normalen projektiven Varietät X, die μ-stabil auf *allen* étalen Galois Überlagerungen prim zur Charakteristik $p \geq 0$ bleiben, sind offen im Modulraum der Gieseker semistabilen Garben. Falls X eine glatte projektive Kurve von Geschlecht mindestens 2 ist, dann ist dieser Ort groß im Modulraum der stabilen Vektorbündel.

Weiterhin können wir den Modulraum der μ-stabilen Vektorbündel mit einer kanonischen Stratifizierung via des Zerfällungsverhalten von Vektorbündeln versehen. Auf einer glatten projektiven Kurve von Geschlecht mindestens 2 finden wir - meist scharfe - Abschätzungen für die Dimension dieser Strata.

Als Anwendung studieren wir den Abschluss der prim zu p trivialisierbaren stabilen Vektorbündel im Modulraum der stabilen Vektorbündel über einer glatten projektiven Kurve von Geschlecht mindestens 2. Im Rang 2 Fall beschreiben wir die irreduziblen Komponenten dieses Abschlusses.

Introduction

Statement of results

Consider the stack of vector bundles on a smooth projective curve C over an algebraically closed field k of characteristic $p \geq 0$. Semistability is a property of vector bundles which is tailored to obtain a moduli space. Via the Harder-Narasimhan-filtration (HN-filtration for short) it also reveals additional structure of the category of vector bundles and immediately implies that semistability is functorial under pullback by finite separable morphisms. Even more structure is revealed via the Jordan-Hölder-filtration (JH-filtration for short). However, in contrast to the HN-filtration the JH-filtration is not unique and thus functoriality fails for stability.

Recently, those morphisms that preserve the stability of vector bundles have been identified: for curves these are exactly the genuinely ramified morphisms, see [3, Theorem 5.3]. In higher dimension, genuinely ramified morphisms also preserve stability, see [2, Theorem 1.2].

The main goal of this thesis is to address a way to measure the failure of stability to be functorial under all finite separable pullbacks. As an application we obtain a very different behaviour of the étale fundamental group in positive versus characteristic 0.

Representations of $\pi_1^{\text{ét}}(C)$ correspond to bundles of degree 0 which are trivialized on some étale cover of C, see [15, 1.2 Proposition]. In positive characteristic these étale trivializable bundles are dense in the moduli space $M_C^{ss,r,0}$ of semistable bundles of rank r and degree 0, see [6, Corollary 5.1]. This no longer holds in characteristic 0 as we show that the general bundle remains stable on all étale covers (avoiding the characteristic). Put another way, the étale fundamental group can recover the moduli space in positive characteristic but not in characteristic 0.

To make our results precise we need a definition. Call a vector bundle on C *prime to p stable* if it remains stable after pullback by all finite Galois morphisms $D \to C$ which have degree prime to p. The locus of prime to p stable vector bundles is open, a direct consequence of the following theorem.

Theorem 1 (Theorem 3.2.7 for curves)**.** Let $r \geq 2$. There exists a prime to p Galois cover $\pi : C_{r,good} \to C$ such that a vector bundle V of rank r is prime to p stable iff π^*V is stable.

An analogous statement holds for μ-stable vector bundles on a normal projective variety, see Theorem 3.2.7. Having identified this locus as open one should also address non-emptiness:

Theorem 2 (Corollary 4.2.6)**.** Let $r \geq 2$. If C has genus $g_C \geq 2$, then the prime to p stable locus $M_C^{p'-s,r,d}$ is big in the moduli space of stable vector bundles $M_C^{s,r,d}$. More precisely, we have

$$\dim(M_C^{s,r,d} \setminus M_C^{p'-s,r,d}) \leq rr_0(g_C - 1) + 1,$$

where r_0 denotes the largest proper divisor of r. If p is not the smallest proper divisor of r, then equality holds.

By considering $d = 0$, we obtain the different behaviour of the étale fundamental group, i.e., the non-density of the étale trivializable bundles in characteristic 0.

Corollary 3. Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Then the stable vector bundles of rank r that are trivialized on a prime to p cover are not dense in $M_C^{s,r,0}$.

In rank 2 and characteristic 0 such a non-density result has been independently obtained by Ghiasabadi and Reppen, see [9, Corollary 4.16].

We also note that the density of the étale trivializable bundles in positive characteristic means that we can not extend Theorem 2 to all covers; the characteristic has to be avoided.

As the prime to p trivializable bundles are not dense in $M_C^{ss,r,0}$ it is natural to ask what their closure is. We give mostly sharp dimension estimates:

Theorem 4 (Theorem 5.0.5)**.** Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Let $Z^{s,r}$ be the closure of the prime to p trivializable stable vector bundles in $M_C^{s,r,0}$ and $Z^{ss,r}$ be the closure of the prime to p trivializable bundles in $M_C^{ss,r,0}$. Then we have the following:

- $\dim(Z^{s,r}) \leq r'(g_C - 1) + 1$, where r' is the prime to p part of r.
- $\dim(Z^{ss,r}) = rg_C$.
- If $p \nmid r$, then $\dim(Z^{s,r}) = r(g_C - 1) + 1$.

In the cases $r = 2$ and $r = p^n$ we can also describe the irreducible components, see Corollary 5.0.6 and Theorem 5.0.7.

The closure of prime to p trivializable bundles is closely related to a canonical stratification of the moduli space of stable vector bundles: the *prime to p decomposition stratification*, see Definition 4.2.1. This stratification is obtained by iterating the cover $C_{r,good}$ to obtain a prime to p Galois cover $C_{r,split}$ with the property that a stable vector bundle V of rank r on X decomposes on $C_{r,split}$ into a direct sum $\bigoplus W_i$ of prime to p stable vector bundles. By the key lemma, Lemma 2.1.13, the stable vector bundles W_i have the same rank m. This induces a stratification $Z^{s,r,d}(m)$ of the moduli space of stable vector bundles $M_C^{s,r,d}$ of rank r and degree d by fixing the rank m. This stratification also works on a normal projective variety. We have mostly sharp dimension estimates:

Theorem 5 (Theorem 4.2.4)**.** Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2, d \in \mathbb{Z}$. Then for $m \mid r$ we have the following:

- $\dim(Z^{s,r,d}(m)) \leq (\frac{r}{m})' m^2(g_C - 1) + 1$, where $(\frac{r}{m})'$ is the prime to p part of $\frac{r}{m}$.
- If $p \nmid \frac{r}{m}$, then we have $\dim(Z^{s,r,d}(m)) = rm(g_C - 1) + 1$.

Strategy of proof

The key observation to prove Theorem 1 is that, while stability is generally not preserved under pullback by a Galois morphism $D \to C$, polystability is preserved. This leads to the key lemma, Lemma 2.1.13: A stable vector bundle V on C decomposes on D into a direct sum $\bigoplus_{i=1}^{n} W_i^{\oplus e}$ of stable vector bundles W_i such that the Galois group of D/C acts transitively on the isomorphism classes of the W_i.

The construction of the cover $C_{r,good}$ checking for prime to p stability is then split into two parts: A cover $C_{r,large}$ checking for the decomposition behaviour if $n \geq 2$ and a cover $C_{r,good}$ including $n = 1$.

The cover $C_{r,large}$ is easily constructed using the transitive action of the Galois group. To include the case $n = 1$ a difficulty arises: while all the conjugates of $W = W_1$ by the Galois group are isomorphic, these isomorphisms might not be compatible. We provide a workaround for descending simple invariant bundles.

Pretending that W descends for now allows for a comparison of the linearizations of V on D and $W^{\oplus e}$. This gives rise to a GL_e representation of the Galois group. Finite subgroups prime to the characteristic of GL_e are well-understood. By Jordan's theorem - which in positive characteristic is due Larsen and Pink - they are close to being abelian. This allows us to find a cover that also checks for this decomposition behaviour.

The same type of cover works in higher dimensions. However, the workaround for descend only works for curves. To obtain Theorem 1 in higher dimensions, we carefully set up the requirements for the workaround of descend and then use a restriction theorem for stability to reduce to dimension 1.

Theorem 2 is obtained by a dimension estimate on the strata defined by the decomposition behaviour of a stable vector bundle with respect to the cover $C_{r,good}$.

The dimension estimate also uses the key lemma. Given a Galois cover $D \to C$ and a stable vector bundle V on C, consider the decomposition of V on D into $\bigoplus_{i=1}^n W_i^{\oplus e}$ of the key lemma. Using the transitive action of the Galois group, V can essentially be recovered from one of the W_i. Furthermore, W_i behaves for the dimension estimate as if it descends to a cover of degree n. This cover is D/H, where H is the stabilizer of the isomorphism class of W_i under the Galois action.

To obtain the mostly sharp estimates of the prime to p decomposition strata one needs to find a way to construct stable vector bundles with prescribed decomposition behaviour. We do this for cyclic covers which suffices for our purpose.

The closure of prime to p trivializable stable vector bundles is contained in the smallest prime to p decomposition stratum. The construction of stable vector bundles using cyclic covers used in the dimension estimates of the prime to p decomposition strata can be modified to yield prime to p trivializable stable vector bundles. This yields similar mostly sharp dimension estimates as for the smallest prime to p decomposition stratum.

Structure

The thesis is structured as follows:

In the first chapter, we collect some preliminaries regarding Gieseker-semistability, semistability, the moduli space of Gieseker semistable sheaves, and Galois morphisms. We advise the reader to skip this chapter and come back to it as needed.

In the second chapter, we collect properties of (semi)stable vector bundles under a Galois pullback. We start with the proof of the key lemma regarding the behaviour of stable vector bundles under a Galois pullback. Then we recall the notion of genuinely ramified morphisms which recently have been shown to preserve stability under pullback, see [2]. We also spell out the proof of this fact. Then we briefly study pushforward and pullback and show that they induce finite morphisms on the level of moduli spaces. For cyclic covers we describe the structure of the direct image of the structure sheaf. Pushforward is closely related to a construction of stable vector bundles given a Galois cover. The other construction is via irreducible representations of the Galois group. We spell out both constructions. Finally, we define the functorial notions of stability and study them for smooth projective curves of genus at most 1.

In the third chapter, we construct the prime to p cover $X_{r,good}$ that checks whether a vector bundle of rank r on a normal projective variety X is prime to p stable.

In chapter 4, we investigate certain strata which arise from the decomposition behaviour of a stable vector bundle on a Galois cover and estimate their dimension if $X = C$ is a smooth projective curve of genus $g_C \geq 2$. This stratification depends on the choice of the cover. Iterating the cover $X_{r,good}$, we obtain a cover $X_{r,split}$ on which a vector bundle of rank r on X decomposes into prime to p stable vector bundles, i.e., for prime to p covers dominating $X_{r,split}$ the decomposition behaviour remains unchanged. This induces the *prime to p decomposition stratification* which is independent of the cover $X_{r,split}$. If $X = C$ is a smooth projective curve, the dimension estimates obtained for arbitrary covers are sharp for the prime to p decomposition if the characteristic is avoided. We obtain Theorem 2 as a direct corollary as the prime to p stable locus is the open prime to p decomposition stratum.

In chapter 5, we study the closure of the prime to p trivializable (semi)stable vector bundles in the moduli space of (semi)stable vector bundles over a smooth projective curve of genus at least 2. We obtain mostly sharp dimension estimates in arbitrary rank. In rank 2 we can also describe the irreducible components.

We include two well-known results in the appendix to which the author could not find a reference in the literature.

Notation

We work over an algebraically closed field k of characteristic $p \geq 0$. A variety is a separated integral scheme of finite type over k. A curve is a variety of dimension 1.

Let X be a variety. The function field of X is denoted by $\kappa(X)$. We call an open subset U of X *big* if $X \setminus U$ has codimension at least 2.

If X is a projective variety we implicitly choose an ample bundle $\mathcal{O}_X(1)$ on X. Given a finite morphism $\pi : Y \to X$ we equip Y with the polarization $\mathcal{O}_Y(1) = \pi^*\mathcal{O}_X(1)$. By (semi)stability we mean μ-(semi)stability with respect to $\mathcal{O}_X(1)$.

We denote the moduli space of (semi)stable vector bundles of rank r and degree d on a smooth projective curve C by $M_C^{s,r,d}$ (resp. $M_C^{ss,r,d}$). On a projective variety X the stable vector bundles with Hilbert polynomial P form an open $M_X^{s,P}$ in the moduli space of Gieseker semistable sheaves $M_X^{G-ss,P}$.

Given a morphism $\pi : Y \to X$ of varieties and a sheaf F on X we denote the pullback π^*F also by $F_{|Y}$.

By a Galois morphism $Y \to X$ of varieties we mean a finite surjective separable morphism such that the extension of function fields $\kappa(Y)/\kappa(X)$ is Galois. A (Galois) cover is a finite étale (Galois) morphism $Y \to X$. A cyclic Galois cover is a Galois cover with cyclic Galois group.

We consider a finite abstract group G also as a finite étale group scheme over k.

For a natural number $r > 1$ the smallest proper divisor is the smallest divisor r' of r such that $r' \neq 1$. The largest proper divisor is the largest divisor r' of r such that $r' \neq r$.

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Contents

1 Preliminaries

We collect the basic properties of (semi)stable reflexive sheaves on normal projective varieties. To define (semi)stability we recall the Hilbert polynomial and its properties. Then we introduce the maximal destabilizing subsheaf and the socle for slope (semi)stability. While the maximal destabilizing subsheaf makes sense for torsion-free sheaves the socle requires that we work with reflexive sheaves on normal projective varieties. In the last section we recall the moduli space of Gieseker-semistable sheaves as well as some of its properties over a smooth projective curve of genus at least 2.

1.1 Saturated and reflexive sheaves

Reflexive sheaves are the right candidates when considering slope-stability on a normal projective variety. We begin by recalling their basic properties:

Lemma 1.1.1. *Let X be a normal variety. Then a torsion-free coherent sheaf F is reflexive iff the morphism induced by adjunction*

$$F \to j_* j^* F$$

is an isomorphism for all big open subschemes $j : U \subseteq X$.

Proof. Let η be the generic point of X. Recall the characterization of reflexive, [31, Tag 0AVB], which asserts that a torsion-free coherent sheaf F can be recovered from its stalks $F_x \subseteq F_\eta$ at codimension 1 points x iff it is reflexive. As a big open $j : U \subseteq X$ contains all codimension 1 points we find that $F \to j_* j^* F$ is an isomorphism if F is reflexive.

Conversely, note that $ev : F \to (F^\vee)^\vee$ is an isomorphism at all codimension 1 points as there torsion-free is the same as locally free. Then ev is also an isomorphism on a big open $j : U \subseteq X$. Using the commutativity of

$$\begin{array}{ccc} F & \xrightarrow{\quad ev \quad} & (F^\vee)^\vee \\ \wr\downarrow & & \wr\downarrow \\ j_* j^* F & \xrightarrow[\sim]{j_*(ev)} & j_*((j^*F)^\vee)^\vee \end{array}$$

and that three out of the four arrows are isomorphisms we conclude that F is reflexive. □

Lemma 1.1.2. *Let X be a variety. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of coherent sheaves on X. If F_2 is reflexive and F_3 is torsion-free, then F_1 is reflexive.*

Proof. This follows from the Snake lemma applied to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \\ 0 & \longrightarrow & (F_1^\vee)^\vee & \longrightarrow & (F_2^\vee)^\vee & \longrightarrow & (F_3^\vee)^\vee & \longrightarrow & 0 \end{array}$$

as for a torsion-free sheaf ev is injective, see also [31, Tag 0EB8]. □

Lemma 1.1.3. *Let $\pi : Y \to X$ be a flat morphism of varieties. Then the pullback of a torsion-free coherent sheaf by π is again torsion-free. The same holds for a saturated subsheaf of a coherent sheaf.*

Proof. The lemma is Zariski-local on X and we can assume that $X = \mathrm{Spec}(A)$ is affine. Let F be a torsion-free coherent sheaf on X. To show that π^*F is torsion-free it suffices to show this on an affine covering of Y. Thus, we may assume that $Y = \mathrm{Spec}(B)$ is affine as well.

The torsion-free coherent sheaf F corresponds to a finite torsion-free A-module M. As M is torsion-free and A is a domain, we have

$$M \subseteq M \otimes_A Q(A) \cong Q(A)^{\oplus r}$$

for $r = \dim(M \otimes_A Q(A))$. The morphism $A \to B$ is flat by assumption and we obtain

$$M \otimes_A B \subseteq Q(A)^{\oplus r} \otimes_A B \subseteq Q(B)^{\oplus r}.$$

The claim follows.

Let $F \subseteq G$ be a saturated subsheaf of a coherent sheaf G on X. By definition $Q := G/F$ is torsion-free. It is coherent as G is coherent. As π is flat the short exact sequence

$$0 \to F \to G \to Q \to 0$$

induces a short exact sequence

$$0 \to \pi^*F \to \pi^*G \to \pi^*Q \to 0.$$

As we already know that π^* preserves torsion-free, π^*F is saturated in π^*G. □

1.2 Numerical invariants of coherent sheaves

1.2.1 The Hilbert polynomial

Definition 1.2.1. Let X be a projective scheme of dimension d. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let $F \neq 0$ be a coherent sheaf on X. Then the Hilbert polynomial P_F of F with respect to $\mathcal{O}_X(1)$ is

$$P_F(n) := \chi(F \otimes \mathcal{O}_X(n)) = \sum_{i=0}^{d} (-1)^i h^i(X, F \otimes \mathcal{O}_X(n)),$$

where $h^i(X, -)$ denotes the dimension of $H^i(X, -)$ over k. We usually suppress the dependence on the choice of $\mathcal{O}_X(1)$. If we want to emphasize the dependence, then we write $P_{F,\mathcal{O}_X(1)}$ instead of P_F.

We briefly give the reason for the name, i.e., P_F is a numerical polynomial of degree equal to the dimension of the support of F, see also [12, Proposition 1.2.1].

Recall that a map $f : \mathbb{N} \to \mathbb{Z}$ is a *numerical polynomial with coefficients in* $\mathbb{Z}$ if there exists a polynomial $P \in \mathbb{Q}[x]$ such that $P(n) = \sum_{i=0}^{r} a_i \binom{n}{i}$ for some $a_i \in \mathbb{Z}$ and $f(n) = P(n)$ for $n \gg 0$, see [31, Tag 00JX]. Note that the condition $P(n) = \sum_{i=0}^{r} a_i \binom{n}{i}$ determines the coefficients of P and P is of degree r.

Lemma 1.2.2. *Let X be a projective scheme. Let $F \neq 0$ be a coherent sheaf on X. Then the Hilbert polynomial of F is a numerical polynomial of degree $d = \dim(\operatorname{supp}(F))$. The leading coefficient of F is positive.*

Proof. Observe that $F = i_* i^* F$, where $i : \operatorname{supp}(F) \to X$ is the closed immersion. As i_* is exact, we have $H^i(X, F) = H^i(\operatorname{supp}(F), i^*F)$ and we can assume that $X = \operatorname{supp}(F)$.

We prove the lemma by induction on the dimension of X. In dimension 0 the Hilbert polynomial is constant. We assume $\dim(X) \geq 1$ in the following. As $\mathcal{O}_X(1)$ is ample some tensor power is very ample and globally generated. By Lemma [12, Lemma 1.1.12], there exists a section s_n of $\mathcal{O}_X(n)$ and a section s_{n+1} of $\mathcal{O}_X(n+1)$ for some $n \gg 0$ such that we have short exact sequences

$$0 \to F \otimes \mathcal{O}_X(-i) \xrightarrow{id \otimes s_i^{-1}} F \to F_{|H_i} \to 0,$$

where H_i is the closed subscheme cut out by s_i^{-1} for $i = n, n+1$. The Euler-characteristic is additive and we obtain

$$P_F(N) - P_F(N-n) = P_{F_{|H_n}}(N) \text{ and}$$

$$P_F(N) - P_F(N-n-1) = P_{F_{|H_{n+1}}}(N)$$

for $N \geq n+1$, where the Hilbert polynomial of $F_{|H}$ is with respect to $\mathcal{O}_X(1)_{|H}$ for $H = H_n, H_{n+1}$. Thus,

$$P_F(N-n) - P_F(N-n-1) = \sum_{i=0}^{d-1} a_i \binom{N}{i}, N \gg 1,$$

is a numerical polynomial with coefficients in $\mathbb{Z}$ by induction.

Consider the polynomial $P'(N) := \sum_{i=0}^{d-1} a_i \binom{N+1}{i+1}$. Then as in [31, Tag 00JZ] we find

$$P'(N) - P'(N-1) = \sum_{i=0}^{d-1} a_i \Big(\binom{N+1}{i+1} - \binom{N}{i+1} \Big) = \sum_{i=0}^{d-1} a_i \binom{N}{i}.$$

We obtain

$$a_{-1} := P'(N) - P_F(N-n) = P'(N-1) - P_F(N-n-1)$$

and conclude

$$P_F(N-n) = P'(N) - a_{-1}$$

for $N \gg 1$. Thus, for $N \gg 1$ we have

$$\begin{aligned} P_F(N) &= P'(N+n) - a_{-1} \\ &= \sum_{i=0}^{d-1} a_i \binom{N+n+1}{i+1} - a_{-1} \\ &= \sum_{i=0}^{d} b_i \binom{N}{i}, \end{aligned}$$

for some $b_i \in \mathbb{Z}$ as $\binom{N+n+1}{i+1}$ is a numerical polynomial with coefficients in $\mathbb{Z}$.

As $\mathcal{O}_X(1)$ is ample and F is coherent, we have $H^i(X, F(N)) = 0$ for $i > 0$ and $N \gg 0$. Thus, we have $P_F(N) = H^0(X, F(N))$ for $N \gg 0$. Furthermore, $F(N)$ is globally generated for $N \gg 0$ and we find that $H^0(X, F(N)) \neq 0$ for $N \gg 0$ since $F \neq 0$. Therefore, the leading coefficient of P_F is positive. □

Definition 1.2.3. Let X be a projective scheme. Let $F \neq 0$ be a coherent sheaf on X. The *dimension* of F is the dimension of $\mathrm{supp}(F)$.

Let F be of dimension d on X. We write the coefficients of the Hilbert polynomial as

$$P_F(n) = \sum_{i=0}^{d} \frac{\alpha_{i,F,\mathcal{O}_X(1)}}{i!} n^i, n \gg 0,$$

where $\alpha_{i,F,\mathcal{O}_X(1)} \in \mathbb{Q}$. If F and $\mathcal{O}_X(1)$ are clear from the context we suppress the index and just write α_i.

The *reduced Hilbert polynomial* of F is $p_F := P_F/\alpha_d$.

Definition 1.2.4. Let X be a projective variety of dimension d. Let F be a coherent sheaf of dimension d on X. The rank of F is the rank of the $\kappa(X) = \mathcal{O}_{X,\eta}$ vector space F_η, where η denotes the generic point of X.

Lemma 1.2.5. *Let X be a projective variety of dimension d. Let $F \neq 0$ be a coherent sheaf of dimension d. Then we have* $\mathrm{rk}(F) = \alpha_{d,F}/\alpha_{d,\mathcal{O}_X}$.

Proof. If the dimension of X is 0, then $X = \mathrm{Spec}(k)$ and F is a finite dimensional k-vector space. Then the lemma is clear.

Assume that $\dim(X) \geq 1$. Replacing the ample line bundle $\mathcal{O}_X(1)$ by a positive tensor power $\mathcal{O}_X(n)$ changes the coefficients α_d of F and $\mathcal{O}_X$ both by n^d. Thus, the quotient $\frac{\alpha_{d,F}}{\alpha_{d,\mathcal{O}_X}}$ is invariant under such a replacement and we can assume that $\mathcal{O}_X(1)$ is globally generated and very ample.

For the general $s \in H^0(X, \mathcal{O}_X(1))$ we obtain a short exact sequence

$$(*) \quad 0 \to F \otimes_{\mathcal{O}_X} \mathcal{O}_X(-1) = F(-H) \xrightarrow{s^{-1}} F \to F_{|H} \to 0,$$

where H is the closed subscheme cut out by s^{-1}, see [12, Lemma 1.1.12].

Note that a coherent sheaf of dimension d is a vector bundle on a non-empty open subset. If X has dimension 1, then the general such H lies in the open locus where F is a vector bundle. As H has dimension 0, we find $F_{|H} \cong \mathcal{O}_H^{\mathrm{rk}(F)}$. Comparing coefficients of the Hilbert polynomial yields

$$\alpha_{1,F} = \alpha_{0,F_{|H}} = \mathrm{rk}(F)\alpha_{0,\mathcal{O}_H} = \mathrm{rk}(F)\alpha_{1,\mathcal{O}_X}$$

and the claim follows if the dimension is 1.

Assume in the following that $\dim(X) \geq 2$. By Bertini's theorem, see [14, Corollaire 6.11 (3)], the general $s \in H^0(X, \mathcal{O}_X(1))$ also satisfies that H is a variety. Furthermore, the general $s \in H^0(X, \mathcal{O}_X(1))$ intersects the non-empty open locus where F is a vector bundle non-trivially. Then $F_{|H}$ is of dimension $\dim(H) = d - 1$.

By induction, we have

$$\frac{\alpha_{d-1,F_{|H}}}{\alpha_{d-1,\mathcal{O}_H}} = \mathrm{rk}(F_{|H}) = \mathrm{rk}(F).$$

Furthermore, we have $\alpha_{d-1,F_{|H}} = \alpha_{d,F}$ by comparing the coefficients of the Hilbert polynomial using the short exact sequence $(*)$. Similarly, we can compute $\alpha_{d,\mathcal{O}_X}$ and the lemma follows. □

1.2.2 Degree and slope

We are particularly interested in the invariants obtained from the first two highest terms of the Hilbert polynomial of a torsion-free coherent sheaf F as these are used to define slope-(semi)stability.

Definition 1.2.6. Let X be a projective variety of dimension $d \geq 1$. Let F be a coherent sheaf F of dimension d on X. The *degree* of F is defined as

$$\deg(F) := \alpha_{d-1}(F) - \mathrm{rk}(F)\alpha_{d-1}(\mathcal{O}_X).$$

The *slope* of F is defined as

$$\mu(F) := \frac{\deg(F)}{\mathrm{rk}(F)}.$$

Note that on a smooth projective curve this definition of the degree agrees with the definition of the degree via Riemann-Roch.

Remark 1.2.7. Let X be a projective variety and $F \neq 0$ a coherent sheaf on X. Replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(N)$ for some $N \geq 1$ changes the coefficients of the Hilbert polynomial of F by

$$\alpha_{i,F,\mathcal{O}_X(N)} = N^i \alpha_{i,F,\mathcal{O}_X(1)}, 0 \leq i \leq \dim(F).$$

Thus, $N^{d-1}\mu_{\mathcal{O}_X(1)}(F) = \mu_{\mathcal{O}_X(N)}(F)$ if F is of dimension $\dim(X) \geq 1$.

Definition 1.2.8. Let X be a projective variety. Let F be a coherent sheaf on X. Then F is called *torsion-free in codimension* 1 if there exists a big open $U \subseteq X$ such $F_{|U}$ is torsion-free on U.

Remark 1.2.9. If X is a normal projective variety of dimension $d \geq 2$, then we can compute the slope on a smooth projective curve C in X: By Bertini's theorem the general hyperplane section H defined by a global section in $\mathcal{O}_X(N), N \gg 0$, is a normal projective variety, see [14, Corollaire 6.11 (3)] and [27, Theorem 7]. As replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(N)$ changes the slope by a non-zero scalar, we can assume in the following that $N = 1$.

Consider for a coherent sheaf F torsion-free in codimension 1 its reflexive hull $G := (F^\vee)^\vee$. Then $\mu(G) = \mu(F)$ as $ev : F \to G$ is an isomorphism on the big open where F is locally free. Furthermore, $G_{|H}$ is torsion-free in codimension 1 for the general hyperplane section H as the general such H intersects the locus where G is not a vector bundle transversally.

We claim that $\mu(G_{|H}) = \mu(G)$. Indeed, we have a short exact sequence

$$(*) \quad 0 \to G(-H) \to G \to G_{|H} \to 0$$

as $G(-H)$ is torsion-free and $\mathrm{Tor}^1(G, \mathcal{O}_H)$ is torsion. Comparing the coefficients of the Hilbert polynomial we obtain

$$\alpha_{d-2,G_{|H}} = \alpha_{d-1,G} - \frac{\alpha_{d,G}}{2}, \text{ and similarly}$$

$$\alpha_{d-1,\mathcal{O}_H} = \alpha_{d-1,\mathcal{O}_X} - \frac{\alpha_{d,\mathcal{O}_X}}{2}.$$

Thus,

$$\begin{aligned}\mu(G_{|H}) &= \frac{\alpha_{d-1,G} - \frac{\alpha_{d,G}}{2}}{\mathrm{rk}(G)} - \alpha_{d-1,\mathcal{O}_X} + \frac{\alpha_{d,\mathcal{O}_X}}{2} \\ &= \mu(G) - \frac{\alpha_{d,\mathcal{O}_X}}{2} + \frac{\alpha_{d,\mathcal{O}_X}}{2} = \mu(G).\end{aligned}$$

This can be used to study the behaviour of the slope under tensoring. Let F and F' be torsion-free sheaves on X. Then we claim that

$$\mu(F \otimes F') = \mu(F) + \mu(F').$$

First observe that we can replace all sheaves by their reflexive hull as this leaves the slope invariant. Then the short exact sequence $(*)$ applied to the sheaves $(F^\vee)^\vee, (F'^\vee)^\vee$, and $((F \otimes F')^\vee)^\vee$, shows that we can reduce to the case where X is a smooth projective curve. In this case F and F' are vector bundles and the statement follows from

$$\deg(F \otimes F') = \mathrm{rk}(F')\deg(F) + \mathrm{rk}(F)\deg(F'),$$

see [31, Tag 0AYX].

Similarly one can show that $\deg(V) = \deg(\det(V))$ for a vector bundle V on X.

1.3 (Semi)stability

1.3.1 Gieseker-(semi)stability

Gieseker-semistability is a property of torsion-free sheaves tailored to obtain a moduli space. It also reveals more structure of the category of torsion-free sheaves via the Harder-Narasimhan and Jordan-Hölder filtration.

Definition 1.3.1. Let X be a projective variety. A torsion-free sheaf F is a called *Gieseker semistable* if for all subsheaves $0 \neq G \subseteq F$ we have $p_G \leq p_F$, where we use the lexicographic order on the coefficients of the reduced Hilbert polynomials. It is called *Gieseker stable* if the inequality is strict for $0 \neq G \subsetneq F$.

The notion of Gieseker-(semi)stability depends in general on the choice of the ample line bundle $\mathcal{O}_X(1)$. For us $\mathcal{O}_X(1)$ is fixed and we suppress the dependence in the notation.

We recall the basic properties of Gieseker-(semi)stability.

Lemma 1.3.2 ([12], Theorem 1.3.4. and Proposition 1.5.2)**.** *Let X be a projective variety. Let $F \neq 0$ be a coherent torsion-free sheaf on X.*

Then F has a unique filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = F$$

such that the successive quotients F_{i+1}/F_i are Gieseker-semistable with reduced Hilbert polynomial p_{i+1} such that $p_n < p_{n-1} < \cdots < p_1$. This filtration is called the Harder-Narasimhan-filtration *or* HN-filtration *for short.*

If F is Gieseker-semistable with reduced Hilbert polynomial p, then there exists a filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n$$

such that the successive quotients F_{i+1}/F_i are Gieseker-stable with reduced Hilbert polynomial p. This filtration is called the Jordan-Hölder-filtration *or* JH-filtration *for short. Furthermore, the associated graded object $\bigoplus_{i=0}^{n-1}(F_{i+1}/F_i)$ is independent of the choice the JH-filtration of F.*

Lemma 1.3.3 ([12], Proposition 1.2.7 and Corollary 1.2.8)**.** *Let X be a projective variety. Let F and G be Gieseker-stable sheaves with the same reduced Hilbert polynomial. Then a non-zero morphism $F \to G$ is an isomorphism.*

In particular, Gieseker-stable vector bundles are simple.

Definition 1.3.4. Let X be a projective variety. A torsion-free coherent sheaf $F \neq 0$ with reduced Hilbert polynomial p is called *Gieseker-polystable* if F is a direct sum of Gieseker-stable sheaves with reduced Hilbert polynomial p.

Two semistable sheaves $F \neq 0$ and $F' \neq 0$ are *S-equivalent* if the associated graded objects of the JH-filtrations of F and F' are isomorphic. We denote this by $F \cong_S F'$.

1.3.2 Slope-(semi)stability

Slope-semistability coincides with Gieseker-semistability on a smooth projective curve. It is the right candidate for our functorial approach as it is a property in codimension 1 and a finite separable morphism of normal varieties is flat in codimension 1. We recall the definition and spell out the proofs for the destabilizing subsheaf as well as the socle as these are essential in studying (semi)stability from a functorial perspective.

Definition 1.3.5. Let X be a projective variety. A coherent sheaf $F \neq 0$ is called *slope-semistable* (also *μ-semistable*) if it is torsion-free in codimension 1 and for all $G \subseteq F$ of rank $0 < \mathrm{rk}(G) \leq \mathrm{rk}(F)$ we have $\mu(G) \leq \mu(F)$.

Furthermore, F is called *slope-stable* (also *μ-stable*) if it is slope-semistable and the inequality is strict for subsheaves $G \subseteq F$ of rank $0 < \mathrm{rk}(G) < \mathrm{rk}(F)$.

Note that we could have alternatively required that

$$\frac{\alpha_{d-1,G}}{\alpha_{d,G}} \leq \frac{\alpha_{d-1,F}}{\alpha_{d,F}}$$

to define μ-semistability and similarly for μ-stability.

Slope-(semi)stability is our main interest and we abbreviate it to *(semi)stability.*

As for Gieseker-(semi)stability, the notion of (semi)stability depends in general on the choice of the ample line bundle $\mathcal{O}_X(1)$. We work with a fixed $\mathcal{O}_X(1)$ and usually suppress this dependence in the notation.

Note that neither (semi)stability nor Gieseker (semi)stability change under the replacement of $\mathcal{O}_X(1)$ with $\mathcal{O}_X(N), N \geq 1$.

(Semi)stability is similarly behaved to Gieseker-(semi)stability. We spell out the proofs of the existence of the maximal destabilizing subsheaf, the socle, and that stable vector bundles are simple.

Let X be a projective variety of dimension $d \geq 1$. (Semi)stability can be viewed as a property in the category $\mathrm{Coh}_{d,d-1}$ of coherent sheaves torsion-free in codimension 1 up to inverting morphisms with kernel and cokernel supported in codimension 2 or higher, see also [12, Section 1.6]. We are interested in (semi)stability on normal projective varieties. In this case, every coherent sheaf torsion-free in codimension 1 is isomorphic in $\mathrm{Coh}_{d,d-1}$ to its reflexive hull. Thus, instead of working in the category $\mathrm{Coh}_{d,d-1}$ we work with reflexive sheaves.

Lemma 1.3.6. *Let X be a projective variety of dimension $d \geq 1$. Consider a short exact sequence*

$$0 \to K \to F \to Q \to 0$$

of coherent sheaves on X. If K, F, and Q are of dimension d, then the following are equivalent:

(i) $\mu(K) < \mu(F)$,

(ii) $\mu(K) < \mu(Q)$, *and*

(iii) $\mu(F) < \mu(Q)$.

Furthermore, if K and F are of dimension d and $\mathrm{supp}(Q)$ *has codimension* 1, *then* $\mu(K) < \mu(F)$.

Proof. The equivalence of (i) - (iii) follows directly from

$$\alpha_{d,F} = \alpha_{d,K} + \alpha_{d,Q} \text{ and } \alpha_{d-1,F} = \alpha_{d-1,K} + \alpha_{d-1,Q}.$$

If K and F are of dimension d and $\operatorname{supp}(Q)$ has codimension 1, then we have

$$\alpha_{d,F} = \alpha_{d,K} \text{ and } \alpha_{d-1,F} = \alpha_{d-1,K} + \alpha_{d-1,Q}.$$

As $\alpha_{d-1,Q}$ is positive we conclude $\mu(K) < \mu(F)$. □

Lemma 1.3.7. *Let X be a projective variety of dimension $d \geq 1$. Let V and W be stable vector bundles on X with the same slope μ. Then every morphism $V \to W$ is either trivial or an isomorphism.*

In particular, stable vector bundles are simple.

If X is in addition normal, then the same holds for stable reflexive sheaves.

Proof. Let $\varphi : V \to W$ be a morphism. If φ is non-zero, then we claim that φ is injective. Indeed, if the kernel K of φ were non-zero, then by stability of V the torsion-free sheaf K has $\mu(K) < \mu$ or the same rank as V. Thus, the image of φ would have slope greater than μ or be torsion. Both cases are impossible as W is stable of slope μ and torsion-free.

As φ is injective and V and W have the same slope, the stability of W implies that V and W have the same rank. The support of the quotient $Q := W/\varphi(V)$ is cut out by the ideal

$$\det(\varphi) \otimes \det(W)^{-1} : \det(V) \otimes \det(W)^{-1} \hookrightarrow \mathcal{O}_X.$$

If Q is non-zero, then Q is supported on the effective Cartier divisor defined by $\det(\varphi) \otimes \det(W)^{-1}$. Thus, we find $\mu(V) < \mu(W)$ - a contradiction.

The endomorphism algebra $\operatorname{End}_{\mathcal{O}_X}(V)$ of a stable vector bundle V is a skew field by the above discussion. It is also a finite k-algebra as

$$\operatorname{End}_{\mathcal{O}_X}(V) = H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(E))$$

and X is projective. Thus, the k-algebra generated by an endomorphism of V in $\operatorname{End}(V)$ is a finite field extension of k. As k is algebraically closed we conclude $\operatorname{End}(V) = k$, i.e., V is simple. See also the proof of [12, Corollary 1.2.8].

Assume that X is in addition normal. Let $\varphi : F \to G$ be a non-zero morphism of stable reflexive sheaves of the same slope on X. Analogous to the vector bundle case we find that φ is injective and that F and G have the same rank. Let $Q := G/F$. Then Q is torsion of dimension $\leq d-2$, as F and G have the same slope and rank. Restricting to the big open $U := X \setminus \operatorname{supp}(Q)$ we find that $\varphi_{|U}$ is an isomorphism. Using the reflexivity of F and G we conclude that φ is an isomorphism.

We find $\operatorname{End}(F) = k$ by an analogous argument to the vector bundle case. □

Remark 1.3.8. Note that Lemma 1.3.7 also holds for Gieseker-stable torsion-free sheaves with the same reduced Hilbert polynomial, see Lemma 1.3.3. While stability implies Gieseker-stability this does not mean that the lemma also holds for torsion-free stable sheaves. For example, the ideal sheaf I_x of a point x on a projective variety X of dimension ≥ 2 is torsion-free and stable of slope 0, but clearly $I_x \subsetneq \mathcal{O}_X$.

Lemma 1.3.9. *Let X be a projective variety. Let $F, F' \subseteq G$ be subsheaves of a torsion-free coherent sheaf G such that $F \cap F' \neq 0$.*
If $\mu(F + F') \leq \mu(F) < \mu(F')$, then $\mu(F') < \mu(F \cap F')$.

Proof. Consider the short exact sequence of torsion-free sheaves

$$0 \to F \cap F' \to F \oplus F' \to F + F' \to 0.$$

As the degree is additive, we obtain

$$\deg(F \oplus F') - \deg(F + F') = \deg(F \cap F').$$

In terms of slopes, we have

$$\frac{\mu(F)\operatorname{rk}(F) + \mu(F')\operatorname{rk}(F') - \mu(F + F')\operatorname{rk}(F + F')}{\operatorname{rk}(F \cap F')} = \mu(F \cap F').$$

Applying the assumptions we find

$$\begin{aligned} \mu(F \cap F') &\geq \frac{\mu(F)\operatorname{rk}(F) + \mu(F')\operatorname{rk}(F') - \mu(F)\operatorname{rk}(F + F')}{\operatorname{rk}(F \cap F')} \\ &= \frac{\mu(F')\operatorname{rk}(F') - \mu(F)(\operatorname{rk}(F') - \operatorname{rk}(F \cap F'))}{\operatorname{rk}(F \cap F')} \\ &> \mu(F'). \end{aligned}$$

□

Lemma 1.3.10 (Analogue of Lemma 1.3.5, [12])**.** *Let X be a projective variety. Let $F \neq 0$ be a torsion-free coherent sheaf on X. There exists a unique subsheaf $0 \neq G \subseteq F$ such that $\mu(G) \geq \mu(G')$ for all $0 \neq G' \subseteq F$ and if $\mu(G) = \mu(G')$ for some $0 \neq G' \subseteq F$, then $G' \subseteq G$.*
In particular, G is semi-stable and saturated.

Proof. As already stated in [12, Theorem 1.6.6] the proof [12, Lemma 1.3.5] carries over to our situation. We spell this out in the following. Uniqueness is clear by the defining properties of G.

We order the non-zero subsheaves of F by $G \leq G'$ iff $G \subseteq G'$ and $\mu(G) \leq \mu(G')$. As F is coherent ascending chains of subsheaves are eventually constant. In particular, every ascending chain with respect to $\leq$ has an upper bound. Also note that the set of non-zero subsheaves contains F and we can apply Zorn's lemma to find a maximal subsheaf of F with respect to $\leq$.

Observe that a maximal subsheaf G of F with respect to $\leq$ is saturated, as its saturation G^{sat} has slope $\mu(G) \leq \mu(G^{sat})$.

Let G be a maximal subsheaf with minimal rank among the maximal subsheaves. We claim that G has the desired properties.

We first show that G is semi-stable. Assume that there exists a non-trivial subsheaf H of G such that $\mu(G) < \mu(H)$. Again by Zorn's lemma there also exists a maximal such H with respect to $\leq$ for subsheaves in G. Similarly, there exists a maximal $H' \subseteq F$ with respect to $\leq$ for subsheaves in F containing H.

Then $\mu(G) < \mu(H) \leq \mu(H')$ by definition. The maximality of G implies $\mu(G + H') \leq \mu(G)$. Note that H' is not contained in G as G has minimal rank among maximal subsheaves and F/H' is torsion-free. Thus, we obtain a strict inequality $\mu(G + H') < \mu(G)$.

By Lemma 1.3.9, we obtain $\mu(H) \leq \mu(H') < \mu(G \cap H')$. This contradicts the maximality of H as $H \subseteq G \cap H' \subseteq G$.

We now check that $\mu(G)$ has maximal slope among all non-trivial subsheaves of F. Assume there was $0 \neq H \subseteq F$ such that $\mu(G) < \mu(H)$. Replacing H by a maximal subsheaf with respect to $\leq$ in F, we can assume that H is maximal. Then H is not contained in G as G has minimal rank among maximal subsheaves and F/H is torsion-free.

Thus, $G \subsetneq H + G$ and by maximality of G we find $\mu(H + G) < \mu(G)$. By Lemma 1.3.9, we obtain $\mu(G) < \mu(H) < \mu(H \cap G)$ if $H \cap G \neq 0$. This is not possible as we have already shown that G is semistable. If $H \cap G = 0$, then $G + H = G \oplus H$ and by Lemma 1.3.6 we obtain $\mu(G \oplus H) > \mu(G)$ contradicting the maximality of G.

Let $0 \neq G' \subseteq F$ with $\mu(G') = \mu(G)$. As every subsheaf of G' is also a subsheaf of F, the maximality of $\mu(G)$ implies that G' is semistable. Then $G \oplus G'$ is semistable with slope $\mu(G + G') = \mu(G)$ as well. As $G \oplus G' \twoheadrightarrow G + G'$ we find that $\mu(G) \leq \mu(G + G')$. By the maximality of G we conclude $G = G + G'$ and thus $G' \subseteq G$. $\square$

Definition 1.3.11. Let X be a projective variety. Let F be a non-trivial torsion-free coherent sheaf on X. The subsheaf G of Lemma 1.3.10 is called the *maximal destabilizing subsheaf* of F.

Definition 1.3.12. Let X be a projective variety. A reflexive sheaf $F \neq 0$ on X is called *polystable* if it is a direct sum of stable torsion-free sheaves of the same slope.

Remark 1.3.13. Note that, as a direct summand of a vector bundle is again a vector bundle, a vector bundle is polystable iff it is a direct sum of stable vector bundles of the same slope.

Lemma 1.3.14. *Let X be a normal projective variety. Let $F \neq 0$ be coherent sheaf torsion-free in codimension* 1*. Then F is (semi)stable if and only iff the reflexive hull of F is (semi)stable.*

Proof. As X is normal and F is torsion-free in codimension 1, the locus where F is a vector bundle is a big open. Thus, $ev : F \to (F^\vee)^\vee$ is an isomorphism on a big open. Then for a subsheaf $G \subseteq F$ torsion-free in codimension 1, the image $G' := ev(G)$ is torsion-free and has the same slope as G. Similarly, for a subsheaf $G' \subseteq (F^\vee)^\vee$ the preimage $ev^{-1}(G')$ is torsion-free in codimension 1 and has the same slope as G'.

We conclude that F contains a subsheaf of larger (or equal) slope iff $(F^\vee)^\vee$ contains a subsheaf of larger (or equal) slope. □

Our next goal is to show the existence of the socle, i.e., the maximal saturated polystable subsheaf of a reflexive sheaf.

Lemma 1.3.15. *Let X be a normal projective variety. Let F be a reflexive semistable sheaf on X. Then there exists a polystable saturated subsheaf $S \subseteq F$ maximal among polystable subsheaves in F. In particular, S is unique and we call it the* socle *of F.*

Proof. Let $G \subseteq F$ be a non-trivial subsheaf of the same slope μ as F and minimal rank among such subsheaves. Clearly, G is semistable. We claim that G is stable. Indeed, if $G' \subseteq G$ is a non-trivial subsheaf of the same slope, then by definition of G they have the same rank.

As G is stable of slope $\mu(F)$, so is its reflexive hull, see Lemma 1.3.14. The reflexive hull coincides with the saturation $G^{sat} \subseteq F$ as F is reflexive, see Lemma 1.1.1. Thus, there is a saturated stable subsheaf $G \subseteq F$ of slope $\mu(F)$.

Let G' be a polystable saturated subsheaf of slope $\mu(F)$ of F not containing G. Then we claim that $G + G'$ is polystable and saturated.

Denote the stable direct summands of G' by $G'_1, \ldots, G'_n$. As G' is reflexive so are the $G'_i, i = 1, \ldots, n$.

If $G \cap G'_i \neq 0$ for some $1 \leq i \leq n$, then using the stability of G and G'_i and $\mu(G) = \mu(F) = \mu(G')$, we find that $G \cap G'_i$ has the same rank and slope as G as well as G'_i. Furthermore, the quotients $G/G \cap G'_i$ and $G'_i/G \cap G'_i$ vanish on a big open as $\mu(G) = \mu(G'_i) = \mu(G \cap G'_i)$. Thus, G'_i and G are both equal to $G \cap G'$ on some big open. As G'_i and G are both reflexive, we conclude $G'_i = G$. By assumption G' does not contain G - a contradiction. This shows that $G + G' \cong G \oplus G'$ is polystable.

It remains to show that $G + G'$ is saturated. Let T be the torsion subsheaf of $F/(G + G')$. As F and $G + G'$ have the same slope and F is semistable, the support of T has codimension at least 2. Let $U = X \setminus \operatorname{supp}(T)$ and denote the open immersion by $j : U \to X$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G \oplus G' & \longrightarrow & F & \longrightarrow & F/(G \oplus G') & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow ad_{F/(G\oplus G')} & & \\ 0 & \longrightarrow & j_*j^*(G \oplus G') & \longrightarrow & j_*j^*F & \longrightarrow & j_*j^*(F/(G \oplus G')), & & \end{array}$$

where the rows are exact and the first two vertical arrows are isomorphisms as F, G, and G' are reflexive. By the snake lemma $ad_{F/G}$ is injective. As U is disjoint from the support, T lies in the kernel of $ad_{F/(G\oplus G')}$. Thus, T vanishes showing that $G + G'$ is saturated.

The process of adding a saturated stable subsheaf of slope $\mu(F)$ to a saturated polystable subsheaf of slope $\mu(F)$ increases the rank and terminates after finitely many steps. The constructed sheaf is polystable, saturated, and maximal among such sheaves as otherwise we could add another saturated stable sheaf in the above manner. □

1.4 The moduli space of Gieseker-semistable sheaves

We briefly recall the definition of the moduli space of Gieseker-semistable sheaves on a projective variety X.

Let (Sch/k) denote the category of schemes of finite type over k. Let $S \in (Sch/k)$. A *family of Gieseker-semistable sheaves over* S is a coherent sheaf $\mathcal{F}$ on $X \times_k S$ flat over S such that for a (not necessarily closed) point $s \in S$ the fibre $\mathcal{F}_s := \mathcal{F}_{|X \times_k s}$ is Gieseker-semistable.

Consider for families of Gieseker-semistable sheaves $\mathcal{F}, \mathcal{F}'$ on $X \times_k S$ the equivalence relation $\mathcal{F} \sim \mathcal{F}'$ if there exists a line bundle L on S such that $\mathcal{F} \cong \mathcal{F}' \otimes \operatorname{pr}_S^* L$, where $\operatorname{pr}_S : X \times_k S \to S$ is the projection.

Let $P \in \mathbb{Q}[t]$ be a polynomial. The moduli space of Gieseker-semistable sheaves on X with Hilbert polynomial P is denoted by $M_X^{G-ss,P}$ and corepresents the contravariant functor

$$\mathcal{M}_X^{G-ss,P} : (Sch/k) \to Sets$$

mapping S to the set of families of Gieseker-semistable sheaves $\mathcal{F}$ on $X \times_k S$ with Hilbert polynomial P up to equivalence by $\sim$, see [16, Theorem 0.2] and [12, Theorem 4.3.4].

The moduli space $M_X^{G-ss,P}$ is a projective scheme over X and its closed points correspond to S-equivalence classes of Gieseker semistable torsion-free sheaves on X. In each S-equivalence class there is a unique Gieseker-polystable representative.

Gieseker-stability is an open condition, see [12, Proposition 2.3.1], and the moduli space $M_X^{G-ss,P}$ contains the open subscheme $M_X^{G-s,P}$ of Gieseker-stable torsion-free sheaves with Hilbert polynomial P.

Similarly, slope-stability as well as being locally-free are open conditions and $M_X^{G-s,P}$ contains the open subscheme $M_X^{s,P}$ of slope-stable vector bundles with Hilbert polynomial P. The openness of slope-stability is not stated directly in [12, Proposition 2.3.1], but an analogous argument works:

For an S-flat family $\mathcal{F}$ on $X \times_k S$ of torsion-free coherent sheaves the points $s \in S$ where $\mathcal{F}_s$ is slope-stable can be described by the non-existence of quotients of $\mathcal{F}_s \to Q$ with slope $\mu(Q) < \mu(\mathcal{F}_s)$ and $\mathrm{rk}(Q) < \mathrm{rk}(\mathcal{F}_s)$. The family of such Q is bounded as they have bounded slope by definition, see Lemma [12, Lemma 1.7.9]. This can be expressed for $s \in S$ to not lie in the image of a finite union of $\mathcal{Q}uot_{\mathcal{F},P'} \to S$ for some polynomial P'. As the quot-scheme is projective over S we conclude the openness of slope-stability.

Let X be a normal projective variety. To define morphisms between moduli spaces of Gieseker semistable sheaves it suffices to define a morphism of the functors they corepresent. For example, if $X = C$ is a smooth projective curve, then

$$M_X^{G-ss,P} \times_k \mathrm{Pic}_X^0 \to M_X^{G-ss,P}, (F, L) \mapsto F \otimes L,$$

is short-hand for the morphism defined via the morphism of functors

$$\mathcal{M}_X^{G-ss,P}(S) \times \mathcal{P}ic_X^0(S) \to \mathcal{M}_X^{G-ss,P}(S), (\mathcal{F}, \mathcal{L}) \mapsto \mathcal{F} \otimes \mathcal{L}.$$

To check that this defines a morphism of functors one has to check compatibility with pullback along $S' \to S$ and that $\mathcal{F} \otimes \mathcal{L}$ is a family of Gieseker-semistable sheaves of Hilbert polynomial P. Then we obtain the desired morphism of moduli spaces by using that $M_X^{G-ss,P} \times_k \mathrm{Pic}_X^0$ and $M_X^{G-ss,P}$ corepresent the corresponding functor.

Similarly, we can define a morphism

$$M_X^{G-ss,r_1P} \times_k M_X^{G-ss,r_2P} \to M_X^{G-ss,(r_1+r_2)P}, (\mathcal{F}_1, \mathcal{F}_2) \mapsto \mathcal{F}_1 \oplus \mathcal{F}_2,$$

as a direct sum of Gieseker-semistable sheaves with the same reduced Hilbert polynomial p is Gieseker semistable with reduced Hilbert polynomial p.

If $X = C$ is a smooth projective curve of genus $g_C \geq 2$ much more can be said, see [28, Theoreme 17 and 18, p.22]: The moduli space $M_C^{s,r,d}$ of stable vector bundles of rank r and degree d is dense in the moduli space $M_C^{ss,r,d}$ of semistable vector bundles of rank r and degree d. Furthermore, $M_C^{s,r,d}$ is a smooth variety, whereas $M_C^{ss,r,d}$ is a normal projective variety. The dimension of $M_C^{s,r,d}$ is $r^2(g_C - 1) + 1$.

Let L be a line bundle on C of degree d. Then moduli space $M_C^{s,r,L}$ (resp. $M_C^{ss,r,L}$) of stable (resp. semistable) bundles of rank r and determinant L are defined as the fibre over L of

$$M_C^{s,r,d} \to \mathrm{Pic}_C^d, V \mapsto \det(V), \text{ and resp. of}$$

$$M_C^{ss,r,d} \to \mathrm{Pic}_C^d, V \mapsto \det(V)$$

and have dimension $(r^2-1)(g_C-1)$. This can be found for characteristic 0 in [12, Corollary 4.5.5]. In arbitrary characteristic, the dimension can be computed by considering the finite surjective morphisms

$$M_L^{ss,r} \times_k \mathrm{Pic}^0_{C/k} \to M_C^{ss,r,d}, (V, M) \mapsto V \otimes M, \text{ and}$$

$$M_L^{s,r} \times_k \mathrm{Pic}^0_{C/k} \to M_C^{s,r,d}, (V, M) \mapsto V \otimes M.$$

1.5 Actions and Covers

As our definition of a Galois cover with Galois group G is slightly nonstandard, we show that it coincides with the notion of a connected G-torsor. We also collect some preliminary results on Galois morphisms of normal varieties which are needed to formulate a descend theory for such morphisms.

Lemma 1.5.1. *Let $\pi : Y \to X$ be a Galois morphism of normal varieties with Galois group G. Then the following hold:*

(i) *There is a canonical action of G on Y/X, i.e., an action of G on Y leaving the morphism $Y \to X$ invariant.*

(ii) *$X = Y/G$ is the quotient under the action obtained in (i), i.e., $\mathcal{O}_X = (\pi_*\mathcal{O}_Y)^G$ as subsheaves of $\pi_*\mathcal{O}_Y$ and on underlying topological spaces $|X| = |Y|/G$.*

(iii) $\mathrm{Aut}(Y/X) = G$.

(iv) *An irreducible component of $Y \times_X Y$ equipped with the reduced subscheme structure is isomorphic to Y.*

(v) *Equip $G \times_k Y$ with the group action*

$$G \times_k G \times_k Y \to G \times_k Y, (\tau, \sigma, y) \mapsto (\tau\sigma, y)$$

and $Y \times_X Y$ with the group action

$$G \times_k Y \times_X Y \to Y \times_X Y, (\sigma, (y, y')) \mapsto (y, \sigma y').$$

Then the morphism

$$\varphi : G \times_k Y \to Y \times_X Y, (\sigma, y) \mapsto (y, \sigma y)$$

is surjective, G-invariant, and identifies the irreducible components of $Y \times_X Y$ with elements of G.

Proof. (i): The action of G on Y/X is obtained via thinking of Y as the normal closure of X in $\kappa(Y)$. Then the Galois action of G on $\kappa(Y)/\kappa(X)$ induces the action of G on Y/X.

(ii): As X is a normal variety and Y is its normal closure in $\kappa(Y)$, we have that G acts on the fibres over a (not necessarily closed) point $x \in X$ transitively, see [31, Tag 0BRK]. Furthermore, $Y \to X$ is finite and surjective. Thus, $|X| = |Y|/G$. It remains to check

$$\mathcal{O}_X = (\mathcal{O}_Y)^G \subseteq \pi_* \mathcal{O}_Y.$$

The inclusion $\mathcal{O}_X \subseteq (\mathcal{O}_Y)^G$ is immediate. Let $X = \bigcup_{i \in I} \operatorname{Spec}(A_i)$ be an affine open covering. As $Y \to X$ is finite, $\operatorname{Spec}(B_i) := Y \times_X \operatorname{Spec}(A_i)$ is an affine open cover of Y. Furthermore, G acts on $\operatorname{Spec}(B_i)/\operatorname{Spec}(A_i)$ and it suffices to prove the statement in the affine case.

Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Consider $b \in B^G$ as an element of $Q(B)$. We find that $b \in Q(A)$ as $Q(B)/Q(A)$ is Galois with Galois group G and $\sigma b = b$ for all $\sigma \in G$. By assumption $A \to B$ is finite. Thus, b is the root of some monic polynomial $f \in A[t]$. As A is normal and $b \in Q(A)$, we conclude $b \in A$.

(iii): By (i) we have a natural morphism $G \to \operatorname{Aut}(Y/X)$. An automorphism Y/X induces an automorphism of $\kappa(Y)/\kappa(X)$. As G is the Galois group of $\kappa(Y)/\kappa(X)$, we obtain a morphism $\operatorname{Aut}(Y/X) \to G$. By construction of the action of G on Y/X, we have that

$$G \to \operatorname{Aut}(Y/X) \to G$$

is the identity. It remains to show that

$$\operatorname{Aut}(Y/X) \to G$$

is injective. As $\mathcal{O}_Y \subseteq \kappa(Y)$, an automorphism of Y/X is trivial iff it is trivial on $\kappa(Y)$ and the claim follows.

(iv) and (v): By (ii) we have $Y/G = X$ and the surjectivity of φ is immediate from the transitive action of G on the fibres. Since Y is a variety, the scheme theoretic image Y^σ of $Y \cong \{\sigma\} \times_k Y$ in $Y \times_X Y$ is a variety. Furthermore, Y is isomorphic to Y^σ as the inverse of $Y \to Y^\sigma$ is given by $\operatorname{pr}_1 : Y \times_X Y \to Y$. We obtain (iv).

Alternatively, we can describe Y^σ as the graph of σ considered as an automorphism of Y/X.

It remains to show that for $\sigma \neq \tau \in G$ the components Y^σ and Y^τ are distinct. If $Y^\sigma = Y^\tau$, then $\sigma = \tau$ as automorphisms of Y over X. By (iii) we obtain $\sigma = \tau$.

The G-invariance is clear from the definition of φ. □

Remark 1.5.2. Let $Y \to X$ be a Galois morphism of normal projective varieties with Galois group G. The action of G on Y induces an action of G on $M_Y^{G-ss,P}$ as the Hilbert polynomial is computed with respect to $\mathcal{O}_X(1)_{|Y}$ which is invariant under the action. The same holds for $M_Y^{G-s,P}$ and $M_Y^{s,P}$.

Our notion of a Galois cover agrees with the notion of a connected G-torsor:

Lemma 1.5.3. *Let G be a finite group. Let $Y \to X$ be a finite separable morphism of normal varieties. Then $Y \to X$ is a Galois cover with Galois group G iff $Y \to X$ is a G-torsor.*

Proof. If $Y \to X$ is a G-torsor, then so is $\mathrm{Spec}(\kappa(Y)) \to \mathrm{Spec}(\kappa(X))$ and $Y \to X$ is Galois with Galois group G. Furthermore, $Y \to X$ is étale as G is étale over k.

Assume that $Y \to X$ is a Galois cover with Galois group G. By Lemma 1.5.1 there is a canonical action of G on Y/X. Furthermore, the morphism

$$\varphi : G \times_k Y \to Y \times_X Y, (y, \sigma) \mapsto (y, \sigma y)$$

is surjective, G-invariant, and the irreducible components of $Y \times_X Y$ are identified with the irreducible components of $G \times_k Y$ under this morphism, see Lemma 1.5.1.

As normal is an ascending property under étale morphisms, the fibre product $Y \times_X Y$ is normal, see [31, Tag 033C]. In particular, $Y \times_X Y$ is the disjoint union of its irreducible components, i.e., $\varphi : G \times_k Y \to Y \times_X Y$ is an isomorphism. □

Lemma 1.5.4. *Let $Y \to X$ be a Galois cover of a normal variety X with Galois group G. Then there is a $1:1$-correspondence between subgroups of G and intermediate covers $Y \to Y' \to X$, where Y' is a normal variety.*

Under this correspondence normal subgroups $N \subseteq G$ correspond to intermediate Galois covers $Y' \to X$.

Proof. This is a special case of [30, Proposition 5.3.8].

For a subgroup $H \subseteq G$ the quotient Y/H exists in the sense of Lemma 1.5.1 (ii) and $Y/H \to X$ is étale, see [30, Proposition 5.3.7]. Then then Galois correspondence for the Galois extension $\kappa(Y)/\kappa(X)$ induces the desired correspondence by taking the integral closure of X in an intermediate field extension. □

Lemma 1.5.5. *Let $\pi : Y \to X$ be a cover of a normal variety X. Let $Z \to Y$ be the normal closure of Y in the Galois hull of $\kappa(Y)/\kappa(X)$. Then $Z \to X$ is a Galois cover, we call it the* Galois closure of Y/X.

Proof. Let $Z' \to X$ be a connected G'-torsor dominating $Y \to X$ for some finite group G', see [30, Proposition 5.3.9 and Proposition 5.3.16]. As $Z' \to X$ is Galois it also dominates $Z \to X$.

Let H be the Galois group of $Z' \to Z$. Then $H \subseteq G'$ is a subgroup and $Z = Z'/H$ is a cover of X by Lemma 1.5.4. □

2 Functoriality

In this chapter we collect several results related to (non-)functoriality of stability.

The starting observation is that, while stability is in general not preserved under pullback by a Galois morphism, polystability is preserved, see Lemma 2.1.9. We obtain the key lemma, Lemma 2.1.13: an action of the Galois group on the isomorphism classes of the stable direct summands of the pullback.

Genuinely ramified morphisms behave very nicely from our functorial perspective: they preserve stability under pullback, see [2, Theorem 2.5]. We spell out the proof. As in the original proof we first prove an alternative description of genuinely ramified morphisms. We give a shorter proof of this fact in the Galois case using different techniques. Then we continue with the strategy of [3], as also stated in the proof of [2, Theorem 2.5], to show that stability is preserved under pullback by genuinely ramified morphisms.

Using that genuinely ramified morphisms preserve stability and basic properties of finite étale morphisms, we show that pullback along a separable morphism induces a finite morphism on the level of moduli spaces. This result is known and can be deduced from [11, Theorem 4.2] which goes back to [7, Theorem I.4]. We do note that our proof is more direct and also gives an understanding of the cardinality of the fibres at a stable vector bundle. Similarly, we can show that pushforward along a finite étale morphism induces a finite morphism on the level of moduli spaces.

We briefly describe the direct image of the structure sheaf of a cyclic cover.

We continue to construct stable vector bundles given a Galois cover. There are two main constructions: via representations and via orbits. The orbit construction is closely related to pushforward and we identify the open where pushforward preserves stability.

Finally, we introduce functorial notions of stability and give a complete description for smooth projective curves of genus ≤ 1.

2.1 The key lemma

In this section we make first observations on the behaviour of a stable vector bundle after a Galois pullback. We recall that semi- and polystability are preserved under such a pullback. This is essentially because the destabilizing subsheaf and the socle are unique and thus descend. Using that polystability is preserved under pullback, we obtain an action of the Galois group on the stable direct summands of the pullback.

To spell this out properly we first prove a descend lemma for Galois morphisms (in codimension 1). While certainly well-known and used in the literature we could not find a reference for this and so include the proof.

2.1.1 Descend of vector bundles

We recall the notions of G-invariance and G-linearization and prove a descend lemma under Galois morphisms for the latter.

Definition 2.1.1. Let $Y \to X$ be a Galois morphism of normal varieties with Galois group G. Thinking of Y as the normal closure of X in $\kappa(Y)$ we obtain an action of G on Y/X, see Lemma 1.5.1. Let $H \subseteq G$ be a subgroup.

A quasi-coherent sheaf V on Y is said to be *H-invariant* if for every $\sigma \in H$ we have an isomorphism $\varphi_\sigma : V \xrightarrow{\sim} \sigma^* V$.

A subsheaf $W \subseteq V$ of an H-invariant quasi-coherent sheaf V is called *H-invariant* if the isomorphisms $\varphi_\sigma : V \xrightarrow{\sim} \sigma^* V$ induce isomorphisms $W \xrightarrow{\sim} \sigma^* W$ of subsheaves.

A quasi-coherent sheaf V on Y is said to admit an *H-linearization* if for all $\sigma \in H$ there exists an isomorphism $\varphi_\sigma : V \xrightarrow{\sim} \sigma^* V$ such that the following diagram

$$\begin{array}{ccccc} V & \xrightarrow{\varphi_\tau} & \tau^* V & \xrightarrow{\tau^*\varphi_\sigma} & \tau^*\sigma^* V \\ & \searrow^{\varphi_{\sigma\tau}} & & & \downarrow{\scriptstyle \alpha_{\sigma,\tau}} \\ & & & & (\sigma\tau)^* V \end{array}$$

commutes for $\sigma, \tau \in H$, where $\alpha_{\sigma,\tau}$ denotes the unique isomorphism $\tau^*\sigma^* \to (\sigma\tau)^*$ of functors induced by the fact that both are left adjoint to $(\sigma\tau)_* = \sigma_*\tau_*$. We suppress $\alpha_{\sigma,\tau}$ and write

$$\tau^*\varphi_\sigma \circ \varphi_\tau = \varphi_{\sigma\tau}$$

instead of the commutativity of the above diagram. This is justified as $\alpha_{\sigma,\tau}$ satisfies all sorts of compatibilities, for example, we have

$$\alpha_{\rho,\sigma\tau}\alpha_{\sigma,\tau} = \alpha_{\sigma,\tau\rho}\alpha_{\tau,\rho} : \rho^*\tau^*\sigma^* \to (\sigma\tau\rho)^*$$

for $\sigma, \tau, \rho \in H$.

Remark 2.1.2. By definition, an H-invariant subsheaf $W \subseteq V$ of a quasi-coherent sheaf admitting an H-linearization admits an H-linearization as well.

Remark 2.1.3. Let $\pi : Y \to X$ be a Galois cover of a normal variety X with Galois group G. As $Y \to X$ is a G-torsor, see Lemma 1.5.3, the cocycle condition for the étale cover $Y \to X$ translates to the notion of a G-linearization and we can descend a (quasi-)coherent sheaf V on Y with a G-linearization to a (quasi-)coherent sheaf W on X, i.e., $\pi^*W \cong V$ compatible with the linearization of V and the natural one of π^*W induced by $\sigma^*\pi^* \cong \pi^*$ for $\sigma \in G$.

We give some natural examples of bundles admitting a linearization:

Example 2.1.4. Let $\pi : Y \to X$ be a Galois morphism of normal varieties with Galois group G. Let $H \subseteq G$ be a subgroup. Let V be a vector bundle on Y. Then $\bigoplus_{\rho\in H} \rho^*V$ and $\bigotimes_{\rho\in H} \rho^*V$ both admit a natural H-linearization. The isomorphisms for $\sigma \in H$ are given by

$$\varphi_\sigma^\oplus : \bigoplus_{\rho\in H} \rho^*V \xrightarrow{(v_\rho)_{\rho\in H} \mapsto (v_{\rho\sigma})_{\rho\in H}} \bigoplus_{\rho\in H} (\rho\sigma)^*V \cong \sigma^* \bigoplus_{\rho\in H} \rho^*V,$$

$$\varphi_\sigma^\otimes : \bigotimes_{\rho\in H} \rho^*V \xrightarrow{\otimes_{\rho\in H} v_\rho \mapsto \otimes_{\rho\in H} v_{\rho\sigma}} \bigotimes_{\rho\in H} (\rho\sigma)^*V \cong \sigma^* \bigotimes_{\rho\in H} \rho^*V.$$

To prove descend along a Galois morphism of normal varieties we restrict our attention to the generic point. There, descend is just descend under an étale Galois cover. This can be extended to descend in codimension 1.

Lemma 2.1.5 ([18], Proposition 1)**.** *Let X be a variety. Let F be a torsion-free sheaf on X. Denote the generic point of X by η. Then a subspace of $W \subseteq F_\eta$ can be uniquely extended to a saturated subsheaf $G \subseteq F$ inducing the inclusion $W \subseteq F_\eta$.*

Proof. The proof can be found in [18] with the superfluous assumption that X is smooth and projective. We briefly give the argument.

We first show the existence of such a G. Define $G(U) := F(U) \cap W$ in F_η on an affine open U. This is compatible with restrictions and extends uniquely to a subsheaf $G \subseteq F$. By construction, we have $G_\eta = W \subseteq F_\eta$. It remains to show that $G \subseteq F$ is saturated. Let $G \subseteq G' \subseteq F$ be the saturation. Then $G_\eta = G'_\eta \subseteq F_\eta$ as torsion vanishes at the generic point. By definition of G, we obtain $G' \subseteq G$ and thus equality.

To show uniqueness, let $G' \subseteq F$ be another saturated subsheaf inducing the inclusion $W = G_\eta \subseteq F_\eta$. As above we obtain $G' \subseteq G$. Note that $T := G/G'$ is torsion. Let $Q' := F/G'$ and $Q = F/G$. Applying the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \longrightarrow & F & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

we obtain that $T \subseteq Q'$. As $G' \subseteq F$ is saturated, the quotient Q' is torsion-free and we find that $T = 0$. This shows the desired equality $G' = G$. □

Definition 2.1.6. Let $\pi : Y \to X$ be a Galois morphism of normal varieites. A coherent sheaf F on Y *descends in codimension* 1 if there exists a big open $U \subseteq X$ and a coherent sheaf G on U such that $G_{|\pi^{-1}(U)} \cong F$.

Lemma 2.1.7. *Let $Y \to X$ be a Galois morphism of normal varieties with Galois group G. Let V be a torsion-free sheaf on X. Then a G-invariant saturated subsheaf W of $V_{|Y}$ in codimension* 1 *descends to a saturated subsheaf of V in codimension* 1.

If $Y \to X$ is in addition to the other assumptions flat, then a G-invariant saturated subsheaf $W \subseteq V_{|Y}$ descends to a saturated subsheaf of V.

Proof. Let η_Y be the generic point of Y and η_X the generic point of X. A finite separable morphism of normal varieties is flat at all codimension 1 points. Thus, we can replace X by the flat locus and Y by the pre-image of the flat locus.

Consider a G-invariant saturated subsheaf $W \subseteq V_{|Y}$. Restricting the inclusion to η_Y we obtain a G-invariant subspace $W_{|\eta_Y} \subseteq (V_{\eta_X})_{|\eta_Y}$.

The field extension $\kappa(Y)/\kappa(X)$ is a G-torsor and we can apply descent theory. We obtain $W'_{\eta_X} \subseteq V_{\eta_X}$ such that $W'_{\eta_X} \otimes_{\kappa(X)} \kappa(Y) = W_{\eta_Y}$ as subspaces of $(V_{\eta_X})_{|\eta_Y}$.

By Lemma 2.1.5, there exists a unique saturated subsheaf $W' \subseteq V$ inducing the inclusion $W'_{\eta_X} \subseteq V_{\eta_X}$. Pulling back along the flat morphism $Y \to X$ we obtain a saturated subsheaf $W'_{|Y} \subseteq V_{|Y}$, see Lemma 1.1.3, which agrees with the inclusion $W_{\eta_Y} \subseteq (V_{\eta_X})_{|\eta_Y}$ on the generic point. By another application of Lemma 2.1.5, we conclude $W'_{|Y} = W$. □

We also observe that descending in codimension 1 is the same as descending for Galois covers:

Lemma 2.1.8. *Let $\pi : Y \to X$ be a Galois cover of a normal variety X with Galois group G. Let W be a vector bundle on Y. If W descends to X in codimension* 1, *then W descends to X.*

Proof. Let $U \subseteq X$ be a big open such that $W_{|\pi^{-1}(U)}$ descends to U. Note that $U' := \pi^{-1}(U)$ is a big open subset of Y. As Y is normal and U' is big we can apply Hartog's lemma to obtain

$$\mathrm{Hom}_{\mathcal{O}_Y}(W, \sigma^* W) \cong \mathrm{Hom}_{\mathcal{O}_{U'}}(W_{|U'}, \sigma^*(W_{|U'}))$$

for all $\sigma \in G$. In particular, any G-linearization of $W_{|U'}$ extends to a G-linearization of W. For Galois covers a coherent sheaf admits a G-linearization if and only if it descends. The lemma follows. □

2.1.2 Proof of the key lemma

Stability is in general not preserved under a Galois pullback. To study stability from a functorial perspective, the key observation is that stable becomes polystable after pullback. Furthermore, there is a transitive action of the Galois group on the isomorphism classes of the stable direct summands.

We first recall the behaviour of the degree under pushforward and pullback as well as the basic properties of semi- and polystability under pullback:

Lemma 2.1.9 (Lemma 3.2.1-3.2.3, [12])**.** *Let $\pi : Y \to X$ be a finite separable morphism of normal projective varieties of degree d. Let $F \neq 0$ be coherent torsion free in codimension 1 on X and G be coherent torsion free in codimension 1 on Y. Then we have the following:*

(i) $\mu(G) = d(\mu(\pi_* G) - \mu(\pi_* \mathcal{O}_Y))$.

(ii) $F_{|Y}$ *is torsion-free in codimension* 1.

(iii) $\mu(F_{|Y}) = d\mu(F)$.

(iv) If F *is semistable iff* $F_{|Y}$ *is semistable.*

Furthermore, if F is reflexive and $F' := ((F_{|Y})^\vee)^\vee$, then we have:

(v) If F is polystable and $Y \to X$ is Galois, then F' is polystable.

(vi) If π is prime to p, then F' is polystable iff F is polystable.

(vii) If F' is stable, then so is F.

Proof. (i) - (vi) are proven in [12], Lemma 3.2.1, Lemma 3.2.2, and Lemma 3.2.3 under the assumption that the characteristic is 0. The proofs don't change except for (vi), where the trace still is a section of $\mathcal{O}_X \subseteq \pi_* \mathcal{O}_Y$ if π is prime to p.

We spell out the proofs. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let $\mathcal{O}_Y(1) := \pi^* \mathcal{O}_X(1)$.

(i): By the projection formula we have

$$\pi_*(G \otimes \mathcal{O}_Y(n)) \cong \pi_*(G) \otimes \mathcal{O}_X(n)$$

for $n \geq 1$. As π is finite, $G \otimes \mathcal{O}_Y(n)$ and $\pi_*(G \otimes \mathcal{O}_Y(n))$ have the same Euler characteristic. Thus, G and $\pi_* G$ have the same Hilbert polynomial. Furthermore, we have $d\,\mathrm{rk}(G) = \mathrm{rk}(\pi_* G)$ and (i) follows.

(ii): Note that F is a vector bundle on a big open as at a codimension 1 point of X the notions of torsion-free and locally free agree. Thus, $F_{|Y}$ is a vector bundle on a big open which implies the claim.

(iii): Let $n = \dim(X)$. We have $\alpha_{n-1}(G) = \alpha_{n-1}(\pi_* G)$ by the argument given in (i). Applying the affine projection formula, we obtain

$$\begin{aligned}\alpha_{n-1}(\pi^* F) &= \alpha_{n-1}(\pi_* \pi^* F) \\ &= \alpha_{n-1}(F \otimes \pi_* \mathcal{O}_Y) \\ &= \mu(F \otimes \pi_* \mathcal{O}_Y) \operatorname{rk}(F) \deg(\pi) + \alpha_{n-1}(\mathcal{O}_X) \operatorname{rk}(F) \deg(\pi).\end{aligned}$$

By the behaviour of the slope under tensoring, see Remark 1.2.9, we find

$$\begin{aligned}\mu(\pi^* F) &= (\mu(F) + \mu(\pi_* \mathcal{O}_Y)) \deg(\pi) + \alpha_{n-1}(\mathcal{O}_X) \deg(\pi) - \alpha_{n-1}(\mathcal{O}_Y) \\ &= \deg(\pi)\mu(F) + (\mu(\pi_* \mathcal{O}_Y) + \alpha_{n-1}(\mathcal{O}_X)) \deg(\pi) - \alpha_{n-1}(\mathcal{O}_Y) \\ &= \deg(\pi)\mu(F) + \alpha_{n-1}(\pi_*(\mathcal{O}_Y)) - \alpha_{n-1}(\mathcal{O}_Y) \\ &= \deg(\pi)\mu(F).\end{aligned}$$

(iv): First assume that $Y \to X$ is Galois. If F is not semistable, consider the maximal destabilizing subsheaf $D \subsetneq F$, see Lemma 1.3.10. As π is flat in codimension 1, D pulls back to a subsheaf $D_{|Y}$ of $F_{|Y}$ in codimension 1. Furthermore, $D_{|Y}$ extends to a coherent subsheaf D' of $F' := (F_{|Y}^{\vee})^{\vee}$. The slope can be computed on a large open and we obtain $\mu(D') > \mu(F')$. As $F_{|Y}$ is semistable iff its reflexive hull F' is semistable, we conclude that $F_{|Y}$ is not semistable, see Lemma 1.3.14.

If $F_{|Y}$ is not semistable, consider the maximal destabilizing subsheaf $D' \subsetneq F_{|Y}$, see Lemma 1.3.10. As the maximal destabilizing subsheaf D' is unique, it is a G-invariant subsheaf of $F_{|Y}$. Furthermore, $D' \subsetneq F'$ is saturated and we can apply Lemma 2.1.7. Thus, $D' \subsetneq F_{|Y}$ descends to a subsheaf of F in codimension 1. This can be extended to a subsheaf $D \subsetneq (F^{\vee})^{\vee}$. The slope can be computed on a big open and we conclude that the reflexive hull of F is not semistable as $\mu(D) > \mu(F)$ by (ii). By Lemma 1.3.14, we obtain that F is not semistable.

If $Y \to X$ is not Galois, consider the normalization Z of Y in the Galois closure of $\kappa(Y)/\kappa(X)$. Then $Z \to Y$ and $Z \to X$ are Galois. Then F is semistable iff $F_{|Z}$ is semistable and similarly $F_{|Y}$ is semistable iff $(F_{|Y})_{|Z}$ is semistable and we conclude.

(v): Assume that $Y \to X$ is Galois. As a polystable reflexive sheaf is a direct sum of stable reflexive sheaves by definition, it suffices to show (v) if F is stable. By (iv) we already know that $\pi^* F$ is semistable. Then the same holds for its reflexive hull F', see Lemma 1.3.14. Note that the G-linearization of $F_{|Y}$ induced by F induces a G-linearization of F'.

Let S' be the socle of F'. By Lemma 1.3.15, the socle is the maximal polystable saturated subsheaf of F'. Thus, it is a G-invariant subsheaf of F' and descends to a subsheaf S of F in codimension 1, see Lemma 2.1.7. As F is reflexive, S extends uniquely to a saturated subsheaf W of F of the same slope as S.

By (ii), we have $\mu(W) = \mu(F)$ and the stability of F implies $W = F$. Thus, $\mathrm{rk}(W) = \mathrm{rk}(S') = \mathrm{rk}(F')$ and we conclude $S' = F'$.

(vi): Assume that π is prime to p. Denote by $\pi' : Y' \to X$ the normalization of X in the Galois closure of $\kappa(Y)/\kappa(X)$. Then π' is prime to p by definition. Furthermore, $Y' \to Y$ and $Y' \to X$ are prime to p and Galois. Thus, if suffices to show (vi) in the Galois case.

We already know by (v) that if F is polystable, then F' is polystable. For the converse, assume that F' is polystable. Consider the trace map $\mathrm{tr} : \pi_*\mathcal{O}_Y \to \mathcal{O}_X$. The composition $\mathcal{O}_X \to \pi_*\mathcal{O}_Y \to \mathcal{O}_X$ is multiplication by $\deg(\pi)$ which is prime to p. We find that $\mathrm{tr} / \deg(\pi)$ is a section of the inclusion $\mathcal{O}_X \subseteq \pi_*\mathcal{O}_Y$.

As F' is polystable, we find that F is semistable by (iv) and Lemma 1.3.14. Let $G \subseteq F$ be a saturated stable subsheaf of the same slope as F. Note that by induction on the rank of F is suffices to find a section of $G \subseteq F$.

Note that $Y \to X$ is flat in codimension 1. Then the reflexive hull $G' := (G^\vee_{|Y})^\vee$ is polystable by (v) and a subsheaf of the polystable sheaf F'. Furthermore, they have the same slope by (iii).

Let G_i be a stable summand of G' and let $F' \cong \bigoplus_{j=1}^n F_j$ be the decomposition of F' into stable sheaves. Then there is a non-zero morphism $G_i \to F_j$ for some j. As G_i (resp. F_j) is saturated and G' (resp. F') is reflexive, G_i (resp. F_j) is reflexive, see Lemma 1.1.1. Furthermore, $\mu(G_i) = \mu(F_j)$ and we conclude $G_i \cong F_j$ by Lemma 1.3.7. Thus, $G' \subseteq F'$ admits a section.

Let U be the big open where F and G are vector bundles and $Y \to X$ is flat. Then on $U' := \pi^{-1}(U)$ we have that $ev : G_{|U'} \xrightarrow{\sim} G'_{|U'}$ and $ev : F_{|U'} \xrightarrow{\sim} F'_{|U'}$ are isomorphisms. Furthermore, $G_{|U'} \subseteq F_{|U'}$ and the trace induces a section of the unit $F \to \pi_*F_{|Y}$ on U. Similarly, we have a section of the unit $G \to \pi_*G_{|Y}$ on U.

We claim that the section of $G' \subseteq F'$ induces a section of $G \subseteq F$. First observe that the section of $G' \subseteq F'$ induces a section of $G_{|U'} \subseteq F_{|U'}$. This in turn induces a section of $G_{|U} \subseteq F_{|U}$ via the commutative diagram

$$\begin{array}{ccc}
G_{|U} & \longrightarrow & F_{|U} \\
\downarrow & & \downarrow \\
(\pi_*G_{|Y})_{|U} & \longrightarrow & (\pi_*F_{|Y})_{|U} \\
{\scriptstyle \frac{\mathrm{tr}}{\deg(\pi)}}\downarrow & & \downarrow{\scriptstyle \frac{\mathrm{tr}}{\deg(\pi)}} \\
G_{|U} & \longrightarrow & F_{|U}.
\end{array}$$

As G and F are reflexive, the section on the big open U induces a section on X and we conclude.

(vii): A proper subsheaf in codimension 1 of F of slope $\geq \mu(F)$ pulls back to a proper subsheaf in codimension 1 of π^*F of slope $\geq \mu(\pi^*F)$ by (ii). The claim follows. □

Remark 2.1.10. The statement of Lemma 2.1.9 simplifies for vector bundles:

- A vector bundle is semistable iff it is semistable after a separable pullback.
- A vector bundle is polystable iff it is polystable after a prime to p separable pullback.
- Polystablility of vector bundles is preserved under pullback by a Galois morphism.

Remark 2.1.11. Let $\pi : D \to C$ be a finite separable morphism of smooth projective curves of degree n. We claim that pullback by π induces a morphism

$$\pi^* : M_C^{ss,r,d} \to M_D^{ss,r,dn}, V \mapsto V_{|D}.$$

Indeed, pullback to D preserves semistability by Lemma 2.1.9 and is also well-behaved in families. We show that this morphism is finite, see Theorem 2.3.2.

We provide examples which show that neither "saturated" nor "subsheaf of a sheaf which descends" can be removed in Lemma 2.1.7.

Example 2.1.12. Let E be an elliptic curve and $\pi : E \to \mathbb{P}^1$ be a $2:1$ Galois morphism ramified at 4 points. Denote the non-trivial element of the Galois group $G = \mathbb{Z}/2$ by σ.

Consider a line bundle L of degree 1 on E. Then $L \oplus \sigma^* L$ admits a G-linearization, but does not descend to $\mathbb{P}^1$. Indeed, if there was a vector bundle V on $\mathbb{P}^1$ such that $V_{|E} \cong L \oplus \sigma^* L$, then V is semistable of slope $\frac{1}{2}$ by Lemma 2.1.9. Grothendieck's classification of vector bundles on $\mathbb{P}^1$ does not allow for such a bundle, see e.g. [10].

Consider a point $e \in E$ at which π is ramified. Let $\mathcal{O}_E(-e)$ be the effective Cartier divisor cutting out $e \in E$. Then $\mathcal{O}_E(-e)$ is a G-invariant subsheaf of $\mathcal{O}_E$ but does not descend to a subsheaf I' of $\mathcal{O}_C$. Indeed, by Lemma 2.1.9 such a subsheaf I' would be a line bundle of slope $\frac{1}{2}$ which is impossible.

The key observation for the (non-)functoriality of stability is the following: The Galois group acts transitively on the isomorphism classes of the stable direct summands of the pullback. In particular, a stable vector bundle can only decompose in a very special way after a Galois pullback. We note that the proof given in [12, Lemma 3.2.3] uses the same construction to obtain a G-invariant subsheaf:

Lemma 2.1.13. *Let $\pi : Y \to X$ be a finite Galois morphism of normal projective varieties with Galois group G. Let V be a stable vector bundle on X of rank r. Then $V_{|Y} \cong (\bigoplus_{i=1}^n W_i)^{\oplus e}$ for some pairwise non-isomorphic stable vector bundles W_i on Y and G acts transitively on the set of isomorphism classes $\{W_i \mid i = 1, \dots, n\}$.*

In particular, the W_i have the same rank m and $mne = r$.

Proof. By Lemma 2.1.9, the vector bundle $V_{|Y}$ is polystable and we find that there is an isomorphism $\psi : V_{|Y} \xrightarrow{\sim} \bigoplus_{i=1}^{n} W_i^{\oplus e_i}$ for pairwise non-isomorphic stable vector bundles W_i on Y.

Let $\iota : W \to V_{|Y}$ be the inclusion of one of the W_i. Denote the G-linearization of $V_{|Y}$ associated to V by $\varphi_\sigma : V_{|Y} \xrightarrow{\sim} \sigma^* V_{|Y}, \sigma \in G$. We claim that the image E' of

$$\bigoplus_{\sigma \in G} \sigma^* W \xrightarrow{\sum (\varphi_\sigma)^{-1} \sigma^* \iota} V_{|Y}$$

is a G-invariant subbundle. Indeed, using that φ_σ is a G-linearization we find that

$$\tau^* (\sum_{\sigma \in G} (\varphi_\sigma)^{-1} \sigma^* \iota) = \varphi_\tau \sum_{\sigma \in G} \varphi_{\sigma\tau}^{-1} (\sigma\tau)^* \iota.$$

Thus, $\tau^* E' = \varphi_\tau(E') \subseteq \tau^* V_{|Y}$ and $E' \subseteq V_{|Y}$ is G-invariant.

To see that $E' \subseteq V_{|Y}$ is subbundle note that $\sigma^* W$ is a stable vector bundle of the same slope as W. Projecting to a direct summand W_i we find that

$$\sigma^* W \to E' \subseteq V_{|Y} \xrightarrow{\psi} \bigoplus_{i=1}^{n} W_i^{\oplus e_i} \to W_i$$

is either an isomorphism or zero by Lemma 1.3.7. Thus, E' maps to to a direct sum of some of the W_i under ψ.

By Lemma 2.1.7, the subbundle $E' \subseteq V_{|Y}$ descends to a saturated subsheaf E of V in codimension 1. As V is reflexive, E extends to a saturated subsheaf of V. As E' has the same slope as $V_{|Y}$, the same is true for $E \subseteq V$, see Lemma 2.1.9. Then the stability of V implies $E = V$. We obtain that $\bigoplus_{\sigma \in G} \sigma^* W \to V_{|Y}$ is surjective.

As W_i and $\sigma^* W$ are stable of the same slope the only morphisms between $\sigma^* W$ and W_i are 0 or isomorphisms, see Lemma 1.3.7. We find that the group G acts transitively on the isomorphism classes of the W_i. Clearly, $\mathrm{rk}(\sigma^* W) = \mathrm{rk}(W)$ for all $\sigma \in G$.

Let $e = e_{i_0}$ be the smallest index among the e_i and $W = W_{i_0}$. For each W_i there is a $\sigma_i \in G$ such that $\sigma_i^* W \cong W_i$. The inclusion $W_i^{\oplus e_i} \to V_{|Y}$ induces an inclusion $W^{\oplus e_i} \to V_{|Y}$ after pullback by σ_i^{-1}. We obtain $e_i \leq e$. By definition of e we have equality. □

2.2 Genuinely Ramified Morphisms

Recently the finite separable morphisms of curves preserving stability have been identified by Biswas and Parameswaran, [3, Theorem 5.3]. These morphisms are the genuinely ramified ones and are - next to étale morphisms - a basic building block of finite separable morphisms. In higher dimensions similar results hold as shown by Biswas, Das, and Parameswaran in [2, Theorem 2.5].

In this subsection we give a short self-contained proof of this theorem. As in [2], we first give an alternative description of genuinely ramified morphisms. Our proof only works in the Galois case but has the advantage of being more concise than the original. It also uses different methods. Then we deduce the main theorem regarding genuinely ramified morphism from this description applying the strategy of [3], as already stated in the proof of [2, Theorem 2.5], with some minor adjustments.

We recall the definition of a genuinely ramified morphism:

Definition 2.2.1. Let $f : Y \to X$ be a morphism of varieties. We say that f is *genuinely ramified* if it is finite separable and every factorization $Y \to Y' \to X$ of f such that $Y' \to X$ is an étale morphism of varieties satisfies that $Y' \to X$ is an isomorphism.

For us the main result regarding genuinely ramified morphisms is the following:

Theorem 2.2.2 ([2], Theorem 2.5)**.** *Let $Y \to X$ be a finite genuinely ramified morphism of normal projective varieties. Then the pullback of a stable vector bundle on X to Y is stable.*

We recall the proof following the strategy of [3]. To do so we need the following alternative description of genuinely ramified morphisms:

Theorem 2.2.3 ([2], Theorem 2.4. (1) - (3) in the Galois case)**.** *Let $\pi : Y \to X$ be a Galois morphism of normal varieties. Then the following are equivalent:*

(i) *π is genuinely ramified.*

(ii) *for every connected cover $X' \to X$ the fibre product $Y \times_X X' \to Y$ is connected.*

(iii) *$Y \times_X Y$ is connected.*

Proof. The implications $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (i)$ are as in [2, Theorem 2.4] and are included for the convenience of the reader. The implication $(i) \Rightarrow (iii)$ uses different techniques than the original.

$(i) \Rightarrow (ii)$: Note that (ii) is equivalent to $\pi_1^{\text{ét}}(Y) \to \pi_1^{\text{ét}}(X)$ being surjective, by general principles, see [31, Tag 0BN6].

The image of $\pi_1^{\text{ét}}(Y) \to \pi_1^{\text{ét}}(X)$ is a closed subgroup and is the intersection of all open normal subgroups $N \subseteq \pi_1^{\text{ét}}(X)$ containing it.

Let N be such an open normal subgroup and consider $G := \pi_1^{\text{ét}}(X)/N$. Then G corresponds to a connected Galois cover $X' \to X$ with Galois group G. As the composition $\pi_1^{\text{ét}}(Y) \to G$ is trivial, we find that $X' \times_X Y \cong G \times_k Y$. We obtain a factorization $Y \to X' \to X$ of $Y \to X$. If $Y \to X$ is genuinely ramified, then $X' = X$, i.e., $N = \pi_1^{\text{ét}}(X)$ for all normal open subgroups containing the image of $\pi_1^{\text{ét}}(Y) \to \pi_1^{\text{ét}}(X)$. Thus, $\pi^*(Y) \to \pi^*(X)$ is surjective.

(ii) $\Rightarrow$ *(i)*: If $Y \to X$ is not genuinely ramified, then it factors over a non-trivial étale cover. Then there is also a non-trivial étale Galois cover $X' \to X$ over which $Y \to X$ factors as $Y \to X$ is Galois. As $X' \times_X X'$ is not connected, the same is true for $Y \times_X X'$.

(iii) $\Rightarrow$ *(i)* : If $Y \to X$ is not genuinely ramified, then as in *(ii)* $\Rightarrow$ *(i)* we find a non-trivial étale Galois cover $X' \to X$ over which $Y \to X$ factors. Clearly, $X' \times_X X'$ is not connected and then neither is $Y \times_X Y$.

(i) $\Rightarrow$ *(iii)* : Assume that $Y \times_X Y$ is not connected. We show that $Y \to X$ has a non-trivial étale part. Denote the Galois group of Y/X by G. Consider the action of G on $Y \times_X Y$ given by $\sigma(y, y') = (y, \sigma y')$ for $\sigma \in G$.

Let Σ be the set of connected components of $Y \times_X Y$. Observe that there is an induced action of G on Σ as well as on the set of irreducible components. As G acts transitive on the set of irreducible components, see Lemma 1.5.1, G also acts transitive on Σ. Thus, there are $\#(G/H)$ many connected components, where H denotes the stabilizer of the connected component containing the diagonal - call this components $(Y \times_X Y)^0$.

We claim that $H \subset G$ is normal and that $Y/H \to X$ is étale.

To see that $H \subset G$ is normal, let $\tau \in H$ and $\sigma \in G$. Observe that for $(y, y) \in (Y \times_X Y)^0$ we have that $(\sigma y, \tau\sigma y) \in (Y \times_X Y)^0$ as τ induces by definition an automorphism of $(Y \times_X Y)^0$ and $(\sigma y, \sigma y)$ still lies in $(Y \times_X Y)^0$. Applying the automorphism

$$(Y \times_X Y)^0 \to (Y \times_X Y)^0, (y, y') \mapsto (\sigma^{-1} y, \sigma^{-1} y')$$

yields that $(y, \sigma^{-1}\tau\sigma y) \in (Y \times_X Y)^0$. Thus, $\sigma^{-1}\tau\sigma$ is restricts to an automorphism of $(Y \times_X Y)^0$, i.e., $\sigma^{-1}\tau\sigma \in H$.

To show that $Y/H \to X$ is étale, first observe that it is Galois with Galois group G/H as H is normal in G. By Lemma 1.5.1,

$$G/H \times_k Y/H \to Y/H \times_X Y/H, (\overline{\sigma}, \overline{y}) \mapsto (\overline{y}, \overline{\sigma y})$$

is surjective and identifies irreducible components of $Y/H \times_X Y/H$ with G/H. We obtain a surjection

$$\varphi : G/H \times_k Y \to Y \times_X (Y/H) = Y \times_{Y/H} (Y/H) \times_X (Y/H)$$

via base change. Note that φ maps $\overline{\sigma} \times_k Y$ onto an irreducible component. Thus, one the one hand, $Y \times_k (Y/H)$ has at most $\#(G/H)$ many irreducible components.

On the other hand, $Y \times_X (Y/H)$ surjects onto $Y/H \times_X Y/H$ and we obtain that $Y \times_X (Y/H)$ has $\#(G/H)$ many irreducible components. Furthermore, φ identifies irreducible components of $Y \times_X Y/H$ with elements of G/H.

There is a natural morphism

$$(Y \times_X Y)/H \to Y \times_X (Y/H)$$

which topologically is a homeomorphism, see [23, §12, p. 111, Theorem 1], where the action of H on $Y \times_X Y$ is given by $\sigma(y, y') = (y, \sigma y')$. By construction of H, the quotient $(Y \times_X Y)/H$ has the same number of connected components as $Y \times_X Y$ - namely $\#(G/H)$ many. As $(Y \times_X Y)/H \to Y \times_X (Y/H)$ is a homeomorphism we find that the irreducible components of $Y \times_X Y/H$ are disjoint.

Thus, $\varphi : G/H \times_k Y \to Y \times_X Y/H$ is topologically a homeomorphism. We find that the fibre of $Y/H \to X$ at a point $x \in X$ has cardinality $\#(G/H)$ and G/H acts transitively on each fibre.

The orbit of a point $\overline{y} \in Y/H$ under G/H is contained in an affine open as $Y/H \to X$ is finite and $X = (Y/H)/(G/H)$, see Lemma 1.5.1. We conclude by [23, §7, p.66, Theorem] that $Y/H \to X$ is étale. □

Using the description of a genuinely ramified Galois morphism $\pi : Y \to X$ obtained in Theorem 2.2.3 we implement to strategy of [3]. We can show that the morphisms between stable vector bundles of the same slope remain the same under such a pullback. To do so we first study the unit $\mathcal{O}_Y \to \pi^*\pi_*\mathcal{O}_Y$.

Lemma 2.2.4 (Analogue of [3], Proposition 3.5)**.** *Let $\pi : Y \to X$ be a Galois morphism of normal varieties. Consider the Cartesian diagram*

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{\mathrm{pr}_2} & Y \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\pi} & X. \end{array}$$

Then we have:

(i) $\pi^*\pi_*\mathcal{O}_Y = \mathrm{pr}_{1,*}\, \mathcal{O}_{Y \times_X Y}$.

(ii) $\pi^*\pi_*\mathcal{O}_Y/T = \mathrm{pr}_{1,*}\, \mathcal{O}_{(Y \times_X Y)_{red}}$*, where T denotes the torsion submodule of* $\pi^*\pi_*\mathcal{O}_Y$.

If in addition $Y \to X$ is genuinely ramified of degree d, then

(iii) $(\pi^*\pi_*\mathcal{O}_Y)/(T + \mathcal{O}_Y) \subseteq \bigoplus_{i=1}^{d-1} \mathcal{L}_i$*, where for $i = 1, \ldots, d-1$ the $\mathcal{L}_i \subsetneq \mathcal{O}_Y$ are proper subsheaves.*

Proof. (i): This is a direct consequence of affine base change.

(ii): Let G be the Galois group of $Y \to X$. The irreducible components of $Y \times_X Y$ are in one-to-one correspondence to elements of G and equipped with the reduced subscheme structure they are all isomorphic to Y via pr_1, see Lemma 1.5.1. As the intersection of the minimal primes of a ring is the nilpotent radical, we have $\mathcal{O}_{(Y\times_X Y)_{red}} \subseteq \bigoplus i_* \mathcal{O}_Z$, where the direct sum is taken over all irreducible components $i : Z \to Y \times_X Y$ equipped with the reduced subscheme structure. We find that $\mathrm{pr}_{1,*}\,\mathcal{O}_{(Y\times_X Y)_{red}}$ is contained in $\bigoplus_{\sigma\in G} \mathcal{O}_Y$. Thus, $\mathrm{pr}_{1,*}\,\mathcal{O}_{(Y\times_X Y)_{red}}$ is torsion free over $\mathcal{O}_Y$.

We obtain a natural surjective morphism

$$\pi^*\pi_*\mathcal{O}_Y/T \twoheadrightarrow \mathrm{pr}_{1,*}\,\mathcal{O}_{(Y\times_X Y)_{red}}.$$

Being an isomorphism is a Zariski-local property and we can assume that X (resp. Y) is the spectrum of a normal domain A (resp. B).

Then all that remains to show is that the nilradical η of $B \otimes_A B$ is torsion considered as a module over $B \to B \otimes_A B, b \mapsto b \otimes 1$. We claim that η is contained in the kernel K of

$$B \otimes_A B \to Q(B) \otimes_{Q(A)} (Q(A) \otimes_A B) \subseteq Q(B) \otimes_{Q(A)} Q(B).$$

Indeed, $Q(B) \otimes_{Q(A)} Q(B)$ is reduced as $Q(B)/Q(A)$ is Galois. Thus, η is torsion as K is torsion.

(iii): Assume that $\pi : Y \to X$ is a genuinely ramified morphism of normal varieties and denote the degree of π by d.

In (ii) we are labeling the irreducible components of $Y \times_X Y$ by

$$Y^\sigma := \{\sigma\} \times_k Y \to Y \times_X Y, (\sigma, y) \mapsto (y, \sigma y)$$

for $\sigma \in G$, see Lemma 1.5.1. We explain a different way to label them by $\{1, \dots, n\}$, where $n = \#G$, together with an order preserving map $\eta : \{1, \dots n\} \to \{0, \dots, n-1\}$ and such that $Y^i \cap Y^{\eta(i)}$ is non-empty for $1 < i \leq n$, see also [3, Lemma 3.4]. Every irreducible component has a shortest path to the diagonal Y^{e_G} setting the distance of irreducible components to 1 if they are distinct and intersect. Then the labeling and η are defined inductively as

- Y^1 is defined to be the diagonal, $\eta(1) := 0$,
- for $1 < i \leq n$ we define Y^i to be an unlabeled component of the shortest distance among unlabeled components to Y^1 and choose $\eta(i)$ such that $Y^{\eta(i)}$ has distance 1 to Y^i and shorter distance to Y^1.

Using this labeling we obtain

$$(\pi^*\pi_*\mathcal{O}_Y)/T \subseteq \bigoplus_{i=1}^{n} \mathcal{O}_{Y^i} \cong \bigoplus_{i=1}^{n} \mathcal{O}_Y.$$

Consider the morphism

$$\varphi : \bigoplus_{i=1}^{n} \mathcal{O}_Y \to \bigoplus_{i=2}^{n} \mathcal{O}_Y, (s_i) \mapsto (s_i - s_{\eta(i)}).$$

Clearly,

$$\mathcal{O}_Y \subseteq (\pi^*\pi_*\mathcal{O}_Y)/T \cong \mathrm{pr}_{1,*}\,\mathcal{O}_{(Y\times_X Y)_{red}} \subseteq \bigoplus_{i=1}^{n} \mathcal{O}_Y$$

is the kernel of φ and thus

$$(\pi^*\pi_*\mathcal{O}_Y)/(T + \mathcal{O}_Y) \subseteq \bigoplus_{i=2}^{n} \mathcal{O}_Y.$$

By construction there exists a closed point $y_i \in Y^i \cap Y^{\eta(i)}$ for $1 < i \leq n$. Consider y_i as a point of Y via $\mathrm{pr}_1 : Y \times_X Y \to Y$. Projecting to i-th component we claim that

$$(\pi^*\pi_*\mathcal{O}_Y)/(T + \mathcal{O}_Y) \to \bigoplus_{i=2}^{n} \mathcal{O}_Y \to \mathcal{O}_Y \to \kappa(y_i)$$

is trivial, i.e.,

$$(\pi^*\pi_*\mathcal{O}_Y)/(T + \mathcal{O}_Y) \subseteq \bigoplus_{i=2}^{n} \mathcal{O}_Y(-y_i).$$

Indeed, a local section s of $\pi^*\pi_*\mathcal{O}_Y \cong \mathrm{pr}_{1,*}\,\mathcal{O}_{Y\times_X Y}$ satisfies $s_i = s_{\eta(i)}$ at y_i as $y_i \in Y^i \cap Y^{\eta(i)}$. □

Remark 2.2.5. Note that in Lemma 2.2.4, in contrast to [3, Proposition 3.5], we do not obtain the information that the slope of the $\mathcal{L}_i$ is negative as they are the ideal sheaves of a point.

Our next step is to show that the morphisms between stable vector bundles of the same slope remain unchanged after a genuinely ramified pullback $\pi : Y \to X$. In the proof the quotient $\pi_*\mathcal{O}_Y/\mathcal{O}_X$ appears and we briefly recall that it is torsion-free:

Lemma 2.2.6. *Let $\pi : Y \to X$ be a finite morphism of varieties. If X is normal, then $\pi_*\mathcal{O}_Y/\mathcal{O}_X$ is torsion-free.*

Proof. It suffices to show the lemma in case that $X = \mathrm{Spec}(A)$ is the spectrum of an affine normal domain A. Then $Y = \mathrm{Spec}(B)$ for some domain finite over A. We want to show that B/A is a torsion-free A-module. Let $0 \neq a \in A$ and $b \in B$ such that $a\bar{b} = 0 \in B/A$. Then $ab \in A$ by definition. As $a \neq 0$, we find $b \in Q(A)$. As B is finite over A and A is normal, we conclude $b \in A$. Thus, $\bar{b} = 0 \in B/A$. □

Lemma 2.2.7 (Analogue of [3], Lemma 4.3)**.** *Let $\pi : Y \to X$ be a genuinely ramified Galois morphism of normal projective varieties. Let V and W be stable vector bundles on X of the same slope. Then pullback defines an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\pi^* V, \pi^* W), \varphi \mapsto \pi^* \varphi.$$

Proof. By adjunction we have

$$\mathrm{Hom}_{\mathcal{O}_Y}(\pi^* V, \pi^* W) \cong \mathrm{Hom}_{\mathcal{O}_X}(V, \pi_* \pi^* W).$$

Applying $\mathrm{Hom}_{\mathcal{O}_X}(V, -)$ to the short exact sequence

$$0 \to W \to \pi_* \pi^* W \to (\pi_* \pi^* W)/W \to 0,$$

it suffices to show the vanishing of

$$\mathrm{Hom}_{\mathcal{O}_X}(V, (\pi_* \pi^* W)/W) \cong \mathrm{Hom}_{\mathcal{O}_X}(V, W \otimes (\pi_* \mathcal{O}_Y / \mathcal{O}_X)).$$

Let d be the degree of π and $T \subseteq \pi^* \pi_* \mathcal{O}_Y$ the torsion submodule. Applying Lemma 2.2.4, we find subsheaves $\mathcal{L}_i \subsetneq \mathcal{O}_Y, i = 1, \dots, d-1$, such that

$$(\pi^* \pi_* \mathcal{O}_Y)/(T + \mathcal{O}_Y) \subseteq \bigoplus_{i=1}^{d-1} \mathcal{L}_i.$$

By the affine projection formula, we have

$$\pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} (\pi_* \mathcal{O}_Y / \mathcal{O}_X) \cong \pi_* (\pi^* \pi_* \mathcal{O}_Y / \mathcal{O}_Y).$$

Tensoring the short exact sequence

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_Y \to \pi_* \mathcal{O}_Y / \mathcal{O}_X \to 0$$

with the torsion-free module $\pi_* \mathcal{O}_Y / \mathcal{O}_X$, see Lemma 2.2.6, we obtain

$$\pi_* \mathcal{O}_Y / \mathcal{O}_X \subseteq \pi_* (\pi^* \pi_* \mathcal{O}_Y / \mathcal{O}_Y),$$

since $\mathrm{Tor}^1(\pi_* \mathcal{O}_Y / \mathcal{O}_X, \pi_* \mathcal{O}_Y / \mathcal{O}_X)$ is torsion. As π is a finite morphism of varieties, $\pi_* T \subseteq \pi_* \pi^* \pi_* \mathcal{O}_Y$ is the torsion submodule. By the torsion-freeness of $\pi_* \mathcal{O}_Y / \mathcal{O}_X$, we obtain

$$\pi_* \mathcal{O}_Y / \mathcal{O}_X \subseteq \pi_* (\pi^* \pi_* \mathcal{O}_Y / (T + \mathcal{O}_Y)) \subseteq \bigoplus_{i=1}^{d-1} \pi_* \mathcal{L}_i.$$

For V', W' stable vector bundles on Y of same slope we have

$$\mathrm{Hom}_{\mathcal{O}_Y}(V', W' \otimes \mathcal{L}_i) = 0$$

for $i = 1, \dots, d-1$, as a direct consequence of Lemma 1.3.7. Then the same vanishing holds for polystable vector bundles V' and W' of the same slope. The pullback of V and W to Y is polystable by Lemma 2.1.13. Thus, we conclude

$$\mathrm{Hom}_{\mathcal{O}_X}(V, W \otimes \pi_*\mathcal{L}_i) = \mathrm{Hom}_{\mathcal{O}_Y}(\pi^*V, \pi^*W \otimes \mathcal{L}_i) = 0$$

for $i = 1, \dots, d-1$. This implies the desired vanishing

$$\mathrm{Hom}_{\mathcal{O}_X}(V, W \otimes (\pi_*\mathcal{O}_Y/\mathcal{O}_X)) = 0$$

as $\pi_*\mathcal{O}_Y/\mathcal{O}_X \subseteq \bigoplus_{i=1}^{d-1} \pi_*\mathcal{L}_i$.

□

We are now in a position to complete the proof of Theorem 2.2.2.

Proof of Theorem 2.2.2. Let $\pi : Y \to X$ be a genuinely ramified morphism of normal projective varieties. Consider a stable vector bundle V on X. We wish to show that π^*V is stable.

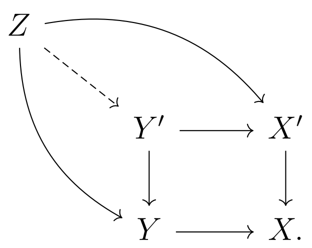
If π is Galois, then we have $k = \mathrm{Hom}_{\mathcal{O}_X}(V, V) = \mathrm{Hom}_{\mathcal{O}_Y}(\pi^*V, \pi^*V)$ by Lemma 2.2.7 and Lemma 1.3.7. As we already know that π^*V is polystable, see Lemma 2.1.9, we conclude that π^*V is stable in the Galois case.

Let Z be the normal closure of X in the Galois closure $\kappa(Z) := Gal(\kappa(Y)/\kappa(X))$ and $Z \to Y \to X$ be the associated morphism.

Consider the factorization of $Z \to X$ into a genuinely ramified morphism $Z \to X'$ and an étale morphism $X' \to X$. Note that $Z \to X'$ is Galois and so is $X' \to X$ as the maximal étale part of a Galois morphism. Denote the Galois group of X'/X by G.

Note that as $Y \to X$ is genuinely ramified, the fibre product $Y' := Y \times_X X'$ is a Galois cover with Galois group G, see Theorem 2.2.3.

There is a morphism $Z \to Y'$ induced by $Z \to Y$ and $Z \to X'$ - in a picture



Let V be a stable vector bundle on X. Then $V_{|X'}$ is polystable by Lemma 2.1.9. Consider the decomposition $V_{|X'} \cong \bigoplus_{i \in I} V_i'$ into stable vector bundles on X'. By the Galois case $V'_{i|Z}$ is stable and then so is $W_i' := V'_{i|Y'}$.

Consider a saturated stable subsheaf $W \subset V_{|Y}$ of slope $\mu(W) = \mu(V_{|Y})$ on Y. Then $W_{|Y'} \subset V_{|Y'}$ is saturated and polystable of slope $\mu(V_{|Y'})$. Thus, there is some subset $J \subset I$ such that $W_{|Y'} \cong \bigoplus_{j \in J} W'_j$. Identifying V'_i with its image in $V_{|X'}$ we find some $W'_i \subset V_{|Y'}$ contained in $W_{|Y'}$.

Then on the one hand, the pullback of the image E of

$$\bigoplus_{\sigma \in G} \sigma^* V'_i \xrightarrow{\oplus \varphi_\sigma^{-1} \sigma^* \iota} V_{|X'}$$

to Y' is contained in $W_{|Y'}$, where $\iota : V'_i \to V_{|X'}$ denotes the inclusion and φ_σ the G-linearization associated to V. On the other hand, E is a G-invariant saturated subsheaf of $V_{|X'}$ and descends to a saturated stable subsheaf of V of the same slope, see also the proof of Lemma 2.1.13. Thus, $E = V_{|X'}$, as V is stable and therefore $W_{|Y'} = V_{|Y'}$. We conclude $W = V_{|Y}$, i.e., $V_{|Y}$ is stable. □

2.3 Push and pull are finite

As semistable vector bundles remain semistable after a finite separable pullback $\pi : D \to C$, we obtain a morphism $\pi^* : M_C^{ss,r,d} \to M_D^{ss,r,\deg(\pi)d}$. The finiteness of π^* can be proven using the degree of the theta divisor. This can be found in [11, Theorem 4.2] and goes back to [7, Theorem I.4].

Here we give a shorter proof only using [3, Lemma 4.3] and basic properties of finite étale morphisms. We first observe that semistability is preserved under an étale pushforward:

Lemma 2.3.1. *Let $\pi : Y \to X$ be a cover of a normal projective variety X of dimension ≥ 1. Then we have the following:*

(i) $\pi_* \mathcal{O}_Y$ has slope 0.

(ii) The pushforward $\pi_* W$ of a semistable vector bundle W on Y is semistable of degree $\deg(W)$.

(iii) Let V be a semistable vector bundle on X. Then $\pi_* \mathcal{O}_Y \otimes V$ is semistable of slope $\mu(V)$.

Proof. (i): If X is a curve, then $\pi_* \mathcal{O}_Y$ has degree 0 by Riemann-Hurzwitz.

We claim that the general case reduces to the curve case. By Bertini's theorem the general complete intersection curve C on X is irreducible and the same holds for $D := Y \times_X C$, see [14, Corollaire 6.11 (3)]. Furthermore, the general such C is normal, see [27, Theorem 7]. As the projection of $\pi' : D = Y \times_X C \to C$ is étale, D is normal.

We can compute the degree of $\pi_* \mathcal{O}_Y$ on C, see Remark 1.2.9. As π is finite, we have $(\pi_* \mathcal{O}_Y)_{|C} \cong \pi'_* \mathcal{O}_D$ by affine base change and the result follows.

(ii): This short argument can already be found in the proof of [3, Proposition 5.1] for line bundles of degree 1 on a smooth projective curve.

Let W be a semistable vector bundle of slope μ and rank r on Y. By (i) we have $\deg \pi_* \mathcal{O}_Y = 0$. By Lemma 2.1.9, the pushforward $\pi_* W$ has slope $\mu / \deg(\pi)$. If $\pi_* W$ was not semistable, consider the maximal destabilizing subsheaf V of $\pi_* W$. By adjunction $\pi^* V \to W$ is a non-zero morphism of semistable torsion-free sheaves. This is a contradiction as

$$\mu(\pi^* V) = \deg(\pi)\mu(V) > \deg(\pi)\mu(\pi_* W) = \mu(W).$$

(iii): Let V be a semistable vector bundle on X. Using (i) we obtain

$$\mu(V) = \mu(\pi_*(\mathcal{O}_D)) + \mu(V) = \mu(\pi_*(\mathcal{O}_D) \otimes V).$$

By the projection formula we have $\pi_* \mathcal{O}_Y \otimes V \cong \pi_* \pi^* V$ which is semistable by (ii) and Lemma 2.1.9 (iii). □

Theorem 2.3.2. *Let $\pi : D \to C$ be a finite separable morphism of smooth projective curves. Let $r \geq 1$ and $d \in \mathbf{Z}$. Then the induced morphism*

$$\pi^* : M_C^{ss,r,d} \to M_D^{ss,r,\deg(\pi)d}$$

is finite. If e denotes the degree of the étale part of π, then the fibre of π^* at a stable vector bundle W on D has cardinality at most e.

Proof. We first show the finiteness. As π^* is a morphism of projective varieties it suffices to show that it is quasi-finite. A finite separable morphism factors as a finite étale and a genuinely ramified morphism. Thus, it suffices to show the quasi-finiteness for these two types of morphisms separately.

The genuinely ramified case immediately follows from [3, Lemma 4.3] which asserts that two semistable vector bundles of the same slope on C are isomorphic iff they are isomorphic after pullback to D. In fact, this tells us that π^* is injective on points.

It remains to consider the case where π is a cover. As cover is dominated by a Galois cover it suffices to prove the quasi-finiteness under the additional assumption that π is Galois. Let V be a polystable vector bundle on C. Consider the polystable vector bundle $\pi^* V \cong \bigoplus W_i$, where the W_i are stable on D, see Lemma 2.1.9. By Lemma 2.3.1, the bundles $\pi_* W_i$ are semistable of slope $\mu(V)$. The projection formula implies that $\pi_*(\mathcal{O}_D) \otimes V \cong \pi_* \pi^* V$. Thus, $V \subseteq \pi_* \pi^* V$ appears in the JH-filtration of $\bigoplus \pi_* W_i$. As the graded object associated to the JH-filtration is unique, there are only finitely many choices for V if we fix $\bigoplus W_i$.

We now estimate the cardinality of the fibre at a stable vector bundle. Let $D \xrightarrow{\pi'} C' \to C$ a factorization of $D \to C$ in a genuinely ramified and étale morphism. As π'^* is injective on points, it remains to estimate the cardinality of the fibre if $D \to C$ is étale. Let W be a stable vector bundle on D. Let V be a semistable vector bundle on C such that $V_{|D} \cong W$ is stable on D. Then V is stable by Lemma 2.1.9. Comparing the ranks of $V \subseteq \pi_* W$, we find that there can be at most $\deg(\pi)$ many different such V. □

Lemma 2.3.3. *Let $\pi : D \to C$ be a cover of degree n of a smooth projective curve C of genus $g_C \geq 2$. Let $r \geq 1$ and $d \in \mathbb{Z}$. Then pushforward induces a finite morphism*

$$\pi_* : M_D^{ss,r,d} \to M_C^{ss,rn,d}, W \mapsto \pi_* W.$$

Proof. By Lemma 2.3.1, the pushforward of a semistable vector bundle on D is semistable on C and of the same degree. As pushforward along a finite flat morphism is well-behaved in families, we obtain the morphism on the level of moduli spaces.

To show finiteness it suffices to show quasi-finiteness as π_* is a morphism of projective varieties. Consider $V \in M_C^{ss,rn,d}$. For W semistable on D such that $\pi_* W \cong_S V$ we find that $\pi^*\pi_* W \cong_S \pi^* V$. As π is affine, the counit $\pi^*\pi_* W \to W$ is surjective. Furthermore, $\pi^*\pi_* W$ is semistable of the same slope as W. We conclude that W is S-equivalent to a direct sum of stable vector bundles appearing in the JH-filtration of $\pi^* V$. As the associated graded of the JH-filtration is unique, there are only finitely many possibilities for the S-equivalence class of W. We conclude that π_* is quasi-finite and thus finite. □

2.4 Cyclic covers

In this section, we collect some basic properties of cyclic covers. They correspond to line bundles and the pushforward of the structure sheaf is of a particularly nice form:

Lemma 2.4.1. *Let X be a variety. Let $r \geq 2$ be prime to p. Then a line bundle of order r on X is trivialized by a cyclic prime to p cover.*

If X is in addition normal and projective, then cyclic covers $\pi : Y \to X$ of order r of X correspond to line bundles of order r on X.

Let L be the corresponding line bundle to a cyclic cover $\pi : Y \to X$ of order r. Then L is trivialized by $Y \to X$ and $\pi_* \mathcal{O}_Y \cong \bigoplus_{i=1}^r L^{\otimes i}$.

Proof. As $p \nmid r$, there is an isomorphism $\mu_r \cong \mathbb{Z}/r\mathbb{Z}$ of abelian sheaves on $X_{\text{ét}}$. In particular, cyclic covers of order r correspond to connected μ_r-torsors.

Let L be a line bundle on X of order r. The short exact sequence

$$0 \to \mu_r \to \mathbf{G}_m \xrightarrow{\lambda \mapsto \lambda^r} \mathbf{G}_m \to 0$$

of abelian sheaves on $X_{\text{ét}}$ induces a long exact cohomology sequence

$$\check{H}^0_{\text{ét}}(X, \mathbf{G}_m) \to \check{H}^0_{\text{ét}}(X, \mathbf{G}_m) \to \check{H}^1_{\text{ét}}(X, \mu_r) \to \check{H}^1_{\text{ét}}(X, \mathbf{G}_m) \to \check{H}^1_{\text{ét}}(X, \mathbf{G_m}).$$

After identifying $\check{H}^1_{\text{ét}}(X, \mathbf{G}_m) \cong \operatorname{Pic}(X)$, we find a μ_r-torsor $Y \to X$ mapping to L. As a torsor trivializes itself, L is trivialized by $Y \to X$.

Let Y_0 be a connected component of Y. Then L is also trivialized by $Y_0 \to X$. Consider the stabilizer $H \subseteq \mu_r$ of Y_0. Then $Y_0 \cong Y/H$ and $Y_0 \to X$ is a connected $\mu_{r'}$-torsor for some $r' \mid r$.

Assume that X is a normal projective variety. Then $\check{H}^0_{\text{ét}}(X, \mathbf{G}_m) = k^*$. Under this identification the morphism $\check{H}^0_{\text{ét}}(X, \mathbf{G}_m) \to \check{H}^0_{\text{ét}}(X, \mathbf{G}_m)$ is given by the r-th power map on k^*, which is surjective as k is algebraically closed. We obtain a left-exact sequence

$$0 \to \check{H}^1_{\text{ét}}(X, \mu_r) \to \check{H}^1_{\text{ét}}(X, \mathbf{G}_m) \to \check{H}^1_{\text{ét}}(X, \mathbf{G_m}).$$

After identifying $\check{H}^1_{\text{ét}}(X, \mathbf{G}_m) \cong \operatorname{Pic}(X)$ we obtain a left-exact sequence

$$0 \to \operatorname{Pic}(X)[r] \to \operatorname{Pic}(X) \xrightarrow{L \mapsto L^{\otimes r}} \operatorname{Pic}(X),$$

where $\operatorname{Pic}(X)[r]$ denotes the line bundles which have trivial r-th tensor-power. This gives us an identification of μ_r-torsors with line bundles of order dividing r. Furthermore, the μ_r-torsor associated to the line bundle trivializes the line bundle as a torsor always trivializes itself.

It remains to show this correspondence sends a connected μ_r-torsor to a line bundle of order r and vice versa as well as the description of the direct image of the structure sheaf.

Let $Y \to X$ be the μ_r-torsor associated to a line bundle L of order r. Let Y_0 be a connected component of Y. As above, we find that $\pi : Y_0 \to X$ is a cyclic cover of order $r' \mid r$ trivializing L. Then $L^{\otimes i}$ is also trivialized by $Y_0 \to X$ for $i \in \mathbb{Z}$. By adjunction we find $L^{\otimes i} \subseteq \pi_* \mathcal{O}_{Y_0}$. As $L^{\otimes i}, i = 0, \ldots, r-1$ are pairwise non-isomorphic and stable of degree 0 and $\pi_* \mathcal{O}_{Y_0}$ is a semistable vector bundle of degree 0, see Lemma 2.3.1, we find that $\pi_* \mathcal{O}_{Y_0} \cong \bigoplus_{i=0}^{r-1} L^{\otimes i}$. As the rank of $\pi_* \mathcal{O}_{Y_0}$ is r', we conclude $r' = r$, i.e., $Y = Y_0$.

Let $Y \to X$ be a connected μ_r-torsor. Let L be the line bundle of order $r' \mid r$ associated to $Y \to X$. Then there is also a $\mu_{r'}$-torsor $Y' \to X$ associated to L. Via $\check{H}^1_{\text{ét}}(X, \mu_{r'}) \to \check{H}^1_{\text{ét}}(X, \mu_r)$ the torsor Y' is mapped to Y. In particular, $Y' \to X$ trivializes $Y \to X$, i.e., $Y \times_X Y' \cong \mu_r \times_k Y'$. We obtain a factorization of $Y' \to X$ via $Y \to X$. Thus, $r \leq r'$ and we conclude $r = r'$.

This also shows the description of $\pi_* \mathcal{O}_Y$ for a cyclic cover $\pi : Y \to X$. $\square$

2.5 Constructing stable vector bundles

There are two ways to construct vector bundles given a Galois cover. One is via representations of the Galois group and the other is via orbits under the Galois action. We specialize these constructions to construct stable vector bundles as well.

2.5.1 Representations

Recall that étale trivializable bundles correspond to representations of the étale fundamental group, see [15, 1.2 Proposition]. Our approach to descend is via linearizations. This point of view allows for a more general correspondence:

Lemma 2.5.1. *Let $Y \to X$ be a Galois cover of a normal variety X with Galois group G. Let W be a vector bundle on X such that $W_{|Y}$ is simple. Then there is a one-to-one correspondence of isomorphism classes of vector bundles on X that become isomorphic to $W_{|Y}^{\oplus r}$ after pullback to Y and representations $G \to \mathrm{GL}_r$ up to conjugation.*

Furthermore, if X and Y are projective and $W_{|Y}$ is stable, then the above correspondence sends irreducible representations to stable vector bundles and vice versa.

Proof. Consider a vector bundle V on X such that $V_{|Y} \cong W_{|Y}^{\oplus r}$. Denote the G-linearization associated to W by $\varphi_\sigma^W : W_{|Y} \xrightarrow{\sim} \sigma^* W_{|Y}, \sigma \in G$; similarly for V. Consider for an isomorphism $\psi : W_{|Y}^{\oplus r} \xrightarrow{\sim} V_{|Y}$ and $\sigma \in G$ the (possibly non-commutative) diagram

$$\begin{array}{ccc} W_{|Y}^{\oplus r} & \xrightarrow{\psi} & V_{|Y} \\ {\scriptstyle (\varphi_\sigma^W)^{\oplus r}} \downarrow & & \downarrow {\scriptstyle \varphi_\sigma^V} \\ \sigma^* W_{|Y}^{\oplus r} & \xrightarrow{\sigma^*\psi} & \sigma^* V_{|Y}. \end{array}$$

As $W_{|Y}$ is simple, the composition

$$\rho^V(\sigma) := ((\varphi_\sigma^W)^{-1})^{\oplus r} \circ \sigma^*\psi^{-1} \circ \varphi_\sigma^V \circ \psi$$

corresponds to a matrix $\rho(\sigma) \in \mathrm{GL}_r$. We claim that this gives rise to a representation

$\rho^V : G \to \mathrm{GL}_r, \sigma \mapsto \rho(\sigma)$. Indeed, for $\sigma, \tau \in G$ we have

$$\begin{aligned}
&\rho^V(\tau)\rho^V(\sigma) = \\
&\Big((\varphi_\tau^W)^{-1}\Big)^{\oplus r}\tau^*(\psi^{-1})\varphi_\tau^V\psi \circ \Big((\varphi_\sigma^W)^{-1}\Big)^{\oplus r}\sigma^*(\psi^{-1})\varphi_\sigma^V\psi = \\
&\Big((\varphi_\sigma^W)^{-1}\Big)^{\oplus r}\sigma^*\Bigg(\Big((\varphi_\tau^W)^{-1}\Big)^{\oplus r}\tau^*(\psi^{-1})\varphi_\tau^V\psi\Bigg)\sigma^*(\psi^{-1})\varphi_\sigma^V\psi = \\
&\Big((\varphi_\sigma^W)^{-1}\Big)^{\oplus r}\sigma^*\Big((\varphi_\tau^W)^{-1}\Big)^{\oplus r}\sigma^*\tau^*(\psi^{-1})\sigma^*(\varphi_\tau^V)\varphi_\sigma^V\psi = \\
&\Big((\varphi_{\tau\sigma}^W)^{-1}\Big)^{\oplus r}(\tau\sigma)^*(\psi^{-1})\varphi_{\tau\sigma}^V\psi = \\
&\rho^V(\tau\sigma),
\end{aligned}$$

where only the second and fourth equality require an explanation. For the second equality we use that $(\varphi_\sigma^W)^{\oplus r}$ commutes with matrices and for the fourth that φ_σ^W and φ_σ^V are G-linearizations. Note that the conjugacy class of ρ^V does not depend on ψ.

A representation $\rho : G \to \mathrm{GL}_r$ gives rise to a G-linearization via

$$W_{|Y}^{\oplus r} \xrightarrow{\rho(\sigma)} W_{|Y}^{\oplus r} \xrightarrow{(\varphi_\sigma^W)^{\oplus r}} \sigma^* W_{|Y}^{\oplus r}.$$

This G-linearization corresponds to a vector bundle $V(\rho)$ on X such that there exists an isomorphism $V(\rho)_{|Y} \cong W_{|Y}^{\oplus r}$ compatible with the natural G-linearization of $V(\rho)_{|Y}$ and the one associated to ρ. Observe that the isomorphism class of $V(\rho)$ only depends on the conjugacy class of ρ.

We claim that these constructions are inverse to each other. Keep the notation of the above constructions and choose an isomorphism $\psi' : W_{|Y}^{\oplus r} \xrightarrow{\sim} V(\rho)_{|Y}$ compatible with the natural G-linearization of $V(\rho)_{|Y}$ and the one associated to ρ. Using this compatibility of ψ', we obtain

$$\begin{aligned}
\rho^{V(\rho)}(\sigma) &= ((\varphi_\sigma^W)^{-1})^{\oplus r} \circ \sigma^*(\psi')^{-1} \circ \varphi_\sigma^{V(\rho)} \circ \psi' \\
&= (\varphi_\sigma^W)^{-1})^{\oplus r} \circ (\varphi_\sigma^W)^{\oplus r} \circ \rho(\sigma) \\
&= \rho(\sigma).
\end{aligned}$$

Similarly, one can verify that $V(\rho^V) \cong V$.

Let $Y \to X$ be a Galois cover of a normal projective variety X. Let W be a stable vector bundle on X such that $W_{|Y}$ is stable. We continue to show that irreducible representations correspond to stable vector bundles.

First observe, that a vector bundle V on Y such that $V_{|X} \cong W_{|Y}^{\oplus r}$ is semistable by Lemma 2.1.9.

Let $\rho : G \to \mathrm{GL}_r$ be a representation. Denote the associated vector bundle via the above construction for $\mathcal{O}_X$ by $V(\rho)$. Then $V(\rho)$ is semistable. If $V(\rho)$ is not stable, then there exists a stable saturated subsheaf $V' \subseteq V(\rho)$ of the same slope. The pullback $V'_{|Y}$ is polystable by Lemma 2.1.9. Furthermore, $V'_{|Y} \subseteq V_{|Y}$ is saturated and in particular reflexive as a saturated subsheaf of a reflexive sheaf.

Let W' be a stable direct summand of $V'_{|Y}$. As $V_{|Y} \cong W_{|Y}^{\oplus r}$ there is a non-zero morphism $W' \to W$. Both are reflexive, stable, and have the same slope. We find that $W' \cong W$, see Lemma 1.3.7. Thus, we have $V'_{|Y} \cong W_{|Y}^{\oplus r'}$ for some $r' < r$. Denote the representation associated to V' by $\rho' : G \to \mathrm{GL}_r$. As $V' \subseteq V$ the G-linearizations induced by V' and V are compatible. Thus, ρ' is a non-trivial subrepresentation of ρ and ρ is reducible.

Conversely, if ρ is reducible, then any non-trivial subreprensentation ρ' of ρ defines a semistable vector bundle $V(\rho')$ of the same slope and smaller rank of V. As ρ' is a subrepresentation, the G-linearizations associated to $V(\rho')$ and $V(\rho)$ are compatible, i.e., $V(\rho') \subsetneq V(\rho)$. Thus, V is not stable. □

Remark 2.5.2. In the setting of Lemma 2.5.1 consider a representation $\rho : G \to \mathrm{GL}_r$ and assume that X is proper. Then $\mathcal{O}_Y$ is simple and the lemma applies for $W = \mathcal{O}_X$.

Denote by $V(\rho)$ the associated bundle to ρ using $W = \mathcal{O}_X$ and by $V(\rho, W)$ the associated bundle for some W on X such that $W_{|Y}$ is simple. Then both vector bundles recover ρ and thus $V(\rho) \otimes W \cong V(\rho, W)$. We think of this as twisting the representation ρ by W.

If X is projective and ρ is irreducible, then tensoring with $V(\rho)$ preserves stability of bundles that remain stable on Y.

2.5.2 Orbits

We construct (stable) vector bundles via orbits as follows:

Lemma 2.5.3. *Let $\pi : Y \to X$ be a Galois cover of a normal projective variety X with Galois group G. Let $P \in \mathbb{Q}[x]$. Then for $\sigma \in G \setminus \{e_G\}$*

$$U_\sigma^P := \{W \in M_Y^{G-ss,P} \mid \sigma^* W \not\cong_S W\} \subseteq M_Y^{G-ss,P}$$

and $U^P := \bigcap_{\sigma \in G \setminus \{e_G\}} U_\sigma \cap M_X^{s,P}$ are open. We have the following:

(i) For $W \in U^P$ the direct image $\pi_* W$ is a stable vector bundle on X.

(ii) $W \in U^P$ iff $W \in M_Y^{s,P}$ and W does not descend to a stable vector bundle W' on an intermediate cover $Y \to Y' \to X$ such that $Y' \not\cong Y$.

Furthermore, if $\pi : D \to C$ is a Galois cover of a smooth projective curve C of genus ≥ 2, then the Hilbert polynomial P is determined by the rank r and degree d and we use the notation $U^{r,d}$ instead of U^P. We have the following:

(iii) $U^{r,d} \subseteq M_D^{s,r,d}$ *is big.*

(iv) If d *and* $\deg(\pi)$ *are coprime, then* $U^{r,d} = M_D^{s,r,d}$.

Note that Lemma 2.5.3 (i) is a generalization of [3, Proposition 5.1]. To obtain the alternative description of U^P given in (ii) we need that for cyclic covers the notions of G-invariance and G-linearization coincide. We postpone the proof of Lemma 2.5.3 and first describe descend for cyclic covers:

Lemma 2.5.4. *Let* $Y \to X$ *be a cyclic cover of a variety* X *with Galois group* G*. Let* V *be a simple sheaf on* Y*. Then* V *descends to* X *iff* V *is* G*-invariant.*

Proof. The "only if" implication is trivial. For the "if" implication let σ be a generator of G of order n. Fix an isomorphism $\varphi_\sigma : V \xrightarrow{\sim} \sigma^* V$. For $2 \leq l < n$ define $\varphi_{\sigma^l} : V \xrightarrow{\sim} (\sigma^l)^* V$ inductively as the composition $\sigma^* \varphi_{\sigma^{l-1}} \circ \varphi_\sigma$. Further define $\varphi_{e_G} = \mathrm{id}_V$, where e_G denotes the neutral element of G.

Consider $\sigma^* \varphi_{\sigma^{n-1}} \circ \varphi_\sigma$. This is an automorphism of V. As V is simple it corresponds to a scalar $\lambda \in k^*$. Since k is algebraically closed we can find an n-th root $\lambda^{1/n}$ of λ. We claim that the automorphisms $\psi_{\sigma^l} := \lambda^{-l/n} \varphi_{\sigma^l}$ define a G-linearization of V. Indeed, for $1 \leq l, l'$ such that $l' + l < n$ we have

$$(\sigma^l)^* \psi_{\sigma^{l'}} \circ \psi_{\sigma^l} = \lambda^{(-l-l')/n} \cdot (\sigma^{l+l'-1})^* \varphi_\sigma \circ \cdots \circ \sigma^* \varphi_\sigma \circ \varphi_\sigma = \psi_{\sigma^{l+l'}}$$

by definition. It remains to check this property for $l + l' = n$. We have

$$(\sigma^l)^* \psi_{\sigma^{l'}} \circ \psi_{\sigma^l} = \lambda^{-1} \cdot (\sigma^{n-1})^* \varphi_\sigma \circ \cdots \circ \sigma^* \varphi_\sigma \circ \varphi_\sigma = \lambda^{-1} \lambda = 1$$

by definition of λ. □

We can now prove the orbit-construction for stable vector bundles:

Proof of Lemma 2.5.3. Let $\sigma \in G$. To see that U_σ is open, consider the pullback square

$$\begin{array}{ccc} Z_\sigma & \longrightarrow & M_Y^{G-ss,P} \\ \downarrow & & \downarrow{\scriptstyle (\sigma^*, \mathrm{id})} \\ M_Y^{G-ss,P} & \xrightarrow{\Delta} & M_Y^{G-ss,P} \times_k M^{G-ss,P}. \end{array}$$

As $M_Y^{G-ss,P}$ is projective, Δ is a closed immersion and Z_σ is closed. Clearly, Z_σ is the complement of U_σ and we find the openness.

Note that $W \in U^P$ iff $W \in M_Y^{s,P}$ and $\sigma^* W \not\cong W$ for $\sigma \in G \setminus \{e_G\}$ as for stable vector bundles S-equivalence is the same as being isomorphic.

(i): Let $W \in U^P$. Then $V := \pi_* W$ is semistable of slope $\mu(W)/\deg(\pi)$ by Lemma 2.3.1. Consider a stable saturated subsheaf $V' \subseteq V$. Then $V'_{|Y}$ is a saturated subsheaf of $\pi^*\pi_*W \cong \bigoplus_{\sigma\in G}\sigma^*W$ and thus reflexive. Furthermore, $V'_{|Y}$ is polystable of the same slope as W. By Lemma 1.3.7, we find that $V'_{|Y} \cong \bigoplus_{\sigma\in\Sigma}\sigma^*W$ for some $\Sigma \subseteq G$. In particular, V' is a stable vector bundle. By Lemma 2.1.13, G acts transitively on the isomorphism classes of the $\sigma^*W, \sigma \in \Sigma$. As $\sigma^*W \ncong W$ for $\sigma \neq e_G$, we conclude $V'_{|Y} = V_{|Y}$ and thus $V' = V$.

(ii): Let $W \in M_Y^{s,P}$ such that $W \notin U^P$. Then there exists $\sigma \in G \setminus \{e_G\}$ such that $\sigma^*W \cong W$. Let G' be the subgroup of G generated by σ. Then G' is cyclic, non-trivial, and Lemma 2.5.4 applies to $Y \to Y/G'$. We conclude that W descends to Y/G'.

Conversely, if W descends to some intermediate cover $Y \to Y'$ such that $Y \ncong Y'$ with Galois group G', then there exists W' on Y' such that $W'_{|Y} \cong W$. Thus, $\sigma'^*W \cong W$ for $\sigma' \in G'$. This concludes (ii).

(iii)-(iv): Let $r \geq 1, d \in \mathbb{Z}$ and $D \to C$ be a Galois cover of smooth projective curves of genus ≥ 2.

(iii): Then by the alternative description given in (ii), the open $U^{r,d}$ is obtained by removing vector bundles that pullback from an intermediate cover, i.e.,

$$U^{r,d} = \bigcap_{D \xrightarrow{\pi'} D' \to C} \left(M_D^{ss,r,d} \setminus \pi'^*(M_{D'}^{ss,r,\frac{d}{\deg(\pi')}})\right) \cap M_D^{s,r,d},$$

where the intersection is taken over intermediate covers $D' \to C$ such that $D \neq D'$. If $M_{D'}^{ss,r,\frac{d}{\deg(\pi')}}$ is non-empty, then it has dimension

$$r^2(g_{D'} - 1) + 1 = \frac{r^2}{\deg(\pi')}(g_D - 1) + 1$$

by Riemann-Hurwitz. Thus, we find that $U^{r,d}$ is big.

(iv): If d and $\deg(\pi)$ are coprime, then the moduli spaces $M_{D'}^{ss,r,\frac{d}{\deg(\pi')}}$ considered in (iii) are empty and we obtain (iv). □

Lemma 2.5.5. *Let $\pi : D \to C$ be a Galois cover of a smooth projective curve C with Galois group G. Let $r \geq 1$ and d be integers. For the open*

$$U^{r,d} = \{V \in M_D^{s,r,d} \mid \sigma^*V \ncong V, \text{ for all } \sigma \in G \setminus \{e_G\}\} \subseteq M_D^{s,r,d}$$

defined in Lemma 2.5.3 pushforward along π induces a Cartesian diagram

$$\begin{array}{ccc} M_D^{ss,r,d} & \xrightarrow{\pi_*} & M_C^{ss,\#(G)r,d} \\ \uparrow & & \uparrow \\ U^{r,d} & \xrightarrow{\pi_*} & M_C^{s,\#(G)r,d}. \end{array}$$

In particular, the morphism $\pi_* : U^{r,d} \to M_C^{s,\#(G)r,d}$ is finite.

Proof. By Lemma 2.5.3, we have $U^{r,d} \subseteq \pi_*^{-1}(M_C^{s,\#(G)r,d})$. For the other inclusion, let $V \in M_D^{ss,r,d}$ such that $\pi_* V$ is stable. Then V is stable, as a subsheaf $W \subseteq V$ of the same slope yields a subsheaf $\pi_*(W) \subseteq \pi_* V$ of the same slope. By the description of $U^{r,d}$ given in Lemma 2.5.3 (ii), we find that it suffices to show that V does not descend to an intermediate cover $D \to D' \to C$ such that $D' \ncong D$.

Assume by way of contradiction that $V \cong V'_{|D}$ for some vector bundle V' on D', where $D \xrightarrow{\pi'} D' \to C$ is an intermediate cover such that $D \ncong D'$. Then $V' \subsetneq \pi'_* V$ is a proper subsheaf of the same slope. Thus, $\pi'_* V$ is not stable and neither is $\pi_* V$.

Finiteness is preserved under base change and we conclude by Lemma 2.3.3. □

2.6 Functorial notions of stability

In this section we define several functorial notions of stability and study them for curves of small genus. The most prominent notion is *prime to p* stability. We define the notion prime to p as follows:

Definition 2.6.1. A finite group G is called *prime to p* if $p \nmid \#(G)$. A finite separable morphism (resp. cover) $\pi : Y \to X$ of varieties is *prime to p* if the Galois hull of $\kappa(Y)/\kappa(X)$ (resp. of Y/X) has Galois group prime to p.

Observe that prime to p morphisms are well-behaved under composition, i.e., the composition of two such morphisms is again prime to p. This is a direct consequence of the following lemma:

Lemma 2.6.2. *Fix a prime q. Let $M/L/K$ be a tower of field extensions. Assume that M/L and L/K are Galois and $q \nmid [M : K]$. Then the Galois closure F of M/K satisfies $q \nmid [F : K]$.*

Proof. As M/K is separable there exists an $\alpha \in M$ such that $M = K(\alpha)$, see [31, Tag 030N]. Clearly, $M = L(\alpha)$. Let f be the minimal polynomial of α over K and g the minimal polynomial of α over L. We have $g \mid f$ as polynomials over L. For $\sigma \in \mathrm{Gal}(L/K)$ we obtain $\sigma^* g \mid \sigma^* f = f$.

We claim that $\prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma^* g = f$ is the prime factorization of f in $L[x]$. Indeed, if $g = \sigma^* g$ for some $\sigma \in G$, then the coefficients of g lie in L^H, where H is the subgroup of G generated by σ. In particular, $[M : L] = [M : L^H]$ and we obtain $L = L^H$, i.e., σ must be trivial. Thus, the prime factorization of f is of desired form up to a unit $u \in L$. However, all polynomials in question are monic and we obtain $u = 1$.

Define $M_\sigma := L[x]/\sigma^* g$. The isomorphism $\sigma : L \to L$ induces an isomorphism $M = L[x]/g \to M_\sigma$. As M/L is Galois so is M_σ/L. The Galois closure F of M/K

is the composite $\prod_{\sigma\in\mathrm{Gal}(L/K)} M_\sigma$. Inductively, we obtain $q \nmid [M_{\sigma_1}\dots M_{\sigma_k} : K]$ as the Galois group is a subgroup

$$\mathrm{Gal}\Big(M_{\sigma_1}\dots M_{\sigma_k}/K\Big) \subseteq \mathrm{Gal}\Big(M_{\sigma_2}\dots M_{\sigma_k}/K\Big) \times \mathrm{Gal}\Big(M_{\sigma_1}/K\Big).$$

□

We now introduce our functorial notions of stability.

Definition 2.6.3. Let X be a projective variety. A sheaf V on X is called *separable-stable*, (resp. *étale-stable*, resp. *prime to p stable*) if for every finite separable, (resp. finite étale, resp. finite étale prime to p) morphism $\pi : Y \to X$ of varieties the pullback π^*V is stable with respect to $\pi^*\mathcal{O}_X(1)$.

Example 2.6.4. Every line bundle is separable-stable. If $p > 0$, then a semistable vector bundle of rank $r = p^n$, $n \geq 1$, and degree coprime to p is prime to p stable.

As a direct consequence of Theorem 2.2.2 we obtain the following:

Corollary 2.6.5. *On a normal projective variety the notions of étale-stability and separable-stability agree for vector bundles.*

Remark 2.6.6. Being able to go back and forth between covers and separable morphisms yields several advantages. On the one hand, it is easier to construct Galois morphisms than Galois covers. On the other hand, descent theory is simpler for Galois covers and there are - up to isomorphism - only finitely many covers of fixed degree. This is a direct consequence of the étale fundamental group of a normal projective variety being topologically finitely generated, see [26, Satz 13.1]. We spell this out in the next lemma.

Lemma 2.6.7. *Let X be a normal projective variety. Up to isomorphism there are only finitely many covers $Y \to X$ of fixed degree.*

Proof. A cover $Y \to X$ corresponds to a finite continuous $\pi_1^{\text{ét}}(X)$-set. A finite continuous $\pi_1^{\text{ét}}(X)$-set $S \cong \{1,\dots,n\}$ is - up to isomorphism - given by a morphism $\pi_1^{\text{ét}}(X) \to \mathbf{S}_n$ of profinite groups, where $\mathbf{S}_n$ denotes the symmetric group of $\{1,\dots,n\}$ equipped with the discrete topology. As $\pi_1^{\text{ét}}(X)$ is topologically finitely generated, see [26, Satz 13.1], there are only finitely many morphisms $\pi_1^{\text{ét}}(X) \to \mathbf{S}_n$ and we conclude. □

Étale-stability on a smooth projective curve C is only interesting if $g_C \geq 2$.

Lemma 2.6.8. *Let C be a smooth projective curve of genus $g_C \leq 1$. Then we have:*

(i) If $g_C = 0$, then the only stable vector bundles are line bundles.

(ii) *If $g_C = 1$, then a stable vector bundle of rank r and degree d is prime to p stable iff $(r, d) = (1)$ and r is a power of p.*

(iii) *If C is an ordinary elliptic curve, then the only étale stable vector bundles are line bundles.*

(iv) *If C is supersingular, then the notions of prime to p stable and étale stable agree.*

Proof. If $g_C = 0$, then (i) follows from Grothendieck's classification of vector bundles on $\mathbb{P}^1$, see e.g. [10].

In the following we use that semistability is preserved under pullback by a finite separable morphism and the behaviour of the degree under pullback, see Lemma 2.1.9.

If $g_C = 1$, we use [1, Theorem 5 and Theorem 7], which are both valid in arbitrary characteristic. These theorems immediately imply that there are no stable vector bundles of rank $r > 1$ and integral slope over an elliptic curve. In fact more can be said: a semistable vector bundle of rank r and degree d is stable iff $(r, d) = (1)$, a direct consequence of [25, Corollary 2.5].

Consider a stable vector bundle V of rank $r > 1$ and degree d such that $(r, d) = (1)$. On a cover of degree non-coprime to r the pullback of V can not be stable by the previous discussion. This proves the claim (iii) for ordinary elliptic curves as they have covers of any square degree. Indeed, for d not divisible by p multiplication by d is of degree d^2. For $d = p$ the dual of the Frobenius $F^\vee : E \to E^{(p)}$ is étale of degree p, see [29, Theorem 3.1].

If r is a power of p and $(r, d) = (1)$, then on all prime to p covers we still have coprime rank and degree. This proves (ii).

If C is supersingular, then every cover is prime to p and we obtain (iv), see [29, Theorem 3.1]. □

3 Proof of Theorem 1

The idea to prove Theorem 1 is simple: There are two types of failure for a stable bundle to remain stable after pullback. Both of these failures can be detected on a single cover. We make this more precise on a smooth projective curve C.

The key observation is that a stable vector bundle V of rank r on C decomposes on a Galois cover $D \to C$ as $V_{|D} \cong \bigoplus_{i=1}^{n} W_i^{\oplus e}$ for some pairwise non-isomorphic stable vector bundles W_i on D such that the Galois group acts transitively on the isomorphism classes of the W_i, see Lemma 2.1.13. This is somewhat similar to the decomposition of a prime ideal in a Galois extension of number fields.

If $n \geq 2$, this decomposition behaviour can already be detected on a Galois cover $C_{r,large}$, a cover dominating all covers of degree dividing r, see Lemma 3.1.2.

If V remains stable on $C_{r,large}$, then for any Galois cover $D \to C$ the decomposition is $V_{|D} \cong W^{\oplus e}$. Pretending that W descends to a stable vector bundle M on C (this is not clear at all but we provide a technical workaround, see Lemma 3.2.1) we can compare the descent data associated to $M^{\oplus e}$ and V to obtain a Gl_e-representation ρ of the Galois group $\mathrm{Gal}(D/C) = G$. The descent data agree on the kernel of ρ and we are reduced to G being a finite subgroup of Gl_e.

If G is prime to p, then Jordan's theorem - which also has a positive characteristic version due Larsen and Pink- has a particularly nice form:

Theorem 3.0.1 ([13] p.114 for characteristic 0, [19] Theorem 0.4 for positive characteristic)**.** *Let $r \geq 1$. There exists a constant $J(r)$ such that for every finite prime to p subgroup $G \subset \mathrm{Gl}_r$ there exists a normal abelian subgroup $N \subseteq G$ of index $\leq J(r)$.*

Thus, there exists a normal abelian subgroup $N \subseteq G$ of index $\leq J(e)$, where $J(e)$ denotes the constant from Jordan's theorem. As a finite abelian subgroup is simultaneously triagonalizable the decomposition $V_{|D} \cong W^{\oplus e}$ can already be detected on D/N. We obtain a prime to p Galois cover $C_{r,good}$ which detects the stability of $V_{|D}$ as a cover dominating all prime to p covers of degree $\leq rJ(r)$.

We split the construction of $C_{r,good}$ into two parts. First, we construct $C_{r,large}$. This construction can also be carried out over any normal projective variety.

Then we continue with the workaround for descending W and finally construct $C_{r,good}$. The same type of cover works over a normal projective variety X. However, the workaround for descent only works for curves. Thus, one has to complete the descent setup on the level of X and then restrict the setup to a large curve.

3.1 A large cover

There are two fundamentally different ways for a stable V bundle to decompose on a Galois cover $Y \to X$: in the decomposition $V_{|Y} \cong \bigoplus_{i=1}^{n} W_i^{\oplus}$ of Lemma 2.1.13 we distinguish the cases $n = 1$ and $n \geq 2$. We first find a cover that checks for $n \geq 2$ using that this decomposition can already be seen on a cover of degree n.

Lemma 3.1.1. *Let $\pi : Y \to X$ be a Galois cover of a normal projective variety X with Galois group G. Further, let V be a stable vector bundle of rank r on X such that the decomposition $V_{|Y} \cong \bigoplus_{i=1}^{n} W_i^{\oplus e}$ of Lemma 2.1.13 satisfies $n \geq 2$. Then there is a factorization of $Y \to X$ into $Y \to Y' \xrightarrow{\pi'} X$ such that $\deg(\pi') = n$ and $V_{|Y'}$ is not stable.*

Proof. By assumption there are at least two different W_i. Consider the stabilizer H of $W := W_i^{\oplus e}$ for some i and fix an inclusion $\iota : W \to V_{|Y}$. Let $\varphi_\sigma : V_{|Y} \to \sigma^* V_{|Y}$ be the G-linearization associated to V. The image E of $\bigoplus_{\sigma \in H} \sigma^* W \xrightarrow{\oplus \varphi_\sigma^{-1} \sigma^* \iota} V_{|Y}$ is an H-invariant subsheaf, see also the proof of Lemma 2.1.13. Using the stability of the W_j we find that E is isomorphic to W. Therefore, the direct summand W of $V_{|Y}$ descends to a direct summand W' of $V_{|Y'}$, where $Y' = Y/H$ and $Y \to Y' \xrightarrow{\pi'} X$ are the induced morphisms. Note that π' has degree $\#(G/H) = n$ and $W' \subsetneq V_{|Y'}$, i.e., $V_{|Y'}$ is not stable. □

As a direct consequence we obtain the large cover checking for decomposition of a stable vector bundle into at least two non-isomorphic stable vector bundles on some cover:

Lemma 3.1.2. *Let X be a normal projective variety and $r \geq 2$. Then we have:*

(i) There exists a Galois cover $X_{r,large} \to X$ satisfying the following:

If V is a vector bundle of rank r on X such that $V_{|X_{r,large}}$ is stable, then for all Galois covers $Y \to X$ we have $V_{|Y} \cong W^{\oplus e}$ for some stable vector bundle W on Y and $e \geq 1$.

(ii) There is a prime to p Galois cover $X'_{r,large} \to X$ satisfying the following:

If V is a vector bundle of rank r on X such that $V_{|X'_{r,large}}$ is stable, then for all prime to p Galois covers $Y \to X$ we have $V_{|Y} \cong W^{\oplus e}$ for some stable vector bundle W on Y and $e \geq 1$.

Proof. (i): Decomposing into different stable vector bundles descends to some cover of degree n such that $n \mid r$, see Lemma 3.1.1. There are only finitely many such covers up to isomorphism, see Lemma 2.6.7. In particular, there is a Galois cover $X_{r,large}$ dominating all covers of degree dividing r. This is the desired cover.

(ii): Define $X'_{r,large}$ as a prime to p Galois cover dominating all prime to p covers of degree dividing r. This is the desired cover. □

3.2 A good cover

To construct the cover $X_{r,good}$ detecting prime to p stability it remains to deal with decomposition behaviour of the form $V_{|Y} \cong W^{\oplus e}$, where $Y \to X$ is a Galois cover of a normal projective variety X and V a stable vector bundle. We start with the workaround for descent of G-invariant stable vector bundles. This requires working on curves and the mild assumption that $\det(W)$ already descends. The determinant-descent can be set up on arbitrary varieties and we are then able to derive the main theorem by reducing to the case of curves via a restriction theorem for stability.

3.2.1 The workaround for descend

To prove the workaround for descent we use techniques from group cohomology. If one is only interested in the case of curves, then there is an honest descent lemma one could use instead, see Lemma 4.1.4.

Lemma 3.2.1 (Workaround for descend)**.** *Let $D \to C$ be a finite Galois morphism of smooth projective curves with Galois group G. Let V be a simple G-invariant vector bundle of rank r on D. Further, assume that* $\det(V)$ *admits a G-linearization.*

Then there exists a lift of the G-linearization of $\det(V)$ *to a system of isomorphisms $\psi_\sigma : V \xrightarrow{\sim} \sigma^* V$. Furthermore, there exists a finite cyclic Galois morphism $\varphi : D' \to D$ such that*

(i) φ is prime to p of degree $\deg(\varphi)$ *such that* $\deg(\varphi) \mid r$,

(ii) $D' \to D \to C$ is a Galois morphism,

(iii) $\mathrm{Gal}(D'/D) \subseteq \mathrm{Gal}(D'/C)$ *is central, and*

(iv) there exists a 1-cocycle $\alpha : \mathrm{Gal}(D'/C) \to \mu_r$ such that

$$\varphi^*\Big(\psi_\sigma\Big) \cdot \alpha(\sigma')^{-1} : V_{|D'} \xrightarrow{\sim} \sigma'^* V_{|D'}$$

defines a $\mathrm{Gal}(D'/C)$*-linearization of $V_{|D'}$, where σ denotes the image of σ' under the natural morphism* $\mathrm{Gal}(D'/C) \to G$.

Proof. We claim that for two simple isomorphic bundles V and W we have a surjective morphism $\mathrm{Hom}(V, W) \xrightarrow{\det} \mathrm{Hom}(\det(V), \det(W))$. Indeed, after identifying $\mathrm{Hom}(V, V)$ with k the determinant corresponds to the r-th power map. Thus, the G-linearization of $\det(V)$ lifts to isomorphisms $\psi_\sigma : V \xrightarrow{\sim} \sigma^* V$ such that $\psi_{\sigma\tau}^{-1} \circ \tau^*\psi_\sigma \circ \psi_\tau = \lambda_{\sigma,\tau} \in \mu_r$.

A computation shows that the family $\lambda_{\sigma,\tau}$ defines an inhomogeneous 2-cocycle: For $\sigma, \tau, \rho \in G$ we have

$$\begin{aligned} \lambda_{\sigma\tau,\rho}\lambda_{\sigma,\tau} = \\ \psi_{\sigma\tau\rho}^{-1} \circ \rho^*(\psi_{\sigma\tau}) \circ \psi_\rho \circ \psi_{\sigma\tau}^{-1} \circ \tau^*(\psi_\sigma) \circ \psi_\tau = \\ \psi_{\sigma\tau\rho}^{-1} \circ \rho^*(\psi_{\sigma\tau}) \circ \rho^*\Big(\psi_{\sigma\tau}^{-1} \circ \tau^*(\psi_\sigma) \circ \psi_\tau\Big) \circ \psi_\rho = \\ \psi_{\sigma\tau\rho}^{-1} \circ \rho^*\tau^*(\psi_\sigma) \circ \rho^*(\psi_\tau) \circ \psi_\rho = \\ \psi_{\sigma\tau\rho}^{-1} \circ (\tau\rho)^*(\psi_\sigma) \circ \psi_{\tau\rho} \circ \psi_{\tau\rho}^{-1} \circ \rho^*(\psi_\tau) \circ \psi_\rho = \\ \lambda_{\sigma,\tau\rho}\lambda_{\tau,\rho}, \end{aligned}$$

see also [4, Proposition 2.8]. Let $p^n r' = r$ with r' coprime to p and $\lambda'_{\sigma,\tau} = \lambda_{\sigma,\tau}^{p^n}$. The 2-cocycle condition for $\lambda_{\sigma,\tau}$ implies the 2-cocycle condition for $\lambda'_{\sigma,\tau}$. We obtain an element $\lambda' = (\lambda'_{\sigma,\tau}) \in H^2(G, \mu_{r'})$.

Let Gal be the absolute Galois group of $\kappa(C)$. As C is a curve over an algebraically closed field, $\kappa(C)$ is a C_1 field by Tsen's Theorem, see [24, Corollary 6.5.5]. In particular, $H^2(\mathrm{Gal}, (\kappa(C)^{sep})^*)$ vanishes, see [24, Proposition 6.5.8]. By Hilbert 90 we also have vanishing of $H^1(\mathrm{Gal}, (\kappa(C)^{sep})^*)$, see [24, Theorem 6.2.1]. Applying these two vanishing results to the long exact cohomology sequence of the short exact sequence

$$0 \to \mu_{r'} \to (\kappa(C)^{sep})^* \xrightarrow{x \mapsto x^{r'}} (\kappa(C)^{sep})^* \to 0,$$

we obtain $H^2(\mathrm{Gal}, \mu_{r'}) = 0$.

By Schreier's theorem, see [24, Theorem 1.2.4], the element $\lambda' \in H^2(G, \mu_{r'})$ corresponds to an extension

$$0 \to \mu_{r'} \to G' \to G \to 0$$

inducing the action of G on $\mu_{r'}$. As the action of G on $\mu_{r'}$ is trivial, we find that $\mu_{r'}$ is central in G'. Write G as a quotient of Gal. Since $H^2(\mathrm{Gal}, \mu_{r'}) = 0$, we obtain that the central extension

$$0 \to \mu_{r'} \to \mathrm{Gal} \times_G G' \to \mathrm{Gal} \to 0,$$

is trivial, i.e., $\mathrm{Gal} \times_G G' \cong \mathrm{Gal} \times \mu_{r'}$. In particular, there exists a surjection $\mathrm{Gal} \times \mu_{r'} \to G'$. Let H denote the image of $\mathrm{Gal} \times 0$ under this morphism. By construction $H \to G' \to G$ is surjective. As $H \subseteq G'$ we find that

$$0 \to \mu_{r'} \to H \times_G G' \to H \to 0$$

is a central split extension and thus trivial.

The kernel K of $H \twoheadrightarrow G$ is a subgroup of $\mu_{r'}$. In particular, $K \subseteq H$ is central and cyclic. Denote by $\kappa(D')$ the field extension of $\kappa(C)$ corresponding to $\mathrm{Gal} \twoheadrightarrow H$ and by D' the associated curve. We obtain Galois morphisms $D' \xrightarrow{\varphi} D \to C$ such that $\mathrm{Gal}(D'/D) \subseteq \mathrm{Gal}(D'/C)$ is central and cyclic. Furthermore, the obstruction $\lambda' \in H^2(G, \mu_{r'})$ vanishes in $H^2(H, \mu_{r'})$.

The triviality of the 2-cocycle $\varphi^*\lambda' \in H^2(H, \mu_{r'})$ means that there is a 1-cocycle $\alpha' : H \to \mu_{r'}$ such that $\partial(\alpha')(\sigma, \tau) = \lambda'_{f(\sigma),f(\tau)}$, where $f : H \twoheadrightarrow G$ denotes the surjection constructed above.

Recall that in positive characteristic p-th roots are unique. Thus, there is a 1-cocycle $\alpha : H \to \mu_r, \sigma \mapsto \alpha'(\sigma)^{1/p^n}$ such that $\partial(\alpha)(\sigma, \tau) = \lambda_{f(\sigma),f(\tau)}$. By construction the isomorphisms $\varphi^*\psi_{f(\sigma)} \cdot \alpha(\sigma)^{-1}, \sigma \in H$, define a linearization. Indeed, we have

$$\left(\varphi^*\psi_{f(\sigma\tau)} \cdot \alpha(\sigma\tau)^{-1}\right)^{-1} \circ \tau^*\varphi^*\psi_{f(\sigma)} \cdot \alpha(\sigma)^{-1} \circ \varphi^*\psi_{f(\tau)} \cdot \alpha(\tau)^{-1} = \lambda_{f(\sigma),f(\tau)} \cdot (\partial(\alpha)(\sigma, \tau))^{-1} = 1.$$

□

Remark 3.2.2. A shorter (but less precise) argument is the following: Recall that $H^2(\mathrm{Gal}, \mu_{r'}) = \operatorname{colim} H^2(G', \mu_{r'})$, see [24, Proposition 1.2.5], where the colimit is taken over all finite Galois extensions of $\kappa(C)$ and G' denotes the Galois group. We obtain $\mathrm{Gal} \twoheadrightarrow G' \twoheadrightarrow G$ such that the obstruction λ vanishes on the associated curve. However, this does not give us a way to control the kernel which is crucial.

Note that Lemma 3.2.1 only works for curves and requires the mild assumption that the determinant descends. Given that the determinant descends we can detect decomposition on a cover of degree bounded by the constant of Jordan's theorem. We do this in the following lemma. This would already allow us to deduce Theorem 1 for curves but we only give the general proof later.

Lemma 3.2.3. *Let $D \to C$ be a prime to p Galois cover with Galois group G. Let V be a vector bundle on C such that $V_{|D} \cong W^{\oplus e}$ for some simple G-invariant vector bundle W satisfying that* $\det(W)$ *descends to C. Denote the constant from Jordan's theorem, see Theorem 3.0.1, by $J(e)$.*

Then there exists a normal subgroup $N \subseteq G$ of index $\leq J(e)$ and $W' \subseteq V_{|C'}$ such that $W'_{|D} \cong W$, where $D \to C' := D/N \to C$ are the natural morphisms.

Proof. Denote the rank of W by r. Let $\psi^W_\sigma : W \xrightarrow{\sim} \sigma^*W, \sigma \in G$, be a system of isomorphisms lifting the descent datum of $\det(W)$, see Lemma 3.2.1. By the same lemma there is a Galois morphism $D' \xrightarrow{\varphi} D$ with prime to p cyclic Galois group H such that $D' \to D \to C$ is a Galois morphism with Galois group G'. Further, there exists a 1-cocycle $\alpha : G' \to \mu_r$ such that $\varphi^*(\psi_\sigma^W) \cdot \alpha(\sigma')^{-1}$ is a G'-linearization, where σ denotes the image of σ' in G. Furthermore, $H \subseteq G'$ is cyclic.

Our goal is to find a normal subgroup $N' \subseteq G'$ of index $\leq J(e)$ containing H and an N'-invariant subbundle $W_{|D'} \subseteq V_{|D'}$. By Lemma 2.1.7 the inclusion $W_{|D'} \subseteq V_{|D'}$ descends to C', where C' denotes the normal closure of C in the fixed field $\kappa(D')^{N'}$. Then the lemma follows as $C' = C/N$, where N is the image of N' in G.

Let $\psi_\sigma^V : V_{|D} \xrightarrow{\sim} \sigma^* V_{|D}$ be the descent datum associated to V. Choose an isomorphism $\psi : V_{|D} \xrightarrow{\sim} W^{\oplus e}$ which exists by assumption. Define a map

$$\rho : G' \to \mathrm{Gl}_e, \sigma' \mapsto \mathrm{diag}(\alpha(\sigma'))((\psi_\sigma^W)^{-1})^{\oplus e} \circ \sigma^*(\psi) \circ \psi_\sigma^V \circ \psi^{-1},$$

where σ denotes the image of σ' in G, i.e., ρ measures the failure of the following diagram

$$\begin{array}{ccc} W^{\oplus e} & \xleftarrow{\quad \psi \quad} & V_{|D} \\ {\scriptstyle ((\psi_\sigma^W)^{-1})^{\oplus e}} \big\uparrow & & \big\downarrow {\scriptstyle \psi_\sigma^V} \\ \sigma^* W^{\oplus e} & \xleftarrow[\sigma^*(\psi)]{} & \sigma^* V_{|D} \end{array}$$

to commute twisted by $\mathrm{diag}(\alpha(\sigma'))$. Another way to put this is that ρ compares the G'-linearizations $(\varphi^*(\psi_\sigma^W)^{-1})^{\oplus e}\mathrm{diag}(\alpha(\sigma'))$ and $\varphi^*(\psi_\sigma^V)$ on D'.

We claim that ρ defines a group morphism. Indeed, for $\sigma', \tau' \in G'$ mapping to σ (resp. τ) in G we have

$$\begin{aligned} &\rho(\tau')\rho(\sigma') = \\ &\mathrm{diag}(\alpha(\tau'))((\psi_\tau^W)^{-1})^{\oplus e}\tau^*(\psi)\psi_\tau^V\psi^{-1}\mathrm{diag}(\alpha(\sigma'))((\psi_\sigma^W)^{-1})^{\oplus e}\sigma^*(\psi)\psi_\sigma^V\psi^{-1} = \\ &\mathrm{diag}(\alpha(\tau')\alpha(\sigma'))((\psi_\tau^W)^{-1})^{\oplus e}\tau^*(\psi)\psi_\tau^V\psi^{-1}((\psi_\sigma^W)^{-1})^{\oplus e}\sigma^*(\psi)\psi_\sigma^V\psi^{-1} = \\ &\mathrm{diag}(\alpha(\tau')\alpha(\sigma'))((\psi_\sigma^W)^{-1})^{\oplus e}\sigma^*\left(((\psi_\tau^W)^{-1})^{\oplus e}\tau^*(\psi)\psi_\tau^V\psi^{-1}\right)\sigma^*(\psi)\psi_\sigma^V\psi^{-1} = \\ &\mathrm{diag}(\alpha(\tau')\alpha(\sigma'))((\psi_\sigma^W)^{-1})^{\oplus e}\sigma^*((\psi_\tau^W)^{-1})^{\oplus e}\sigma^*\tau^*(\psi)\sigma^*(\psi_\tau^V)\psi_\sigma^V\psi^{-1} = \\ &\mathrm{diag}(\alpha(\tau'\sigma'))((\psi_{\tau\sigma}^W)^{-1})^{\oplus e}(\tau\sigma)^*(\psi)\psi_{\tau\sigma}^V\psi^{-1} = \\ &\rho(\tau'\sigma'), \end{aligned}$$

where only the third and fifth equality require an explanation. We use that $((\psi_\sigma^W)^{-1})^{\oplus e}$ commutes with matrices and that matrices with entries in k do not change under pullback to obtain the third equality. To obtain the fifth equality we note that by construction of α $\varphi^*(\psi_\sigma^W)\alpha(\sigma')^{-1}$ defines a G'-linearization, see Lemma 3.2.1.

Replacing D' by $D'/\ker(\rho)$ we can assume that G' is a subgroup of Gl_e. By Jordan's theorem, see Theorem 3.0.1, there is a normal abelian subgroup $N' \subseteq G'$ such that G'/N' has cardinality at most $J(e)$. As H is central in G' the subgroup

$N' + H$ is normal, abelian, and contains H. As a finite abelian subgroup of Gl_e is simultaneously triagonalizable, we find the desired $(N' + H)$-invariant inclusion $W_{|D'} \subseteq V_{|D'}$. □

3.2.2 Setting up determinant descend

To be able to apply Lemma 3.2.3 we need to find a way to descent the determinant bundle. For such a construction we need to take roots of line bundles. If we avoid the characteristic, then this is always possible up to a finite cyclic Galois morphism.

Lemma 3.2.4. *Let X be a normal projective variety of dimension n. Let d be an integer prime to p. Further, let L be a line bundle on X. Then there exists a finite cyclic Galois morphism $\varphi : X' \to X$ such that $\deg(\varphi) \mid d$ and $L_{|X'}$ admits a d-th root on a big open.*

Proof. Let $\mathcal{O}_X(1)$ be an ample line bundle. Clearly, it suffices to find a morphism $X' \to X$ as in the statement such that $L_{|X'} \otimes \mathcal{O}_{X'}(1)^{\otimes nd}$ has a d-th root for some n. Thus, we can assume that L admits a non-zero global section, i.e., $L = \mathcal{O}_X(D)$ for some effective Cartier divisor D. Observe that it suffices to prove the Lemma for $\mathcal{O}_X(-D)$ instead of L.

Choose an affine open U containing the generic point of D in X such that $D_{|U} = V(f)$ for some non-zero divisor $f \in \mathcal{O}_U$. Consider the field extension $K/\kappa(X)$ generated by a d-th root of f. As $p \nmid d$ the extension $K/\kappa(X)$ is cyclic of order $d' \mid d$. Let X' denote the normalization of X in K. Note that there is a canonical finite morphism $\varphi : X' \to X$ of normal projective varieties. It is also separable by construction. As we only want to find an a d-th root on a big open and $X' \to X$ is flat at all codimension 1 points, we can assume that $X' \to X$ is flat.

Consider $U' := \varphi^{-1}(U) \cup \varphi^{-1}(X \setminus D)$. By construction U' is big. We show that $\mathcal{O}_X(-D)_{|X'}$ admits a d-th root on U'. Denote the d-th root of f on $\varphi^{-1}(U)$ by t. Then t defines an effective Cartier divisor D' on U'. Thus, we conclude $\mathcal{O}_{U'}(-D')^{\otimes d} = \mathcal{O}_X(-D)_{|U'}$ as $t^d = f$ on $\varphi^{-1}(U)$ and by construction both are trivial on $\varphi^{-1}(X \setminus D)$. □

Definition 3.2.5. A morphism $\pi : Y \to X$ of varieties is called *quasi-étale* if there is some big open $U \subseteq X$ such that $\pi^{-1}(U) \to U$ is étale. If $\pi^{-1}(U) \to U$ is a Galois cover with Galois group G, then we also say that $\pi : Y \to X$ is a quasi-étale Galois cover with Galois group G.

We can now set up the determinant descent needed to apply Lemma 3.2.3 on a normal projective variety.

Lemma 3.2.6. *Let X be a normal projective variety. Let $Y \to X$ be a prime to p Galois cover with Galois group G. Further, let V be a stable vector bundle of rank r on X such that V is stable on $X'_{r,large}$. Then there exists a commutative diagram of normal projective varieties*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \qquad \textit{such that}$$

(i) we have $V_{|Y'} \cong W'^{\oplus e'}$ such that W' is stable and $\det(W')$ descends on some big open,

(ii) $Y' \to X$ is a prime to p Galois morphism,

(iii) $X' \to X$ is cyclic of degree dividing r, and

(iv) $Y' \to X'$ is a quasi-étale Galois cover.

Proof. Consider the decomposition $V_{|Y} \cong W^{\oplus e}$ of Lemma 2.1.13. Clearly, $\det(W)^{\otimes e}$ and $\bigotimes_{\sigma \in G} \sigma^* \det(W) \cong \det(W)^{\otimes \#(G)}$ descend to X. Therefore, $\det(W)^{\otimes d}$ descends to X as well, where $d = \gcd(e, \#(G))$. Thus, there exists a line bundle L on X such that $L_{|Y} \cong \det(W)^{\otimes d}$. Note that $p \nmid d$ since G is prime to p.

We can apply Lemma 3.2.4 to find a cyclic morphism $X' \to X$ of normal varieties such that $L_{|X'}$ has a d-th root L' on a big open U' of X'. Let Y'' be connected component of the normalization of the reduced fibre product $(Y \times_X X')_{red}$ such that $Y'' \to X'$ is surjective. Note that the natural morphism $\psi : Y'' \to X'$ is prime to p and Galois. Then

$$W'' := \det(W)_{|\psi^{-1}(U')} \otimes L'^{-1}_{|\psi^{-1}(U')}$$

is a line bundle of order dividing d. The cover $U'' \to \psi^{-1}(U')$ associated to W'' trivializes W'', see Lemma 2.4.1.

Let Y' denote the normalization of Y'' in K, where K is the Galois hull of $\kappa(U'')/\kappa(X)$. As $\kappa(U'')/\kappa(Y''), \kappa(Y'')/\kappa(X')$, and $\kappa(X')/\kappa(X)$ are prime to p the same holds for $\kappa(Y')/\kappa(X)$. Then the commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

satisfies the conditions (ii), (iii), and (iv) of the Lemma.

If $W_{|Y'}$ is stable, then $V_{|Y'} \cong W_{|Y'}^{\oplus e}$ and we obtain (i) by construction. If $W_{|Y'}$ is not stable, then we repeat the above construction replacing Y by the étale part of Y'/X. Then we have $V_{|Y} \cong W'^{\oplus e'}$ for $e' > e$ and W' stable. As the integer e' is at most r, this process stops after finitely many iterations. □

3.2.3 Proof of Theorem 1

We can now prove the main theorem.

Theorem 3.2.7. *Let X be a normal projective variety of dimension at least* 1*. Let $r \geq 2$. Then there exists a prime to p Galois cover $X_{r,good} \to X$ such that a vector bundle V of rank r on X is prime to p stable iff $V_{|X_{r,good}}$ is stable.*

In particular, prime to p stability is an open property in the moduli space of Gieseker semistable sheaves on X.

Proof. Let $X_{r,good}$ be a prime to p Galois cover dominating $X'_{r,large}$ from Lemma 3.1.2 and all prime to p covers of degree $\leq J(r)r$, where $J(r)$ is the bound from Jordan's theorem, see Theorem 3.0.1.

The "only if" part is trivial. For the "if" part let V be a vector bundle of rank r on X such that $V_{|X_{r,good}}$ is stable. Consider a prime to p Galois cover $Y \to X$ and let $V_{|Y} \cong W^{\oplus e}$ be the decomposition of Lemma 2.1.13. Applying Lemma 3.2.6 we obtain a commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

satisfying the properties (i) - (iv) of Lemma 3.2.6. In particular, we have an isomorphism $V_{|Y'} \cong W'^{\oplus e'}$ for some stable vector bundle W' such that $\det(W')$ descends on some big open.

Observe that $V' := V_{|X'}$ is stable as the degree of the étale part of φ is at most r.

By Bertini's theorem the general complete intersection curve C' in X' is irreducible and irreducible after pullback to Y', see [14, Corollaire 6.11 (3)]. Furthermore, the general such C' is also normal by [27, Theorem 7]. The general hyperplane section intersects the locus where $Y' \to X'$ is not étale properly. As $Y' \to X'$ is quasi-étale we obtain that the pullback D' of the general such C' is a cover of C'. We also note that $D' \to C'$ is a Galois cover with the same Galois group as $Y' \to X'$.

Observe that there are only finitely many intermediate quasi-étale Galois covers $Y' \to Y'' \to X'$, where Y'' is a normal projective variety. On Y'' the bundle V' decomposes as $V'_{|Y''} \cong W''^{\oplus e''}$ for some stable vector bundle W'' on Y''. Iterating the restriction theorem in arbitrary characteristic for normal projective varieties, see [17, Theorem 0.1] for positive characteristic and [32, Theorem 7.17] for arbitrary characteristic, we find that restricting W'' to $D'' := Y'' \times_{X'} C'$ is stable, where C' is a general complete intersection curve in $c_1(\mathcal{O}_{X'}(-N_1)) \dots c_1(\mathcal{O}_{X'}(-N_{n-1}))$ for $N_i \gg 0$.

Restricting the decomposition $V'_{|Y'} \cong W'^{\oplus e'}$ of V' on Y' to such a $D' := Y' \times_{X'} C'$ we obtain an isomorphism $(V'_{|C'})_{|D'} \cong (W'_{|D'})^{\oplus e'}$. Note that $W'_{|D'}$ is stable and for general C' its determinant $\det(W'_{|D'})$ descends to C' by property (i) of Lemma 3.2.6. Hence, we are in a position to apply Lemma 3.2.3. Thus, there is an intermediate cover $D' \to D'' \to C'$ of degree $\leq J(e')$ such that there is a stable subbundle $M'' \subseteq V'_{|D''}$ pulling back to $W'_{|D'}$ on D'.

We claim that the intermediate cover $D' \to D'' \to C'$ can be lifted to a quasi-étale factorization of $Y' \to Y'' \to X'$. Indeed, let K be the kernel of the natural morphism $\mathrm{Gal}(D'/C') \to \mathrm{Gal}(D''/C')$. As $\mathrm{Gal}(Y'/X') = \mathrm{Gal}(D'/C')$ we can define Y'' to be the normalization of X' in the field extension $\kappa(Y')^K/\kappa(X')$.

Note that $Y'' \to X$ is prime to p of degree at most $rJ(e') \leq rJ(r)$. Consider the factorization $Y'' \to Y''' \to X$ into its étale and genuinely ramified part. We find that $V_{|Y'''}$ is stable by assumption. By Theorem 2.2.2 genuinely ramified morphisms preserve stability and the bundle $V_{|Y''} = V'_{|Y''}$ is stable as well. Thus, $V'_{|Y''} \cong W''$ and we obtain the stability of $V'_{|D''}$. Therefore, $V'_{|D''} \cong M''$ and pulling back to D' we find $V'_{|D'} \cong W'_{|D'}$, i.e., $e' = 1$. Clearly, $e \leq e'$ and we conclude that $V_{|Y}$ is stable. $\square$

4 Strata

Using the decomposition behaviour of a stable vector bundle on a Galois cover, see Lemma 2.1.13, we define a stratification of the moduli space of stable vector bundles by decomposition type. On a smooth projective curve of genus at least 2 we estimate the dimension of these decomposition strata. This stratification depends on the Galois cover.

Iterating the cover $X_{r,good}$ obtained in Theorem 3.2.7, we obtain a prime to p cover $X_{r,split}$ checking for the decomposition behaviour for all large enough prime to p covers. We obtain a canonical stratification independent of the choice of cover. For these *prime to p decomposition strata* the dimension estimates over a smooth projective curve of genus at least 2 are mostly sharp.

4.1 Stratifying by decomposition type

In this section we define a stratification associated to the decomposition type of a stable vector bundle with respect to some fixed Galois cover $Y \to X$ of normal projective varieties. Estimating the dimension of these strata, we obtain the existence of prime to p stable vector bundles on a smooth projective curve of genus at least 2, i.e., part of Theorem 2.

Definition 4.1.1. Let $\pi : Y \to X$ be a Galois cover of a normal projective variety X. Let V be a stable vector bundle of rank r on X. The *decomposition type of* V *with respect to* π is the rank m of the bundles W_i in the decomposition

$$V_{|Y} \cong \bigoplus_{i=1}^{n} W_i^{\oplus e}$$

of Lemma 2.1.13. The *refined decomposition type of* V *with respect to* π is the tuple (n, e).

Note that the refined decomposition type recovers the decomposition type as $mne = r$.

Let $P \in \mathbb{Q}[x]$ be a polynomial. Assume that the moduli space $M_X^{s,P}$ of stable vector bundles with Hilbert polynomial P is non-empty. The Hilbert polynomial determines the rank of the vector bundles in $M_X^{s,P}$ which we denote by r.

For integers $m, n, e, \geq 1$ such that $mne = r$ we define the *refined decomposition strata with respect to* π

$$Z^{s,P}(n,e,\pi) := \left\{ V \in M_X^{s,P} \,\middle|\, \begin{array}{c} V \text{ has refined decomposition} \\ \text{type } (n,e) \text{ with respect to } \pi \end{array} \right\}, \text{ and}$$

$$Z^{s,P}(\cdot \mid n, e \mid \cdot, \pi) := \bigsqcup_{(n',e')} Z^{s,P}(n', e', \pi),$$

where the union is taken over $n', e' \geq 1$ such that $n'e' = ne$ and $n' \mid n$. The *decomposition strata with respect to* π are defined as

$$Z^{s,P}(m,\pi) := \left\{ V \in M_X^{s,P} \,\middle|\, \begin{array}{c} V \text{ has decomposition type } m \\ \text{with respect to } \pi \end{array} \right\}, \text{ and}$$

$$Z^{s,P}(\cdot \mid m, \pi) := \bigsqcup_{m' \mid m} Z^{s,P}(m', \pi).$$

If X is a smooth projective curve, then the Hilbert polynomial is determined by the rank r and degree d and we use the notations

$$Z^{s,r,d}(m,\pi), Z^{s,r,d}(\cdot \mid m, \pi), Z^{s,r,d}(n,e,\pi), \text{ and } Z^{s,r,d}(\cdot \mid n, e \mid \cdot, \pi).$$

We show that the name is justified, i.e., that $Z^{s,P}(m,\pi)$ and $Z^{s,P}(n,e,\pi)$ form a stratification.

Lemma 4.1.2. *Let $\pi : Y \to X$ be a Galois cover of a normal variety X. Let $P \in \mathbb{Q}[x]$ be a polynomial. Assume that $M_X^{s,P}$ is non-empty and denote by r the rank of the vector bundles in $M_X^{s,P}$. Then we have the following:*

(i) $Z^{s,P}(\cdot \mid m, \pi) \subseteq M_X^{s,P}$ is closed for $m \mid r$.

(ii) $Z^{s,P}(\cdot \mid n, e \mid \cdot, \pi) \subseteq Z^{s,P}(m,\pi)$ is closed for $m, n, e \geq 1$ such that $mne = r$.

Proof. As $M_X^{s,P}$ is quasi-projective, it has only finitely many connected components $C_1, \dots, C_l$. On each of these components the Hilbert polynomial $P_j := P(V_{|Y})$ of $V \in C_j$ is independent of V as the Euler characteristic is locally constant, see [23, Corollary, p.50].

We claim that for $V \in C_j$ the bundles W_i in the decomposition $V_{|Y} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ of Lemma 2.1.13 have the same Hilbert-polynomial. Indeed, the Galois group G of $Y \to X$ acts transitively on the isomorphism classes of the W_i. For $\sigma \in G$ we have $P(\sigma^* W_i) = P(W_i)$ as the Hilbert-polynomial is computed with respect to $\pi^*\mathcal{O}_X(1)$ which is invariant under the action of G. Thus, for $V \in Z^{s,P}(m,\pi) \cap C_j$ we have $P(W_i) = \frac{m}{r} P_j$.

Pulling back along π defines a morphism $\pi_j^* : C_j \to M_Y^{G-ss,P_j}$. In M_Y^{G-ss,P_j} the set-theoretic image $\operatorname{im}(\varphi_{n,e})$ of

$$\varphi_{n,e} : \prod_{i=1}^{n} M_Y^{G-ss,\frac{m}{r}P_j} \to M_Y^{G-ss,P_j}, (W_1, \ldots, W_n) \mapsto \bigoplus_{i=1}^{n} W_i^{\oplus e}$$

is closed as it is a morphism of projective schemes, where $m, n, e \geq 1$ such that $mne = r$.

We claim that $Z^{s,P}(\cdot \mid m, \pi) \cap C_j$ is the preimage of $\operatorname{im}(\varphi_{n,1})$ under π_j^*. Indeed, $V \in Z^{s,P}(\cdot \mid m, \pi) \cap C_j$ clearly lies in the preimage. If $V_{|Y} \in \operatorname{im}(\varphi_{n,1})$ for some $V \in C_j$ and n such that $mn = r$, then $V_{|Y}$ is S-equivalent to $\bigoplus_{i=1}^{n} W_i$ for some Gieseker-semistable sheaves W_i of rank m. As $V_{|Y}$ is a direct sum of stable vector bundles with the same Hilbert polynomial, it is Gieseker-polystable. The associated graded object of the JH-filtration is unique and thus V has decomposition type m' with respect to π for some $m' \mid m$.

Then (i) follows from $Z^{s,P}(\cdot \mid m, \pi) = \bigsqcup_{j=1}^{l} Z^{s,P}(\cdot \mid m, \pi) \cap C_j$.

Similarly, (ii) is obtained from

$$(\pi_j^*)^{-1}(\operatorname{im}(\varphi_{n,e})) \cap Z^{s,P}(m, \pi) = Z^{s,P}(\cdot \mid n, e \mid \cdot, \pi) \cap C_j.$$

□

Remark 4.1.3. Ordering $Z^{s,P}(m, \pi), m \mid r$, via division, we obtain a stratification of $M_X^{s,P}$ by $Z^{s,P}(m, \pi)$. Furthermore, the refined decomposition strata $Z^{m,P}(n, e, \pi)$ stratify the decomposition stratum $Z^{s,P}(m, \pi)$ for $mne = r$ if we order them via

$$Z^{s,P}(n', e', \pi) \leq Z^{s,P}(n, e, \pi) :\Leftrightarrow n' \mid n.$$

4.1.1 Dimension estimates

Consider a Galois cover $\pi : D \to C$ of smooth projective curves with Galois group G. To estimate the dimension of the (refined) decomposition strata with respect to π we show that the decomposition $V_{|D} \cong \bigoplus_{i=1}^{n} W_i^{\oplus e}$ of Lemma 2.1.13 can essentially be recovered from W_1. Furthermore, W_1 behaves for the dimension estimates as if it descends to an intermediate cover $D' \to C$ of degree n.

Consider the case $n = 1$. Then there is only one isomorphism class of the conjugates of W_1, i.e., W_1 is G-invariant. This does not necessarily mean that W_1 descends to C. However, it does up to a twist by a line bundle:

Lemma 4.1.4. *Let $\pi : D \to C$ be a finite Galois cover of a smooth projective curve C with Galois group G. Let W be a simple bundle of rank r on D which is G-invariant. Then there exists a line bundle L on D such that $W \otimes L$ descends to C.*

Proof. Note that for a smooth algebraic group G a G-torsor over C corresponds to an element of $\check{H}^1_{\text{ét}}(C,G)$ as a smooth morphism admits étale locally a section. The same holds for D.

We have $H^2_{\text{ét}}(C,\mathbb{G}_m)=0$, see [31, Tag 03RM], similarly for D. By the 5-term exact sequence of the Čech to cohomology spectral sequence, see [21, Corollary 2.10, p.101], we obtain the vanishing of $\check{H}^2_{\text{ét}}$ from the vanishing of $H^2_{\text{ét}}$, i.e.,

$$\check{H}^2_{\text{ét}}(C,\mathbb{G}_m)=0=\check{H}^2_{\text{ét}}(D,\mathbb{G}_m).$$

Consider the short exact sequence of étale sheaves on $C_{\text{ét}}$

$$0\to\mathbb{G}_m\to\mathrm{Gl}_r\to\mathrm{PGl}_r\to 0.$$

Applying the functors $\Gamma(D,-)$ and $\Gamma(C,-)$ we obtain a commutative diagram of exact sequences of pointed sets

$$\begin{array}{ccccccc}
\check{H}^1_{\text{ét}}(D,\mathbb{G}_m) & \longrightarrow & \check{H}^1_{\text{ét}}(D,\mathrm{Gl}_r) & \longrightarrow & \check{H}^1_{\text{ét}}(D,\mathrm{PGl}_r) & \longrightarrow & \check{H}^2_{\text{ét}}(D,\mathbb{G}_m)=0 \\
\uparrow & & \uparrow & & \uparrow & & \| \\
\check{H}^1_{\text{ét}}(C,\mathbb{G}_m) & \longrightarrow & \check{H}^1_{\text{ét}}(C,\mathrm{Gl}_r) & \longrightarrow & \check{H}^1_{\text{ét}}(C,\mathrm{PGl}_r) & \longrightarrow & \check{H}^2_{\text{ét}}(C,\mathbb{G}_m)=0.
\end{array}$$

As $\mathbb{G}_m$ lies in the center of Gl_r this sequence extends to $\check{H}^2$ and exactness at $\check{H}^1_{\text{ét}}(\mathrm{Gl}_r)$ is stronger than usual, see Lemma 6.0.1: If two Gl_r-torsors map to the same PGl_r-torsor they differ by a twist of a line bundle. In particular, we obtain that a PGl_r-torsor can be lifted to a Gl_r-torsor, which also can be found in [5, Chapter III].

The bundle W is an element in $\check{H}^1_{\text{ét}}(D,\mathrm{Gl}_r)$. By definition of G-invariance we have isomorphisms $\psi_\sigma : W \xrightarrow{\sim} \sigma^*W$ for all $\sigma\in G$. The obstruction for descent $\lambda_{\sigma,\tau}:=\psi_{\sigma\tau}^{-1}\circ\tau^*\psi_\sigma\circ\psi_\tau$ is an isomorphism of W. By assumption W is simple and $\lambda_{\sigma,\tau}$ lies in k^*, i.e., considered as a PGl_r-torsor W descends to C, see [8, Theorem 1.4.46]. By the surjectivity of $\check{H}^1_{\text{ét}}(C,\mathrm{Gl_r})\to\check{H}^1_{\text{ét}}(C,\mathrm{PGl}_r)$ we find a vector bundle N on C such that $N_{|D}\cong W$ as PGl_r-torsors. Thus, the vector bundles $N_{|D}$ and W agree up to tensoring with a line bundle L on D. ☐

We can now estimate the dimension of the (refined) decomposition strata. The idea is simple: If a stable vector bundle V decomposes on a Galois cover $D\to C$ of the form $V_{|D}\cong\bigoplus_{i=1}^n W_i^{\oplus e}$, then one of the W_i carries enough information to (essentially) determine V via the action of the Galois group.

Theorem 4.1.5. *Let $\pi: D\to C$ be a Galois cover of a smooth projective curve C of genus $g_C\geq 2$. Let $r\geq 2$ and $d\in\mathbf{Z}$. Let $m,n,e\geq 1$ such that $mne=r$. Let $r=r'$ (resp. m') be the prime to p part of r (resp. m). Then we have the following:*

(i) $\dim(Z^{s,r,d}(n,e,\pi)) \leq nm^2(g_C - 1) + 1$.

(ii) $\dim(Z^{s,r,d}(m,\pi)) \leq rm(g_C - 1) + 1$.

(iii) If π is a prime to p cover, then

$$\dim(Z^{s,r,d}(m,\pi)) \leq \frac{r'}{m'} m^2(g_C - 1) + 1.$$

(iv) If π is a prime to p cover and $r = p^l, l \geq 1$, then $Z^{s,r,d}(n,e,\pi)$ is empty for $n \geq 2$.

Proof. Let G denote the Galois group of $D \to C$. Note that the decomposition strata are locally closed in $M_C^{ss,r,d}$, see Lemma 4.1.2. Thus, the dimension of $Z^{s,r,d}(n,e,\pi)$ is the same as the dimension of its closure in $M_C^{ss,r,d}$.

(i): Consider $V \in Z^{s,r,d}(n,e,\pi)$. Then we have $V_{|D} \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ for some pairwise non-isomorphic stable vector bundles W_i of rank m. Let H be the stabilizer of W_1. Then $D' := D/H$ is an intermediate cover $D \to D' \xrightarrow{\pi'} C$ of degree n.

Fix an inclusion $\iota : W_1^{\oplus e} \to V_{|D}$. Let W be the image of

$$\bigoplus_{\sigma \in H} \sigma^* W_1^{\oplus e} \xrightarrow{\oplus \varphi_\sigma^{-1} \sigma^* \iota} V_{|D},$$

where $\varphi_\sigma : V_{|D} \to \sigma^* V_{|D}$ denotes the G-linearization associated to V. Then W is an H-invariant saturated subsheaf of $V_{|D}$ isomorphic to $W_1^{\oplus e}$. Thus, $W \subseteq V_{|D}$ descends to a saturated subsheaf $W' \subseteq V_{|D'}$ by Lemma 2.1.7.

Since the action of G on the isomorphism classes of the W_i is transitive, we obtain that $V_{|D}$ is isomorphic to a direct sum of conjugates of $W'_{|D}$. As $W'_{|D}$ is semistable, so is W' by Lemma 2.1.9.

By Lemma 4.1.4, there exists a line bundle L on D and a vector bundle W'_1 on D' such that $(W'_1)_{|D} \cong L \otimes W_1$. Then W'_1 is stable by Lemma 2.1.9. Note that $L^{\otimes em}$ descends to D' as

$$\det(W'_1)_{|D} \cong L^{\otimes m} \det(W_1) \text{ and } \det(W_1)^{\otimes e} \cong \det(W')_{|D}.$$

Tensoring W'_1 by a line bundle of degree 1 changes the degree of W'_1 by m and we can assume that W'_1 has degree $d', 0 \leq d' < m$. Fixing the degree of W'_1 also fixes the degree of L. Choose a line bundle L' on D' of degree $d', 0 \leq d' < m$, and denote the moduli space of semistable vector bundles of rank m and determinant L' on D' by $M_{L'}^{ss,m}$. Denote by $P(d')$ the moduli space of line bundles on D of degree f such that their me-th power descends to D' and

$$rf = d\deg(\pi) - e\deg(\pi)d'.$$

Note that if $P(d') \neq \emptyset$, then $\dim(P(d')) = g_{D'}$ as raising a line bundle to its me-th power is a finite morphism $\mathrm{Pic}^f_D \to \mathrm{Pic}^{mef}_D$.

Consider for a finite subset $\Sigma \subseteq G$ of cardinality n the morphism

$$\varphi_{d',\Sigma,D'} : P(d') \times M^{ss,m}_{L'} \to M^{ss,r,d\deg(\pi)}_D, (L, W'_1) \mapsto \bigoplus_{\sigma\in\Sigma} \sigma^*(L \otimes (W'_1)_{|D})^{\oplus e}.$$

Observe that the image $Z_{d',\Sigma,D'}$ of $\varphi_{d',\Sigma,D'}$ is closed as $\varphi_{d',\Sigma,D'}$ is a morphism of projective varieties.

The above discussion shows that

$$\pi^*(Z^{s,r,d}(n, e, \pi)) \subseteq \bigcup_{d'=0}^{m-1} \bigcup_{D'\to C} \bigcup_{\Sigma\subseteq G} Z_{d',\Sigma,D'},$$

where the union is taken over intermediate covers $D \to D' \to C$ of degree n and subsets $\Sigma \subseteq G$ of cardinality n.

We can now estimate the dimension of the refined decomposition stratum. By Theorem 2.3.2, pullback by π is a finite morphism π^* and it suffices to estimate $\dim(\pi^*(Z^{s,r,d}(n, e, \pi)))$. We have

$$\dim(Z_{d',\Sigma,D'}) \leq \dim(P(d')) + \dim(M^{ss,m}_{L'}) = m^2(g_{D'} - 1) + 1.$$

We obtain

$$\dim(Z^{s,r,d}(n, e, \pi)) = \dim(\pi^*(Z^{s,r,d}(n, e, \pi))) \leq m^2 n(g_C - 1) + 1$$

by Riemann-Hurwitz.

(ii): This is a direct consequence of (i) as

$$Z^{s,r,d}(m, \pi) = \bigsqcup_{(n,e)} Z^{s,r,d}(n, e, \pi),$$

where the union is taken over $n, e \geq 1$ such that $mne = r$. Then

$$\dim(Z^{s,r,d}(m, \pi)) = \max_{n,e} \dim(Z^{s,r,d}(n, e, \pi)) \leq rm(g_C - 1) + 1$$

as the maximum is obtained at $e = 1, n = \frac{r}{m}$.

(iii): If π is prime to p and $Z^{s,r,d}(m, \pi)$ is non-empty, then we claim that n is coprime to p as well. Indeed, we have seen in (i) that $n \mid \deg(\pi)$. Then the maximum in the estimate of (ii) is obtained at $e = \frac{rm'}{r'm}, n = \frac{r'}{m'}$. Thus, we conclude

$$\dim(Z^{s,r,d}(m, \pi)) \leq \frac{r'}{m'} m^2(g_C - 1) + 1.$$

(iv): If $Z^{s,r,d}(n, e, \pi)$ is non-empty, then $n \mid \deg(\pi)$. This is impossible under the assumptions of (iv). □

As a direct consequence of the dimension estimate for the decomposition strata we obtain the existence of stable vector bundles that remain stable on a fixed cover.

Corollary 4.1.6. *Let $\pi : D \to C$ be a cover of a smooth projective curve C of genus $g_C \geq 2$. Let $r \geq 2$ and d be integers. Let Z be the closed subset of stable vector bundles in $M_C^{s,r,d}$ that are not stable after pullback to D. Then*

$$\dim(Z) \leq rr_0(g_C - 1) + 1 \text{ and}$$

$$\operatorname{codim}_{M_C^{s,r,d}}(Z) \geq 2,$$

where r_0 denotes the largest proper divisor of r.

In particular, there are stable vector bundles of rank r and degree d on C that remain stable after pull back to D.

Proof. Observe that we can replace $D \to C$ by its Galois closure. Also note that $Z = \bigcup_{m|r, m\neq r} Z^{s,r,d}(m,\pi)$. By Theorem 4.1.5 (ii), we have

$$\dim(Z^{s,r,d}(m,\pi)) \leq rm(g_C - 1) + 1 \leq rr_0(g_C - 1) + 1.$$

As

$$r^2(g_C - 1) + 1 = \dim(M_C^{s,r,d}),$$

$g_C \geq 2$, and $r \geq 2$, we conclude

$$\operatorname{codim}(Z) \geq r(r - r_0)(g_C - 1) \geq 2.$$

□

4.1.2 Existence of prime to p stable vector bundles

Applying Corollary 4.1.6 to the cover $C_{r,good} \to C$ of a smooth projective curve of genus ≥ 2, see Theorem 3.2.7, we obtain the non-emptiness of the prime to p stable locus, i.e., part of Theorem 2. We state this separately:

Corollary 4.1.7. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbb{Z}$. Then the* prime to p stable locus

$$M_C^{p'-s,r,d} := \{V \in M_C^{s,r,d} \mid V \text{ prime to } p \text{ stable}\} \subseteq M_C^{s,r,d}$$

is a big open.

Extending a prime to p stable vector bundle from a large curve to a surrounding smooth projective variety using Mathur's extension theorem, [20, Theorem 1], we obtain the existence of prime to p stable vector bundles in higher dimensions. However, we can not control the numerical data, i.e., which components of the stack of bundles admit prime to p stable vector bundles.

Corollary 4.1.8. *Let X be a smooth projective variety of dimension $n \geq 2$. There are prime to p stable vector bundles of rank $r \geq n$ on X.*

4.2 Prime to p decomposition strata

Let C be a smooth projective curve of genus $g_C \geq 2$. The cover $C_{r,good} \to C$ constructed in Theorem 3.2.7 can be iterated to obtain a cover $C_{r,split}$ checking for the eventual decomposition of a stable vector bundle of rank r on C. This gives rise to the prime to p decomposition strata. Similarly to the decomposition strata with respect to a cover, the dimension of the prime to p decomposition strata can be estimated. The main difference is that these estimates are sharp as long as the characteristic is avoided. To obtain sharp estimates we find a way to construct vector bundles with prescribed prime to p decomposition using cyclic covers.

4.2.1 A split cover

We begin with the definition of the cover $C_{r,split}$. This still works on normal projective varieties:

Definition 4.2.1. Let X be a normal projective variety. Let $r \geq 2$. We define a prime to p Galois cover $\pi_{r,split} : X_{r,split} \to X$ inductively as follows: Set X_1 as $X_{r,good}$. Then define for $1 \leq j < r-1$ the cover X_{j+1} as a prime to p Galois cover dominating $(X_j)_{l,good}$ for $l \leq r-j$ and $l \mid r$. We define $X_{r,split} := X_{r-1}$.

Let V be a stable vector bundle on X with Hilbert polynomial P. The *prime to p decomposition type* (or just *decomposition type*) of V is the decomposition type with respect to the cover $X_{r,split} \to X$.

We call the stratification of $M_X^{s,P}$ with respect to $\pi_{r,split}$ the *prime to p decomposition stratification* (or just *decomposition stratification*) and denote it for $m \mid r$ by

$$Z^{s,P}(m) := Z^{s,P}(m, \pi_{r,split}).$$

If X is a smooth projective curve, the Hilbert polynomial is determined by the rank r and the degree d and we use the notation $Z^{s,r,d}(m)$ instead.

The notations $X_{r,split}$ and $Z^{s,P}(m)$ are justified as the decomposition type stays the same on any prime to p cover dominating $X_{r,split}$ as we show in the next lemma.

Lemma 4.2.2. *Let X be a normal projective variety. Let $r \geq 2$. Then a stable vector bundle of rank r on X decomposes on $X_{r,split}$ into a direct sum of prime to p stable vector bundles.*

In particular, the decomposition types with respect to a prime to p Galois cover $Y \to X$ dominating $X_{r,split} \to X$ and with respect to $X_{r,split} \to X$ agree.

Proof. Let V be stable vector bundle of rank r on X. We follow the behaviour of $V_{|X_j}$ for $j = 1, \ldots, r-1$ of Definition 4.2.1. Let r_j be the decomposition type of V with respect to $X_j \to X$. Then we have $r_{j+1} \leq r_j$ for $1 \leq j < r-1$ and equality holds iff $V_{|X_j}$ is a direct sum of stable vector bundles on X_j that remain stable on

X_{j+1}. By construction of X_{j+1} this is the case iff $V_{|X_j}$ decomposes into a direct sum of prime to p stable vector bundles on X_j.

As there are $r-1$ covers $X_{r-1} \to \cdots \to X_1 \to X$, we find $r_{r-1} = 1$ or $r_{r-1} = r_j$ for some $j < r-1$. In both cases $V_{|X_{r-1}} = V_{|X_{r,split}}$ decomposes into a direct sum of prime to p stable vector bundles.

If $Y \to X$ is a prime to p Galois cover dominating $X_{r,split} \to X$, consider the decomposition of $V_{|X_{r,split}} \cong \bigoplus_{i=1}^n V_i^{\oplus e}$ into pairwise non-isomorphic stable vector bundles of rank m. By the above discussion V_i is prime to p stable. Thus, $V_{|X} \cong \bigoplus_{i=1}^n (V_i)_{|Y}^{\oplus e}$ is a decomposition into stable vector bundles of rank m. □

Remark 4.2.3. Note that the refined decomposition type might still change and is thus not independent of the cover. For example only finitely many prime to p trivializable stable vector bundles of rank r become trivial on $X_{r,split}$. However, there are infinitely many prime to p trivializable stable bundles of rank r if the prime to p completion of the étale fundamental group is large enough; e.g. if $X = C$ is a smooth projective curve of genus at least 2.

4.2.2 Sharp dimension estimates

Theorem 4.2.4. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbb{Z}$. Then for $m \mid r$ we have the following:*

(i) $\dim(Z^{s,r,d}(m)) \leq (\frac{r}{m})' m^2(g_C - 1) + 1$, *where $(\frac{r}{m})'$ denotes the prime to p part of $\frac{r}{m}$.*

(ii) If $p \nmid \frac{r}{m}$, then we have $\dim(Z^{s,r,d}(m)) = rm(g_C - 1) + 1$.

Proof. (i): The estimate follows from Theorem 4.1.5 applied to $C_{r,split} \to C$.

(ii): Let $n = \frac{r}{m}$ and assume that n is prime to p. Cyclic covers of degree n correspond to line bundles of order n, see Lemma 2.4.1. Thus, there exist such cyclic covers. Let $\pi : D \to C$ be cyclic of degree n. Denote the Galois group by G.

By Lemma 2.5.3 and Lemma 2.5.5, the locus $U \subseteq M_D^{s,m,d}$ of stable vector bundles W on D such that $\pi_* W$ is stable is open and non-empty. As $M_D^{s,m,d}$ is irreducible, the intersection $U' := U \cap M_D^{p'-s,m,d}$ is open and non-empty as well.

Pushforward induces a finite morphism $U \xrightarrow{\pi_*} M_C^{s,r,d}$, see Lemma 2.5.5. For $W \in U'$ the direct image $\pi_* W$ has prime to p decomposition type m as by affine base change $\pi^*\pi_* W \cong \bigoplus_{\sigma \in G} \sigma^* W$ which is a direct sum of prime to p stable vector bundles. We obtain

$$\dim(M_D^{ss,m,d}) = \dim(U') \leq \dim(Z^{s,r,d}(m)).$$

By Riemann-Hurwitz and (i), we conclude $\dim(Z^{s,r,d}(m)) = rm(g_C - 1) + 1$. □

Remark 4.2.5. Note that the dimension estimates can be far from being sharp if $p \mid r$: If $r = p^n$ for some $n \geq 1$ and d is prime to p, then every semistable vector bundle of rank r and degree d is prime to p stable. In particular, the decomposition strata $Z^{s,r,d}(m)$ are empty for $m < r$.

As a direct consequence of Theorem 4.2.4, we find a mostly sharp estimate on the codimension of the complement of the prime to p stable locus. Thus, we complete the proof of Theorem 2.

Corollary 4.2.6. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbf{Z}$. Then the following hold:*

(i) If k is uncountable, then there are separable-stable bundles of rank r and degree d.

(ii) The prime to p stable vector bundles of rank r and degree d form a big open subset $M_C^{p'-s,r,d} \subseteq M_C^{s,r,d}$.

(iii) We have

$$\dim(M_C^{s,r,d} \setminus M_C^{p'-s,r,d}) \leq rr_0(g_C - 1) + 1,$$

where r_0 denotes the largest proper divisor of r. If p is not the smallest proper divisor of r, then equality holds.

Proof. As already mentioned separable-stable and étale-stable coincide for curves, see Corollary 2.6.5.

(i): By Corollary 4.1.6 the stable vector bundles that remain stable after pull back to an étale cover form a non-empty open subset of $M_C^{s,r,d}$. There are only countably many étale covers of C up to isomorphism, see Lemma 2.6.7. The intersection of countably many non-empty open subsets of a quasi-projective variety over an uncountable algebraically closed field contains a closed point by Lemma 6.0.2.

(ii) is Corollary 4.1.7.

(iii) The estimate is a direct consequence of Theorem 4.2.4 (i). If $p \nmid \frac{r}{r_0}$, then $Z^{s,r,d}(r_0) \subseteq M_C^{s,r,d} \setminus M_C^{p'-s,r,d}$ has dimension $rr_0(g_C - 1) + 1$ by Theorem 4.2.4 (ii). □

Corollary 4.2.7. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbf{Z}$. Then* $\mathrm{Pic}(M_C^{p'-s,r,d}) = \mathrm{Pic}(M_C^{s,r,d})$.

Proof. By Corollary 4.2.6 we have that $M_C^{p'-s,r,d} \subseteq M_C^{s,r,d}$ is a big open. As $M_C^{s,r,d}$ is smooth, they have the same Picard group. □

As the general bundle is prime to p stable, we obtain:

Corollary 4.2.8. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Then the stable vector bundles of rank r that are trivialized on a prime to p cover are not dense in $M_C^{s,r,0}$.*

5 The closure of prime to p trivializable bundles

As an application of the theory developed we study the closure of the prime to p trivializable stable vector bundles.

In positive characteristic the étale trivializable bundles are dense by a theorem due Ducrohet and Mehta, see [6, Corollary 5.1]. This is no longer the case if we only consider prime to p trivializable bundles as the general bundle remains stable on all prime to p covers. It is natural to ask the question what the closure Z of the prime to p trivializable bundles is. This is closely related to the $Z(1)$-stratum and we can estimate the dimension of Z. This estimate is sharp if the characteristic is avoided. In rank 2 and rank $p^l, l \geq 1$, we also describe the irreducible components.

We begin by recalling the definition of prime to p trivializable bundles and the denseness of the prime to p trivializable bundles of rank 1 in Pic^0. Then we prove the main results.

Definition 5.0.1. Let X be a normal projective variety. We call a vector bundle V of rank r on X *prime to p trivializable* if there exists a prime to p cover $Y \to X$ such that $V_{|Y}$ is the trivial bundle of rank r.

Note that as every prime to p cover is dominated by a prime to p Galois cover, we could have alternatively required that prime to p trivializable bundles become trivial on a prime to p Galois cover.

We specialize the correspondence of étale trivializable bundles to representations of the étale fundamental group to a correspondence prime to p trivializable bundles:

Corollary 5.0.2. *Let X be a normal projective variety. Then the prime to p trivializable bundles (up to isomorphism) correspond to representations of the prime to p completion $\pi_1^{ét,p'}(X)$ of $\pi_1^{ét}(X)$ (up to conjugation).*

Under this correspondence irreducible representations correspond to stable vector bundles.

Proof. As the étale fundamental group is profinite, a representation $\pi_1^{\text{ét},p'}(X) \to \mathrm{GL}_r$ factors via a finite group $\pi_1^{\text{ét},p'}(X) \twoheadrightarrow G \subset \mathrm{GL}_r$. Furthermore, a prime to p Galois cover $Y \to X$ with Galois group G corresponds to a surjection $\pi_1^{\text{ét},p'}(X) \twoheadrightarrow G$.

Then we conclude by Lemma 2.5.1 which asserts that representations of G (up to conjugation) correspond vector bundles (up to isomorphisms) that are trivialized by $Y \to X$.

The correspondence of irreducible representations of G and stable bundles trivialized by $Y \to X$ also holds by Lemma 2.5.1. To conclude the same for representations of $\pi_1^{\text{ét},p'}(X)$ note that a representation of $\pi_1^{\text{ét},p'}(X) \to \mathrm{GL}_r$ is irreducible iff the above factorization $\pi_1^{\text{ét},p'}(X) \twoheadrightarrow G \subset \mathrm{GL}_r$ satisfies that $G \to \mathrm{GL}_r$ is irreducible. □

Recall that denseness of prime to p trivializable bundles holds for the Picard scheme Pic_C^0 of a smooth projective curve C as it is an abelian variety. This can be found in [22] which is at time of writing not yet finished. For the convenience of the reader we include the proof of the denseness theorem.

Theorem 5.0.3 ([22], Theorem 5.30)**.** *Let A be an abelian variety. Then $\bigcup_{n\geq 1} A[q^n]$ is set-theoretically dense in A, where $q \neq p$ is a prime and $A[q^n]$ denotes the finite subgroup of closed points of order dividing q^n.*

Proof. We use that there are exactly $q^{2\dim(A)n}$ points of order dividing q^n on an abelian variety, see [23, Proposition, p.64]. Let B denote the closure of $\bigcup_{n\geq 1} A[q^n]$ equipped with the reduced subscheme structure. Denote the group law on A by $m : A \times A \to A$. As A and $A \times A$ are proper, m is proper as well. The torsion points $\bigcup_{n\geq 1} A[q^n]$ form a subgroup of $A(k)$ and we find that

$$B \times B \to A \times A \xrightarrow{m} A$$

factors over B. The same holds for the connected component B^0 of B containing the neutral element. Similarly, the inversion $i : A \to A$ restricts to B^0. Thus, B^0 is an abelian variety and has $q^{2\dim(B^0)n}$ points of order dividing q^n.

As B is a closed subscheme of A, it is Noetherian and has only finitely many connected components. Let N be the number of connected components. Then the number of points of B of order dividing q^n is $\leq Nq^{2\dim(B^0)n}$. By definition of B, it contains all points of A of order dividing q^n, i.e., we have

$$q^{2\dim(A)n} \leq Nq^{2\dim(B^0)n}$$

for all $n \geq 1$. Considering $n \to \infty$, we obtain $\dim(B^0) = \dim(A)$. Thus, we conclude $B^0 = B = A$. □

Corollary 5.0.4. *Let C be a smooth projective curve. Then the prime to p trivializable bundles of rank 1 are set-theoretically dense in Pic_C^0.*

Proof. As Pic_C^0 is an abelian variety, the torsion points $\bigcup_{N\geq 1} \mathrm{Pic}_C^0[q^N]$ for $p \neq q$ prime are dense by Theorem 5.0.3. A line bundle of order prime to p is trivialized by its associated torsor, see Lemma 2.4.1, and we conclude. □

Theorem 5.0.5. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Let $Z^{s,r}$ be the closure of the prime to p trivializable stable vector bundles in $M_C^{s,r,0}$ and $Z^{ss,r}$ be the closure of the prime to p trivializable vector bundles in $M_C^{ss,r,0}$. Then we have the following:*

(i) $Z^{s,r} \subseteq Z^{s,r,0}(1)$.

(ii) $\dim(Z^{s,r}) \leq r'(g_C - 1) + 1$, *where r' is the prime to p part of r.*

(iii) If $p \nmid r$, then $\dim(Z^{s,r}) = \dim(Z^{s,r,0}(1)) = r(g_C - 1) + 1$.

(iv) $\dim(Z^{ss,r}) = rg_C$.

Proof. (i): The prime to p decomposition type of a prime to p trivializable bundle is 1. Thus, we obtain $Z^{s,r} \subseteq Z^{s,r,0}(1)$ using that $Z^{s,r,0}(1)$ is closed in $M_C^{s,r,0}$, see Lemma 4.1.2.

(ii): This is a direct consequence of (i) and the dimension estimate for the prime to p decomposition strata, see Lemma 4.1.5 (ii).

(iii): Assume that $p \nmid r$. Let $\pi : D \to C$ be a cyclic cover of order r and Galois group $G \cong \mathbf{Z}/r\mathbf{Z}$. Consider the dense open subset $U \subseteq \mathrm{Pic}_D^0$ defined as

$$U := \{L \in \mathrm{Pic}_D^0 \mid \sigma^* L \not\cong L \text{ for all } \sigma \in G \setminus \{e_G\}\},$$

see Lemma 2.5.3. By Lemma 2.5.5, pushforward induces a finite morphism

$$\pi_* : U \to M_C^{s,r,0}, L \mapsto \pi_* L.$$

As the prime to p trivializable line bundles are dense in Pic_D^0, see Corollary 5.0.4, and Pic_D^0 is irreducible, we find that the prime to p trivializable line bundles contained in U are still dense in Pic_D^0.

Note that for a prime to p trivializable line bundle L its conjugates $\sigma^* L, \sigma \in G$, are prime to p trivializable as well. Thus, the pushforward $\pi_* L$ is prime to p trivializable as $\pi^* \pi_* L \cong \bigoplus_{\sigma \in G} \sigma^* L$. Therefore, π_* factors as $U \to Z^{s,r} \to M_C^{s,r,0}$ where we equip $Z^{s,r}$ with the reduced closed subscheme structure.

The image $\pi_*(U)$ in $Z^{s,r}$ has dimension $\dim(U) = g_D$ since π_* is finite. By Riemann-Hurwitz and (ii) we conclude

$$r(g_C - 1) + 1 \leq \dim(Z^{s,r}) \leq \dim(Z^{s,r,0}(1)) \leq r(g_C - 1) + 1.$$

(iv): Observe that $Z^{ss,r}$ contains the image of the finite morphism

$$\prod_{i=1}^{r} \mathrm{Pic}_C^0 \to M_C^{ss,r,0}, (L_1, \ldots, L_r) \mapsto \bigoplus_{i=1}^{r} L_i.$$

Thus, we have $\dim(Z^{ss,r}) \geq rg_C$.

To obtain the other inequality, we claim that $Z^{ss,r}$ is the union of the images of

$$\varphi_{r_1,\dots,r_l} : \prod_{i=1}^{l} \overline{Z^{s,r_i}} \to M_C^{ss,r,0}, (V_1,\dots,V_l) \mapsto \bigoplus_{i=1}^{l} V_i$$

for all possible ways to write $r = \sum_{i=1}^{l} r_i$ and the closure of Z^{s,r_i} is taken in $M_C^{ss,r_i,0}$. Indeed, the image of $\varphi_{r_1,\dots,r_l}$ is closed and so is the finite union $\bigcup_{\sum_{i=1}^{l} r_i = r} \mathrm{im}(\varphi_{r_1,\dots,r_l})$. Furthermore, the union contains all prime to p trivializable bundles of rank r.

By (ii) we have $\dim(\varphi_{r_1,\dots,r_l}) \leq r(g_C - 1) + l \leq r g_C$. □

In the special cases $r = p^n$ and $r = 2$ more can be said:

Corollary 5.0.6. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $p \neq 0$ and $n \geq 1$. Let Z^{s,p^n} be the closure of the prime to p trivializable stable vector bundles in $M_C^{s,p^n,0}$. Then $Z^{s,p^n} = Z^{s,p^n,0}(1)$.*

The irreducible components of Z^{s,p^n} are disjoint and of the form $\mathrm{Pic}_C^0 \otimes V(\rho)$ for some irreducible representations $\rho : G_{p^n,split} \to \mathrm{GL}_{p^n}$, where $G_{p^n,split}$ denotes the Galois group of $C_{p^n,split} \to C$, $V(\rho)$ is the vector bundle associated to ρ, see Lemma 2.5.1, and $\mathrm{Pic}_C^0 \otimes V(\rho)$ denotes the image of $\mathrm{Pic}_C^0 \to M_C^{s,p^n,0}, L \mapsto L \otimes V(\rho)$.

Proof. We already have the inclusion $Z^{s,p^n} \subseteq Z^{s,p^n,0}(1)$ by Theorem 5.0.5 (i). To show the other inclusion consider $V \in Z^{s,p^n,0}(1)$. As p is prime and V has prime to p decomposition type 1, the only possible refined decomposition type of V with respect to the prime to p Galois cover $C_{p^n,split} \to C$ is $(1, p^n)$, i.e., $V_{|C_{p^n,split}} \cong L^{\oplus p^n}$ for some line bundle L on $C_{p^n,split}$, see Theorem 4.1.5 (iv).

Let $G = G_{p^n,split}$. Then $L^{\otimes p^n} \cong \det(V)_{|C_{p^n,split}}$ and $\bigotimes_{\sigma \in G} \sigma^* L \cong L^{\otimes \#(G)}$ descend to C. As G is prime to p, we find that L descends to a line bundle L' on C. We obtain an irreducible representation $\rho : G \to \mathrm{GL}_{p^n}$ using Lemma 2.5.1. Let $V(\rho)$ be the stable vector bundle trivialized by $C_{p^n,split} \to C$ corresponding to ρ, see Lemma 2.5.1. Then we have $V \cong V(\rho) \otimes L'$ which lies in Z^{s,p^n} as tensoring induces a morphism

$$\otimes : \mathrm{Pic}_C^0 \times_k Z^{s,p^n} \to Z^{s,p^n}, (L', V) \mapsto L' \otimes V.$$

Thus, we conclude that $Z^{s,p^n,0}(1) \subseteq Z^{s,p^n}$. The above argument also shows that

$$Z^{s,p^n,0}(1) = \bigcup_{\rho : G \to \mathrm{GL}_{p^n}} \mathrm{Pic}_C^0 \otimes V(\rho),$$

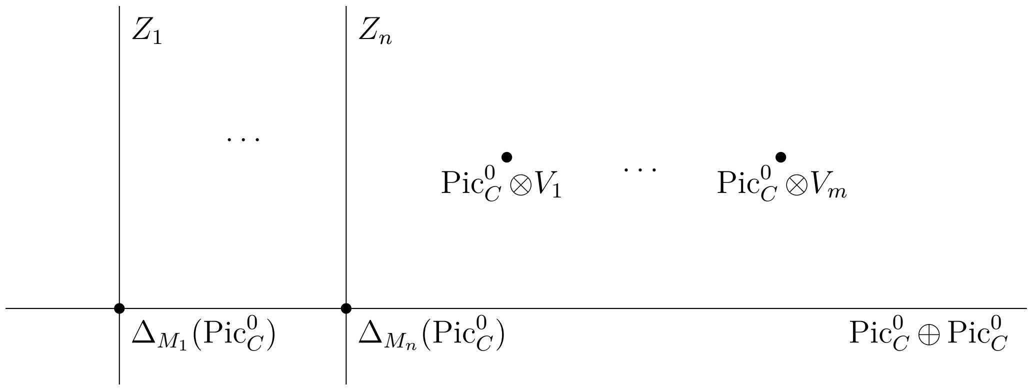
where the union is taken over the irreducible representations.

If $\mathrm{Pic}_C^0 \otimes V(\rho)$ and $\mathrm{Pic}_C^0 \otimes V(\rho')$ are not disjoint, then $V(\rho) \otimes L \cong V(\rho')$ for some line bundle L of degree 0 on C. Thus, $\mathrm{Pic}_C^0 \otimes V(\rho) = \mathrm{Pic}_C^0 \otimes V(\rho')$ and we obtain the description of the irreducible components. □

Theorem 5.0.7. *Let C be a smooth projective curve of genus $g_C \geq 2$. Let $Z^{s,2}$ be the closure of the prime to p trivializable stable vector bundles of rank 2 in $M_C^{s,2,0}$.*
If $p \neq 2$, then $Z^{s,2} = Z^{s,2,0}(1)$ and the irreducible components are as follows:

- *The irreducible components are pairwise disjoint and of dimension g_C or $2(g_C - 1) + 1$.*
- *There are $n = 4g_C - 1$ irreducible components of dimension $2(g_C - 1) + 1$. Denote them by $Z_1, \dots, Z_n$. They correspond to the line bundles on C of order 2.*
- *The closures $\overline{Z_i}$ in $M_C^{ss,2,0}$ are pairwise disjoint and intersect the non-stable locus $\mathrm{Pic}_C^0 \oplus \mathrm{Pic}_C^0$ in $\Delta_{M_i}(\mathrm{Pic}_C^0)$, where*
$$\Delta_{M_i} : \mathrm{Pic}_C^0 \to M_C^{ss,2,0}, L \mapsto L \oplus (L \otimes M_i)$$
and M_i is the corresponding line bundle of order 2.
- *The irreducible components of dimension g_C are of the form $\mathrm{Pic}_C^0 \otimes V_j$ for some stable prime to p trivializable bundles $V_1, \dots, V_m$ of rank 2, where $\mathrm{Pic}_C^0 \otimes V_j$ denotes the image of $\mathrm{Pic}_C^0 \to M_C^{s,2,0}, L \mapsto L \otimes V_j$.*

In a picture:



Proof. The inclusion $Z^{s,2} \subseteq Z^{s,2,0}(1)$ is clear by Theorem 5.0.5 (i).

To show the other inclusion, let $V \in Z^{s,2,0}(1)$. We study the two possibilities for the refined decomposition type of V with respect to $C_{2,split}$ separately.

If the refined decomposition type of V with respect to $C_{2,split}$ is $(1,2)$, then $V_{|C_{2,split}} \cong L^{\oplus 2}$. Let M be a line bundle on C such that $M^{\otimes 2} \cong \det(V)$. Then $L \otimes M_{|D}^{-1}$ has order ≤ 2 and is prime to p trivializable. We find that $V \otimes M^{-1}$ is prime to p trivializable as well. As tensoring with a line bundle of degree 0 defines an automorphism of $Z^{s,2}$, we find $V \in Z^{s,2}$.

Let $D \to C_{2,split}$ be a prime to p cover dominating all connected μ_2-torsors of $C_{2,split}$ and such that $D \to C$ is Galois with Galois group G. Then $V \otimes M^{-1}$ becomes trivial on D and corresponds to an irreducible representation of G. As G is finite there are only finitely many irreducible representation $\rho_1, \ldots, \rho_l$ of G. We obtain $V \in \bigcup_{j=1}^{l} \mathrm{Pic}_C^0 \otimes V(\rho_j)$.

If the refined decomposition type of V with respect to $C_{2,split}$ is $(2,1)$, then there exist non-isomorphic line bundles L_1, L_2 on $C_{2,split}$ such that $V_{|C_{2,split}} \cong L_1 \oplus L_2$. Consider the stabilizer H of the isomorphism class of L_1. Let $G_{2,split}$ be the Galois group of $C_{2,split} \to C$. As $G_{2,split}$ acts transitively on the isomorphism classes of L_1 and L_2, the subgroup $H \subseteq G_{2,split}$ has index 2. Then $\pi : C' := C/H \to C$ is a cyclic cover of order 2. By Lemma 3.1.1, we find that V still has refined decomposition type $(2,1)$ with respect to C'.

Let $V_{|C'} \cong L'_1 \oplus L'_2$ be the decomposition. Then $\sigma^* L'_1 \not\cong L'_1$ for the generator σ of $G_{2,split}/H$. By Lemma 2.5.3, we have $\pi_*(L'_i) \cong V$. In particular, V lies in the image $Z_{C'}$ of the finite morphism

$$\pi_* : U \to M_C^{s,2,0}, L' \mapsto \pi_* L',$$

where $U = U^{2,0} \subseteq \mathrm{Pic}^0_{C'}$ was defined in Lemma 2.5.3. Note that $Z_{C'}$ is irreducible since U is irreducible. As the prime to p trivializable bundles are dense in U, we conclude $V \in Z_{C'} \subset Z^{s,2}$.

From the two cases regarding the refined decomposition type we obtain

$$Z^{s,2,0}(1) \subseteq \bigcup_{\rho: G \to gl_r} \mathrm{Pic}_C^0 \otimes V(\rho) \cup \bigcup_{C' \to C} Z_{C'} \subseteq Z^{s,2},$$

where the union is taken over irreducible representations of G and connected μ_2-torsors $C' \to C$. Thus, we have

$$Z^{s,2,0}(1) = Z^{s,2} = \bigcup_{\rho: G \to gl_r} \mathrm{Pic}_C^0 \otimes V(\rho) \cup \bigcup_{C' \to C} Z_{C'}.$$

In particular, all irreducible components are of the desired form.

By Riemann-Hurwitz and the finiteness of π_*, we obtain the dimension

$$\dim(Z_{C'}) = 2(g_C - 1) + 1 = \dim(Z^{s,2,0}(1)).$$

Thus, $Z_{C'}$ is an irreducible component of dimension $2(g_C - 1) + 1$.

We continue to show that the irreducible components $Z_{C'}$ are disjoint. Let $V \in Z_{C'_1} \cap Z_{C'_2}$ for non-isomorphic connected μ_2-torsors $\pi_i : C'_i \to C, i = 1, 2$. Then $V \cong \pi_{i,*} L'_i$ for some line bundle $L'_i \in \mathrm{Pic}^0_{C'_i}$ such that for the non-trivial element σ_i in the Galois group of $C'_i \to C$ we have $\sigma_i^* L'_i \not\cong L'_i$. Note that $\pi_i^* V \cong L'_i \oplus \sigma^* L'_i$.

By construction $C_{2,split}$ dominates C'_1 and C'_2. Consider the decomposition

$$V_{|C_{2,split}} \cong L'_{i|C_{2,split}} \oplus \sigma_i^* L'_{i|C_{2,split}}, i = 1,2$$

and the stabilizer $H_i \subseteq G_{2,split}$ of $L'_{i|C_{2,split}}$. Note that H_i coincides with the Galois group of $C_{2,split} \to C'_i$. As $L'_{1|C_{2,split}} \cong L'_{2|C_{2,split}}$ or $L'_{1|C_{2,split}} \cong \sigma_2^* L'_{2|C_{2,split}}$, we find that $H_2 \subseteq H_1$ and vice versa. However, $H_1 = H_2$ implies that $C'_1 \cong C'_2$ which was excluded by assumption.

To show that the closures $\overline{Z_{C'}}$ in $M_C^{ss,2,0}$ are disjoint observe that $\overline{Z_{C'}} = \pi_*(\mathrm{Pic}^0_{C'})$ and that $\overline{Z_{C'}} \cap M_C^{s,2,0} = Z_{C'}$. Keep the notation for the disjointness of $Z_{C'}$ and consider $V \in \overline{Z_{C'_1}} \cap \overline{Z_{C'_2}}$. As we already know that $Z_{C'_1} \cap Z_{C'_2} = \emptyset$, it suffices to consider the case where V is not stable. Then $V \cong \pi_* L'_i$ for a line bundle L'_i of degree 0 on C'_i such that $\sigma_i^* L'_i \cong L'_i$ for $i = 1,2$. By Lemma 2.5.4, there exists a line bundle L_i on C such that $L'_i \cong L_{i|C'_i}$ for $i = 1,2$.

Recall that $\pi_{i,*}\mathcal{O}_{C'_i} \cong \mathcal{O}_C \oplus M_i$ for a line bundle M_i of order 2, see Lemma 2.4.1. As $C'_1 \ncong C'_2$, the line bundles M_1 and M_2 are non-isomorphic. By the projection formula, we find

$$V \cong \pi_{i,*}\pi_i^* L_i \cong L_i \oplus (L_i \otimes M_i),\ i = 1,2.$$

As all the line bundles appearing are of degree 0, we conclude

$$L_1 \cong L_2 \text{ and } L_1 \otimes M_1 \cong L_2 \otimes M_2 \text{ or}$$

$$L_1 \cong L_2 \otimes M_2 \text{ and } L_2 \cong L_1 \otimes M_1.$$

In both cases, we find $M_1 \cong M_2$ using that $M_i^{\otimes 2} \cong \mathcal{O}_C$ - a contradiction.

This also shows that

$$(\mathrm{Pic}^0_C \oplus \mathrm{Pic}^0_C) \cap \overline{Z_{C'}} = \Delta_{M'}(\mathrm{Pic}^0_C).$$

Line bundles of order 2 correspond on the one hand to elements in $\mathrm{Pic}^0_C[2] \setminus \{\mathcal{O}_C\}$ and on the other hand to connected μ_2-torsors over C. As $\mathrm{Pic}^0_C[2] \cong (\mathbf{Z}/2\mathbf{Z})^{2g_C}$ we conclude the description of the components of dimension $2(g_C - 1) + 1$.

It remains to show that if $\mathrm{Pic}^0_C \otimes V$ has non-empty intersection with $Z_{C'}$ or $\mathrm{Pic}^0_C \otimes V'$, then it is contained in $Z_{C'}$ or $\mathrm{Pic}^0_C \otimes V'$. Both assertions are clear as tensoring with a line bundle of degree 0 induces an automorphism of $Z_{C'}$ and of $\mathrm{Pic}^0_C \otimes V'$. □

6 Appendix

We spell out [21, Step 3, p.143]:

Lemma 6.0.1. *Let $\mathcal{C}$ be a site and $1 \to A \to B \xrightarrow{\varphi} C \to 1$ be a short exact sequence of sheaves of (possibly non-commutative) groups. Then for $U \in \mathcal{C}$ we obtain a truncated long exact sequence*

$$1 \longrightarrow \check{H}^0(U,A) \longrightarrow \check{H}^0(U,B) \longrightarrow \check{H}^0(U,C) \longrightarrow \check{H}^1(U,A) \longrightarrow \check{H}^1(U,B) \longrightarrow \check{H}^1(U,C),$$

where the 0-th cohomology groups are groups while the first cohomology groups are pointed sets.

If A is central in B, then $\check{H}^1(B)(U)$ admits an $\check{H}^1(A)(U)$-action and two elements $b, b' \in \check{H}^1(B)(U)$ map to the same element in $\check{H}^1(C)(U)$ if and only if there exists an $a \in \check{H}^1(A)(U)$ such that $ab = b'$. Furthermore, the long exact sequence can be extended to $\check{H}^2(A,U)$.

Proof. This is proven in [21, p.122, Proposition 4.5] except for the "A central in B" part. Assume in the following that A is central in B.

Let $\mathcal{U} = (U_i \to U)_{i\in I}$ be a cover of U in $\mathcal{C}$. For i,j in I define $U_{ij} := U_i \times_U U_j$, similarly we define U_{ijk}.

For $a \in \check{H}^1(A)(\mathcal{U})$ and $b \in \check{H}^1(B)(\mathcal{U})$ define $a \cdot b := (a_{ij}b_{ij})_{ij}$, where $(a_{ij})_{ij}$ (resp. $(b_{ij})_{ij}$) is a representative of a resp. b. To see that this is well-defined note that for $(a_i) \in \prod_i A(U_i)$ and $(b_i) \in \prod_i B(U_i)$ we have

$$a_i a a_j^{-1} \cdot b_i b b_j^{-1} = a_i a_{ij} a_j^{-1} b_i b_{ij} b_j^{-1} = a_i b_i a_{ij} b_{ij} (a_j b_j)^{-1}$$

using that A lies central in B. Thus, we have defined an action of $\check{H}^1(A)(\mathcal{U})$ on $\check{H}^1(B)(\mathcal{U})$ which clearly is compatible with restriction along a refinement. Taking colimits we obtain an action of $\check{H}^1(A)(U)$ on $\check{H}^1(B)(U)$.

Let $b, b' \in \check{H}^1(B)(U)$ map to the same element $c \in \check{H}^1(C)(U)$. Then there is a cover $\mathcal{U} = (U_i \to U)$ such that b (resp. b') lift to elements b (resp. b') in $\check{H}^1(B)(\mathcal{U})$ which map to $c \in \check{H}^1(C)(\mathcal{U})$. Choose representatives $b = (b_{ij})_{ij}, b' = (b'_{ij})_{ij}$, and

$c = (c_{ij})_{ij}$. As b and b' both map to c there exists $(c_i) \in \prod_i C(U_i)$ such that $c_i\varphi(b_{ij})c_j^{-1} = \varphi(b'_{ij})$. After refining the cover $\mathcal{U}$ we can assume that there is $(b_i) \in \prod B(U_i)$ mapping to (c_i). We obtain $\varphi(b_i b_{ij} b_j^{-1} b'^{-1}_{ij}) = 1 \in C(U_{ij})$, i.e.,

$$a_{ij} := b_i b_{ij} b_j^{-1} b'^{-1}_{ij} \in A(U_{ij}).$$

As A is central, this defines an element $a = (a_{ij}) \in \check{H}^1(A)(\mathcal{U})$ and thus in $H^1(A)(U)$. We claim $a \cdot b' = b \in \check{H}^1(B)(U)$. Indeed,

$$a_{ij} b'_{ij} = b_i b_{ij} b_j^{-1} b'^{-1}_{ij} b'_{ij} = b_i b_{ij} b_j^{-1} \sim b_{ij}.$$

Conversely, if $a \cdot b' = b$, then b and b' map to the same c.

We define a map $\partial : \check{H}^1(U, C) \to \check{H}^2(U, A)$. Let $\mathcal{U} = (U_i \to U)_{i \in I}$ be a cover of U. An element $c \in \check{H}^1(\mathcal{U}, C)$ is represented by $(c_{ij}) \in \prod_{i,j \in I} C(U_{ij})$. After possibly refining $\mathcal{U}$ we find lifts $b_{ij} \in B(U_{ij})$ of $c_{ij} \in C(U_{ij})$. Then the element $a_{ijk} := b_{ij} b_{jk} b_{ik}^{-1} \in B(U_{ijk})$ maps to $1 \in C(U_{ijk})$ and defines an element in $A(U_{ijk})$. We show that $(a_{ijk}) \in \check{H}^2(\mathcal{U}, A)$ by checking the 2-cocycle condition. For $i, j, k, l \in I$ we have

$$\begin{aligned} a_{jkl} a_{ikl}^{-1} a_{ijl} a_{ijk}^{-1} &= \\ (b_{jk} b_{kl} b_{jl}^{-1})(b_{ik} b_{kl} b_{il}^{-1})^{-1}(b_{ij} b_{jl} b_{il}^{-1})(b_{ij} b_{jk} b_{ik}^{-1})^{-1} &= \\ (b_{jk} b_{kl} b_{jl}^{-1})(b_{il} b_{kl}^{-1} b_{ik}^{-1})(b_{ij} b_{jl} b_{il}^{-1})(b_{ik} b_{jk}^{-1} b_{ij}^{-1}) &= \\ (b_{il} b_{kl}^{-1} b_{ik}^{-1})(b_{ik} b_{jk}^{-1}((b_{jk} b_{kl} b_{jl}^{-1})) b_{ij}^{-1})(b_{ij} b_{jl} b_{il}^{-1}) &= 1 \end{aligned}$$

using that A lies central in B. Observe that $(a_{ijk}) \in \check{H}^2(\mathcal{U}, A)$ is independent of the choice of lifts b_{ij}. Considering (a_{ijk}) as an element in $\check{H}^2(U, A)$ this is also independent of the refinement of $\mathcal{U}$. We obtain a map $\check{H}^1(\mathcal{U}, C) \to \check{H}^2(U, A)$. Passing to the colimit we obtain a the desired map $\partial : \check{H}^1(U, C) \to \check{H}^2(U, A)$.

It remains to check exactness at $\check{H}^1(U, C)$. Consider $c \in \check{H}^1(U, C)$ mapping to the trivial class in $\check{H}^2(U, A)$. We can represent c by $(c_{ij}) \in \prod_{i,j \in I} C(U_{ij})$ on some cover $\mathcal{U} = (U_i \to U)_{i \in I}$ such that there exist lifts $b_{ij} \in B(U_{ij})$. Then (b_{ij}) defines a class in $\check{H}^1(\mathcal{U}, B)$. Indeed, $(a_{ijk}) := \partial(c) = (b_{ij} b_{jk} b_{ik}^{-1})$ measures exactly the failure of (b_{ij}) to be a 2-cocycle. By assumption (a_{ijk}) is trivial as a class in $\check{H}^2(U, A)$, i.e., after possibly refining $\mathcal{U}$ there exists $(a'_{ij}) \in \prod_{i,j \in I} A(U_{ij})$ such that $a_{ijk} = a'_{ij} a'_{jk} (a'_{ik})^{-1} \in A(U_{ijk})$. Using the centrality of A in B we find that $((a'_{ij})^{-1} b_{ij})$ defines a class in $\check{H}^1(\mathcal{U}, B)$ mapping to (c_{ij}). Thus, we find $b \in \check{H}^1(U, B)$ mapping to c and conclude. □

Lemma 6.0.2. *Let X be a scheme of finite type over an algebraically closed field k of cardinality κ. Assume that X has positive dimension. Then for a family $(Z_l)_{l\in\lambda}$ of closed subsets of X of codimension ≥ 1 we have that $X \setminus \bigcup_{l\in\lambda} Z_l$ contains a closed point if $\lambda < \kappa$.*

Proof. As a finite type scheme over k has only finitely many irreducible components, we reduce to the case that X is a irreducible. The lemma is clear for finite λ and we further assume that λ is infinite.

Any closed subset of an algebraic scheme only has finitely many irreducible components and it suffices to show the lemma for a family of irreducible closed subsets of codimension ≥ 1.

Let us first proof the result for $\mathbf{A}_k^n$ and $n \geq 1$ by induction. For $n = 1$ we want to show that a union of $\lambda < \kappa$ many closed points does not contain all closed points of $\mathbf{A}_k^1$. This is true as closed points are in bijection to elements of k which has cardinality κ.

Induction step by contradiction: Assume the lemma for $\mathbf{A}_k^n$ and some $n \geq 1$. Further, assume that $\mathbf{A}_k^{n+1} \setminus \bigcup_{l\in\lambda} Z_l$ contains no closed points, where the Z_l are closed irreducible subsets in $\mathbf{A}_k^{n+1}$ of codimension ≥ 1. Consider for a closed point $x \in \mathbf{A}_k^1$ the morphism

$$\varphi_x : \mathbf{A}_k^n \to \mathbf{A}_k^{n+1}, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x).$$

Then $(\varphi_x^{-1}(Z_l))_{l\in\lambda}$ is a family of closed subsets of cardinality $\lambda < \kappa$ such that $\mathbf{A}_k^n \setminus \bigcup_{l\in\lambda} \varphi_x^{-1}(Z_l)$ contains no closed points. By the induction hypothesis this implies that $\varphi_x^{-1}(Z_{l_x}) = \mathbf{A}_k^n$ for some $l_x \in \lambda$. The irreducibility of Z_{l_x} and $\operatorname{codim}(Z_{l_x}) \geq 1$ imply $Z_{l_x} = \mathbf{A}_k^n \times \{x\}$.

Mapping a closed point $x \in \mathbf{A}_k^1$ to l_x we obtain a map $\varphi : \kappa \to \lambda$. As $\lambda < \kappa$ the map φ can not be injective. A contradiction to $\mathrm{pr}_{n+1}(Z_{l_x}) = \{x\}$, where pr_{n+1} denotes the $n + 1$-th projection.

In the general case, let U be a non-empty affine open of X. Then U has the same dimension $d \geq 1$ as X. By the Noether normalization lemma there is a finite morphism $\varphi : U \to \mathbf{A}_k^d$. We claim that φ is surjective. Indeed, the image of φ has dimension d, is closed, and $\mathbf{A}_k^d$ is irreducible. Then the lemma for $\mathbf{A}_k^d$ implies the lemma for U and thus for X. □

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