ON THE MARTINGALE PROPERTY
OF TIME-HOMOGENEOUS DIFFUSIONS

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Abstract. We study the martingale property of the positive diffusion

$$dX_t = \sigma(X_t)dB_t.$$

We prove that X is a true martingale (not only a local martingale) if and only if the function $x/\sigma^2(x)$ is NOT integrable near infinity.

Key words and phrases. Bubbles, CEV model, local volatility model, martingales, local martingales.

1 Introduction

One of the models for the risk-neutral dynamics of an asset, which is popular both in academy and in industry is the *local volatility model* proposed by Dupire [12]. In this model, the discounted asset price process is the solution of the stochastic differential equation (SDE)

$$dX_t = \sigma(t, X_t)dB_t, \quad X_0 = x_0 \tag{1}$$

with a certain function $\sigma(t, x)$. At least in the theory, this model is able to reproduce exactly the prices of all European call options on any asset with all strikes and maturities.

However, the practical calibration of the local volatility model might be problematic especially for small maturities. In this respect one might be interested in dealing with more robust models. One possibility is to consider the time-averaged variant of (1), i.e. to model the price as the solution of the time-homogeneous SDE

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = x_0. \tag{2}$$

An additional motivation for time-averaging is that it is already known that some models with time-varying parameters can be approximated very accurately by the same models with time-independent parameters; see Piterbarg [24], where this was discovered for the Heston model.

A particular case of (2) is the *constant elasticity of variance (CEV)* model:

$$dX_t = |X_t|^\alpha dB_t, \quad X_0 = x_0. \tag{3}$$

Here $\alpha \in \mathbb{R}$ and $x_0 > 0$ (it is known that there exists a solution of (3) and it is unique among positive solutions). This model was first considered by Cox [6] for $\alpha \leq 1$ and by Emanuel and MacBeth [13] for $\alpha > 1$. It is quite popular in mathematical finance because of its analytical tractability: the prices of various options admit explicit analytic expressions in this model; see Schroder [27], Davydov and Linetsky [7], Delbaen and Shirakawa [10], Carr and Linetsky [2].

Processes (1)–(3) are continuous local martingales. In typical models, they are positive. A positive local martingale is a supermartingale but not necessarily a martingale. It is a martingale if and only if $\mathsf{E}X_t = x_0$ for any $t \geq 0$. The latter property is clearly necessary for any reasonable model since $\mathsf{E}X_t$ means the model price of the contingent claim that pays out the amount X_t at time t. Since this contingent claim is nothing but the underlying asset, its model price should be x_0.

Thus, an important problem is to know whether the process given by the above described dynamics is a martingale. For CEV, the answer is known. Namely, the CEV process is a martingale for $\alpha \leq 1$ and is a strict local martingale for $\alpha > 1$. This was first noted by Emanuel and MacBeth [13]. Their method is based on expressing the CEV process through a Bessel process; for the latter one, the marginal densities are known explicitly, so one can simply test the equality $\mathsf{E}X_t = x_0$.

The goal of this paper is to study the martingale property for diffusions (2). We assume that $x_0 > 0$ and σ satisfies the conditions: $\sigma \neq 0$ on $(0, \infty)$, $\sigma^{-2} \in L^1_{\text{loc}}(0, \infty)$ (i.e. $\int_a^b \sigma^{-2}(x)dx < \infty$ for any $[a, b] \subset (0, \infty)$), and $\sigma = 0$ on $(-\infty, 0]$ (the latter property indicates that we are interested only in positive solutions). These assumptions guarantee the existence and uniqueness of a weak solution (see [15]). It is clearly positive as it stops at the first time it hits zero.

In what follows, we write formulas like $\int^\infty f(x)\,dx < \infty$ in the situations when the convergence or divergence of the integral $\int_c^\infty f(x)\,dx$ does not depend on $c \in (0, \infty)$.

The main result of our paper is

Theorem 1.1. *X is a martingale if and only if* $\int^{\infty} x/\sigma^2(x)\, dx = \infty$.

Remark 1.2. It is known that if $\int^{\infty} x/\sigma^2(x)\, dx = \infty$, then ∞ is a natural boundary for X; if $\int^{\infty} x/\sigma^2(x)\, dx < \infty$, then ∞ is an entrance boundary for X (for example, this can be seen from [26; Ch. V, Th. 51.2 (iv)]). Hence, X is a martingale if and only if ∞ is a natural boundary for X.

In particular, we obtain the following corollary, which in turn implies the result of Emanuel and MacBeth mentioned above.

Corollary 1.3. (i) *If there exists $c > 0$ such that $|\sigma(x)| \leq cx$ for sufficiently large x, then X is a martingale.*
(ii) *If there exist $c > 0$, $\alpha > 1$ such that $|\sigma(x)| \geq cx^{\alpha}$ for sufficiently large x, then X is a strict local martingale.*

After writing this paper we found out that Theorem 1.1 was earlier proved by Delbaen and Shirakawa [11] under a stronger assumption that the functions σ and $1/\sigma$ are locally bounded on $(0, \infty)$. However, our proof is completely different.

2   Martingales Vs Local Martingales

In discrete time models, arbitrage theory suggests densities of equivalent martingale measures as price deflators. In continuous time, however, there is no unanimity about the nature of state price deflators as the two possible classes are local martingales and true martingales. The nature of the price deflators is typically suggested by the fundamental theorem of asset pricing. In continuous time, there exist different variants of this theorem. The papers by Harrison and Kreps [17], Sin [28], Yan [30], and Cherny [3] link the absence of (appropriately defined) arbitrage opportunities to the existence of an equivalent measure, under which the discounted price process is a true martingale. In contrary, the papers by Delbaen and Schachermayer [8], [9] link the absence of (appropriately defined) arbitrage opportunities to the existence of a local martingale measure.[1]

Let us illustrate the difference between local martingale and true martingale measures by two examples.

Example 2.1.[2] Consider the discounted price process $(S_t)_{t \leq 1}$ defined as follows

$$S_t = \begin{cases} \xi_1 \ldots \xi_n, & t < 1, \\ 0, & t = 1, \end{cases}$$

where n is chosen such that $1 - 2^{-n} \leq t < 1 - 2^{-n-1}$. This process is a local martingale under the original measure, but not a martingale. Moreover, there exists no equivalent martingale measure since $S_0 = 1$, while $S_1 = 0$.

This model is not arbitrage free in the framework of [17], [28], [30], and [3]. The arbitrage in those approaches consists in selling the underlying short at time 0 and buying it back at time 1. However, the model is arbitrage-free within the framework of Delbaen and Schachermayer. The above short-selling strategy is ruled out in their framework because its capital process $W_t = -S_t + S_0$ is not bounded below. It is termed non-admissible. But we would like to point out several problems associated with the admissibility condition:

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[1] To be more precise, in the most general case it links the absence of arbitrage to equivalent sigma-martingale measures, but if the price process is positive, any sigma-martingale measure is necessarily a local martingale measure.

[2] This example was proposed to us by Walter Schachermayer.

- This condition prohibits the strategy of selling short the asset at time 0 and buying it back at time 1 in virtually any model, in particular, in the Black-Scholes-Merton one.
- In the exponential Lévy model $S_t = e^{L_t}$, where L is a Lévy process, whose Lévy measure has unbounded support, the admissibility condition would prohibit any short sales. In particular, this is true in the Variance-Gamma model, the CGMY model, etc.
- In the linear Lévy model $S_t = L_t$, where L is a Lévy process, whose Lévy measure has unbounded support, this condition would prohibit all the long and short operations.
- In the discrete-time conditionally Gaussian model (i.e. Law$(\ln S_n/S_{n-1} \mid S_0, \ldots, S_{n-1})$ is Gaussian), the admissibility condition prohibits all the short sales. Moreover, we want to stress that this condition is never employed in discrete time, which would not be the case if it was really economically indispensable.

Let us mention that this example is very similar to the model, in which S is equal to the process $1 + \int_0^t (1-u)^{-1} dB_u$ (B is a Brownian motion) stopped at the first time it hits zero (this example was considered in [18; Sect. 3.3]).

Example 2.2 (CEV). Consider the CEV model (3) with $\alpha > 1$. The filtration $(\mathcal{F}_t)$ is taken as the natural filtration of S. The process S is then a local martingale being a stochastic integral with respect to Brownian motion. However, it is not a martingale because $E[S_t] < S_0$ (see Emanuel and MacBeth [13]). Moreover, there exists no equivalent measure, under which it is a martingale. Indeed, if such a measure $\widetilde{P}$ existed, then under this measure S should have quadratic variation $\langle S \rangle_t = \int_0^t S_u^{2\alpha} du$, so that it should satisfy the SDE $dS_t = S_t^\alpha d\widetilde{B}_t$ with a $\widetilde{P}$-Brownian motion $\widetilde{B}$. But the solution to this SDE is unique in law, so that the distributions of S under $\widetilde{P}$ and the original measure P should coincide. In particular $E_{\widetilde{P}}[S_1] < S_0$, so that S cannot be a $\widetilde{P}$-martingale.

Consider now the contract, which pays out at time 1 the discounted amount $C = S_1$. This contract is replicable at the price $E[C]$. Indeed, the process $M_t = E[C \mid \mathcal{F}_t]$ is an $(\mathcal{F}_t, P)$-martingale. Due to Ito's theorem, S is a strong solution of (3) (as the function $S \mapsto S^\alpha$ is locally Lipschitz), i.e. $(\mathcal{F}_t)$ is included in the natural filtration $(\mathcal{F}_t^B)$ of B. On the other hand, the reverse inclusion is true due to (3), so that $\mathcal{F}_t = \mathcal{F}_t^B$. By the Brownian representation theorem, M can be represented as $M_t = M_0 + \int_0^t H_u dB_u$ with a certain process H. As S is strictly positive (see [13]), this can be further rewritten as $M_t = M_0 + \int_0^t \widetilde{H}_u dS_u$ with $\widetilde{H}_t = H_t/S_t^\alpha$. Thus, the replication price of the contract is $E[C]$.

On the other hand, the payoff of the contract is nothing but the time-1 value of the underlying, so that the common sense (accompanied with the Law of One Price) suggests the price of the contract being equal to S_0. However, as mentioned above, $E[C] < S_0$. Again, in the frameworks of [17], [28], [30], and [3], this "paradox" is resolved very simply: as there is no equivalent martingale measure, this model admits arbitrage in those frameworks.

One possible look at the models of the above examples could be as to an economy with a bubble. Bubbles have recently attracted attention in the mathematical finance literature; see Cox and Hobson [5], Jarrow, Protter, and Shimbo [20], Heston, Loewenstein, and Willard [18]. Option pricing in such models becomes a very delicate task. Taking simply the conditional expectation of the option's payoff (termed the fair price or the fundamental price) might not be reasonable because then, for example, the price of a call option at some intermediate dates might be less than its intrinsic value, which is hardly possible in practice. Emanuel and Mac-Beth [13], Cox and Hobson [5], Heston, Loewenstein and Willard [18], and Madan and Yor [23] propose alternative ways to price call options in models with bubbles. However, then another problem arises that the price of a call does not tend to zero as the strike approaches infinity. Furthermore, as pointed out in [18], in this case American options have no optimal exercise

time, while lookback options have infinite value. As the authors of [18] write, "these counter-factual implications provide a persuasive rationale for avoiding bubbles in many models". In this language, the study we perform in this paper is the study of the absence of bubbles in time-homogeneous diffusion models.

Let us mention that the same problem for stochastic volatility models was studied by Sin [29], Andersen and Piterbarg [1]; see also the discussion in Lewis [22; Ch. 9]. The method of study employed there is based on linking the martingale property of the original process to the non-explosion property of another process. In our proof we also follow a similar path (although the model we are considering is different).

3 Proof of Theorem 1.1

We assume without loss of generality that we work in the following canonical setting. Namely $\Omega = \overline{C}(\mathbb{R}_+)$ is the space of all continuous functions $\omega\colon \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ with the following property: there exists time $\zeta(\omega) \in (0, \infty]$ such that ω is continuous and $\mathbb{R}$-valued on $(0, \zeta(\omega))$ as well as $\omega = +\infty$ on $[\zeta(\omega), \infty)$ and $\lim_{t \uparrow \zeta(\omega)} \omega(t) = +\infty$ whenever $\zeta(\omega) < \infty$ (i.e. Ω is the space of trajectories that may explode near $+\infty$). Further, let X be the coordinate process on Ω (i.e. $X_t(\omega) = \omega(t)$) and ζ be the explosion time of X. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(X_s; s \leq t + \varepsilon)$ and the σ-field $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. Finally, let the measure P be the (unique) solution of the martingale problem associated with SDE (2). In other words, P is the distribution (on $(\Omega, \mathcal{F})$) of a weak solution of (2). Let us note that X does not explode under P (however, it will be convenient below that we work with $\overline{C}(\mathbb{R}_+)$ rather than with $C(\mathbb{R}_+)$). We need to prove that X is an $(\mathcal{F}_t, \mathsf{P})$-martingale if and only if $\int^\infty x/\sigma^2(x)\, dx = \infty$.

For $a \in [0, \infty)$, we set
$$\tau_a(X) = \inf\{t \geq 0\colon X_t = a\}$$
(as usual $\inf \emptyset = \infty$). The process
$$Y_t = \int_0^t \frac{1}{X_u}\, dX_u$$
is well defined under the measure P on the stochastic interval $[0, \tau_0(X))$ and is a continuous local martingale on it. We have

$$X_t = x_0 \exp\left\{Y_t - \frac{1}{2}\langle Y \rangle_t\right\}$$
$$= x_0 \exp\left\{\int_0^t \kappa(X_u)\, dB_u - \frac{1}{2}\int_0^t \kappa^2(X_u)\, du\right\} \quad \text{on } [0, \tau_0(X)) \tag{4}$$

with $\kappa(x) = \sigma(x)/x$. Let us note that $X_t \to 0$ P-a.s. as $t \uparrow \tau_0(X)$ (on the set $\{\tau_0(X) = \infty\}$ this follows from the construction of solutions of (2); see [15]). Hence, we get from (4) by the Dambis-Dubins-Schwartz theorem (see [25; Ch. V, (1.6) and (1.18)]) that

$$\langle Y \rangle_{\tau_0(X)} = \int_0^{\tau_0(X)} \kappa^2(X_u)\, du = \infty \quad \mathsf{P}\text{-a.s.}$$

For $n \in \mathbb{N}$, set
$$\rho_n(X) = \inf\left\{t \geq 0\colon \int_0^t \kappa^2(X_u)\, du \geq n\right\}.$$

We have $\rho_n(X) \uparrow \tau_0(X)$ P-a.s. and $\rho_n(X) < \tau_0(X)$ P-a.s. for any $n \in \mathbb{N}$. The Novikov criterion and (4) yield that the stopped processes $X^{\rho_n(X)} = (X_{t \wedge \rho_n(X)})$ are positive $(\mathcal{F}_t, \mathsf{P})$-martingales.

Let Q^n denote the probability measure on $(\Omega, \mathcal{F})$ that has the density process $X^{\rho_n(X)}/x_0$ with respect to P. This measure exists because we work in the canonical setting. By the Girsanov theorem for local martingales (see [19; Ch. III, Th. 3.11]), the process

$$M_t^n = X_t - \int_0^{t \wedge \rho_n(X)} \frac{\sigma^2(X_u)}{X_u}\, du$$

is a continuous $(\mathcal{F}_t, Q^n)$-local martingale starting from x_0 with

$$\langle M^n \rangle_t = \int_0^t \sigma^2(X_u)\, du.$$

Hence, Q^n is a solution of the martingale problem associated with the SDE

$$dZ_t = \frac{\sigma^2(Z_t)}{Z_t} I(t \le \rho_n(Z))\, dt + \sigma(Z_t)\, dB_t, \quad Z_0 = x_0. \tag{5}$$

Now consider the SDE

$$dZ_t = \frac{\sigma^2(Z_t)}{Z_t}\, dt + \sigma(Z_t)\, dB_t, \quad Z_0 = x_0. \tag{6}$$

Since the functions $\mu(x) = \sigma^2(x)/x$, $x \in (0, \infty)$, and σ satisfy the conditions $\sigma \ne 0$ on $(0, \infty)$ and $(1 + |\mu|)/\sigma^2 \in L^1_{\mathrm{loc}}(0, \infty)$, equation (6) has a unique in law $(0, \infty)$-valued weak solution that may either stop at zero or explode at $+\infty$ (see [14], [16] or [21; Ch. 5, Th. 5.15 and Sec. 5.C]). A simple computation shows that the scale function of a solution of (6) equals $-\infty$ at 0, hence, it does not reach zero. Thus, SDE (6) has a unique in law strictly positive weak solution that may explode at $+\infty$. We denote by Q the probability measure on $(\Omega, \mathcal{F})$ which is the unique solution of the martingale problem associated with (6).

For $n \in \mathbb{N}$, set

$$\eta_n(X) = \inf\left\{ t \ge 0\colon \int_0^t \left(\frac{I(0 < X_u < 1)}{X_u^2} + I(X_u \ge 1) \right) \sigma^2(X_u)\, du \ge n \right\}.$$

Clearly, $\eta_n(X) \le \rho_n(X)$ P, Q-a.s. One can easily see that

$$\eta_n(X) \uparrow \tau_0(X) \quad P\text{-a.s.} \tag{7}$$

We have (see (6))

$$\int_0^\zeta \left(\frac{\sigma^2(X_u)}{X_u} + \sigma^2(X_u) \right) du = \infty \quad Q\text{-a.s.}$$

Since $X_t \to \infty$ Q-a.s. as $t \uparrow \zeta$, we get

$$\int_0^\zeta \sigma^2(X_u)\, du = \infty \quad Q\text{-a.s.}$$

and, consequently,

$$\eta_n(X) \uparrow \zeta \quad Q\text{-a.s. and } \eta_n(X) < \zeta \quad Q\text{-a.s. for any } n \in \mathbb{N}. \tag{8}$$

It follows from (5) and (6) that the restrictions $Q^n|_{\mathcal{F}_{\eta_n(X)}}$ and $Q|_{\mathcal{F}_{\eta_n(X)}}$ are *solutions of* (6) *up to* $\eta_n(X)$ in the sense of Definition 1.31 in [4] (note that $\eta_n(X) < \zeta$ Q, Q^n-a.s. for any $n \in \mathbb{N}$).

Similarly to Theorem 2.11 in [4] one can prove that solution of (6) up to $\eta_n(X)$ is unique. Hence,

$$\mathsf{Q}^n|\mathcal{F}_{\eta_n(X)} = \mathsf{Q}|\mathcal{F}_{\eta_n(X)}.$$

Note that

$$\frac{d\mathsf{Q}^n}{d\mathsf{P}}\bigg|\mathcal{F}_{t\wedge\eta_n(X)} = \frac{1}{x_0}\mathsf{E}_\mathsf{P}(X_{t\wedge\rho_n(X)}|\mathcal{F}_{t\wedge\eta_n(X)}) = \frac{1}{x_0}X_{t\wedge\eta_n(X)}.$$

Applying (7) and (8) we get

$$\mathsf{E}_\mathsf{P}X_t = \mathsf{E}_\mathsf{P}X_t I(\tau_0(X) > t) = \lim_{n\to\infty}\mathsf{E}_\mathsf{P}X_t I(\eta_n(X) > t)$$

$$= x_0\lim_{n\to\infty}\mathsf{Q}^n(\eta_n(X) > t) = x_0\lim_{n\to\infty}\mathsf{Q}(\eta_n(X) > t) = x_0\mathsf{Q}(\zeta > t).$$

Thus, X is an $(\mathcal{F}_t, \mathsf{P})$-martingale if and only if X is nonexplosive under the measure Q. Applying to SDE (6) Feller's test for explosions (see [4; Cor. 4.4] or [21; Ch. 5, Th. 5.29]) we obtain the result.

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