Essays on Regime Uncertainty and Time Inconsistency in Optimal Asset Allocation Problems

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Contents

List of Abbreviations

CE	Certainty equivalent
CM	Constant mix
CRRA	Constant relative risk aversion
DC	Dynamically consistent
EU	Expected utility
EPU	Economic Policy Uncertainty Index
GDP	Gross domestic product
GT	Game theoretic
HD	Hyperbolic discounting
HMC	Hidden Markov chain
MC	Markov chain
MV	Mean variance
OMC	Observable Markov chain
PC	Pre-commitment
QHD	Quasi hyperbolic discounting
RS	Regime switching
RU	Regime uncertainty
TI	Time inconsistency
SR	Savings rate
VAR	Vector autoregression
VIX	CBOE Volatility Index
VoI	Value of information

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Chapter 1

General introduction

1.1 Relevance and problem

Individuals, households, companies and governments can act as investors and make investment decisions with regard to certain asset classes (e.g., stocks, bonds, real estate, cash) and the amount to be invested in these asset classes. Decisions are made by considering individual risk preferences and the risk-return relation of assets, which determines the expected portfolio performance. Investors also consider diversification and hedging. The aim of asset allocation is to improve the risk-return profile of the managed portfolio and construct a portfolio whose characteristics satisfy the investor's demand while achieving a surplus of the invested wealth in form of a return. Due to historically low interest rates and demographic change, asset allocation has become more relevant in the last years. From a real interest rate point of view, it is no longer worthwhile to deposit wealth in a risk-free manner in a bank account. Due to the low pension entitlement, private provision can be necessary and underlines the importance of asset allocation in savings. To build up retirement savings, wealth must be invested in the long term and a suitable portfolio should be generated (cf. Brinson et al. (1991)).

Increasing globalization, internationalization, digitalization and technological change are making interaction in the world (e.g., investment decisions) fast and more complex. As a result, policymakers change the environment for investors more often. This type of uncertainty can be referred to as regime uncertainty and has increased over time (cf. Baker et al. (2014)). Due to global interconnectedness and the resulting correlation of assets and associated risks, uncertainty in one country can have international implications.

On the one hand, decision-makers respond to crises by changing regulatory requirements. In addition to the uncertain size, scope and content of a regulatory change, uncertainty exists about the timing and specific design of future regulatory changes. On the other hand, increasing political polarization and its impact on the political decision-making process increases uncertainty. Nationalism and protectionism lead to uncertainties about changes in regulatory and political discourse and thus to economic divergences. Insufficient knowledge of the impact of regime uncertainty on investors' asset allocation may influence the effectiveness of policy and regulatory measures. There may be unintended consequences of new frameworks and policy targets may be missed (cf. Ranaldo et al. (2021)). Due to global interconnectedness and the resulting correlation of assets and associated risks, uncertainty in one country can have international implications. Spillover effects can also work across markets (stock market, commodity market, and oil market) (cf. Tan et al. (2020), Finta and Aboura (2020)).

Since the last financial crisis and the European sovereign debt crisis, the importance of heightened regime uncertainty for economic development has been discussed more intensively by researchers and the public. According to the International Monetary Fund, uncertainty about regulatory policy contributed to a severe economic downturn in Europe and the United States in 2008 & 2009 and influenced investor behavior to make investments in times of lower uncertainty and higher information (cf. Julio and Yook (2012), Baker et al. (2016)). Regulatory decisions, such as changes in regulatory frameworks (e.g., Basel III, Solvency II), changes in guarantee levels in life insurance or changes in capital requirements of financial institutions have an impact on investment decisions and influence asset prices.

The global spread of COVID-19 shows how regime uncertainty can affect the worldwide economy. The pandemic has affected both financial markets and the real economy around the world and has led to extensive monetary and fiscal policy interventions. Spillover effects have intensified during this period. Furthermore, the pandemic has increased the systemic risk of banks in all countries (cf. Apostolakis et al. (2021), (Goldstein et al. (2021)), Al-Thaqeb et al. (2022), Duan et al. (2021)). Huber et al. (2021) find that extreme events like COVID-19 change investors' risk preferences. Low investments during crisis, when enhanced regime uncertainty prevails, is due to higher investors risk aversion. Also, the Russia-Ukraine war and related policy actions affect asset prices and investor behavior.

Another example of regime uncertainty is the current debate over climate policy.

Country-specific emission programs are being developed to reduce greenhouse gases and encourage the development of low-carbon technologies. Since investments in the energy industry are long-term and tend to only pay off after 15-20 years, regime uncertainty poses major challenges for companies and investors in the energy industry when developing strategies. Stroebel and Wurgler (2021) surveyed economists, finance academics, and regulators on climate finance challenges and conclude that regulatory risk is seen as the greatest climate risk for firms and investors in the upcoming years.

News and announcements are also an important driver of regime uncertainty. Stock return jumps are significantly related to the frequency and content of news. The effects have increased in recent decades (cf. Jeon et al. (2022)). The level of information affects the sentiment of market participants, which in turn affects the dynamics of asset prices. Information is processed individually and individual errors in perception may occur (cf. John and Li (2021)). Decisions have to be made in times of information gaps and uncertainty, which makes risk assessment decisive. The impact of regime uncertainty at the individual and macroeconomic level is crucial for long-term investment decisions. Due to regime uncertainty, asset price dynamics can change significantly and over several periods. Therefore, it is essential to consider possible structural market changes and changing economic conditions in asset allocation decisions (cf. Ang and Bekaert (2002), Ang and Bekaert (2004), Guidolin and Timmermann (2008)).

Decisions under uncertainty, especially regime uncertainty, pose particular challenges to the rationality of decision-makers. In economic decision theory, rationality is usually associated with the transitivity of preferences, i.e., it requires consistency of decisions. However, decision-making is often affected by time inconsistency – an anomaly in behavioral economics which violates the consistency assumption. Time inconsistency is a phenomenon where decision-makers want to revise their initial decision at a later point in time without any information having changed. A future action that is part of an optimal plan today is no longer optimal later. A large number of empirical evidence suggest time-inconsistent behavior in decision-making (cf. Thaler (1981), DellaVigna (2009)).

Time inconsistency leads to biased decisions that are in conflict with the long-term interests of the decision-maker (cf. Strotz (1955)). It can lead to suboptimal decisions as people deviate from initially set plans (e.g., under-saving, over-consumption, postponement). Long-term investment and consumption plans are inconsistent with short-term investment and consumption preferences. To mitigate the negative effects from time-inconsistent behavior, it is important to define long-term investment targets and create a plan for how to achieve them. Commitment instruments can help to avoid short-term decisions that could compromise long-term investment targets.

The behavioral literature has extensively addressed the issue of time inconsistency and proposed that this behavior can be modeled deterministically via the assumption of specific discount functions. However, in theoretical work that attempts to model decision-making, in particular, dynamic asset allocation problems under more real-world conditions such as regime uncertainty, time inconsistency can arise naturally. The problem of time inconsistency can be dealt with via a pre-commitment strategy. A pre-commitment strategy is a deterministic approach and optimizes the objective function at the time when the decision is made ($t = 0$). The investor does not change her strategy over time, thus possible changes in future preferences are simply not taken into account. A pre-commitment strategy does not solve time-inconsistent behavior, but it is a realistic way to deal with the problem of time inconsistency.

In particular, asset allocation decisions involving stock investments are based on estimates and assumptions about the distribution of financial market parameters (expected return, volatility). Merkoulova and Veld (2022) find that more than half the US-population is unable to make a return prediction in this complex world. Decisions are made not only based on factual or estimated information, but also based on cognitive and emotional aspects used to process the information. In the absence of full information, there is uncertainty about the parameters to be used. It is therefore necessary to consider both regime uncertainty and the problem of time inconsistency in decision-making in order to provide appropriate decision support and build up wealth over the long term.

1.2 Aims and research questions

The dissertation aims to fill the research gap on the impact of time inconsistency on an investor's optimal asset allocation and expected utility. Thereby, the relevance of taking regime uncertainty into account as a realistic assumption will be discussed in more detail.

Approaches that measure and model regime uncertainty are necessary to protect investors from financial distress due to unprofitable investments. Investors can decide on the amount of assets they would like to invest over time, whether risky or risk-free. We introduce regime uncertainty as an additional uncertainty component, and use stylized regime-switching models to account for regime uncertainty and assume that the current regime (prevailing state of the capital market and its asset price dynamics) is unobservable to the investor due to a lack of information. Thus, we extend previous asset allocation approaches in continuous time that do not allow for large price movements and structural breaks within a short period of time by considering multiple regimes.

Intertemporal models often make the unrealistic assumption that investors know the predictive model and its parameters for the dynamics of asset returns and state variables. We abstain from the assumption by considering that the investor does not know the true state of the regime and therefore decides on a regime-independent strategy. Thereby, time inconsistency arises naturally in our theoretical works by taking regime uncertainty into account.

In the following chapters of this dissertation the subsequent research questions are investigated: What are the effects of time-inconsistent behavior on decision situations? What are the results of the theoretical and empirical behavioral literature on time inconsistency? Is there a preference in dealing with time inconsistency in optimal asset allocation problems? What are the implications of an investor's time-inconsistent behavior implementing a pre-commitment strategy for optimal asset allocation under regime uncertainty? How does her willingness to pay for the resolution of regime uncertainty (value of information) evolve in this context?

For a brief overview, short summaries of the individual essays are presented in the following section. Chapter 2 discusses the notion of regime uncertainty and motivates its relevance for consideration in asset allocation problems. In Chapter 3, the problem of time inconsistency is examined in more detail from a behavioral perspective using a comprehensive literature review. Further motivation for considering time inconsistency in our setup is provided by evaluating the dealing of time inconsistency in research in the context of asset allocation. Chapter 4 addresses the concrete implications of time inconsistency in a specific optimal asset allocation problem. We consider a stylized setup that can explain the effects of time inconsistency on optimal asset allocation. However, in our study, time inconsistency arises from the assumption of an a priori lottery over two possible regimes and the investor's pre-commitment strategy, which leads to a non-constant savings rate. Chapter 5 complements the research in Chapter 4 since it allows for regime switches and addresses challenges when accounting for a regime-switching environment for a time-inconsistent investor. Utility losses of a time-inconsistent investor implementing a pre-commitment strategy and her willingness to pay for full information are studied. Finally, Chapter 6 concludes and gives an outlook on further research.

1.3 Summary of essays

Essay 1: Time inconsistency and its consideration in asset allocation problems – A literature review

The first essay (Chapter 3) presents a comprehensive literature review on time inconsistency and considers time-inconsistent behavior in asset allocation problems: A well-known anomaly in behavioral economics is the presence of time inconsistency in decision-making of individuals. Time inconsistency describes the fact that individuals want to revise a decision they already made at a later point in time without any change in information. It leads to biased decisions that are in conflict with the long-term interests of the decision-maker (cf. Strotz (1955)). Empirical research has come to the conclusion that this behavior is a phenomenon that occurs in reality (e.g., in numerous consumption and investment decisions), which leads to suboptimal decisions (e.g., overconsumption, excessive borrowing, under-saving, unhealthy lifestyle). Commitment instruments are proposed to mitigate the negative consequences of time inconsistency. However, the demand for commitment instruments depends on the level of information of the time-inconsistent individual: Individuals who do not know that they behave in a time-inconsistent manner do not demand commitment and are called naive. Individuals who are aware of their time inconsistency and therefore demand commitment instruments to mitigate negative consequences of their behavior can be called sophisticated (cf. DellaVigna (2009).

Time-inconsistent behavior is assumed in theoretical research via the assumption of specific discounting functions. Besides the deterministic assumption of time inconsistency, the problem is also considered in many stochastic decision problems where time inconsistency arises through aggregation of non-linear functions. The dealing with this anomaly, especially in the context of dynamic asset allocation, is mainly accomplished by a game-theoretic solution of the problem or by implementing a pre-commitment strategy. A pre-commitment strategy optimizes the objective function at the time the decision is made and is therefore the optimal strategy for $t = 0$. Thus, changes in future preferences are not taken into account. In the game-theoretic approach, the decision problem is interpreted as a game played by "multiple selves" of the same individual, where in each period the investor chooses a strategy that maximizes her objective in that period, taking into account the adjustments she will make in the future. The intrapersonal game is solved by finding a point of subgame perfect Nash equilibrium, e.g., by backward induction (cf. Laibson (1997), Zhao et al. (2016), Becker et al. (2022)).

A preference in dealing with time inconsistency in theoretical research is not trivial. Time consistency is seen as a basic requirement for rational decision-making. Therefore, it seems attractive to find time-consistent strategies (e.g., by using the game-theoretic approach). However, time-consistent strategies are more difficult to implement and specific model assumptions have to be made for the implementation. For example, the game-theoretic approach assumes that the decision-maker has full information about future outcomes and correct beliefs about her future actions. The time-inconsistent pre-commitment strategy is more efficient than other approaches because it optimizes the objective function at the initial time when the decision is made. The strategy does not resolve time inconsistency but is a realistic way to deal with time-inconsistent behavior.

Essay 2: On the impact of time inconsistency in optimal asset allocation problems

The second essay (Chapter 4) investigates the impact of time inconsistency in an optimal asset allocation problem.[1] In the setup used, time inconsistency arises by considering regime uncertainty via a double risk situation that serves as the decision basis of an asset allocation problem: the external risk is given by a simple a priori lottery and the internal risk by a regime that coincides with the classical Merton problem. In the a priori lottery, two regimes with probabilities p and $(1-p)$ can occur, differing by their (μ, σ)-tuples. The second risk dimension implies an external expectation about the outcome of the lottery and an internal expectation about the expected utility within the regimes. The a priori lottery that yields our second risk dimension is a stylized version of a regime-switching model. In particular, our special case of full information resembles a stylized version of a regime-switching model with observable Markov chain (or regime).

The utility aggregation is highly non-linear, which naturally gives rise to time inconsistency in our setup. The time-inconsistent investor does not know the actual state of the regime and follows a pre-commitment strategy which she has decided on at time $t = 0$ and which cannot be revised until maturity. We show that the optimal pre-commitment strategy is in between the regime-dependent Merton solutions and thus is a weighted average of these solutions. In the myopic case, when either the investor has myopic logarithmic pref-

[1] The contents of Chapter 4 are based on a joint work with Antje Mahayni, Nicole Branger and Sascha Offermann.

erences or the time horizon approaches zero, the weighting factors of the regimes depend only on the regime probabilities and volatilities. Since the external risk situation increases with the investment horizon, the investor's optimal decision converges to the worst-case strategy, i.e., the investor chooses the strategy that maximizes the minimum expected utility across the two regimes. Furthermore, we provide a measure δ (normalized to $[0,1]$) that can be used to represent the impact of time inconsistency on the optimal pre-commitment strategy. This measure increases with the length of the investment horizon and with the probability of the good regime (the shift to the worst-case regime is more pronounced the higher the probability of the good regime). However, the influence of the degree of risk aversion on δ is not trivial (the measure increases in the risk aversion of the second risk dimension γ_L and decreases in the risk aversion of the inner risk dimension γ_R).

Regarding the value of information[2], we have the surprising result that the willingness to pay for information about the regime approaches zero not only for an investment horizon of zero, but also for an infinite horizon. Thus, the willingness to pay for the preservation of information about the actual regime reaches a maximum for a finite investment horizon but does not necessarily increase with the length of the time horizon.

Furthermore, a third risk dimension is considered by including ambiguity about the probability of the lottery. Preferences with respect to risk and ambiguity are carried out using the smooth ambiguity approach of Klibanoff et al. (2005). Again, it is possible to separate the effects of the two risk situations as well as the ambiguity aversion. We explain why the impact of time-inconsistency gets more ambiguous since varying the ambiguity situation may also change the risk situation. Although our stylized setup is artificial (in the sense that we do not allow for gradual learning about the regimes), it fits to common problem formulations but allows to separate the outer and inner risk situation.

Essay 3: Optimal asset allocation for a time-inconsistent investor in a regime-switching environment

The third essay (Chapter 5) reviews literature on the consideration of regime-switching models in asset allocation und asset pricing, which justifies our setup that analyzes the value of information in a Markov modulated regime-switching model with two regimes. Our

[2] For more information on the value of information from a decision-theoretical point of view, see Appendix A.1.2.

overall objective is to analyze the effect of regime uncertainty and time inconsistency in an asset allocation problem. We compare the expected utility under an observable Markov chain between two types of investors with different levels of information (full information and no full information). The value of information is obtained as the difference in the certainty equivalents of the investors' strategies. If an investor has no full information about the evolution of future asset price dynamics (e.g., point in time when a regime switch takes place), the problem of time inconsistency arises naturally due to regime uncertainty. In this context, we determine the optimal pre-commitment strategy, which is obtained as a weighted average of the Merton solutions of the regimes.

The optimal pre-commitment strategy depends on the length of the investment horizon. The importance of the first regime, driven by the intensity parameter λ, decreases as the investment horizon increases, so that a time-inconsistent investor wants to adjust her strategy towards the second regime (worst-case Regime). Thus, a long-term investor places increasing weight on the state that will follow in the event of a possible regime switch. However, the certainty equivalent return of the overall optimal strategy under full information and the certainty equivalent return of the optimal pre-commitment strategy are quite similar. Thus, the risk of regime-switching has minimal impact on the optimal certainty equivalent return as long as the pre-commitment strategy is correct on average. Furthermore, we investigate λ^{crit} at which the value of information is highest for a time-inconsistent investor who implements a pre-commitment strategy.

Already for moderate λ the value of information increases scarcely in T. For very small λ the value of information increases with increasing T. We look at other pre-commitment strategies. For other strategies, the loss in the certainty equivalents may be quite substantial, especially for long-term investors. We show a reverse effect on the value of information of a pre-commitment strategy when switching from Regime 2 to Regime 1.

The result shows that asset allocation processes involve complex decisions. Results from models intended to guide investor decision-making depend heavily on the model setup and assumptions. Results should therefore be interpreted cautiously. However, our paper highlights that time inconsistency and regime-switching, as realistic assumptions, should be taken into account in asset allocation decisions. A time-inconsistent investor can mitigate potential utility losses by trying to maximize a pre-commitment strategy. Overall, an investor's investment horizon, the asset price dynamics in the given regimes, the possibility and frequency of regime switches, and the level of information are determinants of an investor's investment and asset allocation decisions.

Chapter 2

The significance of regime uncertainty for decision-making behavior

2.1 Introduction to decision theory and the notion of regime uncertainty

We live in a constantly changing world in which future developments related to health, social, technological, economic, regulatory, and political aspects are uncertain or hard to predict. In the context of asset allocation, this leads to challenges for decision-makers, in particular for long-term investors or for investors making irreversible investments. Due to the longer investment horizon or the fact that the investment cannot be reversed investors' exposure is higher. As part of portfolio optimization, investment decisions must be taken with regard to certain asset classes (e.g., bonds, stocks, cash), their risk-return relation, their proportion in the overall portfolio and investors' risk preferences (cf. Brennan et al. (1997), Guidolin and Timmermann (2007), Wachter (2010)). The expected evolution of future asset prices has to be taken into account in order to assess the value of investment alternatives. While in certain situations asset prices and their future developments are fully determined and known, in reality, investment and portfolio decisions have to be taken under uncertainty, as the financial market is driven by a changing environment. Uncertainty implies a possible deviation from the expected development of the asset value.

The concept of uncertainty dates back to Knight (1921) who distinguishes between

risk and **ambiguity** (often denoted as Knightian uncertainty). In the presence of risk the decision-maker has beliefs expressed as probabilities for the occurrence of possible outcomes (e.g., price developments). The risky asset can be defined as a random variable via a probability function. The probability distribution can be estimated objectively by empirical evidence or by subjective consideration. If the future development of the asset is unknown, a rational investment decision is impossible. No probabilities can be determined about the future development of the asset. Ambiguity refers to situations where probabilities are uncertain or unknown, such that the decision-maker is not able to assign probabilities to possible outcomes. It is already prevalent when probabilities can only be estimated subjectively (cf. Camerer and Weber (1992), Chen and Epstein (2002)). Arrow (1951) already distinguishes between two types of uncertainty: risk describes the uncertainty within a model about the outcome of a random variable that assigns a probability to each possible outcome. Ambiguity exists between models and describes the uncertainty about which probability model should be used. Uncertainty is thus determined by two determinants: the extent of a possible outcome and the associated probability.[1]

Empirical research provides the following results regarding to the existence of ambiguity: Ellsberg (1961) provided with the help of an urn experiment first evidence that individuals are ambiguity averse, i.e., they prefer known probabilities over unknown probabilities. This phenomenon is called the Ellsberg paradox. Mehra and Prescott (1985) support the presence of ambiguity by the equity premium puzzle. Brenner and Izhakian (2018) and Wang and Mu (2019) confirm that ambiguity is priced in the equity market. The following empirical research papers measure ambiguity preferences using surveys based on Ellsberg urn experiments.[2] Dimmock et al. (2016a) find empirical evidence that ambiguity aversion is negatively related to participation in the stock market. Guidolin and Liu (2016) provide evidence that ambiguity-averse investors hold under-diversified portfolios. Antoniou et al. (2015) also confirm, in the context of households, that an increase in ambiguity reduces the probability that a household will invest in stocks. Dimmock et al. (2016b) find that ambiguity aversion is negatively correlated with foreign stock ownership.

[1] Due to the fact that in reality both risk and ambiguity play a role in decision making, we consider both aspects in Chapter 4.

[2] There are other measures of ambiguity. Ambiguity can be derived directly by the current state of information and the quality of information. In addition, ambiguity can also be measured indirectly via disagreement among experts regarding future macroeconomic developments (cf. Antoniou et al. (2015), Anderson et al. (2009)).

The results are stronger for participants who self-assess their knowledge of stock markets as low. Poor and unstable environmental conditions (e.g., the financial crisis) lead market participants with higher levels of ambiguity aversion to sell their stocks.[3]

Dohmen et al. (2011) identify determinants, such as age, gender, and family background, that have an impact on risk attitudes. Bossaerts et al. (2010) find that, as with risk attitudes, attitudes towards ambiguity are heterogeneous. Furthermore, there is a positive correlation between risk aversion and ambiguity aversion. In total, ambiguity, like risk, leads to more cautious behavior and impacts asset allocation.

In model theory the **expected utility** approach has become the standard approach for decision making under uncertainty, in which the Homo economicus as a rational agent aims to maximize her expected utility. Expected utility is obtained as the utility weighted by probabilities from possible outcomes (cf. Neumann and Morgenstern (1947)). The utility function takes into account the subjective preferences of a decision-maker. She can be risk-neutral, risk-seeking and risk-averse. In research, the standard assumption is that the decision-maker behaves in a risk-averse manner.[4] This behavior is often described by a utility function with constant relative risk aversion (CRRA), where the risk aversion parameter γ is constant. Chiappori and Paiella (2011) find that the change in the fraction of risky assets relative to a change in wealth is small and not statistically significant, confirming the assumption of a CRRA function to describe the utility of a risk averse investor.[5]

Over time, **asset allocation** models have emerged in the context of portfolio selection under uncertainty. The construction of a portfolio to increase wealth is a fundamental

[3] In this context Dlugosch and Wang (2020) and Dlugosch and Wang (2022) show that more ambiguity-averse investors increase their foreign investment in contrast to less ambiguity-averse investors after an increase in domestic ambiguity. If ambiguity for foreign investment increases after a shock, it leads to the opposite effect.

[4] A decision-maker is risk averse if she prefers a certain outcome b above a lottery with an expected value of b. The utility function of a risk-averse decision-maker is strictly concave. Research papers that identify risk-averse behavior in the financial market are Sharpe (1965) and Arrow (1951). The St. Petersburg paradox describes a gamble in which the random variable has an infinite expected value and thus the payoff for the gamble is also infinite. Decision-makers nevertheless pay only a small amount to participate in the gamble, which shows that a rational individual does not only decide on the basis of the expected value (cf. Bernoulli (1954)).

[5] In our research papers, we will therefore assume a CRRA investor who wants to maximize her expected utility to model choice under uncertainty.

aspect in modern finance. Brinson et al. (1991) argues that asset allocation has a significant impact on portfolio performance. The objective is to maximize portfolio performance by investing a certain proportion of wealth in risky and risk-free assets. Markowitz (1952) developed the mean-variance framework to solve a portfolio problem in a one-period decision model without consideration of expected utility. The static approach results in the shortcoming that it is unsuitable for an investor who wants to invest and shape her portfolio over a long time period. An advanced intertemporal continuous-time model provides Merton (1971). He solves the dynamic asset allocation problem by maximizing the expected utility of a CRRA investor in a Black-Scholes world. The optimal strategy is given in terms of a constant investment fraction in the risky asset, whereas the price dynamics of the risky asset follows a geometric Brownian motion with constant drift μ and volatility σ (cf. Black and Scholes (1973)).[6]

Accounting for ambiguity in model theory removes the assumption of standard models that the distribution of asset returns is objectively known. Epstein and Schneider (2008) conclude that ambiguity-averse investors take a worst-case view. As a result, they respond more strongly to bad information than to good information. Asset allocation models that account for ambiguity allow for the possibility of multiple distributions for the evolution of assets.[7] Epstein and Schneider (2010), Guidolin and Rinaldi (2013) and Ilut and Schneider (2022) provide an overview on models of ambiguity in asset pricing and asset allocation.

In reality, decision-making situations are complex. In addition to current economic risks, the success of investments often depends on the uncertain environmental development with regard to political and regulatory policies. This additional source of uncertainty is called **regime uncertainty**. The term regime uncertainty (often referred to as policy uncertainty in the literature) is not unified in its definition. Unpredictable exogenous events, such as political and regulatory changes can affect environmental conditions and

[6] For further literature using this model setup see e.g., Brennan et al. (1997), Kole et al. (2006), Branger et al. (2010), Xia (2011). We will follow Merton's expected utility approach in a Black Scholes economy, as we consider a dynamic continuous-time asset allocation problem.

[7] In theoretical papers ambiguity is considered via a second-order distribution of beliefs about the probability of an event. Ambiguity can be modeled via multiple priors or the smooth ambiguity approach. For more information see Gilboa and Schmeidler (2004), Chen and Epstein (2002) and Garlappi et al. (2007) in context of multiple priors and Klibanoff et al. (2005), Klibanoff et al. (2009), Guidolin and Liu (2016) and Suzuki (2018) for smooth ambiguity. We will use the smooth ambiguity approach in Chapter 4. For portfolio selection problems in the mean-variance framework under ambiguity see Garlappi et al. (2007), Boyle et al. (2012) and Maccheroni et al. (2013).

lead to structural breaks in the economy and financial world (cf. Al-Thaqeb et al. (2022), Welling et al. (2015)). Government and regulatory policies have wide-ranging impacts on firms and markets. They set the rules for how firms operate and compete, determine how firms are taxed and subsidized, and influence general macroeconomic conditions and price dynamics of assets (e.g., stocks) in the long-term (cf. Kaviani et al. (2020)). Regime uncertainty is also determined by the economic conditions that lead governments and authorities to adopt new policies. Thus, it is rather an institutional uncertainty that has effects on economic circumstances.

Policy makers respond to crises with new policies. The COVID-19 pandemic is an example. On the one hand, there may be uncertainty about the direction and objective of the change in the political or regulatory framework as well as about its specific design and implementation. In addition, there may be uncertainty about the endurance of the implementation process and its dependence on other rules and regulations. There is also uncertainty about how long the new framework will last and when new changes will occur. Announcement effects also create uncertainty in the financial market. There may be unintended consequences due to the new political or regulatory frameworks (cf. Bloom (2009)).

Baker et al. (2014) notice two factors for the rise in regime uncertainty over time. The first is increasing regulatory requirements due to the interconnected world.[8] The second is increasing political polarization.[9] Due to increasing internationalization, correlations between international stock markets increase, especially in times of crisis and high uncertainty. So-called spillover effects can cause instability or uncertainty in one area or country that spreads to others, leading to a global systemic crisis (cf. Ang and Bekaert (2002), Kole et al. (2006), Yuan et al. (2022)). Indicators of a country's economic uncertainty that drives regime uncertainty are its creditworthiness, its public debt, interest rate levels,

[8] This is particularly relevant for highly regulated industries such as the financial and energy sector. There may be unexpected changes in regulatory requirements and laws issued by the government or supervisory authorities. Minimum capital requirements in the financial industry are an example of a regulatory framwork that changes over time (cf. Gatzert and Kosub (2017)).

[9] There is no full information about the future state of the government and related government activities. Governments and states can intervene in the financial market and trade policy. The political stability can be affected by wars, terrorist attacks, riots, demonstrations and corruption. In addition, there is a risk of ideological change in government leadership associated with policy changes, for example, during elections. Laws can be implemented differently by each government.

currency trends, and unemployment rates. Another determinant of regime uncertainty is technological change, which leads, for example, to a reduction in subsidies for obsolete technologies and an increase in subsidies for alternative technologies. Globalization, internationalization, technologization and digitalization generate new risks that need to be regulated (cf. Hoffmann et al. (2009), Gatzert and Kosub (2017)).

Regime uncertainty can affect investor sentiment. Even if investors are not affected by any policy change (e.g., investors in a different industry or country), it is possible that it may soon be relevant for them as well. There may be unknown interdependencies between industries. Even if new policies intend to strengthen investment in one area, the specific form of the policy may threaten certain investments in a different area (cf. Ranaldo et al. (2021)). Market participants form expectations about market states and dynamics. Therefore, the quality of information, in addition to the risk attitude of market participants, influences stock price movements (cf. Veronesi (2000), Lochstoer and Muir (2022)). In total, regime uncertainty leads to a lack of information and weakens investors' confidence in their ability to predict the extent to which future government or regulatory actions will affect the financial market and its parameters. This has an impact on economic and financial activity, and thus on the price dynamics of stock prices.[10]

To further clarify the concept of regime uncertainty and its relevance, the following two sections will gather further research that deals with the measurement of regime uncertainty and its impact on asset pricing and allocation.

2.2 Quantifiability of regime uncertainty

The quantifiability of regime uncertainty is not subject to a generally accepted measure. There are several approaches to measuring regime uncertainty in the literature that differ in terms of their calculation methods and data basis.

[10] Note that in our research papers the existence of regime uncertainty implies that there is uncertainty regarding the price dynamics of the risky asset, which is defined by the parameters μ and σ. Thus, uncertainty about the current and future state of the regime has implications for the asset allocation decision. Uncertainty with respect to the current state of the regime as well as to possible regime changes can be taken into account in model theory via so-called regime-switching models. The theoretical research in Chapters 4 and 5 deals with an asset allocation problem in this context. Note that only a stylized regime-switching setup is used in Chapter 4.

In one strand of the literature, regime uncertainty is implicitly assumed via exogenous events. Increased regime uncertainty is assumed, for instance, in times of political elections. Upcoming elections trigger uncertainty regarding a new political discourse. Newly elected politicians may hold different views and may seek changes in government and regulation (cf. Giavazzi and McMahon (2012), Julio and Yook (2012), Kelly et al. (2016), Jens (2017)). In addition, there is research that assumes regime uncertainty about exogenous shocks such as wars, attacks, or macroeconomic announcements (cf. Kim and Kung (2017), Kurov and Stan (2018)).

Macroeconomic instability, a driver of regime uncertainty, can be measured by the time-varying volatility of financial market variables, such as stocks and interest rates. The volatility can either be derived from historical data or be estimated implicitly from prices expected by the option market. It is hereby assumed that the risk perception of market participants is reflected in option prices. A well-known volatility index is the VIX, that contains the expected volatility of option prices on stocks of the 500 largest U.S. companies. An increasing index suggests that market participants are becoming more uncertain about future developments in the macroeconomic environment (cf. Bloom (2009), Fernández-Villaverde et al. (2015)). The fluctuation of future expectations of macroeconomic variables (e.g., GDP, production rate, consumer prices, dispersion of corporate profits) is also a proxy for regime uncertainty. It is assumed that the variability of market participants' (survey-based) forecasts increases in times of higher uncertainty. A low degree of uncertainty about future macroeconomic developments leads to more homogeneous expectations. The measures are generated by estimating time series. Assuming that incorrect predictions reflect uncertainty, an increase in ex post prediction errors indicates an increased degree of regime uncertainty at the time the forecast was made (cf. Jurado et al. (2015), Girardi and Reuter (2016)).

To measure specific climate policy uncertainty Boomsma and Linnerud (2015) use the carbon price as a proxy. Other measures that capture regime uncertainty are specific indicators that express, for example, the instability of a government and are formed on the basis of expert views (cf. Julio and Yook (2012), Gatzert and Kosub (2017), Smimou (2014)). Another method to measure regime uncertainty is the text analysis. Hoffmann et al. (2009) and Colombo (2013) examine newspaper articles and company reports to investigate investment behavior at times of heightened regime uncertainty. Baker et al. (2016) develop an uncertainty index (Economic Policy Uncertainty, EPU) to measure regime uncertainty in America. Articles from leading newspapers, which are screened for key words, are used as the basis for determining the index. The EPU increases with the intensity of reporting on regime uncertainty.[11] In addition, the EPU can be decomposed into sub-indices to measure specific uncertainties and country-specific indices have been developed. Thus, measuring uncertainty about national security, health policy, tax policy or financial regulation is also possible.[12]

When comparing the different approaches, it turns out that the measures correlate with each other. Implied volatility increases in the period of exogenous events, such as elections (cf. Goodell and Vähämaa (2013)). The VIX is correlated with the EPU (correlation 0.6). The VIX only takes into account the uncertainty in the stock returns of listed companies. If the EPU is also restricted to the stock market, the correlation of the two rises to 0.7. The increased correlation confirms that there are market- and industry-specific differences with regard to regime uncertainty. If the EPU increases by one unit, then the uncertainty measure of Jurado et al. (2015) increases by about 0.4 units. The stock price volatility and EPU are negatively correlated with GDP. Jeon et al. (2022) find that the magnitude and frequency of stock return jumps are significantly related to the content und frequency of news.

However, the various measures of regime uncertainty used in the literature also have weaknesses. Exogenous shocks (e.g., elections) are indicators of increased uncertainty, but do not reflect the degree of regime uncertainty. It is only indirectly implied regime uncertainty. Increased volatility of market variables such as stock returns may be due to changes in the general sentiment of market participants and not necessarily due to increased regime uncertainty. Therefore, volatility-based measures are only an indicator of regime uncertainty and are not suitable for quantifying regime uncertainty. Some uncertainty measures (e.g., EPU) are ex-post in nature and can only measure uncertainty of past periods. Such measures can help to analyze the impact of uncertainty on asset allocation. However, they are less useful for investors who need to make long-term investment decisions due to the lack of forecasting capabilities. Moreover, there are concerns in the measurement of uncertainty regarding the reliability, accuracy, and bias of newspaper articles.[13]

[11] The EPU follows an increasing trend over time. Thus, the consideration of regime uncertainty is nowadays even more important for decision making (cf. Baker et al. (2014)).

[12] Due to the fact that the number of all published articles varies by newspaper and year, the EPU is scaled to ensure comparability between different newspapers over time.

[13] However, Baker et al. (2016) are able to unravel the weaknesses of text analysis as a measurement tool due to the strong relationship between EPU and the VIX. To show that the political orientation of the newspaper does not bias the EPU, the development of the measure obtained only from politically

It is difficult to quantify and decompose regime uncertainty due to the fact that it is an aggregate measure determined by many complex conditions and factors. Macroeconomic, political and regulatory instability and changes are important drivers of regime uncertainty. The volatility-based measures indicate only macroeconomic uncertainty and do not separate political and regulatory uncertainty from it. The EPU is designed via refined text analysis and can account only for uncertainty about political and regulatory developments.

2.3 The effect of regime uncertainty on asset pricing and asset allocation

The quantifiability of regime uncertainty is not trivial for decision-makers due to the lack of direct observability and anticipation of future developments in a complex world. This chapter discusses the impact of regime uncertainty on asset pricing and asset allocation. First, results are presented from research whose data base was obtained using text analysis. For example, regime uncertainty is presented via the EPU as an independent variable in the regression model. This is followed by research that uses exogenous events, volatility measures, and indicators to identify regime uncertainty.

Some researchers use the EPU developed by Baker et al. (2014) to analyze the effects of regime uncertainty on asset prices and asset allocation. Gulen and Ion (2016) use the EPU as an independent variable in their regression model and find that an increase in EPU decreases the investment rate. Moreover, there is firm heterogeneity: firms that are more dependent on government spending and irreversible investments show a larger decline in the investment rate. For firms operating in industries that are strongly influenced by political and regulatory changes, the effect of a falling investment activity is even more pronounced (cf. Baker et al. (2016)). Delaying investments for too long leads to the fact that foregone cash flows will exceed the benefits of waiting until uncertainty decreases. Bonaime et al. (2018) conclude by developing a logit model that an increase in regime uncertainty leads to a decrease in the probability that a company announces

left-leaning newspapers was compared to the measure developed only from politically right-leaning newspapers. The researchers concluded that the EPU is valid and text analysis methods are particularly suitable for measuring uncertainty in countries and industries for which few data are available because newspapers are a public medium and accessible to everyone.

a M&A transaction. Overall, regime uncertainty affects corporate transactions and can therefore lead to an inefficient and suboptimal allocation of capital to assets. Al-Thaqeb et al. (2022) find that during the COVID-19 crisis period, EPU is high. The effects of regime uncertainty extend to different markets, causing households and firms to postpone financial decisions and invest less. Baker et al. (2020) also use text analysis methods and argue that government-imposed contact restrictions as a reaction to the COVID-19 crisis were mainly responsible for the underperformance of stock prices. In the context of asset pricing, Brogaard and Detzel (2015) find that an increase in EPU leads to a decline in stock returns.

Jens (2017) conducts an analysis using a difference-in-difference model and an instrumental variable approach to identify the causal effect of regime uncertainty on the investment behavior of U.S. firms. It is assumed that enhanced regime uncertainty exists at times of elections. Investment expenditures decline by around 5% in the quarter before a gubernatorial election. The volatility of returns is greatest for firms located in states with an upcoming election. Julio and Yook (2012) come up with similar results. Brogaard et al. (2020) study political uncertainty measured via U.S. elections and conclude that the higher the degree of uncertainty about the outcome of the election, the stronger the following effects: Political uncertainty leads to a decline in stock returns, an increase in market volatility and an increase in government bonds. These results suggest that regime uncertainty increases investors' overall risk aversion, and leads to a shift from risky to safe assets. The effects of regime uncertainty are further amplified when the probability of an election winner changes (cf. Goodell et al. (2020)). Chan et al. (2020) find that investments decline in the period after a president is newly elected due to increased uncertainty about new political discourses. In addition, risk aversion increases. Liu et al. (2017) use an event as political shock to identify the impact of regime uncertainty on asset prices and show a decline in stock prices. Hanke et al. (2020) build portfolios depending on the expected winners and losers of political elections. A correctly anticipated election outcome leads to high positive returns. The result shows that regime uncertainty and the formation of expectations about it based on information have a significant impact on portfolio performance and asset allocation.

Bloom (2009) uses a vector-autoregression (VAR) to analyze the effect of macroeconomic uncertainty shocks on economic activity.[14] An uncertainty shock is represented by

[14] VAR models are used to forecast economic time series in case of simultaneous influences and therefore

a binary variable and takes the value 1 if stock price volatility is 1.65 standard deviations above the mean of the observations. They conclude that uncertainty shocks have a negative effect on consumer prices, output, wages, interest rates, and employment. Fernández-Villaverde et al. (2015) analyze the effect of unexpected changes in fiscal policy on economic activity also via a VAR model and conclude that fiscal policy shocks have negative effects on economic activity.

Julio and Yook (2012) use a country-specific stability indicator that acts as an independent variable in their regression model. They conclude that in times of regime uncertainty, investment decreases and the proportion in liquid assets increase. In times of heightened uncertainty, assets are reallocated, and liquid assets are held as a safety buffer. Papadamou et al. (2020) can analyze direct effects of increased uncertainty due to COVID-19 by constructing a Google-Trends based index. Empirical results suggest that the increased uncertainty mapped by the increased search for the consequences of the COVID-19 pandemic amplifies the negative relationship between stock market returns and their implied volatility.

Many research papers carry out industry- and application-specific studies: Fabrizio (2013) runs a regression in which the annual investment amount in renewable energy is the dependent variable and the proxy for regime uncertainty is a binary variable that takes the value 1 if a regulatory change was introduced first and revised thereafter. Firms are less likely to invest in renewable energy in an unstable environment. The risk of a future local regulatory change lowers the willingness to invest in long-term, regulation-supported assets because, if a change in regulation occurs, the value of the asset may decrease. Ramiah et al. (2013) study green policy announcement effects by using an event study methodology and show that announcements induce uncertainty which results in negative cumulative abnormal returns. Kaviani et al. (2020) use the EPU as a measure of regime uncertainty and demonstrate via OLS panel regressions the effect of regime uncertainty on credit spreads. Heightened regime uncertainty leads to an increase in credit spreads. The effect of regime uncertainty is larger for firms that operate in regulation-intensive industries. Guceri and Albinowski (2021) have conducted an experiment in which two similar investment subsidies were introduced in the same country once in a period of

offer the possibility of capturing the dynamics of macroeconomic variables. It is assumed that exogenous shocks have a lagged effect on macroeconomic variables. One disadvantage of VAR models is that they are not suitable for identifying causal relationships.

economic stability and once in a period of high regime uncertainty. Subsidies have positive effects in times of low uncertainty. In times of high uncertainty, however, some companies reduce their investments. Thus, subsidies may miss their target or have a weaker effect in times of high regime uncertainty. In the context of renewable energy subsidies, Ganhammar (2021) shows that interventions in the market can lead to unintended effects, such as an increase in price risk as well as inhibition of investment in renewable energy. Prices can move contrary to the expectations of policy makers.

In addition to empirical research, theoretical research has focused on analyzing effects of regime uncertainty on asset allocation and asset pricing and is still under development. Pastor and Veronesi (2012) analyze how regime uncertainty affects asset prices using a general equilibrium model. They find that the greater the uncertainty about government policy, the more asset prices fall. Policy changes increase volatility and correlations among stocks. Croce et al. (2012) examine the impact of fiscal policy changes on asset prices when agents are sensitive to regime uncertainty. They conclude that short-term oriented fiscal policy can lead to welfare losses in the long run. Boutchkova et al. (2012) find that industries that are more dependent on trade, contract enforcement, and labor exhibit greater return volatility when domestic political uncertainty is high. Political uncertainty in the countries of the trading partners of trade-dependent industries also leads to higher return volatility.

Regime uncertainty leads to the absence of full information about the future asset performance. This leads to uncertainty about the financial market parameters to be used in model theory and forecasting. Parameter uncertainty plays a significant role with respect to the predictability of returns and asset allocation decisions (cf. Xia (2001)). Pettenuzzo and Timmermann (2011) point out that the parameters of asset pricing and asset allocation models are unstable and subject to structural breaks. Branger and Hansis (2012) argue that the nature of the model has a significant impact on the optimal portfolio, and incorrect model specification can lead to significant utility losses. This implies that model stability is important for an investor's asset allocation decision. We consider this problem in the research presented in Chapters 4 and 5.

Chapter 3

Time inconsistency and its consideration in asset allocation problems – A literature review

3.1 Introduction

This paper presents a comprehensive review of the literature on time inconsistency and considers time-inconsistent behavior in asset allocation problems. The problem of time inconsistency is mainly illuminated in behavioral finance research. Behavioral finance attempts to explain real-world financial phenomena using models that depart from the assumptions of homo economicus as a rationally acting decision-maker. In classical financial market theory a decision-maker has complete information, acts in a utility-oriented manner and makes her decisions free of emotions.[1] However, it is now well known that in reality decision-maker do not always act rational which has led to the publication of research that criticizes the normative model. Researchers note cognitive biases that can cause decision-makers to misjudge either the severity of a possible outcome or event or its probability. For example, the reasons for this can be a lack of information. Emotional and cognitive aspects

[1] For the investigation of intertemporal decisions, the discounted utility model is used as a standard approach, which assumes time consistency. The normative standard for decision-making under uncertainty is expected utility theory. The theory was designed as a model of an idealized decision-maker who behaves in a time-consistent rational manner (cf. Neumann and Morgenstern (1947)).

should therefore be taken into account when analyzing the decision-making process (cf. Kahneman and Tversky (1979), Fishburn (1988), Tversky and Kahneman (1989)).

A well-known anomaly in behavioral economics is time inconsistency in decision-making of individuals. Its presence describes the fact that individuals want to revise a decision they already made at a later point in time without any change in information. A decision-maker's preference for a future outcome A over another outcome B changes over time. Intuitively, the optimal strategy (e.g., investment strategy) found by the decision-maker at the initial time t_0 is no longer optimal at a time t_1 with $t_1 > t_0$ because the decision-maker's preferences change over time. Time inconsistency leads to biased decisions that are in conflict with the long-term interests of the decision-maker (cf. Strotz (1955)). A decision-maker's preference between two payoffs that occur at different times should not change if time passes by but the absolute time interval between these payoffs remains the same. However, experimental evidence suggests that preferences change simply because the point in time in which the decision is made is a different one (cf. Thaler (1981)).

A growing body of economic and financial literature emphasizes the significance of time-inconsistent behavior. Time preferences are heterogeneous, differing from individual to individual due to determinant factors such as age, mortality, wealth, and uncertainty. In addition to heterogeneous time preferences among different individuals, however, an individual's time preferences may also change: Changing time preferences induces time inconsistency and matter for saving, consumption, and investment decisions, and have an impact on asset prices and economic growth. Long-term investment and consumption plans are incompatible with short-term investment and consumption preferences. The time inconsistency problem is also referred to as the self-control problem because the decision-maker must act in a self-controlled manner to resist current temptation and achieve better performance in the long run (cf. Becker and Mulligan (1997), DellaVigna (2009)).

One way to prevent self-control problems and time-inconsistent behavior is the possibility of commitment. The level of information of individuals plays a decisive role: Individuals are heterogeneous and therefore perceive and process information in different ways. Even when all potentially relevant information is available, individuals are not able to process it effectively. Decisions are based not only on factual information, but also on the mental and cognitive aspects used to process the information (cf. Barberis et al. (1998), Brogaard et al. (2022)). There is a distinction between individuals based on their demand for commitment: Individuals who are aware of their time-inconsistent behavior and therefore demand commitment are called sophisticated. Individuals who are not aware of their time-inconsistent behavior are called naive (cf. Gifford Jr (2002), Swem (2022)).

In theoretical works, time inconsistency is assumed deterministically via the assumptions of specific discount functions (cf. Laibson (1997)). The problem is also considered in many stochastic decision problems, which intend to theoretically represent real-world phenomena and provide decision support, especially in the context of dynamic asset allocation. The dealing with time inconsistency is mainly accomplished by two approaches: Time inconsistency can be solved via a game theoretic approach, or by the implementation of a pre-commitment strategy (cf. Zhao et al. (2016), Becker et al. (2022)).

The following sections are based on empirical evidence of time inconsistency and theoretical foundations that also take this behavioral anomaly into account. The aim of Chapter 3 is to examine and evaluate the problem of time inconsistency and to highlight its relevance for asset allocation. It is investigated whether there is a preference in dealing with time inconsistency in theoretical work that considers asset allocation problems. The remainder of Chapter 3 is organized as follows. Section 3.2 provides evidence of time-inconsistent behavior in empirical settings. Furthermore, time inconsistency is examined in more detail from a behavioral point of view. Research that assumes time inconsistency deterministically in a model context is collected. In addition, commitment instruments to mitigate time inconsistency and their demand are highlighted. We also address time inconsistency in the context of collective decision-making. Uncertainty-based approaches are gathered alongside preference-based approaches. The evaluation of dealing with time inconsistency in the context of asset allocation is provided in Section 3.3. Section 3.4 concludes.

3.2 Behavioral research on time inconsistency

There is evidence in psychological and behavioral science that individual preferences are often time-inconsistent. The problem of time inconsistency has been considered first in research dealing with intertemporal choice, thus binary decisions that can be made at different points in time. An intertemporal choice reflects a conflict between a smaller reward that is available immediately and a larger reward that can be obtained later. The following example, based on Thaler (1981), will serve for clarification:

Choice 1:	1.1	100€	today
	1.2	105€	tomorrow
Choice 2:	2.1	100€	in a year or
	2.2	105€	in a year plus one day

The gray background highlights the predominantly chosen option. By asking individuals in choice 1 whether they would rather have 100€ today or 105€ tomorrow, the majority chooses option 1.1 – 100€ today. The immediate prospect of 100€ suppresses the distant prospect of a higher outcome. On the contrary, confronting individuals with choice 2, most would choose 105€ in one year plus one day. Time inconsistency arises as soon as individuals choose option 2.1 and change for option 1.1 in 364 days. Thus, preferences of a decision-maker have only changed because the point in time or time horizon is a different one, even if the information situation does not change. The decision-maker's preference for a payoff at time t over a payoff at time $t+1$ is stronger as time t approaches. A time-inconsistent decision-maker might reconsider her decision at a later point in time and will prefer a different choice when the remaining time horizon shortens.

If the decision-maker would act in a time-consistent manner, she would not change her decision and choose the same amount of money (choice of 1.1 and 2.1). A time-consistent preference order exists when a decision does not change just because time passes, resp. the point in time when the decision is made changes. Thus, the decision-maker holds on to her decision about a future action, regardless of how far in the future it is, as long as she does not receive any new information. Of course, even if preferences are consistent over time, a decision may change over time if the information base changes and new information are available (e.g., information on wages, interest rates, regulations, inflation).

Kirby and Herrnstein (1995) provide further empirical evidence on time-inconsistent behavior by confronting students with intertemporal choices constructed similarly to the example given above. They conclude that about 94% of participants reverse their preference, from a larger, later reward to a smaller but earlier reward, when the time horizon at which participants receive the rewards decreases the same for both rewards.[2]

Hardisty and Pfeffer (2017) examine the effects of uncertainty on individuals' time preferences in terms of monetary gains and losses. In general, individuals try to avoid uncertainty in decision-making situations. Immediate gains and losses are preferred when the future is uncertain. In contrast, future payoffs are preferred when there is present uncertainty. This finding is inconsistent with standard models of intertemporal choice, according to which people should always prefer gains now and losses later.

Several research papers examine time-inconsistent behavior in intertemporal decisions under specific application areas. Read and Van Leeuwen (1998) find under consideration of intertemporal consumption decisions that an individual's current state of appetite has a significant effect on choices that apply to the future. Participants in their study behaved in a time-inconsistent manner as they choose to consume unhealthy food when making a choice over an immediate consumption and announced their intention to eat healthy food when making a choice for consumption at a later time. Gruber and Köszegi (2001) provide evidence that time preferences are inconsistent with respect to smoking. A further example for the fact that preference between two future outcomes may change over time give Gneezy et al. (2014) in the context of negative emotions. Individuals who make an immoral decision are more likely to donate to charity than individuals who do not make an immoral decision. The increase in charitable behavior is the result of a temporary increase in feelings of guilt triggered by past immoral actions. The feeling of guilt decreases over time, resulting in decreasing donations. Schreiber and Weber (2016) show in a survey that individuals are time-inconsistent about their annuity payments. Young individuals prefer a monthly annuity while older people want to receive their annuity as a lump sum payment. For an overview of empirical research on intertemporal decisions regarding monetary payments, resp. financial flows and consumption see Cohen et al. (2020).

Time-inconsistent behavior can generate a variety of consequences for a decision-maker. Individuals with this anomaly can be present-biased. They tend to prefer a current payoff in a trade-off situation between two payoffs at different points in time. Present-bias is

[2] In contrast, Sayman and Öncüler (2009) find empirical evidence for reverse time inconsistency. They can show that decision-makers prefer the smaller, earlier outcome when both options are in the future, but choose the larger, later one when the smaller option becomes immediately available. Reverse temporal inconsistency is more likely to be observed when the time to and between the two options is short.

considered a specific dynamic inconsistency between an individual's preferences for short-term and long-term decisions. Immediate payoffs are overvalued, whereas later payoffs are undervalued. In this context Meier and Sprenger (2010) find that present-biased individuals are more likely to have credit card debt and have it at higher rates than time-consistent individuals. The relationship persists when controlling for demographic characteristics, credit constraints and different interest rates. Moreover, present-biased decision-makers make less optimal savings decisions (cf. Jones and Mahajan (2015)). Some research attempts to explain time-inconsistent behavior via emotional and cognitive aspects. Time inconsistency is seen as a cause of diminishing impatience and self-control problems.[3] In this context Gruber and Köszegi (2001) and Bradford et al. (2017) find that good intentions (e.g., healthier lifestyles, better health care) are not kept by time-inconsistent individuals when making consumption decisions. In addition, field experiments provide evidence that time-inconsistent individuals procrastinate (cf. Ariely and Wertenbroch (2002), Bisin and Hyndman (2020)). Procrastination occurs in the context of tasks that involve costs or negative payoffs. In the financial context, investment decisions (e.g., investment in energy-efficient technologies) are postponed, especially when costs are immediate (cf. Bradford et al. (2017)).

Present-bias can result in high credit card borrowing (cf. Meier and Sprenger (2010)) or less saving and overconsumption (cf. Ameriks et al. (2007), Caliendo and Findley (2013), Jones and Mahajan (2015)). Gill et al. (2018) examine whether present-biased individuals are more likely to make financial mistakes than their time-consistent counterpart. They conclude that time-inconsistent individuals overdraw their bank accounts more often and for longer periods of time, resulting in the payment of high interest rates. Present-biased individuals are more likely to make decisions that trigger immediate benefits and delayed costs. Kuchler and Pagel (2021) find empirical evidence that time-inconsistent individuals fail to keep their self-established debt repayment plans. Thus, time inconsistency can lead to welfare losses and biased non-optimal decisions that are in conflict with the long-run interests of decision-makers.

[3] Takahashi et al. (2012) define impatience as a strong preference for small immediate rewards over large delayed rewards.

Deterministic assumption of time inconsistency:

Time preferences are often expressed in terms of a discount rate, which is the rate at which the value of the discount function declines. There is empirical research that identifies time-inconsistent behavior by determining individual discount factors. Thaler (1981) and Benhabib et al. (2010) try to estimate discount rates through experiments on money-time pairs and find that they are high for close payoff dates and decrease as the time horizon increases (the time until a payment is received is getting longer). Bradford et al. (2017) implicitly determine individual discount factors from surveys of intertemporal decisions. The monetary values between the later, larger payment and the earlier, smaller payment are varied until the decision-maker is indifferent between the two payments. The effect of time inconsistency on specific consumption and investment decisions is studied by regressions. The independent variable denotes the degree of time inconsistency and is determined via the discount rates obtained in the experiments. The researchers conclude that time inconsistency is more pronounced in consumption decisions than in investment decisions. In total the findings suggest that a person's discount rate from today to tomorrow is higher than, for example, from in a year to a year and one day. People grow increasingly impatient as the time horizon shortens, at some point regretting a decision they have already made and want to revise it. Cognitive factors such as impatience but also other factors such as distrust, temptation, inattention, confusion, and beliefs can be an explanation why people change their time preferences (cf. Prelec (2004)). The higher the degree of time inconsistency, the more severe the effects can be. Not only at the individual but also at the firm level, savings may decline, making liquidation more likely (e.g., dividends are paid too early, equity issuance decreases (cf. Yang and Cao (2019)).

In model theory of intertemporal decisions, time preferences are represented by discount functions. The prevailing model for comparing utility at different points in time is the discounted utility model of Samuelson (1937), which assumes that individuals maximize the present value of current and future utility of an outcome. Time preferences are represented via an exponential discount function, which indicates time-consistent behavior due to the fact that the discount rate is constant over time. The discounted utility model follows the assumption of stationarity as a necessary axiom for a rational decision-maker discounting her future utility. The axiom states that a preference for outcomes between two time periods depends only on the absolute time interval between the two time periods, not on how far in the future the two time periods lie. Thus, preferences for two outcomes should remain the same regardless of how far in the future the outcomes lie. However, the model has little empirical support since there is evidence for anomalies like time-inconsistent behavior violating the assumption of stationarity in reality (cf. Kirby and Herrnstein (1995)).

The problem of time inconsistency is therefore considered deterministically in further theoretical research by assuming a non-exponential discount function. Two specific discount functions have emerged: Hyperbolic and quasi-hyperbolic discount functions. A brief overview of selected theoretical literature that assumes time inconsistency deterministically with the assumption of a hyperbolic or quasi-hyperbolic discount function is given in Table 3.1.

Hyperbolic discounting is a functional form of discounting that generates present-bias due to the assumption of a declining discount rate over time (declining rate of time preference). The discount factor can be represented as

$$D_{HD}(t) = \frac{1}{(1 + \alpha \cdot t)^{i/\alpha}},$$

with $i > 0, \alpha \geq 0$. The discount factor $D_{HD}(t)$ is a hyperbolic function of time. α is an index of decreasing impatience and determines how much the function departs from constant discounting. The limiting case $(\lim \alpha \to 0)$ is the exponential discounting function. An outcome in the near future is discounted at a higher discount rate than an outcome in a distant future (cf. Laibson (1997)).

In contrast to hyperbolic-discounting, quasi-hyperbolic discounting accounts only for a bias in $t = 0$ and assumes an otherwise constant discount rate when the present is not considered ($t \neq 0$). Here, the discount factor can be represented as

$$D_{QHD}(t) = \begin{cases} 1 & t = 0 \\ \beta\delta^t & t > 0 \end{cases},$$

with $\beta < 1$ and $\delta \in]0, 1[$. The case $\beta = 1$ corresponds to exponential discounting (δ^t is the exponential discount factor). Thus, there is decresing impatience in $t = 0$ and constant impatience thereafter. Only present outcomes are given a disproportionately higher weight compared to all future outcomes (cf. Frederick et al. (2002), Attema et al. (2010) and O'Donoghue and Rabin (2015)).

Table 3.1: Selected papers incorporating time inconsistency via discount functions

Paper	**Discount function**	**Overview/Research topic**
Loewenstein and Prelec (1992)	HD	propose a hyperbolic discount function model to account for anomalies such as time-inconsistent decision behavior and discuss implications for savings behavior.
Laibson (1997)	QHD	analyzes consumption decisions of a time-inconsistent individual and accounts for time-inconsistent behavior via a quasi-hyperbolic discount function.
Azfar (1999)	HD	considers a probability distribution over discount rates and argues that discount rates should decrease hyperbolically rather than exponentially with time if individuals are uncertain about their discount rates.
Carrillo and Mariotti (2000)	HD	show that hyperbolic discounting can lead to strategic ignorance in which even free information is not acquired.
Frederick et al. (2002)	HD	provide an overview of specific hyperbolic discount functions proposed in the literature to model time-inconsistent behavior.
Diamond and Köszegi (2003)	QHD	use quasi-hyperbolic discounting in a model where a consumer with CRRA utility function makes decisions about retirement and saving.
Laibson et al. (2007)	HD, QHD	find that models that account for consumption and saving, fit field data better when they include short-term discount rates that exceed discount rates for later periods and thus disapprove the restriction of a constant discount rate.
O'Donoghue and Rabin (2008)	HD	study procrastination in projects by representing people's time preferences via hyperbolic discount functions and conclude that procrastination is more likely the greater the cost of project completion.
Rohde (2010)	HD, QHD	develops a hyperbolic factor from indifferences in intertemporal decisions to quantitatively determine the degree of time inconsistency (HD if hyperbolic factor is constant and positive, QHD if hyperbolic factor is is zero for all future points in time except the present).

Takeuchi (2011)	HD, QHD	proposes, in contrast to HD or QHD, a discounting function in the form of an inverse S-curve to consider also reverse time inconsistency (future bias, overvaluation of future payoffs).
Takahashi et al. (2012)	HD	investigate the effect of non-linear time perception on intertemporal choices and account for anomalies in empirically observed intertemporal choice behavior such as decreasing impatience and preference reversal via a hyperbolic discount function.
Yılmaz (2013)	QHD	considers a principal-agent model with moral hazard, in which the agent's time-inconsistent behavior is captured via a quasi-hyperbolic discount function.
Tancheva (2019)	HD	investigates the effect of time inconsistency, modeled by hyperbolic discounting, in a general equilibrium model and concludes that the risk premium in the economy increases when the wealth share of time-inconsistent agents increases.
Turan (2019)	QHD	uses a quasi-hyperbolic discounting structure to represent the consumption-saving decisions of an agent who has time-inconsistent preferences with probability p.
Yang and Cao (2019)	QHD	incorporate a manager's time-inconsistent preferences to study the implications for optimal external financing and dividend payout strategies in a regime-switching economy.
Yoon (2020)	HD, QHD	addresses the relationship between impatience and time inconsistency and concludes that people with intermediate levels of impatience act in a time-inconsistent manner.

HD= Hyperbolic discounting, QHD = Quasi-hyperbolic discounting

As shown in the Table 3.1, there are many different application areas in which specific decision problems are studied in which time-inconsistent behavior is modeled by specific discount functions. However, there is also criticism concerning the use of such discount functions. Read (2001) points out that inconsistent time preferences can also be explained by subadditive discounting. This means that the discount rate for a time horizon is higher as the horizon is divided into subintervals. Discount rates were calculated for a two-year horizon divided into three eight-month intervals (an eight-month interval starting at the same time, an eight-month interval starting eight months later, and an eight-month interval starting sixteen months later). As a result, the discount rate for the two-year horizon is lower than the average discount rate for the three subintervals. In addition, there is no evidence that the discount rates decreased over time since the discount rates for the three eight-month intervals are about the same. This finding argues against hyperbolic discounting.

For a more intensive illumination of the deterministic assumption of time inconsistency, Fernández and Rambaud (2018) provide a suitable overview of specific discount functions that have been used in the literature on intertemporal decisions. On top, they summarize measures in model theory that quantify time inconsistency. Rohde (2019) complements the model-theoretic measures by developing an index that captures the change in discounting and can be used independently of models. Decreasing impatience or present-bias corresponds to positive values of the index whereas reverse time inconsistency is represented by negative values.

The importance of commitment for time-inconsistent decisions:

A frequently discussed approach to mitigate the negative consequences of a time-inconsistent behavior is commitment – a decision instrument or device to constrain one's future decisions (cf. DellaVigna (2009)). To counter the problem of time inconsistency, a decision-maker may be willing to engage self-control and has a demand for commitment. Commitment can be attractive because it eliminates the incentive to deviate from a fixed plan. The decision-maker has self-control when she resists the temptation of a choice reversal. Already Strotz (1955) refers to the example of Odysseus, who gets himself tied to a mast while passing the island of the Sirens. Literature in behavioral finance and economics studies the form and the effectiveness of such self-control mechanisms.

However, individuals may also try to commit to the future by constraining their future choices. In economic applications, commitment can be achieved by holding illiquid assets or through regulatory action. Laibson (1997) analyzes consumption and saving decisions of individuals that exhibit time inconsistency and can invest in an illiquid asset (commitment instrument). He finds that liquid financial products lead to a decline in savings rates because they displace the commitment possibilities of illiquid assets. In this context, a retirement account in which withdrawals from the account are associated with significant costs before retirement may represent a commitment against excessive spending before retirement. If an individual invests a significant portion of her assets in illiquid form, she will find it costly to increase her consumption in future periods by selling these illiquid assets such as real estate (cf. Green (2003)).[4] Carrillo and Mariotti (2000) argue that a time-inconsistent decision-maker may prefer to ignore available information, as action of self-control, because she fears a change in her preferences and therefore wants to revise her decision. DellaVigna and Malmendier (2004) note that individuals may also commit to future actions by imposing a cost on the commitment (e.g., gym membership fee).

Using commitment strategies provides benefits: Strotz (1955) emphasizes the need for commitment instruments to reduce the effects of time inconsistency and refers to these as pre-commitment strategies. Ariely and Wertenbroch (2002) report that students who rely on self-imposed deadlines achieve better grades than those who do not. Procrastination is reduced and work/study performance is increased. Self-imposed deadlines and repetitive tasks that become habits reduce procrastination (cf. Bisin and Hyndman (2020)). In addition, commitment efforts can increase savings: Savings programs counteract negative consequences of time inconsistency by having people commit in advance to use a fraction of their future salary increases to save for retirement. Thaler and Benartzi (2004) find a higher savings rate among those who have signed up for a savings program compared to those who have not. Ashraf et al. (2006) propose a commitment savings product and conclude that savings rates increased significantly for individuals who purchased the product. Depositing the income tax refund into an illiquid account is also a commitment instrument. Immediate incentives to save (e.g., setting up an illiquid account today rather than being willing to do so in the future) can increase the probability of saving by about 2-3 times (cf. Jones and Mahajan (2015)).

Experimental studies show the popularity of commitment instrument to help individuals fulfill plans that would otherwise be difficult to realize due to an existing lack of self-control. There is evidence that people demand commitments and voluntarily impose obligations on themselves that are costly to violate. Ariely and Wertenbroch (2002) deal, in the context of procrastination, with a penalty that has to be paid when a person is late with a task. A decision-maker can prefer to commit, provided she is aware of her self-control problems (cf. Strotz (1955), O'Donoghue and Rabin (1999), Gul and Pesendorfer

[4] Also in the financial context Lien and Yu (2014) study the interplay between a firm's investment and cash flow hedging decisions when the decision-maker has time-inconsistent preferences. In addition to the risk-return relationship of the investment, liquidity risk, commitment effects of financial restrictions, and intertemporal preferences play a role in determining optimal investment and hedging strategies. Financial constraints are disciplining because, for example, firms are less likely to invest in projects with negative net value than firms that are not financially constrained.

(2001)). The demand for commitment instruments depends on the one hand, on individuals' assessment of the extent of their time inconsistency and, on the other hand, on the cost of the instrument. Chemarin and Orset (2011) identify the ignorance of information as a form of commitment. It is intuitive that a decision-maker will acquire information unless the cost of acquiring information exceeds the direct benefit she obtains from that information. Nevertheless, a time-inconsistent decision-maker may remain strategically ignorant, and if she acquires information, she will acquire less information than a time-consistent decision-maker. Carrillo and Mariotti (2000) argues that decision-makers attempt to reduce the divergence between their long- and short-term preferences through cognitive forms of precommitment, such as strategic ignorance. Augenblick et al. (2015) find that present-biased students are more likely to demand commitment than others. Moreover, contractual clauses help to pre-commit.

Commitment instruments differ in terms of their degree of severity and effectiveness. Exley and Naecker (2017) find that commitments made publicly have a better impact and are less likely to be broken than those remaining private. The literature distincts between soft and hard commitments. Bryan et al. (2010) refer to commitment devices that impose a real economic penalty for revising a decision or rewards for success as hard commitments. Soft commitment, on the other hand, is an instrument that has mainly psychological consequences. A clear distinction between the two commitment types is not trivial as some hard commitments also have psychological costs and most soft commitments also have some economic costs. An example of a hard commitment is a savings account where interest payments are cancelled if the monthly deposit remains unpaid. For the time-inconsistent individual, however, there may be psychological costs to this as well, such as loss of self-esteem if she misses a deposit. A soft commitment would be membership fee in a sports club. The individual will incur costs that are primarily psychological, such as disappointment or sense of failure when not attending sports but perhaps also small economic costs, such as opportunity costs.

There is evidence that many people who have problems with self-control are aware of it and demand commitment instruments. However, people tend to be uncertain about the future and therefore demand instruments that also allow flexibility. Research that sheds light on the trade-off between flexibility and commitment are Hendel and Lizzeri (2003) and Amador et al. (2006). The latter conclude that those who do not commit prefer flexibility to be able to adjust plans. Those who do commit may value the certainty of having fixed plans.[5]

Different types of time-inconsistent individuals and their demand for commitment: O'Donoghue and Rabin (1999) make a distinction between two types of time-inconsistent decision-makers based on the level of financial literacy: They divide decision-makers into naive (unaware of how their preferences change over time) and sophisticated (aware of how their preferences change over time) individuals and conclude that naive decision-makers delay activities with immediate costs and perform activities with immediate rewards too soon. Sophistication, however, can mitigate present-bias. Due to the fact that such individuals are aware of their time inconsistency they demand commitment instruments. Liu et al. (2016) find in this context that time-inconsistent preferences lead to underinvestment and overconsumption. These behaviors are more pronounced for naive individuals than for sophisticated. In particular, the sophisticated individual invests more and consumes less than the naive, but invests less and consumes more than a time-consistent individual. Time-consistent and time-inconsistent but naive decision-makers will not choose commitment instruments. A time-consistent decision-maker cannot benefit from pre-commitments that constrain her decision choices while a naive time-inconsistent decision-maker, believing that she will behave well in the future, does not feel that she needs pre-commitment.

Naive decision-makers do not take into account at any time t that their preferences will change in the future and make optimal strategies based only on their preferences at time t. They then constantly revise their decisions throughout the planning horizon. To obtain the strategy for the naive decision-makers, one must solve a standard optimization problem at each time point over the entire planning horizon. Sophisticated individuals make their decisions considering that their preferences will change in the future and try to obtain a time-consistent optimal strategy (cf. Wei et al. (2020)).[6] Mahajan et al. (2020)

[5] Lien and Yu (2014) finds that the optimal hedging strategy forms a trade-off between flexibility and commitment. Galperti (2015) investigate the optimal provision of commitment instruments for people who value both commitment and flexibility and whose preferences differ in the degree of time inconsistency. They argue that the combination of unobservable time inconsistency among agents and preference for flexibility can lead to an adverse selection problem. In the absence of informational asymmetry commitment can solve decision-maker's time inconsistency and avoid trade-offs between commitment and flexibility.

[6] A closer illumination to the solution for the decision-maker in asset allocation problems is considered in Section 3.3.

estimate that time-inconsistent agents make up nearly 80% of their sample, with about 50% of the sample exhibiting a naive form of time inconsistency. Moreover, the present-bias is more pronounced (higher discount factor) for naive individuals than for the sophisticated. Bénabou and Tirole (2004) develop a model of commitment where the degree of self-control increases with the decision-maker's self-confidence in her own willpower.

Wong (2008) examines results of a student midterm exam and recognizes that time-inconsistent sophisticated students performed worse compared to time-consistent ones. Naive students performed the worst. Casari (2009) finds that 60% of their sample are sophisticated individuals and demanded commitment instruments. To avoid a choice reversal, those instruments should have a high commitment character. One has a willingness to pay for commitment and the lower the cost of commitment, the more individuals choose to commit. A minority in the sample was classified as naive and did not ask for any commitment. Grenadier and Wang (2007) consider an investment problem in the real options framework, where time-inconsistent preferences of sophisticated and naive individuals are deterministically given. For a consideration of a principal-agent problem where the agent is a sophisticated time-inconsistent decision-maker, see Yılmaz (2013). Kuchler and Pagel (2021) study the behavior of time-inconsistent individuals in the credit card market. Intuitively, it is attractive for individuals with a present-bias to postpone debt repayment from the current to the next period to avoid reducing consumption in the current period. However, when they face the same decision in the next period, it again seems attractive to postpone repayment to another period. Thus time-inconsistent individuals borrow excessively and often fail to repay later, even though they actually intended to reduce their debt. The researchers find that borrowing and debt repayment depend significantly on the extent to which individuals are aware (sophisticated or naive) of the difference in their short- and long-term time preferences - even if they exhibit the same degree of time inconsistency. In contrast to naive individuals, sophisticated individuals behave more patiently and stick better to their self-established plans to pay off debt. Laibson (1997) already argues that sophisticated individuals are willing to pay money (commitment instrument) to get rid of their credit cards immediately and not have access to them in the future. These findings highlight the importance of distinguishing between the behavior of sophisticated and naive time-inconsistent decision-makers.

Empirical evidence has shown that decision-makers have problems with self-control, recognize them and try to manage them via commitment instruments. Sophisticated decision-makers may anticipate their changing preferences but tend to underestimate the magnitude of these changes – this anomaly is called projection bias. They may therefore make suboptimal commitment decisions.[7] Ariely and Wertenbroch (2002) show that in the context of procrastination and self-imposed deadlines, many decision-makers do not set their deadlines optimally. Della Vigna and Malmendier (2006) find in the context that consumers choose suboptimal gym commitment contracts that do not match their actual frequency of attendance. 80% of gym members in their sample who pay a monthly fee would have been better off if they had chosen to pay per visit. Commitment instruments also reduce flexibility and may be too constraining in some cases.

Time inconsistency in collective decisions:

Collective decisions are subject to the problem of time inconsistency since different individuals with heterogeneous preferences are forced to make a joint decision. Time preferences cannot only change over time for the same person but also vary across individuals. Baddeley (2019) distinguishes therefore between inter- and intra-personal time inconsistency. This intra-personal anomaly occurs naturally in collective investment decisions and can lead to significant welfare losses.

Members of a household make joint consumption and saving decisions, and decision-makers in the corporation for example make joint project, financing, and investment decisions. Adams et al. (2014) argue that any aggregation of heterogeneous time preferences leads to time inconsistency. Household members have different discount factors, which are influenced by factors such as age and the level of financial literacy. Even if all individuals in the group are time-consistent and have an individual exponential discount function, a collective decision is time-inconsistent (cf. Gollier and Zeckhauser (2005)). In this context, Jackson and Yariv (2014) and Jackson and Yariv (2015) show that for any heterogeneity in time preferences, any aggregation of utility functions must be time-inconsistent, even if individuals within the group are perfectly time-consistent. Maximizing a weighted sum of utility functions leads to present-bias and hyperbolic discounting. Hertzberg (2016) uses a consumption and saving model for multi-member households to show that despite the fact that individuals have an exponential discount function, households are time-inconsistent and members spend too much on private consumption goods. The household remains time-inconsistent even if members can save separately but there exists the possibility that one member will transfer wealth to another member in the future.

[7] Prelec (2004) and Gottlieb and Zhang (2021) refer to decision-makers who underestimate their time inconsistency and break their commitments as partially naive.

In the investment and asset allocation context, Garlappi et al. (2017) examine effects of group decisions where members have heterogeneous beliefs and conclude that time inconsistencies can lead to underinvestment. Ebert et al. (2020) introduce weighted discounting functions to study the decision behavior of groups that are uncertain about which discount rate to use. They examine the impact of group diversity on investment in a real options framework and find that greater group diversity leads to delayed investments. This also applies to investments with interpersonal uncertainty about the discount rate to be used: greater parameter uncertainty leads to delayed investments and higher risk taking. Jackson and Yariv (2015) confirm this and find that in addition to heterogeneity in time preferences in collective decisions, time inconsistency is also caused by heterogeneous risk preferences or beliefs. The aggregation of utility functions of household members that have time-consistent discount functions is similar to the aggregation of subjective preferences over lotteries.[8]

Glätzle-Rützler et al. (2021) find in a laboratory experiment that small groups consisting of three persons behave more patiently than time-inconsistent individuals. Depending on group composition, time inconsistency can be reduced. In the insurance context, Chen et al. (2021a) and Chen et al. (2021b) find that in a collectively managed pension fund, the welfare of individual investors does not decline, and individual optimal solutions are attainable when a financially fair sharing rule is applied. Delegating an investment decision to a fund manager or other experts who handle an individual's portfolio management is as commitment tool that can counter time inconsistency and avoid investment mistakes (cf. Malliaris and Malliaris (2021)).

Uncertainty-based view of time inconsistency:

While choice reversal over time may originate from inconsistent time preferences, it may also be the result of uncertainty about future outcomes. According to an uncertainty-based explanation, a decision-maker may revise her choice over time due to the fact that future outcomes or events are associated with default risk. Some researchers have criticized preference-based explanations and put forward uncertainty-based explanations that

[8] Chapter 4 examines this fact in a dynamic asset allocation context. Parameter uncertainty is assumed here via the assumption of an a priori lottery, where the investor is uncertain about the distribution parameters of the currently prevailing regime. The utility aggregation is non-linear and leads to time inconsistency.

interpret time inconsistency as the rational choice of exponential discounters that decide in the presence of uncertainty.[9]

Prelec and Loewenstein (1991) consider a number of parallels between risky and intertemporal decisions. Andersen et al. (2008) argue that time and uncertainty are closely related. Anything that is delayed and in the future is almost by definition uncertain. They conduct experiments to jointly estimate risk and time preferences. Because individuals are generally risk averse, they find that the joint elicitation results in estimates of discount rates that are significantly lower and more reasonable than those found in previous research. Caplin and Leahy (2001) extend expected utility theory by considering psychological aspects, linking lotteries to mental states. They argue that as time passes, anticipatory feelings about the future may change, resulting in time inconsistency. Halevy (2004) distinguishes in a consumption decision between a certain present and a risky future. The planned consumption path can be viewed as a lottery. Present-bias is attributed to the certainty of the present as opposed to the risk associated with any future path. Halevy (2008) shows that the presence of certainty plays a role in the generation of time-inconsistent preferences. Hyperbolic discounting can be reformulated in terms of non-expected utility probability weighting. If individuals prefer certainty, then present certain consumption is preferred to future uncertain consumption. They conclude that discounted expected utility is a good decision rule only in a pure risk situation. If a decision-maker uses a non-expected utility function (non-linear in the continuation probability), then she exhibits decreasing impatience and present-bias.[10]

Machina (1989) identifies as a feature of expected utility that it is linear in its probabilities. This means that each outcome probability pair $u(x_i)p_i$ is independent of the other outcome probability pair. Any non-linear functional forms of individual preference functions over lotteries no longer satisfy this criterion and dynamic inconsistencies occur.[11] Epper et al. (2009) investigate the relationship between probability weighting and

[9] Note that the expected utility model is used to evaluate risky decisions while the discounted utility model is employed in the context of intertemporal decisions.

[10] Allais (1953) and Kahneman and Tversky (1979) argue that when two options are far from certain, individuals effectively act as maximizers of discounted expected utility, whereas when one option is certain and another is uncertain, a disproportionate preference for certainty prevails. Due to the fact that certainty is a feature of the present, a decision-maker will exhibit present-bias. Andreoni and Sprenger (2012) also find that certainty is disproportionately preferred.

[11] We refer to the research presented in Chapter 4 and Chapter 5, in which the utility aggregation is highly

time discounting. They show that uncertainty generates hyperbolic discounting behavior even when time preferences are exponential. Individuals tend to have non-linear probability weighting when making decisions under uncertainty. Larger deviations from linear probability weighting lead to larger decreases in individual impatience. The valuation of monetary outcomes in the expected utility theory should be extended by emotional and cognitive aspects triggered by resolution of uncertainty. Walther (2010) introduces a non-linear probability weighting function in this context. When probabilities are weighted in a non-linear way, a behavior occurs that has already been described by hyperbolic discounting. Ebert and Strack (2015) examine the predictions of cumulative prospect theory in a dynamic context for a naive decision-maker who is unaware of her time inconsistency. The time inconsistency induced by probability weighting causes investors to postpone their investment decisions. Pennesi (2017) also studies the interaction between uncertainty and time preferences in consumption decisions. He introduces a variant of the discounted subjective expected utility model in which time preferences depend on the state. Thus, the individual is uncertain about the discount factor to use. The present values of the states are aggregated by the subjective probability p. The model is able to account for present-bias and decreasing impatience even when the future is exponentially discounted.

Casari (2009) examines time preferences under commitment when the future is risky and the present is known. He conducted an experimental study and argue that preferences for commitment or flexibility reveal whether time inconsistency is a preference-based or uncertainty-based phenomenon. He concludes that time-inconsistent behavior is preference-based for the majority in their sample due to the fact that there is a preference for commitment to limit the amount of choices available in the future. In a scenario without uncertainty, there would be no reason to strictly prefer flexibility. However, since some individuals opted for flexibility and were willing to accept costs in order to have more choice in the future, an uncertainty-based approach also seems to be relevant. Both explanatory approaches are not mutually exclusive. Preference-based approaches generally allow for the possibility that a decision-maker prefers to commit when she is aware of her time inconsistency. In uncertainty-based approaches, decision-makers usually have exponential time preferences and the choice reversal explanation is based on uncertainty. In all these models, commitment is never optimal.

In this section, the topic of time inconsistency was examined based on behavioral

non-linear, causing time inconsistency.

research that identifies time-inconsistent behavior, attempts to model it and proposes commitment instruments to mitigate time inconsistency. The evaluation of dealing with time inconsistency in the context of asset allocation is provided in the following section.

3.3 Accounting for time inconsistency in optimal asset allocation

There are three approaches to deal with time inconsistency in optimal asset allocation problems in the literature. First, there is the option to pre-commit (cf. Strotz (1955)). A pre-commitment strategy is a deterministic approach and is optimal only for time $t = 0$. It optimizes the objective function at the initial time when the decision is being made. A pre-commitment strategy does not resolve the time inconsistency but it is a realistic way to deal with this problem. Only the optimal strategy identified at the initial time is derived, regardless of the fact that it may not be optimal for the objective functionals in the future. Thus, this strategy simply does not take the change of future preferences into account. A pre-commitment strategy should be implemented only if it is effectively binding, i.e., that the investor can make a convincing commitment to stick with her decisions and not revise them at a later time. This can be achieved e.g., by investing in a product based on the pre-commitment strategy. A large number of the literature follows this approach.

An second approach that tries to transform the originally time-inconsistent problem into a time-consistent one is the strategy of consistent planning. Strotz (1955) initially proposed the strategy and argues that it should be implemented when pre-commitment is not possible. A decision-maker who is aware of her time inconsistency rejects the strategy she will not follow. At each decision point t, her problem is then to find the best strategy among all the strategies that she will actually follow. Thus, the decision-maker maximizes dynamic utility under the condition that her future behavior is optimal. A recursive approach is used to solve the problem. The feasible set of options is generated recursively by backward elimination of strictly dominated strategies (cf. Caplin and Leahy (2006)).

However, consistent planning is usually interpreted as a solution concept for a game played by "multiple egos" of the same individual, in which the investor in each period chooses a strategy that maximizes her objective in that period, taking into account the adjustments she will make in the future. Each decision point is assigned to a different self (cf. Siniscalchi (2011)). The intrapersonal game is solved by finding a subgame perfect

Nash equilibrium point, for example by backward induction.[12] In this way, the concept of optimality is replaced by that of Nash equilibrium. However, the approach assumes that the decision-maker has complete information about future outcomes and correct beliefs about her future actions. The goal is to reduce the difference between planned actions and expected temptations in the future. Thus, time inconsistency is solved strategically using a game theoretic approach (cf. Bjork and Murgoci (2010), Zhao et al. (2016)).

A third approach is the dynamically optimal strategy which is introduced in the mean-variance (MV) portfolio selection problem. The intuition behind this approach is the fact that individuals often prefer to use the optimal strategy based on the current state without regard to the plans and goals set at the very beginning. At any time t, a time-consistent investor can be viewed as the reincarnation of a pre-commitment investor who implements the optimal pre-commitment strategy for that time t. She chooses a different optimal strategy that maximizes the objective functional at any time t. The investor forgets about her past and ignores her future. The strategy is intuitive because it takes into account the behavior of a naive decision-maker who constantly reevaluates her position and solves infinitely many problems in an optimal way (cf. Karnam et al. (2017), Vigna (2020)).

Optimal stochastic control problems are generally considered to be time-inconsistent. The optimal strategy chosen at one point in time is then no longer optimal at another point in time. The dynamic programming principle cannot be applied to solve the problem, and Bellman's principle does not hold (cf. Bjork and Murgoci (2010)).

For a further representation of time inconsistency in general stochastic control problems, see Björk and Murgoci (2014), for the discrete-time framework and Björk et al. (2017) for the continuous-time framework. A specific problem that leads to time-inconsistent behavior is the investment-consumption problem with non-exponential discounting (cf. Strotz (1955)). Another time-inconsistent problem is the dynamic MV selection problem, where the time inconsistency is due to the fact that the objective criterion contains a non-linear function of the expectation of the final wealth. The problem cannot be solved directly by dynamic programming due to the non-separable structure of the variance minimization

[12] In a Nash equilibrium, players play each other's best strategies, maximizing their payoff while taking into account each other's decision. Each player chooses exactly one strategy from which it makes no sense for any player to to deviate from. No player can improve her position by deviating from the strategy, so there are no incentives to change behavior.

problem. The conditional variance is a non-linear function of the expected value of the terminal wealth (cf. Bayraktar et al. (2019)). Time inconsistency also arises in various other asset allocation problems whenever an aggregation over non-linear functions (e.g., expected utilities, certainty equivalents) takes place (cf. Desmettre and Steffensen (2021), Becker et al. (2022)). Table 3.2 provides an overview of research that implements the three ways of dealing with time inconsistency in optimal asset allocation problems.

Table 3.2: Selected papers dealing with time inconsistency in optimal asset allocation

Paper	**Resolution**	**Overview/Research topic**
Basak and Chabakauri (2010)	PC, GT	solve the dynamic MV problem in an imperfect market and derive a time-consistent solution via a recursive approach. For short horizons, the pre-commitment solution approximates the time-consistent solution. The strategies coincide when the market price of risk is zero since no assets are invested in risky stocks. The effect of time inconsistency increases for longer time horizons. For the case of a constant market price for risk, the expected terminal value of the pre-commitment strategy is higher for sufficiently long investment horizons than in the case of the time-consistent strategy.
Wang and Forsyth (2011)	PC, GT	compare the optimal pre-commitment with the optimal dynamically consistent strategy under constraints (e.g., no bankruptcy, no short selling) in the dynamic MV framework. Although the investment strategies are different, the efficient frontiers of both strategies are quite similar. The pre-commitment strategy is more efficient because it is a globally optimal strategy.
Cui et al. (2012)	PC	extend a dynamic MV problem in which a pre-commitment strategy is performed by relaxing self-financing constraints. Money withdrawals from the market are allowed, whereby in addition to the final wealth achieved by the pre-commitment solution, the investor can obtain a free cash flow stream during the investment process.

Czichowsky (2013)	GT	extends the problem in Basak and Chabakauri (2010) by considering a MV problem both in discrete time and in continuous time and solves the problam with the consistent planning approach.
Lioui (2013)	PC, GT	also compares the dynamic time-consistent strategy with the pre-commitment strategy in a multi-asset MV framework. The pre-commitment strategy dominates the time-consistent strategy with respect to the certainty equivalent. However, these welfare gains can result from huge and unrealistic positions in the risky assets.
Kronborg and Steffensen (2015)	GT	investigate a time-inconsistent investment and consumption problem for a MV investor where a capital injection in form of a deterministic labor income is considered. For a more realistic model, risk aversion is time and state dependent. The optimal time-consistent solution is derived.
Li et al. (2015)	PC, GT	consider the time-inconsistent reinsurance investment problem under the MV criterion. The pre-commitment strategy is compared with the dynamic time-consistent strategy: The time-consistent reinsurance strategy is independent of current wealth while the pre-commitment reinsurance strategy is a function of the current wealth. The time-consistent investment strategy as well as the pre-commitment investment strategy depend both on current wealth.
Cong and Oosterlee (2016)	PC, GT	consider a MV optimal asset allocation problem and establish a link between a time-consistent and a pre-commitment investment strategy. The investment target of a time-consistent investor varies over time while a pre-commitment investor has a fixed target. A hybrid strategy is defined by introducing a fixed target into the time-consistent strategy and is solved by a numerical algorithm. The hybrid strategy produces a better MV efficient frontier than the time-consistent strategy.

Zhao et al. (2016)	GT	consider an optimal investment and reinsurance problem with a defaultable bond for an insurer under the MV criterion in a jump-diffusion risk model. The optimal time-consistent strategy to maximize the objective function involving the expected value and variance of terminal wealth is obtained via a game-theoretic framework. Closed form solutions are set up via extended Hamilton-Jacobi-Bellman system of equations.
Shi et al. (2017)	PC	formulate a continuous-time MV problem in a jump-diffusion market and derive the optimal pre-commitment strategy. When the investor's wealth level exceeds a certain level, the pre-committed strategy leads to irrational investment behavior. Therefore, a semi-self-financing strategy in which the investor can withdraw part of her wealth from the market is established. The new strategy leads to a better investment performance as it achieves the same mean-variance pair and obtains a non-negative free cash flow stream.
Pun (2018)	GT	considers an ambiguity-averse investor with time-inconsistent preferences in a dynamic MV portfolio problem. The investor is concerned about model uncertainty in the sense that she does not fully trust the reference model of the controlled Markov state process. The problem is solved using a game theoretic framework to characterize the robust dynamic optimal control of the problem.
Van Staden et al. (2018)	PC, GT	study the MV problem under different types of investment constraints (e.g., no trading under bankruptcy, leverage constraints, and different interest rates) where no closed-form solution is known. The fraction invested in the risky asset in the pre-commitment strategy is more affected by the maximum leverage constraint than the fraction in the time-consistent strategy.
Bayraktar et al. (2019)	GT	formulate an infinite horizon MV stopping problem as a subgame perfect Nash equilibrium and determine time-consistent strategies.

Bensoussan et al. (2019)	PC, GT	consider a MV problem with the constraining option that no short selling is allowed. With the addition of the constraint, the payoff can improve within the game-theoretic approach, whereas this is not the case with the pre-commitment strategy.
Christensen and Lindensjö (2020)	GT	develop a game-theoretic framework for time-inconsistent stopping problems where time inconsistency arises due to the consideration of a non-linear function of an expected reward. A subgame perfect Nash equilibrium is found for a MV problem.
Menoncin and Vigna (2020)	PC, DS	deal with the selection of a MV portfolio for a defined contribution pension fund. Numerical results show that the median of the proportion of risky assets is lower for the pre-commitment strategy than for the dynamic optimal strategy and the variance of wealth is lower in case of implementing a pre-commitment strategy. The dynamically optimal strategy may react better to extreme scenarios of market returns due to the continuous adjustment of the final objective.
Strub and Li (2020)	DS	investigate portfolio optimization with reference point updating. There is only one framework that predicts realistic investment behavior: The decision-maker cannot anticipate the reference point update and therefore faces a time-inconsistent problem. A dynamically optimal strategy is solved where the reference point is updated in a non-recursive way.
Vigna (2020)	PC, GT, DS	compares the three different approaches in dealing with time inconsistency for a MV portfolio problem. The pre-commitment strategy beats the other two strategies, while the consistent planning strategy dominates the dynamically optimal strategy up to a time $t^* \in [t_0, T]$ and is dominated by the dynamically optimal strategy from t^* onward.
Alia et al. (2021)	GT	study the Merton portfolio management problem in the context of non-exponential discounting and use a game-theoretic approach to deal with time inconsistency. An explicit representation of equilibrium policies is provided for the special cases of power, logarithmic and exponential utility functions.

Balter and Schweizer (2021)	PC, GT	consider the collective investment decision of a planner whose agents have heterogeneous risk preferences. The planner applies a concave utility function to the distribution of individual certainty equivalents to calculate the welfare that results from different decisions. The pre-commitment strategy beats the dynamically consistent strategy from the t_0 perspective. However, there will always be a point of regret where the investor wishes she had not made a pre-commitment.
Bosserhoff and Stadje (2021)	GT	consider a time-consistent MV portfolio selection problem of an insurer and include basis risk (mortality). The optimal solution is identified using a Nash subgame perfect equilibrium.
Dai et al. (2021)	GT	consider a dynamic portfolio choice model with a MV criterion for the log returns of the portfolio. Using backward stochastic differential equations, which can be solved numerically, the optimal time-consistent strategy can be found.
Desmettre and Steffensen (2021)	GT	solve the problem of an investor who maximizes utility but is uncertain about her preferences. Time inconsistency arises from the aggregation of certainty equivalents. A time-consistent strategy is formed by equilibrium theory approach.

PC= Pre-commitment, GT = Game theoretic, DS= Dynamically strategy

The literature review reveals that many research papers compare the different approaches of dealing with time inconsistency in asset allocation problems. The approach to deal with time inconsistency should be chosen in an individual way and dependent on the type of decision-maker and the nature of the considered optimization problem (cf. Vigna (2020)).

Time consistency of strategies is a basic requirement for rational decision-making in decision theory. Therefore, it seems attractive to find time-consistent strategies. The dynamically optimal strategy may react better to extreme scenarios of market returns due to the continuous adjustment of the final objective. The game theoretic approach is difficult to implement in the context of continuous time (since it is a game with an infinite number of players) (cf. Karnam et al. (2017)). Furthermore, the decision-maker must have full information about future outcomes and correct beliefs about her future actions. However, in reality there are many informational gaps (Zhao et al. (2016)).

The time-inconsistent pre-commitment strategy dominates the other strategies because it optimizes the objective function at the initial time ($t = 0$), that is, at the time the decision is made. However, the decision-maker must be able to pre-commit. Nevertheless, the approach leads to the strategy remaining time-inconsistent (cf. Wang and Forsyth (2011)). Depending on parameter constellations and model assumptions, the pre-commitment strategy may lead to unrealistic positions in the risky asset (cf. Lioui (2013)). The decision on which approach to use in asset allocation problems is therefore not trivial and should depend on the individual setup and information situation (cf. Bensoussan et al. (2019)).

3.4 Conclusion

The paper provides a comprehensive literature review on time inconsistency, in particular how to deal with this anomaly in asset allocation problems. Time inconsistency describes the fact that individuals want to revise a decision they already made at a later point in time without any change in information. Behavioral research concludes that time inconsistency may lead to suboptimal decisions due to the deviation from plans (e.g., undersaving, overconsumption, procrastination). Long-term investment and consumption plans are incompatible with short-term investment and consumption preferences. Commitment strategies can counteract self-control problems such as time inconsistency. The demand for commitment instruments depends on the level of sophistication: Naive decision-makers do not know that their preferences change over time and therefore do not demand commitment instruments. Sophisticated individuals, on the other hand, are aware of their time-inconsistent behavior over time and demand commitment instruments. Mitigating negative effects of time inconsistency can be done through appropriate provision of commitment instruments and reducing uncertainty via more transparent design of policy reforms and should be further explored in research (cf. Baddeley (2019)).

In theoretical works, time inconsistency is assumed deterministically by the assumptions of hyperbolic or quasi-hyperbolic discount functions. In stochastic control problems, the problem of time inconsistency arises naturally via the aggregation of non-linear functions. The evaluation of dealing with time inconsistency in asset allocation research is presented. In this case, time inconsistency is mostly solved by a game theoretic approach or a pre-commitment strategy is implemented. Time consistency is considered as a basic requirement for rational decision-making. Therefore, it seems attractive to find time-consistent strategies (e.g., using the game-theoretic approach). However, time-consistent strategies are more difficult to implement and specific model assumptions have to be made for the implementation.

The time-inconsistent pre-commitment strategy is more efficient than other approaches because it optimizes the objective function at the initial time when the decision is being made. A pre-commitment strategy does not resolve the time inconsistency, but it is a realistic way to deal with time-inconsistent behavior and can mitigate the deviation of decisions that leads to suboptimal outcomes (see behavioral literature on commitment instruments). Thus, in an optimization problem that leads to time inconsistency, the answer in dealing with it is not trivial and depends on model assumptions and subjective factors such as the attitude towards time consistency and preference for commitment.

Chapter 4

On the impact of time inconsistency in optimal asset allocation problems

4.1 Introduction

In this chapter we analyze the impact of time inconsistency on the optimal asset allocation problem of a risk and ambiguity averse investor.[1] It is well understood that the problem of time inconsistency naturally arises by aggregating utilities or certainty equivalents (cf. Jackson and Yariv (2015), and Desmettre and Steffensen (2021)). Such an aggregation results when considering uncertainty about preferences or the welfare of a collective of heterogeneous investors, where heterogeneity can prevail with respect to beliefs, time preferences, and risk preference.

Classical financial market theory assumes that decision-makers act as time consistent rational utility maximizers. Behavioral economics deals with non-rational human behavior in economic situations and examines constellations in which decision-makers act in contradiction to the model of the homo oeconomicus. It thus also accounts for anomalies like time-inconsistent behavior.

[1] Time inconsistency is commonly used to refer to the change of a decision-maker's preference for a particular future outcome over another with the passage of time.

As a consequence of time inconsistency, the behavior of a decision-maker might change depending on the timing of payoffs, and she thus might want to revise a decision already made at a later point in time (cf. Strotz (1955), DellaVigna (2009)).[2]

We consider a stylized model setup which allows for the analysis of time inconsistency in these situations caused by various kinds of utility aggregation. It builds on the Merton problem where an investor with constant relative risk aversion maximizes her expected utility by splitting her wealth between a risky and a risk-free asset. The risk-free asset grows at a constant rate; the price dynamics of the risky asset are given by a geometric Brownian motion with drift μ and volatility σ. We introduce an a priori lottery $(p, 1-p)$ over two regimes which differ in the (μ, σ)-tupel. Thus, we introduce a second dimension of risk which implies an outer expectation about the outcome of the lottery and an inner expectation about the utility within the regimes.[3]

We assume that the investor deals with time inconsistency by following a pre-commitment strategy[4], and we show that the optimal pre-commitment strategy is deterministic in our setup. It is between the regime-dependent Merton solutions and thus can be represented as a weighted average of these solutions. In the myopic case, when either the investor has myopic logarithmic preferences or the planning horizon declines to zero, the weighting factors of the regimes only depend on the regime probabilities and volatilities. The longer the investment horizon, the more the optimal strategy converges to the limiting worst-case strategy, i.e., the investor maximizes the worst-case savings rate over the regimes and thus follows a maximin decision rule which no longer depends on the regime probabilities.

The time inconsistency of the optimal pre-commitment strategy for a level of relative

[2] Thaler (1981) provides first empirical evidence on time-inconsistent behavior. Further empirical research papers that draw attention to the problem of time inconsistency in decision making are Kirby and Herrnstein (1995), Wong (2008) and Schreiber and Weber (2016). Time inconsistency can lead to biased non-optimal decisions that are in conflict with the long-run interests of decision-makers (cf. Read and Van Leeuwen (1998), Gruber and Köszegi (2001), Meier and Sprenger (2010), Ameriks et al. (2007), Jones and Mahajan (2015)).
In theoretical papers, time inconsistency is often considered deterministically by assuming a non-exponential discount function. Yılmaz (2013) examines heterogeneous time preferences in a principal-agent-model. Further papers modeling time inconsistency via a non-exponential discount function are Laibson (1997) , Azfar (1999), Benhabib et al. (2010) and Yoon (2020).

[3] The outer expectation implies an aggregation of utilities.

[4] For a general discussion of strategies in case of time inconsistency, see Strotz (1955).

risk aversion $\gamma > 1$ and an investment horizon $T > 0$ can be traced back to the impact of the investment horizon on the certainty equivalents in the two regimes. These certainty equivalents determine the trade-off between speculating on the better regime (and following the optimal strategy for the good regime) and hedging against the worse regime (and following the optimal strategy for the bad regime). The larger the difference between the two regimes, the stronger the hedging motive, and the more the optimal strategy moves towards the worst-case strategy.[5]

In line with this behavior of the optimal strategy, the optimal savings rate is given by the expected savings rate over the regimes for $T \to 0$ and by the worst-case savings rate over the regimes for $T \to \infty$. In both limiting cases ($T \to 0$, $T \to \infty$), the willingness to pay for information about the regime is zero. This implies that the value of information[6] is maximal for some finite investment horizon but does not necessarily increase in the length of the time horizon over which a suboptimal pre-commitment strategy (instead of the overall optimal strategy for known regimes under full information) is followed.

While our setup is stylized, it captures a variety of different scenarios. As pointed out, we describe a situation of risk with respect to the true regime. We show that our setup also covers the case of state-dependent preferences, when risk preferences differ across regimes. In terms of an aggregation over investors who follow a common investment strategy, our setup allows for heterogeneous beliefs and for heterogeneous preferences.

In our analysis, we abstract from the case of (gradual) learning of the true regime. Note, however, that we include the case of 'maximal learning' in which the investor is told the true regime immediately; the gain in the certainty equivalent in this case goes to zero for both $T \to 0$ and $T \to \infty$. Since this gain is an upper bound for the case of gradual learning, we can conclude that the gain from learning is also zero for $T \to 0$ and $T \to \infty$ when the investor learns over time.

We add a third dimension to our decision problem by considering ambiguity w.r.t. the regime probabilities $(p, 1-p)$.[7] In dealing with ambiguity, we follow Klibanoff et al. (2005)

[5] If there is only a difference in μ but not in σ, the myopic and long term behavior is the same as in the case of gradual learning over time, cf. Bäuerle and Grether (2017).

[6] For more information on the value of information from a decision-theoretical point of view, see Appendix A.1.2.

[7] Accounting for ambiguity dates back to Arrow (1951). There is strong empirical evidence for the existence of ambiguity in decision making: Antoniou et al. (2015), Brenner and Izhakian (2018) and Dimmock

and use the smooth model of ambiguity aversion. The impact of ambiguity preferences over the regime probabilities (third dimension) can then be analyzed analogously to the impact of risk preferences over the regimes (second dimension). Intuitively, the setup gives rise to a trade-off between the amount of ambiguity over probabilities and the amount of risk over the regimes. To simplify the exposition, we assume that there are two possible distributions $(p_a, 1 - p_a)$ and $(p_b, 1 - p_b)$. The amount of ambiguity is increasing in the spread between the probabilities p_a and p_b. In the limit, $p_a = 1$ and $p_b = 0$, the regime is known and the risk in the second dimension has vanished; we are left with risk about the normally distributed return and ambiguity about the true regime. For $p_a = p_b$, on the other hand, ambiguity w.r.t. the probability distribution has vanished, and we are left with risk about the normally distributed return and risk about the true regime.

In summary, our main contribution is to explain the common impacts of aggregating expected utilities of future wealth or certainty equivalents on optimal decisions and utilities. We bring together various strands of the literature and identify the common drivers of time inconsistency under risk and ambiguity. In addition, our analysis of the value of information gives a simple explanation why gradually learning over time is not able to resolve time inconsistency. While the results are presented in an asset allocation context, they straightforwardly carry out to other decision problems which account for state-dependent preferences, risk preferences which differ across regimes, and/or the welfare of a collective of heterogeneous decision-makers. One example are group decisions (of the management) in corporate finance.

The first dimension of risk in our setup already dates back to Merton (1971). He solves the problem of maximizing the expected utility of an investor with constant relevant risk aversion (CRRA) in a Black-Scholes model setup. Our paper relates to the literature on collective decision making and aggregation over (heterogeneous) investors. One way to aggregate preferences is the utilitarian approach of aggregating utilities. Adams et al. (2014) and Jackson and Yariv (2014) show that heterogeneity in time preferences results in a present-bias even if individual preferences are time-consistent. Garlappi et al. (2017) show that group decisions of decision-makers with heterogeneous beliefs are dynamically inconsistent. Another way of accumulation is to aggregate certainty equivalents, as argued

et al. (2016b) find that ambiguity is priced in the equity market. An increase of ambiguity leads to underinvestments (cf. Garlappi et al. (2017)). Ambiguity in portfolio choice is widely spread in the literature (see the literature overview of Guidolin and Rinaldi (2013)).

in Kryger and Steffensen (2010) for their titular problem of 'collective objectives'. Jensen and Steffensen (2015) and Fahrenwaldt et al. (2020) aggregate certainty equivalents to disentangle time and risk preferences. Desmettre and Steffensen (2021) and Balter and Schweizer (2021) aggregate over certainty equivalents to deal with uncertain preferences, while Balter et al. (2021) study the case of drift uncertainty.

Joint decisions for heterogeneous investors might entail utility losses. Alserda et al. (2019) document heterogeneous risk aversion in the pension domain. Jensen and Nielsen (2016) analyze the sub-optimality of linear sharing rules in case of collective investment. Non-linear sharing rules are considered by Chen et al. (2021a) with heterogeneous risk aversion under stochastic volatility and by Chen et al. (2021b) for the case of heterogeneous guarantee levels. Pazdera et al. (2016) study the problem in an incomplete market.

Due to the assumption of a double risk situation our problem is similar to a setup in which two investors with heterogeneous beliefs are restricted to follow the same strategy, thus time inconsistency arises naturally. The strand of literature referring to time inconsistency in optimal asset allocation problems can be traced back to Strotz (1955). He proposes two strategies for dealing with time inconsistency – a strategy of pre-commitment and a strategy of consistent planning. Balter et al. (2021) compare a pre-commitment strategy with a dynamically consistent one in the context of ambiguity and learning and determine a point of regret for a pre-commitment investor. Björk and Murgoci (2014) and Björk et al. (2017) account for time inconsistency in stochastic control problems. They derive a game theoretical solution within a discrete-time and a continuous-time framework. These two papers build on the work of Basak and Chabakauri (2010) who study dynamic portfolio choice under mean-variance preferences. They show that the optimal investment strategy is time-inconsistent and find a distinction between pre-commitment, dynamically consistent, and myopic strategies. Further literature in this context is given by Cong and Oosterlee (2016), Pedersen and Peskir (2017), Dai et al. (2021) and the recent papers of Vigna (2020) and Van Staden et al. (2021) who compare dynamically consistent and pre-commitment strategies in a mean-variance setup.

Biagini and Pınar (2017) derive a robust solution of the Merton problem of an ambiguity averse investor. Borgonovo and Marinacci (2015) give results for certainty equivalents in a multi-event problem in the presence of risk and ambiguity aversion. Jin and Zhou (2015) analyze a portfolio choice problem in an expected utility and mean-variance framework by maximizing the worst Sharpe ratio. Further literature in the context of ambiguity in a mean-variance framework is given by Maccheroni et al. (2013), Pflug and

Wozabal (2007) and Pınar (2014).

Our analysis of the value of information is closely linked to the topic of learning. We refer the interested reader to portfolio optimization problems with unknown drift process (e.g., Karatzas and Zhao (2001)). Our limiting result concerning a long-term investor resembles the findings of Bäuerle and Grether (2017) who consider a Bayesian financial market.

The a priori lottery which gives our second risk dimension is a stylized version of a regime-switching model. In particular, our special case of full information resembles a stylized variant of a regime-switching model with observable Markov chain (regime, respectively). While in our case, it is straightforward that the optimal strategy only depends on the regime, this is also true in a (more sophisticated) dynamic setup. Sotomayor and Cadenillas (2009) and Ocejo (2018) solve the asset allocation problem for diffusion processes in a regime-switching model, Capponi and Figueroa-López (2014) extend the analysis to defaultable bonds. Zhou and Yin (2003) solve Markowitz mean-variance portfolio selection in continuous-time.

The remainder of Chapter 4 is organized as follows. In Section 4.2, we give a brief review of the basic Merton results (optimal strategy, corresponding utility, and savings rate). Subsequently, we introduce our stylized setup with an a priori lottery which results in one of two regimes (characterized by the drift and volatility of the risky asset). Along the way, we comment on the links of our stylized setup to other sources for aggregating utilities and on the link to behavioral biases. In Section 4.3, we derive the optimal precommitment strategy. Comparing the case with observed regimes (full learning) to the case with unobserved regimes gives us the value of information about the regime which we analyze in more detail in Section 4.4. In Section 4.5, we additionally account for ambiguity about the regime probabilities and show the analogies and differences stemming from risk and ambiguity aversion. Section 4.6 concludes.

4.2 Model assumptions

We modify the classic Merton problem (cf. Merton (1971)) by introducing an a priori lottery where the outcome is one of two regimes. Once the regime is known, the investment problem boils down to the Merton problem. To simplify the exposition, we give a review of the Merton problem as well as some basic results.

4.2.1 Merton problem

Throughout the following, we consider an investor with constant relative risk aversion (CRRA), i.e., her utility function is

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \gamma > 1 \\ \ln x & \gamma = 1 \end{cases},$$

where γ denotes her relative risk aversion. The investor is equipped with an initial amount of V_0 which we can normalize to $V_0 = 1$ because of the CRRA framework. We only consider $\gamma \geq 1$ as the usual choice in asset allocation.

The investment decisions of the investor are given in terms of the fraction π of her portfolio wealth which she invests in a risky asset, the stock S. The remaining fraction is invested in a risk-free asset growing with constant interest rate r. In the benchmark model of Black-Scholes, the dynamics of the stock price are given by a geometric Brownian motion with constant drift μ and volatility σ, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t \text{ where } S_0 = s_0.$$

W_t is a Brownian motion under the probability measure P. If the investor with investment horizon T ($T \geq 0$) chooses the weight π_t for the risky investment in the stock at time t ($0 \leq t \leq T$), the dynamics of her wealth are given by

$$dV_t = [r + \pi_t(\mu - r)]V_t dt + \pi_t \sigma V_t dW_t.$$

The optimal investment strategy π_t^* (which maximizes the expected utility of terminal wealth V_T) is given by the well-known constant Merton fraction π^{Mer}, i.e.[8]

$$\pi_t^* = \pi^{\text{Mer}} = \frac{\mu - r}{\gamma \sigma^2}. \tag{4.1}$$

This strategy also maximizes the certainty equivalent CE_T and the savings rate y_T which are in general defined by

$$u(CE_T) = E[u(V_T)] \quad \text{and} \quad y_T = \frac{1}{T} \ln CE_T.$$

[8] The optimal strategy implies a constant investment fraction. With no uncertainty about the future dynamics, there are no state variables to condition on, and with CRRA, there is also no need to condition on current wealth. Furthermore, a time-dependent strategy would increase the variance without increasing the mean, and is thus dominated by a time-independent strategy. For more information on the Merton problem in a Black Scholes setup, see the Appendix A.1.1.

The savings rate y (we drop the index T since the savings rate in our setting is independent of T) for a constant portfolio weight π is given by[9]

$$y(\pi) = r + \pi(\mu - r) - \frac{1}{2}\gamma\pi^2\sigma^2. \tag{4.2}$$

For a positive portfolio weight, it is increasing in μ and decreasing in σ. The maximal savings rate is

$$y(\pi^{\text{Mer}}) = r + \frac{1}{2\gamma} \cdot \frac{(\mu - r)^2}{\sigma^2} = r + \frac{1}{2\gamma} \cdot \lambda^2,$$

where $\lambda = \frac{\mu - r}{\sigma}$ denotes the constant market price of risk. In view of the introduction of two regimes, we also define the loss rate if a strategy π instead of the optimal strategy π^{Mer} is used. It is given by

$$l(\pi) = y(\pi^{\text{Mer}}) - y(\pi) = \frac{1}{2}\gamma\sigma^2\left(\pi - \pi^{\text{Mer}}\right)^2. \tag{4.3}$$

4.2.2 Lottery over regimes

Now, we consider two regimes which reflect different dynamics of the stock price or different beliefs of the investor about these dynamics.[10] In regime i ($i \in \{1, 2\}$), the drift and the volatility of the stock are denoted by μ_i and σ_i. The risk-free rate is constant and equal to r in all regimes. In particular, we now assume

$$dS_{t,i} = \mu_i S_{t,i} dt + \sigma_i S_{t,i} dW_{t,i} \text{ for } i = 1, 2,$$

where $S_{0,1} = S_{0,2} = s_0$ and where $W_{t,i}$ is a Brownian motion under the probability measure P_i. We interpret the two regimes as a good (Regime 1) and a bad (Regime 2) one, and we assume that the investor chooses (in the optimum within a regime) a higher stock weight in the good than in the bad regime:

Assumption 1 (Regimes)
Throughout the paper, we make the following assumptions on the regimes:

[9] Since we have assumed normally distributed log returns, the maximization of utility in the base case is equivalent to a mean-variance portfolio selection problem.

[10] Generalizing our results to n ($n \geq 2$) regimes is straightforward.

1. *Regime 1 is the good regime and Regime 2 is the bad regime, i.e., it holds that*

$$\lambda_1 = \frac{\mu_1 - r}{\sigma_1} \quad > \quad \lambda_2 = \frac{\mu_2 - r}{\sigma_2}.$$

2. *The optimal portfolio weight in the good regime exceeds the optimal portfolio weight in the bad regime, i.e., it holds that*

$$\pi_1^{Mer} = \frac{\mu_1 - r}{\gamma\sigma_1^2} \quad > \quad \pi_2^{Mer} = \frac{\mu_2 - r}{\gamma\sigma_2^2}.$$

For expositional simplicity, we immediately restrict ourselves to constant weights π. Explanations why this is a reasonable simplification which accounts for common problems stemming from utility aggregation are provided in Section 4.3 (cf. Remark 2). We use the convention that $y(\pi, i)$ ($i \in \{1, 2\}$) denotes the savings rate within Regime i, and that $\pi_i^{\text{Mer}} = \frac{\mu_i - r}{\gamma\sigma_i^2}$ denotes the Merton strategy in Regime i. With Eqn. (4.2), it holds (for $i \in \{1, 2\}$)

$$y(\pi, i) = r + \pi(\mu_i - r) - \frac{1}{2}\gamma\pi^2\sigma_i^2.$$

The investment fraction $\pi \neq 0$ for which $y(\pi, 1) = y(\pi, 2)$ is denoted by π^{equal}. Straightforward calculations give

$$y(\pi, 1) - y(\pi, 2) = \begin{cases} \frac{1}{2}\gamma(\sigma_1^2 - \sigma_2^2)\pi(\pi^{\text{equal}} - \pi) & \sigma_1 \neq \sigma_2 \\ \pi\,(\mu_1 - \mu_2) & \sigma_1 = \sigma_2 \end{cases}, \tag{4.4}$$

where, for $\sigma_1 \neq \sigma_2$,

$$\pi^{\text{equal}} = \frac{\mu_1 - \mu_2}{\frac{1}{2}\gamma(\sigma_1^2 - \sigma_2^2)}. \tag{4.5}$$

For $\sigma_1 \neq \sigma_2$, the two regimes thus result in the same savings rate for the trivial choice $\pi = 0$ (then, the savings rates coincide with the risk-free rate) and for $\pi = \pi^{\text{equal}}$. For $\sigma_1 = \sigma_2$, $y(\pi, 1)$ and $y(\pi, 2)$ only coincide for the trivial choice $\pi = 0$.

Combining the results on π^{equal} and Assumption 1 allows us to single out the two cases that are interesting for our analysis later. In the straightforward first case, Regime 1 comes with a higher expected return and a lower (or the same) volatility than Regime 2 – π^{equal} is negative (or does not exist). For positive portfolio weights, it then always holds true that the savings rate is higher in Regime 1 than in Regime 2, i.e., Regime 2 is the worse one for all $\pi > 0$. In the second case, Regime 1 has a larger expected return and a larger volatility (or a lower expected return and a lower volatility) than Regime 2. Then,

Benchmark parameter

μ_1	μ_2	σ_1	σ_2	r
0.1316	0.0769	0.2080	0.2221	0.00

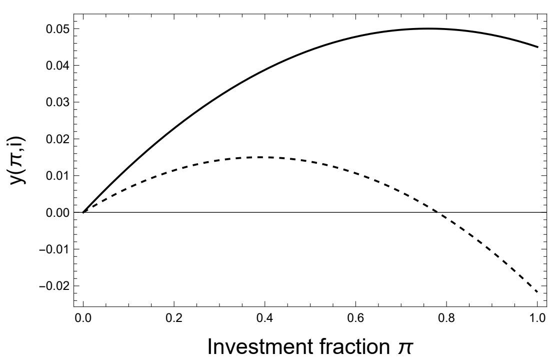
Table 4.1: Benchmark parameter constellation.

the difference of the savings rates switches sign for $\pi = \pi^{\text{equal}} > 0$.[11] π^{equal} is between π_1^{Mer} and π_2^{Mer} for

$$\frac{\sigma_1\sigma_2}{0.5(\sigma_1^2+\sigma_2^2)} < \frac{(\mu_1-r)/\sigma_1}{(\mu_2-r)/\sigma_2} < \frac{0.5(\sigma_1^2+\sigma_2^2)}{\sigma_1\sigma_2}.$$

The fraction in the middle is the ratio λ_1/λ_2 of the market prices of risk. Since the first part of Assumption 1 implies $\lambda_1/\lambda_2 > 1$, the left inequality always holds true. If the right hand side inequality is met, too, then Regime 1 is the worse one for π between π_2^{Mer} and π^{equal}, and Regime 2 is the worse one for π between π^{equal} and π_1^{Mer}. Figure 4.1 illustrates the savings rate $y(\pi, i)$ as a function of π.[12]

Savings rate $y(\pi, i)$ depending on investment fraction π



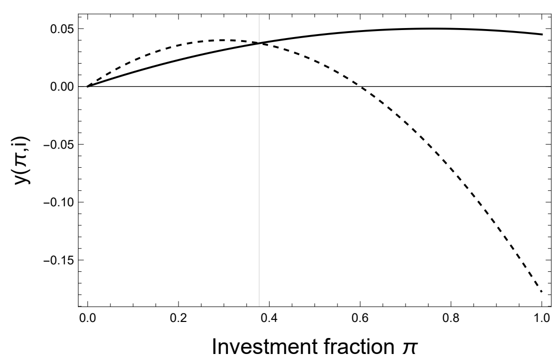


Figure 4.1: The left figure displays the savings rate depending on the investment fraction π. The right figure displays the savings rate for a different parameter constellation, where $\pi^{\text{equal}} = 0.3776$. In comparison to the benchmark parameter setup, $\mu_1 < \mu_2$ ($\mu_1 = 0.1316, \mu_2 = 0.2667$) and $\sigma_1 < \sigma_2$ ($\sigma_1 = 0.2080, \sigma_2 = 0.4714$). In both illustrations for $\gamma_R = 4$ the black graph pictures $y(\pi, 1)$ and the black dashed $y(\pi, 2)$.

[11] For $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$, $y(\pi, 1)$ exceeds $y(\pi, 2)$ when π is between zero and π^{equal}. For $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$, $y(\pi, 1)$ is smaller than $y(\pi, 2)$ when π is between zero and π^{equal}.

[12] Note, that we also run a **robustness analysis** regarding all illustrations with a second parameter constellation. For more information, see the Appendix A.4.

In our stylized modification of the Merton problem (”initial lottery”) the regime is determined by the lottery $L = (p, 1 - p)$ at time 0 and then stays constant over time. The lottery thus adds a second dimension to the risk situation. The expected utility (EU) of terminal wealth V_T (immediately before the lottery L takes place) is[13]

$$EU_{T,p} = p\, E_{P_1}[u(V_T)] + (1-p)E_{P_2}[u(V_T)]. \tag{4.6}$$

For the following analysis, we assume a more general case and distinguish between the levels of relative risk aversion γ_R within a regime and γ_L when aggregating over the regimes. The expected utility of the investor is then given by

$$EU_{T,p} = p\, u_L\left(u_R^{-1}(E_{P_1}[u_R(V_T)])\right) + (1-p)u_L\left(u_R^{-1}(E_{P_2}[u_R(V_T)])\right), \tag{4.7}$$

where u_L (u_R^{-1}) is a CRRA function with relative risk aversion γ_L (γ_R). Distinguishing between these two risk aversions allows us to separately study the impact of uncertainty within each regime and uncertainty about the regime. Setting $\gamma_R = \gamma_L = \gamma$ gives Eqn. (4.6) again.

$EU_{T,p}$ aggregates the savings rates (or certainty equivalents) over the regimes.[14] It holds that

$$EU_{T,p} = \begin{cases} \frac{1}{1-\gamma_L}\left[pe^{y(\pi,1)(1-\gamma_L)T} + (1-p)e^{y(\pi,2)(1-\gamma_L)T}\right] & \gamma_L > 1 \\ \left[py(\pi,1) + (1-p)y(\pi,2)\right]T & \gamma_L = 1 \end{cases}.$$

Instead of expected utility (4.7), one can also look at the certainty equivalent savings rate $y_{T,p}$ which is defined as

$$y_{T,p}(\pi) := \frac{1}{T}\ln\left(u_L^{-1}\left(EU_{T,p}\right)\right), \tag{4.8}$$

i.e.

$$y_{T,p}(\pi) = \begin{cases} \frac{1}{(1-\gamma_L)T}\ln\left[pe^{y(\pi,1)(1-\gamma_L)T} + (1-p)e^{y(\pi,2)(1-\gamma_L)T}\right] & \gamma_L > 1 \\ py(\pi,1) + (1-p)y(\pi,2) & \gamma_L = 1 \end{cases}. \tag{4.9}$$

[13] While the utilities refer to the ones obtained in different regimes, our setup is similar to the problem of a social planner who aggregates the utilities of investors with different beliefs or different levels of risk aversion (cf. literature given in the introduction of the paper).

[14] Eqn. (4.7) emphasizes that we aggregate over certainty equivalents, not over utilities, similar to Desmettre and Steffensen (2021) and the references given herein (cf. comment introduction) who also introduce time inconsistency by aggregation of certainty equivalents.

Due to the double risk situation, i.e., a lottery over two different Merton problems, risk aversion comes into play twice. First, the investor uses a CRRA-utility function with relative risk aversion γ_R to determine the expected utility of a strategy conditional on the regime, so that γ_R captures the aversion towards normally distributed return innovations. The larger γ_R, the lower the savings rate, and the smaller the optimal portfolio weight. Second, the investor uses a CRRA-utility function with risk aversion γ_L to aggregate the certainty equivalents over the two regimes, i.e., to calculate $EU_{T,p}$ given the savings rates $y(\pi, i)$ in the two regimes $i = 1, 2$. The larger γ_L, the lower the savings rate $y_{T,p}(\pi)$ resulting out of $y(\pi, 1)$ and $y(\pi, 2)$.

The setup is analogous to the smooth ambiguity aversion model of Klibanoff et al. (2005) in which u_R and u_L would describe risk preferences and ambiguity preferences, respectively.[15] Here, γ_R is the risk aversion within the regime, while $\gamma_L - \gamma_R$ can be interpreted as the additional aversion with respect to the lottery over the regimes. The setup furthermore reflects the case of a collective investment. Here, a representative investor aggregates over the individual certainty equivalents of investors with heterogeneous beliefs concerning the mean and the volatility of the risky asset. As the next remark shows, our setup can also capture the case when different risk preferences are used to judge the payoffs in Regime 1 and Regime 2.[16]

Remark 1 (State dependent risk aversion) *The problem with state dependent risk aversion $\gamma_{R1} \neq \gamma_{R2}$ is equivalent to a problem with state independent risk aversion $\gamma_R = \gamma_{R1}$ and modified variance $\tilde{\sigma}_2^2 = \frac{\gamma_{R_2}}{\gamma_{R_1}}\sigma_2^2$ in Regime 2. In both cases, the regime-dependent savings rates are*

$$y(\pi, 1) = r + \pi(\mu_1 - r) - \frac{1}{2}\gamma_{R_1}\pi^2\sigma_1^2$$
$$y(\pi, 2) = r + \pi(\mu_2 - r) - \frac{1}{2}\gamma_{R_1}\pi^2\frac{\gamma_{R_2}}{\gamma_{R_1}}\sigma_2^2.$$

Notice that this remark motivates why we do not restrict our analysis to the case

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15 In Section 4.5, we analyze ambiguity w.r.t. the probability distribution over the regimes.

16 Experimental evidence for regime dependent risk aversion includes Cohn et al. (2015) who find that individuals act more risk averse in a bear regime in comparison to a bull regime. Berrada et al. (2018), Gordon and St-Amour (2000), Gordon and St-Amour (2004) study asset pricing with state-dependent risk aversion. Asset allocation problems with state-dependent risk aversion are studied by Wei et al. (2013) and Wei et al. (2020). Björk et al. (2014) and Wang et al. (2021) deal with mean-variance asset allocation and wealth-dependent risk aversion.

$\sigma_1 = \sigma_2$ (and place ourselves within a setup with pure drift uncertainty) but explicitly allow for different volatilities across the regimes.[17]

The aggregation (4.9) over the savings rates in the two regimes is highly non-linear unless $\gamma_L = 1$. For $T > 0$, Jensen's inequality implies

$$y_{T,p}(\pi) \begin{cases} < p\, y(\pi,1) + (1-p)y(\pi,2) & \gamma_L > 1 \\ = p\, y(\pi,1) + (1-p)y(\pi,2) & \gamma_L = 1 \end{cases}.$$

In the special case of log-utility, the savings rate for an initial lottery coincides with the expected savings rate over the Merton problems in the two regimes. The strategy that maximizes expected utility then simultaneously maximizes the expected savings rate. This is not true for $\gamma_L \neq 1$ unless one considers the myopic case $T \to 0$ (cf. following proposition). The savings rates for the limiting cases $T \to 0$ and $T \to \infty$ are given in the following proposition:

Proposition 1 (Limits of savings rate)
For $\gamma_L > 1$, the limiting values of the certainty equivalent savings rate $y_{T,p}(\pi)$ are

$$\lim_{T\to 0} y_{T,p}(\pi) = p\, y(\pi,1) + (1-p)y(\pi,2)$$
$$\lim_{T\to \infty} y_{T,p}(\pi) = \min\{y(\pi,1), y(\pi,2)\}.$$

The proof is given in Appendix A.5.

The left graph of Figure 4.2 illustrates that for $\gamma_L > 1$, the savings rate is decreasing in the investment horizon T. It drops from the expected savings rate (implied by $T \to 0$) to the worst case savings rate (implied by $T \to \infty$).[18]

4.3 Impact of time inconsistency

We subsequently consider the optimal asset allocation problem. With an aggregation of certainty equivalents over regimes, this problem is in general time-inconsistent. Even if the

[17] In Appendix A.3, a brief supplementary analysis for the case of state-dependent risk aversion $\gamma_{R1} < \gamma_{R2}$ is given.

[18] Notice that Assumption 1 only states that Regime 2 implies a lower savings rate than Regime 1 if one compares the within regime optimal savings rates, i.e., the savings rates of Regime 1 and 2 corresponding to the within regime optimal weights π_1^{Mer} and π_2^{Mer}. For a given portfolio weight π, however, the worst case savings rate over the regimes is given by $\min\{y(\pi,1), y(\pi,2)\}$.

Aggregated savings rate $y_{T,p}$

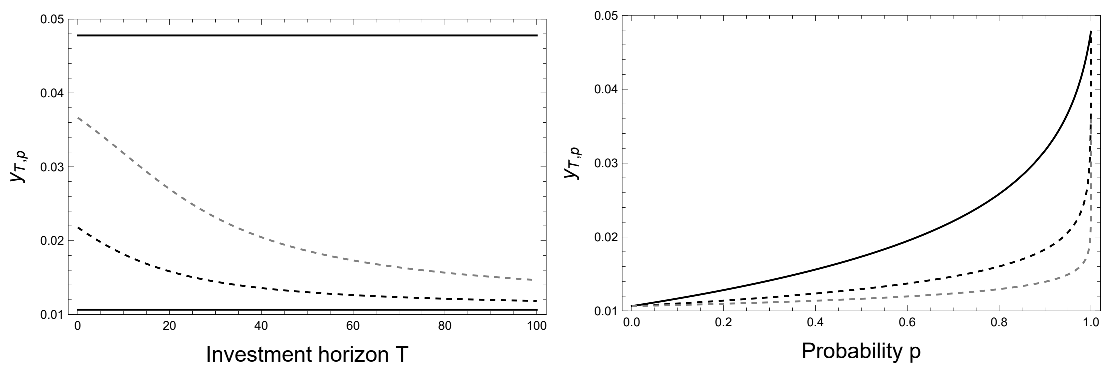

Figure 4.2: The left figure displays the savings rate depending on the investment horizon for $\gamma_L = \gamma_R = 4$ and $\pi = 0.6$. The upper (lower) black line pictures $p = 1$ ($p = 0$). The gray (black) dashed graph refers to $p = 0.7$ ($p = 0.3$). The right figure displays the savings rate depending on the probability for $\pi = 0.6$, $T = 100$ and $\gamma_R = 4$. The black graph refers to $\gamma_L = 2$, while the black (gray) dashed graph refers to $\gamma_L = 4$ ($\gamma_L = 8$).

investor chooses the optimal strategy today (in the sense that it maximizes the expected utility $EU_{T,p}$), she will want to deviate from this strategy at a later point in time.[19] The literature suggests several ways to deal with this problem. The investor can choose the optimal strategy today by solving a game against her future selves (sophisticated strategy), but she can also choose the optimal strategy today without taking the future time inconsistency into account (naive strategy) or pre-commit to not deviating from this strategy later on (pre-commitment strategy).[20] In the following, we will study the optimal pre-commitment strategy.

We limit the analysis to constant portfolio weights, that is to pre-commitment strategies from which the investor is not allowed to (or cannot) deviate later on. The strategies can thus only depend on the initial length of the investment horizon T. Such a pre-commitment strategy is indeed the optimal choice in case of no state variables and CRRA utility.[21] From the perspective of time $t = 0$, changes in the portfolio weight later on would

[19] See Strotz (1955) for a first discussion of time inconsistency. Further references are given in Section 4.1.

[20] The pre-commitment strategy is the optimal decision if the investment decision is irreversible. This is most likely the case for group decisions (of the management) in corporate finance (see e.g., Garlappi et al. (2017)).

[21] For a proof see Balter et al (2021).

only lower the expected utility. Nevertheless, it can still be the case that the investor wants to deviate from the strategy when she reconsiders her choice at a later point in time $t > 0$.

In this section, we refrain from classical learning, i.e., the investor can not gradually update her subjective probabilities of the two regimes and learn the true regime in the long run. In Section 4.4, we will consider the limiting case in which the investor learns about the true regime immediately after the realization of the lottery and can then condition her strategy on the true regime.

Remark 2 (Stylized setup) *In summary, there are various justifications for the stylized setup, and, in particular, the assumption that we do not allow for learning:*
(i) In the special case $\sigma_1 = \sigma_2$, our model subsumes the problem posed by drift uncertainty. Here, we do not account for the possibility to gradually learn about the drift coefficient over time. However, in Section 4.4, it turns out that in both limiting cases ($T \to 0$, (and more importantly) $T \to \infty$), the willingness to pay for information about the regime is zero. Thus, we show later on that learning is not able to resolve the problem of time inconsistency.[22]
(ii) In our setup, the case $\sigma_1 \neq \sigma_2$ implies that an investor who is able to learn obtains full information immediately (cf. (i) and Section 4.4). In addition, recall Remark 1 which gives an additional reason why it is interesting to consider $\sigma_1 \neq \sigma_2$.
(iii) The optimization problem under pre-commitment can alternatively be explained by an irreversible investment where the payoff is given in terms of the payoff corresponding to a constant investment fraction π.

To facilitate the exposition, we directly specify the optimal pre-commitment strategy $\pi^{*,\text{pre}}_{T,p}$ by maximizing the expected utility over a constant (regime independent) investment fraction π, i.e.

$$\pi^{*,\text{pre}}_{T,p} := \arg\max_{\pi} EU_{T,p} = \arg\max_{\pi} y_{T,p}(\pi),$$

where $EU_{T,p}$ is given in Eqn. (4.7) and $y_{T,p}$ is given in Eqn. (4.9).

For the interpretation, we give the optimal pre-commitment strategy as a function of the regime dependent Merton solutions. Since the highest possible savings rate in Regime

[22] Nevertheless, the interested reader is e.g., referred to Bäuerle and Grether (2017) who consider a Bayesian financial market.

i is obtained by π_i^{Mer}, it follows that

$$\pi_{T,p}^{*,\text{pre}} \in \left[\pi_2^{\text{Mer}}, \pi_1^{\text{Mer}}\right] =: \mathcal{A}. \tag{4.10}$$

From the specification of the set $\mathcal{A}$ in Eqn. (4.10) it follows that we can write the optimal pre-commitment strategy $\pi_{T,p}^{*,\text{pre}}$ as a weighted average of the regime dependent Merton solutions, i.e.

$$\pi_{T,p}^{*,\text{pre}} := \alpha_{T,p}^{*}\pi_1^{Mer} + (1-\alpha_{T,p}^{*})\pi_2^{Mer}, \tag{4.11}$$

where $\alpha_{T,p}^{*}$ gives the optimal weight of the Merton solution for Regime 1. We stress the impact of the investment horizon and the regime probabilities by the notation $\alpha_{T,p}^{*}$. In addition, $\alpha_{T,p}^{*}$ may depend on all model and preference parameters. In the following proposition, we give the implicit function for $\alpha_{T,p}^{*}$ which involves $\pi_{T,p}^{*,\text{pre}}$ and which follows from the first order condition for the optimal pre-commitment strategy.[23]

Proposition 2 (Optimal pre-commitment strategy)
Along the lines of Eqn. (5.6), the optimal pre-commitment strategy $\pi_{T,p}^{*,pre}$ is given by the weighting factor $\alpha_{T,p}^{*} = \alpha_{T,p}(\pi_{T,p}^{*,pre})$ where

$$\alpha_{T,p}(\pi) = \frac{p\sigma_1^2 f_1(\pi,T)}{p\sigma_1^2 f_1(\pi,T) + (1-p)\sigma_2^2 f_2(\pi,T)}$$
$$\textit{and } f_i(\pi,T) = e^{y(\pi,i)(1-\gamma_L)T},\ i=1,2.$$

The proof of the Proposition is given in Appendix A.6.

Different from the optimal Merton strategy, the optimal pre-commitment strategy depends on T for $\gamma_L \neq 1$ and is thus time-inconsistent (since the strategy which is optimal today is no longer optimal later on when the remaining investment horizon has shortened). The time inconsistency can be traced back to the dependence of the regime-dependent certainty equivalents on the length of the investment horizon. These certainty equivalents enter the optimal weights of the Merton strategies via f_1 and f_2. When the investor considers the trade-off between speculating on the better regime (and following the optimal strategy for Regime 1) and hedging against the bad regime (and following the optimal strategy for Regime 2), she takes the difference of the certainty equivalents in the two

[23] The result can easily be generalized to more than two regimes.

regimes into account. The larger this difference, the more she tends towards the hedging motive. Since the difference depends on T, the optimal strategy will depend on T, too.

To get an intuition for the behavior of the optimal pre-commitment strategy, we first look at the impact of time T:

Proposition 3 (Limiting values of optimal pre-commitment strategy) *For $\gamma_L > 1$, the limiting values for the optimal pre-commitment strategy are given by*

$$\lim_{T \to 0} \alpha^*_{T,p} = \frac{p\sigma_1^2}{p\sigma_1^2 + (1-p)\sigma_2^2}.$$

and

$$\lim_{T \to \infty} \pi^{*,pre}_{T,p} = \begin{cases} \pi_2^{Mer} & y(\pi,2) < y(\pi,1)\ \forall \pi \in \mathcal{A} \\ \pi^{equal} & \pi^{equal} \in \mathcal{A} \end{cases}, \tag{4.12}$$

where π^{equal} is defined in Eqn. (4.5).

For $T \to 0$ (and also for $\gamma_L = 1$), the savings rate equals the expected savings rate. To get the intuition for the functional form of $\alpha^*_{0,p}$, note that maximizing the expected savings rate is equivalent to minimizing the expected loss rate. The loss rate in regime i, given in Eq. (4.3), scales with the squared volatility σ_i^2. Consequently, the weighting factor of the Merton-strategy π_i^{Mer} in the optimal pre-commitment strategy scales with the variance σ_i^2, too.[24] It is thus the larger the more likely and the riskier a regime is, but does not depend on the savings rate within the regime. The dependence on the volatilities and the probability is also shown in the left graph of Figure 4.3, which plots the limiting $\alpha^*_{0,p}$ as a function of $\sigma_2 - \sigma_1$, and for different values of p.

The other limiting case is given by $T \to \infty$. For an infinite investment horizon and $\gamma_L > 1$, the savings rate of a strategy π is the lower of the regime-dependent savings rates $y(\pi, 1)$ and $y(\pi, 2)$ (cf. Proposition 1). The optimal pre-commitment strategy is thus given by the strategy $\pi \in \mathcal{A}$ that maximizes this worst-case savings rate. It is independent of the regime probabilities as long as $0 < p < 1$. The worst-case strategy can be the Merton-strategy for Regime 2 (if Regime 2 is the worse regime for all $\pi \in \mathcal{A}$), or π^{equal} (if the

[24] Plugging in shows that the optimal strategy can also be interpreted as the Merton strategy for the average expected returns and average variance.

worst regime switches in $\mathcal{A}$).[25]

In the general case where $0 < T < \infty$ and $\gamma_L > 1$, the weighting factor α of a regime i does not only depend on its probability and its volatility, but also on the function f_i which is the (negative) multiple of the utility of the certainty equivalent in this regime. The function f_i is the smaller the larger the savings rate in the regime and thus downplays the weight of the good regime. Consequently, the optimal strategy is shifted towards the worst-case strategy.

To aggregate the impact of the savings rates captured in the functions f_1 and f_2, we define the function δ as[26]

$$\delta(\pi, T) := 1 - \frac{f_1(\pi, T)}{f_2(\pi, T)} = 1 - e^{[y(\pi,1)-y(\pi,2)](1-\gamma_L)T}. \tag{4.13}$$

It describes the relative difference between expected utilities in the two regimes for a given strategy π. For $y(\pi, 1) > y(\pi, 2)$ and $\gamma_L > 1$, we have $\delta(\pi, T) \in [0, 1)$. The lower limit of $\delta = 0$ is attained for $T \to 0$ while the upper limit of one is approached for $T \to \infty$.

With this definition of δ, we can rewrite the portfolio weight of the optimal pre-commitment strategy as

$$\alpha^*_{T,p} = \frac{p\sigma_1^2(1 - \delta(\pi^{*,pre}_{T,p}, T))}{p\sigma_1^2(1 - \delta(\pi^{*,pre}_{T,p}, T)) + (1-p)\sigma_2^2}. \tag{4.14}$$

The auxiliary functions f, δ, and α allow to study the impact of γ_R on the optimal strategy. For the function $f_i(\pi, T) = f_i(\pi, T; \gamma_R, \gamma_L)$, it holds that

$$\begin{aligned} f_i(\pi, T; \gamma_R, \gamma_L) &= e^{\left[\pi(\mu_i - r) - 0.5\gamma_R\pi^2\sigma_i^2\right](1-\gamma_L)T} \\ &= f_i(\gamma_R\pi, T/\gamma_R; 1, \gamma_L), \end{aligned}$$

which implies an analogous relation for δ

$$\delta(\pi, T; \gamma_R, \gamma_L) = \delta(\gamma_R\pi, T/\gamma_R; 1, \gamma_L).$$

[25] An analogous reasoning can be found in Bäuerle and Grether (2017) who study the limiting investment strategy under drift uncertainty and learning. They show that a risk averse (loving) investor bases her decision on the worst (best) case drift. For $\sigma_1 = \sigma_2$ and $T \to \infty$, the optimal pre-commitment strategy thus coincides with the limiting optimal learning strategy.

[26] In case of n regimes, we cannot define one function δ, but would rather look at the ratios $f(\pi, i)/f(\pi, n)$. If n is the worst-case regime over all $\pi \in \mathcal{A}$, it holds that $f(\pi, i)/f(\pi, n)$ goes to zero for $i \neq n$ and to one for $i = n$.

An increase in γ_R can thus be treated like a (proportional) increase in the portfolio weight and an (inversely proportional) decrease in the investment horizon. For the weighting factor α, it follows that

$$\alpha_{T,p}(\pi;\gamma_R,\gamma_L) = \alpha_{T/\gamma_R,p}(\gamma_R\pi;1,\gamma_L).$$

In the implicit function (5.6) which gives the optimal pre-commitment strategy, the weighting factor is applied to the Merton strategies, which inversely scale with γ_R. Overall, this gives the impact of γ_R on the optimal portfolio:

Proposition 4 (Impact of γ_R on optimal strategies) *The optimal Merton strategy π_i^{Mer} depends on γ_R and the optimal pre-commitment strategy $\pi_{T,p}^{*,pre}$ depends on γ_R, γ_L, p, and T. It holds that*

$$\begin{aligned} \pi_i^{Mer}(\gamma_R) &= \frac{1}{\gamma_R}\pi_i^{Mer}(1) \\ \pi_{T,p}^{*,pre}(\gamma_R,\gamma_L) &= \frac{1}{\gamma_R}\pi_{T/\gamma_R,p}^{*,pre}(1,\gamma_L). \end{aligned}$$

For $T = 0$ (and also for $\gamma_L = 1$), we have $\delta = 0$, and the optimal strategy is the myopic one. In the non-myopic case ($T > 0$ and $\gamma_L > 1$), first assume $y(\pi,1) > y(\pi,2)$ for all $\pi \in \mathcal{A}$. When T goes from zero to infinity, δ goes from 0 to 1. Hence, $\alpha_{T,p}^*$ goes from the myopic weight to a weight of zero in the limit, and the pre-commitment strategy approaches the worst-case strategy π_2^{Mer}. Second, assume that $\pi^{equal} \in \mathcal{A}$. Then, the optimal pre-commitment strategy converges from the myopic strategy (which depends on p) to the worst-case strategy π^{equal} (which is independent of the probabilities). δ no longer goes to one, but to the value which sets $\alpha\pi_1^{\text{Mer}} + (1-\alpha)\pi_2^{\text{Mer}}$ equal to π^{equal}.

Thus, the function δ describes the convergence of the myopic strategy to the worst-case strategy. For a given π, δ increases in the difference of the savings rates $y(\pi,1)-y(\pi,2)$ given in Eqn. (4.4). Intuitively, the *force* to the worst-case regime matters the more, the higher the difference between the good and the bad regime is, which is illustrated in the right hand side of Figure 4.3. Another figure in this context can be found in Appendix A.2 (Figure A.2). Moreover, the optimal pre-commitment strategy $\pi_{T,p}^{*,\text{pre}}$ is shown as a function of T and p in Figure A.1.

δ increases in the risk aversion γ_L and decreases in the risk aversion γ_R. This relation is illustrated in Figure 4.4. Intuitively, a higher risk aversion γ_L w.r.t. regime uncertainty causes a faster convergence of the optimal strategy towards the worst-case strategy, which

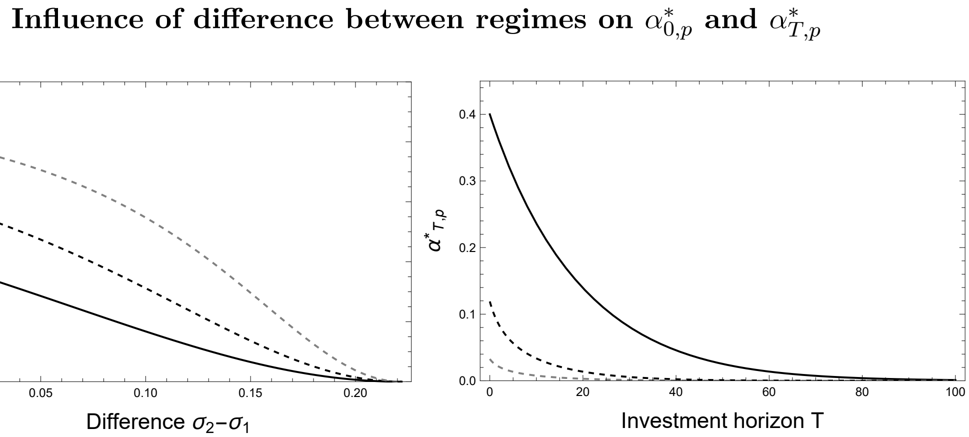

Figure 4.3: The left picture displays the optimal weight on the Merton solution in Regime 1 at $T = 0$ for $\gamma_L = \gamma_R = 4$ depending on the differences $\sigma_2 - \sigma_1$ (where $\sigma_2 = 0.2221$). The black line pictures $p = 0.4$, the black dashed $p = 0.6$ and the gray dashed $p = 0.8$. The right picture shows the optimal weight $\alpha^*_{T,p}$ on the Merton solution in Regime 1 depending on the investment horizon T for $p = 0.4$, $\gamma_L = \gamma_R = 4$ and $\sigma_2 = 0.2$. The black line pictures $\sigma_1 = \sigma_2$. The black (gray) dashed line pictures $\sigma_1 = 0.1$ ($\sigma_1 = 0.05$).

is reached for the limiting value $\delta = 1$. The impact of γ_R is more involved. An increase in γ_R is equivalent to inversely scaling the investment horizon with γ_R (and scaling the investment strategy with γ_R). Since δ is an increasing function of the investment horizon, this implies that δ is a decreasing function of γ_R. The convergence to the limiting worst-case strategy thus takes longer.

Finally, it holds that $\delta(\pi^{*,pre}_{T,p}, T)$ also depends on p via the dependence of the optimal pre-commitment strategy on p. To get the intuition, note that time inconsistency shifts the importance from the good regime towards the bad regime over time. This shift is the more severe, the higher the myopic importance of the good regime is, i.e., the higher p is. This relation is illustrated in Figure 4.4.

δ is not only related to the weights of the optimal pre-commitment strategy but can also be used to rewrite the savings rate. For $\gamma_L > 1$, it holds that

$$y_{T,p}(\pi) = y(\pi, 2) + \frac{1}{(1-\gamma_L)T} \ln\left[1 - p\,\delta(\pi, T)\right]. \tag{4.15}$$

Since Regime 2 is the bad one, the second term is non-negative and measures the additional contribution of the good regime to the savings rate. This contribution depends on p as well as on δ. It increases with the probability p of the good regime. It also increases

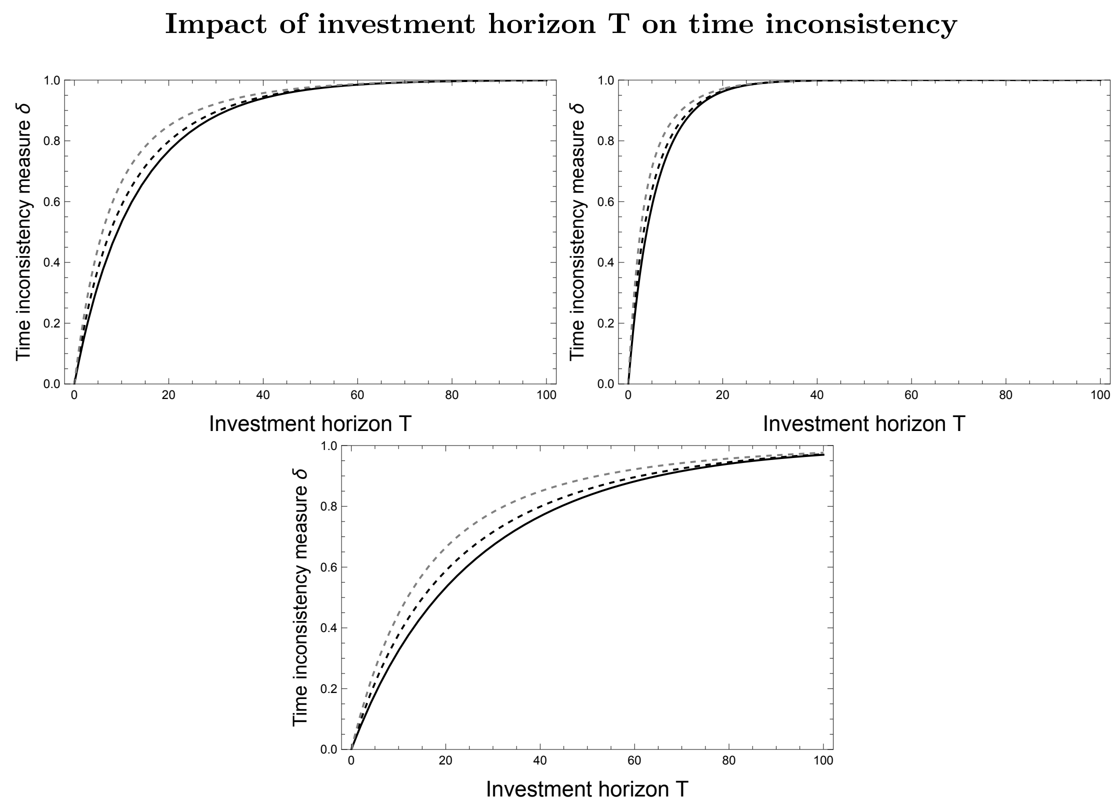

Figure 4.4: The figure gives the time inconsistency measure δ as a function of the investment horizon. The upper left figure refers to a level of risk aversion $\gamma_R = \gamma_L = 4$. The upper right figure refers to $\gamma_R = 4$ and $\gamma_L = 8$. The lower figure refers to $\gamma_R = 8$ and $\gamma_L = 4$. The black graphs refer to the optimal pre-commitment strategy $\pi_{T,p}^{*,\text{pre}}$ for $p = 0.2$. The dashed black (dashed gray) graphs refer to the optimal pre-commitment strategy for $p = 0.5$ ($p = 0.8$).

with δ, which in turn increases with the difference between the savings rates in the good and bad regime.

4.4 Value of information

We now compare the utility of the optimal pre-commitment strategy to the utility of the optimal strategy under full information. In the latter case, the investor can condition her strategy on the regime. The optimal strategy π^* under full information maximizes the expected utility when the outcome of the a priori lottery is known, i.e.,

$$\begin{aligned}\pi^* &= (\pi_1^*, \pi_2^*)\\ &:= \arg\max_{(\pi_1,\pi_2)} \left\{ p\, u_L\left(u_R^{-1}(E_{P_1}[u_R(V_T(\pi_1))])\right) + (1-p)u_L\left(u_R^{-1}(E_{P_2}[u_R(V_T(\pi_2))])\right)\right\}.\end{aligned}$$

Note that the two terms in the weighted sum depend on either π_1 or π_2. With the Merton result it immediately follows:[27]

Proposition 5 (Optimal strategy under full information)
In case of an initial lottery L over Regimes 1 and 2, the expected utility and the savings rate of a CRRA investor who can condition the strategy on the true regime are maximized for

$$\pi^* = (\pi_1^{Mer}, \pi_2^{Mer}). \tag{4.16}$$

The maximal savings rate $y_{T,p}(\pi^*)$ when we can condition on the regime is

$$y_{T,p}(\pi^*) = \begin{cases} \frac{1}{(1-\gamma_L)T} \ln\left[p\, e^{y(\pi_1^{\text{Mer}},1)(1-\gamma_L)T} + (1-p)e^{y(\pi_2^{\text{Mer}},2)(1-\gamma_L)T}\right] & \gamma_L \neq 1 \\ p\, y(\pi_1^{\text{Mer}},1) + (1-p)y(\pi_2^{\text{Mer}},2) & \gamma_L = 1 \end{cases}.$$

The value of the regime information can be measured by the quotient of the certainty equivalents associated with the optimal strategies with and without the regime information, i.e., by the ratio $CE_{T,p}^{*}/CE_{T,p}^{*,\text{pre}}$ of the certainty equivalents. This ratio minus one gives the percentage of wealth that the investor gains if she learns about the regime immediately after the initial lottery has taken place. Alternatively, one can measure the value of information about the regime by the difference $y_{T,p}(\pi^*) - y_{T,p}(\pi_{T,p}^{*,\text{pre}})$ in the savings rates, i.e., by the annual rate of return that the investor gains by this information.

Proposition 6 (Value of full information)

(i) The difference of the savings rates is given by

[27] Intuitively, it is clear that the result can be generalized to a dynamic version of observable regime switches. We refer the interested reader to Sotomayor and Cadenillas (2009) and the references given in the introduction.

$$y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre}_{T,p}) = \begin{cases} \frac{1}{(1-\gamma_L)T} \ln \left[\beta_{T,p}(\pi^{*,pre}_{T,p}) e^{l(\pi^{*,pre}_{T,p},1)(1-\gamma_L)T} + \left(1-\beta_{T,p}(\pi^{*,pre}_{T,p})\right) e^{l(\pi^{*,pre}_{T,p},2)(1-\gamma_L)T}\right] & \gamma_L \neq 1 \\ \beta_{T,p}(\pi^{*,pre}_{T,p})\, l(\pi^{*,pre}_{T,p},1) + (1-\beta_{T,p}(\pi^{*,pre}_{T,p}))\, l(\pi^{*,pre}_{T,p},2) & \gamma_L = 1 \end{cases},$$

where

$$\beta_{T,p}(\pi) = \frac{p\,(1-\delta(\pi,T))}{p\,(1-\delta(\pi,T)) + 1 - p}.$$

(ii) The ratio of the certainty equivalents is given by

$$\frac{CE^*_{T,p}}{CE_{T,p}(\pi^{*,pre}_{T,p})} = e^{\left[y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre}_{T,p})\right]T}$$
$$= \left[\beta_{T,p}(\pi^{*,pre}_{T,p}) e^{l(\pi^{*,pre}_{T,p},1)(1-\gamma_L)T} + \left(1-\beta_{T,p}(\pi^{*,pre}_{T,p})\right) e^{l(\pi^{*,pre}_{T,p},2)(1-\gamma_L)T}\right]^{\frac{1}{1-\gamma_L}}, \gamma_L > 1.$$

The proof is given in Appendix A.7.

The loss in the savings rate and in the certainty equivalent is equal to some weighted average of the regime-specific loss rates, where the weights depend on the savings rates in the two regimes under partial information.

The limiting behavior of the gains from full information is summarized in the following proposition:

Proposition 7 (Limits for value of full information)

(i) For $\gamma_L > 1$, the limits of the difference of the savings rates are

$$\lim_{T\to 0}\left[y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre}_{T,p})\right] = \frac{1}{2}\gamma_R\, p(1-p)\left(\pi^{Mer}_1 - \pi^{Mer}_2\right)^2 \frac{\sigma_1^2\sigma_2^2}{p\sigma_1^2 + (1-p)\sigma_2^2}.$$
$$\lim_{T\to\infty}\left[y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre}_{T,p})\right] = \begin{cases} 0 & \pi^{equal} \notin \mathcal{A} \\ y(\pi^{Mer}_2, 2) - y(\pi^{equal}, \cdot) & \pi^{equal} \in \mathcal{A}. \end{cases}$$

The limit for $T \to 0$ is also the loss for $\gamma_L = 1$, independent of T.

(ii) For $\gamma_L > 1$, the limits of the ratio of the certainty equivalents are

$$\lim_{T\to 0} \frac{CE^*_{T,p}}{CE_{T,p}(\pi^{*,pre}_{T,p})} = 1$$
$$\lim_{T\to\infty} \frac{CE^*_{T,p}}{CE_{T,p}(\pi^{*,pre}_{T,p})} = \begin{cases} 1 & \pi^{equal} \notin \mathcal{A} \\ \infty & \pi^{equal} \in \mathcal{A}. \end{cases}$$

The limit for $T \to 0$ is also the ratio for $\gamma_L = 1$, independent of T.

The proof is given in Appendix A.8.

For $T \to 0$, the gain in the savings rate goes to some finite number. It increases in the difference between the optimal Merton strategies. The corresponding ratio of the certainty equivalents is one, reflecting that the certainty equivalent for a vanishing investment planning horizon is just equal to the initial wealth.

The behavior of the gain for $T \to \infty$ is more surprising. If $y(\pi, 1) > y(\pi, 2)$ for all $\pi \in \mathcal{A}$, the gain in the savings rate goes to zero, and the ratio of the certainty equivalents goes to one. The gain thus vanishes for a long investment horizon, while, intuitively, one would expect it to increase in the length of the time horizon over which the investor can avoid following a suboptimal strategy. The reason is that both savings rates go to the worst-case savings rate. In case of full information, this is due to the dominance of the worst-case utility for $T \to \infty$. In case of no information, the optimal strategy goes to the worst-case strategy, which again implies that the savings rate goes to the worst-case savings rate. By a similar argument, the ratio of the certainty equivalents goes to one. Thus, the losses first increase in T (in line with intuition) and then decrease in T (due to the dominance of the worst-case utility in the long run). If $\pi^{equal} \in \mathcal{A}$, the gain does not vanish in the limit. The optimal pre-commitment strategy goes to π^{equal}. The gain in the savings rates then goes to the difference between the worst-case savings rate $y(\pi_2^{\text{Mer},}, 2)$ and $y(\pi^{equal}, \cdot)$, which is larger than zero. Analogously, the ratio of the certainty equivalents goes to infinity.

Value of information (CE) depending on investment horizon T

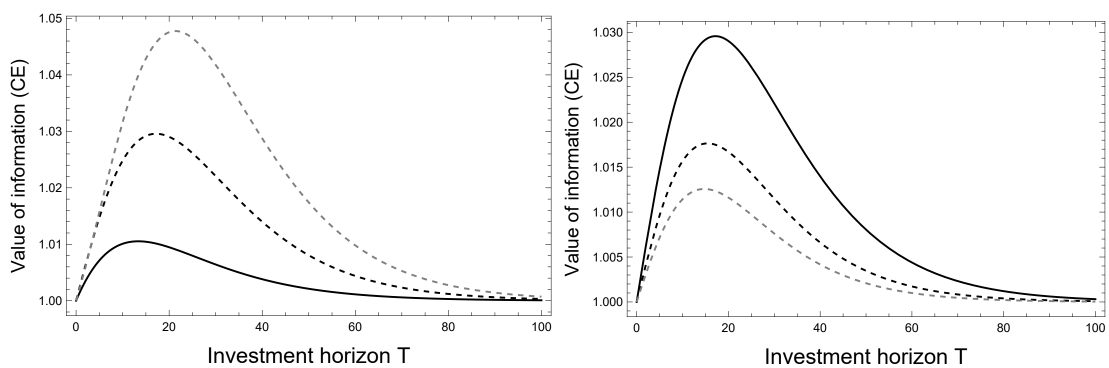

Figure 4.5: Both figures give the value of information by the ratio of the certainty equivalents depending on the investment horizon T. The left figure refers to $p = 0.2$ (black), $p = 0.5$ (black dashed), and $p = 0.7$ (gray dashed) where $\gamma_L = \gamma_R = 4$. The right figure to $\gamma_L = \gamma_R = 4$ (black), $\gamma_L = \gamma_R = 6$ (black dashed), $\gamma_L = \gamma_R = 8$ (gray dashed) where $p = 0.5$

Figure 4.5 gives the ratio of the certainty equivalents as a function of T. If $y(\pi,1) > y(\pi,2)$ for all $\pi \in \mathcal{A}$, then the ratio of the CEs goes to one for $T \to 0$ and for $T \to \infty$. There is thus an investment horizon $\hat{T}$ for which the value of information obtains its maximum.

To study the impact of risk aversion γ_R on the value of information and on $\hat{T}$, we build on Proposition 4. With the results for the impact of γ_R on the optimal strategies, we can derive the following proposition which gives the impact of γ_R on the certainty equivalent and the value of information.

Proposition 8 (Impact of γ_R on value of information) *For the certainty equivalent of a strategy π, it holds that*

$$CE_{T,p}(\pi,\gamma_R,\gamma_L) = CE_{T/\gamma_R,p}(\gamma_R\pi,1,\gamma_L).$$

For the certainty equivalent of the optimal pre-commitment strategy, it holds that

$$CE_{T,p}(\pi^{*,pre}_{T,p}(\gamma_R,\gamma_L),\gamma_R,\gamma_L) = CE_{T/\gamma_R,p}(\pi^{*,pre}_{T/\gamma_R,p}(1,\gamma_L),1,\gamma_L).$$

For the value of information, it then holds that

$$VoI_{T,p}(\gamma_R,\gamma_L) = VoI_{T/\gamma_R,p}(1,\gamma_L).$$

For the time horizon $\hat{T} = T(\gamma_R,\gamma_L)$ which maximizes the value of information, it follows that

$$\hat{T}(\gamma_R,\gamma_L) = \gamma_R\hat{T}(1,\gamma_L),$$

while the maximal value of information does not depend on γ_R:

$$VoI^*(\gamma_R,\gamma_L) = VoI^*(1,\gamma_L).$$

Table 4.2 confirms these results. It gives the maximizing point in time $\hat{T}$ (panel A) and the maximal value of information (panel B) for different combinations of γ_R and γ_L. The maximizing point in time $\hat{T}$ is increasing in γ_R and the maximal value of information is independent of γ_R. The optimal pre-commitment strategy (Panel C) is decreasing in γ_R (as are the optimal Merton strategies). The table also shows the dependence on γ_L. Both the maximizing point in time $\hat{T}$ and the maximal value of information are decreasing in γ_L. To get the intuition, note that a larger value of γ_L implies a faster convergence to the worst-case solution (which lowers $\hat{T}$) and a lower certainty equivalent. γ_L has almost no effect on the optimal pre-commitment strategy for an investment horizon $T = \hat{T}$.

Panel A: Maximizing $\hat{T}(\gamma_R, \gamma_L)$						
$\gamma_R \backslash \gamma_L$	2	4	6	8	10	12
2	25.75	8.60	5.15	3.70	2.85	2.35
4	51.55	17.20	10.30	7.35	5.75	4.70
6	77.30	25.75	15.45	11.05	8.60	7.05
8	103.10	34.35	20.60	14.75	11.45	9.35
10	128.85	42.95	25.75	18.40	14.30	11.70
12	154.60	51.55	30.90	22.10	17.20	14.05
Panel B: Maximal value of information $VoI^*(\gamma_R, \gamma_L)$						
$\gamma_R \backslash \gamma_L$	2	4	6	8	10	12
2	1.0914	1.0296	1.0176	1.0126	1.0098	1.0080
4	1.0914	1.0296	1.0176	1.0126	1.0098	1.0080
6	1.0914	1.0296	1.0176	1.0126	1.0098	1.0080
8	1.0914	1.0296	1.0176	1.0126	1.0098	1.0080
10	1.0914	1.0296	1.0176	1.0126	1.0098	1.0080
12	1.0914	1.0296	1.0176	1.0126	1.0098	1.0080
Panel C: Optimal pre-commitment strategy $\pi^{*,pre}_{\hat{T}(\gamma_R,\gamma_L),p}(\gamma_R, \gamma_L)$						
$\gamma_R \backslash \gamma_L$	2	4	6	8	10	12
2	0.9106	0.9103	0.9106	0.9098	0.9111	0.9101
4	0.4552	0.4552	0.4553	0.4553	0.4550	0.4550
6	0.3035	0.3035	0.3035	0.3035	0.3034	0.3034
8	0.2276	0.2276	0.2276	0.2276	0.2276	0.228
10	0.1821	0.1821	0.1821	0.1821	0.1821	0.1821
12	0.1517	0.1517	0.1518	0.1517	0.1517	0.1518

Table 4.2: The table gives the maximizing time to maturity (Panel A) and the maximal value of information (Panel B) as a function of γ_R and γ_L. The probability p is set to 0.5. Moreover, the optimal pre-commitment strategy is taken into account for $\hat{T}$ (Panel C).

Finally, $\hat{T}$ is increasing in p – the farer away the initial beliefs are from the worst-case (Regime 2), the longer it takes for the solution to converge to the worst-case solution, and the larger the investment horizon for which the value of information is largest.

The results also help to explain the dependence of the value of information for fixed T on γ_R and γ_L. As Figure 4.6 shows, the value of information is a hump-shaped function of γ_R and a decreasing or hump-shaped function of γ_L. To get the intuition for the impact of γ_R, we rely on Proposition 8. It states that a larger risk aversion γ_R can alternatively be written as a smaller time horizon. Furthermore, the value of information is a hump-shaped function of T. If $T < \hat{T}(\gamma_R, \gamma_L) = \gamma_R \hat{T}(1, \gamma_L)$, the value of information is increasing in T. This implies that for $\gamma_R > \frac{T}{\hat{T}(1,\gamma_L)}$, the value of information is decreasing in γ_R, while it is increasing in γ_R for smaller values of γ_R.

Value of Information for varying risk aversion with different probabilities

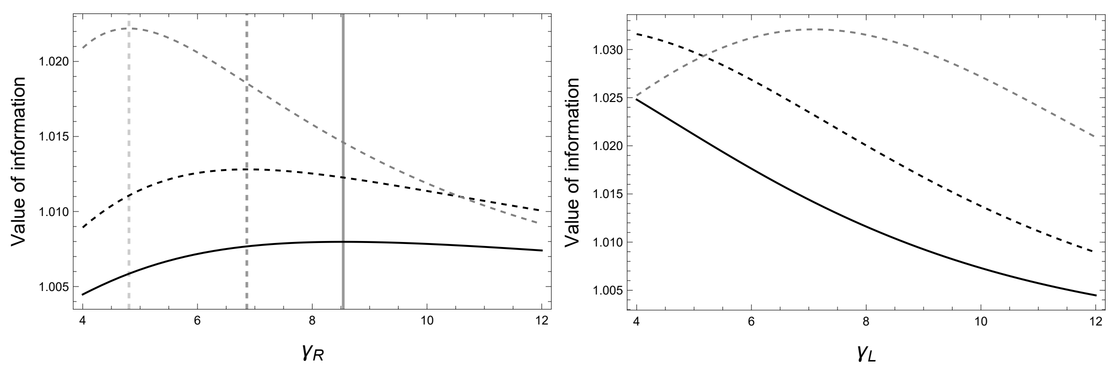

Figure 4.6: The left illustration shows the value of information as a function of risk aversion γ_R for $\gamma_L = 12$. The right illustration shows the value of information as a function of risk aversion γ_L for $\gamma_R = 4$. The black graphs refer to $p = 0.5$, the black dashed to $p = 0.7$ and the gray dashed to $p = 0.9$. The time horizon is 10 years for both illustrations.

Besides risk aversion, the value of information also depends on the probability p of the good regime. VoI is zero for $p = 0$ and $p = 1$, when the second dimension of risk vanishes, and the regime is known. For intermediate values of p, the value of information is an inversely u-shaped function of p.

$y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre})$ **for varying** p **with different investment horizons** T

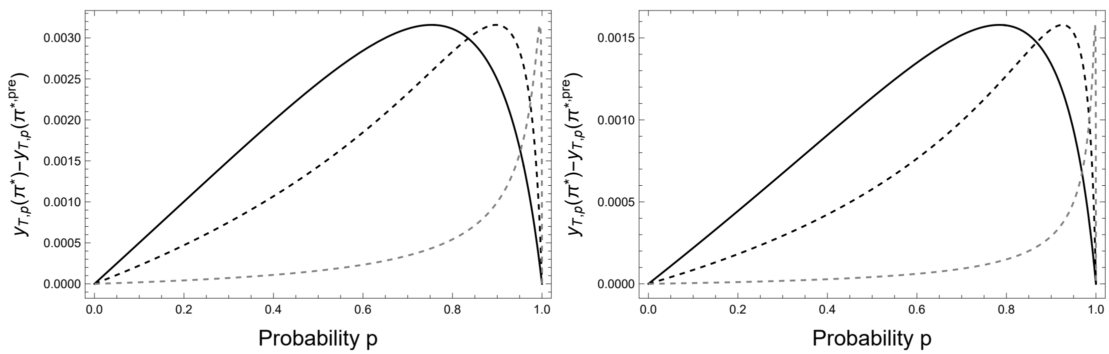

Figure 4.7: The left figure refers to a risk aversion of $\gamma_L = \gamma_R = 4$, the right to $\gamma_L = \gamma_R = 8$. The black lines picture $T = 10$, the black dashed $T = 20$ and the gray dashed $T = 50$.

Figure 4.7 shows the difference in the savings rates as a function of p. When T approaches zero, the value of information is largest for $p = \sigma_2/(\sigma_1 + \sigma_2)$. For equal volatilities, this simplifies to $p = 0.5$ for which uncertainty about the true regime is largest. The dependence on p vanishes for $T \to \infty$ when the savings rates are determined by the worst-case values.

4.5 Accounting for ambiguity

In addition to risk within the regime (first dimension) and the lottery over the regimes (second dimension), we now introduce ambiguity w.r.t. the lottery (third dimension), i.e., there is uncertainty about the probability distribution $(p, 1-p)$ of the a priori lottery. We use the smooth ambiguity approach of Klibanoff et al. (2005) to model the investor's ambiguity aversion. We are interested in the impact of ambiguity on the optimal strategy and on the savings rate.

The investor's time $t = 0$ certainty equivalent of receiving V_T is given by

$$u_A^{-1}\left(E_p\left[u_A\left(u_L^{-1}\left(EU_{T,p}\right)\right)\right]\right) \tag{4.17}$$

$$u_L(x) = \begin{cases} \frac{x^{1-\gamma_L}}{1-\gamma_L} & \gamma_L > 1 \\ \ln x & \gamma_L = 1 \end{cases} \quad \text{and } u_A(x) = \begin{cases} \frac{x^{1-\gamma_A}}{1-\gamma_A} & \gamma_A > 1 \\ \ln x & \gamma_A = 1 \end{cases},$$

for two increasing utility and ambiguity functions u_L and u_A, where γ_L and γ_A capture the (constant) relative aversions towards regime uncertainty and ambiguity about the probability distribution of the regimes. The corresponding savings rate is given by

$$y_T^{\text{amb}}(\pi) = \frac{1}{T}\ \ln\left(u_A^{-1}\left(E_p\left[u_A\left(u_L^{-1}\left(EU_{T,p}\right)\right)\right]\right)\right).$$

The optimal pre-commitment strategy under ambiguity $\pi_{T,\tilde{p}}^{*,\text{pre, amb}}$ is defined by

$$\pi_{T,\tilde{p}}^{*,\text{pre, amb}} := \arg\max_{\pi}\ y_T^{\text{amb}}(\pi).$$

For the sake of simplicity, we model ambiguity by a situation with two different probability distributions $(p_a, 1-p_a)$ and $(p_b, 1-p_b)$ over Regime 1 and Regime 2. The investor assigns the probabilities $\tilde{p}$ and $1-\tilde{p}$ to these two distributions. W.l.o.g., we make the following assumption on p_a and p_b:

Assumption 2 (Probability distribution regimes)
The probability of the good Regime 1 is larger for the first probability distribution than for the second one, i.e., it holds that $p_a \geq p_b$.

Throughout the following, we assume that u_A is a CRRA utility function with constant relative risk aversion parameter γ_A. Without ambiguity aversion, i.e., for $\gamma_A = \gamma_L$, we are back in a decision problem under risk with a lottery $(q, 1-q)$ where

$$q := \tilde{p}\, p_a + (1-\tilde{p})p_b. \tag{4.18}$$

We will use the lottery $(q, 1-q)$ as one benchmark later on.

For a given portfolio weight π, the expected utility of the investor is

$$E_p\left[u_A\left(u_L^{-1}\left(EU_{T,p}\right)\right)\right] = \frac{1}{1-\gamma_A}\left[\tilde{p}e^{y_{T,p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p})e^{y_{T,p_b}(\pi)(1-\gamma_A)T}\right], \tag{4.19}$$

where $y_{T,p_a}(\pi)$ is the savings rate in a risk situation described by the distribution $(p_a, 1-p_a)$ and $y_{T,p_b}(\pi)$ is the savings rate in a risk situation described by the distribution $(p_b, 1-p_b)$. Note that the aggregation over the two probability distributions in case of ambiguity in Eqn. (4.19) has the same functional form as the aggregation over the two regimes in case of risk in Eqn. (4.7).

The savings rate in case of ambiguity is

$$y_{T,\tilde{p}}^{\text{amb}}(\pi) := \frac{1}{(1-\gamma_A)T}\ln\left[\tilde{p}e^{y_{T,p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p})e^{y_{T,p_b}(\pi)(1-\gamma_A)T}\right].$$

Plugging in the corresponding formulas for the savings rates in case of risk over the regimes gives

$$y^{\text{amb}}_{T,\tilde{p}}(\pi) = \begin{cases} \frac{1}{(1-\gamma_A)T} \ln \Big[\tilde{p} \left[p_a e^{y(\pi,1)(1-\gamma_L)T} + (1-p_a) e^{y(\pi,2)(1-\gamma_L)T} \right]^{\frac{1-\gamma_A}{1-\gamma_L}} & \\ \qquad + (1-\tilde{p}) \left[p_b e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b) e^{y(\pi,2)(1-\gamma_L)T} \right]^{\frac{1-\gamma_A}{1-\gamma_L}} \Big] & \gamma_L \neq 1 \\ \frac{1}{(1-\gamma_A)T} \ln \left[\tilde{p} e^{\overline{y}_{p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p}) e^{\overline{y}_{p_b}(\pi)(1-\gamma_A)T} \right] & \gamma_L = 1, \end{cases}$$

where $\bar{y}_p(\pi)$ denotes the average savings rate when the probability of Regime 1 is p.

Proposition 9 (Limiting savings rates in case of ambiguity aversion)
For $\gamma_A > 1$, the savings rate for $T \to 0$ is given by

$$\lim_{T \to 0} y^{amb}_{T,\tilde{p}}(\pi) = q\, y(\pi,1) + (1-q)\, y(\pi,2),$$

and the savings rate for $T \to \infty$ is given by

$$\lim_{T \to \infty} y^{amb}_{T,\tilde{p}}(\pi) = \begin{cases} \min\{y(\pi,1), y(\pi,2)\} & \pi^{equal} \notin \mathcal{A} \\ y(\pi^{equal}, \cdot) & \pi^{equal} \in \mathcal{A} \end{cases}.$$

In both limiting cases, the savings rate no longer depends on the risk aversion γ_L over the regimes and the ambiguity aversion γ_A over the lotteries. It converges to the expected savings rate in the myopic case $T \to 0$ and to the worst-case savings rate for $T \to \infty$.

For $T > 0$ and $\gamma_A > \gamma_L$, Jensen's inequality[28] implies that

$$y^{\text{amb}}_{T,\tilde{p}}(\pi) < \tilde{p}\, y_{T,p_a}(\pi) + (1-\tilde{p}) y_{T,p_b}(\pi) \leq q\ y(\pi,1) + (1-q)\ y(\pi,2).$$

The first inequality captures the impact of ambiguity aversion, the second inequality captures the impact of risk aversion when aggregating over the regimes. Similar to risk aversion, ambiguity aversion thus also reduces the savings rate relative to the expected savings rate under the lottery $(q, 1-q)$.

The optimal pre-commitment strategy of the ambiguity-averse investor is given by

$$\pi^{*,\text{pre},\text{amb}}_{T,\tilde{p}} := \arg\max_{\pi} \frac{1}{(1-\gamma_A)T} \ln \left[\tilde{p} e^{y_{T,p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p}) e^{y_{T,p_b}(\pi)(1-\gamma_A)T} \right].$$

[28] $EU = \frac{1}{1-\gamma_L} e^{y(\pi,i)(1-\gamma_L)T}$ is a concave function of the portfolio weight, thus the expectation $EU_{T,p}$ over the regimes is concave in the portfolio weights. Thus utility under ambiguity is for $\gamma_A > \gamma_L$ concave in the portfolio weights.

The strategies that maximize the two savings rates in the above expression separately from each other are $\pi^{*,\text{pre}}_{T,p_a}$ and $\pi^{*,\text{pre}}_{T,p_b}$. Similar to the optimal pre-commitment strategy in case of risk (which is between the optimal strategies π_1^{Mer} and π_2^{Mer} in the two regimes), the optimal pre-commitment strategy in case of ambiguity is between $\pi^{*,\text{pre}}_{T,p_a}$ and $\pi^{*,\text{pre}}_{T,p_b}$, i.e., it holds that

$$\pi^{*,\text{pre, amb}}_{T,\tilde{p}} \in \left[\min\left\{\pi^{*,\text{pre}}_{T,p_a}, \pi^{*,\text{pre}}_{T,p_b}\right\}, \max\left\{\pi^{*,\text{pre}}_{T,p_a}, \pi^{*,\text{pre}}_{T,p_b}\right\}\right] =: \mathcal{A}^{\text{amb}}_T.$$

In analogy to the previous section, we again state the optimal pre-commitment strategy $\pi^{*,pre,amb}_{T,\tilde{p}}$ under ambiguity as a weighted average of the two regime dependent Merton fractions π_1^{Mer} and π_2^{Mer}.

Proposition 10 (Optimal pre-commitment strategy under ambiguity)
The optimal pre-commitment strategy $\pi^{*,pre,amb}_{T,\tilde{p}}$ under ambiguity aversion γ_A and risk aversion γ_L ($\gamma_A > \gamma_L$) solves the equation

$$\begin{aligned}\pi = {} & \tilde{\alpha}_{T,\tilde{p}}(\pi)\left[\alpha_{T,p_a}(\pi)\pi_1^{Mer} + (1-\alpha_{T,p_a}(\pi))\pi_2^{Mer}\right] \\ & + (1-\tilde{\alpha}_{T,\tilde{p}}(\pi))\left[\alpha_{T,p_b}(\pi)\pi_1^{Mer} + (1-\alpha_{T,p_b}(\pi))\pi_2^{Mer}\right]. \end{aligned} \tag{4.20}$$

The weight $\tilde{\alpha}_{T,\tilde{p}}(\pi)$ is given by

$$\begin{aligned}\tilde{\alpha}_{T,\tilde{p}}(\pi) &= \frac{\tilde{p}(1-\delta^{amb}_T(\pi))}{\tilde{p}(1-\delta^{amb}_T(\pi)) + (1-\tilde{p})}, \\ \delta^{amb}_T(\pi) &= 1 - \frac{p_a\sigma_1^2(1-\delta(\pi,T)) + (1-p_a)\sigma_2^2}{p_b\sigma_1^2(1-\delta(\pi,T)) + (1-p_b)\sigma_2^2}\left[\frac{p_a(1-\delta(\pi,T)) + (1-p_a)}{p_b(1-\delta(\pi,T)) + (1-p_b)}\right]^{\frac{\gamma_A-\gamma_L}{\gamma_L-1}},\end{aligned}$$

the weights $\alpha_{T,p_a}(\pi)$ and $\alpha_{T,p_b}(\pi)$ are given by Eqn. (4.14), $\delta(\pi,T)$ is given in Eqn. (4.13).

The proof of Proposition 10 is given in Appendix A.9.

Again, we first look at the limiting values of the optimal pre-commitment strategies under ambiguity.

Proposition 11 (Limiting optimal portfolio weights under ambiguity)
For $\gamma_A > 1$, the limiting strategy for $T \to 0$ is given by

$$\lim_{T\to 0}\pi^{*,pre,\ amb}_{T,\tilde{p}} = \frac{q\sigma_1^2}{q\sigma_1^2 + (1-q)\sigma_2^2}\pi_1^{Mer} + \frac{(1-q)\sigma_2^2}{q\sigma_1^2 + (1-q)\sigma_2^2}\pi_2^{Mer} = \lim_{T\to 0}\pi^{*,pre}_{T,q},$$

and the limiting strategy for $T \to \infty$ is given by

$$\lim_{T\to\infty}\pi^{*,pre,\ amb}_{T,\tilde{p}} = \begin{cases} \pi_2^{Mer} & y(\pi,2) < y(\pi,1) \quad \forall \pi \in \mathcal{A} \\ \pi^{equal} & otherwise \end{cases} = \lim_{T\to\infty}\pi^{*,pre}_{T,q}.$$

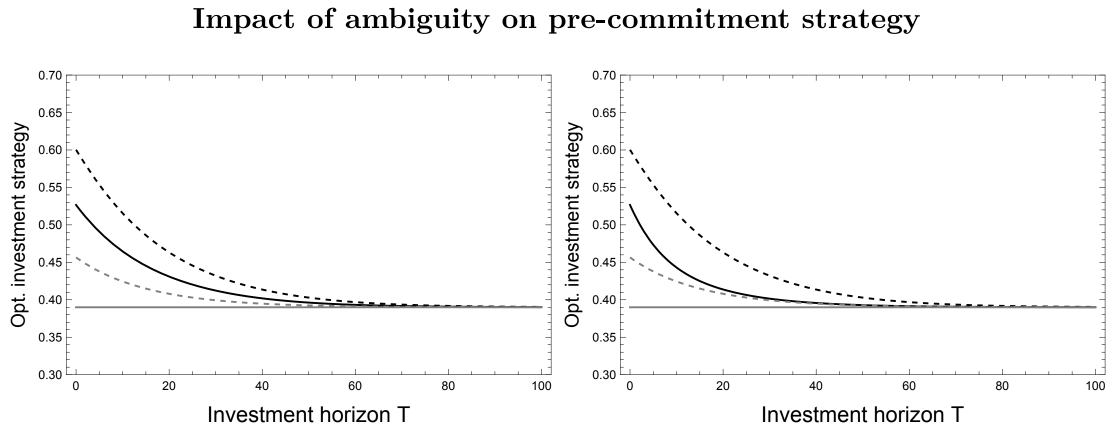

Figure 4.8: The illustrations are plotted for $\gamma_L = \gamma_R = 4$. The gray lines display $\pi_2^{Mer} = 0.39$. The black graphs refer to the optimal pre-commitment strategy under ambiguity aversion $\pi_{T,\tilde{p}}^{*,\text{pre, amb}}$ with $\tilde{p} = 0.5$, $p_a = 0.6$ and $p_b = 0.2$. The dashed black (dashed gray) graphs refer to the optimal pre-commitment strategy without ambiguity $\pi_{T,p}^{*,\text{pre}}$ under the given probability distribution over the regimes with $p = 0.6$ ($p = 0.2$). The left (right) figure refers to a level of ambiguity $\gamma_A = 4$ ($\gamma_A = 16$).

For $T \to 0$, the optimal pre-commitment strategy under risk and ambiguity coincides with the optimal pre-commitment strategy under risk with probability distribution $(q, 1-q)$. For $T \to \infty$, it again holds true that the investor maximizes the worst-case savings rate over the regimes (see Figure 4.8). In both cases the strategy depends on γ_R (via the Merton fraction), but neither on the ambiguity aversion parameter γ_A nor on risk aversion γ_L. Furthermore, the limiting optimal strategies coincide for all combinations $\tilde{p}$, p_a, p_b that imply the same q (in case $T \to 0$) and for all combinations that assign positive probabilities $q, 1-q$ to both regimes (in case $T \to \infty$).

We now compare the optimal strategy $\pi_{T,\tilde{p}}^{*,\text{pre, amb}}$ under risk and ambiguity with the optimal strategy $\pi_{T,q}^{*,\text{pre}}$ under risk. For $T \to 0$, $T \to \infty$, and for an ambiguity-neutral investor with $\gamma_A = \gamma_L$, the optimal pre-commitment strategy under ambiguity coincides with the optimal pre-commitment strategy $\pi_{T,q}^{*,\text{pre}}$. For an ambiguity-averse investor with $\gamma_A > \gamma_L$, it holds that $\pi_{T,\tilde{p}}^{*,\text{pre, amb}}$ (with risk and ambiguity aversion parameters γ_L and $\gamma_A > \gamma_L$) is closer to the limiting worst-case strategy than $\pi_{T,q}^{*,\text{pre}}$ (with risk aversion parameter γ_L)(see Figure 4.9).

In general, the weights of the Merton strategies depend on risk aversion γ_L w.r.t. lotteries and on ambiguity aversion γ_A. They also depend on the amount of risk about the regimes and on the amount of ambiguity over the lotteries. We start with the analysis of

Impact of probability $\tilde{p}$ on optimal strategies $\pi^{*,\mathbf{pre,\ amb}}_{T,\tilde{p}}$ and $\pi^{*,\mathbf{pre}}_{T,q}$

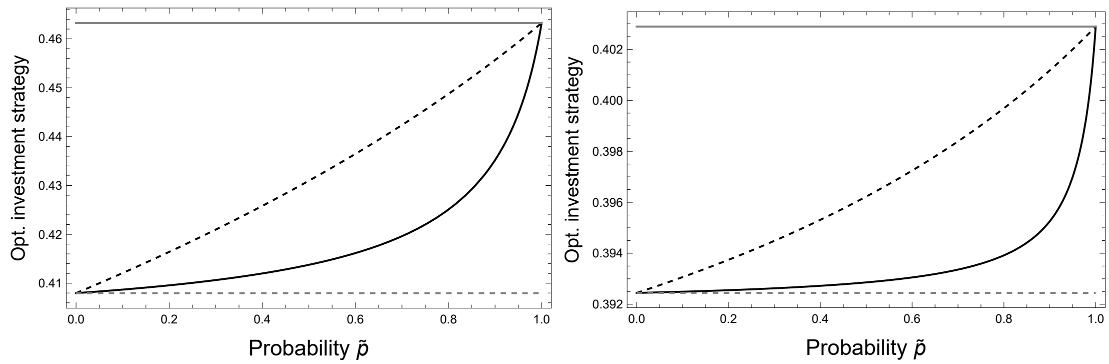

Figure 4.9: The pictures are created for $\gamma_L = \gamma_R = 4$, $\gamma_A = 16$, $p_a = 0.6, p_b = 0.2$. In the left (right) illustration $T = 20$ ($T = 50$) is assumed. The black graphs show the optimal pre-commitment strategy under risk and ambiguity $\pi^{*,\text{pre, amb}}_{T,\tilde{p}}$, the black dashed graphs show the optimal pre-commitment strategy under risk $\pi^{*,\text{pre}}_{T,q}$ with probability q. The gray lines show the optimal pre-commitment strategy under risk with probability p_a and the gray dashed lines show the optimal pre-commitment strategy under risk with probability p_b.

two special cases in which one of these dimensions vanishes.

First, ambiguity (the third dimension) vanishes in the special case $p_a = p_b$. When the lotteries $(p_a, 1-p_a)$ and $(p_b, 1-p_b)$ coincide, it does not matter whether the investor faces lottery a or lottery b. Proposition 10 then gives

$$\pi^{*,pre,amb}_{T,\tilde{p}} = \pi^{*,pre}_{T,q} = \pi^{*,pre}_{T,p_a} = \pi^{*,pre}_{T,p_b}.$$

In line with intuition, this is the optimal pre-commitment strategy of an investor with risk aversion γ_L who faces the lottery $(q, 1-q) = (p_a, 1-p_a) = (p_b, 1-p_b)$. Second, uncertainty about the regime (second dimension) vanishes in the special case $p_a = 1, p_b = 0$. With $\alpha_{T,p_a}(\cdot) = 1$ and $\alpha_{T,p_b}(\cdot) = 0$, Proposition 10 simplifies to

$$\begin{aligned} \pi^{*,pre,amb}_{T,\tilde{p}} &= \hat{\alpha}^*_{T,\tilde{p}} \pi_1^{Mer} + (1-\hat{\alpha}^*_{T,\tilde{p}})\pi_2^{Mer} \\ \text{where } \hat{\alpha}^*_{T,\tilde{p}}(\pi) &= \frac{\tilde{p}\sigma_1^2 e^{y(\pi,1)(1-\gamma_A)T}}{\tilde{p}\sigma_1^2 e^{y(\pi,1)(1-\gamma_A)T} + (1-\tilde{p})\sigma_2^2 e^{y(\pi,2)(1-\gamma_A)T}}. \end{aligned}$$

This strategy coincides with the optimal pre-commitment strategy of an investor with risk aversion $\gamma_L = \gamma_A$ who faces the lottery $(q, 1-q) = (\tilde{p}, 1-\tilde{p})$.

For a given q, i.e., a given probability of Regime 1 after a merge of the lotteries, there is thus a trade-off between ambiguity over the lottery and uncertainty over the regime. For $p_a = p_b$, ambiguity over the lottery vanishes, and the investor is left with uncertainty over the regime. The larger the difference between p_a and p_b, the higher the ambiguity about the lottery, and the lower the uncertainty over the regimes. Finally, uncertainty w.r.t. the regime vanishes and ambiguity is maximized when $p_a = 1$ and $p_b = 0$. For $\gamma_A > \gamma_L$, the optimal portfolio weight moves closer to its limiting level when going from the case with risk over lotteries only to the case with ambiguity over the regime only. Thus, there is an opposing effect of the ambiguity situation over the probabilities and the risk situation about the regimes.

$\pi^{*,pre,amb}_{T,\tilde{p}}$, const. q for different p_a and p_b combinations

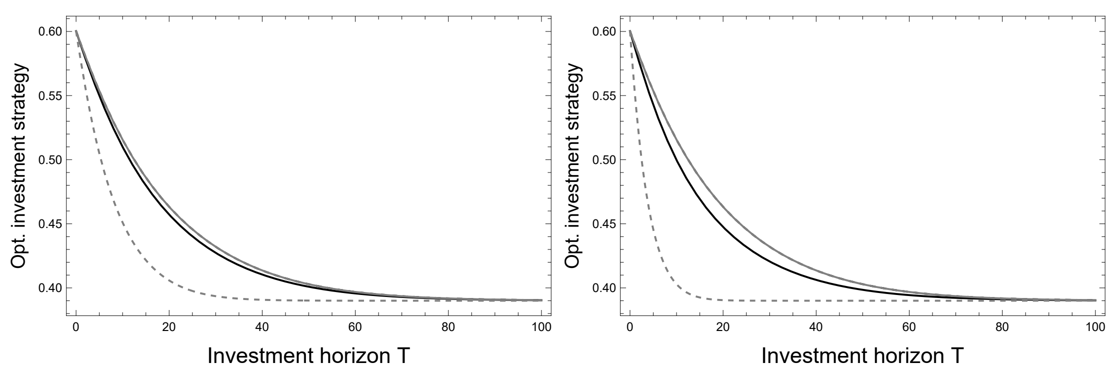

Figure 4.10: The left (right) hand side refers to $\gamma_L = \gamma_R = 4, \gamma_A = 8$ ($\gamma_L = \gamma_R = 4, \gamma_A = 16$). Both illustrations are plotted with a constant $q = \tilde{p} = 0.6$. The black graphs display $\pi^{*,pre,amb}_{T,\tilde{p}}$ for $p_a = 0.7, p_b = 0.45$. For $p_a = p_b = 0.6$ it holds $\pi^{*,pre,amb}_{T,\tilde{p}} = \pi^{*,pre}_{T,q}$ (gray and black dashed graphs). The gray dashed graphs refer to $p_a = 1, p_b = 0$.

In addition, Proposition 10 sheds light on the importance of the levels of ambiguity aversion γ_A and risk aversions γ_L and γ_R. Intuitively it is clear that the optimal pre-commitment strategy $\pi^{*,pre,amb}_{T,\tilde{p}}$ is decreasing in the risk aversion γ_R, since a higher γ_R leads to a reduction of the regime-dependent Merton fractions (first risk dimension). In addition, a higher level of risk aversion γ_L over the regimes yields a faster convergence towards the Merton fraction associated with the worst-case regime (second risk dimension). Concerning the third dimension (ambiguity), the speed of convergence towards the worst-case strategy (maximin strategy) is monotonically increasing in the difference between γ_A and γ_L, i.e., the higher the difference between the two parameters, the faster the convergence. An illustration of the convergence behaviour of the optimal pre-commitment strategy under ambiguity and risk aversion to the worst-case strategy is given in Figure 4.10.

In addition to the value of information about the regime (full information), we consider now the value of information about the lottery, i.e., the willingness to pay for resolving the ambiguity and then knowing whether $(p_a, 1-p_a)$ or $(p_b, 1-p_b)$ applies. In the latter case, the investor knows whether to use the strategy $\pi^{*,\text{pre}}_{T,p_a}$ or $\pi^{*,\text{pre}}_{T,p_b}$. Under full information, she knows whether to use π^{Mer}_1 or π^{Mer}_2. We consider the following ratios of certainty equivalents

$$\text{VoI}^{\text{Lot}}(T) = \frac{\left[\tilde{p}\, CE_{T,p_a}(\pi^{*,pre}_{T,p_a})^{1-\gamma_A} + (1-\tilde{p})CE_{T,p_b}(\pi^{*,pre}_{T,p_b})^{1-\gamma_A}\right]^{\frac{1}{1-\gamma_A}}}{\left[\tilde{p}\, CE_{T,p_a}(\pi^{*,pre,amb}_{T,\tilde{p}})^{1-\gamma_A} + (1-\tilde{p})CE_{T,p_b}(\pi^{*,pre,amb}_{T,\tilde{p}})^{1-\gamma_A}\right]^{\frac{1}{1-\gamma_A}}}$$

and

$$\text{VoI}^{\text{Full}}(T) = \frac{q\, CE_T(\pi^{\text{Mer}}_1, 1) + (1-q)CE_T(\pi^{\text{Mer}}_2, 2)}{\left[\tilde{p}\, CE_{T,p_a}(\pi^{*,pre,amb}_{T,\tilde{p}})^{1-\gamma_A} + (1-\tilde{p})CE_{T,p_b}(\pi^{*,pre,amb}_{T,\tilde{p}})^{1-\gamma_A}\right]^{\frac{1}{1-\gamma_A}}},$$

i.e., $\text{VoI}^{\text{Lot}}(T) - 1$ denotes the willingness to pay for the knowledge of the lottery and $\text{VoI}^{\text{Full}}(T) - 1$ denotes the willingness to pay for the knowledge of the regime.

Value of information: $\pi^{*,pre}_{T,pa}$ and $\pi^{*,pre}_{T,pb}$ known, const. q

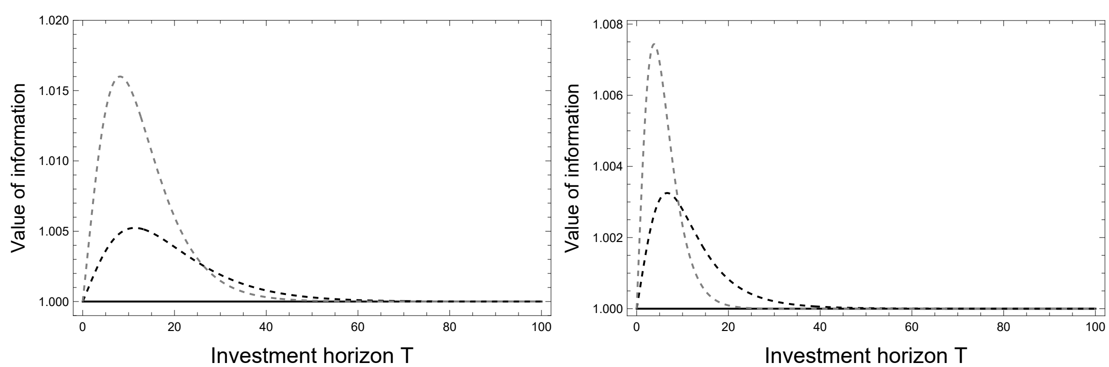

Figure 4.11: The left illustration refers to $\gamma_L = \gamma_R = 4, \gamma_A = 8$, whereas the right illustration refers to $\gamma_L = \gamma_R = 4, \gamma_A = 16$. Both illustrations are plotted with a constant $q = \tilde{p} = 0.6$. The black graphs display $p_a = p_b = 0.6$ (black dashed: $p_a = 0.8, p_b = 0.3$ and gray dashed: $p_a = 1, p_b = 0$).

An illustration of the value of information for resolving the ambiguity situation ($VoI^{\text{Lot}}(T)$), is given in Figure 4.11. For all combinations of p_a and p_b, the limits $T \to 0$

and $T \to \infty$ of $VoI^{\text{Lot}}(T)$ are equal to 1, i.e., the willingness to pay is zero. Intuitively, the same reasoning as for the second risk dimension (see Section 4.4) also applies to the third dimension (ambiguity): the willingness to pay is again zero when the horizon is zero and when the worst-case utility dominates for $T \to \infty$. In the special case $p_a = p_b$ (no ambiguity), it holds that $VoI^{\text{Lot}}(T) = 1$ for all T.

$\hat{T}$ for varying p_a, const. q and different risk aversion parameter γ_R

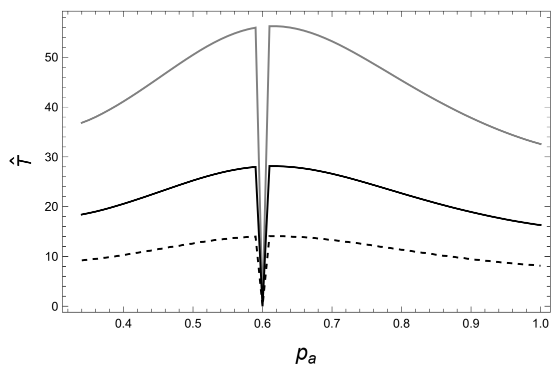

Figure 4.12: The figure shows $\hat{T}$ under ambiguity with constant $q = \tilde{p} = 0.6$, $\gamma_L = 4$ and $\gamma_A = 8$ for varying p_a (notice that this implies also varying p_b). The black graph refers to $\gamma_R = 8$, the black dashed to $\gamma_R = 4$ and the gray one to $\gamma_R = 16$.

Similar to Section 4.4, there is an investment horizon $\hat{T}$ for which the value of information achieves its maximum. The value of information is the highest for $p_a = 1$ and $p_b = 0$ where the influence of ambiguity is the highest. However, beyond $\hat{T}$, the higher the maximal value of information is, the faster is the convergence to the worst case (see Figure 4.11). Furthermore the value of information drops faster for a higher difference between γ_A and γ_L since a higher difference implies a faster convergence to the worst case as already mentioned before. Figure 4.12 illustrates $\hat{T}$ as a function of p_a. $\hat{T} = 0$ for $p_a = p_b = 0.6$, since ambiguity over the lottery vanishes and $VoI^{\text{Lot}}(T) = 1$ for all T. With increasing ambiguity (p_a moves further away from 0.6) $\hat{T}$ decreases. This means that the maximal value of information is reached at smaller investment horizons.

4.6 Conclusion

We consider a stylized setup of an investment decision to shed light on the impacts of time inconsistency. In the first instance, we introduce time inconsistency by means of a double risk situation and an aggregation over certainty equivalents. While the outer risk is given by a simple a priori lottery over two regimes, the inner risk situation in each regime coincides with the classic Merton problem. Although our stylized setup is artificial, it fits many (dynamic) decision problems (cf. introduction), including heterogeneity w.r.t. risk aversion and beliefs.

The double risk situation allows for an intuitive interpretation of the results. Technically, we can separate the outer and inner risk situation. Since the outer risk situation increases in time (the investment horizon), the optimal decision of the investor converges to the worst-case strategy, i.e., the investor chooses the strategy that maximizes the minimum of the savings rates over the two regimes.

For a finite investment horizon, the optimal investment decision can be written as a weighted average of the optimal regime dependent (Merton) solutions. While in the myopic case, the weights basically resemble the probabilities given by the lottery, the weight of the good regime is reduced as the investment horizon increases, so that the worst case regime gains importance.

We provide a measure δ (normalized to $[0,1]$) for the impact of time inconsistency on the optimal strategy. This measure is increasing in the level of risk aversion γ_L and decreasing in γ_R. Furthermore, δ increases in the length of the investment horizon (the longer the investment period, the more risk aversion and thus the shift to the worst-case strategy matters) and in the probability of the good regime (the shift towards the worst-case regime is the more pronounced the higher the probability of the good regime).

Concerning the willingness to pay for information about the current regime, we have the surprising result that the willingness to pay for full information goes to zero not only for an investment horizon of zero, but also for an infinite horizon, and is thus maximal for some investment horizon in between. However, we show that the willingness to pay obtains a maximum, i.e., first increases in the investment horizon and then decreases to zero. Thus, there is an investment horizon $\hat{T}$ where the willingness to pay is maximized.

In addition to the two dimensions of risk, we also introduce an additional dimension stemming from ambiguity about the regime probabilities. Using the smooth ambiguity model of Klibanoff et al. (2005), this implies a further outer expectation (accounting for the ambiguity aversion). Again, we are able to separate the effects of the two risk situations as well as the ambiguity aversion. We explain that the impact of time-inconsistency gets more ambiguous by showing that varying the ambiguity situation may also change the risk situation.

Chapter 5

Optimal asset allocation for a time-inconsistent investor in a regime-switching environment

5.1 Introduction

In this chapter, we analyze the value of information for a time-inconsistent investor who aims to solve a dynamic asset allocation problem in a regime-switching environment. Regime-switching models take into account the impact of exogenous shocks on assets and the financial market. These shocks can lead to changes in market dynamics. In contrast to standard models, the risk-return structure of an asset (e.g., a stock) is not assumed to be deterministic, but modeled in a regime-dependent manner. Therefore, in addition to the economic risk of assets, expressed in terms of its risk-return relation, uncertainty about the current and future state of the environment, which may change as a result of events and affect financial market parameters, is also taken into account.

The switching behavior of the states of the economy can be attributed to structural changes in (macro-)economic conditions and political and regulatory frameworks (cf. Ang and Timmermann (2012)). These changes are usually observable but can induce uncertainty about their implications for future asset price dynamics. On the one hand, the announcement of new policies and frameworks before their implementation induces uncertainty. On the other hand, there is uncertainty after implementation with regard to the specific design and impact of the change in policy. This additional uncertainty influences the economic environment, which is problematic since especially for long-term investors there can be significant variations in economic variables over time (cf. Pastor and Veronesi (2012)). The objective of this chapter is therefore to examine the impact of a lack of information about the effects of this uncertainty on asset price dynamics on an investor's optimal asset allocation.

A strand of the literature therefore deals with the examination of regime changes and its effects on asset allocation: Following the pathbreaking papers of Merton (1969) and Merton (1971), dynamic asset allocation has been investigated for a large variety of assumptions on the assets' dynamics including, among others, regime-switching models.[1] Regime-switching models have their origin in the work of Hamilton (1989), who considers the business cycle in two regimes (boom or recession). Since then, many empirical studies have shown that regime-switching models help predict market dynamics of asset prices. Ang and Bekaert (2002), Ang and Bekaert (2004) and Guidolin and Timmermann (2007) conclude in this regard that accounting for regime-switching in the strategy improves asset allocation. An overview of regime-switching models in asset allocation and asset pricing is given in Ang and Timmermann (2012). Kole et al. (2006) draw attention to investor diversification in international equity markets and find that accounting for systemic crises via regime-switching models strongly influences asset allocation decisions. Taking regime switches into account, especially when combining with the Merton approach, captures the timing and intensity of crises much more effectively than standard models and thus substantially influences optimal asset allocation decisions.[2] Tu (2010) argues that ignoring regime-switching leads to significant losses in the certainty-equivalent. Thus, by taking into account state changes in markets, regime-switching models contribute to a more realistic representation of financial markets and asset price dynamics. Therefore, it can provide a better basis for decision making and improve an investor's asset allocation.

Lu et al. (2021) examine the impact of shocks to oil prices on U.S. stock market

[1] In the classical Merton investment model, return rates and volatilities of risky assets are assumed to be constants. An optimal investment strategy is obtained by solving the problem of maximizing an expected utility function from terminal wealth. However, in a regime-switching model the asset price dynamics are modeled by diffusion processes with regime-dependent drift and diffusion coefficients.

[2] In the Merton problem, the dynamics of the risky asset follows a geometric Brownian motion that implies normally distributed logarithmic returns. However, the assumption is not realistic since in reality exogenous events cause stock returns to exhibit more extreme variations than the normal distribution permits.

volatility and find that oil shocks have time-varying performance, highlighting the importance of accounting for regime-switching. Milidonis et al. (2011) deal with mortality dynamics in the insurance context and find that mortality is characterized by different regimes. Alexander and Kaeck (2008) find evidence that credit default swap spreads exhibit regime-specific behavior. Therefore, the consideration of regime-switching is also important in other application problems.

We consider regime-switching by employing a Markov-modulated approach in a simple two-asset world, in which the decision consists of what fraction to invest in the stock market or in a risk-free asset. Regime 1 embodies an economically good state where drift μ_1 is higher and the volatility σ_1 is lower compared to the second regime. Regime 2 thus represents an unstable state resembling a poor economic condition. Therefore it applies that $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$. We model the two regimes via an observable Markov chain.[3]

Intertemporal models often make the unrealistic assumption that investors know the parameters that determine asset dynamics and state variables. However, we live in a constantly changing and connected world, making this assumption unrealistic. Therefore, we consider regime uncertainty resp. parameter uncertainty about the determination of the value of information in our model and quantify the value of information by comparing the certainty equivalents of two types of investors differing in their level of information.

The investor who has full information – i.e., she knows the respective regimes, their parameters and the point in time when the regime switch takes place – aims at maximizing her expected utility. Assuming that regimes are observable and therefore the parameters of the underlying process, the optimal allocation strategy under expected utility maximization of a CRRA investor is given as the Merton solution of the respective regime (cf. Ocejo (2018), Becker et al. (2022)). This solution is a time-independent constant mix strategy.[4]

[3] Reasons for the use of an observable Markov chain are provided in Section 5.2. We consider two regimes following the research of Hardy (2001), Guidolin and Timmermann (2005) and Ahmad et al. (2015). On the one hand, they provide empirical evidence for the existence of two regimes. On the other hand, they find through model calibration that adding an additional regime does not improve the model. Nevertheless, our model can be extended by increasing the number of more than two regimes, adding and removing regimes. However, it should be noted that the complexity of the calculation increases.

[4] Note that in our setup full information denotes the case where the Markov chain is observable. Thus the investor can condition her strategy on the optimal solution of the current regime which is defined by the model parameters. When a regime switch occurs, she simply switches her optimal investment strategy to the Merton solution of the new regime.

The strategy of an investor who has no full information about the future asset price dynamics turns out to be time-dependent. Time inconsistency arises for the investor due to the regime uncertainty. The investor is uncertain if and when a regime switch will take place. Policy changes affect asset price dynamics. Even if the future regime is known through an policy announcement, investors may not know exactly when the changes will be reflected in the price dynamics of stocks. We assume that the investor cannot learn about it over time either. Therefore she wants to rely on a regime-independent strategy. This implies a constant investment fraction as optimal deterministic strategy. We refer to the maximizing strategy as the optimal pre-commitment strategy. A pre-commitment strategy helps the investor to stick with her initial investment decision. We are interested in such pre-commitment strategies, and the aim of this paper is to shed light on the impact of time inconsistency on investment decisions.[5]

Time inconsistency arises naturally in many (dynamic) decision problems. One example is heterogeneity in time preferences where every method of aggregating utility functions is time-inconsistent (cf. Jackson and Yariv (2015)). Time inconsistency implies that an investor who dynamically invests reconsiders her strategy at a later date. There is limited research so far that considers the presence and problem of time inconsistency as an anomaly in behavioral finance when allocating assets in a regime-switching environment. Yang et al. (2020) investigate a mean-variance portfolio optimization problem under a game-theoretic framework. Liang and Song (2015) consider a similar problem in an insurance economics context. Yang and Cao (2019) analyze optimal time-inconsistent financing and dividend payout strategies in a regime-switching environment where the manager has a hyperbolic discount function. Some papers examine an investment-consumption problem where the discount functions depend on the prevailing regime of the environment (cf. Pirvu and Zhang (2011), Wei et al. (2020)). Thus, time inconsistency is deterministically given via a discount function. Accounting for regime switches, the latter derive optimal solutions for this time-inconsistent problem for different utility functions.[6]

We show that, as in the paper by Becker et al. (2022), who do not consider regime-switching, but assume an a priori lottery over two regimes, the optimal pre-commitment

[5] Note that we do not account for learning in this setup.

[6] Under a game-theoretic framework time-inconsistent control problems have also been studied where the asset price dynamics depend on the regime-switching environment (cf. Bjork and Murgoci (2010), Wei (2017), Pun (2018) and Mei and Yong (2019)).

strategy is given as a weighted average of the Merton solutions of the two regimes. The strategy depends on the length of the investment horizon, such that the optimal fraction that a long-term investor invests in the risky asset differs significantly from the fraction that a short-term investor invests. The regime uncertainty can be expressed by the intensity parameter λ. As the investment horizon increases, a regime switch becomes more likely and the importance of the first regime decreases, so a time-inconsistent investor biases her strategy towards the second regime (worst-case strategy). We find that regime uncertainty has only a slightly impact on the optimal certainty equivalent return as long as the pre-commitment strategy is correct on average. Furthermore, we investigate λ^{crit} at which the value of information is highest for a time-inconsistent investor who implements a pre-commitment strategy. Other (non-optimal) pre-commitment strategies will be discussed.

In this chapter we proceed as follows: First, a literature review of theoretical papers is provided, which use regime-switching in asset allocation and asset pricing and serves as motivation for our setup. In Section 5.3 we will present the decision model in which we want to analyze utilization effects. Section 5.4 gives an analysis of different investment strategies and the value of information in a regime-switching environment resulting from the availability of different levels of information. The value of information is given by the difference of the certainty equivalents associated with the optimal strategies with and without regime information. We conclude in Section 5.4.

5.2 Motivation for setup

5.2.1 Regime-switching in asset allocation and asset pricing

As mentioned before, the consideration of regime shifts in asset pricing and asset allocation has significant implications for optimal investment strategies. Asset price dynamics, which may be affected by regime switches, are critical factors in determining optimal investment strategies. A regime-switching model can be used to extend the classical assumption of Black and Scholes (1973) that the price process of the stock follows a geometric Brownian motion. In the Black-Scholes world, the parameters of der underlying process are constants, whereas in the regime-switching model the parameters take on different values in different time periods. Regime-switching models are more realistic as they take into account the fact that financial market parameters may change over time due to changing environmental conditions triggered, for example, by political and regulatory events.

With regard to the theoretical consideration, the dynamics of the risky stock, i.e., the drift μ and the diffusion coefficient/volatility σ, are stochastically controlled via a Markov chain. Thus, the asset price dynamics are driven by diffusion processes with regime dependent drift μ and volatility σ.[7] A distinction can be made between an **observable** Markov chain and a **hidden** Markov chain.[8] In the following, the difference between the two Markov chains will be briefly discussed from a theoretical perspective:

Observable Markov chain

Assuming that regime switches are independent of the underlying Brownian motion and observable, there are closed form (or quasi closed form) solutions available. Let $(Y_t)_{t\in[0,T]}$ be an observable Markov chain, i.e., the current state of the Markov chain $Y_0 = s_i$ at time $t = 0$ is known. In the context of the financial market let $(W_t)_{t\in[0,T]}$ be a Brownian motion, $(S_t)_{t\in[0,T]}$ the price of the risky asset and $(B_t)_{t\in[0,T]}$ the risk free bond. S and B should be adapted to the filtration $\mathcal{F}^O = (\mathcal{F}_t^O)_{t\geq 0}$, where $\mathcal{F}_t^O = \sigma(W_i, Y_i; 0 \leq i \leq t)$. This means that $\mathcal{F}_t^O$ is the sigma algebra generated by the Brownian motion and the Markov chain. By using this filtration when evaluating the asset evolution, full information about the Brownian motion and the Markov chain is available up to time t, so the current state of the Markov chain is known. For the assumption that $(Y_t)_{t\in[0,T]}$ has two regimes, where the states are s_1 and s_2, the unconditional probability at $t = 0$ of the Markov chain is given by

$$\mathbb{P}(Y_0 = s_1) = p, \quad \mathbb{P}(Y_0 = s_2) = 1 - p,$$

i.e., the Markov chain starts in Regime 1 with probability p and in Regime 2 with $1 - p$. With the transition probabilities $q_{ij}(t, u) := \mathbb{P}(Y_u = s_j | Y_t = s_i)$, the generator A of the continuous-time Markov chain is defined as follows:

$$A_t = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}, \text{ where}$$

$$a_{ij}(t) := \lim_{h\to 0} \frac{q_{ij}(t, t+h) - \delta_{ij}}{h}, \text{ where } \delta_{ij} := \begin{cases} 1, i = j \\ 0, i \neq j. \end{cases}$$

[7] There is also research that allows the price dynamics of the risk-free asset to depend on the current regime at time t (see the literature given in Table 5.1). However, for simplicity and feasibility, we assume that the risk-free interest rate r is constant and equal to 0.

[8] In the literature, a synonymous term used for a hidden Markov chain is an unobservable Markov chain.

The generator indicates the constant, instantaneous intensity of a transition from one regime to another (cf. Capponi and Figueroa-López (2014)). Assuming that the financial market model contains two assets, a risky asset S and a risk-free asset B, both have to be adapted to the filtration $\mathcal{F}^O = (\mathcal{F}_t^O)_{t\geq 0}$. Using this filtration by evaluating the asset evolution, all information about the Brownian motion and the Markov chain up to time t are known. This means the current state of the Markov chain is known, and the dynamics of the risky asset $(S_t)_{t\in[0,T]}$ can be defined by

$$dS_t = S_t\mu(Y_t)dt + S_t\sigma(Y_t)dW_t, \tag{5.1}$$

where the drift μ and the volatility σ both depend on the observable Markov chain $(Y_t)_{t\in[0,T]}$.

<u>*Hidden Markov chain*</u>

Let $(Y_t^H)_{t\in[0,T]}$ be a hidden Markov chain. Due to the fact that the current state of the chain at time $t = 0$ is unknown, the filtration has to be adjusted. In the analysis of the asset evolution S_t or B_t, the filtration $\mathcal{F}^H = (\mathcal{F}_t^H)_{t\geq 0}$ is used, where $\mathcal{F}_t^H = \sigma(S_i; 0 \leq i \leq t)$. In contrast to the filtration $\mathcal{F}_t^O$, where the information about the current state of the Markov chain and the Brownian motion are included, the filtration $\mathcal{F}_t^H$ is the sigma algebra generated by the risky asset S. This means that at time t the investor can only observe the asset price S_t, but neither information about the current regime of the Markov chain nor information about the Brownian motion is known. To overcome this, filtered probabilities have to be used. For the assumption of the existence of two regimes the filtered probabilities are defined by

$$p_t = P(Y_t^H = s_1|\mathcal{F}_t^H) \overset{\text{Markov Property}}{=} P(Y_t^H = s_1|S_t) \tag{5.2}$$

$$1 - p_t = P(Y_t^H = s_2|\mathcal{F}_t^H) \overset{\text{Markov Property}}{=} P(Y_t^H = s_2|S_t), \tag{5.3}$$

where p_0 (resp. $1 - p_0$) is the probability that at time $t = 0$ the hidden Markov chain starts in Regime 1 (resp. Regime 2). Transition probabilities must be determined to calculate the filtered probabilities. Honda (2003) uses the simplified assumption that the transition probabilities are modeled via an exponential distribution with intensity parameter λ. With this assumption the stochastic process of the filtered probabilities evolves as follows:

$$dp_t = \lambda(1 - 2p_t)dt + p_t(1 - p_t)\frac{\mu(s_2) - \mu(s_1)}{\sigma}dW_t = \mu_p(p_t)dt + \sigma_p(p_t)dW_t, \tag{5.4}$$

where $\mu_p(p_t) = \lambda(1 - 2p_t)$ and $\sigma_p(p_t) = p_t(1 - p_t)\frac{\mu(s_2)-\mu(s_1)}{\sigma}$. The price dynamics of the risky stock can now be represented as

$$dS_t = S_t\mu_p(p_t)dt + S_t\sigma_p(p_t)dW_t. \tag{5.5}$$

Combining the risk free asset, the filtered probabilities given in (5.4) and the risky asset given in (5.5) with the filtration $(\mathcal{F}_t^H)_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, P)$, a so-called Markovian equivalent economy is constructed in which asset allocation problems can be considered in a regime-switching environment with unknown regimes.

To shed light on the significance of regime-switching models, Table 5.1 collects some research papers that use observable and hidden Markov chains in an asset pricing and asset allocation context with different application areas (Applic.). The literature is classified by application area (general, default risk, option pricing, insurance and risk measures) and type of Markov chain (hidden: HMC, observable: OMC):[9]

Table 5.1: Selected papers that incorporate RS via Markov chains

Applic.	**MC**	**Paper**	**Overview/Research topic**
General	HMC	Honda (2003)	studies a dynamic optimal consumption and portfolio selection of an investor with power utility for a situation in which the drift of the risky asset follows a regime-switching process. The optimal strategy of an investor with a long time horizon may differ significantly from that of an investor with a short time horizon. This is caused by an investor's demand for hedging against fluctuations in mean returns.
	HMC	Taksar and Zeng (2007)	provide an optimal strategy of a CRRA investor in a multi-stock market where the drift and volatility of stock price dynamics depend on regime states. They find an approximation for the optimal strategy in a recursive way.

[9] The literature presented uses as dynamic asset model choice, as we do, a Black Scholes world, where the stock price follows a geometric Brownian motion with constant drift and constant volatility. There are also other models (e.g., volatility models, jump diffusion models) that incorporate regime-switching processes. For reasons of scope and due to the fact that the Black Scholes model is the most common model, other models are not discussed. For further information see Naik (1993) and Liang and Bayraktar (2014).

	OMC	Sotomayor and Cadenillas (2009)	study a consumption and investment problem assuming different utility functions, where the prices of the risk-free asset and n risky assets depend on the respective regime of the financial market. They show that the optimal investment fraction in the risky asset is greater in a bull market than in a bear market. The optimal consumption to wealth ratio depends not only on the regime, but also on the investor's risk attitude. Risk-averse investors consume more in a bull market than in a bear market.
	HMC	Luo and Zeng (2014)	consider in a multi-stock-model an asset allocation problem of investors with hyperbolic utility functions. The market model is incomplete, and the returns and volatilities of stocks are controlled by a hidden Markov process. The researchers are able to determine the approximate optimal strategy recursively.
	OMC	Shen and Siu (2012)	examine an optimal asset allocation problem in a financial market that can take on two regime states with the possibility of investing money risk-free in the bank account, in a stock or a zero-coupon bond. For the exponential utility case, numerical analyses are conducted with the result that regime switches have significant effects on the optimal investment strategies for stocks and bonds. The market prices of risk are essential factors in determining the optimal investment strategies.
	OMC	Ocejo (2018)	studies the utility maximization problem for the terminal wealth of a CRRA investor who invests in a riskless bond and a risky asset. She finds explicit solutions in a two and three regime world. The optimal fraction of wealth invested in the risky asset remains constant in each regime and is the Merton solution of the regime.
	HMC	Campani et al. (2021)	analyze optimal portfolio and consumption strategies under the assumption of four regimes in which a decision-maker can invest in large and small stocks, long-term government bonds, and a risk-free asset. They develop analytical approximations and find that the optimal strategy has a myopic and a hedging component and also depends on the regime probability. The consumption-to-wealth ratio is largely independent of the state of the economy, unlike the asset allocation strategy.

Default Risk	OMC	Capponi and Figueroa-López (2014)	provide investment strategies where the price dynamics of assets (defaultable bond, stock and bank account) are governed by a regime switch. They obtain explicit constructions of value functions by splitting the utility maximization problem into a pre- and post-default component. As the time of maturity approaches, the bond prices in the different regimes converge, so that the expected return decreases. A logarithmic investor changes her strategy from a long position to a short position sooner than a power utility investor when risk is high.
	HMC	Capponi et al. (2015)	consider an investor with power utility who maximizes her expected utility by allocating her wealth into a stock, a defaultable security, and a bank account. Using a numerical analysis, it can be shown that the investor increases her stock holdings when the probability of being in a economically good regime increases, and decreases her credit risk exposure when the probability of being in regimes with high default risk increases.
	HMC	Choi et al. (2015)	develop a model for the term structure of credit risk spreads, where the borrower's creditworthiness is represented as a regime-switching process. The efficiency of the model-theoretic representation is demonstrated using a callable bond.
	HMC	Bo et al. (2019)	examine asset allocation in a credit market under the assumption that the contagion effect upon default may depend on different regimes. They provide an approximate strategy via dynamic programming and provide foundations for the numerical treatment of risk-sensitive portfolio problems with defaults and stochastic factor processes.
Option Pricing	HMC	Buffington and Elliott (2002)	deal with the valuation of European and American options when the interest rate of the risk-free asset and the drift and volatility of the underlying stock depend on the state of the environment. They propose a valuation for European options and obtain a Black-Scholes equation. In addition, an approximate solution for American options is found.

	HMC	Elliott et al. (2005)	study an option pricing problem, where the drift and volatility of the risky underlying asset and the market interest rate follow a regime-switching process. They develop a method for the valuation of options that can be extended to the valuation of other options, interest rate products, and credit derivatives.
	HMC	Boyle and Draviam (2007)	evaluate European, Asian, and lookback options under the assumption that the volatility of the underlying stock follows regime switches. They perform numerical analyses and conclude that the difference between option prices with and without regime switches is substantial for lookback options and more moderate for European and Asian options.
	HMC	Henriksen (2011)	investigates the pricing of barrier options using a regime-switching model. The regime-switching model can reproduce Norwegian stock market index data better than the traditional Black-Scholes model.
	OMC	Fu et al. (2014)	investigate an asset allocation problem to maximize the expected utility of the terminal wealth of a decision-maker who can invest in an option, an underlying stock, and a risk-free bond in a regime-switching environment. The optimal value function is a solution to a dynamic programming equation, leading to explicit forms for the optimal value function. They show that it is optimal for an investor in a multi-regime-market to allocate her wealth in the same way as in a single-regime-market as long as the current state of the Markov chain is known.
	HMC	Zhu et al. (2019)	consider a pricing problem for a European call option in which both the market interest rate and the volatility process of the underlying stock price follow a regime-switching process. Numerical analyses are performed to evaluate the model. The consideration of regime-switching in option evaluation is useful.

Insurance	HMC	Siu (2005)	develops a model to value participating life insurance policies with surrender options and interest rate guarantees when the asset's market value dynamics are driven by a regime-switching process. A decomposition of the value of the policy into the value of an identical contract without surrender options and the premium of the surrender options has been performed and an approximate solution to the problem has been found.
	OMC	Lin et al. (2009)	develop a model for the valuation of equity-indexed and variable annuities where the dynamics of the underlying asset and the interest rate jointly follow a regime-switching process. Numerical analyses illustrate the importance of regime-switching in the context of guarantee pricing.
	HMC	Korn et al. (2011)	consider an asset allocation problem in a regime-switching environment of a member of a defined contribution pension plan who wants to maximize the expected utility of the terminal wealth. They develop a robust filter for the hidden state of the economy and present an algorithm for estimating the unknown parameters. Explicit investment strategies for an investor with logarithmic utility are derived.
	OMC	Chen and Delong (2015)	investigate an asset allocation problem for a contribution-based pension scheme. They identify an investment strategy that maximizes expected exponential utility of the discounted excess wealth over a target payment (target lifetime annuity).
	OMC	Fan et al. (2015)	evaluate equity-linked annuities with mortality risk using a regime-switching model. Via numerical analysis, it is found that the proposed model beats single-regime-switching models because they may underestimate the prices of equity-linked products.
	HMC	Jang and Kim (2015)	study an asset allocation problem for an insurer that purchases reinsurance and show via numerical analyses that regime changes can affect the optimal reinsurance strategy, asset allocation strategy, and insurance coverage significantly. Insurers opt for a higher reinsurance ratio and lower risky investment during a high volatile regime.

Risk measures	HMC	Billio and Pelizzon (2000)	use a regime-switching approach to predict the return distribution and estimate the Value at Risk of individual stocks as well as portfolios. Using back tests, they find that the regime-switching approach is superior to other methods of calculating Value at Risk values.
	HMC	Elliott et al. (2008)	provide an approach to evaluate risk measures for derivative financial instruments on the basis of a Markov-modulated Black-Scholes model. The approach provides an effective way to value risk measures for European options, barrier options, and American options.
	OMC	Yiu et al. (2010)	considers the optimal asset allocation of a utility maximizer with a Value at Risk constraint when the price dynamics of the risky asset follow a regime-switching process. The maximum value of the portfolio's Value at Risk in a short period of time over different states of the chain is obtained. Numerical results can be provided for the sensitivity analysis of the optimal portfolio and the Value at Risk level with respect to the model parameters.

MC= Markov chain, OMC= observable Markov chain, HMC= hidden Markov chain

Furthermore, there are various strands of literature that deal with extensions in the theoretical analysis (e.g., ambiguity, learning), other optimization problems (mean-variance analysis) and empirical applications (time series).[10] In the following the literature strands are briefly discussed:

A research paper that deals with an investment problem under ambiguity in a regime-switching environment is Kim et al. (2009). Liu (2011) examines a continuous-time intertemporal consumption and portfolio choice problem under ambiguity, where the drift of a risky asset follows a hidden Markov chain.

Literature dealing with regime-switching and learning in the context of asset allocation and pricing is extensive. Tu (2010) provides a framework for portfolio decision-making that accounts for regime change and parameter uncertainty. Berrada et al. (2018) consider regime-switching and learning in the context of asset pricing. Epstein and Schneider (2007),

[10] For the sake of an comprehensive literature review, we briefly address each literature strand based on selected research papers. We abstain from considering ambiguity and learning in our setup due to the fact that we want to focus primarily on analyzing the value of information of a time-inconsistent investor with CRRA-utility in a regime-switching environment.

Ju and Miao (2012) and Liu and Zhang (2022) analyze portfolio choice and asset pricing problems in a regime-switching environment considering ambiguity and learning.

In addition to Merton's portfolio problem (see references for asset allocation in Table 5.1), Markowitz's mean-variance approach is also frequently used in portfolio theory. In this asset allocation setup, some research papers take into account a regime-switching environment. Zhou and Yin (2003) solve a continuous-time version of the Markowitz mean-variance portfolio selection model in which the market consists of a bank account and several stocks that follow an observable Markov chain (for the case of a HMC see Elliott et al. (2010)). Xie (2009) extends the approach by adding a stochastic liability to the decision problem. Using a continuous-time and multi-period regime-switching model Chen et al. (2008) and Chen and Yang (2011) study an asset-liability management problem in the mean-variance framework. Chen and Yam (2013) consider an insurer's optimal investment reinsurance problem in which the claim variable follows a regime-switching process. Frauendorfer et al. (2007) and Bae et al. (2014) develop a portfolio selection problem of a pension fund and find that considering the possibility of regime changes in the mean-variance framework improves portfolio performance, especially in times of a crisis. Collin-Dufresne et al. (2020) solve a portfolio choice problem in a mean-variance framework when expected returns, covariances, and trading costs follow a regime-switching model.

Regime-switching models also find use in the estimation and forecasting of time series. They account for structural breaks and counter biased estimates (cf. Ang and Bekaert (2002)). Some research papers find that regime-switching models can characterize time series behaviour of certain variables (e.g., stock returns) better than models accounting for only one regime (cf. Hamilton and Susmel (1994), Schaller and Norden (1997)). Hamilton (1989) and Erlwein et al. (2012) suggest for the prediction of macroeconomic variables, such as GDP, to let the parameters of an autoregression depend on a Markov chain. Bollen et al. (2000) capture exchange rate movements. Hardy (2001) finds that regime-switching models provide a better fit to stock price data from a historical period than other commonly used econometric models (e.g., GARCH models). Giesecke et al. (2011) use a regime-switching model to forecast corporate bond default rates by financial and macroeconomic variables, such as GDP and stock return volatility. Guidolin and Hyde (2012) compare VAR-models with regime-switching models to make predictions about the future return evolution of stocks and bonds in the context of long-term investments. A regime-switching model is more complex but produces portfolios that better protect investors from possible future changes and produce higher quality predictions.

5.2.2 Challenges in considering regime switches

The consideration of regime-switching leads to challenges in model theory. Regime-switching models with a hidden Markov chain take into account that in reality investors often do not have full information about the price dynamics of future assets. However, the optimization of the expected utility of the terminal wealth of the investor is not straight forward and closed-form solutions are not available. The optimization problem and the determination of an optimal strategy can only be solved numerically, e.g., using Monte Carlo simulations. Explicit solutions for the optimal strategy are only obtained in special cases, e.g., when the investor has a logarithmic utility function (cf. Honda (2003), Taksar and Zeng (2010), Chen and Delong (2015)).[11]

Further, research papers dealing with regime-switching in asset allocation make various assumptions to avoid the problem of time inconsistency. The assumptions are very specific and less likely to be encountered in real life portfolio decisions. For example, Korn et al. (2011) assume an investor who has a logarithmic utility function. Siu (2012) finds a solution for the optimal strategy, assuming risk-neutral asset dynamics. Kim et al. (2009) also consider a risk-neutral entrepreneur. Welling et al. (2015) explicitly point out that their results are affected by the assumption of risk neutrality. Also, by assuming myopic investor behavior (time horizon of T approaching 0), the model setup simplifies, and time inconsistency is avoided (cf. Becker et al. (2022)).

Without these assumptions the optimization problem becomes highly nonlinear due to expected utility aggregation. The optimal strategy results from a non-linear function of the portfolio weights, which is time dependent. This violates Bellman's stochastic optimality principle, which states that a control law that is optimal for the entire time interval $[0, T]$ is also optimal for each subinterval $[t, T]$ (cf. Liang and Song (2015), Mei and Yong (2019)).[12]

To demonstrate the difference between time-inconsistent and time-consistent strategies that can be achieved by a small change in model assumptions, we compare the value

[11] In our setup, we decide to model the regime-switching environment on the basis of an observable Markov chain. However, we will still address the issue of parameter uncertainty by assuming an investor who does not have full information about the Markov chain and therefore has to pre-commit.

[12] The dynamic mean variance framework is also time-inconsistent in the sense that the Bellman optimality principle does not hold. Therefore, the optimal strategy that maximizes the objective function at the initial time does not necessarily optimize the objective function at a later time (cf. Yang et al. (2020)).

of informationin of a CRRA investor obtained in the setup of Becker et al. (2022) with the value of information of an investor with a logarithmic utility function under otherwise identical assumptions.[13] The assumption that the investor has a CRRA utility function whose risk aversion parameter $\gamma > 1$ leads to a time-inconsistent pre-commitment strategy for an investor who does not have full information about the regimes (see Chapter 4). Here, the optimal pre-commitment strategy $\pi^{*,\text{pre}}_{T,p}$ is given by the weighting factor $\alpha^*_{T,p} = \alpha_{T,p}(\pi^{*,\text{pre}}_{T,p})$ where

$$\pi^{*,\text{pre}}_{T,p} := \alpha^*_{T,p}\pi_1^{Mer} + (1-\alpha^*_{T,p})\pi_2^{Mer},$$

$$\alpha_{T,p}(\pi) = \frac{p\sigma_1^2 f_1(\pi,T)}{p\sigma_1^2 f_1(\pi,T) + (1-p)\sigma_2^2 f_2(\pi,T)}$$
$$\text{and } f_i(\pi,T) = e^{y(\pi,i)(1-\gamma_L)T},\ i=1,2.$$

However, by assuming a logarithmic utility function under the identical setup, we show that the optimal pre-commitment strategy is independent of time and given by

$$\frac{\partial \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right]}{\partial \pi} = p\sigma_1^2\pi_1^{Mer}T + (1-p)\sigma_2^2\pi_2^{Mer}T - \pi T(p\sigma_1^2 + (1-p)\sigma_2^2) \overset{!}{=} 0$$
$$\Leftrightarrow \pi^{*,pre} = \frac{p\sigma_1^2\pi_1^{Mer} + (1-p)\sigma_2^2\pi_2^{Mer}}{p\sigma_1^2 + (1-p)\sigma_2^2} \tag{5.6}$$

More detailed explanations and the calculation of the optimal pre-commitment strategy that maximizes the expected utility for an investor with a logarithmic utility function can be found in the Appendix B.2 (in particular Proposition 15). By means of the formula (5.2.2) it can be seen that the optimal strategy does not depend on the investment horizon T. For investors with logarithmic utility function, it turns out that the time horizon in this setup does not play any role in determining the optimal portfolio and the investor acts in a time-consistent manner. In Figure 5.1, we see that the value of information of a time-inconsistent CRRA-investor differs significantly from the value of information of a time-consistent investor with logarithmic utility. For a time-consistent investor with logarithmic utility, the value of information increases as the time horizon T increases. The

[13] Note that the value of information is given by the quotient of the certainty equivalents from a strategy under full information and a pre-commitment strategy under no information. In case of full information the regime is observable and the investor can condition her strategy on the regime.

value of information is greater the more uncertain the investor is about which regime prevails in reality ($p = 0.5$). Comparing the absolute values in the images is not useful because the certainty equivalents of time-consistent and time-inconsistent investor have a different scale due to different utility functions. For the analysis of the time-inconsistent strategy, see the content in Chapter 4.

Value of information depending on time-consistent and time-inconsistent set-up

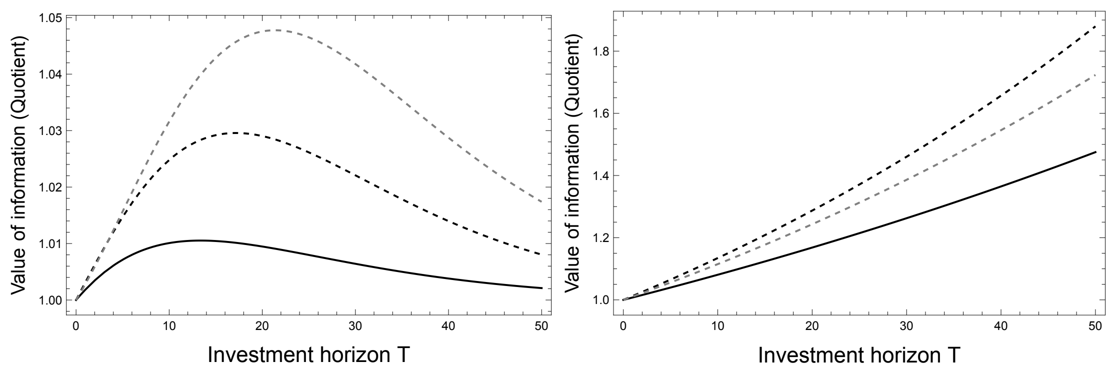

Figure 5.1: On the left hand side of the illustration the value of information is created for a time-inconsistent CRRA utility investor with a $\gamma_R = \gamma_L = 4$. On the right hand side of the illustration the value of information is created for a time-consistent investor with a logarithmic utility function. We use the benchmark parameter set of Chapter 4. The black graphs are plotted for $p = 0.2$, the black dashed for $p = 0.5$ and the gray dashed for $p = 0.7$.

An easy intuition why the assumption of an investor with a logarithmic utility function avoids time inconsistency is that logarithmic utility implies myopic behavior. However, it is often the case that investors are neither myopic nor act risk neutral and want to invest their money in the long run. Due to the fact that time inconsistency is a problem that occurs in reality and influences asset allocation decisions, time inconsistency should be considered in model-theoretic work and not be avoided via simple assumptions. Therefore, we consider the problem of time inconsistency in the following analysis in a regime-switching setup. Many research papers dealing with time inconsistency assume time-inconsistent behavior via a hyperbolic discount function - time inconsistency is thus given deterministically (cf. Wei et al. (2020)). In our approach, time inconsistency naturally arises via the assumption that an investor does not have full information about future market conditions, which may change over time.

5.3 Model setup

The stylized setup is given by a regime-switching model which is represented by an observable Markov chain $(Y_t)_{t\in[0,T]}$ with two regimes and one regime switch can occur on the interval $[0,T]$. Moreover, we assume that the Markov chain starts in Regime 1, i.e., the unconditional probability at $t=0$ of the Markov chain is $p=1$.[14] The point in time of a regime switch which determines the transition probability from the first regime to the second regime develops stochastically, modeled by an exponentially distributed random variable $\tau \sim Exp(\lambda)$, where λ is the intensity parameter. The density and distribution functions of τ are given by

$$f_\tau^\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \; F_\tau^\lambda(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}. \tag{5.7}$$

Thus the process $(Y_t)_{t\in[0,T]}$ starts with Regime 1 and remains there for an exponentially distributed length of time, and then jumps at $t=\tau$ to state s_2. Our financial market model contains two assets, a risky asset S and a risk-free asset B. The evolution of the risk free asset $(B_t)_{t\in[0,T]}$ is given by

$$dB_t = B_t r dt,$$

where r defines the risk-free interest rate. In Regime i with $i \in \{1,2\}$, the drift and the volatility of the stock are denoted by μ_i and σ_i.[15] The dynamics of the risky asset $(S_t)_{t\in[0,T]}$ are defined by

$$dS_t = S_t \left(\mu_1 1_{\{t\leq\tau\}} + \mu_2 1_{\{t>\tau\}}\right) dt + \left(\sigma_1 1_{\{t\leq\tau\}} + \sigma_2 1_{\{t>\tau\}}\right) dW_t, \tag{5.8}$$

where $S_0 = s_0$. On each regime the asset price dynamics are log-normal distributed. We assume that the drift and volatility depend on the Markov chain, but there are also other possibilities to incorporate a regime switch (see Appendix B.1). In total the evolution of

[14] We work on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^O)_{t\in[0,T]}, \mathbb{P})$ with the filtration $(\mathcal{F}_t^O)_{t\in[0,T]}$. For further information, see comments on observable Markov chain in Section 5.2.

[15] For simplicity, we assume that the price dynamics of the risk-free asset remain unchanged by the regime switch. Moreover, we assume that $r=0$. When the current regime is given by the state s_1, we write $\mu_t = \mu(Y_t = 1) = \mu_1$ and $\sigma_t = \sigma(Y_t = 1) = \sigma_1$ (resp. μ_2 and σ_2 for state s_2).

the portfolio wealth of the Investor $(V_t)_{t\in[0,T]}$ is given by the following dynamics:

$$\begin{aligned} dV_t &= V_t \left(\pi_t \frac{dS_t}{S_t} + (1-\pi_t)\frac{dB_t}{B_t} \right) \\ &= V_t \left(\pi_t \mu(Y_t)dt + \pi_t \sigma(Y_t)dW_t + (1-\pi_t)rdt \right) \\ &= V_t \left(\{\pi_t[\mu(Y_t) - r] + r\}dt + \pi_t \sigma(Y_t)dW_t \right), \end{aligned} \tag{5.9}$$

where π_t is the investment fraction of the risky asset S. If the regime switches at the random point in time $\tau \le T$ from Regime 1 to Regime 2 it holds:

$$\begin{aligned} X_1 &= \frac{V_T}{V_0} = \frac{V_\tau}{V_0}\frac{V_T}{V_\tau} \\ &= e^{[\pi(\mu_1-r)+r-\frac{1}{2}\pi^2\sigma_1^2]\tau+\sigma_1\pi W_\tau} e^{[\pi(\mu_2-r)+r-\frac{1}{2}\pi^2\sigma_2^2](T-\tau)+\sigma_2\pi(W_T-W_\tau)} \\ &= e^{[\pi(\mu_2-r)+r-\frac{1}{2}\pi^2\sigma_2^2]T} e^{[\pi(\mu_1-\mu_2)-\frac{1}{2}\pi^2(\sigma_1^2-\sigma_2^2)]\tau+\pi(\sigma_1 W_\tau+\sigma_2(W_T-W_\tau))}. \end{aligned} \tag{5.10}$$

If the regime switches randomly at a point in time τ with $\tau > T$ the terminal wealth can be stated as

$$\begin{aligned} X_2 &= \frac{V_T}{V_0} \\ &= e^{[\pi(\mu_1-r)+r-\frac{1}{2}\pi^2\sigma_1^2]T+\sigma_1\pi W_T}. \end{aligned} \tag{5.11}$$

The aim is to calculate and maximize the expected utility resp. the certainty equivalent of an investor. Therefore, we have to determine how the investor's preferences are modeled in form of a utility function. We consider an investor with constant relative risk aversion (CRRA), i.e., her utility function is given by

$$u(x) := \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, \text{ for } \gamma > 1 \\ \ln x, \text{ for } \gamma = 1 \end{cases}, \tag{5.12}$$

where γ denotes her relative risk aversion. In the context of this research work, we will only consider the case of $\gamma > 1$. We can state the expected utility optimization problem of the investor's terminal wealth by

$$\max_{\pi_t} \mathbb{E}_{\mathbb{P}} \left[u\Big(\frac{V_T}{V_0}\Big) \right]. \tag{5.13}$$

The proof is given in Appendix 13

Assumption 3 (Regime 1 and Regime 2)
We consider two regimes, where $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$. Thus, the financial market parameters in the first regime are more beneficial than in the second regime.

1. *Regime 1 is the good regime and Regime 2 is the bad regime, i.e., it holds that*

$$\lambda_1 = \frac{\mu_1 - r}{\sigma_1} \quad > \quad \lambda_2 = \frac{\mu_2 - r}{\sigma_2}.$$

2. *The optimal portfolio weight in Regime 1 exceeds the optimal portfolio weight in Regime 2, i.e., it holds that*

$$\pi_1^{Mer} = \frac{\mu_1 - r}{\gamma\sigma_1^2} \quad > \quad \pi_2^{Mer} = \frac{\mu_2 - r}{\gamma\sigma_2^2}.$$

With regard to the level of information, there arise different optimization problems. If the investor has **full information**, she can choose the optimal strategy π_t^* in the respective regime. Following Ocejo (2018) the optimal solution of problem (5.13) is then given by

$$\pi_t^* = \frac{\mu(Y_t) - r}{\gamma\sigma(Y_t)^2}.$$

where the parameters of the asset price dynamics are depending on the current regime at time t and the optimal strategy results by adapting the investment fraction to the Merton solution which fits the current parameters in the regime.[16]

Proposition 12

For the assumption that Regime 1 *is present at time* $t = 0$ *and the regime switches from state* s_1 *to state* s_2 *at time* $t = \tau$, *the optimal investment strategy of the investor with* ***full information*** *is given by*

$$\begin{aligned} \pi_t^* &= \mathit{1}_{\{t<\tau\}}\pi^{Mer_1} + \mathit{1}_{\{t\geq\tau\}}\pi^{Mer_2} \\ &= \mathit{1}_{\{t<\tau\}}\frac{\mu_1 - r}{\gamma\sigma_1^2} + \mathit{1}_{\{t\geq\tau\}}\frac{\mu_2 - r}{\gamma\sigma_2^2}, \end{aligned} \tag{5.14}$$

while

$$\pi_t^* := arg\max_{\pi_t} \mathbb{E}_{\mathbb{P}}\left[u\Big(\frac{V_T}{V_0}\Big)\right].$$

[16] The result of Merton (1971) provides the optimal investment fraction π^* in the risky asset in a single-regime-setting. The optimal strategy is given by $\pi_t^* = \pi^* = \pi^{Mer} = \frac{\mu - r}{\gamma\sigma^2}$, which means following a constant mix strategy will maximize the investor's expected utility. Any deviation from the optimal Merton solution leads to a utility loss for the investor. π^* is larger in a market that represents good economic conditions than in a crisis period. If a regime switch occurs, the Merton solution of the previous regime of course cannot be optimal anymore because of the changing drift and volatility parameters.

The savings rate $y(\pi, i)$ within Regime i ($i \in \{1,2\}$) is given by

$$y(\pi, i) = r + \pi(\mu_i - r) - \frac{1}{2}\gamma\pi^2\sigma_i^2.$$

$\pi_i^{Mer} = \frac{\mu_i - r}{\gamma\sigma_i^2}$ *denotes the Merton strategy in Regime i. With Eqn.* (5.14)*, it thus holds*

$$y\left(\pi_i^{Mer}, i\right) = r + \frac{1}{2\gamma} \cdot \frac{(\mu_i - r)^2}{\sigma_i^2} = r + \frac{1}{2\gamma} \cdot \lambda_i^2.$$

Now, we consider an investor who must pre-commit herself to a constant investment fraction π due to the fact that she has **no full information** about future asset price dynamics and the point in time when a regime switch occurs. We are able to calculate the expected utility of the investor's optimal strategy in closed-form:

Proposition 13

*The optimal pre-commitment strategy maximizes expected utility when the investor has **no full information**.*

$$\pi_T^{*,pre} := arg\max_{\pi_t} \mathbb{E}_{\mathbb{P}}\left[u\Big(\frac{V_T}{V_0}\Big)\right] \;\; s.t. \; \pi_t = \pi.$$

Let u be the CRRA utility function with relative risk aversion parameter γ, τ the exponentially distributed random point in time with intensity parameter λ where the regime switches from state s_1 to state s_2 and T the maturity of the investment. Then the expected utility of the terminal wealth V_T can be stated as

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right] &= \frac{1}{1-\gamma}\left[\lambda e^{\xi_2 T}\frac{1}{\nu-\lambda}\left[e^{(\nu-\lambda)T} - 1\right] + e^{(\xi_1-\lambda)T}\right], \;\; where \\
\xi_1 &= \left[\pi(\mu_1 - r) + r - \frac{1}{2}\gamma\sigma_1^2\pi^2\right](1-\gamma) \\
\xi_2 &= \left[\pi(\mu_2 - r) + r - \frac{1}{2}\gamma\sigma_2^2\pi^2\right](1-\gamma), \\
\nu &= \left[\pi(\mu_1 - \mu_2) - \frac{1}{2}\pi^2(\sigma_1^2\gamma - \sigma_2^2\gamma)\right](1-\gamma).
\end{aligned}$$

We can further simplify to

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right] &= \frac{1}{1-\gamma}\left[a(\pi,\lambda)e^{y(\pi,1)(1-\gamma)T}e^{-\lambda T} + (1 - a(\pi,\lambda))e^{y(\pi,2)(1-\gamma)T}\right], \;\; with \\
a(\pi,\lambda) &= \frac{\xi_1 - \xi_2}{\xi_1 - \xi_2 - \lambda} = \frac{y(\pi,1)(1-\gamma) - y(\pi,2)(1-\gamma)}{y(\pi,1)(1-\gamma) - y(\pi,2)(1-\gamma) - \lambda}, \;\; and \\
\xi_1 &= \left[\pi(\mu_1 - r) + r - \frac{1}{2}\gamma\sigma_1^2\pi^2\right](1-\gamma) = y(\pi,1)(1-\gamma) \\
\xi_2 &= \left[\pi(\mu_2 - r) + r - \frac{1}{2}\gamma\sigma_2^2\pi^2\right](1-\gamma) = y(\pi,2)(1-\gamma).
\end{aligned}$$

The proof is given in Appendix B.3.

Remark 3

(i)Notice that the parameter ξ_1 only depends on the first regime, ξ_2 only on the second regime and ν is an interaction term between both regimes with $\nu = \xi_1 - \xi_2$.

(ii) Using the relation $\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right] = u(CE)$ we can easily calculate the certainty equivalent (CE) of the terminal wealth where

$$CE = \left[\frac{\xi_1 - \xi_2}{\xi_1 - \xi_2 - \lambda} e^{(\xi_1 - \lambda)T} + \frac{-\lambda}{\xi_1 - \xi_2 - \lambda} e^{\xi_2 T}\right]^{\frac{1}{1-\gamma}}.$$

(iii) With $\gamma = 0$ we get the closed-form formula for the expected value of the terminal wealth $\mathbb{E}_{\mathbb{P}}(V_T)$.

(iiii) Defining $y_T(\pi) = \frac{1}{T}\ln(CE)$ as the certainty equivalent return we get for $r = 0$ and $\gamma > 1$

$$y_T(\pi) = \frac{1}{(1-\gamma)T}\ln[a(\pi,\lambda)e^{y(\pi,1)(1-\gamma)T}e^{-\lambda T} + (1 - a(\pi,\lambda))e^{y(\pi,2)(1-\gamma)T}].$$

(iv) When the investor is well informed about the parameters that hold in the two possible regimes she is able to invest the optimal investment fractions π^{Mer_1} and π^{Mer_2} in the risky asset. The parameters ξ_1, ξ_2 and ν are then of the following form

$$\begin{aligned}
\xi_1^{Mer} &= \left[\pi^{Mer_1}(\mu_1 - r) + r - \frac{1}{2}\gamma\sigma_1^2(\pi^{Mer_1})^2\right](1-\gamma) \\
\xi_2^{Mer} &= \left[\pi^{Mer_2}(\mu_2 - r) + r - \frac{1}{2}\gamma\sigma_2^2(\pi^{Mer_2})^2\right](1-\gamma) \\
\nu^{Mer} &= \xi_1^{Mer} - \xi_2^{Mer}.
\end{aligned}$$

5.4 Impact of time inconsistency on optimal asset allocation and the value of information

We solve the optimization problems of investors which differ in their information content under a regime-switching environment. Problems stemming from time inconsistency are illustrated and discussed via the value of information. We will analyze and compare the strategies associated with the above-mentioned optimization problems. To perform our analyses, we will use the benchmark parameter setup of Chapter 4.

As it turns out, the pre-commitment strategy $\pi_T^{*,\text{pre}}$ gives rise to a time-inconsistent strategy due to the fact that the strategy depends on the time horizon T. The lack of

Benchmark parameter

μ_1	μ_2	σ_1	σ_2	r
0.1316	0.0769	0.2080	0.2221	0.00

Table 5.2: Benchmark parameter constellation.

information due to regime uncertainty leads to a time-inconsistent behavior of the investor. The proof of Proposition 14 is given in the Appendix B.4.

Proposition 14
For $\gamma > 1$ the optimal pre-commitment strategy is given by the implicit function

$$\pi_T^{*,pre} = \pi_1^{Mer} \frac{\alpha_1(\pi_T^{*,pre})}{1-\alpha_3(\pi_T^{*,pre})} + \pi_2^{Mer} \frac{\alpha_2(\pi_T^{*,pre})}{1-\alpha_3(\pi_T^{*,pre})}, \text{ with } \alpha_1+\alpha_2+\alpha_3 = 1 \text{ where}$$

$$\alpha_1(\pi_T^{*,pre}) = \frac{[\lambda\sigma_1^2(e^{(\xi_1-\lambda)T} - e^{\xi_2 T}) - T(\xi_1-\xi_2-\lambda)(\xi_1-\xi_2)\sigma_1^2 e^{(\xi_1-\lambda)T}}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}$$

$$\alpha_2(\pi_T^{*,pre}) = \frac{\lambda\sigma_2^2(e^{\xi_2 T} - e^{(\xi_1-\lambda)T}) + T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}$$

$$\alpha_3(\pi_T^{*,pre}) = \frac{\left(\lambda e^{\xi_2 T} - \lambda e^{(\xi_1-\lambda)T}\right)(\sigma_1^2-\sigma_2^2) + T(\xi_1-\xi_2-\lambda)(\xi_1-\xi_2)\sigma_1^2 e^{(\xi_1-\lambda)T}}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}.$$

We can further aggregate the weighting factor of the Merton solutions π_1^{Mer} and π_2^{Mer}:

$$\pi_T^{*,pre} = \beta\pi_1^{Mer} + (1-\beta)\pi_2^{Mer}, \text{ where}$$

$$\beta = \frac{\sigma_1^2 g_1}{\sigma_1^2 g_1 + \sigma_2^2 g_2}, 1-\beta = \frac{\sigma_2^2 g_2}{\sigma_1^2 g_1 + \sigma_2^2 g_2} \text{ with}$$

$$g_1 = e^{-\lambda T} e^{y(\pi_T^{*,pre},1)(1-\gamma_L)T}\left(\frac{T(\xi_1-\xi_2)(\xi_1-\xi_2-\lambda)}{\lambda} - 1\right) + e^{y(\pi_T^{*,pre},2)(1-\gamma_L)T}$$

$$g_2 = e^{-\lambda T} e^{y(\pi_T^{*,pre},1)(1-\gamma_L)T} + e^{y(\pi_T^{*,pre},2)(1-\gamma_L)T}(T(\lambda-\xi_1+\xi_2)-1).$$

It Holds: $\mathbb{P}(\tau > T) = e^{-\lambda T}$; $\mathbb{P}(\tau \leq T) = 1 - e^{-\lambda T}$.

The proof is given in Appendix B.4. The intensity parameter λ determines the frequency of the regime switch and the point in time when the switch takes place.[17]

[17] Note that we allow only for one possible regime switch.

It affects the optimal pre-commitment strategy $\pi_T^{*,pre}$: For $\lambda \to 0$ a regime switch is less likely and obtains later in time. Only the first regime is relevant, thus the optimal pre-commitment strategy $\pi_T^{*,pre}$ converges to the Merton solution from the first regime ($\pi_1^{Mer} = 0.76$ with $\gamma = 4$). The proportion of β that is invested in π_1^{Mer} decreases with increasing λ. This means that for $\lambda \to +\infty$ the second regime becomes more relevant, such that $\pi_T^{*,pre}$ approaches π_2^{Mer}. The optimal λ for the strategy is thus reached in the limiting cases and depends on which regime is the stable, resp. good one in the initial state (in our case Regime 1). As the time horizon T increases, the optimal pre-commitment strategy approaches the Merton solution of the bad regime ($\pi_2^{Mer} = 0.39$ with $\gamma = 4$) at a faster rate. This is intuitive, since the investor faces greater exposure as the time horizon increases, and regime-switching becomes more likely. An illustration of this relationship is given on the left side of Figure 5.2.

Impact of parameter λ and time horizon T on optimal pre-commitment strategy $\pi_T^{*,pre}$

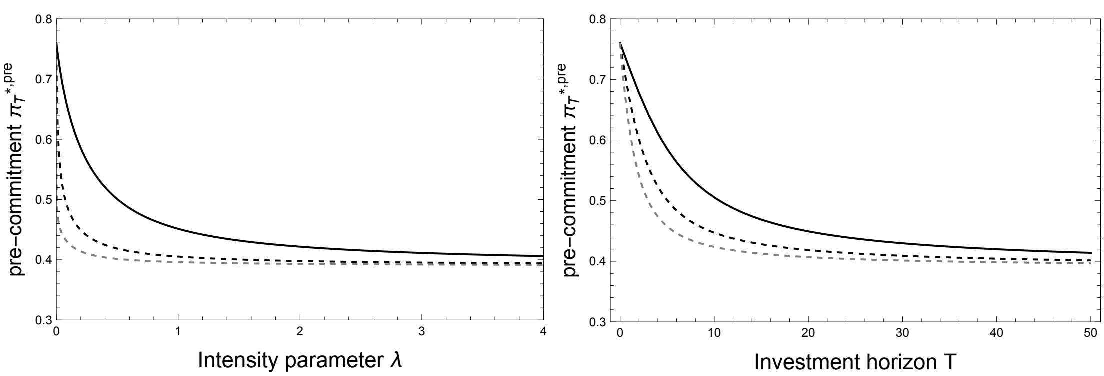

Figure 5.2: The illustrations are created for $\gamma = 4$. The left figure shows the optimal pre-commitment strategy $\pi_T^{*,pre}$ depending on λ. The black graph shows the strategy for $T = 5$, the black dashed graph for $T = 20$ and the gray dashed for $T = 50$. The right figure shows the optimal pre-commitment strategy $\pi_T^{*,pre}$ depending on the investment horizon T. The black graph shows the strategy for $\lambda = 0.2$, the black dashed for $\lambda = 0.5$ and the gray dashed for $\lambda = 0.9$.

The optimal pre-commitment strategy $\pi_T^{*,pre}$ is obtained as a mixture of the two Merton solutions of the two possible regimes (see Proposition 14). The longer the investment horizon, the greater the influence of the second regime. Thus, a long-term investor places increasing weight on the state that will follow in the event of a possible regime switch. In our case, she puts more weight on the bad state. The optimal portfolio of an investor with a long-time horizon differs significantly from an investor's portfolio with a short-time horizon due to time inconsistency. The difference is due to the investor's need to hedge against fluctuations in the second regime, which represents bad economic conditions. A higher λ implies an earlier switch to the bad regime and thus an increase of the percentage of time spent in the bad regime, which leads to an earlier approach to the worst case strategy (see right side of Figure 5.2).

Optimal certainty equivalent return

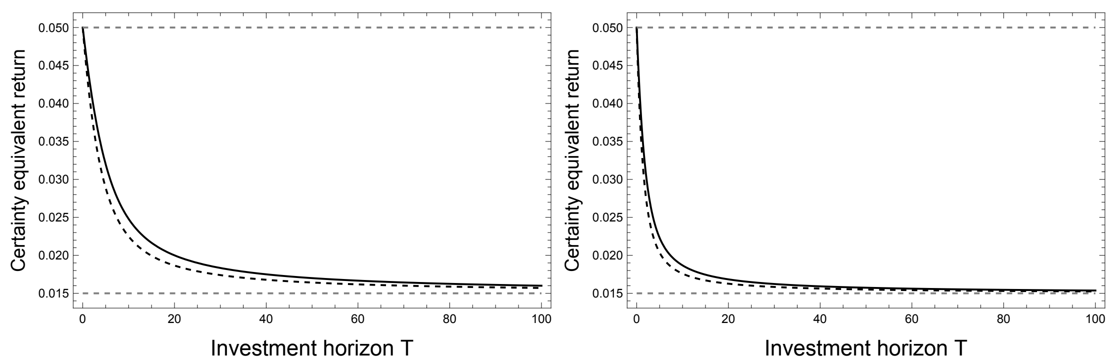

Figure 5.3: The illustrations are created for $\gamma = 4$. The left (right) figure shows the optimal certainty equivalent return of the strategies for $\lambda = 0.2$ ($\lambda = 0.9$). The black graphs show the certainty equivalent return for π_t^* in case of full information (overall optimal strategy). The dashed black graphs show the certainty equivalent return for the optimal pre-commitment strategy $\pi_T^{*,pre}$. The dashed gray lines picture the certainty equivalent return of the merton solutions in the respective regime.

The certainty equivalent return of the overall optimal strategy under full information and the optimal pre-commitment strategy is given in Figure 5.3. For $T \to \infty$ the certainty equivalent return converges for both strategies towards the optimal certainty equivalent return in the bad regime ($y(\pi^{Mer2}, 2) = r + \pi^{Mer2}(\mu_2 - r) - 0.5\gamma\pi^{Mer2}\sigma_2^2$). The larger T , the longer the investor will be in the bad second regime, and the smaller her overall certainty equivalent return. It is straightforward to show that the optimal pre-commitment strategy is between the two Merton solutions. Similarly, the optimal certainty equivalent return is between the certainty equivalent returns in the economies with only Regime 1 and only Regime 2. The higher the intensity parameter λ, the faster converges the certainty equivalent return to the certainty equivalent return of the bad regime. The certainty equivalent return using the optimal pre-commitment strategy is only slightly smaller than the certainty equivalent return using the optimal regime-dependent Merton solutions under full information. Since the regime switch to the bad second regime also affects an investor with full information, the certainty equivalent return of the overall strategy (regime-dependent Merton solutions) also converges to the certainty equivalent return of the bad regime. Thus, the risk of regime-switching has minimal impact on the optimal certainty equivalent return as long as the strategy is correct on average.

In a next step we analyze the certainty equivalent (CE) losses which occur from a time-inconsistent pre-commitment strategy. Intuitively, one would assume that the longer the horizon for which the information is relevant, the higher would be it's value. A poor investment decision by the investor due to lack of information would be more severe the longer the investment horizon in which she cannot obtain information and therefore is not able to adjust her decision on the basis of new information. However, we show that this is not necessarily the case. The value of information is given by the difference in the certainty equivalents associated with the optimal strategies under pre-commitment and under full information about the regimes (point in time of the regime switch and the asset price dynamics within the regimes). We want to identify λ^{crit} at which the value of information is highest, i.e.

$$\lambda^{crit} := \arg\max_{\lambda} \; \mathrm{CE}^{\mathrm{Mer}} - \mathrm{CE}^{\pi_T^{*,pre}}.$$

The influence of λ is not trivial. For a long-term investor (high value of T), the value of information seems to decrease with increasing λ. This insight can be taken from Table 5.3. Since Regime 1 represents an economically good state, the smaller λ, the higher the certainty equivalents of both optimal strategies ($\mathrm{CE}^{\mathrm{Mer}}$ and $\mathrm{CE}^{\pi_T^{*,pre}}$) should be. Thus the certainty equivalents are monotonically decreasing in λ for a fix T. Additionally, the change in the difference between the two certainty equivalents gets smaller as λ increases. It is to be noted that an increasing λ means that it is more likely that a regime switch will occur and that the point in time of the regime switch will arrive sooner, i.e., for a high T Regime 2 holds for a long time. For this reason, the optimal strategy of an investor with full information, who can refer to the Merton solution of the prevailing regime at the current time, also turns out to be smaller since in the second regime the investments do not yield such high returns due to the worse price dynamics of the risky asset. Thus, both strategies converge. For $\lambda \to 0$ the value of information is highest as Regime 1 prevails for a long time and a pre-commitment investor invests too little in stocks although good environmental conditions prevail.

If the time horizon T is small, λ^{crit} at which the value of information (difference in certainty equivalents) is maximal, cannot be reached in the limiting case ($\lambda \to 0$). The results can be seen in Table 5.4. With a small T, the certainty equivalents are obviously smaller due to the shorter investment horizon, which means that the absolute value of information is also smaller. The value of information first increases with increasing λ. At a certain λ (for $T = 2$: $\lambda^{crit} = 0.336$), the value of information is maximal, after that the difference in the certainty equivalents decreases as λ increases. The smaller the investment horizon T, the larger λ^{crit} tends to be. Figure 5.4 provides an illustration that displays these relations. For long investment horizons, the value of information is higher and monotonically decreasing with increasing λ. For small investment horizons, the willingness to pay for full information approaches 0 for $\lambda \to 0$. This is quite intuitive because a small investment horizon and a small λ means that no regime switch will take place and only Regime 1 is relevant. The investor knows the first regime and the pre-commitment strategy converges to the Merton solution of the first regime. As λ increases, the value of information first increases and decreases after λ^{crit} is reached. The maximal value of information shifts for varying small T. The smaller T, the higher λ^{crit} and the smaller the value of information.

Value of information depending intensity parameter λ

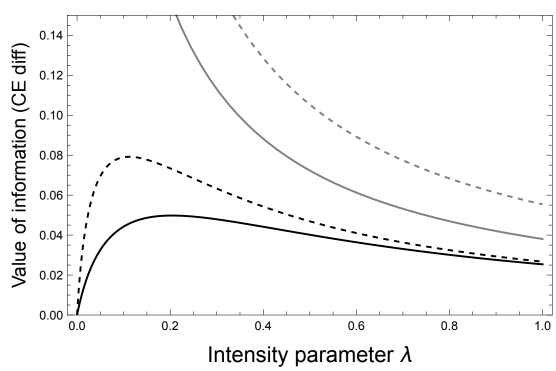

Figure 5.4: In this illustration we assume $\gamma = 4$. The black graph gives the value of information as difference in the certainty equivalents for $T = 4$ and the black dashed for $T = 6$. Furthermore, the gray (gray dashed) shows the value of information for $T = 50$ ($T = 100$).

Value of information for varying λ with fixed time horizon (1/2)

λ	$\mathrm{CE}^{\mathrm{Mer}}$	$\mathrm{CE}^{\pi_T^{*,pre}}$	$\mathrm{CE}^{\mathrm{Mer}} - \mathrm{CE}^{\pi_T^{*,pre}}$	$\pi_T^{*,pre}$
0.001	8.70596	7.44088	1.26508	0.55798
0.1	2.68926	2.54161	0.147649	0.42731
0.2	2.43675	2.34479	0.0919615	0.41391
0.3	2.33973	2.27261	0.0671176	0.40754
0.4	2.28805	2.23515	0.0528969	0.40383
0.5	2.25588	2.21222	0.0436638	0.40141
0.6	2.23392	2.19674	0.0371803	0.39971
0.7	2.21796	2.18558	0.0323757	0.39845
0.8	2.20584	2.17717	0.0286719	0.39748
0.9	2.19632	2.17059	0.0257291	0.39671
1.0	2.18864	2.16531	0.0233345	0.39608

Table 5.3: The table is constructed for $T = 50$ and $\gamma = 4$. The loss in the certainty equivalent of a pre-commitment investor, resp. the difference in the certainty equivalents of the two optimal strategies under full information and under pre-commitment $\mathrm{CE}^{\mathrm{Mer}} - \mathrm{CE}^{\pi_T^{*,pre}}$ gives the value of information.

Value of information for varying λ with fixed time horizon (2/2)

λ	$\mathrm{CE}^{\mathrm{Mer}}$	$\mathrm{CE}^{\pi_T^{*,pre}}$	$\mathrm{CE}^{\mathrm{Mer}} - \mathrm{CE}^{\pi_T^{*,pre}}$	$\pi_T^{*,pre}$
0.001	1.0509	1.10505	0.00003	0.75949
0.1	1.09661	1.09469	0.00193	0.71454
0.2	1.0893	1.08637	0.00293	0.67793
0.3	1.08301	1.07964	0.00338	0.64790
0.4	1.07758	1.0741	0.00348	0.62276
0.5	1.07285	1.06948	0.00337	0.60142
0.6	1.06872	1.06559	0.00313	0.58311
0.7	1.06510	1.06229	0.00281	0.56727
0.8	1.06190	1.05945	0.00245	0.55346
0.9	1.05908	1.05700	0.00207	0.54135
1.0	1.05657	1.05488	0.00169	0.53065

Table 5.4: The table is constructed for $T = 2$ and $\gamma = 4$. The loss in the certainty equivalent of a pre-commitment investor, resp. the difference in the certainty equivalents of the two optimal strategies under full information and under pre-commitment $\mathrm{CE}^{\mathrm{Mer}} - \mathrm{CE}^{\pi_T^{*,pre}}$ gives the value of information.

Value of information depending on time horizon T (1/2)

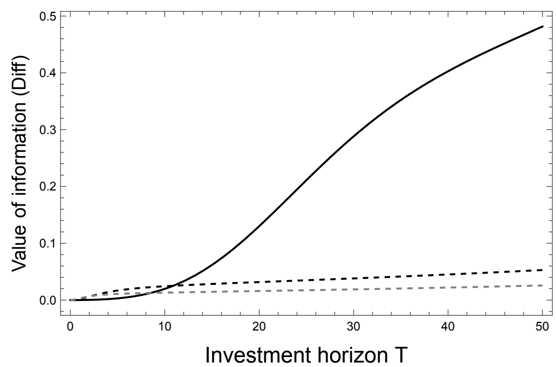

Figure 5.5: The figure shows the value of information as the difference in the certainty equivalents of the overall optimal strategy under full information and the pre-commitment strategy depending on the investment horizon T. The image was created for $\gamma = 4$. The black graph has a $\lambda = 0.01$, the black dashed a $\lambda = 0.4$ and the gray dashed a $\lambda = 0.9$.

Thus, time horizon effects determine the significance of the intensity parameter λ for the value of information of a time-inconsistent pre-commitment investor. When λ is high, the value of information approaches zero for both short and long investment periods. The second regime, where worse economic conditions prevail, becomes more significant for higher λ, so that the certainty equivalents of the two optimal strategies converge. When λ is small, i.e., a regime switch is unlikely in the investment horizon, the value of information is high at long investment horizons, since the pre-commitment investor misses out on returns by investing too little in stocks in a good economic environment. At short time horizons, the value of information has to build up first since potential utility losses are not so high at short investment horizons.

To better highlight the influence of the time component, we plot the evolution of the value of information ($CE^{\text{Mer}} - CE^{\pi_T^{*,pre}}$) as a function of the investment horizon T in Figure 5.5. For already moderate λ (e.g., $\lambda = 0.4$), the value of information increases scarcely with increasing T. Only for very small λ (e.g., $\lambda = 0.01$) the value of information increases with increasing T. A low λ means that a regime switch will occur at a later point in time and thus becomes less likely. Staying longer in the economically good state leads to higher utility losses for the pre-commitment investor whose strategy consists of a mixture of the two Merton solutions. Overall, the optimal asset allocation of the pre-commitment strategy, and thus the evolution of the value of information, depends on the interacting determinants of time and regime uncertainty (intensity parameter λ). Making optimal decisions under more realistic, behavioral assumptions is not trivial.

We illustrate the value of information (certainty equivalent losses in contrast to overall optimal strategy) for an investor who can implement the optimal pre-commitment strategy $\pi_T^{*,pre}$ in the presence of regime uncertainty. This strategy is time-inconsistent. To illustrate that this strategy is the best constant mix strategy, we discuss other constant investment strategies. A rather unsophisticated investor who knows the current regime may be unaware or ignorant of the existence of a second regime and thus does not anticipate a regime switch. She invests over the entire investment horizon π^{Mer1} in the stock. A different investment strategy arises when the investor knows the parameters of the regimes but does not know when the regime switch will take place and therefore invests the average of the Merton solutions ($\frac{\pi^{Mer1}+\pi^{Mer2}}{2} = 0.575$). The value of information for these strategies (difference in the certainty equivalent of the overall optimal strategy and the certainty equivalents of the unsophisticated strategies) is illustrated in Figure 5.6. For both non-optimal constant strategies, the value of information increases with increasing investment horizon T. The value of information under the first strategy $\pi = \pi^{Mer1}$ is higher for moderate and higher λ than under strategy $\pi = 0.575$, because the first strategy differs significantly from the Merton solution in Regime 2 and leads to losses from the moment when the regime switch takes place. For low λ, the first strategy is better for short investment horizons (lower value of information), since Regime 1 is relevant, and the strategy is similar to the Merton solution of Regime 1. At a certain point, however, the value of the information rises sharply, indicating a regime switch.

In the presence of regime uncertainty, the optimal pre-commitment strategy is the best strategy that an investor who does not have full information can implement. For other strategies, the loss in the certainty equivalents may be quite substantial, especially for long-term investors.

One might also consider an investor who pre-commits to switch at a deterministic point in time s ($0 \leq s \leq T$) from the optimal solution in Regime 1 to Regime 2. She thus solves

$$\pi_t^{*,\text{pre, s}} := \arg\max_{\pi_t} \; E_P\left[u\left(\frac{V_T}{V_0}\right)\right] \text{ s.t. } \pi_t = \pi_1^{\text{Mer}} 1_{\{t\leq s\}} + \pi_2^{\text{Mer}} 1_{\{t>s\}}.$$

This is a more sophisticated version of the originally pre-commitment strategy. While the investor who pre-commits herself to a constant investment fraction basically mitigates between the Merton solutions of the regimes (if no corner solution results, she is wrong

Value of information depending on time horizon T (2/2)

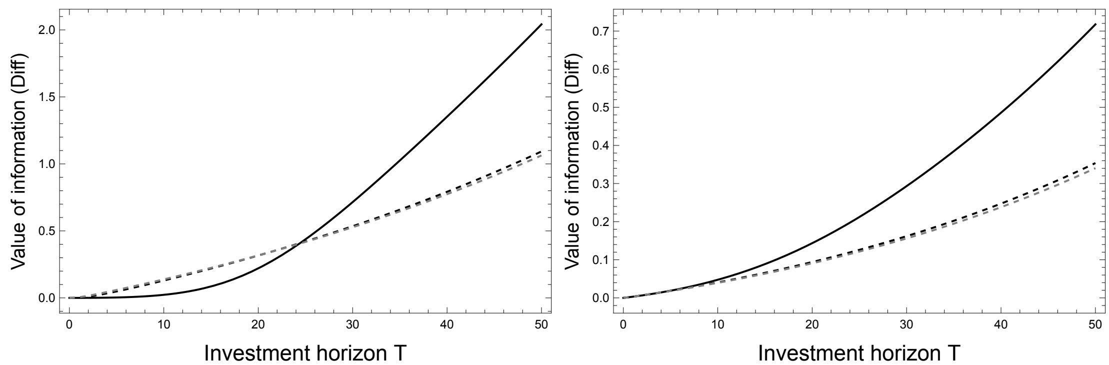

Figure 5.6: In both illustrations $\gamma = 4$. The left image shows the value of information for an investor with a non-optimal investment fraction of $\pi = \pi^{Mer1} = 0.76$. The right image shows the value of information for an investor with a non-optimal investment fraction of $\pi = 0.575$. The black graphs illustrate the value of information for $\lambda = 0.01$, the black dashed for $\lambda = 0.4$ and the gray dashed for $\lambda = 0.9$.

all the time), the investor who switches from π^{Mer1} to π^{Mer2} is only wrong if $s < \tau$ (i.e., she uses the Merton solution of Regime 2 on $[s, \tau]$ instead of π^{Mer1}) and for $s > \tau$ (i.e., she uses the Merton solution of Regime 1 on $[\tau, s]$ instead of π^{Mer2}). The optimal time s^* to switch the strategy could be compared to a more naive strategy that assumes that a regime switch occurs at the expected switch time $s = \frac{1}{\lambda}$. Depending on the length of the investment horizon and the size of λ, the answer as to which pre-commitment strategy is better will vary. The strategy where the investor pre-commits to switch at a deterministic point in time should dominate the originally pre-commitment strategy until time s if $s \leq \tau$. After that point in time it depends on how well s is chosen. If $\tau < s$ the originally pre-commitment strategy should dominate for the time interval from s to τ. The strategy of switching at a deterministic point in time is not discussed in more detail in this research work, but it might be interesting to investigate this strategy intensively in future research.

Now let us briefly consider the case where investors make their asset allocation decisions at unstable times. Thus, the currently prevailing environmental state is given by the parameters of Regime 2. The investor has no full information, so that regime uncertainty exists regarding the exact switch from Regime 2 to Regime 1. Figure 5.7 shows the pre-commitment strategy $\pi_T^{*,pre}$ and the value of the information for this case as a function of T. With increasing intensity parameter λ, the pre-commitment strategy for increasing time horizons approaches the Merton solution of Regime 1 faster. A high

λ indicates that a regime switch to the good state (Regime 1) is likely. For very small λ, the pre-commitment strategy behaves differently as a function of T. The strategy remains around the low Merton solution of the first regime (Regime 2). By a very small λ, switching to the good state is unlikely and Regime 1 becomes insignificant. The pre-commitment strategy converges to the optimal Merton solution in Regime 2. The certainty equivalent of the pre-commitment strategy thus converges to the certainty equivalent of the Merton solution. In general, the certainty equivalents of the pre-commitment strategy and the overall optimal strategy are lower if the unstable regime lasts for a long time due to the low investment fraction in the risky stock, which decreases the difference in the certainty equivalents and results in a value of the information close to zero.

There is a reverse effect on the value of information of a pre-commitment strategy compared to the case when the investor starts in Regime 1 and the economy can switch to Regime 2. The higher the probability of a regime switch to a good state and the sooner the switch takes place (incorporated by moderate to high λ) the higher the value of information for a time-inconsistent pre-commitment investor who primarily invests her wealth in the long run. The impact on the value of information is more powerful for an investor who starts in a bad regime than in a good regime if a regime switch is likely to occur in the investment horizon.

Value of information and strategies depending on time horizon T when switching from Regime 2 to Regime 1

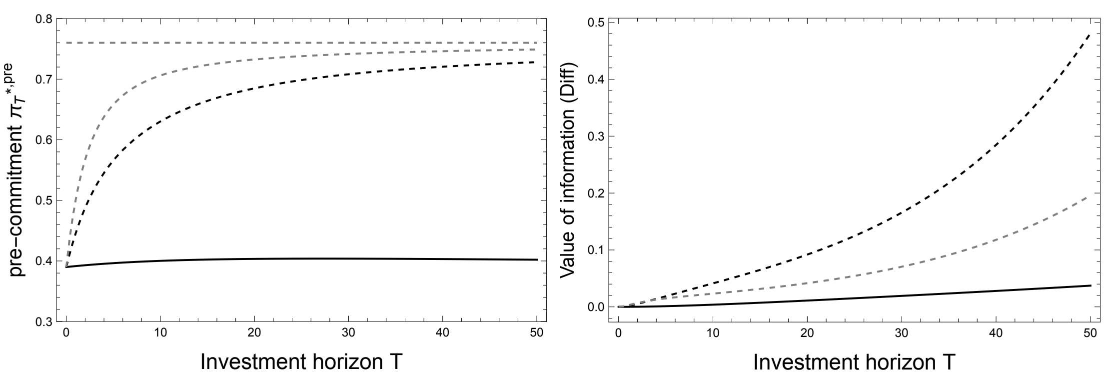

Figure 5.7: In both illustrations $\gamma = 4$. The left image shows the optimal pre-commitment strategy $\pi_T^{*,pre}$ depending on T. The right image shows the value of information depending on the investment horizon T. For both illustrations the black graphs picture the case $\lambda = 0.01$, the black dashed picture $\lambda = 0.4$ and the gray dashed picture $\lambda = 0.9$.

5.5 Conclusion

We give a literature review on the consideration of regime-switching models in asset allocation und asset pricing, which justifies our model setup. An analysis of the optimal asset allocation problem of a time-inconsistent investor in a regime-switching environment is conducted. We compare the expected utility of optimal asset allocation under an observable Markov chain between two types of investors with different levels of information. The value of information is obtained as the difference in the certainty equivalents of the investors' strategies. If an investor has no full information about the regimes, the problem of time inconsistency arises naturally due to regime uncertainty. To deal with time inconsistency we determine the optimal pre-commitment strategy, which is obtained as a weighted average of the Merton solutions of the regimes.

The optimal pre-commitment strategy depends on the length of the investment horizon and the intensity parameter λ. The optimal fraction invested in the risky asset by a time-inconsistent long-term investor is significantly different compared to the fraction invested by a time-inconsistent short-term investor. The importance of the first regime, driven by the intensity parameter λ, decreases as the investment horizon increases. With increasing investment horizon the pre-commitment strategy converges to the Merton solution of the second regime (worst case strategy). While the pre-commitment strategy differs from the overall optimal strategy under full information, altogether the risk of regime-switching appears to have minimal impact on the optimal certainty-equivalent return as long as the pre-commitment strategy is correct on average.

We examine λ^{crit}, at which the value of information is highest for a time-inconsistent investor implementing a precommitment strategy. For a long-term investor the value of information seems to decrease with increasing λ, thus λ^{crit} is reached in the limiting case $\lambda \to 0$. The smaller the investment horizon T, the larger λ^{crit} tends to be.

Other pre-commitment strategies are also considered. Here the loss in the certainty equivalents can be quite significant, especially for investors aiming for a long-term optimal asset allocation. We also investigate the reverse case, where the unstable Regime 2 prevails and a switch to the stable Regime 1 is possible. The results show that asset allocation decisions are complex and depend heavily on model assumptions. We show that time inconsistency and regime-switching should be considered as realistic assumptions in asset allocation decisions. A time-inconsistent investor can mitigate potential utility losses by trying to maximize a pre-commitment strategy.

Chapter 6

General conclusion

The main focus of this thesis is the investigation of regime uncertainty with the interplay of time inconsistency in optimal asset allocation problems. Increasing global interconnectedness, digitalization and technological change make worldwide interaction complex. Policymakers change the environment for investors more often which leads to regime uncertainty – uncertainty about future asset price dynamics. Decision-making under uncertainty poses particular challenges to the rationality of decision-makers and is often affected by time inconsistency which violates the consistency assumption in rational decision-making. Time inconsistency is an behavioral phenomenon where decision-makers want to revise their decision at a later point in time although the information basis has not changed. This anomaly leads to biased decisions that are in conflict with the long-run interests of decision-makers. In theoretical work that attempts to model decision-making, especially dynamic asset allocation problems under more real-world conditions like regime uncertainty, time inconsistency arises via the aggregation of non-linear functions. We focus on a pre-commitment strategy as approach to deal with time inconsistency, since the strategy optimizes the objective function at the initial time and it is a realistic way to deal with time-inconsistent behavior.

Under the assumption of an a priori lottery over possible prevailing regimes we come to the conclusion that the willingness to pay for full information approaches zero not only for an investment horizon of zero but also for an infinite horizon. In the regime-switching set-up we find that as long as the pre-commitment strategy is correct on "average", the certainty equivalent losses of a time-inconsistent investor exposed to regime uncertainty are small. Other non-optimal strategies may lead to greater losses in the certainty equivalent. Results depend very much on the model assumptions. Implementing an optimal strategy seems to mitigate extreme losses due to regime uncertainty and time inconsistency. Overall, regime uncertainty and time-inconsistency exist in reality and should therefore also be considered in theoretical approaches that study asset allocation decisions. Regulatory changes and announcements should be made as transparent as possible to mitigate negative consequences of regime uncertainty.

We would like to give a brief research outlook. The research in Chapter 4 and Chapter 5 could be extended by allowing for gradual learning. We only consider the case where an investor with a lack of information is willing to pay for full information, i.e., the case of 100% learning. Under real-world conditions, at least sophisticated investors are able to obtain information through a wide variety of channels. Referring to Chapter 5, potential future research could consider multiple regime states and recurrent regime switches. This would consider the fact that policymakers introduce and change increasingly new regulations and policies that affect asset price dynamics. Due to the current interest rate levels it would be worthwhile to consider a risk-free interest rate not equal to zero. Since in reality investments are often made in different stocks to account for diversification, the setup could be extended to include several stocks or funds and their correlations with each other. It should also be noted that in reality any rebalancing of the portfolio will result in transaction costs and taxes. An adjustment based on new information is therefore not possible free of charge Furthermore, the intensity parameter λ could be modeled in a time-dependent manner, i.e., during periods of high regime uncertainty, λ could increase. It would also be interesting to intensively study an investor who commits to a deterministic switching point. Since we live in uncertain times, the topic of ambiguity with respect to the distribution of τ could be explored.

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Appendix A

Appendix to Chapter 4

A.1 Supplementary comments

A.1.1 Black-Scholes set-up and Merton problem

The subsection serves as a general and more intensive illustration of the Merton problem under assumptions of a Black-Scholes market model.[1] Black and Scholes (1973) assume a complete and arbitrage free financial market consisting of a risk-free bond and a risky asset.[2] The major assumption is that the stock price follows a geometric Brownian motion – the stock price S_t develops according to the following continuous-time stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{A.1}$$

The parameter μ represents the expected rate of return resp. drift and σ the volatility of the process. Both parameters are constant. W_t is a Wiener process – a continuous-time stochastic process with independent and normally distributed increments, respectively random variables. It expresses uncertainty about the change in the value of a variable

[1] The Black-Scholes model is a financial mathematical model for the valuation of financial options. In the analysis, any derivative that has a non-dividend paying stock as underlying is considered.

[2] The following assumptions are made in the model: No taxes, no dividends or further transaction costs are taken into account. The risk-free interest rate r is constant and known over time. All assets are dividable without restriction. The option is exercisable only at maturity. There are no sanctions for short selling. Trading activity is continuous.

over time. A Wiener process represents the motion of variables as a Brownian motion. The value changes of a variable are normally distributed and according to the Markov property, the probability distributions of value changes are independent of each other.

The stock price process in (A.1) is a process without memory, i.e., the stock price history plays no role in the expectation about future price developments. The Markov property states that the probability distribution for a future point in time does not depend on the past price trend, but only on the current price. This confirms the hypothesis of capital market efficiency: The current stock price already prices in information about past stock prices. Due to this property, the process is also called Markov-process. The unique solution of the above SDE is given by:

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}, \tag{A.2}$$

where S_0 is the initial asset price and the exponential functional of the standard Brownian motion is defined as a geometric Brownian motion. Furthermore, the Black-Scholes model focuses on the stock price's log-return. A positive random variable X is log-normally distributed with the two parameters μ and σ if $\ln(X)$ is normal distributed with $N(\mu, \sigma^2)$. It follows that S_t is lognormal distributed

$$\ln(S_t) = \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \sim \mathcal{N}\left(\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right),$$

with mean $\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t$ and variance $\sigma^2 t$. Thus, the stock price's log-return is given by

$$\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right). \tag{A.3}$$

The geometric Brownian motion and the corresponding log-normal distribution of stock returns form the basis for the Black-Scholes market. In this context, Merton (1971) solves the problem of maximizing the expected utility of the terminal portfolio wealth of a CRRA-investor $E(u(V_T))$ by identifying an optimal strategy for a dynamic investment problem that implies a constant investment fraction. The evolution of the portfolio wealth, denoted by the stochastic process $(V_t)_{t\in[0,T]}$, is described by

$$dV_t = V_t\left(\pi\frac{dS_t}{S_t} + (1-\pi)\frac{dB_t}{B_t}\right), V_0 = 1. \tag{A.4}$$

The investment fraction π is restricted to values between zero and one ($\pi \in [0,1]$), i.e., no short selling and borrowing is allowed. The remaining part $(1-\pi)$ is invested in a risk-free asset B. The solution of Equation (A.4) is again given by a geometric Brownian motion. It holds under the real world measure $\mathbb{P}$:

$$V_t = V_0 e^{\left(\mu_A - \frac{1}{2}\sigma_A^2\right)t + \sigma_A W_t},$$

where the drift μ_A and volatility σ_A is given by

$$\begin{aligned} \mu_A &:= \pi\mu + (1-\pi)r = r + \pi(\mu - r) \\ \sigma_A &:= \pi\sigma. \end{aligned} \tag{A.5}$$

The optimal investment strategy is called Merton solution π^{Mer}. It is a constant-mix strategy, where the investor allocates wealth in the risky asset S and maximizes the expected utility of the investor's terminal wealth $E(u(V_T))$. It has the following form:

$$\pi^{Mer} = \frac{\mu - r}{\gamma_R \sigma^2}. \tag{A.6}$$

The Merton solution is given by the quotient of the excess return $(\mu - r)$ and the squared asset volatility σ^2 weighted by the level of relative risk aversion γ. Since the strategy is optimal, any deviation from the Merton solution results in a utility loss of the investor. The utility loss is measured by the loss rate l_2 which determines the loss in the certainty equivalent due to the sub-optimal investment fraction. Due to the relationship $u(CE_T) = E(u(V_T))$, the strategy also maximizes the certainty equivalent CE_T. The certainty equivalent of an uncertain or random wealth is the amount of certainty (in monetary units) whose utility to the investor is equal to the expected utility of the uncertain wealth (indifference of the investor). An analysis on the certainty equivalent is more common because in contrast to utility this quantity is easier to interpret.

A.1.2 Value of information in decision theory

The following comments in this subsection are based on Laux et al. (2005). Decisions are made on the basis of information, where the level of information can usually be improved. A decision situation under risk exists when future outcomes are uncertain, but the decision-maker can form a probability distribution about possible environmental states shaping the outcomes. The quantification of probabilities depends on the level of information. Information is the result of the examination and analysis of facts that serve as the basis for forecasting data relevant to decision-making. If the decision-maker does not obtain further information, she forms probabilities about future states based on her previous level of information – these probabilities are called **a priori** probabilities.

Information acquisition is a trade-off between the cost of acquiring information and the value of information. The value of information can also be expressed as the willingness to pay for information, because it is the cost amount at which obtaining the information has neither advantages nor disadvantages. If the costs are lower than the value of the information, the acquisition of information is advantageous. It is important that the value of information must be determined **before** the information outcome is known. The information valuation thus takes place ex ante. After all, since it is a decision under risk the decision whether to obtain certain information and what strategy to choose must also be made before the information is made available, resp. must take place before knowledge of the environmental state.

In the context of expected utility as a decision-making tool for decisions under uncertainty, this means the following: If the decision-maker does not obtain the information, she chooses the strategy or alternative that maximizes the expected utility on the basis of her a priori probability distribution. In this case, the value of information corresponds to the cost amount for which the expected utility with information and the expected utility without information are identical. The maximum value of information results under consideration of full information. With full information, the decision-maker knows with certainty that a particular environmental state will occur.

The optimal decision under full information is made according to the principle of flexible planning: The decision-maker has to determine for each possible result of information acquisition (certain environmental state) which strategy she would choose afterwards and which expected utility she would achieve thereby – for different information results different conditional plans are obtained. Since with full information the strategy is chosen with certainty that gives the highest profit or expected utility in the actually realized environmental state, the decision-maker can condition herself on the possible states and choose the optimal strategy in each state.

The value of information is subjective construct, as it depends on the decision-maker's goals, expectations, attitudes, and possible alternatives. It also depends on what a priori expectations the decision-maker has about environmental states. The more precise these expectations are, the lower the value of information. The expectations are most precise when they are certain, i.e., the decision-maker can assign the probability $p = 1$ to an environmental state. The information value is then equal to zero.

The contents of Chapter 4 deal with an a priori lottery – a probability distribution over possible regimes (environmental states). At the time when the decision is being made, a decision-maker has no full information about the current regime that will occur in reality and must therefore commit to a strategy.[3]

A.2 Supplementary figures

Impact of probability p and investment horizon T on optimal pre-commitment strategy $\pi_{T,p}^{*,\mathbf{pre}}$

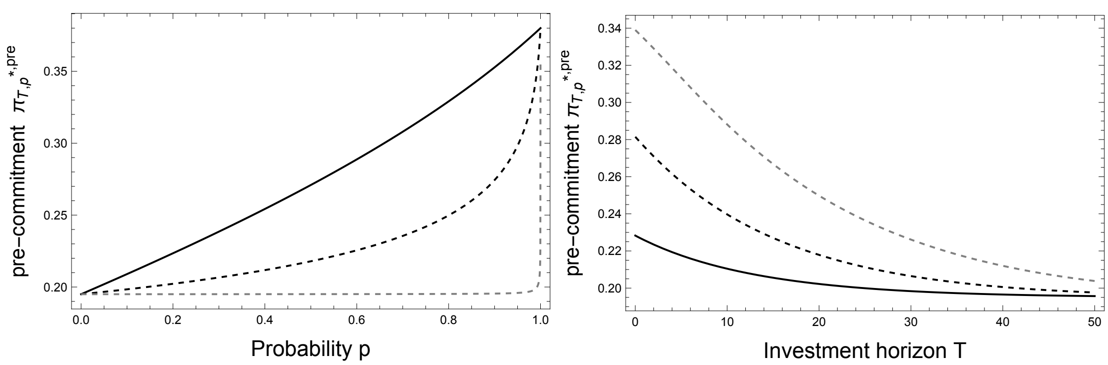

Figure A.1: Both illustrations are plotted for $\gamma_L = \gamma_R = 8$. The left illustration shows $\pi_{T,p}^{*,\text{pre}}$ depending on the probability p. The black graph pictures $T = 2$, the black dashed $T = 20$ and the gray $T = 100$. The right illustration shows $\pi_{T,p}^{*,\text{pre}}$ depending on the investment horizon T. The black graph pictures $p = 0.2$, the black dashed $p = 0.5$ and the gray $p = 0.8$. With increasing probability p for the good regime, the optimal pre-commitent strategy approaches the Merton solution of the first regime. The effects are mitigated the larger the investment horizon is. For $T \to \infty$, the optimale pre-commitment strategy approaches the worst case strategy (Merton solution of the second regime).

[3] Of course, a decision-maker can update her probability judgement by obtaining new information at a cost. However, we only consider the case of full information. Updating or gradual learning is not discussed further in the dissertation. Hansen et al. (2002) conclude that the effect of learning is small. Chen and Epstein (2002) argue that after a certain point, not much can be learned. in general, the process of learning in an uncertain and changing environment is complex.

Impact of difference in the regimes and investment horizon T on $\alpha^*_{T,p}$ for varying p

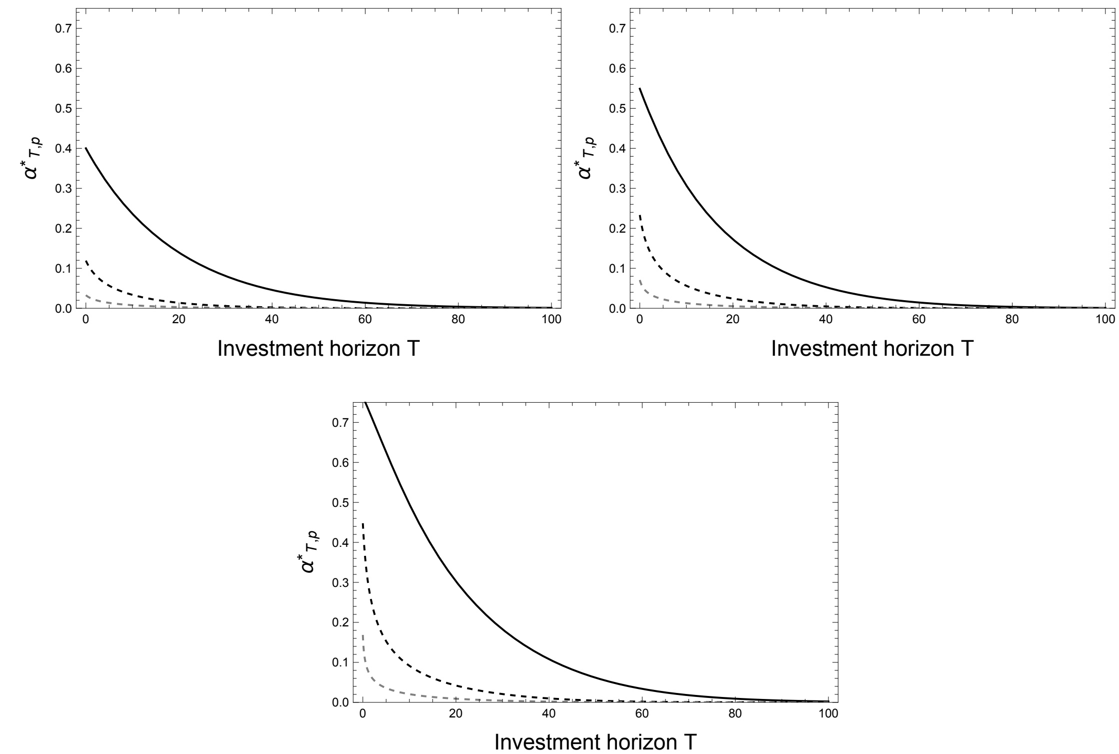

Figure A.2: The upper left picture shows the optimal weight $\alpha^*_{T,p}$ on the Merton solution in Regime 1 depending on the investment horizon T for $p = 0.4$, $\gamma_L = \gamma_R = 4$ and $\sigma_2 = 0.2$. The black line pictures $\sigma_1 = \sigma_2$. The black (gray) dashed line pictures $\sigma_1 = 0.1$ ($\sigma_1 = 0.05$). The upper right pictures shows the relations for $p = 0.6$ and the lower picture for $p = 0.8$. The greater the difference between the regimes, the faster the optimal solution converges to the worst case. The effects are also amplified by the regime probability. If probability p for the good regime is high, $\alpha^*_{T,p}$ is higher for small T. However, as T increases, $\alpha^*_{T,p}$ falls more sharply for higher than for lower p.

A.3 State-dependent risk aversion

This section shows the effects of state-dependent risk aversion in the optimal asset allocation problem considered in Chapter 4. The justification for the assumption of state-dependent risk aversion can be found in the literature: Cohn et al. (2015) and Guiso et al. (2018) find empirical evidence that in times of uncertainty and distress, investors' risk appetite decreases. In the research papers, the degree of risk aversion is measured by surveying risk-return combinations and decision behavior in lottery situations. In addition, risk behavior was measured by priming financial professionals with a boom or bust scenario. Theoretical papers, such as Wei et al. (2013), Wei et al. (2020), Li et al. (2022), also consider state-dependent risk aversion in the context of portfolio selection in the regime-switching environment.

In the asset allocation problem of Chapter 4, the Merton solutions on the respective regime under the assumption of state-dependent risk aversion are given as follows:

$$\pi^{Mer_1} = \frac{\mu(s_1) - r}{\gamma_{R1}\sigma(s_1)^2}, \quad \text{resp.}$$
$$\pi^{Mer_2} = \frac{\mu(s_2) - r}{\gamma_{R2}\sigma(s_2)^2}.$$

The regime-dependent savings rates are

$$y(s_1) = r + \pi^{Mer1}(\mu_1 - r) - \frac{1}{2}\gamma_{R1}(\pi^{Mer1})^2\sigma_1^2$$
$$y(s_2) = r + \pi^{Mer2}(\mu_2 - r) - \frac{1}{2}\gamma_{R2}(\pi^{Mer2})^2\sigma_2^2.$$

γ_{R1} and γ_{R2} are the state-dependent risk aversion parameters within the current regime. Due to the fact that the second regime describes a worse market condition compared to the first regime, we assume that the investor acts more risk averse in this regime. Thus, it holds that $\gamma_{R1} < \gamma_{R2}$.

With state-dependent risk aversion the internal risk situation increases. The spread between the Merton solutions increases as the Merton solution of the second regime gets smaller. The assumption leads to the fact that the optimal pre-commitment strategy for a given p converges faster towards the worst case (cf. Figure A.3 in contrast to Figure A.1 without state-dependent risk aversion). The effects increase the higher the difference between γ_{R1} and γ_{R2} (see upper right image in comparison to the lower image of Figure A.3). The optimal pre-commitment strategy is lower in the case of state-dependent risk aversion and thus closer to the worst case for a given T (see Figure A.4). The value of information under state-dependent risk aversion as a function of T turns out to be even higher (see right image of Figure A.5) and increases even faster for small investment horizons and slower for longer horizons than in the case of constant risk aversion parameters. The maximum value of the information is also reached at an earlier point in time. Thus, concerning the willingness to pay for full information about the regimes, we have the result that the higher the inner risk aversion of the second regime is (resp. the higher the difference in the state-dependent risk aversions), the higher is the willingness to pay for information.

Impact of probability p on the optimal pre-commitment strategy $\pi_{T,p}^{*,\mathbf{pre}}$ under state-dependent risk aversion

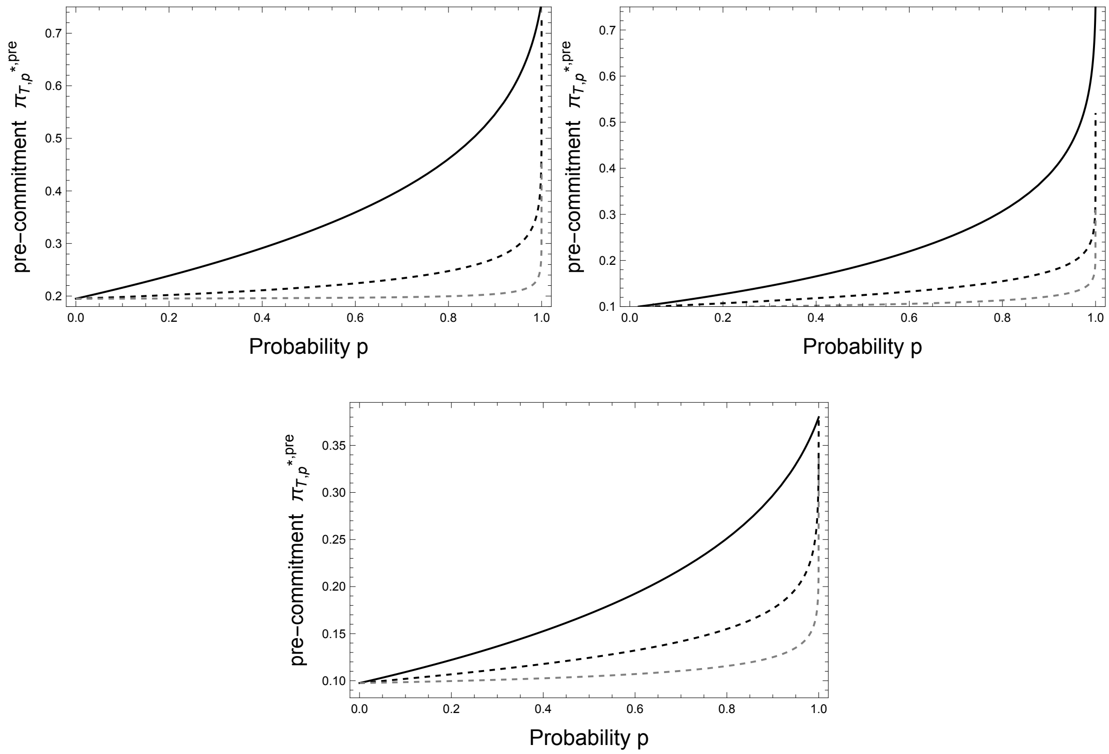

Figure A.3: The upper left picture shows the optimal pre-commitment strategy $\pi_{T,p}^{*,\text{pre}}$ for $\gamma_L = 8$, $\gamma_{R1} = 4$ and $\gamma_{R2} = 8$. The upper right picture shows $\pi_{T,p}^{*,\text{pre}}$ for $\gamma_L = 8$, $\gamma_{R1} = 4$ and $\gamma_{R2} = 16$. The lower picture shows $\pi_{T,p}^{*,\text{pre}}$ for $\gamma_L = 8$, $\gamma_{R1} = 8$ and $\gamma_{R2} = 16$. The black graphs picture the optimal pre-commitment strategy for $T = 2$, the dashed black for $T = 20$ and the dashed gray for $T = 50$.

lmpact of investment horizon T on the optimal pre-commitment strategy $\pi_{T,p}^{*,\mathbf{pre}}$ under state-dependent risk aversion

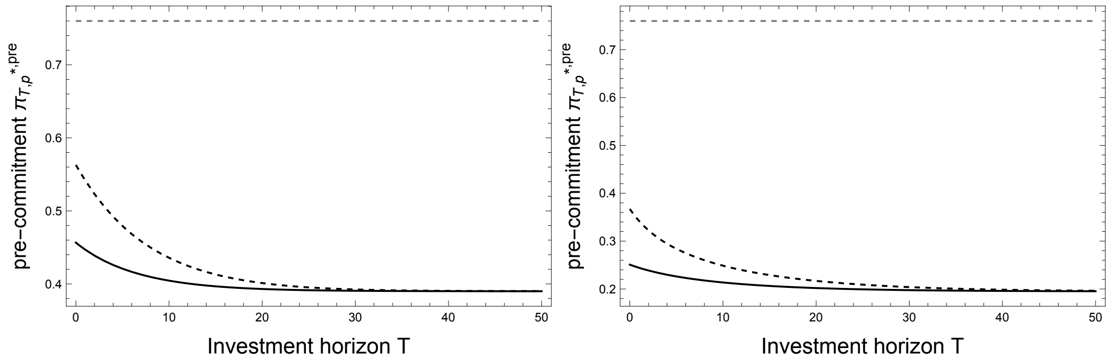

Figure A.4: The left image shows the optimal pre-commitment strategy $\pi_{T,p}^{*,\text{pre}}$ depending on T for $\gamma_L = 8$, $\gamma_{R1} = 4$ and $\gamma_{R2} = 4$. The right image shows $\pi_{T,p}^{*,\text{pre}}$ depending on T for $\gamma_L = 8$, $\gamma_{R1} = 4$ and $\gamma_{R2} = 8$. The black graphs refer to $p = 0.2$, the dashed black to $p = 0.5$ and the dashed gray to $p = 1$.

Value of information depending on investment horizon T under state-dependent risk aversion

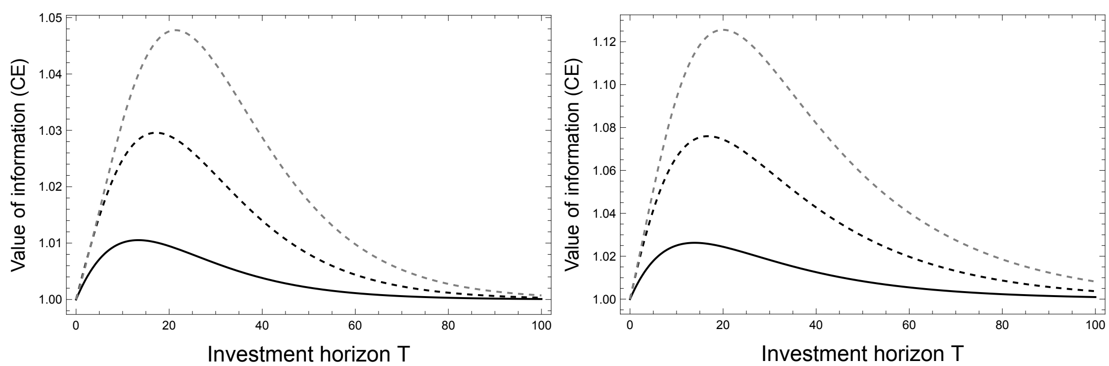

Figure A.5: The left illustration shows the value of information depending on T for $\gamma_L = 4$, $\gamma_{R1} = 4$ and $\gamma_{R2} = 4$. The right illustration shows the value of information depending on T for $\gamma_L = 4$, $\gamma_{R1} = 4$ and $\gamma_{R2} = 8$. The black graphs refer to $p = 0.2$, the dashed black to $p = 0.5$ and the dashed gray to $p = 0.7$.

A.4 Robustness analysis

This section contains a robustness analysis by using a new parameter constellation (Table A.1) for all generated illustrations in Chapter 4. The originally used parameter constellation (Table 4.1) yields for $\gamma_R = 4$ the Merton solutions $\pi^{Mer1} = 0.76$, $\pi^{Mer2} = 0.39$ ($y(\pi^{Mer1}, 1) = 0.05$, $y(\pi^{Mer2}, 2) = 0.015$). In the second benchmark parameter constellation, on the contrary, the Merton solutions are also given by $\pi^{Mer1} = 0.70$, $\pi^{Mer2} = 0.30$ ($y(\pi^{Mer1}, 1) = 0.05$, $y(\pi^{Mer2}, 2) = 0.02$). Note that in the first benchmark parameter constellation the difference in μ_1 and μ_2 is larger whereas the difference in σ_1 and σ_2 is small. In the second benchmark parameter constellation, the opposite is true. The difference in the volatilities is larger and the difference in the drift coefficients is small. Due to $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$ the first regime remains the good one. Overall, the Merton solutions are smaller for the second benchmark parameter constellation, but the difference between π^{Mer1} and π^{Mer2} is larger.

Of course, the variation of the regime parameters leads to other absolute values in the sensitivity analysis (e.g., the value of information is higher when using the second parameter set), but the results do not change with respect to the impact of time inconsistency in the optimal asset allocation problem. The effects of individual variables, resp. parameters, on the strategy and the value of information and their relations to each other remain the same. This can be seen from the fact that the shape of the graphs in all figures is similar. In total our results are robust to changes in regime parameters.

Benchmark parameter 2

μ_1	μ_2	σ_1	σ_2	r
0.1426	0.1333	0.2259	0.3333	0.00

Table A.1: Second benchmark parameter constellation.

Savings rate $y(\pi, i)$ depending on investment fraction $\pi - 2$

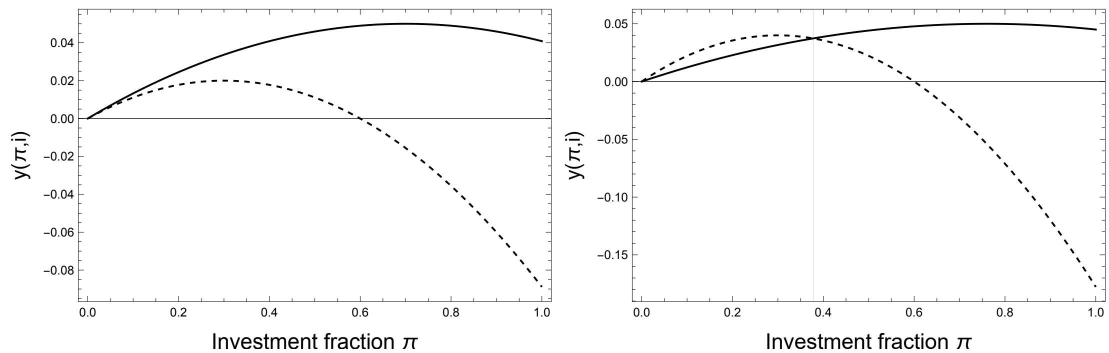

Figure A.6: The left figure displays the savings rate depending on the investment fraction π. The right figure displays the savings rate for a different parameter constellation, where $\pi^{\text{equal}} = 0.3776$. In comparison to the second benchmark parameter setup, $\mu_1 < \mu_2$ ($\mu_1 = 0.1316, \mu_2 = 0.2667$) and $\sigma_1 < \sigma_2$ ($\sigma_1 = 0.2080, \sigma_2 = 0.4714$). In both illustrations for $\gamma_R = 4$ the black graph pictures $y(\pi, 1)$ and the black dashed $y(\pi, 2)$.

Aggregated savings rate $y_{T,p} - 2$

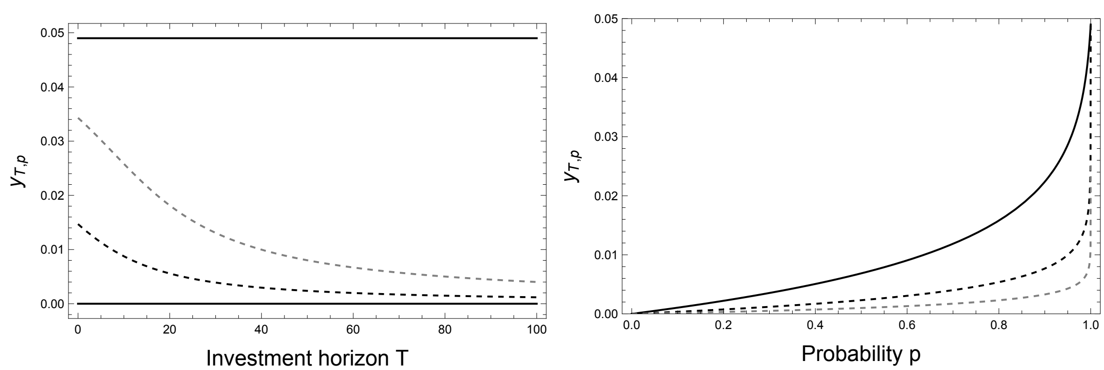

Figure A.7: The left figure displays the savings rate depending on the investment horizon for $\gamma_L = \gamma_R = 4$ and $\pi = 0.6$. The upper (lower) black line pictures $p = 1$ ($p = 0$). The gray (black) dashed graph refers to $p = 0.7$ ($p = 0.3$). The right figure displays the savings rate depending on the probability for $\pi = 0.6$, $T = 100$ and $\gamma_R = 4$. The black graph refers to $\gamma_L = 2$, while the black (gray) dashed graph refers to $\gamma_L = 4$ ($\gamma_L = 8$).

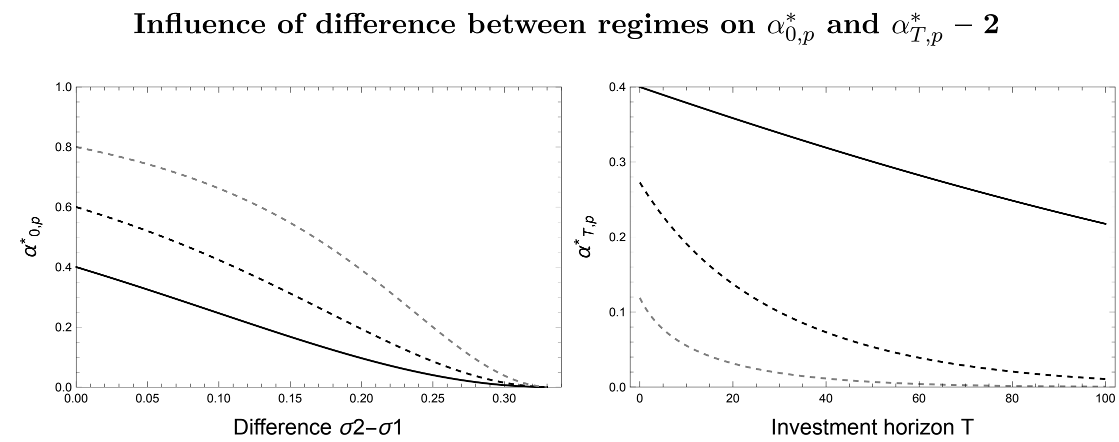

Figure A.8: The left picture displays the optimal weight on the Merton solution in Regime 1 at $T = 0$ for $\gamma_L = \gamma_R = 4$ depending on the differences $\sigma_2 - \sigma_1$ (where $\sigma_2 = 0.3333$). The black line pictures $p = 0.4$, the black dashed $p = 0.6$ and the gray dashed $p = 0.8$. The right picture shows the optimal weight $\alpha^*_{T,p}$ on the Merton solution in Regime 1 depending on the investment horizon T for $p = 0.4$, $\gamma_L = \gamma_R = 4$ and $\sigma_2 = 0.3333$. The black line pictures $\sigma_1 = \sigma_2$. The black (gray) dashed line pictures $\sigma_1 = 0.25$ ($\sigma_1 = 0.15$).

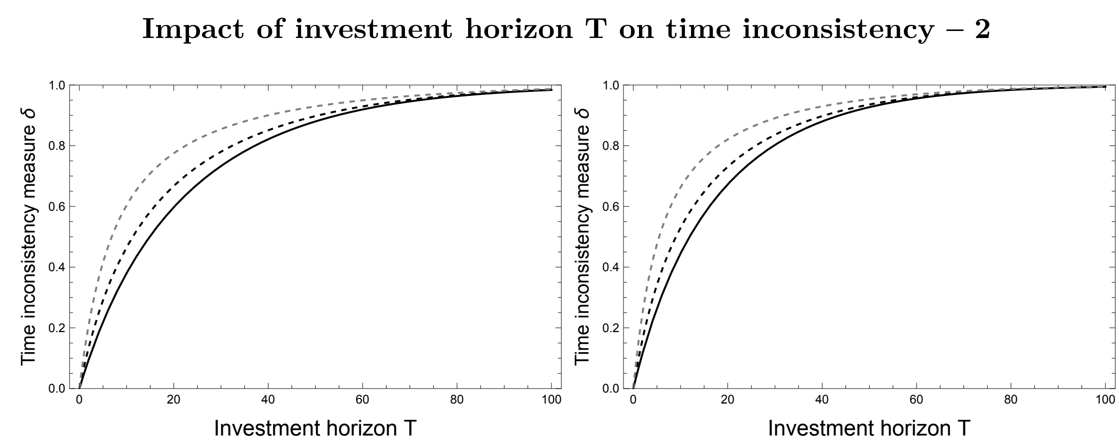

Figure A.9: The figure gives the time inconsistency measure δ as a function of the investment horizon. The left (right) figure refers to a level of risk aversion $\gamma_L = \gamma_R = 4$ ($\gamma_L = \gamma_R = 16$). The black graphs refer to the optimal pre-commitment strategy $\pi^{*,\text{pre}}_{T,p}$ for $p = 0.2$. The dashed black (dashed gray) graphs refer to the optimal pre-commitment strategy for $p = 0.5$ ($p = 0.8$).

Value of information (CE) depending on investment horizon T – 2

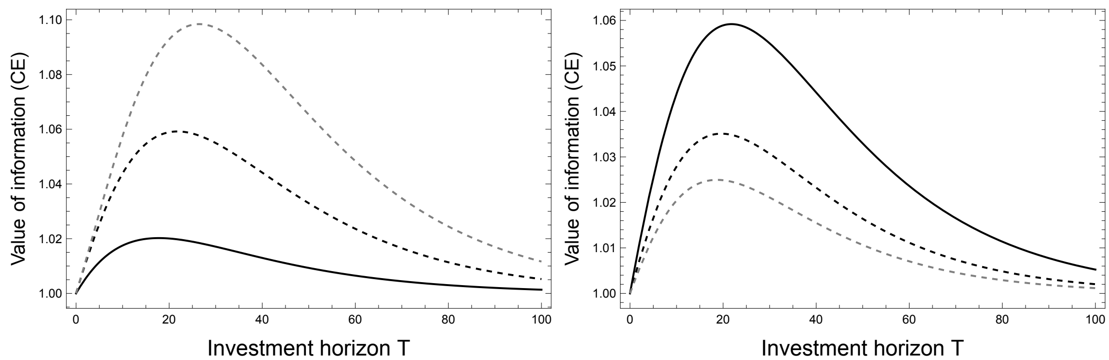

Figure A.10: Both figures give the value of information by the ratio of the certainty equivalents depending on the investment horizon T. The left figure refers to $p = 0.2$ (black), $p = 0.5$ (black dashed), and $p = 0.7$ (gray dashed) where $\gamma_L = \gamma_R = 4$. The right figure to $\gamma_L = \gamma_R = 4$ (black), $\gamma_L = \gamma_R = 6$ (black dashed), $\gamma_L = \gamma_R = 8$ (gray dashed) where $p = 0.5$

Value of Information for varying risk aversion with different probabilities – 2

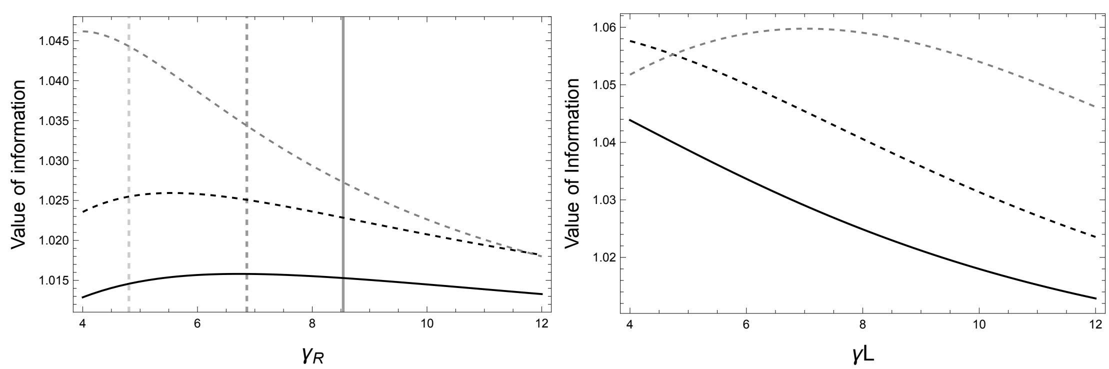

Figure A.11: The left illustration shows the value of information as a function of risk aversion γ_R for $\gamma_L = 12$. The right illustration shows the value of information as a function of risk aversion γ_L for $\gamma_R = 4$. The black graphs refer to $p = 0.5$, the black dashed to $p = 0.7$ and the gray dashed to $p = 0.9$. The time horizon is 10 years for both illustrations.

Panel A: Maximizing $\hat{T}(\gamma_R, \gamma_L) - 2$						
$\gamma_R \backslash \gamma_L$	2	4	6	8	10	12
2	32.70	10.90	6.55	4.65	3.65	2.95
4	65.40	21.80	13.10	9.35	7.25	5.95
6	98.10	32.70	19.60	14.00	10.9	8.90
8	130.80	43.60	26.15	18.70	14.55	11.90
10	163.50	54.50	32.70	23.35	18.15	14.85
12	196.20	65.40	39.25	28.05	21.80	17.85
Panel B: Maximal value of information $VoI^*(\gamma_R, \gamma_L) - 2$						
$\gamma_R \backslash \gamma_L$	2	4	6	8	10	12
2	1.1883	1.0592	1.0351	1.0250	1.0194	1.0158
4	1.1883	1.0592	1.0351	1.0250	1.0194	1.0158
6	1.1883	1.0592	1.0351	1.0250	1.0194	1.0158
8	1.1883	1.0592	1.0351	1.0250	1.0194	1.0158
10	1.1883	1.0592	1.0351	1.0250	1.0194	1.0158
12	1.1883	1.0592	1.0351	1.0250	1.0194	1.0158
Panel C: Optimal pre-commitment strategy $\pi^{*,pre}_{\hat{T}(\gamma_R,\gamma_L),p}(\gamma_R, \gamma_L) - 2$						
$\gamma_R \backslash \gamma_L$	2	4	6	8	10	12
2	0.6991	0.6991	0.6990	0.6994	0.6987	0.6997
4	0.3496	0.3496	0.3495	0.3495	0.3496	0.3495
6	0.2330	0.2330	0.2330	0.2330	0.2330	0.2331
8	0.1748	0.1748	0.1748	0.1748	0.1747	0.1748
10	0.1398	0.1398	0.1398	0.1398	0.1398	0.1398
12	0.1165	0.1165	0.1165	0.1165	0.1165	0.1165

Table A.2: The table gives the maximizing time to maturity (Panel A) and the maximal value of information (Panel B) as a function of γ_R and γ_L. The probability p is set to 0.5. Moreover the optimal pre-commitment strategy is taken into account for $\hat{T}$ (Panel C).

$y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre})$ for varying p with different investment horizons T – 2

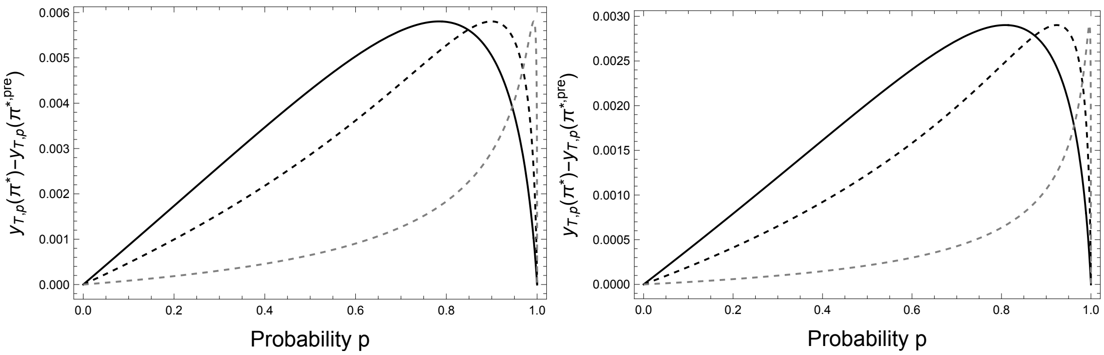

Figure A.12: The left figure refers to a risk aversion of $\gamma_L = \gamma_R = 4$, the right to $\gamma_L = \gamma_R = 8$. The black lines picture $T = 10$, the black dashed $T = 20$ and the gray dashed $T = 50$.

Impact of ambiguity on pre-commitment strategy – 2

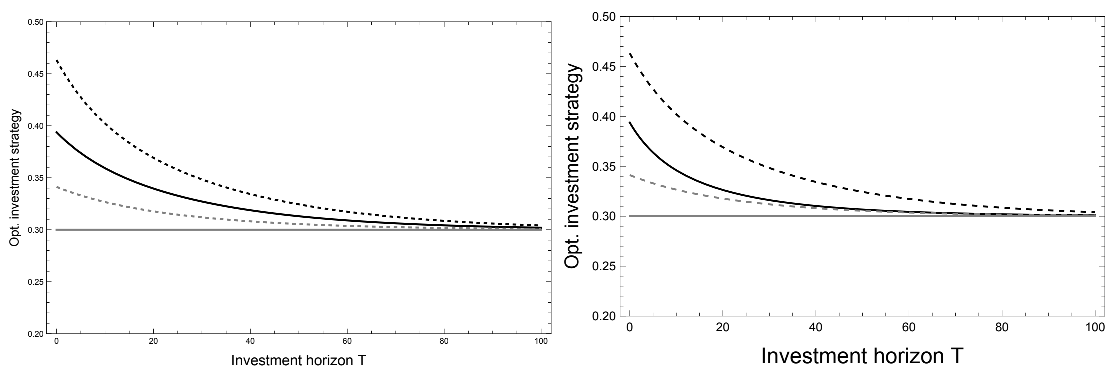

Figure A.13: The illustrations are plotted for $\gamma_L = \gamma_R = 4$. The gray lines display $\pi_2^{Mer} = 0.30$. The black graphs refer to the optimal pre-commitment strategy under ambiguity aversion $\pi_{T,\tilde{p}}^{*,\text{pre, amb}}$ with $\tilde{p} = 0.5$, $p_a = 0.6$ and $p_b = 0.2$. The dashed black (dashed gray) graphs refer to the optimal pre-commitment strategy without ambiguity $\pi_{T,p}^{*,\text{pre}}$ under the given probability distribution over the regimes with $p = 0.6$ ($p = 0.2$). The left (right) figure refers to a level of ambiguity $\gamma_A = 4$ ($\gamma_A = 16$).

Impact of probability $\tilde{p}$ on optimal strategies $\pi^{*,\mathbf{pre,\ amb}}_{T,\tilde{p}}$ and $\pi^{*,\mathbf{pre}}_{T,q}$ – 2

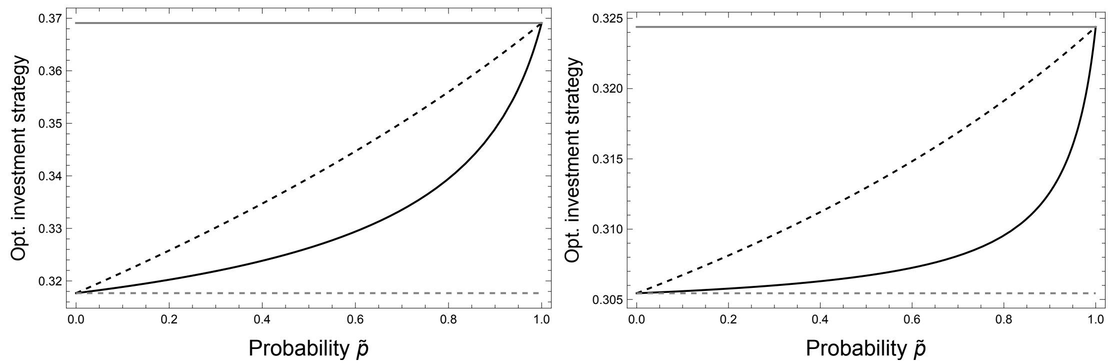

Figure A.14: The pictures are created for $\gamma_L = \gamma_R = 4$, $\gamma_A = 16$, $p_a = 0.6, p_b = 0.2$. In the left (right) illustration $T = 20$ ($T = 50$) is assumed. The black graphs show the optimal pre-commitment strategy under risk and ambiguity $\pi^{*,\text{pre, amb}}_{T,\tilde{p}}$, the black dashed graphs show the optimal pre-commitment strategy under risk $\pi^{*,\text{pre}}_{T,q}$ with probability q. The gray lines show the optimal pre-commitment strategy under risk with probability p_a and the gray dashed lines show the optimal pre-commitment strategy under risk with probability p_b.

$\pi^{*,pre,amb}_{T,\tilde{p}}$, const. q for different p_a and p_b combinations – 2

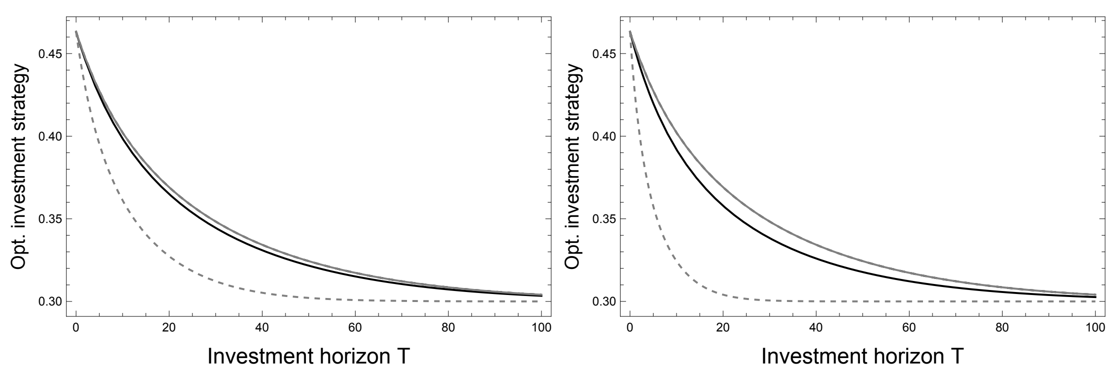

Figure A.15: The left (right) hand side refers to $\gamma_L = \gamma_R = 4, \gamma_A = 8$ ($\gamma_L = \gamma_R = 4, \gamma_A = 16$). Both illustrations are plotted with a constant $q = \tilde{p} = 0.6$. The black graphs display $\pi^{*,pre,amb}_{T,\tilde{p}}$ for $p_a = 0.7, p_b = 0.45$. For $p_a = p_b = 0.6$ it holds $\pi^{*,pre,amb}_{T,\tilde{p}} = \pi^{*,pre}_{T,q}$ (gray and black dashed graphs). The gray dashed graphs refer to $p_a = 1, p_b = 0$.

Value of information: $\pi^{*,pre}_{T,pa}$ and $\pi^{*,pre}_{T,pb}$ known, const. q – 2

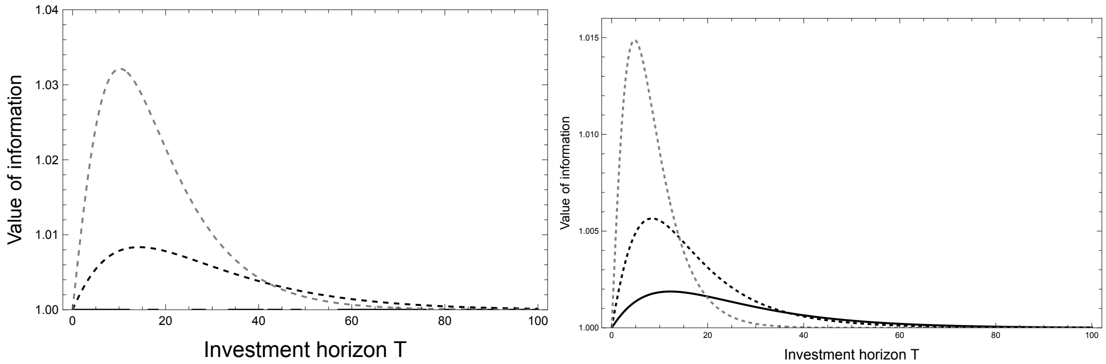

Figure A.16: The left illustration refers to $\gamma_L = \gamma_R = 4, \gamma_A = 8$, whereas the right illustration refers to $\gamma_L = \gamma_R = 4, \gamma_A = 16$. Both illustrations are plotted with a constant $q = \tilde{p} = 0.6$. The black graphs display $p_a = p_b = 0.6$ (black dashed: $p_a = 0.8, p_b = 0.3$ and gray dashed: $p_a = 1, p_b = 0$).

$\hat{T}$ for varying p_a, const. q and different risk aversion parameter γ_R – 2

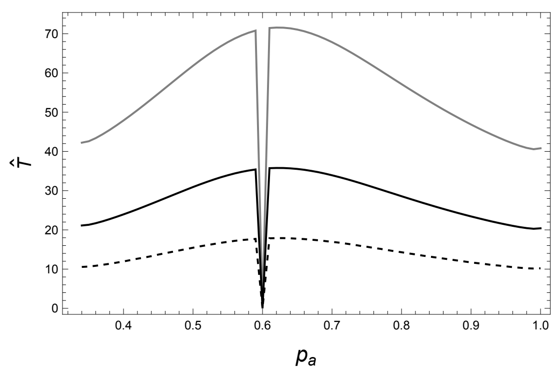

Figure A.17: The figure shows $\hat{T}$ under ambiguity with constant $q = \tilde{p} = 0.6$, $\gamma_L = 4$ and $\gamma_A = 8$ for varying p_a (notice that this implies also varying p_b). The black graph refers to $\gamma_R = 8$, the black dashed to $\gamma_R = 4$ and the gray one to $\gamma_R = 16$.

A.5 Proof of Proposition 1

For $\gamma_L > 1$, the certainty equivalent savings rate $y_{T,p}(\pi)$ is given by

$$y_{T,p}(\pi) = \frac{1}{(1-\gamma_L)T} \ln \left[p e^{y(\pi,1)(1-\gamma_L)T} + (1-p) e^{y(\pi,2)(1-\gamma_L)T} \right].$$

For the partial derivative with respect to T, it holds that

$$\begin{aligned}
\frac{\partial y_{T,p}(\pi)}{\partial T} &= \frac{1}{T} \frac{p e^{y(\pi,1)(1-\gamma_L)T} y(\pi,1) + (1-p) e^{y(\pi,2)(1-\gamma_L)T} y(\pi,2)}{p e^{y(\pi,1)(1-\gamma_L)T} + (1-p) e^{y(\pi,2)(1-\gamma_L)T}} \\
&\quad - \frac{1}{(1-\gamma_L)T^2} \ln \left[p e^{y(\pi,1)(1-\gamma_L)T} + (1-p) e^{y(\pi,2)(1-\gamma_L)T} \right] \\
&= \frac{1}{(1-\gamma_L)T^2} \frac{p e^{y(\pi,1)(1-\gamma_L)T} y(\pi,1)(1-\gamma_L)T + (1-p) e^{y(\pi,2)(1-\gamma_L)T} y(\pi,2)(1-\gamma_L)T}{p e^{y(\pi,1)(1-\gamma_L)T} + (1-p) e^{y(\pi,2)(1-\gamma_L)T}} \\
&\quad - \frac{1}{(1-\gamma_L)T^2} \ln \left[p e^{y(\pi,1)(1-\gamma_L)T} + (1-p) e^{y(\pi,2)(1-\gamma_L)T} \right] \\
&= \frac{1}{(1-\gamma_L)T^2} \frac{p e^{x_1} x_1 + (1-p) e^{x_2} x_2}{p e^{x_1} + (1-p) e^{x_2}} - \frac{1}{(1-\gamma_L)T^2} \ln \left[p e^{x_1} + (1-p) e^{x_2} \right],
\end{aligned}$$

where we define $x_i = y(\pi, i)(1-\gamma_L)T$. We can then write the derivative as a function of the random variable X with realizations x_1 (with probability p) and x_2 (with probability $(1-p)$):

$$\begin{aligned}
\frac{\partial y_{T,p}(\pi)}{\partial T} &= \frac{1}{(1-\gamma_L)T^2} \left[\frac{E\left[e^X X\right]}{E\left[e^X\right]} - \ln E\left[e^X\right] \right] \\
&= \frac{1}{(1-\gamma_L)T^2 E\left[e^X\right]} \Big\{ \underbrace{E\left[e^X X\right] - E\left[e^X\right] \ln E\left[e^X\right]}_{>0 \quad (z \ln z \text{ is convex fct. for } z>0)} \Big\} \\
&< 0.
\end{aligned}$$

Next, we turn to the certainty equivalent savings rate for the optimal pre-commitment strategy, where we have to take into account that the optimal pre-commitment strategy depends on T, too:

$$\begin{aligned}
\frac{\partial y_{T,p}(\pi^{*,pre}_{T,p})}{\partial T} &= \frac{\partial y_{T,p}(\pi)}{\partial T}\Big|_{\pi=\pi^{*,pre}_{T}} + \underbrace{\frac{\partial y_{T,p}(\pi)}{\partial \pi}\Big|_{\pi=\pi^{*,pre}_{T,p}}}_{=0 \quad \text{(FOC)}} \frac{\partial \pi^{*,pre}_{T,p}}{\partial T} \\
&< 0.
\end{aligned}$$

A.6 Proof of Proposition 2

We need to show that $\pi_{T,p}^{*,pre} = \alpha_{T,p}^{*}\pi_1^{Mer} + (1-\alpha_{T,p}^{*})\pi_2^{Mer}$, where $\pi_{T,p}^{*,pre} = \underset{\pi}{\text{argmax}}\{y_{T,p}(\pi)\}$. For $\gamma_L > 1$ the certainty equivalent savings rate $y_{T,p}(\pi)$ is given by

$$y_{T,p}(\pi) = \frac{1}{(1-\gamma_L)T} \ln\left(pe^{y(\pi,1)(1-\gamma_L)T} + (1-p)e^{y(\pi,2)(1-\gamma_L)T}\right).$$

Calculating the FOC we receive

$$\begin{aligned}
\frac{\partial y_{T,p}}{\partial \pi} &= \frac{p\gamma_R\sigma_1^2(\pi_1^{Mer}-\pi)e^{y(\pi,1)(1-\gamma_L)T} + (1-p)\gamma_R\sigma_2^2(\pi_2^{Mer}-\pi)e^{y(\pi,2)(1-\gamma_L)T}}{pe^{y(\pi,1)(1-\gamma_L)T} + (1-p)e^{y(\pi,2)(1-\gamma_L)T}} \overset{!}{=} 0 \\
\Leftrightarrow \pi &\overset{!}{=} \frac{p\sigma_1^2\pi_1^{Mer}e^{y(\pi,1)(1-\gamma_L)T} + (1-p)\sigma_2^2\pi_2^{Mer}e^{y(\pi,2)(1-\gamma_L)T}}{p\sigma_1^2e^{y(\pi,1)(1-\gamma_L)T} + (1-p)\sigma_2^2e^{y(\pi,2)(1-\gamma_L)T}},
\end{aligned} \tag{A.7}$$

i.e., the optimal pre-commitment strategy $\pi_{T,p}^{*,pre}$ is implicitly defined as solution of (A.7). Separating the fraction leads to

$$\begin{aligned}
\pi_{T,p}^{*,pre} =& \frac{p\sigma_1^2e^{y(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T}}{p\sigma_1^2e^{y(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T} + (1-p)\sigma_2^2e^{y(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}}\pi_1^{Mer} + \\
& \frac{p\sigma_1^2e^{y(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T}}{p\sigma_1^2e^{y(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T} + (1-p)\sigma_2^2e^{y(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}}\pi_2^{Mer} \\
\Leftrightarrow \pi_{T,p}^{*,pre} =& \alpha_{T,p}^{*}\pi_1^{Mer} + (1-\alpha_{T,p}^{*})\pi_2^{Mer}.
\end{aligned}$$

A.7 Proof of Proposition 6

The value of information is given by the difference of the certainty equivalent savings rates $y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,\text{pre}})$. For $\gamma_L > 1$ it holds

$$y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,\text{pre}}) = \frac{1}{(1-\gamma_L)T} \ln\left[\frac{pe^{y_{T,p}(\pi_1^{Mer},1)(1-\gamma_L)T} + (1-p)e^{y_{T,p}(\pi_2^{Mer},2)(1-\gamma_L)T}}{pe^{y_{T,p}(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T} + (1-p)e^{y_{T,p}(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}}\right].$$

The inner part of the log function can be written as

$$\begin{aligned}
& \frac{pe^{y_{T,p}(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T}}{pe^{y_{T,p}(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T} + (1-p)e^{y_{T,p}(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}} \frac{pe^{y_{T,p}(\pi_1^{Mer},1)(1-\gamma_L)T}}{pe^{y_{T,p}(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T}} + \\
& \frac{(1-p)e^{y_{T,p}(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}}{pe^{y_{T,p}(\pi_{T,p}^{*,pre},1)(1-\gamma_L)T} + (1-p)e^{y_{T,p}(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}} \frac{(1-p)e^{y_{T,p}(\pi_2^{Mer},2)(1-\gamma_L)T}}{(1-p)e^{y_{T,p}(\pi_{T,p}^{*,pre},2)(1-\gamma_L)T}} \\
=& \beta_{T,p}(\pi_{T,p}^{*,pre})e^{l\left(\pi_{T,p}^{*,pre},1\right)(1-\gamma_L)T} + (1-\beta_{T,p}(\pi_{T,p}^{*,pre}))e^{l\left(\pi_{T,p}^{*,pre},2\right)(1-\gamma_L)T}.
\end{aligned}$$

For $\gamma_L = 1$ the value of information $y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,\text{pre}})$ is given by

$$\begin{aligned} y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,\text{pre}}) =& p(y_{T,p}(\pi_1^{Mer}, 1) - y_{T,p}(\pi_{T,p}^{*,pre}, 1)) + \\ & (1-p)(y_{T,p}(\pi_2^{Mer}, 2) - y_{T,p}(\pi_{T,p}^{*,pre}, 2)). \end{aligned}$$

Using the fact that $\beta_{T,p}(\pi) = p$ for $\gamma_L = 1$ gives the claimed representation.

A.8 Proof of Proposition 7

For $\gamma_L > 1$ it holds

$$\lim_{T\to 0} \left\{ y_{T,p}(\pi^*) - y_{T,p}(\pi_{T,p}^{*,pre}) \right\} = \lim_{T\to 0} y_{T,p}(\pi^*) - \lim_{T\to 0} y_{T,p}(\pi_{T,p}^{*,pre}).$$

Notice that

$$\begin{aligned} \lim_{T\to 0} y_{T,p}(\pi_{T,p}^{*,pre}) &= \lim_{T\to 0} \left\{ py(\pi_{T,p}^{*,pre}, 1) + (1-p)y(\pi_{T,p}^{*,pre}, 2) \right\} \\ &= py\left(\lim_{T\to 0} \pi_{T,p}^{*,pre}, 1 \right) + (1-p)y\left(\lim_{T\to 0} \pi_{T,p}^{*,pre}, 2 \right) \end{aligned}$$

and

$$\begin{aligned} &\lim_{T\to 0} \pi_{T,p}^{*,pre} = \lim_{T\to 0} \alpha_{T,p}^* \pi_1^{Mer} + (1 - \lim_{T\to 0} \alpha_{T,p}^*)\pi_2^{Mer}, \text{ where} \\ &\lim_{T\to 0} \alpha_{T,p}^* = \frac{p\sigma_1^2}{p\sigma_1^2 + (1-p)\sigma_2^2}. \end{aligned}$$

Using the results of Proposition 1 and the fact that

$$l(\pi_{T,p}^{*,pre}, i) = y(\pi_i^{Mer}, i) - y(\pi_{T,p}^{*,pre}, i) = \frac{1}{2}\gamma_R \sigma_i^2 (\pi_{T,p}^{*,pre} - \pi_i^{Mer})^2,$$

we get

$$\lim_{T\to 0} \left\{ y_{T,p}(\pi^*) - y_{T,p}(\pi_{T,p}^{*,pre}) \right\} = pl(\lim_{T\to 0} \pi_{T,p}^{*,pre}, 1) + (1-p)l(\lim_{T\to 0} \pi_{T,p}^{*,pre}, 2). \tag{A.8}$$

Calculating the two loss rates we receive with the above stated results

$$l(\lim_{T\to 0} \pi_{T,p}^{*,pre}, 1) = \frac{1}{2}\gamma_R (1-p)^2 \sigma_2^2 \left(\frac{\sigma_1 \sigma_2}{p\sigma_1^2 + (1-p)\sigma_2^2} \right)^2 \left(\pi_1^{\text{Mer}} - \pi_2^{\text{Mer}} \right)^2 \tag{A.9}$$

$$l(\lim_{T\to 0} \pi_{T,p}^{*,pre}, 2) = \frac{1}{2}\gamma_R p^2 \sigma_1^2 \left(\frac{\sigma_1 \sigma_2}{p\sigma_1^2 + (1-p)\sigma_2^2} \right)^2 \left(\pi_1^{\text{Mer}} - \pi_2^{\text{Mer}} \right)^2. \tag{A.10}$$

Combining (A.8), (A.9) and (A.10) we finally get

$$\lim_{T\to 0}\left\{y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre}_{T,p})\right\} = \frac{1}{2}\gamma_R\left(\pi_1^{\mathrm{Mer}} - \pi_2^{\mathrm{Mer}}\right)^2 \frac{p^2(1-p)\sigma_1^2 + p(1-p)^2\sigma_2^2}{\left(p\sigma_1^2 + (1-p)\sigma_2^2\right)^2}\ \sigma_1^2\sigma_2^2$$
$$= \frac{1}{2}\gamma_R p(1-p)\left(\pi_1^{\mathrm{Mer}} - \pi_2^{\mathrm{Mer}}\right)^2 \frac{\sigma_1^2\sigma_2^2}{p\sigma_1^2 + (1-p)\sigma_2^2}.$$

For $\gamma_L = 1$ we are immediately in the situation of equation (A.8), s.t. the same result holds.

For the case $T \to \infty$ we distinguish between:

$$\lim_{T\to\infty} \pi^{*,pre}_{T,p} \neq \pi^{equal} \text{ and } \lim_{T\to\infty} \pi^{*,pre}_{T,p} = \pi^{equal}.$$

For $\lim_{T\to\infty} \pi^{*,pre}_{T,p} \neq \pi^{equal}$ it holds

$$\lim_{T\to\infty}\{y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre})\}$$
$$= \lim_{T\to\infty}\left\{\frac{1}{(1-\gamma_L)T}\ln\left[\frac{pe^{(y(\pi_1^{Mer},1) - y(\pi^{*,pre}_{T,p},1))(1-\gamma_L)T} + (1-p)e^{(y(\pi_2^{Mer},2) - y(\pi^{*,pre}_{T,p},2))(1-\gamma_L)T}}{1 - p\delta(\pi^{*,pre}_{T,p}, T)}\right]\right\}$$
$$= \lim_{T\to\infty}\left\{\frac{1}{(1-\gamma_L)T}\right\}\ln\left[\frac{1}{1-p}\right] = 0.$$

For $\pi^{*,pre}_{T,p} = \pi^{equal}$ it holds that $y_{T,p}(\pi^{*,pre}_{T,p}, 1) = y_{T,p}(\pi^{*,pre}_{T,p}, 2)$ (for this we write $y(\pi^{equal}, \cdot)$ because the regime i does not matter here), s.t. $\delta(\pi^{*,pre}_{T,p}, T) = 0$, for all T and thus it holds

$$\lim_{T\to\infty}\{y_{T,p}(\pi^*) - y_{T,p}(\pi^{*,pre})\} = \lim_{T\to\infty}\left\{y_{T,p}(\pi^*) - y(\pi^{equal}, \cdot)\right\}$$
$$= \min\left\{y(\pi_1^{Mer}, 1), y(\pi_2^{Mer}, 2)\right\} - y(\pi^{equal}, \cdot).$$

A.9 Proof of Proposition 10

We want to show that $\pi_T^{*,pre,amb} = \underset{\pi}{\operatorname{argmax}}\left\{y^{amb}_{T,\tilde{p}}(\pi)\right\}$ is given by

$$\alpha^*_{T,\tilde{p}}\left(\alpha_{T,p_a}\pi_1^{Mer} + (1-\alpha_{T,p_a})\pi_2^{Mer}\right) + (1-\alpha_{T,\tilde{p}})\left(\alpha_{T,p_b}\pi_1^{Mer} + (1-\alpha_{T,p_b})\pi_2^{Mer}\right).$$

The certainty equivalent savings rate for $\gamma_L > 1$ with ambiguity $y^{amb}_{T,p}(\pi)$ is given by

$$y^{\mathrm{amb}}_{T,\tilde{p}}(\pi) := \frac{1}{(1-\gamma_A)T}\ln\left[\tilde{p}e^{y_{T,p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p})e^{y_{T,p_b}(\pi)(1-\gamma_A)T}\right].$$

Calculating the FOC we receive

$$\frac{\partial y^{\mathrm{amb}}_{T,\tilde{p}}(\pi)}{\partial\pi} = \frac{\tilde{p}\frac{\partial y_{T,p_a}(\pi)}{\partial\pi}e^{y_{T,p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p})\frac{\partial y_{T,p_b}(\pi)}{\partial\pi}e^{y_{T,p_b}(\pi)(1-\gamma_A)T}}{\tilde{p}e^{y_{T,p_a}(\pi)(1-\gamma_A)T} + (1-\tilde{p})e^{y_{T,p_b}(\pi)(1-\gamma_A)T}} \overset{!}{=} 0. \qquad \text{(A.11)}$$

Within the results in the proof of Proposition 2 it furthermore holds

$$\frac{\partial y_{T,p_i}(\pi)}{\partial \pi} = \frac{p_i \gamma_L \sigma_1^2 (\pi_1^{Mer} - \pi) e^{y(\pi,1)(1-\gamma_L)T} + (1-p_i)\gamma_L \sigma_2^2 (\pi_2^{Mer} - \pi) e^{y(\pi,2)(1-\gamma_L)T}}{p_i e^{y(\pi,1)(1-\gamma_L)T} + (1-p_i) e^{y(\pi,2)(1-\gamma_L)T}}, \text{ for } i = a, b.$$

Using this result we can solve the FOC (A.11) for π and can formulate that $\pi_T^{*,pre,amb}$ has to fulfill the equation

$$\begin{aligned}
\pi =& \frac{\tilde{p}}{1-\tilde{p}} \left(\frac{p_a e^{y(\pi,1)(1-\gamma_L)T} + (1-p_a) e^{y(\pi,2)(1-\gamma_L)T}}{p_b e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b) e^{y(\pi,2)(1-\gamma_L)T}} \right)^{\frac{\gamma_A - \gamma_L}{\gamma_L - 1}} \times \\
& \frac{p_a \sigma_1^2 e^{y(\pi,1)(1-\gamma)T} (\pi_1^{Mer} - \pi) + (1-p_a)\sigma_2^2 .6^{y(\pi,2)(1-\gamma_L)T} (\pi_2^{Mer} - \pi)}{p_b \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b)\sigma_2^2 e^{y(\pi,1)(1-\gamma_L)T}} + \\
& \frac{p_b \sigma_1^2 \pi_1^{Mer} e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b)\sigma_2^2 \pi_2^{Mer} e^{y(\pi,2)(1-\gamma_L)T}}{p_b \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T}}.
\end{aligned}$$

Simplifying this equation we receive

$$\begin{aligned}
\pi =& \frac{\tilde{p}\xi_a}{\tilde{p}\xi_a + (1-\tilde{p})\xi_b} \left[\frac{p_a \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} \pi_1^{Mer} + (1-p_a)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T} \pi_2^{Mer}}{p_a \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} + (1-p_a)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T}} \right] + \\
& \left(1 - \frac{\tilde{p}\xi_a}{\tilde{p}\xi_a + (1-\tilde{p})\xi_b}\right) \left[\frac{p_b \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} \pi_1^{Mer} + (1-p_b)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T} \pi_2^{Mer}}{p_b \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T}} \right], \text{ where}
\end{aligned}$$

$$\xi_a = \left(p_a \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} + (1-p_a)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T} \right) \left[p_a e^{y(\pi,1)(1-\gamma_L)T} + (1-p_a) e^{y(\pi,2)(1-\gamma_L)T} \right]^{\frac{\gamma_A - \gamma_L}{\gamma_L - 1}},$$

$$\xi_b = \left(p_b \sigma_1^2 e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b)\sigma_2^2 e^{y(\pi,2)(1-\gamma_L)T} \right) \left[p_b e^{y(\pi,1)(1-\gamma_L)T} + (1-p_b) e^{y(\pi,2)(1-\gamma_L)T} \right]^{\frac{\gamma_A - \gamma_L}{\gamma_L - 1}}.$$

Recall that $\delta^{pre}(\pi, T) = 1 - e^{(y(\pi,1) - y(\pi,2))(1-\gamma_L)T}$ and define $\delta^{amb}(\pi, T) := 1 - \frac{\xi_a}{\xi_b}$, s.t. we can finally write the equation as

$$\begin{aligned}
\pi =& \frac{\tilde{p}(1-\delta_T^{amb}(\pi))}{\tilde{p}(1-\delta_T^{amb}(\pi)) + (1-\tilde{p})} \left[\frac{p_a \sigma_1^2 (1-\delta_T^{pre}(\pi))\pi_1^{Mer}}{p_a \sigma_1^2 (1-\delta_T^{pre}(\pi)) + (1-p_a)\sigma_2^2} + \frac{(1-p_a)\sigma_2^2 \pi_2^{Mer}}{p_a \sigma_1^2 (1-\delta_T^{pre}(\pi)) + (1-p_a)\sigma_2^2} \right] + \\
& \frac{1-\tilde{p}}{\tilde{p}(1-\delta_T^{amb}(\pi)) + (1-\tilde{p})} \left[\frac{p_b \sigma_1^2 (1-\delta_T^{pre}(\pi))\pi_1^{Mer}}{p_b \sigma_1^2 (1-\delta_T^{pre}(\pi)) + (1-p_b)\sigma_2^2} + \frac{(1-p_b)\sigma_2^2 \pi_2^{Mer}}{p_b \sigma_1^2 (1-\delta_T^{pre}(\pi)) + (1-p_b)\sigma_2^2} \right] \\
=& \alpha_{T,\tilde{p}}^* \left(\alpha_{T,p_a} \pi_1^{Mer} + (1-\alpha_{T,p_a})\pi_2^{Mer} \right) + (1-\alpha_{T,\tilde{p}}) \left(\alpha_{T,p_b} \pi_1^{Mer} + (1-\alpha_{T,p_b})\pi_2^{Mer} \right).
\end{aligned}$$

Appendix B

Appendix to Chapter 5

B.1 Possibilities to incorporate a regime switch

Regime-switching models are useful for several reasons: They are able to track and replicate behavioral patterns found in stock market and macroeconomic time series data. They are analytically tractable: Despite the fact that, asset allocation under regime uncertainty is a complex problem, regime-switching models are able to capture nonlinear (stylized) dynamics. Appropriate mixing of conditional normal distributions (or other types of distributions) can produce large amounts of nonlinear effects. Even if the true model is unknown, regime-switching models can provide a good approximation of complicated real-world processes that determine stock returns.

In the literature there are several ways to consider a regime switch. On the one hand, the interest rate r of the risk-free asset (e.g., bond) may be subject to a regime change. The price dynamics of the risk-free asset $(B_t)_{t\in[0,T]}$ would then be given by:

$$dB_t = B_t r dt \text{ or with regime switch: } dB_t = B_t r(Y_t)dt$$

The price dynamics of the risky asset (e.g., stock) following the assumption of Black and Scholes (1973) follows a geometric Brownian motion with drift μ and volatility σ. Both parameters may depend on a regime switch modeled via Markov chains. The stock price $(S_t)_{t\in[0,T]}$ develops according to the following continuous-time stochastic differential equation (SDE):

$$dS_t = S_t\mu dt + S_t\sigma dW_t \text{ or with regime switch: } dS_t = S_t\mu(Y_t)dt + S_t\sigma(Y_t)dW_t$$

The wealth process $(X_t)_{t\in[0,T]}$ results as follows:

$$dX_t = X_t\pi_t\frac{dS_t}{S_t} + X_t(1-\pi_t)\frac{dB_t}{B_t} \text{ or with regime switch in } r,\ \mu \text{ and } \sigma:$$

$$\frac{dX_t}{X_t} = \pi_t\left(\mu(Y_t)dt + \sigma(Y_t)dW_t\right) + (1-\pi_t)r(Y_t)dt$$

Thus, an observable or hidden Markov chain can be modeled in the interest rate r and/or in the drift μ and/or in the volatility σ.

B.2 Assumptions leading to time consistency

As already shown in the research work in Chapter 4, the utility aggregation is highly non-linear for a CRRA-investor with risk aversion paramter $\gamma_L > 1$, which leads to time inconsistency. However, the assumption of a **logarithmic utility function** ($\gamma_L = 1$) in this setup leads to the asset allocation problem becoming time-consistent. The expected utility of the terminal wealth of the investor $EU_{T,p}$ aggregates over utilities under assumption of an a priori lottery over two regimes. Thus with

$$E_{P_i}[u(V_T)] = \begin{cases} \frac{1}{1-\gamma_L} e^{y(\pi,i)(1-\gamma_L)T} & \gamma_L > 1 \\ y(\pi,i)T & \gamma_L = 1, \end{cases}$$

it follows

$$EU_{T,p} = \begin{cases} \frac{1}{1-\gamma_L}\left[pe^{y(\pi,1)(1-\gamma_L)T} + (1-p)e^{y(\pi,2)(1-\gamma_L)T}\right] & \gamma_L > 1 \\ \left[py(\pi,1) + (1-p)y(\pi,2)\right]T & \gamma_L = 1 \end{cases},$$

$$y_{T,p}(\pi) = \begin{cases} \frac{1}{(1-\gamma_L)T}\ln\left[pe^{y(\pi,1)(1-\gamma_L)T} + (1-p)e^{y(\pi,2)(1-\gamma_L)T}\right] & \gamma_L \neq 1 \\ py(\pi,1) + (1-p)y(\pi,2) & \gamma_L = 1 \end{cases}. \tag{B.1}$$

For logarithmic utility, the savings rate with an initial lottery coincides with the expected savings rate of the Merton problems in the two regimes. Thus, maximizing expected utility of terminal wealth is equivalent to maximizing the expected savings rate. The expected utility maximizing strategy also yields the highest expected savings rate. However, this is not true for $\gamma_L \neq 1$.

Proposition 15 *When p is the probability for Regime 1 and $(1-p)$ the probability for Regime 2, the expected utility of the terminal wealth V_T for an investor with a logarithmic utility function and $V_0 = 1$ can be calculated in terms of*

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right] &= p\ \mathbb{E}_{\mathbb{P}}\left[\ln\left(\frac{V_{T,1}}{V_0}\right)\right] + (1-p)\ \mathbb{E}_{\mathbb{P}}\left[\ln\left(\frac{V_{T,2}}{V_0}\right)\right] \\ &= p\left\{(\sigma_1^2\pi\pi_1^{Mer} + r - \frac{1}{2}\sigma_1^2\pi^2)T + \sigma_1\pi\mathbb{E}_{\mathbb{P}}\left[W_T\right]\right\} \\ &+ (1-p)\left\{(\sigma_2^2\pi\pi_2^{Mer} + r - \frac{1}{2}\sigma_2^2\pi^2)T + \sigma_2\pi\mathbb{E}_{\mathbb{P}}\left[W_T\right]\right\} \\ &= \left[p\sigma_1^2\pi\pi_1^{Mer} + (1-p)\sigma_2^2\pi\pi_2^{Mer} - \frac{1}{2}\pi^2(p\sigma_1^2 + (1-p)\sigma_2^2) + r\right]T.\end{aligned}$$

To calculate the optimal pre-commitment strategy that maximizes the expected utility of the investor we have to calculate the first order condition:

$$\begin{aligned}\frac{\partial\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right]}{\partial\pi} &= p\sigma_1^2\pi_1^{Mer}T + (1-p)\sigma_2^2\pi_2^{Mer}T - \pi T(p\sigma_1^2 + (1-p)\sigma_2^2) \overset{!}{=} 0 \\ \Leftrightarrow \pi^{pre,*} &= \frac{p\sigma_1^2\pi_1^{Mer} + (1-p)\sigma_2^2\pi_2^{Mer}}{p\sigma_1^2 + (1-p)\sigma_2^2} \\ \Leftrightarrow \pi^{pre,*} &== \frac{p(\mu_1 - r) + (1-p)(\mu_2 - r)}{\gamma_R(p\sigma_1^2 + (1-p)\sigma_2^2)}.\end{aligned} \tag{B.2}$$

The optimal pre-commitment strategy follows from the solution of an implicit function and is independent of T. Thus, time consistency can be obtained by $\gamma_L = 1$ (log-investor). As soon as we deviate from the logarithmic utility case, the strategy is not time-consistent, i.e., an investor would want to switch to the optimal pre-commitment strategy for a shortened investment horizon as time passes.

An easy intuition why the assumption of a logarithmic utility function avoids problems stemming from time inconsistency is that logarithmic utility implies myopic behavior. Note that while considerung an investor with $\gamma_L > 1$, myopia implies that the investor acts risk neutral, w.r.t. the regime dependent savings rates (i.e., the decision can be formulated by means of the expected savings rate $py(\pi,1)+(1-p)y(\pi,2)$. The investor thus always chooses the stratgey that is optimal over the next instant and neither takes the remaining investment horizon nor the continuation utility into account. Thus, also for a myopic investor $(T \to 0)$ the optimal strategy is time-consistent.

Moreover, for $\gamma_L > 1$ (and $T > 0$), the savings rate of a strategy π equals the expected savings rate over the regimes for the boundary cases $p = 0$ and $p = 1$, i.e., if the second dimension of the risk situation vanishes. The difference between $y_{T,p}(\pi)$ and $p\,y(\pi,1)+(1-p)y(\pi,2)$ also depends on p and is largest for

$$p^*_{SR}(\pi,T) = \frac{1}{1-e^{(y(\pi,1)-y(\pi,2))(1-\gamma_L)T}} + \frac{1}{(y(\pi,1)-y(\pi,2))(1-\gamma_L)T}. \tag{B.3}$$

It holds that $p^*_{SR}(\pi,T)$ is increasing in T. With the savings rate $y_{T,p}(\pi)$ approaching the lower of the two savings rate, the difference to the higher expected savings rate is maximized when the latter has more and more weight on the larger of the two savings rates.

B.3 Proof of Proposition 13

With the results of equations (5.10) and (5.11) we can state the expected utility of a CRRA investor in terms of

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{V_T}{V_0}\right)\right] &= \mathbb{E}_{\mathbb{P}}\left[u\left(1_{\{\tau\le T\}}X_1 + 1_{\{\tau>T\}}X_2\right)\right] \\ &= \frac{1}{1-\gamma}\left\{\mathbb{E}_{\mathbb{P}}\left[1_{\{\tau\le T\}}X_1^{(1-\gamma)}\right] + \mathbb{E}_{\mathbb{P}}\left[1_{\{\tau>T\}}X_2^{(1-\gamma)}\right]\right\}.\end{aligned} \tag{B.4}$$

Let's start with the calculation of $E_1 := \mathbb{E}_{\mathbb{P}}\left[1_{\{\tau\le T\}}X_1^{(1-\gamma)}\right]$ in (B.4):

$$\begin{aligned}E_1 &= \mathbb{E}_{\mathbb{P}}[1_{\{\tau\le T\}}e^{[\pi(\mu_2-r)+r-\frac{1}{2}\pi^2\sigma_2^2]T(1-\gamma)} \\ &\quad\cdot e^{[\pi(\mu_1-\mu_2)-\frac{1}{2}\pi^2(\sigma_1^2-\sigma_2^2)]\tau(1-\gamma)+\pi(\sigma_1 W_\tau+\sigma_2(W_T-W_\tau))(1-\gamma)}] \\ &= e^{[\pi(\mu_2-r)+r-\frac{1}{2}\pi^2\sigma_2^2]T(1-\gamma)} \\ &\quad\cdot \mathbb{E}_{\mathbb{P}}\left[1_{\{\tau\le T\}}e^{[\pi(\mu_1-\mu_2)-\frac{1}{2}\pi^2(\sigma_1^2-\sigma_2^2)]\tau(1-\gamma)}\mathbb{E}_{\mathbb{P}}\left[e^{\pi(\sigma_1 W_\tau+\sigma_2(W_T-W_\tau))(1-\gamma)}|\tau\right]\right],\end{aligned} \tag{B.5}$$

where the last equation holds because of the basic properties of the conditional expectation. Furthermore the Brownian motion W_τ is independent from $W_T - W_\tau$. Together with the fact that $\mathbb{E}\left[e^X\right] = e^{\mu+\frac{1}{2}\sigma^2}$ for

$X \sim N(\mu, \sigma^2)$, equation (B.5) can be written as

$$\begin{aligned}
&e^{[\pi(\mu_2-r)+r-\frac{1}{2}\pi^2\sigma_2^2]T(1-\gamma)}\mathbb{E}_{\mathbb{P}}[1_{\{\tau\leq T\}}e^{[\pi(\mu_1-\mu_2)-\frac{1}{2}\pi^2(\sigma_1^2-\sigma_2^2)]\tau(1-\gamma)} \\
&\quad\cdot \mathbb{E}_{\mathbb{P}}[e^{\pi(\sigma_1 W_\tau+\sigma_2(W_T-W_\tau))(1-\gamma)}|\tau]] \\
&= e^{[\pi(\mu_2-r)+r-\frac{1}{2}\pi^2\sigma_2^2]T(1-\gamma)}\mathbb{E}_{\mathbb{P}}[1_{\{\tau\leq T\}}e^{[\pi(\mu_1-\mu_2)-\frac{1}{2}\pi^2(\sigma_1^2-\sigma_2^2)]\tau(1-\gamma)} \\
&\quad\cdot e^{\frac{1}{2}\pi^2\sigma_1^2(1-\gamma)^2\tau+\frac{1}{2}\pi^2\sigma_2^2(1-\gamma)^2(T-\tau)}] \\
&= e^{\xi_2 T}\mathbb{E}_{\mathbb{P}}\left[1_{\{\tau\leq T\}}e^{\nu\tau}\right]. \qquad \text{(B.6)}
\end{aligned}$$

Now it holds that the function $g(x) := 1_{\{x\leq T\}}e^{\nu x}$ is measurable, so the transformation $g(\tau)$ is still a random variable. Together with $\tau \sim Exp(\lambda)$ and its absolute continuous density function $f_\tau^\lambda(x)$we get

$$\mathbb{E}_{\mathbb{P}}[g(\tau)] = \int_{-\infty}^{\infty} g(x) f_\tau^\lambda(x)dx. \qquad \text{(B.7)}$$

Combining (B.6) and (B.7) we get

$$\begin{aligned}
e^{\xi_2 T}\mathbb{E}_{\mathbb{P}}\left[1_{\{\tau\leq T\}}e^{\nu\tau}\right] &= e^{\xi_2 T}\int_{-\infty}^{\infty} 1_{\{x\leq T\}}e^{\nu x} f_\tau^\lambda(x)dx \\
&= e^{\xi_2 T}\int_0^T e^{\nu x}\lambda e^{-\lambda x}dx \\
&= \lambda e^{\xi_2 T}\frac{1}{\nu-\lambda}\left[e^{(\nu-\lambda)T}-1\right]. \qquad \text{(B.8)}
\end{aligned}$$

For the calculation of $E_2 := \mathbb{E}_{\mathbb{P}}\left[1_{\{\tau>T\}}X_2^{(1-\gamma)}\right]$ in (B.4) it holds:

$$\begin{aligned}
E_2 &= \mathbb{E}_{\mathbb{P}}\left[1_{\{\tau>T\}}e^{[\pi(\mu_1-r)+r-\frac{1}{2}\pi^2\sigma_1^2]T(1-\gamma)+\sigma_1\pi W_T(1-\gamma)}\right] \\
&= e^{[\pi(\mu_1-r)+r-\frac{1}{2}\pi^2\sigma_1^2]T(1-\gamma)}\mathbb{E}_{\mathbb{P}}\left[1_{\{\tau>T\}}e^{\sigma_1\pi W_T(1-\gamma)}\right] \\
&= e^{[\pi(\mu_1-r)+r-\frac{1}{2}\pi^2\sigma_1^2]T(1-\gamma)}\mathbb{E}_{\mathbb{P}}\left[1_{\{\tau>T\}}\mathbb{E}_{\mathbb{P}}\left[e^{\sigma_1\pi W_T(1-\gamma)}|\tau\right]\right] \\
&= e^{\xi_1 T}\mathbb{E}_{\mathbb{P}}\left[1_{\{\tau>T\}}\right] \\
&= e^{\xi_1 T}\left[1-F_\tau^\lambda(T)\right] \\
&= e^{(\xi_1-\lambda)T}. \qquad \text{(B.9)}
\end{aligned}$$

Combining (B.8) and (B.9) gives the final result.

B.4 Proof of Proposition 14

To maximize the expected utility of a CRRA investor that follows a pre-commitment strategy we have to minimize the expression

$$\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}e^{(\xi_1-\lambda)T} + \frac{-\lambda}{\xi_1-\xi_2-\lambda}e^{\xi_2 T}.$$

The first order condition is given by

$$\begin{aligned}
&\frac{\partial}{\partial\pi}\left\{\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\right\}e^{(\xi_1-\lambda)T} + \frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\frac{\partial}{\partial\pi}\left\{e^{(\xi_1-\lambda)T}\right\} + \\
&\quad\frac{\partial}{\partial\pi}\left\{\frac{-\lambda}{\xi_1-\xi_2-\lambda}\right\}e^{\xi_2 T} + \frac{-\lambda}{\xi_1-\xi_2-\lambda}\frac{\partial}{\partial\pi}\left\{e^{\xi_2 T}\right\} \stackrel{!}{=} 0.
\end{aligned}$$

Let us calculate the corresponding derivatives first (r=0):

- $\frac{\partial \xi_1}{\partial \pi} = (1-\gamma)\gamma\sigma_1^2(\pi_1^{Mer} - \pi)$
- $\frac{\partial \xi_2}{\partial \pi} = (1-\gamma)\gamma\sigma_2^2(\pi_2^{Mer} - \pi)$
- $\frac{\partial(\xi_1-\xi_2)}{\partial \pi} = (1-\gamma)\gamma\left[\sigma_1^2\pi_1^{Mer} - \sigma_2^2\pi_2^{Mer} - \pi(\sigma_1^2 - \sigma_2^2)\right]$
- $\frac{\partial}{\partial \pi}\left\{\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\right\} = \frac{\frac{\partial(\xi_1-\xi_2)}{\partial \pi}(\xi_1-\xi_2-\lambda)-(\xi_1-\xi_2)\frac{\partial(\xi_1-\xi_2-\lambda)}{\partial \pi}}{(\xi_1-\xi_2-\lambda)^2} = \frac{-\lambda\frac{\partial(\xi_1-\xi_2)}{\partial \pi}}{(\xi_1-\xi_2-\lambda)^2}$
  $= \frac{-\lambda(1-\gamma)\gamma\left[\sigma_1^2(\pi_1^{Mer}-\pi)-\sigma_2^2(\pi_2^{Mer}-\pi)\right]}{(\xi_1-\xi_2-\lambda)^2}$
- $\frac{\partial}{\partial \pi}\left\{e^{(\xi_1-\lambda)T}\right\} = \frac{\partial \xi_1 T}{\partial \pi}e^{(\xi_1-\lambda)T} = (1-\gamma)\gamma\sigma_1^2(\pi_1^{Mer} - \pi)Te^{(\xi_1-\lambda)T}$
- $\frac{\partial}{\partial \pi}\left\{\frac{-\lambda}{\xi_1-\xi_2-\lambda}\right\} = \frac{\lambda\frac{\partial(\xi_1-\xi_2-\lambda)}{\partial \pi}}{(\xi_1-\xi_2-\lambda)^2} = \frac{\lambda\frac{\partial(\xi_1-\xi_2)}{\partial \pi}}{(\xi_1-\xi_2-\lambda)^2} = -\frac{\partial}{\partial \pi}\left\{\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\right\}$
- $\frac{\partial}{\partial \pi}\left\{e^{\xi_2 T}\right\} = \frac{\partial \xi_2 T}{\partial \pi}e^{\xi_2 T} = (1-\gamma)\gamma\sigma_2^2(\pi_2^{Mer} - \pi)Te^{\xi_2 T}$

Because of the fact that $\frac{\partial}{\partial \pi}\left\{\frac{-\lambda}{\xi_1-\xi_2-\lambda}\right\} = -\frac{\partial}{\partial \pi}\left\{\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\right\}$ and $\frac{-\lambda}{\xi_1-\xi_2-\lambda} = 1 - \frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}$ we can write the first order condition as follows

$$\frac{\partial}{\partial \pi}\left\{\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\right\}(e^{(\xi_1-\lambda)T} - e^{\xi_2 T}) + \frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\left(\frac{\partial}{\partial \pi}\left\{e^{(\xi_1-\lambda)T}\right\} - \frac{\partial}{\partial \pi}\left\{e^{\xi_2 T}\right\}\right) +$$
$$\frac{\partial}{\partial \pi}\left\{e^{\xi_2 T}\right\} \stackrel{!}{=} 0$$
$$\Leftrightarrow \frac{-\lambda(1-\gamma)\gamma\left[\sigma_1^2(\pi_1^{Mer}-\pi)-\sigma_2^2(\pi_2^{Mer}-\pi)\right]}{(\xi_1-\xi_2-\lambda)^2}(e^{(\xi_1-\lambda)T} - e^{\xi_2 T}) +$$
$$\frac{\xi_1-\xi_2}{\xi_1-\xi_2-\lambda}\left((1-\gamma)\gamma\sigma_1^2(\pi_1^{Mer}-\pi)Te^{(\xi_1-\lambda)T} - (1-\gamma)\gamma\sigma_2^2(\pi_2^{Mer}-\pi)Te^{\xi_2 T}\right) +$$
$$(1-\gamma)\gamma\sigma_2^2(\pi_2^{Mer}-\pi)Te^{\xi_2 T} \stackrel{!}{=} 0$$

$$\Leftrightarrow \pi^{pre,*} = \frac{[\sigma_1^2(\pi_1^{Mer}-\pi^{pre,*}) - \sigma_2^2(\pi_2^{Mer}-\pi^{pre,*})](e^{(\xi_1-\lambda)T} - e^{\xi_2 T})}{Te^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}$$
$$-\frac{(\xi_1-\xi_2)\sigma_1^2(\pi_1^{Mer}-\pi^{pre,*})e^{(\xi_1-\lambda)T}}{\lambda\sigma_2^2 e^{\xi_2 T}} + \pi_2^{Mer}$$
$$\Leftrightarrow \pi^{pre,*} = \pi_1^{Mer}\frac{[\lambda\sigma_1^2(e^{(\xi_1-\lambda)T} - e^{\xi_2 T}) - T(\xi_1-\xi_2-\lambda)(\xi_1-\xi_2)\sigma_1^2 e^{(\xi_1-\lambda)T}}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)} +$$
$$\pi_2^{Mer}\frac{\lambda\sigma_2^2(e^{\xi_2 T} - e^{(\xi_1-\lambda)T}) + T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)} +$$
$$\pi^{pre,*}\frac{\left(\lambda e^{\xi_2 T} - \lambda e^{(\xi_1-\lambda)T}\right)(\sigma_1^2-\sigma_2^2) + T(\xi_1-\xi_2-\lambda)(\xi_1-\xi_2)\sigma_1^2 e^{(\xi_1-\lambda)T}}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}$$
$$\Leftrightarrow \pi^{pre,*} = \alpha_1\pi_1^{Mer} + \alpha_2\pi_2^{Mer} + \alpha_3\pi^{pre,*}$$
$$\Leftrightarrow \pi^{pre,*} = \pi_1^{Mer}\frac{\alpha_1}{1-\alpha_3} + \pi_2^{Mer}\frac{\alpha_2}{1-\alpha_3}, \text{where } \alpha_1+\alpha_2+\alpha_3 = 1 \text{ with}$$
$$\alpha_1 = \frac{[\lambda\sigma_1^2(e^{(\xi_1-\lambda)T} - e^{\xi_2 T}) - T(\xi_1-\xi_2-\lambda)(\xi_1-\xi_2)\sigma_1^2 e^{(\xi_1-\lambda)T}}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}$$
$$\alpha_2 = \frac{\lambda\sigma_2^2(e^{\xi_2 T} - e^{(\xi_1-\lambda)T}) + T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}$$
$$\alpha_3 = \frac{\left(\lambda e^{\xi_2 T} - \lambda e^{(\xi_1-\lambda)T}\right)(\sigma_1^2-\sigma_2^2) + T(\xi_1-\xi_2-\lambda)(\xi_1-\xi_2)\sigma_1^2 e^{(\xi_1-\lambda)T}}{T\lambda e^{\xi_2 T}\sigma_2^2(\xi_1-\xi_2-\lambda)}.$$

Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig und ohne unerlaubte Hilfe angefertigt und andere als die in der Dissertation angegebenen Hilfsmittel nicht benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht.

Goch, 14. April 2023 Lara Becker

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