Virtual Localization Formula for Witt Theory

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"Brevity is the soul of wit."

W. Shakespear, *Hamlet*

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Abstract

In this thesis, we extend the Virtual Localization Formula to a wide class of motivic ring spectra, obtaining in particular a localization formula for virtual fundamental classes in Witt theory KW. We extend to general spectra the notions and operations of equivariant bivariant theories. Then we specialise to $SL[\eta^{-1}]$-oriented spectra A, obtaining a presentations of $\mathrm{A}(BN)$ (where N is the normaliser of the torus in SL_2), and an Atiyah-Bott localization theorem for $\mathrm{A}_N(X)$, for X a scheme equipped with an N-action. Of independent interest, we also describe the ring structure of $\mathrm{KW}(BN)$ and give a complete computation of KW-valued Euler classes for all rank two vector bundles over BN.

Zusammenfassung

In dieser Arbeit erweitern wir die virtuelle Lokalisierungsformel auf eine breite Klasse von motivischen Ringspektren, und erhalten insbesondere eine Lokalisierungsformel für virtuelle Fundamentalklassen in der Witt-Theorie KW. Wir erweitern die Begriffe und Operationen der äquivarianten bivarianten Theorien auf allgemeine Spektren. Dann spezialisieren wir uns auf $SL[\eta^{-1}]$-orientierte Spektren A, und erhalten eine Darstellung von $\mathrm{A}(BN)$ (wobei N der Normalisator des Torus in SL_2 ist) und einen Atiyah-Bott-Lokalisationssatz für $\mathrm{A}_N(X)$, für X ein Schema mit einer N-Aktion. Von unabhängigem Interesse ist auch die Beschreibung der Ringstruktur von $\mathrm{KW}(BN)$, und eine vollständige Berechnung der KW-wertigen Eulerklassen für alle Rang-zwei-Bündel über BN.

Contents

Introduction

In the presence of a torus action, many intersection theoretic and enumerative problems get simplified, looking at the fixed point locus of the problem. But to do so, one has to precisely relate the equivariant intersection theory to the intersection theory of the fixed points. In algebraic geometry, using Chow groups and the Bott residue theorem of [EG98b], one can get a complete description of equivariant characteristic classes of a scheme X in terms of invariants of the fixed locus X^T. For algebraic K-theory analogous computations were made in [Tho86]. Often the spaces we are interested in are not smooth and we need to take into account deformation theoretic information to treat them. Schemes (or even algebraic stacks) equipped with a perfect obstruction theory, in the sense of [BF97], give rise to virtual fundamental classes and can basically be studied as if they were (almost) smooth (or quasi-smooth, using the terminology from derived geometry).

In [GP99], Graber-Pandharipande, using techniques from [EG98b], proved a "Virtual Localization Formula" for virtual classes. This formula relates the virtual fiundamental class of a scheme X, equipped with a torus action and a perfect obstruction theory, to a suitable virtual class associated to the fixed point locus. Extensions of Atiyah-Bott and virtual localization theorems, in the context of motivic homotopy theory, were recently made in [Lev22a], [Lev22b] and, with different techniques (applied to algebraic stacks), in [Ara+22]. One of the main reasons to extend intersection theoretic techniques to motivic homotopy theory is that, one can get much richer invariants. Indeed, in the motivic homotopy category $\mathrm{SH}(\Bbbk)$, by a celebrated theorem of Morel, we have that the endomorphisms of the sphere spectrum corresponds to quadratic forms in the Grothendieck-Witt ring: $\mathrm{End}_{\mathrm{SH}(\Bbbk)}(\mathbb{1}_\Bbbk) \simeq \mathrm{GW}(\Bbbk)$. In the recent years, this led to much progress in the field, now called $\mathbb{A}^1$-enumerative geometry, with a lot of new interesting results by Kass, Levine, McKean, Pauli, Solomon, Wendt, Wickelgren and many others (for example one can look in [BW21] and references therein).

The purpose of this thesis is to extend the results in [Lev22b] to the case of $SL[\eta^{-1}]$-oriented ring spectra (see section 2.1)). These are just SL-oriented spectra where the algebraic Hopf map η is invertible. One of the main examples of such spectra is given by the Witt spectrum, representing Balmer's derived Witt groups. One can think of Witt sheaf cohomology, used in *loc. cit.*, and the Witt spectrum as quadratic analogues of Chow groups and K-theory. Indeed, Witt sheaf cohomology and Witt theory capture the essentially quadratic information in Chow-Witt groups and Hermitian K-theory, which are quadratic refinements of Chow groups and K-theory respectively. We hope that with the result in this thesis, it will be possible to derive a virtual localization formula in Hermitian K-theory too.

While Witt sheaf cohomology is an homotopy module, and hence it is easier to work with, from an homotopic point of view: it is *bounded* (in the sense of [Lev22a, Definition 4.13]). But the $SL[\eta^{-1}]$-oriented spectra (like Witt theory KW) are not bounded. One needs to be careful while working with general spectra on algebraic stacks or ind-schemes, like the ones approximating quotient stacks following the works of Totaro, Edidin-Graham and Morel-Voevodsky. We dedicated the whole first chapter to take care of these issues. While writing the thesis, we were informed by Lorenzo Mantovani ([Man22]) about his work in progress on equivariant (motivic) intersection theory for unbounded spectra. The development of such tools was made completely independently, but there could be some unforeseen overlaps.

Relying on the foundational works of Ananyevskiy (cf. [Ana15; Ana16b; Ana19]), we are now able to extend most of the results in [Lev22a; Lev22b] to any $SL[\eta^{-1}]$-oriented ring spectrum. As already noted in [Lev22a], the localization theorem for $\mathbb{G}_m$-actions is not interesting in the case of spectra where η is invertible. In those cases, the Euler classes we need to invert will already be zero (see corollary 2.2.6). The closest natural candidate for a localization theorem is then N, the normaliser of the torus in SL_2. Indeed we obtained a nice presentation of the cohomology of $\mathcal{B}N$:

Proposition 1 (proposition 2.2.16)**.** *For any $SL[\eta^{-1}]$-oriented ring spectrum, we get the following isomorphisms of graded $\mathrm{A}^\bullet(S)$-modules:*

$$A^\bullet(\mathcal{B}N) \simeq \mathrm{A}^\bullet(\mathcal{B}SL_2) \oplus \mathrm{A}^\bullet(S)$$

$$A^\bullet(\mathcal{B}N; \gamma_N) \simeq \mathrm{A}^{\bullet-2}(\mathcal{B}SL_2)e(\mathcal{T}) \oplus \mathrm{A}^\bullet(S)$$

where $\mathcal{T}$ is the tangent bundle of $\left[\mathbb{P}(\mathrm{Sym}^2(F))/SL_2\right]$ over $\mathcal{B}SL_2$ and γ_N is the generator of $\mathrm{Pic}(\mathcal{B}N)$.

With this description at our disposal, we will be able to extend the Atiyah-Bott localization theorem to our context:

Theorem 2 (theorem 4.1.18)**.** *Let $X \in \mathbf{Sch}^N_{/\Bbbk}$ be a scheme with an N-action and let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented spectrum. Let $\iota : |X|^N \hookrightarrow X$ be the closed immersion. Let $L \in \mathrm{Pic}(X)$ be an N-linearised line bundle. Suppose the N-action is semi-strict. Then there is a non-zero integer M such that:*

$$\iota_* : \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(|X|^N; \iota^*L\right)\left[(M\cdot e)^{-1}\right] \longrightarrow \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(X; L\right)\left[(M\cdot e)^{-1}\right]$$

is an isomorphism.

Finally, for schemes X equipped with special N-actions, and an N-linearised obstruction theory, we have:

Theorem 3 (theorem 4.3.10). *Let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Let $\iota : X \hookrightarrow Y$ be a closed immersion in $\mathbf{Sch}^N_{/\Bbbk}$, with Y a smooth N-scheme. Let $\varphi_\bullet : \mathcal{E}_\bullet \to \mathbb{L}_{X/\Bbbk}$ be an N-linearised perfect obstruction theory. Then we have:*

$$[X, \varphi]^{vir}_N = \sum_{j=1}^{s} \iota_{j*} \left(\left[|X|^N_j, \varphi^{(j)} \right]^{vir}_N \cap e_N \left(N^{vir}_{\iota_j} \right)^{-1} \right) \in \mathrm{A}^{\mathrm{BM}}_N (X, E_\bullet) \left[(M \cdot e)^{-1} \right]$$

where $[-,-]^{vir}_N$ denote the appropriate virtual fundamental class.

Outline of the Thesis

All along this work, we closely followed proofs and strategies from [Lev19; Lev22a; Lev22b]: the importance of those papers in our work cannot be overstated. Our original contribution was to provide with additional or different arguments where needed for our generalisations. In particular, we made a crucial use of the machinery developed in [Cho21a] and in [KR21]. The proofs of proposition 2.2.16, lemma 4.1.10 are new and rely on the fact, proved in [Ana16b], that the special linear algebraic cobordism MSL is the universal SL-oriented theory. These two theorems, 2.2.16 and 4.1.10, will allow us to generalise almost all we need for our final virtual localization theorem.

Of independent interest, in Chapter 3, we also developed a theory of *twisted symplectic* orientations on motivic spectra. We used the theory of twisted symplectic bundles and orientations, to extend results in [Ana17] and explicitly compute the Euler classes in KW of all irreducible rank two representations of BN. At the time of writing, we were not aware that the notion of twisted symplectic bundles was also studied by Asok-Fasel-Hopkins in one of their upcoming works: what we do here is rather pedestrian compared to their work, nonetheless we hope it might be useful for future computations. We are grateful to J. Fasel, who informed us about their work.

To summarise:

- In Chapter 1: we prove and state all the preliminary notions and properties of equivariant Borel-Moore homology, cohomology and virtual fundamental classes. These will be extensively used in the rest of the thesis.
- In Chapter 2: we recall the notions of SL-orientations and Euler classes; we give a complete additive description of $\mathrm{A}^\bullet(BN)$ for A an $SL[\eta^{-1}]$-oriented spectrum and we end with a complete multiplicative description of $\mathrm{KW}^\bullet(BN)$ (corollary 2.3.6).

- In Chapter 3: we develop the notion of twisted symplectic orientations and we extend [Ana17, Lemma 8.2] to the twisted case; this will allow us to explicitly compute the Euler classes in Witt theory for any rank two vector bundle over BN in proposition 3.4.3.

- In Chapter 4: we finally deal with our main theorems. Using lemma 4.1.10, we generalise, to all $SL[\eta^{-1}]$-oriented spectra, the Atiyah-Bott localization theorem (theorem 4.1.18) and the Virtual Localization Theorem (theorem 4.3.10), obtaining as a special case the Virtual Localization Theorem for KW in corollary 4.3.12.

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Notations and Conventions

1. The categories $\mathbf{Sch}_{/B}, \mathbf{Sch}^G_{/B}$ will always denote quasi-projective schemes over a base scheme B of finite Krull dimension, without or with a (left) G-action. If we write $\mathbf{Sm}_{/B}, \mathbf{Sm}^G_{/B}$ then we only consider smooth quasi-projective schemes.

2. We will denote by $\mathcal{ASt}_{/S}$ the ∞-category of algebraic stacks over some base S (that could be either a scheme or a stack, but for us will always be a scheme). Moreover given $X \in \mathbf{Sch}^G_{/B}$, we will denote the associated quotient stack as $[X/G]$.

3. If not specified otherwise, whenever we are working over a field $\Bbbk$, we will assume it is of characteristic different from 2. If we work over a general base scheme B, we will always assume $\dfrac{1}{2} \in \mathcal{O}_B^\times$.

4. Recall that a morphism of quasi-projective schemes $f : X \to S$ in $\mathbf{Sch}_{/B}$ is called *lci* (that stands for *local complete intersection*) if there exists a factorization of f as $X \overset{i}{\hookrightarrow} M \overset{p}{\to} S$ with i a regular closed immersion and p a smooth map. In the conventions of [DJK21], these are called *smoothable lci* maps, but we do not need such distinction.

5. If not specified otherwise, G will always denote a closed sub-group scheme inside GL_n for some n.

6. Given a group S-scheme G, we will always denote by $\mathfrak{g}_S^\vee$ the sheaf associated to the co-Lie algebra of G. If the base scheme is clear from the context, we will only write $\mathfrak{g}^\vee$.

7. In [MV99, §4], some ind-schemes were introduced to approximate quotient stacks. For a given algebraic group G, those ind-schemes were denoted in *loc. cit.* as *geometric* classifying spaces $B_{gm}G$. To distinguish between actual quotient classifying stacks $[S/G] \in \mathcal{ASt}_{/S}$, over some base S, and geometric classifying spaces (that are just ind-schemes), we will use a dual notation:

$$\mathcal{B}G := [S/G] \in \mathcal{ASt}_{/S}$$

$$BG := B_{gm}G \in \mathrm{Ind}(\mathbf{Sch}_{/S})$$

   We will come back to these notations, with more details, in Chapter 1. In general, we will try to use the calligraphic font $\mathcal{X}, \mathcal{Y}, \mathcal{B}G$, etc., for algebraic stacks.

8. Given a scheme (or an algebraic stack) X, we will denote its Thomason-Trobaugh K-theory space as $\mathrm{K}(X) = \mathrm{K}(\mathcal{P}erf(X))$, where $\mathcal{P}erf(X)$ is the infinity category of perfect complexes on X. Given $X \in \mathbf{Sch}^G_{/B}$ with associated quotient stack $\mathcal{X} := [X/G]$, the *genuine* equivariant K-theory of $\mathcal{X}$ will be denoted as $\mathrm{K}^G(X) := \mathrm{K}(\mathcal{P}erf^{\mathcal{G}}(X)) = \mathrm{K}(\mathcal{X})$.

9. Given infinity category $\mathcal{C}$, we will denote by $\mathrm{Map}_{\mathcal{C}}(-,-)$ the mapping space of $\mathcal{C}$. If moreover $\mathcal{C}$ is closed monoidal, the internal mapping space of $\mathcal{C}$ we will denoted as $\underline{\mathrm{Map}}_{\mathcal{C}}(-,-)$.

10. When dealing with *bivariant* theories, as known as Borel-Moore homology for us, we will have to take into account *twists* by perfect complexes. For example given a scheme (or an algebraic stack) X and a perfect complex $v \in \mathcal{P}erf(X)$, we can define the v-twisted Borel-Moore homology $\mathbb{E}(X, [v])$ as in definition 1.1.2 (following [DJK21]), where $[v] \in K_0(X) = K_0(\mathcal{P}erf(X))$. When $\mathcal{V}$ is a locally free sheaf with associated class $v := [\mathcal{V}] \in K_0(X)$, to the automorphism in $\mathrm{SH}(X)$ given by *v-suspension* Σ^v, it will then correspond the element $\mathrm{Th}_X(V) := \Sigma^v \mathbb{1}_X$, where $V := \mathbb{V}_X(\mathcal{V}) := \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathcal{V}^\vee))$ is the vector bundle given by $\mathcal{V}$. We will always use calligraphic letters $\mathcal{V}, \mathcal{E}$, etc., for perfect complexes and Roman letters for vector bundles. We are also using calligraphic letters for algebraic stacks, but this should not be source of any confusion since it will be clear from the context to which kind of object we are referring to.

11. We will stick with the conventions and notation of [DJK21] for motivic bivariant theories as much as possible (see later on for some recap). Be aware of the clash of conventions between [Lev19], [Lev22b], [Lev22a] and [DJK21] regarding twists by line bundles: what in the former three papers is denoted $\mathbb{E}^{a,b}(X; L)$ is actually $\mathbb{E}^{a+2,b+1}(\mathrm{Th}_X(L))$, that in the notation of the latter one is $\mathbb{E}^{a+2,b+1}(X; -[\mathcal{L}])$ where $\mathcal{L}$ is the invertible sheaf corresponding to the line bundle L. More precisely we have:

$$\mathbb{E}^{a,b}_{\mathrm{Levine}}(X; L) := \mathbb{E}^{a+2,b+1}(\mathrm{Th}_X(L)) =: \mathbb{E}^{a+2,b+1}_{\mathrm{DJK}}(X, -[\mathcal{L}])$$

where the double index conventions on the right hand side are defined in definition 1.1.2. Although slightly confusing, both notation have their pros and cons. It should be clear what we are using from the context and from the fact that we will always use the semi-colon for one and just a comma for the other (and square brackets if we want to stress that we are considering K-theoretic classes). The rule of the thumb should be: if we are working with cohomology theories and if it is a twist by a line bundle (and not by an invertible sheaf), we are using the $\mathbb{E}_{\mathrm{Levine}}(-;-)$-convention, otherwise we are following [DJK21].

Chapter 1

Equivariant Cohomological Intersection Theory

1.1 Borel-Moore Bivariant Theories and Operations

Let us briefly recall the notions of Borel-Moore motives, bivariant theories and their operations for schemes (for details see [DJK21]). For this section we will drop the square brackets on K-theory classes of vector bundles since there will not be confusion with other notations.

Definition 1.1.1. We will denote by $\mathbb{1}_S \in \mathrm{SH}(S)$ the sphere spectrum. Given any $X \in \mathbf{Sch}_{/S}$, with structure map $\pi_X : X \longrightarrow S$, and given $v \in \mathrm{K}_0(X)$, then we define its twisted Borel-Moore motive over S as:

$$(X/S)^{\mathrm{BM}}(v) := \pi_{X!}(\Sigma^v \pi_X^* \mathbb{1}_S)$$

where Σ^v is the v-suspension functor.

$\left(\begin{array}{c}\textit{Proper}\\ \textit{Pullback}\end{array}\right)$ Given a proper map $p : X \to Y$ in $\mathbf{Sch}_{/S}$ and $v \in \mathrm{K}_0(Y)$, then the unit $\epsilon^*_*(p) : Id \to p_*p^* \simeq p_!p^*$ induces a map:

$$p^* : (Y/S)^{\mathrm{BM}}(v) \longrightarrow (X/S)^{\mathrm{BM}}(p^*v)$$

Indeed, the map p^* is given by the following composition:

$$\pi_{Y!}\Sigma^v\pi_Y^*\mathbb{1}_S \overset{\epsilon^*_*(p)}{\to} \pi_{Y!}p_*p^*\Sigma^v\pi_Y^*\mathbb{1}_S \simeq \pi_{Y!}p_!\Sigma^{p^*v}p^*\pi_Y\mathbb{1}_S \simeq \pi_{X!}\Sigma^{p^*v}\pi_X^*\mathbb{1}_S$$

$\left(\begin{array}{c}\textit{Smooth}\\ \textit{Pushforward}\end{array}\right)$ Given a smooth map $f : X \to Y$ in $\mathbf{Sch}_{/S}$, $v \in \mathrm{K}_0(Y)$, the adjunction co-unit $\eta^*_{\#}(f) : f_\# f^* \to Id$ together with the purity isomorphism $\mathfrak{p}_f : f_! \simeq f_\#\Sigma^{-\mathbb{L}_f}$ induces a map:

$$f_* : (X/S)^{\mathrm{BM}}(\mathbb{L}_f + f^*v) \longrightarrow (Y/S)^{\mathrm{BM}}(v)$$

Indeed, the map f_* is defined as the following composition:

$$\pi_{X!}\Sigma^{f^*v+\mathbb{L}_f}\pi_X^*\mathbb{1}_S \simeq \pi_{Y!}f_!\Sigma^{\mathbb{L}_f}f^*\Sigma^v\pi_Y^*\mathbb{1}_S \overset{\mathfrak{p}_f}{\simeq} \pi_{Y!}f_\# f^*\Sigma^v\pi_Y^*\mathbb{1}_S \xrightarrow{\eta^*_\#(f)} \pi_{Y!}\Sigma^v\pi_Y^*\mathbb{1}_S$$

$\left(\begin{array}{c}\textit{Gysin}\\ \textit{Pushforward}\end{array}\right)$ Given a lci map $f : X \to Y$ in $\mathbf{Sch}_{/S}$, $v \in \mathrm{K}_0(Y)$, via deformation to the normal bundle we get a map:

$$f_! : (X/S)^{\mathrm{BM}}(\mathbb{L}_f + f^*v) \longrightarrow (Y/S)^{\mathrm{BM}}(v)$$

Let $\mathfrak{p}_f : \Sigma^{\mathbb{L}_f}f^* \to f^!$ be the purity transformation of [DJK21, §4.3] and let $\eta_!^!(f) : f_!f^! \to Id$ be the co-unit of the adjunction. Then $f_!$ is defined as the composition:

$$\pi_{X!}\Sigma^{\mathbb{L}_f+f^*v}\pi_X^*\mathbb{1}_S \simeq \pi_{Y!}f_!\Sigma^{\mathbb{L}_f}f^*\Sigma^v\pi_Y^*\mathbb{1}_Y \xrightarrow{\mathfrak{p}_f} \pi_{Y!}f_!f^!\Sigma^v\pi_Y^*\mathbb{1}_S \xrightarrow{\eta_!^!(f)} \pi_{Y!}\Sigma^v\pi_Y^*\mathbb{1}_S$$

Definition 1.1.2. Denote again by $\mathbb{1}_S \in \mathrm{SH}(S)$ the sphere spectrum. Using the same conventions as in [DJK21], for any (unital, commutative) ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$, we can define the following bivariant theories:

(*Cohomology*) For any scheme $\pi : X \to S$, $v \in \mathrm{K}_0(X)$, we define its cohomology (over S) as:

$$\mathbb{E}(X, v) := \underline{\mathrm{Map}}_{\mathrm{SH}(S)}(\mathbb{1}_S, \pi_*(\Sigma^v\pi^*\mathbb{E}))$$

where $\underline{\mathrm{Map}}_{\mathrm{SH}(S)}(-,-)$ is the internal mapping spectrum in $\mathrm{SH}(S)$. Its associated cohomology groups will be:

$$\mathbb{E}^{a,b}(X, v) := \pi_0\left(\Sigma^{a,b}\mathbb{E}(X, v)\right)$$

$\left(\begin{array}{c}\textit{Borel-Moore}\\ \textit{Homology}\end{array}\right)$ Given $\pi : X \in \mathbf{Sch}_{/S}$, $v \in \mathrm{K}_0(X)$, we define its Borel-Moore homology (sometimes referred to also as Borel-Moore bivariant theory) over S as:

$$\mathbb{E}^{\mathrm{BM}}(X/S, v) := \underline{\mathrm{Map}}_{\mathrm{SH}(S)}\left((X/S)^{\mathrm{BM}}(v), \mathbb{E}\right) \simeq \underline{\mathrm{Map}}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \pi_*\left(\Sigma^{-v}\pi^!\mathbb{E}\right)\right)$$

Its associated Borel-Moore homology groups will be:

$$\mathbb{E}^{\mathrm{BM}}_{a,b}(X/S, v) := \pi_0\left(\Sigma^{-a,-b}\mathbb{E}^{\mathrm{BM}}(X/S, v)\right) = \mathrm{Hom}_{\mathrm{SH}(S)}\left(\Sigma^{a,b}(X/S)^{\mathrm{BM}}(v), \mathbb{E}\right)$$

Remark 1.1.3. The operations on Borel-Moore motives, namely proper pullbacks, smooth and Gysin pushforwards, induce on the Borel-Moore theory analogous operations respectively called *proper pushforwards, smooth pullbacks and Gysin morphisms* that will be denoted $f_*, f^*, f^!$ for $f : X \to Y$ a proper, smooth, lci map respectively.

Remark 1.1.4. Recall from [Lev17, Lemma 2.2] that the Gysin pushforward on Borel-Moore motives, defined whenever we have a section $s : X \to V$ of a smooth map $f : V \to X$, is the inverse of the smooth pushforward map if f is a vector bundle. Thus, in case f is a vector bundle, we get for free that the Gysin pushforward $s_!$ on Borel-Moore motives commutes with proper pullbacks and smooth pushforwards since $f_* = (s_!)^{-1}$ does (for example using [Lev17, Lemma 2.3] and functoriality of smooth pushforwards).

Notation 1.1.5. When $\mathbb{E} = \mathbb{1}$, we simply denote its associated (absolute) Borel-Moore bivariant theory, cohomology, homology and compactly supported cohomology simply as $\mathrm{H}^{\mathrm{BM}}(\cdot), \mathrm{H}, \mathrm{H}^c, \mathrm{H}_c$ respectively.

Recall that given any cartesian square in $\mathbf{Sch}_{/B}$:

$$\begin{tikzcd} X_T \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

we get a *base change* map (cf. [DJK21, §2.2.7]) on bivariant theories associated to some spectrum $\mathbb{E}$:

$$\Delta^* : \mathbb{E}^{\mathrm{BM}}(X/S, v) \longrightarrow \mathbb{E}^{\mathrm{BM}}(X_T/T, g^*v)$$

Moreover for a lci map of schemes $f : X \to Y$ and for any motivic ring spectrum $\mathbb{E}$ we can construct fundamental classes $\eta_f^{\mathbb{E}}$ that will satisfy various compatibility and functoriality properties (cf. [DJK21, p. 4.1.4]). The Gysin pushforward defined on Borel-Moore motive induces a Gysin pullback map on Borel-Moore homology that corresponds with the multiplication by the fundamental class:

Theorem 1.1.6 ([DJK21, p. 4.2.1])**.** *Let $\mathbb{E} \in \mathrm{SH}(S)$ a motivic ring spectrum. For any lci map $f : X \to Y$ in $\mathbf{Sch}_{/S}$, and any $v \in \mathrm{K}_0(X)$, there is a Gysin pullback map given by:*

$$\begin{array}{cccc} f^! : & \mathbb{E}^{\mathrm{BM}}(Y/S, v) & \longrightarrow & \mathbb{E}^{\mathrm{BM}}(X/S, [\mathbb{L}_f] + f^*v) \\ & x & \mapsto & \eta_f^{\mathbb{E}} \cdot x \end{array}$$

satisfying functoriality and transverse base change as in [DJK21, p. 2.4.2].

Again from *loc. cit.* let us recall how refined fundamental classes and refined Gysin maps are defined:

Definition 1.1.7 ([DJK21, p. 4.2.5])**.** Suppose we have:

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

a cartesian square of in $\mathbf{Sch}_{/S}$ where f is lci.

$\left(\begin{array}{c}\textit{Refined}\\ \textit{Fundamental}\\ \textit{Class}\end{array}\right)$ The refined fundamental class with respect to the square Δ, with coefficients in $\mathbb{E}$, is given by:

$$\eta_{\Delta}^{\mathbb{E}} := \Delta^*(\eta_f^{\mathbb{E}}) \in \mathbb{E}^{\mathrm{BM}}\left(X'/Y', q^*[\mathbb{L}_f]\right)$$

$\left(\begin{array}{c}\textit{Refined}\\ \textit{Gysin Map}\end{array}\right)$ Similarly to theorem 1.1.6, we get a *refined Gysin map* defined as:

$$\begin{array}{rccc} g_{\Delta}^{!}: & \mathbb{E}^{\mathrm{BM}}(Y'/S, w) & \longrightarrow & \mathbb{E}^{\mathrm{BM}}(X'/S, q^*[\mathbb{L}_f] + g^*w) \\ & x & \mapsto & \eta_{\Delta}^{\mathbb{E}} \cdot x \end{array}$$

for $w \in \mathrm{K}_0(Y')$.

Remark 1.1.8. If the cartesian square of definition 1.1.7 is Tor-independent, then this means that both horizontal arrows are lci, and the refined Gysin map $g_{\Delta}^{!} = g^{!}$ is just the usual Gysin map by transverse base change [DJK21, 4.2.6(iii)]. In particular this holds when the vertical maps are identities and then the refined Gysin map will just be the usual Gysin map.

Remark 1.1.9. If it is clear from the context, we will omit the pullbacks from the twists in Borel-Moore homology and cohomology.

Refined Gysin maps satisfy some important compatibilities and functoriality properties that we will need later on, in particular we have:

Proposition 1.1.10 ([DJK21])**.** *Consider $R \in \mathbf{Sch}_{/S}$ and $v \in \mathrm{K}_0(R)$ and consider the following diagram of cartesian squares in $\mathbf{Sch}_{/S}$:*

$$\begin{tikzcd} Z \arrow[r, "q_2"] \arrow[rr, bend left, "q"] \arrow[d, "h"'] \arrow[dr, phantom, "\Delta_2"] & X \arrow[r, "q_1"] \arrow[d, "g"] \arrow[dr, phantom, "\Delta_1"] & T \arrow[d, "f"] \\ W \arrow[r, "p_2"'] \arrow[rr, bend right, "p"'] & Y \arrow[r, "p_1"'] & R \end{tikzcd} \qquad (\star)$$

where we will denote as Δ the big cartesian square given by composing Δ_1, Δ_2. Then:

1. (*Compatibility*) *Suppose f is lci:*

1a. $\left(\begin{array}{c}\textit{Proper}\\ \textit{Pushforward}\end{array}\right)$ *Suppose p_2 is proper, then the following diagram commutes (in the ∞-category* $\mathrm{SH}(S)$*):*

$$\begin{tikzcd} \mathbb{E}^{\mathrm{BM}}(W/S,v) \arrow[r, "h^!_\Delta"] \arrow[d, "p_{2*}"'] & \mathbb{E}^{\mathrm{BM}}(Z/S,v+\mathbb{L}_f) \arrow[d, "q_{2*}"] \\ \mathbb{E}^{\mathrm{BM}}(Y/S,v) \arrow[r, "g^!_{\Delta_1}"'] & \mathbb{E}^{\mathrm{BM}}(X/S,v+\mathbb{L}_f) \end{tikzcd}$$

where we omitted the pullbacks from the classes in K-theory to lighten the notation.

1b. $\begin{pmatrix} \textit{Smooth} \\ \textit{Pullback} \end{pmatrix}$ *Suppose p_2 is smooth, then the following diagram commutes (in the ∞-category* $\mathrm{SH}(S)$*)::*

$$\begin{tikzcd} \mathbb{E}^{\mathrm{BM}}(Y/S,v) \arrow[r, "g^!_{\Delta_1}"] \arrow[d, "p_2^!"'] & \mathbb{E}^{\mathrm{BM}}(X/S,v+\mathbb{L}_f) \arrow[d, "q_2^!"] \\ \mathbb{E}^{\mathrm{BM}}(W/S,v+\mathbb{L}_{p_2}) \arrow[r, "h^!_\Delta"'] & \mathbb{E}^{\mathrm{BM}}(Z/S,v+\mathbb{L}_f+\mathbb{L}_{p_2}) \end{tikzcd}$$

where we omitted the pullbacks from the classes in K-theory to lighten the notation.

2.(Functoriality) Consider the same diagram given in eq. ($\star$) and suppose now that p_1, p_2 are lci, then:

$$\left(q^!_{2,\Delta_2} \circ q^!_{1,\Delta_1}\right) \simeq q^!_\Delta : \mathbb{E}^{\mathrm{BM}}(Z/S,v) \longrightarrow \mathbb{E}^{\mathrm{BM}}(T/S,v+\mathbb{L}_p)$$

3. $\begin{pmatrix} \textit{Excess} \\ \textit{Intersection} \end{pmatrix}$ *Consider the cartesian square Δ in eq. ($\star$), suppose p, q are lci and that factor p as a regular closed immersion p_2 and a smooth map p_1. Then we have an excess bundle ξ given by the quotient bundle obtained through the natural map of normal bundles $N_{Z/X} \to h^* N_{W/Y}$. Then given any diagram of cartesian squares like:*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "h'"'] \arrow[dr, phantom, "\ulcorner\ \Delta_3"] & Z \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\ulcorner\ \Delta"] & T \arrow[d, "f"] \\ W' \arrow[r] & W \arrow[r] & R \end{tikzcd}$$

where the outer square will be denoted as Δ_4. Then we have, up to homotopy, that:

$$h'^!_{\Delta_4} \simeq e^{\mathbb{E}}(\xi) \cdot h'^!_{\Delta_3}$$

with $e^{\mathbb{E}}(\xi)$ the $\mathbb{E}$-Euler class of ξ.

4. $\left(\begin{array}{c}\textit{Fundamental}\\ \textit{Classes,}\\ \textit{Poincaré}\\ \textit{Duality}\end{array}\right)$ *For $\pi_X : X \to S \in \mathbf{Sch}_{/S}$ a lci S-scheme, we can define the $\mathbb{E}$-fundamental class of X by:*

$$[X]_{\mathbb{E}} := \pi_X^! [S]_{\mathbb{E}} = \eta_{\pi_X}^{\mathbb{E}}$$

where $[S]_{\mathbb{E}} \in \mathbb{E}^{0,0}(S) \simeq \mathbb{E}_{0,0}^{\mathrm{BM}}(S/S)$ is the class induced by $Id_{\mathbb{1}_S} \in \mathrm{H}^{0,0}(S)$ under the unit map $\mathbb{1}_S \to \mathbb{E}$.

(i) *For $f : Y \to X$ an lci map in $\mathbf{Sch}_{/S}$, we have $f^![Y]_{\mathbb{E}} = [X]_{\mathbb{E}}$.*

(ii) *For $X \in \mathbf{Sm}_{/S}$, $\mathbb{E} \in \mathrm{SH}(S)$ and $v \in \mathrm{K}_0(X)$, the cap product:*

$$[X]_{\mathbb{E}} \cap \cdot : \mathbb{E}^{a,b}(X, v) \longrightarrow \mathbb{E}_{-a,-b}^{\mathrm{BM}}\left(X/S, \mathbb{L}_{X/S} - v\right)$$

defines an isomorphism, the same one induced by the purity transformation.

Proof. The compatibilities and functoriality follow from [DJK21, 4.2.6(i), 4.1.4]. The excess intersection formula is [DJK21, 4.2.6(ii)]. The last statement is [DJK21, 4.3.8(i), 4.3.9]. □

Recall from [DJK21, §3.1] that the Thom space construction on a scheme $X \in \mathbf{Sch}_{/S}$ is functorial with respect to monomorphisms of vector bundles. In particular, associated to the zero section $s : X \hookrightarrow E$ of a vector bundle E on X, we get a map in pointed motivic spaces $\mathrm{H}_+(X)$:

$$s_* : X_+ \longrightarrow \mathrm{Th}_X(E)$$

Definition 1.1.11 ([DJK21, Def.3.1.2])**.** We will call *Euler class* the map:

$$s_* : X_+ \longrightarrow \mathrm{Th}_X(E)$$

and we will denote it as $e(E)$. Under the natural map:

$$\mathrm{Map}_{\mathrm{H}_+(X)}(X_+, \mathrm{Th}_X(E)) \longrightarrow \mathrm{Map}_{\mathrm{SH}(X)}(\mathbb{1}_X, \mathrm{Th}_X(E))$$

We will often interpret the Euler class as an element in twisted cohomology $e(V) \in \mathrm{H}(X, [\mathcal{E}])$ where $\mathcal{E}$ is the locally free sheaf associated to E.

Remark 1.1.12. We will see later on a similar, but different notion of Euler classes for SL-oriented theories. Under appropriate Thom isomorphism these different Euler classes will coincide (when both are defined), so there is no need to use different names to distinguish between them. The definition we just gave, taken from [DJK21], works without assuming any kind of orientation.

1.2 Geometric Approximations

We will now work over a base scheme S and we will denote by GL_n the group scheme $GL_{n,S} := GL_n \times_{\mathbb{Z}} S$ defined over S, where GL_n is the usual group scheme of invertible $(n \times n)$-matrices defined over $\mathrm{Spec}(\mathbb{Z})$. Whenever we will encounter a group scheme G over S, we will always assume that G is smooth.
We describe a version of the construction of the classifying space of G, found in [MV99, §4.2]. We can consider $V_m = \mathbb{A}_S^{n(n+m)} \simeq \mathrm{Hom}(\mathbb{A}_S^{n+m}, \mathbb{A}_S^n)$ equipped with the natural (left) GL_n-action (hence we also have an induced natural G-action). Once we identify V_m with the scheme of $(n, n+m)$-matrices, we can restrict to the open subset $U_m \subseteq V_m$ made of those matrix with rank n. On U_m we have a free GL_n-action (hence a free G-action too) . We can define a map $s_m : U_m \hookrightarrow U_{m+1}$ sending an element $B \in U_m$ to a matrix of the form:

$$\left(\begin{array}{c|c} & 0 \\ B & \vdots \\ & 0 \end{array} \right)$$

The image of this map will be a closed subscheme factoring through the open $U^\circ_{m+1} \subseteq U_{m+1}$ formed by matrices $A \in U_{m+1}$ such that the submatrix A_1 obtained forgetting the $(n+m+1)^{th}$-column has rank n. The map that forgets the last row $p_m : U^\circ_{m+1} \longrightarrow U_m$ is a vector bundle with zero section the map s_m just described above. In this way we get that the inclusion $U_m \hookrightarrow U_{m+1}$ factors as:

$$U_m \overset{s_m}{\hookrightarrow} U^\circ_{m+1} \overset{u^\circ_{m+1}}{\hookrightarrow} U_{m+1} \tag{1.1}$$

where s_m is the zero section of p_m and u°_{m+1} denote the open immersion of $U^\circ_{m+1} \subseteq U_{m+1}$.

Notation 1.2.1. Since we will closely follow [Lev22a] and [Lev22b], we will denote from now on $E_mG = E_mSL_n = E_mGL_n := U_m$ the open subscheme of $M_{n,n+m} \simeq \mathbb{A}_S^{n(n+m)}$ we saw before, given by matrices of maximal rank n. We will choose as a base point $x_0 = (I_n, 0_n, \dots, 0_n)$ where I_n is the $(n \times n)$ identity matrix and 0_n is the zero vector column of length n. We will denote by $ESL_n = EGL_n$ the presheaf on $\mathbf{Sm}_S$ given by $\operatorname{colim}_m E_mSL_n$. For any closed subgroup G of GL_n or SL_n, we will denote the quotient scheme $B_mG := E_mGL_n/G$ whose limit gives us the approximation $BG := \operatorname{colim}_m B_mG$ for the quotient stack $\mathcal{B}G := [S/G]$. For $G = GL_n$, the schemes $B_mGL_n \simeq \mathrm{Gr}_S(n, n+j)$ are represented by the Grassmannians; for a general smooth closed subgroup $G \subseteq GL_n$ we still have that B_jG exists as a quasi-projective scheme.

Remark 1.2.2. Considering $F \simeq \mathbb{A}_S^n$ the fundamental representation of GL_n, we again have that $\iota_j : B_mG \hookrightarrow B_{m+1}G$ factors as in (1.1) through a zero section of a vector bundle $B^\circ_m G := (E_m G \times F)\,/G$, followed by an open immersion $\iota^\circ_m : B^\circ_m G \hookrightarrow B_{m+1}G$. We will denote $E^\circ_m G := E_m G \times F$.

Remark 1.2.3. It is not difficult to see that if G is smooth over S, the schemes $B_m G$ are smooth over S too (cf. [Lev22a, Lemma 4.1]). Moreover the construction of the BG's depends on some choices, like the choice of the *admissible gadget* used to define them. But it was proved in [MV99, §4.3] that the resulting object of $\mathrm{H}(S)$ is independent of all the choices made.

Definition 1.2.4. We will denote by $\mathbf{Sch}^G_{q/S}$ the full subcategory of $\mathbf{Sch}^G_{/S}$ made by schemes X such that the fppf quotient $(X \times E_m G)\,/G$ is representable by an element of $\mathbf{Sch}_{/S}$. If $S = \mathrm{Spec}(\Bbbk)$, then by [EG98a, Proposition 23] we have $\mathbf{Sch}^G_{q/\Bbbk} = \mathbf{Sch}^G_{/S}$. In general any quasi-projective scheme X with a G-linearised action will be in $\mathbf{Sch}^G_{q/S}$ by [MFK94, Proposition 7.1].

Remark 1.2.5. Given a G-scheme $X \in \mathbf{Sch}^G_{q/S}$, we can consider $Y_m := X \times^G E_m G := (X \times E_m G)\,/G$, with inclusions $y_m : Y_m \hookrightarrow Y_{m+1}$. Using the factorization in (1.2.2), we get an induced factorization of y_m as:

$$Y_m \overset{\sigma_m}{\hookrightarrow} Y^\circ_{m+1} := X \times E^\circ_{m+1}G/G \overset{y^\circ_{m+1}}{\hookrightarrow} X_{m+1} = X \times E_{m+1}G/G$$

where σ_m is a zero section of a vector bundle and y°_{m+1} is an open immersion.

Notation 1.2.6. Given a $X \in \mathbf{Sch}^G_{q/S}$, with G a closed subgroup of GL_n or SL_n, we will denote $X_m := X \times^G E_m GL_n$ and as an approximation for $\mathcal{X} := [X/G]$ we can consider the presheaf on $\mathbf{Sm}_S$ given by the ind-scheme $\widetilde{X} := \operatorname{colim}_m X_m$. We will abuse the notation and also denote $\widetilde{X}$ the image of the presheaf under motivic localization, i.e. as a motivic space or spectra in $\mathrm{H}(S)$ or $\mathrm{SH}(S)$. We will often refer to the X_m's as the *finite-level approximations* of $\widetilde{X}$, or the *Totaro approximations* of $\widetilde{X}$.

1.3 Equivariant Borel-Moore Homology and Cohomology

1.3.1 Six Operations for Algebraic Stacks

To define and talk about equivariant Borel-Moore motives and homology for $X \in \mathbf{Sch}^G_{q/S}$, we will now use the limit extended motivic homotopy category $\mathrm{SH}^{\triangleleft}([X/G])$ recently constructed in [KR21] and in [Cho21a]. The two different approaches in [Cho21a] and [KR21] agree when both are defined (cf. [KR21, Corollary 12.28] or [Cho21b, Corollary 2.5.4]), so we will use results from both sides without making any explicit distinction. For our purposes we will only need the limit extended functor $\mathrm{SH}^{\triangleleft}$, and not the genuine theory developed in [Hoy17], thus we will drop the superscript $\triangleleft$ from the notation.

In [Cho21a] the author introduced the notion of maps, between algebraic stacks, admitting sections Nisnevich-locally and used these to extend the motivic category to those algebraic stacks that admit a Nisnevich-local cover. We will briefly recall some notions and some of the main features of SH for algebraic stacks following *loc.cit.*.

Definition 1.3.1. 1. Let $f : X \to Y$ be a map between schemes, we say that f admits Nisnevich local sections if there exists a Nisnevich cover $g : Y' \to Y$ and a map $s : Y' \to X$ such that the following commutes:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y' \arrow[ur, dashed, "s"] \arrow[r, "g"] & Y \end{tikzcd}$$

2. Given an algebraic stack $\mathcal{X} \in \mathcal{ASt}$, we say that $\mathcal{X}$ is a *NL*-stack (for "Nisnevich Local") if it admits a smooth atlas $x : X \to \mathcal{X}$, where X is a scheme, such that for any scheme Y and any map $Y \to \mathcal{X}$, the map:

$$Y \times_{\mathcal{X}} X \to Y$$

has Nisnevich local sections. We call this atlas a NL-atlas for $\mathcal{X}$ and we denote the category of NL-stacks as $\mathcal{ASt}^{\mathrm{NL}}$.

3. A map $f : \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathcal{ASt}^{\mathrm{NL}}$ is said to admit Nisnevich local sections if there exists a NL-atlas $y : Y \to \mathcal{Y}$ such that we have the following commutative diagram:

$$\begin{tikzcd} & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[ur, dashed, "s"] \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

The Grothendieck topology on $\mathcal{ASt}^{NL}$ generated by NL-maps will be called the NL-topology τ_{NL}.

Remark 1.3.2. If we restrict to schemes (or even algebraic spaces), the NL-topology is equivalent to the classical Nisnevich topology.

Denote by $\mathcal{Pr}^{\mathrm{L}}_{stb}$ the ∞-category whose objects are presentable stable ∞-categories, and whose morphisms are colimit-preserving functors. Let $\mathcal{CAlg}(\mathcal{Pr}^{\mathrm{L}}_{stb})$ be the ∞-category of presentable stable symmetric monoidal ∞-categories. In [Cho21a, Theorem 5.5.1], via NL-descent, the following ∞-functor was constructed:

$$\begin{array}{cccc} \mathrm{SH}^* : & (\mathcal{ASt}^{\mathrm{NL}})^{op} & \longrightarrow & \mathcal{CAlg}(\mathcal{Pr}^{\mathrm{L}}_{stb}) \\ & \mathcal{X} & \mapsto & \mathrm{SH}(\mathcal{X}) \\ & f & \mapsto & f^* \end{array}$$

that, informally speaking, sends an object $\mathcal{X}$ to its motivic homotopy category $\mathrm{SH}(\mathcal{X})$ and sends a map $f : \mathcal{X} \to \mathcal{Y}$ to the monoidal pullback functor $f^* : \mathrm{SH}(\mathcal{Y}) \to \mathrm{SH}(\mathcal{X})$. Together with the pullback map f^*, we have its right adjoint f_* for any map in $\mathcal{ASt}^{NL}$, the tensor-hom adjunction $- \otimes - \dashv \underline{\mathrm{Map}}(-,-)$ and for any representable map f we get a pair of adjoint exceptional functors $f_! \dashv f^!$.
For any representable smooth map $f : \mathcal{X} \to \mathcal{Y}$, a left adjoint$f_\#$ to f^* was also constructed. We will now show that with little to no effort we can extend this construction to any smooth map:

Lemma 1.3.3. *Given $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{ASt}^{\mathrm{NL}}$, the pullback functor f^* admits a left adjoint $f_\#$.*

Proof. This is widely known, we will write it here only for completeness, but no claim on originality is made.
Since SH^* lands in the ∞-category of stable presentable ∞-categories $\mathcal{Pr}_{stb}$, to check that f^* admits a left adjoint we just need to check that f^* preserves limits, by the Adjoint Functor Theorem [HTT, Corollary 5.5.2.9]. Let $x : X \to \mathcal{X}$ be a NL-atlas. Then since x^* is conservative by [Cho21a, Lemma 5.1.1], to check that f^* preserves limits it is the same as checking that x^*f^* preserves limits. But $f \circ x : X \to \mathcal{Y}$ is a smooth map from a scheme to an algebraic stack, hence it is smooth representable and by [Cho21a, Proposition 5.1.2] we know x^*f^* has a left adjoint. Again by the Adjoint Functor Theorem, this means that x^*f^* preserves limits. Hence f_* has a left adjoint $f_\#$. □

We can then summarise the properties of SH in the following:

Theorem 1.3.4 ([Cho21a, Theorem 5.5.1])**.** *We have a functor:*

$$\mathrm{SH}^* : \left(\mathcal{ASt}^{\mathrm{NL}}\right)^{op} \longrightarrow \mathcal{Pr}^{\mathrm{L}}_{stb}$$

extending the usual functor defined on schemes.

1. *For every $\mathcal{X}$ NL-stack, we have the tensor- hom adjunction in $\mathrm{SH}(\mathcal{X})$:*

$$- \otimes - \dashv \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{X})}(-,-)$$

2. *For any map $f : \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathcal{ASt}^{\mathrm{NL}}$, we have a pair of adjoint functors:*

$$f^* \dashv f^*$$

and if f is smooth we also have:

$$f_\# \dashv f^* \dashv f_*$$

3. *For $f : \mathcal{X} \longrightarrow \mathcal{Y}$ a representable, separated, of finite type map in $\mathcal{ASt}^{\mathrm{NL}}$, we have another pair of adjoint functors:*

$$f_! \dashv f^!$$

Moreover these functors, when defined, satisfy the usual compatibilities including projection formulas, (representable) smooth and proper base change, (representable) purity isomorphism and localization triangles.

Remark 1.3.5. In this thesis we will not make use of the exceptional functors $f_!, f^!$ for non-representable morphisms. We will come back to the construction and the properties of those functors in a forthcoming work joint with C. Chowdhury [CD23]. We are very thankful to C. Chowdhury for all the conversations the author had with him about the motivic homotopy theory of stacks. Most of the results in this section are restatements of results already in the literature, but any new results found in this section should be considered as a joint work.

The natural transformations of *base change* $Ex^*_!$, *smooth base change* $Ex^*_{\#}$, and *proper base change* Ex^*_* were constructed respectively in [Cho21a, Theorem 4.1.1, Proposition 5.1.2, Proposition 5.1.4]. We are still missing the natural transformations $Ex_{\#*}$, $Ex_{!*}$, $Ex^{*!}$ and $Ex^!_*$ that we have at our disposal for schemes (see for example [Hoy17, Proposition 6.12 and §6.2]), but the hard work is already done for us: we will deduce all the missing exchange transformation (for representable maps) from the one in [Cho21a].

Proposition 1.3.6. *Consider the following cartesian diagram of representable maps in $\mathcal{ASt}^{NL}$:*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & \mathcal{Y} \arrow[d, "p"] \\ \mathcal{Z} \arrow[r, "f"'] & \mathcal{X} \end{tikzcd}$$

(i) *Suppose f is smooth and p is proper. Then we have a natural exchange transformation:*
$$Ex_{\#*} : f_\# q_* \longrightarrow p_* g_\#$$
and moreover this is an equivalence.

(ii) *If p and q are separated of finite type, then we have an exchange equivalence:*
$$Ex^!_* : p^! f_* \simeq g_* q^!$$

(iii) *If p is separated of finite type, then we have a natural transformation:*
$$Ex_{!*} : p_! g_* \longrightarrow f_* q_!$$
If f is proper the above transformation is an equivalence.

(iv) If p is separated of finite type, then we have a natural transformation:

$$Ex^{*!} : g^*p^! \longrightarrow q^!f^*$$

If f is smooth the above transformation is an equivalence.

(v) If p, q are separated of finite type and f, g are smooth, we have a natural exchange transformation:

$$Ex_{!\#} : f_\# q_! \longrightarrow p_! g_\#$$

and moreover this is an equivalence.

Proof. The exchange transformation in (ii) is obtained via adjunctions from $Ex_!^*$ and so we get the result from [Cho21a, Theorem 4.1.1].

The construction of the natural transformations is exactly like in [Hoy17, §6] and in [CD19, §2.4]. For (i), $Ex_{\#*}$ is given by the following composition:

$$Ex_{\#*} : f_\# q_* \xrightarrow{\eta_*^*(p)} p_*p^*f_\# q_* \overset{Ex_\#^*}{\simeq} p_*g_\# q^* q_* \xrightarrow{\epsilon_*^*(q)} p_*g_\#$$

For (iii) and (iv), the exchange transformations are given by:

$$Ex_{!*} : p_!g_* \xrightarrow{\eta_*^*(f)} f_*f^*p_!g_* \overset{Ex_!^*}{\simeq} f_*q_!g^*g_* \xrightarrow{\epsilon_*^*(g)} f_*q_!$$

$$Ex^{*!} : g^*p^! \xrightarrow{\eta_*^*(f)} g^*p^!f_*f^* \overset{Ex_*^!}{\simeq} g^*g_*q^!f^* \xrightarrow{\epsilon_*^*(g)} q^!f^*$$

For (v), the exchange transformation $Ex_{!\#}$ is given by the composition:

$$Ex_{!\#} : f_\# q_! \xrightarrow{\epsilon_\#^*(g)} f_\# q_! g^* g_\# \overset{Ex_!^*}{\simeq} f_\# f^* p_! g_\# \xrightarrow{\eta_\#^*(f)} p_! g_\#$$

Now we only need to prove that the exchange transformations are equivalences (under the appropriate hypothesis).

Consider the following diagram:

$$\begin{tikzcd}
W \arrow[rr, "\tilde{g}"] \arrow[dr, "\tilde{q}"] \arrow[dd, "w"'] & & Y \arrow[dr, "\tilde{p}"] \arrow[dd, "y"] & \\
& Z \arrow[rr, "\tilde{f}"] \arrow[dd, "z"] & & X \arrow[dd, "x"] \\
\mathcal{W} \arrow[rr, "g"] \arrow[dr, "q"] & & \mathcal{Y} \arrow[dr, "p"] & \\
& \mathcal{Z} \arrow[rr, "f"'] & & \mathcal{X}
\end{tikzcd} \tag{1.2}$$

where every square is cartesian and the vertical maps x, y, w, z are NL-atlases of $\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z}$ respectively. The induced maps between the atlases will be denoted by $\tilde{f}, \tilde{g}, \tilde{p}, \tilde{q}$ and notice that every single map in the diagram is representable.

(i) Let us start proving that $Ex_{\#*}$ is an equivalence. Assume that f is smooth and p is proper. By the conservativity of x^*, proving that:

$$Ex_{\#*} : f_\# q_* \xrightarrow{\eta^*_*(p)} p_* p^* f_\# q_* \overset{Ex^*_\#}{\simeq} p_* g_\# q^* q_* \xrightarrow{\epsilon^*_*(q)} p_* g_\# \tag{1.3}$$

is an equivalence, is the same as proving that:

$$x^* Ex_{\#*} : x^* f_\# q_* \xrightarrow{x^*\eta^*_*(p)} x^* p_* p^* f_\# q_* \overset{x^* Ex^*_\#}{\simeq} x^* p_* g_\# q^* q_* \xrightarrow{x^*\epsilon^*_*(q)} x^* p_* g_\# \tag{1.4}$$

is an equivalence. But we can rewrite $x^* Ex_{\#*}$ as:

$$\begin{tikzcd}
x^* f_\# q_* \arrow[r, "\epsilon^*_*(p)"] \arrow[d, "\wr\, Ex^*_\#"'] \arrow[ddr, "\epsilon^*_*(\tilde{p})"] & x^* p_* p^* f_\# q_* \arrow[r, phantom, "\overset{Ex^*_\#}{\simeq}"] \arrow[d, "\wr\, Ex^*_*"] & x^* p_* g_\# q^* q_* \arrow[r, "\eta^*_*(q)"'] \arrow[d, "\wr\, Ex^*_*"] & x^* p_* g_\# \arrow[d, "\wr\, Ex^*_*"] \\
\tilde{f}_\# z^* q_* \arrow[d, "\wr\, Ex^*_*"'] \arrow[ddr, "\epsilon^*_*(\tilde{p})"] & \tilde{p}_* y^* p^* f_\# q_* \arrow[r, phantom, "\overset{Ex^*_\#}{\simeq}"] \arrow[d, "\wr"] & \tilde{p}_* y^* g_\# q^* q_* \arrow[r, "\eta^*_*(q)"'] \arrow[d, "\wr\, Ex^*_\#"] & \tilde{p}_* y^* g_\# \arrow[d, "\wr\, Ex^*_\#"] \\
\tilde{f}_\# \tilde{q}_* w^* \arrow[ddr, "\epsilon^*_*(\tilde{p})"] & \tilde{p}_* \tilde{p}^* x^* f_\# q_* \arrow[d, "\wr\, Ex^*_\#"] & \tilde{p}_* \tilde{g}_\# w^* q^* q_* \arrow[r, "\eta^*_*(q)"'] \arrow[d, "\wr"] & \tilde{p}_* \tilde{g}_\# w^* \arrow[dd, equal] \\
 & \tilde{p}_* \tilde{p}^* \tilde{f}_\# z^* q_* \arrow[r, phantom, "\overset{Ex^*_\#}{\simeq}"] \arrow[d, "\wr\, Ex^*_*"] & \tilde{p}_* \tilde{g}_\# \tilde{q}^* z^* q_* \arrow[d, "\wr\, Ex^*_*"] & \\
 & \tilde{p}_* \tilde{p}^* \tilde{f}_\# \tilde{q}_* w^* \arrow[r, phantom, "\overset{Ex^*_\#}{\simeq}"] & \tilde{p}_* \tilde{g}_\# \tilde{q}^* \tilde{q}_* w^* \arrow[r, "\eta^*_*(\tilde{q})"'] & \tilde{p}_* \tilde{g}_\# w^*
\end{tikzcd}$$

We can fill each cell of the diagram by the naturality of the adjunctions and the exchange transformations we already have. Hence looking at the bottom row of the big diagram above, $x^* Ex_{\#*}$ in (1.4) becomes:

$$x^* Ex_{\#*} : \tilde{f}_\# \tilde{q}_* w^* \xrightarrow{\epsilon^*_*(\tilde{p})} \tilde{p}_* \tilde{p}^* \tilde{f}_\# \tilde{q}_* w^* \overset{Ex^*_\#}{\simeq} \tilde{p}_* \tilde{g}_\# \tilde{q}^* \tilde{q}_* w^* \xrightarrow{\eta^*_*(\tilde{q})} \tilde{p}_* \tilde{g}_\# w^* \tag{1.5}$$

But we already know, from [Hoy17, Proposition 6.12], that the smooth-proper exchange transformation (induced by the top square made by the atlases in (1.2)):

$$Ex_{\#*} : \tilde{f}_\# \tilde{q}_* \xrightarrow{\epsilon^*_*(\tilde{p})} \tilde{p}_* \tilde{p}^* \tilde{f}_\# \tilde{q}_* \overset{Ex^*_\#}{\simeq} \tilde{p}_* \tilde{g}_\# \tilde{q}^* \tilde{q}_* \xrightarrow{\eta^*_*(\tilde{q})} \tilde{p}_* \tilde{g}_\#$$

is an equivalence. Thus (1.5) is an equivalence too and, by conservativity of x^*, we get that the exchange transformation:

$$Ex_{\#*} : f_\# q_* \xrightarrow{\eta^*_*(p)} p_* p^* f_\# q_* \overset{Ex^*_\#}{\simeq} p_* g_\# q^* q_* \xrightarrow{\epsilon^*_*(q)} p_* g_\#$$

is an equivalence as well.

(ii) As we already said, that $Ex^!_*$ is an equivalence follows by adjunction from the fact that $Ex^*_!$ is an equivalence as proved in [Cho21a, Theorem 4.1.1] .

(iii) Use the same notation of (1.2). Suppose p is separated of finite type and f is proper. To show that $Ex_{!*}$ is an equivalence, by conservativity of x^*, it is enough to show that:

$$x^* Ex_{!*} : x^* p_! g_* \longrightarrow x^* f_* q_!$$

is an equivalence. Similarly to what we did for (ii), it is possible to rewrite $x^* Ex_{!*}$ as:

$$\tilde{p}_! \tilde{g}_* w^* \xrightarrow{Ex_{!*} w^*} \tilde{f}_* \tilde{q}_! w^*$$

and this is an equivalence since $Ex_{!*} : \tilde{p}_! \tilde{g}_* \xrightarrow{\sim} \tilde{f}_* \tilde{q}_!$ by [Hoy17, §6.2].

(iv) Use the same notation as in the diagram (1.2). Suppose p is separated of finite type and f is smooth. To show that $Ex^{*!}$ is an equivalence, by conservativity of x^*, it is enough to show that:

$$x^* Ex^{*!} : x^* g^* p^! \longrightarrow x^* q^! f^*$$

is an equivalence. Once again, it is possible to rewrite $x^* Ex_{!*}$ as:

$$\tilde{g}_! \tilde{p}_* w^* \xrightarrow{Ex_{!*} w^*} \tilde{q}_* \tilde{f}_! w^*$$

and this is an equivalence since $Ex_{!*} : \tilde{g}_! \tilde{p}_* \xrightarrow{\sim} \tilde{q}_* \tilde{f}_!$ by [Hoy17, §6.2].

(v) Use the same notation of (1.2). Suppose p, q are separated of finite type and f, g are smooth. Similarly to all the other cases, we only need to show that:

$$x^* Ex_{!\#} : x^* f_\# q_! \longrightarrow x^* p_! g_\#$$

is an equivalence. We can rewrite $x^* Ex_{!\#}$ as:

$$\tilde{f}_\# \tilde{q}_! w^* \xrightarrow{Ex_{!\#} w^*} \tilde{p}_! \tilde{g}_\# w^*$$

But $Ex_{!\#} : \tilde{f}_\# \tilde{q}_! \longrightarrow \tilde{p}_! \tilde{g}_\#$ is already an equivalence by [CD19, Theorem 2.4.26].

□

Borel J-Homomorphism

Let $\mathcal{X}$ be an NL-stack and let $\mathcal{E}$ be a locally free sheaf on $\mathcal{X}$. Denote by $p : E \to \mathcal{X}$ be a vector bundle associated to $\mathcal{E}$, with zero section given by $s : \mathcal{X} \to E$. Notice that both p and s are representable maps. Following [Hoy17, §5.2], we have adjoint functors:

$$p_{\#}s_* : \mathrm{SH}(\mathcal{X}) \leftrightarrows \mathrm{SH}(\mathcal{X}) : s^!p^*$$

Lemma 1.3.7. *The adjoint functors:*

$$p_{\#}s_* : \mathrm{SH}(\mathcal{X}) \leftrightarrows \mathrm{SH}(\mathcal{X}) : s^!p^*$$

are equivalence of ∞-categories, inverse to one another.

Proof. Choose a NL-atlas $x : X \to \mathcal{X}$ and consider:

$$\begin{tikzcd} E_X \arrow[r, "p_X"'] \arrow[d, "y"'] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\Delta"] & X \arrow[l, bend right, "s_X"'] \arrow[d, "x"] \\ E \arrow[r, "p"] & \mathcal{X} \arrow[l, bend left, "s"] \end{tikzcd}$$

Pulling back via x^* the unit and co-unit of the adjunction $(p_{\#}s_* \dashv s^!p^*)$, we get:

$$x^*\epsilon : x^* \longrightarrow x^*(s^!p^*)(p_{\#}s_*) \simeq (s^!_X p^*_X)(p_{X\#}s_{X*})x^*$$

$$x^*\eta : x^*(p_{\#}s_*)(s^!p^*) \simeq (p_{X\#}s_{X*})(s^!_X p^*_X)x^* \longrightarrow x^*$$

and both are equivalences since the natural transformations $Id_{\mathrm{SH}(X)} \overset{\sim}{\to} (p_{\#}s_*) \simeq (s^!_X p^*_X)(p_{X\#}s_{X*})$ and $(p_{X\#}s_{X*})(s^!_X p^*_X) \overset{\sim}{\to} Id_{\mathrm{SH}(X)}$ are equivalences themselves. By conservativity of x^*, we get that:

$$\epsilon : Id_{\mathrm{SH}(\mathcal{X})} \longrightarrow (s^!p^*)(p_{\#}s_*)$$

$$\eta : (p_{\#}s_*)(s^!p^*) \longrightarrow Id_{\mathrm{SH}(\mathcal{X})}$$

are equivalences and hence $p_{\#}s_*$ and $s^!p^*$ are inverse to one another. □

Definition 1.3.8. In the same situation as above, the adjoint functors:

$$\Sigma^{\mathcal{E}} := p_{\#}s_* : \mathrm{SH}(\mathcal{X}) \leftrightarrows \mathrm{SH}(\mathcal{X}) : s^!p^* =: \Sigma^{-\mathcal{E}}$$

are equivalences of ∞-categories and are called *Thom transformations.* In particular we have $\Sigma^{\mathcal{E}} \simeq \Sigma^{\mathcal{E}}\mathbb{1}_{\mathcal{X}} \otimes (-)$. Denote by $\mathrm{Pic}(\mathrm{SH}(\mathcal{X}))$ the ∞-category of invertible objects in $\mathrm{SH}(\mathcal{X})$, and denote by:

$$\mathrm{Th}_{\mathcal{X}}(E) := \Sigma^{\mathcal{E}}\mathbb{1}_{\mathcal{X}} \in \mathrm{Pic}(\mathrm{SH}(\mathcal{X}))$$

the *Thom space* of E, with inverse $\Sigma^{-\mathcal{E}}\mathbb{1}_{\mathcal{X}}$.

We can make a further upgrade of the Thom space construction. Thanks to [BH21b, §16.2], we have a J-homomorphism functor:

$$J : \mathrm{K} \longrightarrow \mathrm{Pic}(\mathrm{SH})$$

where $K : \mathbf{Sch}^{qcqs} \to \mathcal{S} \subseteq \mathcal{C}at_\infty$ it the *cdh*-sheaf assigning to each qcqs scheme X the Thomason-Trobaugh K-theory space and sending each map f to f^* (in K-theory), while Pic(SH) is the *cdh*-sheaf that assigns to each scheme the space of invertible objects Pic(SH(X)) and sends a map f to f^*. Since both are *cdh*-sheaves, and in particular Nisnevich sheaves, we can Kan extend J via NL-sheafification to a natural transformation:

$$J^{\triangleleft} : \mathrm{K}^{\triangleleft}(-) \longrightarrow \mathrm{Pic}(SH^{\triangleleft}(-))$$

where $\mathrm{K}^{\triangleleft}$ and $\mathrm{Pic}(SH^{\triangleleft})$ are the functors defined on $\left(\mathcal{ASt}^{NL}\right)^{op}$ via [Cho21a, Theorem 3.4.1].

On the other hand, the genuine K-theory for an algebraic stack $\mathcal{X} \in \mathcal{ASt}$ is defined as:

$$\mathrm{K}\left(\mathcal{X}\right) := \Omega^{\infty}\mathrm{K}\left(\mathcal{P}erf\left(\mathcal{X}\right)\right)$$

where the right hand side is the Thomason-Trobaugh K-theory space of the ∞-category of perfect complexes (cf. [Kha22]). For any map of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, we have a pullback map $f^* : \mathrm{K}(\mathcal{Y}) \to \mathrm{K}(\mathcal{X})$ induced by the pullback at the level of perfect complexes. If $x : X \to \mathcal{X}$ is a NL-atlas, then for any map $f_n : X^n_{\mathcal{X}} := X \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} X \to \mathcal{X}$ in the Čech nerve of the atlas, we will get a pullback map $f_n^* : \mathrm{K}(\mathcal{X}) \to \mathrm{K}(X^n_{\mathcal{X}})$. For any NL-stack $\mathcal{X}$, the space $\mathrm{K}^{\triangleleft}(\mathcal{X})$ is a limit over the Čech nerve of one of his NL-atlases (by construction) and this is functorial in $\mathcal{X}$. Hence by the universal property of the limit we get a canonical map:

$$j : \mathrm{K}(-) \longrightarrow \mathrm{K}^{\triangleleft}(-)$$

between functors $\mathrm{K}, \mathrm{K}^{\triangleleft} : \mathcal{ASt}^{op} \to \mathcal{S}$.

Definition 1.3.9. We define the *Borel J-homomorphism* as:

$$J_{Bor} := J^{\triangleleft} \circ j : \mathrm{K} \longrightarrow \mathrm{Pic}(\mathrm{SH}^{\triangleleft})$$

For a given NL-stack $\mathcal{X}$ and a given $v \in \mathrm{K}_0(\mathcal{X})$, we will denote the associated automorphism of $\mathrm{SH}(\mathcal{X})$ as Σ^v, with inverse Σ^{-v}.

Remark 1.3.10. 1. If $\mathcal{V}$ is a locally free sheaf on a NL-stack $\mathcal{X}$, with associated vector bundle V and with $v := [\mathcal{V}] \in \mathrm{K}_0(\mathcal{X})$, then the J_{Bor} will send v to $\mathrm{Th}_{\mathcal{X}}(V) = \Sigma^v \mathbb{1}_{\mathcal{X}}$ defined before: this is true at the level of schemes and the same claim follows for NL-stacks by uniqueness of Kan extensions.

2. Notice that a fiber sequence in $\mathcal{P}erf(\mathcal{X})$ of the form:

$$\mathcal{E} \to \mathcal{F} \to \mathcal{G}$$

gives us a canonical path $[\mathcal{F}] \simeq [\mathcal{E}] + [\mathcal{G}]$ in the space $K(\mathcal{X})$. Then, by construction of the J_{Bor} natural transformation, we get that:

$$\Sigma^{\mathcal{F}} \simeq \Sigma^{\mathcal{E}}\Sigma^{\mathcal{G}}$$

1.4 Equivariant Bivariant Theories and Totaro Approximations

From now on we can restrict ourselves to very specific NL-stacks: quotient stacks. Fix a base scheme S, a smooth linear algebraic group $G \subseteq GL_{n,S}$ and equip S with the trivial G-action. Let X be any G-scheme such that the fppf quotient $X \times^G GL_n := (X \times GL_n)/G$ is represented by a scheme. Then $[X/G] = [X \times^G GL_n/GL_n]$ is a NL-stack with NL-atlas given by $X \times^G GL_{n,S}$, since $GL_{n,S}$ is special.

Remark 1.4.1. Any quasi-projective scheme X with a G-linearised action gives rise to a NL-stack $[X/G] = [X \times^G GL_n/GL_n]$.

Disclaimer 1.4.2. From now on, even if we will not specify it, we will only consider schemes $X \in \mathbf{Sch}^G_{/S}$ such that $[X/G]$ is a NL-stack. In practice we will just restrict ourselves to quasi-projective schemes with linearised G-action, but this assumption is not strictly necessary in general. Since $E_0G = E_0GL_n = GL_n$, any scheme $\mathbf{Sch}^G_{q/S}$ will give rise to a NL-stack.

Notation 1.4.3. To distinguish between the classifying quotient stack and the geometric model in $\mathrm{H}(S)$, we will use $\mathcal{B}G := [S/G]$ and $BG := \operatorname{colim}_m E_mG/G$. We will see later on that this distinction will not be really necessary when working in $\mathrm{SH}(S)$ and from the second chapter on we will drop the double notation and use the simpler BG for both.

Notation 1.4.4. Let $X \in \mathbf{Sch}^G_{/S}$ and let $\mathfrak{g}^\vee$ be the sheaf associated to the co-Lie algebra of G. Then we denote by $\mathfrak{g}^\vee_X \in \mathbf{QCoh}^G(X)$ the G-linearised sheaf associated to $\mathfrak{g}^\vee \otimes_S \mathcal{O}_X$. With a little abuse of notation, for $X = S$ we will just write $\mathfrak{g}^\vee$ instead of $\mathfrak{g}^\vee_S$, but it should be clear from the context to which sheaf we are referring to.

Definition 1.4.5. Let $g : \mathcal{X} \to \mathcal{B}$ be a representable, separated, of finite type map of NL-stacks and $w \in \mathrm{K}_0(\mathcal{X})$ and let $\mathbb{E} \in \mathrm{SH}(\mathcal{B})$.

(*Borel-Moore Motive*) The twisted Borel-Moore motive over $\mathcal{B}$ is defined as:

$$(\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(w) := g_!\Sigma^w \mathbb{1}_{\mathcal{X}}$$

(Borel-Moore Homology) The twisted Borel-Moore homology over $\mathcal{B}$ is defined as:

$$\mathbb{E}^{\mathrm{BM}}(\mathcal{X}/\mathcal{B}, w) := \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{B})}\left((\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(w), \mathbb{E}\right)$$

The BM-homology groups will be then defined as:

$$\mathbb{E}^{\mathrm{BM}}_{a,b}(\mathcal{X}/\mathcal{B}, w) := \pi_0\left(\Sigma^{-a,-b}\mathbb{E}^{\mathrm{BM}}(\mathcal{X}/\mathcal{B}, w)\right)$$

(Generalised Cohomology) The twisted cohomology of $\mathcal{X}$ is defined as:

$$\mathbb{E}(\mathcal{X}, w) := \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{B})}\left(\mathbb{1}_{\mathcal{B}}, g_*\Sigma^w g^*\mathbb{E}\right)$$

and its twisted cohomology groups as:

$$\mathbb{E}^{a,b}(\mathcal{X}, w) := \pi_0\left(\Sigma^{a,b}\mathbb{E}(\mathcal{X}, w)\right)$$

Now let us consider a more specific setting:

Definition 1.4.6. Let $B \in \mathbf{Sch}^G_{/S}$ and $X \in \mathbf{Sch}^G_{/B}$. Denote the respective quotient stacks as $\mathcal{B} := [B/G]$ and $\mathcal{X} := [X/G]$, and consider $v \in \mathrm{K}_0^G(X) = \mathrm{K}_0(\mathcal{X})$. Denote by $\pi_X : X \to B$ the structure map of X, by $f : \mathcal{X} \to \mathcal{B}$ the induced map of quotient stacks, by $\pi_{\mathcal{B}} : \mathcal{B} \to B$ the structure map (over B) of $\mathcal{B}$ and by $\pi_{\mathcal{X}} = \pi_{\mathcal{B}} \circ f : \mathcal{X} \to B$ the structure map (over B) of $\mathcal{X}$. Then we can define for any motivic ring spectrum $\mathbb{E} \in \mathrm{SH}(B)$:

(Equivariant Borel-Moore Motive) The equivariant Borel-Moore motive of X twisted by v is defined as:

$$(X/B)^{\mathrm{BM}}_G(v) := (\pi_{\mathcal{B}})_\# \Sigma^{\mathfrak{g}^\vee} f_! \Sigma^v \mathbb{1}_{\mathcal{X}}$$

(Equivariant Borel-Moore Homology) The equivariant Borel-Moore homology (or bivariant theory) of X twisted by v is then:

$$\mathbb{E}^{\mathrm{BM}}_G(X/B, v) := \underline{\mathrm{Map}}_{\mathrm{SH}(B)}\left((X/B)^{\mathrm{BM}}_G(v), \mathbb{E}\right)$$

The equivariant BM-homology groups will be then defined as:

$$\mathbb{E}^{\mathrm{BM}}_{a,b,G}(X/B, v) := \pi_0\left(\Sigma^{-a,-b}\mathbb{E}^{\mathrm{BM}}_G(X/B, v)\right)$$

$\left(\begin{array}{c}\textit{Equivariant}\\ \textit{Cohomology}\end{array}\right)$ As a special case of twisted cohomology, we can define the *equivariant cohomology* of X as:

$$\mathbb{E}_G(X,v) := \mathbb{E}(\mathcal{X},v) = \underline{\mathrm{Map}}_{\mathrm{SH}(B)}\left(\mathbb{1}_B, \pi_{\mathcal{X}*}\Sigma^v\pi_{\mathcal{X}}^*\mathbb{E}\right)$$

and its equivariant cohomology groups as:

$$\mathbb{E}_G^{a,b}(X,v) := \pi_0\left(\Sigma^{a,b}\mathbb{E}_G(X,v)\right)$$

Remark 1.4.7. We will mainly apply the above definition to the special case of $B = S$ and $\mathcal{B} = \mathcal{B}G = [S/G]$, but it will be useful sometimes to work over a more general $\mathbf{Sch}^G_{/S}$ with a non-trivial G-action. For example, this will be useful every time we need to mimic proofs or constructions from [DJK21] and apply them to the case of representable maps of NL-stacks.

Remark 1.4.8. If we had at our disposal the exceptional functors for non-representable maps, then the definitions of equivariant motives and theories would have been much cleaner, if not redundant. But nonetheless, these ad hoc definitions will behave as expected. For example, let us consider a smooth scheme $M \in \mathbf{Sch}^G_{/S}$. The cotangent complex of $\mathcal{M} := [M/G]$ will be $\mathbb{L}_{\mathcal{M}/S} \simeq \left[\mathbb{L}^G_{M/S} \to \mathfrak{g}^\vee_M\right]$, where $\mathbb{L}^G_{M/S} \in \mathcal{P}\!\mathit{erf}^G(M)$, sitting in (homological) degree zero, is the equivariant sheaf obtained from $\mathbb{L}_{M/S} = \Omega_{M/S}$ and its natural G-linearization. Let $f : \mathcal{M} \to \mathcal{B}G$. Then we have:

$$\begin{aligned}(M/S)^{\mathrm{BM}}_G\left(\mathbb{L}_{\mathcal{M}/S}\right) &= (\pi_{\mathcal{B}G})_\#\Sigma^{\mathfrak{g}^\vee} f_!\Sigma^{\mathbb{L}_{\mathcal{M}/S}}\mathbb{1}_{\mathcal{M}} \simeq\\ &\simeq (\pi_{\mathcal{B}G})_\#\Sigma^{\mathfrak{g}^\vee} f_\#\Sigma^{-\mathbb{L}^G_{M/S}}\Sigma^{\mathbb{L}_{\mathcal{M}/S}} f^*\mathbb{1}_S \simeq\\ &\simeq (\pi_{\mathcal{B}G})_\# f_\#\Sigma^{\mathfrak{g}^\vee_M}\Sigma^{-\mathbb{L}^G_{M/S}}\Sigma^{\mathbb{L}_{\mathcal{M}/S}} f^*\mathbb{1}_S \simeq\\ &\simeq (\pi_{\mathcal{M}})_\#\pi^*_{\mathcal{M}}\mathbb{1}_S\end{aligned}$$

where we used relative purity for the representable map f (cf [Cho21a, Theorem 5.4.1]) and the functoriality of the Thom automorphism. This tells us that the Borel-Moore motive of a smooth quotient is indeed its classical motive. We will soon give a more explicit description of $\pi_{\mathcal{M}\#}\mathbb{1}_{\mathcal{M}}$, justifying the name *classical motive*.

Remark 1.4.9. Up to a twist by $\mathfrak{g}^\vee$, we can identify the Borel-Moore homology of the quotient stacks with the equivariant Borel-Moore homology. Indeed, let $B \in \mathbf{Sch}^G_{/S}$, then we have:

$$\begin{aligned}\mathbb{E}^{\mathrm{BM}}_G(-/B, v) &= \underline{\mathrm{Map}}_{\mathrm{SH}(B)}\left(\pi_{\mathcal{B}\#}\Sigma^{\mathfrak{g}^\vee}([-/G]/\mathcal{B})^{\mathrm{BM}}(v), \mathbb{E}\right) \simeq\\ &\simeq \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{B})}\left(([-/G]/\mathcal{B})^{\mathrm{BM}}(v+\mathfrak{g}^\vee), \pi^*_{\mathcal{B}}\mathbb{E}\right) =\\ &= \pi^*_{\mathcal{B}}\mathbb{E}^{\mathrm{BM}}\left([-/G]/\mathcal{B}, v+\mathfrak{g}^\vee\right)\end{aligned}$$

where $\pi_{\mathcal{B}} : \mathcal{B} = [B/G] \to B$ is the structure map.
Similarly for equivariant cohomology we have:

$$\mathbb{E}_G(-, v) = \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{B})}(\mathbb{1}_{\mathcal{B}}, \pi_{[-/G]*}\Sigma^v \pi_{[-/G]}{}^* (\pi_{\mathcal{B}}^*\mathbb{E})) = \pi_{\mathcal{B}}^*\mathbb{E}([-/G], v)$$

In the special case $B = S$, the above computations tells us that:

$$\mathbb{E}_G^{\mathrm{BM}}(-/S, v) \simeq \pi_{\mathcal{B}G}^* \mathbb{E}^{\mathrm{BM}}\left([-/G]/\mathcal{B}G, v + \mathfrak{g}^\vee\right)$$

and:

$$\mathbb{E}_G(-, v) \simeq \pi_{\mathcal{B}G}^*\mathbb{E}([-/G], v)$$

Bivariant Operations

As for the case of schemes, we can talk about smooth pushforwards, proper pullbacks and Gysin maps for representable maps of NL-Stacks, and associated operations in BM-homology and in cohomology. We will omit the pullback maps on the twists to make the notation lighter, but it should be clear from the context what twists we are really using.

Definition 1.4.10. Suppose $\pi_{\mathcal{X}} : \mathcal{X} \to \mathcal{B}$ and $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{B}$ are representable, separated of finite type maps in $\mathcal{ASt}^{NL}_{/B}$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a representable map between them.

(Smooth Pushforward) If f is smooth and $v \in \mathrm{K}_0(\mathcal{Y})$, then we have a smooth pushforward map between BM-motives:

$$f_! : (\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(v + \mathbb{L}_f) \longrightarrow (\mathcal{Y}/\mathcal{B})^{\mathrm{BM}}(v)$$

induced by $f_!\Sigma^{\mathbb{L}_f} f^* \overset{\mathfrak{p}_f}{\simeq} f_\# f^* \xrightarrow{\eta^*_\#(f)} Id$. Indeed, the map f_* is defined as the following composition:

$$\pi_{\mathcal{X}!}\Sigma^{f^*v + \mathbb{L}_f}\pi_{\mathcal{X}}^*\mathbb{1}_{\mathcal{B}} \simeq \pi_{\mathcal{Y}!} f_! \Sigma^{\mathbb{L}_f} f^* \Sigma^v \pi_{\mathcal{Y}}^*\mathbb{1}_{\mathcal{B}} \overset{\mathfrak{p}_f}{\simeq} \pi_{\mathcal{Y}!} f_\# f^* \Sigma^v \pi_{\mathcal{Y}}^* \mathbb{1}_{\mathcal{B}} \overset{\eta^*_\#(f)}{\to} \pi_{\mathcal{Y}!}\Sigma^v\pi_{\mathcal{Y}}^*\mathbb{1}_S$$

(Proper Pullback) If f is proper, then we have a proper pullback map between BM-motives:

$$f^* : (\mathcal{Y}/\mathcal{B})^{\mathrm{BM}}(v) \longrightarrow (\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(v)$$

induced by the identification $\alpha_f : f_! \overset{\sim}{\to} f_*$ by [Cho21a, Proposition 5.3.3] and the natural transformation given by the unit $\epsilon^*_*(f) : Id \to f_* f^*$. Indeed, the map f^* is given by the following composition:

$$\pi_{\mathcal{Y}!}\Sigma^v\pi_{\mathcal{Y}}^*\mathbb{1}_{\mathcal{B}} \overset{\epsilon^*_*(f)}{\to} \pi_{\mathcal{Y}!} f_* f^* \Sigma^v \pi_{\mathcal{Y}}^*\mathbb{1}_{\mathcal{B}} \simeq \pi_{\mathcal{Y}!} f_! \Sigma^{f^*v} f^* \pi_{\mathcal{Y}} \mathbb{1}_{\mathcal{B}} \simeq \pi_{\mathcal{X}!}\Sigma^{f^*v}\pi_{\mathcal{X}}^*\mathbb{1}_{\mathcal{B}}$$

$\left(\begin{array}{c}\textit{Gysin}\\ \textit{Pushforward}\end{array}\right)$ If f is smooth and it admits a closed section $s : \mathcal{Y} \to \mathcal{X}$, then we have a natural transformation:

$$Id \simeq (f \circ s)_!(f \circ s)^! = f_! s_! s^! f^! \xrightarrow{\eta_!^!(s)} f_! f^! \simeq f_! \Sigma^{\mathbb{L}_f} f^*$$

This natural transformation induces then a *Gysin pushforward* on BM-motives:

$$s_! : (\mathcal{Y}/\mathcal{B})^{\mathrm{BM}}(v) \longrightarrow (\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(v + \mathbb{L}_f)$$

Indeed, the map $s_!$ is given by the following composition:

$$\pi_{\mathcal{Y}!}\Sigma^v \pi_{\mathcal{Y}}^* \mathbb{1}_{\mathcal{B}} \xrightarrow{\eta_!^!(s)} \pi_{\mathcal{Y}!} f_! f^! \Sigma^v \pi_{\mathcal{Y}}^* \mathbb{1}_{\mathcal{B}} \simeq \pi_{\mathcal{Y}} f_! \Sigma^{f^*v + \mathbb{L}_f} f^* \pi_{\mathcal{Y}}^* \mathbb{1}_{\mathcal{B}} \simeq \pi_{\mathcal{X}!} \Sigma^{f^*v+\mathbb{L}_f} \pi_{\mathcal{X}}^* \mathbb{1}_{\mathcal{B}}$$

When f is a vector bundle, then by homotopy invariance the smooth pushforward f_* is an isomorphism on BM-motives with inverse given by the Gysin pushforward $s_!$ (the same argument in [Lev17, Lemma 2.2] works verbatim).

The operations we just defined on BM-motives will respectively induce smooth pullbacks, proper pushforwards and Gysin pullbacks on BM-homology as in the case of schemes. Moreover everything works in the same way also for equivariant BM-motives and equivariant BM-homology using the identifications in remark 1.4.9.

Given a regular embedding of NL-stacks, we can define a Gysin pushforward map closely following the construction in the schematic case. This is also known to the experts and already appeared in [Kha19]. We will use a very special and easier case of the construction in *loc. cit.*, so for the reader convenience we will repeat the construction here. The construction of a deformation space for algebraic stacks already appeared in different places (with slightly different flavours), namely [Kre99], [KR19], [AP19]. Working with the deformation space over $\mathbb{A}^1$ or $\mathbb{P}^1$ does not really matter for us, but since in [Lev17] and [DJK21] the former is preferred, we will stick with this choice. For this reason, we will slightly modify the construction of [AP19, §7]. Consider a closed embedding of schemes $X \hookrightarrow Y$ and consider the classical deformation space of Fulton:

$$\mathrm{Def}_{X/Y} := \mathrm{Bl}_{X\times\{0\}}\left(Y \times \mathbb{A}^1\right) \setminus \mathrm{Bl}_{X\times\{0\}}(Y \times \{0\})$$

The deformation space $\mathrm{Def}_{X/Y}$ is flat over $\mathbb{A}^1$ and the same proofs as in [AP19, Lemma 7.1, Theorem 7.2] tell us that for any locally finite type map $\mathcal{Z} \to \mathcal{Y}$ of algebraic stacks we have a commutative diagram:

$$\begin{tikzcd} \mathfrak{C}_{\mathcal{Z}/\mathcal{Y}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Def}_{\mathcal{Z}/\mathcal{Y}} \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{Y} \times \mathbb{G}_m \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l, hook'] \end{tikzcd} \tag{1.6}$$

where $\mathfrak{C}_{\mathcal{Z}/\mathcal{Y}}$ is the intrinsic normal cone of [AP19, Theorem 6.2]. If $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is a closed immersion of NL-stacks, with associated ideal sheaf $\mathcal{I}_Z$, then $\mathfrak{C}_{\mathcal{Z}/\mathcal{Y}} \simeq \mathrm{Spec}\left(\bigoplus_{k\geq 0} \mathcal{I}_{\mathcal{Z}}^k/\mathcal{I}_{\mathcal{Z}}^{k+1}\right)$. With the deformation space at hand, we are now ready to state the following:

Proposition 1.4.11 (Gysin for Regular Embeddings)**.** *Let $\iota : \mathcal{X} \hookrightarrow \mathcal{M}$ be a regular closed embedding in $\mathcal{ASt}^{NL}_{/\mathcal{B}}$, with $\pi_{\mathcal{X}} : \mathcal{X} \to \mathcal{B}$ and $\pi_{\mathcal{M}} : \mathcal{M} \to \mathcal{B}$ representable, separated of finite type maps. Then for any $v \in \mathcal{P}erf(\mathcal{M})$ we have a natural transformation:*

$$\iota_! : (\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(\iota^* v - \mathcal{N}_{\mathcal{X}/\mathcal{M}}) \longrightarrow (\mathcal{M}/\mathcal{B})^{\mathrm{BM}}(v)$$

where $\mathcal{N}_{\mathcal{X}/\mathcal{M}} \simeq \mathcal{I}_{\mathcal{X}}/\mathcal{I}_{\mathcal{X}}^2$ is the normal sheaf of the regular embedding.

Proof. Once we get a natural transformation:

$$\iota_! : (\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(-\mathcal{N}_{\mathcal{X}/\mathcal{M}}) \longrightarrow (\mathcal{M}/\mathcal{B})^{\mathrm{BM}}$$

it is easy to get the twisted version just following the same steps, so we will only deal with the untwisted case.

Given a closed regular embedding $\iota : \mathcal{X} \hookrightarrow \mathcal{M}$, we have that the normal cone $\mathfrak{C}_{\mathcal{X}/\mathcal{M}}$ is isomorphic to the normal bundle $N_{\mathcal{X}/\mathcal{M}} \simeq \mathbb{V}_{\mathcal{X}}\left(\mathbb{L}_{\mathcal{X}/\mathcal{M}}[-1]\right) = \mathbb{V}_{\mathcal{X}}\left(\mathcal{I}/\mathcal{I}^2\right)$. By [Cho21a, Corollary 3.3.10], the deformation space $\mathrm{Def}_{\mathcal{X}/\mathcal{M}}$ and the normal bundle $N_{\mathcal{X}/\mathcal{M}}$ are both NL-stacks. Hence the deformation to the normal cone construction gives us a closed and open embedding of NL-stacks:

$$N_{\mathcal{X}/\mathcal{M}} \overset{\iota_N}{\hookrightarrow} \mathrm{Def}_{\mathcal{X}/\mathcal{M}} \underset{j}{\hookleftarrow} \mathcal{M} \times \mathbb{G}_m$$

We hence have a localization sequence by [Cho21a, Proposition 5.2.1]:

$$j_\# j^* \to Id_{\mathrm{SH}(\mathrm{Def}_{\mathcal{X}/\mathcal{M}})} \to \iota_{N*}\iota_N^* \tag{1.7}$$

By the localization sequence associated to $\mathbb{G}_{m,\mathcal{M}} \hookrightarrow \mathbb{A}^1_{\mathcal{M}} \hookleftarrow \{0\}$ we get a natural map:

$$\partial : \mathrm{Th}_{\mathcal{M}}\left(\mathbb{A}^1_{\mathcal{M}}\right)[-1] \to p_{1\#}\mathbb{1}_{\mathbb{G}_{m,\mathcal{M}}}$$

where $p_1 : \mathbb{G}_{m,\mathcal{M}} = \mathcal{M} \times \mathbb{G}_m \to \mathcal{M}$ is the projection map. Pulling back ∂ along a NL-atlas $\mu : M \to \mathcal{M}$, and denoting by $\tilde{p}_1 : \mathbb{G}_{m,M} \to M$ the projection, we get:

$$\mathrm{Th}_M\left(\mathbb{A}^1_M\right)[-1] \longrightarrow \tilde{p}_{1\#}\mathbb{1}_{\mathbb{G}_{m,M}}$$

But this map is an equivalence in $\mathrm{SH}(M)$ since:

$$\mathrm{Th}_M\left(\mathbb{A}^1_M\right)[-1] \simeq \Sigma_{\mathbb{P}^1}\mathbb{1}_M[-1] \simeq \Sigma_{\mathbb{G}_m}\mathbb{1}_M \simeq \tilde{p}_{1\#}\mathbb{1}_{\mathbb{G}_{m,M}}$$

and by conservativity of μ^* ([Cho21a, Proposition 5.1.1]) this means that ∂ was an equivalence too. Desuspending ∂, we get:

$$\mathbb{1}_{\mathcal{M}} \xrightarrow{\sim} \Sigma^{-\mathcal{O}_{\mathcal{M}}} p_{1\#}\mathbb{1}_{\mathbb{G}_{m,\mathcal{M}}}[1] \simeq p_{1\#}\Sigma^{-\Omega_{\mathbb{G}_{m,\mathcal{M}}/\mathcal{M}}}\mathbb{1}_{\mathbb{G}_{m,\mathcal{M}}}[1] \simeq p_{1!}\mathbb{1}_{\mathbb{G}_{m,\mathcal{M}}}[1] \tag{1.8}$$

where we used purity for p_1 (cf. [Cho21a, Theorem 5.4.1]) and the fact that $\Omega_{\mathbb{G}_{m,\mathcal{M}}/\mathcal{M}} \simeq \mathcal{O}_{\mathbb{G}_{m,\mathcal{M}}}$. In particular the map above will induce an equivalence:

$$(\mathbb{G}_{m,\mathcal{M}}/\mathcal{B})^{\mathrm{BM}}[1] = \pi_{\mathbb{G}_{m,\mathcal{M}}!}\mathbb{1}_{\mathbb{G}_{m,\mathcal{M}}}[1] \simeq \pi_{\mathcal{M}!}\mathbb{1}_{\mathcal{M}} = (\mathcal{M}/\mathcal{B})^{\mathrm{BM}} \tag{1.9}$$

where $\pi_{\mathbb{G}_{m,\mathcal{M}}} : \mathbb{G}_{m,\mathcal{M}} \to \mathcal{B}$ is the structure map of $\mathbb{G}_{m,\mathcal{M}}$.

From the localization sequence associated to the deformation to the normal cone, we get a boundary map:

$$\partial_N : \pi_{N_{\mathcal{X}/\mathcal{M}}!}\mathbb{1}_N \longrightarrow \pi_{\mathbb{G}_{m,\mathcal{M}}!}\mathbb{1}_{\mathbb{G}_{m,\mathcal{M}}}[1] \tag{1.10}$$

where $\pi_{N_{\mathcal{X}/\mathcal{M}}} : N_{\mathcal{X}/\mathcal{M}} \to \mathcal{B}$ is the structure map. Using the Gysin map associated to the zero section $s_0 : \mathcal{X} \hookrightarrow N_{\mathcal{X}/\mathcal{M}}$, we get:

$$s_{0!} : \pi_{\mathcal{X}!}\Sigma^{\mathbb{L}_{\mathcal{X}/\mathcal{M}}}\mathbb{1}_X \longrightarrow \pi_{N_{\mathcal{X}/\mathcal{M}}!}\mathbb{1}_{N_{\mathcal{X}/\mathcal{M}}} \tag{1.11}$$

Composing together the maps from (1.9), (1.10) and (1.11), we get:

$$\iota_! : (\mathcal{X}/\mathcal{B})^{\mathrm{BM}}(-\mathcal{N}_{\mathcal{X}/\mathcal{M}}) = \pi_{X!}\Sigma^{\mathbb{L}_{\mathcal{X}/\mathcal{M}}}\mathbb{1}_X \longrightarrow (\mathcal{M}/\mathcal{B})^{\mathrm{BM}}$$

□

Remark 1.4.12. In the proposition above, if we set $\mathcal{B} = \mathcal{M}$, we get as a special case a natural map:

$$\iota_! : (\mathcal{X}/\mathcal{M})^{\mathrm{BM}}(\mathbb{L}_\iota) = (\mathcal{X}/\mathcal{M})^{\mathrm{BM}}(-\mathcal{N}_{\mathcal{X}/\mathcal{M}}) \to \mathbb{1}_{\mathcal{M}}$$

For any ring spectrum $\mathbb{F} \in \mathrm{SH}(\mathcal{M})$, we get a natural map:

$$\mathbb{F}^{\mathrm{BM}}(\mathcal{M}/\mathcal{M}, v) \quad\longrightarrow\quad \mathbb{F}^{\mathrm{BM}}(\mathcal{X}/\mathcal{M}, v + \mathbb{L}_\iota)$$

and we denote the image of $\mathbb{1}_{\mathcal{M}}$ under the above map as η_ι. Then for any ring spectrum $\mathbb{E} \in \mathrm{SH}(\mathcal{B})$, we have a map:

$$\begin{array}{cccc} \iota^! : & \mathbb{E}^{\mathrm{BM}}(\mathcal{M}/\mathcal{B}, v) & \longrightarrow & \mathbb{E}^{\mathrm{BM}}(\mathcal{X}/\mathcal{B}, v + \mathbb{L}_\iota) \\ & x & \mapsto & \eta_\iota \cdot x \end{array}$$

where the product $\eta_\iota \cdot x$ is defined exactly as in [DJK21, §2.2.7(4)]. This map is the same map induced by the Gysin pushforward on Borel-Moore motives of proposition 1.4.11.

Definition 1.4.13. The element $\eta_\iota^{\mathbb{F}} \in \mathbb{F}^{\mathrm{BM}}(\mathcal{X}/\mathcal{M}, v + \mathbb{L}_\iota)$ is said to be *an* $\mathbb{F}$-*orientation* of ι (following the conventions in [DJK21, Definition 2.3.2]). We will often omit the superscript from the orientations.

1.4.1 Colimit Motives à la Edidin-Graham-Totaro

We want now to relate the equivariant Borel-Moore motives with the colimit motive one can obtain from the ind-scheme given by the Totaro approximations $X \times^G E_m G$ as introduced in 1.2.6.

Notation 1.4.14. Given $X \in \mathbf{Sch}^G_{q/S}$, any G-linearised perfect complex $v \in \mathcal{P}erf^G(X)$ gives us for each m the pullback complex $p_1^* v_m \in \mathcal{P}erf^G(X \times E_m G)$ via the map $p_1 : X \times E_m G \to X$. By descent this gives us an element $v_m \in \mathcal{P}erf(X_m)$ such that $\iota_m^* v_{m+1} \simeq v_m$ for $\iota_m : X_m \hookrightarrow X_{m+1}$. Thus we have compatible maps:

$$\begin{array}{cccc} (\cdot)_m : & \mathcal{P}erf^G(X) & \longrightarrow & \mathcal{P}erf(X_m) \\ & v & \mapsto & v_m \end{array}$$

The pullback maps ι_m^* on perfect complexes give rise to a pro-object in stable ∞-categories that we can name $\mathcal{P}erf(\widetilde{X}) := \lim_m \mathcal{P}erf(X_m)$ (this is a very much simpler and special case of [Hen17, Def. 1.2.2, Rmk. 1.2.3]). Sometimes we will abuse the notation and denote by v the perfect complex in $\mathcal{P}erf(\widetilde{X})$ induced by the objects v_m. The same notations and remarks hold true for when we talk about locally free sheaves or vector bundles.

Remark 1.4.15. Let $X \in \mathbf{Sch}^G_{q/S}$, then by remark 1.2.5 the transition maps of $\widetilde{X}$, denoted as $\iota_m : X_m \hookrightarrow X_{m+1}$, are lci maps that can be factored through a zero section of a vector bundle and an open immersion. In this special case we get nice functorialities between bivariant theories. Indeed let us suppose we have $Z \xrightarrow{s} E \xrightarrow{j} Y$ ion $\mathbf{Sch}_{/S}$ where s is the zero section of a vector bundle $p : E \to X$, j is an open immersion and let $f : Z \to Y$ be the composition. We will denote by $\pi_- : - \to S$ the structure map for $- = Z, E, Y$ and by $\mathcal{E}$ the locally free sheaf associated to E. Let us consider any $v \in \mathrm{K}_0(Y)$, then the open immersion will in particular induce a smooth pushforward map:

$$j_! : (E/S)^{\mathrm{BM}}(v) \to (Y/S)^{\mathrm{BM}}(v)$$

via the unit map $j_\# j^* \to Id$. Since $p : E \to Z$ is a vector bundle (with relative cotangent complex $\mathbb{L}_p = p^*\mathcal{E}$), the smooth pushforward map:

$$p_! : (E/S)^{\mathrm{BM}}(v) \longrightarrow (Z/S)^{\mathrm{BM}}(v - \mathcal{E})$$

has an inverse given by:

$$s_! : (Z/S)^{\mathrm{BM}}(v - \mathcal{E}) \longrightarrow (E/S)^{\mathrm{BM}}(v)$$

by [Lev17, Lemma 2.2]. So we can define a Gysin map (for smooth maps with a section) as:

$$f_! := s_! \circ j_! : \mathbb{E}^{\mathrm{BM}}(Y/S, v) \longrightarrow \mathbb{E}^{\mathrm{BM}}(Z/S, v - \mathcal{E})$$

Consider now our Totaro approximations for quotient stacks. We have a fiber sequence of cotangent complexes:

$$\pi_{G,m}^{*}\mathbb{L}_{B_mG/S} \longrightarrow \mathbb{L}_{E_mG/S} \longrightarrow \mathfrak{g}^{\vee} \otimes \mathcal{O}_{E_mG}$$

where $\pi_{G_m} : E_mG \to B_mG$, $\mathfrak{g}^{\vee}$ is the co-Lie algebra of G over B. Taking the respective G-linearised sheaves, the former fiber sequence gives us a fiber sequence on B_mG:

$$\mathbb{L}_{B_mG/S} \longrightarrow \mathbb{L}^{G}_{E_mG/S} \longrightarrow \mathfrak{g}_m^{\vee}$$

where $\mathfrak{g}_m := \mathfrak{g}^{\vee}_{E_mG}$. Using the factorization 1.2.5 and the functoriality of Borel-Moore motives as described above, for $X \in \mathbf{Sch}^{G}_{q/S}$, $v \in \mathrm{K}_0^G(X)$ and $v = \{v_m\}$ as in notation 1.4.14, we get maps Gysin pushforward maps:

$$\iota_{m!} : (X_m/S)^{\mathrm{BM}}\left(v_m + \mathbb{L}^{G}_{E_mG/S}\right) \longrightarrow X_{m+1}/S^{\,\mathrm{BM}}\left(v_{m+1} + \mathbb{L}^{G}_{E_{m+1}G/S}\right);$$

here we have abused notation and omitted the pullbacks on the twists by $\mathbb{L}^{G}_{E_mG/S}$. But replacing v_m with $v_m - \mathfrak{g}_m^{\vee}$ we also get:

$$\iota_{m!} : (X_m/S)^{\mathrm{BM}}\left(v_m + \mathbb{L}_{B_mG/S}\right) \longrightarrow (X_{m+1}/S)^{\mathrm{BM}}\left(v_{m+1} + \mathbb{L}_{B_{m+1}G/S}\right)$$

This will give us a map:

$$\iota_m^{!} : \mathbb{E}^{\mathrm{BM}}\left(X_{m+1}/S, v_{m+1} + \mathbb{L}_{B_mG/S}\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}\left(X_{m+1}/S, v_m + \mathbb{L}_{B_{m+1}G/S}\right)$$

For the cohomology theory, we can consider the map $Id \to \iota_{m*}\iota_m^{*}$ and this will induce:

$$\iota_m^{*} : \mathbb{E}\left(X_{m+1}, v_{m+1}\right) \to \mathbb{E}\left(X_m, v_m\right)$$

Definition 1.4.16. Let $\mathcal{B} \in \mathcal{ASt}^{NL}_{/S}$ and let $\mathcal{X}, \mathcal{Y} \in \mathcal{ASt}^{NL}_{/\mathcal{B}}$ be smooth NL-stacks over $\mathcal{B}$, with maps:

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "g"] \arrow[dr, "f_1"'] & & \mathcal{Y} \arrow[dl, "f_2"] \\ & \mathcal{B} & \end{tikzcd}$$

Then we have a natural transformation:

$$g_* : f_{1\#}g^{*} \longrightarrow f_{2\#} \tag{1.12}$$

given by the composition:

$$f_{1\#}g_* \xrightarrow{\epsilon^{*}_{\#}(f_2)} f_{1\#}g^{*}f_2^{*}f_{2\#} \simeq f_{1\#}f_1^{*}f_{2\#} \xrightarrow{\eta^{*}_{\#}(f_1)} f_{2\#}$$

We will refer to g_* as *relative pushforward* along g.

Lemma 1.4.17. *Relative pushforwards are functorial with respect to compositions, that is, given composable maps in* $\mathcal{ASt}^{NL}_{/\mathcal{B}}$*:*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "g"] \arrow[dr, "f_1"'] & \mathcal{Y} \arrow[r, "h"] \arrow[d, "f_2"] & \mathcal{Z} \arrow[dl, "f_3"] \\
& \mathcal{B} &
\end{tikzcd}$$

then we have that $(hg)_* \simeq h_* g_*$*.*

Proof. The natural homotopy $(hg)_* \simeq h_* g_*$ is given by the following diagram:

$$\begin{tikzcd}
f_{1\#}(hg)^* = f_{1\#}g^*h^* \arrow[r, "\epsilon^*_\#(f_3)"] \arrow[dd, "\epsilon^*_\#(f_2)"'] & f_{1\#}(hg)^*f_3^*f_{3\#} = f_{1\#}f_1^*f_{3\#} \arrow[r, "\eta^*_\#(f_1)"] \arrow[d, equal] & f_{3\#} \arrow[dddd, equal] \\
& f_{1\#}g^*h^*f_3^*f_{3\#} \arrow[d, "\epsilon^*_\#(f_2)"'] & \\
f_{1\#}g^*f_2^*f_{2\#}h^* \arrow[r, "\epsilon^*_\#(f_3)"] \arrow[d, equal] & f_{1\#}g^*f_2^*f_{2\#}h^*f_3^*f_{3\#} \arrow[d, equal] & \\
f_{1\#}f_1^*f_{2\#}h^* \arrow[r, "\epsilon^*_\#(f_3)"] \arrow[d, "\eta^*_\#(f_1)"'] & f_{1\#}f_1^*f_{2\#}h^*f_3^*f_{3\#} \arrow[d, "\eta^*_\#(f_1)"'] & \\
f_{2\#}h^* \arrow[r, "\epsilon^*_\#(f_3)"'] & f_{2\#}h^*f_3^*f_{3\#} = f_{2\#}f_2^*f_{3\#} \arrow[r, "\eta^*_\#(f_2)"'] & f_{3\#}
\end{tikzcd}$$

where the squares on the left clearly commutes, and the square on the right commutes using the fact that $\eta^*_\#(f_2) \circ \epsilon^*_\#(f_2) \simeq Id$ by the usual property of unit and co-unit relative to the adjunction $f_{2\#} \dashv f_2^*$. Indeed, we can fill the right square (forgetting about the $f_{3\#}$ everywhere) as:

$$\begin{tikzcd}
f_{1\#}g^*h^*f_3^* \arrow[r, equal] \arrow[d, "\epsilon^*_{\#}(f_2)"'] & f_{1\#}g^*f_2^* \arrow[r, equal] & f_{1\#}f_1^* \arrow[r, "\eta^*_{\#}(f_1)"] \arrow[dddll, "\epsilon^*_{\#}(f_2)"'] & Id \arrow[dddd, equal] \arrow[ddddlll, "\epsilon^*_{\#}(f_2)"'] \\
f_{1\#}g^*f_2^*f_{2\#}h^*f_3^* \arrow[d, equal] & & & \\
f_{1\#}g^*f_2^*f_{2\#}f_2^* \arrow[d, equal] \arrow[uur, "\eta^*_{\#}(f_2)"] & & & \\
f_{1\#}f_1^*f_{2\#}f_2^* \arrow[d, "\eta^*_{\#}(f_1)"'] & & & \\
f_{2\#}f_2^* \arrow[rrr, "\eta^*_{\#}(f_2)"'] & & & Id
\end{tikzcd}$$

□

Remark 1.4.18. Consider again:

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "g"] \arrow[dr, "f_1"'] & & \mathcal{Y} \arrow[dl, "f_2"] \\
& \mathcal{B} &
\end{tikzcd}$$

in $\mathcal{ASt}^{NL}_{/\mathcal{B}}$. If $f_2 = Id$, then the relative pushforward is just the map induced by the co-unit $f_{1\#}f_1^* \to Id$.

Definition 1.4.19. Let $p : \mathcal{W} \to \mathcal{Z}$ be a representable, separated finite type map of NL-stacks and let $g : \mathcal{Z} \to \mathcal{B}$ be a smooth map of NL-stacks. Then we define the Borel-Moore motive of $\mathcal{W}$ over $\mathcal{Z}$, relative to $\mathcal{B}$, as:

$$(\mathcal{W}/\mathcal{Z})^{\mathrm{BM}}_{\mathcal{B}} := g_{\#}p_!\mathbb{1}_Z$$

If $w \in \mathcal{Perf}(\mathcal{W})$, the twisted version of the relative Borel-Moore motive will be:

$$(\mathcal{W}/\mathcal{Z})^{\mathrm{BM}}_{\mathcal{B}}(w) := g_{\#}p_!\Sigma^w\mathbb{1}_Z$$

Remark 1.4.20. Consider the following diagram in $\mathcal{Ast}^{NL}$:

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X} \arrow[d, "q"] \\
\mathcal{Z} \arrow[r, "g"'] & \mathcal{B}
\end{tikzcd}$$

where all the maps are representable, finite type, separated and where f, g are smooth.

In the definition above, if all the maps are representable, we can use the exchange transformation $Ex_{!\#}$ (cf. proposition 1.3.6) and see that:

$$(\mathcal{W}/\mathcal{Z})^{\mathrm{BM}}_{\mathcal{B}} := g_{\#}p_!\mathbb{1}_Z \overset{Ex_{!\#}}{\simeq} q_!f_{\#}\mathbb{1}_{\mathcal{W}}$$

A similar statement of course holds also for the twisted relative Borel-Moore motive.

Proposition 1.4.21. *Given a diagram of NL-stacks:*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "g"] \arrow[dr, "f_1"'] & & \mathcal{Y} \arrow[dl, "f_2"] \\ & \mathcal{B} & \end{tikzcd}$$

where g is smooth, then the relative pushforward g_* is the same as:

$$f_{1\#}g^* \simeq f_{2\#}g_{\#}g^* \longrightarrow f_{2\#}$$

induced by $\eta^*_{\#}(g) : g_{\#}g^* \to Id$.

Proof. To get the claim it is enough to check the commutativity of the following diagram:

$$\begin{tikzcd} f_{\#}g^* \arrow[r, "\epsilon^*_{\#}(f_2)"] \arrow[d, equal] & f_{1\#}g^*f_2^*f_{2\#} \arrow[r, equal] \arrow[d, equal] & f_{1\#}f_1^*f_{2\#} \arrow[dd, "\eta^*_{\#}(f_1)"] \\ f_{2\#}g_{\#}g^* \arrow[r, "\epsilon^*_{\#}(f_2)"'] \arrow[d, "\eta^*_{\#}(g)"'] & f_{2\#}g_{\#}g^*f_2^*f_{2\#} \arrow[d, "\eta^*_{\#}(g)"'] \arrow[dr, "\eta^*_{\#}(f_1)"] & \\ f_{2\#} \arrow[r, "\epsilon^*_{\#}(f_2)"'] & f_{2\#}f_2^*f_{2\#} \arrow[r, "\eta^*_{\#}(f_2)"'] & f_{2\#} \end{tikzcd}$$

The rectangles on the left are clearly commutative, and the rectangle on the right commutes by the naturality of the co-unit transformations $\eta^*_{\#}(f_2) \circ \eta^*_{\#}(g) \simeq \eta^*_{\#}(f_1)$. $\square$

Remark 1.4.22. By lemma 1.4.17, the relative pushforward induces a *motive functor* that can be informally be described as:

$$\begin{array}{rccc} M(-): & \mathbf{Sm}/B & \longrightarrow & \mathrm{SH}(B) \\ & \pi_X : X \to B & \mapsto & \pi_{X\#}\mathbb{1}_X \\ & g : X \to Y & \mapsto & g_* \end{array}$$

We also have the infinity suspension functor $\Sigma^\infty : \mathbf{Sm}_{/B} \to \mathrm{SH}(B)$ that sends a smooth B-scheme X to Σ^∞ and a map $g : X \to Y$ in $\mathbf{Sm}_{/B}$ to $\Sigma^\infty g$.

Consider the category $\mathbf{Sm}^{sm}_{/B}$ of smooth B-schemes with smooth maps between them. Let $g : X \to Y$ be a map in $\mathbf{Sm}^{sm}_{/B}$, then $\Sigma^\infty g$ is just the smooth pushforward of motives, that, by proposition 1.4.21, is equivalent to $g_*(\mathbb{1}_Y) : M(X) \to M(Y)$. In particular this implies that:

$$\Sigma^\infty\big|_{\mathbf{Sm}^{sm}_{/B}} \simeq M(-)\big|_{\mathbf{Sm}^{sm}_{/B}} \tag{1.13}$$

The functors Σ^∞ and $M(-)$ are left Kan extended from their restrictions on $\mathbf{Sm}^{sm}_{/B}$, therefore it follows from (1.13) (and from the uniqueness of Kan extensions) that we have a natural equivalence $\Sigma^\infty \simeq M(-)$ as functors over $\mathbf{Sm}_{/B}$. In particular, for any map g in $\mathbf{Sm}_{/B}$ we get a natural equivalence:

$$\Sigma^\infty g \simeq g_*$$

Remark 1.4.23. Proposition 1.4.21 implies that if $g : \mathcal{X} \to \mathcal{Y}$ is a vector bundle, then the relative pushforward g_* will be an equivalence by homotopy invariance (i.e. since $g_\# g^* \xrightarrow{\sim} Id$). Given any vector bundle section $s : \mathcal{Y} \to \mathcal{X}$, the relative pushforward:

$$s_* : f_{2\#} = f_{2\#}(g \circ s)^* = f_{2\#} s^* g^* \longrightarrow f_{1\#} g^*$$

will be the inverse of g_*.

Consider now the following diagram of representable maps in $\mathcal{ASt}^{NL}$:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[drr, "f_Z"'] \arrow[dd, "p_Z"'] & Y \arrow[dr, "f_Y"] \arrow[dd, "p_Y"'] & \\
& & \mathcal{X} \arrow[dd, "p"] \\
B_Z \arrow[r, "g"] \arrow[drr, "g_Z"'] & B_Y \arrow[dr, "g_Y"] & \\
& & \mathcal{B}
\end{tikzcd}$$

where all the squares are cartesian, p, p_Z, p_Y are separated of finite type, and f_Z, f_Y, g_Z, g_Y are smooth.

Proposition 1.4.24. *With the same notation as above, suppose $f : Z \to Y$ and $g : B_Z \to B_Y$ factorise as zero section of a vector bundle together with an open immersion. Then the relative pushforward f_* induces a map:*

$$f_* : (Z/B_Z)^{\mathrm{BM}}_{\mathcal{B}} \longrightarrow (Y/B_Y)^{\mathrm{BM}}_{\mathcal{B}}$$

that we can identify with $g_{Z\#}$ applied to the Gysin pushforward $f_! := j_! \circ s_!$ as defined in remark 1.4.15.

Proof. By definition we have:

$$(Z/B_Z)^{\mathrm{BM}}_{\mathcal{B}} := g_{Z\#}\,(Z/B_Z)^{\mathrm{BM}} = g_{Z\#}p_{Z!}\mathbb{1}_Z \overset{Ex_{!\#}}{\simeq} p_! f_{Z\#}\mathbb{1}_Z$$

$$(Y/B_Y)^{\mathrm{BM}}_{\mathcal{B}} := g_{Y\#}\,(Y/B_Y)^{\mathrm{BM}} = g_{Y\#}p_{Y!}\mathbb{1}_Y \overset{Ex_{!\#}}{\simeq} p_! f_{Y\#}\mathbb{1}_Y$$

Therefore the relative pushforward f_* gives us a map on the relative Borel-Moore motives defined as the following composition:

$$(Z/B_Z)^{\mathrm{BM}}_{\mathcal{B}} \simeq p_! f_{Z\#}\mathbb{1}_Z \simeq p_! f_{Z\#}f^*\mathbb{1}_Y \xrightarrow{f_*} p_! f_{Y\#}\mathbb{1}_Y \simeq (Y/B_Y)^{\mathrm{BM}}_{\mathcal{B}}$$

Suppose we can factorise f and g as $f = \omega \circ s_f$ and $g = j \circ s_g$ where s_f, s_g are zero sections of vector bundles $\nu_f : W \to Z$ and $v_g : B_W \to B_Z$. Thus we have:

$$\begin{tikzcd}
Z \arrow[r, "s_f"'] \arrow[drr, "f_Z"'] \arrow[dd, "p_Z"'] & W \arrow[l, bend right, "\nu_f"'] \arrow[rr, "\omega"] \arrow[dr, "f_W"] \arrow[dd, "p_W"] & & Y \arrow[dl, "f_Y"'] \arrow[dd, "p_Y"] \\
& & \mathcal{X} \arrow[dd, "p"] & \\
B_Z \arrow[r, "s_g"'] \arrow[drr, "g_Z"'] & B_W \arrow[l, bend right, "v_g"'] \arrow[rr, "j"] \arrow[dr, "g_W"] & & B_Y \arrow[dl, "g_Y"] \\
& & \mathcal{B} &
\end{tikzcd}$$

where again all the squares of the diagram are cartesian. Then the relative pushforwards $(\nu_f)_*, (v_g)_*$, for what we said in proposition 1.4.21, can be identified with the natural transformation associated to the adjunctions $\nu_{f\#}\nu_f \to Id$ and $v_{g\#}v_g^* \to Id$ respectively. But using purity for ν_g, we get:

$$\begin{aligned}
g_{W\#}p_{W!} &\simeq g_{Z\#}v_{g\#}p_{W!} \simeq \\
&\simeq g_{Z\#}v_{g!}\Sigma^{\Omega_{v_g}}p_{W!} \simeq \\
&\simeq g_{Z\#}v_{g!}p_{W!}\Sigma^{\Omega_{\nu_f}} \simeq \\
&\simeq g_{Z\#}(v_g \circ p_W)_!\Sigma^{\Omega_{\nu_f}}
\end{aligned}$$

that evaluated at $\mathbb{1}_W$ gives us:

$$(W/B_W)^{\mathrm{BM}}_{\mathcal{B}} = g_{W\#}p_{W!}\mathbb{1}_W \simeq g_{Z\#}\,(W/B_Z)^{\mathrm{BM}}\,(\Omega_{\nu_f})$$

This means that the relative pushforward $(\nu_f)_*$ is just $g_{Z\#}$ applied to the smooth pushforward of Borel-Moore motives:

$$(\nu_f)_! : (W/B_Z)^{\mathrm{BM}}\,(\Omega_{\nu_f}) \longrightarrow (Z/B_Z)^{\mathrm{BM}}$$

induced by:

$$(v_g \circ p_W)_! \Sigma^{\Omega_{\nu_f}} \mathbb{1}_W \simeq (v_g \circ p_W)_! \Sigma^{\Omega_{\nu_f}} \nu_f^* \mathbb{1}_Z \simeq p_{Z!} \nu_{f!} \nu_f^! \mathbb{1}_Z \xrightarrow{\eta_!^!(\nu_f)} p_{Z!}$$

This implies that the relative pushforward $(s_f)_* = ((\nu_f)_*)^{-1}$ (cf. remark 1.4.23) is just $g_{Z\#}$ applied to the Gysin map:

$$(s_f)_! : (Z/B_Z)^{\mathrm{BM}} \left(-s_f^* \Omega_{\nu_f}\right) \longrightarrow (W/B_Z)^{\mathrm{BM}}$$

In other words:

$$(s_f)_* = g_{Z\#}(s_f)_! \tag{1.14}$$

Since the relative pushforward f_* is given by the composition $\omega_* \circ s_{f*}$ (by lemma 1.4.17), from proposition 1.4.21 and from (1.14) we get that f_* is just $g_{Z\#}$ applied to the Gysin pushforward:

$$f_! : (Z/B_Z)^{\mathrm{BM}} \left(-s_f^* \Omega_{\nu_f}\right) \longrightarrow (Y/B_Y)^{\mathrm{BM}}$$

and we are done. □

We want now to apply the functoriality of relative pushforwards to the relative Borel-Moore motives attached to the Totaro approximations of quotient stacks. In this way we will get a colimit motive approximating the motives of quotient stacks. Namely, consider $X \in \mathbf{Sch}^G_{q/S}$ and $v \in \mathcal{P}\!\mathit{erf}^G(X)$ and write $\mathcal{X} := [X/G]$. Set $X_m := X \times^G E_m G$, $v_m \in \mathcal{P}\!\mathit{erf}(X_m)$ as in notation 1.4.14, $B_m G := E_m G/G$ with maps:

$$\begin{tikzcd}
\cdots \arrow[r] & X_m \arrow[r, "\iota_m"] \arrow[d, "p_m"'] \arrow[rrr, bend left, "f_m"] \arrow[dr, phantom, "\ulcorner", very near start] & X_{m+1} \arrow[r] \arrow[d, "p_{m+1}"'] \arrow[rr, bend left, "f_{m+1}"] & \cdots \arrow[r] & \mathcal{X} \arrow[d, "p"] \\
\cdots \arrow[r] & B_m G \arrow[r, "i_m"'] \arrow[rrr, bend right, "g_m"'] & B_{m+1} G \arrow[r] \arrow[rr, bend right, "g_{m+1}"'] & \cdots \arrow[r] & \mathcal{B}G
\end{tikzcd}$$

The relative pushforwards associated to triangles:

$$\begin{tikzcd}
X_m \arrow[rr, "\iota_m"] \arrow[dr, "f_m"'] & & X_{m+1} \arrow[dl, "f_{m+1}"] \\
& \mathcal{X} &
\end{tikzcd}$$

will give us maps:

$$(\iota_m)_* : f_{m\#}\iota_m^* \longrightarrow f_{m+1\#}$$

This natural transformations will induce the following ones:

$$p_! f_{m\#}\Sigma^{v_m}\mathbb{1}_{X_m} \simeq p_! f_{m\#}\iota_m^*\Sigma^{v_{m+1}}\mathbb{1}_{X_{m+1}} \longrightarrow p_! f_{m+1\#}\Sigma^{v_{m+1}}\mathbb{1}_{X_{m+1}} \tag{1.15}$$

But since:

$$(X_k/B_kG)^{\mathrm{BM}}_{\mathcal{B}G}(v_k) := g_{k\#}p_k!\Sigma^{v_k}\mathbb{1}_{X_k} \overset{Ex_{!\#}}{\simeq} p_! f_{k\#}\Sigma^{v_k}\mathbb{1}_{X_k} \quad \forall\ k$$

from (1.15) we get a map:

$$(\iota_m)_* : (X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) \longrightarrow (X_{m+1}/B_{m+1}G)^{\mathrm{BM}}_{\mathcal{B}G}(v_{m+1})$$

On the other hand, if we consider:

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[dr, "f_m"'] & \mathcal{X} \arrow[dl, equal] \\ & \mathcal{X} \end{tikzcd}$$

the relative pushforward, that in this case is simply the co-unit $f_{m\#}f_m^* \to Id$ (cf. remark 1.4.23), will induce a natural transformation:

$$g_{m\#}p_m^*\Sigma^{v_m}\mathbb{1}_{X_m} \overset{Ex_{!\#}}{\simeq} p_! f_{m\#}f_m^*\Sigma^{v}\mathbb{1}_{\mathcal{X}} \longrightarrow p_!\Sigma^{v}\mathbb{1}_{\mathcal{X}}$$

and hence a map:

$$(f_m)_* : (X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) \longrightarrow (\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}(v)$$

By lemma 1.4.17, looking at:

$$\begin{tikzcd} X_m \arrow[r, "\iota_m"] \arrow[dr, "f_m"'] & X_{m+1} \arrow[r, "f_{m+1}"] \arrow[d, "f_{m+1}"'] & \mathcal{X} \arrow[dl, equal] \\ & \mathcal{X} & \end{tikzcd}$$

we have that $(f_m)_* = (f_{m+1})_*(\iota_m)_*$ and hence we get a well defined map:

$$(f_\infty)_* := \operatorname*{colim}_m (f_m)_* : \operatorname*{colim}_m\ (X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) \longrightarrow (\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}(v) \tag{1.16}$$

Proposition 1.4.25. *In the notation above, the following hold:*

(*i*) *We have a natural equivalence:*

$$\operatorname*{colim}_m\ f_{m\#}f_m^* \xrightarrow{\sim} Id$$

(ii) *We have a natural isomorphism:*

$$(f_\infty)_* : \operatorname*{colim}_m \ (X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) \longrightarrow (\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}(v)$$

Proof. First of all, we can already assume G is special replacing X with $X \times^G GL_n$ and rewriting $\mathcal{X}$ as $\left[X \times^G GL_n/GL_n\right]$. If G is special, we can take $x : X \to \mathcal{X}$ as our NL-atlas.

(i) Let:

$$\eta_\infty : \operatorname*{colim}_m f_{m\#}f_m^* \longrightarrow Id$$

the natural map induced by the co-units $\eta_m : f_{m\#}f_m^* \to Id$. Consider the following diagram, where c_m is the projection:

$$\begin{tikzcd} X \times E_mG \arrow[r, "c_m"] \arrow[d, "q_m^X"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "x"] \\ X_m \arrow[r, "f_m"'] & \mathcal{X} \end{tikzcd}$$

By our assumption, x is a NL-atlas. To show that η_∞ is an equivalence, by [Cho21a, Lemma 5.1.1], it is enough to show that:

$$x^*\eta_\infty : x^*\left(\operatorname*{colim}_m f_{m\#}f_m^*\right) \longrightarrow x^*Id_{\mathrm{SH}(\mathcal{X})} \simeq Id_{\mathrm{SH(X)}}$$

is an equivalence. Since x^* is a left adjoint, it commutes with colimits; moreover we have that:

$$x^*f_{m\#}f_m^* \overset{Ex^*_\#}{\simeq} c_{m\#}q_m^{X*}f_m^* \simeq c_{m\#}c_m^*x^* \tag{1.17}$$

where $Ex^*_\#$ is the exchange transformation of [Cho21a, Proposition 5.1.2]. Hence our map becomes:

$$x^*\eta_\infty : \operatorname*{colim}_m c_{m\#}c_m^*x^* \longrightarrow Id_{\mathrm{SH}(X)} \tag{1.18}$$

But in $\mathrm{SH}(X)$, the map $\operatorname*{colim}_m c_{m\#}c_m^*\mathbb{1}_X \simeq \operatorname*{colim}_m \Sigma^\infty(X \times E_mG) \longrightarrow \mathbb{1}_X$ is an $\mathbb{A}^1$-equivalence by [MV99, §4 Proposition 2.3]. By the projection formula [Cho21a, Theorem 5.5.1], for any $\mathbb{E} \in \mathrm{SH}(X)$ we have that:

$$\operatorname*{colim}_m c_{m\#}c_m^*\mathbb{E} = \operatorname*{colim}_m c_{m\#}\left(\mathbb{1}_{X\times E_mG} \otimes c_m^*\mathbb{E}\right) \simeq \left(\operatorname*{colim}_m c_{m\#}\mathbb{1}_{X\times E_mG}\right) \otimes \mathbb{E} \simeq$$
$$\simeq \left(\operatorname*{colim}_m c_{m\#}c_m^*\mathbb{1}_X\right) \otimes \mathbb{E} \xrightarrow{\sim} \mathbb{E}$$

where the last equivalence follows from $x^*\eta_\infty$ evaluated at $\mathbb{1}_X$. Thus we have an equivalence:

$$x^*\left(\operatorname*{colim}_m\ f_{m\#}f_m^*\right) \simeq \operatorname*{colim}_m\ c_{m\#}c_m^*x^* \xrightarrow{\sim} Id_{\mathrm{SH}(X)} \simeq x^*Id_{\mathrm{SH}(\mathcal{X})} \tag{1.19}$$

By conservativity of x^*, as we already said, this is enough to conclude that η_∞ is an equivalence too.

(ii) Now we want to show that:

$$(f_\infty)_* : \operatorname*{colim}_m\ (X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) \longrightarrow (\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}(v)$$

is an equivalence. Each element in the colimit can be rewritten as:

$$(X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) := g_{m\#}p_m!\Sigma^{v_m}\mathbb{1}_{X_m} \overset{Ex_{!\#}}{\simeq} p_!f_{m\#}f_m^*\Sigma^v\mathbb{1}_{\mathcal{X}}$$

where we are using the cartesian diagram:

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d, "p_m"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X} \arrow[d, "p"] \\ B_mG \arrow[r, "g_m"'] & \mathcal{B}G \end{tikzcd}$$

So the colimit can be rewritten as:

$$\operatorname*{colim}_m\ (X_m/B_mG)^{\mathrm{BM}}_{\mathcal{B}G}(v_m) \simeq p_!\left(\operatorname*{colim}_m\ f_{m\#}f_m^*\right)\Sigma^v\mathbb{1}_{\mathcal{X}}$$

since $p_!$ is a left adjoint and hence commutes with colimits. Then if we prove that:

$$\eta_\infty : \operatorname*{colim}_m\ f_{m\#}f_m^* \longrightarrow Id$$

is an equivalence, we will get that $(f_\infty)_*$ is an equivalence too. Therefore the claim follows from part (i).

□

1.4.2 Some Comparisons of Motives

Lemma 1.4.26. *Let $M \in \mathbf{Sm}^G_{/S}$ such that the (fppf) quotient $[M/G] \simeq Q_M \in \mathbf{Sm}_{/S}$ is represented by a scheme. Let π_{Q_M} and $\pi_{\mathcal{B}G}$ be the structure maps of Q_M and $\mathcal{B}G$, with f_M the map between them fitting in the following diagram:*

$$\begin{tikzcd} Q_M \arrow[r, "f_M"'] \arrow[dr, bend right, "\pi_{Q_M}"'] & \mathcal{B}G \arrow[d, "\pi_{\mathcal{B}G}"] \\ & S \end{tikzcd}$$

Then we have:

(*i*) $\pi_{Q_M!} \simeq \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{M!}$

(*ii*) $(Q_M/S)^{\mathrm{BM}} \simeq (M/S)^{\mathrm{BM}}_G$

Proof. (*i*) The quotient $Q_M \simeq [M/G]$ is smooth and its cotangent complex is:

$$\Omega_{Q_M} \simeq \left[\mathbb{L}^G_{M/S} \to \mathfrak{g}^\vee_M\right]$$

where $\mathbb{L}^G_{M/S}$ sitting in (homological) degree zero is the locally free sheaf obtained from $\Omega_{M/S}$ (and its G-linearisation) and where $\mathfrak{g}^\vee_M$ is the locally free sheaf obtained from $\mathfrak{g}^\vee \otimes \mathcal{O}_M$, with $\mathfrak{g}^\vee$ the sheaf of the co-Lie algebra of G. We have equivalences:

$$\mathfrak{g}^\vee \simeq \mathbb{L}_{\mathcal{B}G}[+1]$$
$$\mathfrak{g}^\vee_M \simeq f_M^*\mathfrak{g}^\vee$$
$$\mathbb{L}_{f_M} \simeq \mathbb{L}^G_{M/S}$$

where the last equivalence follows from descent and the fact that $Q_M \simeq \mathcal{B}G \times_S M$. By functoriality of the motivic J-homomorphism, we have:

$$\Sigma^{\Omega_{Q_M}} \simeq \Sigma^{\mathbb{L}^G_{M/S}}\Sigma^{-\mathfrak{g}^\vee_M} \tag{1.20}$$

By functoriality of $(-)_\#$ we have:

$$\pi_{Q_M\#} \simeq \pi_{\mathcal{B}G\#}f_{M\#} \tag{1.21}$$

Piecing everything together, we then get:

$$\begin{aligned} \pi_{Q_M!} &\simeq \pi_{Q_M\#}\Sigma^{-\Omega_{Q_M}} \simeq \\ &\overset{(1.20)}{\simeq} \pi_{Q_X\#}\Sigma^{-\mathbb{L}^G_{M/S}}\Sigma^{\mathfrak{g}^\vee_M} \simeq \\ &\overset{(1.21)}{\simeq} \pi_{\mathcal{B}G\#}f_{M\#}\Sigma^{-\mathbb{L}^G_{M/S}}\Sigma^{\mathfrak{g}^\vee_M} \simeq \\ &\simeq \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee}f_{M\#}\Sigma^{-\mathbb{L}^G_{M/S}} \simeq \\ &\simeq \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee}f_{M!} \end{aligned}$$

where the first and last equivalences follow from (representable) relative purity ([Cho21a, Proposition 5.4.1]), and the second to last follows from the commutativity of suspension transformations with the appropriate six functors $f_{M\#}\Sigma^{\mathfrak{g}^\vee_M} = f_{M\#}\Sigma^{f_M^*\mathfrak{g}^\vee} \simeq \Sigma^{\mathfrak{g}^\vee}f_{M\#}$.

(ii) We can deduce the equality on Borel-Moore motives from the previous point. Indeed, evaluating at $\mathbb{1}_{Q_M}$ the equivalence proved before we get:

$$(Q_M/S)^{\mathrm{BM}} = \pi_{Q_M!}\mathbb{1}_{Q_M} \simeq \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{M!}\mathbb{1}_{Q_M} = (M/S)^{\mathrm{BM}}_G$$

□

Remark 1.4.27. The previous lemma works in greater generality. Indeed, we only need to require Q_M to be represented by a quasi-separated algebraic space and the proof works verbatim. This requirement was taken only to assure that the exceptional functors and purity statements hold, and this is the case for quasi-separated algebraic spaces by [Cho21a, Theorem 5.5.1] applied to this very special case.

If we have a closed immersion $X \hookrightarrow M$ in $\mathbf{Sch}^G_{/S}$, such that $[M/G] \simeq Q_M$ is represented by a scheme, then $[X/G] \simeq Q_X$ is represented by a scheme too. This holds in general if we relax the representability requirement: whenever we have a representable map of algebraic stacks, such that the target of the map is an algebraic space, then the source is represented by an algebraic space too (cf. [Alp23, Lemma 3.3.1]).

Proposition 1.4.28. *Let $\iota : X \hookrightarrow M$ a closed immersion in $\mathbf{Sch}^G_{/S}$, such that M is smooth and the quotient $[M/G] \simeq Q_M$ is representable by a scheme $Q_M \in \mathbf{Sch}_{/S}$. Denote by $Q_X := [X/G]$ the scheme representing the $G-$quotient of X. Then there is a natural isomorphism:*

$$(Q_X/S)^{\mathrm{BM}} \simeq (X/S)^{\mathrm{BM}}_G$$

Proof. Let us fix some notations. Let $U := M\setminus X$ be the open complement of X. The map $[U/G] \hookrightarrow [M/G] \simeq Q_M$ is a representable map. Since $[M/G] \simeq Q_M \in \mathbf{Sch}_{/S}$, then $Q_U := [U/G]$ is an algebraic space by [Alp23, Lemma 3.3.1] and hence a scheme since Q_U is open inside Q_M (that was a scheme by assumption). We will denote the various maps in the following diagram as:

$$\begin{tikzcd}
\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "q^\bullet_m"'] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d, "a"] & \\
Q_\bullet \arrow[r, "f_\bullet"'] \arrow[rrd, bend right, "\pi_{Q_\bullet}"'] & \mathcal{B}G \arrow[rd, "\pi_{\mathcal{B}G}"'] & \\
 & & S
\end{tikzcd}$$

for $\bullet = M, X, U$. Let $j_G : Q_U \simeq [U/G] \hookrightarrow Q_M \simeq [M/G]$ and $\iota_G : Q_X \simeq [X/G] \hookrightarrow Q_M \simeq [M/G]$ be the induced open and closed immersions on the respective quotients. The map $j_G : Q_U \hookrightarrow Q_M$ gives us the co-unit map $\eta^*_\#(j_G) : j_\# j^*_G \simeq j_{G!}h^*_G \to Id$ and hence we get an induced natural transformation:

$$\begin{array}{ccccc}
\pi_{Q_U!}\pi^*_{Q_U} & \simeq & \pi_{Q_M!}j_{G!}j_G^*\pi^*_{Q_M} & \xrightarrow{j_{G!}} & \pi_{Q_M!}\pi^*_{Q_M} \\
\wr\| & & & & \wr\| \\
\pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{U!}\pi^*_{Q_U} & \simeq & \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{M!}j_{G!}j_G^*\pi^*_{Q_M} & \xrightarrow{j_{G!}} & \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{M!}\pi^*_{Q_M}
\end{array} \tag{1.22}$$

where the vertical equivalences follow from the first statement in lemma 1.4.26. But using the localization sequence relative to the pair (j_G, ι_G), we know that we have a fiber sequence:

$$j_{G\#}j_G^* \longrightarrow Id \longrightarrow \iota_{G*}\iota_G^*$$

and thus we have fiber sequences:

$$\pi_{Q_U!}\pi^*_{Q_U} \xrightarrow{j_{G!}} \pi_{Q_M!}\pi^*_{Q_M} \xrightarrow{\iota_G^*} \pi_{Q_X!}\pi^*_{Q_X}$$

$$\pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{U!}\pi^*_{Q_U} \xrightarrow{j_{G!}} \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{M!}\pi^*_{Q_M} \xrightarrow{\iota_G^*} \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{X!}\pi^*_{Q_X}$$

Using (1.22), we then get:

$$\begin{aligned}
\pi_{Q_X!}\pi^*_{Q_X} &\simeq \mathrm{Cofib}\left(\pi_{Q_U!}\pi^*_{Q_U} \xrightarrow{j_{G!}} \pi_{Q_M!}\pi^*_{Q_M}\right) \simeq \\
&\overset{(1.22)}{\simeq} \mathrm{Cofib}\left(\pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{U!}\pi^*_{Q_U} \xrightarrow{j_{G!}} \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{M!}\pi^*_{Q_M}\right) \simeq \\
&\simeq \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee} f_{X!}\pi^*_{Q_X}
\end{aligned}$$

and this concludes our proof. □

Corollary 1.4.29. *Let $X \in \mathbf{Sch}^G_{q/S}$ and $X_m := X \times^G E_mG$. Then we have for each m we have a natural isomorphism:*

$$(X_m/S)^{\mathrm{BM}} \simeq (X \times E_mG/S)^{\mathrm{BM}}_G$$

Proof. Just apply proposition 1.4.28 to $X \times E_mG$ with quotient $Q_{X\times E_mG} = X_m$. □

The following is a well known fact (cf. for example [KR21, Theorem 12.16]):

Proposition 1.4.30. *Let $M \in \mathbf{Sm}^G_{q/S}$. Let $\pi_{M_m} : M_m := M \times^G E_mG \to S$ and $\pi_{\mathcal{M}} : \mathcal{M} \to S$ be the structure maps. Let $v \in \mathcal{P}erf^G(M)$ and let v_m be the corresponding object in $\mathcal{P}erf(M_m)$, as in notation 1.4.14. Then there exists a natural equivalence:*

$$\operatorname*{colim}_m \pi_{M_m\#}\Sigma^{v_m}\mathbb{1}_{M_m} \longrightarrow \pi_{\mathcal{M}\#}\Sigma^{v}\mathbb{1}_{\mathcal{M}}$$

Proof. We can assume that G is special, otherwise replace M with $M \times^G GL_n$. Consider the following diagram:

$$\begin{tikzcd} M \times E_m G \arrow[r, "b_m"] \arrow[d, "q_m"'] \arrow[dr, phantom, "\ulcorner\ \Delta"] & M \arrow[d, "a"] \\ M_m \arrow[r, "\beta_m"'] & \mathcal{M} \end{tikzcd}$$

Notice that every map in the square above is representable and a is a NL-atlas. Since β_m is smooth, by smooth pushforward for each m we have a map:

$$\pi_{M_m\#}\Sigma^{v_m}\mathbb{1}_{M_m} \simeq \pi_{\mathcal{M}\#}\beta_{m\#}\beta_m^*\Sigma^v\pi_{\mathcal{M}}\mathbb{1}_{\mathcal{M}} \longrightarrow \pi_{\mathcal{M}\#}\Sigma^v\mathbb{1}_{\mathcal{M}}$$

induced by $\eta^*_{\#}(\beta_m) : \beta_{m\#}\beta_m^* \to Id$. By naturality of the co-unit transformations, we get a map a map:

$$\operatorname*{colim}_m \pi_{M_m\#}\Sigma^{v_m}\mathbb{1}_{M_m} \longrightarrow \pi_{\mathcal{M}\#}\Sigma^v\mathbb{1}_{\mathcal{M}}$$

Since $\pi_{M_m} = \pi_{\mathcal{M}}\circ\beta_m$ and since $\pi_{\mathcal{M}\#}$ is a left adjoint (hence commutes with colimits), we have:

$$\operatorname*{colim}_m \pi_{M_m\#}\Sigma^{v_m}\mathbb{1}_{M_m} \simeq \pi_{\mathcal{M}\#}\operatorname*{colim}_m \beta_{m\#}\beta_m^*\Sigma^v\mathbb{1}_{\mathcal{M}}$$

To prove our claim is enough to show that the natural map:

$$\operatorname*{colim}_m \beta_{m\#}\beta_m^* \longrightarrow Id \tag{1.23}$$

induced by smooth pushforward along the β_m's is an equivalence. This follows by the first part of proposition 1.4.25.

□

Corollary 1.4.31. *Let $M \in \mathbf{Sm}^G_{q/S}$. Let $\pi_{M_m} : M_m := M \times^G E_m G \to S$ and $\pi_{\mathcal{M}} : \mathcal{M} \to S$ be the structure maps.Then there exists a natural equivalence:*

$$\operatorname*{colim}_m \pi_{M_m\#}\mathbb{1}_{M_m} \longrightarrow \pi_{\mathcal{M}\#}\mathbb{1}_{\mathcal{M}}$$

Corollary 1.4.32. *Let $\mathbb{E} \in \mathrm{SH}(S)$ a motivic ring spectrum and let $M \in \mathbf{Sm}^G_{q/S}$. Denote by $\mathcal{M} := [M/G]$ and by $M_m := M \times^G E_m G$, with respective structure maps over S denoted as $\pi_{\mathcal{M}}$ and π_{M_m}. Let $v \in \mathcal{P}erf^G(M)$ and let $\{v_m\}$ the induced compatible system of perfect complexes on each M_m. Then we have:*

$$\mathbb{E}(\mathcal{M}, v) = \mathbb{E}_G(M, v) \simeq \lim_m \mathbb{E}(M_m, v_m)$$

Proof. It is just a consequence of proposition 1.4.30, indeed:

$$\begin{aligned}
\mathbb{E}_G(M,v) &:= \mathrm{Map}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \pi_{\mathcal{M}*}\Sigma^v\pi_{\mathcal{M}}^*\mathbb{E}\right) \simeq \\
&\simeq \mathrm{Map}_{\mathrm{SH}(S)}\left(\pi_{\mathcal{M}\#}\Sigma^{-v}\pi_{\mathcal{M}}^*\mathbb{1}_S, \mathbb{E}\right) \simeq \\
&\simeq \mathrm{Map}_{\mathrm{SH}(S)}\left(\operatorname*{colim}_m \pi_{M_m\#}\Sigma^{-v_m}\pi_{M_m}^*\mathbb{1}_S, \mathbb{E}\right) \simeq \\
&\simeq \lim_m \mathrm{Map}_{\mathrm{SH}(S)}\left(\pi_{M_m\#}\Sigma^{-v_m}\pi_{M_m}^*\mathbb{1}_S, \mathbb{E}\right) \simeq \\
&\simeq \lim_m \mathrm{Map}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \pi_{M_m*}\Sigma^{v_m}\pi_{M_m}^*\mathbb{E}\right) =: \lim_m \mathbb{E}(M_m, v_m)
\end{aligned}$$

□

Remark 1.4.33. Notice that, by remark 1.4.22, the transition maps of $\mathbb{E}(BG) \simeq \lim_m \mathbb{E}(B_mG)$ are given by $\Sigma^\infty i_m$ where $i_m : B_mG \hookrightarrow B_{m+1}G$ is the natural inclusion map. In particular this means that we can compute $\mathbb{E}(\mathcal{B}G)$ using the classical limit of mapping spectra given by $\mathbb{E}(B_mG)$.

Remark 1.4.34. Given $\mathbb{E} \in \mathrm{SH}(S)$, we know by [Hoy17, Corollary 6.25] that the functor:

$$\begin{array}{ccc}
\left(\mathbf{Sch}_{/S}\right)^{op} & \longrightarrow & \mathrm{SH}(S) \\
f : X \to S & \mapsto & f_*f^*\mathbb{E}
\end{array}$$

is a *cdh*-sheaf (hence a Nisnevich sheaf). By the construction of $\mathrm{SH}^{\triangleleft}$, obtained as NL-sheafification (cf. [Cho21a, Theorem 3.4.1]), this implies that for any NL-stack $\mathcal{Y}$ and any $\mathbb{F} \in \mathrm{SH}(\mathcal{Y})$, the functor:

$$\begin{array}{ccc}
\left(\mathcal{ASt}^{NL}_{/\mathcal{Y}}\right)^{op} & \longrightarrow & \mathrm{SH}(\mathcal{Y}) \\
g : \mathcal{X} \to \mathcal{Y} & \mapsto & g_*g^*\mathbb{F}
\end{array}$$

is a NL-sheaf. Then, if $X \to \mathcal{X}$ is a NL-atlas and $g_n : X^n_{\mathcal{X}} := X \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} X \to \mathcal{Y}$ are the structure maps, we have:

$$g_*g^* \simeq \lim_{n\in\Delta} (g_n)_*(g_n)^*$$

by NL-descent.

Remark 1.4.35. The corollary 1.4.31 was already proved in a different form in [Kri12, Proposition 3.2]. Indeed, consider M as in corollary 1.4.31. Let $\pi_{M^n_{\mathcal{M}}} : M^n_{\mathcal{M}} := M \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} M \to S$ and $\pi_{\mathcal{M}} : \mathcal{M} \longrightarrow S$ be the structure maps. By remark 1.4.34, we have:

$$\pi_{\mathcal{M}*}\pi_{\mathcal{M}}^* \simeq \lim_{n\in\Delta} \pi_{M^n_{\mathcal{M}}*}\pi^*_{M^n_{\mathcal{M}}}$$

For any $\mathbb{E} \in \mathrm{SH}(S)$, we have:

$$\begin{aligned}\mathrm{Map}_{\mathrm{SH}(S)}\left(\pi_{\mathcal{M}\#}\mathbb{1}_{\mathcal{M}}, \mathbb{E}\right) &\simeq \mathrm{Map}_{\mathrm{SH(S)}}\left(\mathbb{1}_S, \pi_{\mathcal{M}\#}\pi_{\mathcal{M}}^*\mathbb{E}\right) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \lim_{n\in\Delta} \pi_{M^n_{\mathcal{M}}*}\pi^*_{M^n_{\mathcal{M}}}\mathbb{E}\right) \simeq \\ &\simeq \lim_{n\in\Delta} \mathrm{Map}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \pi_{M^n_{\mathcal{M}}*}\pi^*_{M^n_{\mathcal{M}}}\right) \simeq \\ &\simeq \lim_{n\in\Delta} \mathrm{Map}_{\mathrm{SH}(S)}\left(\pi_{M^n_{\mathcal{M}}\#}\mathbb{1}_{Mn_{\mathcal{M}}}, \mathbb{E}\right) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(S)}\left(\operatorname*{colim}_{n\in\Delta} \pi_{M^n_{\mathcal{M}}\#}\mathbb{1}_{Mn_{\mathcal{M}}}, \mathbb{E}\right)\end{aligned}$$

This means that:

$$\pi_{\mathcal{M}\#}\pi^*_{\mathcal{M}} \simeq \operatorname*{colim}_{n\in\Delta} \pi_{M^n_{\mathcal{M}}\#}\pi^*_{M^n_{\mathcal{M}}}$$

by the (co-)Yoneda Lemma. But in $\mathrm{SH}(S)$ we have that $\pi_{M^n_{\mathcal{M}}\#}\pi^*_{M^n_{\mathcal{M}}}\mathbb{1}_S \simeq \Sigma^\infty M^n_{\mathcal{M}}$ and hence:

$$\operatorname*{colim}_{n\in\Delta} \pi_{M^n_{\mathcal{M}}\#}\pi^*_{M^n_{\mathcal{M}}}\mathbb{1}_S \simeq |\check{C}_\bullet\left(M/\mathcal{M}\right)|$$

where the right hand side is the geometric realization of the Čech nerve given by the atlas $M \to \mathcal{M}$. Then the content of [Kri12, Proposition 3.2] is equivalent to the content of our corollary 1.4.31.

1.5 Properties of Equivariant Bivariant Theories

Proposition 1.5.1. *Let $f : X \longrightarrow Y$ in $\mathbf{Sch}^G_{q/S}$, $v \in \mathrm{K}^G_0(Y)$ and $\mathbb{E} \in \mathrm{SH}(S)$ a motivic ring spectrum. Let $f_G : \mathcal{X} := [X/G] \to \mathcal{Y} := [Y/G]$ be the representable map induced on the quotient stacks.*

1. *If f is a regular embedding, then f_G is a regular embedding and induces a map:*

$$f^!_G : \mathbb{E}^{\mathrm{BM}}_G\left(Y/S, v\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\left(X/S, v + \mathbb{L}_f\right)$$

2. *If f is a proper morphism, then f_G is a representable proper morphism and induces a map:*

$$(f_G)_* : \mathbb{E}^{\mathrm{BM}}_G\left(X/S, v\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\left(Y/S, v\right)$$

3. *For f smooth, the morphism f_G is smooth and induces a map:*

$$f^!_G : \mathbb{E}^{\mathrm{BM}}_G\left(Y/S, v\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\left(X/S, v + \mathbb{L}_f\right)$$

4. *For any f, the morphism f_G induces a map:*

$$f^*_G : \mathbb{E}_G\left(Y, v\right) \longrightarrow \mathbb{E}_G\left(X, v\right)$$

5. *For $v \in \mathrm{K}_0^G(X), w \in \mathrm{K}_0^G(Y)$ we have cup product and cap product maps:*

$$\mathbb{E}_G^{a,b}(X,v) \times \mathbb{E}_G^{c,d}(X,w) \xrightarrow{\cup} \mathbb{E}_G^{a+c,b+d}(X,v+w)$$

$$\mathbb{E}^{\mathrm{BM}}_{a,b,G}(X/S,v) \times \mathbb{E}_G^{c,d}(X,w) \xrightarrow{\cap} \mathbb{E}^{\mathrm{BM}}_{a-c,b-d,G}(X/S,v-w)$$

6. *Let $[S]_G \in \mathbb{E}_G^{0,0}(S)$ be the element arising from the natural map $\eta_{\mathbb{B}G} : \pi_{\mathbb{B}G\#}\pi^*_{\mathbb{B}G}\mathbb{1}_S \to \mathbb{1}_S$ composed with the unit map $\mathbb{1}_S \to \mathbb{E}$. Let $\pi_X : X \to S$ the structure map of a scheme in $\mathbf{Sm}^G_{q/S}$. Then we define the equivariant fundamental class of X as:*

$$[X]_G := (\pi_X)^!_G([S]_G)$$

Moreover, given $v \in \mathrm{K}_0^G(X)$, we have a Poincaré duality isomorphism given by the cap product with the fundamental class:

$$[X]_G \cap \cdot : \mathbb{E}_G^{a,b}(X,v) \xrightarrow{\sim} \mathbb{E}^{\mathrm{BM}}_{-a,-b,G}\left(X/S, \mathbb{L}_{[X/G]/S} - v\right)$$

Proof. The claims in (1),(2) and (3) follow directly from the respective operations of Borel-Moore motives in 1.4.11, 1.4.10. Indeed it is enough to notice that:

$$(X/S)^{\mathrm{BM}}_G = \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee}\pi^!_{\mathcal{X}}\mathbb{1}_{\mathcal{X}} = \pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee}(\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}$$

where $\pi_{\mathbb{B}G} : \mathcal{B}G \to S$ is the structure map of $\mathcal{B}G$ and where $\pi^G_{\mathcal{X}} : \mathcal{X} := [X/G] \to \mathcal{B}G$. A similar description holds also for $(Y/S)^{\mathrm{BM}}_G$ (and for the twisted versions as well). Applying the operations on Borel-Moore motives over $\mathcal{B}G$, we get the smooth and Gysin pullback and the proper pushforward on equivariant Borel-Moore homology as we wanted.
The pullback map in (4) is easily defined using the adjunction $f_{G*}f^*_G \to Id$. Denote by $\pi_{\mathcal{X}} : \mathcal{X} \to S$ and $\pi_{\mathcal{Y}} : \mathcal{Y} \to S$ the structure maps. The map:

$$\pi_{\mathcal{X}*}\pi^*_{\mathcal{X}} \simeq \pi_{\mathcal{Y}*}f_{G*}f^*_G\pi^*_{\mathcal{Y}} \xrightarrow{\eta^*_*(f_G)} \pi_{\mathcal{Y}*}\pi^*_{\mathcal{Y}}$$

will induce the map:

$$\mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{1}_S, \pi_{\mathcal{Y}*}\Sigma^v\pi^*_{\mathcal{Y}}\mathbb{E}) \longrightarrow \mathrm{Map}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \pi_{\mathcal{X}*}\Sigma^{f^*_G v}\pi^*_{\mathcal{X}}\mathbb{E}\right)$$

that we define to be our map:

$$f^*_G : \mathbb{E}_G(Y,v) \longrightarrow \mathbb{E}_G(X,v)$$

For (5) we can consider maps $\pi^G_{\mathcal{X}} : \mathcal{X} \to \mathcal{B}G$ and $\pi^G_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{B}G$. Recall from remark 1.4.9, the equivariant cohomology and Borel-Moore homology can be rewritten as:

$$\mathbb{E}_G(X,v) = \mathrm{Map}_{\mathrm{SH}(\mathcal{B}G)}(\mathbb{1}_{\mathcal{B}G}, \pi^G_{\mathcal{X}*}\Sigma^v\pi^{G*}_{\mathcal{X}}(\pi^*_{\mathcal{B}G}\mathbb{E})) = \pi^*_{\mathcal{B}G}\mathbb{E}(\mathcal{X},v)$$

$$
\begin{aligned}
\mathbb{E}_G^{\mathrm{BM}}(X/S, v) &= \mathrm{Map}_{\mathrm{SH}(S)}\left(\pi_{\mathcal{B}G\#}\Sigma^{\mathfrak{g}^\vee}(\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}(v), \mathbb{E}\right) \simeq \\
&\simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{B}G)}\left((\mathcal{X}/\mathcal{B}G)^{\mathrm{BM}}(v+\mathfrak{g}^\vee), \pi_{\mathcal{B}G}^*\mathbb{E}\right) = \\
&= \pi_{\mathcal{B}G}^*\mathbb{E}^{\mathrm{BM}}\left(\mathcal{X}/\mathcal{B}G, v+\mathfrak{g}^\vee\right)
\end{aligned}
$$

Similar descriptions hold for $\mathcal{Y}$ too. Hence we can construct cup products and cap products for representable maps of stacks over $\mathcal{B}G$ and use those as definition for the equivariant theories under the above identification. But for representable maps of NL-stacks the constructions in [DJK21, §2.2.7(4), 2.2.9] work formally also for our case, using the six functors of [Cho21a, Theorem 5.5.1] and the exchange transformations in proposition 1.3.6.

The Poincaré duality statement in (6) follows from the purity isomorphism for $\pi_{\mathcal{X}}^G : \mathcal{X} \to \mathcal{B}G$ ([Cho21a, Theorem 5.5.1]). Indeed, using the purity isomorphism $(\pi_{\mathcal{X}}^G)^* \simeq \Sigma^{-\mathbb{L}_{\mathcal{X}/\mathcal{B}G}}(\pi_{\mathcal{X}}^G)^!$ we get:

$$
\begin{aligned}
\mathbb{E}_G(\mathcal{X}, v) &\simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{B}G)}\left(\mathbb{1}_{\mathcal{B}G}, (\pi_{\mathcal{X}}^G)_*\Sigma^v(\pi_{\mathcal{X}}^G)^*\pi_{\mathcal{B}G}^*\mathbb{E}\right) \simeq \\
&\simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{B}G)}\left(\mathbb{1}_{\mathcal{B}G}, (\pi_{\mathcal{X}}^G)_*\Sigma^v\Sigma^{-\mathbb{L}_{\mathcal{X}/\mathcal{B}G}}(\pi_{\mathcal{X}}^G)^!\pi_{\mathcal{B}G}^*\mathbb{E}\right) = \\
&= \pi_{\mathcal{B}G}^*\mathbb{E}^{\mathrm{BM}}\left(\mathcal{X}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/\mathcal{B}G} - v\right)
\end{aligned}
\tag{1.24}
$$

But $\mathbb{L}_{\mathcal{X}/S} \simeq \left[\mathbb{L}_{\mathcal{X}/\mathcal{B}G} \to (\pi_{\mathcal{X}}^G)^*\mathfrak{g}^\vee\right]$, hence $\Sigma^{\mathbb{L}_{\mathcal{X}/S}}\Sigma^{\mathfrak{g}^\vee} \simeq \Sigma^{\mathbb{L}_{\mathcal{X}/\mathcal{B}G}}$. This implies that:

$$
\pi_{\mathcal{B}G}^*\mathbb{E}^{\mathrm{BM}}\left(\mathcal{X}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/\mathcal{B}G} - v\right) \simeq \pi_{\mathcal{B}G}^*\mathbb{E}^{\mathrm{BM}}\left(\mathcal{X}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/S} + \mathfrak{g}^\vee - v\right)
$$

From the identifications remark 1.4.9, we can write:

$$
\mathbb{E}_G^{\mathrm{BM}}\left(X/S, \mathbb{L}_{\mathcal{X}/S} - v\right) \simeq \pi_{\mathcal{B}G}^*\mathbb{E}^{\mathrm{BM}}\left(\mathcal{X}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/S} + \mathfrak{g}^\vee - v\right)
$$

Therefore using (1.24), we get our Poincaré duality isomorphism:

$$
\mathbb{E}_G(\mathcal{X}, v) \simeq \mathbb{E}_G^{\mathrm{BM}}\left(X/S, \mathbb{L}_{\mathcal{X}/S} - v\right)
$$

□

Proposition 1.5.2 (Equivariant Localization Sequence)**.** *Let $\iota : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$ in $\mathbf{Sch}_{q/S}^G$. For $v \in \mathrm{K}_0^G(X)$, we have the following fiber sequence:*

$$
(U/S)_G^{\mathrm{BM}}(v) \xrightarrow{j_!^G} (X/S)_G^{\mathrm{BM}}(v) \xrightarrow{\iota_G^*} (Z/S)_G^{\mathrm{BM}}(v)
$$

For $\mathbb{E} \in \mathrm{SH}(S)$, we also have a twisted fiber sequence on bivariant homology:

$$
\mathbb{E}_G^{\mathrm{BM}}(Z/S, v) \xrightarrow{\iota_*^G} \mathbb{E}_G^{\mathrm{BM}}(X/S, v) \xrightarrow{j_G^*} \mathbb{E}_G^{\mathrm{BM}}(U/S, v) \tag{1.25}
$$

Proof. As we did in remark 1.4.9, we can rewrite all the equivariant Borel-Moore motives as motives over $\mathcal{B}G$ (up to a twist). Then we just apply the localization sequence of [Cho21a, Proposition 5.2.1]. Evaluating at $\mathcal{E}$ we get the localization sequence for Borel-Moore homology. $\square$

Corollary 1.5.3. *Let $\iota : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$ with $\mathcal{Z}, \mathcal{X}, \mathcal{U} \in \mathbf{Sm}^G_{q/S}$. For $\mathbb{E} \in \mathrm{SH}(S)$ and $v \in \mathrm{K}_0^G(X)$, we have the following localization sequence in equivariant cohomology:*

$$\mathbb{E}_G\left(Z, v + \mathcal{N}_{\mathcal{Z}/\mathcal{X}}\right) \xrightarrow{\iota_*^G} \mathbb{E}_G\left(X, v\right) \xrightarrow{j_G^*} \mathbb{E}_G\left(U, v\right) \tag{1.26}$$

where $\mathcal{N}_{\mathcal{Z}/\mathcal{X}}$ is conormal sheaf associated to the closed immersion $\iota : \mathcal{Z} := [Z/G] \hookrightarrow \mathcal{X} := [X/G]$.

Proof. This is a consequence of proposition 1.5.2 and purity. By remark 1.4.9, we can rephrase everything in terms of quotient stacks $\tilde{\iota} : \mathcal{Z} := [Z/G] \hookrightarrow \mathcal{X} := [X/G]$ and $\tilde{j} : \mathcal{U} := [U/G] \hookrightarrow \mathcal{X}$. Since Z, X are smooth, the map $\tilde{\iota} : \mathcal{Z} \hookrightarrow \mathcal{X}$ is lci and the normal sheaf $\mathcal{N}_{\mathcal{Z}/\mathcal{X}} \simeq \mathcal{N}^G_{Z/X}$ can be obtained by from $\mathcal{N}_{Z/X}$ (with its natural G-linearisation). Moreover we have that $\mathbb{L}_{\mathcal{Z}/\mathcal{B}G} \simeq \left[\mathcal{N}_{\mathcal{Z}/\mathcal{X}}[1] \to \mathbb{L}_{\mathcal{X}/\mathcal{B}G}\right]$. Using purity we get:

$$\begin{aligned}
\mathbb{E}^{\mathrm{BM}}\left(\mathcal{Z}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/\mathcal{B}G} - v\right) &\simeq \mathbb{E}\left(\mathcal{Z}, v + \mathcal{N}_{\mathcal{Z}/\mathcal{X}}\right)\\
\mathbb{E}^{\mathrm{BM}}\left(\mathcal{X}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/\mathcal{B}G} - v\right) &\simeq \mathbb{E}\left(\mathcal{X}, v\right)\\
\mathbb{E}^{\mathrm{BM}}\left(\mathcal{U}/\mathcal{B}G, \mathbb{L}_{\mathcal{X}/\mathcal{B}G} - v\right) &\simeq \mathbb{E}\left(\mathcal{U}, v\right)
\end{aligned}$$

Thus, from the localization sequence in Borel-Moore homology we get:

$$\mathbb{E}\left(\mathcal{Z}, v + \mathcal{N}_{\mathcal{Z}/\mathcal{X}}\right) \xrightarrow{\iota_*^G} \mathbb{E}\left(\mathcal{X}, v\right) \xrightarrow{j_G^*} \mathbb{E}\left(\mathcal{U}, v\right)$$

$\square$

Proposition 1.5.4 (Equivariant refined Gysin Pullbacks)**.** *Let:*

$$\begin{tikzcd}
W \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & Z \arrow[d, "p"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

be a cartesian square in $\mathbf{Sch}^G_{q/S}$ with f a regular embedding. Consider $v \in \mathrm{K}_0^G(Z)$ and $\mathbb{E} \in \mathrm{SH}(S)$ a motivic ring spectrum. We have a well defined map:

$$(g)^!_{\Delta,G} : \mathbb{E}^{\mathrm{BM}}_G\left(Z/S, v\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\left(W/S, v + \mathbb{L}_f\right)$$

induced by the equivariant operations on the quotient stacks $\mathcal{W} := [W/G]$ and $\mathcal{Z} := [Z/G]$.

Proof. We will just apply the construction in [DJK21, Definition 4.2.5] to our case. Consider the cartesian square of NL-Stacks:

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{q}"'] \arrow[dr, phantom, "\tilde{\Delta}"] & \mathcal{Z} \arrow[d, "\tilde{p}"] \\ \mathcal{X} \arrow[r, "\tilde{f}"'] & \mathcal{Y} \end{tikzcd}$$

where $\mathcal{W} = [W/G]$, $\mathcal{Z} = [Z/G]$, $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/G]$. For any $w \in \mathrm{K}_0^G(X)$ and any $\mathbb{F} \in \mathrm{SH}(Y)$, we have a base change canonical transformation:

$$\tilde{\Delta}^* : \mathbb{F}^{\mathrm{BM}}(\mathcal{X}/\mathcal{Y}, w) \longrightarrow \mathbb{F}^{\mathrm{BM}}(\mathcal{W}/\mathcal{Z}, \tilde{q}^* w) \tag{1.27}$$

induced by:

$$\tilde{f}_* \Sigma^{-w} \tilde{f}^! \xrightarrow{\epsilon^*_*(\tilde{p})} \tilde{f}_* \Sigma^{-w} \tilde{f}^! \tilde{p}_* \tilde{p}^* \overset{Ex^!_*}{\simeq} \tilde{f}_* \Sigma^{-w} \tilde{q}_* \tilde{g}^! \tilde{p}^* \simeq \tilde{f}_* \tilde{q}_* \Sigma^{-\tilde{q}^* w} \tilde{g}^! \tilde{p}^* \simeq \tilde{p}_* \tilde{g}_* \Sigma^{-\tilde{q}^* w} \tilde{g}^! \tilde{p}^*$$

Indeed, evaluating the above natural transformation on $\mathbb{F}$ and taking the mapping spaces everywhere we get:

$$\tilde{\Delta}^* : \mathrm{Map}_{\mathrm{SH}(\mathcal{Y})}\left(\mathbb{1}_{\mathcal{Y}}, \tilde{f}_* \Sigma^{-w} \tilde{f}^! \mathbb{F}\right) \longrightarrow \mathrm{Map}_{\mathrm{SH}(\mathcal{Y})}\left(\mathbb{1}_{\mathcal{Y}}, \tilde{p}_* \tilde{g}_* \Sigma^{-\tilde{q}^* w} \tilde{g}^! \tilde{p}^* \mathbb{F}\right)$$

where the left hand side is:

$$\mathrm{Map}_{\mathrm{SH}(\mathcal{Y})}\left(\mathbb{1}_{\mathcal{Y}}, \tilde{f}_* \Sigma^{w} \tilde{f}^! \mathbb{F}\right) \simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{Y})}\left(\tilde{f}_! \Sigma^{w} \mathbb{1}_{\mathcal{X}}, \mathbb{F}\right) = \mathbb{F}^{\mathrm{BM}}(\mathcal{X}/\mathcal{Y}, w)$$

and where the right hand side is:

$$\mathrm{Map}_{\mathrm{SH}(\mathcal{Y})}\left(\mathbb{1}_{\mathcal{Y}}, \tilde{p}_* \tilde{g}_* \Sigma^{-q^* w} \tilde{g}^! \tilde{p}^* \mathbb{F}\right) \simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{Y})}\left(\tilde{g}_! \Sigma^{\tilde{q}^* w} \mathbb{1}_{\mathcal{W}}, \tilde{p}^* \mathbb{F}\right) = \tilde{p}^* \mathbb{F}^{\mathrm{BM}}(\mathcal{W}/\mathcal{Z}, \tilde{q}^* w)$$

Since f is a regular embedding, we have an orientation $\eta_f \in \mathbb{E}^{\mathrm{BM}}(\mathcal{X}/\mathcal{Y}, \mathbb{L}_f)$ by remark 1.4.12 and hence an element $\tilde{\Delta}^*(\eta_f) \in \mathbb{E}^{\mathrm{BM}}(\mathcal{W}/\mathcal{Z}, \mathbb{L}_f)$. For any $v \in \mathrm{K}_0^G(Z)$, we then get a map:

$$\begin{array}{rccc} (\tilde{g}_{\tilde{\Delta}})^! : & \mathbb{E}^{\mathrm{BM}}(\mathcal{Z}/\mathcal{B}G, v) & \longrightarrow & \mathbb{E}^{\mathrm{BM}}(\mathcal{W}/\mathcal{B}G, v + \mathbb{L}_f) \\ & x & \mapsto & \tilde{\Delta}^*(\eta_f) \cdot x \end{array}$$

But $(W/S)_G^{\mathrm{BM}} = \pi_{\mathcal{B}G\#} \Sigma^{\mathfrak{g}^\vee} (\mathcal{W}/\mathcal{B}G)^{\mathrm{BM}}$ and similarly $(Z/S)_G^{\mathrm{BM}} = \pi_{\mathcal{B}G\#} \Sigma^{\mathfrak{g}^\vee} (\mathcal{Z}/\mathcal{B}G)^{\mathrm{BM}}$. Thus we can rewrite:

$$\begin{aligned} \mathbb{E}_G^{\mathrm{BM}}(-/S, v) &= \mathrm{Map}_{\mathrm{SH}(S)}\left(\pi_{\mathcal{B}G\#} \Sigma^{\mathfrak{g}^\vee} ([-/G]/\mathcal{B}G)^{\mathrm{BM}}(v), \mathbb{E}\right) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{B}G)}\left(([-/G]/\mathcal{B}G)^{\mathrm{BM}}(v + \mathfrak{g}^\vee), \pi_{\mathcal{B}G}^* \mathbb{E}\right) = \\ &= \pi_{\mathcal{B}G}^* \mathbb{E}^{\mathrm{BM}}\left([-/G]/\mathcal{B}G, v + \mathfrak{g}^\vee\right) \end{aligned}$$

Hence, from $(\tilde{g}_{\tilde{\Delta}})^!$ we get a map:

$$(g)^!_{\Delta,G} : \mathbb{E}^{\mathrm{BM}}_G\,(Z/S\,, v) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\,(W/S\,, v + \mathbb{L}_f)$$

□

Remark 1.5.5. If we formally apply the argument of [DJK21, Proposition 4.2.6(iii)] to the equivariant setting (under the usual identification remark 1.4.9), we see that if the square Δ is also Tor-independent then the refined Gysin map corresponds to the Gysin map obtained in equivariant Borel-Moore homology via proposition 1.4.11 (cf. remark 1.4.12).

Remark 1.5.6. Now that we have in our equivariant setting all the usual operations of Borel-Moore homology such as smooth pullbacks, proper pushforwards, localization sequences and also refined Gysin pullbacks, we get for free all the properties they satisfy in the non-equivariant case. Indeed, the proofs in [DJK21] and [Lev17] can be formally adapted to our case using the machinery of six functors developed in [Cho21a].

Remark 1.5.7. Suppose we have $Z, W, X, Y, T, R \in \mathbf{Sch}^G_{/S}$, together with their associated quotient stacks $\mathcal{Z}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{T}, \mathcal{R}$ respectively. Moreover suppose we have maps $q_1 : X \to T$, $q_2 : Z \to X$ (with composite $q := q_1 \circ q_2$), and adjacent Cartesian squares:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\tilde{q}_2"] \arrow[d, "\tilde{h}"'] \arrow[rr, bend left, "\tilde{q}"] \arrow[dr, phantom, "\widetilde{\Delta_2}"] & \mathcal{X} \arrow[r, "\tilde{q}_1"] \arrow[d, "\tilde{g}"] \arrow[dr, phantom, "\widetilde{\Delta_1}"] & \mathcal{T} \arrow[d, "\tilde{f}"] \\
\mathcal{W} \arrow[r, "\tilde{p}_2"'] \arrow[rr, bend right, "\tilde{p}"'] & \mathcal{Y} \arrow[r, "\tilde{p}_1"'] & \mathcal{R}
\end{tikzcd}$$

Denote by $\widetilde{\Delta}$ the big outer Cartesian square. Then the pullback transformation we defined in (1.27):

$$\widetilde{\Delta}^* : \mathbb{F}^{\mathrm{BM}}\,(\mathcal{T}/\mathcal{R}, -) \longrightarrow \mathbb{F}^{\mathrm{BM}}\,(\mathcal{Z}/\mathcal{W}, -)$$

is actually given by the composite $\widetilde{\Delta_2}^* \circ \widetilde{\Delta_1}^*$. This fact together with the functoriality of Gysin maps for regular embeddings (cf. [DJK21, Theorem 3.2.21]), tells us that we have an homotopy:

$$(q)^!_{p,\Delta} \simeq (q_2)^!_{p_2,\Delta_2} \circ (q_1)^!_{p_1,\Delta_1}$$

External and Refined Intersection Products

Let $Z, X \in \mathbf{Sch}^G_{q/S}$, and let $\mathbb{E} \in \mathrm{SH}(S)$. Consider the cartesian diagram:

$$\begin{tikzcd} X \times_S Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ \Delta"] & Z \arrow[d] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and consider the associated cartesian diagram of quotient stacks:

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{B}G} \mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ \tilde{\Delta}"] & \mathcal{Z} \arrow[d] \\ \mathcal{X} \arrow[r, "\tilde{f}"'] & \mathcal{B}G \end{tikzcd}$$

Then following [DJK21, §2.2.7(4)] (as we did in the proof of proposition 1.5.4), we get a base change natural transformation:

$$\tilde{\Delta}^* : \mathbb{E}^{\mathrm{BM}}_{a,b}\left(\mathcal{Z}/\mathcal{B}G\,, v\right) \longrightarrow \tilde{f}^*\mathbb{E}^{\mathrm{BM}}_{a,b}\left(\mathcal{X} \times_{\mathcal{B}G} \mathcal{Z}/\mathcal{X}\,, v\right)$$

Up to a twist by $\mathfrak{g}^\vee$, we can identify the Borel-Moore homology of the quotient stacks with the equivariant Borel-Moore homology (cf. proof of proposition 1.5.4):

$$\begin{aligned} \mathbb{E}^{\mathrm{BM}}_G\left(-/B\,, v\right) &= \mathrm{Map}_{\mathrm{SH}(B)}\left(q_{B\#}\Sigma^{\mathfrak{g}^\vee}\left([-/G]/\mathcal{B}\right)^{\mathrm{BM}}(v), \mathbb{E}\right) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(\mathcal{B})}\left(\left([-/G]/\mathcal{B}\right)^{\mathrm{BM}}(v + \mathfrak{g}^\vee), \pi^*_{\mathcal{B}}\mathbb{E}\right) = \\ &= \pi^*_{\mathcal{B}}\mathbb{E}^{\mathrm{BM}}\left([-/G]/\mathcal{B}, v + \mathfrak{g}^\vee\right) \end{aligned}$$

where $q_B : B \to [B/G] = \mathcal{B}$ is the quotient map for some scheme $B \in \mathbf{Sch}^G_{q/S}$ and where $\pi_{\mathcal{B}} : \mathcal{B} \to S$ is the structure map. Hence the map $\tilde{\Delta}^*$ gives rise to a base change transformation:

$$\Delta^* : \mathbb{E}^{\mathrm{BM}}_{G,a,b}\left(Z/S\,, v\right) \longrightarrow f^*\mathbb{E}^{\mathrm{BM}}_{G,a,b}\left(X \times_S Z/X\,, v\right)$$

We also have a composition product (induced by [DJK21, 2.2.7(4)]):

$$- \odot^G - : f^*\mathbb{E}^{\mathrm{BM}}_{G,a,b}\left(X \times_S Z/X\,, v\right) \times \mathbb{E}^{\mathrm{BM}}_{G,c,d}\left(X/S\,, w\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}_{G,a+c,b+d}\left(X \times_S Z/S\,, v + w\right)$$

Definition 1.5.8. We define the *external product* as the composition $- \boxtimes^G_{Z,X} - := (- \odot^G -) \circ (\Delta^* \times Id)$:

$$- \boxtimes_{Z,X} - : \mathbb{E}^{\mathrm{BM}}_{G,a,b}\left(Z/S\,, v\right) \times \mathbb{E}^{\mathrm{BM}}_{G,c,d}\left(X/S\,, w\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}_{G,a+c,b+d}\left(X \times_S Z/S\,, v + w\right)$$

Let $Y \in \mathbf{Sm}^G_{q/S}$ and let $\delta_Y : Y \to Y \times_S Y$ be the diagonal. Consider:

$$\begin{tikzcd}
X \times_S Z \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\Delta_\delta"] & X \times_Y Z \arrow[d, "p \times q"] \\
Y \arrow[r, "\delta_Y"'] & Y \times_S Y
\end{tikzcd}$$

Definition 1.5.9. With the notation above, we define the *refined intersection product*, with respect to $p : X \to Y$ and $q : Z \to Y$, as the composition $- *^G_{p,q} - := g^!_{\Delta_\delta, G} \circ (- \boxtimes^G_{Z,X} -)$:

$$- *^G_{p,q} - : \mathbb{E}^{\mathrm{BM}}_{G,a,b}(Z/S, v) \times \mathbb{E}^{\mathrm{BM}}_{G,c,d}(X/S, w) \longrightarrow \mathbb{E}^{\mathrm{BM}}_{G,a+c,b+d}\left(X \times_Y Z/S, v + w - \Omega_{Y/S}\right)$$

Proposition 1.5.10. *In the same situation as above. Then:*

1. *Let $\iota_1 : Z' \to Z$ and $\iota_2 : X' \to X$ be closed immersions in $\mathbf{Sch}^G_{q/S}$. Let $q' = q \circ \iota_1, p' = p \circ \iota_2$. For any $\alpha \in \mathbb{E}^{\mathrm{BM}}_{G,a,b}(Z'/S, v)$ and $\beta \in \mathbb{E}^{\mathrm{BM}}_{G,c,d}(X'/S, w)$, we have:*
$$(\iota_1)^G_*(\alpha) \boxtimes^G_{Z,X} (\iota_2)^G_*(\beta) \simeq (\iota_1 \times \iota_2)^G_*(\alpha \boxtimes^G_{Z',X'} \beta)$$
$$(\iota_1)^G_*(\alpha) *^G_{p,q} (\iota_2)^G_*(\beta) \simeq (\iota_1 \times \iota_2)^G_*(\alpha *^G_{p',q'} \beta)$$

2. *Suppose that we have lci maps $f : Z' \to Z, g : X' \to X$ in $\mathbf{Sch}^G_{q,/S}$, and suppose we have Tor-independent squares:*
$$\begin{tikzcd}
Z \times_S X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta_1"] & X' \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd} \qquad \begin{tikzcd}
Z' \times_S X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta_2"] & Z' \arrow[d] \\
Z \times_S X' \arrow[r] & Z
\end{tikzcd}$$
Then for any $\alpha \in \mathbb{E}^{\mathrm{BM}}_{G,a,b}(Z/S, v)$ and $\beta \in \mathbb{E}^{\mathrm{BM}}_{G,c,d}(X/S, w)$, we have:
$$f^!_G(\alpha) \boxtimes^G_{Z',X'} g^!_G(\beta) \simeq (f \times g)^!_G(\alpha \boxtimes_{Z,X} \beta)$$

Proof. The proof in [Lev22b, Lemma 1.6] formally applies to our case as well. □

1.6 Equivariant VFC

1.6.1 Graber-Pandharipande Construction

Let us quickly recall how the virtual fundamental class in [GP99] was constructed in the non-equivariant case for schemes. Let us first recall the notion of abelian cone:

Definition 1.6.1. Let X be a scheme and $\mathcal{F} \in \mathcal{QCoh}(X)$ an element of the (derived) $(\infty, 1)$-category of quasi-coherent sheaves (cf. for example [AP19, Definition 2.11]). The abelian cone associated to $\mathcal{F}$ is the prestack over X defined by the mapping space:

$$\mathbb{V}_X\left(\mathrm{Spec}(A) \xrightarrow{f} X\right) := \mathrm{Map}_{\mathcal{D}(A)}\left(f^*\mathcal{F}, A\right)$$

where $\mathcal{D}(A) \simeq \mathcal{QCoh}(\mathrm{Spec}(A))$ is the derived category of A. For $\mathcal{F} \in \mathcal{QCoh}(X)^{\heartsuit} \simeq \mathbf{QCoh}(X)$ a quasi-coherent sheaf in the classical sense, we recover the classical definition $\mathbb{V}_X(\mathcal{F}) = \mathrm{Spec}_X\left(\mathrm{Sym}^{\bullet}(\mathcal{F})\right)$ (cf. [AP19, Ex. 3.2]).

Definition 1.6.2. A perfect obstruction theory on a scheme Z is given by a map $\varphi_{\bullet} : \mathcal{E} \to \mathbb{L}_{Z/S}$ in $\mathcal{QCoh}(Z)$, such that $\mathcal{E}$ is a perfect complex of Tor-amplitude $[0, 1]$, $h_0(\varphi)$ is an isomorphism and $h_1(\varphi)$ is surjective.

Let $\varphi_{\bullet} : \mathcal{E} \to \mathbb{L}_{Z/S}$ a perfect obstruction theory on a scheme Z. Let us assume Z is quasi-projective over S, in particular it will admit a closed immersion $\iota_Z : Z \hookrightarrow M$ where M is quasi-projective and smooth over S. We can then take a representative of $\varphi_{\bullet}$ of the form:

$$\varphi_{\bullet} : (\mathcal{F}_1 \to \mathcal{F}_0) \longrightarrow \left(\mathcal{I}/\mathcal{I}^2 \to \iota_Z^*\Omega_{M/S}\right) \simeq \tau_{\leq 1}\mathbb{L}_{Z/S}$$

where the $\mathcal{F}_i$'s are locally free and $\mathcal{I}$ is the ideal sheaf associated to the closed immersion $Z \hookrightarrow M$. The assumption that $\varphi_{\bullet}$ is a perfect obstruction theory, implies that we have an exact sequence:

$$\mathcal{F}_1 \to \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{F}_0 \xrightarrow{\gamma} \iota_Z^*\Omega_{M/S} \to 0 \tag{1.28}$$

Hence, if $Q := \mathrm{Ker}(\gamma)$, we also have a surjection $\mathcal{F}_1 \to Q$. Let $N_{Z/M} := \mathbb{V}_Z\left(\mathcal{I}/\mathcal{I}^2\right)$, $F_0 := \mathbb{V}_Z(\mathcal{F}_0)$ and $F_1 := \mathbb{V}_Z(\mathcal{F}_1)$ be the abelian cones associated to $\mathcal{I}/\mathcal{I}^2$, $\mathcal{F}_0$ and $\mathcal{F}_1$ respectively.

The exact sequence (1.28) tells us that we have $\iota_Z^* T_{M/S}$ as a subcone inside $N_{Z/M} \times_Z F_0$ and hence we can consider:

$$\mathbb{V}_Z(Q) \simeq \left(N_{Z/M} \times_Z F_0\right)/\iota_Z^* T_{M/S}$$

Moreover, again using (1.28), we have a closed immersion:

$$\mathbb{V}_Z(Q) \hookrightarrow F_1$$

Inside the abelian cone $N_{Z/M}$ we have the normal cone $\mathfrak{C}_{Z/M}$, which contains the image of $\iota_Z^* T_{M/S}$. So denoting by:

$$D := \mathfrak{C}_{Z/M} \times_Z F_0$$

the induced closed subscheme of $N_{Z/M} \times_Z F_0$, we can define a closed subcone of $\mathbb{V}_Z(\mathcal{Q})$ by:

$$D^{vir} := D/\iota_Z^* T_{M/S}$$

Composing with the closed immersion $\mathbb{V}_Z(Q) \hookrightarrow F_1$, we get the closed immersion:

$$\iota_{D^{vir}/F_1} : D^{vir} \hookrightarrow F_1$$

We also have the projection maps:

$$\pi_1 : D \longrightarrow \mathfrak{C}_{Z/M}$$

$$\pi_2 : D \longrightarrow D^{vir}$$

exhibiting D as an affine space bundle over $\mathfrak{C}_{Z/M}$ and D^{vir} respectively. Hence, for any ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$, the smooth pullback maps:

$$\pi_1^! : \mathbb{E}^{\mathrm{BM}}\left(\mathfrak{C}_{Z/M}/S, v\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}\left(D/S, v + \mathcal{F}_0\right) \tag{1.29}$$

$$\pi_2^! : \mathbb{E}^{\mathrm{BM}}\left(D^{vir}/S, v\right) \longrightarrow \mathbb{E}^{\mathrm{BM}}\left(D/S, v + \Omega_{M/S}\right) \tag{1.30}$$

are isomorphisms. Thus the fundamental class of the normal cone (as defined in [Lev17]) :

$$\left[\mathfrak{C}^{st}_{Z/M}\right] \in \mathbb{E}^{\mathrm{BM}}\left(\mathfrak{C}_{Z/M}/S, \Omega_{M/S}\right)$$

defines the class:

$$\left[D^{vir}\right] := \pi_2^!\left((\pi_1^!)^{-1}\left[\mathfrak{C}^{st}_{Z/M}\right]\right) \in \mathbb{E}^{\mathrm{BM}}\left(D^{vir}/S, \mathcal{F}_0\right)$$

Definition 1.6.3. Let $\mathbb{E} \in \mathrm{SH}(S)$ be a motivic ring spectrum, Z a quasi-projective scheme over S with a closed embedding into $M \in \mathbf{Sm}_{/S}$. Let $\varphi_\bullet : \mathcal{E} \to \mathbb{L}_{Z/S}$ be a perfect obstruction theory on Z, with $\mathcal{F}_\bullet \to \tau_{\leq 1}\mathbb{L}_{Z/S}$ a representative of $\tau_{\leq 1}\varphi$. Let $s_{F_1} : Z \to F_1$ be the zero section of $F_1 := \mathbb{V}(\mathcal{F}_1)$. Then the virtual fundamental class is defined as:

$$[Z, \varphi_\bullet]^{vir}_{\mathbb{E}} := s_{F_1}^!\left(\iota_{D^{vir}/F_1 *}\left[D^{vir}\right]\right) \in \mathbb{E}^{\mathrm{BM}}\left(Z/S, \mathcal{E}\right)$$

Remark 1.6.4. As shown in [Lev22b, Proposition 4.2], this construction coincides with the virtual fundamental class defined in [Lev17], hence it is independent of all the choices made along the way and is thus a well-defined element of $\mathbb{E}^{\mathrm{BM}}(Z/S, \mathcal{E})$, depending only on Z, $\varphi_\bullet : \mathcal{E} \to \mathbb{L}_{Z/S}$ and $\mathcal{E}$.

1.6.2 Equivariant VFC after Edidin-Graham-Totaro

Now we would like to extend the Graber-Pandharipande construction to our equivariant setting. To do so, we first need a class of the normal cone. Along this section we will work with a motivic ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$.

Let us recall in the non-equivariant case how the class of the cone for for a closed immersion $Z \hookrightarrow M$ was constructed. Given a scheme $Z \in \mathbf{Sch}_{/S}$ and a closed embedding $Z \hookrightarrow M$ with $M \in \mathbf{Sm}_{/S}$, we can construct a specialization to the normal cone map as follows. We will denote with $\pi_\star : \star \to S$ the structure maps of our schemes living over S. First consider the map:

$$(M/S)^{\mathrm{BM}}(\Omega_{M/S}) := \pi_{M!}(\mathrm{Th}_M(T_M)) \simeq \pi_{M\#}\pi_M^* \mathbb{1}_S \longrightarrow \mathbb{1}_S$$

where T_M is the tangent bundle of M. This map induces a map:

$$\pi_M^! : \mathbb{E}(S) \longrightarrow \mathbb{E}^{\mathrm{BM}}\left(M/S, \Omega_{M/S}\right)$$

that is simply the Gysin map associated to the smooth map π_M. Then we can consider:

$$\begin{tikzcd} \mathbb{G}_{m,M} := M \times_S \mathbb{G}_m \arrow[rr, "p_1"] \arrow[dr, "\pi := \pi_{\mathbb{G}_{m,M}}"'] & & M \arrow[dl, "\pi_M"] \\ & S & \end{tikzcd}$$

and by a simple computation we have:

$$\begin{aligned}
(\mathbb{G}_{m,M}/S)^{\mathrm{BM}}(\Omega_{M/S})[1] &\simeq \pi_!\left(\Sigma^{p_1^*\Omega_{M/S}} \mathbb{1}_{\mathbb{G}_{m,M}}\right)[1] \\
&\simeq \pi_\#\left(\Sigma^{-p_2^*\Omega_{\mathbb{G}_m/S}} \mathbb{1}_{\mathbb{G}_{m,M}}\right)[1] \simeq \\
&\simeq \Sigma_T^{-1}\Sigma_{\mathbb{S}^1}(M \times \mathbb{G}_m) \simeq \\
&\simeq \Sigma_{\mathbb{G}_m}^{-1}(M \times \mathbb{G}_m) \simeq \\
&\simeq M = \pi_\#\pi^*\mathbb{1}_S \\
&\simeq (M/S)^{\mathrm{BM}}(\Omega_{M/S})
\end{aligned}$$

that will induce an isomorphism:

$$\sigma_{\mathbb{G}_m} : \mathbb{E}^{\mathrm{BM}}(M/S, w) \xrightarrow{\sim} \mathbb{E}^{\mathrm{BM}}(\mathbb{G}_{m,M}/S, w)[-1]$$

for any $w \in \mathrm{K}_0(M)$. Then back to our original closed embedding $\iota_Z : Z \hookrightarrow M$, we can consider the deformation space $\mathrm{Def}_{Z/M}$ associated to ι_Z with closed and open immersions:

$$\mathfrak{C}_{Z/M} \xhookrightarrow{\iota_{\mathfrak{C}}} \mathrm{Def}_{Z/M} \xhookleftarrow{j} \mathbb{G}_{m,M}$$

Taking the boundary of the localization sequence associated to $\iota_{\mathfrak{C}}$, we get a map:

$$\partial_{\mathfrak{C}} : \mathbb{E}^{\mathrm{BM}}(\mathbb{G}_{m,M}/S, w)[-1] \longrightarrow \mathbb{E}^{\mathrm{BM}}\left(\mathfrak{C}_{Z/M}/S, w\right)$$

Definition 1.6.5. Given a scheme $Z \in \mathbf{Sch}_{/S}$ and a closed embedding $\iota_Z : Z \hookrightarrow M$ with $M \in \mathbf{Sm}_{/S}$, the *specialization to the normal cone* map with respect to ι_Z is given by:

$$\mathrm{sp}_{\iota_Z} := \partial_{\mathfrak{C}} \circ \sigma_{\mathbb{G}_m} \circ \pi_M^! : \mathbb{E}^{\mathrm{BM}}(S/S) \longrightarrow \mathbb{E}^{\mathrm{BM}}\left(\mathfrak{C}_{Z/M}/S, \Omega_{M/S}\right)$$

The *class of the cone* will then be:

$$\left[\mathfrak{C}_{Z/M}\right] := \mathrm{sp}_{\iota_Z}([S]) \in \mathbb{E}^{\mathrm{BM}}\left(\mathfrak{C}_{Z/M}/S, \Omega_{M/S}\right)$$

where $[S] \in \mathbb{E}^{\mathrm{BM}}(S/S)$ is just the class of of the unit of $\mathbb{E}$.

Now we would like to get an equivariant equivalent of this specialization to the normal cone map. First, we start pointing out that the motivic space $BG \in \mathrm{SH}(S)$ has a natural structure map to the unit in $\mathrm{SH}(S)$ Indeed, by corollary 1.4.31 we have $BG = \operatorname{colim}_m \pi_{B_mG\#}\mathbb{1}_{B_mG} \simeq \pi_{\mathcal{B}G\#}\mathbb{1}_{\mathcal{B}G} = \pi_{\mathcal{B}G\#}\pi_{\mathcal{B}G}^*\mathbb{1}_S$ and hence:

$$\eta_{BG} : BG \simeq \pi_{\mathcal{B}G\#}\pi_{\mathcal{B}G}^*\mathbb{1}_S \overset{\eta_{\#}^*(\pi_{\mathcal{B}G})}{\longrightarrow} \mathbb{1}_S$$

Moreover:

$$BG \simeq \pi_{\mathcal{B}G\#}\mathbb{1}_{\mathcal{B}G} \simeq (S/S)_G^{\mathrm{BM}}(-\mathfrak{g}^\vee) = (S/S)_G^{\mathrm{BM}}(\mathbb{L}_{\mathcal{B}G})$$

This induces a map:

$$\eta_{BG}^* : \mathbb{E}^{\mathrm{BM}}(S/S) \longrightarrow \mathbb{E}_G^{\mathrm{BM}}\left(S/S, -\mathfrak{g}^\vee\right) \tag{1.31}$$

Proposition 1.6.6. *Let $B \in \mathbf{Sch}^G_{q,/S}$, let $\iota_X : X \hookrightarrow M$ be in $\mathbf{Sch}^G_{/B}$ and let $v \in \mathrm{K}_0^G(M)$. For $\mathbb{E} \in \mathrm{SH}(\mathcal{B})$, then there exists a well defined (equivariant) specialization to the normal cone map:*

$$\mathrm{sp}^G_{\iota_X} : \mathbb{E}_G^{\mathrm{BM}}(M/B, v) \longrightarrow \mathbb{E}_G^{\mathrm{BM}}\left(\mathfrak{C}_{X/M}/B, v\right)$$

Proof. Using remark 1.4.9, we can identify the equivariant Borel-Moore homology and with the Borel-Moore homology of the respective quotient stacks. To lighten the notation, we will prove the statement of the proposition for $v = 0$, the general case can be obtained by minor modifications of our arguments and it is left to the reader. Denote by $\pi_M : M \to B$ the structure map, with induced quotient map $(\pi_M)_G : \mathcal{M} = [M/G] \to \mathcal{B}$.

As we did in proposition 1.4.11, we have a canonical equivalence:

$$\mathbb{1}_{\mathcal{M}} \simeq (\mathbb{G}_{m,\mathcal{M}}/\mathcal{M})[+1]$$

This will induce an isomorphism:

$$\tilde{\sigma}_{\mathcal{M}} : \mathbb{E}^{\mathrm{BM}}(\mathcal{M}/\mathcal{B}) \overset{\sim}{\to} \mathbb{E}^{\mathrm{BM}}(\mathbb{G}_{m,\mathcal{M}}/\mathcal{B})[-1]$$

and hence, under the identification in remark 1.4.9, an equivalence:

$$\sigma^G_{\mathbb{G}_m} : \mathbb{E}^{\mathrm{BM}}_G (M/B) \xrightarrow{\sim} \mathbb{E}^{\mathrm{BM}}_G (\mathbb{G}_{m,M}/B)[-1]$$

Set $\mathcal{X} := [X/G]$. Using the deformation to the normal cone as we did in proposition 1.4.11:

$$\begin{array}{ccccc} \mathfrak{C}_{\mathcal{X}/\mathcal{M}} & \hookrightarrow & \mathrm{Def}_{\mathcal{X}/\mathcal{M}} & \hookleftarrow & \mathbb{G}_{m,\mathcal{M}} \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & \mathbb{G}_m \end{array}$$

we get a localization sequence:

$$\mathbb{E}^{\mathrm{BM}} (\mathbb{G}_{m,\mathcal{M}}/\mathcal{B}) \to \mathbb{E}^{\mathrm{BM}} \left(\mathrm{Def}_{\mathcal{X}/\mathcal{M}}/\mathcal{B}\right) \to \mathbb{E}^{\mathrm{BM}} \left(\mathfrak{C}_{\mathcal{X}/\mathcal{M}}/\mathcal{B}\right)$$

Since $\mathfrak{C}_{\mathcal{X}/\mathcal{M}} \simeq \left[\mathfrak{C}_{X/M}/G\right]$ and $\mathrm{Def}_{\mathcal{X}/\mathcal{M}} = \left[\mathrm{Def}_{X/M}/G\right]$, by remark 1.4.9, we get an equivariant localization sequence of the form:

$$\mathbb{E}^{\mathrm{BM}}_G (\mathbb{G}_{m,M}/B) \to \mathbb{E}^{\mathrm{BM}}_G \left(\mathrm{Def}_{X/M}/B\right) \to \mathbb{E}^{\mathrm{BM}}_G \left(\mathfrak{C}_{X/M}/B\right)$$

Therefore, we get a boundary map:

$$\partial^G_{\mathfrak{C}} : \mathbb{E}^{\mathrm{BM}}_G (\mathbb{G}_{m,M}/B)[-1] \longrightarrow \mathbb{E}^{\mathrm{BM}}_G \left(\mathfrak{C}_{X/M}/B\right)$$

The composition of $\sigma^G_{\mathbb{G}_m}$ and $\partial^G_{\mathfrak{C}}$ gives us the map:

$$\mathrm{sp}_{\iota_X} := \partial^G_{\mathfrak{C}} \circ \sigma^G_{\mathbb{G}_m} : \mathbb{E}^{\mathrm{BM}}_G (M/B) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G \left(\mathfrak{C}_{X/M}/B\right)$$

□

Corollary 1.6.7. *In the same situation as in proposition 1.6.6, suppose $M \in \mathbf{Sm}^G_{/B}$ is smooth. Then we have a well defined (equivariant) specialization to the normal cone map relative to B:*

$$\mathrm{sp}^G_{\iota_X/\mathcal{B}} : \mathbb{E}^{\mathrm{BM}}_G (B/B, v) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G \left(\mathfrak{C}_{X/M}/B, \Omega^G_{M/B} + v\right)$$

Proof. Suppose for simplicity that $v = 0$ and let us identify the equivariant Borel-Moore homology with the Borel-Moore homology associated to the quotient stacks using remark 1.4.9. Denote by $\pi_M : M \to B$ the structure map, with induced quotient map $(\pi_M)_G : \mathcal{M} = [M/G] \to \mathcal{B}$.

By smooth pushforward of Borel-Moore motives we get:

$$(\pi_M)^!_G : \mathbb{E}^{\mathrm{BM}} (\mathcal{B}/\mathcal{B}) \longrightarrow \mathbb{E}^{\mathrm{BM}} \left(\mathcal{M}/\mathcal{B}, \mathbb{L}_{\mathcal{M}/\mathcal{B}}\right)$$

inducing the map:

$$(\pi_M)^!_G : \mathbb{E}^{\mathrm{BM}}_G (B/B) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G \left(M/B, \Omega^G_{M/B}\right)$$

under the identification $\mathbb{L}_{\mathcal{M}/\mathcal{B}} \simeq \Omega^G_{M/B}$.

Composing $(\pi_M)^!_G$ with sp_{ι_X} of proposition 1.6.6 we get our desired map:

$$\mathrm{sp}^G_{\iota_X/\mathcal{B}} := \partial^G_{\mathfrak{C}} \circ \sigma^G_{\mathbb{G}_m} \circ (\pi_M)^!_G : \mathbb{E}^{\mathrm{BM}}_G(B/B) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\left(\mathfrak{C}_{X/M}/B, \Omega^G_{M/B}\right)$$

The general case for $v \neq 0$ is left to the reader. □

Remark 1.6.8. As a special case of the prvious corollary and of proposition 1.6.6 $B = S$ with a trivial action we recover the specialization map over $\mathcal{B}G$ as a special case.

Definition 1.6.9. Let $\iota_X : X \hookrightarrow M$ in $\mathbf{Sch}^G_{q/S}$ with $M \in \mathbf{Sm}^G_{q,/S}$. Let $\mathcal{M} := [M/G]$ and let $[S] \in \mathbb{E}^{\mathrm{BM}}_{0,0}(S/S)$ be the element given by the identity map on the sphere spectrum $\mathbb{1}_S$ composed with the unit map of $\mathbb{E}$. Let η^*_{BG} be the map in (1.31). We define the *equivariant class of the normal cone* by:

$$\left[\mathfrak{C}_{X/M}\right]_G := \mathrm{sp}^G_{\iota_X/S}([S]) \in \mathbb{E}^{\mathrm{BM}}_{G,0,0}\left(\mathfrak{C}_{X/M}/S, \mathbb{L}_{\mathcal{M}/S}\right)$$

where:

$$\mathrm{sp}^G_{\iota_X/S} := \mathrm{sp}^G_{\iota_X/\mathcal{B}G} \circ \eta^*_{BG} : \mathbb{E}^{\mathrm{BM}}(S/S) \longrightarrow \mathbb{E}^{\mathrm{BM}}_G\left(\mathfrak{C}_{X/M}/S, \mathbb{L}_{\mathcal{M}/S}\right)$$

is the *equivariant specialization to the normal cone* map.

Let $X \in \mathbf{Sch}^G_{/S}$, then $\mathbb{L}_{X/S}$ has a natural G-linearization.

Definition 1.6.10. Let $\mathcal{QC}oh^G(X)$ be the derived $(\infty,1)$-category of G-linearised complexes on X. We say that $\varphi_\bullet : \mathcal{E} \to \mathbb{L}_{X/S}$ in $\mathcal{QC}oh^G(X)$ is a G-linearised perfect obstruction theory if $\varphi_\bullet$ is an obstruction theory after forgetting the G-action, i.e. if $\mathcal{E}$ is of Tor-amplitude $[0,1]$ and if $h_0(\varphi)$ is an isomorphism and $h_1(\varphi)$ is surjective.

Remark 1.6.11. Since the virtual fundamental class associated to a perfect obstruction theory only depends on its 1-truncation (cf. [AP19, Proposition 8.2]), we will often refer to the induced map $\varphi : \mathcal{E} \to \tau_{\leq 1}\mathbb{L}_{X/S}$ as the obstruction theory.

Assumption 1.6.12. We will always assume from now on when dealing with perfect obstruction theories, that the schemes we are working with satisfy the G-resolution property. In other words for any $\varphi : \mathcal{E} \to \mathbb{L}_{X/S}$, we will always assume that there exists a representative of $\tau_{\leq 1}\varphi$ of the form $(\mathcal{F}_1 \to \mathcal{F}_0) \longrightarrow \tau_{\leq 1}\mathbb{L}_{X/S}$ with $\mathcal{F}_0, \mathcal{F}_1$ two G-linearised locally free sheaves on X.

Let $\iota_X : X \hookrightarrow M$ be in $\mathbf{Sch}^G_{q/S}$ with $M \in \mathbf{Sm}^G_{q,/S}$, equipped with a G-linearised perfect obstruction theory $\varphi : \mathcal{E} \longrightarrow \mathbb{L}_{X/S}$. Take a representative of φ of the form $(\mathcal{F}_1 \to \mathcal{F}_0) \longrightarrow \tau_{\leq 1}\mathbb{L}_{X/S}$. As done in the non-equivariant case section 1.6.1, using the same notation, we can define the following objects and maps in $\mathbf{Sch}^G_{q/S}$:

$$D := \mathfrak{C}_{Z/M} \times F_0 \qquad\qquad D^{vir} := D/\iota_X^* T_{M/S}$$

$$\iota_{D^{vir}/F_1} : D^{vir} \hookrightarrow F_1 \qquad\qquad \begin{tikzcd} & D \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathfrak{C}_{X/M} & & D^{vir} \end{tikzcd}$$

where ι_{D^{vir}/F_1} is a closed immersion and π_1, π_2 are affine space bundles. Denote also by $s_{F_1} : X \longrightarrow F_1$ the G-equivariant zero-section of F_1.

Definition 1.6.13. With $X, M, \varphi : \mathcal{E} \to \mathbb{L}_{X/S}$ as above, let $\mathbb{E} \in \mathrm{SH}(S)$ be a motivic ring spectrum. Let:

$$\left[D^{vir}\right]_G := \left((\pi_2)^!_G\right)^{-1} (\pi_1)^!_G \left[\mathfrak{C}_{X/M}\right]_G \in \mathbb{E}^{\mathrm{BM}}_G \left(D^{vir}/S, \mathcal{F}_0\right)$$

Then we define the *equivariant virtual fundamental class* of X as:

$$[X, \varphi]^{vir}_G := (s_{F_1})^!_G (\iota_{D^{vir}/F_1})^G_* \left[D^{vir}\right]_G \in \mathbb{E}^{\mathrm{BM}}_G \left(X/S, \mathcal{E}_\bullet\right)$$

Remark 1.6.14. The above definition depends a priori on the choices we made, like for example the choice of the closed embedding $X \hookrightarrow M$. But formally applying the proofs in [Lev17], plus [Lev22a, Proposition 4.2], we get that the construction is actually independent of all the choices and that the virtual fundamental class only depends on the G-scheme, the perfect obstruction theory, and the choice of a motivic ring spectrum $\mathbb{E}$.

Equivariant Vistoli's Lemma

For later use, we record here an equivariant version of the Vistoli's lemma proved in [Lev22b, Proposition 2.1]; we will closely follow the construction in *loc. cit.* and no claim of originality is made here. We could consider representable maps of NL-stacks, but for the sake of simplicity we will only present the Vistoli's lemma for quotient stacks. Consider the following cartesian diagram in $\mathcal{ASt}_{/S}$:

$$\begin{tikzcd} \mathcal{Z} := [Z/G] \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\Delta"] & \mathcal{X}_1 := [X_1/G] \arrow[d, hook, "\iota_1"] \\ \mathcal{X}_2 := [X_2/G] \arrow[r, hook, "\iota_2"'] & \mathcal{Y} := [Y/G] \end{tikzcd}$$

with $Z, X_1, X_2, Y \in \mathbf{Sch}^G_{/S}$ and ι_1, ι_2 closed immersions. For any closed immersion $g : \mathcal{W} \hookrightarrow \mathcal{T}$ of locally finite type algebraic stacks, we denote by Def_g or by $\mathrm{Def}_{\mathcal{W}/\mathcal{T}}$ the associated deformation space, as the one in (1.6), and by $\mathfrak{C}_g$ or by $\mathfrak{C}_{\mathcal{W}/\mathcal{T}}$ the associated normal cone contained in Def_g (cf. [AP19, §6]). The cartesian square Δ, gives rise to the following diagram:

$$\begin{tikzcd}
& \mathfrak{C}_{\alpha_1} \arrow[d, "\beta_1"'] \arrow[rd, bend left] & & \\
\mathfrak{C}_{\alpha_2} \arrow[r, "\beta_2"] \arrow[rd, bend right] & f_1^*\mathfrak{C}_{\iota_2} \times_Y f_2^*\mathfrak{C}_{\iota_1} \arrow[r] \arrow[d] \arrow[rd, phantom, very near start, "\ulcorner"] & f_2^*\mathfrak{C}_{\iota_2} \arrow[r, "\alpha_1"] \arrow[d] \arrow[rd, phantom, very near start, "\ulcorner"] & \mathfrak{C}_{\iota_2} \arrow[d] \\
& f_1^*\mathfrak{C}_{\iota_2} \arrow[r] \arrow[d, "\alpha_2"'] \arrow[rd, phantom, very near start, "\ulcorner"] & \mathcal{Z} \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[rd, phantom, very near start, "\ulcorner"] & \mathcal{X}_1 \arrow[d, "\iota_1"] \\
& \mathfrak{C}_{\iota_2} \arrow[r] & \mathcal{X}_2 \arrow[r, "\iota_2"'] & \mathcal{Y}
\end{tikzcd} \tag{1.32}$$

Notice that β_1 and β_2 are closed immersions (cf. [Lev22b, §2]).

Construction 1.6.15. Define:

$$\mathrm{Def}_1 := \mathrm{Def}_{\mathcal{X}_1/\mathcal{Y}} \times \mathbb{A}^1$$

$$\mathrm{Def}_2 := \mathrm{Def}_{\mathcal{X}_2 \times \mathbb{A}^1/\mathcal{Y} \times \mathbb{A}^1}$$

Both of them have structure maps $\pi_i : \mathrm{Def}_i \longrightarrow \mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1$. Let:

$$\mathrm{Def}_{12} := \mathrm{Def}_1 \times_{\mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1} \mathrm{Def}_2$$

be the *double deformation space*. The stack Def_{12} comes equipped with a structure map:

$$\pi_{12} : \mathrm{Def}_{12} \longrightarrow \mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1$$

and projection maps:

$$p_j : \mathrm{Def}_{12} \longrightarrow \mathrm{Def}_j$$

Notice that we have $\pi_1^{-1}(\mathcal{X}_1 \times \{0\} \times \mathbb{A}^1) \simeq \mathfrak{C}_{\iota_1} \times \mathbb{A}^1$ and $\pi_2^{-1}(\mathcal{X}_2 \times \mathbb{A}^1 \times \{0\}) \simeq \mathfrak{C}_{\iota_2} \times \mathbb{A}^1$, therefore we have complementary closed an open immersions:

$$\mathfrak{C}_{\iota_1} \times \mathbb{A}^1 \xhookrightarrow{\sigma_1} \mathrm{Def}_1 \overset{\eta_1}{\hookleftarrow} \mathcal{Y} \times \mathbb{G}_m \times \mathbb{A}^1 \qquad \mathfrak{C}_{\iota_2} \times \mathbb{A}^1 \xhookrightarrow{\sigma_2} \mathrm{Def}_2 \overset{\eta_2}{\hookleftarrow} \mathcal{Y} \times \mathbb{A}^1 \times \mathbb{G}_m$$

We also have a closed immersion:

$$\hat{\sigma}_1 : \mathrm{Def}_{\alpha_1} = \mathrm{Def}_{f_2^*\mathfrak{C}_{\iota_1}/\mathfrak{C}_{\iota_1}} \hookrightarrow \mathrm{Def}_{12}$$

Morover $\hat{\sigma}_1$ fits in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Def}_{\alpha_1} \setminus \mathfrak{C}_{\alpha_1} \arrow[r, hook, "\circ" marking] \arrow[d, "\wr"'] & \mathrm{Def}_{\alpha_1} \arrow[r] \arrow[d] & \mathrm{Def}_{12} \arrow[d, "p_1"] \arrow[dd, bend left, "\pi_{12}"] \\
\mathfrak{C}_{\iota_1} \times \mathbb{G}_m \arrow[r, hook, "\circ" marking] \arrow[d] & \mathfrak{C}_{\iota_1} \times \mathbb{A}^1 \arrow[r] \arrow[d] & \mathrm{Def}_1 \arrow[d, "\pi_1"] \\
\mathcal{Y} \times \{0\} \times \mathbb{G}_m \arrow[r, hook, "\circ" marking] & \mathcal{Y} \times \{0\} \times \mathbb{A}^1 \arrow[r] & \mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1
\end{tikzcd}$$

Similarly we have a closed immersion $\hat{\sigma}_2 : \mathrm{Def}_{\alpha_2} = \mathrm{Def}_{f_1^*\mathfrak{C}_{\iota_2}/\mathfrak{C}_{\iota_2}} \hookrightarrow \mathrm{Def}_{12}$, fitting in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Def}_{\alpha_2} \setminus \mathfrak{C}_{\alpha_1} \arrow[r, hook, "\circ" marking] \arrow[d, "\wr"'] & \mathrm{Def}_{\alpha_2} \arrow[r] \arrow[d] & \mathrm{Def}_{12} \arrow[d, "p_2"] \arrow[dd, bend left, "\pi_{12}"] \\
\mathfrak{C}_{\iota_2} \times \mathbb{G}_m \arrow[r, hook, "\circ" marking] \arrow[d] & \mathfrak{C}_{\iota_2} \times \mathbb{A}^1 \arrow[r] \arrow[d] & \mathrm{Def}_2 \arrow[d, "\pi_2"] \\
\mathcal{Y} \times \mathbb{G}_m \times \{0\} \arrow[r, hook, "\circ" marking] & \mathcal{Y} \times \mathbb{A}^1 \times \{0\} \arrow[r] & \mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1
\end{tikzcd}$$

Therefore the closed immersions:

$\mathrm{Def}_{\alpha_1} \hookrightarrow \pi_{12}^{-1}(\mathcal{Y} \times \{0\} \times \mathbb{A}^1)$ $\qquad\qquad$ $\mathrm{Def}_{\alpha_2} \hookrightarrow \pi_{12}^{-1}(\mathcal{Y} \times \mathbb{A}^1 \times \{0\})$

are both isomorphism when restricted over $(\mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1) \setminus (\mathcal{Y} \times \{0\} \times \{0\})$. Moreover since:

$$\pi_{12}^{-1}(\mathcal{Y} \times \{0\} \times \{0\}) \simeq \pi_1^{-1}(\mathcal{Y} \times \{0\} \times \mathbb{A}^1) \times_{\mathcal{Y}} \pi_2^{-1}(\mathcal{Y} \times \mathbb{A}^1 \times \{0\}) = f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2}$$

we get that:

$$\pi_{12}^{-1}(\mathcal{Y} \times \{0\} \times \mathbb{A}^1) = \mathrm{Def}_{\alpha_1} \cup (f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2})$$

and:

$$\pi_{12}^{-1}(\mathcal{Y} \times \mathbb{A}^1 \times \{0\}) = \mathrm{Def}_{\alpha_2} \cup (f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2})$$

Putting everything together, we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & \mathrm{Def}_{\alpha_1} \arrow[rrrr, "\hat{\sigma}_1"'] \arrow[dddd] & & & & \mathrm{Def}_{12} \arrow[dddd, "\pi_{12}"'] \\
\mathfrak{C}_{\alpha_1} \arrow[ur, "\sigma_{\alpha_1}"] \arrow[drr, "\beta_1"] \arrow[ddddrr, bend right] & & & & & \\
 & & f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2} \arrow[uurrr] \arrow[ddd] & & & \\
 & & & \mathfrak{C}_{\alpha_2} \arrow[ul, "\beta_2"'] \arrow[r, "\sigma_{\alpha_2}"] \arrow[ddl] & \mathrm{Def}_{\alpha_2} \arrow[uuur, "\hat{\sigma}_2"] \arrow[ddd] & \\
 & \mathcal{Y} \times \{0\} \times \mathbb{A}^1 \arrow[rrrr] & & & & \mathcal{Y} \times \mathbb{A}^1 \times \mathbb{A}^1 \\
 & & \mathcal{Y} \times \{0\} \times \{0\} \arrow[ul] \arrow[urrr] \arrow[drr] & & & \\
 & & & & \mathcal{Y} \times \mathbb{A}^1 \times \{0\} \arrow[uur] &
\end{tikzcd}
\tag{1.33}
$$

The restriction of π_{12} to the complement of $\pi_{12}^{-1}((\mathcal{Y} \times \{0\} \times \mathbb{A}^1) \cup (\mathcal{Y} \times \mathbb{A}^1 \times \{0\}))$ gives us an isomorphism:

$$\mathrm{Def}_{12} \backslash \pi_{12}^{-1}((\mathcal{Y}\times\{0\}\times\mathbb{A}^1)\cup(\mathcal{Y}\times\mathbb{A}^1\times\{0\})) \xrightarrow{\sim} \mathcal{Y}\times\mathbb{A}^1\times\mathbb{A}^1 \backslash((\mathcal{Y}\times\{0\}\times\mathbb{A}^1)\cup(\mathcal{Y}\times\mathbb{A}^1\times\{0\}))$$

Moreover we have the following equalities:

$$\pi_{12}^{-1}((\mathcal{Y} \times \{0\} \times \mathbb{A}^1) \cup (\mathcal{Y} \times \mathbb{A}^1 \times \{0\})) = \hat{\sigma}_1\,(\mathrm{Def}_{\alpha_1}) \cup f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2} \cup \hat{\sigma}_2(\mathrm{Def}_{\alpha_2})$$

$$(f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2}) \cap \hat{\sigma}_1(\mathrm{Def}_{\alpha_1}) = \beta_1(\mathfrak{C}_{\alpha_1})$$

$$(f_2^*\mathfrak{C}_{\iota_1} \times_{\mathcal{Y}} f_1^*\mathfrak{C}_{\iota_2}) \cap \hat{\sigma}_2(\mathrm{Def}_{\alpha_2}) = \beta_2(\mathfrak{C}_{\alpha_2})$$

Proposition 1.6.16 (Vistoli's Lemma)**.** *Consider the following diagram in* $\mathbf{Sch}^G_{/S}$*:*

$$
\begin{tikzcd}
 & \mathfrak{C}_{a_1} \arrow[d, "b_1"'] \arrow[rd, bend left] & & \\
\mathfrak{C}_{a_2} \arrow[r, "b_2"] \arrow[rd, bend right] & g_1^*\mathfrak{C}_{i_2} \times_Y g_2^*\mathfrak{C}_{i_1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & g_2^*\mathfrak{C}_{i_2} \arrow[r, "a_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{C}_{i_1} \arrow[d] \\
 & g_1^*\mathfrak{C}_{i_2} \arrow[r] \arrow[d, "a_2"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[r, "g_2"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & X_1 \arrow[d, "\iota_1"] \\
 & \mathfrak{C}_{i_2} \arrow[r] & X_2 \arrow[r, "\iota_2"'] & Y
\end{tikzcd}
$$

Let $v \in \mathrm{K}_0^G(Y)$ and let $\mathbb{E} \in \mathrm{SH}(S)$. Using the same notation as in proposition 1.6.6 for the equivariant specialization maps, we have:

$$(b_1)_*^G \circ \mathrm{sp}_{a_1}^G \circ \mathrm{sp}_{i_1}^G = (b_2)_*^G \circ \mathrm{sp}_{a_2}^G \circ \mathrm{sp}_{i_2}^G$$

as maps $\mathbb{E}_G^{\mathrm{BM}}(Y/S, v) \to \mathbb{E}_G^{\mathrm{BM}}(g_1^*\mathfrak{C}_{i_2} \times_Y g_2^*\mathfrak{C}_{i_1}/S)$

Proof. The proof is formal and follows using the same arguments as in [Lev22b, Proposition 2.1]. Indeed, once we identify the equivariant Borel-Moore homology with the Borel-Moore homology of the quotient stacks, we can use construction 1.6.15 and the formalism of the six-functors to just repeat the proof in *loc. cit.* in our context. □

Chapter 2

Some Computations on $\boldsymbol{SL}[\boldsymbol{\eta}^{-1}]$-Theories of Classifying Spaces

Assumption 2.0.1. Let $\mathbb{k}$ be a field with $\mathrm{char}(\mathbb{k}) \neq 2$. We work over a base-scheme S that is a $\mathbb{k}$-scheme of finite Krull dimension.

2.1 Quick Recap on the Background

Recall that we are distinguishing between the classifying stacks $\mathcal{B}G$ and the ind-scheme approximating them BG. But in light of corollary 1.4.32, we will not distinguish between $\mathbb{E}(\mathcal{B}G, -)$ and $\mathbb{E}(BG, v)$.

Let N be the normaliser of the standard diagonal torus $T \subseteq SL_2$. Note that

$T \simeq \mathbb{G}_m$, where for R a ring, we map $t \in \mathbb{G}_m(R) = R^\times$ to the diagonal matrix:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

We often simply write this matrix as t, when there is no cause of confusion. Notice that N is generated by T plus the element:

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

As showed by the computation in [Lev19, §2], we have:

$$SL_2/_N \simeq GL_2/_{N'} \simeq \mathbb{P}^2 \setminus C$$

where N' is the normaliser of the diagonal torus in GL_2 and C is the conic defined by the equation $Q := T_1^2 - 4T_0T_2$.

Closely following [Lev19, §5], the aim of this section is to compute $\mathrm{A}(\mathcal{B}N)$ for A an $SL[\eta^{-1}]$-oriented theory (cf. notation 2.1.7 for the convention) and get as a special case the computation of the Witt theory of $\mathcal{B}N$.

For the reader convenience, let us recall how to view the left SL_2 action on SL_2/N under the above identification. Consider $F = \mathbb{A}^2$ with the standard left SL_2 action. We get a map:

$$sq : \mathbb{P}(F) \longrightarrow \mathbb{P}\left(\mathrm{Sym}^2(F)\right)$$

induced by the squaring map:

$$\begin{array}{rccc} sq: & F & \longrightarrow & \mathrm{Sym}^2(F) \\ & v & \mapsto & v^2 \end{array}$$

This map is constructed in the following standard way. The map $\varphi : F \to F \otimes F$ sending $v \mapsto v \otimes v$ is SL_2-equivariant (where the SL_2-action on $F \otimes F$ is the diagonal one given by $g \cdot (v \otimes w) := g \cdot v \otimes g \cdot w$). Post-composing φ with the quotient map $F \otimes F \to \mathrm{Sym}^2(F)$ that sends $a \otimes b$ to ab, we get the SL_2-equivariant map sq we wanted.

Using sq we can identify $C \subseteq \mathbb{P}^2$ with $sq(\mathbb{P}(F)) \subseteq \mathbb{P}\left(\mathrm{Sym}^2(F)\right)$. Since sq is SL_2-invariant, this means that C is SL_2-invariant. In particular Q is SL_2-invariant up to a scalar, so considering the multiplication morphism $SL_2 \ni g\cdot : C \to C$ with associated map on the global section denoted by $(g\cdot)^*$, we get:

$$\begin{array}{ccc} SL_2 & \longrightarrow & \mathbb{G}_m \\ g & \mapsto & \frac{(g\cdot)^*Q}{Q} \end{array}$$

But we know SL_2 is a simple algebraic group, so it only admits a trivial character and hence we get that Q is actually SL_2-invariant. As an ind-scheme approximation for $\mathcal{B}N$ we choose as in [Lev19, §2, §5]:

$$BN := {}^{SL_2}/_{N} \times^{SL_2} EGL_2 \simeq EGL_2/N \simeq ESL_2/N$$

We have that SL_2 is special, so $ESL_2 \longrightarrow BSL_2$ is a Zariski locally trivial bundle and so it is $BN \longrightarrow BSL_2$ (cf. [Lev19, §2]). From the description of SL_2/N we recalled above, we can realise BN as an open subscheme of the $\mathbb{P}^2$-bundle $\mathbb{P}\left(\mathrm{Sym}^2(F)\right) \times^{SL_2} ESL_2 \to BSL_2$, with closed complement $\mathbb{P}(F) \times^{SL_2} ESL_2$:

$$\begin{tikzcd} BN \arrow[r, hook, "\circ" marking] \arrow[dr] & \mathbb{P}(\mathrm{Sym}^2(F)) \times^{SL_2} ESL_2 \arrow[d] & \mathbb{P}(F) \times^{SL_2} ESL_2 \arrow[l, hook', "/" marking] \arrow[dl] \\ & BSL_2 & \end{tikzcd} \tag{2.1}$$

Remember that we have a short exact sequence:

$$1 \to T \longrightarrow N \longrightarrow \{\pm 1\} \to 1$$

where T is our torus in SL_2. The normaliser as we already said is generated by T and the element σ, so sending T to 1 and σ to -1 give us a representation $\rho^- : N \longrightarrow \mathbb{G}_m$.

The description we just gave for the approximating ind-scheme BN is also useful to give a different presentation of the quotient stack $\mathcal{B}N$. Indeed, since $SL_2/N \simeq \mathbb{P}(\mathrm{Sym}^2(F)) \setminus \mathbb{P}(F)$ we have:

$$\begin{tikzcd}
\mathcal{B}N = [(SL_2/N)/SL_2] \arrow[r, hook, "j"] \arrow[dr, "p"'] & \left[\mathbb{P}(\mathrm{Sym}^2(F))/SL_2\right] \arrow[d, "p_2"] & {[\mathbb{P}(F)/SL_2]} \arrow[l, hook', "\iota"'] \arrow[dl, "\bar{p}"] \\
& \mathcal{B}SL_2 &
\end{tikzcd} \tag{2.2}$$

Since $SL_2/N \simeq \mathbb{P}^2 \setminus C$, a line bundle over $\mathcal{B}N$ can be described as a line bundle over $\mathbb{P}^2 \setminus C$ together with a SL_2-linearisation. The line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ is equipped with a natural SL_2-linearisation (induced by the natural action of SL_2 on $\mathbb{P}^2$), hence its restriction to $\mathbb{P}^2 \setminus C$ gives us a well defined element $\gamma_N \in \mathrm{Pic}^{SL_2}(\mathbb{P}^2 \setminus C) \simeq \mathrm{Pic}(\mathcal{B}N)$.

Lemma 2.1.1. *Let $\Bbbk$ be a field. The Picard group* $\mathrm{Pic}(\mathcal{B}N)$ *of* $\mathcal{B}N \in \mathcal{ASt}_{/\Bbbk}$ *is generated by the line bundle* $\gamma_N \in \mathrm{Pic}^{SL_2}(\mathbb{P}^2 \setminus C) \simeq \mathrm{Pic}(\mathcal{B}N)$ *coming from* $\mathcal{O}_{\mathbb{P}^2}(1)$*, wioth its natural SL_2-linearisation. Moreover* $\mathrm{Pic}(\mathcal{B}N) \simeq \mathbb{Z}/2\mathbb{Z}$.

Proof. Since we work over a field, by [Bri15, Proposition 2.10], we can identify $\mathrm{Pic}(\mathcal{B}N) = \mathrm{Pic}^N(\Bbbk) \simeq \mathcal{X}(N) = \mathrm{Hom}(N, \mathbb{G}_m)$, where $\mathcal{X}(N)$ denotes the character group. Let $\chi \in \mathcal{X}(N)$ and let t be the parameter of the diagonal torus $T \subseteq N$ and $\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N$. Then we must have that $\chi(t) = t^n$ for some integer n while $\chi(\sigma) \in \{\pm 1\}$. But we also have:

$$\chi\left(\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = \chi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \sigma\right)$$

This implies:

$$\chi(\sigma) \cdot t^n = t^{-n} \cdot \chi(\sigma)$$

and hence $n = 0$. Therefore there can be only two characters for the group N: the trivial one sending σ to the identity and the non trivial one sending σ to -1. This proves that $\mathrm{Pic}(\mathcal{B}N) \simeq \mathbb{Z}/2\mathbb{Z}$. To see that γ_N is the generator, it is enough to notice that the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2 \setminus C$, with its SL_2-trivialization, cannot be trivial since there are no non-vanishing global sections of $\mathcal{O}(1)$ on $\mathbb{P}^2 \setminus C$. Thus γ_N must be the generator. Moreover we do have a non-vanishing global section for $\mathcal{O}(2)$, given by $Q = T_1^2 - 4T_0T_2$, and hence γ_N is indeed a 2-torsion element as expected. $\square$

Remark 2.1.2. The representation $\rho^- : N \to \mathbb{G}_m$ sending σ to -1 corresponds exactly to the line bundle γ_N generating $Pic(BN)$.

SL- and $SL[\eta^{-1}]$-Orientations

From now on in this chapter, we will assume our base field $\Bbbk$ to be perfect. We will denote by S our general base scheme. We have a very special element in $\mathrm{H}^{-1,-1}(S)$, the algebraic Hopf map:

$$\eta : \mathbb{A}^2_S \setminus \{0\} \longrightarrow \mathbb{P}^1_S$$

sending $(x,y) \mapsto [x : y]$, giving us an element $\eta : \Sigma_{\mathbb{G}_m} \mathbb{1}_S \to \mathbb{1}_S \in \mathrm{H}^{-1,-1}(S)$. For any motivic ring spectrum $\mathbb{E}$, via the unit map $\mu : \mathbb{1}_S \to \mathbb{E}$, we get a corresponding element $\eta_{\mathbb{E}} \in \mathbb{E}^{-1,-1}(S)$.

Definition 2.1.3. A motivic ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$ is said to be η-invertible if multiplication by $\eta_{\mathbb{E}}$, $- \times \eta_{\mathbb{E}} : \mathbb{E}^{0,0}(S) \to \mathbb{E}^{-1,-1}(S)$, is an isomorphism.

Definition 2.1.4 ([Ana15, Def.1])**.** An SL_n-vector bundle (E,θ) on some $X \in \mathbf{Sch}_{/S}$ is the data given by a vector bundle E of rank n, together with a trivialization of the determinant, $\theta : \det(E) \xrightarrow{\sim} \mathcal{O}_X$. We will often denote the SL_n-vector bundle by just the underlying vector bundle E. If there is no need to specify the rank of the bundle, we will say that E is a SL-vector bundle.

After Panin-Walter, we will use the following definition:

Definition 2.1.5 (cf. [Ana19])**.** Let $\mathbf{C}$ be a full subcategory of $\mathbf{Sch}_{/S}$. Given a ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$, an *SL-orientation with respect to* $\mathbf{C}$ for $\mathbb{E}$ is a rule which assigns to each SL_n-vector bundle V, over $X \in \mathbf{C}$, an element:

$$\mathrm{th}(V) \in \mathbb{A}^{2n,n}(\mathrm{Th}_X(V))$$

with the following properties:

1. For any isomorphism $\varphi : V_1 \to V_2$ of SL-vector bundles over $X \in \mathbf{C}$, we have:

$$\varphi^* \mathrm{th}(V_2) = \mathrm{th}(V_1)$$

   where φ^* is the pullback map induced by φ.

2. For any morphism $f : X \to Y$ in $\mathbf{C}$, and V an SL-vector bundle over Y, we have:

$$f^* \mathrm{th}(V) = \mathrm{th}(f^* V)$$

3. For $\mathcal{V}_1, V_2$ SL-vector bundles on some $X \in \mathbf{C}$, we have:

$$\mathrm{th}(V_1 \oplus V_2) = p_1^* \mathrm{th}(V_1) \cup p_2^* \mathrm{th}(V_2)$$

   where $p_i : V_1 \oplus V_2 \to V_i$ are the projection maps.

4. We have:
$$\mathrm{th}(\mathbb{A}^1_S) = \Sigma_T 1 \simeq [\Sigma_T \mathbb{1}_S \xrightarrow{\Sigma_T u_{\mathbb{E}}} \Sigma_T \mathbb{E}] \in \mathbb{E}^{2,1}(\mathbb{P}^1_S)$$
where $u_{\mathbb{E}} : \mathbb{1}_S \to \mathbb{E}$ is the unit map of the ring spectrum.

We refer to the elements $\mathrm{th}(V)$ as *Thom classes*. If a ring spectrum $\mathbb{E}$ has a normalised SL-orientation with respect to $\mathbf{C} := \mathbf{Sm}_{/S}$, we simply say that $\mathbb{E}$ has an SL-orientation, and we will say that $\mathbb{E}$ is SL-oriented. If $\mathbf{C} := \mathbf{Sch}_{/S}$, then a normalised SL-orientation with respect to $\mathbf{C}$ will be called an *absolute SL-orientation*, and $\mathbb{E}$ will be said to be absolutely SL-oriented (following the conventions in [DF21]).

Remark 2.1.6. We will consider basically just absolute SL-orientations. What will follow is already well known to the experts and it will be very similar to the material already presented in [BW21, §4.3]. We only need to concentrate on SL-oriented spectra and, by [Ana16b, Theorem 4.7], we know that those are strongly SL-oriented in the sense of [BW21], so the reader can safely refer to the latter if they prefer.

Notation 2.1.7. Given an SL-oriented spectrum $\mathrm{A} \in \mathrm{SH}(S)$ that is also η-invertible, we will say for short that A is $SL[\eta^{-1}]$-oriented. We will use the letter A whenever we want to stress the fact that we are working with $SL[\eta^{-1}]$-oriented spectra.

Remark 2.1.8. Following the conventions of [Ana15, Def.19], any η-invertible spectrum A will be regarded just as a graded theory through the isomorphisms $\mathrm{A}^{\bullet} := \mathrm{A}^{a-b,0} \simeq \mathrm{A}^{a,b}$ induced by η.

Definition 2.1.9. Let $\mathbf{C}$ be a full subcategory of $\mathbf{Sch}_{/S}$ and let $\mathbb{E} \in \mathrm{SH}(S)$ be a motivic ring spectrum. A *system of SL-Thom isomorphism* for $\mathbb{E}$ (over $\mathbf{C}$) is the data given by a collection of isomorphism $\tau_V : \mathbb{E}^{\bullet,\bullet}(X) \xrightarrow{\sim} \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_X(V))$, for $X \in \mathbf{C}$ and V SL_n-vector bundle on X, such that:

1. Given a map $f : X \to Y$ in $\mathbf{C}$ and V an SL_n-vector bundle on X, we have the following commutative diagram:
$$\begin{tikzcd} \mathbb{E}^{\bullet,\bullet}(Y) \arrow[r, "\tau_{f^*V}", "\sim"'] \arrow[d] & \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_Y(f^*V)) \arrow[d] \\ \mathbb{E}^{\bullet,\bullet}(X) \arrow[r, "\sim", "\tau_V"'] & \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_X(V)) \end{tikzcd}$$
where the vertical arrows are induced by the pullback on cohomology.

2. Given an isomorphism $\varphi : V \xrightarrow{\sim} W$ of SL_n-vector bundles on $X \in \mathbf{C}$, we get a commutative diagram:
$$\begin{tikzcd} & \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_X(V)) \arrow[d, "\varphi^*", "\wr"'] \\ \mathbb{E}^{\bullet,\bullet}(X) \arrow[ur, "\tau_V"] \arrow[r, "\tau_W"'] & \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_X(W)) \end{tikzcd}$$

3. Given V_1, V_2 SL-vector bundles of rank j, k over $X \in \mathbf{C}$, the Thom isomorphism are *multiplicative*, that is, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{E}^{\bullet,\bullet}(X) \times \mathbb{E}^{\bullet,\bullet}(X) \arrow[r, "\tau_{V_1} \times \tau_{V_2}", "\sim"'] \arrow[d] & \mathbb{E}^{\bullet+2j,\bullet+j}\left(\mathrm{Th}_X\left(V_1\right)\right) \times \mathbb{E}^{\bullet+2k,\bullet+k}\left(\mathrm{Th}_X\left(V_2\right)\right) \arrow[d] \\ \mathbb{E}^{\bullet,\bullet}(X) \arrow[r, "\sim", "\tau_{V_1 \oplus V_2}"'] & \mathbb{E}^{\bullet+2(j+k),\bullet+j+k}\left(\mathrm{Th}_X\left(V_1 \oplus V_2\right)\right) \end{tikzcd}$$

where the vertical arrows are induced by multiplication map of $\mathbb{E}$, together with the identification $\mathrm{Th}_X(V_1) \otimes \mathrm{Th}_X(V_2) \simeq \mathrm{Th}_X(V_1 \oplus V_2)$.

4. If V is an SL_n-vector bundle on $X \in \mathbf{C}$ isomorphic to the trivial SL_n-vector bundle $\mathbb{A}^n_X$, then we have that $\tau_V \simeq \Sigma^{2n,n}$.

Remark 2.1.10. If we had a collection of maps:

$$\left\{\tau_V : \mathbb{E}^{\bullet,\bullet}(X) \to \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_X(V))\right\}_{X\in\mathbf{C}}$$

satisfying (1),(2), and (4) in the previous proposition, then we automatically get that τ_V are isomorphism by a Mayer-Vietoris argument (cf. [Ana21, Lemma 3.7]).

Remark 2.1.11. Notice that working with a special group like SL, giving Thom classes $\mathrm{th}(V)$, for SL-vector bundles V over $X \in \mathbf{C}$, amounts to the same data as giving a system of Thom isomorphisms:

$$\mathbb{E}^{\bullet,\bullet}(X) \xrightarrow{\sim} \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_X(V))$$

From a system of Thom isomorphism, we can get a family of Thom classes just taking $\mathrm{th}(V) := \tau_V(1)$. Vice versa, giving a family of Thom classes $\{\mathrm{th}(V)\}$, we can define $\tau_V(-) := \mathrm{th}(V) \cup p^*-$ with $p : V \to X$ the projection map and $- \cup - :$ $\mathbb{E}^{\bullet,\bullet}(\mathrm{Th}_X(X)) \times \mathbb{E}^{\bullet,\bullet}(V) \to \mathbb{E}^{\bullet,\bullet}(\mathrm{Th}_X(V))$ the cup usual product map.
So an SL-orientation will correspond to giving (a system of) Thom isomorphism for all $X \in \mathbf{Sm}_{/S}$, while an absolute SL-orientation will correspond to giving Thom isomorphisms for all $X \in \mathbf{Sch}_{/S}$.

By [BH21b, Example 16.30] (applied to $G = (SL_n)_n$), for any scheme $X \in \mathbf{Sch}_{/S}$ and any V vector SL-bundle of rank n, we have an isomorphism $\tau_V : \Sigma^V \mathrm{MSL}_X \xrightarrow{\sim} \Sigma^{2n,n}\mathrm{MSL}_X$, where $\mathrm{MSL}_X := f^*MSL_S$ is the pullback of the special linear algebraic cobordism spectrum of [PW22] along the structure map $f : X \to S$. If we denote $u_{\mathrm{MSL}_X} : \mathbb{1}_X \to \mathrm{MSL}_X$ the unit map of MSL_X, then we have:

$$\Sigma^V u_{\mathrm{MSL}_X} : \Sigma^V \mathbb{1}_X \longrightarrow \Sigma^V \mathrm{MSL}_X \overset{\tau_V}{\simeq} \Sigma^{2n,n}\mathrm{MSL}_X$$

Notice that $\Sigma^V u_{\mathrm{MSL}_X}$ lives in $\mathrm{MSL}_X^{2n,n}(\mathrm{Th}_X(V)) \simeq \mathrm{MSL}_S^{2n,n}(X,-[V])$ and it is not hard to check that these elements satisfy all the properties in definition 2.1.5.

Definition 2.1.12. For any X, V as above, we will denote the elements $\mathrm{th}_{\mathrm{MSL}}(V) := \Sigma^V u_{\mathrm{MSL}_X} \in \mathrm{MSL}_X^{2n,n}(\mathrm{Th}_X(V)) \simeq \mathrm{MSL}_S^{2n,n}(X,-[V])$ and we will call $\mathrm{th}_{\mathrm{MSL}}(V)$ the *canonical* MSL-*Thom class* of V.

Canonical MSL-Thom classes give us an absolute orientation for MSL_S that restricts on smooth schemes to the usual SL-orientation of MSL_S. For any SL-oriented ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$, by [Ana16b, Theorem 4.7, Lemma 4.9], there exists a ring spectrum map $\varphi : \mathrm{MSL}_S \to \mathbb{E}$ such that $\varphi(\mathrm{th}_{\mathrm{MSL}}(V)) = \mathrm{th}_{\mathbb{E}}(V)$ for each smooth $X \in \mathbf{Sm}_{/S}$ and any vector SL-bundle V on X.

Remark 2.1.13. Once we have Thom classes and Thom isomorphism in cohomology for some $\mathbb{E} \in \mathrm{SH}$, we will get Thom isomorphisms also in Borel-Moore homology using the cohomology product action on Borel-Moore homology (cf. [Lev22a, §3.3]). In particular if $\mathbb{E}$ is SL-oriented with respect to $\mathbf{C}$, for any V vector SL-bundle over $X \in \mathbf{C} \subseteq \mathbf{Sch}_{/S}$, of rank r, we will have:

$$\mathbb{E}^{\mathrm{BM}}_{a+2r,b+r}(X/S) \xrightarrow{\sim} \mathbb{E}^{\mathrm{BM}}_{a,b}(X/S,[V])$$

Definition 2.1.14. Given $\mathbb{E} \in \mathrm{SH}(S)$ an SL-oriented ring spectrum and a map $\varphi : \mathrm{MSL} \to \mathbb{E}$ of ring spectra, we call φ *an* SL*-orientation map.*

Since φ is a map of ring spectra, if we define for any $X \in \mathbf{Sch}_{/S}$ and any vector SL-bundle V on X:

$$\mathrm{th}^{\varphi}_{\mathbb{E}}(V) := \varphi(\mathrm{th}_{\mathrm{MSL}}(V)) \in \mathbb{E}^{2n,n}(\mathrm{Th}_X(V))$$

we get an absolute SL-orientation on $\mathbb{E}$ extending the given SL-orientation we already had.

Definition 2.1.15. Consider $\mathbb{E} \in \mathrm{SH}(S)$ an SL-oriented ring spectrum, and suppose we are given an SL-orientation map $\varphi : \mathrm{MSL} \to \mathbb{E}$. Then we call the *$\varphi$-induced absolute SL-orientation* the orientation data given by Thom classes:

$$\mathrm{th}^{\varphi}_{\mathbb{E}}(V) := \varphi(\mathrm{th}_{\mathrm{MSL}}(V)) \in \mathbb{E}^{2n,n}(\mathrm{Th}_X(V))$$

for any $X \in \mathbf{Sch}_{/S}$ and any vector SL-bundle V on X. For short we will just say *φ-induced SL-orientation.*

We do not know a priori if the $\varphi : \mathrm{MSL} \to \mathbb{E}$ that can be associated with an SL-orientation is unique. While a similar unicity statement holds true for GL- and Sp-orientations by [DF21, Remark 2.1.5], for SL-orientations is still open: there could be an obstruction preventing the uniqueness of φ living in $\lim^1 \mathbb{E}^{2n-1,n}(\mathrm{MSL}_n^{fin})$ by [PW22, Theorem 5.8], where MSL_n^{fin} are the Thom spaces associated to the tangent bundle of the special linear Grassmannian $SGr(n,n^2)$ (cf. [PW22, §5],[Ana16b, Def. 4.5]).

Proposition 2.1.16. *Let* $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ *be an* η*-invertible motivic ring spectrum. Then* SL*-orientations are in one to one correspondence with* SL*-orientation maps* $\varphi : \mathrm{MSL}_{\Bbbk} \longrightarrow \mathrm{A}$.

Proof. Let $SGr_{\Bbbk}(n,m)$ the special linear Grassmannian, defined as the complement of the zero section of the determinant bundle associated to the universal bundle $E(n,m)$ over the Grassmannian $Gr_{\Bbbk}(n,m)$ (see [Ana15] for more details). Let us denote $\mathrm{Th}(n,m) := \mathrm{Th}_{SGr(n,m)}\left(\mathcal{T}(n,m)\right)$ the Thom space associated to the tautological bundle $\mathcal{T}(n,m)$ of the special linear Grassmannian $SGr(n,m)$. Since A is η-invertible, we can adopt the single graded convention $\mathrm{A}^{\bullet}$. By [PW22, Theorem 5.8], we know that $\varphi : \mathrm{MSL}_{\Bbbk} \to \mathrm{A}$ as in our claim exists and the obstruction to the uniqueness of φ lies in $\lim^1 \mathrm{A}^{n-1}(\mathrm{MSL}_n^{fin})$, where $\mathrm{MSL}_n^{fin} = \mathrm{Th}(n,n^2)$ are the finite approximation spaces for $\mathrm{MSL}_{\Bbbk}$. By a cofinality argument, we have that the same proof as in *loc. cit.* works also if we use, as finite level approximation for $\mathrm{MSL}_{\Bbbk}$,the spaces MSL_{2n}^{fin}. So it turns out that φ is unique if:

$$\lim{}^1 \ \mathrm{A}^{2n-1}(\mathrm{MSL}_{2n}^{fin}) = 0$$

Notice that it is enough to show the surjectivity of the maps:

$$\ldots \to \mathrm{A}^p(\mathrm{MSL}_{2(n+1)}^{fin}) \xrightarrow{i_n^*} \mathrm{A}^p(\mathrm{MSL}_{2n}) \to \ldots$$

induced by the maps of the direct system:

$$\ldots \to \mathcal{T}(2n,4n^2) \xrightarrow{i_n} \mathcal{T}(2(n+1),4(n+1)^2) \to \ldots$$

But since A is SL-oriented, for every k, we have isomorphisms:

$$\mathrm{A}^{\bullet-2n}(SGr(2n,k)) \xrightarrow{\cup \mathrm{th}(2n,k)} \mathrm{A}^{\bullet}(\mathrm{Th}_{SGr(2n,k)}\left(\mathcal{T}(2n,k)\right))$$

where $\mathrm{th}(2n,k)$ denotes the Thom class of $\mathcal{T}(2n,k)$. Using the last Thom isomorphism together with the computations in [Ana15, Theorem 9], we get that i_n^* are surjective and hence $\lim^1 \mathrm{A}^{2n-1}(\mathrm{MSL}_{2n}^{fin}) = 0$, giving us the uniqueness of φ. $\square$

Given an SL-oriented ring spectrum $\mathbb{E}$, with an SL-orientation map $\varphi : \mathrm{MSL} \to \mathbb{E}$, the φ-induced SL-orientation is uniquely determined. On the other hand, given an absolute SL-orientation, its restriction to smooth schemes $X \in \mathbf{Sm}_{/S}$ uniquely determines an associated SL-orientation, thus we get the following:

Corollary 2.1.17. *Let* $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ *be an* η*-invertible motivic ring spectrum. Then we have a one to one correspondence between the following data:*

1. SL*-orientations on* A*;*
2. *maps of ring spectra* $\varphi : \mathrm{MSL} \longrightarrow \mathrm{A}$ *such that* $\varphi(\mathrm{th}_{\mathrm{MSL}}(V)) = \mathrm{th}_{\mathrm{A}}(V)$ *for any* V *vector* SL*-bundle over* $X \in \mathbf{Sm}_{/\Bbbk}$*;*

3. *absolute SL-orientations.*

Remark 2.1.18. We will only deal with $SL[\eta^{-1}]$-oriented theories over smooth $\Bbbk$-schemes or just over some field $\Bbbk$. So from now on, with a slight abuse of notation, we will just say SL-orientation instead of *absolute* SL-orientation. According to the corollary above, this will make no harm in the case we are working over a field. Using a Leray spectral sequence argument we can also extend proposition 2.1.16 to smooth $\Bbbk$-scheme S (cf. proposition 3.2.1), but for most of our applications we will just work over a field, hence we will not need this result in such generality. Thanks to [DF21, Remark 2.1.5], we also need no distinction between Sp-oriented and absolutely Sp-oriented theories.

Recall from [PW18] that there exists a spectrum $\mathrm{BO}_S \in SH(S)$, whenever $\frac{1}{2} \in \mathcal{O}_S^\times$, that represents Hermitian K-theory[1].

Definition 2.1.19. Let $\mathrm{KW}_S := \mathrm{BO}_{S,\eta}$ be the Witt theory (absolute) spectrum defined by inverting the element $\eta \in \mathrm{BO}_S^{-1,-1}(S)$ as done in detail in [Ana16b, §6 and Theorem 6.5].

Remark 2.1.20. The spectrum BO_S is Sp-oriented (cf. [PW18]) and hence SL-oriented. This induces an SL-orientation on KW, and indeed KW will be our main example and focus point as an $SL[\eta^{-1}]$-oriented theory.

Thom Isomorphism and Euler Classes

For any SL-oriented theory, we can then talk about Euler classes $e(E, \theta)$ for SL-bundles E.

Notation 2.1.21. As already mentioned in the introduction, we will adopt the convention of [Lev19] for twisted cohomology theories. That means that given $\mathbb{E}$ an SL-oriented theory and $L \to X$ a line bundle over some $X \in \mathbf{Sch}_{/S}$, we denote the L-twisted $\mathbb{E}$- cohomology by:

$$\mathbb{E}^{a,b}(X; L) := \mathbb{E}^{a+2,b+1}(\mathrm{Th}_X(L))$$

Similarly, given a vector bundle $V \to X$, we will denote the L-twisted $\mathbb{E}$-cohomology on $\mathrm{Th}_X(V)$ as:

$$\mathbb{E}^{a,b}(\mathrm{Th}_X(V); L) := \mathbb{E}^{a+2,b+1}(\mathrm{Th}_X(V) \otimes \mathrm{Th}_X(L)) \simeq \mathbb{E}^{a+2,b+1}(\mathrm{Th}_X(V \oplus L))$$

Remark 2.1.22. Notice that if $L \simeq \mathbb{A}^1_X$ is the trivial line bundle, then $\mathbb{E}^{a,b}(X; L) := \mathbb{E}^{a+2,b+1}(\mathbb{P}^1_X) \simeq \mathbb{E}^{a,b}(X)$.

[1] There are recent works towards possible extension to more general schemes where 2 is not invertible in the ring of regular functions. It is worth mentioning for example [Kum20].

Given any vector bundle V of rank r on $X \in \mathbf{Sch}_{/S}$, if $L := \det(V)$, we can construct the associated SL-vector bundle given by $V \oplus L^{-1}$ with its canonical trivialization of the determinant $\omega_{can} : V \oplus L^{-1} \to \mathcal{O}_X$.

Definition 2.1.23. Let $\mathbf{C}$ be a full subcategory of $\mathbf{Sch}_{/S}$ and let $\mathbb{E} \in \mathrm{SH}(S)$ be a ring spectrum with an SL-orientation with respect to $\mathbf{C}$. Let $p : V \to X$ be a rank r vector bundle on $X \in \mathbf{C}$ with determinant $L := \det(V)$.

1. We define the Thom class in L^{-1}-twisted cohomology by:

$$\mathrm{th}(V) := \mathrm{th}_{V \oplus L^{-1}} \in \mathbb{E}^{2r,r}(\mathrm{Th}_X(V); L^{-1}) := \mathbb{E}^{2r+2,r+1}(\mathrm{Th}_X\left(V \oplus L^{-1}\right))$$

2. Let $s_{0,L} : X \oplus L^{-1} \xrightarrow{s_0 \oplus Id} V \oplus L^{-1}$ be the map induced by the zero section s_0 of V, then we define the (twisted) Euler class as:

$$e(E) := s_{0,L}^{*}\mathrm{th}^{\varphi}(V) \in \mathbb{E}^{2n+2,n+1}(\mathrm{Th}_X\left(L^{-1}\right)) = \mathbb{E}^{2n,n}(X; L^{-1})$$

Proposition 2.1.24 (Twisted Thom Isomorphism)**.** *Let $p : V \to X$ be a rank r vector bundle over a scheme X, and let $\mathbb{E} \in \mathrm{SH}(S)$ be a SL-oriented ring spectrum together with an SL-orientation map φ. Then we have an isomorphism:*

$$\vartheta_V^{\varphi} := p^*(-) \cup \mathrm{th}^{\varphi}(V) : \mathbb{E}^{*,*}(X; \det(V)) \longrightarrow \mathbb{E}^{*+2r,*+r}(\mathrm{Th}(V))$$

Proof. Denote by $L := \det(V)$ the determinant bundle of V and let $\mathcal{V}$, $\mathcal{L}$ be the locally free sheaves associated to V and L. Using the absolute SL-orientation induced by φ, the construction in [LR20, §3.10] works verbatim in our case. Let us briefly sketch how one should proceed (more details can be found in *loc. cit*). The Thom class $\mathrm{th}^{\varphi}_{V \oplus L^{-1}}$ gives us a Thom isomorphism:

$$q^*(-) \cup \mathrm{th}^{\varphi}_{V \oplus L^{-1}} : \mathbb{E}^{\bullet,\bullet}(X) \longrightarrow \mathbb{E}^{\bullet+2(r+1),\bullet+r+1}(\mathrm{Th}_X(V); L^{-1})$$

This means that we have an equivalence of spectra:

$$\Sigma^{[\mathcal{O}^{r+1}]-[\mathcal{V}]-[\mathcal{L}]}\mathbb{E} \simeq \mathbb{E}$$

Similarly we have $\Sigma^{[\mathcal{O}^2]-[\mathcal{L}]-[\mathcal{L}^{-1}]}\mathbb{E} \simeq \mathbb{E}$, and hence:

$$\Sigma^{[\mathcal{O}^r]-[\mathcal{V}]}\mathbb{E} \simeq \Sigma^{[\mathcal{O}]-[\mathcal{L}]}\mathbb{E} \tag{2.3}$$

The equivalence of eq. (2.3) (together with homotopy invariance for $p : V \to X$) gives us our isomorphism ϑ_V^{φ}. □

Remark 2.1.25. If $\mathbb{E} \in \mathrm{SH}(\Bbbk)$ is an $SL[\eta^{-1}]$-oriented motivic spectrum, we will drop the φ from the notation in virtue of corollary 2.1.17. Notice also that the Euler classes defined by the SL-orientations will coincide, under the relevant Thom isomorphism, with the Euler classes defined in Chapter 1 using the formalism of [DJK21].

References with more details for Euler classes in SL-oriented theories can be found in [Ana19, §3] and [LR20, §3] (even if they work with SL-orientations, everything can be adapted to our φ-induced, absolute SL-oriented case). A treatment of Euler classes closer to the one given here can also be found in [BW21, §5].

Remark 2.1.26. Consider an SL-oriented spectrum $\mathbb{E} \in \mathrm{SH}(S)$, with an SL-orientation map φ, and V a vector bundle over $X \in \mathbf{Sch}_{/S}$ with determinant $L := \det(V)$ (and associated locally free sheaves denoted by $\mathcal{V}$ and $\mathcal{L}$). Similarly to remark 2.1.13, using L-twisted Borel-Moore homology:

$$\mathbb{E}^{\mathrm{BM}}_{a,b}(X/S; L) := \mathbb{E}^{\mathrm{BM}}_{a-2,b-1}(X/S, -[\mathcal{L}])$$

we get L^{-1}-twisted Thom isomorphism:

$$\mathbb{E}^{\mathrm{BM}}_{a-2r,b-r}\left(X/S; L^{-1}\right) \xrightarrow{\sim} \mathbb{E}^{\mathrm{BM}}_{a,b}(X/S, [V])$$

using the φ-induced Thom classes $\mathrm{th}^{\varphi}(V) \in \mathbb{E}^{2r,r}(X; L^{-1})$. To remember how twisted Thom isomorphism works (both for cohomology and Borel-Moore homology), it is enough to remember that:

$$\Sigma^{[\mathcal{V}]\oplus[\mathcal{L}^{-1}]}\mathbb{E} \simeq \Sigma^{2r+2,r+1}\mathbb{E}$$

or equivalently:

$$\Sigma^{-2(r+1),-(r+1)}\Sigma^{[\mathcal{V}]}\mathbb{E} \simeq \Sigma^{2,1}\Sigma^{-[\mathcal{L}^{-1}]}\mathbb{E}$$

Construction 2.1.27. We will now construct a *symbol* element associated to a section of a line bundle, using the construction of a symbol associated to an invertible function on a scheme X as done in [Ana19, Definition 6.1]. For simplicity we will restrict to the case of a scheme, but the same procedure will work for any NL-stack without changing a word. Recall from *loc. cit.* that given $u \in \Gamma(X, \mathcal{O}_X^{\times})$, for $X \in \mathbf{Sm}_{/S}$ and $\mathbb{E} \in \mathrm{SH}(X)$, we have a well defined element $\langle u \rangle \in \mathbb{E}^{0,0}(X)$ induced by the multiplication by u on $T = \mathbb{A}^1_X/\mathbb{G}_{m,X}$. This element $\langle u \rangle$ is called the *symbol* associated to u. Consider now a line bundle $p : L \to X$, and consider $\lambda : X \to L$ a section. Denote by $\mathcal{Z}(\lambda)$ the vanishing locus of λ:

$$\begin{tikzcd} \mathcal{Z}(\lambda) \arrow[r, hook, "\iota_\lambda"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook, "s_0"] \\ X \arrow[r, hook, "\lambda"'] & L \end{tikzcd}$$

Let $j_\lambda : U(\lambda) \hookrightarrow X$ be the open complement in X of $\mathcal{Z}(\lambda)$. Then λ induces a non vanishing section $j^*\lambda : U \to j_\lambda^* L^{\times}$ of $j_\lambda^* L$. But this means we can trivialise $j_\lambda^* L$, i.e. we have:

$$\tau_{j_\lambda^*} : \mathbb{A}^1_{U(\lambda)} \xrightarrow{\sim} j_\lambda^* L$$

with associated inverse:

$$\left(\tau_{j^*_\lambda}\right)^{-1} : j^*_\lambda L \xrightarrow{\sim} \mathbb{A}^1_{U(\lambda)}$$

Taking the associated Thom spaces, we get:

$$\mathrm{Th}(\tau^{-1}_{j^*_\lambda}) : \mathrm{Th}_{U(\lambda)}\left(j^*_\lambda L\right) \simeq \Sigma^{j^*_\lambda \mathcal{L}} \mathbb{1}_{U(\lambda)} \longrightarrow \mathrm{Th}_{U(\lambda)}\left(\mathbb{A}^1\right) \simeq \Sigma^{\mathcal{O}} \mathbb{1}_{U(\lambda)}$$

Twisting by $\Sigma^{-\mathcal{O}}$, we have:

$$\Sigma^{-\mathcal{O}}\mathrm{Th}(\tau^{-1}_{j^*_\lambda}) : \Sigma^{-\mathcal{O}}\Sigma^{j^*_\lambda \mathcal{L}} \mathbb{1}_{U(\lambda)} \longrightarrow \mathbb{1}_{U(\lambda)}$$

Definition 2.1.28. With the same notation above, let $\mathbb{E} \in \mathrm{SH}(S)$ be a ring spectrum with unit $u : \mathbb{1} \to \mathbb{E}$, then we define the $\mathbb{E}$-*symbol* associated to $\lambda : X \to L$ to be:

$$\langle \lambda \rangle_{\mathbb{E}} := u \circ \Sigma^{-\mathcal{O}}\mathrm{Th}(\tau^{-1}_{j^*_\lambda}) : \Sigma^{-\mathcal{O}}\mathrm{Th}_{U(\lambda)}\left(j^*_\lambda L\right) \to \mathbb{E} \in \mathbb{E}^{0,0}(U(\lambda); j^*_\lambda L)$$

Example 2.1.29. Consider the section of $\mathcal{O}_{\mathbb{P}^2}(2)$ given by $Q = T_1^2 - 4T_0T_2$, then $U(Q) = \mathbb{P}^2 \setminus C$ with C the zero-locus of Q. Then for any $\mathbb{E} \in \mathrm{SH}(S)$ we have:

$$\langle Q \rangle \in \mathbb{E}^{0,0}(\mathbb{P}^2 \setminus C; \mathcal{O}(2))$$

Suppose that $\mathbb{E}$ is either an element of $\mathrm{SH}(S)[\eta^{-1}]$ or it is SL-oriented. Then for any scheme X and any line bundle L over X, by [Hau23, Proposition 3.3.1] or [Ana19, Theorem 4.3] respectively, there exists an isomorphism:

$$\varphi : \mathbb{E}(X; L^{\otimes 2}) \xrightarrow{\sim} \mathbb{E}(X)$$

Hence we get a well defined element:

$$q_0 := \varphi(\langle Q \rangle) \in \mathbb{E}^{0,0}(\mathbb{P}^2 \setminus C)$$

Example 2.1.30. Consider $p : L \to X$ a line bundle over X. Let:

$$t_{can} : L \to p^* L$$

be the tautological section. Then $U(t_{can}) = L^\times = L \setminus 0$ and for any $\mathbb{E} \in \mathrm{SH}(S)$ we get:

$$\langle t_{can} \rangle \in \mathbb{E}^{0,0}(L^\times; L)$$

Consider $X = BGL_n$ and $L = \mathcal{O}(1)$. Then $L^\times \simeq BSL_n$ and we get:

$$\langle t_{can} \rangle \in \mathbb{E}^{0,0}(BSL_n; \mathcal{O}(1))$$

Sometimes we will refer to $\langle t_{can} \rangle$ as the *tautological symbol* associated to L.

2.1.1 SL-Orientations for NL-Stacks

We will now present an easy way to get Thom classes and Euler classes on NL-stacks. The methods used here can be adapted to most of the common G-orientations used in the literature, but since we will need to specialise to SL-oriented spectra anyway, we will only talk about those.

Consider $\mathcal{U}_n \to \mathcal{B}SL_n$ the universal bundle over $\mathcal{B}SL_n$ (the one corresponding, under Yoneda, to the identity map of $\mathcal{B}SL_n$.

Proposition 2.1.31. *Let $\mathbb{E} \in \mathrm{SH}(S)$ be an SL-oriented ring spectrum. Then we have a natural equivalence of mapping spectra:*

$$\tau : \mathbb{E}(\mathcal{B}SL_n) \longrightarrow \Sigma^{2n,n}\mathbb{E}(\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n))$$

Proof. First of all we need to construct the map τ and then we will prove it is indeed an isomorphism. Let $SGr_S(j,k)$ be the special linear Grassmannian, with tautological bundle $\mathcal{T}(j,k)$. For each double index (j,k), we have natural maps $\sigma_{j,k}; SGr_S(j,k) \to \mathcal{B}SL_n$ classifying the tautological bundles, that is, we have cartesian squares:

$$\begin{tikzcd} \mathcal{T}(j,k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_n \arrow[d] \\ SGr_S(j,k) \arrow[r, "\sigma_{j,k}"'] & \mathcal{B}SL_n \end{tikzcd}$$

By proposition 1.4.30, we have a natural equivalence:

$$\beta_\infty : \operatorname*{colim}_m \pi_{SGr_S(n,m)\#}\mathrm{Th}_{SGr_S(n,m)}\big(\mathcal{T}(n,m)\big) \xrightarrow{\sim} \pi_{\mathcal{B}SL_n\#}\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n)$$

But the left hand side is by definition $\mathrm{MSL}_n := \operatorname*{colim}_m \pi_{SGr_S(n,m)\#}\mathrm{Th}_{SGr_S(n,m)}\big(\mathcal{T}(n,m)\big)$ as defined in [PW22, §4]. By construction of MSL as a spectrum (cf. [PW22, §4]), we have a natural maps:

$$u_n : \Sigma^{-2n,-n}\mathrm{MSL}_n \to \mathrm{MSL}$$

By [PW22, Theorem 5.9] we have a map of motivic spectra $\varphi : \mathrm{MSL} \to \mathbb{E}$. Consider the following composition of maps:

$$\mathrm{th}_{\mathcal{U}_n} := \Sigma^{2n,n}(\varphi \circ u_n) \circ \beta_\infty^{-1} : \pi_{\mathcal{B}SL_n\#}\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n) \longrightarrow \Sigma^{2n,n}\mathbb{E}$$

This means that we have an element $\mathrm{th}_{\mathcal{U}_n} \in \mathbb{E}^{2n,n}(\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n))$. Let $p_n : \mathcal{U}_n \to \mathcal{B}SL_n$ be the projection map and let us finally define the map we are looking for:

$$\begin{array}{cccc} \tau : & \mathbb{E}^{\bullet,\bullet}(\mathcal{B}SL_n) & \longrightarrow & \mathbb{E}^{\bullet+2n,\bullet+n}(\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n)) \\ & x & \mapsto & \mathrm{th}_{\mathcal{U}_n} \cup p^*x \end{array}$$

Rewriting $\mathbb{E}(\mathcal{B}SL_n)$ and $\Sigma^{2n,n}\mathbb{E}(\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n))$ in terms of mapping spectra, we want to show that the map:

$$\tau : \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{B}SL_n)}(\mathbb{1}_{\mathcal{B}SL_n}, \pi^*_{\mathcal{B}SL_n}\mathbb{E}) \longrightarrow \underline{\mathrm{Map}}_{\mathrm{SH}(\mathcal{B}SL_n)}(\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n), \Sigma^{2n,n}\pi^*_{\mathcal{B}SL_n}\mathbb{E})$$

is indeed an equivalence. Since SL_n is special, we can take $a : S \to \mathcal{B}SL_n$ as our NL-atlas. By [Cho21a, Lemma 5.1.1], the map $a^* : \mathrm{SH}(\mathcal{B}SL_n) \to \mathrm{SH}(S)$ is conservative, hence τ is an equivalence if and only if $a^*\tau$ is an equivalence. Notice that $a^*\mathcal{U}_n \simeq \mathbb{A}^n_S$ and denote by $q_n := a^*p_n : \mathbb{A}^n_S \to S$ the projection map. Then we have:

$$a^*\tau(-) = a^*\mathrm{th}_{\mathcal{U}_n} \cup q_n^* - = \mathrm{th}_{\mathbb{A}^n_S} \cup q_n^* -$$

But this means that $a^*\tau$ is the Thom isomorphism map associated to $\mathbb{A}^n_S$, so it is indeed an equivalence as we wanted to show. □

Definition 2.1.32. We define the *canonical* Thom class of $\mathcal{U}_n \to \mathcal{B}SL_n$ as the element:

$$\mathrm{th}(\mathcal{U}_n) := \tau(1_{\mathcal{B}SL_n}) \in \mathbb{E}^{2n,n}(\mathrm{Th}_{\mathcal{B}SL_n}(\mathcal{U}_n))$$

where $1_{\mathcal{B}SL_n} \in \mathbb{E}^{0,0}(\mathcal{B}SL_n)$ is the identity element in the $\mathbb{E}$-cohomology of $\mathcal{B}SL_n$.

Now, let $\mathcal{X} \in \mathcal{ASt}^{NL}_{/S}$ be a NL-stack. Let $v : V \to \mathcal{X}$ be a vector bundle of rank n with trivialised determinant. The vector bundle V is classified by a map f_V such that:

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_n \arrow[d] \\ \mathcal{X} \arrow[r, "f_V"'] & \mathcal{B}SL_n \end{tikzcd}$$

Definition 2.1.33. We define the Thom class of the special linear vector bundle $V \to \mathcal{X}$ with values in a SL-oriented ring spectrum $\mathbb{E} \in \mathrm{SH}(S)$ as:

$$\mathrm{th}(V) := f_V^*\mathrm{th}(\mathcal{U}_n) \in \mathbb{E}^{2n,n}(\mathrm{Th}_{\mathcal{X}}(V))$$

where $f_V : \mathcal{X} \to \mathcal{B}SL_n$ is the map classifying the special linear bundle V.

As we did for schemes, once we have Thom classes for vector bundles with trivialised determinants, we can define Thom classes for general vector bundles living in twisted cohomology. Indeed if $V \to \mathcal{X}$ is a vector bundle of rank n with determinant $L := \det(V)$, then $V \oplus L^{-1}$ is a vector bundle of rank $n+1$ with trivial determinant. We can define:

Definition 2.1.34. The Thom class for V in L^{-1}-twisted $\mathbb{E}$-cohomology is the element:

$$\mathrm{th}_V := \mathrm{th}_{V \oplus L^{-1}} \in \mathbb{E}^{2n,n}(\mathrm{Th}_{\mathcal{X}}(V); L^{-1}) := \mathbb{E}^{2n+2,n+1}\left(\mathrm{Th}_{\mathcal{X}}\left(V \oplus L^{-1}\right)\right)$$

Definition 2.1.35. In the same notation as above, we defined the $\mathbb{E}$-valued Euler class of a vector bundle $V \to \mathcal{X}$ of rank n as:

$$e(V) := s^*\mathrm{th}_V \in \mathbb{E}^{2n,n}(X; L^{-1}) := \mathbb{E}^{2n+2,n+1}(\mathrm{Th}_{\mathcal{X}}\left(L^{-1}\right))$$

where s^* is the pullback map induced by the zero section $s_0 : \mathcal{X} \to V$.

Remark 2.1.36. Using remark 1.4.9, we can translate everything we did in this section to the equivariant setting.

2.2 The Additive Structure of $SL[\eta^{-1}]$-Oriented Theories

Notation 2.2.1. To make the notation more compact, we will write:

$$\widetilde{-} := [-/G]$$

to denote the quotient stack of G-equivariant objects, whenever the group G is clear from the context.

Lemma 2.2.2 ([Lev19, Lemma 5.1])**.** *Let F be the tautological rank two representation of SL_2 and let $\mathrm{A} \in \mathrm{SH}(S)$ be an η-invertible spectrum. Then the structure map $\pi_{\widetilde{\mathbb{P}(F)}} : \widetilde{\mathbb{P}(F)} \to S$ induces an isomorphism:*

$$\pi^*_{\widetilde{\mathbb{P}(F)}} : \mathrm{A}^\bullet(S) \xrightarrow{\sim} \mathrm{A}^\bullet_{SL_2}(\mathbb{P}(F))$$

Proof. We will denote by $\pi_- : - \to S$ the structure maps of our schemes and stacks. Give to $F = \mathbb{A}^2$ the standard left SL_2-action, and equip $\mathbb{A}^2 \setminus \{0\}$ with the induced (left) action. Equip SL_2 with the left SL_2-action coming from matrix multiplication, then we have a SL_2-equivariant map:

$$\begin{array}{cccc} r: & SL_2 & \longrightarrow & \mathbb{A}^2 \setminus \{0\} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & (a,b) \end{array}$$

that realises SL_2 as a (SL_2-equivariant) $\mathbb{A}^1$-bundle over $\mathbb{A}^2 \setminus \{0\}$, with zero section map $s_0 : \mathbb{A}^2 \setminus \{0\} \to SL_2$. This means that once we pass to the quotient stacks we get an $\mathbb{A}^1$-bundle map:

$$\tilde{r} : \left[{}^{SL_2}/_{SL_2}\right] \simeq S \longrightarrow \left[{}^{\mathbb{A}^2 \setminus \{0\}}/_{SL_2}\right] =: \widetilde{\mathbb{A}^2 \setminus \{0\}}$$

with zero section $\tilde{s_0} : \widetilde{\mathbb{A}^2 \setminus \{0\}} \to S$. By homotopy invariance, we get an equivalence of mapping spectra:

$$\tilde{s_0}^* : \mathrm{A}(S) = \underline{\mathrm{Map}}_{\mathrm{SH}(S)}(\mathbb{1}_S, \mathrm{A}) \xrightarrow{\sim} \mathrm{A}\left(\widetilde{\mathbb{A}^2 \setminus \{0\}}\right) = \underline{\mathrm{Map}}_{\mathrm{SH}(S)}\left(\mathbb{1}_S, \pi_{\widetilde{\mathbb{A}^2\setminus\{0\}}*}\pi^*_{\widetilde{\mathbb{A}^2\setminus\{0\}}}\mathrm{A}\right)$$

On the other hand, the algebraic Hopf map:

$$\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}(F)$$

induces a map on the quotients stacks:

$$\tilde{\eta} : \widetilde{\mathbb{A}^2 \setminus \{0\}} \to \widetilde{\mathbb{P}(F)} := \left[\mathbb{P}(F)/SL_2\right]$$

Hence we get a well defined map:

$$\tilde{\eta}^* : \mathrm{A}\left(\widetilde{\mathbb{P}(F)}\right) \longrightarrow \mathrm{A}\left(\widetilde{\mathbb{A}^2 \setminus \{0\}}\right)$$

By remark 1.4.34, for any NL-stack $\mathcal{X} \in \mathcal{ASt}^{NL}_{/S}$ and any NL-atlas $X \to \mathcal{X}$, we can compute the cohomology of $\mathcal{X}$ using its Čech nerve $X^n_{\mathcal{X}} := \check{C}_n\left(X/\mathcal{X}\right)$:

$$\begin{aligned}
\mathrm{A}(\mathcal{X}) &= \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{1}_S, \pi_{\mathcal{X}*}\pi^*_{\mathcal{X}}\mathrm{A}) \simeq \\
&\simeq \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{1}_S, \lim_{n\in\Delta} \pi_{X^n_{\mathcal{X}}*}\pi^*_{X^n_{\mathcal{X}}}\mathrm{A}) \simeq \\
&\simeq \lim_{n\in\Delta} \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{1}_S, \pi_{X^n_{\mathcal{X}}*}\pi^*_{X^n_{\mathcal{X}}}\mathrm{A}) \simeq \\
&\simeq \lim_{n\in\Delta} \mathrm{A}(X^n_{\mathcal{X}})
\end{aligned}$$

For $\widetilde{\mathbb{P}(F)} = [\mathbb{P}(F)/SL_2]$ a NL-atlas is given by $\mathbb{P}(F) \to \widetilde{\mathbb{P}(F)}$ with Čech nerve given levelwise by $\check{C}_n\left(\mathbb{P}(F)/\widetilde{\mathbb{P}(F)}\right) = \mathbb{P}(F) \times SL^n_n$, hence:

$$\mathrm{A}\left(\widetilde{\mathbb{P}(F)}\right) \simeq \lim_{n\in\Delta} \mathrm{A}\left(\mathbb{P}(F) \times SL^n_2\right)$$

Similarly, for $\widetilde{\mathbb{A}^2 \setminus \{0\}}$ we get:

$$\mathrm{A}\left(\widetilde{\mathbb{A}^2 \setminus \{0\}}\right) \simeq \lim_{n\in\Delta} \mathrm{A}\left((\mathbb{A}^2 \setminus \{0\}) \times SL^n_2\right)$$

Therefore the map $\tilde{\eta}^*$ restricted levelwise on the Čech nerves gives us a map:

$$\tilde{\eta}^*_n : \mathrm{A}\left(\mathbb{P}(F) \times SL^n_2\right) \longrightarrow \mathrm{A}\left((\mathbb{A}^2 \setminus \{0\}) \times SL^n_2\right)$$

that is just the pullback map along $\eta_n = \eta \times Id : (\mathbb{A}^2 \setminus \{0\}) \times SL^n_2 \to \mathbb{P}(F) \times SL^n_2$. But since η is invertible in A, all the maps $\tilde{\eta}^*_n$ are invertible and hence:

$$\tilde{\eta}^* : \mathrm{A}\left(\widetilde{\mathbb{P}(F)}\right) \longrightarrow \mathrm{A}\left(\widetilde{\mathbb{A}^2 \setminus \{0\}}\right)$$

is an equivalence.

Now consider the following commutative diagram of quotient stacks:

$$\begin{tikzcd} \widetilde{\mathbb{A}^2 \setminus \{0\}} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{s}_0"'] & \widetilde{\mathbb{P}(F)} \arrow[dl, "\pi_{\widetilde{\mathbb{P}(F)}}"] \\ S & \end{tikzcd}$$

This means that we have:

$$(\tilde{s}_0)^* \simeq \tilde{\eta}^* \pi^*_{\widetilde{\mathbb{P}(F)}}$$

And hence, since both $(\tilde{s}_0)^*$ and $\tilde{\eta}^*$ are equivalences, the pullback map:

$$(\pi_{\widetilde{\mathbb{P}(F)}})^* \simeq (\tilde{\eta}^*)^{-1}(\tilde{s}_0)^* : \mathrm{A}(S) \longrightarrow \mathrm{A}\left(\widetilde{\mathbb{P}(F)}\right)$$

is an equivalence too.

□

To prove the full statement of the additive presentation of $\mathrm{A}^\bullet(BN)$, for an $SL[\eta^{-1}]$-oriented A, we will need various intermediate steps.

Lemma 2.2.3 ([Lev23])**.** *Given $A \in \mathrm{SH}(S)$ and η-invertible spectrum, then the pullback of the structure map $\pi_{\mathbb{P}^2_S} : \mathbb{P}^2_S \to S$ induces an isomorphism:*

$$\mathrm{A}^\bullet(S) \longrightarrow \mathrm{A}^\bullet(\mathbb{P}^2_S)$$

Proof. We can cover $\mathbb{P}^2_S$ by two opens U, V where $U = \mathbb{P}^2_S \setminus \{p\}$ with $p = [1:0:0]$, and $V = \mathbb{P}^2_S \setminus \{x_0 = 0\} \simeq \mathbb{A}^2_S$. Then we have a Mayer-Vietoris diagram:

$$\begin{tikzcd} U \cap V \simeq \mathbb{A}^2 \setminus \{0\} \arrow[r] \arrow[d] & U \arrow[d, "j_U"] \\ V \simeq \mathbb{A}^2_S \arrow[r, "j_V"'] & \mathbb{P}^2_S \end{tikzcd}$$

We can identify U with the bundle $\mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1_S$ that sends $[x_0 : x_1 : x_2] \mapsto [0 : x_1 : x_2]$, hence U is $\mathbb{A}^1$-equivalent to $\mathbb{P}^1_S$. Then up to $\mathbb{A}^1$-invariance, the Mayer-Vietoris diagram becomes:

$$\begin{tikzcd} \mathbb{A}^2 \setminus \{0\} \arrow[r, "\eta"] \arrow[d] & \mathbb{P}^1_S \arrow[d] \\ \mathbb{A}^2_S \arrow[r, "j_V"'] & \mathbb{P}^2_S \end{tikzcd}$$

Hence $\mathbb{P}^2_S$ is $\mathbb{A}^1$-equivalent to $\mathbb{P}^1_S \amalg_{\mathbb{A}^2\setminus\{0\}} \mathbb{A}^2_S$. But if we invert the (unstable) Hopf map, we can replace $\mathbb{A}^2_S \setminus \{0\}$ with $\mathbb{P}^1_S$ and hence $\mathbb{P}^2_S$ becomes equivalent to the $\mathbb{A}^2_S$ in $\mathrm{SH}(S)[\eta^{-1}]$. Consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "j_V"] \arrow[dr, "\pi_{\mathbb{A}^2}"'] & \mathbb{P}^2_S \arrow[d, "\pi_{\mathbb{P}^2}"] \\ & S \end{tikzcd}$$

Then we have that $\pi^*_{\mathbb{A}^2} \simeq j^*_V \pi^*_{\mathbb{P}^2_S}$, but $\pi^*_{\mathbb{A}^2}$ is an equivalence by homotopy invariance and j^*_V becomes an equivalence once we invert η, therefore $\pi^*_{\mathbb{P}^2_S}$ is an equivalence too in $\mathrm{SH}(S)[\eta^{-1}]$. This means that we have an isomorphism:

$$\pi^*_{\mathbb{P}^2_S} : \mathrm{A}^\bullet(S) \xrightarrow{\sim} \mathrm{A}^\bullet(\mathbb{P}^2_S)$$

as claimed. $\square$

Lemma 2.2.4 ([Lev23]). *Let* $\mathrm{A} \in \mathrm{SH}(S)$ *be an* $SL[\eta^{-1}]$*-oriented spectrum. Let* $T \to \mathbb{P}^2_S$ *be the tangent bundle of* $\mathbb{P}^2_S$ *and let* $e(T) \in \mathrm{A}^2(\mathbb{P}^2_S; \mathcal{O}_{\mathbb{P}^2}(3)) \simeq \mathrm{A}^2(\mathbb{P}^2_S; \mathcal{O}_{\mathbb{P}^2}(1))$ *be its Euler class. The cup product with* $e(T)$ *induces an isomorphism:*

$$\mathrm{A}^{n-2}(S) \simeq \mathrm{A}^n(\mathbb{P}^2_S; \mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathrm{A}^n(\mathbb{P}^2_S; \mathcal{O}_{\mathbb{P}^2}(3))$$

Proof. Consider the localization sequence associated to a point $\{p\} \hookrightarrow \mathbb{P}^2$:

$$\mathrm{A}^{\bullet-2}(p) \xrightarrow{\iota_*} \mathrm{A}^\bullet(\mathbb{P}^2_S; \mathcal{O}(1)) \xrightarrow{j^*} \mathrm{A}^\bullet(\mathbb{P}^2_S \setminus \{p\}; \mathcal{O}(1))$$

where we identified the twist by $\mathcal{O}(3)$ with the one by $\mathcal{O}(1)$ since A is SL-oriented. As in lemma 2.2.3,homotopy invariance gives us an isomorphism:

$$\mathrm{A}^\bullet(\mathbb{P}^2_S \setminus \{p\}; \mathcal{O}(1)) \simeq \mathrm{A}^\bullet(\mathbb{P}^1_S; \mathcal{O}(1))$$

By definition we have $\mathrm{A}^\bullet(\mathbb{P}^1_S; \mathcal{O}(1)) = \mathrm{A}^{\bullet+1}(\mathrm{Th}_{\mathbb{P}^1_S}(\mathcal{O}(1)))$. But by homotopy invariance, $\mathcal{O}(1)$ can be identified with $\mathbb{P}^1_S$ and after inverting η, we can also identify $\mathbb{A}^2_S \setminus \{0\} \simeq \mathcal{O}(1) \setminus \{0\} \subseteq \mathcal{O}(1)$ with $\mathbb{P}^1_S$. Hence $\mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(1))$ becomes just a (pointed) point, so $A^\bullet(\mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(1))) = 0$. Then from the localization sequence we get an isomorphism:

$$\iota_* : \mathrm{A}^{\bullet-2}(S) \xrightarrow{\sim} \mathrm{A}^\bullet(\mathbb{P}^2_S; \mathcal{O}(1))$$

The pushforward map $\pi_{\mathbb{P}^2 *} : \mathrm{A}^\bullet(\mathbb{P}^2_S; \mathcal{O}(1)) \longrightarrow \mathrm{A}^\bullet(S)$, with $\pi_{\mathbb{P}^2_S} : \mathbb{P}^2_S \to S$ the structure map, is an isomorphism too since $\pi_{\mathbb{P}^2_S *}\iota_* = Id$.

Now we make the following claim: if T is the tangent bundle of $\mathbb{P}^2_S$, then $\pi_{\mathbb{P}^2_S} e(T)$ is a unit in $\mathrm{A}^0(S)$. To see this, we use the Leray spectral sequences for $\mathrm{A}^\bullet(S)$ and $\mathrm{A}^\bullet(\mathbb{P}^2_S; \mathcal{O}(1))$ (cf. [ADN18, §4]) to reduce to computations along the fibers $\mathrm{A}^\bullet(\kappa(x))$ and $\mathrm{A}^\bullet(\mathbb{P}^2_{\kappa(x)})$. By the motivic Gauß-Bonnet theorem (cf. [LR20] or

[DJK21, p. 4.6.1]), we have that $\chi\left(\mathbb{P}^2_{\kappa(x)}/\kappa(x)\right) = \pi_{\kappa(x)*}(e^{\mathbb{S}}(T_{\mathbb{P}^2_{\kappa(x)}}))$, where $e^{\mathbb{S}}(\cdot)$ denotes the Euler characteristic relative to the sphere spectrum. We know that the Euler-characteristic of $\mathbb{P}^2_{\kappa(x)}$ is equal to $\langle 2\rangle + \langle -1\rangle \in \mathrm{GW}(\kappa(\mathrm{x}))$ by the computation in [Hoy14, Ex.1.8], but this means that $\pi_{\kappa(x)*}(e(T_{\mathbb{P}^2_{\kappa(x)}}))$ is the image of $\langle 1\rangle \in \mathrm{W}(\kappa(x))$ under the unit map $\mathrm{W}(\kappa(x)) \to \mathrm{A}^0(\kappa(x))$. So we have that $\pi_{\mathbb{P}^2_{\kappa(x)}*}(\iota_{\kappa(x)*}1_{\mathbb{P}^2}) = \pi_{\mathbb{P}^2*}(e(T_{\mathbb{P}^2}))$ defines an isomorphism and is indeed a unit. Therefore:

$$\left(\cdot \cup e(T_{\mathbb{P}^2_{\kappa(x)}})\right) \circ \pi^*_{\mathbb{P}^2_{\kappa(x)}} : \mathrm{A}^{\bullet-2}(\kappa(x)) \longrightarrow \mathrm{A}^{\bullet}(\mathbb{P}^2_{\kappa(x)}; \mathcal{O}(1))$$

defines an isomorphism too, as we wanted.

□

We will now go on a brief excursus computing $SL[\eta^{-1}]$-oriented theories of $\mathbb{P}^n_S$, just for the sake of completeness. The following lemma (as well as the last two lemmas before) is already known (cf. [Ana16a, Theorem 2] for a more general statement):

Lemma 2.2.5. *Given* $\mathrm{A} \in \mathrm{SH}(S)$ *with* A *an* $SL[\eta^{-1}]$*-oriented spectrum. Then we have:*

$$\mathrm{A}^{\bullet}(\mathbb{P}^{2k+1}) \simeq \mathrm{A}^{\bullet}(S) \oplus \mathrm{A}^{\bullet-2k-1}(S)$$
$$\mathrm{A}^{\bullet}(\mathbb{P}^{2k+1}; \mathcal{O}(1)) \simeq 0$$
$$\mathrm{A}^{\bullet}(\mathbb{P}^{2n}) \simeq \mathrm{A}^{\bullet}(S)$$
$$\mathrm{A}^{\bullet}(\mathbb{P}^{2n}; \mathcal{O}(1)) \simeq \mathrm{A}^{\bullet-2n}(S)$$

Proof. We will proceed by induction. Let us start considering $n = 1$, then take the localization sequence associated to $p \hookrightarrow \mathbb{P}^1$ with open complement given by $\mathbb{A}^1$:

$$\ldots \to \mathrm{A}^{\bullet-1}(S) \overset{(\iota_1)_*}{\to} \mathrm{A}^{\bullet}(\mathbb{P}^1_S) \overset{j_1^*}{\to} \mathrm{A}^{\bullet}(S) \overset{\partial}{\to} \mathrm{A}^{\bullet}(S) \to \ldots$$

Then the projection map $\pi_{\mathbb{P}^1} : \mathbb{P}^1_S \to S$ induces the pullback map π^* that exhibits j_1^* as a split surjection, telling us that $\partial = 0$ and that:

$$\mathrm{A}^{\bullet}(\mathbb{P}^1) \simeq \mathrm{A}^{\bullet}(S) \oplus \mathrm{A}^{\bullet-1}(S)$$

For the twisted theory of $\mathbb{P}^1$, we already saw in the proof of lemma 2.2.4 that:

$$\mathrm{A}^{\bullet}(\mathbb{P}^1; \mathcal{O}(1)) \simeq 0$$

Moreover lemma 2.2.3 and lemma 2.2.4 already took care of the case $n = 2$. Let us assume we know the result for any $m < 2n+1$, then we just need to show it also holds for $2n+1$ and $2n+2$. Let us start with $\mathbb{P}^{2n+1}_S$, and again consider $p \hookrightarrow \mathbb{P}^{2n+1}$. Similarly to what we did in lemma 2.2.3, we can identify $\mathbb{P}^{2n+1} \setminus \{p\}$ with $\mathcal{O}_{\mathbb{P}^{2n}}(1)$

and hence, up to $\mathbb{A}^1$-equivalence, with $\mathbb{P}^{2n}$. Now the localization sequence associated to $p \hookrightarrow \mathbb{P}^{2n+1}$ reads as:

$$\ldots \to \mathrm{A}^{\bullet-2n-1}(S) \overset{(\iota_{2n+1})_*}{\to} \mathrm{A}^\bullet(\mathbb{P}^{2n+1}_S) \overset{j^*_{2n+1}}{\to} \mathrm{A}^\bullet(\mathbb{P}^{2n}) \overset{\partial_{2n+1}}{\to} \mathrm{A}^{\bullet-2n}(S) \to \ldots$$

By induction hypothesis, we have $\mathrm{A}^\bullet(\mathbb{P}^{2n}) = \mathrm{A}^\bullet(S)$, and once again the projection map $\pi_{\mathbb{P}^{2n+1}} : \mathbb{P}^{2n+1}_S \to S$ induces a map $\pi^*_{\mathbb{P}^{2n+1}}$ that makes j^*_{2n+1} into a split surjection, giving us:

$$\mathrm{A}^\bullet(\mathbb{P}^{2n+1}) \simeq \mathrm{A}^\bullet(S) \oplus \mathrm{A}^{\bullet-2n-1}(S)$$

Considering again the localization sequence associated to $p \hookrightarrow \mathbb{P}^{2n+1}$, but this time with the twist by $\mathcal{O}(1)$, we get:

$$\ldots \mathrm{A}^{\bullet-2n-1}(S) \overset{(\iota(1)_{2n+1})_*}{\to} \mathrm{A}^\bullet(\mathbb{P}^{2n+1}_S; \mathcal{O}(1)) \overset{j(1)^*_{2n+1}}{\to} \mathrm{A}^\bullet(\mathbb{P}^{2n}; \mathcal{O}(1)) \overset{\partial_{2n+1}(1)}{\to} \mathrm{A}^{\bullet-2n}(S) \to \ldots$$

By induction hypothesis $\partial_{2n+1}(1)$ must be an isomorphism and thus $\mathrm{A}^\bullet(\mathbb{P}^{2n+1}_S; \mathcal{O}(1)) \simeq 0$. The case $2n+2$ is completely similar, and we will leave it to the reader. □

Corollary 2.2.6. *For any η-inverted spectrum $\mathrm{A} \in \mathrm{SH}(S)$, we have:*

$$\mathrm{A}^\bullet\left(B\mathbb{G}_{m,S}\right) \simeq \mathrm{A}^\bullet(S)$$

$$\mathrm{A}^\bullet\left(B\mathbb{G}_{m,S}; \mathcal{O}(1)\right) \simeq 0$$

Proof. Recall from [MV99], that the model for $B\mathbb{G}_m$ is given by $\mathbb{P}^\infty$. We have that:

$$B\mathbb{G}_m \simeq \mathbb{P}^\infty = \operatorname*{colim}_m \ \mathbb{P}^m$$

For the untwisted case, we can take the colimit over the even dimensional projective spaces (since this is a cofinal system). Then the structure map $\pi_{\mathcal{B}\mathbb{G}_m} : \mathcal{B}\mathbb{G}_m \to S$, under the identification of corollary 1.4.31, induces a map:

$$\pi^*_{\mathcal{B}\mathbb{G}_m} : \mathrm{A}^\bullet(S) \to \mathrm{A}^\bullet(\mathcal{B}\mathbb{G}_m) \simeq \mathrm{A}^\bullet(B\mathbb{G}_m) \simeq \lim_k \mathrm{A}^\bullet(\mathbb{P}^{2k})$$

If we consider $\mathrm{A}^\bullet(S)$ as a limit spectrum over a constant pro-system, then the map $\pi^*_{\mathcal{B}\mathbb{G}_m}$ levelwise becomes $\pi^*_{\mathbb{P}^{2k}} : \mathrm{A}(S) \to \mathrm{A}(\mathbb{P}^{2k})$. By lemma 2.2.5, $\pi^*_{\mathbb{P}^{2k}}$ is an equivalence for every k and therefore $\pi^*_{\mathcal{B}\mathbb{G}_m}$ is an equivalence too, proving the first claim of our corollary. For the second claim, we just write $B\mathbb{G}_m$ as a colimit of odd dimensional projective spaces and again by lemma 2.2.5 we get:

$$\mathrm{A}^\bullet(B\mathbb{G}_m; \mathcal{O}(1)) \simeq \lim_k \mathrm{A}^\bullet(\mathbb{P}^{2k+1}; \mathcal{O}(1)) = 0$$

□

The previous corollary can also be deduced from the following stronger result proved in [Hau23, Theorem 6.1.3]:

Theorem 2.2.7 (Haution)**.** *The map $B\mathbb{G}_m \to S$ is induces an isomorphism in* $\mathrm{SH}(S)[\eta^{-1}]$.

Let $\theta : T \longrightarrow \mathbb{P}(\mathrm{Sym}^2(F))$ be the tangent bundle of $\mathbb{P}(\mathrm{Sym}^2(F))$. We have an induced map:

$$\mathcal{T} := [T/SL_2] \to \mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)) := \left[\mathbb{P}(\mathrm{Sym}^2(F))/SL_2\right]$$

where $\mathcal{T}$ is the relative tangent bundle of $\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F))$ over $\mathcal{B}SL_2$.

Lemma 2.2.8 ([Lev19, Lemma 5.2])**.** *Let* $\mathrm{A} \in \mathrm{SH}(S)$ *be an* $SL[\eta^{-1}]$*-oriented motivic spectrum. Then:*

(*a*) *The map* $\tilde{p} : \mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)) \longrightarrow \mathcal{B}SL_2$ *induces an isomorphism:*

$$\tilde{p}^* : \mathrm{A}^\bullet\left(\mathcal{B}SL_2\right) \xrightarrow{\sim} \mathrm{A}^\bullet_{SL_2}\left(\mathbb{P}(\mathrm{Sym}^2(F))\right)$$

(*b*) $\mathrm{A}^\bullet_{SL_2}\left(\mathbb{P}(\mathrm{Sym}^2(F)); \mathcal{O}(1)\right)$ *is a free* $\mathrm{A}^\bullet(\mathcal{B}SL_2)$*-module generated by* $e(\mathcal{T})$*.*

Proof. (*a*) Let us consider the cohomological motivic Leray spectral sequence as in [ADN18, §4], applied to $p : \mathbb{P}(\mathrm{Sym}^2(F)) \times Z \to Z$ for any scheme $Z \in \mathbf{Sch}_{/S}$. Then we have:

$$E_1^{p,q} = \bigoplus_{x \in Z^{(p)}} \mathrm{A}^{p+q}\left(p^{-1}(x); N_x\right) \Rightarrow \mathrm{A}^{p+q}\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z\right)$$

where p^*N_x is the normal bundle of the inclusion $\iota_x : p^{-1}(x) \hookrightarrow \mathbb{P}_m(\mathrm{Sym}^2(F))$, that is, the pullback of the normal bundle N_x of $x : \mathrm{Spec}(\kappa(x)) \hookrightarrow Z$. Since A is SL-oriented, we actually have a spectral sequence of the form:

$$E_1^{p,q} = \bigoplus_{x \in Z^{(p)}} \mathrm{A}^{q}\left(p^{-1}(x); \det(p^*N_x)\right) \Rightarrow \mathrm{A}^{p+q}\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z\right)$$

The bundle $p : \mathbb{P}(\mathrm{Sym}^2(F)) \times Z \to Z$ has fibers $p^{-1}(x) \simeq \mathbb{P}^2_{\kappa(x)}$ for all $x \in Z$. By lemma 2.2.4, we have:

$$p_x^* : \mathrm{A}^q(x; \det(N_x)) \xrightarrow{\sim} \mathrm{A}^q\left(\mathbb{P}^2_{\kappa(x)}; \det(p^*N_{m,x})\right)$$

But we also have the Gersten spectral sequence for Z:

$$E_1^{p,q} = \bigoplus_{x \in Z^{(p)}} \mathrm{A}^q(\kappa(x); \det(N_x)) \Rightarrow \mathrm{A}^{p+q}(Z)$$

and by the functoriality of the Leray spectral sequences[2] [ADN18, Proposition 4.2.10] we get that the pullback map:

$$p^* : \mathrm{A}^\bullet(Z) \xrightarrow{\sim} \mathrm{A}^\bullet(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z)$$

[2]For Z, the Leray spectral sequence is just the Gersten one.

is an isomorphism, since the induced map p_x^* on the E_1-terms is so. This implies that:

$$\begin{aligned} \mathrm{A}(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z) &= \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{1}_S, (\pi_{\mathbb{P}(\mathrm{Sym}^2(F))\times Z})_*(\pi_{\ \mathbb{P}(\mathrm{Sym}^2(F))\times Z})^*\mathrm{A}) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{1}_Z, \pi_Z^*\mathrm{A}) = \\ &= \mathrm{A}(Z) \end{aligned} \tag{2.4}$$

The pullback map:

$$\tilde{p}^* : \mathrm{A}(\mathcal{B}SL_2) \longrightarrow \mathrm{A}(\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)))$$

by remark 1.4.34 gives us a map of limits of mapping spectra over the Čech nerves:

$$\tilde{p}^* : \lim_{n\in\Delta} \mathrm{A}\left(SL_2^n\right) \longrightarrow \lim_{n\in\Delta} \mathrm{A}\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times SL_2^n\right)$$

But by the computation (2.4) we made above, this is a levelwise equivalence and thus we get an isomorphism:

$$\tilde{p}^* : \mathrm{A}^\bullet\left(\mathcal{B}SL_2\right) \xrightarrow{\sim} \mathrm{A}^\bullet_{SL_2}\left(\mathbb{P}(\mathrm{Sym}^2(F))\right)$$

where we identified $\mathrm{A}(\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)))$ with the SL_2-equivariant cohomology by remark 1.4.9.

(*b*) By a similar argument, for any scheme $Z \in \mathbf{Sch}_{/S}$, we can use again the Leray spectral sequence converging to $\mathrm{A}^{p+q}\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z; \mathcal{O}(1)\right)$ and the isomorphisms:

$$\mathrm{A}^\bullet\left(\mathbb{P}^2_{\kappa(x)}; \mathcal{O}(1)\right) \simeq \mathrm{A}^{\bullet-2}(\kappa(x)) \cdot e(T_{\mathbb{P}^2_{\kappa(x)}})$$

from lemma 2.2.4. So this time the spectral sequence is telling us that $e(\mathcal{T}_Z) \in \mathrm{A}^2\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z\right)$, where $\mathcal{T}_Z$ is the tangent bundle of $\mathbb{P}(\mathrm{Sym}^2(F)) \times Z$ over Z, is a generator for $\mathrm{A}^\bullet\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z; \mathcal{O}(1)\right)$ as a free $\mathrm{A}^\bullet(Z)$-module. Therefore we get an equivalence of mapping spectra:

$$\mathrm{A}(Z) \xrightarrow{\sim} \Sigma^3\mathrm{A}(\mathrm{Th}_{\mathbb{P}(\mathrm{Sym}^2(F))}(\mathcal{O}(1)) \times Z) \tag{2.5}$$

where $\Sigma^3\mathrm{A}(\mathrm{Th}_{\mathbb{P}(\mathrm{Sym}^2(F))}(\mathcal{O}(1)) \times Z) = \Sigma^2\mathrm{A}(\mathbb{P}(\mathrm{Sym}^2(F)) \times Z; \mathcal{O}(1))$ by definition.

By remark 1.4.34, using the Čech nerve of $\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F))$, we can rewrite:

$$\mathrm{A}(\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F))) \simeq \lim_{n\in\Delta} \mathrm{A}(\mathbb{P}(\mathrm{Sym}^2(F)) \times SL_2^n)$$

The isomorphism of (2.5) tells us that:

$$\mathrm{A}(\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F))) \simeq \lim_{n\in\Delta} \Sigma^{-2}\mathrm{A}(SL_2^n) \cdot e(\mathcal{T}_{SL_2^n})$$

But for each n, the bundles $\mathcal{T}_{SL_2^n}$ is the pullback of the relative tangent bundle $\mathcal{T}$ of $\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)))$ over $\mathcal{B}SL_2$, hence $e(\mathcal{T})$ gets pulled back to $e(\mathcal{T}_{SL_2^n})$ along the maps of the Čech nerve. Since we also have that $\mathrm{A}(\mathcal{B}SL_2) \simeq \lim_{n\in\Delta} \mathrm{A}(SL_2^n)$, we get:

$$\mathrm{A}(\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F))) \simeq \Sigma^{-2}\left(\lim_{n\in\Delta} \mathrm{A}(SL_2^n)\right) \cdot e(\mathcal{T}) \simeq \Sigma^{-2}\mathrm{A}(\mathcal{B}SL_2)\cdot e(\mathcal{T})$$

After identifying $\mathrm{A}(\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)))$ with the equivariant cohomology by remark 1.4.9, we get our claim.

□

Now we want to use the localization sequence relative to:

$$\begin{tikzcd}
\mathcal{B}N \arrow[r, hook, "j"] \arrow[dr, "p"'] & \mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)) \arrow[d, "p_2"] & \widetilde{\mathbb{P}(F)} \arrow[l, hook', "\iota"'] \arrow[dl, "\bar{p}"] \\
& \mathcal{B}SL_2 &
\end{tikzcd} \tag{2.6}$$

to compute $\mathrm{A}^\bullet(BN)$. But first we will need the following:

Definition 2.2.9. Let $\mathbb{E} \in \mathrm{SH}(S)$, we say that $\mathbb{E}$ is even if $\mathbb{E}^{a,b}(\Bbbk) = 0$ for all fields $\Bbbk$ and for all integers a, b such that $a - b$ is odd.

Remark 2.2.10. If $\mathrm{A} \in \mathrm{SH}(S)$ is η-invertible, since we follow the same convention $\mathrm{A}^n(\cdot) := \mathrm{A}^{n,0}(\cdot) \simeq \mathrm{A}^{n+i,i}(\cdot)$ as in [Ana15], then A is even if and only if $\mathrm{A}^n(\Bbbk) = 0$ for all fields $\Bbbk$ and all odd n. In particular by [BH21a, p. 8.11], MSL_η is even.

Example 2.2.11. Witt theory KW and Witt cohomology $H\mathcal{W}$ are also examples of even spectra.

Remark 2.2.12. Let us assume that our base scheme $S = \Bbbk$ is a field. It follows from [Ana15, Theorem 10] that, for A an $SL[\eta^{-1}]$-oriented even spectrum, we have $\mathrm{A}^n(BSL_k) = 0$ for any integer k and any odd integer n. Hence using lemma 2.2.2,lemma 2.2.8, for those kind of spectra we have that:

$$\mathrm{A}^n_{SL_2}(\mathbb{P}(F)) \simeq 0$$
$$\mathrm{A}^n_{SL_2}\left(\mathbb{P}(\mathrm{Sym}^2(F))\right) \simeq 0$$
$$\mathrm{A}^n_{SL_2}\left(\mathbb{P}(\mathrm{Sym}^2(F)); \mathcal{O}(1)\right) \simeq 0$$

whenever n is odd (and always working over a field $\Bbbk$).

Remark 2.2.13. In the following proposition we are considering quotient stacks over a base field $\Bbbk$. We should write $\mathcal{B}G_\Bbbk$ to stress this and to avoid confusion with the classifying stack $\mathcal{B}G_S$ over some more general base S, but since it will be clear from the context we will just denote them as $\mathcal{B}G$ (and the same applies to the ind-schemes BG).

Proposition 2.2.14 ([Lev19, Proposition 5.3]). *Let $\Bbbk$ be a field and consider $\mathcal{B}N$, $\widetilde{\mathbb{P}(\mathrm{Sym}^2(F))}$, $\widetilde{\mathbb{P}(F)} \in \mathcal{ASt}^{NL}_{/\Bbbk}$. Let $\mathcal{T}$ be the tangent bundle of $\widetilde{\mathbb{P}(\mathrm{Sym}^2(F))}$ over $\mathcal{B}SL_2$ and let γ_N be the generator of $\mathrm{Pic}(\mathcal{B}N)$. For any $SL[\eta^{-1}]$-oriented ring spectrum $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ and any integers n, k, we get split exact sequences:*

$$0 \to A^{2n}_{SL_2}(\mathbb{P}(\mathrm{Sym}^2(F)); \mathcal{O}(k)) \xrightarrow{j^*} \mathrm{A}^{2n}(\mathcal{B}N; \mathcal{O}(k)) \xrightarrow{\partial} \mathrm{A}^{2n}(\Bbbk) \to 0$$

yielding the following isomorphisms of graded $\mathrm{A}^\bullet(\Bbbk)$-modules:

$$A^\bullet(\mathcal{B}N) \simeq \mathrm{A}^\bullet(\mathcal{B}SL_2) \oplus \mathrm{A}^\bullet(\Bbbk)$$

$$A^\bullet(\mathcal{B}N; \gamma_N) \simeq \mathrm{A}^{\bullet-2}(\mathcal{B}SL_2)e(\mathcal{T}) \oplus \mathrm{A}^\bullet(\Bbbk)$$

Proof. Let us begin noticing via the inclusion ι, the line bundle $\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(F))}(k) \in \mathrm{Pic}^{SL_2}(\mathbb{P}(\mathrm{Sym}^2(F)))$ pulls back to $\iota^*\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(F))}(k) \simeq \mathcal{O}_{\mathbb{P}(F)}(2k)$ over $\mathbb{P}(F)$ and that the normal bundle of $\mathbb{P}(F) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2(F))$ is isomorphic to $\mathcal{O}_{\mathbb{P}(F)}(2)$, so that all the twists will get trivialised over $\mathbb{P}(F)$.
Using the localization sequence associated to eq. (2.6), we get:

$$\ldots \to \mathrm{A}^{2n-1}_{SL_2}(\mathbb{P}(F)) \xrightarrow{\iota_{k*}} A^{2n}_{SL_2}(\mathbb{P}(\mathrm{Sym}^2(F)); \mathcal{O}(k)) \xrightarrow{j_k^*} \mathrm{A}^{2n}(\mathcal{B}N; \mathcal{O}(k)) \xrightarrow{\partial_k^{\mathrm{A}}} \mathrm{A}^{2n}_{SL_2}(\mathbb{P}(F)) \to \ldots \tag{2.7}$$

Now let us for a moment work with the universal $SL[\eta^{-1}]$-oriented ring spectrum (cf. [Ana16b, Theorem 4.7]), that is, MSL_η; we will drop the super-script on the boundary maps for the moment and we will use it again at the end for the general case. By remark 2.2.10, we know MSL_η is even and hence, by remark 2.2.12, our long exact sequence above gives rise to short exact sequences involving only even terms:

$$0 \to \mathrm{MSL}^{2n}_{\eta,SL_2}(\mathbb{P}(\mathrm{Sym}^2(F)); \mathcal{O}(k)) \xrightarrow{j_k^*} \mathrm{MSL}^{2n}_\eta(\mathcal{B}N; \mathcal{O}(k)) \xrightarrow{\partial_k} \mathrm{MSL}^{2n}_{\eta,SL_2}(\mathbb{P}(F)) \to 0$$

We have that $\mathrm{Pic}^{SL_2}(\mathbb{P}(\mathrm{Sym}^2(F))) = \mathbb{Z}$ is generated by $\mathcal{O}(1)$, while $j^*\mathcal{O}(1) \simeq \gamma_N$ is the generator of $\mathrm{Pic}(\mathcal{B}N)$, so, by lemma lemma 2.2.8 we can further reduce to the following two kinds of exact sequences:

$$0 \to \mathrm{MSL}^{2n}_\eta(\mathcal{B}SL_2) \xrightarrow{j^*} \mathrm{MSL}^{2n}_\eta(\mathcal{B}N) \xrightarrow{\partial_0} \mathrm{MSL}^{2n}_\eta(\Bbbk) \to 0 \tag{2.8}$$

$$0 \to \mathrm{MSL}^{2n-2}_\eta(\mathcal{B}SL_2) \cdot e(\mathcal{T}) \xrightarrow{j^*} \mathrm{MSL}^{2n}_\eta(\mathcal{B}N; \gamma_N) \xrightarrow{\partial_1} \mathrm{MSL}^{2n}_\eta(\Bbbk) \to 0 \tag{2.9}$$

Recall we wrote $\mathcal{B}N$ as $\left[(\mathbb{P}^2 \setminus C)/SL_2\right]$, where C was the conic given by the zero locus of the section $Q = T_1^2 - 4T_0T_2$ of $\mathcal{O}_{\mathbb{P}^2}(2)$. Applying our construction 2.1.27 to the section $\lambda_Q : \left[\mathbb{P}^2/SL_2\right] \to \mathcal{O}_{[\mathbb{P}^2/SL_2]}(2)$ induced by Q, we get a well defined element $\langle\lambda_Q\rangle \in \mathrm{MSL}^0_\eta(\mathcal{B}N)$. To get an $\mathrm{A}^\bullet(\Bbbk)$-module splitting for eq. (2.8) and eq. (2.9), it is enough to send $1 \in \mathrm{MSL}^0_\eta(\Bbbk)$ to some elements $q_0^{\mathrm{MSL}_\eta}, q_1^{\mathrm{MSL}_\eta}$ such that their boundary will be 1 again. For eq. (2.9), we are just content to choose any $q_1^{\mathrm{MSL}_\eta}$

such that $\partial_1(q_0^{\mathrm{MSL}_\eta}) = 1$ (this will exists by surjectivity of ∂_1). For eq. (2.8), we can make a clever choice: set $q_0^{\mathrm{MSL}_\eta} := \langle\lambda_Q\rangle$ constructed before. First let us show that $\partial_0(q_0^{\mathrm{MSL}_\eta})$ is invertible. By a Mayer-Vietoris argument we can reduce to the case of a trivial vector bundle, hence to the case where λ_Q is just the standard coordinate section t of $\mathbb{A}^1$. But in this case $\partial(\langle t\rangle) = \eta$ by [Ana19, Lemma 6.4] applied levelwise to the Čech nerves. Via the isomorphism $\mathrm{MSL}_\eta^{-1,-1}(\Bbbk) \simeq \mathrm{MSL}_\eta^0(\Bbbk)$ given by multiplication with η^{-1}, the boundary $\partial(\langle t\rangle)$ is really sent to 1 and hence $\partial_0(q_0^{\mathrm{MSL}_\eta})$ is invertible. Let us prove that this boundary is not only invertible, but it is indeed 1. Let U be some dense open of $\mathbb{P}^2$ where $\mathcal{O}(2)$ gets trivialised, let V be the open dense subset of $\mathbb{P}(F)$ corresponding to $U \cap C$ under the identification $C \simeq \mathbb{P}(F)$. Let $j_V : V \hookrightarrow \mathbb{P}(F) \to \widetilde{\mathbb{P}(F)}$ be the map defined by the composition of the open immersion $V \hookrightarrow \mathbb{P}(F)$ together with the standard quotient map given by the atlas of $\widetilde{\mathbb{P}(F)}$. Since $\mathrm{MSL}_\eta^\bullet(\Bbbk) \simeq \mathrm{MSL}_\eta^\bullet(\widetilde{\mathbb{P}(F)})$ via the pullback along the structure map $\pi_{\widetilde{\mathbb{P}(F)}}$, we can identify j_V^* with π_V^*, where $\pi_V : V \to \mathrm{Spec}(\Bbbk)$ is the structure map of V. Since on V we can trivialise $\mathcal{O}(2)$, we get that $j_V^*(\partial_0(q_0^{\mathrm{MSL}_\eta}) - 1) = 0$. But π_V^* (and hence j_V^*) is injective, so this implies that $\partial_0(q_0^{\mathrm{MSL}_\eta}) = 1$ as claimed.

For a general $SL[\eta^{-1}]$-oriented ring spectrum A, by [Ana16b, Theorem 4.7, Lemma 4.9], we have a map $\varphi_\eta^{SL} : \mathrm{MSL}_\eta \to \mathrm{A}$ of SL-oriented ring spectra. Consider again the long exact sequence eq. (2.7) and consider elements $q_0^{\mathrm{A}} := \varphi_\eta^{SL}(q_0^{\mathrm{MSL}_\eta}), q_1^{\mathrm{A}} := \varphi_\eta^{SL}(q_1^{\mathrm{MSL}_\eta})$. Set $i \in \{0,1\}$. Since $\partial_i^{\mathrm{A}}(q_i^{\mathrm{A}}) = \partial_i^{\mathrm{A}}(\varphi_\eta^{SL}(q_i^{\mathrm{MSL}_\eta})) = \varphi_\eta^{Sl}(\partial_i^{\mathrm{MSL}_\eta}(q_i^{\mathrm{MSL}_\eta}))$ and since φ_η^{SL} is a map of ring spectra, sending $1_{\mathrm{MSL}_\eta} \in \mathrm{MSL}_\eta^0(\Bbbk)$ to $1_{\mathrm{A}} \in \mathrm{A}^0(\Bbbk)$, we have that the boundaries of q_i^{A} are both 1. Hence ∂_i^{A} are split surjective maps of $\mathrm{A}^\bullet(\Bbbk)$-modules as in the MSL_η case and we are done.

□

Let us generalise the previous proposition to the case of a general smooth $\Bbbk$-scheme S. But before doing that we will need to prove the following lemma:

Lemma 2.2.15. *Let $\mathcal{X} \in \mathcal{ASt}_{/B}^{NL}$ be a smooth NL-stack and let $v \in \mathrm{K}_0(\mathcal{X})$. Then there exists a spectrum $\mathbb{E}_{\mathcal{X},v} \in \mathrm{SH}(B)$ such that there is a natural equivalence of functors:*

$$\theta : \mathbb{E}_{\mathcal{X},v}(-) \simeq \mathbb{E}(\mathcal{X} \times_B -, v) : \mathbf{Sm}_{/B} \longrightarrow \mathrm{SH}(B)$$

Proof. Let $Y \in \mathbf{Sm}_{/B}$. Denote by $\pi_\mathcal{X} : \mathcal{X} \to B$ and by $\pi_Y : Y \to B$ the structure maps of $\mathcal{X}$ and Y. We claim that:

$$\mathbb{E}_{\mathcal{X},v} := \underline{\mathrm{Map}}_{\mathrm{SH}(B)}(\mathbb{1}_B, \pi_{\mathcal{X}*}\Sigma^v\pi_\mathcal{X}^*\mathbb{E}) \in \mathrm{SH}(B)$$

is the spectrum we are looking for, where $\underline{\mathrm{Map}}$ denotes the internal mapping space in $\mathrm{SH}(S)$. Consider the following Tor-independent cartesian square:

$$\begin{tikzcd} \mathcal{X}\times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "\pi_Y"] \\ \mathcal{X} \arrow[r, "\pi_{\mathcal{X}}"'] & S \end{tikzcd}$$

Notice that since the square is Tor-independent we have $p_2^*\Omega_X \simeq \mathbb{L}_{\mathcal{X}\times Y/\mathcal{X}}$, moreover p_1 is clearly a smooth representable map. By [Cho21a, Theorem 4.1.1(1)] we have $Ex_!^* : \pi_{\mathcal{X}}^*\pi_{X!} \simeq p_{1!}p_2^*$ and hence we get:

$$\begin{aligned} \pi_{\mathcal{X}\#}\Sigma^{-v}\pi_{\mathcal{X}}^*\pi_{Y\#}\mathbb{1}_Y &\simeq \pi_{\mathcal{X}\#}\Sigma^{-v}\pi_{\mathcal{X}}^*\pi_{Y!}\Sigma^{\Omega_Y}\mathbb{1}_Y \simeq \\ &\overset{Ex_!^*}{\simeq} \pi_{\mathcal{X}\#}p_1\Sigma^{-p_1^*v}{}_!p_2^*\Sigma^{\Omega_Y}\mathbb{1}_Y \simeq \\ &\simeq \pi_{\mathcal{X}\#}p_{1!}\Sigma^{-p_1^*v}\Sigma^{p_2^*\Omega_Y}p_2^*\mathbb{1}_Y \simeq \\ &\simeq \pi_{\mathcal{X}\#}p_{1!}\Sigma^{-p_1^*v}\Sigma^{\mathbb{L}_{\mathcal{X}\times Y/\mathcal{X}}}p_2^*\mathbb{1}_Y \simeq \\ &\simeq \pi_{\mathcal{X}\#}p_{1\#}\Sigma^{-p_1^*v}p_2^*\mathbb{1}_Y \simeq \\ &\simeq \pi_{\mathcal{X}\times Y\#}\Sigma^{-p_1^*v}\mathbb{1}_{\mathcal{X}\times Y} \end{aligned} \tag{2.10}$$

where we used purity (for representable maps) twice, once for π_Y and once for p_1. But this means that:

$$\begin{aligned} \mathbb{E}_{\mathcal{X},v}(Y) &= \mathrm{Map}_{\mathrm{SH}(B)}\left(\mathbb{1}_B, \pi_{Y*}\pi_Y^*\mathbb{E}_{\mathcal{X},v}\right) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(B)}\left(\pi_{Y\#}\mathbb{1}_Y, \mathbb{E}_{\mathcal{X},v}\right) = \\ &\simeq \mathrm{Map}_{\mathrm{SH}(B)}\Big(\pi_{Y\#}\mathbb{1}_Y, \underline{\mathrm{Map}}_{\mathrm{SH}(B)}(\mathbb{1}_B, \pi_{\mathcal{X}*}\Sigma^v\pi_{\mathcal{X}}^*\mathbb{E})\Big) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(B)}\left(\pi_{Y\#}\mathbb{1}_Y, \pi_{\mathcal{X}*}\Sigma^v\pi_{\mathcal{X}}^*\mathbb{E}\right) \simeq \\ &\simeq \mathrm{Map}_{\mathrm{SH}(B)}\left(\pi_{\mathcal{X}\#}\Sigma^{-v}\pi_{\mathcal{X}}^*\pi_{Y\#}\mathbb{1}_Y, \mathbb{E}\right) \simeq \\ &\overset{(2.10)}{\simeq} \mathrm{Map}_{\mathrm{SH}(B)}\Big(\pi_{\mathcal{X}\times Y\#}\Sigma^{-p_1^*v}\mathbb{1}_{\mathcal{X}\times Y}, \mathbb{E}\Big) \simeq \\ &\simeq \mathbb{E}(\mathcal{X}\times Y, v) \end{aligned}$$

This identification $\mathbb{E}_{\mathcal{X},v}(Y) \simeq \mathbb{E}(\mathcal{X}\times_B Y, v)$ is moreover functorial in the Y, where $f : Y_1 \to Y_2$ is sent to the pullback map:

$$f^* : \mathbb{E}_{\mathcal{X},v}(Y_2) \simeq \mathbb{E}(\mathcal{X}\times_B Y_2, v) \longrightarrow \mathbb{E}_{\mathcal{X},v}(Y_1) \simeq \mathbb{E}(\mathcal{X}\times_B Y_1, v)$$

Hence we have a natural equivalence of functors:

$$\theta : \mathbb{E}_{\mathcal{X},v}(-) \simeq \mathbb{E}(\mathcal{X}\times_B -, v) : \mathbf{Sm}_{/B} \longrightarrow \mathrm{SH}(B)$$

□

Proposition 2.2.16. *Let $S \in \mathbf{Sm}_{/\Bbbk}$. For any $SL[\eta^{-1}]$-oriented ring spectrum $\mathrm{A} \in \mathrm{SH}(S)$ and any integers n, k, we get split exact sequences:*

$$0 \to A^{2n}_{SL_2}(\mathbb{P}(\mathrm{Sym}^2(F)); \mathcal{O}(k)) \xrightarrow{j^*} \mathrm{A}^{2n}(\mathcal{B}N; \mathcal{O}(k)) \xrightarrow{\partial} \mathrm{A}^{2n}(S) \to 0$$

yielding the following isomorphisms of graded $\mathrm{A}^\bullet(S)$-modules:

$$A^\bullet(\mathcal{B}N_S) \simeq \mathrm{A}^\bullet(\mathcal{B}SL_{2,S}) \oplus \mathrm{A}^\bullet(S)$$

$$A^\bullet(\mathcal{B}N_S; \gamma_N) \simeq \mathrm{A}^{\bullet-2}(\mathcal{B}SL_{2,S})e(\mathcal{T}) \oplus \mathrm{A}^\bullet(S)$$

where $\mathcal{T}$ is the tangent bundle of $\mathbb{P}(\widetilde{\mathrm{Sym}^2}(F))$ over $\mathcal{B}SL_{2,S}$ and γ_N is the generator of $\mathrm{Pic}(\mathcal{B}N)$.

Proof. Since S is smooth, using corollary 1.5.3, from (2.6) we get a localization sequence for $\mathcal{B}N_S, \mathbb{P}_S(\widetilde{\mathrm{Sym}^2}(F)) := \mathbb{P}(\widetilde{\mathrm{Sym}^2}(F)) \times S$ and $\widetilde{\mathbb{P}_S(F)} := \widetilde{\mathbb{P}(F)} \times S$. For any integer k we get:

$$\ldots \to A^{2n}_{SL_2}(\mathbb{P}_S(\mathrm{Sym}^2(F)); \mathcal{O}(k)) \xrightarrow{j_k^*} \mathrm{A}^{2n}(\mathcal{B}N_S; \mathcal{O}(k)) \xrightarrow{\partial_k^{\mathrm{A}}} \mathrm{A}^{2n}_{SL_2}(\mathbb{P}_S(F)) \to \ldots$$

Using the SL-orientation, we can just consider $k = 0, 1$. Recall that $\mathcal{B}N = \left[\mathbb{P}^2 \setminus C / SL_2\right]$, where C is the conic given by the zero locus of the section $Q = T_1^2 - 4T_0T_2$ of $\mathcal{O}_{\mathbb{P}^2}(2)$. Applying our construction 2.1.27 to the section $\lambda_Q : \left[\mathbb{P}^2/SL_2\right] \to \mathcal{O}_{[\mathbb{P}^2/SL_2]}(2)$ induced by Q, we get a well defined element $q_0 \in \mathrm{A}^0(\mathcal{B}N_S)$ (cf. proof proposition 2.2.14). The structure map $\pi_S : S \to \Bbbk$ induces a map $g : \mathcal{B}N_S \to \mathcal{B}N_\Bbbk$ and hence a pullback map:

$$g^* : \mathrm{A}^\bullet(\mathcal{B}N_\Bbbk; \gamma_N) \longrightarrow \mathrm{A}^\bullet(\mathcal{B}N_S; \gamma_N)$$

Consider $q_{1,\Bbbk} \in \mathrm{A}^0(\mathcal{B}N_\Bbbk; \gamma_N)$ constructed in the proof of proposition 2.2.14, and set:

$$q_1 := g^* q_{1,\Bbbk} \mathrm{A}^0(\mathcal{B}N_S; \gamma_N)$$

We then have two maps:

$$(j^*, \sigma_0) : \quad \mathrm{A}^\bullet(\mathcal{B}SL_{2,S}) \oplus \mathrm{A}^\bullet(S) \quad \longrightarrow \quad \mathrm{A}^\bullet(\mathcal{B}N_S) \tag{2.11}$$

$$(j^*(-)e(\mathcal{T}), \sigma_1) : \quad \mathrm{A}^{\bullet-2}(\mathcal{B}SL_{2,S}) \oplus \mathrm{A}^\bullet(S) \quad \longrightarrow \quad \mathrm{A}^\bullet(\mathcal{B}N_S; \gamma_N) \tag{2.12}$$

where $\sigma_0 : \mathrm{A}(S) \to \mathrm{A}(\mathcal{B}N)$ sends $1 \mapsto q_0$ and $\sigma_1 : \mathrm{A}(S) \to \mathrm{A}(\mathcal{B}N; \gamma_N)$ sends $1 \mapsto q_1$.

We want now to apply the homotopy Leray (or Gersten) spectral sequence of [ADN18] to both sides of (2.11) and (2.12), but the cautious reader might object that we are dealing with algebraic stacks and not schemes any more. But by lemma 2.2.15, we can then apply the results in [ADN18] to the motivic spectra $\mathrm{A}_{\mathcal{B}N}$ and $\mathrm{A}_{\mathcal{B}SL_2}$ representing $\mathrm{A}(\mathcal{B}N_S)$ and $\mathrm{A}(\mathcal{B}SL_{2,S})$ for $S \in \mathbf{Sm}_{/\Bbbk}$. Namely, we have that:

$$\mathrm{A}_{\mathcal{B}SL_2}(S) = \mathrm{A}(BSL_{2,S})$$

$$\mathrm{A}_{\mathcal{B}N}(S) = \mathrm{A}(\mathcal{B}N_S)$$

$$\mathrm{A}_{\mathcal{B}N,\gamma_N}(S) = \mathrm{A}(\mathcal{B}N_S; \gamma_N)$$

and we can apply the results in [ADN18] to these spectra. By [ADN18, Theorem 4.2.9] (with $f = Id$ in *loc. cit.*), we have spectral sequences:

$$E_1^{p,q} := \bigoplus_{s \in S^{(p)}} \mathrm{A}^q_{\mathcal{B}N}(\kappa(s)) \Rightarrow \mathrm{A}^{p+q}_{\mathcal{B}N}(S) = \mathrm{A}(\mathcal{B}N_S)$$

$$'E_1^{p,q} := \bigoplus_{s \in S^{(p)}} \left(\mathrm{A}^q_{\mathcal{B}SL_2}(\kappa(s)) \oplus \mathrm{A}^q(\kappa(s))\right) \Rightarrow \mathrm{A}^{p+q}(\mathcal{B}SL_{2,S}) \oplus \mathrm{A}^{p+q}(S)$$

The map $(j^*, \sigma_0) : \mathrm{A}^\bullet(\mathcal{B}SL_{2,S}) \oplus \mathrm{A}^\bullet(S) \longrightarrow \mathrm{A}^\bullet(\mathcal{B}N_S)$ induces a map between spectral sequences E_1 and $'E_1$, but the latter is an isomorphism by proposition 2.2.14 and hence (j^*, σ_0) is an isomorphism too. By a similar argument, applying again proposition 2.2.14 at the level of spectral sequences, we get that:

$$(j^*(-)e(\mathcal{T}), \sigma_1) : \mathrm{A}^{\bullet-2}(\mathcal{B}SL_{2,S}) \oplus \mathrm{A}^\bullet(S) \longrightarrow \mathrm{A}^\bullet(\mathcal{B}N_S; \gamma_N)$$

is an isomorphism too and we are done.

□

Remark 2.2.17. 1. From [Ana15, Theorem 10] (plus corollary 1.4.32 and remark 1.4.33), we have that $\mathrm{A}^\bullet(\mathcal{B}SL_2) \simeq \mathrm{A}^\bullet(S)[\![e]\!]$, where $e = e(E_2)$ is the Euler class of the bundle associated to the tautological rank two bundle on $\mathcal{B}SL_2$. Denoting by $p : \mathcal{B}N \to \mathcal{B}SL_2$, from the computations we just made, we then have that $\mathrm{A}^\bullet(\mathcal{B}N) \simeq \mathrm{A}^\bullet(S)[\![p^*e]\!] \oplus q_0^{\mathrm{A}} \cdot \mathrm{A}^\bullet(S)$. Then if $\xi : S \to \mathcal{B}N$ is the base-point of $\mathcal{B}N$, the projection of $\mathrm{A}^\bullet(\mathcal{B}N; \cdot)$ onto the second factor $\mathrm{A}^\bullet(S)$ in our previous theorem, is just ξ^*. Hence computing:

$$\partial_1((1 + q_0^{\mathrm{A}}) \cdot e(\mathcal{T})) = \partial_0(q_0^{\mathrm{A}}) \cdot \xi^* e(\mathcal{T}) = 1 \cdot e(\xi^* \mathcal{T}) = 0$$

since $\mathcal{T}$ gets trivialised once pulled back to the base-point. Since $\partial_1((1 + q_0^{\mathrm{A}}) \cdot e(\mathcal{T})) = 0$, $(1 + q_0^{\mathrm{A}}) \cdot e(\mathcal{T})$ must live in the factor $\mathrm{A}^{-2}(\mathcal{B}SL_2) \cdot e(\mathcal{T})$ of $\mathrm{A}^2(\mathcal{B}N; \gamma_N)$. Thus we have that $(1 + q_0^{\mathrm{A}}) \cdot e(\mathcal{T}) = \lambda_1^{\mathrm{A}} \cdot e(\mathcal{T})$ for some $\lambda_1^A \in \mathrm{A}^0(\mathcal{B}SL_2)$. This is particular true for MSL_η, and hence, by universality of MSL as SL-oriented theory, we have $\lambda_1^A = \varphi_\eta^{SL}(\lambda_1^{\mathrm{MSL}_\eta})$.

2. For $A = H\mathcal{W}$, our $q_0^{H\mathcal{W}}$ is exactly $\langle \bar{q} \rangle$ constructed in [Lev19, §5] (under the identification of corollary 1.4.32).

2.3 The Multiplicative Structure of $\mathrm{KW}^{\bullet}(BN)$

The results in the previous subsection gave us the additive description of $\mathrm{A}^{\bullet}(\mathcal{B}N_S)$ for an $SL[\eta^{-1}]$-oriented ring spectrum A and S a smooth $\Bbbk$-scheme. From now on, up to the end of this chapter, if not otherwise specified, we will always work over a smooth $\Bbbk$-scheme S, so we will omit the subscript from the notation.

Let us now proceed with the computation of the multiplicative structure of $\mathrm{KW}^{\bullet}(\mathcal{B}N)$ and the $\mathrm{KW}^{\bullet}(\mathcal{B}SL_2)$-module structure of $\mathrm{KW}^{\bullet}(\mathcal{B}N;\gamma_N)$. Let $q_0 := q_0^{\mathrm{KW}}$, $q_1 := q_1^{\mathrm{KW}}$ be the elements constructed in the proof of proposition 2.2.16. We have that $q_0^2 = 1$ (cf. [Ana19, p. 6.2]).

Recall from remark 2.2.17 (for A = KW), we have that $\mathrm{KW}^{\bullet}(\mathcal{B}N) \simeq \mathrm{KW}^{\bullet}(S)[\![p^*e]\!] \oplus q_0 \cdot \mathrm{KW}^{\bullet}(S)$.

The only relations left to compute are:

$$q_0{\cdot}p^*e \in \mathrm{KW}^2(BN) \qquad q_0{\cdot}j^*e(\mathcal{T}) \in \mathrm{KW}^2(\mathcal{B}N;\gamma_N) \qquad q_0\cdot q_1 \in q_1\cdot \mathrm{KW}^0(S)$$

To do this we can consider the inclusion $\mathbb{G}_m \hookrightarrow N$ and the corresponding map $\mathcal{B}\mathbb{G}_m \longrightarrow \mathcal{B}N$, where $\mathcal{B}\mathbb{G}_m \simeq [(SL_2/\mathbb{G}_m)/SL_2]$. The description of $\mathcal{B}N$ as the quotient $\left[\left(\mathbb{P}\left(\mathrm{Sym}^2(F)\right)\setminus \mathbb{P}(F)\right)/SL_2\right]$, gives us a section j^*Q of $j^*\mathcal{O}(2)$ coming from the section $Q := T_1^2 - 4T_0T_2$ of $\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2 F)}(2)$, via the restriction along $j : \mathcal{B}N \hookrightarrow \widetilde{\mathbb{P}(\mathrm{Sym}^2)}(F)$.

Given an algebraic stack X, a line bundle $\mathbb{V}(\mathcal{L})$ on X and a section s of $\mathcal{L}^{\otimes 2}$, we can construct another algebraic stack $X(\sqrt{s})$ as the fiber product:

$$\begin{tikzcd} X(\sqrt{s}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & L \arrow[d, "sq"] \\ X \arrow[r, hook, "\sigma"'] & L^{\otimes 2} \end{tikzcd}$$

where $sq : L \longrightarrow L^{\otimes 2}$ is the squaring map and $\sigma : X \longrightarrow L^{\otimes 2}$ is the map induced by the section s.

In particular, on $\mathcal{B}N$ we have a section j^*Q of $j^*\mathcal{O}(2) = j^*\mathcal{O}(1)^{\otimes 2}$ and we can consider the stack $\mathcal{B}N(\sqrt{j^*Q})$. Similarly to what was showed in [Lev19, End of Sect. §2 and §5], we can identify $\mathcal{B}\mathbb{G}_m \simeq \mathcal{B}N(\sqrt{Q}) \to \mathcal{B}N$ as a double cover.

Lemma 2.3.1. *Consider $S = \mathrm{Spec}(\Bbbk)$. Then we have $(1+q_0)\cdot p^*e = 0$ in $\mathrm{KW}^2(\mathcal{B}N)$.*

Proof. We will freely use the notation employed in the proof of proposition 2.2.16 and we will closely follow the proof in [Lev19, Lemma 5.4]. Let us recall here the underlying geometry of our objects. We have a rank two tautological bundle $E_2 =$

$[F/SL_2] \longrightarrow \mathcal{B}SL_2$ and its pullback $\hat{E}_2 := p^*E_2 \longrightarrow \mathcal{B}N$. The class $e \in \mathrm{KW}^2(\mathcal{B}SL_2)$ was the Euler class of E_2, under the appropriate identification (cf. corollary 1.4.32), and so we have $p^*e = e(\hat{E}_2)$. We have the double cover $\varphi : \mathcal{B}\mathbb{G}_m \longrightarrow \mathcal{B}N$. The pullback $\varphi^*\hat{E}_2$ splits as $\varphi^*\hat{E}_2 \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$ corresponding to the decomposition of F into the eigenspaces relative to t and t^{-1} under the $\mathbb{G}_m$-action. The $\mathbb{Z}/2\mathbb{Z} \simeq N/\mathbb{G}_m$-action sends $t \mapsto t^{-1}$ and thus swaps the two factors $\mathcal{O}(1)$ and $\mathcal{O}(-1)$. Considering the cone $(\mathcal{O}(1) \times 0) \cup (0 \times \mathcal{O}(-1)) \subseteq \varphi^*\hat{E}_2$, we get by descent under the $\mathbb{Z}/2\mathbb{Z}$-action the corresponding cone $\mathfrak{C} \subseteq \hat{E}_2$. We have a map $\nu : \mathfrak{C}^N := \mathcal{O}(1) \longrightarrow \mathfrak{C}$, induced by the normalization of the atlas of $\mathfrak{C}$[3]. So we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{C}^N := \mathcal{O}(1) \arrow[r, "\iota_{\mathfrak{C}^N}"] \arrow[d, "\nu"'] & \varphi^*\hat{E}_2 \arrow[r, "\rho"] \arrow[d, "\hat{\varphi}"'] & \mathcal{B}\mathbb{G}_m \arrow[d, "\varphi"] \\
\mathfrak{C} \arrow[r, "\iota_{\mathfrak{C}}"'] & \hat{E}_2 \arrow[r, "r"'] & \mathcal{B}N
\end{tikzcd}$$

Considering the localization sequences associated to:

$$0_{\mathfrak{C}} = \mathcal{B}N \hookrightarrow \mathfrak{C} \hookleftarrow \mathfrak{C} \setminus 0 =: \mathfrak{C}^\circ \qquad 0_{\hat{E}_2} = \mathcal{B}N \xhookrightarrow{s_0} \hat{E}_2 \hookleftarrow \hat{E}_2 \setminus 0 =: \hat{E}_2^\circ$$

$$0_{\mathfrak{C}^N} = \mathcal{B}\mathbb{G}_m \hookrightarrow \mathfrak{C}^N \hookleftarrow \mathfrak{C}^N \setminus 0 =: (\mathfrak{C}^N)^\circ \simeq \mathfrak{C}^\circ \qquad 0_{\varphi^*\hat{E}_2} = \mathcal{B}\mathbb{G}_m \xhookrightarrow{\sigma_0} \varphi^*\hat{E}_2 \hookleftarrow \varphi^*\hat{E}_2^\circ$$

we get the following diagram:

$$\begin{tikzcd}
\mathrm{KW}^{0,0}(\mathfrak{C}^\circ; \mathcal{O}(-1)) \arrow[rr, "\partial_{\mathfrak{C}^N}"] \arrow[dr, "\iota_{\mathfrak{C}^N/\varphi^*E^*}"] \arrow[dd, "\hat{\varphi}_*"'] & & \mathrm{KW}^{-1,-1}(\mathcal{B}\mathbb{G}_m) \arrow[dr, "Id"] \arrow[dd, "\varphi_*" near end] & \\
& \mathrm{KW}^{2,1}\left(\varphi^*\hat{E}_2^\circ\right) \arrow[rr, "\partial_{\varphi^*\hat{E}_2}" near start] \arrow[dd] & & \mathrm{KW}^{-1,-1}(\mathcal{B}\mathbb{G}_m) \arrow[dd] \\
\mathrm{KW}^{0,0}(\mathfrak{C}^\circ; \mathcal{O}(-1)) \arrow[rr, "\partial_{\mathfrak{C}}" near end] \arrow[dr, "\iota_{\mathfrak{C}/\hat{E}_2}"] & & \mathrm{KW}^{-1,-1}(\mathcal{B}N) \arrow[dr, "Id"] & \\
& \mathrm{KW}^{2,1}\left(\hat{E}_2^\circ\right) \arrow[rr, "\partial_{\hat{E}_2}"'] & & \mathrm{KW}^{-1,-1}(\mathcal{B}N)
\end{tikzcd} \tag{2.13}$$

[3]All the geometry we have done so far is the geometry of SL_2-quotient stacks, we can just work with their standard atlas and then by descent pass to the SL_2-quotients.

where we used the fact that the normal bundle of $\mathfrak{C}^N$ inside $\varphi^*\hat{E}_2$ is $N_{\mathfrak{C}^N/\varphi^*\hat{E}_2} \simeq \rho^*\mathcal{O}_{\mathcal{B}\mathbb{G}_m}(-1)\big|_{\mathfrak{C}^N}$.

Let $\pi : \mathfrak{C}^N \simeq \mathcal{O}_{\mathcal{B}\mathbb{G}_m}(1) \to \mathcal{B}\mathbb{G}_m$ be the line bundle map, then we can consider on $\mathcal{O}_{\mathfrak{C}^N}(1) \simeq \pi^*\mathcal{O}(1)$ the tautological section $can : \mathfrak{C}^N \simeq \mathcal{O}_{\mathcal{B}\mathbb{G}_m}(1) \longrightarrow \mathcal{O}_{\mathfrak{C}^N}(1)$[4]. By definition 2.1.28, we get a well defined element $\langle t_{can}\rangle \in \mathrm{KW}^{0,0}(\mathfrak{C}^\circ; \mathcal{O}(1))$.
By a Mayer-Vietoris argument, to compute $\partial_{\mathfrak{C}^N}(\langle t_{can}\rangle)$ we can restrict to trivialising open subsets, and we have that $\partial_{\mathfrak{C}^N}(\langle t_{can}\rangle) = \eta \in \mathrm{KW}^{-1,-1}(\mathcal{O}_{\mathcal{B}\mathbb{G}_m}(1))$. But by homotopy invariance we get $\mathrm{KW}^{-1,-1}(\mathcal{O}_{\mathcal{B}\mathbb{G}_m}(1)) \simeq \mathrm{KW}^{-1,-1}(\mathcal{B}\mathbb{G}_m)$. Using the isomorphism induced by η, we can identify $\mathrm{KW}^{-1,-1}(\mathcal{B}\mathbb{G}_m) \simeq \mathrm{KW}^0(\mathcal{B}\mathbb{G}_m)$: under this isomorphism η is sent to 1, so we have $\partial_{\mathfrak{C}^N}(\langle t_{can}\rangle) = 1 \in \mathrm{KW}^0(\mathcal{B}\mathbb{G}_m)$.

If we push forward through φ_* the boundary of $\langle t_{can}\rangle$, we get $\varphi_*\langle 1\rangle = \langle 2\rangle(1+q_0)$ by lemma 2.3.3 (see below).
Using the commutativity of the diagram eq. (2.13), we see that $\langle 2\rangle(1+q_0) = \varphi_*\left(\partial_{\mathfrak{C}^N}\left(\langle t_{can}\rangle\right)\right) \simeq \partial_{\hat{E}_2}\left(\hat{\varphi}_*(\iota_{\mathfrak{C}^N})_*\langle t_{can}\rangle\right)$ is a boundary in $\mathrm{KW}^0\left(BN\right)$, so it is sent to zero via $(s_0)_*$, where $s_0 : BN \to \hat{E}_2$ is the zero section of the bundle. But if we post-compose with the pullback map s_0^*, we also get $(s_0)^*(s_0)_*\varphi_*\langle 1\rangle = 0$, and spelling out the element on the left we have $\langle 2\rangle(1+q_0)\cdot e(\hat{E}_2) = 0$ as we wanted.

□

We need to complete the previous proof, but before doing that we need another technical lemma. Recall that by [Ana15, Corollary 4], for $m = 2k+1$ and for any $SL[\eta^{-1}]$-oriented ring spectrum $\mathrm{A} \in \mathrm{SH}(\Bbbk)$, we have:

$$\mathrm{A}^\bullet(B_mSL_2) = \mathrm{A}^\bullet(\Bbbk)[e]/(e^{m-1})$$

where e is the Euler class of the tautological bundle.

Lemma 2.3.2. *Let* $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ *be a* $SL[\eta^{-1}]$*-oriented ring spectrum. Denote by* $\nu_m : B_mN \to \mathcal{B}N$ *the natural map to the quotient stack and by* $p : \mathcal{B}N \to \mathcal{B}SL_2$ *the structure map over* $\mathcal{B}SL_2$*. Let* e *be the Euler class of the tautological bundle over* $\mathcal{B}SL_2$*. Then, for each odd integer* m*,the pullback map:*

$$\nu_m^* : \mathrm{A}^\bullet(\mathcal{B}N) \simeq \mathrm{A}^\bullet(\Bbbk)[\![p^*e]\!] \oplus q_0 \cdot \mathrm{A}^\bullet(\Bbbk) \longrightarrow \mathrm{A}^\bullet(B_mN)$$

is surjective with kernel $\mathrm{Ker}(\nu_m^*) = (p^*e^{m-1})\cdot \mathrm{A}^\bullet(\Bbbk)[\![p^*e]\!]$.

Proof. Denote for short:

$$\mathbb{P}_k(F) := \mathbb{P}(F) \times^{SL_2} E_mSL_2$$

$$\mathbb{P}_m(\mathrm{Sym}^2(F)) := \mathbb{P}(\mathrm{Sym}^2(F)) \times^{SL_2} E_mSL_2$$

[4]For a line bundle $p : L \to X$, we get a tautological section on $p^*L \to L$ from the fact that for every $y = p(l)$ we have $(p^*L)_l \simeq L_y \ni l$, so the section in this case is just $l \mapsto (l, l)$.

and let $j_m : B_m N \hookrightarrow \mathbb{P}_m(\mathrm{Sym}^2(F))$ and $\iota_m : \mathbb{P}_m(F) \hookrightarrow \mathbb{P}_m(\mathrm{Sym}^2(F))$ be the associated open and closed immersions. Let:

$$\pi_m : \mathbb{P}_m(\mathrm{Sym}^2(F)) \longrightarrow B_m SL_2$$

be the structure map over $B_m SL_2$.

We will start proving the following:

Claim 1. *The map induced by proper pushforward along ι_m:*

$$(\iota_m)_* : \mathrm{A}^\bullet(\mathbb{P}_m(F)) \longrightarrow \mathrm{A}^{\bullet+1}(\mathbb{P}_m(\mathrm{Sym}^2(F)))$$

is the zero-map.

Proof of the Claim. Consider T_{π_m} the relative tangent bundle of $\pi_m : \mathbb{P}_m(\mathrm{Sym}^2(F)) \to B_m SL_2$. The fibers of π_m are $\pi_m^{-1}(x) \simeq \mathbb{P}^2_{\kappa(x)}$, hence comparing the Leray spectral sequences it is easy to see that:

$$\pi_m^* : \mathrm{A}^\bullet(B_m SL_2) \xrightarrow{\sim} \mathrm{A}^\bullet\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times^{SL_2} E_m SL_2\right)$$

is an equivalence. Indeed, the map between spectral sequences, induced by π_m^* above, is an isomorphism by lemma 2.2.3. We also have an isomorphism given by:

$$(- \cup e(T_{\pi_m})) \circ \pi_m^* : \mathrm{KW}^{\bullet-2}(B_m SL_2) \xrightarrow{\sim} \mathrm{A}^\bullet\left(\mathbb{P}(\mathrm{Sym}^2(F)) \times^{SL_2} E_m SL_2\right)$$

This follows again from the comparison of Leray spectral sequences, using the fact that on the fibers the map induced by $(- \cup e(T_{\pi_m})) \circ \pi_m^*$ is an isomorphism (cf. end of proof of lemma 2.2.4). In particular we get that the map:

$$(- \cup e(T_{\pi_m})) : \mathrm{A}^\bullet(\mathbb{P}_m(\mathrm{Sym}^2(F))) \longrightarrow \mathrm{A}^{\bullet+2}(\mathbb{P}_m(\mathrm{Sym}^2(F)); \mathcal{O}(1))$$

is an isomorphism. But $i_m^* T_{\pi_m}$ fits in a short exact sequence:

$$0 \to T_{\pi_m \circ i_m} \to i_m^* T_{\pi_m} \to N_{i_m} \to 0$$

where N_{i_m} is the normal bundle of the closed immersion. This implies that $i_m^* e(T_{\pi_m}) = 0$, indeed $T_{\pi_m \circ i_m}$ and N_{i_m} are line bundles and Euler classes of line bundles are trivial for $SL[\eta^{-1}]$-oriented spectra (see [Lev19, Lemma 4.3]). Then by the projection formula, for any $x \in \mathrm{KW}^{\bullet-1}$ we have:

$$(i_m)_*(x) \cup e(T_{\pi_m}) = (i_m)_*(x \cup i_m^* e(T_{\pi_m})) = 0$$

But $(- \cup e(T_{\pi_m}))$ is an isomorphism, in particular it is injective and this implies that we must have $i_m^*(x) = 0$ as claimed.

(Claim 1)
■

From claim 1, it follows that for each integer k the localization sequence:

$$\dots \stackrel{(\iota_m)_*}{\to} \mathrm{A}^k(\mathbb{P}_m(\mathrm{Sym}^2(F))) \xrightarrow{j_m^*} \mathrm{A}^k(B_mN) \xrightarrow{\partial_m} \mathrm{A}^k(\mathbb{P}_m(F)) \stackrel{(\iota_m)_*}{\to} \dots \tag{2.14}$$

splits into short exact sequences:

$$0 \to \mathrm{A}^k(\mathbb{P}_m(\mathrm{Sym}^2(F))) \xrightarrow{j_m^*} \mathrm{A}^k(B_mN) \xrightarrow{\partial_m} \mathrm{A}^k(\mathbb{P}_m(F)) \to 0$$

Let x_0 be the base point of B_mSL_2 and consider $y_0 \in \pi_m^{-1}(x_0)$ the $\Bbbk$-point y_0 : $\mathrm{Spec}(\Bbbk) \to \mathbb{P}_m(\mathrm{Sym}^2(F))$ given by $[1:0]$ in the fiber of x_0. Then the map:

$$\pi^*_{\mathbb{P}_m(\mathrm{Sym}^2(F))} : \mathrm{A}^\bullet(\Bbbk) \longrightarrow \mathrm{A}^\bullet(\mathbb{P}_m(\mathrm{Sym}^2(F))) \tag{2.15}$$

is injective, and splits via y_0^*. Let $q_0 := q_0^{\mathrm{A}} \in \mathrm{A}^0(\mathcal{B}N)$ be the element we used to get the splitting of proposition 2.2.14 and denote by $q_{0,m}$ its pullback to B_mN. Recall that the map $p_m : B_mN \to B_mSL_2$ is given by the composition of j_m and π_m (and similarly for $p : \mathcal{B}N \to \mathcal{B}SL_2$). Then by (2.14) and (2.15), we deduce that the map:

$$\psi_m := (p_m^*, q_{0,m} \cdot \pi^*_{B_mN}) : \mathrm{A}^\bullet(B_mSL_2) \oplus \mathrm{A}^\bullet(\Bbbk) \longrightarrow \mathrm{A}^\bullet(B_mN)$$

is injective. Therefore, denoting by $\sigma_m : B_mSL_2 \to \mathcal{B}SL_2$ the natural map to the quotient stack, we get a commutative diagram:

$$\begin{tikzcd}
\mathrm{A}^\bullet(\mathcal{B}SL_2) \oplus \mathrm{A}^\bullet(\Bbbk) \arrow[r, "\sim"] \arrow[d, "{(\sigma_m^*, Id)}"'] & \mathrm{A}^\bullet(\mathcal{B}N) \arrow[d, "\nu_m^*"] \\
\mathrm{A}^\bullet(B_mSL_2) \oplus \mathrm{A}^\bullet(\Bbbk) \arrow[r, hook, "\psi_m"'] & \mathrm{A}^\bullet(B_mN)
\end{tikzcd} \tag{2.16}$$

But this implies that $\mathrm{Ker}(\nu_m^*) \simeq \mathrm{Ker}(\sigma_m^*)$. For m odd, by [Ana15, Corollary 4, Theorem 10], we have:

$$\mathrm{Ker}(\sigma_m^*) \simeq (e^{m-1}) \cdot \mathrm{A}^\bullet(\Bbbk)[\![e]\!]$$

and we are done.

$\square$

As promised, let us now complete the proof of lemma 2.3.1 where we used the following computation:

Lemma 2.3.3. *Let $S = \mathrm{Spec}(\Bbbk)$ and let $\varphi : \mathcal{B}\mathbb{G}_m \longrightarrow \mathcal{B}N$ be the double cover we already introduced. Then we have:*

$$\varphi_*\langle 1\rangle = \langle 2\rangle(1 + q_0) \in \mathrm{KW}^0(\mathcal{B}N)$$

Proof. As already mentioned, we have $\mathcal{B}\mathbb{G}_m \simeq \mathcal{B}N(\sqrt{Q})$ and we can use Grothendieck-Serre Duality to compute $\varphi_* 1$ for KW (cf. [LR20, §8D]). We will reduce the computation on $\mathcal{B}N$ to a computation on the finite level approximations given by $B_m N$. From proposition 2.2.14, we know that:

$$\mathrm{KW}^0(\mathcal{B}N) \simeq (\mathrm{KW}^\bullet(\Bbbk)[\![p^*e]\!])^0 \oplus \mathrm{KW}^0(\Bbbk) \cdot q_0$$

Hence:

$$\varphi_*\langle 1\rangle = \sum_{i=0}^{\infty} a_i e^{2i} + b \cdot q_0$$

where $a_i \in \mathrm{KW}^{-4i}(\Bbbk)$ and $b \in \mathrm{KW}^0(\Bbbk)$. Once we determine the coefficients a_i's and b we are done. We have a natural map:

$$\alpha : \mathrm{KW}^0(\mathcal{B}N) \longrightarrow H\mathcal{W}^0(\mathcal{B}N)$$

from the 0^{th} Witt theory to the 0^{th} Witt sheaf cohomology, induced by sheafification. Under this map, we have:

$$\alpha(\varphi_*\langle 1\rangle) = a_0 + b \cdot q_0^{H\mathcal{W}}$$

But we also have:

$$\alpha(\varphi_*\langle 1\rangle) = \varphi_*^{H\mathcal{W}}\langle 1\rangle$$

where $\varphi_*^{H\mathcal{W}}$ is the pushforward map on Witt sheaf cohomology. By [Lev19, Proof of Proposition 5.3], we know that $\varphi_*^{H\mathcal{W}}\langle 1\rangle = \langle 2\rangle(1 + q_0^{H\mathcal{W}})$ and hence:

$$\alpha(\varphi_*\langle 1\rangle) = a_0 + b \cdot q_0^{H\mathcal{W}} = \langle 2\rangle(1 + q_0^{H\mathcal{W}})$$

implying that:

$$a_0 = b = \langle 2\rangle \tag{2.17}$$

It remains to determine all the remaining a_i's for $i > 0$. By lemma 2.3.2, these coefficients are determined via the pullback map to $B_m N$ for m going to infinity. By construction $B_m\mathbb{G}_m = B_m N(\sqrt{Q_m})$, where Q_m is the section obtained via pullback from the $\mathcal{O}_{\mathbb{P}^2\setminus C}(2)$-section $Q = T_1^2 - 4T_0T_2$. Indeed, if t denotes the tautological section of $\pi^*\mathcal{O}(1)$, associated to $\pi : \mathcal{O}(1) \to B_m N$, then $B_m\mathbb{G}_m \simeq \mathcal{V}(t^2 - \pi^*Q_m) \subseteq \pi^*\mathcal{O}(1)$ is the zero locus of the section $(t^2 - \pi^*Q_m)$. The map on locally free sheaves $\mathcal{O}(-1) \to \mathcal{O}_{B_m\mathbb{G}_m}$ sends a local section y of $\mathcal{O}(-1)$ to yt restricted to $B_m\mathbb{G}_m$. Let $\varphi_m : B_m\mathbb{G}_m \to B_m N$ be the map between the finite approximations, then we get that $(\varphi_m)_*\mathcal{O}_{B_m\mathbb{G}_m} = \mathcal{O}_{B_m N} \oplus \mathcal{O}(-1)$. A local section of $\varphi_{m*}\mathcal{O}_{B_m\mathbb{G}_m}$ will then be of the for $x + yt$, with x, y local sections of $\mathcal{O}_{B_m N}$ and $\mathcal{O}(-1)$ respectively. Due to the fact that φ_m is étale, we have a well defined trace map $\mathrm{Tr}_{\varphi_m} : (\varphi_m)_*\mathcal{O}_{B_m\mathbb{G}_m} \to \mathcal{O}_{B_m N}$. Then for $x + yt$ local section of $\varphi_{m*}\mathcal{O}_{B_m\mathbb{G}_m}$ we get that:

$$\mathrm{Tr}_{\varphi_m}((x + yt)^2) = \mathrm{Tr}_{\varphi_m}(x^2 + Q_m y^2 + 2xyt) = 2x^2 + 2Q_m y^2$$

In other words:
$$\mathrm{Tr}_{\varphi_m}(\langle 1\rangle) = \langle 2\rangle(1+q_{0,m})$$
with $q_{0,m}$ the pullback of q_0.

Thanks to [LR20, §8D], we can identify $(\varphi_m)_*\langle 1\rangle$ with the quadratic form given by $\mathrm{Tr}_{\varphi_m}(\langle 1\rangle)$, that is:
$$(\varphi_m)_*\langle 1\rangle = \langle 2\rangle(1+q_{0,m}) \tag{2.18}$$
Now let m be an odd integer. Notice that the difference $\varphi_*\langle 1\rangle - \langle 2\rangle(1+q_0)$ is sent to $(\varphi_m)_*\langle 1\rangle - \langle 2\rangle(1+q_{0,m})$ in $\mathrm{KW}^0(B_mN)$, thus by (2.18) we get:
$$\nu_m^*\left(\varphi_*\langle 1\rangle - \langle 2\rangle(1+q_0)\right) = (\varphi_m)_*\langle 1\rangle - \langle 2\rangle(1+q_{0,m}) = 0 \tag{2.19}$$
By lemma 2.3.2 (for A = KW), this implies that:
$$\nu_m^*\left(\varphi_*\langle 1\rangle - \langle 2\rangle(1+q_0)\right) = \sum_{i>0}^{m-1} a_i e^{2i} = 0$$
and therefore $a_i = 0$ for all $0 < i < m$. For bigger and bigger m, this gives us that $a_i = 0$ for all $i > 0$. Hence, together with (2.17), we have that:
$$\varphi_*\langle 1\rangle = \sum_i a_i e^{2i} + b\cdot q_0 = \langle 2\rangle(1+q_0)$$
as claimed.

□

Proposition 2.3.4 ([Lev19, Lemma 5.4]). *Consider $S = \mathrm{Spec}(\Bbbk)$. We have $(1+q_0)\cdot p^*e = 0$ in $\mathrm{KW}^2(\mathcal{B}N)$, $(1+q_0)\cdot j^*e(\mathcal{T}) = 0$ in $\mathrm{KW}^2(\mathcal{B}N;\gamma_N)$ and $(1+q_0)\cdot q_1 = q_1 \in q_1\cdot \mathrm{A}^0(\Bbbk)$.*

Proof. We will freely use the notation employed in the proof of proposition 2.2.16. We already proved that $(1+q_0)\cdot p^*e = 0$ in lemma 2.3.1. Let us prove the second statement, that is, $(1+q_0)\cdot j^*e(\mathcal{T}) = 0$. By remark 2.2.17, there exists a unique $\lambda := \lambda_1^{\mathrm{KW}} \in \mathrm{KW}^0(\mathcal{B}SL_2)$ such that $(1+q_0)\cdot j^*e(\mathcal{T}) = \lambda\cdot e(\mathcal{T})$. But then multiplying both sides by p^*e, since we already know $(1+q_0)\cdot p^*e = 0$, we get $0 = (1+\bar{q})\cdot p^*e\cdot j^*e(\mathcal{T}) = \lambda\cdot p^*e\cdot j^*e(\mathcal{T})$. An element in $\mathrm{KW}^0(\mathcal{B}SL_2)$ is a power series $\sum_i a_i p^*e^i$ with $a_i \in \mathrm{KW}^{-2i}(S)$, so $\lambda\cdot p^*e\cdot j^*e(\mathcal{T}) = 0$ implies that $\sum_i a_i p^*e^{i+1} = 0 \in \mathrm{KW}^2(\mathcal{B}SL_2)$, that is, $a_i = 0\ \forall\ i$ and hence $\lambda = 0$.

Let us now show that $(1+q_0)\cdot q_1 = q_1$. Recall that q_1 in proposition 2.2.14 was chosen to be any element that was sent to 1 under ∂_1. Let $\tilde{q}_1 \in \mathrm{KW}^0(\mathcal{B}N;\gamma_N)$ be another element such that $\partial_1(\tilde{q}_1) = 1$. Then $q_1 - \tilde{q}_1 \in \mathrm{Ker}(\partial_1)$, that is:
$$q_1 - \tilde{q}_1 = \alpha\cdot j^*e(\mathcal{T})$$

for $\alpha \in \mathrm{KW}^{-2}(\mathcal{B}SL_2)$. Then we have:

$$(1+q_0)q_1 = (1+q_0)(\alpha \cdot j^*e(\mathcal{T}) + \tilde{q}_1) = (1+q_0)\tilde{q}_1 \tag{2.20}$$

since $(1+q_0)j^*e(\mathcal{T}) = 0$. Without loss of generality we can therefore replace q_1 with any other $\tilde{q}_1 \in \mathrm{KW}^0(\mathcal{B}N;\gamma_N)$ such that $\partial_1(\tilde{q}_1) = 1$. On $\mathbb{P}^2$ with coordinates $[T_0 : T_1 : T_2]$, consider the quadratic form:

$$-(T_0x^2 + T_1xy + T_2y^2)$$

and set $\tilde{q}_1 \in \mathrm{KW}^0(\mathcal{B}N)$ to be the corresponding element. Let us check that the boundary of $\tilde{q}_1$ is indeed 1. Let $\pi_{\widetilde{\mathbb{P}(F)}} : \widetilde{\mathbb{P}(F)} \to \mathrm{Spec}(\Bbbk)$ be the structure map of $\widetilde{\mathbb{P}(F)}$, then we know that $\mathrm{KW}^\bullet(\Bbbk)$ is isomorphic to $\mathrm{KW}^\bullet(\widetilde{\mathbb{P}(F)})$ via $\pi^*_{\widetilde{\mathbb{P}(F)}}$. Let U be any dense open subset of $\mathbb{P}^2$ where $\mathcal{O}(2)$ gets trivialised, denote by $V := U \cap C$ the corresponding dense open in C and denote by $\pi_V : V \to \mathrm{Spec}(\Bbbk)$ the structure map of V. Since there always exists a rational $\Bbbk$-point in V, the map:

$$\pi_V^* : \mathrm{KW}^\bullet(\Bbbk) \to \mathrm{KW}^\bullet(V)$$

must be injective and this implies that the map $\pi_V^*(\pi^*_{\widetilde{\mathbb{P}(F)}})^{-1}$ is injective too. Now take $U = \{T_0 \neq 0\}$, then we have $\tilde{q}_1\big|_U = -(x^* + t_1xy, t_2y^")$, with $t_1 = \frac{T_1}{T_0}$ and $t_2 = \frac{T_2}{T_0}$. Diagonalising $\tilde{q}_1\big|_U$, we get:

$$\begin{aligned}\tilde{q}_1\big|_U &= -\left[(x+\frac{t_1}{2}y)^2 + (t_2 - \frac{t_1^2}{4})y^2\right] = \\ &= -\left[(x+\frac{t_1}{2}y)^2 + q_0\big|_U y^2\right]\end{aligned}$$

So $\tilde{q}_1\big|_U = -1 + q_0\big|_U$ and hence $\partial(\tilde{q}_1\big|_U) = 1$ since $\partial(q_0) = 1$ on U. But this implies that $\pi_V^*(\pi^*_{\widetilde{\mathbb{P}(F)}})^{-1}(\partial_1(\tilde{q}_1) - 1) = \partial(\tilde{q}_1\big|_U) - 1 = 0$, and, being $\pi_V^*(\pi^*_{\widetilde{\mathbb{P}(F)}})^{-1}$ injective, we deduce that $\partial_1(\tilde{q}_1) = 1$ as we wanted to show.
Now that we know that $\partial_1(\tilde{q}_1) = 1$, by (2.20), we can replace q_1 in $(1+q_0)q_1$ with $\tilde{q}_1$. Since on U we have $\tilde{q}_1\big|_U = -1 + q_0\big|_U$, we have that $(1+q_0\big|_U)\tilde{q}_1\big|_U = 0$. But this implies:

$$\pi_V^*(\pi^*_{\widetilde{\mathbb{P}(F)}})^{-1}((1+q_0)\tilde{q}_1) = (1+q_0\big|_U)\tilde{q}_1\big|_U = 0$$

and since $\pi_V^*(\pi^*_{\widetilde{\mathbb{P}(F)}})^{-1}$ is an injective map, we get that $(1+q_0)\tilde{q}_1 = 0$ and we are done.

$\square$

Putting together all the result we got so far, we get:

Corollary 2.3.5. *Let* $\mathrm{KW}^\bullet(\mathbb{k})[\![x_0, x_2]\!]$ *be the graded algebra over* $\mathrm{KW}^\bullet(\mathbb{k})$*, freely generated by* x_0, x_2 *in degrees* $deg(x_i) = i$*. Sending* x_0 *to* $q_0 := q_0^{\mathrm{KW}}$ *and* x_2 *to* p^*e *defines a* $\mathrm{KW}(\mathbb{k})$*-algebra isomorphism:*

$$\psi_{\mathbb{k}} : \mathrm{KW}^\bullet(\mathbb{k})[\![x_0, x_2]\!] \Big/ \big(x_0^2 - 1, (1 + x_0)x_2\big) \longrightarrow \mathrm{KW}^\bullet(\mathcal{B}N)$$

Moreover $\mathrm{KW}^\bullet(\mathcal{B}N; \gamma_N)$ *is the quotient of the free* $\mathrm{KW}^\bullet(\mathcal{B}N)$*-module* $\mathrm{KW}^{\bullet-2}(\mathcal{B}SL_2)\cdot e(\mathcal{T}) \oplus q_1 \cdot \mathrm{KW}^\bullet(\mathbb{k})$ *modulo the relations* $(1 + q_0)j^*e(\mathcal{T}) = 0$*,* $(1 + q_0)q_1 = 0$*.*

Proof. It is just a consequence of proposition 2.2.16 and proposition 2.3.4. The relation $x_0^2 = 1$ comes from $q_0^2 = 1$ that holds for any quadratic form $\langle u\rangle$ where u is a unit. □

Corollary 2.3.6. *Let* S *be a smooth* $\mathbb{k}$*-scheme. Let* $\mathrm{KW}^\bullet(S)[\![x_0, x_2]\!]$ *be the graded algebra over* $\mathrm{KW}^\bullet(S)$*, freely generated by* x_0, x_2 *in degrees* $deg(x_i) = i$*. Sending* x_0 *to* $q_0 := q_0^{\mathrm{KW}}$ *and* x_2 *to* p^*e *defines a* $\mathrm{KW}(S)$*-algebra isomorphism:*

$$\psi_S : \mathrm{KW}^\bullet(S)[\![x_0, x_2]\!] \Big/ \big(x_0^2 - 1, (1 + x_0)x_2\big) \longrightarrow \mathrm{KW}^\bullet(\mathcal{B}N)$$

Moreover $\mathrm{KW}^\bullet(\mathcal{B}N; \gamma_N)$ *is the quotient of the free* $\mathrm{KW}^\bullet(\mathcal{B}N)$*-module* $\mathrm{KW}^{\bullet-2}(\mathcal{B}SL_2)\cdot e(\mathcal{T}) \oplus q_1 \cdot \mathrm{KW}^\bullet(S)$ *modulo the relations* $(1 + q_0)j^*e(\mathcal{T}) = 0$*,* $(1 + q_0)q_1 = 0$*.*

Proof. We will prove the corollary reducing to the case over a field, hence we will use subscripts to indicate over which base are we working with. By proposition 2.2.16 for $\mathrm{A} = \mathrm{KW}$ (and by [Ana15, Theorem 10]), additively we have the following isomorphism:

$$\mathrm{KW}^\bullet(\mathcal{B}N_S) \simeq \mathrm{KW}^\bullet(\mathcal{B}SL_{2,S}) \oplus \mathrm{KW}^\bullet(S) \cdot q_{0,S}$$

Therefore we have a natural map:

$$\varphi_S : \mathrm{KW}^\bullet(S)[\![x_{0,S}, x_{2,S}]\!] \longrightarrow \mathrm{KW}^\bullet(\mathcal{B}N_S)$$

sending x_2 to $p^*e_S \in \mathrm{KW}(\mathcal{B}SL_{2,S})$, i.e. the Euler class of the tautological bundle of $\mathcal{B}SL_{2,S}$, and sending x_0 to $q_{0,S}$. By corollary 2.3.5, the natural map:

$$\varphi_{\mathbb{k}} : \mathrm{KW}^\bullet(\mathbb{k})[\![x_{0,\mathbb{k}}, x_{2,\mathbb{k}}]\!] \longrightarrow \mathrm{KW}^\bullet(\mathcal{B}N_{\mathbb{k}})$$

passes to the quotient giving us the isomorphism:

$$\psi_{\mathbb{k}} : \mathrm{KW}^\bullet(\mathbb{k})[\![x_{0,\mathbb{k}}, x_{2,\mathbb{k}}]\!] \Big/ \Big(x_{0,\mathbb{k}}^2 - 1, (1 + x_{0,\mathbb{k}})x_{2,\mathbb{k}}\Big) \xrightarrow{\sim} \mathrm{KW}^\bullet(\mathcal{B}N_{\mathbb{k}})$$

The structure map $\pi : S \to \mathrm{Spec}(\mathbb{k})$ induces, via the pullback π_S^*, a map:

$$\theta_S : \mathrm{KW}^\bullet(S)[\![x_{0,S}, x_{2,S}]\!] \longrightarrow \mathrm{KW}^\bullet(\mathbb{k})[\![x_{0,\mathbb{k}}, x_{2,\mathbb{k}}]\!]$$

sending $x_{0,S}$ to $x_{0,\Bbbk}$ and $x_{2,S}$ to $x_{2,\Bbbk}$. We also get a map:

$$\theta_{\mathcal{B}N} : \mathrm{KW}^\bullet(\mathcal{B}N_S) \longrightarrow \mathrm{KW}^\bullet(\mathcal{B}N_\Bbbk)$$

induced again by π_S^* and sending p^*e_S to $p^*e_\Bbbk$ and $q_{0,S}$ to $q_{0,\Bbbk}$. This implies that φ_S passes to the quotient, giving us a map:

$$\psi_S : \mathrm{KW}^\bullet(S)[\![x_{0,S}, x_{2,S}]\!] \Big/ \left(x_{0,S}^2 - 1, (1 + x_{0,S})x_{2,S}\right) \longrightarrow \mathrm{KW}^\bullet(\mathcal{B}N)$$

Since ψ_S is an isomorphism on the underlying modules, we get that ψ_S is also an isomorphism of $\mathrm{KW}^\bullet(S)$-algebras.

The twisted case $\mathrm{KW}^\bullet(\mathcal{B}N_S; \gamma_N)$ is completely analogous and left to the reader.

$\square$

Chapter 3

Euler Classes Computations

3.1 $SL[\eta^{-1}]$-Theories on BGL_n

In this chapter, we are going to compute the KW-Euler classes for some special rank 2 vector bundles of the ind-scheme BN. By the identifications proposition 1.4.30 and corollary 1.4.32, this Euler classes can be used to better understand the Witt theory of $\mathcal{B}N$ and give some computational insight for a possible Grothendieck-Riemann-Roch formula for KW. We will follow the notations as in [Lev19, §6]. Before diving into these enumerative formulas, we will see how we can reduce computations of characteristic classes from general vector bundles to special linear ones. Then we will need to introduce *twisted symplectic bundles* to handle formal ternary laws (in the sense of [DF21]) that will be crucial for our last computation, eq. (3.11). Throughout we will be identifying the motives and the cohomology theories of the ind-schemes BG with those of the quotient stacks $\mathcal{B}G$, using proposition 1.4.30 and corollary 1.4.32.

Recall from [Lev19, §4] that for any SL-oriented ring spectrum $\mathbb{E}$ we have a map:

$$\pi^* : \mathrm{A}^\bullet(BGL_n) \oplus \mathrm{A}^\bullet(BGL_n; det(E_n)) \longrightarrow \mathrm{A}^\bullet(BSL_n) \tag{3.1}$$

Indeed we have the pullback map:

$$\pi_0^* : \mathrm{A}^\bullet(BGL_n) \longrightarrow \mathrm{A}^\bullet(BSL_n)$$

induced by $\pi_0 : BSL_n \longrightarrow BGL_n$. Now consider the tautological rank n vector bundle $E_n \to BGL_n$ with $\det(E_n) = \mathcal{O}(1)$, then its pullback $\pi_0^* E_n$ will be the tautological special linear bundle over BSL_n, so we have a canonical trivialization $\theta : \det(\pi_0^* E_n) \simeq \pi_0^*\mathcal{O}(1) \xrightarrow{\sim} \mathcal{O}_{BSL_n}$. Composing the pullback map on the twisted theories with $\theta_* := (\tau_{\pi_0^*\mathcal{O}(1)})^{-1}$, given by the inverse Thom isomorphism (cf. 2.1.9), we get:

$$\mathrm{A}^\bullet(BGL_n; \mathcal{O}(1)) \xrightarrow{\pi_0^*} \mathrm{A}^\bullet(BSL_n; \pi_0^*\mathcal{O}(1)) \xrightarrow{\theta_*} \mathrm{A}^\bullet(BSL_n)$$

and we denote this map as $\pi_1^* := \theta_* \circ \pi_0^*$. Then putting together π_0^* and π_1^* we get our desired π^*.

We would like to reduce the computations of characteristic classes from general vector bundles to special linear ones. With a minor adaptation of the arguments in [Lev19, Proposition 4.1] we can prove the following:

Proposition 3.1.1. *Let $S \in \mathbf{Sm}_{/\Bbbk}$. Let E_n be the universal tautological bundle of BGL_n, and let $\mathrm{A} \in \mathrm{SH}(S)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then the map* (3.1)*:*

$$\pi^* : \mathrm{A}^\bullet(BGL_n) \oplus \mathrm{A}^\bullet(BGL_n; det(E_n)) \longrightarrow \mathrm{A}^\bullet(BSL_n)$$

is an isomorphism. In particular we have:

($n = 2m$)

$$\pi^*(\mathrm{A}^\bullet(BGL_n)) \simeq \mathrm{A}^\bullet(S)[\![p_1, \dots, p_{m-1}, e^2]\!]$$

$$\pi^*(\mathrm{A}^\bullet(BGL_n; \det(E_n))) \simeq e \cdot \mathrm{A}^\bullet(S)[\![p_1, \dots, p_m, e^2]\!]$$

where $p_i = p_i(\mathcal{T}(n, \infty))$ are the Pontryagin classes (cf. [Ana15, Theorem 10]) of to the tautological bundle $\mathcal{T}(n, \infty)$ over BSL_n and $e = e(\mathcal{T}(n, \infty))$ is the Euler class of the tautological bundle.

($n = 2m + 1$) Then $\mathrm{A}^\bullet(BGL_n; \det(E_n)) \simeq 0$ and:

$$\pi^*(\mathrm{A}^\bullet(BGL_n)) \longrightarrow \mathrm{A}^\bullet(BSL_n)$$

is an isomorphism.

Proof. Let us consider the line bundle $q : \mathcal{O}(1) := \det(E_n) \longrightarrow BGL_n$. We can identify the map $\pi_0 : BSL_n \longrightarrow BGL_n$ with the $\mathbb{G}_m$-principal bundle $q \circ j : \mathcal{O}(1) \setminus \{0\} \longrightarrow BGL_n$ where $j : \mathcal{O}(1) \setminus \{0\} \hookrightarrow \mathcal{O}(1)$. By homotopy invariance we can also identify $\mathrm{A}(BGL_n) \simeq \mathrm{A}(\mathcal{O}(1))$, thus we have the localization sequences:

$$\begin{aligned} \ldots \to \mathrm{A}^{a,b}(BGL_n) \to & \mathrm{A}^{a,b}(BSL_n) \longrightarrow \ldots \\ & \ldots \overset{\partial_{a,b}}{\to} \mathrm{A}^{a-1,b-1}(BGL_n; det(E_n)) \overset{e(\det(E_n))\cup\cdot}{\to} \mathrm{A}^{a+1,b}(BGL_n) \to \ldots \end{aligned} \tag{3.2}$$

$$\ldots \to \mathrm{A}^{a,b}(BGL_n; \mathcal{O}(-1)) \to \mathrm{A}^{a,b}(BSL_n; \pi_0^*\mathcal{O}(-1)) \overset{\bar{\partial}_{a,b}}{\to} \mathrm{A}^{a-1,b-1}(BGL_n) \to \ldots \tag{3.3}$$

By [Lev19, lemma 4.3] the cup product with $e(L)$ for any line bundle L is zero in any η-inverted SL-oriented ring spectrum, so the long exact sequence (3.2) splits in short exact sequences:

$$0 \to \mathrm{A}^{a,b}(BGL_n) \to \mathrm{A}^{a,b}(BSL_n) \overset{\partial_{a,b}}{\to} \mathrm{A}^{a-1,b-1}(BGL_n; det(E_n)) \to 0$$

We want now to find a splitting for ∂ and we already have a natural candidate π_1^*, in the notation used above the proposition. Let us consider the tautological section $can : \mathcal{O}(1) \to q^*\mathcal{O}(1)$. Using definition 2.1.28, the tautological section defines an element $\langle t_{can}\rangle \in \mathrm{A}^{0,0}(BSL_n, \pi_0^*\mathcal{O}(1)) \simeq \mathrm{A}^{0,0}(BSL_n, \pi_0^*\mathcal{O}(-1))$. The map π_1^* we constructed before was:

$$\mathrm{A}^\bullet(BGL_n; \mathcal{O}(1)) \xrightarrow{\pi_0^*} \mathrm{A}^\bullet(BSL_n; \pi_0^*\mathcal{O}(1)) \xrightarrow{\theta_*} \mathrm{A}^\bullet(BSL_n)$$

but θ_* is just multiplication by $\langle t_{can}\rangle$. To show that the short exact sequence actually splits we need to prove that $\partial \circ \pi_1^*$ is an isomorphism. As a first step we claim that for any $x \in \mathrm{A}^\bullet(BGL_n; \mathcal{O}(1))$ we have:

$$\partial(\pi_1^*(x)) = \bar{\partial}(\langle t_{can}\rangle) \cup x$$

Indeed:

$$\partial(\pi_1^*(x)) = \partial(\langle t_{can}\rangle \cup \pi_0^*(x)) = \partial(\langle t_{can}\rangle \cup q^* j^*(x))$$

and ∂ is $\mathrm{A}^\bullet(BGL_n; \mathcal{O}(1))$-linear, with $\mathrm{A}^\bullet(BGL_n; \mathcal{O}(1))$ acting on $E^\bullet(BSL_n)$ via multiplication through $q^* j^*$ and on $E^\bullet(BGL_n; \mathcal{O}(1))$ just via multiplication, and this proves the claim.
We want now to prove that multiplication by $\bar{\partial}(\langle t_{can}\rangle) \in \mathrm{A}^{-1,-1}(BGL_n)$ is an isomorphism:

$$\bar{\partial}(\langle t_{can}\rangle) \cup \cdot : \mathrm{A}^{a,b}(BGL_n; \mathcal{O}(1)) \longrightarrow \mathrm{A}^{a-1,b-1}(BGL_n; \mathcal{O}(1))$$

Using a Mayer-Vietoris argument on the finite approximation pieces $B_m GL_n$ we can reduce ourselves to a local computation, where we already know the result by [Ana19, Lemma 6.4]. Indeed given two open sets U, V of $B_m GL_n$, we get a Nisnevich excision square related to $\{U, V, U \cup V\}$ and a Mayer-Vietoris exact sequence for any Nisnevich sheaf. In particular multiplication by an element $u \in \mathrm{A}^{-1,-1}(B_m GL_n)$ gives us a map of Mayer-Vietoris sequences:

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] \\
\mathrm{A}^{a,b}(U \cup V; \mathcal{O}(1)) \arrow[r, "u \cup \cdot"] \arrow[d] & \mathrm{A}^{a-1,b-1}(U \cup V; \mathcal{O}(1)) \arrow[d] \\
\mathrm{A}^{a,b}(U; \mathcal{O}(1)) \oplus \mathrm{A}^{a,b}(V; \mathcal{O}(1)) \arrow[r, "(u \cup \cdot, u \cup \cdot)"] \arrow[d] & \mathrm{A}^{a-1,b-1}(U; \mathcal{O}(1)) \oplus \mathrm{A}^{a-1,b-1}(V; \mathcal{O}(1)) \arrow[d] \\
\mathrm{A}^{a,b}(U \cap V; \mathcal{O}(1)) \arrow[r, "u \cup \cdot"] \arrow[d] & \mathrm{A}^{a-1,b-1}(U \cap V; \mathcal{O}(1)) \arrow[d] \\
\mathrm{A}^{a+1,b}(U \cup V; \mathcal{O}(1)) \arrow[r, "u \cup \cdot"] \arrow[d] & \mathrm{A}^{a,b-1}(U \cup V; \mathcal{O}(1)) \arrow[d] \\
\cdots & \cdots
\end{tikzcd}$$

So to show that multiplication by $\bar{\partial}(\langle t_{can} \rangle)$ is an isomorphism, it is enough to show that $\bar{\partial}(\langle t_{can} \rangle)$ restricts to an invertible element when passing to open sets U_i covering $B_m GL_n$. But in local coordinates $\bar{\partial}(\langle t_{can} \rangle)$ restricts to η by [Ana19, Lemma 6.4] and thus we get our desired splitting for $B_m GL_n$. This implies that for any k we have:

$$\pi_m^* : \mathrm{A}^k(B_m GL_n) \oplus \mathrm{A}^k(B_m GL_n; \mathcal{O}(1)) \xrightarrow{\sim} \mathrm{A}^k(B_m SL_n)$$

where π_m^* is the map induced by π^* at the finite level approximations. Since the isomorphism holds for any k, we have an equivalence of mapping spectra:

$$\pi_m^* : \mathrm{A}(B_m GL_n) \oplus \mathrm{A}(B_m GL_n; \mathcal{O}(1)) \xrightarrow{\sim} \mathrm{A}(B_m SL_n)$$

where $\mathrm{A}(B_m G) = \underline{\mathrm{Map}}(\Sigma^\infty B_m G, \mathrm{A})$ (and similarly for the twisted version). But since $\mathrm{A}(BG) = \underline{\mathrm{Map}}(BG, \mathrm{A}) = \lim_m \underline{\mathrm{Map}}(B_m G, \mathrm{A})$, from the equivalence of mapping spectra π_m^*, we get:

$$\pi^* : \mathrm{A}(BGL_n) \oplus \mathrm{A}(BGL_n; \mathcal{O}(1)) \xrightarrow{\sim} \mathrm{A}(BSL_n)$$

again at the level of mapping spectra, and this gives us our claim.

For the explicit presentation of the image of π^* the same argument as in the proof of [Lev19, Proposition 4.1] will give us the result, using the statement of [Ana15, Theorem 10] generalised over any smooth $\Bbbk$-scheme S in theorem 3.2.3 (we will actually generalised it to smooth NL-stacks).

□

Remark 3.1.2. The previous proposition was already proved independently in [Hau23, Remark 6.3.7] using stronger results. Indeed in *loc. cit.* it is proved that $BGL_{2r} \simeq BGL_{2r+1}$ and $BGL_{2r+1} \simeq BSL_{2r+1}$ in $\mathrm{SH}(S)[\eta^{-1}]$ (this are respectively [Hau23, Theorem 6.3.3, Theorem 6.3.6]).

The proof of proposition 3.1.1, used as a key input that BSL_n can be seen as the complement of the zero section of the determinant tautological bundle of BGL_n. And that is all we actually need: the same proof goes through verbatim if we consider $BN \to BN_G$ where N_G is the normaliser of the torus inside GL_2. The model for BN_G is given by $(GL_2/N_G) \times^{GL_2} EGL_2$, where again $GL_2/N_G \simeq SL_2/N \simeq \mathbb{P}^2 \setminus C$. We then have:

Proposition 3.1.3. *Let $S \in \mathbf{Sm}_{/\Bbbk}$. For any $SL[\eta^{-1}]$-oriented ring spectrum $\mathrm{A} \in \mathrm{SH}(S)$ we have an induced isomorphism:*

$$\pi^* : \mathrm{A}^\bullet(BN_G) \oplus \mathrm{A}^\bullet(BN_G; \mathcal{O}(1)) \longrightarrow \mathrm{A}^\bullet(BN)$$

Proof. Use the same exact proof as in proposition 3.1.1, replacing BSL_n with BN and BGL_n with BN_G and pulling back the appropriate maps along the morphisms $BN \to BSL_2$ and $BN_G \to BGL_2$ induced by $N \hookrightarrow SL_2$ and $N_G \hookrightarrow GL_2$ respectively. □

Remark 3.1.4. The previous proposition was also proved in [Vie23, Proposition 2.5.10] under the assumption that the unit map of A makes the $\mathrm{A}^\bullet(-)$-cohomology into a module for the sheaf of Witt groups $H\mathcal{W}$.

Remark 3.1.5. As a corollary of proposition 3.1.3, it is not hard to get an additive description for $\mathrm{A}^\bullet(BN_G)$, using proposition 2.2.16, and a multiplicative description of $\mathrm{KW}^\bullet(BN_G)$ using corollary 2.3.6. We are very grateful to Annaloes Viergever that made us realise we could improve and apply our results to the case of BN_G.

3.2 Künneth Formulas

We want now to prove some Künneth formulas for $\mathcal{B}SL_n, \mathcal{B}GL_n$ and $\mathcal{B}N$. Let us start with an extension of [Ana15, Theorem 9]. Notice that using our model for the approximations B_mSL_n we have that $B_mSL_n \simeq SGr(n, n+m)$. Since we will work over different base schemes we will denote the special Grassmannian over some scheme S as $SGr_S(n,k)$.

Proposition 3.2.1. *Let $S \in \mathbf{Sm}_{/\Bbbk}$ and let $\mathrm{A} \in \mathrm{SH}(S)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Let $T_1 := \mathcal{T}(2n, 2k+1)$ be the tautological bundle of rank $2n$ on $SGr_{\Bbbk}(2n, 2k+1)$, with Pontryagin classes $p_i(T_1)$. Then there is an isomorphism of $\mathrm{A}^\bullet(S)$-algebras:*

$$\varphi : \mathrm{A}^\bullet(S)[p_1, p_2, \dots, p_k, e]\Big/J_{2n,2k+1} \longrightarrow \mathrm{A}^\bullet(SGr_S(2n, 2k+1))$$

where:

$$J_{2n,2k+1} := \big(e^2 - p_n, g_{k-n+1}(p_1,\dots,p_n), g_{k-n+2}(p_1,\dots,p_n),\dots,g_k(p_1,\dots,p_m)\big)$$

is defined with the same polynomials as in [Ana15, Theorem 9] and φ is defined by sending $\varphi(p_i) := p_i(T_1)$ and $\varphi(e) := e(T_1)$.

Proof. If $S = \mathrm{Spec}(\Bbbk)$, then this is just [Ana15, Theorem 9]. For a general smooth $\Bbbk$-scheme S, we have a map:

$$\varphi : \mathrm{A}^\bullet(S)[p_1,\dots,e] \longrightarrow \mathrm{A}^\bullet(SGr_S(2n,2k+1))$$

sending p_i to $p_i(T_1)$ and e to $e(T_1)$. The structure map $\pi_S : S \to \mathrm{Spec}(\Bbbk)$ induces a pullback map:

$$\pi_S^* : \mathrm{A}^\bullet(SGr_{\Bbbk}(2n,2k+1)) \to \mathrm{A}^\bullet(SGr_S(2n,2k+1))$$

The tautological bundle T_1 on $SGr_S(2n,2k+1)$ is the pullback of the tautological bundle $T_{1,\Bbbk}$ on $SGr_{\Bbbk}(2n,2k+1)$, hence the Pontryagin and Euler classes of T_1 satisfy all the relations in $J_{2m,2k+1}$ since they hold for the classes of $T_{1,\Bbbk}$ (by [Ana15, Theorem 9]). This implies that φ passes to the quotient, that is, we get a map:

$$\varphi : \mathrm{A}^\bullet(S)[p_1,p_2,\dots,p_n,e]\Big/J_{2n,2k+1} \longrightarrow \mathrm{A}^\bullet(SGr_S(2n,2k+1))$$

By [ADN18, Theorem 4.2.9], we have Leray spectral sequences both for source and target of φ:

$$E_1^{p,q} = \bigoplus_{s\in S^{(p)}} \mathrm{A}^q(\kappa(s))[p_1,p_2,\dots,p_n,e]\Big/J_{2n,2k+1} \Rightarrow \mathrm{A}^{p+q}(S)[p_1,p_2,\dots,p_n,e]\Big/J_{2n,2k+1}$$

$${}'E_1^{p,q} = \bigoplus_{s\in S^{(p)}} \mathrm{A}^q(SGr_{\kappa(s)}(2n,2k+1)) \Rightarrow \mathrm{A}^{p+q}(SGr_S(2n,2k+1))$$

The map between spectral sequences by φ is an isomorphism by [Ana15, Theorem 9], hence φ is an isomorphism too as claimed. □

Theorem 3.2.2. *Let $S \in \mathbf{Sm}_{/\Bbbk}$ and let $\mathcal{X} \in \mathcal{ASt}^{NL}_{/S}$ be a smooth NL-algebraic stack. Let $\mathrm{A} \in \mathrm{SH}(S)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then we have a map:*

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[p_1,p_2,\dots,p_n,e]\Big/J_{2n,2k+1} \longrightarrow \mathrm{A}^\bullet(\mathcal{X}\times_S SGr_S(2n,2k+1))$$

that is an isomorphism, where:

$$J_{2n,2k+1} := \big(e^2 - p_n, g_{k-n+1}(p_1,\dots,p_n), g_{k-n+2}(p_1,\dots,p_n),\dots,g_k(p_1,\dots,p_n)\big)$$

and φ is defined by sending $\varphi(p_i) := p_i(T_1)$ and $\varphi(e) := e(T_1)$ with T_1 the tautological bundle of $SGr_S(2n,2k+1)$.

Proof. Via the pullback map induced by the first projection $p_1 : \mathcal{X} \times_S SGr_S(2n, 2k+1)$, we get an $\mathrm{A}^\bullet(\mathcal{X})$-algebra structure on $\mathrm{A}^\bullet(\mathcal{X} \times_S SGr_S(2n, 2k+1))$. Since the Pontryagin and Euler classes in $\mathrm{A}^\bullet(\mathcal{X} \times_S SGr_S(2n, 2k+1))$ are the pullback of the respective classes in $\mathrm{A}^\bullet(SGr_S(2n, 2k+1))$, then there exists a unique map of $\mathrm{A}^\bullet(\mathcal{X})$-algebras:

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[p_1, p_2, \ldots, p_n, e]\Big/ J_{2n,2k+1} \longrightarrow \mathrm{A}^\bullet(\mathcal{X} \times_S SGr_S(2n, 2k+1))$$

defined by sending p_i to $p_i(T_1)$ and e to $e(T_1)$. Let $x : X \to \mathcal{X}$ be a NL-atlas and let $X^r_{\mathcal{X}} := \check{C}_r(X/\mathcal{X})$ be the scheme at the r^{th}-level of the Čech nerve. By proposition 3.2.1 we have isomorphisms:

$$\varphi_{X^r_{\mathcal{X}}} : \mathrm{A}^{p,q}(X^r_{\mathcal{X}})[p_1, \ldots, p_n, e]\Big/ J_{2n,2k+1} \xrightarrow{\sim} \mathrm{A}^{p+q}(X^r_{\mathcal{X}} \times_S SGr_S(2n, 2k+1))$$

for each r and for each bi-degree (p, q). This implies that we have an equivalence of mapping spectra:

$$\underline{\mathrm{Map}}(X^r_{\mathcal{X}}, A_{J_{2n,2k+1}}) \xrightarrow{\sim} \underline{\mathrm{Map}}(SGr_{X^r_{\mathcal{X}}}(2n, 2k+1), \mathrm{A}) \tag{3.4}$$

where $A_{J_{2n,2k+1}}$ is the ring spectrum representing $\mathrm{A}^\bullet(-)[p_1, \ldots, p_n, e]/\, J_{2n,2k+1}$. The equivalence (3.4), by remark 1.4.34, implies that we have:

$$\begin{aligned}
\mathrm{A}_{J_{2n,2k+1}}(\mathcal{X}) &\simeq \lim_{r\in\Delta} \mathrm{A}_{J_{2n,2k+1}}(X^r_{\mathcal{X}}) \simeq \\
&\simeq \lim_{r\in\Delta} \underline{\mathrm{Map}}(X^r_{\mathcal{X}}, A_{J_{2n,2k+1}}) \simeq \\
&\simeq \lim_{r\in\Delta} \underline{\mathrm{Map}}(SGr_{X^r_{\mathcal{X}}}(2n, 2k+1), \mathrm{A}) \simeq \\
&\simeq \lim_{r\in\Delta} \mathrm{A}(SGr_{X^r_{\mathcal{X}}}(2n, 2k+1)) \simeq \\
&\simeq \mathrm{A}(\mathcal{X} \times_S SGr_S(2n, 2k+1))
\end{aligned}$$

In other words:

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[p_1, p_2, \ldots, p_n, e]\Big/ J_{2n,2k+1} \longrightarrow \mathrm{A}^\bullet(\mathcal{X} \times_S SGr_S(2n, 2k+1))$$

is an equivalence as claimed. □

Theorem 3.2.3 (Künneth for BSL)**.** *Let $S \in \mathbf{Sm}_{/\Bbbk}$ and let $\mathcal{X} \in \mathcal{ASt}^{NL}_{/S}$ be a smooth NL-algebraic stack. Denote by $\mathcal{U}_r$ the universal tautological rank r bundle over $\mathcal{B}SL_{r,S}$. Let $\mathrm{A} \in \mathrm{SH}(S)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then there are unique $\mathrm{A}^\bullet(\mathcal{X})$-algebra maps:*

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[\![p_1, p_2, \ldots, p_{n-1}, e]\!] \longrightarrow \mathrm{A}^\bullet(\mathcal{X} \times_S \mathcal{B}SL_{2n,S})$$

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[\![p_1, p_2, \ldots, p_n]\!] \longrightarrow \mathrm{A}^\bullet(\mathcal{X} \times_S \mathcal{B}SL_{2n+1,S})$$

that are continuous with respect to the topology given by the restriction to the finite level approximations $\mathcal{X} \times_S SGr_S(2n, 2k+1)$, resp. $\mathcal{X} \times_S SGr_S(2n+1, 2k+1)$, and with $\varphi_{\mathcal{X}}(p_i) = p_i(\mathcal{U}_{2n})$, resp. $\varphi_{\mathcal{X}}(p_i) = p_i(\mathcal{U}_{2n+1})$ and $\varphi_{\mathcal{X}}(e) = e(\mathcal{U}_{2n})$.
Moreover the maps $\varphi_{\mathcal{X}}$ are isomorphisms. In particular we get an isomorphism:

$$\mathrm{A}^\bullet(\prod_{j=0}^{s} \mathcal{B}SL_{k_j,S}) \simeq \mathrm{A}^\bullet(\mathcal{B}SL_{k_0,S})\widehat{\otimes}_{\mathrm{A}^\bullet(S)} \ldots \widehat{\otimes}_{\mathrm{A}^\bullet(S)}\mathrm{A}^\bullet(\mathcal{B}SL_{k_s,S}) = \widehat{\bigotimes_j}\mathrm{A}^\bullet(\mathcal{B}SL_{k_j,S})$$

Proof. Let us start with the even case (the only case we are actually interested in for future applications). We know that the ind-scheme $BSL_{n,S} = \operatorname{colim}_k SGr_S(n, 2k+1)$, since the system made of $SGr_S(n, 2k+1)$ is cofinal inside the system made by all special linear Grassmannians. Then theorem 3.2.2 tells us that the system $\{\mathrm{A}^\bullet(\mathcal{X} \times_S SGr_S(n, 2k+1))\}$ satisfies the Mittag-Leffler condition. Hence by corollary 1.4.32, applied to the spectrum $\mathrm{A}_{\mathcal{X}}$ as constructed in the proof of proposition 2.2.16, we have:

$$A^\bullet(\mathcal{X} \times_S \mathcal{B}SL_{n,S}) \simeq \lim_k \mathrm{A}^\bullet(\mathcal{X} \times SGr_S(n, 2k+1))$$

By theorem 3.2.2 this gives us the isomorphism:

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[\![p_1, p_2, \ldots, p_{n-1}, e]\!] \longrightarrow \mathrm{A}^\bullet(\mathcal{X} \times_S \mathcal{B}SL_{2n,S})$$

For the odd case, it is enough to use the identification $\mathrm{A}^\bullet(SGr_S(2n+1, 2k+1)) \simeq \mathrm{A}^\bullet(SGr_S(2k-2n, 2k+1))$ as pointed out in [Ana15, Remark 14] and hence reduce to the same argument used in the even case.
Now for the last statement of the theorem, it is enough to show that:

$$\mathrm{A}^\bullet(\mathcal{B}SL_{k_1,S} \times \mathcal{B}SL_{k_2,S}) \simeq \mathrm{A}^\bullet(\mathcal{B}SL_{k_1,S})\widehat{\otimes}_{\mathrm{A}^\bullet(S)}\mathrm{A}^\bullet(\mathcal{B}SL_{k_2,S})$$

and then iterate. Without loss of generality, we can suppose $k_1 = 2n$ and $k_2 = 2m+1$. Let us denote the Pontryagin and Euler classes of $\mathcal{B}SL_{k_1,S}$ by $p_{i,1}, e_1$ and denote by $p_{i,2}$ the Pontryagin classes of $\mathcal{B}SL_{k_2,S}$. Then by what we just proved, for $\mathcal{X} = \mathcal{B}SL_{k_1,S}$, we have:

$$\begin{aligned}
\mathrm{A}^\bullet(\mathcal{B}SL_{k_1,S} \times \mathcal{B}SL_{k_2,S}) &\simeq \mathrm{A}^\bullet(\mathcal{B}SL_{k_1,S})[\![p_{1,2}, \ldots, p_{m,2}]\!] \simeq \\
&\simeq (\mathrm{A}^\bullet(S)[\![p_{1,1}, \ldots, p_{n-1,1}, e_1]\!])\,[\![p_{1,2}, \ldots, p_{m,2}]\!] \simeq \\
&\simeq \mathrm{A}^\bullet(S)[\![p_{1,1}, \ldots, p_{n-1,1}, e_1]\!]\widehat{\otimes}_{\mathrm{A}^\bullet(S)}(\mathrm{A}^\bullet(S)[\![p_{1,2}, \ldots, p_{m,2}]\!] \simeq \\
&\simeq \mathrm{A}^\bullet(\mathcal{B}SL_{k_1,S})\widehat{\otimes}_{\mathrm{A}^\bullet(S)}\mathrm{A}^\bullet(\mathcal{B}SL_{k_2,S})
\end{aligned}$$

as we wanted. All the other cases are totally analogous and then we can iterate the computations for the general case $\prod_{j=0}^{s} \mathcal{B}SL_{k_j}$. □

Corollary 3.2.4 (Künneth for BGL)**.** *Let $S \in \mathbf{Sm}_{/\Bbbk}$ and let $\mathcal{X} \in \mathcal{ASt}^{NL}_{/S}$ be a smooth NL-algebraic stack. Let $\mathcal{U}_r$ be the universal tautological bundle over $\mathcal{B}GL_{r,S}$. Let $\mathrm{A} \in \mathrm{SH}(S)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then there is a unique $\mathrm{A}^\bullet(\mathcal{X})$-algebra map:*

$$\varphi_{\mathcal{X}} : \mathrm{A}^\bullet(\mathcal{X})[\![p_1, p_2, \ldots, p_n]\!] \longrightarrow \mathrm{A}^\bullet(\mathcal{X} \times_S \mathcal{B}GL_{2n,S}) \simeq \mathrm{A}^\bullet(\mathcal{X} \times_S \mathcal{B}GL_{2n+1,S})$$

that is continuous with respect to the topology given by the finite level approximations $Gr_S(2n, k)$, resp. $Gr_S(2n+1, k)$, and such that the elements p_i are sent to the Pontryagin classes $p_i(\mathcal{U}_{2n})$, resp. $p_i(\mathcal{U}_{2n+1})$.

Moreover $\varphi_{\mathcal{X}}$ is an isomorphism. In particular, we get an equivalence:

$$\mathrm{A}^\bullet(\prod_{j=0}^{s} \mathcal{B}GL_{k_j,S}) \simeq \mathrm{A}^\bullet(\mathcal{B}GL_{k_0,S}) \widehat{\otimes}_{\mathrm{A}^\bullet(S)} \ldots \widehat{\otimes}_{\mathrm{A}^\bullet(S)} \mathrm{A}^\bullet(\mathcal{B}GL_{k_s,S}) = \widehat{\bigotimes_j} \mathrm{A}^\bullet(\mathcal{B}GL_{k_j,S})$$

Proof. Under the identifications $BGL_{2n,S} \simeq BGL_{2n+1,S} \simeq BSL_{2r+1}$ of [Hau23, Theorem 6.3.3, Theorem 6.3.7] in $\mathrm{SH}(S)[\eta^{-1}]$, the corollary follows from theorem 3.2.3 (implicitly using corollary 1.4.32). ☐

Proposition 3.2.5. *Let $S \in \mathbf{Sm}_{/\Bbbk}$. Let A be an $SL[\eta^{-1}]$-oriented ring spectrum. Let $\mathcal{X} = \prod_{i=1}^{s} \mathcal{B}GL_{n_i,S} \times \prod_{j=s+1}^{s+r} \mathcal{B}SL_{n_j,S}$. Then we have:*

$$\mathrm{A}^\bullet(X; L) \simeq \widehat{\bigotimes_{i=1}^{s}} \mathrm{A}^\bullet\left(\mathcal{B}GL_{n_i}; L_i\right) \widehat{\otimes}_{W(\Bbbk)} \widehat{\bigotimes_{j=s+1}^{s+r}} \mathrm{A}^\bullet\left(\mathcal{B}SL_{n_j}; L_j\right)$$

with $L := L_1 \boxtimes \ldots \boxtimes L_{s+r}$ and $L_i \in \mathrm{Pic}(\mathcal{B}GL_{n_i})$ for $i = 1, \ldots, s$ and $L_j \in \mathrm{Pic}(\mathcal{B}SL_{n_j})$ for $j = s+1, \ldots, s+r$ and where $\widehat{\otimes}$ denotes the completed tensor product.

Proof. The proposition follows by a Mayer-Vietoris argument from the untwisted cases in theorem 3.2.3 and corollary 3.2.4. ☐

3.3 Twisted Borel Classes

To aid in computing Euler classes of certain interesting rank 2 bundles on $\mathcal{B}N$, we introduce the notion of twisted Borel classes. We will always work over some base scheme $S \in \mathbf{Sm}_{/\Bbbk}$.

In analogy with the fact that any rank two SL-bundle has a canonical symplectic structure, we would like to consider a general rank two vector bundle with non-trivial determinant as a *twisted symplectic bundle*:

Definition 3.3.1. Given a vector bundle V over a scheme X and a line bundle $L \in \mathrm{Pic}(X)$, an *L-twisted symplectic form* on V is a non-degenerate alternating form $\omega^L : \Lambda^2 V \longrightarrow L$. A vector bundle V equipped with an L-twisted symplectic form will be an L-twisted symplectic bundle (V, ω^L).

Remark 3.3.2.
1. Any rank 2 vector bundle V over any scheme X has a canonical $\det(V)$-twisted symplectic form $\omega_{can} : \Lambda^2 V \xrightarrow{\sim} \det(V)$ given by the identification of its second exterior power with its determinant.
2. For trivial twists $L \simeq \mathcal{O}_X$, we recover the notion of symplectic bundles.

Definition 3.3.3. A *twisted symplectic Thom structure* on an SL-oriented ring cohomology theory $\mathbb{E}$ is a rule that assigns to each rank two L-twisted symplectic bundle (E, ω_L) over X, a class:

$$\mathrm{th}(E, \omega^L) \in \mathbb{E}^{4,2}\left(\mathrm{Th}_X(E); L\right) := \mathbb{E}^{6,3}_{\mathrm{DJK}}\left(X, -[L \oplus E]\right)$$

such that:

1. For an isomorphism of twisted symplectic bundles $u : (E_1, \omega_{L_1}) \xrightarrow{\sim} (E_2, \omega_{L_2})$, we have $u^*\mathrm{th}(E_1, \omega^{L_1}) = \mathrm{th}(E_2, \omega^{L_2})$.
2. For a map $f : X \longrightarrow Y$ and a twisted symplectic bundle (E, ω^L) over Y, we have $f^*\mathrm{th}(E, \omega^L) = \mathrm{th}(f^*E, f^*\omega^L)$.
3. For $can : \Lambda^2 \mathcal{O}_X^2 \xrightarrow{\sim} \mathcal{O}_X$ the canonical isomorphism, the class $\mathrm{th}(\mathcal{O}_X^2, can)$ is the image of $1 \in \mathbb{E}^{0,0}(X)$ under the suspension isomorphism:
$$\mathbb{E}^{0,0}(X) \simeq \mathbb{E}^{4,2}(\Sigma^{4,2}\mathbb{1}_X) = \mathbb{E}^{4,2}(\mathrm{Th}_X\left(\mathcal{O}_X^2\right))$$

Notation 3.3.4. To distinguish between Thom classes coming from (twisted) symplectic bundles and the ones coming from SL-vector bundle, we will denote the latter as $th^{SL}(\cdot)$.

The following is basically due to Ananyevskiy [Ana16a, Corollary 1]:

Proposition 3.3.5. *Any SL-oriented ring spectrum $\mathbb{E}$ admits a twisted symplectic Thom structure.*

Proof. For a rank 2 vector bundle V over some $X \in \mathbf{Sch}_{/S}$, we get the canonical identification $\omega_{can} : \Lambda^2 V \longrightarrow \det(V)$. Then we define:

$$\mathrm{th}(V, can) := \mathrm{th}^{SL}(V) \in \mathbb{E}^{4,2}(\mathrm{Th}_X\left(V\right); \det{}^{-1}(V))$$

where $\mathrm{th}^{SL}(V)$ is the SL-Thom class living in the twisted cohomology as defined in definition 2.1.23.

It is not difficult to check that this class satisfies the desired relations, see for example [LR20, §3] for a detailed account of these classes (in the case $X \in \mathbf{Sm}_{/S}$). □

Definition 3.3.6. Given a scheme X, $L \in \operatorname{Pic}(X)$ and an L-twisted symplectic bundle (V, ω^L) of rank $2n+2$, we define the L-twisted quaternionic Grassmannian as:

$$\mathrm{HGr}^L_X(k, V) := \left\{ W \in \mathrm{Gr}(k, V) \;\middle|\; \omega^L\big|_W : \Lambda^2 W \to L \text{ is non-degenerate} \right\}$$

We will often denote the corresponding tautological rank k bundle as $\mathcal{U}^L_V$ if not specified otherwise. When $k = 2$, we will call this the twisted quaternionic projective space and we will denote it with $\mathrm{HP}^L_X(V)$.
Let (V, ω^L) be a twisted symplectic bundle of dimension $2n+2$ over a scheme X, then if the base scheme X is clear from the context we will just write:

$$\mathrm{HP}^n_L := \mathrm{HP}^L_X(V)$$

Consider now any rank 2 vector bundle V over X with its zero section $s_0 : X \to V$, $\mathbb{E}$ an SL-oriented ring spectrum. By the twisted Thom isomorphism (cf. proposition 2.1.24), we get an induced map $s_* : \mathbb{E}^{0,0}(X) \to \mathbb{E}^{4,2}(\mathrm{Th}_X(V); \det^{-1}(V))$ such that $s_*(\mathbb{1}) = \mathrm{th}(V, \omega_{can})$. Post-composing with the map forgetting supports, we also get $\bar{s}_* : \mathbb{E}^{0,0}(X) \to \mathbb{E}^{4,2}(V; \det^{-1}(V))$. In particular the element $\bar{s}_*(\mathbb{1})$ will be the image of the class $(\mathrm{th}(V), \omega_{can})$ through the map forgetting supports.

Definition 3.3.7. Let $\mathbb{E}$ be an SL-oriented ring spectrum with a twisted symplectic Thom structure, and let (V, ω_L) be an L-twisted symplectic, rank 2, bundle over X with zero section $s : X \to V$. Let $\alpha^* : \mathbb{E}^{\bullet,\bullet}(\mathrm{Th}_X(V); L) \to \mathbb{E}^{\bullet,\bullet}(V; L)$ be the map forgetting supports. We define the *twisted Borel class* of V as:

$$b^{\mathbb{E}}_L(V, \omega_L) := -s^* \alpha^* \mathrm{th}(V, \omega_L) \in \mathbb{E}^{4,2}(X; L)$$

If the ring spectrum $\mathbb{E}$ is clear from the context we will drop it from the notation.

Remark 3.3.8. If V is an SL-bundle of rank 2 over X, then its twisted Borel class $b_{\mathcal{O}_X}(V, \omega_{can}) \in \mathbb{E}^{4,2}(\mathrm{Th}_X(V); \mathcal{O}_X) = \mathbb{E}^{6,3}(\mathrm{Th}_X(V) \otimes \mathrm{Th}_X(\mathbb{A}^1_X)) \simeq \mathbb{E}^{4,2}(\mathrm{Th}_X(V))$ corresponds to the classical untwisted Borel class coming from the SL-orientation of $\mathbb{E}$, so no harm is done if we just refer to Borel class of V.

Theorem 3.3.9 (Twisted Quaternionic Projective Bundle Theorem)**.** *Let $\mathbb{E}$ be an SL-oriented ring spectrum with a twisted Thom structure. Let (V, ω^L) be a twisted symplectic bundle of rank $2n$ over a scheme $X \in \mathbf{Sm}_{/S}$, let $(\mathcal{U}, \omega^L|_{\mathcal{U}})$ be the tautological rank 2 bundle over HP^n_L and let $\zeta := b(\mathcal{U}, \omega^L|_{\mathcal{U}})$ be its Borel class. Write $\pi : \mathrm{HP}^n_L \to X$ for the projection map. Then for any closed subset $Z \subseteq X$ we have the isomorphism of $\mathbb{E}(X)$-modules:*

$$(1, \zeta, \zeta^2, \ldots, \zeta^{n-1}) : \bigoplus_{j=0}^{n-1} \mathbb{E}_Z(X; L^{\otimes -j}) \longrightarrow \mathbb{E}_{\pi^{-1}(Z)}(\mathrm{HP}^n_L)$$

Moreover there are unique classes $b^L_j(\mathcal{E},\omega^L) \in \mathbb{E}^{4j,2j}(X; L^{\otimes j})$ *such that:*

$$\zeta^n - b_1(E,\omega^L) \cup \zeta^{n-1} + \dots b_2(E,\omega^L) \cup \zeta^{n-2} - \dots + (-1)^n b_n(E,\omega^L) = 0$$

and if (E,ω^L) *is the trivial symplectic bundle then* $b_j(E,\omega^L) = 0$ *for* $j = 1, \dots$.

Proof. The map we are looking for is given by:

$$\begin{array}{rccc} (1,\zeta,\zeta^2,\dots,\zeta^{n-1}): & \bigoplus_{j=0}^{n-1} \mathbb{E}_Z(X; L^{\otimes -j}) & \longrightarrow & \mathbb{E}_{\pi^{-1}(Z)}(\mathrm{HP}^n_L) \\ & (a_0,\dots,a_{n-1}) & \mapsto & \sum_0^{n-1} a_j \zeta^j \end{array}$$

Find opens that trivialise L over X and use them with a Mayer-Vietoris argument to reduce to the untwisted case proved in [PW18, Theorem 8.2]. □

Definition 3.3.10. The classes $b^L_i(\mathcal{E},\omega^L)$ of the theorem above are the Borel classes associated to (V,ω^L) with respect to the twisted Thom structure defined over $\mathbb{E}$. For $i > n$ and $i < 0$ set $b^L_i(\mathcal{E},\omega^L) = 0$ and set $b^L_0(\mathcal{E},\omega^L) = 1$. We define the *total twisted Borel class* as:

$$b^L_t(V,\omega^L) := 1 + b^L_1(V,\omega^L)t + \dots + b^L_n(V,\omega^L)t^n$$

Construction 3.3.11. Let (E,ω^L) be L-twisted symplectic bundle of rank $2r$: we want now to construct a twisted version of the *quaternionic flag bundle*. First consider:

$$\pi_1 : \mathrm{HFlag}^L_X(1, r-1; E) := \mathrm{HGr}^L_X(2; E) \longrightarrow X$$

Then over $\mathrm{HFlag}^L_X(1,r-1;E)$ we have the tautological rank 2 twisted bundle $U_{1,1}$ (the one classified by the identity map of $\mathrm{HGr}^L_X(2;E)$). The bundle $U_{1,1}$ is a sub-bundle of $\pi_1^* E$, therefore we can consider:

$$U^{\perp}_{1,1} := \left\{ w \in E \;\middle|\; \omega^L(w,u) = 0 \;\forall\; u \in U_{1,1} \right\}$$

that is the tautological sub-bundle of rank $2r-2$ over $\mathrm{HFlag}^L_X(1, r-1; E)$. Let us rename $U^{\perp}_{1,1}$ as:

$$W_1 := U^{\perp}_{1,1}$$

This gives us a decomposition:

$$(\pi_1^* E, \pi_1^* \omega^L) \simeq (U_{1,1}, \omega_{1,1}) \perp (W_1, \omega_{W_1})$$

where $\omega_{1,1} := \pi_1^*\omega^L\big|_{U_{1,1}}$ and $\omega_{W_1} := \pi_1^*\omega^L\big|_{W_1}$.

Now we iterate; to ease the notation we will drop the upper and lower scripts from the twisted hyperbolic Grassmannians HGr. Consider the natural map:

$$p_2 : \mathrm{HFlag}^L_X(1^2, r-2; E) := \mathrm{HGr}(2; W_1) \longrightarrow \mathrm{HFlag}^L_X(1, r-1; E)$$

and denote by $U_{2,2}$ the universal tautological rank 2 bundle over $\mathrm{HFlag}_X^L(1^2, r-2; E)$. This bundle is a natural sub-bundle of $p_2^* W_1$ and again, naming $W_2 := U_{2,2}^{\perp}$, we get:

$$(p_2^* W_1, p_2^* \omega_{W_1}) \simeq (U_{2,2}, \omega_{2,2}) \perp (W_2, \omega_{W_2})$$

where $\omega_{2,2}$ and ω_{W_2} are the restrictions of $p_2^*\omega_{W_1}$ to $U_{2,2}$ and W_2 respectively. In particular, if $\pi_2 : \mathrm{HFlag}_X^L(1^2, r-2; E) \to X$ is the map given by $\pi_1 \circ p_2$, we have the following decomposition:

$$(\pi_2^* E, \pi_2^* \omega^L) \simeq p_2^* (U_{1,1}, \omega_{1,1}) \perp (U_{1,2}, \omega_{1,1}) \perp (U_{2,2}, \omega_{2,1})$$

Notice that on $\mathrm{HFlag}_X^L(1^2, r-2; E)$ we have three tautological bundles, namely $U_{1,2} := p_2^* U_{1,1}$, $U_{2,2}$ and W_2.
Proceeding again in the same way, we can inductively construct:

$$p_k : \mathrm{HFlag}_X^L(1^k, r-k; E) := \mathrm{HGr}(2; W_{k-1}) \longrightarrow \mathrm{HFlag}_X^L(1^{k-1}, r-k+1; E)$$

together with a natural map:

$$\pi_k : \mathrm{HFlag}_X^L(1^k, r-k; E) \longrightarrow X$$

Over $\mathrm{HFlag}_X^L(1^k, r-k; E)$ we have the following (inductively defined) tautological bundles:

$$(U_{i,k}, \omega_{i,k}) := p_k^*(U_{i,k-1}, \omega_{i,k-1}) \qquad i = 1, \dots, k-1$$

$$(U_{k,k}, \omega_{k,k})$$

$$(W_k, \omega_{W_k}) := (U_{k,k}^{\perp}, \omega_{U_{k,k}^{\perp}})$$

where $U_{k,k}$ is the tautological rank 2 bundle of $\mathrm{HGr}(2; W_{k-1})$. By construction, we have:

$$\pi_k^*(E, \omega^L) \simeq \bigperp_{i=1}^{k} (U_{i,k}, \omega_{i,k}) \perp (W_k, \omega_{W_k})$$

Definition 3.3.12. Let (E, ω^L) be an L-twisted symplectic bundle of rank $2r$, then we define the *complete twisted quaternionic flag bundle* as:

$$\mathrm{HFlag}_X^L(E, \omega^L) := \mathrm{HFlag}_X^L(1^r, 0; E)$$

and we denote its tautological rank 2 bundles as $(\mathcal{U}_i, \omega_i^L) := (U_{1,r}, \omega_{i,r})$ for $i = 1, \dots, r$.

Let $q : \mathrm{HFlag}_X^L(E, \omega^L) \to X$ be the projection map, then by construction we have:

$$q^*(E, \omega^L) \simeq \bigoplus_{i=1}^{r} (\mathcal{U}_i, \omega_i^L)$$

Definition 3.3.13. Let (E, ω^L) be an L-twisted symplectic bundle of rank $2r$ and let $\mathrm{HFlag}_X^L(E, \omega^L)$ be its associated complete flag variety. Given $\mathcal{U}_i$ tautological rank 2 bundle over $\mathrm{HFlag}_X^L(E, \omega^L)$, we define the i^{th} *twisted Borel root* of (E, ω^L) to be:

$$u_i := b\left(\mathcal{U}_i, \omega_i^L\right)$$

Definition 3.3.14. Let $GL_{2,\det}^{\times r}$ be the following sub-group of $GL_2^{\times r}$:

$$GL_{2,\det}^{\times r} := \left\{(g_1, \ldots, g_r) \in GL_2^{\times r} \;\middle|\; \det(g_1) = \ldots = \det(g_r)\right\} \subseteq GL_2^{\times r}$$

We have an inclusion $SL_2^{\times r} \subseteq GL_{2,\det}^{\times r}$ fitting into the following exact sequence:

$$1 \to SL_2^{\times r} \longrightarrow GL_{2,\det}^{\times r} \xrightarrow{\det(-)} \mathbb{G}_m \to 1$$

Remark 3.3.15. Let $\mathcal{D}_r$ be the universal bundle over $\mathcal{B}GL_{2,\det}^{\times r}$. The quotient stack $\mathcal{B}GL_{2,\det}^{\times r}$ is the stack classifying r-tuples of rank 2 vector bundles V_i, over some scheme X, together with isomorphisms $\rho_i : \det(V_i) \xrightarrow{\sim} L$ for $L \in \mathrm{Pic}(X)$. In other words, if $\mathcal{V} := (\{V_i\}_i, L, \{\rho_i\}_i)$ denotes a collection of such objects over X, then there exists a classifying map $f_{\mathcal{V}}$ fitting in the following cartesian diagram:

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{D}_r \arrow[d] \\ \mathcal{X} \arrow[r, "f_{\mathcal{V}}"'] & \mathcal{B}GL_{2,\det}^{\times r} \end{tikzcd}$$

The map:

$$GL_{2,\det}^{\times r} \xrightarrow{\det(-)} \mathbb{G}_m$$

induces a map:

$$\widetilde{\det} : \mathcal{B}GL_{2,\det}^{\times r} \longrightarrow \mathcal{B}\mathbb{G}_m$$

Denote by $\mathcal{U}_{\mathcal{B}\mathbb{G}_m}$ the universal bundle of $\mathcal{B}\mathbb{G}_m$. The universal bundle $\mathcal{D}_r$ is given by an r-tuple of rank 2 bundles $\mathcal{V}_i$, together with isomorphisms $\det(\mathcal{V}_i) \simeq \widetilde{\det}^* \mathcal{U}_{\mathcal{B}\mathbb{G}_m}$.

The inclusion $SL_2^{\times r} \subseteq GL_{2,\det}^{\times r}$ gives rise to a natural map:

$$\mathfrak{i}_0 : BSL_2^{\times r} \longrightarrow BGL_{2,\det}^{\times r}$$

Let $\mathcal{O}(1)$ be the tautological bundle of $B\mathbb{G}_m$; with a little abuse of notation we will denote with $\mathcal{O}(1)$ also the pullback to $BGL_{2,\det}^{\times r}$ of the tautological bundle of $B\mathbb{G}_m$ along the natural map $BGL_{2,\det}^{\times r} \longrightarrow B\mathbb{G}_m$. The pullback $\mathfrak{i}_0^*\mathcal{O}(1)$ is isomorphic to the determinant of the rank 2 tautological bundles of $BSL_2^{\times r}$ and therefore it gets trivialised. Let $\vartheta : \det(\mathfrak{i}_0^*\mathcal{O}(1)) \xrightarrow{\sim} \mathcal{O}_{BSL_2^{\times r}}$ be such trivialization. For any SL-oriented spectrum A, via the (inverse of the) Thom isomorphism associated to ϑ, we have an identification map:

$$\vartheta_* : \mathrm{A}^{\bullet,\bullet}(BSL_2^{\times r}; \mathfrak{i}_0^*\mathcal{O}(1)) \xrightarrow{\sim} \mathrm{A}^{\bullet,\bullet}(BSL_2^{\times r})$$

Then the pullback $\mathfrak{i}_0^*$ induces the two following maps:

$$\mathfrak{i}_0^* : \mathrm{A}^{\bullet,\bullet}(BGL_{2,\det}^{\times r}) \longrightarrow \mathrm{A}^{\bullet,\bullet}(BSL_2^{\times r})$$

$$\mathfrak{i}_1^* := \vartheta_* \circ \mathfrak{i}_0^* : \mathrm{A}^{\bullet,\bullet}(BGL_{2,\det}^{\times r}; \mathcal{O}(1)) \longrightarrow \mathrm{A}^{\bullet,\bullet}(BSL_2^{\times r})$$

Considering $\mathfrak{i}_0^*$ and $\mathfrak{i}_1^*$ together we get the map:

$$\mathfrak{i}^* : \mathrm{A}^{\bullet,\bullet}(BGL_{2,\det}^{\times r}) \oplus \mathrm{A}^{\bullet,\bullet}(BGL_{2,\det}^{\times r}; \mathcal{O}(1)) \longrightarrow \mathrm{A}^{\bullet,\bullet}(BSL_2^{\times r}) \tag{3.5}$$

Proposition 3.3.16. *Let* A *be an* $SL[\eta^{-1}]$*-oriented ring spectrum. Then the map* (3.5)*:*

$$\mathfrak{i}^* : \mathrm{A}^{\bullet}(\mathcal{B}GL_{2,\det}^{\times r}) \oplus \mathrm{A}^{\bullet}(\mathcal{B}GL_{2,\det}^{\times r}; \mathcal{O}(1)) \longrightarrow \mathrm{A}^{\bullet}(\mathcal{B}SL_2^{\times r})$$

is an isomorphism.

Proof. Recall that we have:

$$1 \to SL_2^{\times r} \longrightarrow GL_{2,\det}^{\times r} \overset{\det(-)}{\longrightarrow} \mathbb{G}_m \to 1$$

The bundle $\mathcal{O}(1)$ is the tautological bundle of $B\mathbb{G}_m$ pulled back to $BGL_{2,\det}^{\times r}$, and we have that $BSL_2^{\times r} \simeq \mathcal{O}(1) \setminus \{0\}$. Then, we can identify $BSL_2^{\times r} \longrightarrow BGL_{2,\det}^{\times r}$ with the natural map $\mathcal{O}(1) \setminus \{0\} \to BGL_{2,\det}^{\times r}$ given by the composition of the open immersion $j : \mathcal{O}(1) \setminus \{0\} \hookrightarrow \mathcal{O}(1)$ with the projection map from $\mathcal{O}(1)$. From here, the same argument used to prove the first part of proposition 3.1.1 applies verbatim replacing BSL_n with $BSL_2^{\times r}$ and BGL_n with $BGL_{2,\det}^{\times r}$. We will sketch the proof again for more clarity, but we will leave the details to the reader.
By homotopy invariance we can identify $\mathrm{A}(BGL_{2,\det}^{\times r}) \simeq \mathrm{A}(\mathcal{O}(1))$, and we get a localization sequence:

$$\begin{aligned} \ldots \to \mathrm{A}^{a,b}\left(BGL_{2,\det}^{\times r}\right) \to \mathrm{A}^{a,b}\left(BSL_2^{\times r}\right) \longrightarrow \ldots \\ \ldots \overset{\partial_{a,b}}{\to} \mathrm{A}^{a-1,b-1}\left(BGL_{2,\det}^{\times r}; \mathcal{O}(1)\right) \overset{e(\mathcal{O}(1))\cup\cdot}{\to} \mathrm{A}^{a+1,b}\left(BGL_{2,\det}^{\times r}\right) \to \ldots \end{aligned} \tag{3.6}$$

By [Lev19, Lemma 4.3] this sequence splits into short exact sequences:

$$0 \to \mathrm{A}^{a,b}\left(BGL_{2,\det}^{\times r}\right) \to \mathrm{A}^{a,b}\left(BSL_2^{\times r}\right) \overset{\partial_{a,b}}{\to} \mathrm{A}^{a-1,b-1}\left(BGL_{2,\det}^{\times r}; \mathcal{O}(1)\right) \to 0$$

To conclude, we need to find a splitting for $\partial_{a,b}$ and the candidate map is given by:

$$\mathfrak{i}_1^* := \vartheta_* \circ \mathfrak{i}_0^* : \mathrm{A}^\bullet(BGL_{2,\det}^{\times r}; \mathcal{O}(1)) \longrightarrow \mathrm{A}^\bullet(BSL_2^{\times r})$$

We now want to prove that $\partial \circ \mathfrak{i}_1^*$ is an isomorphism. Let $\bar{\partial}$ be the boundary map in the twisted localization sequence:

$$\ldots \to \mathrm{A}^{a,b}\left(BGL_n; \mathcal{O}(-1)\right) \to \mathrm{A}^{a,b}\left(BSL_n; \mathfrak{i}_0^*\mathcal{O}(-1)\right) \xrightarrow{\bar{\partial}_{a,b}} \mathrm{A}^{a-1,b-1}\left(BGL_n\right) \to \ldots$$

Let $\langle t_{can} \rangle \in \mathrm{A}^0(BSL_2^{\times r}; \mathfrak{i}_0^*\mathcal{O}(1)) \simeq \mathrm{A}^0(BSL_2^{\times r}; \mathfrak{i}_0^*\mathcal{O}(-1))$ given by the tautological section of $\mathcal{O}(1)$. Then we have:

$$\partial \circ \mathfrak{i}_1^*(-) = \bar{\partial}(\langle t_{can} \rangle) \cup -$$

To check that multiplication by $\bar{\partial}(\langle t_{can} \rangle)$ is an isomorphism, by the Milnor's $\lim^1$-exact sequence, we can reduce to prove the claim on the finite level approximations $B_m SL_2^{\times r}$ and $B_m GL_{2,\det}^{\times r}$. By a Mayer-Vietoris argument, we can further restrict to opens of $B_m GL_{2,\det}^{\times r}$ where $\mathcal{O}(1)$ gets trivialised. But then the restriction of $\bar{\partial}(\langle t_{can} \rangle)$ becomes just η by [Ana19, Lemma 6.4], and hence the map $\bar{\partial}(\langle t_{can} \rangle) \cup -$ is invertible as claimed. □

Theorem 3.3.17 (Twisted Symplectic Splitting Principle)**.** *Let $\mathbb{E}$ be an $SL[\eta^{-1}]$-oriented ring spectrum with its canonical twisted Thom structure and let $X \in \mathbf{Sm}_{/S}$ be a smooth S-scheme. Take $L \in \mathrm{Pic}(X)$, let (E, ω^L) be a twisted symplectic bundle and let $q : \mathrm{HFlag}_X^L(E, \omega^L) \to X$ be the associated complete twisted quaternionic flag bundle. Then the map:*

$$q^* : \mathbb{E}^\bullet(X) \longrightarrow \mathbb{E}^\bullet\left(\mathrm{HFlag}_X^L(E, \omega^L)\right)$$

is injective. Moreover we have that:

$$q^* b_t^L(E, \omega^L) = \prod_{j=1}^{r} b_t(\mathcal{U}_i, \omega_i^L)$$

with $(\mathcal{U}_i, \omega_i^L)$ the universal rank 2 bundles of $\mathrm{HFlag}_X^L(E, \omega^L)$.

Proof. The first claim is just an iterated application of the Twisted Projective Bundle theorem 3.3.9. For the remaining claim, we already noticed that $q^*E \simeq \bigoplus_{i=1}^r \left(\mathcal{U}_i, \omega_i^L\right)$. Now these $(\mathcal{U}_i, \omega_i^L)$ are twisted symplectic bundles of rank 2, but that means they are just rank 2 vector bundles together with isomorphisms $\omega_i^L : \det(\mathcal{U}_i) \xrightarrow{\sim} L\big|_{\mathcal{U}_i}$. Therefore, we get a natural map:

$$p : \mathrm{HFlag}_X^L(E, \omega^L) \longrightarrow \mathcal{B}GL_{2,\det}^{\times r}$$

classifying $(\{\mathcal{U}_i\}, L, \{\omega_i\})$ (cf. remark 3.3.15). The universal bundle $\mathcal{D}_r$ of $\mathcal{B}GL_{2,\det}^{\times r}$ is given by an r-tuple of rank 2 bundles $\mathcal{V}_i$, together with isomorphisms $\rho_i : \det(\mathcal{V}_i) \xrightarrow{\sim}$

$\mathcal{O}(1)$, where $\mathcal{O}(1)$ is the pullback of the tautological bundle over $\mathcal{B}\mathbb{G}_m$. We can regard the $\mathcal{V}_i$ as twisted symplectic bundles, with twisted symplectic forms given by $\omega_{i,can} : \Lambda^2\mathcal{V}_i \simeq \det(\mathcal{V}_i) \xrightarrow{\rho_i} \mathcal{O}(1)$. Then by construction of p, we get that:

$$p^*(\mathcal{V}_i, \omega_{i,can}) \simeq (\mathcal{U}_i, \omega_i^L)$$

If we prove the claim of our theorem for the bundle $\bigoplus(\mathcal{V}_i, \omega_{i,can})$, then, via the pullback map p^*, we get the relation we want for $q^*(E, \omega^L)$. This means that it is enough to prove the following:

$$b_t^{\mathcal{O}(1)}\left(\bigoplus(\mathcal{V}_i, \omega_{i,can})\right) = \prod b_t^{\mathcal{O}(1)}((\mathcal{V}_i, \omega_{i,can}))$$

By proposition 3.3.16, we have an injective map:

$$\mathfrak{i}_1^* : \mathrm{A}^\bullet\left(\mathcal{B}GL_{2,\det}^{\times r}; \mathcal{O}(1)\right) \hookrightarrow \mathrm{A}^\bullet\left(\mathcal{B}SL_2^{\times r}\right) \tag{3.7}$$

induced by $\mathfrak{i}_0 : \mathcal{B}SL_2^{\times r} \to \mathcal{B}GL_{2,\det}^{\times r}$. For each i, the pullback $\mathfrak{i}_0^*\mathcal{V}_i$ is isomorphic to the universal rank 2 SL-vector bundle $\mathcal{W}_i$ of $\mathcal{B}SL_2^{\times r}$. Recall that the map $\mathfrak{i}_1^*$ was given by the pullback map $\mathfrak{i}_0^*$, together with the inverse of the Thom isomorphism associated to the determinant of the SL-vector bundles $\mathfrak{j}^*\mathcal{V}_i$. By injectivity of (3.7), we can therefore reduce ourselves to prove:

$$b_t\left(\bigoplus(\mathcal{W}_i, \psi_{i,can})\right) = \prod(\mathcal{W}_i, \psi_{i,can})$$

with $\psi_{i,can}$ the canonical symplectic form given by their SL-structure. But this amounts to the usual Cartan Sum formula for (untwisted) symplectic bundles, and therefore we can conclude by [PW18, Theorem 10.5]. □

Corollary 3.3.18 (Twisted Cartan Sum Formula)**.** *If we have* $\left(F, \psi^L\right) \simeq \left(E_1, \omega_1^L\right) \oplus (E_2, \omega_2^L)$ *a direct sum of* L*-twisted symplectic bundles, then:*

$$b_t^L(F, \psi^L) = b_t^L(E_1, \omega_1^L) b_t^L(E_2, \omega_2^L)$$

$$b_i^L(F, \psi^L) = \sum_{j=0}^{i} b_{i-j}^L(E_1, \omega_1^L) b_j^L(E_2, \omega_2^L)$$

Proof. Using the twisted symplectic splitting principle, we can just follow the same steps as in [PW18, Theorem 10.5]. □

Remember that the spectrum $\mathrm{BO}^{\bullet,\bullet}$ defined in [PW18] is $(8,4)$-periodic, with periodicity isomorphism:

$$\mathrm{BO}^{\bullet+8,\bullet+4} \xrightarrow{\cdot\cup\Sigma^{8,4}\gamma} \mathrm{BO}^{\bullet,\bullet}$$

with $\gamma \in \mathrm{BO}^{-8,-4}(pt)$ the element corresponding to $\mathbb{1} \in \mathrm{BO}^{0,0}(pt)$ under the periodicity isomorphism:

$$\mathrm{BO}^{0,0}(pt) \simeq \mathrm{GW}_0^{[0]}(pt) \simeq \mathrm{GW}_0^{[-4]}(pt) \simeq \mathrm{BO}^{-8,-4}(pt)$$

and $\mathrm{GW}_i^{[n]}$ are the higher Grothendieck-Witt groups of [Sch10]. We will call γ the *Bott element of Hermitian K-Theory.* This element will induce the 4 periodicity on KW once we invert the η-map.

Now to explicitly compute the Euler classes of $\widetilde{\mathcal{O}^{\pm}}(m)$, we need a twisted version of [Ana17, Lemma 8.2]:

Lemma 3.3.19 (Ananyevskiy)**.** *Let E_1, E_2, E_3 be rank 2 bundles over some scheme $X \in \mathbf{Sm}_{/S}$, with the determinants L_1, L_2, L_3 respectively, together with their canonical twisted symplectic structures; let $E := E_1 \otimes E_2 \otimes E_3$ be the $L_1 \otimes L_2 \otimes L_3$-twisted symplectic bundle of rank 8, with the induced twisted symplectic structure. Let $L := L_1 \otimes L_2 \otimes L_3$, $\xi_i := b_1^L(E_i) \in \mathrm{KW}^{4,2}(X; L_i)$ and denote with $\sigma(n_1, n_2, n_3)$ the sum of all the monomials lying in the orbit of $\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3}$ under the action of S_3. Then:*

$$
\begin{aligned}
b_1^L(E) &= \gamma\sigma(1,1,1)\\
b_2^L(E) &= \gamma\sigma(2,2,0) - 2\sigma(2,0,0)\\
b_3^L(E) &= \gamma\sigma(3,1,1) - 8\sigma(1,1,1)\\
b_4^L(E) &= \gamma\sigma(2,2,2) + \sigma(4,0,0) - 2\sigma(2,2,0)
\end{aligned}
$$

Remark 3.3.20. Notice that (mod 2), the sequences (n_1, n_2, n_3) (and their permutations) appearing in the formulas above for $b_i^L(E)$ are the same. This means that the terms land in the same twists, under the canonical identifications:

$$\mathrm{KW}^{\bullet}(X; L_1^{a_1} \otimes L_2^{a_2} \otimes L_3^{a_3}) \simeq \mathrm{KW}^{\bullet}(X; L_1^{b_1} \otimes L_2^{b_2} \otimes L_3^{b_3})$$

for:

$$(a_1, a_2, a_3) \equiv (b_1, b_2, b_3) \quad (mod\ 2)$$

Proof of 3.3.19. It is enough to prove the theorem for $X = \mathcal{B}GL_2^{\times 3}$, but again by the Künneth formula 3.2.5 and proposition 3.1.1 we can reduce to $\mathcal{B}SL_2^{\times 3}$ case. Taking the finite level approximation $\mathcal{B}_m SL_2$ we can use Ananyevskiy's result [Ana17, Lemma 8.2] and since Witt theory for $\mathcal{B}SL_2$ satisfies the Mittag-Leffler condition by [Ana15, Theorem 9] (or by proposition 3.2.1), taking the limit of the theories on the finite level approximation we get the desired result for BSL_2. □

3.4 Euler classes for $\widetilde{\mathcal{O}}^{\pm}(m)$

Before actually computing the Euler classes we are interested in, let us recall some facts on about the rank 2 vector bundles of BN. Let us identify:

$$
\begin{array}{cccc}
\iota: & \mathbb{G}_m & \hookrightarrow & T_{SL_2}\\
& t & \mapsto & \begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}
\end{array}
$$

where T_{SL_2} is the torus in SL_2. We will denote as done before:

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N$$

Definition 3.4.1. For $m \geq 1$ an integer, let $F^{\pm}(m) := \Bbbk^2$, then we define representations $(F^{\pm}(m), \rho_m^{\pm})$ by:

$$\rho_m^{\pm}(\iota(t)) := \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}$$

$$\rho_m^{\pm}(\sigma) := \pm \begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix}$$

We call (F_0, ρ_0) the trivial one-dimensional representation, and (F_0, ρ_0^-) the one defined by $\rho_0^-(t) = 1$ and $\rho_0^-(\sigma) = -1$.
We denote with $p^{(\pm m)} : \widetilde{\mathcal{O}}^{\pm}(m) \longrightarrow BN$ the rank 2 vector bundle:

$$p^{(\pm m)} : F_m^{\pm} \times^N ESL_2 \longrightarrow ESL_2/N = BN$$

corresponding to $\rho_m^{\pm}$, with zero section $s_0^{(\pm m)} : BN \longrightarrow \widetilde{\mathcal{O}}^{\pm}(m)$ and $\mathrm{th}^{(\pm m)} \in \mathrm{KW}\left(\widetilde{\mathcal{O}}^{\pm}(m), \det(\widetilde{\mathcal{O}}^{\pm}(m))\right)$ the Thom class defined as $(s_0^{(\pm m)})_*(\mathbb{1})$.

Remark 3.4.2. We recall that $\mathrm{Pic}\,(BN) \simeq \mathbb{Z}/2\mathbb{Z}$ can be generated by $\gamma := \widetilde{\mathcal{O}}^{-}(0)$. For the one dimensional representation $\det(\rho_m^{\pm})$ we choose the generator given by $e_1 \wedge e_2$, with $e_1 = (1,0)$ and $e_2 = (0,1)$ the standard basis of $\Bbbk^2$. Then we have canonical isomorphisms $\det\left(\widetilde{\mathcal{O}}^{\pm}(m)\right) \simeq \mathcal{O}_{BN}$ for m odd and $\det\left(\widetilde{\mathcal{O}}^{\pm}(m)\right) \simeq \gamma$ for $m > 0$ even.

Given a triple tensor product of rank two bundles $U_1 \otimes U_2 \otimes U_3$, we denote their bases as $\{e_1, e_2\}, \{f_1, f_2\}, \{g_1, g_2\}$. We have the isomorphism:

$$\widetilde{\mathcal{O}}^{+}(1) \otimes \widetilde{\mathcal{O}}^{+}(1) \otimes \widetilde{\mathcal{O}}^{+}(1) \simeq \widetilde{\mathcal{O}}^{+}(3) \oplus \widetilde{\mathcal{O}}^{+}(1)^{\oplus 3}$$

where the base chosen for this identification is given by dual pairs $\{v_i, w_i\}$ that are perpendicular to all other vectors:

$$\begin{aligned} v_1 &:= e_1 \otimes f_1 \otimes g_1 & w_1 &= e_2 \otimes f_2 \otimes g_2 \\ v_2 &:= -e_1 \otimes f_1 \otimes g_2 & w_2 &= e_2 \otimes f_2 \otimes g_1 \\ v_3 &:= -e_1 \otimes f_2 \otimes g_1 & w_3 &= e_2 \otimes f_1 \otimes g_2 \\ v_4 &:= -e_2 \otimes f_1 \otimes g_1 & w_4 &= e_1 \otimes f_2 \otimes g_2 \end{aligned}$$

In a similar way, for any $m > 1$ we have:

$$\widetilde{\mathcal{O}}^{+}(m) \otimes \widetilde{\mathcal{O}}^{+}(1) \otimes \widetilde{\mathcal{O}}^{+}(1) \simeq \widetilde{\mathcal{O}}^{+}(m+2) \oplus \widetilde{\mathcal{O}}^{+}(m)^{\oplus 2} \oplus \widetilde{\mathcal{O}}^{+}(m-2) \tag{3.8}$$

where the base is given by:

$$\begin{array}{llll} v_1 := & e_1 \otimes f_1 \otimes g_1 & w_1 = e_2 \otimes f_2 \otimes g_2 \\ v_2 := & -e_1 \otimes f_1 \otimes g_2 & w_2 = e_2 \otimes f_2 \otimes g_1 \\ v_3 := & -e_1 \otimes f_2 \otimes g_1 & w_3 = e_2 \otimes f_1 \otimes g_2 \\ v_4 := & e_1 \otimes f_2 \otimes g_2 & w_4 = e_2 \otimes f_1 \otimes g_1 \end{array}$$

We also have:

$$\widetilde{\mathcal{O}}^+(2) \otimes \widetilde{\mathcal{O}}^+(2) \otimes \widetilde{\mathcal{O}}^+(1) \simeq \widetilde{\mathcal{O}}^+(5) \oplus \widetilde{\mathcal{O}}^+(3)^{\oplus 2} \oplus \widetilde{\mathcal{O}}^-(1) \tag{3.9}$$

with with bases:

$$\begin{array}{llll} v_1 := & e_1 \otimes f_1 \otimes g_1 & w_1 = e_2 \otimes f_2 \otimes g_2 \\ v_2 := & -e_1 \otimes f_1 \otimes g_2 & w_2 = e_2 \otimes f_2 \otimes g_1 \\ v_3 := & e_1 \otimes f_2 \otimes g_1 & w_3 = e_2 \otimes f_1 \otimes g_2 \\ v_4 := & e_2 \otimes f_1 \otimes g_1 & w_4 = e_1 \otimes f_2 \otimes g_2 \end{array}$$

Proposition 3.4.3. *We have the following:*

1. *For any m:*
$$e(\widetilde{\mathcal{O}}^-(m)) = -e(\widetilde{\mathcal{O}}^+(m))$$

2. *For $m > 1$ the recurrence relation:*
$$b_1(\widetilde{\mathcal{O}}^+(m+2)) = (\gamma e^2 - 2) b_1(\widetilde{\mathcal{O}}^+(m)) - b_1(\widetilde{\mathcal{O}}^+(m-2))$$
In particular $b_1(\widetilde{\mathcal{O}}^+(m))$ is always a multiple of $e := e(\widetilde{\mathcal{O}}^+(1))$ for m odd and $\tilde{e} := e(\widetilde{\mathcal{O}}^+(2))$ for m even respectively.

3. *For $m = 2n+1$:*
$$b_1\left(\widetilde{\mathcal{O}}^+(m)\right) = \sum_{k=0}^{n} (-1)^{n-k} \alpha_{k,n} \gamma^k e^{2k+1} \tag{3.10}$$
with $e = e(\widetilde{\mathcal{O}}^+(1))$ and $\alpha_{k,n}$ defined by the recurrence relation:
$$\begin{cases} \alpha_{0,n} = 2n+1 & \forall\ n \geq 0 \\ \alpha_{k,n} = \sum_{j=1}^{n} j \cdot \alpha_{k-1,n-j} & \forall\ n \geq k > 0 \\ \alpha_{k,n} = 0 & \textit{else} \end{cases}$$

4. *For $m = 2n$:*

$$b_1\left(\widetilde{\mathcal{O}}^+(m)\right) = \tilde{e}\left(\sum_{k=0}^{n-1}(-1)^{n-k+1}\beta_{k,n}\gamma^k e^{2k}\right) \tag{3.11}$$

with $e = e(\widetilde{\mathcal{O}}^+(1))$, $\tilde{e} = e(\widetilde{\mathcal{O}}^+(2))$ and $\beta_{k,n}$ defined by the recurrence relation:

$$\begin{cases} \beta_{0,n} = n & \forall\ n \geq 0 \\ \beta_{k,n} = \sum_{j=1}^{n} j \cdot \beta_{k-1,n-j} & \forall\ n \geq k > 0 \\ \beta_{k,n} = 0 & \textit{else} \end{cases}$$

5. *We also have:*

$$\tilde{e}^2 = -4e + \gamma e^4 \tag{3.12}$$

Proof. First let us notice that from our computations in Chapter 2, it follows that we can compute the Witt theory of BN by its finite level approximations B_mN. In particular, we get that the Cartan Sum Formula and Ananyevskiy's Lemma hold true for BN too.
The first assertion of the proposition follows from the fact that ρ_m^+ and ρ_m^- are isomorphic as representation through $(x, y) \mapsto (-x, y)$, but this map induces a (-1) on the determinant, so we get $e(\widetilde{\mathcal{O}}^-(m)) = -e(\widetilde{\mathcal{O}}^+(m))$.
The decomposition eq. (3.8) and the Cartan Sum Formula 3.3.18, for $m > 1$, gives us:

$$b_1\left(\widetilde{\mathcal{O}}^+(m) \otimes \widetilde{\mathcal{O}}^+(1)^{\otimes 2}\right) = b_1(\widetilde{\mathcal{O}}^+(m+2)) + 2b_1(\widetilde{\mathcal{O}}^+(m)) + b_1(\widetilde{\mathcal{O}}^+(m))$$

while for $m = 1$ we have:

$$b_1\left(\widetilde{\mathcal{O}}^+(1) \otimes \widetilde{\mathcal{O}}^+(1)^{\otimes 2}\right) = b_1(\widetilde{\mathcal{O}}^+(3)) + 3b_1(\widetilde{\mathcal{O}}^+(1))$$

At the same time, using Ananyevskiy's lemma 3.3.19 we have:

$$b_1(\widetilde{\mathcal{O}}^+(m) \otimes \widetilde{\mathcal{O}}^+(1)^{\otimes 2}) = \gamma b_1(\widetilde{\mathcal{O}}^+(m))b^2$$

Putting this all together we get:

$$b_1(\widetilde{\mathcal{O}}^+(3)) = -3e + \gamma e^3$$

and for $m > 1$:

$$b_1(\widetilde{\mathcal{O}}^+(m+2)) = (\gamma e^2 - 2)b_1(\widetilde{\mathcal{O}}^+(m)) - b_1(\widetilde{\mathcal{O}}^+(m-2)) \tag{3.13}$$

From this by induction is not difficult to see that e and $\tilde{e}$ divides $b_1(\widetilde{\mathcal{O}}^+(m+2))$ for odd or even $m > 1$ respectively, but it will be even more clear from the recursive formulas we are going to show.

Now let us consider $m = 2n+1$. We want to prove the recursive formula eq. (3.10). We just saw the formula for $m = 3$. Now we proceed by induction on n, let us suppose we know the formula for $m = 2n+1$ and we want to prove it for $m+2 = 2(n+1)+1 = 2n+3$. Using the induction hypothesis on eq. (3.13), we have:

$$
\begin{aligned}
& b_1(\widetilde{\mathcal{O}}^+(m+2)) = (\gamma b^2 - 2) b_1(\widetilde{\mathcal{O}}^+(m)) - b_1(\widetilde{\mathcal{O}}^+(m-2)) = \\
&= \sum_{k=0}^{n} (-1)^{n-k} \alpha_{k,n} \gamma^{k+1} b^{2k+3} - 2 \sum_{k=0}^{n} (-1)^{n-k} \alpha_{k,n} \gamma^k b^{2k+1} - \sum_{k=0}^{n-1} (-1)^{n-1-k} \alpha_{k,n-1} \gamma^k b^{2k+1} = \\
&= (-1)^{n+1} (-\alpha_{0,n-1} + 2\alpha_{0,n}) b + \sum_{k=1}^{n} (-1)^{n+1-k} (-\alpha_{k,n-1} + 2\alpha_{k,n}) \gamma^k b^{2k+1} + \\
&\quad + \sum_{h=1}^{n+1} (-1)^{n+1-h} \alpha_{h-1,n} \gamma^h b^{2h+1} + \alpha_{n,n} \gamma^{n+1} b^{2n+3} = \\
&= (-1)^{n+1} (2n+3) b + \sum_{k=1}^{n} (-1)^{n+1-k} \left(-\alpha_{k,n-1} + 2\alpha_{k,n} + \alpha_{k-1,n}\right) \gamma^k b^{2k+1} + \alpha_{n,n} \gamma^{n+1} b^{2n+3}
\end{aligned}
$$

and it is easy to see that the coefficients $\alpha_{0,n+1}$ and $\alpha_{n+1,n+1}$ are already of the desired form. The only thing left to check is that:

$$-\alpha_{k,n-1} + 2\alpha_{k,n} + \alpha_{k-1,n}$$

is actually:

$$\sum_{j=1}^{n+1} j \cdot \alpha_{k-1,n+1-j}$$

for $k = 1, \ldots, n$. using the induction hypothesis:

$$
\begin{aligned}
& -\alpha_{k,n-1} + 2\alpha_{k,n} + \alpha_{k-1,n} = \\
&= -\sum_{j=1}^{n-1} j \cdot \alpha_{k-1,n-1-j} + 2 \sum_{j=1}^{n} j \cdot \alpha_{k-1,n-j} + \alpha_{k-1,n} = \\
&= -\sum_{j=1}^{n-1} j \cdot \alpha_{k-1,n-1-j} + 2 \sum_{h=1}^{n-1} (h+1) \cdot \alpha_{k-1,n-1-h} + 2\alpha_{k-1,n-1} + \alpha_{k-1,n} = \\
&= \sum_{j=1}^{n-1} (-j + 2j + 2) \cdot \alpha_{k-1,n-1-j} + 2 \cdot \alpha_{k-1,n-1} + \alpha_{k-1,n} = \\
&= \sum_{j=0}^{n-1} (j+2) \cdot \alpha_{k-1,n-1-j} + 2 \cdot \alpha_{k-1,n+1-2} + \alpha_{k-1,n+1-1} = \sum_{r=1}^{n+1} r \cdot \alpha_{k-1,n+1-r}
\end{aligned}
$$

And this completes the proof of the claim for the odd case $m = 2n+3$.

For the even case $m = 2n$ the proof is basically the same and we will leave it as an exercise.

Lastly, again using Ananyevskiy's lemma 3.3.19 and the Cartan sum formula on the decomposition from eq. (3.9), we have:

$$\gamma\tilde{e}^2 e = b_1\left(\widetilde{\mathcal{O}}^+(2)^{\otimes 2}\otimes\widetilde{\mathcal{O}}^+(1)\right) = b_1(\widetilde{\mathcal{O}}^+(5)) + b_1\left(\widetilde{\mathcal{O}}^+(3)\right) + b_1\left(\widetilde{\mathcal{O}}^-(1)\right)$$

And from what we proved before, this means:

$$\gamma\tilde{e}^2 e = (5e - 5\gamma e^3 + \gamma e^5) + (-3e + \gamma e^3) - 2e = -4\gamma e^3 + \gamma e^5$$

but since γ is an isomorphism, multiplication by γe is injective on the first summand of $\mathrm{KW}^\bullet(BN) \simeq \mathrm{KW}^\bullet(S)[\![e]\!] \oplus q_0 \cdot \mathrm{KW}^\bullet(S)$ and hence:

$$\tilde{e}^2 = -4e^2 + \gamma e^4$$

□

Remark 3.4.4. Here is a table of easy examples of coefficients $\alpha_{k,n}$ of the odd Euler classes:

n \ k	0	1	2	3	4	5	6
0	1						
1	3	1					
2	5	5	1				
3	7	14	7	1			
4	9	30	27	9	1		
5	11	55	77	44	11	1	
6	13	91	182	156	65	13	1

It is not hard to show that $\alpha_{1,n} = \sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}, \alpha_{n-2,n} = n+1(2n+1), \alpha_{n-1,n} = 2n+1, \alpha_{n,n} = 1$. Recall that the coefficient of the lowest degree term in the recursive formula will be $(-1)^n(2n+1) = (-1)^n \cdot m$ for $m = 2n+1$, so the lowest degree term of $e^{\mathrm{KW}}\left(\widetilde{\mathcal{O}}^+(m)\right)$ will have the same form of the Witt cohomology Euler class computed in [Lev19, Theorem 7.1].
For even Euler classes, the table of the first coefficients looks like:

n \ k	0	1	2	3	4	5	6
1	1						
2	2	1					
3	3	4	1				
4	4	10	6	1			
5	5	20	21	8	1		
6	6	35	56	36	10	1	
7	7	56	126	120	55	12	1

Again the coefficient of the lowest degree term in the recursive formula will be $(-1)^{n-1}n = (-1)^{n-1}\frac{m}{2}$ for $m = 2n$, so the lowest degree term of $e^{\mathrm{KW}}\left(\widetilde{\mathcal{O}}^{+}(m)\right)$ will have the same form as the formula obtained for the Witt cohomology Euler class in [Lev19, Theorem 7.1]. But this is not just a coincidence:

Corollary 3.4.5 ([Lev19, Theorem 7.1])**.** *Let $H\mathcal{W}$ be the spectrum representing Witt-sheaf cohomology. Denote by $e^{H\mathcal{W}} := e^{H\mathcal{W}}\left(\widetilde{\mathcal{O}}^{+}(1)\right)$ and $\tilde{e}^{H\mathcal{W}} := e^{H\mathcal{W}}\left(\widetilde{\mathcal{O}}^{+}(2)\right)$. Then:*

$$e^{H\mathcal{W}}\left(\widetilde{\mathcal{O}}^{+}(m)\right) = \begin{cases} (-1)^{\frac{m-1}{2}} \cdot m \cdot e^{H\mathcal{W}} & \textit{for } m \textit{ odd} \\ (-1)^{\frac{m+2}{2}} \cdot \frac{m}{2} \cdot \tilde{e}^{H\mathcal{W}} & \textit{for } m \textit{ even} \end{cases}$$

Proof. By [FH23, Remark 6.7], we know that the formal ternary law of Hermitian K-theory recovers the computations of [Ana17, Lemma 8.2]. Setting $\gamma = 0$, we recover the formal ternary law for Witt sheaf cohomology by [DF21, Theorem 3.3.2]. Using the formulas for Witt sheaf cohomology obtained from lemma 3.3.19 for $\gamma = 0$, we can proceed by induction as we did for KW in proposition 3.4.3. In this way we recover the (closed form) formulas of [Lev19, Theorem 7.1] as claimed. □

Remark 3.4.6. Clearly $\widetilde{\mathcal{O}}^{+}(1) \simeq p^{*}E_2$ where $p : BN \to BSL_2$ and E_2 is the bundle arising from the universal bundle of $\mathcal{B}SL_2$, so the class $e := e\left(\widetilde{\mathcal{O}}^{+}(1)\right)$ is the free generator of $\mathrm{KW}^{\bullet}(BN)$ by corollary 2.3.6. Similarly the class $\tilde{e} := e\left(\widetilde{\mathcal{O}}^{+}(2)\right)$ is the free generator of $\mathrm{KW}^{\bullet}(BN; \gamma_N)$ again by corollary 2.3.6 and [Lev19, Lemma 6.1].
The reason we are so interested in these rank 2 vector bundles $\widetilde{\mathcal{O}}^{\pm}(m)$ is that the $\rho_m^{\pm}$'s classify irreducible representations of BN with a choice of the determinant. This together with [Ana15, *Splitting Principle, §9*], proposition 3.1.1, and the structure theorem corollary 2.3.6, tells us that the characteristic classes of these bundles, and their symmetric powers, will recover all the characteristic classes of any vector bundle on BN.

Remark 3.4.7. Let us notice that given any rank two vector bundle $E \to X$ over some X, we have $e(\mathrm{Sym}^k E) = 0 \in \mathrm{KW}^{k+1}(X; \det^{-1} \mathrm{Sym}^k E)$ for k even since $\mathrm{Sym}^k E$ will have rank $k+1$ and thus we can apply [Lev19, Lemma 4.3]. For $k = 2r+1$ odd, $e(\mathrm{Sym}^k E)$, seen as a polynomial in $e := e(\widetilde{\mathcal{O}}^{+}(1))$, will have as lowest degree term $k!! \cdot e^{r+1}$, where $k!! := k \cdot (k-2) \cdot \ldots \cdot 3 \cdot 1$. The proof of this fact is basically the same proof used in [Lev19, Theorem 8.1] using the concrete description we have for the lowest degree term of $e(\widetilde{\mathcal{O}}^{+}(m))$ given by proposition 3.4.3 and the fact we remarked above that they recover the same computations done in [Lev19, Theorem 7.1]).
Moreover it is worth noticing that this correlation between Euler classes in Witt theory and Witt cohomology is reminiscent of a Grothendieck-Riemann-Roch formula for the negative Borel character of [DF21]. By [Bac18, Corollary 38], a good heuristic principle is to look in classical stable homotopy for GRR formulas. For example we expect formulas similar to the ones in [LM89, Prop. 11.14, 11.22, 11.24] (and also

[Hu22, Thm. 2.23]) to hold true in the motivic context. We hope to come back to these questions in the future.

Chapter 4

Virtual Localization

4.1 Atiyah-Bott Localization

We are now ready to extend the Atiyah-Bott localization theorem to any $SL[\eta^{-1}]$-oriented ring spectrum A, closely following [Lev22a]. Once again, we remind the reader that we will freely interchange the quotient stack $\mathcal{B}N$ with its ($\mathbb{A}^1$-equivalent) ind-scheme approximation BN, implicitly using proposition 1.4.30.

Disclaimer 4.1.1. To avoid burdening the notation too much, in the following chapter we will drop the G from the upper- and lower-superscripts of the equivariant operations on Borel-Moore homology and cohomology.

Definition 4.1.2. Let $K \supset \Bbbk$ be a field, $\chi : \mathbb{G}_m \to \mathbb{G}_m$ a $\mathbb{G}_m$-character such that $\chi(-Id) = 1$, and take $\lambda \in K^\times$. Define the subgroup scheme $\Lambda(\chi, \lambda) \subseteq N_K$ as:

$$\Lambda(\chi, \lambda) := \chi^{-1}(1) \amalg \chi^{-1}(\lambda^{-1}) \cdot \sigma$$

Definition 4.1.3. As done in [Lev22a], consider again $K \supset \Bbbk$ a field, a character $\chi : \mathbb{G}_m \to \mathbb{G}_m$ and the following types of N-homogenous spaces X over K (where K_X will denote the N-invariants of $\mathcal{O}_X(X)$):

(a) $X = \big(N/\chi^{-1}(1)\big)_K$;

(b) Suppose $\chi(-Id) = 1$ and let $X = N_K/\Lambda(\chi, \lambda)$ for some $\lambda \in K^\times$;

(c±) Let $K' \supset K$ a degree two extension:

- (c+) Suppose $\chi(-Id) = 1$ and take some $\lambda \in K^\times$. Let τ be the conjugation of K' over K. We can choose an isomorphism of $\mathbb{G}_m$-homogeneous spaces $\mathbb{G}_m/\chi^{-1}(1)_{K'} \simeq \mathbb{G}_{m,K'}$ inducing the isomorphism:

$$\big(N/\chi^{-1}(1)\big)_{K'} \simeq \mathbb{G}_{m,K'} \amalg \sigma \cdot \mathbb{G}_{m,K'}$$

Let $\rho(\sigma^i \cdot x) := \sigma \cdot \sigma^i \cdot \frac{\lambda}{x}$, for $i = 0, 1$, be the K'-automorphism of $\big(N/\chi^{-1}(1)\big)_{K'}$. Composing with τ, we get the F-automorphism, $\tau \circ \rho$ which gives us a $\mathbb{Z}/2\mathbb{Z}$-action on $\big(N/\chi^{-1}(1)\big)_{K'}$. The left N-action on N descends to a left action on the quotient and hence we get an N-homogeneous space:

$$X := \big(N/\chi^{-1}(1)\big)_{K'}/\langle \tau \circ \rho \rangle$$

(c-) If $\chi(-Id) = -1$, we can take some $\lambda_0 \in K^\times$, then choose a generator $\sqrt{a}$ for K' over K (with $a \in K^\times$) and take $\lambda := \lambda_0 \sqrt{a}$. With this λ we can define as before $\rho(\sigma^i \cdot x) := \sigma \cdot \sigma^i \cdot \frac{\lambda}{x}$, for $i = 0, 1$, and consider the N-homogeneous space:

$$X := \big(N/\chi^{-1}(1)\big)_{K'}/\langle \tau \circ \rho \rangle$$

These will be all the N-homogeneous spaces we have to consider, using a special case of [Lev22a, Lemma 7.4]:

Lemma 4.1.4 ([Lev22a, Lemma 7.4]). *Let X be an N-homogeneous space. Suppose $\mathbb{G}_m$ acts non-trivially on X and that X is smooth over $k_X := H^0(X, \mathcal{O}_X)^N$. Let $Y := X/\mathbb{G}_m$ with its induced structure as a $\mathbb{Z}/2\mathbb{Z}$-homogeneous space over k_X. Then:*

1. *As a Y-scheme $X \simeq \mathbb{G}_{m,Y}$;*

2. *Fix a closed point $y_0 \in Y$, let $A \subseteq \mathbb{Z}/2\mathbb{Z}$ be the isotropy group of y_0, let B be the kernel of the action map $A \to \mathrm{Aut}_{k_X}(k_X(y_0))$. Then there are three cases:*

(Case a) $B = A = \{0\}$*;*

(Case b) $A = B = \mathbb{Z}/2\mathbb{Z}$*;*

(Case c) $A = \mathbb{Z}/2\mathbb{Z}$ *and* $B = \{0\}$.

3. *In (Case a) X is a homogeneous space of type (a); in case (b) then X is of type (b); in case (c) then X is of type $(c\pm)$, depending on whether $\chi(-Id) = \pm 1$, and $\kappa(y_0)$ is the degree two extension of $k_{X'}$ referred to in definition 4.1.3.*

Via the N-action, the map $\pi_X : X \to \mathrm{Spec}(k_X) \to \mathrm{Spec}(\Bbbk)$ induces a map:

$$\pi_X^N : \mathcal{X} = \left[{}^{X}/_{N}\right] \to \mathcal{B}N$$

For any $\mathrm{A} \in \mathrm{SH}(\Bbbk)$, this induces the pullback maps:

$$(\pi_X^N)^* : \mathrm{A}^\bullet(BN) \longrightarrow \mathrm{A}_N^\bullet(X)$$

$$(\pi_X^N)^* : \mathrm{A}^\bullet(BN; \gamma_N) \longrightarrow \mathrm{A}_N^\bullet(X; \pi_X^* \gamma_N)$$

where γ_N is the generator of $\mathrm{Pic}(BN)$.

Notation 4.1.5. Similarly to remark 3.4.6, we will denote by e the Euler class of $(\pi_X^N)^* \widetilde{\mathcal{O}}^+(1)$ living in $\mathrm{A}^2_N(X) \simeq \mathrm{A}\left(X \times^N EN\right)$. We will omit the appropriate pullbacks from the twists in our equivariant theories, if it is clear from the context what twist should actually be used; the same will happen for pullbacks on bundles and associated Euler classes.

Lemma 4.1.6 ([Lev22a, Lemma 7.6])**.** *Let X, χ be as in lemma 4.1.4. Suppose $\chi(t) = t^m$ for some integer $m \geq 1$. Let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then we have:*

(a) In case (a), e goes to zero in $\mathrm{A}^\bullet_N(X)$.

(b) In case (b), m is even and $e(\widetilde{\mathcal{O}}(m))$ goes to zero in $\mathrm{A}^\bullet_N(X; \pi_X^* \gamma_N)$.

(c+) In case (c+), i.e. m even, $e(\widetilde{\mathcal{O}}(m))$ goes to zero in $\mathrm{A}^\bullet_N(X; \pi_X^* \gamma_N)$.

(c-) In case (c−), i.e. m odd, $e(\widetilde{\mathcal{O}}(2m))$ goes to zero in $\mathrm{A}^\bullet_N(X; \pi_X^* \gamma_N)$.

Remark 4.1.7. In the cases (*b*) and (*c*+), the pullback of γ_N gets trivialised. Hence the Euler classes, considered in the previous lemma, live in $\mathrm{A}^\bullet_N(X)$.

Proof. The same proof used in [Lev22a, Lemma 7.6] works verbatim for a generic A, since only the vanishing of Euler classes for odd rank vector bundles and multiplicativity of Euler classes was used there, and these properties still hold true for any $SL[\eta^{-1}]$-oriented spectrum. □

Notation 4.1.8. From now on, we will denote the Euler classes of the $\widetilde{\mathcal{O}}^\pm(m)$ as $\tilde{e}^\pm(m) := e\left(\widetilde{\mathcal{O}}^\pm(m)\right)$.

Remark 4.1.9. Consider $X \in \mathbf{Sch}^N_{/\Bbbk}$, $L \in \mathrm{Pic}(X)$ an N-linearised line bundle, and let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum.. Then products in A-cohomology induce a structure of graded commutative ring on $\mathrm{A}^\bullet_N(X)$ and a structure of $\mathrm{A}^\bullet_N(X)$-module on $\mathrm{A}^\bullet_N(X; L)$. Identifying $\mathrm{A}^\bullet_N(X; L^{\otimes 2}) \simeq \mathrm{A}^\bullet_N(X)$, we get a commutative graded product:

$$\mathrm{A}^\bullet_N(X; L) \otimes \mathrm{A}^\bullet_N(X; L) \longrightarrow \mathrm{A}^\bullet_N(X)$$

For $s \in \mathrm{A}^{2\bullet}_N(X)$, we say that an element $x \in \mathrm{A}^{2\bullet}_N(X; L)[s^{-1}]$ is invertible if there exists $y \in \mathrm{A}^{2\bullet}_N(X; L)[s^{-1}]$ such that $xy = 1 \in \mathrm{A}^{2\bullet}_N(X)[s^{-1}]$. A homogeneous element $x \in \mathrm{A}^{2\bullet}_N(X; L)$ is invertible if and only if $x^2 \in \mathrm{A}^{2\bullet}_N(X)$ is invertible in the usual sense as an element of the commutative graded ring.

Lemma 4.1.10 (Key Lemma)**.** *Let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. For any odd integer m, the Euler class $\tilde{e}^+(m)$ is invertible in:*

$$\mathrm{A}^\bullet(\mathcal{B}N)\left[(m \cdot e)^{-1}\right]$$

For any even integer $m = 2n$, the Euler class $\tilde{e}^+(m)$ is invertible in:

$$\mathrm{A}^\bullet(\mathcal{B}N; \gamma_N)\left[(m \cdot e)^{-1}\right]$$

Proof. For this proof we will make use of the ind-scheme BN approximating $\mathcal{B}N$, by proposition 1.4.30 we already know that all the computations will carry over to $\mathcal{B}N$. To prove our claim, it is enough to consider the universal case MSL_η: the general case will follow considering the SL-orientation map $\varphi_\eta^A : \mathrm{MSL}_\eta \to A$ associated to the $SL[\eta^{-1}]$-oriented ring spectrum.
Since we are working in $\mathrm{SH}(\Bbbk)$, by [BH21a, Theorem 8.8], we have:

$$\mathrm{MSL}_\eta(\Bbbk) \simeq W(\Bbbk)[y_2, y_4, y_6, \dots]$$

where y_{2i} has degree $-4i$. But then by [Ana15, Theorem 10] and our proposition 2.2.16, we get:

$$\mathrm{MSL}_\eta^\bullet(BN) \simeq (W(\Bbbk)[y_2, y_4, \dots]) [\![e]\!] \oplus q_0^{\mathrm{MSL}} \cdot W(\Bbbk)[y_2, y_4, \dots]$$

Let us suppose m is odd. Since $\tilde{e}^+(m)$ has degree two, we have:

$$\tilde{e}^+(m) = \sum_{i=0}^{\infty} a_i \cdot e^{2i+1}$$

where $a_i = a_i(y_2, y_4, \dots) \in (W(\Bbbk)[y_2, y_4, \dots])^{-4i}$. Notice that $a_0 \in W(\Bbbk)$, since it has degree zero. The map $\varphi_\eta^{H\mathcal{W}} : \mathrm{MSL}_\eta \to H\mathcal{W}$, induced by the SL-orientation, is a map of ring spectra given by sending each $y_i \mapsto 0$. Thus we have:

$$\varphi_\eta^{H\mathcal{W}}(a_0 \cdot e) = \overline{a_0} \cdot \varphi_\eta^{H\mathcal{W}}(e^{\mathrm{MSL}_\eta})$$

where $\overline{a_0} := \varphi_\eta^{H\mathcal{W}}(a_0) = a_0(0, 0, \dots)$. But comparing the Euler classes in MSL_η and $H\mathcal{W}$, we get:

$$\varphi_\eta^{H\mathcal{W}}(\tilde{e}^+(m)) = \overline{a_0} \cdot e^{H\mathcal{W}} = \pm m \cdot e^{H\mathcal{W}}$$

thus $a_0(0, 0, \dots) = \overline{a_0} = \pm m$ by the computations made in [Lev19, Theorem 7.1]. But a_0, as we said before, as degree zero, hence $a_0 = \pm m$. But this implies that, in $\mathrm{MSL}_\eta^\bullet(BN)\left[m^{-1}\right]$, we can write:

$$\tilde{e}^+(m) = \pm m \cdot e(1 + \sum_{j=1} b_j \cdot e^j)$$

and $(1 + \sum_{j=1} b_j \cdot e^j)$ is already invertible in $\mathrm{MSL}_\eta^\bullet(BN)\left[m^{-1}\right]$. Hence $\tilde{e}^+(m)$ is invertible in $\mathrm{MSL}_\eta^\bullet(BN)\left[(m \cdot e)^{-1}\right]$. The even case $m = 2n$ is completely analogous. We still have that:

$$\mathrm{MSL}_\eta^\bullet(BN; \gamma_N) \simeq e(\mathcal{T}) \cdot (W(\Bbbk)[y_2, y_4, \dots]) [\![e]\!] \oplus q_1^{\mathrm{MSL}} \cdot W(\Bbbk)[y_2, y_4, \dots]$$

by proposition 2.2.16 (and [Ana15, Theorem 10]), where $\mathcal{T}$ is the pullback of the tangent bundle of $\mathbb{P}(\mathrm{Sym}^2(F)) \times^{SL_2} ESL_2$ relative to BSL_2. From [Lev19, Lemma

6.1], we know $\tilde{e} := \tilde{e}^+(2) = e(\mathcal{T})$.

As before we have:

$$\tilde{e}^+(2n) = \tilde{e} \cdot \left(\sum_{i=0}^{\infty} c_i \cdot e^{2i} \right)$$

where $c_i = c_i(y_2, y_4, \dots) \in (W(\Bbbk)[y_2, y_4, \dots])^{-4i}$. Again we can make a comparison between the Euler classes of MSL_η and $H\mathcal{W}$, and we get that $\varphi_\eta^{H\mathcal{W}}(c_0) = c_0(0, 0, \dots) = \pm n$ by [Lev19, Theorem 7.1]. But then $c_0 = \pm n$, since it is a degree zero element. Thus, in $\mathrm{MSL}_\eta^\bullet(BN)\left[n^{-1}\right]$,we can write:

$$\tilde{e}^+(2n) = \tilde{e} \cdot (\pm n \cdot e) \left(1 + \sum_{i=1}^{\infty} d_i \cdot e^{2i-1} \right)$$

where the element $\left(1 + \sum_{i=1}^{\infty} d_i \cdot e^{2i-1}\right)$ is already invertible in $\mathrm{MSL}_\eta^\bullet(BN)\left[n^{-1}\right]$. If we consider $(\tilde{e}^+(2n))^2 \in \mathrm{MSL}_\eta^4(BN)$, we have:

$$\left(\tilde{e}^+(2n)\right)^2 = \tilde{e}^2 \cdot (\pm n \cdot e)^2 \left(1 + \sum_{i=1}^{\infty} d_i \cdot e^{2i-1} \right)^2$$

Using the fact that $(\tilde{e}^{H\mathcal{W}})^2 = -4e \in H\mathcal{W}^4(BN)$ (cf. [Lev19, Theorem 7.1]), again by looking at the formal power series of $\tilde{e}^2$ compared to $(\tilde{e}^{H\mathcal{W}})^2$, we get that:

$$\left(\tilde{e}^+(m)\right)^2 = e^2 \cdot (\pm m \cdot e)^2 \,(1 + \dots) \in \mathrm{MSL}_\eta^4(BN)$$

So in the end it is enough to invert $m \cdot e$ and we also get that $\tilde{e}^+(m)$ is invertible in $\mathrm{MSL}_\eta^\bullet(BN; \gamma_N)\left[(m \cdot e)^{-1}\right]$. ☐

Proposition 4.1.11. *Let X be an N-homogeneous space, and let A be an $SL[\eta^{-1}]$-oriented ring spectrum. Then there exists an integer $M > 0$ such that:*

$$\mathrm{A}_N^\bullet(X)[(Me)^{-1}] \simeq 0$$

Proof. Using lemma 4.1.6, if X is of type (a), then we just invert e and $M = 1$. For case (b) and $(c\pm)$, again by lemma 4.1.6 we have that $\tilde{e}^+(m)$ will go to zero for an appropriate m depending on the type of X. Then we conclude by lemma 4.1.10, noticing that if the Euler class going to zero is some $\tilde{e}^+(2k)$ living in the twisted theory, then its square $\tilde{e}^+(2k)^2 \in \mathrm{A}_N^\bullet(X)$ will also go to zero.

☐

Remark 4.1.12. Let T be the standard torus inside N. Since $N/T \simeq \mathbb{Z}/2\mathbb{Z}$, given X an N-equivariant scheme over $\Bbbk$, we have that its T-fixed points $X^T \subseteq X$ are the union of the 0-dimensional N-orbits.

Theorem 4.1.13 (Torus Localization, cf. [Lev19, Theorem 7.10]). *Let $X \in \mathbf{Sch}^N_{/\Bbbk}$ and let $\iota : X^T \hookrightarrow X$ the the associated closed immersion. Then for any $SL[\eta^{-1}]$-oriented ring spectrum* A*, and for any $L \in \operatorname{Pic}(X)$ with an N-linearization, there exists an integer $M > 0$ such that:*

$$\iota_* : \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(X^T/\Bbbk; \iota^* L\right)\left[(M \cdot e)^{-1}\right] \longrightarrow \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(X/\Bbbk; L\right)\left[(M \cdot e)^{-1}\right]$$

is an isomorphism.

Proof. Remember that $\mathrm{char}(\Bbbk) > 2$. If we invert the exponential characteristic p of $\Bbbk$, then taking the perfect closure of $\Bbbk^{perf}$ of $\Bbbk$ induces an isomorphism $\mathrm{A}^{\mathrm{BM}}_{\bullet,N}(\cdot/\Bbbk;\cdot) \xrightarrow{\sim} \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(\cdot/\Bbbk^{perf};\cdot\right)$ by [EK19, p. 2.1.5], so we can assume $p \mid M$ and $\Bbbk$ to be perfect.
Using the localization sequence for equivariant Borel-Moore homology, we reduce ourselves to show that if X has no zero-dimensional orbits, that is, $X^T = \emptyset$, then $\mathrm{A}^{\mathrm{BM}}_{\bullet,N}(X/\Bbbk; L) \simeq 0$. By [Lev22a, Proposition 1.1], there exists a finite stratification $X = \amalg_\alpha X_\alpha$ such that for each α we have that X_α/N exists as a quasi-projective $\Bbbk$-scheme and we can take $\pi_\alpha : X_\alpha \to X_\alpha/N$ to be smooth. So applying again a localization sequence argument, we can reduce to the case of $Z := X/N$ is a quasi-projective $\Bbbk$-scheme and $\pi : X \longrightarrow Z$ is smooth. Again by localization sequence argument, we can assume Z is integral and π is equi-dimensional. We have to show that if π has strictly positive relative dimension, then $\mathrm{A}^{\mathrm{BM}}_{\bullet,N}(X/\Bbbk; L) \simeq 0$, so we can assume π has strictly positive dimension.
Let us consider the finite level approximation for N-equivariant Borel-Moore homology:

$$\mathrm{A}^{\mathrm{BM}}_{\bullet,N^{(m)}}(X/\Bbbk; L) := \mathrm{A}^{\mathrm{BM}}_{\bullet}\left(X \times^N E_m N/\Bbbk; L\right)$$

For each of these we have a Leray homological spectral sequence (cf. [ADN18, p. 4.1.2]):

$$E^1_{p,q} = \bigoplus_{z \in Z_{(q)}} \mathrm{A}^{\mathrm{BM}}_{p+q,N^{(m)}}(X_z/\Bbbk; L_z) \Rightarrow \mathrm{A}^{\mathrm{BM}}_{p+q,N^{(m)}}(X/\Bbbk; L)$$

But by purity for $\Bbbk \to \kappa(z)$, since $z \in Z_{(q)}$, we have:

$$\mathrm{A}^{\mathrm{BM}}_{p,N^{(m)}}(X_z/\kappa(z); L_z) \simeq \mathrm{A}^{\mathrm{BM}}_{p+q,N^{(m)}}(X_z/\Bbbk; L)$$

and hence the spectral sequence becomes:

$$E^1_{p,q} = \bigoplus_{z \in Z_{(q)}} \mathrm{A}^{\mathrm{BM}}_{p,N^{(m)}}(X_z/\kappa(z); L_z) \Rightarrow \mathrm{A}^{\mathrm{BM}}_{p+q,N^{(m)}}(X/\Bbbk; L)$$

If ξ is a generic point of Z, then there exists an open neighbourhood U_ξ of ξ such that each fiber X_z of $\pi : X \to Z$ has the same same type, as homogeneous space for N over $\kappa(z)$, as does X_ξ. So after taking a further stratification, we can assume that all the fibers X_z, for $z \in Z$, have the same type. So by proposition 4.1.11 there exists an M such that $\mathrm{A}^{\mathrm{BM}}_{p,N^{(m)}}\left(X_z/\kappa(z)\right)\left[(Me)^{-1}\right] \simeq 0$, since $\mathrm{A}^{\mathrm{BM}}_{p,N^{(m)}}\left(X_z/\kappa(z)\right)$ is a module over cohomology $\mathrm{A}^{\bullet}_{N}\left(X_z\right) \simeq \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(X_z/\kappa(z)\right)$. The module map is obtained through the pullback induced by the natural map $E_m N \to \mathrm{Spec}(\mathbb{k})$, after identifying Borel-Moore Witt homology and Witt theory (up to some re-indexing) via purity, relative to $X_z \to z$ (which is a smooth map, since it is a fiber of $X \to Z$). Notice that the integer M only depends on the type of X_z, so it is the same integer for any $z \in Z$. But again, since $\mathrm{A}^{\mathrm{BM}}_{p,N^{(m)}}\left(X_z/\kappa(z); L_z\right)$ is a also module over cohomology $\mathrm{A}^{\bullet}_{N}\left(X_z\right)$, we have $\forall\ z \in Z$:

$$\mathrm{A}^{\bullet}_{N}\left(X_z\right)\left[(Me)^{-1}\right] \simeq 0 \Rightarrow \mathrm{A}^{\mathrm{BM}}_{p,N^{(m)}}\left(X_z/\kappa(z)\,; L_z\right)\left[(Me)^{-1}\right] \simeq 0$$

and hence the spectral sequence $E^1_{p,q}$ tells us that:

$$\mathrm{A}^{\mathrm{BM}}_{p+q,N^{(m)}}\left(X/\mathbb{k}\,; L)\right)\left[(Me)^{-1}\right] \simeq 0\ \forall\ p, q\ \forall\ m$$

In particular, this means that:

$$\left\{\mathrm{A}^{\mathrm{BM}}_{p+q,N^{(m)}}\left(X/\mathbb{k}\,; L)\right)\left[(Me)^{-1}\right]\right\}_{m\in\mathbb{N}}$$

satisfies the Mittag-Leffler condition. Therefore the limit of those groups actually computes $\mathrm{A}^{\mathrm{BM}}_{p+q,N}\left(X/\mathbb{k}; L)\right)\left[(Me)^{-1}\right]$. Hence we get:

$$\mathrm{A}^{\mathrm{BM}}_{p+q,N}\left(X/\mathbb{k}\,; L)\right)\left[(Me)^{-1}\right] \simeq 0$$

as we wanted to prove.

□

Before we finally get the Atiyah-Bott localization for an N-action, we need to recall some more theorems and definitions from [Lev22a]:

Definition 4.1.14 ([Lev22a, Def. 8.2])**.** Let $\bar{\sigma}$ be the image of $\sigma \in N$ in $N/T \simeq \mathbb{Z}/2\mathbb{Z} = \langle\bar{\sigma}\rangle$.

1. Let us denote $|X|^N$ the union of the irreducible components $Z \subseteq X^T$ such that the generic point ξ_Z is fixed by $\bar{\sigma}$, and let us denote X^T_{ind} the union of irreducible components $W \subseteq X^T$ such that $\bar{\sigma} \cdot W \cap W = \emptyset$.
2. We say that the N-action is *semi-strict* if $X^T_{red} = |X|^N \cup X^T_{ind}$.
3. A semi-strict N-action is said to be *strict* if $|X|^N \cap X^T_{ind} = \emptyset$ and we can decompose $|X|^N$ as disjoint union of two N-stable closed subschemes:

$$|X|^N = X^N \amalg X^T_{fr}$$

where the N/T-action on X^N_{fr} is free.

Notation 4.1.15. For any group G, a character χ will correspond to a line bundle over BG that we will denote as L_χ.

Proposition 4.1.16. *Let $X \in \mathbf{Sch}^N_{/\mathbb{k}}$ with a trivial N-action, and let L be a line bundle on X also with trivial N-action. Let $\chi : N \to \mathbb{G}_m$ be a character with associated line bundle L_χ. Let A be an $SL[\eta^{-1}]$-oriented ring spectrum. Then, up to inverting the exponential characteristic of $\mathbb{k}$, we have an isomorphism:*

$$\mathrm{A}^{-\bullet}(\mathcal{B}N; L_\chi) \otimes \mathrm{A}^{\mathrm{BM}}_{\bullet}(X/\mathbb{k}; L) \xrightarrow{\sim} \mathrm{A}^{\mathrm{BM}}_{N,\bullet}(X/\mathbb{k}; L_\chi \otimes L)$$

Proof. We can assume that $\mathbb{k}$ is perfect, if it is not we will invert its exponential characteristic and use [EK19, Corollary 2.1.7]. Notice that since A is SL-oriented, if we twist or untwist $\mathrm{A}^{\mathrm{BM}}_{N,\bullet}(X/\mathbb{k}; L_\chi \otimes L)$ by the determinant of the co-Lie algebra $\mathfrak{n}^\vee$ of N, nothing changes (up to isomorphism). To simplify the notation, we give the proof in the untwisted case (i.e. for trivial L and L_χ); the proof in the twisted case is the same and is left to the reader. Notice that $[X/N] \simeq X \times \mathcal{B}N$. Consider:

$$\begin{tikzcd}
X \times \mathcal{B}N \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ \Delta"] & \mathcal{B}N \arrow[d] \\
X \arrow[r] & \mathrm{Spec}(\mathbb{k})
\end{tikzcd}$$

Using the composition product, under the identification $\mathrm{A}^{-\bullet}(\mathcal{B}N) \simeq \mathrm{A}^{\mathrm{BM}}_{\bullet}(\mathcal{B}N/\mathcal{B}N)$, together with the base change $\Delta^* : \mathrm{A}^{\mathrm{BM}}_{\bullet}(X/\mathbb{k}, v) \to \mathrm{A}^{\mathrm{BM}}_{\bullet}(X \times \mathcal{B}N/\mathcal{B}N)$, we get a map:

$$\psi := (- \odot -) \circ (\Delta^* \times Id) : \mathrm{A}^{\mathrm{BM}}_{\bullet}(X/\mathbb{k}) \otimes \mathrm{A}^{-\bullet}(\mathcal{B}N) \longrightarrow \mathrm{A}^{\mathrm{BM}}_{\bullet}(X \times \mathcal{B}N/\mathcal{B}N)$$

Under the identification remark 1.4.9, the target of ψ is just $\mathrm{A}^{\mathrm{BM}}_{N,\bullet}(X/\mathbb{k}, -\mathfrak{n}^\vee)$.

Using the Leray spectral sequence ([ADN18, Theorem 4.1.2]), we can show that ψ is an equivalence. Indeed, we have:

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathrm{A}^{\mathrm{BM}}_{p+q}(\kappa(x)/\mathbb{k}) \Rightarrow \mathrm{A}^{\mathrm{BM}}_{p+q}(X/\mathbb{k})$$

$$'E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathrm{A}^{\mathrm{BM}}_{p+q}(\mathcal{B}N_{\kappa(x)}/\mathcal{B}N) \Rightarrow \mathrm{A}^{\mathrm{BM}}_{p+q}(X \times \mathcal{B}N/\mathcal{B}N)$$

But $\mathrm{A}^\bullet(\mathcal{B}N_{\mathbb{k}})$ is a free $\mathrm{A}^\bullet(\mathbb{k})$-module by corollary 2.3.5 and hence from E_1 we get:

$$''E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathrm{A}^{\mathrm{BM}}_{p+q}(\kappa(x)/\mathbb{k}) \otimes \mathrm{A}^\bullet(\mathcal{B}N) \Rightarrow \mathrm{A}^{\mathrm{BM}}_{p+q}(X/\mathbb{k}) \otimes \mathrm{A}^\bullet(\mathcal{B}N)$$

But the map ψ at the level of spectral sequences induces a map $''E_1^{*,*} \to' E_1^{*,*}$ that becomes an isomorphism. Let us go into more details on why that is true. Since we assumed that $\Bbbk$ is perfect, for any $x \in X^{(p)}$ the extension $\kappa(x)/\Bbbk$ is separable, that is, $\mathrm{Spec}(\kappa(x))$ is smooth over $\Bbbk$. Then we can use purity to see that $\mathrm{A}^{\mathrm{BM}}(\kappa(x)/\Bbbk) \simeq \mathrm{A}(\kappa(x))$ and $\mathrm{A}^{\mathrm{BM}}(\mathcal{B}N_{\kappa(x)}/\mathcal{B}N) \simeq \mathrm{A}(\mathcal{B}N_{\kappa(x)})$. But by proposition 2.2.16 for $S = \mathrm{Spec}(\kappa(x))$ with $x \in X^{(p)}$, we have that:

$$\mathrm{A}^{\bullet}(\kappa(x)) \otimes_{\mathrm{A}^{\bullet}(\Bbbk)} \mathrm{A}^{\bullet}(\mathcal{B}N) \simeq \mathrm{A}^{\bullet}(\mathcal{B}N_{\kappa(x)}) \tag{4.1}$$

To see why this equivalence it is true, proposition 2.2.16 tells us that $\mathrm{A}^{\bullet}(\mathcal{B}N) \simeq \mathrm{A}^{\bullet}(\mathcal{B}SL_2) \oplus \mathrm{A}^{\bullet}(\Bbbk)$. Then to check that (4.1) holds, it is equivalent to check that the same equality holds for $\mathcal{B}SL_2$ instead of $\mathcal{B}N$. By [Ana15, Theorem 9] (or proposition 3.2.1), we know that we can compute $\mathrm{A}^{\bullet}(\mathcal{B}SL_{2,k})$ using the finite approximations $B_m SL_{2,k}$ for any field k. Since filtered limits commute with finite colimits, we have:

$$\mathrm{A}^{\bullet}(\kappa(x)) \otimes_{\mathrm{A}^{\bullet}(\Bbbk)} \left(\lim_m \mathrm{A}^{\bullet}(B_m SL_2)\right) \simeq \lim_m \mathrm{A}^{\bullet}(\kappa(x)) \otimes_{\mathrm{A}^{\bullet}(\Bbbk)} \mathrm{A}^{\bullet}(B_m SL_2)$$

Clearly the right hand side is just $\mathrm{A}^{\bullet}(\mathcal{B}SL_{2,\kappa(x)})$ and we have our claim.

Therefore ψ induces an isomorphism on spectral sequences and thus it is an isomorphism itself and we are done.

□

Proposition 4.1.17. *Let $X \in \mathbf{Sch}_{/\Bbbk}$ with an N-action, let $L \in \mathrm{Pic}(X)$ an N-linearised line bundle, and let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum.*

1. *Let $\iota_0 : X^N \hookrightarrow X$ the closed immersion. For each connected component X_i^N of X^N there is a line bundle L_i on X_i^N with trivial N-action and a character χ_i such that $\iota_0^* L\big|_{X_i^N} \simeq L_i \otimes L_{\chi_i}$ as N-linearised line bundle. Moreover:*

$$\mathrm{A}^{\mathrm{BM}}_{N,k}\left(X^N; L\right) \simeq \bigoplus_{p+q=k} \mathrm{A}^{\mathrm{BM}}_p(X_i^N; L_i) \otimes_{W(\Bbbk)} \mathrm{A}^{-q}\left(BN; L_{\chi_i}\right)$$

2. *Let $\iota_{ind} : X^T_{ind} \hookrightarrow X$ be the inclusion, then:*

$$\mathrm{A}^{\mathrm{BM}}_N\left(X^T_{ind}/\Bbbk; \iota^*_{ind} L\right)\left[e^{-1}\right] = 0$$

where e is the Euler class associated to (the pullback of) $\widetilde{\mathcal{O}}^+(1)$.

Proof. For the first assertion we may assume X^N is connected. The line bundle $\iota_0^* L$ lives over a space with trivial N-action, so N will act with a character χ on $\iota_0^* L$ (corresponding to a line bundle L_χ) so that we will have an isomorphism of N-linearised line bundles $\iota_0^* L \simeq L_0 \otimes L_\chi$ for some line bundle L_0 having the trivial N-action. Then the isomorphism of the first assertion in the proposition will follow from proposition 4.1.16.
For the second assertion, consider $C_1, \ldots, C_{2r}$ the irreducible components of X^T_{ind}

with $\bar{\sigma}C_{2i-1} = C_{2i}$. We will proceed by induction on r. Let us start with $r = 1$, as an N-scheme $X^T_{ind} = C_1 \amalg C_2 \simeq (N/T) \times C_1$ with the trivial action on C_1. Hence:

$$\left[X^T_{ind}/N\right] \simeq C_1 \times \mathcal{B}T$$

and we get:

$$\mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(X^T_{ind}/\Bbbk;\iota^*_{ind}L\right) \simeq \mathrm{A}^{\mathrm{BM}}_{\bullet}\left(C_1/\Bbbk;L_1\right) \otimes_{W(\Bbbk)} \mathrm{A}^{-\bullet}\left(BT;L_{\chi_{C_1}}\right)$$

where L_1 is a line bundle over C_1 and $L_{\chi_{C_1}}$ is the bundle associated to a character χ_{C_1} of T. Let $q : B\mathbb{G}_m \to BN$, we have $q^*\widetilde{\mathcal{O}}^+(1) \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$, and so $e = q^*e(\widetilde{\mathcal{O}}^+(1)) \simeq e\left(q^*\widetilde{\mathcal{O}}^+(1)\right) = e\left(\mathcal{O}(1) \oplus \mathcal{O}(-1)\right) = 0$. But $\mathrm{A}^{-\bullet}\left(BT;L_{\chi_{C_1}}\right)$ is a $\mathrm{A}^{-\bullet}(BT)$-module and $\mathrm{A}^{-\bullet}(BT)\left[e^{-1}\right] = 0$, hence $\mathrm{A}^{-\bullet}\left(BT;L_{\chi_{C_1}}\right)\left[e^{-1}\right] = 0$ and we are done for the initial inductive step.

For $r > 1$, consider $X^T_{ind} = C \cup C'$ where $C := C_1 \amalg C_2$ and $C' := C_3 \cup \ldots \cup C_{2r}$. By induction:

$$\mathrm{A}^{\mathrm{BM}}_N\left(C/\Bbbk;\iota^*_{ind}L\right)\left[e^{-1}\right] = \mathrm{A}^{\mathrm{BM}}_N\left(C'/\Bbbk;\iota^*_{ind}L\right)\left[e^{-1}\right] = 0$$

Moreover $C \cap C' = (C_1 \cap C') \amalg (C_2 \cap C') \simeq (C_1 \cap C') \amalg \bar{\sigma} \cdot (C_1 \cap C')$, hence $(C \cap C') \times^N EN \simeq (C_1 \cap C') \times BT$ similar to the computation made for $r = 1$. But using the same argument as before, we get in this way that $\mathrm{A}^{\mathrm{BM}}_N\left(C \cap C'/\Bbbk;\iota^*_{ind}L\right)\left[e^{-1}\right] = 0$. Then using the localization sequence, it easy to see that:

$$\mathrm{A}^{\mathrm{BM}}_N\left(X^T_{ind}/\Bbbk;\iota^*_{ind}L\right)\left[e^{-1}\right] = 0$$

□

Theorem 4.1.18 (Atiyah-Bott Localization for N-action)**.** *Let $X \in \mathbf{Sch}^N_{/\Bbbk}$ be a scheme with an N-action and let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented spectrum. Let $\iota : |X|^N \hookrightarrow X$ be the closed immersion. Let $L \in \mathrm{Pic}(X)$ be an N-linearised line bundle. Suppose the N-action is semi-strict. Then there is a non-zero integer M such that:*

$$\iota_* : \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(|X|^N;\iota^*L\right)\left[(M \cdot e)^{-1}\right] \longrightarrow \mathrm{A}^{\mathrm{BM}}_{\bullet,N}(X;L)\left[(M \cdot e)^{-1}\right]$$

is an isomorphism.

Proof. Since the action is semi-strict, we can consider the closed immersion $X^T \simeq X^T_{ind} \cup |X|^N \hookrightarrow X$, that has $X \setminus X^T$ as open complement. By proposition 4.1.17, we have $\mathrm{A}^{\mathrm{BM}}_N\left(X^T_{ind}/\Bbbk;\iota^*_{ind}L\right)\left[e^{-1}\right] = 0$. Applying again proposition 4.1.17, this time to the scheme $|X|^N$, we get:

$$\mathrm{A}^{\mathrm{BM}}_N\left(X^T_{ind} \cap |X|^N/\Bbbk;\iota^*_{ind}L\right)\left[e^{-1}\right] = 0$$

since $\left(|X|^N\right)^T_{ind} \simeq X^T_{ind} \cap |X|^N$. But using a localization sequence, this implies that:

$$\mathrm{A}^{\mathrm{BM}}_N \left(X^T_{ind} \setminus |X|^N/\Bbbk\,;\iota^*_{ind}L\right)\left[e^{-1}\right] = 0$$

as well. So we have that the inclusion $|X|^N \hookrightarrow X^T$ induces an isomorphism on N-equivariant Borel-Moore homologies, and so we get the final claim using theorem 4.1.13. □

4.2 Bott Residue Formula

Consider now $\iota : Y \hookrightarrow X$ a regular embedding in $\mathbf{Sch}^N_{/\Bbbk}$. Let $N_{Y/X}$ be the normal bundle associated to ι, it has a natural N-linearisation, so we can consider $e_N(N_{Y/X}) \in \mathrm{A}^{\mathrm{BM}}_{r,N}\left(Y/\Bbbk; \det^{-1}\left(N_{Y/X}\right)\right)$, where r is the rank of $N_{Y/X}$.

Lemma 4.2.1. *Let $L \in \mathrm{Pic}(X)$ and let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then:*

$$\iota^!\iota_* : \mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(Y/\Bbbk\,;\iota^*L\right) \longrightarrow \mathrm{A}^{\mathrm{BM}}_{\bullet-r,N}\left(Y/\Bbbk\,;\iota^*L \otimes \det^{-1}\left(N_{Y/X}\right)\right)$$

is the cup product with $e_N\left(N_{Y/X}\right)$.

Proof. Rewriting the equivariant Borel-Moore homology as Borel-Moore homology over $\mathcal{B}N$ (cf. remark 1.4.9), the claim follows using the same arguments as in [DJK21, Corollary 4.2.3]. □

Following [Lev22a, Lemma 9.3, Construction 2.7], we have that the Euler class of an N-linearised vector bundle V on some connected $Y \in \mathbf{Sch}^N_{/\Bbbk}$ gets inverted once we invert some Euler classes coming from representations of N over $\Bbbk$. Let us briefly recall such construction (for details refer to *loc. cit.*). Let $\mathcal{V}$ be an N-linearised locally free sheaf on some connected scheme $Y \in \mathbf{Sch}^N_{/\Bbbk}$.

(*Case 1*) Suppose N acts trivially on Y. Then for each point $y \in Y$, we can find N-stable open neighbourhoods $j_{U_y} : U_y \hookrightarrow Y$ of y such that our locally free sheaf trivialises as $\psi_y : j^*_{U_y}\mathcal{V} \xrightarrow{\sim} \mathcal{O}_{U_y}\otimes_\Bbbk V(f)$ for some $\Bbbk$-representation f of N. Taking the decomposition of $V(f)$ into isotypical components, we get the corresponding decomposition of $j^*_{U_y}\mathcal{V}$. We can actually find a global decomposition of $\mathcal{V}$ into isotypical components, indexed by $\Bbbk$-irreducible representations of N:

$$\mathcal{V} = \bigoplus_\phi \mathcal{V}_\phi$$

in such a way that on each trivialising open subset $j_U : U \hookrightarrow Y$ we have:

$$j^*_U\mathcal{V}_\phi = \mathcal{O}_U \otimes V(\phi)^{n_\phi}$$

where $V(\phi)$ is a $\Bbbk$-irreducible representation of N. The $\bigoplus_\phi V(\phi)^{n_\phi}$ turns out to be completely determined by $\mathcal{V}$, up to isomorphism. We denote its isomorphism class as $[\mathcal{V}^{gen}]$.

(Case 2) Suppose $T = \mathbb{G}_m \subseteq N$ acts trivially on Y. We can decompose $\mathcal{V}$ into weight spaces $\mathcal{V} = \bigoplus_m \mathcal{V}_m$, and let $\mathcal{V}^{\mathfrak{m}} := \bigoplus_{m \neq 0} \mathcal{V}_m$. Then for any N-trivialised open $j_U : U \hookrightarrow Y$, we have:

$$j_U^* \mathcal{V} \simeq \mathcal{V}_0 \oplus \bigoplus_{m>0} \mathcal{O}_U \otimes^{\sigma,\tau} V(\rho_m)^{n_m}$$

where ρ_i's are the rank two N-representations introduced in Chapter 3, and $\cdot \otimes^{\sigma,\tau} V(\rho_i)$ denotes the $\mathcal{O}_U$-semi-linear extension of the representations in the sense of [Lev22a, Def.2.5]. Then isomorphism class of the representation $\bigoplus_{m>0} V(\rho_m)^{n_m}$ is uniquely determined by $\mathcal{V}$ and hence we denote this class by $[\mathcal{V}^{gen}]$.

(Case 3) Suppose $\mathbb{G}_m = T \subseteq N$ acts trivially on Y and that $q : Y \to Y/N \simeq Y/\langle\bar{\sigma}\rangle$ is a degree 2 étale cover, where $\bar{\sigma}$ is the image of σ in $N/T \simeq \mathbb{Z}/2\mathbb{Z} \simeq \langle\bar{\sigma}\rangle$. Then for any N-trivialised open $j_U : U \hookrightarrow Y$, we have:

$$j_U^* \mathcal{V} \simeq \mathcal{O}_U \otimes^{\sigma,\tau} V(\rho_0)^{n_0} \oplus \bigoplus_{m>0} \mathcal{O}_U \otimes^{\sigma,\tau} V(\rho_m)^{n_m}$$

where the notation is the same as in *Case 2*. This time the isomorphism class of $V(\rho_0)^{n_0} \oplus \bigoplus_{m>0} V(\rho_m)^{n_m}$ is uniquely determined by $\mathcal{V}$, and we denote this class by $[\mathcal{V}^{gen}]$.

Definition 4.2.2 ([Lev22a, Def. 4.7])**.** Let V an N-linearised vector bundle on a connected $Y \in \mathbf{Sch}^N_{/\Bbbk}$. Suppose we are in one of the three cases when $[\mathcal{V}^{gen}]$ is defined. Choose a representative $V^{gen} \in [\mathcal{V}^{gen}]$ and suppose it has even rank $2r$ (this is always true in *Case 2*). Let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then we have:

$$e_N(V^{gen}) \in \mathrm{A}^{2r}\left(BN; \det{}^{-1}(V^{gen})\right)$$
$$e_N(V^{gen})^2 \in \mathrm{A}^{4r}\left(BN\right)$$

We then define the *generic Euler class* as the subset:

$$[e_N(V^{gen})] := \left\{ u \cdot e_N(V^{gen})^2 \;\middle|\; u \in \left(\mathrm{A}^0(\Bbbk)\right)^{\times} \right\} \subseteq \mathrm{A}^{4r}(BN)$$

and this depends by construction only on the isomorphism class of V as an N-linearised bundle. The localization:

$$\mathrm{A}^{\bullet}_N\left(Y\right)\left[[e_N(V^{gen})]^{-1}\right]$$

will denote the localization with respect to any element $y \in [e_N(V^{gen})]$ seen as an element in $\mathrm{A}_N(Y)$ through the $\mathrm{A}(BN)$-module map. If the representative V^{gen} has a trivialization of its determinant, then it will give us an actual class $e_N(V^{gen}) \in \mathrm{A}^{2r}(BN)$ and then it becomes a localization by $e_N(V^{gen})$ in the usual sense.

Lemma 4.2.3 ([Lev22a, Lemma 9.3]). *Let V an N-linearised vector bundle on a connected scheme $Y \in \mathbf{Sch}^N_{/\mathbb{k}}$ of rank $2r$. Let $\mathrm{A} \in \mathrm{SH}(\mathbb{k})$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Let us suppose that assumptions of [Lev22a, Construction 2.7] are satisfied, so we are in Case 1,2,3 in loc.cit. and hence we get a generic Euler class $[e^{gen}_N(V)] \subseteq \mathrm{A}^{4r}(BN)$. Choose an element $e_N(V^{gen}) \in [e_N(V^{gen})]$. Then $e_N(V) \in \mathrm{A}^{2r}_N(Y;\det^{-1}(V))$ is invertible in $\mathrm{A}^{2\bullet}_N(Y;\det^{-1}(V))\left[\left(e^{gen}_N(V)\right)^{-1}\right]$.*

Proof. We need to show that $e_N(V)$ is invertible in $\mathrm{A}^{2\bullet}_N(Y;\det^{-1}(V))\left[\left(e^{gen}_N(V)\right)^{-1}\right]$, but this is equivalent to show that multiplication by $e_N(V)$:

$$\cdot e_V(N) : \mathrm{A}^{2\bullet}_N(Y;\det^{-1}(V))\left[\left(e^{gen}_N(V)\right)^{-1}\right] \longrightarrow \mathrm{A}^{2\bullet+2r}_N(Y;\det^{-1}(V))\left[\left(e^{gen}_N(V)\right)^{-1}\right]$$

is an isomorphism. By assumptions, we can find an open cover made by N-stable opens $U_i \subseteq Y$ such that V trivialises over each U_i. Then by a Mayer-Vietoris argument, we can reduce ourselves to prove our claim on all the intersections of our U_i's. But there the claim is obvious, since on each U_i we have an isomorphism of vector bundles (with an N-action) $V\big|_{U_i} \simeq \pi_Y^* V^{gen}$, where $\pi_Y : Y \to \mathrm{Spec}(\mathbb{k})$ is the structure map.

□

Theorem 4.2.4 (Bott Residue Formula). *Let $X \in \mathbf{Sch}^N_{/\mathbb{k}}$, $L \in \mathrm{Pic}(X)$ an N-linearised line bundle, and $\mathrm{A} \in \mathrm{SH}(\mathbb{k})$ an $SL[\eta^{-1}]$-oriented ring spectrum. Let us suppose that each connected component $\iota_j : |X|^N_j \hookrightarrow X$ of $|X|^N$ is a regular embedding. Moreover for each normal bundle N_j associated to ι_j, assume that the hypothesis of Case 1,2,3 in [Lev22a, Construction 2.7] are satisfied for $V = N_j$. Denote by $e^{gen}_N(N_{fix})$ the products of $e^{gen}_N(N_j)$'s. Finally assume that the N-action on X is semi-strict.*
Denote $P := p \cdot M \cdot e$, where p is the exponential characteristic of the ground field $\mathbb{k}$ and M is the same integer as in theorem 4.1.13.
Under the identification induced by the decomposition $|X|^N = \bigcup_j |X|^N_j$ in its connected components:

$$\mathrm{A}^{\mathrm{BM}}_N\left(|X|^N/\mathbb{k};\iota^* L\right) \simeq \prod_j \mathrm{A}^{\mathrm{BM}}_N\left(|X|^N_j/\mathbb{k};\iota_j^* L\right)$$

the inverse of the isomorphism:

$$\iota_* : \mathrm{A}^{\mathrm{BM},\bullet}_N\left(|X|^N/\mathbb{k};\iota^* L\right)\left[\left(P\cdot e^{gen}_N(N_{fix})\right)^{-1}\right] \longrightarrow \mathrm{A}^{\mathrm{BM},\bullet}_N\left(X/\mathbb{k};L\right)\left[\left(P\cdot e^{gen}_N(N_{fix})\right)^{-1}\right]$$

is given by:

$$x \mapsto \prod_j \iota_j^!(x) \cap e_N(N_j)^{-1}$$

Proof. After inverting P, we can use theorem 4.1.18 and hence see $e_N^{gen}(N_{fix})$ and $e_N^{gen}(N_j)$ as elements in $\mathrm{A}_N^{\mathrm{BM}}(X/\Bbbk;L)$. Inverting said elements will invert also $e_N(N_j)$ by lemma 4.2.3. By theorem 4.1.18 we can find elements:

$$y_j \in \mathrm{A}_N^{\mathrm{BM}}\left(|X|_j^N/\Bbbk;\iota_j^* L\right)\left[\left(P\cdot e_N^{gen}(N_j)\right)^{-1}\right]$$

such that:

$$x=\sum_j \iota_{j*}y_j$$

for any $x \in \mathrm{A}_N^{\mathrm{BM}}(X/\Bbbk;L)\left[\left(P\cdot e_N^{gen}(N_{fix})\right)^{-1}\right]$. Since by lemma 4.2.1 we have:

$$\iota_j^!(x)=\iota_j^!\iota_{j*}y_j=y_j\cap e_N(N_j)$$

the map we defined sending x to $\prod_j \iota_j^!(x)\cap e_N(N_j)^{-1}$ will be indeed the inverse to ι_*. □

Remark 4.2.5. If $X \in \mathbf{Sch}_{/\Bbbk}^N$ is smooth, then X^T is smooth too, the action will be semi-strict, and each $\iota_j : |X|_j^N \hookrightarrow X$ will be a regular embedding. Moreover the respective normal bundles will be of the form $N_j = N_j^{\mathrm{m}}$, so we are in Case 2 of [Lev22a, Construction 2.7] and we can indeed apply the previous theorem.

4.3 Virtual Localization Formula

We now have all the formal properties we need to prove the virtual localization formula of virtual fundamental classes of N-equivariant schemes following [Lev22b] and hence [GP99].

For this section, let us fix $X \in \mathbf{Sch}/\Bbbk^N$ with a closed immersion $\iota : X \hookrightarrow Y$ in $\mathbf{Sch}_{/\Bbbk}^N$, where Y is a smooth $\Bbbk$-scheme. Let us also suppose X is equipped with an N-equivariant perfect obstruction theory represented by a two term complex $\mathcal{E}_\bullet := (\mathcal{E}_1 \to \mathcal{E}_0)$, of N-linearised locally free sheaves, together with an N-equivariant map $\varphi_\bullet : \mathcal{E}_\bullet \longrightarrow \mathbb{L}_{X/\Bbbk}$.

Now consider the maximal subtorus $T := \mathbb{G}_m \subseteq N$. We have the fixed T-schemes $X^T \subseteq Y^T$, where we give X^T the scheme structure $X^T := X \cap Y^T$. Notice that Y^T is smooth (this is true in much greater generality, for example see [Rom22, Theorem 4.3.6]); consider its connected components $Y_1,\dots,Y_s$ and inclusion maps $\iota_i^Y : Y_i \hookrightarrow Y$.

Let us denote $\iota_j : X_j := Y_j \cap X \hookrightarrow X$ so that $X^T = \coprod_j X_j$.

Let $\mathcal{F}$ be a T-linearised coherent sheaf on X_j. The T-action on the X_j is trivial, so we can decompose $\mathcal{F}$ into its weight spaces for the T-action:

$$\mathcal{F} \simeq \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_m$$

If $\mathcal{F}$ is locally free, then so are the $\mathcal{F}_m$'s.

Notation 4.3.1. Let $\mathcal{F}$ be a T-linearised coherent sheaf on some scheme Z with a trivial T-action. Let $\mathcal{F} \simeq \bigoplus_{m \in \mathbb{Z}} \mathcal{F}_m$ be its decomposition in weight spaces, then we will denote:

$$\mathcal{F}^{\mathfrak{m}} := \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathcal{F}_m$$

the *moving part*, and by:

$$\mathcal{F}^{\mathfrak{f}} := \mathcal{F}_0$$

the *fixed part* of $\mathcal{F}$.

In the situation described above where we have $\iota_j : X_j \hookrightarrow X$ and a perfect obstruction theory $\varphi_\bullet : \mathcal{E} \to \mathbb{L}_{X/\mathbb{k}}$ on X, then $\varphi_\bullet$ induces maps:

$$\varphi_\bullet^{(j)} : \iota_j^* \mathcal{E}_\bullet^{\mathfrak{f}} \longrightarrow \mathbb{L}_{X_j/\mathbb{k}}$$

that by [GP99, Proposition 1] are perfect obstruction theories for the X_j's.

Definition 4.3.2. The virtual conormal sheaf of each X_j is defined to be the perfect complex $\mathcal{N}_j^{vir} := \iota_j^* \mathcal{E}_\bullet^{\mathfrak{m}}$.

By [Lev22b, Lemma 6.2] we have:

Lemma 4.3.3 ([Lev22b, Lemma 6.2])**.** *For each j, the perfect obstruction theory $\varphi_\bullet^{(j)} : \iota_j^* \mathcal{E}_\bullet^{\mathfrak{f}} \to \mathbb{L}_{X_j/\mathbb{k}}$ and $\mathcal{N}_j^{vir}$ have a natural N-linearisation.*

Remark 4.3.4. If Y is smooth then the N-action is semi-strict [Lev22a, Remark 9.6]. By [Lev22b, Remark 6.4], if the N-action on Y is strict, then the N action on X will also be strict.

We will assume the N-action on X is strict.

Definition 4.3.5. By conventions set before $X_j := X \cap Y_j$ with Y_j connected components of Y^T. Let us denote:

$$|X|_j^N := |X|^N \cap X_j$$

and thus we have:

$$|X|^N \simeq \coprod_j |X|_j^N$$

where each $|X|_j^N$ has its own perfect obstruction theory $\varphi_\bullet^{(j)}$ with associated virtual normal cone given by $\mathcal{N}_j^{vir}$. Each $|X|_j^N$ will decompose in connected component that we will denote as $|X|_{j,k}^N$

Remark 4.3.6. In the case of a strict N-action, consider an N-linearised locally free sheaf $\mathcal{V}$ on some connected components of X^T. Suppose that $\mathcal{V} = \mathcal{V}^{\mathfrak{m}}$. Then $\mathcal{V}$ admits a *generic representation type* $[\mathcal{V}^{gen}]$ in the sense of [Lev22a, Construction 2.7], which is an isomorphism class of N-representations over $\mathbb{k}$.

Lemma 4.3.7. *Let* $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ *be an* $SL[\eta^{-1}]$*-oriented ring spectrum. Consider* $X \in \mathbf{Sch}^N_{/\Bbbk}$*. Let* $\mathcal{V}$ *an* N*-linearised sheaf on* X *and suppose the* N*-action on* X *is strict and we have* $\mathcal{V} = \mathcal{V}^{\mathfrak{m}}$ *on* $|X|^N$*. The restriction of* $\mathcal{V}$ *on the connected components* $|X|^N_{j,k}$ *will be denoted by* $\mathcal{V}_{j,k}$*. For fixed* j,k *we have:*

1. *there exists integers* M, n *such that* $e_N(\mathcal{V}^{gen}_{j,k})$ *is invertible in* $\mathrm{A}^\bullet(BN)\left[(M \cdot e^n)^{-1}\right]$.
2. *the class* $e_N(V_{j,k})$ *is invertible in* $\mathrm{A}^\bullet_N\left(|X|^N_{j,k}; \det^{-1}(V_{j,k})\right)\left[(M \cdot e^n)^{-1}\right]$

Proof. The second claim is a consequence of lemma 4.2.3. For the first claim it is enough to notice that $\mathcal{V}^{gen}_{j,k}$ is given by irreducible $\Bbbk$-representations of N, and we know that we can recover any irreducible representation of N by tensor products of the representations ρ^+_m, ρ^-_0. Then we conclude by applying multiple times lemma 4.1.10. □

Let us recall again our setting: we are working with $\iota : X \hookrightarrow Y$ an N-equivariant closed immersion with Y smooth over $\Bbbk$; we are given an N-linearised perfect obstruction theory $\varphi_\bullet : \mathcal{E}_\bullet \longrightarrow \mathbb{L}_{X/\Bbbk}$. We suppose that the N-action on X is strict.

Definition 4.3.8. Let $\mathrm{A} \in \mathrm{SH}(\Bbbk)$ be an $SL[\eta^{-1}]$-oriented ring spectrum. Then we have:

(i) For each connected component $\iota^Y_j : Y_j \hookrightarrow Y$ of Y^T, by lemma 4.3.7, there exists an integer M^Y_j such that $e_N(\iota^{Y*}_j T^{\mathfrak{m}}_Y)$ will be invertible in:

$$\mathrm{A}_N\left(Y_j; \det^{-1}(\iota^{Y*}_j T^{\mathfrak{m}}_Y)\right)\left[\left(M^Y_j \cdot e\right)^{-1}\right]$$

(ii) For each component $\iota_{j,k} : |X|^N_{j,k} \hookrightarrow X$ of $|X|^N$, by lemma 4.3.7, there exists an integer $M^X_{j,k}$ such that $e_N\left(\left(\iota^*_{j,k}E^{\mathfrak{m}}_1\right)^{gen}\right)$ is invertible in:

$$\mathrm{A}_N\left(|X|^N_{j,k}; \det^{-1}(\iota^*_{j,k}E^{\mathfrak{m}}_1)\right)\left[\left(M^X_{j,k} \cdot e\right)^{-1}\right]$$

Hence we can define:

$$e_N\left(N^{vir}_{\iota_{j,k}}\right) := e_N((\iota^*_{j,k}E^{\mathfrak{m}}_0)) \cdot e_N((\iota^*_{j,k}E^{\mathfrak{m}}_1))^{-1}$$

living in $\mathrm{A}_N\left(|X|^N_{j,k}; \det^{-1}\left(N^{vir}_{\iota_{j,k}}\right)\right)\left[\left(M^X_{j,k} \cdot e\right)^{-1}\right]$.
Denoting $M^X_j := \prod_k M^X_{j,k}$ we can also define:

$$e_N\left(N^{vir}_{\iota_j}\right) := \left\{e_N\left(N^{vir}_{\iota_{j,k}}\right)\right\}_k \in \mathrm{A}_N\left(|X|^N_j; \det^{-1}\left(N^{vir}_{\iota_j}\right)\right)\left[\left(M^X_j \cdot e\right)^{-1}\right]$$

where we use the identification:

$$\mathrm{A}_N\left(|X|_j^N;\det{}^{-1}\left(N_{\iota_j}^{vir}\right)\right)\left[\left(M_j^X\cdot e\right)^{-1}\right]\simeq$$
$$\simeq\prod_k \mathrm{A}_N\left(|X|_{j,k}^N;\det{}^{-1}\left(N_{\iota_{j,k}}^{vir}\right)\right)\left[\left(M_{j,k}^X\cdot e\right)^{-1}\right]$$

Remark 4.3.9. From theorem 4.1.18, we also have an integer used in the Atiyah-Bott localization theorem for X that we will denote as M_0.

Theorem 4.3.10 (Virtual Localization Formula)**.** *Let* $\mathrm{A}\in\mathrm{SH}(\Bbbk)$ *be an* $SL[\eta^{-1}]$*-oriented ring spectrum. Let* $\iota: X\hookrightarrow Y$ *be a closed immersion in* $\mathbf{Sch}^N_{/\Bbbk}$*, with* Y *a smooth* N*-scheme. Let* $\varphi_\bullet:\mathcal{E}_\bullet\to\mathbb{L}_{X/\Bbbk}$ *be an* N*-linearised perfect obstruction theory. Suppose the* N*-action on* X *is strict. With the notation introduced in definition 4.3.8, we set:*

$$M:=M_0\cdot\prod_{i,j}M_i^X\cdot M_j^Y$$

Let $\left[|X|_j^N,\varphi_\bullet^{(j)}\right]_N^{vir}\in\mathrm{A}^{\mathrm{BM}}_{\bullet,N}\left(|X|_j^N/\Bbbk,\iota_j^*\mathcal{E}_\bullet^{\mathfrak{f}}\right)$ *the* N*-equivariant virtual fundamental class for the* N*-linearised perfect obstruction theory* $\varphi_\bullet^{(j)}$ *on* $|X|_j^N$*. Then we have:*

$$[X,\varphi]_N^{vir}=\sum_{j=1}^s\iota_{j*}\left(\left[|X|_j^N,\varphi^{(j)}\right]_N^{vir}\cap e_N\left(N_{\iota_j}^{vir}\right)^{-1}\right)\in\mathrm{A}_N^{\mathrm{BM}}\left(X,\mathcal{E}_\bullet\right)\left[(M\cdot e)^{-1}\right]$$

Proof. We have developed along the way all the necessary tools used in the proof [Lev22b, Theorem 6.7], upgrading them to the case of an $SL[\eta^{-1}]$-oriented ring spectrum. Then the very same strategy used in *loc. cit.* works also in our case. For completeness, we will sketch now the proof for a general $SL[\eta^{-1}]$-oriented ring spectrum, but no claim of originality is made here.

Now consider $X_j=Y_j\cap X^T$. By proposition 4.1.17, the localised Borel-Moore homology of X^T_{ind} vanishes. By assumption, our action is strict and thus we have $X_j=|X|_j^N\amalg X_j\cap X^T_{ind}$. By a localization sequence argument, we can replace X with $X\setminus X^T_{ind}$ and Y with $Y\setminus X^T_{ind}$, so without loss of generality we can assume $X^T_{ind}=\emptyset$ and $X_j=|X|_j^N$. Let us denote the following inclusions:

$$\iota: X\hookrightarrow Y \qquad\qquad i_j: X_j\hookrightarrow X$$
$$\iota_j: X_j\hookrightarrow Y_j \qquad\qquad i_j^Y: Y_j\hookrightarrow Y$$

The N-equivariant fundamental class $[Y]_N$ of Y lives in $\mathrm{A}_N^{\mathrm{BM}}\left(Y/S,\Omega_{Y/\Bbbk}\right)$. The fixed locus Y^T is smooth over $\Bbbk$, with connected components $Y_1,\dots,Y_s$. Let $[Y_j]_N\in\mathrm{A}_N^{\mathrm{BM}}(Y_j/\Bbbk,\Omega_{Y_j/\Bbbk})$ be the fundamental classes of the connected components. Since the normal bundle associated to $\iota_j^Y: Y_j\hookrightarrow Y$ is given by $(\iota_j^Y)^*T^{\mathfrak{m}}_{Y/\Bbbk}$, by theorem 4.2.4 and remark 4.2.5 we have that:

$$[Y]_N=\sum_{j=1}^s(\iota_j^Y)_*\left([Y_j]_N\cap e_N\left((\iota_j^Y)^*T^{\mathfrak{m}}_{Y/\Bbbk}\right)^{-1}\right)\in\mathrm{A}_N^{\mathrm{BM}}\left(Y/\Bbbk,\Omega_{Y/\Bbbk}\right)\left[(M_Ye)^{-1}\right]$$

where we used the functoriality of lci fundamental classes (cf. [DJK21, Theorem 4.2.1]) to identify $(\iota_j^Y)^![Y]_N = [Y_j]_N$. Consider the following cartesian square:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "Id_Y"] \\ Y_j \arrow[r, hook, "\iota_j^Y"'] & Y \end{tikzcd}$$

If we take the refined intersection product with respect to $\iota : X \hookrightarrow Y$ and $Id_Y : Y \to Y$, by compatibility with proper pushforwards (proposition 1.5.10), from the equation above we get:

$$[X,\varphi]^{vir}_N = [X,\varphi]^{vir}_N *_{\iota, Id_Y} [Y]_N = \sum_{j=1}^{s} (\iota_j^Y)_* \left([X,\varphi]^{vir}_N *_{\iota,\iota_j^Y} [Y_j]_N \cap e_N \left((\iota_j^Y)^* T^{\mathfrak{m}}_{Y/\Bbbk} \right)^{-1} \right)$$

We will then only need to show that:

Claim 2.

$$[X,\varphi]^{vir}_N *_{\iota,\iota_j^Y} [Y_j]_N \cap e_N \left((\iota_j^Y)^* T^{\mathfrak{m}}_{Y/\Bbbk} \right)^{-1} = [X_j, \varphi^{(j)}]^{vit}_N \cap e_N \left(N^{vir}_{\iota_j} \right)^{-1}$$

Notation 4.3.11. To make the rest of the proof easier to read, we will change our notation a bit. Given a cartesian square:

$$\begin{tikzcd} X \arrow[r, hook, "g"] \arrow[d] \arrow[dr, phantom, "\Delta"] & Y \arrow[d] \\ Z \arrow[r, hook, "f"'] & W \end{tikzcd}$$

where f is lci, we will denote the refined Gysin map as $f^! := g^!_\Delta$. And for any given vector bundle $E \to W$, we will denote its zero section as $s_E : W \hookrightarrow E$.

Let us briefly recall our construction of the equivariant virtual fundamental class from section 1.6.2. We have the cone $D := \mathfrak{C}_{X/Y} \times \mathbb{V}(\mathcal{E}_0) = C_{X/Y} \times E_0$, with quotient $D^{vir} := D/\iota^* T_{Y/\Bbbk}$. The virtual cone D^{vir} has a closed immersion $\iota_\varphi : D^{vir} \hookrightarrow E_1 := \mathbb{V}(\mathcal{E}_1)$ in $\mathbf{Sch}^N_{/\Bbbk}$. Then the virtual class was defined as:

$$[X,\varphi]^{vir}_N := s^!_{E_1} \left((\iota_\varphi)_* [D^{vir}]_N \right) \in \mathrm{A}^{\mathrm{BM}}_N (X/\Bbbk, E_\bullet)$$

For each X_j the N-linearised obstruction theory is given by $\varphi^{(j)} : i_j^* \mathcal{E}^{\mathfrak{f}}_\bullet \to \tau_{\leq 1} \mathbb{L}_{X_j/\Bbbk}$. Denoting by $D_j := \mathfrak{C}_{X_j/Y_j} \times \iota_j^* E^{\mathfrak{f}}_0$ and $D_j^{vir} := D_j^{vir} / \iota_j^* T_{Y_j}$ the corresponding cones, we have:

$$[X_j, \varphi^{(j)}]^{vir}_N := s^!_{i_j^* E^{\mathfrak{f}}_1} \left((\iota_{\varphi^{(j)}})_* [D_j^{vir}]_N \right) \in \mathrm{A}^{\mathrm{BM}}_N \left(X_j/\Bbbk, E^{\mathfrak{f}}_\bullet \right)$$

Let us start with the proof of the following:

Claim 3. *Consider the cartesian squares:*

$$
\begin{tikzcd}
\iota^*T_Y \arrow[r, "t_Y"] \arrow[d] \arrow[dr, phantom, "\Delta_1"] & D \arrow[d] \\
X \arrow[r] \arrow[u, bend left, "s_{\iota^*T_Y}"] & E_1
\end{tikzcd}
\quad
\begin{tikzcd}
\iota_j^*T_{Y_j} \arrow[r, "t_{Y_j}"] \arrow[d] \arrow[dr, phantom, "\Delta_2"] & D_j \arrow[d] \\
X_j \arrow[r] & i_j^*E_1^{\mathfrak{f}}
\end{tikzcd}
\quad
\begin{tikzcd}
i_j^*\iota^*T_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta_{Y_j}"] & D_j \times i_j^*E_0^{\mathfrak{m}} \arrow[d] \\
X_j \arrow[r] \arrow[u, bend left] & i_j^*E_1
\end{tikzcd}
$$

The refined Gysin pullbacks associated to the squares above will give us:

$$[X, \varphi]_N^{vir} = s^!_{\iota^*T_Y} s^!_{E_1}[D]_N$$

$$[X_j, \varphi^{(j)}]_N^{vir} = s^!_{\iota_j^*T_{Y_j}} s^!_{i_j^*E_1^{\mathfrak{f}}}[D_j]_N$$

$$[X, \varphi]_N^{vir} *_{\iota, i_j^Y} [Y_j]_N = s^!_{i_j^*\iota^*T_Y} s^!_{i_j^*E_1} \left[D_j \times i_j^*E_0^{\mathfrak{m}}\right]$$

The first two equations in claim 3 follow from the squares Δ_1, Δ_2 and the functoriality of (refined) Gysin pullbacks (remark 1.5.7). Let $\beta_j^D : D_j \times i_j^*E_0^{\mathfrak{m}} \hookrightarrow D$ be the closed immersion of cones, then an application of the equivariant Vistoli's lemma proposition 1.6.16 tells us that:

$$(\beta_j^D)_* \left[D_j \times i_j^*E_0^{\mathfrak{m}}\right]_N = \pi^!_{Y_j}[D]_N \tag{4.2}$$

where $\pi^!_{Y_j}$ denotes the refined Gysin pullback with respect to $\pi_{Y_j} : Y_j \to S$. Then using eq. (4.2) and the commutativity of refined Gysin pullbacks, we get the third equation in claim 3 (see [Lev22b, Theorem 6.7, Proof: Step 4] for more details).

(Claim 2)
$\blacksquare$

Notice that $(i_j^Y)^*T_Y^{\mathfrak{f}} \simeq T_{Y_j}$, hence $(i_j^Y)^*T_Y \simeq T_{Y_j} \oplus (i_j^Y)^*T_Y^{\mathfrak{m}}$. Then we have:

$$s^!_{i_j^*\iota^*T_Y} s^!_{i_j^*E_1} \left[D_j \times i_j^*E_0^{\mathfrak{m}}\right] = s^!_{i_j^*\iota^*T_Y^{\mathfrak{m}}} s^!_{i_j^*E_1} \left[D_j^{vir} \times i_j^*E_0^{\mathfrak{m}}\right]$$

Claim 4. *We have that:*

$$[X, \varphi]_N^{vir} *_{\iota, i_j^Y} [Y_j]_N \cap e_N\left(i_j^*E_0^{\mathfrak{m}}\right) = s^!_{i_j^*E_0^{\mathfrak{m}}}\left(s^!_{i_j^*E_1}[D_j^{vir} \times i_j^*E_0^{\mathfrak{m}}]_N\right) \cap e_N\left(i_j^*\iota^*T_Y^{\mathfrak{m}}\right)$$

We have a closed immersion $D_j^{vir} \times_{X_j} i_j^*E_0^{\mathfrak{m}} \hookrightarrow i_j^*D/\iota_j^*T_{Y_j}$ and composing this map with the natural map $i_j^*D/\iota_j^*T_{Y_j} \to i_j^*E_1$, we get a map $\sigma : D_j^{vir} \times_{X_j} i_j^*E_0^{\mathfrak{m}} \to i_j^*E_1$. Denote by $\mathcal{Z}_\sigma\left(D_j^{vir} \times_{X_j} i_j^*E_0^{\mathfrak{m}}\right)$ the scheme-theoretic pullback of $D_j^{vir} \times_{X_j} i_j^*E_0^{\mathfrak{m}}$ along the zero section of $i_j^*E_1$. There exists a commutative diagram in $\mathbf{Sch}^N_{/\Bbbk}$:

$$\begin{tikzcd}
\mathcal{Z}_\sigma\left(D_j^{vir}\times_{X_j} i_j^*E_0^{\mathfrak{m}}\right) \arrow[r, hook, "f"] \arrow[d, hook, "g"'] & i_j^*E_0^{\mathfrak{m}} \arrow[d] \\
i_j^*\iota^*T_Y^{\mathfrak{m}} \arrow[r] & X_j
\end{tikzcd}$$

Let $\alpha := s^!_{i_j^*E_1}\left[D_j^{vir}\times_{X_j} i_j^*E_0^{\mathfrak{m}}\right]$. By the (equivariant) excess intersection formula (cf. [DJK21, Prop. 3.3.4]), we have:

$$s_{i_j^*\iota^*T_Y^{\mathfrak{m}}}\left(f_*(\alpha)\cap e_N\left(i_j^*E_0^{\mathfrak{m}}\right)\right) = s_{i_j^*E_0^{\mathfrak{m}}}\left(g_*(\alpha)\cap e_N\left(i_j^*\iota^*T_Y^{\mathfrak{m}}\right)\right)$$

and this gives us the formula of claim 4.

(Claim 3) ■

Now we can finally prove claim 2. We have a natural map $(\iota_{\varphi(j)}, d^{\mathfrak{m}}) : D_j^{vir}\times i_j^*E_0^{\mathfrak{m}} \to i_j^*E_1^{\mathfrak{m}}$, induced by the inclusion $\iota_{\varphi(j)}$ and by the "moving" differential $d^{\mathfrak{m}}$ on $\mathcal{E}_\bullet^{\mathfrak{m}}$. By $\mathbb{A}^1$-homotopy invariance, we can suppose that $(\iota_{\varphi(j)}, d^{\mathfrak{m}})$ factors via the first coordinate projection $D_j^{vir}\times i_j^*E_0^{\mathfrak{m}} \xrightarrow{p_1} D_j^{vir}$, and the natural inclusion map $D_j^{vir}\hookrightarrow i_j^*E_1^{\mathfrak{m}}\subseteq i_j^*E_1$. By the equivariant excess intersection formula, we have:

$$\begin{aligned}
s^!_{i_j^*E_0^{\mathfrak{m}}}s^!_{i_j^*E_1^{\mathfrak{m}}}\left[D_j^{vir}\times i_j^*E_0^{\mathfrak{m}}\right]_N &= \left(s^!_{i_j^*E_1^{\mathfrak{m}}}\left[D_j^{vir}\times i_j^*E_0^{\mathfrak{m}}\right]\right)\cap e_N\left(i_j^*E_0^{\mathfrak{m}}\right) = \\
&= \left[X_j,\varphi^{(j)}\right]_N^{vir}\cap e_N\left(i_j^*E_1^{\mathfrak{m}}\right)
\end{aligned}$$

Putting together all the statements we proved, we got:

$$\begin{aligned}
&[X,\varphi]_N^{vir} *_{\iota,\iota_j^Y} [Y_j]_N\cap e_N\left((\iota_j^Y)^*T^{\mathfrak{m}}_{Y/\Bbbk}\right)^{-1} = \\
&\qquad = s^!_{i_j^*\iota^*T_Y}s^!_{i_j^*E_1}\left[D_j\times i_j^*E_0^{\mathfrak{m}}\right]\cap e_N\left((\iota_j^Y)^*T^{\mathfrak{m}}_{Y/\Bbbk}\right)^{-1} = \\
&\qquad = s^!_{i_j^*\iota^*T_Y^{\mathfrak{m}}}s^!_{i_j^*E_1}\left[D_j^{vir}\times i_j^*E_0^{\mathfrak{m}}\right]\cap e_N\left((\iota_j^Y)^*T^{\mathfrak{m}}_{Y/\Bbbk}\right)^{-1} = \\
&\qquad = s^!_{i_j^*E_0^{\mathfrak{m}}}\left(s^!_{i_j^*E_1}[D_j^{vir}\times i_j^*E_0^{\mathfrak{m}}]_N\right)\cap\left(e_N\left(i_j^*E_0^{\mathfrak{m}}\right)\right)^{-1} = \\
&\qquad = \left[X_j,\varphi^{(j)}\right]_N^{vir}\cap e_N\left(i_j^*E_1^{\mathfrak{m}}\right)\cap\left(e_N\left(i_j^*E_0^{\mathfrak{m}}\right)\right)^{-1}
\end{aligned}$$

that is exactly our claim 2. Thus we have just proved our virtual localization theorem. □

Corollary 4.3.12 (Virtual Localization For Witt Theory)**.** *In the same situation as in theorem 4.3.10, for* $\mathrm{A}=\mathrm{KW}$ *we get:*

$$[X,\varphi]_N^{vir} = \sum_{j=1}^{s}\iota_{j*}\left(\left[|X|_j^N,\varphi^{(j)}\right]_N^{vir}\cap e_N\left(N_{\iota_j}^{vir}\right)^{-1}\right)\in \mathrm{KW}_N^{\mathrm{BM}}(X,E_\bullet)\left[(M\cdot e)^{-1}\right]$$

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