# On parahoric $(\mathcal{G}, \mu)$-displays and the EKOR stratification for Shimura varieties of Hodge type

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# Abstract

We extend the definition of parahoric (Dieudonné) $(\mathcal{G}, \boldsymbol{\mu})$-displays given by Pappas to not necessarily $p$-torsionfree base rings. We also introduce the notion of an $(m, n)$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-display. Then we study the deformation theory of Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays. When $\mathcal{G}$ is a parahoric group scheme for a general linear or a general symplectic group we give an explicit moduli description of $(\mathcal{G}, \boldsymbol{\mu})$-displays and their variants in terms of (homogeneously polarized) lattice chains.

As an application we realize the EKOR stratification of the special fiber of a Kisin-Pappas integral Shimura variety of Hodge type as the fibers of a smooth morphism into the algebraic stack of $(2, 1\text{-rdt})$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays.



# Zusammenfassung

Wir erweitern Pappas' Definition von parahorischen (Dieudonné) $(\mathcal{G}, \boldsymbol{\mu})$-Displays auf nicht notwendigerweise $p$-torsionsfreie Basisringe. Wir definieren außerdem den Begriff eines $(m, n)$-abgeschnittenen $(\mathcal{G}, \boldsymbol{\mu})$-displays. Dann studieren wir die Deformationstheorie von Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-Displays. Im Fall, dass $\mathcal{G}$ ein parahorisches Gruppenschema für eine allgemeine lineare Gruppe oder eine allgemeine symplektische Gruppe ist geben wir eine explizite modultheoretische Beschreibung von $(\mathcal{G}, \boldsymbol{\mu})$-Displays und ihren Varianten in Termen von (homogen polarisierten) Gitterketten.

Als Anwendung realisieren wir die EKOR-Stratifizierung auf der speziellen Faser einer Kisin-Pappas ganzzahligen Shimura-Varietät von Hodge Typ als die Fasern einer glatten Abbildung in den algebraischen Stack von $(2, 1\text{-rdt})$-abgeschnittenen $(\mathcal{G}, \boldsymbol{\mu})$-Displays.

# Introduction

Fix a rational prime $p$, an integer $g \geq 1$ and a subset $J \subseteq \mathbf{Z}$ with $J + 2g\mathbf{Z} \subseteq J$ and $-J \subseteq J$. Denote by $\mathbf{L}_p \subseteq \mathrm{GSp}_{2g}(\mathbf{Q}_p)$ the parahoric subgroup that is the stabilizer of some self-dual $\mathbf{Z}_p$-lattice chain of type $J$ in $\mathbf{Q}_p^{2g}$. Also fix a (small enough) compact open subgroup $\mathbf{L}^p \subseteq \mathrm{GSp}_{2g}(\mathbf{A}_f^p)$ and set $\mathbf{L} := \mathbf{L}_p\mathbf{L}^p$. The associated Siegel Shimura variety $\mathrm{Sh}_{\mathbf{L}} = \mathrm{Sh}_{\mathbf{L}}(\mathrm{GSp}_{2g}, S^{\pm})$ then has a moduli description that was first given by de Jong in [Jon93] and then in full generality by Rapoport and Zink in [RZ96]; it parametrizes certain polarized chains of $g$-dimensional Abelian varieties of type $J$ with $\mathbf{L}^p$-level structure. This moduli description gives rise to a natural integral model $\mathscr{S}_{\mathbf{L}}$ of $\mathrm{Sh}_{\mathbf{L}}$ over $\mathbf{Z}_p$ whose special fiber is typically not smooth.

There exists a natural map, called the central leaves map,

$$\Upsilon\colon \mathscr{S}_{\mathbf{L}}\big(\overline{\mathbf{F}}_p\big) \to \breve{\mathbf{L}}_{p,\sigma}\backslash \mathrm{GSp}_{2g}(\breve{\mathbf{Q}}_p);$$

here $\breve{\mathbf{L}}_p = \mathcal{G}(\breve{\mathbf{Z}}_p)$ is the group of $\breve{\mathbf{Z}}_p$-valued points of the parahoric group scheme $\mathcal{G}$ over $\mathbf{Z}_p$ corresponding to $\mathbf{L}_p$ and $\breve{\mathbf{L}}_{p,\sigma}$ denotes the twisted conjugation action $g.x = gx\sigma^{-1}(g)^{-1}$. It is roughly given by sending a polarized chain of Abelian varieties to the twisted conjugacy class corresponding to the Frobenius $\Phi^{\mathrm{cov}}$ of the associated covariant rational Dieudonné module (more precisely to $p \cdot \sigma^{-1}(\Phi^{\mathrm{cov},-1})$), see the work of Oort ([Oor04]) and He and Rapoport ([HR17]). The image of this map is given by $\breve{\mathbf{L}}_{p,\sigma}\backslash X$ where

$$X = \breve{\mathbf{L}}_p \operatorname{Adm}_{\mathcal{G}}(\boldsymbol{\mu}_g)\breve{\mathbf{L}}_p \subseteq \mathrm{GSp}_{2g}(\breve{\mathbf{Q}}_p)$$

is a finite union of double cosets.

The fibers of the composition

$$\lambda\colon \mathscr{S}_{\mathbf{L}}\big(\overline{\mathbf{F}}_p\big) \xrightarrow{\Upsilon} \breve{\mathbf{L}}_{p,\sigma}\backslash X \to \breve{\mathbf{L}}_p\backslash X/\breve{\mathbf{L}}_p = \operatorname{Adm}_{\mathcal{G}}(\boldsymbol{\mu}_g)$$

define a stratification of $\mathscr{S}_{\mathbf{L},\overline{\mathbf{F}}_p}$ by smooth locally closed subschemes that is called the Kottwitz-Rapoport (KR) stratification. In fact Rapoport and Zink construct the following data.

- A flat projective scheme $\mathbb{M}^{\mathrm{loc}} = \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}_g}$ over $\mathbf{Z}_p$ with a $\mathcal{G}$-action called the local model, that parametrizes isotropic chains of $g$-dimensional subspaces of the given self-dual lattice chain. It satisfies $\mathbb{M}^{\mathrm{loc}}(\overline{\mathbf{F}}_p) = X/\breve{\mathbf{L}}_p$.

- A smooth morphism of algebraic stacks

$$\mathscr{S}_{\mathbf{L}} \to \big[\mathcal{G}\backslash\mathbb{M}^{\mathrm{loc}}\big]$$

that parametrizes the Hodge filtration in the de Rham cohomology of a polarized chain of Abelian varieties and gives back the map $\mathscr{S}_{\mathbf{L}}(\overline{\mathbf{F}}_p) \to \mathrm{Adm}_{\mathcal{G}}(\boldsymbol{\mu}_g)$ after taking $\overline{\mathbf{F}}_p$-valued points.

He and Rapoport also consider the composition

$$\upsilon\colon \mathscr{S}_{\mathbf{L}}(\overline{\mathbf{F}}_p) \xrightarrow{\Upsilon} \breve{\mathbf{L}}_{p,\sigma}\backslash X \to \breve{\mathbf{L}}_{p,\sigma}\backslash (X/\breve{\mathbf{L}}_{p,1}) = \breve{\mathbf{L}}_{p,\sigma}(\breve{\mathbf{L}}_{p,1} \times \breve{\mathbf{L}}_{p,1})\backslash X$$

where $\breve{\mathbf{L}}_{p,1} \subseteq \breve{\mathbf{L}}_p$ is the pro-unipotent radical, and call the (finitely many) fibers of $\upsilon$ Ekedahl-Kottwitz-Oort-Rapoport (EKOR) strata. These EKOR strata can be shown to be smooth by comparing them to the KR strata at Iwahori level $J = \mathbf{Z}$, using a result of Görtz and Hoeve (see [GH12]).

In the hyperspecial case $J = 2g\mathbf{Z}$ the EKOR stratification is also just called the Ekedahl-Oort (EO) stratification and was first considered by Oort (see [Oor01]). Two points in $\mathscr{S}_{\mathbf{L}}(\overline{\mathbf{F}}_p)$ corresponding to two polarized Abelian varieties $A$ and $A'$ lie in the same EO stratum if and only if $A[p] \cong A'[p]$. Viehmann and Wedhorn (see [VW13]) realize the EO stratification as the fibers of a smooth morphism from $\mathscr{S}_{\mathbf{L},\mathbf{F}_p}$ into a certain algebraic stack of zips with symplectic structure (in the sense of Moonen-Wedhorn and Pink-Wedhorn-Ziegler, see [MW04] and [PWZ15]).

Thus it is natural to ask the following question.

**Question 0.1.** *Is it possible to naturally realize the map $\upsilon$ (or maybe even $\Upsilon$) as a smooth morphism from $\mathscr{S}_{\mathbf{L},\mathbf{F}_p}$ to some algebraic stack that is defined in terms of linear algebraic/group theoretic data?*

The existence of such a smooth morphism would in particular give a new proof of the smoothness of the EKOR strata and the closure relations between them. More importantly it could also provide a tool for studying the geometry of $\mathscr{S}_{\mathbf{L},\mathbf{F}_p}$ and cycles on it.

The goal of this thesis is to give an affirmative answer to Question 0.1 and its natural generalization to Shimura varieties of Hodge type. To this end we develop a theory of certain truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays, extending work of Pappas ([Pap23]). Then, using results of Hamacher and Kim ([HK19]) and Pappas, we construct a morphism from the $p$-completion of the Shimura variety to the moduli space of these objects and show that it is smooth.

Let us give an overview of results that have been obtained so far, also considering more general Shimura varieties than the Siegel modular variety.

- Moonen and Wedhorn ([MW04]) introduce the notion of an $F$-zip that is a characteristic $p$ analog of the notion of a Hodge structure. Given an Abelian variety $A$ over some $\mathbf{F}_p$-algebra $R$ the de Rham cohomology $H^1_{\mathrm{dR}}(A/R)$ naturally is equipped the structure of an $F$-zip. If $R$ is perfect then the datum of $H^1_{\mathrm{dR}}(A/R)$ with its $F$-zip structure is equivalent to the datum of the Dieudonné module of the $p$-torsion $A[p]$.

  Pink, Wedhorn and Ziegler ([PWZ15]) define a group theoretic version of the notion of an $F$-zip (that is called $\mathcal{G}$-zip).

- Viehmann and Wedhorn ([VW13]) define a moduli space of $F$-zips with polarization and endomorphism structure (that they call $\mathcal{D}$-zips) in a PEL-type situation with hyperspecial level structure (that in particular includes the hyperspecial Siegel case). They construct a morphism from the special fiber of the associated Shimura variety to this stack of $\mathcal{D}$-zips that parametrizes the EO stratification. Then they show that this morphism is flat and use this to deduce that the EO strata are non-empty and quasi-affine and to compute their dimension (smoothness of the EO strata was already shown by Vasiu in [Vas06]).

  Zhang ([Zha18]) constructs a morphism from the special fiber of the Kisin integral model (see [Kis10]) of a Hodge type Shimura variety at hyperspecial level to the group-theoretic stack of $\mathcal{G}$-zips. They then show that this morphism is smooth and thus gives an EO stratification with the desired properties. Shen and Zhang ([SZ22]) later generalize this to Shimura varieties of Abelian type.

  Hesse ([Hes20]) considers an explicit moduli space of chains of $F$-zips with polarization and constructs a morphism from the Siegel modular variety at parahoric level into this stack. However it appears that such a morphism is not well-behaved in a non-hyperspecial situation.

- Xiao and Zhu ([XZ17]) construct a perfect stack of local shtukas $\mathsf{Sht}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ as well as truncated versions $\mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$ in a hyperspecial situation. For a Shimura variety of Hodge type (still at hyperspecial level) they construct a morphism from the perfection of its special fiber into $\mathsf{Sht}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ that gives a central leaves map $\Upsilon$. They claim that the induced morphisms to $\mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$ are perfectly smooth (although it seems as if the argument in their proof does not work). They also construct a natural perfectly smooth forgetful morphism from $\mathsf{Sht}^{\mathrm{loc},(2,1)}_{\mathcal{G},\boldsymbol{\mu}}$ to the perfection of the stack of $\mathcal{G}$-zips and in particular recover the smoothness result of Zhang after perfection.

- Shen, Yu and Zhang ([SYZ21]) generalize the previous work of Zhang, Shen-Zhang and Xiao-Zhu to Shimura varieties of Abelian type at parahoric level (where the construction of integral models is due to Kisin and Pappas, see [KP18]). They give two constructions.

  - They construct a morphism from each KR stratum into a certain stack of $G$-zips, parametrizing the EKOR strata contained in this KR stratum. They also show that this morphism is smooth, thus establishing the smoothness of the EKOR strata.
  - They also construct perfect stacks of local shtukas $\mathsf{Sht}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ and truncated versions $\mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$ in the parahoric situation and a morphism from the perfection of the special fiber of the Shimura variety to $\mathsf{Sht}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$, realizing $\Upsilon$, such that the induced morphisms to $\mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$ are perfectly smooth (the proof is by the same argument as in [XZ17] and there seems to be the same problem).

- Zink ([Zin02]) defines a stack of (not necessarily nilpotent) displays over $\mathrm{Spf}(\mathbf{Z}_p)$. Bültel and Pappas ([BP20]) define a group theoretic version of this notion in a hyperspecial situation (that is called $(\mathcal{G},\mu)$-display). In characteristic $p$ this gives a deperfection of the notion of local shtuka from [XZ17].

  In the parahoric Hodge type situation there is a definition of $(\mathcal{G},\boldsymbol{\mu})$-display due to Pappas (see [Pap23]) but this definition is only over $p$-torsionfree $p$-adic rings. Using results of Hamacher and Kim ([HK19]) Pappas also constructs a parahoric display on the $p$-completion of the integral model and shows that it is locally universal.

- In [Hof22] I define the notion of a homogeneously polarized chain of displays and certain truncated variants. Then I construct a smooth morphism from the special fiber of a Siegel modular variety at parahoric level to the moduli stack of these truncated objects.

# Content

Let $p$ be a rational prime not equal to 2.

## $(m,n)$-truncated displays

Let $R$ be a $p$-complete ring. Denote by $W(R)$ the associated ring of Witt vectors, by $I_R \subseteq W(R)$ the augmentation ideal, i.e. the kernel of the projection $W(R) \to R$, by $\sigma\colon W(R) \to W(R)$ the Frobenius morphism and by $\sigma^{\mathrm{div}}\colon I_R^\sigma \to W(R)$ the divided Frobenius, i.e. the linearization of the inverse of the Verschiebung morphism. Having set up this notation, we now recall Zink's definition of a (not necessarily nilpotent) display.

**Definition 0.1** ([Zin02, Definition 1])**.** A *display over $R$* is a tuple $(M, M_1, \Phi, \Phi_1)$ that is given as follows.

1. $M$ is a finite projective $W(R)$-module.

2. $M_1 \subseteq M$ is a $W(R)$-submodule such that we have $I_R M \subseteq M$ and such that $M_1/I_R M \subseteq M/I_R M$ is a direct summand.

3. $\Phi\colon M^\sigma \to M$ and $\Phi_1\colon M_1^\sigma \to M$ are morphisms of $W(R)$-modules such that $\Phi_1$ is surjective and the diagram

$$\begin{tikzcd} (I_R M)^\sigma \arrow[r] \arrow[d] & M_1^\sigma \arrow[d, "\Phi_1"] \\ I_R^\sigma \otimes_{W(R)} M^\sigma \arrow[r, "\sigma^{\mathrm{div}} \otimes \Phi"] & M \end{tikzcd}$$

   is commutative.

There is a natural way of associating to a tuple $(M, M_1)$ satisfying 1. and 2. above (we call such a tuple a *pair*) a finite projective $W(R)$-module $\widetilde{M_1}$ that is given by $\widetilde{M_1} = \mathrm{im}(M_1^\sigma \to M^\sigma)$ in the case that $W(R)$ is $p$-torsionfree. Giving $(\Phi, \Phi_1)$ is then equivalent to giving an isomorphism of $W(R)$-modules $\Psi\colon \widetilde{M_1} \to M$. So we see that displays can also be viewed as tuples $(M, M_1, \Psi)$. This is the point of view that we will use.

When $R$ is a complete Noetherian local ring with perfect residue field of characteristic $p$ then Zink also defines the notion of a *Dieudonné display over* $R$ in [Zin01]. The definition of a Dieudonné display is the same as that of a display, up to replacing the ring of Witt vectors $W(R)$ with the Zink ring $\widehat{W}(R) \subseteq W(R)$.

Now also fix a tuple $(m, n)$ of positive integers such that $m \geq n + 1$. Denote by $W_m(R)$ and $W_n(R)$ the rings of truncated Witt vectors of length $m$ and $n$ respectively and by $I_{m,R} \subseteq W_m(R)$ and $I_{n,R} \subseteq W_n(R)$ their augmentation ideals. The Frobenius $\sigma$ on $W(R)$ then induces $\sigma\colon W_m(R) \to W_n(R)$. We can now make the following definition.

**Definition 0.2** (Definition 2.1.9)**.** An $(m, n)$*-truncated display over* $R$ is a tuple $(M, M_1, \Psi)$ that is given as follows.

- $M$ is a finite projective $W_m(R)$-module.
- $M_1 \subseteq M$ is a $W_m(R)$-submodule such that we have $I_{m,R}M \subseteq M_1$ and such that $M_1/I_{m,R}M \subseteq M/I_{m,R}$ is a direct summand.
- $\Psi\colon \widetilde{M_1} \to W_n(R) \otimes_{W_m(R)} M$ is an isomorphism of $W_n(R)$-modules; here $\widetilde{M_1}$ is a finite projective $W_n(R)$-module that is naturally attached to $(M, M_1)$ similarly as in the non-truncated situation.

There already exists the notion of an *n-truncated display* that is due to Lau in characteristic $p$ ([Lau13]) and due to Lau and Zink in general ([LZ18]). The two notions can be related; there exists a natural forgetful functor

$$\big\{(m, n)\text{-truncated displays}\big\} \to \big\{n\text{-truncated displays}\big\}$$

that can be thought of as forgetting some of the information that is contained in the $W_m(R)$-module $M$ (but it remembers more than just the base change $W_n(R) \otimes_{W_m(R)} M$). Our $(m, n)$-truncated displays can be thought of as more naive versions of $n$-truncated displays. But it seems to us that for defining truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays in a parahoric situation, these $(m, n)$-truncated objects are better suited.

## $(\mathcal{G}, \boldsymbol{\mu})$-displays

Let $(\mathcal{G}, \boldsymbol{\mu})$ be a local model datum, i.e. a tuple consisting of a parahoric $\mathbf{Z}_p$-group scheme $\mathcal{G}$ with generic fiber $G = \mathcal{G}_{\mathbf{Q}_p}$ and a minuscule $G(\overline{\mathbf{Q}}_p)$-conjugacy class $\boldsymbol{\mu}$ of cocharacters $\mu\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to G_{\overline{\mathbf{Q}}_p}$. Denote by $E$ the reflex field of $(\mathcal{G}, \boldsymbol{\mu})$ and by $\mathbf{F}_q$ its residue field. By work of Pappas-Zhu, Anschütz-Gleason-Lourenço-Richarz, Fakhruddin-Haines-Lourenço-Richarz and many others we then have a natural associated local model $\mathbb{M}^{\mathrm{loc}} = \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ that is a flat projective $\mathcal{O}_E$-scheme with a $\mathcal{G}$-action whose generic fiber identifies with the homogeneous space $X_{\boldsymbol{\mu}}(G)$ of parabolic subgroups of $G_E$ of type $\boldsymbol{\mu}^{-1}$. We denote by $\mathrm{M}^{\mathrm{loc}}$ the $p$-completion of $\mathbb{M}^{\mathrm{loc}}$. Let $R$ be a $p$-complete $\mathcal{O}_E$-algebra.

The following definition is given implicitly in Pappas' article ([Pap23]).

**Definition 0.3** (Definition 3.1.1)**.** A $(\mathcal{G}, \boldsymbol{\mu})$-*pair over* $R$ is a tuple $(\mathcal{P}, q)$ consisting of a $\mathcal{G}$-torsor $\mathcal{P}$ over $W(R)$ and a $\mathcal{G}$-equivariant morphism $q\colon \mathcal{P}_R \to \mathrm{M}^{\mathrm{loc}}$.

Note that $(\mathcal{G}, \boldsymbol{\mu})$-pairs $(\mathcal{P}, q)$ are group theoretic versions of pairs $(M, M_1)$ as considered above; the $\mathcal{G}$-torsor $\mathcal{P}$ corresponds to the module $M$ and the morphism $q$ corresponds to the filtration $M_1$. In the case $\mathcal{G} = \mathrm{GL}_{h,\mathbf{Z}_p}$ and $\boldsymbol{\mu} = \boldsymbol{\mu}'_d = (0^{(h-d)}, (-1)^{(d)})$ we have an equivalence

$$\left\{\text{pairs } (M, M_1) \;\middle|\; \begin{array}{c} M \text{ is of rank } h, \\ M_1/I_R M \text{ is of rank } d \end{array}\right\} \to \left\{(\mathrm{GL}_{h,\mathbf{Z}_p}, \boldsymbol{\mu}'_d)\text{-displays}\right\}.$$

When $(\mathcal{G}, \boldsymbol{\mu})$ is of Hodge type (see Notation 3.0.1 for more details) and $R$ is $p$-torsionfree then Pappas constructs a natural functor

$$\left\{(\mathcal{G}, \boldsymbol{\mu})\text{-pairs}\right\} \to \left\{\mathcal{G}\text{-torsors over } W(R)\right\}, \qquad (\mathcal{P}, q) \mapsto (\mathcal{P}, q)^{\sim}$$

that generalizes the construction $(M, M_1) \mapsto \widetilde{M_1}$ to the group-theoretic situation, and consequently is able to define the notion of a $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$.

**Definition 0.4** ([Pap23, Definition 4.2.2], see also Definition 3.1.8)**.** A $(\mathcal{G}, \boldsymbol{\mu})$-*display over* $R$ is a tuple $(\mathcal{P}, q, \Psi)$ where $(\mathcal{P}, q)$ is a $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$ and $\Psi\colon (\mathcal{P}, q)^{\sim} \to \mathcal{P}$ is an isomorphism of $\mathcal{G}$-torsors over $W(R)$.

Similarly, when $R$ is a $p$-torsionfree complete Noetherian local $\mathcal{O}_E$-algebra with perfect residue field of characteristic $p$, then Pappas also defines the notion of a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display.

We observe that Pappas' construction extends to the situation when $R$ is not necessarily $p$-torsionfree, allowing us to also extend the definition of $(\mathcal{G}, \boldsymbol{\mu})$-display.

We also define the notion of an $(m, n)$-*truncated* $(\mathcal{G}, \boldsymbol{\mu})$-*display* for $(m, n)$ as before. The groupoids of $(m, n)$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays over varying $p$-complete $\mathcal{O}_E$-algebras form a $p$-adic formal algebraic stack

$$\mathsf{Disp}^{(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$$

of finite presentation over $\mathrm{Spf}(\mathcal{O}_E)$.

When $R$ is an $\mathbf{F}_q$-algebra and $m \geq 2$ then we also define the notion of an $(m, 1\text{-rdt})$-*truncated* $(\mathcal{G}, \boldsymbol{\mu})$-*display over* $R$; here the symbol 1-rdt indicates that the isomorphism $\Psi$ is between $\mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p}$-torsors over $R$ (where $\mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p}$ denotes the reductive quotient of the special fiber of $\mathcal{G}$) instead of between $\mathcal{G}$-torsors over $R$ as it would be for $(m, 1)$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays. Again, the groupoids of $(m, 1\text{-rdt})$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays over varying

$\mathbf{F}_q$-algebras form an algebraic stack $\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}^{(m,1\text{-rdt})}$ of finite presentation over $\mathbf{F}_q$, and we have a natural bijection

$$\left|\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}^{(m,1\text{-rdt})}(\overline{\mathbf{F}}_p)\right| \to \breve{\mathbf{K}}_{p,\sigma}(\breve{\mathbf{K}}_{p,1} \times \breve{\mathbf{K}}_{p,1})\backslash \breve{\mathbf{K}}_p \operatorname{Adm}_{\mathcal{G}}(\boldsymbol{\mu})\breve{\mathbf{K}}_p$$

where $|\text{-}|$ denotes taking the set of isomorphism classes of a groupoid and $\breve{\mathbf{K}}_p = \mathcal{G}(\breve{\mathbf{Z}}_p)$ similarly as before.

When $\mathcal{G}$ is a parahoric group scheme for either a general linear or a general symplectic group then we give an explicit description of $(\mathcal{G}, \boldsymbol{\mu})$-displays in terms of (homogeneously polarized) chains of displays, see also our preprint [Hof22]. Here we use the description of $\mathcal{G}$-torsors given by Rapoport and Zink in [RZ96].

We then study the deformation theory of (Dieudonné) $(\mathcal{G}, \boldsymbol{\mu})$-displays. Here our main result is the following.

**Theorem 0.5** (Theorem 3.3.11)**.** *Let $\mathcal{P}_0 = (\mathcal{P}_0, q_0, \Psi_0)$ be a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$. Then $\mathcal{P}_0$ admits a universal deformation with an explicit description.*

This theorem can be viewed as a more conceptual version of [KP18, Proposition 3.2.17] and some of the arguments we use in the proof are inspired by the arguments of Kisin and Pappas.

Actually we should remark that our result is not unconditional. We assume a certain hypothesis (see Hypothesis 3.3.3) to be satisfied in order to construct the universal deformation. In fact we believe that this hypothesis is always satisfied, but we are unable to prove this. Note that the same hypothesis is also assumed implicitly in [KP18, Subsection 3.2.12] and that in [Pap23, Proposition 4.5.3] it is claimed that the hypothesis is always satisfied (but i think the proof contains a mistake).

## Application to Shimura varieties

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type (see Notation 1.8.13 for details), let $\mathbf{K} = \mathbf{K}_p\mathbf{K}^p \subseteq \mathbf{G}(\mathbf{A}_f)$ be a small enough compact open subgroup such that $\mathbf{K}_p$ is a parahoric stabilizer and denote by $\mathscr{S}_{\mathbf{K}}$ the associated integral Shimura variety over the ring of integers $\mathcal{O}_E$ of the local reflex field $E$ as defined by Kisin and Pappas in [KP18]. There is a local model datum $(\mathcal{G}, \boldsymbol{\mu})$ attached to the Shimura datum $(\mathbf{G}, \mathbf{X})$ and the parahoric subgroup $\mathbf{K}_p \subseteq \mathbf{G}(\mathbf{Q}_p)$ and by work of Hamacher and Kim ([HK19]) and Pappas the $p$-completion $\widehat{\mathscr{S}_{\mathbf{K}}}$ naturally supports a $(\mathcal{G}, \boldsymbol{\mu})$-display.

Using results of from [KP18] and [Pap23] we show the following theorem.

**Theorem 0.6** (Theorem 3.5.4, Theorem 4.8.3)**.** *Let $(m, n)$ be a tuple of positive integers with $m \geq n + 1$. Then the morphism*

$$\widehat{\mathscr{S}_{\mathbf{K}}} \to \mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}^{(m,n)}$$

*is smooth. Similarly, for a positive integer $m \geq 2$ also the morphism*

$$\mathscr{S}_{\mathbf{K},\mathbf{F}_q} \to \mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}^{(m,1\text{-rdt})}$$

*is smooth.*

The geometric fibers of the morphism $\mathscr{S}_{\mathbf{K},\mathbf{F}_q} \to \mathsf{Disp}^{(m,1\text{-rdt})}_{\mathcal{G},\boldsymbol{\mu}}$ are precisely the EKOR strata, so that this in particular answers Question 0.1 affirmatively.

After passing to to the perfection of the special fiber our theorem recovers the result [SYZ21, Theorem 4.4.3] by Shen, Yu and Zhang (whose proof seems problematic to us as was already remarked earlier), see Corollary 3.5.5, at least up to a slightly different normalization in the definition of $(m, n)$-restricted local shtukas.

Similarly, after restricting to a single KR stratum, we also recover [SYZ21, Theorem 3.4.11], see Corollary 3.5.6.

# Outline

- Chapter 1 is a collection of preliminary material; its sections can be read separately. We advise the reader to skip this chapter in the beginning and come back to it when needed.

- In Chapter 2 we review the theory of (Dieudonné) displays and introduce the new notion of an $(m, n)$-truncated display.

- Chapter 3 is the heart of this thesis. We define $(\mathcal{G}, \boldsymbol{\mu})$-displays, generalizing the definition from [Pap23], as well as Dieudonné and $(m, n)$-truncated variants. We study the deformation theory of Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays and then apply the theory to the Kisin-Pappas integral models for Shimura varieties of Hodge type at parahoric level.

- In Chapter 4 we give explicit linear algebraic descriptions of $(\mathcal{G}, \boldsymbol{\mu})$-displays when $\mathcal{G}$ is a parahoric group scheme for either a general linear or a general symplectic group and then give an application to the Rapoport-Zink integral models of Siegel modular varieties at parahoric level.

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# Notation

- The rings in this text are always assumed to be commutative and unital.
- Let $R$ be a local ring. We write $\mathfrak{m}_R \subseteq R$ for its maximal ideal.
- Let $R$ be a ring. We write $R[\varepsilon] \coloneqq R[x]/x^2$. For $\ell \in \mathbf{Z}_{>0}$ we similarly write $R[\varepsilon_1, \dots, \varepsilon_\ell] \coloneqq R[x_1, \dots, x_\ell]/(x_i x_j \mid i, j)$.
- The symbols $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ denote the fields of rational, real and complex numbers. We write $\overline{\mathbf{Q}}$ for the algebraic closure of $\mathbf{Q}$ inside $\mathbf{C}$.
- Let $K$ be a field that is endowed with a discrete valuation. We write $\mathcal{O}_K \subseteq K$ for its valuation subring and $\mathfrak{m}_K \subseteq \mathcal{O}_K$ for its maximal ideal.
- We fix a rational prime $p$ not equal to 2 and denote by $\mathbf{Q}_p$, $\mathbf{Z}_p$ and $\mathbf{F}_p$ the field of $p$-adic numbers, its ring of integers and its residue field. We fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ with residue field $\overline{\mathbf{F}}_p$ and write $\breve{\mathbf{Q}}_p$ and $\breve{\mathbf{Z}}_p$ for the completion of the maximal unramified extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ and its ring of integers. Finally we fix an embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$.
- Let $q = p^f$ be a power of $p$. We write $\mathbf{F}_q \subseteq \overline{\mathbf{F}}_p$ for the unique subfield of cardinality $q$ so that $\mathbf{F}_q/\mathbf{F}_p$ is a finite extension of degree $f$.
- Let $K/\mathbf{Q}_p$ be a finite extension that is contained in $\overline{\mathbf{Q}}_p$. We write $\breve{K} = K \cdot \breve{\mathbf{Q}}_p$ for the completion of its maximal unramified extension inside $\overline{\mathbf{Q}}_p$.
- Let $R$ be a $\mathbf{Z}_p$-algebra. We write $W(R)$ for the associated ring of Witt vectors (that naturally is again a $\mathbf{Z}_p$-algebra) and $I_R \subseteq W(R)$ for its augmentation ideal, i.e. the kernel of the natural morphism $W(R) \to R$. We also write $\sigma\colon W(R) \to W(R)$ for the Frobenius and $\sigma^{\mathrm{div}}\colon I_R^\sigma \to W(R)$ for the linearization of the inverse of the Verschiebung (that we also call the divided Frobenius because it satisfies $p\sigma^{\mathrm{div}}(1 \otimes x) = \sigma(x)$ for $x \in I_R$).

  Let $n$ be a positive integer. We write $W_n(R)$ for the ring of $n$-truncated Witt vectors and $I_{n,R} \subseteq W_n(R)$ for its augmentation ideal. When $m$ is a second positive integer with $m \geq n+1$ we again have a Frobenius $\sigma\colon W_m(R) \to W_n(R)$ and a divided Frobenius $\sigma^{\mathrm{div}}\colon W_n(R) \otimes_{\sigma, W_m(R)} I_{m,R} \to W_n(R)$.
- Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. We write $\widehat{W}(R) \subseteq W(R)$ for the associated Zink ring (see [Zin01]), $\widehat{I}_R \subseteq \widehat{W}(R)$ for its augmentation ideal and $\sigma\colon \widehat{W}(R) \to \widehat{W}(R)$ and $\sigma^{\mathrm{div}}\colon (\widehat{I}_R)^\sigma \to \widehat{W}(R)$ for its Frobenius and divided Frobenius similarly as before.

- Let $G$ be a connected reductive group over a field $k$ and let $\nu\colon \mathbf{G}_{m,k} \to G$ be a cocharacter. Then we have the associated parabolic subgroup $P_\nu \subseteq G$ that is given by
$$P_\nu(R) = \Big\{ g \in G(R) \;\Big|\; \lim_{t\to 0} \mu(t) g \mu(t)^{-1} \text{ exists} \Big\}.$$
Given an algebraic closure $\overline{k}$ of $k$ and a $G(\overline{k})$-conjugacy class $\boldsymbol{\mu}$ of cocharacters $\mu\colon \mathbf{G}_{m,\overline{k}} \to G_{\overline{k}}$ with field of definition $k' \subseteq \overline{k}$ we denote by $X_{\boldsymbol{\mu}}(G)$ the smooth projective geometrically connected $k'$-scheme that parametrizes parabolic subgroups $P \subseteq G_{k'}$ of type $\boldsymbol{\mu}^{-1}$.

- Let $k$ be a field and fix an algebraic closure $\overline{k}$ of $k$. Let $V$ be a vector space of dimension $h \in \mathbf{Z}_{\geq 0}$ over $k$ and let $0 \leq d \leq h$. Then we denote by $\boldsymbol{\mu}_d$ the $\mathrm{GL}(V)(\overline{k})$-conjugacy class of cocharacters of $\mathrm{GL}(V)_{\overline{k}}$ that induce a weight decomposition $V_{\overline{k}} = V_0 \oplus V_1$ with $V_1$ of dimension $d$.

  Likewise we denote by $\boldsymbol{\mu}'_d$ the $\mathrm{GL}(V)(\overline{k})$-conjugacy class of cocharacters of $\mathrm{GL}(V)_{\overline{k}}$ that induce a weight decomposition $V_{\overline{k}} = V_0 \oplus V_{-1}$ with $V_{-1}$ of dimension $d$.

- Let $\mathcal{G}$ be a parahoric group scheme over $\mathbf{Z}_p$ with generic fiber $G = \mathcal{G}_{\mathbf{Q}_p}$. We denote by $L^+\mathcal{G}$ the (Witt vector) positive loop group of $\mathcal{G}$, i.e. the affine group scheme over $\mathbf{Z}_p$ given by $(L^+\mathcal{G})(R) = \mathcal{G}(W(R))$. For a positive integer $n$ we also write $L^{(n)}\mathcal{G}$ for the $n$-truncated positive loop group of $\mathcal{G}$, i.e. the smooth affine group scheme over $\mathbf{Z}_p$ given by $(L^{(n)}\mathcal{G})(R) = \mathcal{G}(W_n(R))$. Finally we write $L^{(1\text{-rdt})}\mathcal{G} = \mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p}$ for the quotient of the special fiber of $\mathcal{G}$ by its unipotent radical.

  We write $\mathbb{L}^+\mathcal{G}$ and $\mathbb{L}^{(n)}\mathcal{G}$ for the perfection of the special fibers of $L^+\mathcal{G}$ and $L^{(n)}\mathcal{G}$ respectively and we write $\mathbb{L}G$ for the loop groop, i.e. for the sheaf of groups on the category of perfect $\mathbf{F}_p$-algebras (that we equip with the flat topology) given by $(\mathbb{L}G)(R) \coloneqq G(W(R)[1/p])$.

- Whenever possible, we normalize group actions so that they are left actions.

- Let $\Lambda$ be a finite free $\mathbf{Z}_p$-module of rank $h$ and let $\mathcal{G} \subseteq \mathrm{GL}(\Lambda)$ be a smooth closed $\mathbf{Z}_p$-subgroup scheme. Then by a $\mathcal{G}$-structure on a finite projective $R$-module of rank $h$ (for some $\mathbf{Z}_p$-algebra $R$) we mean a choice of reduction of the associated $\mathrm{GL}(\Lambda)$-torsor to a $\mathcal{G}$-torsor. We have an equivalence of groupoids
$$\Big\{ \mathcal{G}\text{-torsors over } R \Big\} \to \left\{ \begin{matrix} \text{finite projective } R\text{-modules of rank } h \\ \text{with a } \mathcal{G}\text{-structure} \end{matrix} \right\}$$
and we usually identify corresponding objects on both sides.

- In some parts of this thesis $V$ will denote a finite-dimensional $\mathbf{Q}_p$-vector space (or a $\mathbf{Q}$-vector space). We will then denote $\mathbf{Z}_p$-lattices in $V$ (or in $V_{\mathbf{Q}_p}$) by $\Xi$ and the corresponding dual lattices inside the dual vector space $W = V^\vee$ by $\Lambda = \Xi^\vee$. The main focus is then on the objects $W$ and $\Lambda$ (when $V$ is the underlying $\mathbf{Q}$-vector space of a Siegel Shimura datum then it is $W$ that is naturally related to the cohomology of the universal Abelian variety on the associated Siegel modular variety).

# Contents

# 1 Preliminaries

## 1.1 Witt vectors

**Proposition 1.1.1.** *Let $\mathcal{G}$ be a smooth affine $\mathbf{Z}_p$-group scheme and let $R$ be a $p$-complete ring. Then we have an equivalence of groupoids*

$$\begin{array}{rcl} \{\mathcal{G}\text{-torsors over } W(R)\} & \to & \{L^+\mathcal{G}\text{-torsors over } R\}, \\ X & \mapsto & L^+X, \end{array}$$

*where $L^+X$ is defined by $(L^+X)(R') = X(W(R'))$. For a positive integer $n$ we similarly have an equivalence of groupoids*

$$\begin{array}{rcl} \{\mathcal{G}\text{-torsors over } W_n(R)\} & \to & \{L^{(n)}\mathcal{G}\text{-torsors over } R\}, \\ X & \mapsto & L^{(n)}X. \end{array}$$

*Proof.* See [BH20, Lemma 2.12]. □

**Lemma 1.1.2.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. Let $\mathfrak{a} \subseteq R$ be an ideal that is equipped with divided powers. We assume that these divided powers are compatible with the canonical divided powers on $pR \subseteq R$ and that they are continuous in the sense that there exists a fundamental system of open ideals $\mathfrak{b} \subseteq R$ such that the divided power structure on $R$ extends to the quotient $R/\mathfrak{b}$.*

*Then we have a natural isomorphism*

$$\widehat{W}(\mathfrak{a}) \to \widehat{\bigoplus_{r \geq 0}} \mathfrak{a}.$$

*Under this isomorphism the Frobenius $\sigma \colon \widehat{W}(\mathfrak{a}) \to \widehat{W}(\mathfrak{a})$ corresponds to the map*

$$(y_0, y_1, y_2, \dots) \mapsto (py_1, py_2, py_3, \dots).$$

*Proof.* See [Zin01, Section 2] and [Zin02, Section 1.4]. □

**Lemma 1.1.3.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$ such that $\mathfrak{m}_R = \sqrt{pR}$. Then the topology on $\widehat{W}(R)$ induced by writing*

$$\widehat{W}(R) = \varprojlim_{i,j} \widehat{W}(R/\mathfrak{m}_R^i)/p^j \widehat{W}(R/\mathfrak{m}_R^i)$$

*coincides with the $p$-adic topology. In particular $\widehat{W}(R)$ is $p$-complete.*

*Proof.* We have to show that for every $j \in \mathbf{Z}_{>0}$ there exists some $i \in \mathbf{Z}_{>0}$ such that

$$\widehat{W}(\mathfrak{m}_R^i) \subseteq p^j \widehat{W}(R).$$

So fix $j$ and choose $i$ such that $\mathfrak{m}_R^i \subseteq p^{j+1}R$ (this is possible by our assumption $\mathfrak{m}_R = \sqrt{pR}$). Then we have

$$\widehat{W}(\mathfrak{m}_R^i) \subseteq \widehat{W}(p^{j+1}R) = p^j \widehat{W}(pR) \subseteq p^j \widehat{W}(R)$$

where we use the natural isomorphism $\widehat{W}(p^k R) \cong \widehat{\bigoplus}_{r \geq 0} p^k R$ from Lemma 1.1.2 for the equality in the middle. $\square$

**Remark 1.1.4.** In fact $\widehat{W}(R)$ is $p$-complete for any complete Noetherian local ring $R$ with residue field $\overline{\mathbf{F}}_p$. This can be deduced from the case of Artinian local rings with residue field $\overline{\mathbf{F}}_p$ (that is a special case of Lemma 1.1.3).

**Lemma 1.1.5** (Pappas)**.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$ that is additionally $p$-torsionfree and satisfies $\mathfrak{m}_R = \sqrt{pR}$. Topologize $\widehat{W}(R)[1/p]$ by declaring $\widehat{W}(R) \subseteq \widehat{W}(R)[1/p]$ to be an open subring that carries its $p$-adic topology.*

*Let $x \in \widehat{W}(\mathfrak{m}_R)[1/p]$. Then we have*

$$\lim_{i \to \infty} p^{-i} \sigma^i(x) = 0.$$

*Proof.* On $\widehat{W}(\mathfrak{m}_R/pR)$ multiplication by $p$ is given by

$$p \cdot (x_0, x_1, x_2, \dots) = (0, x_0^p, x_1^p, \dots).$$

The assumption $\mathfrak{m}_R = \sqrt{pR}$ thus implies that $\widehat{W}(\mathfrak{m}_R/pR)$ is $p^\infty$-torsion so that

$$\widehat{W}(\mathfrak{m}_R)[1/p] = \widehat{W}(pR)[1/p].$$

This means that in proving the claim we may assume without loss of generality that $x \in \widehat{W}(pR)$. Write

$$(y_0, y_1, y_2, \dots) \in \widehat{\bigoplus_{r \geq 0}} pR$$

for the image of $x$ Under the isomorphism from Lemma 1.1.2. Then the image of $p^{-i}\sigma^i(x)$ is given by

$$(y_i, y_{i+1}, y_{i+2}, \dots).$$

As $\lim_{r \to \infty} y_r = 0$ in the $\mathfrak{m}_R$-adic (or equivalently the $p$-adic) topology on $R$ this implies that $\lim_{i \to \infty} p^{-i}\sigma^i(x) = 0$ as desired. $\square$

**Lemma 1.1.6.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$ and let $x \in \widehat{W}(R)$ with $\sigma(x) = x$. Then we already have $x \in \mathbf{Z}_p$.*

*Proof.* Write $x = (x', x'') \in \breve{\mathbf{Z}}_p \oplus \widehat{W}(\mathfrak{m}_R)$ so that $\sigma(x') = x'$ and $\sigma(x'') = x''$. Then it follows that $x' \in \mathbf{Z}_p$ and as $\sigma$ is topologically nilpotent on $\widehat{W}(\mathfrak{m}_R)$ it also follows that $x'' = 0$. □

**Lemma 1.1.7.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. Then the morphism*

$$\widehat{W}(R)^\times \to \widehat{W}(R)^\times, \qquad x \mapsto x\sigma(x)^{-1}$$

*is surjective.*

*Proof.* Let $y \in \widehat{W}(R)^\times$. We can certainly find $x_0 \in \breve{\mathbf{Z}}_p^\times$ such that $x_0\sigma(x_0)^{-1} = y \in \breve{\mathbf{Z}}_p^\times$. As $\sigma$ is topologically nilpotent on $\widehat{W}(\mathfrak{m}_R) = \ker(\widehat{W}(R) \to \breve{\mathbf{Z}}_p)$ this $x_0$ lifts uniquely to an element $x \in \widehat{W}(R)^\times$ with $x\sigma(x)^{-1} = y \in \widehat{W}(R)^\times$ as desired. □

# 1.2 Some category theory

## 1.2.1 Preadditive categories with duality

**Definition 1.2.1.** A *preadditive category with duality* is a tuple $(\mathcal{C}, (\text{-})^\vee, \eta)$ consisting of a preadditive category $\mathcal{C}$, an additive equivalence $(\text{-})^\vee \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and a biduality isomorphism $\eta\colon \mathrm{id}_{\mathcal{C}} \to (\text{-})^{\vee\vee}$ such that $(\eta_x)^\vee \eta_{x^\vee} = \mathrm{id}_{x^\vee}$ for all $x \in \mathcal{C}$.

Let $\mathcal{C}, \mathcal{D}$ be preadditive categories with duality. A *functor between preadditive categories with duality (from $\mathcal{C}$ to $\mathcal{D}$)* is a tuple $(F, \alpha)$ consisting of an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ and an isomorphism $\alpha\colon F \circ (\text{-})^\vee \to (\text{-})^\vee \circ F$ such that the diagram

$$\begin{tikzcd} & F(x) \arrow[dl, "F(\eta_{\mathcal{C}})"'] \arrow[dr, "\eta_{\mathcal{D}}"] & \\ F(x^{\vee\vee}) \arrow[r, "\alpha"] & F(x^\vee)^\vee \arrow[r, "\alpha"] & F(x)^{\vee\vee} \end{tikzcd}$$

is commutative for all $x \in \mathcal{C}$.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between preadditive categories with duality. An *isomorphism (from $F$ to $G$)* is an isomorphism between functors $\omega\colon F \to G$ such that the diagram

$$\begin{tikzcd} F(x^\vee) \arrow[r, "\alpha_F"] \arrow[d, "\omega"] & F(x)^\vee \arrow[d, "\omega^{\vee,-1}"] \\ G(x^\vee) \arrow[r, "\alpha_G"] & G(x)^\vee \end{tikzcd}$$

is commutative for all $x \in \mathcal{C}$.

**Definition 1.2.2.** Let $\mathcal{C}$ be a preadditive category with duality. A morphism $f\colon x \to x^\vee$ is called *symmetric* (resp. *antisymmetric*) if we have $f^\vee = f\colon x^{\vee\vee} \cong x \to x^\vee$ (resp. if $f^\vee = -f$).

### 1.2.2 Actions of symmetric monoidal categories on preadditive categories with duality

**Remark 1.2.3.** Symmetric monoidal categories are always implicitly required to be endowed with a preadditive structure and rigid.

Note that every symmetric monoidal category $\mathcal{M}$ naturally is a commutative monoid object in the $(2,1)$-category of preadditive categories with duality by using the duality defined by $a^\vee \coloneqq \mathcal{H}om(a, 1)$ for $a \in \mathcal{M}$.

In the following definition we use the theory of operads and algebras over them. A reference for this material is Section 2.1 in the book [Lur17] by Lurie.

**Definition 1.2.4.** A *preadditive category with an action of a symmetric monoidal category* is an algebra over the operad $\mathbf{Pf}$ introduced by Liu and Zheng in [LZ12, Definition 1.5.6] in the $(2,1)$-category of preadditive categories, that we equip with the cartesian symmetric monoidal structure, such that the underlying commutative monoid object is a symmetric monoidal category in the sense of Remark 1.2.3. More explicitly, an action of a symmetric monoidal category $\mathcal{M}$ on a preadditive category $\mathcal{C}$ is a biadditive functor

$$\otimes \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$

that is equipped with natural isomorphisms

$$\big((a \otimes b) \otimes x \to a \otimes (b \otimes x)\big)_{a,b \in \mathcal{M}, x \in \mathcal{C}} \quad \text{and} \quad \big(1 \otimes x \to x\big)_{x \in \mathcal{C}}.$$

These isomorphisms are required to satisfy some coherence conditions that are encoded in the operad $\mathbf{Pf}$.

Similarly, a *preadditive category with duality with an action of a symmetric monoidal category* is an algebra over $\mathbf{Pf}$ in the $(2,1)$-category of preadditive categories with duality, that we equip with the cartesian symmetric monoidal structure, again such that the underlying commutative monoid object is a symmetric monoidal category in the sense of Remark 1.2.3. More explicitly, an action of a symmetric monoidal category $\mathcal{M}$ on a preadditive category with duality $\mathcal{C}$ is an action of $\mathcal{M}$ on $\mathcal{C}$ in the above sense that is further equipped with a natural isomorphism

$$\big((a \otimes x)^\vee \to a^\vee \otimes x^\vee\big)_{a \in \mathcal{M}, x \in \mathcal{C}}$$

that is required to satisfy some further coherence conditions.

**Definition 1.2.5.** Let $\mathcal{C}$ be a preadditive category with duality that is equipped with an action of a symmetric monoidal category $\mathcal{M}$. Then a morphism $f \colon x \to a \otimes x^\vee$ with $x \in \mathcal{C}$ and $a \in \mathcal{M}$ invertible is called *(anti-)symmetric* if it agrees with the (negative of the) composition

$$x \cong a \otimes (a \otimes x^\vee)^\vee \xrightarrow{\mathrm{id}_a \otimes f^\vee} a \otimes x^\vee.$$

**Remark 1.2.6.** Note that functors between preadditive categories with duality with an action of a symmetric monoidal category preserve (anti-)symmetric morphisms. Also note that putting $a = 1$ in Definition 1.2.5 recovers Definition 1.2.2.

**Definition 1.2.7.** Let $\mathcal{C}$ be a preadditive category with duality that is equipped with an action of a symmetric monoidal category $\mathcal{M}$. Then an *($\mathcal{M}$-)homogeneously polarized object in* $\mathcal{C}$ is a tuple $(x, a, \lambda)$ consisting of an object $x \in \mathcal{C}$, an invertible object $a \in \mathcal{M}$ and an antisymmetric isomorphism $\lambda\colon x \to a \otimes x^\vee$. An isomorphism between twe homogeneously polarized objects $(x, a, \lambda)$ and $(x', a', \lambda')$ is a tuple $(f, g)$ consisting of isomorphisms $f\colon x \to x'$ and $g\colon a \to a'$ such that the diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\lambda"] & x' \arrow[d, "\lambda'"] \\ a \otimes x^\vee \arrow[r, "g \otimes f^{\vee,-1}"] & a' \otimes x'^\vee \end{tikzcd}$$

is commutative.

## 1.3 Some deformation theory

A reference for the material in this section is [Sch68]. Note that some of the results we state here are actually slightly more general than the corresponding results in the reference, but all the arguments given there generalize accordingly.

**Notation 1.3.1.** We fix a finite extension $F/\breve{\mathbf{Q}}_p$. The letters $R, R', R''$ always denote complete Noetherian local $\mathcal{O}_F$-algebras with residue field $\overline{\mathbf{F}}_p$.

**Definition 1.3.2.** A *deformation problem (over $\mathcal{O}_F$)* is a functor Def on the category of complete Noetherian local $\mathcal{O}_F$-algebras with residue field $\overline{\mathbf{F}}_p$ such that for every $R$ the natural map $\mathrm{Def}(R) \to \varprojlim_i \mathrm{Def}(R/\mathfrak{m}_R^i)$ is a bijection.

Given a complete Noetherian local $\mathcal{O}_F$-algebra $R'$ with residue field $\overline{\mathbf{F}}_p$ we have an associated deformation problem

$$\mathrm{Spf}(R')\colon R \mapsto \mathrm{Spf}(R')(R) = \mathrm{Hom}_{\mathcal{O}_F}(R', R).$$

Let Def be a deformation problem. An element $x^{\mathrm{univ}} \in \mathrm{Def}(R^{\mathrm{univ}})$ for some complete Noetherian local $\mathcal{O}_F$-algebra $R^{\mathrm{univ}}$ with residue field $\overline{\mathbf{F}}_p$ is called a *universal deformation for* Def if the corresponding morphism $\mathrm{Spf}(R^{\mathrm{univ}}) \to \mathrm{Def}$ is an isomorphism. In this case we also say that Def is *pro-representable* and in fact *pro-represented by* $(R^{\mathrm{univ}}, x^{\mathrm{univ}})$.

**Lemma 1.3.3.** *Let* Def *be a deformation problem and suppose we are given $R$ such that the maps*

$$\mathrm{Def}(R[\varepsilon_1, \varepsilon_2]) \to \mathrm{Def}(R[\varepsilon]) \times_{\mathrm{Def}(R)} \mathrm{Def}(R[\varepsilon])$$

*and*

$$\mathrm{Def}(R[\varepsilon_1, \varepsilon_2, \varepsilon_3]) \to \mathrm{Def}(R[\varepsilon_1, \varepsilon_2]) \times_{\mathrm{Def}(R)} \mathrm{Def}(R[\varepsilon])$$

*are bijective.*

*Let $x \in \mathrm{Def}(R)$. Then tho set*

$$T_x \mathrm{Def} \coloneqq \mathrm{fib}_x\big(\mathrm{Def}(R[\varepsilon]) \to \mathrm{Def}(R)\big)$$

*(the fiber over $x$ of the map $\mathrm{Def}(R[\varepsilon]) \to \mathrm{Def}(R)$) naturally carries the structure of an $R$-module that is given as follows.*

- *The addition* $T_x \operatorname{Def} \times T_x \operatorname{Def} \to T_x \operatorname{Def}$ *is induced by the composition*

$$\operatorname{Def}(R[\varepsilon]) \times_{\operatorname{Def}(R)} \operatorname{Def}(R[\varepsilon]) \xleftarrow{\sim} \operatorname{Def}(R[\varepsilon_1, \varepsilon_2]) \xrightarrow{\varepsilon_i \mapsto \varepsilon} \operatorname{Def}(R[\varepsilon]).$$

- *The zero element* $0 \in T_x \operatorname{Def}$ *is the image of* $x$ *under the morphism*

$$\operatorname{Def}(R) \to \operatorname{Def}(R[\varepsilon]).$$

- *Let* $a \in R$*. The scalar multiplication by this element* $T_x \operatorname{Def} \to T_x \operatorname{Def}$ *is induced by the morphism*

$$\operatorname{Def}(R[\varepsilon]) \xrightarrow{\varepsilon \mapsto a\varepsilon} \operatorname{Def}(R[\varepsilon]).$$

*Proof.* See [Sch68, Lemma 2.10]. □

**Lemma 1.3.4.** *Let* Def *be a deformation problem and suppose we are given a diagram*

$$\begin{tikzcd} R' \arrow[rr] \arrow[dr] & & R'' \arrow[dl] \\ & R & \end{tikzcd}$$

*where all the maps are surjective and* $\mathfrak{a} := \ker(R' \to R'')$ *is a free* $R$*-module of rank* 1 *with generator* $\zeta \in \mathfrak{a}$*. Suppose furthermore that the assumptions from Lemma 1.3.3 are satisfied so that we can talk about the tangent spaces* $T_x \operatorname{Def}$ *at points* $x \in \operatorname{Def}(R)$ *and that furthermore also the maps*

$$\operatorname{Def}(R[\varepsilon] \times_R R') \to \operatorname{Def}(R[\varepsilon]) \times_{\operatorname{Def}(R)} \operatorname{Def}(R'),$$

$$\operatorname{Def}(R[\varepsilon_1, \varepsilon_2] \times_R R') \to \operatorname{Def}(R[\varepsilon_1, \varepsilon_2]) \times_{\operatorname{Def}(R)} \operatorname{Def}(R')$$

*and*

$$\operatorname{Def}(R' \times_{R''} R') \to \operatorname{Def}(R') \times_{\operatorname{Def}(R'')} \operatorname{Def}(R')$$

*are bijective.*

*Let* $y \in \operatorname{Def}(R'')$ *with image* $x \in \operatorname{Def}(R)$*. Then the composition*

$$\operatorname{Def}(R[\varepsilon]) \times_{\operatorname{Def}(R)} \operatorname{Def}(R') \xleftarrow{\sim} \operatorname{Def}(R[\varepsilon] \times_R R') \xrightarrow{(a+a'\varepsilon, b) \mapsto b + a'\zeta} \operatorname{Def}(R')$$

*induces a group action*

$$T_x \operatorname{Def} \times \operatorname{fib}_y\big(\operatorname{Def}(R') \to \operatorname{Def}(R'')\big) \to \operatorname{fib}_y\big(\operatorname{Def}(R') \to \operatorname{Def}(R'')\big)$$

*and in fact* $\operatorname{fib}_y(\operatorname{Def}(R') \to \operatorname{Def}(R''))$ *is either empty or this action is simply transitive (i.e.* $\operatorname{fib}_y(\operatorname{Def}(R') \to \operatorname{Def}(R''))$ *is a* $T_x \operatorname{Def}$*-pseudotorsor).*

*Proof.* See [Sch68, Remark 2.15]. □

**Theorem 1.3.5** (Schlessinger)**.** *Let* Def *be a deformation problem. Then* Def *is pro-representable if and only if the following conditions are satisfied.*

- $\mathrm{Def}(\overline{\mathbf{F}}_p) = \{*\}$.
- *The map* $\mathrm{Def}(R' \times_R R'') \to \mathrm{Def}(R') \times_{\mathrm{Def}(R)} \mathrm{Def}(R'')$ *is a bijection for all diagrams*

$$\begin{tikzcd} & R'' \arrow[d] \\ R' \arrow[r] & R \end{tikzcd}$$

*of Artinian local* $\mathcal{O}_F$*-algebras with residue field* $\overline{\mathbf{F}}_p$ *such that* $R' \to R$ *is surjective.*

- *The tangent space* $T_* \mathrm{Def}$ *of* $\mathrm{Def}$ *at* $* \in \mathrm{Def}(\overline{\mathbf{F}}_p)$ *is a finite-dimensional* $\overline{\mathbf{F}}_p$*-vector space.*

*Proof.* See [Sch68, Theorem 2.11]. $\square$

## 1.4 Local models

### 1.4.1 Local model data and local models

**Definition 1.4.1.** A *local model datum* is a tuple $(\mathcal{G}, \boldsymbol{\mu})$ that is given as follows.

- $\mathcal{G}$ is a parahoric group scheme over $\mathbf{Z}_p$. We write $G = \mathcal{G}_{\mathbf{Q}_p}$ for its generic fiber (that is a connected reductive algebraic group over $\mathbf{Q}_p$).
- $\boldsymbol{\mu}$ is a minuscule $G(\overline{\mathbf{Q}}_p)$-conjugacy class of cocharacters $\mu\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to G_{\overline{\mathbf{Q}}_p}$.

Associated to such a local model datum we have its *reflex field* $E(\mathcal{G}, \boldsymbol{\mu})$ that is the field of definition of $\boldsymbol{\mu}$, a finite extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$.

**Notation 1.4.2.** Fix a local model datum $(\mathcal{G}, \boldsymbol{\mu})$. Write $G = \mathcal{G}_{\mathbf{Q}_p}$ and $E \coloneqq E(\mathcal{G}, \boldsymbol{\mu})$ and let $q$ be the cardinality of the residue field $\mathcal{O}_E/\mathfrak{m}_E = \mathbf{F}_q$.

**Definition 1.4.3.** The *(Witt vector) affine flag variety* $\mathcal{F}\ell_{\mathcal{G}}$ *of* $\mathcal{G}$ is the quotient

$$\mathcal{F}\ell_{\mathcal{G}} \coloneqq \mathbb{L}G/\mathbb{L}^+\mathcal{G}$$

on the category of perfect $\mathbf{F}_p$-algebras with respect to the flat topology (or equivalently the étale topology). It comes equipped with a natural left $\mathbb{L}^+\mathcal{G}$-action.

We have the $\boldsymbol{\mu}$*-admissible locus*

$$\mathcal{A}_{\mathcal{G},\boldsymbol{\mu}} \subseteq \mathcal{F}\ell_{\mathcal{G},\mathbf{F}_q}$$

that is the union of those $\mathbb{L}^+\mathcal{G}$-orbits corresponding to elements in the (finite) $\boldsymbol{\mu}$-*admissible set*

$$\mathrm{Adm}(\boldsymbol{\mu})_{\mathcal{G}} \subseteq \mathcal{G}(\breve{\mathbf{Z}}_p)\backslash G(\breve{\mathbf{Q}}_p)/\mathcal{G}(\breve{\mathbf{Z}}_p)$$

(see [AGLR22, Definition 3.11] and [HR17] for more details).

**Remark 1.4.4.** The $\mathbb{L}^+\mathcal{G}$-action on $\mathcal{A}$ factors through $\mathcal{G}_{\mathbf{F}_p}^{\mathrm{pf}}$.

**Theorem 1.4.5** (Zhu, Bhatt-Scholze)**.** *The affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is an ind-projective ind-(perfect scheme) over $\mathbf{F}_p$. Consequently the $\boldsymbol{\mu}$-admissible locus $\mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}$ is a projective perfect scheme over $\mathbf{F}_q$.*

*Proof.* See [Zhu17] and [BS17]. □

The following theorem is due to Anschütz-Gleason-Lourenço-Richarz and Fakhruddin-Haines-Lourenço-Richarz, building on the work of Pappas-Zhu and many others (many cases were already known earlier).

**Theorem 1.4.6** (AGLR, FHLR)**.** *Associated to $(\mathcal{G},\boldsymbol{\mu})$ there is a natural flat projective weakly normal $\mathcal{O}_E$-scheme $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ that comes equipped with an action of $\mathcal{G}$ and has the following properties.*

- *We have an identification*
$$\big(\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}\big)_E = X_{\boldsymbol{\mu}}(G).$$
- *The special fiber $(\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}})_{\mathbf{F}_q}$ is reduced and we have an identification*
$$\big(\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}\big)^{\mathrm{pf}}_{\mathbf{F}_q} = \mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}.$$

*Proof.* See [AGLR22] and [FHLR22]. □

**Remark 1.4.7.** When $\mathcal{G}$ is reductive then $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ identifies with the homogeneous space $X_{\boldsymbol{\mu}}(\mathcal{G})$ and is in particular smooth.

### 1.4.2 Local models for $\mathrm{GL}(V)$

**Notation 1.4.8.** Fix integers $h \in \mathbf{Z}_{>0}$ and $0 \le d \le h$ and a $\mathbf{Q}_p$-vector space $V$ of dimension $h$. Also fix a nonempty subset $J \subseteq \mathbf{Z}$ such that $J + h\mathbf{Z} \subseteq J$ and a tuple $(\Xi_i)_{i\in J}$ of $\mathbf{Z}_p$-lattices $\Xi_i \subseteq V$ such that the following conditions are satisfied.

- For $i \le j$ we have $\Xi_i \supseteq \Xi_j$ and the $\mathbf{Z}_p$-module $\Xi_i/\Xi_j$ is of length $j - i$.
- $\Xi_{i+h} = p\Xi_i$.

We write $W \coloneqq V^\vee$ and $\Lambda_i \coloneqq \Xi_i^\vee \subseteq W$. For $i \le j$ we then have that $\Lambda_i \subseteq \Lambda_j$ and that the $\mathbf{Z}_p$-module $\Lambda_j/\Lambda_i$ is of length $j-i$, and $\Lambda_{i+h} = p^{-1}\Lambda_i$. For $i \le j$ we write $\rho_{i,j} \colon \Lambda_i \to \Lambda_j$ for the inclusion and we also write $\theta_i \colon \Lambda_i \to \Lambda_{i+h}$ for the isomorphism given by multiplication $p^{-1}$.

Associated to the tuple $(V, (\Xi_i)_i, d)$ we have a local model datum
$$\big(\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d\big).$$
Here $\mathrm{GL}((\Xi_i)_i)$ denotes the parahoric group scheme over $\mathbf{Z}_p$ with generic fiber $\mathrm{GL}(V)$ that corresponds to the lattice chain $(\Xi_i)_i$, or more explicitly the automorphism group scheme of the tuple $((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$, and $\boldsymbol{\mu}_d$ denotes the $\mathrm{GL}(V)(\overline{\mathbf{Q}}_p)$-conjugacy class of those cocharacters of $\mathrm{GL}(V)_{\overline{\mathbf{Q}}_p}$ that induce a weight decomposition $V_{\overline{\mathbf{Q}}_p} = V_0 \oplus V_1$ with $V_1$ of dimension $d$. The reflex field of this local model datum is $\mathbf{Q}_p$.

**Theorem 1.4.9.** *The local model $\mathbb{M}^{\mathrm{loc}}_{\mathrm{GL}((\Xi_i)_i),\boldsymbol{\mu}_d}$ is the moduli space of tuples $(C_i)_i$ consisting of direct summands $C_i \subseteq \Lambda_{i,R}$ of rank $d$ such that $\rho_{i,j}(C_i) \subseteq C_j$ and $\theta_i(C_i) = C_{i+h}$.*

*Proof.* This follows from the main result of [Gör01] by Görtz. □

### 1.4.3 Local models for $\mathrm{GSp}(V)$

**Notation 1.4.10.** Fix $g \in \mathbf{Z}_{>0}$ and a tuple $(V, \overline{\psi})$ consisting of a $\mathbf{Q}_p$-vector space $V$ of dimension $h \coloneqq 2g$ and a $\mathbf{Q}_p^\times$-class $\overline{\psi}$ of symplectic forms $\psi \colon V \times V \to \mathbf{Q}_p$ and set $d \coloneqq g$. Also fix a non-empty subset $J \subseteq \mathbf{Z}$ such that $J + 2g\mathbf{Z} \subseteq J$ and $-J \subseteq J$ and a tuple $(\Xi_i)_{i \in J}$ of $\mathbf{Z}_p$-lattices in $V$ as in Notation 1.4.8, but with the following additional requirement.

- There exists a representative $\psi \in \overline{\psi}$ that restricts to a perfect pairing

  $$\Xi_i \times \Xi_{-i} \to \mathbf{Z}_p$$

  for all $i$.

The set of representatives $\psi \in \overline{\psi}$ that satisfy the above condition forms precisely one $\mathbf{Z}_p^\times$-subclass of $\overline{\psi}$. From now on $\overline{\psi}$ denotes this $\mathbf{Z}_p^\times$-subclass.

As before we have $W$, $\Lambda_i$, $\rho_{i,j}$ and $\theta_i$. We now also write $\overline{\psi}$ for the $\mathbf{Z}_p^\times$-class of symplectic forms on $W$ induced by the one on $V$ and we write $\overline{\psi_i}$ for the induced $\mathbf{Z}_p^\times$-class of perfect pairings $\Lambda_i \times \Lambda_{-i} \to \mathbf{Z}_p$.

Associated to the tuple $(V, \overline{\psi}, (\Xi_i)_i)$ we have a local model datum

$$\big(\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_g\big).$$

Here $\mathrm{GSp}((\Xi_i)_i)$ denotes the parahoric group scheme over $\mathbf{Z}_p$ with generic fiber $\mathrm{GSp}(V)$ the connected reductive group of symplectic similitudes of $(V, \overline{\psi})$ that corresponds to the self-dual lattice chain $(\Xi_i)_i$, or more explicitly the automorphism group scheme of the tuple $((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (\overline{\psi_i})_i)$, and $\boldsymbol{\mu}_g$ denotes the $\mathrm{GSp}(V)(\overline{\mathbf{Q}}_p)$-conjugacy class of those cocharacters of $\mathrm{GSp}(V)_{\overline{\mathbf{Q}}_p}$ that induce a weight decomposition $V_{\overline{\mathbf{Q}}_p} = V_0 \oplus V_1$ where $V_0, V_1 \subseteq V_{\overline{\mathbf{Q}}_p}$ are Lagrangian subspaces. The reflex field of this local model datum is $\mathbf{Q}_p$.

**Theorem 1.4.11.** *The local model $\mathbb{M}^{\mathrm{loc}}_{\mathrm{GSp}((\Xi_i)_i),\boldsymbol{\mu}_g}$ is the moduli space of tuples $(C_i)_i$ consisting of direct summands $C_i \subseteq \Lambda_{i,R}$ of rank $g$ such that $\rho_{i,j}(C_i) \subseteq C_j$, $\theta_i(C_i) = C_{i+2g}$ and $\psi_i(C_i, C_{-i}) = 0$.*

*Proof.* This follows from the main result of [Gör03] by Görtz. □

## 1.5 Local shtukas

**Notation 1.5.1.** Fix a local model datum $(\mathcal{G}, \boldsymbol{\mu})$, see Definition 1.4.1. In this section $m$ always denotes a positive integer and $(m, n)$ always denotes a tuple of positive integers with $m \geq n+1$, where we allow $n$ to take the additional value 1-rdt, in which case we require $m \geq 2$.

**Construction 1.5.2.** Set

$$\mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}} \coloneqq \mathcal{A}_{\mathcal{G},\boldsymbol{\mu}} \times_{\mathcal{F}\ell_{\mathcal{G},\mathbf{F}_q}} (\mathbb{L}G)_{\mathbf{F}_q},$$

where we refer to Definition 1.4.3 for the definition of the affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ and the admissible locus $\mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}$. Equip $\mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}}$ with the left action

$$\big(\mathbb{L}^+\mathcal{G} \times \mathbb{L}^+\mathcal{G}\big) \times \mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}} \to \mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}}, \qquad \big((k_1,k_2),g\big) \mapsto k_2 \cdot g \cdot \sigma^{-1}(k_1)^{-1}.$$

This action makes $\mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}} \to \mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}$ into an $L^+\mathcal{G}$-equivariant $L^+\mathcal{G}$-torsor. We write

$$\mathcal{A}^{(n)}_{\mathcal{G},\boldsymbol{\mu}} \to \mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}$$

for its reduction to an $\mathbb{L}^{(n)}\mathcal{G}$-torsor. The $\mathbb{L}^+\mathcal{G}$-equivariant structure on $\mathcal{A}^{(n)}_{\mathcal{G},\boldsymbol{\mu}} \to \mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}$ then factors through $\mathbb{L}^{(m)}\mathcal{G}$ by the argument in [SYZ21, Lemma 4.2.2].

**Definition 1.5.3** (Xiao-Zhu, Shen-Yu-Zhang)**.** We define the *stack of local shtukas for* $(\mathcal{G},\boldsymbol{\mu})$ as the quotient stack

$$\mathsf{Sht}^{\mathrm{loc}}_{\mathcal{G},\mu} \coloneqq \big[(\mathbb{L}^+\mathcal{G})_\Delta \backslash \mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}}\big].$$

Similarly we also define the *stack of* $(m,n)$*-restricted local shtukas for* $(\mathcal{G},\mu)$ as

$$\mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\mu} \coloneqq \big[(\mathbb{L}^{(m)}\mathcal{G})_\Delta \backslash \mathcal{A}^{(n)}_{\mathcal{G},\boldsymbol{\mu}}\big].$$

**Remark 1.5.4.** We note that our normalizations are different than the ones in [SYZ21].

## 1.6 $(\mathcal{G},\mu)$-displays in the sense of Bültel-Pappas

**Notation 1.6.1.** Let $\mathcal{G}$ be a reductive group scheme over $\mathbf{Z}_p$ and write $G = \mathcal{G}_{\mathbf{Q}_p}$ for its generic fiber. Let $F/\mathbf{Q}_p$ be a finite unramified extension and let $\mu\colon \mathbf{G}_{m,\mathcal{O}_F} \to \mathcal{G}_{\mathcal{O}_F}$ be a minuscule cocharacter. Set

$$H_\mu \coloneqq P_{\mu^{-1}} \times_{\mathcal{G}_{\mathcal{O}_F}} (L^+\mathcal{G})_{\mathcal{O}_F},$$

where $P_{\mu^{-1}} \subseteq \mathcal{G}_{\mathcal{O}_F}$ denotes the parabolic subgroups associated to $\mu^{-1}$.

In this section $R$ always denotes a $p$-complete $\mathcal{O}_F$-algebra.

**Proposition 1.6.2** (Bültel-Pappas)**.** *There exists a morphism of $\mathcal{O}_F$-group schemes*

$$\varphi\colon H_\mu \to (L^+\mathcal{G})_{\mathcal{O}_F}$$

*that is characterized by the property*

$$\varphi(h) = \sigma\big(\mu(p)^{-1} \cdot h \cdot \mu(p)\big) \in G\big(W(R)[1/p]\big)$$

*for all $h \in H_\mu(R)$.*

*Proof.* See [BP20, Proposition 3.1.2]. □

**Definition 1.6.3.** A $(\mathcal{G}, \mu)$*-display over $R$ in the sense of Bültel-Pappas* is a tuple $(\mathcal{P}, \mathcal{Q}, \Psi)$ where $\mathcal{P}$ is an $L^+\mathcal{G}$-torsor over $R$, $\mathcal{Q}$ is a choice of reduction of $\mathcal{P}$ to an $H_\mu$-torsor and

$$\Psi \colon L^+\mathcal{G} \times^{\varphi, H_\mu} \mathcal{Q} \to \mathcal{P}$$

is an isomorphism of $L^+\mathcal{G}$-torsors over $R$.

**Remark 1.6.4.** As we are normalizing our display functors contravariantly (see Theorem 2.5.1), our normalizations in this section are different than the ones in [BP20].

## 1.7 Zips

**Notation 1.7.1.** In this section $R$ always denotes an $\overline{\mathbf{F}}_p$-algebra.

Let us start with the following definition of zip data and zips. It is essentially the same as in [PWZ11, Definition 3.1] and [PWZ15, Definition 3.1], but slightly more general.

**Definition 1.7.2.** A *zip datum* is a tuple $\mathcal{Z} = (H, P_+, P_-, L_-, \varphi)$ where $H$ is an affine group scheme over $\overline{\mathbf{F}}_p$, $P_+, P_- \subseteq H$ are subgroup schemes, $L_-$ is a quotient group scheme of $P_-$ and $\varphi \colon P_+ \to L_-$ is a morphism of group schemes.

Let $\mathcal{Z}$ be a zip datum as above. A *zip for $\mathcal{Z}$ over $R$* is a tuple $(I, I_+, I_-, \Psi)$ where $I$ is an $H$-torsor over $R$, $I_+$ (resp. $I_-$) is a reduction of $I$ to a $P_+$-torsor (resp. a $P_-$-torsor) and

$$\Psi \colon L_- \times^{\varphi, P_+} I_+ \to L_- \times^{P_-} I_-$$

is an isomorphism of $L_-$-torsors.

**Proposition 1.7.3.** *Let $\mathcal{Z} = (H, P_+, P_-, L_-, \varphi)$ be a zip datum and let*

$$E_{\mathcal{Z}} \coloneqq P_+ \times_{\varphi, L_-} P_-$$

*the associated zip group.*

*The groupoids of zips for $\mathcal{Z}$, for varying $\overline{\mathbf{F}}_p$-algebras, form a stack that is naturally equivalent to the quotient stack $[E_{\mathcal{Z}} \backslash H]$ for the action*

$$E_{\mathcal{Z}} \times H \to H, \qquad \big((p_+, p_-), h\big) \mapsto p_+ h p_-^{-1}.$$

*Proof.* See [PWZ15, Proposition 3.11]. □

**Example 1.7.4.** Let $G$ be a connected reductive group over $\mathbf{F}_p$ and let

$$\mu \colon \mathbf{G}_{m, \overline{\mathbf{F}}_p} \to G_{\overline{\mathbf{F}}_p}$$

be a minuscule cocharacter. Then we have an associated zip datum $(H, P_+, P_-, L_-, \varphi)$ that is given as follows.

- $H \coloneqq G_{\overline{\mathbf{F}}_p}$.
- $P_+ \coloneqq P_{\mu^{-1}} \subseteq H$.
- $P_- \coloneqq P_\mu^\sigma \subseteq H$.
- $L_- = (P_-)^{\mathrm{rdt}}$ is the reductive quotient of $P_-$. Note that $\mu$ gives a Levi decomposition $P_\mu = U_\mu \rtimes L_\mu$ so that we can identify $L_-$ with $L_\mu^\sigma$ and thus also with $(P_+)^{\mathrm{rdt},\sigma}$.
- $\varphi\colon P_+ \to L_-$ is the composition

$$\varphi\colon P_+ \to (P_+)^{\mathrm{rdt}} \xrightarrow{\sigma} (P_+)^{\mathrm{rdt},\sigma} \cong L_-.$$

Zips for this zip datum are then the same as zips for $(G, \mu^{-1})$ as in [PWZ15] (see also [Zha18, Subsection 1.2]).

## 1.8 Shimura varieties

### 1.8.1 Shimura data and Shimura varieties

**Definition 1.8.1.** The *Deligne torus* $\mathbf{S}$ is the torus over $\mathbf{R}$ given by $\mathbf{S} \coloneqq \operatorname{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{m,\mathbf{C}}$. Explicitly we have

$$\mathbf{S}(R) = \left\{ z = a + bi \in R \oplus Ri \;\middle|\; a^2 + b^2 \in R^\times \right\}$$

where $i$ denotes a formal symbol, and the group structure is given by

$$(a + bi) \cdot (a' + b'i) = (aa' - bb') + (ab' + a'b)i.$$

**Remark 1.8.2.** The characters

$$\mathbf{S}_\mathbf{C} \to \mathbf{G}_{m,\mathbf{C}}, \qquad z = a + bi \mapsto z = a + bi$$

and

$$\mathbf{S}_\mathbf{C} \to \mathbf{G}_{m,\mathbf{C}}, \qquad z = a + bi \mapsto \overline{z} = a - bi$$

form a $\mathbf{Z}$-basis of the character lattice $X^*(\mathbf{S}_\mathbf{C})$, i.e. give rise to an isomorphism $\mathbf{S}_\mathbf{C} \to \mathbf{G}_{m,\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}}$ with inverse

$$\mathbf{G}_{m,\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}} \to \mathbf{S}_\mathbf{C}, \qquad (r_1, r_2) \mapsto \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2i} i.$$

Given a $\mathbf{S}$-representation on an $\mathbf{R}$-vector space $V$ we denote by $V^{p,q} \subseteq V_\mathbf{C}$ the eigenspace of the character $z \mapsto z^{-p} \cdot \overline{z}^{-q}$. This turns $V$ into a real Hodge structure and in fact this construction gives an equivalence between the categories of $\mathbf{S}$-representations and real Hodge structures.

**Definition 1.8.3** (Deligne)**.** A *Shimura datum* is a tuple $(\mathbf{G}, \mathbf{X})$ consisting of a connected reductive group $\mathbf{G}$ over $\mathbf{Q}$ and a $\mathbf{G}(\mathbf{R})$-conjugacy class $\mathbf{X}$ of morphisms $h\colon \mathbf{S} \to \mathbf{G}_{\mathbf{R}}$, such that the following conditions are satisfied for all (or equivalently one) $h \in \mathbf{X}$.

- The representation
$$\mathbf{S} \xrightarrow{h} \mathbf{G}_{\mathbf{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(\mathbf{G}_{\mathbf{R}}))$$
corresponds to a real Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$.
- Conjugation by $h(i)$ induces a Cartan involution of the adjoint group $\mathbf{G}^{\mathrm{ad}}_{\mathbf{R}}$.
- The adjoint group $\mathbf{G}^{\mathrm{ad}}$ has no non-trivial $\mathbf{Q}$-factor $\mathbf{H}$ such that the composition
$$\mathbf{S} \xrightarrow{h} \mathbf{G}_{\mathbf{R}} \to \mathbf{H}_{\mathbf{R}}$$
is trivial.

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum. To each $h \in \mathbf{X}$ we have an associated cocharacter $\mu_h$ of $\mathbf{G}_{\mathbf{C}}$ that is given as the composition
$$\mu_h\colon \mathbf{G}_{m,\mathbf{C}} \xrightarrow{\iota_1} \mathbf{G}_{m,\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}} \to \mathbf{S}_{\mathbf{C}} \xrightarrow{h} \mathbf{G}_{\mathbf{C}},$$
where $\iota_1$ denotes the inclusion of the first factor and the second morphism is the isomorphism from Remark 1.8.2. We write $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{G}, \mathbf{X})$ for the $\mathbf{G}(\mathbf{C})$-conjugacy class of cocharacters of $\mathbf{G}_{\mathbf{C}}$ such that $\mu_h \in \boldsymbol{\mu}$ for all $h \in \mathbf{X}$. Finally we write $\mathbf{E} = \mathbf{E}(\mathbf{G}, \mathbf{X})$ for the field of definition of $\boldsymbol{\mu}$, a finite extension of $\mathbf{Q}$ inside $\mathbf{C}$, and call it the *reflex field of* $(\mathbf{G}, \mathbf{X})$.

**Remark 1.8.4.** Note that the first condition in Definition 1.8.3 implies that the $\mathbf{G}(\mathbf{C})$-conjugacy class $\boldsymbol{\mu}$ of cocharacters of $\mathbf{G}_{\mathbf{C}}$ attached to a Shimura datum $(\mathbf{G}, \mathbf{X})$ is minuscule.

**Theorem 1.8.5** (Deligne, Milne)**.** *Let* $(\mathbf{G}, \mathbf{X})$ *be a Shimura datum. Then* $\mathbf{X}$ *naturally is a complex manifold (and in fact a finite disjoint union of Hermitian symmetric domains).*

*Let* $\mathbf{K} \subseteq \mathbf{G}(\mathbf{A}_f)$ *be a small enough compact open subgroup. Then there is a naturally defined smooth quasi-projective scheme* $\mathrm{Sh}_{\mathbf{K}} = \mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, \mathbf{X})$ *over the reflex field* $\mathbf{E}$ *with*
$$\mathrm{Sh}_{\mathbf{K}}(\mathbf{C}) = \mathbf{G}(\mathbf{Q})\backslash\big(\mathbf{X} \times \mathbf{G}(\mathbf{A}_f)/\mathbf{K}\big)$$
*as complex manifolds. We call* $\mathrm{Sh}_{\mathbf{K}}$ *the* Shimura variety attached to $(\mathbf{G}, \mathbf{X})$ at level $\mathbf{K}$.

*The formation of* $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, \mathbf{X})$ *is functorial in the tuple* $(\mathbf{G}, \mathbf{X}, \mathbf{K})$ *and given small enough compact open subgroups* $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{G}(\mathbf{A}_f)$ *the induced morphism*
$$\mathrm{Sh}_{\mathbf{K}'} \to \mathrm{Sh}_{\mathbf{K}}$$
*is finite étale.*

*Proof.* See [Mil05, Section 14]. □

### 1.8.2 Siegel modular varieties

**Notation 1.8.6.** Fix $g \in \mathbf{Z}_{>0}$ and a tuple $(V, \overline{\psi})$ consisting of a $\mathbf{Q}$-vector space $V$ of dimension $h = 2g$ and a $\mathbf{Q}^\times$-class $\overline{\psi}$ of symplectic forms $\psi\colon V \times V \to \mathbf{Q}$. We write $W := V^\vee$ and also write $\overline{\psi}$ for the $\mathbf{Q}^\times$-class of symplectic forms on $W$ induced by $\overline{\psi}$.

Let $\mathrm{GSp}(V)$ be the connected reductive group of symplectic similitudes of $(V, \overline{\psi})$ over $\mathbf{Q}$. Let $S^\pm$ be the set of complex structures $J$ on $V_\mathbf{R}$ that are compatible with $\overline{\psi}$ in the sense that for every (or equivalently one) representative $\psi \in \overline{\psi}$ the induced bilinear form

$$V_\mathbf{R} \times V_\mathbf{R} \to \mathbf{R}, \qquad (u, v) \mapsto \psi(u, Jv)$$

is symmetric and definite. To $J \in S^\pm$ we associate the morphism

$$h_J \colon \mathbf{S} \to \mathrm{GSp}(V)_\mathbf{R}, \qquad a + bi \mapsto a + bJ.$$

The map $J \mapsto h_J$ defines a bijection between $S^\pm$ and precisely one $\mathrm{GSp}(V)(\mathbf{R})$-conjugacy class of morphisms $\mathbf{S} \to \mathrm{GSp}(V)_\mathbf{R}$. We identify $S^\pm$ with this conjugacy class.

**Theorem 1.8.7.** *The tuple* $(\mathrm{GSp}(V), S^\pm)$ *is a Shimura datum with reflex field* $\mathbf{Q}$ *(that we call the* Siegel Shimura datum attached to $(V, \overline{\psi})$*).*

*Let* $\mathbf{L} \subseteq \mathrm{GSp}(V)(\mathbf{A}_f)$ *be a small enough compact open subgroup. The associated Shimura variety* $\mathrm{Sh}_\mathbf{L} = \mathrm{Sh}_\mathbf{L}(\mathrm{GSp}(V), S^\pm)$ *over* $\mathbf{Q}$*, also called the* Siegel modular variety attached to $(V, \overline{\psi})$ at level $\mathbf{L}$*, is the moduli space of tuples*

$$\big(A, \overline{\lambda}, \overline{\eta}\big)$$

*that are given as follows.*

- *$A$ is an Abelian variety up to isogeny of dimension $g$.*
- *$\overline{\lambda}$ is a $\mathbf{Q}$-homogeneous polarization of $A$, i.e. a $\mathbf{Q}^\times$-class of symmetric isomorphisms $\lambda\colon A \to A^\vee$ such that there exists an Abelian variety representative of $A$ and $\lambda \in \overline{\lambda}$ such that $\lambda$ defines a polarization of that representative.*
- *$\overline{\eta}$ is an $\mathbf{L}$-class of isomorphisms $\eta\colon V_{\mathbf{A}_f} \to V_f(A)$ identifying $\overline{\psi}$ with the $\mathbf{A}_f^\times$-class of symplectic forms on $V_f(A)$ induced by $\overline{\lambda}$.*

*Proof.* See [Mil05, Theorem 6.11]. □

**Remark 1.8.8.** The $\mathbf{G}(\mathbf{C})$-conjugacy class $\boldsymbol{\mu}$ of cocharacters of $\mathrm{GSp}(V)_\mathbf{C}$ attached to the Siegel Shimura datum $(\mathrm{GSp}(V), S^\pm)$ is the one of those cocharacters that induce a weight decomposition $V_\mathbf{C} = V_0 \oplus V_1$ where $V_0, V_1 \subseteq V_\mathbf{C}$ are both Lagrangian subspaces, i.e. the one that we called $\boldsymbol{\mu}_g$ in Notation 1.4.10.

**Remark 1.8.9.** Let us explain how the moduli-theoretic description of the Siegel modular variety $\mathrm{Sh}_\mathbf{L}$ given in Theorem 1.8.7 matches the description of its $\mathbf{C}$-valued points as

$$\mathrm{Sh}_\mathbf{L}(\mathbf{C}) = \mathrm{GSp}(V)(\mathbf{Q}) \backslash \big(S^\pm \times \mathrm{GSp}(V)(\mathbf{A}_f)/\mathbf{L}\big).$$

Let us consider tuples $(A, \overline{\lambda}, \xi)$ consisting of an Abelian variety $A$ up to isogeny over $\mathbf{C}$, a $\mathbf{Q}$-homogeneous polarization $\overline{\lambda}$ of $A$ and an isomorphism $\xi\colon H_1(A, \mathbf{Q}) \to V$ that is compatible with the $\mathbf{Q}^\times$-class of symplectic forms on $H_1(A, \mathbf{Q})$ induced by $\overline{\lambda}$ and $\overline{\psi}$. Then we have a bijection

$$\big\{(A, \overline{\lambda}, \xi)\big\} \to S^\pm$$

that is given by sending $(A, \overline{\lambda}, \xi)$ to the complex structure $J$ on $V_{\mathbf{R}}$ that corresponds to the natural complex structure on $H_1(A, \mathbf{R})$ under the isomorphism $\xi$. It is equivariant with respect to the natural $\mathrm{GSp}(V)(\mathbf{Q})$-actions on both sides.

Now let us consider tuples $(A, \overline{\lambda}, \xi, \eta)$ where $A, \overline{\lambda}$ and $\xi$ are as before and in addition $\eta\colon V_{\mathbf{A}_f} \to V_f(A)$ is an isomorphism that is compatible with $\overline{\psi}$ and the $\mathbf{A}_f^\times$-class of symplectic forms on $V_f(A)$ induced by $\overline{\lambda}$. Then we have a bijection

$$\big\{(A, \overline{\lambda}, \xi, \eta)\big\} \to S^\pm \times \mathrm{GSp}(V)(\mathbf{A}_f)$$

that is given by sending $(A, \overline{\lambda}, \xi, \eta)$ to $(J, x)$ where $J$ is as before and $x$ is the composition

$$V_{\mathbf{A}_f} \xrightarrow{\eta} V_f(A) \to \mathbf{A}_f \otimes_{\mathbf{Q}} H_1(A, \mathbf{Q}) \xrightarrow{\xi} V_{\mathbf{A}_f}.$$

It is equivariant with respect to the natural $\mathrm{GSp}(V)(\mathbf{Q}) \times \mathrm{GSp}(V)(\mathbf{A}_f)$-actions.

After taking the quotient by $\mathrm{GSp}(V)(\mathbf{Q})$ we obtain a bijection

$$\big\{(A, \overline{\lambda}, \eta)\big\} \to \mathrm{GSp}(V)(\mathbf{Q})\backslash\big(S^\pm \times \mathrm{GSp}(V)(\mathbf{A}_f)\big)$$

that is still equivariant for the $\mathrm{GSp}(V)(\mathbf{A}_f)$-actions. Finally, after further taking the quotient by a small enough compact open subgroup $\mathbf{L} \subseteq \mathrm{GSp}(V)(\mathbf{A}_f)$, we obtain a bijection

$$\big\{(A, \overline{\lambda}, \overline{\eta})\big\} \to \mathrm{GSp}(V)(\mathbf{Q})\backslash\big(S^\pm \times \mathrm{GSp}(V)(\mathbf{A}_f)/\mathbf{L}\big) = \mathrm{Sh}_{\mathbf{L}}(\mathbf{C})$$

where now the tuples $(A, \overline{\lambda}, \overline{\eta})$ are as in Theorem 1.8.7.

### 1.8.3 Integral models for Siegel modular varieties at hyperspecial level

**Notation 1.8.10.** We use Notation 1.8.6. Additionally we fix a self-dual $\mathbf{Z}_p$-lattice $\Xi \subseteq V_{\mathbf{Q}_p}$. Denote by $\mathbf{L}_p \subseteq \mathrm{GSp}(V)(\mathbf{Q}_p)$ the stabilizer of $\Xi$ and write $\mathrm{GSp}(\Xi)$ for the connected reductive group scheme of symplectic similitudes of $(\Xi, \overline{\psi})$ over $\mathbf{Z}_p$. Note that we then have $\mathbf{L}_p = \mathrm{GSp}(\Xi)(\mathbf{Z}_p)$.

**Theorem 1.8.11** (Mumford, Kottwitz)**.** *Let $\mathbf{L}^p \subseteq \mathrm{GSp}(V)(\mathbf{A}_f^p)$ be a small enough compact open subgroup and set $\mathbf{L} \coloneqq \mathbf{L}_p\mathbf{L}^p \subseteq \mathrm{GSp}(V)(\mathbf{A}_f)$. Then the Siegel modular variety $\mathrm{Sh}_{\mathbf{L}}$ has a natural smooth quasi-projective integral model $\mathscr{S}_{\mathbf{L}} = \mathscr{S}_{\mathbf{L}}(\mathrm{GSp}(V), S^\pm)$ over $\mathbf{Z}_p$ that is the moduli space of tuples*

$$\big(A, \overline{\lambda}, \overline{\eta}\big)$$

*that are given as follows.*

- *$A$ is an Abelian variety up to prime-to-p isogeny of dimension $g$.*
- *$\overline{\lambda}$ is a $\mathbf{Z}_{(p)}$-homogeneous principal polarization of $A$, i.e. a $\mathbf{Z}_{(p)}^{\times}$-class of symmetric isomorphisms $\lambda\colon A \to A^{\vee}$ such that there exists an Abelian variety representative of $A$ and $\lambda \in \overline{\lambda}$ such that $\lambda$ defines a polarization of that representative.*
- *$\overline{\eta}$ is a $\mathbf{L}^p$-class of isomorphisms $\eta\colon V_{\mathbf{A}_f^p} \to V_f^p(A)$ identifying $\mathbf{A}_f^{p,\times}$-classes of symplectic forms similarly as in Theorem 1.8.7.*

*Given small enough compact open subgroups $\mathbf{L}'^p \subseteq \mathbf{L}^p \subseteq \mathrm{GSp}(V)(\mathbf{A}_f^p)$ with associated compact open subgroups $\mathbf{L}' \subseteq \mathbf{L} \subseteq \mathrm{GSp}(V)(\mathbf{A}_f)$ the natural morphism $\mathrm{Sh}_{\mathbf{L}'} \to \mathrm{Sh}_{\mathbf{L}}$ extends to a morphism*

$$\mathscr{S}_{\mathbf{L}'} \to \mathscr{S}_{\mathbf{L}}$$

*that is still finite étale.*

*Proof.* See [Kot92, Section 5]. □

**Remark 1.8.12.** Over a $\mathbf{Q}_p$-algebra $R$ the functor

$$\begin{aligned} \left\{\begin{matrix}\text{Abelian varieties up to}\\ \text{prime-to-}p\text{ isogeny}\end{matrix}\right\} &\to \left\{ \big(A,T\big) \;\middle|\; \begin{matrix} A \text{ an Abelian variety up to isogeny,}\\ T \subseteq V_p(A) \text{ a } \mathbf{Z}_p\text{-lattice}\end{matrix}\right\}, \\ A &\mapsto \big(A, T_p(A)\big) \end{aligned}$$

is an equivalence. From this it then follows that the moduli description in Theorem 1.8.11 gives back the one in Theorem 1.8.7 after base changing to $\mathbf{Q}_p$.

### 1.8.4 Integral models for Shimura varieties of Hodge type

**Notation 1.8.13.** Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type (i.e. a Shimura datum that embeds into a Siegel Shimura datum). Assume that $G \coloneqq \mathbf{G}_{\mathbf{Q}_p}$ splits over a tamely ramified extension of $\mathbf{Q}_p$ and $p \nmid |\pi_1(G^{\mathrm{der}})|$. We then have the associated reflex field $\mathbf{E}$ and the associated conjugacy class $\boldsymbol{\mu}$ as in Definition 1.8.3 and denote by $E$ the completion of $\mathbf{E}$ at the place corresponding to the embedding $\mathbf{E} \subseteq \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. Let $\mathbf{K}_p \subseteq G(\mathbf{Q}_p)$ be a parahoric stabilizer and let $\mathcal{G}$ be the corresponding parahoric group scheme over $\mathbf{Z}_p$, so that $\mathcal{G}_{\mathbf{Q}_p} = G$ and $\mathcal{G}(\mathbf{Z}_p) = \mathbf{K}_p$.

By results of Kisin and Pappas (see [KP18]) there then exist $(V, \overline{\psi})$ and $\Xi \subseteq V_{\mathbf{Q}_p}$ as in Notation 1.8.6 and Notation 1.8.10 and an embedding of Shimura data

$$(\mathbf{G}, \mathbf{X}) \to (\mathrm{GSp}(V), S^{\pm})$$

such that $G \to \mathrm{GSp}(V)_{\mathbf{Q}_p}$ extends to a closed immersion

$$\mathcal{G} \to \mathrm{GSp}(\Xi)$$

of group schemes over $\mathbf{Z}_p$ and such that $X_{\boldsymbol{\mu}}(G) \to X_{\boldsymbol{\mu}_g}(\mathrm{GSp}(V)_{\mathbf{Q}_p})_E = \mathrm{LGrass}_{V,E}$ extends to a closed immersion of local models

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}} \to (\mathbb{M}^{\mathrm{loc}}_{\mathrm{GSp}(\Xi),\boldsymbol{\mu}_g})_{\mathcal{O}_E} = \mathrm{LGrass}_{\Xi,\mathcal{O}_E}$$

(see Theorem 1.4.6 for the notation $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$).

**Definition 1.8.14** (Kisin-Pappas). Let $\mathbf{K}^p \subseteq \mathbf{G}(\mathbf{A}_f^p)$ be a small enough compact open subgroup and set $\mathbf{K} \coloneqq \mathbf{K}_p\mathbf{K}^p \subseteq \mathbf{G}(\mathbf{A}_f)$. Choose a small enough compact open subgroup $\mathbf{L}^p \subseteq \mathrm{GSp}(V)(\mathbf{A}_f^p)$ such that $\mathbf{K}^p$ maps into $\mathbf{L}^p$ and such that the induced morphism of Shimura varieties

$$\mathrm{Sh}_{\mathbf{K}}(\mathbf{G},\mathbf{X}) \to \mathrm{Sh}_{\mathbf{L}}\big(\mathrm{GSp}(V), S^{\pm}\big)_{\mathbf{E}}$$

is a closed immersion (where we have set $\mathbf{L} \coloneqq \mathbf{L}_p\mathbf{L}^p$). Note that this is always possible by [Kis10, Lemma 2.1.2].

We then define an integral model $\mathscr{S}_{\mathbf{K}} = \mathscr{S}_{\mathbf{K}}(\mathbf{G},\mathbf{X})$ of $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G},\mathbf{X})$ over $\mathcal{O}_E$ as the normalization of the closure of $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G},\mathbf{X})_E$ inside $\mathscr{S}_{\mathbf{L}}(\mathrm{GSp}(V), S^{\pm})_{\mathcal{O}_E}$. Note that this is independent of the choice of $\mathbf{L}^p$.

**Remark 1.8.15.** The integral model $\mathscr{S}_{\mathbf{K}}(\mathbf{G},\mathbf{X})$ from Definition 1.8.14 is in fact independent of the choice of Hodge embedding $(\mathbf{G},\mathbf{X}) \to (\mathrm{GSp}(V), S^{\pm})$. This is shown by Pappas in [Pap23, Theorem 8.1.6].

### 1.8.5 Integral models for Siegel modular varieties at parahoric level

**Notation 1.8.16.** We use Notation 1.8.6. Additionally we fix $J$ and a tuple $(\Xi_i)_{i\in J}$ of $\mathbf{Z}_p$-lattices $\Xi_i \subseteq V_{\mathbf{Q}_p}$ with associated data

$$\Lambda_i \subseteq W_{\mathbf{Q}_p}, \quad \rho_{i,j}, \quad \theta_i, \quad \overline{\psi_i} \quad \text{and} \quad \mathrm{GSp}((\Xi_i)_i)$$

as in Notation 1.4.10. Set $\mathbf{L}_p \coloneqq \mathrm{GSp}((\Xi_i)_i)(\mathbf{Z}_p) \subseteq \mathrm{GSp}(V)(\mathbf{Q}_p)$.

**Theorem 1.8.17** (de Jong, Rapoport-Zink). *Let $\mathbf{L}^p \subseteq \mathrm{GSp}(V)(\mathbf{A}_f^p)$ be a small enough compact open subgroup and set $\mathbf{L} \coloneqq \mathbf{L}_p\mathbf{L}^p \subseteq \mathrm{GSp}(V)(\mathbf{A}_f)$. Then the Siegel modular variety $\mathrm{Sh}_{\mathbf{L}}$ has a natural quasi-projective integral model $\mathscr{S}_{\mathbf{L}}$ over $\mathbf{Z}_p$ that is the moduli space of tuples*

$$\Big(\big(A_i\big)_{i\in J}, (\rho_{i,j})_{i,j\in J, i\leq j}, \overline{(\lambda_i)_{i\in J}}, \overline{\eta}\Big)$$

*that are given as follows.*

- *$((A_i)_i, (\rho_{i,j})_{i,j})$ is a diagram of shape $J^{\mathrm{op}}$ of Abelian varieties up to prime-to-p isogeny of dimension $g$ such that $\rho_{i,j} \colon A_j \to A_i$ is an isogeny of degree $p^{j-i}$ and* $\ker(\rho_{i-2g,i}) = A_i[p]$.
- *$\overline{(\lambda_i)_i}$ is a $\mathbf{Z}_{(p)}^{\times}$-class of isomorphisms*

$$(\lambda_i)_i \colon \big((A_i)_i, (\rho_{i,j})_{i,j}\big) \to \big((A_{-i}^{\vee})_i, (\rho_{-j,-i}^{\vee})_{i,j}\big)$$

*that is symmetric (i.e. $\lambda_i^{\vee} = \lambda_{-i}$) and such that for all (or equivalently one) $i \geq 0$ the composition*

$$A_i \xrightarrow{\rho_{-i,i}} A_{-i} \xrightarrow{\lambda_{-i}} A_i^{\vee}$$

*is a $\mathbf{Z}_{(p)}$-homogeneous polarization, i.e. there exists an Abelian variety representative of $A_i$ such that $\lambda_{-i}\rho_{-i,i}$ defines a polarization of that representative.*

- *$\overline{\eta}$ is a $\mathbf{L}^p$-class of isomorphisms $\eta\colon V_{\mathbf{A}_f^p} \to V_f^p(A)$ identifying $\mathbf{A}_f^{p,\times}$-classes of symplectic forms similarly as in Theorem 1.8.7. Here $V_f^p(A)$ denotes the common rational prime-to-p Tate module of the $A_i$.*

*Proof.* See [RZ96, Chapter 6]. □

**Remark 1.8.18.** In fact the integral models from Theorem 1.8.17 agree with the ones defined in Definition 1.8.14 by [Pap23, Theorem 8.1.6].

# 2 Displays

The notion of a (not necessarily nilpotent) display was introduced by Zink in [Zin02] and studied extensively by Lau and Zink. It gives a generalization of the theory of Dieudonné modules to arbitrary $p$-complete base rings. Just as in classical Dieudonné theory, there is a natural way to associate a display to a $p$-divisible group. However, this construction typically does not yield an equivalence of categories anymore (although it does if one restricts to formal $p$-divisible groups).

If one considers only complete Noetherian local rings with perfect residue field of characteristic $p$ as base rings there exists a theory of Dieudonné displays, that refines the theory of (usual) displays. There is again a natural way to associate a Dieudonné display to a $p$-divisible groups and Zink showed in [Zin01] that this in fact does give an equivalence of categories.

In this chapter we recall the theory of displays and Dieudonné displays. However we phrase it in a slightly non-standard way, similarly as in [KP18]. The equivalence of our definitions and the ones from [Zin02] and [Zin01] follows from the argument in [KP18, Lemma 3.1.5]. We also introduce the new related notion of an $(m, n)$-truncated display (for positive integers $m$ and $n$ with $m \geq n+1$) that is inspired by the definition of $(m, n)$-restricted local shtukas given by Xiao and Zhu in [XZ17] (see also Section 1.5). Our $(m, n)$-truncated displays are in some sense more naive versions of the $n$-truncated displays introduced by Lau and Zink in [Lau13] and [LZ18], but they are better suited for our purposes.

**Notation 2.0.1.** We fix integers $0 \leq d \leq h$. The letter $R$ always denotes a $p$-complete ring, $m$ always denotes a positive integer and $(m, n)$ always denotes a tuple of positive integers with $m \geq n+1$.

## 2.1 Pairs and displays

**Definition 2.1.1.** A *pair (of type $(h, d)$) over $R$* is a tuple $(M, M_1)$ consisting of a finite projective $W(R)$-module $M$ (of rank $h$) and a $W(R)$-submodule $M_1 \subseteq M$ with $I_R M \subseteq M_1$ and such that $M_1/I_R M \subseteq M/I_R M$ is a direct summand (of rank $d$).

An *$m$-truncated pair over $R$* is a tuple $(M, M_1)$ consisting of a finite projective $W_m(R)$-module $M$ and a $W_m(R)$-submodule $M_1 \subseteq M$ with $I_{m,R} M \subseteq M_1$ and such that $M_1/I_{m,R} M \subseteq M/I_{m,R} M$ is a direct summand.

Now let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. A *Dieudonné pair over $R$* is a tuple $(M, M_1)$ consisting of a finite projective $\widehat{W}(R)$-module $M$ and a $\widehat{W}(R)$-submodule $M_1 \subseteq M$ with $\widehat{I}_R M \subseteq M_1$ and such that $M_1/\widehat{I}_R M \subseteq M/\widehat{I}_R M$ is a direct summand.

**Definition 2.1.2.** Let $(M, M_1)$ and $(M', M'_1)$ be two pairs over $R$. Then a morphism $(M, M_1) \to (M', M'_1)$ is a morphism of $W(R)$-modules $f\colon M \to M'$ such that $f(M_1) \subseteq M'_1$. In the same way we also define morphisms of $m$-truncated pairs and Dieudonné pairs.

**Remark 2.1.3.** Let $(M, M_1)$ be a pair over $R$ and let $R \to R'$ be a morphism of $p$-complete rings. Then we can form the base change $(M', M'_1) = (M, M_1)_{R'}$ that is a pair over $R'$. It is characterized by

$$M' = W(R') \otimes_{W(R)} M \quad \text{and} \quad M'_1/I_{R'}M' = R' \otimes_R (M_1/I_R M).$$

Similarly we can also base change $m$-truncated pairs and Dieudonné pairs.

**Remark 2.1.4.** There are natural truncation functors

$$\bigl\{\text{pairs over } R\bigr\} \to \bigl\{m\text{-truncated pairs over } R\bigr\}$$

and

$$\bigl\{m'\text{-truncated pairs over } R\bigr\} \to \bigl\{m\text{-truncated pairs over } R\bigr\}$$

for $m \leq m'$. There also is a natural functor

$$\bigl\{\text{Dieudonné pairs over } R\bigr\} \to \bigl\{\text{pairs over } R\bigr\}.$$

**Remark 2.1.5.** Let $(M, M_1)$ be a pair over $R$. Then $(M, M_1)$ always has a *normal decomposition* $(L, T)$, i.e. a direct sum decomposition $M = L \oplus T$ such that $M_1 = L \oplus I_R T$. Given a second pair $(M', M'_1)$ over $R$ with normal decomposition $(L', T')$, every morphism of pairs $f\colon (M, M_1) \to (M', M'_1)$ can be written in matrix form $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with

$$a\colon L \to L', \quad b\colon T \to L', \quad c\colon L \to I_R T', \quad d\colon T \to T'.$$

The same is true for $m$-truncated pairs and Dieudonné pairs.

**Proposition 2.1.6.** *There are unique functors*

$$\begin{aligned} \Bigl\{\textit{pairs (of type } (h,d)\textit{) over } R\Bigr\} &\to \begin{Bmatrix} \textit{finite projective } W(R)\textit{-modules} \\ \textit{(of rank } h\textit{)} \end{Bmatrix}, \\ (M, M_1) &\mapsto \widetilde{M_1} \end{aligned}$$

*together with natural isomorphisms $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ that are compatible with base change in $R$ and that in the case that $W(R)$ is $p$-torsionfree are given by*

$$\widetilde{M_1} = \operatorname{im}\bigl(M_1^\sigma \to M^\sigma\bigr).$$

*There also exist unique functors*

$$\begin{aligned} \bigl\{m\textit{-truncated pairs over } R\bigr\} &\to \bigl\{\textit{finite projective } W_n(R)\textit{-modules}\bigr\}, \\ (M, M_1) &\mapsto \widetilde{M_1} \end{aligned}$$

*that are compatible with base change in $R$ and with passing from pairs to $m$-truncated pairs (where we recall that $m$ and $n$ are positive integers with $m \geq n+1$).*

*When we restrict to complete Noetherian local rings $R$ with residue field $\overline{\mathbf{F}}_p$ there also exist unique functors*

$$\begin{array}{rcl} \{\textit{Dieudonné pairs over } R\} & \to & \{\textit{finite projective } \widehat{W}(R)\textit{-modules}\}, \\ (M, M_1) & \mapsto & \widetilde{M_1} \end{array}$$

*together with natural isomorphisms $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ that are compatible with base change in $R$ and that in the case that $\widehat{W}(R)$ is $p$-torsionfree are again given by the formula $\widetilde{M_1} = \operatorname{im}(M_1^\sigma \to M^\sigma)$. This construction is compatible with passing from Dieudonné pairs to pairs.*

*Proof.* Let $(M, M_1)$ be a pair over $R$ and choose a normal decomposition $(L, T)$, see Remark 2.1.5. We then define

$$\widetilde{M_1} \coloneqq L^\sigma \oplus T^\sigma.$$

Given a second pair $(M', M_1')$ with normal decomposition $(L', T')$, a morphism of pairs $f\colon (M, M_1) \to (M', M_1')$ can be written in matrix form $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with

$$a\colon L \to L', \quad b\colon T \to L', \quad c\colon L \to I_R T', \quad d\colon T \to T'.$$

We then define $\widetilde{f}\colon \widetilde{M_1} \to \widetilde{M_1'}$ by the matrix $\left(\begin{smallmatrix} a^\sigma & pb^\sigma \\ \dot{c} & d^\sigma \end{smallmatrix}\right)$ where $\dot{c}$ denotes the composition

$$\dot{c}\colon L^\sigma \xrightarrow{c^\sigma} I_R^\sigma \otimes_{W(R)} T'^\sigma \xrightarrow{\sigma^{\mathrm{div}} \otimes \mathrm{id}} T'^\sigma.$$

Finally we define the natural isomorphism

$$\widetilde{M_1}[1/p] = L^\sigma[1/p] \oplus T^\sigma[1/p] \to M^\sigma[1/p] = L^\sigma[1/p] \oplus T^\sigma[1/p], \qquad l + t \mapsto l + pt.$$

One can now check that this is well-defined and has the required properties.

For $m$-truncated pairs the construction of $(M, M_1) \mapsto \widetilde{M_1}$ is essentially the same as for pairs; here

$$\dot{c}\colon W_n(R) \otimes_{\sigma, W_m(R)} L \to W_n(R) \otimes_{\sigma, W_m(R)} T'$$

is given as the composition

$$\begin{aligned} W_n(R) \otimes_{\sigma, W_m(R)} L \xrightarrow{c^\sigma} \big(W_n(R) \otimes_{\sigma, W_m(R)} I_{m,R}\big) \otimes_{W_n(R)} \big(W_n(R) \otimes_{\sigma, W_m(R)} T'\big) \\ \xrightarrow{\sigma^{\mathrm{div}} \otimes \mathrm{id}} W_n(R) \otimes_{\sigma, W_m(R)} T'. \end{aligned}$$

Finally, for Dieudonné pairs the construction of $(M, M_1) \mapsto \widetilde{M_1}$ is exactly the same as for pairs. □

**Remark 2.1.7.** Let $(M, M_1)$ be a pair over $R$. Then there is a natural surjective morphism of $W(R)$-modules $M_1^\sigma \to \widetilde{M_1}$. Typically this morphism is not an isomorphism but it is when $R$ is a perfect $\mathbf{F}_p$-algebra.

**Remark 2.1.8.** For a pair $(M, M_1)$ over $R$ the isomorphism $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ can be refined to natural morphisms

$$M^\sigma \to \widetilde{M_1} \quad \text{and} \quad \widetilde{M_1} \to M^\sigma$$

coming from the inclusions $pM^\sigma \subseteq \widetilde{M_1} \subseteq M^\sigma$ in the case that $W(R)$ is $p$-torsionfree. The same remark also applies to Dieudonné pairs.

**Definition 2.1.9.** A *display over* $R$ is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is a pair over $R$ and $\Psi\colon \widetilde{M_1} \to M$ is an isomorphism of $W(R)$-modules.

Similarly, an $(m, n)$*-truncated display over* $R$ is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is an $m$-truncated pair over $R$ and $\Psi\colon \widetilde{M_1} \to W_n(R) \otimes_{W_m(R)} M$ is an isomorphism of $W_n(R)$-modules.

Now let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. A *Dieudonné display over* $R$ is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is a Dieudonné pair over $R$ and $\Psi\colon \widetilde{M_1} \to M$ is an isomorphism of $\widehat{W}(R)$-modules.

**Definition 2.1.10.** Let $(M, M_1, \Psi)$ be a display over $R$. Then the *Frobenius of* $(M, M_1, \Psi)$ is the composition

$$\Phi\colon M^\sigma \to \widetilde{M_1} \xrightarrow{\Psi} M,$$

where $M^\sigma \to \widetilde{M_1}$ is the morphism from Remark 2.1.8. Note that $\Phi$ becomes an isomorphism after inverting $p$.

We make the same definition for Dieudonné displays.

## 2.2 Duals and twists

**Definition 2.2.1.** Let $(M, M_1)$ be a pair over $R$. Then we define its *dual*

$$(M, M_1)^\vee \coloneqq (M^\vee, M_1^*)$$

as follows.

- $M^\vee = \mathrm{Hom}_{W(R)}(M, W(R))$ is the dual of the finite projective $W(R)$-module $M$.
- $M_1^* \subseteq M$ is the $W(R)$-submodule of all $\omega\colon M \to W(R)$ such that $\omega(M_1) \subseteq I_R$. Equivalently it is the preimage under

  $$M^\vee \to M^\vee/I_R M^\vee \cong (M/I_R M)^\vee$$

  of the orthogonal complement $(M_1/I_R M)^\perp \subseteq (M/I_R M)^\vee$.

This endows the category of pairs over $R$ with a duality in the sense of Definition 1.2.1. We similarly define duals of $m$-truncated pairs and Dieudonné pairs.

**Remark 2.2.2.** Note that if $(M, M_1)$ is a pair of type $(h, d)$ over $R$ then $(M, M_1)^\vee$ is of type $(h, h - d)$.

**Lemma 2.2.3.** *Given a pair $(M, M_1)$ over $R$ there exists a unique natural isomorphism*

$$\widetilde{M_1^*} \to \widetilde{M_1}^\vee$$

*that is compatible with base change in $R$ and makes the diagram*

$$\begin{tikzcd}
\widetilde{M_1^*}[1/p] \arrow[r] \arrow[d] & \widetilde{M_1}^\vee[1/p] \arrow[d] \\
M^{\sigma,\vee}[1/p] \arrow[r, "p^{-1}"] & M^{\sigma,\vee}[1/p]
\end{tikzcd}$$

*commutative. The same is true for Dieudonné pairs.*

*Given an $m$-truncated pair $(M, M_1)$ over $R$ there exists a unique natural isomorphism*

$$\widetilde{M_1^*} \to \widetilde{M_1}^\vee$$

*that is compatible with base change in $R$ and with passing from pairs to $m$-truncated pairs.*

*In other words, the various functors $(M, M_1) \mapsto \widetilde{M_1}$ from Proposition 2.1.6 are compatible with dualities in the sense of Definition 1.2.1.*

*Proof.* Let $(L, T)$ be a normal decomposition of $(M, M_1)$. Then $(T^\vee, L^\vee)$ is a normal decomposition of $(M^\vee, M_1^*)$. We then define the desired natural isomorphism to be the composition

$$\widetilde{M_1^*} \cong T^{\vee,\sigma} \oplus L^{\vee,\sigma} \cong (L^\sigma \oplus T^\sigma)^\vee \cong \widetilde{M_1}^\vee. \qquad \square$$

**Definition 2.2.4.** Let $(M, M_1, \Psi)$ be a display over $R$. Then we define its *dual*

$$(M, M_1, \Psi)^\vee := (M^\vee, M_1^*, \Psi^{\vee,-1}),$$

where $\Psi^{\vee,-1}$ really denotes the composition

$$\widetilde{M_1^*} \to \widetilde{M_1}^\vee \xrightarrow{\Psi^{\vee,-1}} M^\vee,$$

the first isomorphism being the one from Lemma 2.2.3. This endows the category of pairs over $R$ with a duality. We similarly define duals of $(m, n)$-truncated displays and Dieudonné displays.

**Definition 2.2.5.** Let $(M, M_1)$ be a pair over $R$ and let $I$ be an invertible $W(R)$-module. Then we define the *twist*

$$I \otimes (M, M_1) := (I \otimes_{W(R)} M, I \otimes_{W(R)} M_1).$$

This endows the category of pairs over $R$ with an action of the symmetric monoidal category of invertible $W(R)$-modules in the sense of Definition 1.2.4 and this action is compatible with the duality from Definition 2.2.1. We similarly define twists of $m$-truncated displays and Dieudonné displays.

**Lemma 2.2.6.** *Given a pair* $(M, M_1)$ *over* $R$ *and an invertible* $W(R)$*-module* $I$ *there exists a unique natural isomorphism*

$$(I \otimes_{W(R)} M_1)^{\sim} \cong I^{\sigma} \otimes_{W(R)} \widetilde{M_1}$$

*that is compatible with base change in* $R$ *and makes the diagram*

$$\begin{array}{ccc} (I \otimes_{W(R)} M_1)^{\sim}[1/p] & \longrightarrow & \big(I^{\sigma} \otimes_{W(R)} \widetilde{M_1}\big)[1/p] \\ \downarrow & & \downarrow \\ (I \otimes_{W(R)} M)^{\sigma}[1/p] & \longrightarrow & I^{\sigma}[1/p] \otimes_{W(R)[1/p]} M^{\sigma}[1/p] \end{array}$$

*commutative. The same is true for Dieudonné pairs.*

*Given an* $m$*-truncated pair* $(M, M_1)$ *over* $R$ *and an invertible* $W_m(R)$*-module* $I$ *there exists a unique natural isomorphism*

$$(I \otimes_{W_m(R)} M_1)^{\sim} \cong (W_n(R) \otimes_{\sigma, W_m(R)} I) \otimes_{W_n(R)} \widetilde{M_1}.$$

*that is compatible with base change in* $R$ *and with passing from pairs to* $m$*-truncated pairs.*

*In other words, the various functors* $(M, M_1) \mapsto \widetilde{M_1}$ *from Proposition 2.1.6 are equivariant with respect to the functors of symmetric monoidal categories*

$$\begin{array}{rcl} \{\textit{invertible } W(R)\textit{-modules}\} & \to & \{\textit{invertible } W(R)\textit{-modules}\}, \\ I & \mapsto & I^{\sigma}, \end{array}$$

$$\begin{array}{rcl} \{\textit{invertible } W_m(R)\textit{-modules}\} & \to & \{\textit{invertible } W_n(R)\textit{-modules}\}, \\ I & \mapsto & W_n(R) \otimes_{\sigma, W_m(R)} I, \end{array}$$

*and, when* $R$ *is a complete Noetherian local ring with residue field* $\overline{\mathbf{F}}_p$*,*

$$\begin{array}{rcl} \{\textit{invertible } \widehat{W}(R)\textit{-modules}\} & \to & \{\textit{invertible } \widehat{W}(R)\textit{-modules}\}, \\ I & \mapsto & I^{\sigma}. \end{array}$$

*Proof.* Let $(L, T)$ be a normal decomposition of $(M, M_1)$. Then $(I \otimes_{W(R)} L, I \otimes_{W(R)} T)$ is a normal decomposition of $I \otimes (M, M_1)$. We then define the desired natural isomorphism to be the composition

$$(L \otimes_{W(R)} M_1)^{\sim} \cong (I \otimes_{W(R)} L)^{\sigma} \oplus (I \otimes_{W(R)} T)^{\sigma} \cong I^{\sigma} \otimes_{W(R)} (L^{\sigma} \oplus T^{\sigma}) \cong I^{\sigma} \otimes_{W(R)} \widetilde{M_1}.$$

□

**Definition 2.2.7.** Let $(M, M_1, \Psi)$ be a display over $R$ and let $(I, \iota)$ be a tuple consisting of an invertible $W(R)$-module $I$ and an isomorphism $\iota \colon I^{\sigma} \to I$. Then we define the twist

$$(I, \iota) \otimes (M, M_1, \Psi) := (I \otimes_{W(R)} M, I \otimes_{W(R)} M_1, \iota \otimes \Psi),$$

where $\iota \otimes \Psi$ really denotes the composition

$$(I \otimes_{W(R)} M_1)^{\sim} \to I^{\sigma} \otimes_{W(R)} \widetilde{M_1} \xrightarrow{\iota \otimes \Psi} M,$$

the first isomorphism being the one from Lemma 2.2.6. This endows the category of displays over $R$ with an action of the symmetric monoidal category of tuples $(I, \iota)$ as above and this action is compatible with the duality from Definition 2.2.4.

We similarly define twists of $(m,n)$-truncated displays and Dieudonné displays. For $(m,n)$-truncated displays over $R$ the twists are by tuples $(I, \iota)$ consisting of an invertible $W_m(R)$-module $I$ and an isomorphism $\iota\colon W_n(R) \otimes_{\sigma, W_m(R)} I \to I$.

## 2.3 Grothendieck-Messing Theory for Dieudonné displays

**Notation 2.3.1.** In this section $S \to R$ always denotes a surjection of complete Noetherian local rings with residue field $\overline{\mathbf{F}}_p$ such that its kernel $\mathfrak{a} \subseteq S$ is equipped with nilpotent divided powers that are compatible with the canonical divided powers on $pS \subseteq S$ and continuous (see Lemma 1.1.2).

**Definition 2.3.2.** A *Dieudonné pair for $S/R$* is a tuple $(M, M_1)$ consisting of a finite free $\widehat{W}(S)$-module $M$ and a $\widehat{W}(R)$-submodule $M_1 \subseteq \widehat{W}(R) \otimes_{\widehat{W}(S)} M$ such that $(\widehat{W}(R) \otimes_{\widehat{W}(S)} M, M_1)$ is a Dieudonné pair over $R$.

**Lemma 2.3.3.** *The functor*

$$\begin{array}{rcl} \{\textit{Dieudonné pairs over } S\} & \to & \{\textit{finite free } \widehat{W}(S)\textit{-modules}\}, \\ (M, M_1) & \mapsto & \widetilde{M_1} \end{array}$$

*admits a natural factorization over the category of Dieudonné pairs for $S/R$. We denote the induced functor again by*

$$\begin{array}{rcl} \{\textit{Dieudonné pairs for } S/R\} & \to & \{\textit{finite free } \widehat{W}(S)\textit{-modules}\}, \\ (M, M_1) & \mapsto & \widetilde{M_1}. \end{array}$$

*This construction is compatible with base change in $S/R$.*

*Proof.* Write $\widehat{I}_{S/R}$ for the kernel of the projection $\widehat{W}(S) \to R$. As described in [Zin01, Section 2] the divided powers give rise to a $\widehat{W}(S)$-linear inclusion $\mathfrak{a} \subseteq \widehat{W}(S)$ and $\widehat{I}_{S/R}$ decomposes as $\widehat{I}_{S/R} = \widehat{I}_S \oplus \mathfrak{a}$. We can therefore define a divided Frobenius

$$\sigma^{\mathrm{div}}\colon \widehat{I}_{S/R}^{\sigma} = \widehat{I}_S^{\sigma} \oplus \mathfrak{a}^{\sigma} \xrightarrow{(\sigma^{\mathrm{div}}, 0)} \widehat{W}(S).$$

As $\mathfrak{a}$ is contained in the kernel of $\sigma\colon \widehat{W}(S) \to \widehat{W}(S)$ we still have $p\sigma^{\mathrm{div}}(1 \otimes x) = \sigma(x)$ for $x \in \widehat{I}_{S/R}$. At this point we can now proceed as in the proof of Proposition 2.1.6 to construct the desired functor. $\square$

**Definition 2.3.4.** A *Dieudonné display for $S/R$* is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is a Dieudonné pair for $S/R$ and $\Psi\colon \widetilde{M_1} \to M$ is an isomorphism of $\widehat{W}(S)$-modules.

**Theorem 2.3.5** (Zink)**.** *The natural forgetful functor*

$$\{\textit{Dieudonné displays for } S/R\} \to \{\textit{Dieudonné displays over } R\}$$

*is an equivalence of categories.*

*Proof.* This is a reformulation of [Zin01, Theorem 3]. ☐

## 2.4 The universal deformation of a display over $\overline{\mathbf{F}}_p$

The goal of this section is to give a construction of a universal deformation of a display over $\overline{\mathbf{F}}_p$ to a Dieudonné display, see Theorem 2.4.4. Here we roughly follow [KP18, Subsection 3.1].

**Notation 2.4.1.** Let $(M_0, M_{0,1}, \Psi_0)$ be a display of type $(h, d)$ over $\overline{\mathbf{F}}_p$ and fix a finite extension $F/\breve{\mathbf{Q}}_p$.

We consider the deformation problem Def over $\mathcal{O}_F$ given by

$$\mathrm{Def}\colon R \mapsto \left\{ \begin{matrix} \text{deformations } (M, M_1, \Psi) \text{ of } (M_0, M_{0,1}, \Psi_0) \text{ over } R \\ \text{as a Dieudonné display} \end{matrix} \right\}.$$

**Proposition 2.4.2.** *The deformation problem* Def *is pro-representable.*

*Proof.* We check the conditions from Theorem 1.3.5.

- Clearly we have $\mathrm{Def}(\overline{\mathbf{F}}_p) = \{*\}$.
- Suppose that we are given a diagram

$$\begin{tikzcd} & R'' \arrow[d] \\ R' \arrow[r] & R \end{tikzcd}$$

  of Artinian local $\mathcal{O}_F$-algebras with residue field $\overline{\mathbf{F}}_p$ with $R' \to R$ surjective and assume without loss of generality that $R' \to R$ is a square zero extension. Also set $R''' := R' \times_R R''$ and equip the kernels of the morphisms $R' \to R$ and $R''' \to R''$ with the trivial divided powers.

  We have to show that the map

$$\mathrm{Def}(R''') \to \mathrm{Def}(R') \times_{\mathrm{Def}(R)} \mathrm{Def}(R'')$$

  is a bijection.

So let $(M'', M_1'', \Psi'') \in \mathrm{Def}(R'')$ and write $(M, M_1, \Psi)$ for its base change to $R$. By Theorem 2.3.5 $(M'', M_1'', \Psi'')$ lifts uniquely to a Dieudonné display $(M''', M_1'', \Psi''')$ for $R'''/R''$. In the same way $(M, M_1, \Psi)$ lifts uniquely to a Dieudonné display $(M', M_1, \Psi')$ for $R'/R$ and in fact $(M', M_1, \Psi')$ is the base change of $(M''', M_1'', \Psi''')$ along the morphism $(R''' \to R'') \to (R' \to R)$.

Now lifting $(M'', M_1'', \Psi'')$ to an object in $\mathrm{Def}(R''')$ is the same as giving a lift of $M_1''$ to a filtration $M_1''' \subseteq M'''$ (making $(M''', M_1''')$ a Dieudonné pair) while lifting $(M, M_1, \Psi)$ to an object in $\mathrm{Def}(R')$ is the same as giving a lift of $M_1$ to a filtration $M_1' \subseteq M'$. Thus the claim follows because $R''' = R' \times_R R''$.

- Theorem 2.3.5 implies that the tangent space $T_* \mathrm{Def}$ at the base point $* \in \mathrm{Def}(\overline{\mathbf{F}}_p)$ is naturally isomorphic to the tangent space of the Grassmannian $\mathrm{Grass}_{M_0,d}$ at the point
$$\big(M_{0,1}/pM_0 \subseteq M_0/pM_0 \cong M_{0,\overline{\mathbf{F}}_p}\big) \in \mathrm{Grass}_{M_0,d}(\overline{\mathbf{F}}_p).$$
In particular it is finite-dimensional. □

**Construction 2.4.3.** Write $R^{\mathrm{univ}}$ for the completed local ring of $\mathrm{Grass}_{M_0,d,\mathcal{O}_F}$ at the $\overline{\mathbf{F}}_p$-point corresponding to $M_{0,1}/pM_0 \subseteq M_{0,\overline{\mathbf{F}}_p}$. Set

$$M^{\mathrm{univ}} := \widehat{W}(R^{\mathrm{univ}}) \otimes_{\breve{\mathbf{Z}}_p} M_0$$

and let

$$M_1^{\mathrm{univ}} \subseteq M^{\mathrm{univ}}$$

be the preimage of the universal direct summand of $M^{\mathrm{univ}}/\widehat{I}_{R^{\mathrm{univ}}} M^{\mathrm{univ}} \cong M_{0,R^{\mathrm{univ}}}$. We thus have a Dieudonné pair

$$\big(M^{\mathrm{univ}}, M_1^{\mathrm{univ}}\big)$$

over $R^{\mathrm{univ}}$ that is a deformation of $(M_0, M_{0,1})$. Set

$$\mathfrak{a} := \mathfrak{m}_{R^{\mathrm{univ}}}^2 + \mathfrak{m}_F R^{\mathrm{univ}} \subseteq R^{\mathrm{univ}}$$

and equip $\mathfrak{m}_{R^{\mathrm{univ}}}/\mathfrak{a} \subseteq R^{\mathrm{univ}}/\mathfrak{a}$ with the trivial divided powers.

Now consider the composition

$$\begin{aligned}\widehat{W}(R^{\mathrm{univ}}/\mathfrak{a}) \otimes_{\widehat{W}(R^{\mathrm{univ}})} \widetilde{M_1^{\mathrm{univ}}} \to \big(\widetilde{M_{0,1}}\big)_{\widehat{W}(R^{\mathrm{univ}}/\mathfrak{a})} &\xrightarrow{\Psi_0} M_{0,\widehat{W}(R^{\mathrm{univ}}/\mathfrak{a})} \\ &\to \widehat{W}(R^{\mathrm{univ}}/\mathfrak{a}) \otimes_{\widehat{W}(R^{\mathrm{univ}})} M^{\mathrm{univ}},\end{aligned}$$

where the first isomorphism is the one given by Lemma 2.3.3. We choose a lift of this composition to an isomorphism

$$\Psi^{\mathrm{univ}} \colon \widetilde{M_1^{\mathrm{univ}}} \to M^{\mathrm{univ}}$$

so that we obtain a deformation

$$\big(M^{\mathrm{univ}}, M_1^{\mathrm{univ}}, \Psi^{\mathrm{univ}}\big) \in \mathrm{Def}(R^{\mathrm{univ}}).$$

**Theorem 2.4.4** (Kisin-Pappas). *The deformation* $(M^{\mathrm{univ}}, M_1^{\mathrm{univ}}, \Psi^{\mathrm{univ}}) \in \mathrm{Def}(R^{\mathrm{univ}})$ *from Construction 2.4.3 is universal.*

*Proof.* By Proposition 2.4.2 we know that Def is pro-representable. From the construction and Theorem 2.3.5 it follows that the morphism $\mathrm{Spf}(R^{\mathrm{univ}}) \to \mathrm{Def}$ induces an isomorphism on tangent spaces at the respective unique $\overline{\mathbf{F}}_p$-points (see also the third bullet point in the proof of Proposition 2.4.2). As $R^{\mathrm{univ}}$ is a power series ring over $\mathcal{O}_F$ this is enough to conclude that in fact $\mathrm{Spf}(R^{\mathrm{univ}}) \to \mathrm{Def}$ is an isomorphism. □

## 2.5 Displays and $p$-divisible groups

**Theorem 2.5.1** (Lau, Zink). *There exists a natural functor*

$$\big\{p\text{-divisible groups over } R\big\}^{\mathrm{op}} \to \big\{\text{displays over } R\big\}$$

*that is compatible with dualities and coincides with (contravariant) Dieudonné theory when $R$ is a perfect $\mathbf{F}_p$-algebra. If $X$ is a $p$-divisible group of height $h$ and dimension $d$ then the associated display is of type $(h, d)$.*

*Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. Then there is a natural equivalence*

$$\big\{p\text{-divisible groups over } R\big\}^{\mathrm{op}} \to \big\{\text{Dieudonné displays over } R\big\}$$

*that is again compatible with dualities. The two constructions are compatible in the sense that the diagram*

$$\begin{tikzcd} & \big\{\text{Dieudonné displays over } R\big\} \arrow[d] \\ \big\{p\text{-divisible groups over } R\big\}^{\mathrm{op}} \arrow[ur] \arrow[r] & \big\{\text{displays over } R\big\} \end{tikzcd}$$

*is commutative.*

*Proof.* This is proven by Lau and Zink in [Lau13, Proposition 2.1] and [Zin01]. □

**Remark 2.5.2.** The functors in Theorem 2.5.1 are not exactly the ones from [Lau13] and [Zin01], as we choose the contravariant normalization (following [KP18]) instead of the covariant one.

# 3 Parahoric $(\mathcal{G}, \boldsymbol{\mu})$-displays

In [BP20], Bültel and Pappas introduced the notion of a $(\mathcal{G}, \mu)$-display for a reductive group scheme $\mathcal{G}$ over $\mathbf{Z}_p$ and a minuscule cocharacter $\mu$ of $\mathcal{G}$ defined over the ring of integers of some finite unramified extension of $\mathbf{Q}_p$ (see Section 1.6). One recovers the usual notion of a display by setting $\mathcal{G} = \mathrm{GL}_{h,\mathbf{Z}_p}$. In a Hodge type situation, Pappas generalized this in [Pap23] to the case where $\mathcal{G}$ is a parahoric group scheme, but he only works over $p$-torsionfree $p$-complete base rings.

In this chapter we recall the definitions of (Dieudonné) $(\mathcal{G}, \boldsymbol{\mu})$-displays from [Pap23] and generalize them to general $p$-complete base rings (respectively complete Noetherian local rings with residue field $\overline{\mathbf{F}}_p$). We also introduce the new notion of an $(m, n)$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-display. Then we study the deformation theory of Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays and in particular explicitly construct a universal deformation for a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$. Finally we give an application to the Kisin-Pappas integral models of Shimura varieties of Hodge type at parahoric level defined in [KP18] (see Subsection 1.8.4) and in particular to the EKOR stratification that was introduced by He and Rapoport in [HR17].

**Notation 3.0.1.** We fix a local model datum $(\mathcal{G}, \boldsymbol{\mu})$, i.e. a tuple consisting of a parahoric group scheme $\mathcal{G}$ over $\mathbf{Z}_p$ with (connected reductive) generic fiber $G = \mathcal{G}_{\mathbf{Q}_p}$ and a minuscule geometric conjugacy class of cocharacters of $G$ (see Definition 1.4.1). Write $E = E(\mathcal{G}, \boldsymbol{\mu}) \subseteq \overline{\mathbf{Q}}_p$ for its reflex field and let $q$ be the cardinality of the residue field $\mathcal{O}_E/\mathfrak{m}_E = \mathbf{F}_q$. Denote by $\mathbb{M}^{\mathrm{loc}} = \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\boldsymbol{\mu}}$ the local model associated to $(\mathcal{G}, \boldsymbol{\mu})$ (see Theorem 1.4.6) and write $\mathrm{M}^{\mathrm{loc}}$ for its $p$-completion.

We also fix a finite free $\mathbf{Z}_p$-module $\Lambda$ of rank $h$ and a closed immersion of $\mathbf{Z}_p$-algebraic groups $\iota\colon \mathcal{G} \to \mathrm{GL}(\Lambda)$ such that the following conditions are satisfied.

- $\iota(\boldsymbol{\mu}) = \boldsymbol{\mu}'_d$ for some $0 \leq d \leq h$. Here $\boldsymbol{\mu}'_d$ denotes the conjugacy class of cocharacters of $\mathrm{GL}(\Lambda)_{\overline{\mathbf{Q}}_p}$ that induce a weight decomposition $\Lambda_{\overline{\mathbf{Q}}_p} = \Lambda_0 \oplus \Lambda_{-1}$ with $\Lambda_{-1}$ of dimension $d$.
- $\mathbf{G}_{m,\mathbf{Z}_p} \subseteq \iota(\mathcal{G})$.
- The induced closed immersion $X_{\boldsymbol{\mu}}(G) \to X_{\boldsymbol{\mu}'_d}(\mathrm{GL}(\Lambda))_E$ extends to a closed immersion
$$\mathbb{M}^{\mathrm{loc}} \to \left(\mathbb{M}^{\mathrm{loc}}_{\mathrm{GL}(\Lambda),\boldsymbol{\mu}'_d}\right)_{\mathcal{O}_E} = \mathrm{Grass}_{\Lambda,d,\mathcal{O}_E}.$$
This morphism is then automatically equivariant for $\iota\colon \mathcal{G} \to \mathrm{GL}(\Lambda)$.

Whenever it is convenient we omit $\iota$ from our notation and simply view it as an inclusion $\mathcal{G} \subseteq \mathrm{GL}(\Lambda)$.

In this chapter $R$ always denotes a $p$-complete $\mathcal{O}_E$-algebra. Moreover, $m$ always denotes a positive integer and $(m, n)$ always denotes a tuple of positive integers with $m \geq n+1$. Sometimes (with explicit indication) we allow $n$ to take the additional value 1-rdt (recall that we write $L^{(\text{1-rdt})}\mathcal{G}$ for the reductive quotient of $\mathcal{G}_{\mathbf{F}_p}$), in which case we require $m \geq 2$.

**Remark 3.0.2.** Note that in Notation 3.0.1 the existence of $(\Lambda, \iota)$ with the required properties restricts the possible choices for $(\mathcal{G}, \boldsymbol{\mu})$. By [Pap23, Proposition 3.1.6], the following conditions are sufficient for the existence of $(\Lambda, \iota)$.

- There exists a finite-dimensional $\mathbf{Q}_p$-vector space $W$ and an injective morphism of $\mathbf{Q}_p$-algebraic groups $\kappa\colon G \to \mathrm{GL}(W)$ such that $\kappa(\boldsymbol{\mu}) = \boldsymbol{\mu}'_d$ for some $d$ and $\mathbf{G}_{m,\mathbf{Q}_p} \subseteq \kappa(G)$.
- $G$ splits over a tamely ramified extension of $\mathbf{Q}_p$.
- $p$ does not divide the order of $\pi_1(G_{\mathrm{der},\overline{\mathbf{Q}}_p})$.
- $\mathcal{G}$ is a parahoric stabilizer.

## 3.1 $(\mathcal{G}, \boldsymbol{\mu})$-pairs and $(\mathcal{G}, \boldsymbol{\mu})$-displays

**Definition 3.1.1.** A *$(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$* is a tuple $(\mathcal{P}, q)$ consisting of a $\mathcal{G}$-torsor $\mathcal{P}$ over $W(R)$ and a $\mathcal{G}$-equivariant morphism $q\colon \mathcal{P}_R \to \mathrm{M}^{\mathrm{loc}}$.

An *$m$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$* is a tuple $(\mathcal{P}, q)$ consisting of a $\mathcal{G}$-torsor $\mathcal{P}$ over $W_m(R)$ and a $\mathcal{G}$-equivariant morphism $q\colon \mathcal{P}_R \to \mathrm{M}^{\mathrm{loc}}$.

Now let $R$ be a complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebra with residue field $\overline{\mathbf{F}}_p$. A *Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$* is a tuple $(\mathcal{P}, q)$ consisting of a $\mathcal{G}$-torsor $\mathcal{P}$ over $\widehat{W}(R)$ and a $\mathcal{G}$-equivariant morphism $q\colon \mathcal{P}_R \to \mathrm{M}^{\mathrm{loc}}$.

**Remark 3.1.2.** Giving a $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$ is the same as giving a pair $(M, M_1)$ of type $(h, d)$ over $R$ (see Definition 2.1.1) together with a $\mathcal{G}$-structure on $M$ such that for every (or equivalently one) local trivialization of $M$ the $R$-point of $(\mathrm{Grass}_{\Lambda,d,\mathcal{O}_E})^\wedge_p$ parametrizing $M_1$ is contained in $\mathrm{M}^{\mathrm{loc}}$. Similar statements also hold for truncated $(\mathcal{G}, \boldsymbol{\mu})$-pairs and Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pairs.

In the following we will abuse notation and refer to a pair $(M, M_1)$ that is implicitly equipped with a $\mathcal{G}$-structure as above as a $(\mathcal{G}, \boldsymbol{\mu})$-pair.

**Lemma 3.1.3.** *The groupoids of $(\mathcal{G}, \boldsymbol{\mu})$-pairs over varying $p$-complete $\mathcal{O}_E$-algebras naturally form a stack $\mathsf{Pair}_{\mathcal{G},\boldsymbol{\mu}}$ over $\mathrm{Spf}(\mathcal{O}_E)$ and we have a natural equivalence*

$$\mathsf{Pair}_{\mathcal{G},\boldsymbol{\mu}} \cong \left[L^+\mathcal{G} \backslash \mathrm{M}^{\mathrm{loc}}\right].$$

*Similarly, also $m$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-pairs form a ($p$-adic formal algebraic) stack $\mathsf{Pair}^{(m)}_{\mathcal{G},\boldsymbol{\mu}}$ over $\mathrm{Spf}(\mathcal{O}_E)$ that is naturally equivalent to the quotient stack $[L^{(m)}\mathcal{G} \backslash \mathrm{M}^{\mathrm{loc}}]$.*

*Proof.* This is immediate from the definition and Proposition 1.1.1. ☐

**Proposition 3.1.4** (Kisin-Pappas, Pappas). *There are unique functors*

$$\begin{array}{rcl} \{(\mathcal{G},\boldsymbol{\mu})\text{-pairs over } R\} & \to & \{\mathcal{G}\text{-torsors over } W(R)\}, \\ (M, M_1) & \mapsto & \widetilde{M_1} \end{array}$$

*together with natural isomorphisms $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ that are compatible with base change in $R$ and with passing to pairs (without $(\mathcal{G},\boldsymbol{\mu})$-structure), see Remark 3.1.2 and Proposition 2.1.6.*

*There exist unique functors*

$$\begin{array}{rcl} \{m\text{-truncated } (\mathcal{G},\boldsymbol{\mu})\text{-pairs over } R\} & \to & \{\mathcal{G}\text{-torsors over } W_n(R)\}, \\ (M, M_1) & \mapsto & \widetilde{M_1} \end{array}$$

*that are compatible with base change and with passing from $(\mathcal{G},\boldsymbol{\mu})$-pairs to $m$-truncated $(\mathcal{G},\boldsymbol{\mu})$-pairs. These functors are also compatible with passing from $m$-truncated $(\mathcal{G},\boldsymbol{\mu})$-pairs to $m$-truncated pairs.*

*When we restrict to complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebras $R$ with residue field $\overline{\mathbf{F}}_p$ there also exist unique functors*

$$\begin{array}{rcl} \{\text{Dieudonné } (\mathcal{G},\boldsymbol{\mu})\text{-pairs over } R\} & \to & \{\mathcal{G}\text{-torsors over } \widehat{W}(R)\}, \\ (M, M_1) & \mapsto & \widetilde{M_1} \end{array}$$

*together with natural isomorphisms $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ that are compatible with base change in $R$ and with passing to Dieudonné pairs. This construction is compatible with passing from Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-pairs to $(\mathcal{G},\boldsymbol{\mu})$-pairs.*

*Proof.* The statement about truncated $(\mathcal{G},\boldsymbol{\mu})$-pairs follows immediately from the one about non-truncated $(\mathcal{G},\boldsymbol{\mu})$-pairs. The remaining part of the claim is essentially [Pap23, Proposition 4.1.10]; for completeness we give a sketch of the argument.

After passing to the universal case and applying [Pap23, Corollary 3.2.6 and Remark 3.2.7] we are reduced to considering Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-pairs in the case $R = \mathcal{O}_K$ for some finite extension $K/\breve{E}$. We then conclude by the following key lemma that is a reformulation of [KP18, Lemma 3.2.9].

**Lemma 3.1.5.** *Let $K/\breve{E}$ be a finite extension and let $(M, M_1)$ be a Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-pair over $\mathcal{O}_K$. Then there exists a (unique) $\mathcal{G}$-structure on $\widetilde{M_1}$ such that the isomorphism $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ is compatible with $\mathcal{G}$-structures.*

*Proof.* Choose a trivialization $M \cong \Lambda_{\widehat{W}(\mathcal{O}_K)}$. After possibly enlarging $K$ we may assume that there exists a representative $\mu\colon \mathbf{G}_{m,K} \to G_K$ of $\boldsymbol{\mu}$ that gives rise to the point in $\mathbb{M}^{\mathrm{loc}}(K) \cong \mathrm{M}^{\mathrm{loc}}(\mathcal{O}_K)$ corresponding to $M_1$.

At this point we set up some notation. Set $\mathfrak{S} \coloneqq \breve{\mathbf{Z}}_p[\![u]\!]$ and let $\sigma\colon \mathfrak{S} \to \mathfrak{S}$ be the Frobenius lift given by

$$\sigma\Big(\sum_i a_i u^i\Big) \coloneqq \sum_i \sigma(a_i) u^{pi}.$$

Fix a uniformizer $\varpi_K \in K$ and let $P(u) \in \mathfrak{S}$ be its minimal polynomial over $\breve{\mathbf{Q}}_p$. Then we have the $\sigma$-equivariant morphism

$$\alpha\colon \mathfrak{S} \to \widehat{W}(\mathcal{O}_K), \qquad \sum_i a_i u^i \mapsto \sum_i a_i[\varpi_K^i].$$

Here the infinite sum in $\widehat{W}(\mathcal{O}_K)$ is taken with respect to the complete linear topology given by writing $\widehat{W}(\mathcal{O}_K) \cong \varprojlim_j \widehat{W}(\mathcal{O}_K/\varpi_K^j\mathcal{O}_K)$.

Note that we have $\alpha(P(u)) = P([\varpi_K]) \in \widehat{I}_{\mathcal{O}_K}$ because its image in $\mathcal{O}_K$ is $P(\varpi_K) = 0$. Moreover we have $\sigma^{\mathrm{div}}(1 \otimes P([\varpi_K])) \in \widehat{W}(\mathcal{O}_K)^\times$. Indeed its image in $\breve{\mathbf{Z}}_p$ is given by $\sigma^{\mathrm{div}}(1 \otimes b_0) \in \breve{\mathbf{Z}}_p^\times$ where $b_0 \in p\breve{\mathbf{Z}}_p$ is the zeroth coefficient of $P(u)$ (that has additive $p$-adic valuation 1 because $P(u)$ is Eisenstein). Thus we see that

$$\sigma\big(P([\varpi_K])\big)\widehat{W}(\mathcal{O}_K) = p\widehat{W}(\mathcal{O}_K).$$

Now set $N \coloneqq \Lambda_{\mathfrak{S}}$ and let $N_1 \subseteq N$ be the preimage of the direct summand of $N/P(u)N \cong \Lambda_{\mathcal{O}_K}$ corresponding to the given point in $M^{\mathrm{loc}}(\mathcal{O}_K)$. Then $N_1$ is again a free $\mathfrak{S}$-module of rank $h$ and the above computation implies that the tautological isomorphism $\widehat{W}(R) \otimes_{\mathfrak{S}} N^\sigma \cong M^\sigma$ restricts to an isomorphism $\widehat{W}(R) \otimes_{\mathfrak{S}} N_1^\sigma \cong \widetilde{M_1}$. Thus to prove the claim it suffices to show that there exists a $\mathcal{G}$-structure on $N_1$ such that the isomorphism $N_1[1/P(u)] \cong N[1/P(u)]$ is compatible with $\mathcal{G}$-structures.

Next we observe that the morphism $\mathfrak{S} \to \mathfrak{S}/P(u)\mathfrak{S} \cong \mathcal{O}_K$ induces a morphism

$$\widehat{\mathfrak{S}}_0 \coloneqq \mathfrak{S}[1/p]^\wedge_{P(u)} \to K$$

that admits a unique section by Hensel's Lemma, making $\widehat{\mathfrak{S}}_0$ into a $K$-algebra. Thus it makes sense to consider the element $P(u)\mu(P(u)) \in G(\widehat{\mathfrak{S}}_0[1/P(u)])$. By definition of $N_1$ this element induces an isomorphism

$$P(u)\mu(P(u))\colon \widehat{\mathfrak{S}}_0 \otimes_{\mathfrak{S}} N \to \widehat{\mathfrak{S}}_0 \otimes_{\mathfrak{S}} N_1$$

that we use to define a $\mathcal{G}$-structure on $\widehat{\mathfrak{S}}_0 \otimes_{\mathfrak{S}} N_1$. Over $\widehat{\mathfrak{S}}_0[1/P(u)]$ this $\mathcal{G}$-structure agrees with the one coming from the isomorphism $N_1[1/P(u)] \cong N[1/P(u)]$ so that we can apply Beauville-Laszlo gluing (see for example [Stacks, Tag 05ET]) to obtain a $\mathcal{G}$-structure on $N_1[1/p]$ that is compatible with the given one on $N_1[1/P(u)]$. By the purity result [Ans22, Proposition 10.3] by Anschütz it follows that these $\mathcal{G}$-structures extend to a $\mathcal{G}$-structure on $N_1$ as desired. $\square$

$\square$

**Remark 3.1.6.** The functors $(M, M_1) \mapsto \widetilde{M_1}$ from Proposition 3.1.4 are independent of the choice of embedding $\iota$ (see [Pap23, Remark 4.1.11]).

**Remark 3.1.7.** By Lemma 3.1.3 the datum of the functors

$$\big\{(\mathcal{G}, \boldsymbol{\mu})\text{-pairs over } R\big\} \to \big\{\mathcal{G}\text{-torsors over } W(R)\big\}$$

from Proposition 3.1.4 for varying $R$ is equivalent to an $L^+\mathcal{G}$-equivariant $L^+\mathcal{G}$-torsor over $\mathrm{M}^{\mathrm{loc}}$ that we denote by $\mathrm{M}^{\mathrm{loc},(\infty)}$.

We write $\mathrm{M}^{\mathrm{loc},(n)}$ for the reduction of this $L^+\mathcal{G}$-torsor to an $L^{(n)}\mathcal{G}$-torsor. The $L^+\mathcal{G}$-equivariant structure on $\mathrm{M}^{\mathrm{loc},(n)}$ factors through $L^{(n+1)}\mathcal{G}$.

We also write $\mathrm{M}^{\mathrm{loc},(1\text{-rdt})}$ for the reduction of $\mathrm{M}^{\mathrm{loc},(\infty)}_{\mathbf{F}_q}$ to an $L^{(1\text{-rdt})}\mathcal{G}$-torsor over $\mathrm{M}^{\mathrm{loc}}_{\mathbf{F}_q}$ (where we recall that $L^{(1\text{-rdt})}\mathcal{G} = \mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p}$ denotes the reductive quotient of the special fiber of $\mathcal{G}$). The $L^+\mathcal{G}$-equivariant structure on $\mathrm{M}^{\mathrm{loc},(1\text{-rdt})}$ factors through $L^{(2)}\mathcal{G}$.

**Definition 3.1.8.** A *$(\mathcal{G}, \boldsymbol{\mu})$-display over $R$* is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is a $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$ and $\Psi\colon \widetilde{M_1} \to M$ is an isomorphism of $\mathcal{G}$-torsors over $W(R)$.

An *$(m, n)$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$* is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is an $m$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$ and $\Psi\colon \widetilde{M_1} \to M_{W_n(R)}$ is an isomorphism of $\mathcal{G}$-torsors over $W_n(R)$.

Similarly, when $R$ is an $\mathbf{F}_q$-algebra, an *$(m, 1\text{-rdt})$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$* is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is an $m$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$ and $\Psi\colon \mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p} \times^{\mathcal{G}_{\mathbf{F}_p}} (\widetilde{M_1})_R \to \mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p} \times^{\mathcal{G}_{\mathbf{F}_p}} M_R$ is an isomorphism of $\mathcal{G}^{\mathrm{rdt}}_{\mathbf{F}_p}$-torsors over $R$.

Let $R$ be a complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebra with residue field $\overline{\mathbf{F}}_p$. A *Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$* is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$ and $\Psi\colon \widetilde{M_1} \to M$ is an isomorphism of $\mathcal{G}$-torsors over $\widehat{W}(R)$.

**Remark 3.1.9.** Giving a $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$ is the same as giving a display $(M, M_1, \Psi)$ of type $(h, d)$ over $R$ (see Definition 2.1.9) together with a $\mathcal{G}$-structure on $M$ such that $(M, M_1)$ is actually a $(\mathcal{G}, \boldsymbol{\mu})$-pair and the isomorphism $\Psi\colon \widetilde{M_1} \to M$ is compatible with the $\mathcal{G}$-structures on both sides.

In the situation where $W(R)$ is $p$-torsionfree this last condition is equivalent to saying that the Frobenius $\Phi\colon M^\sigma[1/p] \to M[1/p]$ (see Definition 2.1.10) is compatible with the $\mathcal{G}$-structures on both sides.

The same remark also applies to Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays.

**Lemma 3.1.10.** *The groupoids of $(\mathcal{G}, \boldsymbol{\mu})$-displays over varying $p$-complete $\mathcal{O}_E$-algebras naturally form a stack $\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}$ over $\mathrm{Spf}(\mathcal{O}_E)$ and we have a natural equivalence*

$$\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}} \cong \left[(L^+\mathcal{G})_\Delta \backslash \mathrm{M}^{\mathrm{loc},(\infty)}\right]$$

*where the subscript $\Delta$ indicates that we take the quotient by the diagonal action of $L^+\mathcal{G}$ on $\mathrm{M}^{\mathrm{loc},(\infty)}$.*

*Similarly, also $(m, n)$-truncated $(\mathcal{G}, \boldsymbol{\mu})$-displays form a $p$-adic formal algebraic stack $\mathsf{Disp}^{(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$ over $\mathrm{Spf}(\mathcal{O}_E)$ and we have a natural equivalence*

$$\mathsf{Disp}^{(m,n)}_{\mathcal{G},\boldsymbol{\mu}} \cong [(L^{(m)}\mathcal{G})_\Delta \backslash \mathrm{M}^{\mathrm{loc},(n)}].$$

*This also applies when $n = 1\text{-rdt}$, in which case $\mathsf{Disp}^{(m,1\text{-rdt})}_{\mathcal{G},\boldsymbol{\mu}}$ is an algebraic stack over $\mathrm{Spec}(\mathbf{F}_q)$.*

**Lemma 3.1.11.** *For $(m,n) \leq (m',n')$ the natural forgetful morphism*

$$\mathsf{Disp}^{(m',n')}_{\mathcal{G},\boldsymbol{\mu}} \to \mathsf{Disp}^{(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$$

*is smooth. This is also true when $n$ and/or $n'$ take the value* 1-rdt*, in which case the domain of the morphism may need to be base changed to $\mathbf{F}_q$.*

*Proof.* This follows directly from the smoothness of the projection $\mathrm{M}^{\mathrm{loc},(n')} \to \mathrm{M}^{\mathrm{loc},(n)}$. □

**Lemma 3.1.12.** *Let $R' \to R$ be a surjection of complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebras with residue field $\overline{\mathbf{F}}_p$. Suppose that we are given a Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-display $(M, M_1, \Psi)$ over $R$, an $(m,n)$-truncated $(\mathcal{G},\boldsymbol{\mu})$-display $(\overline{M'}, \overline{M'_1}, \overline{\Psi'})$ over $R'$ and an isomorphism between the two induced $(m,n)$-truncated $(\mathcal{G},\boldsymbol{\mu})$-displays over $R$. Then $(M, M_1, \Psi)$ and $(\overline{M'}, \overline{M'_1}, \overline{\Psi'})$ admit a compatible lift to a Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-display over $R'$.*

*The statement is also true for $n =$* 1-rdt *when $R, R'$ are $\overline{\mathbf{F}}_p$-algebras.*

*Proof.* We prove the statement in the case $n \neq$ 1-rdt (that case then follows from Lemma 3.1.11). Using the smoothness of $\mathcal{G}$ we can certainly find a compatible lift of $(M, M_1)$ and $(\overline{M'}, \overline{M'_1})$ to a Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-pair $(M', M'_1)$ over $R'$. Then $\Psi$ and $\overline{\Psi'}$ together induce an isomorphism

$$\big(\Psi, \overline{\Psi'}\big) \colon \big(\widehat{W}(R) \times_{W_n(R)} W_n(R')\big) \otimes_{\widehat{W}(R')} \widetilde{M'_1} \to \big(\widehat{W}(R) \times_{W_n(R)} W_n(R')\big) \otimes_{\widehat{W}(R')} M'.$$

Again using the smoothness of $\mathcal{G}$ we can lift this isomorphism to an isomorphism $\Psi' \colon \widetilde{M'_1} \to M'$ to obtain the desired Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-display $(M', M'_1, \Psi')$ over $R'$. □

## 3.2 Grothendieck-Messing Theory for Dieudonné $(\mathcal{G},\boldsymbol{\mu})$-displays

**Proposition 3.2.1** (Kisin-Pappas)**.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$ that is additionally $p$-torsionfree and satisfies $\mathfrak{m}_R = \sqrt{pR}$. Let $(M_0, M_{0,1}, \Psi_0)$ be a display over $\overline{\mathbf{F}}_p$ and let $(M, M_1, \Psi)$ be a deformation to a Dieudonné display over $R$. Then there exists a unique isomorphism*

$$\widehat{W}(R)[1/p] \otimes_{\breve{\mathbf{Q}}_p} M_0[1/p] \to M[1/p]$$

*that lifts the identity on $M_0[1/p]$ and is compatible with the respective Frobenii $\Phi_0$ and $\Phi$, i.e. makes the diagram*

$$\begin{tikzcd}
\widehat{W}(R)[1/p] \otimes_{\breve{\mathbf{Q}}_p} M_0[1/p]^{\sigma} \arrow[r] \arrow[d, "\Phi_0"] & M[1/p]^{\sigma} \arrow[d, "\Phi"] \\
\widehat{W}(R)[1/p] \otimes_{\breve{\mathbf{Q}}_p} M_0[1/p] \arrow[r] & M[1/p]
\end{tikzcd}$$

*commutative.*

*Now let $R$ be a complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebra with residue field $\overline{\mathbf{F}}_p$ that is additionally $p$-torsionfree and satisfies $\mathfrak{m}_R = \sqrt{pR}$ as before. Let $(M_0, M_{0,1}, \Psi_0)$ be a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$ and let $(M, M_1, \Psi)$ be a deformation to a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$. Then the isomorphism $\widehat{W}(R)[1/p] \otimes_{\breve{\mathbf{Q}}_p} M_0[1/p] \to M[1/p]$ from the first part of the proposition is compatible with the $\mathcal{G}$-structures on both sides.*

*Proof.* This is [KP18, Lemma 3.1.17 and Lemma 3.2.13]. Note that $\widehat{W}(R)[1/p]$ with the topology considered in the reference is typically not complete. However, the argument still works when one uses the $p$-adic topology on $\widehat{W}(R)$ instead, see Lemma 1.1.3 and Lemma 1.1.5. This was communicated to us by Pappas. ☐

**Corollary 3.2.2.** *Let $R$ be a complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebra with residue field $\overline{\mathbf{F}}_p$ that is additionally $p$-torsionfree and satisfies $\mathfrak{m}_R = \sqrt{pR}$. Let $(M_0, M_{0,1}, \Psi_0)$ be a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$ and let $(M, M_1, \Psi)$ be a deformation to a Dieudonné display over $R$.*

*Then there exists at most one $\mathcal{G}$-structure on $M$ that is compatible with the given $\mathcal{G}$-structure on $M_0$ and makes $(M, M_1, \Psi)$ into a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$.*

*Proof.* This is immediate from Proposition 3.2.1. ☐

**Notation 3.2.3.** For the rest of this section $S \to R$ always denotes a surjection of complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebras with residue field $\overline{\mathbf{F}}_p$ such that $S$ is $p$-torsionfree and satisfies $\mathfrak{m}_S = \sqrt{pS}$ and such that the kernel $\mathfrak{a}$ of $S \to R$ has nilpotent divided powers (that are automatically continuous).

**Definition 3.2.4.** A *Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair for $S/R$* is a tuple $(M, M_1)$ consisting of a finite free $\widehat{W}(S)$-module $M$ that is equipped with a $\mathcal{G}$-structure and a $\widehat{W}(R)$-submodule $M_1 \subseteq \widehat{W}(R) \otimes_{\widehat{W}(S)} M$ such that $(\widehat{W}(R) \otimes_{\widehat{W}(S)} M, M_1)$ is a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair over $R$.

A Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair for $S/R$ (or over $R$) is called *liftable (to $S$)* if it admits a lift to a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair over $S$. A Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display over $R$ is called *liftable (to $S$)* if its underlying $(\mathcal{G}, \boldsymbol{\mu})$-pair has this property.

**Lemma 3.2.5.** *There are unique functors*

$$\begin{aligned} \{\textit{liftable Dieudonné } (\mathcal{G}, \boldsymbol{\mu})\textit{-pairs for } S/R\} &\to \{\mathcal{G}\textit{-torsors over } \widehat{W}(S)\}, \\ (M, M_1) &\mapsto \widetilde{M}_1 \end{aligned}$$

*together with natural isomorphisms $\widetilde{M}_1[1/p] \to M^\sigma[1/p]$ that are compatible with passing to Dieudonné pairs for $S/R$, see Lemma 2.3.3. This construction is compatible with base change in $S/R$.*

*Proof.* As $\widehat{W}(S)$ is $p$-torsionfree, given a liftable Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair $(M, M_1)$ for $S/R$ there exists at most one $\mathcal{G}$-structure on the finite free $\widehat{W}(S)$-module $\widetilde{M}_1$ from Lemma 2.3.3 that is compatible with the $\mathcal{G}$-structure on $M^\sigma$ under the isomorphism

$\widetilde{M_1}[1/p] \to M^\sigma[1/p]$. The existence of such a $\mathcal{G}$-structure follows immediately from the liftablility of $(M, M_1)$ to $S$ and Proposition 3.1.4. $\square$

**Definition 3.2.6.** A *liftable Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display for $S/R$* is a tuple $(M, M_1, \Psi)$ where $(M, M_1)$ is a liftable Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair for $S/R$ and $\Psi\colon \widetilde{M_1} \to M$ is an isomorphism of $\mathcal{G}$-torsors over $\widehat{W}(S)$.

**Theorem 3.2.7.** *The natural forgetful functor*

$$\left\{\begin{matrix}\textit{liftable Dieudonné } (\mathcal{G}, \boldsymbol{\mu})\textit{-displays} \\ \textit{for } S/R\end{matrix}\right\} \to \left\{\begin{matrix}\textit{liftable Dieudonné } (\mathcal{G}, \boldsymbol{\mu})\textit{-displays} \\ \textit{over } R\end{matrix}\right\}$$

*is an equivalence.*

*Proof.* The essential surjectivity of the functor is clear, so that it remains to check fully faithfulness. So suppose we are given two liftable Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays $(M, M_1, \Psi)$ and $(M', M_1', \Psi')$ for $S/R$ and an isomorphism between the induced Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays over $R$. By Theorem 2.3.5 this isomorphism lifts uniquely to an isomorphism of Dieudonné displays $(M, M_1, \Psi) \to (M', M_1', \Psi')$ for $S/R$ that automatically is compatible with $\mathcal{G}$-structures by Corollary 3.2.2. $\square$

**Remark 3.2.8.** We expect that there exist natural functors

$$\begin{array}{rcl}\big\{\text{liftable Dieudonné } (\mathcal{G}, \boldsymbol{\mu})\text{-pairs for } S/R\big\} & \to & \big\{\mathcal{G}\text{-torsors over } \widehat{W}(S)\big\}, \\ (M, M_1) & \mapsto & \widetilde{M_1}\end{array}$$

as in Lemma 3.2.5, but without the assumption that $S$ is $p$-torsionfree and $\mathfrak{m}_S = \sqrt{pS}$ and that Theorem 3.2.7 still holds true in that setting.

The main obstruction to doing that seems to be the following. Given a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair $(M, M_1)$ for $S/R$ and two lifts $(M, M_1')$ and $(M, M_1'')$ to a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair over $S$ we don't know if the natural isomorphism of finite free $\widehat{W}(S)$-modules

$$\widetilde{M_1'} \to \widetilde{M_1''}$$

given by Lemma 2.3.3 is compatible with $\mathcal{G}$-structures.

## 3.3 The universal deformation of a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$

The goal of this section is to give a construction of a universal deformation of a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$ to a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display, see Theorem 3.3.11, similarly to what was done in Section 2.4. Some of our arguments are inspired by [KP18, Subsection 3.2].

**Notation 3.3.1.** Let $(M_0, M_{0,1}, \Psi_0)$ be a $(\mathcal{G}, \boldsymbol{\mu})$-display over $\overline{\mathbf{F}}_p$ and choose a trivialization $M_0 \to \Lambda_{\breve{\mathbf{Z}}_p}$.

We consider the deformation problem $\mathrm{Def}_{\mathcal{G}}$ over $\mathcal{O}_{\breve{E}}$ given by

$$\mathrm{Def}_{\mathcal{G}} \colon R \mapsto \left\{ \begin{matrix} \text{deformations } (M, M_1, \Psi) \text{ of } (M_0, M_{0,1}, \Psi_0) \text{ over } R \\ \text{as a Dieudonné } (\mathcal{G}, \boldsymbol{\mu})\text{-display} \end{matrix} \right\}.$$

We also write $\mathrm{Def}_{\mathrm{GL}(\Lambda)}$ for the deformation problem that was called Def in Section 2.4 (for $F = \breve{E}$) and is given by

$$\mathrm{Def}_{\mathrm{GL}(\Lambda)} \colon R \mapsto \left\{ \begin{matrix} \text{deformations } (M, M_1, \Psi) \text{ of } (M_0, M_{0,1}, \Psi_0) \text{ over } R \\ \text{as a Dieudonné display} \end{matrix} \right\}.$$

Note that we have a natural morphism $\mathrm{Def}_{\mathcal{G}} \to \mathrm{Def}_{\mathrm{GL}(\Lambda)}$.

In this section $R$ and $R'$ always denote complete local Noetherian $\mathcal{O}_{\breve{E}}$-algebras with residue field $\overline{\mathbf{F}}_p$ and $K$ always denotes a finite extension of $\breve{E}$ with uniformizer $\varpi_K$.

**Construction 3.3.2** (continued on p. 38)**.** Write

$$R_{\mathrm{GL}(\Lambda)}, \qquad \big(M^{\mathrm{GL}(\Lambda)}, M_1^{\mathrm{GL}(\Lambda)}\big) \quad \text{and} \quad \mathfrak{a}_{\mathrm{GL}(\Lambda)}$$

for the objects that were called $R^{\mathrm{univ}}$, $(M^{\mathrm{univ}}, M_1^{\mathrm{univ}})$ and $\mathfrak{a}$ respectively in Construction 2.4.3 and equip $\mathfrak{a}_{\mathrm{GL}(\Lambda)}$ with the trivial divided powers. Equip $M^{\mathrm{GL}(\Lambda)}$ with its natural $\mathcal{G}$-structure (coming from the defining isomorphism $M^{\mathrm{GL}(\Lambda)} = \Lambda_{R_{\mathrm{GL}(\Lambda)}}$).

Write $R_{\mathcal{G}}$ for the completed local ring of $\mathrm{M}^{\mathrm{loc}}_{\mathcal{O}_{\breve{E}}}$ at the $\overline{\mathbf{F}}_p$-point corresponding to $M_{0,1}/pM_0 \subseteq M_{0,\overline{\mathbf{F}}_p}$ so that $R_{\mathcal{G}}$ is a quotient of $R_{\mathrm{GL}(\Lambda)}$. Write

$$\big(M^{\mathcal{G}}, M_1^{\mathcal{G}}\big) \coloneqq \big(M^{\mathrm{GL}(\Lambda)}, M_1^{\mathrm{GL}(\Lambda)}\big)_{R_{\mathcal{G}}}$$

for the base change of the Dieudonné pair $(M^{\mathrm{GL}(\Lambda)}, M_1^{\mathrm{GL}(\Lambda)})$ to $R_{\mathcal{G}}$ and note that it is a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair that is a deformation of $(M_0, M_{0,1})$. Finally write

$$\mathfrak{a}_{\mathcal{G}} \coloneqq \mathfrak{m}_{R_{\mathcal{G}}}^2 + \mathfrak{m}_{\breve{E}} R_{\mathcal{G}}$$

(or equivalently $\mathfrak{a}_{\mathcal{G}} \coloneqq \mathfrak{a}_{\mathrm{GL}(\Lambda)} R_{\mathcal{G}}$) and equip it with the trivial divided powers as well.

**Hypothesis 3.3.3.** The natural isomorphism of finite free $\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}})$-modules

$$\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}}) \otimes_{\widehat{W}(R_{\mathcal{G}})} \widetilde{M_1^{\mathcal{G}}} \to \big(\widetilde{M_{0,1}}\big)_{\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}})}$$

given by Lemma 2.3.3 is compatible with $\mathcal{G}$-structures.

**Remark 3.3.4.** Hypothesis 3.3.3 is implicitly assumed in [KP18, Subsection 3.2]. We expect that it always holds true (see also Remark 3.2.8).

It would be sufficient for the hypothesis to be true if every $\overline{\mathbf{F}}_p[\varepsilon]$-point of $\mathrm{M}^{\mathrm{loc}}$ could be lifted to an $\mathcal{O}_K[\varepsilon]$-point for some finite extension $K/\breve{E}$, see Lemma 3.2.5. In fact it would even be sufficient to only lift a generating set of tangent vectors at every $\overline{\mathbf{F}}_p$-point of $\mathrm{M}^{\mathrm{loc}}$.

**Construction 3.3.2** (continuing from p. 37)**.** Assuming Hypothesis 3.3.3 the composition

$$\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}}) \otimes_{\widehat{W}(R_{\mathcal{G}})} \widetilde{M_1^{\mathcal{G}}} \to \big(\widetilde{M_{0,1}}\big)_{\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}})} \xrightarrow{\Psi_0} M_{0,\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}})}$$
$$\to \widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}}) \otimes_{\widehat{W}(R_{\mathcal{G}})} M^{\mathcal{G}}$$

is an isomorphism of $\mathcal{G}$-torsors over $\widehat{W}(R_{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}})$. We choose a lift of this isomorphism to an isomorphism of $\mathcal{G}$-torsors over $\widehat{W}(R_{\mathcal{G}})$

$$\Psi^{\mathcal{G}} \colon \widetilde{M_1^{\mathcal{G}}} \to M^{\mathcal{G}}$$

so that we obtain a deformation

$$\big(M^{\mathcal{G}}, M_1^{\mathcal{G}}, \Psi^{\mathcal{G}}\big) \in \mathrm{Def}_{\mathcal{G}}(R_{\mathcal{G}}).$$

Now $\Psi^{\mathcal{G}}$ and the composition

$$\widehat{W}(R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)}) \otimes_{\widehat{W}(R_{\mathrm{GL}(\Lambda)})} \widetilde{M_1^{\mathrm{GL}(\Lambda)}} \to \big(\widetilde{M_{0,1}}\big)_{\widehat{W}(R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)})}$$
$$\xrightarrow{\Psi_0} M_{0,\widehat{W}(R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)})} \to \widehat{W}(R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)}) \otimes_{\widehat{W}(R_{\mathrm{GL}(\Lambda)})} M^{\mathrm{GL}(\Lambda)}$$

together induce an isomorphism of finite free $\widehat{W}(R_{\mathcal{G}} \times_{R^{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}}} R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)})$-modules

$$\widehat{W}\big(R_{\mathcal{G}} \times_{R^{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}}} R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)}\big) \otimes_{\widehat{W}(R^{\mathrm{GL}(\Lambda)})} \widetilde{M_1^{\mathrm{GL}(\Lambda)}}$$
$$\to \widehat{W}\big(R_{\mathcal{G}} \times_{R^{\mathcal{G}}/\mathfrak{a}_{\mathcal{G}}} R_{\mathrm{GL}(\Lambda)}/\mathfrak{a}_{\mathrm{GL}(\Lambda)}\big) \otimes_{\widehat{W}(R^{\mathrm{GL}(\Lambda)})} M^{\mathrm{GL}(\Lambda)}.$$

We further choose a lift of this isomorphism to an isomorphism of finite free $R_{\mathrm{GL}(\Lambda)}$-modules

$$\Psi^{\mathrm{GL}(\Lambda)} \colon \widetilde{M_1^{\mathrm{GL}(\Lambda)}} \to M^{\mathrm{GL}(\Lambda)}$$

so that we obtain a deformation

$$\big(M^{\mathrm{GL}(\Lambda)}, M_1^{\mathrm{GL}(\Lambda)}, \Psi^{\mathrm{GL}(\Lambda)}\big) \in \mathrm{Def}_{\mathrm{GL}(\Lambda)}(R_{\mathrm{GL}(\Lambda)})$$

that is universal by Theorem 2.4.4.

In summary we have now constructed a commutative diagram

$$\begin{array}{ccc} \mathrm{Spf}(R_{\mathcal{G}}) & \longrightarrow & \mathrm{Spf}\big(R_{\mathrm{GL}(\Lambda)}\big) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{\mathcal{G}} & \longrightarrow & \mathrm{Def}_{\mathrm{GL}(\Lambda)} \end{array}$$

where the right vertical morphism is an isomorphism. In the following we will abuse notation by identifying $\mathrm{Spf}(R_{\mathrm{GL}(\Lambda)})$ with $\mathrm{Def}_{\mathrm{GL}(\Lambda)}$ and viewing $\mathrm{Spf}(R_{\mathcal{G}})$ as a sub-deformation problem of $\mathrm{Def}_{\mathcal{G}}$.

The goal is now to show that $(M^{\mathcal{G}}, M_1^{\mathcal{G}}, \Psi^{\mathcal{G}}) \in \mathrm{Def}_{\mathcal{G}}(R_{\mathcal{G}})$ is a universal deformation, i.e. that $\mathrm{Def}_{\mathcal{G}} = \mathrm{Spf}(R_{\mathcal{G}})$.

**Lemma 3.3.5.** *Let*

$$\begin{array}{ccc} R''' & \longrightarrow & R'' \\ \downarrow & & \downarrow \\ R' & \longrightarrow & R \end{array}$$

*be a fiber product diagram of complete Noetherian local $\mathcal{O}_{\breve{E}}$-algebras with residue field $\overline{\mathbf{F}}_p$ such that the following conditions are satisfied.*

- *$R'$ and $R'''$ are p-torsionfree and $\mathfrak{m}_{R'} = \sqrt{pR'}$ and $\mathfrak{m}_{R'''} = \sqrt{pR'''}$.*
- *The ideal $\ker(R' \to R) \subseteq R'$ has nilpotent divided powers (this implies that $\ker(R''' \to R'') \subseteq R'''$ has nilpotent divided powers as well).*

*Then the map*

$$\mathrm{Def}_{\mathcal{G}}(R''') \to \mathrm{Def}_{\mathcal{G}}(R') \times_{\mathrm{Def}_{\mathcal{G}}(R)} \mathrm{Def}_{\mathcal{G}}(R'')$$

*is a bijection.*

*Proof.* This follows from the argument that was given in the second bullet point of the proof of Proposition 2.4.2, using Theorem 3.2.7 instead of Theorem 2.3.5. □

**Remark 3.3.6.** The assumptions in Lemma 3.3.5 are in particular satisfied for the fiber products

$$\mathcal{O}_K[\varepsilon] \times_{\mathcal{O}_K} \mathcal{O}_K[t]/t^\ell, \qquad \mathcal{O}_K[\varepsilon_1, \varepsilon_2] \times_{\mathcal{O}_K} \mathcal{O}_K[t]/t^\ell,$$
$$\mathcal{O}_K[t]/t^{\ell+1} \times_{\mathcal{O}_K[t]/t^\ell} \mathcal{O}_K[t]/t^{\ell+1},$$

for $\ell \in \mathbf{Z}_{>0}$.

Thus we have well-defined tangent spaces

$$T_x \mathrm{Def}_{\mathcal{G}}$$

at points $x \in \mathrm{Def}_{\mathcal{G}}(\mathcal{O}_K)$, see Lemma 1.3.3, and given a point $y \in \mathrm{Def}_{\mathcal{G}}(\mathcal{O}_K[t]/t^\ell)$ with image $x \in \mathrm{Def}_{\mathcal{G}}(\mathcal{O}_K)$, the fiber

$$\mathrm{fib}_y\Big(\mathrm{Def}_{\mathcal{G}}\big(\mathcal{O}_K[t]/t^{\ell+1}\big)\Big)$$

naturally has the structure of a $T_x \mathrm{Def}_{\mathcal{G}}$-pseudotorsor, see Lemma 1.3.4.

**Lemma 3.3.7.** *Let $x \in \mathrm{Spf}(R_{\mathcal{G}})(\mathcal{O}_K)$. Then we have*

$$T_x \mathrm{Spf}(R_{\mathcal{G}}) = T_x \mathrm{Def}_{\mathcal{G}}.$$

*Proof.* Theorem 3.2.7 produces an isomorphism

$$T_x \mathrm{Spf}(R_{\mathcal{G}}) = T_x \mathrm{M}^{\mathrm{loc}} \to T_x \mathrm{Def}_{\mathcal{G}}$$

(that does not need to agree with the inclusion $T_x \operatorname{Spf}(R_{\mathcal{G}}) \subseteq T_x \operatorname{Def}_{\mathcal{G}}$). This implies that $T_x \operatorname{Def}_{\mathcal{G}}$ is a finite free $\mathcal{O}_K$-module of the same rank as $T_x \operatorname{Spf}(R_{\mathcal{G}})$. Now the composition

$$T_x \operatorname{Spf}(R_{\mathcal{G}}) \subseteq T_x \operatorname{Def}_{\mathcal{G}} \to T_x \operatorname{Def}_{\mathrm{GL}(\Lambda)} = T_x \operatorname{Spf}(R_{\mathrm{GL}(\Lambda)})$$

is a split monomorphism of $\mathcal{O}_K$-modules because $R_{\mathrm{GL}(\Lambda)} \to R_{\mathcal{G}}$ is surjective. But this now implies that $T_x \operatorname{Spf}(R_{\mathcal{G}}) = T_x \operatorname{Def}_{\mathcal{G}}$ as desired. $\square$

**Lemma 3.3.8.** *Let* $(M, M_1, \Psi) \in \operatorname{Def}_{\mathcal{G}}(\mathcal{O}_K[\![t]\!])$ *such that*

$$x = (M, M_1, \Psi)_{\mathcal{O}_K, t \mapsto 0} \in \operatorname{Spf}(R_{\mathcal{G}})(\mathcal{O}_K).$$

*Then we have*

$$(M, M_1, \Psi) \in \operatorname{Spf}(R_{\mathcal{G}})\big(\mathcal{O}_K[\![t]\!]\big).$$

*Proof.* We inductively show that

$$(M, M_1, \Psi)_{\mathcal{O}_K[t]/t^\ell} \in \operatorname{Spf}(R_{\mathcal{G}})\big(\mathcal{O}_K[t]/t^\ell\big)$$

for all $\ell \in \mathbf{Z}_{>0}$.

So assume that this holds for a fixed $\ell$. Setting $y = (M, M_1, \Psi)_{\mathcal{O}_K[t]/t^\ell}$ we then have

$$(M, M_1, \Psi)_{\mathcal{O}_K[t]/t^{\ell+1}} \in \operatorname{fib}_y\Big(\operatorname{Def}_{\mathcal{G}}\big(\mathcal{O}_K[t]/t^{\ell+1}\big)\Big)$$

so that it suffices to show that the map

$$\operatorname{fib}_y\Big(\operatorname{Spf}(R_{\mathcal{G}})\big(\mathcal{O}_K[t]/t^{\ell+1}\big)\Big) \to \operatorname{fib}_y\Big(\operatorname{Def}_{\mathcal{G}}\big(\mathcal{O}_K[t]/t^{\ell+1}\big)\Big)$$

is a bijection. As it is in fact a morphism of $(T_x \operatorname{Spf}(R_{\mathcal{G}}) = T_x \operatorname{Def}_{\mathcal{G}})$-pseudotorsors (see Lemma 3.3.7) it is furthermore sufficient to show that $\operatorname{fib}_y(\operatorname{Spf}(R_{\mathcal{G}})(\mathcal{O}_K[t]/t^{\ell+1}))$ is non-empty.

Now the assumption $y \in \operatorname{Spf}(R_{\mathcal{G}})(\mathcal{O}_K[t]/t^\ell)$ provides us with a trivialization

$$\widehat{W}\big(\mathcal{O}_K[t]/t^\ell\big) \otimes_{\widehat{W}(\mathcal{O}_K[\![t]\!])} M \to \widehat{W}\big(\mathcal{O}_K[t]/t^\ell\big) \otimes_{y, \widehat{W}(R_{\mathcal{G}})} M^{\mathcal{G}} \to \Lambda_{\widehat{W}(\mathcal{O}_K/t^\ell)}$$

such that $y \in \operatorname{Spf}(R_{\mathcal{G}})(\mathcal{O}_K[t]/t^\ell)$ corresponds to the direct summand

$$\begin{aligned} &\widehat{W}\big(\mathcal{O}_K[t]/t^\ell\big) \otimes_{\widehat{W}(\mathcal{O}_K[\![t]\!])} \big(M_1/\widehat{I}_{\mathcal{O}_K[\![t]\!]} M\big) \\ &\subseteq \widehat{W}\big(\mathcal{O}_K[t]/t^\ell\big) \otimes_{\widehat{W}(\mathcal{O}_K[\![t]\!])} \big(M/\widehat{I}_{\mathcal{O}_K[\![t]\!]} M\big) \cong \Lambda_{\mathcal{O}_K[t]/t^\ell}. \end{aligned}$$

Lifting this trivialization to one of

$$\widehat{W}\big(\mathcal{O}_K[t]/t^{\ell+1}\big) \otimes_{\widehat{W}(\mathcal{O}_K[\![t]\!])} M$$

then gives rise to a lift $y' \in \operatorname{Spf}(R_{\mathcal{G}})(\mathcal{O}_K[t]/t^{\ell+1})$ of $y$ corresponding to

$$\begin{aligned} &\widehat{W}\big(\mathcal{O}_K[t]/t^{\ell+1}\big) \otimes_{\widehat{W}(\mathcal{O}_K[\![t]\!])} \big(M_1/\widehat{I}_{\mathcal{O}_K[\![t]\!]} M\big) \\ &\subseteq \widehat{W}\big(\mathcal{O}_K[t]/t^{\ell+1}\big) \otimes_{\widehat{W}(\mathcal{O}_K[\![t]\!])} \big(M/\widehat{I}_{\mathcal{O}_K[\![t]\!]} M\big) \cong \Lambda_{\mathcal{O}_K[t]/t^{\ell+1}} \end{aligned}$$

so that $\operatorname{fib}_y(\operatorname{Spf}(R_{\mathcal{G}})(\mathcal{O}_K[t]/t^{\ell+1}))$ is non-empty as desired. $\square$

**Lemma 3.3.9.** *Let $(M', M_1', \Psi'), (M'', M_1'', \Psi'') \in \mathrm{Def}_{\mathcal{G}}(\mathcal{O}_K)$ such that there exists an isomorphism of Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pairs over $\mathcal{O}_K$*

$$(M', M_1') \to (M'', M_1'')$$

*inducing the identity on $(M_0, M_{0,1})$. Then there exists*

$$(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}\big(\mathcal{O}_K[\![t]\!]\big)$$

*that specializes to $(M', M_1', \Psi')$ respectively $(M'', M_1'', \Psi'')$ under the morphism*

$$\mathcal{O}_K[\![t]\!] \to \mathcal{O}_K, \qquad t \mapsto \varpi_K \quad \textit{respectively} \quad t \mapsto 0.$$

*Proof.* Choose an isomorphism $(M', M_1') \cong (M'', M_1'')$ as above and define $(M, M_1)$ as the base change of any of these two Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pairs along $\mathcal{O}_K \to \mathcal{O}_K[\![t]\!]$. Now consider the surjective morphism

$$\mathcal{O}_K[\![t]\!] \to \mathcal{O}_K \times_{\overline{\mathbf{F}}_p} \mathcal{O}_K, \qquad t \mapsto (\varpi_K, 0).$$

Then the isomorphisms $\Psi'$ and $\Psi''$ induce an isomorphism

$$(\Psi', \Psi'') \colon (\mathcal{O}_K \times_{\overline{\mathbf{F}}_p} \mathcal{O}_K) \otimes_{\mathcal{O}_K[\![t]\!]} \widetilde{M_1} \to (\mathcal{O}_K \times_{\overline{\mathbf{F}}_p} \mathcal{O}_K) \otimes_{\mathcal{O}_K[\![t]\!]} M$$

that we can lift to an isomorphism $\Psi \colon \widetilde{M_1} \to M$ to obtain an object

$$(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}\big(\mathcal{O}_K[\![x]\!]\big)$$

with the desired properties. □

**Lemma 3.3.10.** *Let $(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}(R)$. Then there exists a morphism $R' \to R$ with $R'$ reduced and $p$-torsionfree and a preimage*

$$(M', M_1', \Psi') \in \mathrm{Def}_{\mathcal{G}}(R')$$

*of $(M, M_1, \Psi)$.*

*Proof.* Using flatness and formal reducedness of $\mathrm{M}^{\mathrm{loc}}$ we can find $R' \to R$ with $R'$ reduced and $p$-torsionfree and a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pair $(M', M_1')$ over $R'$ together with an isomorphism $(M', M_1')_R \cong (M, M_1)$. Replacing $R'$ with $R'[\![x_1, \dots, x_\ell]\!]$ for a suitable $\ell \in \mathbf{Z}_{>0}$ if necessary, we may even assume that $R' \to R$ is surjective. Then $\Psi$ can be lifted to an isomorphism $\Psi' \colon \widetilde{M_1'} \to M'$ so that we obtain our desired lift. □

**Theorem 3.3.11.** *Assume that Hypothesis 3.3.3 is satisfied. Then the deformation $(M^{\mathcal{G}}, M_1^{\mathcal{G}}, \Psi^{\mathcal{G}}) \in \mathrm{Def}_{\mathcal{G}}(R_{\mathcal{G}})$ from Construction 3.3.2 is universal.*

*Proof.* We first show that

$$\mathrm{Spf}(R_{\mathcal{G}})(\mathcal{O}_K) = \mathrm{Def}_{\mathcal{G}}(\mathcal{O}_K).$$

Given $(M', M'_1, \Psi') \in \mathrm{Def}_{\mathcal{G}}(\mathcal{O}_K)$ we can choose a trivialization $M' \to \Lambda_{\widehat{W}(\mathcal{O}_K)}$ lifting the identity on $M_0$ and then find a (unique) deformation

$$(M'', M''_1, \Psi'') \in \mathrm{Spf}(R_{\mathcal{G}})(\mathcal{O}_K)$$

such that the composition $M' \to \Lambda_{\widehat{W}(\mathcal{O}_K)} \to M''$ defines an isomorphism of Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-pairs over $\mathcal{O}_K$

$$(M', M'_1) \to (M'', M''_1).$$

Applying Lemma 3.3.9 we obtain a deformation

$$(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}\big(\mathcal{O}_K[\![t]\!]\big)$$

that specializes to $(M', M'_1, \Psi')$ and $(M'', M''_1, \Psi'')$ under $t \mapsto \varpi_K$ and $t \mapsto 0$ respectively. By Lemma 3.3.8 we then have $(M, M_1, \Psi) \in \mathrm{Spf}(R_{\mathcal{G}})(\mathcal{O}_K[\![t]\!])$, so that in particular

$$(M', M'_1, \Psi') \in \mathrm{Spf}(R_{\mathcal{G}})(\mathcal{O}_K)$$

as desired.

Next we claim that the morphism

$$\mathrm{Def}_{\mathcal{G}} \to \mathrm{Def}_{\mathrm{GL}(\Lambda)} = \mathrm{Spf}(R_{\mathrm{GL}(\Lambda)})$$

has image inside $\mathrm{Spf}(R_{\mathcal{G}})$ so that we obtain a retraction $r\colon \mathrm{Def}_{\mathcal{G}} \to \mathrm{Spf}(R_{\mathcal{G}})$ of the inclusion. By Lemma 3.3.10 it suffices to show that the map

$$\mathrm{Def}_{\mathcal{G}}(R) \to \mathrm{Spf}(R_{\mathrm{GL}(\Lambda)})(R)$$

has image inside $\mathrm{Spf}(R_{\mathcal{G}})(R)$ for $R$ reduced and $p$-torsionfree. By noting that such $R$ always injects into a product of $\mathcal{O}_K$'s we are further reduced to the case $R = \mathcal{O}_K$ where we have already established the claim.

Now let $x = (M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}(R)$. Then the point $r(x) \in \mathrm{Spf}(R_{\mathcal{G}})(R)$ gives rise to a second $\mathcal{G}$-structure on the finite free $\widehat{W}(R)$-module $M$ that also makes $(M, M_1, \Psi)$ into a Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-display that is a deformation of $(M_0, M_{0,1}, \Psi_0)$. We have to show that $r(x) = x$, which is the same as showing that both $\mathcal{G}$-structures agree. Lemma 3.3.10 once more allows us to reduce to the case $R = \mathcal{O}_K$ where this follows from Corollary 3.2.2. ☐

**Remark 3.3.12.** Let us recall the notions of *sections rigid in the first order* and *locally universal Dieudonné* $(\mathcal{G}, \boldsymbol{\mu})$*-displays* given by Pappas in [Pap23, Definitions 4.5.8 and 4.5.10].

- Let $(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}(R)$, set $\mathfrak{a}_R := \mathfrak{m}_R^2 + \mathfrak{m}_{\breve{E}} R \subseteq R$ and equip $\mathfrak{m}_R/\mathfrak{a}_R \subseteq R/\mathfrak{a}_R$ with the trivial divided powers. A trivialization $M \cong \Lambda_{\widehat{W}(R)}$ (compatible with the given trivialization of $M_0$) is called *rigid in the first order* if the diagram

$$\begin{array}{ccc} \widehat{W}(R/\mathfrak{a}_R) \otimes_{\widehat{W}(R)} \widetilde{M_1} & \xrightarrow{\Psi} & \widehat{W}(R/\mathfrak{a}_R) \otimes_{\widehat{W}(R)} M \\ \downarrow & & \downarrow \\ \big(\widetilde{M_{0,1}}\big)_{\widehat{W}(R/\mathfrak{a}_R)} & \xrightarrow{\Psi_0} & M_{0,\widehat{W}(R/\mathfrak{a}_R)} \end{array}$$

is commutative. Here the vertical isomorphisms are induced by the trivialization and Lemma 2.3.3 (and both depend on the choice of trivialization).

Trivializations that are rigid in the first order always exist (the standard trivialization of $M^{\mathcal{G}}$ is rigid in the first order essentially by definition of $\Psi^{\mathcal{G}}$ and this is the universal case).

- A deformation $(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}(R)$ is called *locally universal* if there exists a trivialization $M \cong \widehat{W}(R) \otimes_{\mathbf{Z}_p} \Lambda$ that is rigid of the first order and such that the morphism $R_{\mathcal{G}} \to R$ induced by the filtration $M_1$ is an isomorphism.

From Construction 3.3.2 and Theorem 3.3.11 it follows that a given deformation $(M, M_1, \Psi) \in \mathrm{Def}_{\mathcal{G}}(R)$ is locally universal if and only if it is a universal deformation, i.e. if the corresponding morphism $\mathrm{Spf}(R) \to \mathrm{Def}_{\mathcal{G}}$ is an isomorphism.

## 3.4 Comparison

### 3.4.1 Comparison to restricted local shtukas

In this subsection we compare our ($(m,n)$-truncated) $(\mathcal{G}, \mu)$-displays with the ($(m,n)$-restricted) local shtukas studied by Xiao and Zhu in [XZ17, Section 5] and by Shen, Yu and Zhang in [SYZ21, Section 4], see Section 1.5.

**Notation 3.4.1.** Write $\mathcal{F}\ell$, $\mathcal{A}$ and $\mathcal{A}^{(\infty)}$ for the objects $\mathcal{F}\ell_{\mathcal{G}}$, $\mathcal{A}_{\mathcal{G},\boldsymbol{\mu}}$ and $\mathcal{A}^{(\infty)}_{\mathcal{G},\boldsymbol{\mu}}$ from Definition 1.4.3 and Construction 1.5.2.

**Proposition 3.4.2.** *There is a natural $(\mathbb{L}^+\mathcal{G} \times \mathbb{L}^+\mathcal{G})$-equivariant isomorphism*

$$\mathcal{A}^{(\infty)} \to \big(\mathrm{M}^{\mathrm{loc},(\infty)}\big)^{\mathrm{pf}}_{\mathbf{F}_q}$$

*that is compatible with the identification $\mathcal{A} = (\mathrm{M}^{\mathrm{loc}})^{\mathrm{pf}}_{\mathbf{F}_q}$ from Theorem 1.4.6. Consequently we obtain equivalences*

$$\big(\mathsf{Disp}_{\mathcal{G},\mu}\big)^{\mathrm{pf}}_{\mathbf{F}_q} \cong \mathsf{Sht}^{\mathrm{loc}}_{\mathcal{G},\mu} \quad \textit{and} \quad \big(\mathsf{Disp}^{(m,n)}_{\mathcal{G},\mu}\big)^{\mathrm{pf}}_{\mathbf{F}_q} \cong \mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\mu}.$$

*Proof.* Write $\mathcal{A}_{\Lambda,d}$, $\mathcal{A}^{(\infty)}_{\Lambda,d}$, $\mathrm{M}^{\mathrm{loc}}_{\Lambda,d}$ and $\mathrm{M}^{\mathrm{loc},(\infty)}_{\Lambda,d}$ for the objects analogous to $\mathcal{A}$, $\mathcal{A}^{(\infty)}$, $\mathrm{M}^{\mathrm{loc}}$ and $\mathrm{M}^{\mathrm{loc},(\infty)}$ when $(\mathcal{G}, \boldsymbol{\mu})$ is replaced by $(\mathrm{GL}(\Lambda), \boldsymbol{\mu}'_d)$. For a $p$-complete ring $R$ we then make the identifications

$$\mathrm{M}^{\mathrm{loc}}_{\Lambda,d}(R) \cong \big\{M_1 \subseteq \Lambda_{W(R)} \bigm| (\Lambda_{W(R)}, M_1) \text{ is a pair of type } (h,d)\big\}$$

and

$$\mathrm{M}^{\mathrm{loc},(\infty)}_{\Lambda,d}(R) \cong \big\{(M_1, \xi) \bigm| M_1 \in \mathrm{M}^{\mathrm{loc}}_{\Lambda,d}(R),\, \xi \colon \widetilde{M_1} \to \Lambda_{W(R)} \text{ is an isomorphism}\big\}.$$

Then the identification $\mathcal{A}_{\Lambda,d} = (\mathrm{M}^{\mathrm{loc}}_{\Lambda,d})^{\mathrm{pf}}_{\mathbf{F}_p}$ is explicitly given by

$$\mathcal{A}_{\Lambda,d} \to \big(\mathrm{M}^{\mathrm{loc}}_{\Lambda,d}\big)^{\mathrm{pf}}_{\mathbf{F}_p}, \qquad g \cdot \mathbb{L}^+ \mathrm{GL}(\Lambda) \mapsto M_1^g \coloneqq pg\Lambda_{W(R)} \subseteq \Lambda_{W(R)}$$

and we can define an isomorphism

$$\mathcal{A}^{(\infty)}_{\Lambda,d} \to \big(\mathrm{M}^{\mathrm{loc},(\infty)}_{\Lambda,d}\big)^{\mathrm{pf}}_{\mathbf{F}_p}, \qquad g \mapsto \Big(M_1^g,\ \xi^g \colon \widetilde{M_1^g} \cong p\sigma(g)\Lambda_{W(R)} \xrightarrow{p^{-1}\sigma(g)^{-1}} \Lambda_{W(R)}\Big)$$

that has all the required properties. We now claim that this isomorphism restricts to an isomorphism

$$\mathcal{A}^{(\infty)} \to \big(\mathrm{M}^{\mathrm{loc},(\infty)}\big)^{\mathrm{pf}}_{\mathbf{F}_q}.$$

This amounts to showing that for a perfect $\mathbf{F}_q$-algebra $R$ and $g \in \mathcal{A}^{(\infty)}(R)$ the isomorphism

$$\xi^g \colon \widetilde{M_1^g} \to \Lambda_{W(R)}$$

preserves $\mathcal{G}$-structures. As the $\mathcal{G}$-structure on $\widetilde{M_1^g}$ is characterized by the identification $\widetilde{M_1^g}[1/p] \cong \Lambda^{\sigma}_{W(R)}[1/p]$ this boils down to noting that $p^{-1}\sigma(g)^{-1} \in G(W(R)[1/p])$. $\square$

### 3.4.2 Comparison to $(\mathcal{G}, \mu)$-displays in the sense of Bültel-Pappas

In this subsection we compare our $(\mathcal{G}, \boldsymbol{\mu})$-displays with the $(\mathcal{G}, \mu)$-displays defined by Bültel and Pappas in [BP20], see Section 1.6.

**Notation 3.4.3.** Suppose that $\mathcal{G}$ is reductive and fix a representative $\mu$ of $\boldsymbol{\mu}$ that is defined over $\mathcal{O}_F$ for some finite unramified extension $F/\mathbf{Q}_p$. Then $\mu$ induces a weight decomposition $\Lambda_{\mathcal{O}_F} = \Lambda_{-1} \oplus \Lambda_0$ and we define

$$M_1^{\mathrm{std}} \subseteq \Lambda_{W(\mathcal{O}_F)}$$

to be the preimage of $\Lambda_{-1}$ under $\Lambda_{W(\mathcal{O}_F)} \to \Lambda_{\mathcal{O}_F}$ so that $(\Lambda_{W(\mathcal{O}_F)}, M_1^{\mathrm{std}})$ is a $(\mathcal{G}, \boldsymbol{\mu})$-pair over $\mathcal{O}_F$. Note that $H_\mu$ (introduced in Notation 1.6.1) naturally is the automorphism group scheme of $(\Lambda_{W(\mathcal{O}_F)}, M_1^{\mathrm{std}})$.

In this subsection $R$ always denotes a $p$-complete $\mathcal{O}_F$-algebra.

**Proposition 3.4.4.** *There is a natural equivalence*

$$\begin{aligned} \mathsf{Pair}_{\mathcal{G},\boldsymbol{\mu},\mathcal{O}_F} &\to \big\{H_\mu\text{-}torsors\big\}, \\ (M, M_1) &\mapsto \mathcal{I}som\big((M, M_1), (\Lambda_{W(\mathcal{O}_F)}, M_1^{\mathrm{std}})\big). \end{aligned}$$

*Under this equivalence there is a natural identification between the constructions $(M, M_1) \mapsto \widetilde{M_1}$ and $\mathcal{Q} \mapsto L^+\mathcal{G} \times^{\varphi, H_\mu} \mathcal{Q}$ (implicitly using Proposition 1.1.1). Consequently we obtain a natural equivalence*

$$\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu},\mathcal{O}_F} \cong \big\{(\mathcal{G}, \mu)\text{-}displays\ in\ the\ sense\ of\ Bültel\text{-}Pappas\big\}.$$

*Proof.* The first claim is immediate because $\mathcal{G}$ acts transitively on $\mathrm{M}^{\mathrm{loc}}$ so that every $(\mathcal{G}, \boldsymbol{\mu})$-pair is locally isomorphic to $(\Lambda_{W(\mathcal{O}_{E'})}, M_1^{\mathrm{std}})$.

For the second claim we note that the element $p \cdot \sigma(\mu(p)) \in G(W(\mathcal{O}_{E'})[1/p])$ induces an isomorphism $\Lambda_{W(\mathcal{O}_{E'})} \to \widetilde{M_1^{\mathrm{std}}}$. If we denote the inverse of this isomorphism by $\xi$ then $\varphi$ can be described as the morphism

$$h \mapsto \left(\Lambda_{W(R)} \xrightarrow{\xi^{-1}} \left(\widetilde{M_1^{\mathrm{std}}}\right)_{W(R)} \xrightarrow{\tilde{h}} \left(\widetilde{M_1^{\mathrm{std}}}\right)_{W(R)} \xrightarrow{\xi} \Lambda_{W(R)}\right).$$

The claim then follows formally from this. □

### 3.4.3 Comparison to the theory of zips

In this subsection we compare our $(\mathcal{G}, \boldsymbol{\mu})$-displays with certain zips as considered by Shen, Yu and Zhang in [SYZ21, Section 3], see Section 1.7 for the definition of zip data and zips that we are using.

**Notation 3.4.5.** Throughout this subsection $R$ always denotes on $\overline{\mathbf{F}}_p$-algebra.

Recall that $\mathrm{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$ decomposes into finitely many $\mathcal{G}$-orbits that are called Kottwitz-Rapoport (KR) strata and are enumerated by the set $\mathrm{Adm}(\mu)_{\mathcal{G}}$ (see [HR17]). Let us fix $w \in \mathrm{Adm}(\mu)_{\mathcal{G}}$ and denote the corresponding KR-stratum by $\mathrm{M}^{\mathrm{loc},w} \subseteq \mathrm{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$.

**Definition 3.4.6.** We say that a $(\mathcal{G}, \mu)$-pair $(\mathcal{P}, q)$ over $R$ is *of type* $w$ if the morphism $q \colon \mathcal{P} \to \mathrm{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$ has image inside $\mathrm{M}^{\mathrm{loc},w}$ and write $\mathsf{Pair}^w_{\mathcal{G},\mu,\overline{\mathbf{F}}_p} \subseteq \mathsf{Pair}_{\mathcal{G},\mu,\overline{\mathbf{F}}_p}$ for the substack of $(\mathcal{G}, \mu)$-pairs of type $w$.

We make the same definition for $(\mathcal{G}, \mu)$-displays and the truncated variants.

Let us now also fix $M_1^{\mathrm{std}} \subseteq \Lambda_{\breve{\mathbf{Z}}_p}$, making $(\Lambda_{\breve{\mathbf{Z}}_p}, M_1^{\mathrm{std}})$ into a $(\mathcal{G}, \boldsymbol{\mu})$-pair of type $w$ over $\overline{\mathbf{F}}_p$, and an isomorphism $\xi \colon \widetilde{M_1^{\mathrm{std}}} \to \Lambda_{\breve{\mathbf{Z}}_p}$. Note that the tuple $(M_1^{\mathrm{std}}, \xi)$ defines a point $w \in \mathrm{M}^{\mathrm{loc},(\infty)}(\overline{\mathbf{F}}_p) \subseteq G(\breve{\mathbf{Q}}_p)$ (see Section 3.4.1). We then have the following data.

- Write $H_{w,+} \subseteq (L^+\mathcal{G})_{\overline{\mathbf{F}}_p}$ for the stabilizer of $w \in \mathrm{M}^{\mathrm{loc}}(\overline{\mathbf{F}}_p)$. Note that $H_{w,+}$ naturally is the automorphism group scheme of the $(\mathcal{G}, \boldsymbol{\mu})$-pair $(\Lambda_{\breve{\mathbf{Z}}_p}, M_1^{\mathrm{std}})$. We also write $H^{(m)}_{w,+} \subseteq (L^{(m)}\mathcal{G})_{\overline{\mathbf{F}}_p}$ for the image of $H_{w,+}$.

- Write $H_{w,-} \subseteq (L^+\mathcal{G})_{\overline{\mathbf{F}}_p}$ for the stabilizer of the *conjugate filtration*, i.e. the image of the composition
$$\Lambda_{\breve{\mathbf{Z}}_p} \to \widetilde{M_1^{\mathrm{std}}} \xrightarrow{\xi} \Lambda_{\breve{\mathbf{Z}}_p} \to \Lambda_{\overline{\mathbf{F}}_p}.$$
Here the first map is the one from Remark 2.1.8 and the action of $(L^+\mathcal{G})_{\overline{\mathbf{F}}_p}$ on $\Lambda_{\overline{\mathbf{F}}_p}$ is the usual one through the quotient $\mathcal{G}_{\overline{\mathbf{F}}_p}$. We also write $H^{(n)}_{w,-} \subseteq (L^{(n)}\mathcal{G})_{\overline{\mathbf{F}}_p}$ for the image of $H_{w,-}$ and $H^{(m,n)}_{w,-} \subseteq (L^{(m)}\mathcal{G})_{\overline{\mathbf{F}}_p}$ for the preimage of $H^{(n)}_{w,-}$. This last bit of notation is a bit awkward as we always have $H^{(m,n)}_{w,-} = H^{(m)}_{w,-}$ when $n \neq 1$-rdt.

- We now consider the morphism

$$\begin{aligned} \varphi \colon H_{w,+} &\to (L^+\mathcal{G})_{\overline{\mathbf{F}}_p}, \\ h &\mapsto \Big(\Lambda_{W(R)} \xrightarrow{\xi^{-1}} \big(\widetilde{M_1^{\mathrm{std}}}\big)_{W(R)} \xrightarrow{\widetilde{h}} \big(\widetilde{M_1^{\mathrm{std}}}\big)_{W(R)} \xrightarrow{\xi} \Lambda_{W(R)}\Big). \end{aligned}$$

  One can characterize $\varphi$ by the property

$$\varphi(h) = \sigma(w^{-1}hw) \in G\big(W(R)[1/p]\big)$$

  for all $h \in H_{w,+}(R)$.

  Also note that in fact $\varphi$ has image inside $H_{w,-}$. Indeed, given $h \in H_{w,+}(R)$ the automorphism $\widetilde{h}$ will always stabilize the image of $\Lambda_{W(R)} \to (\widetilde{M_1^{\mathrm{std}}})_{W(R)}$. In the following we will always think of $\varphi$ as a morphism $\varphi \colon H_{w,+} \to H_{w,-}$.

  The morphism $\varphi$ clearly induces morphisms $H_{w,+}^{(m)} \to H_{w,-}^{(n)}$ that we again denote by $\varphi$.

At this point we have constructed zip data

$$\big((L^+\mathcal{G})_{\overline{\mathbf{F}}_p}, H_{w,+}, H_{w,-}, H_{w,-}, \varphi\big) \quad \text{and} \quad \big((L^{(m)}\mathcal{G})_{\overline{\mathbf{F}}_p}, H_{w,+}^{(m)}, H_{w,-}^{(m,n)}, H_{w,-}^{(n)}, \varphi\big).$$

**Remark 3.4.7.** When $\mathcal{G}$ is reductive there is only one KR-stratum and we can choose $w = \mu(p)$ for some representative of $\mu$ defined over $\mathcal{O}_F$ for some finite unramified extension $F/\mathbf{Q}_p$. Then $H_{w,+}$ and $\varphi$ as defined here are precisely the base changes to $\overline{\mathbf{F}}_p$ of the objects $H_\mu$ and $\varphi$ from Notation 1.6.1 and Proposition 1.6.2.

**Proposition 3.4.8.** *There is a natural equivalence*

$$\mathsf{Pair}^w_{\mathcal{G},\mu,\overline{\mathbf{F}}_p} \cong \big\{H_{w,+}\text{-torsors}\big\}, \qquad (M, M_1) \mapsto \mathcal{I}som\big((M, M_1), (\Lambda_{\breve{\mathbf{Z}}_p}, M_1^{\mathrm{std}})_R\big).$$

*Under this equivalence there is a natural identification between the constructions $(M, M_1) \mapsto \widetilde{M}_1$ and $\mathcal{Q} \mapsto (L^+\mathcal{G})_{\overline{\mathbf{F}}_p} \times^{\varphi, H_{w,+}} \mathcal{Q}$. Consequently we obtain a natural equivalence*

$$\mathsf{Disp}^w_{\mathcal{G},\mu,\overline{\mathbf{F}}_p} \cong \Big\{\text{zips for } \big((L^+\mathcal{G})_{\overline{\mathbf{F}}_p}, H_{w,+}, H_{w,-}, H_{w,-}, \varphi\big)\Big\}.$$

*Similarly we also have natural equivalences*

$$\mathsf{Pair}^{(m),w}_{\mathcal{G},\mu,\overline{\mathbf{F}}_p} \cong \big\{H^{(m)}_{w,+}\text{-torsors}\big\}$$

*and*

$$\mathsf{Disp}^{(m,n),w}_{\mathcal{G},\mu,\overline{\mathbf{F}}_p} \cong \Big\{\text{zips for } \big((L^{(m)}\mathcal{G})_{\overline{\mathbf{F}}_p}, H_{w,+}^{(m)}, H_{w,-}^{(m,n)}, H_{w,-}^{(n)}, \varphi\big)\Big\}.$$

*Proof.* This is immediate from the definitions. $\square$

In [SYZ21, Section 1.3] the authors construct a zip datum $\mathcal{Z}_w = (\mathcal{G}^{\mathrm{rdt}}_{\overline{\mathbf{F}}_p}, P_+, P_-, L_-, \varphi)$ associated to $w$ (we warn the reader that we do not fully understand this construction and treat it as a black box). As in [SYZ21, Section 3.3] we have natural morphisms of zip data

$$\big((L^+\mathcal{G})_{\overline{\mathbf{F}}_p}, H_{w,+}, H_{w,-}, H_{w,-}, \varphi\big), \big((L^{(m)}\mathcal{G})_{\overline{\mathbf{F}}_p}, H^{(m)}_{w,+}, H^{(m,n)}_{w,-}, H^{(n)}_{w,-}, \varphi\big) \to \mathcal{Z}_w$$

that are compatible in the obvious sense. If we write $\mathsf{Zip}_{\mathcal{G},\mu,w}$ for the stack of zips for $\mathcal{Z}_w$ we thus obtain natural (compatible) smooth morphisms of algebraic stacks

$$\mathsf{Disp}^{(m,n),w}_{\mathcal{G},\mu,\overline{\mathbf{F}}_p} \to \mathsf{Zip}_{\mathcal{G},\mu,w}.$$

For $n = 1$-rdt this morphism is a homeomorphism on the level of underlying topological spaces by [SYZ21, Proposition 4.2.6].

**Remark 3.4.9.** When $\mathcal{G}$ is reductive and $w = \mu(p)$ as in Remark 3.4.7 then the zip datum $\mathcal{Z}_w$ is the one attached to $(\mathcal{G}_{\mathbf{F}_p}, \mu)$, see Example 1.7.4.

## 3.5 Application to Shimura varieties

**Notation 3.5.1.** In this section we change our setup.

We use Notation 1.8.13 and set $\Lambda \coloneqq \Xi^\vee$. Then we have the local model datum $(\mathcal{G}, \boldsymbol{\mu})$ and the closed immersion

$$\iota\colon \mathcal{G} \to \mathrm{GSp}(\Xi) \to \mathrm{GL}(\Xi) \cong \mathrm{GL}(\Lambda)$$

satisfies the assumptions from Notation 3.0.1 (for $(h, d) = (2g, g)$). Let $\mathbf{K}^p$, $\mathbf{K}$, $\mathbf{L}^p$ and $\mathbf{L}$ be as in Definition 1.8.14 so that we have the morphism of integral models of Shimura varieties

$$\mathscr{S}_{\mathbf{K}} = \mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{X}) \to \mathscr{S}_{\mathbf{L}}\big(\mathrm{GSp}(V), S^\pm\big)_{\mathcal{O}_E}.$$

Write $\widehat{\mathscr{S}_{\mathbf{K}}}$ for the $p$-completion of $\mathscr{S}_{\mathbf{K}}$. We assume that Hypothesis 3.3.3 holds true.

**Construction 3.5.2.** Let

$$A \to \mathscr{S}_{\mathbf{K}}$$

be the pullback of the ($g$-dimensional) Abelian variety up to prime-to-$p$ isogeny over $\mathscr{S}_{\mathbf{L}}(\mathrm{GSp}(V), S^\pm)$ coming from its moduli description, see Theorem 1.8.11, and let

$$X \coloneqq A\big[p^\infty\big]$$

be its associated $p$-divisible group (that is of height $2g$ and dimonsion $g$). By Theorem 2.5.1 $X$ gives rise to the following data.

- A display $(M, M_1, \Psi)$ of type $(2g, g)$ over $\widehat{\mathscr{S}_{\mathbf{K}}}$.

- For every point $x \in \mathscr{S}_{\mathbf{K}}(\overline{\mathbf{F}}_p)$ a Dieudonné display $(M_x, M_{x,1}, \Psi_x)$ of type $(2g, g)$ over the completed local ring of $\mathscr{S}_{\mathbf{K},\mathcal{O}_{\breve{E}}}$ at $x$. These are compatible with $(M, M_1, \Psi)$ in the obvious sense.

**Theorem 3.5.3** (Hamacher-Kim, Pappas)**.** *There are naturally defined compatible $\mathcal{G}$-structures on $M$ and $M_x$ for $x \in \mathscr{S}_{\mathbf{K}}(\overline{\mathbf{F}}_p)$ that make $(M, M_1, \Psi)$ and $(M_x, M_{x,1}, \Psi_x)$ into (Dieudonné) $(\mathcal{G}, \boldsymbol{\mu})$-displays.*

*Moreover the Dieudonné $(\mathcal{G}, \boldsymbol{\mu})$-displays $(M_x, M_{x,1}, \Psi_x)$ are locally universal (see Remark 3.3.12).*

*Proof.* See [Pap23, Theorem 8.1.4]. ☐

**Theorem 3.5.4.** *The morphism*

$$\widehat{\mathscr{S}_{\mathbf{K}}} \to \mathsf{Disp}^{(m,n)}_{\mathcal{G},\boldsymbol{\mu}} \qquad \textit{(respectively } \mathscr{S}_{\mathbf{K},\mathbf{F}_q} \to \mathsf{Disp}^{(m,1\text{-rdt})}_{\mathcal{G},\boldsymbol{\mu}} \textit{ for } n = 1\text{-rdt}\textit{)}$$

*induced by $(M, M_1, \Psi)$ is smooth.*

*Proof.* Both $\mathscr{S}_{\mathbf{K}}$ and $\mathsf{Disp}^{(m,n)}_{\mathcal{G},\mu}$ are of finite type over $\mathrm{Spf}(\mathcal{O}_E)$. Thus by [Stacks, Tag 02HX] it suffices to show that every lifting problem

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r] \arrow[d] & \mathscr{S}_{\mathbf{K}} \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] \arrow[ur, dashed] & \mathsf{Disp}^{(m,n)}_{\mathcal{G},\mu}, \end{tikzcd}$$

where $R' \to R$ is a surjection of Artinian local $\mathcal{O}_{\breve{E}}$-algebras with residue field $\overline{\mathbf{F}}_p$, admits a solution. This follows from Theorem 3.5.3, i.e. the local universality of the Dieudonné $(\mathcal{G}, \mu)$-displays $(M_x, M_{x,1}, \Psi_x)$, together with Lemma 3.1.12. ☐

**Corollary 3.5.5** ([SYZ21, Theorem 4.4.3])**.** *There is a natural perfectly smooth morphism*

$$\mathscr{S}^{\mathrm{pf}}_{\mathbf{K},\mathbf{F}_q} \to \mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$$

*(where we allow $n = 1\text{-rdt}$); see Section 1.5 for the definition of $\mathsf{Sht}^{\mathrm{loc},(m,n)}_{\mathcal{G},\boldsymbol{\mu}}$.*

*Proof.* This follows from Theorem 3.5.4 and Proposition 3.4.2. ☐

**Corollary 3.5.6** ([Zha18, Theorem 3.1.2], [SYZ21, Theorem 3.4.11])**.** *Fix an element $w \in \mathrm{Adm}(\boldsymbol{\mu})_{\mathcal{G}}$ and a representative of $w$ in $\mathrm{M}^{\mathrm{loc},(\infty)}(\overline{\mathbf{F}}_p)$. Then there is a natural smooth morphism*

$$\mathscr{S}^{w}_{\mathbf{K},\overline{\mathbf{F}}_p} \to \mathsf{Zip}_{\mathcal{G},\boldsymbol{\mu},w}$$

*where $\mathscr{S}^{w}_{\mathbf{K},\overline{\mathbf{F}}_p} \subseteq \mathscr{S}_{\mathbf{K},\overline{\mathbf{F}}_p}$ denotes the KR-stratum corresponding to $w$.*

*Proof.* This follows from Theorem 3.5.4 and the discussion in Subsection 3.4.3. ☐

**Remark 3.5.7.** The underlying topological space

$$\left|\left(\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}^{(m,1\text{-rdt})}\right)_{\overline{\mathbf{F}}_p}\right| = \left|\left(\mathsf{Sht}_{\mathcal{G},\boldsymbol{\mu}}^{\mathrm{loc},(m,1\text{-rdt})}\right)_{\overline{\mathbf{F}}_p}\right|$$

has finitely many points whose closure relations can be described explicitly, see [SYZ21, Lemma 4.2.4]. The (finitely many) fibers of

$$\mathscr{S}_{\mathbf{K},\overline{\mathbf{F}}_p} \to \left(\mathsf{Disp}_{\mathcal{G},\boldsymbol{\mu}}^{(m,1\text{-rdt})}\right)_{\overline{\mathbf{F}}_p}$$

are precisely the EKOR strata introduced by He and Rapoport in [HR17]. Thus Theorem 3.5.4 solves the problem of realizing the EKOR-stratification as a smooth morphism from the special fiber of the integral model $\mathscr{S}_{\mathbf{K}}$ to a naturally defined algebraic stack.

# 4 (Homogeneously polarized) chains of displays

In this chapter we give an explicit moduli description for $(\mathcal{G}, \boldsymbol{\mu})$-displays in the case where $\mathcal{G}$ is a parahoric group scheme for either a general linear or a general symplectic group. We apply this to the integral models of Siegel modular varieties at parahoric level defined by Rapoport and Zink in [RZ96], see Subsection 1.8.5.

**Notation 4.0.1.** In this chapter $R$ always denotes a $p$-complete ring. As in Notation 3.0.1, $m$ and $n$ always denote positive integers and $(m, n)$ always denotes a tuple of positive integers such that $m \geq n+1$. Again, we sometimes (with explicit indication) allow $n$ to take the additional value 1-rdt, in which case we require $m \geq 2$.

## 4.1 Chains

**Notation 4.1.1.** We use Notation 1.4.8. Let $\mathcal{E}$ be the set of subsets $\{i, j\} \subseteq J$ with $i, j$ being consecutive in $J$. We think of the elements of $\mathcal{E}$ as edges. For $e = \{i, j\} \in \mathcal{E}$ with $i \leq j$ we set $|e| \coloneqq j - i$, $\Xi_e \coloneqq \Xi_i / \Xi_j$ (so that $\Xi_e$ is an $\mathbf{F}_p$-vector space of dimension $|e|$) and $\Lambda_e \coloneqq \Xi_e^\vee$. Note that we have a natural isomorphism

$$\Lambda_e \cong \ker\bigl(\rho_{i,j} \colon \Lambda_i / p\Lambda_i \to \Lambda_j / p\Lambda_j\bigr).$$

We write $\theta_e \colon \Lambda_e \to \Lambda_{e+h}$ for the isomorphism induced by $\theta_i$ and $\theta_j$.

The reductive quotient $\mathrm{GL}((\Xi_i)_i)^{\mathrm{rdt}}_{\mathbf{F}_p}$ of the special fiber of $\mathrm{GL}((\Xi_i)_i)$ identifies with

$$\mathrm{GL}((\Xi_e)_e) = \prod_{e \in \mathcal{E}/h\mathbf{Z}} \mathrm{GL}(\Xi_e),$$

the automorphism group of the tuple $((\Lambda_e)_e, (\theta_e)_e)$.

Note that by Remark 3.0.2 the local model datum $(\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)$ satisfies the assumptions from Notation 3.0.1.

**Theorem 4.1.2.** *Let $A$ be a $p$-complete ring. Consider tuples*

$$\bigl((M_i)_{i \in J}, (\rho_{i,j})_{i,j \in J, i \leq j}, (\theta_i)_{i \in J}\bigr)$$

*that are given as follows.*

- $((M_i)_i, (\rho_{i,j})_{i,j})$ *is a diagram of finite projective $A$-modules of rank $h$ of shape $J$ such that the morphism of $A/pA$-modules*

$$M_i / pM_i \to M_j / pM_j$$

*induced by $\rho_{i,j}$ is of constant rank $h - (j - i)$ for all $i \leq j \leq i + h$.*

- $\theta_i \colon M_i \to M_{i+h}$ *are isomorphisms such that we have the compatibility*

$$\theta_j \circ \rho_{i,j} = \rho_{i+h,j+h} \circ \theta_i$$

*and such that* $\rho_{i,i+h} = p \cdot \theta_i$.

*Then we have an equivalence of groupoids*

$$\begin{aligned} \Big\{\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big)\Big\} &\to \Big\{\mathrm{GL}((\Xi_i)_i)\text{-}\textit{torsors over } A\Big\}, \\ \big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big) &\mapsto \mathcal{I}som\Big(\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big), \big((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big)\Big). \end{aligned}$$

*Proof.* This is proven by Rapoport and Zink in [RZ96, Theorem 3.11]. □

**Definition 4.1.3.** A *chain (of type $(h, J)$) over $R$* is a tuple

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big)$$

as in Theorem 4.1.2 for the ring $A = W(R)$. We similarly define the notions of *$n$-truncated chains over $R$* and *Dieudonné chains over $R$* by replacing $W(R)$ with $W_n(R)$ and $\widehat{W}(R)$ respectively (the latter only makes sense when $R$ is a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$).

**Remark 4.1.4.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ be a chain over $R$. Then we have an associated finite projective $W(R)[1/p]$-module $M[1/p]$ that is given by $M_i[1/p]$ for any choice of $i$ (for different choices of $i$ these are canonically identified via $(\rho_{i,j})_{i,j}$). This construction makes the diagram

$$\begin{array}{ccc} \big\{\text{chains over } R\big\} & \longrightarrow & \big\{\mathrm{GL}((\Xi_i)_i)\text{-torsors over } W(R)\big\} \\ \downarrow & & \downarrow \\ \big\{\text{finite projective } W(R)[1/p]\text{-modules}\big\} & \longrightarrow & \big\{\mathrm{GL}(V)\text{-torsors over } W(R)[1/p]\big\} \end{array}$$

commutative.

The same works for Dieudonné chains.

**Definition 4.1.5.** Let $R$ be an $\mathbf{F}_p$-algebra. A 1-rdt-*truncated chain (of type $(h, J)$) over $R$* is a tuple $((N_e)_{e \in \mathcal{E}}, (\theta_e)_{e \in \mathcal{E}})$ that is given as follows.

- $N_e$ is a finite projective $R$-module of rank $|e|$.
- $\theta_e \colon N_e \to N_{e+h}$ is an isomorphism.

We also define a restriction functor

$$\big\{\text{1-truncated chains over } R\big\} \to \big\{\text{1-rdt-truncated chains over } R\big\}$$

by sending a 1-truncated chain $(M, \rho, \theta)$ to the 1-rdt-truncated chain $(N, \theta)$, where $N_e \coloneqq \ker(\rho_{i,j}) \subseteq M_i$ and $\theta_e \colon N_e \to N_{e+h}$ is the isomorphism induced by $\theta_i$.

**Proposition 4.1.6.** *Let $R$ be an $\mathbf{F}_p$-algebra. Then we have an equivalence of groupoids*

$$\begin{aligned}\big\{1\text{-rdt-}\textit{truncated chains over } R\big\} &\to \big\{\mathrm{GL}((\Xi_e)_e)\text{-}\textit{torsors over } R\big\},\\ \big((N_e)_e, (\theta_e)_e\big) &\mapsto \mathcal{I}som\Big(\big((N_e)_e, (\theta_e)_e\big), \big((\Lambda_e)_e, (\theta_e)_e\big)\Big)\end{aligned}$$

*that makes the diagram*

$$\begin{tikzcd}\big\{1\text{-}\textit{truncated chains over } R\big\} \arrow[r] \arrow[d] & \big\{\mathrm{GL}((\Xi_i)_i)\text{-}\textit{torsors over } R\big\} \arrow[d] \\ \big\{1\text{-rdt-}\textit{truncated chains over } R\big\} \arrow[r] & \big\{\mathrm{GL}((\Xi_e)_e)\text{-}\textit{torsors over } R\big\}\end{tikzcd}$$

*commutative.*

*Proof.* This is immediate from the definitions. □

## 4.2 Chains of pairs

**Definition 4.2.1.** A *chain of pairs (of type $(h, J, d)$) over $R$* is a tuple

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i\big)$$

that is given as follows.

- $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ is a chain over $R$.
- $M_{i,1} \subseteq M_i$ is a $W(R)$-submodule such that $(M_i, M_{i,1})$ is a pair of type $(h, d)$, see Definition 2.1.1, and such that we have $\rho_{i,j}(M_{i,1}) \subseteq M_{j,1}$ and $\theta_i(M_{i,1}) = M_{i+h,1}$.

We similarly define the notions of *$m$-truncated chains of pairs* and *Dieudonné chains of pairs.*

**Remark 4.2.2.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ be a chain of pairs over $R$. Then

$$\rho_{i,j} \colon (M_i, M_{i,1}) \to (M_j, M_{j,1})$$

is a morphism of pairs and

$$\theta_i \colon (M_i, M_{i,1}) \to (M_{i+h}, M_{i+h,1})$$

is an isomorphism of pairs.

The same is true for $m$-truncated chains of pairs and Dieudonné chains of pairs.

**Proposition 4.2.3.** *We have natural equivalences of groupoids*

$$\big\{\textit{chains of pairs over } R\big\} \to \big\{(\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)\text{-}\textit{pairs over } R\big\},$$

$$\left\{\begin{matrix}m\text{-}\textit{truncated}\\ \textit{chains of pairs over } R\end{matrix}\right\} \to \left\{\begin{matrix}m\text{-}\textit{truncated}\\ (\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)\text{-}\textit{pairs over } R\end{matrix}\right\}$$

*and, when $R$ is a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$,*

$$\big\{\textit{Dieudonné chains of pairs over } R\big\} \to \big\{\textit{Dieudonné } (\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)\text{-}\textit{pairs over } R\big\}.$$

*Proof.* This is immediate from Theorem 4.1.2 and comparing Definition 4.2.1 with the description of the local model $\mathbb{M}^{\mathrm{loc}}_{\mathrm{GL}((\Xi_i)_i),\boldsymbol{\mu}_d}$ given in Theorem 1.4.9. ☐

## 4.3 Chains of displays

**Lemma 4.3.1.** *Suppose that we are given ring $A$ that is complete with respect to a descending sequence of ideals $(\mathfrak{b}_\lambda)_{\lambda\in\mathbf{Z}_{>0}}$ such that $A/\mathfrak{b}_{\lambda'} \to A/\mathfrak{b}_\lambda$ has nilpotent kernel for all $\lambda \leq \lambda'$ and such that $p \in \mathfrak{b} := \mathfrak{b}_1$. Let $M, M'$ be finite projective $A$-modules of rank $h$ and let $\rho\colon M \to M'$ and $\rho\colon M' \to M$ be morphisms of $A$-modules such that $\rho'\rho = p\cdot \mathrm{id}_M$ and $\rho\rho' = p \cdot \mathrm{id}_{M'}$ and such that the induced morphisms of $A/\mathfrak{b}$-modules*

$$\rho\colon M/\mathfrak{b}M \to M'/\mathfrak{b}M' \quad \textit{and} \quad \rho'\colon M'/\mathfrak{b}M' \to M/frakbM$$

*are of constant rank $\ell$ and $\ell'$ with $\ell + \ell' = h$.*

*Then also the induced morphisms of $A/pA$-modules*

$$\rho\colon M/pM \to M'/pM' \quad \textit{and} \quad \rho'\colon M'/pM' \to M/pM$$

*are of constant rank $\ell$ and $\ell'$.*

*Proof.* Modulo $\mathfrak{b}$ the morphisms $\rho$ and $\rho'$ compose to 0 in both directions and are of complementary constant rank. Thus there exist direct sum decompositions

$$M/\mathfrak{b}M = P \oplus P' \quad \text{and} \quad M'/\mathfrak{b}M' = P \oplus P'$$

with respect to which we have $\rho = \left(\begin{smallmatrix} \mathrm{id}_P & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{id}_{P'} \end{smallmatrix}\right)$. Now lift these decompositions to

$$M = Q \oplus Q' \quad \text{and} \quad M' = Q \oplus Q'.$$

After possibly changing the lifts we may then assume that

$$\rho = \begin{pmatrix} \mathrm{id}_Q & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad \rho' = \begin{pmatrix} * & * \\ * & \mathrm{id}_{Q'} \end{pmatrix}.$$

The assumptions $\rho'\rho = p \cdot \mathrm{id}_M$ and $\rho\rho' = p \cdot \mathrm{id}_{M'}$ then imply that we in fact have

$$\rho = \begin{pmatrix} \mathrm{id}_Q & 0 \\ 0 & p\cdot \mathrm{id}_{Q'} \end{pmatrix} \quad \text{and} \quad \rho' = \begin{pmatrix} p\cdot\mathrm{id}_Q & 0 \\ 0 & \mathrm{id}_{Q'} \end{pmatrix}$$

so that the claim follows. ☐

**Corollary 4.3.2.** *In the definition of chains (respectively $n$-truncated chains), see Definition 4.1.3, one can replace the condition that the morphism of $W(R)/pW(R)$-modules (respectively $W_n(R)/pW_n(R)$-modules)*

$$\rho_{i,j}\colon M_i/pM_i \to M_j/pM_j$$

*is of constant rank $h-(j-i)$ with the condition that the morphism of $R/pR$-modules*

$$\rho_{i,j}\colon R/pR\otimes_{W(R)}M_i\to R/pR\otimes_{W(R)}M_j$$
$$\big(\textit{respectively}\quad \rho_{i,j}\colon R/pR\otimes_{W_n(R)}M_i\to R/pR\otimes_{W_n(R)}M_j\big)$$

*is of constant rank $h-(j-i)$.*

*Similarly, in the definition of Dieudonné chains, see also Definition 4.1.3, one can replace the condition that the morphism of $\widehat{W}(R)/p\widehat{W}(R)$-modules*

$$\rho_{i,j}\colon \widehat{W}(R)/p\widehat{W}(R)\otimes_{\widehat{W}(R)}M_i\to \widehat{W}(R)/p\widehat{W}(R)\otimes_{\widehat{W}(R)}M_j$$

*is of constant rank $h-(j-i)$ with the condition that the morphism of $\overline{\mathbf{F}}_p$-vector spaces*

$$\rho_{i,j}\colon \overline{\mathbf{F}}_p\otimes_{\widehat{W}(R)}M_i\to \overline{\mathbf{F}}_p\otimes_{\widehat{W}(R)}M_j$$

*is of rank $h-(j-i)$.*

*Proof.* This follows directly from applying Lemma 4.3.1 with

$$A=W(R)=\varprojlim_{\lambda}W_\lambda(R/p^\lambda R)\quad\text{and}\quad A=\widehat{W}(R)=\varprojlim_{\lambda}\widehat{W}(R/\mathfrak{m}_R^\lambda)/p^\lambda\widehat{W}(R/\mathfrak{m}_R^\lambda)$$

respectively. □

**Proposition 4.3.3.** *Let $((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i)$ be a chain of pairs over $R$. Then the tuple*

$$\big((\widetilde{M_{i,1}})_i,(\widetilde{\rho_{i,j}})_{i,j},(\widetilde{\theta}_i)_i\big)$$

*is a chain over $R$ (for the definition of $\widetilde{M_{i,1}}$ we refer to Proposition 2.1.6).*

*The same is true for $m$-truncated chains of pairs and Dieudonné chains of pairs.*

*Proof.* We only need to check that the morphism of $W(R)/pW(R)$-modules

$$\widetilde{\rho_{i,j}}\colon \widetilde{M_{i,1}}/p\widetilde{M_{i,1}}\to\widetilde{M_{j,1}}/p\widetilde{M_{j,1}}$$

is of constant rank $h-(j-i)$ for all $i\le j\le i+h$. By Corollary 4.3.2 it now actually suffices to check that the morphism of $R/pR$-modules

$$\widetilde{\rho_{i,j}}\colon R/pR\otimes_{W(R)}\widetilde{M_{i,1}}\to R/pR\otimes_{W(R)}\widetilde{M_{j,1}}$$

is of constant rank $h-(j-i)$. Passing to the universal case we may assume that $R/pR$ is a reduced $\mathbf{F}_p$-algebra of finite type (see Theorem 1.4.6). Applying [Stacks, Tag 0FWG] we may then reduce to the case $R=\overline{\mathbf{F}}_p$. The morphism $\rho_{i,j}$ then gives rise to a commutative diagram

$$\begin{array}{ccc} M_{i,1} & \xrightarrow{\rho_{i,j}} & M_{j,1}\\ \downarrow & & \downarrow\\ M_i & \xrightarrow{\rho_{i,j}} & M_j\end{array}$$

of free $\breve{\mathbf{Z}}_p$-modules of rank $h$ where all the morphisms are injective. Moreover the cokernels of the vertical morphisms have length $h-d$ (over $\breve{\mathbf{Z}}_p$) and the cokernel of the lower vertical map has length $j-i$. Thus also the cokernel of the upper vertical morphism has length $j-i$. As the image of $M_{i,1} \to M_{j,1}$ contains $pM_{j,1}$ we can conclude that the morphism $\rho_{i,j}\colon M_{i,1}/pM_{i,1} \to M_{j,1}/pM_{j,1}$ of $\overline{\mathbf{F}}_p$-vector spaces is of rank $h-(j-i)$. Twisting by $\sigma$ then gives the result (see Remark 2.1.7).

The proof for $m$-truncated chains of pairs and Dieudonné chains of pairs works in the same way. $\square$

**Remark 4.3.4.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ be a chain of pairs over $R$ and denote by $\widetilde{M_1}[1/p]$ the $W(R)[1/p]$-module attached to the chain $((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta}_i)_i)$, see Remark 4.1.4. Then we have a natural isomorphism

$$\widetilde{M_1}[1/p] \to M^\sigma[1/p]$$

that is given by the isomorphism $\widetilde{M_{i,1}}[1/p] \to M_i^\sigma[1/p]$ for any $i$.

The same is true for Dieudonné chains of pairs.

**Definition 4.3.5.** A *chain of displays (of type $(h,J,d)$) over $R$* is a tuple

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i\big)$$

that is given as follows.

- $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ is a chain of pairs over $R$.
- $$(\Psi_i)_i\colon \big((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta}_i)_i\big) \to \big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big)$$
  is an isomorphism of chains over $R$.

We similarly define the notions of $(m,n)$*-truncated chains of displays* (where we allow $n = 1$-rdt) and *Dieudonné chains of displays* (see also Definition 2.1.9 and Definition 3.1.8).

**Remark 4.3.6.** For the convenience of the reader we spell out the definition of an $(m, 1\text{-rdt})$-truncated chain of displays over an $\mathbf{F}_p$-algebra $R$.

Such a $(m, 1\text{-rdt})$-truncated chain of displays over $R$ is a tuple

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_e)_e\big)$$

where

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i\big)$$

is an $m$-truncated chain of pairs over $R$ and

$$(\Psi_e)_e\colon \big((\widetilde{N_e})_e, (\widetilde{\theta_e})_e\big) \to \big((N_e)_e, (\theta_e)_e\big)$$

is an isomorphism of 1-rdt-truncated chains over $R$; here the tuples $((\widetilde{N_e})_e, (\widetilde{\theta_e})_e)$ and $((N_e)_e, (\theta_e)_e)$ are the 1-rdt-truncated chains associated to $((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta}_i)_i)$ and $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ respectively.

**Remark 4.3.7.** Let $\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i\big)$ be a chain of displays. Then $(M_i, M_{i,1}, \Psi_i)$ is a display of type $(h, d)$ and moreover

$$\rho_{i,j} \colon (M_i, M_{i,1}, \Psi_i) \to (M_j, M_{j,1}, \Psi_j)$$

is a morphism of displays and

$$\theta_i \colon (M_i, M_{i,1}, \Psi_i) \to (M_{i+h}, M_{i+h,1}, \Psi_{i+h})$$

is an isomorphism of displays. The same is true for $(m, n)$-truncated chains of displays (where we allow $n = 1$-rdt) and Dieudonné chains of displays.

**Proposition 4.3.8.** *The constructions*

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i\big) \mapsto \big((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta}_i)_i\big)$$

*from Proposition 4.3.3 and the group-theoretic ones from Proposition 3.1.4 are identified under the equivalences from Proposition 4.2.3 and Theorem 4.1.2.*

*Consequently we obtain natural equivalences*

$$\big\{\textit{chains of displays over } R\big\} \to \big\{(\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)\textit{-displays over } R\big\},$$

$$\left\{\begin{matrix} (m,n)\textit{-truncated} \\ \textit{chains of displays over } R \end{matrix}\right\} \to \left\{\begin{matrix} (m,n)\textit{-truncated} \\ (\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)\textit{-displays over } R \end{matrix}\right\}$$

*and, when $R$ is a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$,*

$$\left\{\begin{matrix} \textit{Dieudonné} \\ \textit{chains of displays over } R \end{matrix}\right\} \to \left\{\begin{matrix} \textit{Dieudonné} \\ (\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)\textit{-displays over } R \end{matrix}\right\}.$$

*Proof.* Set $J' := J \cap \{0, \dots, h-1\}$. We have a closed immersion of $\mathbf{Z}_p$-algebraic groups

$$\iota \colon \mathrm{GL}((\Xi_i)_i) \to \mathrm{GL}(\Lambda), \qquad \Lambda := \bigoplus_{i \in J'} \Lambda_i.$$

that satisfies the assumptions from Notation 3.0.1. With respect to $\iota$, the underlying module of the $\mathrm{GL}((\Xi_i)_i)$-torsor corresponding to a chain $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ is

$$\bigoplus_{i \in J'} M_i$$

and the underlying pair of the $(\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)$-pair corresponding to a chain of pairs $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ is

$$\left(\bigoplus_{i \in J'} M_i, \bigoplus_{i \in J'} M_{i,1}\right).$$

From this it is clear that the functor

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i\big) \mapsto \big((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta}_i)_i\big)$$

satisfies the properties characterizing the functor $(M, M_1) \mapsto \widetilde{M_1}$ for $(\mathrm{GL}((\Xi_i)_i), \boldsymbol{\mu}_d)$-pairs.

The claim about $m$-truncated chains of pairs follows from this and the claim for Dieudonné chains of pairs follows from the same argument. □

## 4.4 Homogeneously polarized chains

**Notation 4.4.1** (continued on p. 60)**.** We use Notation 1.4.10 and let $\mathcal{E}$, $\Xi_e$, $\Lambda_e$, $\theta_e$ be as in Notation 4.1.1.

Fix a free $\mathbf{Z}_p$-module $\Gamma$ of rank 1 and set $U := \Gamma[1/p]$. Let

$$\lambda\colon W \to U \otimes_{\mathbf{Q}_p} W^\vee$$

be an isomorphism of $\mathbf{Q}_p$-vector spaces that induces the $\mathbf{Z}_p^\times$-class $\overline{\psi}$ after trivializing $\Gamma$ in all possible ways and write

$$\lambda_i\colon \Lambda_i \to \Gamma \otimes_{\mathbf{Z}_p} \Lambda_{-i}^\vee$$

for the isomorphism of $\mathbf{Z}_p$-modules induced by $\lambda$. Note that the datum $(\lambda_i)_i$ carries the same information as the datum $(\overline{\psi_i})_i$.

Note that by Remark 3.0.2 the local model datum $(\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_g)$ satisfies the assumptions from Notation 3.0.1.

**Definition 4.4.2.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ be a chain (of type $(h, J, d) = (2g, J, g)$) over $R$. We define its *dual* as

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big)^\vee := \big((M_{-i}^\vee)_i, (\rho_{-j,-i}^\vee)_{i,j}, (\theta_{-i-h}^\vee)_i\big).$$

This endows the category of chains over $R$ with a duality in the sense of Definition 1.2.1. We similarly define duals of $n$-truncated chains and Dieudonné chains.

Let $R$ be an $\mathbf{F}_p$-algebra. Then we define the *dual* of a 1-rdt-truncated chain $((N_e)_e, (\theta_e)_e)$ over $R$ as

$$\big((N_e)_e, (\theta_e)_e\big)^\vee := \big((N_{-e}^\vee)_e, (\theta_{-e-h}^\vee)_e\big).$$

This again turns the category of 1-rdt-truncated chains over $R$ into a preadditive category with duality.

We also equip the restriction functor

$$\big\{\text{1-truncated chains over } R\big\} \to \big\{\text{1-rdt-truncated chains over } R\big\}$$

with a coherence datum for the respective dualities. For this, let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ be a 1-truncated chain over $R$. Let $((N_e)_e, (\theta_e)_e)$ be the associated 1-rdt-truncated chain and let $((N'_e)_e, (\theta'_e)_e)$ be the 1-rdt-truncated chain associated to $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)^\vee$. Then we want to construct a natural isomorphism

$$\big((N'_e)_e, (\theta'_e)_e\big) \cong \big((N_e)_e, (\theta_e)_e\big)^\vee.$$

To do this, we make the following observations (where $e = \{i, j\} \in \mathcal{E}$ with $i \le j$).

- Dualizing the exact sequence

  $$M_{-j} \xrightarrow{\rho_{-j,-i}} M_{-i} \to \operatorname{coker}(\rho_{-j,-i}) \to 0$$

  gives an isomorphism

  $$\ker(\rho^\vee_{-j,-i}) \cong \operatorname{coker}(\rho_{-j,-i})^\vee.$$

- We have an exact sequence

  $$M_{-j} \xrightarrow{\rho_{-j,-i}} M_{-i} \xrightarrow{\rho_{-i,-j+h}} M_{-j+h} \xrightarrow{\rho_{-j+h,-i+h}} M_{-i+h}$$

  so that $\rho_{-i,-j+h}$ gives rise to an isomorphism

  $$\operatorname{coker}(\rho_{-j,-i}) \cong \ker(\rho_{-j+h,-i+h}).$$

  Note that this uses that $pR = 0$.

- The isomorphisms $\theta_{-j}$ and $\theta_{-i}$ together induce an isomorphism

  $$\ker(\rho_{-j,-i}) \cong \ker(\rho_{-j+h,-i+h}).$$

Now we can finally define the components of the desired isomorphism as

$$N'_e = \ker(\rho^\vee_{-j,-i}) \cong \operatorname{coker}(\rho_{-j,-i})^\vee \cong \ker(\rho_{-j+h,-i+h})^\vee \cong \ker(\rho_{-j,-i})^\vee = N^\vee_{-e}.$$

**Definition 4.4.3.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ be a chain over $R$ and let $I$ be an invertible $W(R)$-module. Then we define the *twist*

$$I \otimes \big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big) := \big((I \otimes_{W(R)} M_i)_i, (\mathrm{id}_I \otimes \rho_{i,j})_{i,j}, (\mathrm{id}_I \otimes \theta_i)_i\big).$$

This endows the category of chains over $R$ with an action of the symmetric monoidal category of invertible $W(R)$-modules in the sense of Definition 1.2.4 and this action is compatible with the duality from Definition 4.4.2.

We similarly define twists of $n$-truncated chains (where we allow $n =$ 1-rdt) and Dieudonné chains.

**Definition 4.4.4.** A *homogeneously polarized chain (of type $(g, J)$) over $R$* is a tuple

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, I, (\lambda_i)_i\big)$$

that is given as follows.

- $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)$ is a chain over $R$.
- $I$ is an invertible $W(R)$-module.
- $(\lambda_i)_i \colon ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i) \to I \otimes ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i)^\vee$ is an antisymmetric isomorphism of chains over $R$ (see Definition 1.2.5 for the definition of "antisymmetric").

We similarly define the notions of *$n$-truncated homogeneously polarized chains* (where we allow $n = 1$-rdt) and *Dieudonné homogeneously polarized chains*.

**Remark 4.4.5.** Note that a homogeneously polarized chain over $R$ is the same thing as a homogeneously polarized object in the category of chains over $R$ in the sense of Definition 1.2.7. The same is true for $n$-truncated homogeneously polarized chains (where we allow $n = 1$-rdt) and Dieudonné homogeneously polarized chains.

**Notation 4.4.1** (continuing from p. 58)**.** We denote by

$$(\lambda_e)_e \colon \big((\Lambda_e)_e, (\theta_e)_e\big) \to \Gamma/p\Gamma \otimes \big((\Lambda_e)_e, (\theta_e)_e\big)^\vee$$

the isomorphism of 1-rdt-truncated chains over $\mathbf{F}_p$ induced by the isomorphism

$$(\lambda_i)_i \colon \big((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big) \to \Gamma \otimes \big((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i\big)^\vee.$$

The reductive quotient $\mathrm{GSp}((\Xi_i)_i)^{\mathrm{rdt}}_{\mathbf{F}_p}$ of the special fiber of $\mathrm{GSp}((\Xi)_i)$ identifies with $\mathrm{GSp}((\Xi_e)_e)$, the automorphism group of the tuple $((\Lambda_e)_e, (\theta_e)_e, \Gamma/p\Gamma, (\lambda_e)_e)$.

**Remark 4.4.6.** The group $\mathrm{GSp}((\Xi_e)_e)$ has the following explicit description.

- Suppose that $0, g \in J$. Then we have
$$\mathrm{GSp}((\Xi_e)_e) = \mathbf{G}_{m,\mathbf{F}_p} \times \prod_{i \in J/(h\mathbf{Z}, \pm)} \mathrm{GL}(\Xi_e).$$
- Suppose that $0 \notin J$ and $g \in J$ and write $e_0 \in \mathcal{E}$ for the edge $\{i, j\}$ with $i < 0 < j$. Then we have
$$\mathrm{GSp}((\Xi_e)_e) = \mathrm{GSp}(\Xi_{e_0}) \times \prod_{i \in (J/(h\mathbf{Z}, \pm)) \setminus \{e_0\}} \mathrm{GL}(\Xi_e).$$
- Suppose that $0 \in J$ and $g \notin J$ and write $e_g \in \mathcal{E}$ for the edge $\{i, j\}$ with $i < g < j$. Then we similarly have
$$\mathrm{GSp}((\Xi_e)_e) = \mathrm{GSp}(\Xi_{e_g}) \times \prod_{i \in (J/(h\mathbf{Z}, \pm)) \setminus \{e_g\}} \mathrm{GL}(\Xi_e).$$

- Suppose that $0, g \notin J$ and write $e_0, e_g \in \mathcal{E}$ as before. Then we have
$$\mathrm{GSp}((\Xi_e)_e) = \mathrm{GSp}(\Xi_{e_0}, \Xi_{e_g}) \times \prod_{i \in (J/(h\mathbf{Z}, \pm)) \setminus \{e_0, e_g\}} \mathrm{GL}(\Xi_e);$$
here $\mathrm{GSp}(\Xi_{e_0}, \Xi_{e_g}) = \mathrm{GSp}(\Xi_{e_0}) \times_{\mathbf{G}_{m,\mathbf{F}_p}} \mathrm{GSp}(\Xi_{e_g})$ denotes the group of tuples of symplectic similitudes of $\Xi_{e_0}$ and $\Xi_{e_g}$ with the same similitude factor.

**Theorem 4.4.7.** *We have an equivalence of groupoids*
$$\left\{\begin{matrix} 1\text{-}\textit{truncated} \\ \textit{homogeneously polarized chains over } R \end{matrix}\right\} \to \left\{\mathrm{GSp}((\Xi_i)_i)\text{-}\textit{torsors over } R\right\}$$
*that is given by mapping a* 1*-truncated homogeneously polarized chain*
$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, I, (\lambda_i)_i\big)$$
*over $R$ to*
$$\mathcal{I}som\Big(\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, I, (\lambda_i)_i\big), \big((\Lambda_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, \Gamma, (\lambda_i)_i\big)\Big).$$
*Let $R$ be an $\mathbf{F}_p$-algebra. Then we have an equivalence of groupoids*
$$\left\{\begin{matrix} 1\text{-rdt-}\textit{truncated} \\ \textit{homogeneously polarized chains over } R \end{matrix}\right\} \to \left\{\mathrm{GSp}((\Xi_e)_e)\text{-}\textit{torsors over } R\right\}$$
*that is given by mapping a* 1-rdt-*truncated homogeneously polarized chain*
$$\big((N_e)_e, (\theta_e)_e, I, (\lambda_e)_e\big)$$
*over $R$ to*
$$\mathcal{I}som\Big(\big((N_e)_e, (\theta_e)_e, I, (\lambda_e)_e\big), \big((\Lambda_e)_e, (\theta_e)_e, \Gamma/p\Gamma, (\lambda_e)_e\big)\Big)$$
*and makes the diagram*
$$\begin{array}{ccc} \left\{\begin{matrix} 1\text{-}\textit{truncated} \\ \textit{homogeneously polarized chains over } R \end{matrix}\right\} & \longrightarrow & \left\{\mathrm{GSp}((\Xi_i)_i)\text{-}\textit{torsors over } R\right\} \\ \downarrow & & \downarrow \\ \left\{\begin{matrix} 1\text{-rdt-}\textit{truncated} \\ \textit{homogeneously polarized chains over } R \end{matrix}\right\} & \longrightarrow & \left\{\mathrm{GSp}((\Xi_e)_e)\text{-}\textit{torsors over } R\right\} \end{array}$$
*commutative.*

*Proof.* The first part is proven by Rapoport and Zink in [RZ96, Theorem 3.16]. The second part is immediate from the definitions. ☐

**Remark 4.4.8.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, I, (\lambda_i)_i)$ be a homogeneously polarized chain over $R$. Then the associated finite projective $W(R)[1/p]$-module $M[1/p]$ from Remark 4.1.4 is naturally equipped with an antisymmetric isomorphism

$$\lambda\colon M[1/p] \to I[1/p] \otimes_{W(R)[1/p]} M[1/p]^\vee$$

so that $(M[1/p], I[1/p], \lambda)$ is a homogeneously polarized finite projective $W(R)[1/p]$-module. This construction makes the diagram

$$\begin{array}{ccc} \left\{\begin{matrix}\text{homogeneously polarized}\\ \text{chains over } R\end{matrix}\right\} & \longrightarrow & \left\{\mathrm{GSp}((\Xi_i)_i)\text{-torsors over } W(R)\right\} \\ \downarrow & & \downarrow \\ \left\{\begin{matrix}\text{homogeneously polarized}\\ \text{finite projective } W(R)[1/p]\text{-modules}\end{matrix}\right\} & \longrightarrow & \left\{\mathrm{GSp}(V)\text{-torsors over } W(R)[1/p]\right\} \end{array}$$

commutative.

The same works for Dieudonné chains.

## 4.5 Homogeneously polarized chains of pairs

**Definition 4.5.1.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ be a chain of pairs over $R$ and let $I$ be an invertible $W(R)$-module.

- We define the *dual of* $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ as

  $$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i\big)^\vee \coloneqq \big((M_{-i}^\vee)_i, (\rho_{-j,-i}^\vee)_{i,j}, (\theta_{-i-h}^\vee)_i, (M_{-i,1}^*)_i\big)$$

  where we refer to Definition 2.2.1 for the notation $M_{-i,1}^*$.

- We define the *twist*

  $$\begin{aligned} &I \otimes \big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i\big) \\ \coloneqq &\big((I \otimes_{W(R)} M_i)_i, (\mathrm{id}_I \otimes \rho_{i,j})_{i,j}, (\mathrm{id}_I \otimes \theta_i)_i, (I \otimes_{W(R)} M_{i,1})_i\big), \end{aligned}$$

  see also Definition 2.2.5.

This compatibly endows the category of chains of pairs over $R$ with a duality and an action of the symmetric monoidal category of invertible $W(R)$-modules. We similarly define duals and twists of $m$-truncated chains of pairs and Dieudonné chains of pairs.

**Definition 4.5.2.** A *homogeneously polarized chain of pairs over* $R$ is a tuple

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, I, (\lambda_i)_i\big)$$

that is given as follows.

- $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)$ is a chain of pairs over $R$.
- $I$ is an invertible $W(R)$-module.
- $(\lambda_i)_i \colon ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i) \to I \otimes ((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i)^\vee$ is an antisymmetric isomorphism of chains of pairs over $R$.

We similarly define the notions of *$m$-truncated homogeneously polarized chains of pairs* and *Dieudonné homogeneously polarized chains of pairs*.

**Remark 4.5.3.** Note that a homogeneously polarized chain of pairs over $R$ is the same thing as a homogeneously polarized object in the category of chains of pairs over $R$. The same is true for $m$-truncated homogeneously polarized chains of pairs and Dieudonné homogeneously polarized chains of pairs.

**Remark 4.5.4.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, I, (\lambda_i)_i)$ be a homogeneously polarized chain of pairs over $R$. Then

$$\lambda_i \colon (M_i, M_{i,1}) \to I \otimes (M_{-i}, M_{-i,1})^\vee$$

is an isomorphism of pairs over $R$ and the diagram

$$\begin{tikzcd}
I \otimes \left(I \otimes (M_{-i}, M_{-i,1})^\vee\right)^\vee \arrow[d] \arrow[dr, "\mathrm{id}_I \otimes \lambda_i^\vee"] & \\
(M_{-i}, M_{-i,1}) \arrow[r, "\lambda_{-i}"'] & I \otimes (M_i, M_{i,1})^\vee
\end{tikzcd}$$

is commutative.

The same is true for $m$-truncated homogeneously polarized chains of pairs and Dieudonné homogeneously polarized chains of pairs.

**Proposition 4.5.5.** *We have natural equivalences*

$$\left\{\begin{matrix}\textit{homogeneously polarized}\\ \textit{chains of pairs over } R\end{matrix}\right\} \to \left\{(\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_g)\textit{-pairs over } R\right\},$$

$$\left\{\begin{matrix}m\textit{-truncated homogeneously polarized}\\ \textit{chains of pairs over } R\end{matrix}\right\} \to \left\{\begin{matrix}m\textit{-truncated}\\ (\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_g)\textit{-pairs over } R\end{matrix}\right\}$$

*and, when $R$ is a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$,*

$$\left\{\begin{matrix}\textit{Dieudonné homogeneously polarized}\\ \textit{chains of pairs over } R\end{matrix}\right\} \to \left\{\begin{matrix}\textit{Dieudonné}\\ (\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_g)\textit{-pairs over } R\end{matrix}\right\}.$$

*Proof.* This is immediate from Theorem 4.4.7 and comparing Definition 4.5.2 with the description of the local model $\mathbb{M}^{\mathrm{loc}}_{\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_g}$ given in Theorem 1.4.11. ☐

## 4.6 Homogeneously polarized chains of displays

**Proposition 4.6.1.** *Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, I, (\lambda_i)_i)$ be a homogeneously polarized chain of pairs over $R$. Then the tuple*

$$\big((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta_i})_i, I^\sigma, (\widetilde{\lambda_i})_i\big),$$

*where $\widetilde{\lambda_i}$ really denotes the composition*

$$\widetilde{M_{i,1}} \xrightarrow{\widetilde{\lambda_i}} (I \otimes_{W(R)} M^*_{-i,1})^\sim \to I^\sigma \otimes_{W(R)} \widetilde{M^*_{-i,1}} \to I^\sigma \otimes_{W(R)} \widetilde{M_{-i,1}}^\vee,$$

*the second and third isomorphism being the ones from Lemma 2.2.6 and Lemma 2.2.3, is a homogeneously polarized chain over $R$.*

*The same is true for $m$-truncated homogeneously polarized chains of pairs and Dieudonné homogeneously polarized chains of pairs.*

*Proof.* This is immediate from Proposition 4.3.3 and the definitions. □

**Remark 4.6.2.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, I, (\lambda_i)_i)$ be a homogeneously polarized chain of pairs over $R$. Denote by $\widetilde{M_1}[1/p]$ the $W(R)[1/p]$-module attached to the chain $((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta_i})_i)$ as in Remark 4.3.4 and denote by

$$\lambda\colon \widetilde{M_1}[1/p] \to I^\sigma[1/p] \otimes_{W(R)[1/p]} \widetilde{M_1}[1/p]^\vee$$

the antisymmetric isomorphism from Remark 4.4.8. Then the natural isomorphism $\widetilde{M_1}[1/p] \to M^\sigma[1/p]$ defines an isomorphism of homogeneously polarized finite projective $W(R)[1/p]$-modules

$$\big(\widetilde{M_1}[1/p], I^\sigma[1/p], \widetilde{\lambda}\big) \to \big(M^\sigma[1/p], I^\sigma[1/p], \lambda^\sigma\big).$$

The same is true for Dieudonné chains of pairs.

**Definition 4.6.3.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i)$ be a chain of displays over $R$ and let $(I, \iota)$ be a tuple consisting of an invertible $W(R)$-module $I$ and an isomorphism $\iota\colon I^\sigma \to I$.

- We define the *dual of* $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i)$ as

$$\begin{aligned} &\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i\big)^\vee \\ &:= \big((M^\vee_{-i})_i, (\rho^\vee_{-j,-i})_{i,j}, (\theta^\vee_{-i-h})_i, (M^*_{-i,1})_i, (\Psi^{\vee,-1}_{-i})_i\big) \end{aligned}$$

  where $\Psi^{\vee,-1}_{-i}$ really denotes the composition

$$\widetilde{M^*_{-i,1}} \to \widetilde{M_{-i,1}}^\vee \xrightarrow{\Psi^{\vee,-1}_{-i}} M^\vee_{-i}.$$

- We define the *twist*

$$
\begin{aligned}
&(I,\iota)\otimes\big((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,(\Psi_i)_i\big)\\
\coloneqq\ &\big((I\otimes_{W(R)}M_i)_i,(\mathrm{id}_i\otimes\rho_{i,j})_{i,j},(\mathrm{id}_I\otimes\theta_i)_i,(I\otimes_{W(R)}M_{i,1})_i,\mathrm{id}_{I^\sigma}\otimes\Psi_i\big)
\end{aligned}
$$

  where $\mathrm{id}_{I^\sigma}\otimes\Psi_i$ really denotes the composition

$$(I\otimes_{W(R)}M_{i,1})^{\sim}\to I^\sigma\otimes_{W(R)}\widetilde{M_{i,1}}\xrightarrow{\mathrm{id}_{I^\sigma}\otimes\Psi_i}I\otimes_{W(R)}M.$$

This compatibly endows the category of chains of displays over $R$ with a duality and an action of the symmetric monoidal category of tuples $(I,\iota)$ as above. We similarly define duals and twists of $(m,n)$-truncated chains of displays and Dieudonné chains of displays, see also Definition 2.2.4 and Definition 2.2.7.

**Definition 4.6.4.** A *homogeneously polarized chain of displays over* $R$ is a tuple

$$\big((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,(\Psi_i)_i,I,\iota,(\lambda_i)_i\big)$$

that is given as follows.

- $((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,I,(\lambda_i)_i)$ is a homogeneously polarized chain of pairs over $R$.
- $$((\Psi_i)_i,\iota)\colon((\widetilde{M_{i,1}})_i,(\widetilde{\rho_{i,j}})_{i,j},(\widetilde{\theta}_i)_i,I^\sigma,(\widetilde{\lambda}_i)_i)\to((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,I,(\lambda_i)_i)$$

  is an isomorphism of of homogeneously polarized chains over $R$.

We similarly define the notions of $(m,n)$-*truncated homogeneously polarized chains of displays* (where we allow $n=$ 1-rdt) and *Dieudonné homogeneously polarized chains of displays*.

**Remark 4.6.5.** A homogeneously polarized chain of displays over $R$ is the same as a tuple

$$\big((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,(\Psi_i)_i,I,\iota,(\lambda_i)_i\big)$$

that is given as follows.

- $((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,(\Psi_i)_i)$ is a chain of displays over $R$.
- $(I,\iota)$ is a tuple consisting of a finite projective $W(R)$-module $I$ and an isomorphism $\iota\colon I^\sigma\to I$.
- $$
\begin{aligned}
&(\lambda_i)_i\colon\big((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,(\Psi_i)_i\big)\\
&\to(I,\iota)\otimes\big((M_i)_i,(\rho_{i,j})_{i,j},(\theta_i)_i,(M_{i,1})_i,(\Psi_i)_i\big)^\vee
\end{aligned}
$$

  is an antisymmetric isomorphism of chains of displays over $R$.

Thus a homogeneously polarized chain of displays over $R$ is the same thing as a homogeneously polarized object in the category of chains of displays over $R$.

The same is true for $(m, n)$-truncated homogeneously polarized chains of displays (where we allow $n = 1\text{-rdt}$) and Dieudonné homogeneously polarized chains of displays.

**Remark 4.6.6.** Let $((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i, I, \iota, (\lambda_i)_i)$ be a homogeneously polarized chain of displays. Then

$$\lambda_i\colon (M_i, M_{i,1}, \Psi_i) \to (I, \iota) \otimes (M_{-i}, M{-i}, 1, \Psi_{-i})^\vee$$

is an isomorphism of displays. The same is true for $(m, n)$-truncated homogeneously polarized chains of displays (where we allow $n = 1\text{-rdt}$) and Dieudonné homogeneously polarized chains of displays.

**Proposition 4.6.7.** *The constructions*

$$\big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, I, (\lambda_i)_i\big) \mapsto \big((\widetilde{M_{i,1}})_i, (\widetilde{\rho_{i,j}})_{i,j}, (\widetilde{\theta}_i)_i, I^\sigma, (\widetilde{\lambda}_i)_i\big)$$

*from Proposition 4.6.1 and the group-theoretic ones from Proposition 3.1.4 are identified under the equivalences from Proposition 4.5.5 and Theorem 4.4.7.*

*Consequently we obtain natural equivalences*

$$\left\{\begin{matrix}\textit{homogeneously polarized}\\ \textit{chains of displays over } R\end{matrix}\right\} \to \left\{(\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_d)\textit{-displays over } R\right\},$$

$$\left\{\begin{matrix}(m,n)\textit{-truncated homogeneously polarized}\\ \textit{chains of displays over } R\end{matrix}\right\} \to \left\{\begin{matrix}(m,n)\textit{-truncated}\\ (\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_d)\textit{-displays over } R\end{matrix}\right\}$$

*and, when $R$ is a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$,*

$$\left\{\begin{matrix}\textit{Dieudonné homogeneously polarized}\\ \textit{chains of displays over } R\end{matrix}\right\} \to \left\{\begin{matrix}\textit{Dieudonné}\\ (\mathrm{GSp}((\Xi_i)_i), \boldsymbol{\mu}_d)\textit{-displays over } R\end{matrix}\right\}.$$

*Proof.* This is completely analogous to Proposition 4.3.8. □

## 4.7 (Homogeneously polarized) chains of $p$-divisible groups

In the first half of this section we use Notation 4.1.1.

**Definition 4.7.1.** A *chain of $p$-divisible groups (of type $(h, J, d)$) over $R$* is a diagram

$$\big((X_i)_i, (\rho_{i,j})_{i,j}\big)$$

of shape $J^{\mathrm{op}}$ of $p$-divisible groups of height $h$ and dimension $d$ over $R$ such that $\rho_{i,j}\colon X_j \to X_i$ is an isogeny of height $j - i$ and $\ker(\rho_{i-h,i}) = X_i[p]$.

**Proposition 4.7.2.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. Then we have a natural equivalence*

$$\{\textit{chains of } p\textit{-divisible groups over } R\}^{\mathrm{op}} \to \{\textit{Dieudonné chains of displays over } R\}.$$

*Proof.* This follows from Theorem 2.5.1 and Corollary 4.3.2 and the fact that a morphism of $p$-divisible groups is an isogeny of some height $r \in \mathbf{Z}_{\geq 0}$ if and only if the corresponding morphism of contravariant Dieudonné modules is injective with cokernel of finite length $r$ over $\breve{\mathbf{Z}}_p$. □

Now we switch back to using Notation 4.4.1.

**Definition 4.7.3.** A *homogeneously polarized chain of $p$-divisible groups (of type $(g, J)$) over $R$* is a tuple

$$\Big((X_i)_i, (\rho_{i,j})_{i,j}, \overline{(\lambda_i)_i}\Big)$$

that is given as follows.

- $((X_i)_i, (\rho_{i,j})_{i,j})$ is a chain of $p$-divisible groups of type $(2g, J, g)$.
- $\overline{(\lambda_i)_i}$ is a $\mathbf{Z}_p^\times$-class of isomorphisms

$$(\lambda_i)_i \colon \Big(\big(X_i\big)_i, \big(\rho_{i,j}\big)_{i,j}\Big) \to \Big(\big(X_{-i}^\vee\big)_i, \big(\rho_{-j,-i}^\vee\big)_{i,j}\Big)$$

  that are antisymmetric (i.e. $\lambda_i^\vee = -\lambda_{-i}$).

**Lemma 4.7.4.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. Consider the functor*

$$\begin{aligned}\{\textit{invertible } \mathbf{Z}_p\textit{-modules}\} &\to \left\{(I, \iota) \;\middle|\; \begin{array}{c} I \textit{ an invertible } \widehat{W}(R)\textit{-module,} \\ \iota \colon I^\sigma \to I \textit{ an isomorphism}\end{array}\right\}, \\ I_0 &\mapsto \big(\widehat{W}(R) \otimes_{\mathbf{Z}_p} I_0, \iota^{\mathrm{can}}\big)\end{aligned}$$

*where $\iota^{\mathrm{can}}$ denotes the canonical isomorphism*

$$\iota^{\mathrm{can}} \colon \big(\widehat{W}(R) \otimes_{\mathbf{Z}_p} I_0\big)^\sigma \to \widehat{W}(R) \otimes_{\mathbf{Z}_p} I_0, \qquad a' \otimes (a \otimes x) \mapsto (a'\sigma(a)) \otimes x.$$

*This functor is an equivalence.*

*Proof.* The fully faithfulness follows directly from Lemma 1.1.6 while the essential surjectivity follows from Lemma 1.1.7. □

**Proposition 4.7.5.** *Let $R$ be a complete Noetherian local ring with residue field $\overline{\mathbf{F}}_p$. Then we have a natural equivalence*

$$\left\{\begin{array}{c}\textit{homogeneously polarized} \\ \textit{chains of } p\textit{-divisible groups over } R\end{array}\right\}^{\mathrm{op}} \to \left\{\begin{array}{c}\textit{Dieudonné homogeneously polarized} \\ \textit{chains of displays over } R\end{array}\right\}.$$

*Proof.* This follows from Proposition 4.7.2 and Lemma 4.7.4. □

## 4.8 Application to Siegel modular varieties

**Notation 4.8.1.** We use Notation 1.8.16 and Notation 4.4.1. Let $\mathbf{L}^p$ be as in Theorem 1.8.17 so that we have the associated integral Siegel modular variety $\mathscr{S}_{\mathbf{L}}$. Write $\widehat{\mathscr{S}_{\mathbf{L}}}$ for the $p$-completion of $\mathscr{S}_{\mathbf{L}}$.

**Construction 4.8.2.** Let

$$\Big(\big(A_i\big)_i, (\rho_{i,j})_{i,j}, \overline{(\lambda_i)_i}\Big)$$

be the universal object over $\mathscr{S}_{\mathbf{L}}$ coming from its moduli description, see Theorem 1.8.17, and let

$$\Big((X_i)_i, (\rho_{i,j})_{i,j}, \overline{(\lambda_i)_i}\Big)$$

be the associated homogeneously polarized chain of $p$-divisible groups. Applying Theorem 2.5.1 we obtain the following data.

- A homogeneously polarized chain of displays

$$M = \big((M_i)_i, (\rho_{i,j})_{i,j}, (\theta_i)_i, (M_{i,1})_i, (\Psi_i)_i, I, \iota, (\lambda_i)_i\big)$$

  over $\widehat{\mathscr{S}_{\mathbf{L}}}$; here we argue as in the proofs of Proposition 4.3.3 and Proposition 4.7.2 to verify the rank condition for

$$\rho_{i,j} \colon \big(\mathcal{O}_{\widehat{\mathscr{S}_{\mathbf{L}}}}/p\mathcal{O}_{\widehat{\mathscr{S}_{\mathbf{L}}}}\big) \otimes_{W(\mathcal{O}_{\widehat{\mathscr{S}_{\mathbf{L}}}})} M_i \to \big(\mathcal{O}_{\widehat{\mathscr{S}_{\mathbf{L}}}}/p\mathcal{O}_{\widehat{\mathscr{S}_{\mathbf{L}}}}\big) \otimes_{W(\mathcal{O}_{\widehat{\mathscr{S}_{\mathbf{L}}}})} M_j,$$

  see Corollary 4.3.2.

- For every point $x \in \mathscr{S}_{\mathbf{L}}(\overline{\mathbf{F}}_p)$ a Dieudonné homogeneously polarized chain of displays

$$M_x = \big((M_{x,i})_i, (\rho_{x,i,j})_{i,j}, (\theta_{x,i})_i, (M_{x,i,1})_i, (\Psi_{x,i})_i, I_x, \iota_x, (\lambda_{x,i})_i\big)$$

  over the completed local ring of $\mathscr{S}_{\mathbf{L},\breve{\mathbf{Z}}_p}$ at $x$. These are compatible with $M$ in the obvious sense.

**Theorem 4.8.3.** *For $x \in \mathscr{S}_{\mathbf{L}}(\overline{\mathbf{F}}_p)$ the Dieudonné homogeneously polarized chain of displays $M_x$ is a universal deformation of its base change to $\overline{\mathbf{F}}_p$.*

*Consequently the morphism*

$$\widehat{\mathscr{S}_{\mathbf{L}}} \to \mathsf{Disp}^{(m,n)}_{\mathrm{GSp}((\Xi_i)_i),\boldsymbol{\mu}_g} \qquad \textit{(respectively } \mathscr{S}_{\mathbf{L},\mathbf{F}_p} \to \mathsf{Disp}^{(m,1\text{-rdt})}_{\mathrm{GSp}((\Xi_i)_i),\boldsymbol{\mu}_g} \textit{ for } n = 1\text{-rdt)}$$

*induced by $M$ is smooth.*

*Proof.* The first part follows from Proposition 4.7.5 and the Serre-Tate theorem, see for example [Mes72, Theorem V.2.3]. The second part then follows as in the proof of Theorem 3.5.4. ☐

**Remark 4.8.4.** For a suitable Hodge embedding the data $M$ and $M_x$ from Construction 4.8.2 agree with the group-theoretic ones from Theorem 3.5.3.

Thus Theorem 4.8.3 is a special case of Theorem 3.5.4. Note however that the result in the Hodge type case relies on Hypothesis 3.3.3 while we did not need any such hypothesis here.

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