The quadratic Euler characteristic of a smooth projective same-degree complete intersection and motivic Donaldson-Thomas invariants of $\mathbb{P}^3$.

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Introduction

This thesis consists of two chapters. The first one is about computing the quadratic Euler characteristic of a smooth projective complete intersection of hypersurfaces of the same degree and the second one is about computing motivic Donaldson-Thomas invariants for $\mathbb{P}^3$.

The quadratic Euler characteristic of a smooth projective complete intersection of hypersurfaces of the same degree

The quadratic Euler characteristic of a smooth projective scheme over a perfect field of characteristic not equal to 2 is a refined or motivic analogue of the usual topological Euler characteristic, and of Euler characteristics defined using étale or De Rham cohomology. It comes from a very general definition of a categorical Euler characteristic associated to a dualizable object of a symmetric monoidal category, living in the endomorphism ring of the unit. Motivic homotopy theory, introduced by Morel and Voevodsky, constructs the stable motivic homotopy category $\mathrm{SH}(k)$ of a field k; a symmetric monoidal category in which a smooth projective scheme over k has an image which is dualizable. A deep theorem by Morel (see [42, Theorem 6.4.1]) states that if k is perfect and not of characteristic 2, then the endomorphism ring of the unit in $\mathrm{SH}(k)$ is isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(k)$ of k. This is the group completion of the ring of all isometry classes of nondegenerate quadratic forms over k. Therefore, we obtain the *quadratic Euler characteristic* $\chi(X/k) \in \mathrm{GW}(k)$ of any smooth projective scheme X over k, which is a (virtual) quadratic form.
These quadratic Euler characteristics carry a lot of information within them: if $k \subset \mathbb{R}$ then the rank of $\chi(X/k)$ is equal to the topological Euler characteristic of the $\mathbb{C}$-points $X(\mathbb{C})$, while the signature of $\chi(X/k)$ is the topological Euler characteristic of the real points $X(\mathbb{R})$. Quadratic Euler characteristics are often used in the fast-growing field of refined enumerative geometry, which aims to obtain "quadratic enrichments" of results in classical enumerative geometry. However, they are in general hard to compute.
The motivic Gauss-Bonnet Theorem (see [35]) proven by Levine and Raksit gives a rather explicit way to compute quadratic Euler characteristics. Namely, consider a smooth projective scheme X over a perfect field k which is not of characteristic 2 as before. For $a \in k^*$, let $\langle a \rangle$ be the quadratic form $x \mapsto ax^2 \in \mathrm{GW}(k)$. Then we can compute $\chi(X/k) \in \mathrm{GW}(k)$ as follows:

- If $\dim(X)$ is odd, then $\chi(X/k) = C \cdot H$ for some $C \in \mathbb{Z}$, where H is the hyperbolic form $\langle 1 \rangle + \langle -1 \rangle$.
- If $\dim(X) = 2n$ is even, then $\chi(X/k) = C \cdot H + Q$ for some $C \in \mathbb{Z}$, where Q is the quadratic form given by the composition

$$H^n(X, \Omega_X^n) \times H^n(X, \Omega_X^n) \xrightarrow{\cup} H^{2n}(X, \Omega_X^{2n}) \xrightarrow{\text{Trace}} k.$$

  Here, Ω_X denotes the sheaf of differential forms on X, the first map is the cup product on cohomology and we write $\Omega_X^q = \wedge^q \Omega_X$.

The constant C can be computed in practice. Therefore, one can compute the quadratic Euler characteristic of a smooth projective scheme if one understands the form Q in the even dimensional case. This form has been computed successfully in the case of hypersurfaces by Levine, Lehalleur and Srinivas in [33]. Given a smooth projective hypersurface $X = V(F) \subset \mathbb{P}^n$, the authors use inspiration from the paper [11] by Carlson and Griffiths to describe an isomorphism from the primitive cohomology of Ω^q_X to certain graded pieces of the Jacobian ring

$$J_X = k[X_0, \cdots, X_n]/\left(\frac{\partial F}{\partial X_0}, \cdots, \frac{\partial F}{\partial X_n}\right).$$

This is then applied to compare the cup product on cohomology with the usual ring multiplication of J_X. In doing this, the authors represent the result on an open cover of X and compare this with a representation of $c_1(\mathcal{O}(m))^n$ to compute the trace, which then allows them to compute the form Q explicitly.
The purpose of this paper is to show a similar result for smooth projective complete intersections of hypersurfaces which are of the same degree. For this, some work in the style of [11] has already been done by Konno in the paper [27] and Terasoma in the paper [50]. Using inspiration from those papers, given a smooth projective complete intersection $X = V(F_0, \cdots, F_r) \subset \mathbb{P}^n$ where the F_i are of the same degree $m > 1$ and $n \geq r+2$, we consider the hypersurface

$$\mathcal{X} = V(F) \subset \mathbb{P}^r \times \mathbb{P}^n$$

where $F = Y_0F_0 + \cdots + Y_rF_r$ and show that one can compute $\chi(X/k)$ from $\chi(\mathcal{X}/k)$. After this, we study isomorphisms from the primitive cohomology groups of $\Omega^q_{\mathcal{X}}$ to graded parts of the Jacobian ring

$$J = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]/\left(F_0, \cdots, F_r, \frac{\partial F}{\partial X_0}, \cdots, \frac{\partial F}{\partial X_n}\right).$$

Unlike the Jacobian ring for the hypersurface, this Jacobian ring is infinite dimensional over k. One can link the cup product on cohomology to the usual product in the Jacobian ring, namely, following [50], we show the following result.

Proposition (See Corollary 1.4.10)**.** *Consider the bidegree*

$$\rho = (n-r-1, (n+r+1)m - 2(n+1)).$$

There is a surjective homomorphism $\phi : J^\rho \to H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}) \cong k$*, such that the diagram*

$$\begin{array}{ccc} H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim} \otimes H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})_{prim} & \xrightarrow{i_* \circ \cup} & H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}) \\ \uparrow & & \uparrow \phi \\ J^{q-r,(q+1)m-(n+1)} \otimes J^{p-r,(p+1)m-(n+1)} & \longrightarrow & J^\rho \end{array}$$

commutes.

A slight generalization of an argument from [27] shows that the map ϕ is in fact an isomorphism unless if $\mathcal{X}$ is odd dimensional, $r = 1$ and $m = 2$. The one exception will not matter for our purposes, because we know from the Motivic Gauss-Bonnet Theorem that the quadratic Euler characteristic of $\mathcal{X}$ is hyperbolic in this case. Furthermore, we will study a slight variant of the Jacobian ring, given by

$$\tilde{J} = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]/(Y_0F_0, \cdots, Y_rF_r, X_0\bar{F}_0, \cdots, X_n\bar{F}_n)$$

and show that $\tilde{J}^{\rho+(r+1,n+1)}$ is one dimensional.
If we make some extra assumptions, we can compute the trace map. Assume that $m+1$ is invertible in k, that $V(F_i)$ is smooth for all $i \in \{0, \cdots, r\}$ and that $V(F_0, \cdots, F_r)$ is smooth and of codimension $r+1$. Assume moreover that these assumptions remain true after setting any subset of the X_i equal to zero. For $A \in J^{q-r,(q+1)m-(n+1)}$ and $B \in J^{p-r,(p+1)m-(n+1)}$, write $\omega_A \in H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim}$ and $\omega_B \in H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})_{prim}$ for their images. Write

$$G_0 = Y_0F_0, \cdots, G_r = Y_rF_r, G_{r+1} = X_0\bar{F}_0, \cdots, G_{n+r} = X_n\bar{F}_n$$

and $Z_0 = Y_0, \cdots, Z_r = Y_r, Z_{r+1} = X_0, \cdots, Z_{n+r+1} = X_n$. Let M be the Jacobian matrix given by $\{\frac{\partial G_i}{\partial Z_j}\}_{i,j}$. Let $M_{i|j}$ be its minor with the i'th row and the j'th column missing.

Theorem (See Lemma 1.5.7 and Theorem 1.5.9)**.** *There is a unique element* $\tilde{C} \in k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]$ *such that*

$$(m+1)Y_iX_j\tilde{C} = (-1)^j \det(M_{0|j+r+1})Y_i + (-1)^{r+i} \det(M_{0|i})X_j$$

for all $i \in \{1, \cdots, r\}$ *and* $j \in \{r+1, \cdots, n+r+1\}$. *Assume that we are not in the situation that* $\dim(\mathcal{X})$ *is odd,* $r = 1$ *and* $m = 2$. *Then the map*

$$\psi : J^\rho \to \tilde{J}^{\rho+(r+1,n+1)}, D \mapsto D\prod_{i=0}^{r} Y_i \prod_{j=0}^{n} X_j$$

is an isomorphism. Therefore, we have that $\tilde{C} = \psi(C)$ *for a unique* $C \in J^\rho$. *Write* $AB = \lambda C$ *in* J^ρ *for some* $\lambda \in k^*$. *Then*

$$Tr(\omega_A \cup \omega_B) = (-1)^{r+1}m^{n+1}\binom{n+r}{r}\lambda.$$

Even though this does not give a completely explicit formula to compute the quadratic Euler characteristic, it may provide a useful algorithm in concrete cases.
Also, we will see from the proof of the above theorem that if $\binom{n+r}{r}$ is invertible in k, we have that $Cm^{-n}\binom{n+r}{r}^{-1}$ has trace 1. We call this the *Scheja-Storch generator*, and conjecture that there is a way to define this without the assumption that $\binom{n+r}{r}$ is invertible in k.
As an application, we compute the quadratic Euler characteristic of a complete intersection of two generalized Fermat hypersurfaces.

Theorem (See Corollary 1.6.3)**.** *Let $F_0 = \sum_{i=0}^n a_i X_i^m, F_1 = \sum_{i=0}^n b_i X_i^m$ be two generalized Fermat hypersurfaces in $\mathbb{P}^n$. Assume that $a_i, b_i \in k^*$ and that $a_i b_j - a_j b_i \neq 0$ for all $i \neq j$. Let $X = V(F_0, F_1)$. Then*

$$\chi(X/k) = \begin{cases} B_{n,m} \cdot H & \text{if } n \text{ is odd} \\ B_{n,m} \cdot H + \langle 1 \rangle & \text{if } n \text{ is even, } m \text{ odd} \\ B_{n,m} \cdot H + \langle 1 \rangle + \sum_{i=0}^n \langle \prod_{j \neq i} (a_i b_j - a_j b_i) \rangle & \text{if } n, m \text{ are even} \end{cases}$$

where $B_{n,m} \in \mathbb{Z}$ is given by

$$B_{n,m} = \begin{cases} \frac{1}{2} \deg(c_{n-2}(T_X)) & \text{if } n \text{ odd} \\ \frac{1}{2} \deg(c_{n-2}(T_X)) - 1 & \text{if } n \text{ even, } m \text{ odd} \\ \frac{1}{2} \deg(c_{n-2}(T_X)) - n - 1 & \text{if } n, m \text{ even} \end{cases}$$

We note that in the paper [7] by Cox and Batyrev, there is a more general construction of an isomorphism between primitive Hodge cohomology groups and certain graded parts of a Jacobian ring in the setting of toric varieties. This follows the methods of [11] as do we, but they do not consider the multiplicative structure.

It would be interesting to extend these results to the case where the hypersurfaces do not necessarily have the same degrees. This might be possible by extending the above arguments to the situation where $\mathbb{P}^r$ is a replaced by a weighted r-dimensional projective space, and will be explored in future work.

Motivic Donaldson-Thomas invariants: an analogue of the results in [39] and [40] for cohomology of Witt sheaves

If one tries to count a certain type of objects for which there exists a reasonable moduli space, one could do that by doing intersection theory on that moduli space. However, these moduli spaces often have all sorts of bad singularities, or may not have the expected dimension. Virtual fundamental classes, introduced by Behrend and Fantechi in their paper [8], provide a way to make reasonable computations in spite of these problems. For example, if we try to count ideal sheaves on a smooth projective scheme which are of a given length, the corresponding moduli space is the Hilbert scheme. The construction of those goes back to Grothendieck, see [22], and see for instance [23] for more details. Hilbert schemes often have all sorts of weird singularities, see for instance Vakil's paper [54]. The degrees of the corresponding virtual fundamental classes in this situation are called *Donaldson-Thomas invariants*. They were first constructed and defined by Donaldson and Thomas in [16] and [51].

A particular example of a computation of Donaldson-Thomas invariants is done in the papers [39] and [40], by Maulik, Nekrasov, Pandharipande and Okounkov. If we take a smooth projective threefold X over $\mathbb{C}$ with an action of the three dimensional torus $\mathbb{T}$ on it, we can look at the Hilbert scheme $\mathrm{Hilb}^n(X)$ of ideal sheaves on X of length n. To this, one associates a virtual fundamental class, of which the degree $I_n \in \mathbb{Z}$ is the Donaldson-Thomas invariant. In [39] and [40], there is a proof of the fact that

$$\sum_{n\geq 0} I_n q^n = M(-q)^{\deg(c_3(T_X\otimes K_X))}$$

where $M(q) = \prod_{n\geq 1}(1-q^n)^{-n}$ is the MacMahon function (see MacMahon's book [38, Article 43] and Stanley's book [49, Corollary 7.20.3]), T_X is the tangent bundle on X and K_X is the canonical line bundle.

The proof uses the virtual localization formula by Graber and Pandharipande, see [19], from which one can deduce in this particular case that

$$I_n = \sum_{[\mathcal{I}]\in \mathrm{Hilb}^n(X)^{\mathbb{T}}} \frac{e(\mathrm{Ext}^2(\mathcal{I},\mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I},\mathcal{I}))}.$$

Here, the sum is over all ideal sheaves $\mathcal{I}$ on X of length n which are fixed under the induced action of $\mathbb{T}$ on $\mathrm{Hilb}^n(X)$. These are locally given by monomial ideals. The authors then compute the trace of the virtual tangent space

$$\mathrm{Ext}^1(\mathcal{I},\mathcal{I}) - \mathrm{Ext}^2(\mathcal{I},\mathcal{I})$$

of each fixed ideal sheaf $\mathcal{I}$. From this trace, one can read off the corresponding equivariant virtual Euler class and express this in terms of the Euler classes of the standard line bundles $\mathcal{O}(1,0,0), \mathcal{O}(0,1,0)$ and $\mathcal{O}(0,0,1)$ on $B\mathbb{T}$. Here, $B\mathbb{T}$ is the classifying space of $\mathbb{T}$, see Totaro's paper [53]. To deduce the above formula, the Bott residue formula, see [10] is used.

We study an analogue using the notion of motivic virtual fundamental classes with values in cohomology of Witt sheaves defined by Levine in [31] and the corresponding virtual localization formula which is proven by Levine in the same paper. In this setting, we take the base field to be $\mathbb{R}$ (see Notation 2.3.1). Let N_S be the normalizer of the torus in SL_2 over $\mathbb{R}$, generated by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \text{ for } t\in\mathbb{R}^* \text{ and } \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

Let $a,b\in\mathbb{Z}$ be odd such that $a, b, 3a-b, 3b-a, 3a+b, 3b+a, a-b, a+b \in \mathbb{R}^*$. There is an action of N_S on $\mathbb{P}^3$ given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\cdot [X_0 : X_1 : X_2 : X_3] = [t^a X_0 : t^{-a}X_1 : t^b X_2 : t^{-b}X_3]$$
$$\sigma\cdot [X_0 : X_1 : X_2 : X_3] = [-X_1 : X_0 : -X_3 : X_2].$$

This action does not have fixed points, but there are the two fixed couples $\{[1:0:0:0],[0:1:0:0]\}$ and $\{[0:0:1:0],[0:0:0:1]\}$, so that we can study fixed ideal sheaves of even length.

Levine has constructed an orientation for $\mathrm{Hilb}^n(X)$ for X any smooth projective threefold with a given isomorphism $K_X \cong L^{\otimes 2}$ for some invertible sheaf L

on X, i.e. an isomorphism from the determinant of this obstruction theory to the square of a line bundle. This implies that there is a well defined quadratic degree $\tilde{I}_n \in \mathcal{W}(\mathrm{Spec}(\mathbb{R})) \cong \mathbb{Z}$ of the virtual fundamental class of $\mathrm{Hilb}^n(\mathbb{P}^3)$. Using [31, Theorem 6.7], if the torus inside N_S acts with isolated fixed points, we have that

$$\tilde{I}_n = \sum_{[\mathcal{I}] \in \mathrm{Hilb}^n(\mathbb{P}^3)^{N_S}} \frac{e(\mathrm{Ext}^2(\mathcal{I},\mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I},\mathcal{I}))}.$$

We therefore compute the trace of the representation $\mathrm{Ext}^2(\mathcal{I},\mathcal{I}) - \mathrm{Ext}^1(\mathcal{I},\mathcal{I})$ of N_S for ideal sheaves $\mathcal{I}$ that are fixed by the N_S-action, using the strategy of [39] and [40]. We can compute the Euler classes from that in terms of those of canonical rank two bundles using the results [28, Proposition 5.5 and Theorem 7.1]. We apply this to compute that $\tilde{I}_2 = 10, \tilde{I}_4 = 25$ and $\tilde{I}_6 = -50$, with some help of SAGE, see the attached code here. This leads us to the following conjecture.

Conjecture. *Let $X = \mathbb{P}^3$ and equip this with the natural action of the normalizer of the torus in SL_2. For $n \geq 0$, let $\tilde{I}_n$ be the degree of the motivic virtual fundamental class associated to $Hilb^n(X)$. Then we have that*

$$\sum_{n\geq 0} \tilde{I}_n q^n = M(-q^2)^{\widetilde{\deg}(e(V))}.$$

where V is a certain locally free sheaf on $Hilb^2(\mathbb{P}^3)$ with quadratic Euler class $e(V)$ and $\widetilde{\deg}$ is the quadratic degree map.

For $n = 8$ and higher, the present method does not work to compute the corresponding motivic Donaldson-Thomas invariant. Namely, choose coordinates x, y, z on $\{X_0 \neq 0\} \subset \mathbb{P}^3$ then the ideal sheaves which are locally given by $(x + \lambda yz, y^2, z^2)$ for $\lambda \in \mathbb{R}^*$ on this open are of length four. But the ideals are not monomial, so these are not isolated fixed points.
The results for $\mathbb{P}^3$ also lead to the following conjecture.

Conjecture. *Let X be a smooth projective scheme over $\mathbb{R}$ of dimension 3, with an action of the normalizer of the torus in SL_2, together with an isomorphism $K_X \cong L^{\otimes 2}$ for some invertible sheaf L on X, and let $\tilde{I}_n$ be the degree of the motivic virtual fundamental class associated to $Hilb^n(X)$. Then we have that*

$$\sum_{n\geq 0} \tilde{I}_n t^n = M(-q^2)^{\widetilde{\deg}(e(V_X))}$$

where V_X is a certain locally free sheaf on $Hilb^2(X)$.

We also show how to extend some of the localization methods that go into the above formula for actions by the normalizer N_G in GL_2. More precisely, we prove an analogue of [28, Proposition 5.5], which is a computation of the cohomology of Witt sheaves on BN_G, from which one can deduce what the Euler classes of canonical rank two bundles on BN_G are. Obtaining the full result for this action was not possible so far, due to some technical obstructions, but it would be interesting to see if one can find a way around these problems.

Structure

Chapter 1 is devoted to computing the quadratic Euler characteristic of a smooth projective complete intersection of hypersurfaces of the same degree. Section 1.1 contains the definition of a quadratic Euler characteristic and some of its basic properties, after which we give a more detailed summary of the results in [33]. Section 1.2 is devoted to some results on cohomology of differential forms and first Chern classes which will be needed later on. Then in Section 1.3 we construct isomorphisms from bigraded parts of the Jacobian ring to primitive cohomology. In Section 1.4 we compare the cup product and the product in the Jacobian ring, and we show that J^{ρ} and $\tilde{J}^{\rho+(r+1,n+1)}$ are one dimensional, after which we compute the trace in Section 1.5. Finally, we work out the example of intersecting two generalized Fermat hypersurfaces of the same degree in Section 1.6.

Chapter 2 contains the results on motivic Donaldson-Thomas invariants. In Section 2.1, we give a summary of the construction of a classifying space following [53] and of some results in [28] which are needed later. In Section 2.2, we summarize the results of [39] and [40] and compute I_1 and I_2 for $\mathbb{P}^3$. In Section 2.3 we study how to define the motivic virtual fundamental class of interest, and a strategy to compute the degree. Then in Section 2.4, we compute $\tilde{I}_n$ for $n \leq 6$. Finally, in Section 2.5 we compute the cohomology of Witt sheaves on BN_G, i.e. show an analogue of [28, Proposition 5.5] for BN_G.

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The titlepage was created by myself on http://canva.com, with some help of Paint 3D. The design was inspired by the window structure in office WSC-O-3.71, where I have done most of the work on this thesis, and the color scheme is representative of the writing map which I used to make notes in seminars and other events during my PhD in Essen.
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Contents

Chapter 1

The quadratic Euler characteristic of a smooth projective complete intersection

Throughout, let k be a perfect field which is not of characteristic 2.

1.1 Quadratic Euler characteristics

In this section we give the definition and some basic properties of quadratic Euler characteristics and then discuss the computation of the quadratic Euler characteristic of a smooth projective hypersurface from [33].

1.1.1 Quadratic Euler characteristics

We give the definition of a quadratic Euler characteristic following the one by Levine in [29, Section 1]. The quadratic Euler characteristic will be a particular case of a more general definition of Euler characteristic, introduced by Dold and Puppe in [14].
Let $\mathcal{C}$ be a symmetric monoidal category, and denote $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for the tensor product and $1 \in \mathcal{C}$ for the unit. Let τ be the symmetry isomorphism from the tensor product $\otimes$ to $\otimes \circ t$, where $t : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the usual symmetry given by $t(a, b) = (b, a)$. The following definition is taken from [14, Definition 1.2 and Theorem 1.3].

Definition 1.1.1. An object $X \in \mathcal{C}$ is *strongly dualizable* if there exists an object $X^\vee \in \mathcal{C}$ and morphisms $\delta_X : 1 \to X \otimes X^\vee$ and $\mathrm{ev}_X : X^\vee \otimes X \to 1$ in $\mathcal{C}$

such that the compositions

$$X \cong 1 \otimes X \xrightarrow{\delta_X \otimes \mathrm{Id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{Id} \otimes \mathrm{ev}_X} X \otimes 1 \cong X$$

and

$$X^\vee \cong X^\vee \otimes 1 \xrightarrow{\mathrm{Id} \otimes \delta_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}_X \otimes \mathrm{Id}} 1 \otimes X^\vee \cong X^\vee$$

are the identity morphisms.

Remark 1.1.2. If X is strongly dualizable, the triple $(X^\vee, \delta_X, \mathrm{ev}_X)$ is unique up to unique isomorphism. We usually call $X^\vee$ the *dual* of X, with the morphisms δ_X and ev_X being understood.

Now let $X \in \mathcal{C}$ be a strongly dualizable object. The following definition is a special case of [14, Definition 4.1].

Definition 1.1.3. The *categorical Euler characteristic* of X is the composition

$$1 \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{\tau} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} 1.$$

To k, we can associate the *motivic stable homotopy category* $\mathrm{SH}(k)$, see for instance Morel's book [43] or Hoyois' paper [25] for its construction and properties. We have that $\mathrm{SH}(k)$ is a symmetric monoidal category, with the "smash product" as its tensor product. For a smooth projective scheme X over k, we have the suspension spectrum $\Sigma^\infty_T X_+ \in \mathrm{SH}(k)$ and this is a strongly dualizable object, see for instance [25, Theorem 5.22 and Corollary 6.13].

Definition 1.1.4. The *Grothendieck-Witt ring* $\mathrm{GW}(k)$ of k is the group completion of the monoid (under orthogonal direct sum) of isometry classes of non-degenerate quadratic forms over k.

Remark 1.1.5. One can think of $\mathrm{GW}(k)$ as the group generated by the forms

$$\langle a \rangle : x \mapsto ax^2$$

for $a \in k^*$ modulo the relations

- $\langle ab^2 \rangle = \langle a \rangle$ for $a, b \in k^*$.
- $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$ for $a, b, a+b \in k^*$.
- $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$ for $a \in k^*$.

Note that $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for $a, b \in k^*$, by definition.
This presentation originally goes back to Witt, see [55, Section 1]. In the form above, it is [43, Lemma 2.9], where the result is deduced from the statement for Witt rings, see [41, Lemma (1.1) in Chapter 4].

Definition 1.1.6. The form $H = \langle 1 \rangle + \langle -1 \rangle$ is called the *hyperbolic form*.

By a deep result of Morel (see [42, Theorem 6.4.1]) we have that

$$\mathrm{End}(1_{\mathrm{SH}(k)}) \cong \mathrm{GW}(k).$$

Combining the above, the quadratic Euler characteristic of a smooth projective scheme over k can now be defined as follows.

Definition 1.1.7. The *quadratic Euler characteristic* $\chi(X/k) \in \mathrm{GW}(k)$ of a smooth projective scheme X over k is the categorical Euler characteristic of X in $\mathrm{SH}(k)$.

Quadratic Euler characteristics satisfy several nice relations, which one can find in e.g. [29]. One property which we will need is the following.

Proposition 1.1.8 (See [29], Proposition 1.4(3))**.** *Let X be a smooth projective scheme over k and let Z be a smooth closed subscheme of pure codimension c with complement U. Then*

$$\chi(X/k) = \chi(U/k) + \langle -1\rangle^c \chi(Z/k).$$

Remark 1.1.9. Even though U in the above statement is not projective, it has a quadratic Euler characteristic in $\mathrm{GW}(k)$. Namely, it has been proven by Riou in [36] that a quasi-projective scheme is dualizable in the stable motivic homotopy category if we invert the characteristic of k. If $\mathrm{char}(k) = p > 0$ we have an injective morphism $\mathrm{GW}(k) \to \mathrm{GW}(k)[p^{-1}]$ and one can show that the categorical Euler characteristic always lands in $\mathrm{GW}(k)$. See [29, Remark 1.1.2] for more details.

Example 1.1.10 (See [29], Proposition 1.4(4))**.** We have that

$$\chi(\mathbb{P}^n/k) = \sum_{i=0}^{n} \langle -1\rangle^i.$$

One way to prove this is to use Proposition 1.1.8 together with induction on n and the fact that $\mathbb{A}^n$ is equivalent to a point in $\mathrm{SH}(k)$. Note that this is a multiple of H if n is odd. Also, the rank of this form is $n + 1$ (which is the topological Euler characteristic of complex projective space) and its signature is either 0 or 1, depending on the parity of n (which is the topological Euler characteristic of real projective space).

Remark 1.1.11. This is true in general: if $k \subset \mathbb{R} \subset \mathbb{C}$ then we have that the rank of $\chi(X/k)$ is equal to the topological Euler characteristic of $X(\mathbb{C})$ while the signature is equal to the topological Euler characteristic of $X(\mathbb{R})$. See [29, Remark 1.4.1].

Remark 1.1.12. For a smooth quasi-projective scheme U over k we have that $\chi(\mathbb{P}^n \times U/k) = \chi(\mathbb{P}^n/k)\chi(U/k)$ by [29, Proposition 1.4(4)].

1.1.2 The quadratic Euler characteristic of a smooth projective hypersurface

Notation 1.1.13. For a scheme X over k, we denote the sheaf of differential forms on X over k by Ω_X. We write $\Omega_X^q = \wedge^q \Omega_X$ for $q \in \mathbb{Z}_{\geq 0}$.

Note that by [23, Exercise II.4.5], for a smooth projective scheme Y we have that $H^1(Y, \mathcal{O}_Y^*) \cong \operatorname{Pic}(Y)$, where $\operatorname{Pic}(Y)$ is the Picard group of Y, i.e. the group of line bundles on Y. There is the canonical "dlog morphism"

$$\mathcal{O}_{XY}^* \to \Omega_Y, f \mapsto \frac{df}{f}$$

inducing the map

$$c_1 : H^1(Y, \mathcal{O}_Y^*) \to H^1(Y, \Omega_Y).$$

Definition 1.1.14. The *first Chern class* of a line bundle L on a scheme Y is the image $c_1(L) \in H^1(Y, \Omega_Y)$ of the class of L under the above map.

Notation 1.1.15. We write $c_1(L)^i \in H^i(Y, \Omega_Y^i)$ for the i-fold cup product of $c_1(L)$ with itself.

An important computational tool is the motivic Gauss-Bonnet Theorem for SL-oriented cohomology theories proven by Levine and Raksit in their paper [35]. A more general motivic Gauss-Bonnet theorem has been proven by Déglise, Jin and Khan in [12], and the theorem of Levine-Raksit can be viewed as a special case of their statement. A version where the scheme does not need to be smooth has been proven by Azouri, see [5].
We do not state the theorem in all of its generality here, but rather one of its applications which provides a way to compute a quadratic Euler characteristic in practice.

Theorem 1.1.16 (See [35], Corollary 8.7)**.** *Let X be a smooth projective scheme over k. Then:*

- *If $\dim(X)$ is odd, then $\chi(X/k) = C \cdot H$ for some $C \in \mathbb{Z}$, where H is the hyperbolic form.*
- *If $\dim(X) = 2n$ is even, then $\chi(X/k) = C \cdot H + Q$ for some $C \in \mathbb{Z}$, where Q is the quadratic form given by the composition*

$$H^n(X, \Omega_X^n) \times H^n(X, \Omega_X^n) \xrightarrow{\cup} H^{2n}(X, \Omega_X^{2n}) \xrightarrow{Trace} k.$$

Here the first map is the cup product on cohomology.

Remark 1.1.17. By [35, Theorem 5.3] the rank of $\chi(X/k)$ is equal to the degree of $c_n(T_X)$ where T_X is the tangent bundle of X. This gives a way to determine the constant C in the above theorem in practice. There is also a formula for C in terms of dimensions of cohomology groups of Ω_X^q given in [35, Corollary 8.7].

The form Q has been computed successfully in the case of hypersurfaces by Levine, Lehalleur and Srinivas in [33], using inspiration from the paper [11] by Carlson and Griffiths. We now summarize their strategy.
Consider a smooth hypersurface $X = V(F) \subset \mathbb{P}^n$ where $F \in k[X_0, \cdots, X_n]$ is a homogeneous polynomial of degree $m \in \mathbb{Z}_{\geq 2}$. Assume that the characteristic of k is coprime to m.

Definition 1.1.18. The *Jacobian ring of* X is

$$J_X = k[X_0, \cdots, X_n] / \left(\frac{\partial F}{\partial X_0}, \cdots, \frac{\partial F}{\partial X_n} \right).$$

Note that J_X has a natural grading induced by the grading of $k[X_0, \cdots, X_n]$. Furthermore, J_X is a finite dimensional k-algebra. The top nonzero graded part is $J_X^{(m-2)(n+1)}$, which is a one dimensional vector space over k; for a proof, see [26, Lemma 4], where the result is deduced from the fact that J_X is Gorenstein together with the proof of [46, (4.7) Korrolar].

Construction 1.1.19. There is a canonical choice of generator e_F of $J_X^{(m-2)(n+1)}$ called the *Scheja-Storch generator*. Namely, as $m \geq 2$, for $i \in \{0, \cdots, n\}$ we can write

$$\frac{\partial F}{\partial X_i} = \sum_{j=0}^{n} a_{ij} X_j$$

for some (non-unique) $a_{ij} \in k[X_0, \cdots, X_n]$. One defines $e_F = \det((a_{ij})_{i,j})$. One can show that this is independent of the choice of a_{ij}, see [46, (1.2)(α)].

Example 1.1.20. Let $F = \sum_{i=0}^{n} a_i X_i^m$ where $a_0, \cdots, a_n \in k^*$. Then X is a *generalized Fermat hypersurface*. We have that $\frac{\partial F}{\partial X_i} = m a_i X_i^{m-1}$ and so one computes e_F as

$$e_F = m^{n+1} \prod_{i=0}^{n} a_i X_i^{m-2}.$$

Let $i : X \to \mathbb{P}^n$ be the natural inclusion. This induces a pushforward map $i_* : H^q(X, \Omega_X^p) \to H^{q+1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{p+1})$ for all $p, q \in \mathbb{Z}_{\geq 0}$ as defined by Srinivas in [47].

Definition 1.1.21. The *primitive cohomology of* X *with respect to* $p, q \in \mathbb{Z}_{\geq 0}$ *such that* $p + q = n - 1$ is defined by $H^q(X, \Omega_X^p)_{prim} = \ker(i_*)$.

Remark 1.1.22. Let $c_1(\mathcal{O}(1)) \in H^1(X, \Omega_X)$ be the first Chern class of $\mathcal{O}(1)$. The Hard Lefschetz Theorem tells us that for $0 < i \leq n - 1$, the map

$$(-) \cup c_1(\mathcal{O}(1))^i : \bigoplus_{p+q=n-i} H^q(X, \Omega_X^p) \to \bigoplus_{p+q=n+i} H^q(X, \Omega_X^p)$$

is an isomorphism. Classically, for $0 \leq i \leq n - 1$, the primitive cohomology of X is defined to be the kernel of the morphism

$$(-) \cup c_1(\mathcal{O}(1))^{i+1} : \bigoplus_{p+q=n-i} H^q(X, \Omega_X^p) \to \bigoplus_{p+q=n+i+2} H^q(X, \Omega_X^p).$$

For $i = 0$ and p, q such that $p + q = n - 1$, this definition coincides with the one above. To see this, note that multiplication with $c_1(\mathcal{O}(1))$ takes $H^q(X, \Omega_X^p)$ to $H^{q+1}(X, \Omega_X^{p+1})$. The pullback $i^* : H^{q+1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{p+1}) \to H^{q+1}(X, \Omega_X^{p+1})$ is an isomorphism by the Weak Lefschetz Theorem, so we can view multiplication with $c_1(\mathcal{O}(1))$ as a morphism to $H^{q+1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{p+1})$. Now as $i^* i_*$ is multiplication with $c_1(\mathcal{O}(1))$, we see that the kernel coincides with $\ker(i_*)$.

One can show that $H^q(X, \Omega_X^p)_{prim} = H^q(X, \Omega_X^p)$ whenever $p \neq q$. In [33], the authors prove the following result.

Proposition 1.1.23 ([33], Proposition 3.2)**.** *For each $q \geq 0$, there is a canonical isomorphism $\psi_q : J_X^{(q+1)m-n-1} \to H^q(X, \Omega_X^{n-1-q})_{prim}$.*

This result originally goes back to Dolgachev, see [15], and the characteristic zero case is due to Griffiths, see [21]. In [33], there is then a comparison of the cup product on cohomology with the usual ring multiplication of J_X, leading up to the following result.

Proposition 1.1.24 ([33], Proposition 3.7)**.** *Consider $p, q \in \mathbb{Z}_{\geq 0}$ be such that $p + q = n - 1$ and let $A \in J_X^{(q+1)m-n-1}$ and $B \in J_X^{(p+1)m-n-1}$. Let*

$$\omega_A = \psi_q(A) \in H^q(X, \Omega_X^p)_{prim} \text{ and } \omega_B = \psi_p(B) \in H^p(X, \Omega_X^q)_{prim}$$

be their images. Write $F_i = \frac{\partial F}{\partial X_i}$. Cover $\mathbb{P}^n$ by the open cover $\mathcal{U} = \{U_0, \cdots, U_n\}$ where $U_i = \{F_i \neq 0\}$, and let $C^i(\mathcal{U}, \Omega_{\mathbb{P}^n}^n)$ denote the i'th group in the Čech complex corresponding to $\mathcal{U}$. Furthermore, let $\bar{\omega} = \sum_{i=0}^n (-1)^i X_i d\hat{X}^i$ be the generator of $\Omega_{\mathbb{P}^n}^n(n+1)$, where we write $d\hat{X}^i = dX_0 \cdots dX_{i-1} dX_{i+1} \cdots dX_n$. Then the element $i_*(\omega_A \cup \omega_B) \in H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^n)$ is represented by

$$\frac{-mAB\bar{\omega}}{F_0 \cdots F_n} \in C^n(\mathcal{U}, \Omega_{\mathbb{P}^n}^n).$$

They then compare this with a representation of $c_1(\mathcal{O}(m))^n$ on the same cover to compute the trace, which yields the following result.

Theorem 1.1.25 ([33], Theorem 3.9)**.** *In the situation of the above proposition, suppose that $AB = \lambda e_F$ for $\lambda \in k^*$. Then*

$$Tr(\omega_A \cup \omega_B) = -m\lambda.$$

Example 1.1.26. In the case of a generalized Fermat hypersurface X as before, if $n = 2p + 1$ is odd, we need to calculate the form Q. One can show that $H^p(X, \Omega_X^p) = H^p(X, \Omega_X^p)_{prim} \oplus c_1(\mathcal{O}(1))^p$ and compute that $c_1(\mathcal{O}(1))^p$ contributes a form $\langle m \rangle$. For the primitive cohomology, we evaluate the form on basis elements of $H^p(X, \Omega_X^p)_{prim}$, i.e. on the corresponding parts of the Jacobian ring. If $AB = \lambda e_F$ for some $\lambda \in k^*$ and two distinct basis elements A and B, we also have that $BA = \lambda e_F$ and one can check that this yields a hyperbolic form. If m is odd, there are no basis elements that square to a nonzero multiple of e_F. If m is even, we have that

$$\left(X_0^{\frac{m-2}{2}} \cdots X_n^{\frac{m-2}{2}}\right)^2 = \frac{e_F}{m^{n+1}\prod_{i=0}^n a_i}.$$

Using Theorem 1.1.25, this gives rise to the form $\langle -m\prod_{i=0}^n a_i\rangle$. One can see from [35, Theorem 5.3] that the rank of $\chi(X/k)$ is equal to $\deg(c_n(T_X))$ where T_X is the tangent bundle on X. Putting everything together, we find that

$$\chi(X/k) = \begin{cases} A_{n,m}\cdot H & \text{if } n \text{ even} \\ A_{n,m}\cdot H + \langle m\rangle & \text{if } n, m \text{ odd} \\ A_{n,m}\cdot H + \langle m\rangle + \langle -m\prod_{i=0}^n a_i\rangle & \text{if } n \text{ odd}, m \text{ even} \end{cases}$$

for integers $A_{n,m} \in \mathbb{Z}$ given by

$$A_{n,m} = \begin{cases} \frac{1}{2}\deg(c_n(T_X)) & \text{if } n \text{ even} \\ \frac{1}{2}\deg(c_n(T_X)) - 1 & \text{if } n, m \text{ odd} \\ \frac{1}{2}\deg(c_n(T_X)) - 2 & \text{if } n \text{ odd, } m \text{ even} \end{cases}$$

This is also [29, Theorem 11.1], but there it is proven in a different way, namely using Levine's quadratic Riemann-Hurwitz formula from [29].

1.2 Setup, cohomology of differential forms and primitive cohomology

In the next sections, we will be working with the following setup.

Notation 1.2.1. Let $n, m, r \in \mathbb{Z}_{\geq 1}$ be such that $n \geq r+2$ and $m \geq 2$. Assume that m is coprime to the characteristic of k, if this is positive. Let $F_0, \cdots, F_r \in k[X_0, \cdots, X_n]$ be homogeneous polynomials of the same degree m. Let $X = V(F_0, \cdots, F_r) \subset \mathbb{P}^n$ be the intersection of the $V(F_i)$ and assume that this is a smooth complete intersection. We define $F = Y_0F_0 + \cdots + Y_rF_r$ and consider the hypersurface

$$\mathcal{X} = V(F) \subset \mathbb{P}^r \times \mathbb{P}^n.$$

We write $i : \mathcal{X} \to \mathbb{P}^r \times \mathbb{P}^n$ for the inclusion. Note that $\mathcal{X}$ is of bidegree $(1, m)$ and that it has dimension $n + r - 1$. We note that $\frac{\partial F}{\partial Y_i} = F_i$ for $i \in \{0, \cdots, r\}$ and write $\bar{F}_j = \frac{\partial F}{\partial X_j}$ for $j \in \{0, \cdots, n\}$.

Notation 1.2.2. We denote the canonical projections by $\pi_n : \mathbb{P}^r \times \mathbb{P}^n \to \mathbb{P}^n$ and $\pi_r : \mathbb{P}^r \times \mathbb{P}^n \to \mathbb{P}^r$.

Remark 1.2.3. Note that $\mathcal{X}$ is smooth: suppose $(y_0, \cdots, y_r, x_0, \cdots, x_n) \in \mathcal{X}$ is a point where all F_i and $\bar{F}_j$ vanish, then $(y_0, \cdots, y_r, x_0, \cdots, x_n)$ is in $\mathbb{P}^r \times X$.

As X is smooth, the vectors

$$\left(\frac{\partial F_i}{\partial X_0}(x), \cdots, \frac{\partial F_i}{\partial X_n}(x)\right)$$

for $i \in \{0, \cdots, r\}$ are linearly independent for any $x \in X$. Now as

$$\bar{F}_j(y_0, \cdots, y_r, x_0, \cdots, x_n) = \sum_{i=0}^{r} y_i \frac{\partial F_i}{\partial X_j}(x_0, \cdots, x_n) = 0$$

for all $j \in \{0, \cdots, n\}$, we have that

$$\sum_{i=0}^{r} y_i \left(\frac{\partial F_i}{\partial X_0}(x_0, \cdots, x_n), \cdots, \frac{\partial F_i}{\partial X_n}(x_0, \cdots, x_n)\right) = 0$$

and so $y_0 = \cdots = y_r = 0$, but this is impossible.

Remark 1.2.4. Note that we have the two Euler equations

$$F = \sum_{i=0}^{r} Y_i F_i \text{ and } mF = \sum_{j=0}^{n} X_j \bar{F}_j.$$

We now observe that we can compute $\chi(X/k)$ from $\chi(\mathcal{X}/k)$.

Lemma 1.2.5. *We have that*

$$\chi(\mathcal{X}/k) = \chi(\mathbb{P}^{r-1}/k)\chi(\mathbb{P}^n/k) + \langle -1\rangle^r \chi(X/k).$$

Proof. Let U be the complement of X in $\mathbb{P}^n$ and let $\pi_n|_{\mathcal{X}} : \mathcal{X} \to \mathbb{P}^n$ be the restriction of π_n to $\mathcal{X}$. Then $(\pi_n|_{\mathcal{X}})^{-1}(X) = \mathbb{P}^r \times X$ and $(\pi_n|_{\mathcal{X}})^{-1}(U) \to U$ is a Zariski locally trivial $\mathbb{P}^{r-1}$-bundle. Using [29, Proposition 1.4(4)], we have that

$$\chi((\pi_n|_{\mathcal{X}})^{-1}(U)/k) = \chi(\mathbb{P}^{r-1}/k) \cdot \chi(U/k)$$

and

$$\chi(\mathbb{P}^r \times X/k) = \chi(\mathbb{P}^r/k)\chi(X/k).$$

By Proposition 1.1.8, we also have that

$$\chi(\mathbb{P}^n/k) = \chi(U/k) + \langle -1\rangle^{r+1}\chi(X/k).$$

Recalling Example 1.1.10, this yields

$$\begin{aligned}
\chi(\mathcal{X}/k) &= \chi(\mathbb{P}^{r-1}/k)\chi(U/k) + \langle -1\rangle^r \chi(\mathbb{P}^r/k)\chi(X/k) \\
&= \chi(\mathbb{P}^{r-1}/k)\chi(\mathbb{P}^n/k) + (\langle -1\rangle^r \chi(\mathbb{P}^r/k) - \langle -1\rangle^{r+1}\chi(\mathbb{P}^{r-1}/k))\chi(X/k) \\
&= \chi(\mathbb{P}^{r-1}/k)\chi(\mathbb{P}^n/k) + \left(\langle -1\rangle^r \sum_{i=0}^{r} \langle -1\rangle^i - \langle -1\rangle^{r+1} \sum_{i=0}^{r-1} \langle -1\rangle^i\right)\chi(X/k) \\
&= \chi(\mathbb{P}^{r-1}/k)\chi(\mathbb{P}^n/k) + \langle -1\rangle^r \chi(X/k)
\end{aligned}$$

as desired. □

In the coming sections, the strategy will be to adapt the arguments of [33] to a hypersurface in $\mathbb{P}^r \times \mathbb{P}^n$, using inspiration from [50]. In this section, we start by introducing two exact sequences which we will use in what follows, and we study the cohomology groups of differential forms for a product of projective spaces, which will be needed later on. We also study first Chern classes of line bundles on $\mathbb{P}^r \times \mathbb{P}^n$ and primitive cohomology.

Notation 1.2.6. The Picard group of $\mathbb{P}^r \times \mathbb{P}^n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with generators coming from the canonical sheaves $\mathcal{O}_{\mathbb{P}^r}(a)$ and $\mathcal{O}_{\mathbb{P}^n}(b)$ for $a, b \in \mathbb{Z}$. For a sheaf $\mathcal{F}$ on $\mathbb{P}^r \times \mathbb{P}^n$, we denote

$$\mathcal{F}(a,b) = \mathcal{F} \otimes \pi_r^* \mathcal{O}_{\mathbb{P}^r}(a) \otimes \pi_n^* \mathcal{O}_{\mathbb{P}^n}(b).$$

There are thus canonical sheaves of the form $\mathcal{O}(a,b)$ on $\mathbb{P}^r \times \mathbb{P}^n$.
For $a \in \mathbb{Z}_{\geq 0}$, we write $\mathcal{O}(a\mathcal{X})$ for the sheaf with sections having poles of order at most a on $\mathcal{X}$, which are regular everywhere else. We set $\mathcal{F}(a\mathcal{X}) = \mathcal{F} \otimes \mathcal{O}(a\mathcal{X})$. For $a < 0$, we write $\mathcal{F}(a\mathcal{X}) = \mathcal{F} \otimes \mathcal{I}_{\mathcal{X}}^{-a}$ where $\mathcal{I}_{\mathcal{X}}$ is the ideal sheaf of $\mathcal{X}$.

1.2.1 Two exact sequences

Recall from e.g. [23, Theorem II.8.17] that there is an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{X}) \xrightarrow{dF/F \wedge (-)} i^* \Omega_{\mathbb{P}^r \times \mathbb{P}^n} \to \Omega_{\mathcal{X}} \to 0. \tag{1.1}$$

Here, the second map is the natural surjection. There is another useful exact sequence

$$0 \to \Omega^{p+1}_{\mathbb{P}^r \times \mathbb{P}^n} \to \Omega^{p+1}_{\mathbb{P}^r \times \mathbb{P}^n}(\log(\mathcal{X})) \xrightarrow{res_X} i_* \Omega^p_{\mathcal{X}} \to 0 \tag{1.2}$$

for any $p \in \mathbb{Z}_{\geq 0}$ which is called the *residue sequence*. We will need the following statement in the next sections.

Lemma 1.2.7. *For $p, q \in \mathbb{Z}_{\geq 0}$, the boundary map*

$$\delta^{p,q} : H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}) \to H^{q+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r \times \mathbb{P}^n})$$

induced from the long exact cohomology sequence of the exact sequence (1.2) coincides with the pushforward map $i_* : H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}) \to H^{q+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r \times \mathbb{P}^n})$, which is again the pushforward as defined in [47].

The proof works exactly the same as that of [33, Lemma 2.2].

Remark 1.2.8. We will use in what follows that for an exact sequence of vector spaces $0 \to V \to W \to Z \to 0$ with V one dimensional, the induced sequence $0 \to V \otimes \wedge^{k-1} Z \to \wedge^k W \to \wedge^k Z \to 0$ is again exact for any $k \in \mathbb{Z}_{\geq 1}$. For an exact sequence $0 \to V \to W \to Z \to 0$ with Z a line bundle, we similarly have that the sequence $0 \to \wedge^k V \to \wedge^k W \to \wedge^{k-1} V \otimes Z \to 0$ is again exact for any $k \in \mathbb{Z}_{\geq 1}$.

1.2.2 Cohomology of differential forms

We will need Bott's theorem in what follows.

Theorem 1.2.9 (Bott's theorem for projective space, see [15], Theorem 2.3.2)**.** *Let $m \in \mathbb{Z}$. The cohomology of $\Omega^q_{\mathbb{P}^n}(m)$ satisfies $H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(m)) = 0$ for:*

- $p > 0$ *and* $m \geq q - n, m \neq 0$.
- $p > 0, m = 0$ *and* $p \neq q$.
- $p = 0$ *and* $m \leq q$*, except for* $m = p = q = 0$.

We will also make use of the following statement, which is useful to apply Bott's theorem to twisted sheaves of differentials on $\mathbb{P}^r \times \mathbb{P}^n$.

Proposition 1.2.10. *Let $p, q \in \mathbb{Z}_{\geq 0}$ and let $a, b \in \mathbb{Z}$. Then*

$$H^p(\mathbb{P}^r \times \mathbb{P}^n, \Omega^q_{\mathbb{P}^r \times \mathbb{P}^n}(a,b)) = \bigoplus_{i+j=q} \bigoplus_{k+l=p} H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(a)) \otimes H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(b)).$$

Proof of Proposition 1.2.10. We have that $\Omega_{\mathbb{P}^r \times \mathbb{P}^n} \cong \pi_r^* \Omega_{\mathbb{P}^r} \oplus \pi_n^* \Omega_{\mathbb{P}^n}$, by e.g. [23, Exercise II.8.3]. This implies that

$$\Omega^q_{\mathbb{P}^r \times \mathbb{P}^n} \cong \bigoplus_{i+j=q} \pi_r^* \Omega^i_{\mathbb{P}^r} \otimes \pi_n^* \Omega^j_{\mathbb{P}^n}$$

and tensoring with $\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^n}(a,b) = \pi_r^* \mathcal{O}_{\mathbb{P}^r}(a) \otimes \pi_n^* \mathcal{O}_{\mathbb{P}^n}(b)$ yields that

$$\Omega^q_{\mathbb{P}^r \times \mathbb{P}^n}(a,b) \cong \bigoplus_{i+j=q} \pi_r^* \Omega^i_{\mathbb{P}^r}(a) \otimes \pi_n^* \Omega^j_{\mathbb{P}^n}(b).$$

Now using [48, Tag 0BED], we have that

$$\begin{aligned} H^p(\mathbb{P}^r \times \mathbb{P}^n, \Omega^q_{\mathbb{P}^r \times \mathbb{P}^n}(a,b)) &= \bigoplus_{i+j=q} H^p(\mathbb{P}^r \times \mathbb{P}^n, \pi_r^* \Omega^i_{\mathbb{P}^r}(a) \otimes \pi_n^* \Omega^j_{\mathbb{P}^n}(b)) \\ &= \bigoplus_{i+j=q} \bigoplus_{k+l=p} H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(a)) \otimes H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(b)) \end{aligned}$$

which is the desired result. □

1.2.3 Primitive cohomology

The primitive cohomology of $\mathcal{X}$ is defined in the same way as in the case of a smooth hypersurface in $\mathbb{P}^n$, see Definition 1.1.21.

Definition 1.2.11. Let $p, q \geq 0$ be such that $p + q = n + r - 1$. The *primitive cohomology* of $\mathcal{X}$ with respect to p, q is $H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})_{\text{prim}} = \ker(i_*) \subset H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})$ where i_* is the pushforward $i_* : H^p(\mathcal{X}, \Omega^q_{\mathcal{X}}) \to H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{q+1}_{\mathbb{P}^r \times \mathbb{P}^n})$ as defined in [47].

Remark 1.2.12. The same arguments as in Remark 1.1.22 with $\mathcal{O}(1)$ replaced by $\mathcal{O}(1,m)$ show that this definition coincides with the classical definition of primitive cohomology.

The following result is probably standard, but we include a proof here for the reader's convenience.

Lemma 1.2.13. *The map* $i_* : H^p(\mathcal{X}, \Omega^q_{\mathcal{X}}) \to H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{q+1}_{\mathbb{P}^r \times \mathbb{P}^n})$ *is surjective if either* $p \neq q$ *or* $p = q$ *and* $p \geq r$*. Furthermore, if* $p \neq q$*, we have that* $H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{q+1}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0$*, so that* i_* *is the zero map.*

Proof. Using Proposition 1.2.10, we have that

$$H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{q+1}_{\mathbb{P}^r \times \mathbb{P}^n}) \cong \bigoplus_{i+j=p+1} \bigoplus_{k+l=q+1} H^i(\mathbb{P}^r, \Omega^k_{\mathbb{P}^r}) \otimes H^j(\mathbb{P}^n, \Omega^l_{\mathbb{P}^n}).$$

Suppose $p \neq q$. Using Theorem 1.2.9 we see that all terms in the above sum are zero except for those with $i = k$ and $j = l$. However, for such terms we have that $i + j = k + l$ which cannot be true as $p \neq q$. This implies that $H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{q+1}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0$. Therefore, i_* is the zero map, so in particular, i_* is surjective.
If $p = q$, then we see that $H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r \times \mathbb{P}^n}) \cong k^{N_{p+1}}$ for some $N_{p+1} \in \mathbb{Z}_{\geq 0}$. A basis is given by

$$\alpha_i^{p+1} = c_1(\mathcal{O}_{\mathbb{P}^r}(1))^i \otimes c_1(\mathcal{O}_{\mathbb{P}^n}(1))^j$$

where $i + j = p + 1$, $i \leq r$ and $j \leq n$. We now assume that $p \geq r$, then we have that $N_{p+1} \leq N_p$. As

$$i_* i^* 1 = c_1(\mathcal{O}(1,m)) = c_1(\mathcal{O}(1,0)) + m c_1(\mathcal{O}(0,1))$$

using the projection formula we see that

$$\begin{aligned} i_* i^* \alpha_i^p &= i_*(i^* \alpha_i^p \otimes i^* 1) \\ &= \alpha_i^p \otimes (c_1(\mathcal{O}(1,0)) + m c_1(\mathcal{O}(0,1))) \\ &= \alpha_{i+1}^{p+1} + m \alpha_i^{p+1} \end{aligned}$$

and the latter term is nonzero as m is coprime to the characteristic of k. We see from this that the matrix of $i_* i^*$ has an $N_{p+1} \times N_{p+1}$ minor with determinant a power of m, hence invertible. We conclude from this that i_* is surjective. $\square$

Corollary 1.2.14. *For* $p, q \in \mathbb{Z}_{\geq 0}$ *such that* $p + q = n + r - 1$*, we have that* $H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})_{prim} = H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})$ *as long as* $p \neq q$*.*

1.3 Isomorphism from the Jacobian ring to primitive cohomology

We keep the notation that was set up at the beginning of the previous section.

Definition 1.3.1. The *Jacobian ring* of F is given by

$$J = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n] / \left(F_0, \cdots, F_r, \bar{F}_0, \cdots, \bar{F}_n\right)$$

This definition is taken from [50]. Note that J has a natural bigrading where $J^{a,b}$ has degree a in the variables Y_i and b in the variables X_i. In this section, we will follow the argumentation of [33] with this Jacobian ring in order to show that certain graded pieces are isomorphic to certain primitive Hodge cohomology groups.

Notation 1.3.2. The line bundle $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(r+1, n+1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n}$ with global generator $\omega \wedge \bar{\omega}$ where

$$\omega = \sum_{i=0}^{r} (-1)^i Y_i dY^i$$

where we write $dY^i = dY_0 \wedge \cdots \wedge d\hat{Y}_i \wedge \cdots \wedge dY_r$ and

$$\bar{\omega} = \sum_{j=0}^{n} (-1)^j X_j dX^j$$

where we write $dX^j = dX_0 \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge dX_n$. We will more generally write $dX^{i_0,\cdots,i_k}$ to mean the wedge product of all dX_i with $dX_{i_0}, \cdots, dX_{i_k}$ removed, and use the notation $dY^{i_0,\cdots,i_k}$ similarly.

Notation 1.3.3. We fix integers $p, q \in \mathbb{Z}_{\geq 0}$ satisfying $p + q = n + r - 1$.

Construction 1.3.4. Consider the exact sequence (1.1). For $j \in \{0, \cdots, q-1\}$, we take a wedge product and use Remark 1.2.8 to obtain the exact sequence

$$0 \to \Omega^{n+r-j-2}_{\mathcal{X}}(-\mathcal{X}) \to i^*\Omega^{n+r-1-j}_{\mathbb{P}^r\times\mathbb{P}^n} \to \Omega^{n+r-1-j}_{\mathcal{X}} \to 0.$$

Now twisting by the line bundle $i^*\mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n}((q-j)\mathcal{X})$, we obtain the exact sequences

$$0 \to \Omega^{n+r-j-2}_{\mathcal{X}}((q-j-1)\mathcal{X}) \to i^*\Omega^{n+r-1-j}_{\mathbb{P}^r\times\mathbb{P}^n}((q-j)\mathcal{X}) \to \Omega^{n+r-1-j}_{\mathcal{X}}((q-j)\mathcal{X}) \to 0 \tag{1.3}$$

where the first map is induced by $dF/F \wedge (-)$. Patching those together, we see that there is an exact sequence

$$\begin{aligned} 0 \to \Omega^p_{\mathcal{X}} \xrightarrow{dF/F\wedge(-)} i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X}) \xrightarrow{dF/F\wedge(-)} i^*\Omega^{p+2}_{\mathbb{P}^r\times\mathbb{P}^n}(2\mathcal{X}) \xrightarrow{dF/F\wedge(-)} \cdots \\ \cdots \to i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X}) \xrightarrow{\pi_q} \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X}) \to 0 \end{aligned}$$

of sheaves on $\mathcal{X}$.

Notation 1.3.5. Let $\mathcal{C}(p)$ be the complex

$$0 \to i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X}) \xrightarrow{dF/F\wedge(-)} i^*\Omega^{p+2}_{\mathbb{P}^r\times\mathbb{P}^n}(2\mathcal{X}) \xrightarrow{dF/F\wedge(-)} \cdots$$

$$\cdots \to i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X}) \xrightarrow{\pi_q} \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X}) \to 0$$

where we put $i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X})$ in degree zero. This gives rise to the map

$$\delta : H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \to \mathbb{H}^q(\mathcal{X}, \mathcal{C}(p)) \cong H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}).$$

Notation 1.3.6. Note that a section ξ of $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}$ over $\mathbb{P}^r \times \mathbb{P}^n \setminus \mathcal{X}$ has a pole on $\mathcal{X}$ of order at most a if and only if it is of the form $\xi = \frac{A\omega\wedge\bar{\omega}}{F^a}$ where $A \in k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]$ is of bidegree $(a-(r+1), am-(n+1))$. This gives an isomorphism

$$\tilde{\psi_a} : k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^{a-(r+1),am-(n+1)} \to H^0(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X}))$$

sending an element A to $\frac{A\omega\wedge\bar{\omega}}{F^a}$.

Remark 1.3.7. As $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(\log(\mathcal{X}))$ is isomorphic to $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X})$, we can view the residue map in the exact sequence (1.2) in this degree as a morphism $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X}) \to i_*\Omega^{n+r-1}_{\mathcal{X}}$.

The purpose of this section is to prove the following statement, which is a generalization of [33, Proposition 3.2] to products of projective spaces. The argument is taken from [33], with some adaptations.

Proposition 1.3.8. *Suppose that $q \geq r$. The composition*

$$k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^{(q+1)-(r+1),(q+1)m-(n+1)} \xrightarrow{\tilde{\psi}_{q+1}}$$
$$H^0(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X})) \xrightarrow{res} H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \xrightarrow{\delta} H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})$$

descends to an isomorphism

$$\psi_q : J^{(q+1)-(r+1),(q+1)m-(n+1)} \to H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim}.$$

Remark 1.3.9. Let $X = V(F_0) \subset \mathbb{P}^n$ be a smooth hypersurface defined by a homogeneous polynomial $F_0 \in k[X_0, \cdots, X_n]$ of degree m. Then we can form the hypersurface $\mathcal{X} = V(Y_0F_0) \subset \mathbb{P}^0 \times \mathbb{P}^n$ which is isomorphic to X. The corresponding Jacobian ring is given by

$$J = k[Y_0, X_0 \cdots, X_n]/\left(F_0, Y_0\frac{\partial F_0}{\partial X_0}, \cdots, Y_0\frac{\partial F_0}{\partial X_n}\right).$$

Consider the usual Jacobian ring

$$J_X = k[X_0, \cdots, X_n]/\left(\frac{\partial F_0}{\partial X_0}, \cdots, \frac{\partial F_0}{\partial X_n}\right)$$

then for $a, b \in \mathbb{Z}$ we have a natural map

$$J^{a,b} \to J^b_X, Y_0 \mapsto 1, X_j \mapsto X_j.$$

For fixed a, there is the section

$$g_a : J^b_X \to J^{a,b}, f \mapsto Y_0^a f$$

which is an isomorphism. Now let $p, q \in \mathbb{Z}_{\geq 0}$ be such that $p + q = n - 1$. If we set $r = 0$ (which is not possible with the assumptions we made, but we still do it for a moment) in Proposition 1.3.8, we find an isomorphism

$$\psi_q \circ g_q : J_X^{(q+1)m-n-1} \to H^q(X, \Omega^p_X)_{prim}.$$

We therefore find that the above statement is in accordance with [33, Proposition 3.2].

1.3.1 An exact sequence relating δ and π_q

In order to prove Proposition 1.3.8, we will need the following proposition, which is a generalization of [33, Lemma 3.1(2)]. The proof is more or less the same, but included here for the reader's convenience.

Proposition 1.3.10. *The map*

$$\delta : H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \to \mathbb{H}^q(\mathcal{X}, \mathcal{C}(p)) \cong H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})$$

gives rise to an exact sequence

$$H^0(\mathcal{X}, i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) \xrightarrow{\pi_q} H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \xrightarrow{\delta} H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}) \tag{1.4}$$

where δ is surjective if $p \neq q$ and has image $H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim}$ in case that $p = q$.

In order to prove Proposition 1.3.10, we will use the hypercohomology spectral sequence

$$E_1^{a,b} = H^b(\mathcal{X}, \mathcal{C}(p)^a) \implies \mathbb{H}^{a+b}(\mathcal{X}, \mathcal{C}(p)) \cong H^{a+b}(\mathcal{X}, \Omega^p_{\mathcal{X}}).$$

We first prove two lemmas.

Lemma 1.3.11. *$E_1^{a,b} = 0$ for all $b > 0$ and $a < q$, except for $a = 0$ and $b = p$.*

Proof. Let $a, b \in \mathbb{Z}$ be such that $b > 0$ and $0 \leq a < q$. Note that there is the standard exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n}(-\mathcal{X}) \to \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n} \to i_*\mathcal{O}_{\mathcal{X}} \to 0.$$

Tensoring the above exact sequence with the sheaf $\Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X})$ and noting that this sheaf is locally free gives the exact sequence

$$0 \to \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X}) \to \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}) \to i_*i^*\Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}) \to 0 \tag{1.5}$$

where we used the projection formula to see that

$$i_*\mathcal{O}_{\mathcal{X}} \otimes \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}) = i_*(\mathcal{O}_{\mathcal{X}} \otimes i^*\Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}))$$

$$= i_* i^* \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}).$$

Part of the long exact cohomology sequence of (1.5) is

$$\cdots \to H^b(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X})) \to H^b(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X})) \to \quad (1.6)$$
$$H^b(\mathbb{P}^r\times\mathbb{P}^n, i_*i^*\Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X})) \to H^{b+1}(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X})) \to \cdots$$

We have by Proposition 1.2.10 that

$$H^b(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X})) = \bigoplus_{i+j=p+a+1}\bigoplus_{k+l=b} H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(a+1))\otimes H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(m(a+1))).$$

As $a+1>0$, by Theorem 1.2.9 one has that $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(a+1)) = 0$ provided that $k>0$ and $a+1\geq i-r$ or $k=0$ and $a+1\leq i$. We note that:

- If $k>0$ and $a+1<i-r$ we see in particular that $i-r>0$ and so $i>r$ implying that $\Omega^i_{\mathbb{P}^r}=0$ and so $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(a+1))=0$.
- If $k=0$ and $a+1>i$ then $l>0$ as $b>0$. If $m(a+1)\geq j-n$ then $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(m(a+1)))=0$. Otherwise, $m(a+1)<j-n$ implies that $j>n$ and so $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(m(a+1)))=0$ in this case as well.

This proves that $H^b(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}))=0$ for $b>0$ and $a\geq 0$. Furthermore, using Proposition 1.2.10 again we have that

$$H^{b+1}(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X})) = \bigoplus_{i+j=p+a+1}\bigoplus_{k+l=b+1} H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(a))\otimes H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(ma)).$$

We distinguish between two cases:

- If $a>0$ then $H^{b+1}(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X}))=0$ in a similar way as above.
- If $a=0$ then using Theorem 1.2.9 we have that $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r})=0$ provided that $k\neq i$ and $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n})=0$ provided that $l\neq j$, so the only case where this is nonzero is if $k=i$ and $l=j$, i.e. if $p=b$.

So if we assume that $p\neq b$ or $a>0$, then $H^{b+1}(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}(a\mathcal{X}))=0$. From the sequence (1.6), we see that

$$E_1^{a,b} = H^b(\mathcal{X}, i^*\Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X})) = H^b(\mathbb{P}^r\times\mathbb{P}^n, i_*i^*\Omega^{p+a+1}_{\mathbb{P}^r\times\mathbb{P}^n}((a+1)\mathcal{X}))=0$$

which is precisely what we needed to show. □

Construction 1.3.12. Now suppose that $a=0, b=p$ and $p>0$.
We have seen in the proof of Lemma 1.3.11 that

$$H^p(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X})) = H^{p+1}(\mathbb{P}^r\times\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X}))=0.$$

From the exact sequence (1.6) it follows that

$$E_1^{0,p} = H^p(\mathcal{X}, i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X})) \cong H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}). \tag{1.7}$$

Note that $E_\infty^{0,p}$ is nonzero. Indeed, $E_1^{a,b} = 0$ if $a + b = p + 1$ and $b > 0$, and one can compute that $E_1^{p+1,0} = 0$. Also, we have that

$$H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}) \cong \bigoplus_{i+j=p+1} \bigoplus_{k+l=p+1} H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}) \otimes H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}) \cong k^{p+1}$$

and so $E_1^{0,p}$ is nonzero by (1.7).
There is a surjection $H^p(\mathcal{X}, \Omega^p_{\mathcal{X}}) \to E_\infty^{0,p}$. From this, we can define the map

$$\beta : H^p(\mathcal{X}, \Omega^p_{\mathcal{X}}) \to E_1^{0,p} = H^p(\mathcal{X}, i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X})).$$

Using (1.7) again, we can view β as a map

$$H^p(\mathbb{P}^r \times \mathbb{P}^n, i_*\Omega^p_{\mathcal{X}}) \to H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n})$$

after identifying $H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})$ with $H^p(\mathbb{P}^r \times \mathbb{P}^n, i_*\Omega^p_{\mathcal{X}})$.

Lemma 1.3.13. *The morphism β is precisely the coboundary map from the exact sequence (1.2). As a consequence, β is surjective if $p \geq r$.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n} & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\log(\mathcal{X})) & \xrightarrow{\text{res}} & i_*\Omega^p_{\mathcal{X}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle dF/F\wedge(-)} & & \\
0 & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n} & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X}) & \longrightarrow & i_*i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X}) & \longrightarrow & 0
\end{array}$$

with the top row coming from the exact sequence (1.2) and the lower row coming from the exact sequence (1.5).
This diagram commutes, because we know from the exact sequence (1.1) that the map $dF/F \wedge (-)$ is precisely the inclusion of $\mathcal{O}_{\mathcal{X}}(-\mathcal{X})$ into $i^*\Omega_{\mathbb{P}^r\times\mathbb{P}^n}$. By Lemma 1.2.7, the coboundary map of the long exact sequence associated to the upper sequence is precisely i_*. Therefore, we have a commutative diagram

$$\begin{array}{ccc}
H^p(\mathbb{P}^r \times \mathbb{P}^n, i_*\Omega^p_{\mathcal{X}}) & \xrightarrow{i_*} & H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}) \\
\downarrow{\scriptstyle dF/F\wedge(-)} & \searrow^{\beta} & \| \\
H^p(\mathbb{P}^r \times \mathbb{P}^n, i_*i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}(\mathcal{X})) & \xrightarrow{\cong} & H^{p+1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n})
\end{array}$$

Here we note that we constructed the spectral sequence from the exact sequence (1.1), so that β is by construction the map induced from $dF/F \wedge (-)$, composed with the isomorphism coming from the coboundary map in the sequence (1.5). It follows that $\beta = i_*$. We note that i_* is surjective if $p \geq r$ by Lemma 1.2.13, which gives the last part of the statement. $\square$

Lemma 1.3.14. *All differentials going into or out of $E_s^{0,p}$ are zero for $p \geq r$.*

Proof. Note that all incoming differentials to the terms $E_s^{0,p}$ are zero by reason of degree, and we have that $E_s^{0,p} = \ker(d_{s-1}^{0,p}) \subset E_{s-1}^{0,p}$ for all $s \geq 2$. Now the fact that the edge map β is surjective for $p \geq r$ by Lemma 1.3.13 shows that all outgoing differentials are zero. □

Proof of Proposition 1.3.10. Note that if $q = 0$, then we have the exact sequence

$$H^0(\mathcal{X}, i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}) \to H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}) \to H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}})$$

where the last map is the identity and

$$H^0(\mathcal{X}, i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}) = H^0(\mathbb{P}^r \times \mathbb{P}^n, i_*i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}).$$

Note that

$$\begin{aligned} H^0(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}) &\cong H^0(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}) \otimes H^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) \\ &\quad \oplus H^0(\mathbb{P}^r, \Omega^{r-1}_{\mathbb{P}^r}) \otimes H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}) \\ &= 0 \end{aligned}$$

using Proposition 1.2.10 and Theorem 1.2.9 again. Similarly, we have that

$$\begin{aligned} H^1(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(-\mathcal{X})) &\cong H^0(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}(-1)) \otimes H^1(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(-m)) \\ &\quad \oplus H^1(\mathbb{P}^r, \Omega^{r-1}_{\mathbb{P}^r}(-1)) \otimes H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(-m)) \\ &\quad \oplus H^0(\mathbb{P}^r, \Omega^{r-1}_{\mathbb{P}^r}(-1)) \otimes H^1(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(-m)) \\ &\quad \oplus H^1(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}(-1)) \otimes H^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(-m)) \\ &= 0 \end{aligned}$$

so using the long exact sequence (1.6) we find that $H^0(\mathcal{X}, i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}) = 0$. Therefore, we assume that $q > 0$ from now on.

First assume that $p \neq q$. We note that the contributions to $H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})$ come from all $E_1^{a,b}$ satisfying $a + b = q$. These are all zero except possibly for $E_1^{q,0}$, by Lemma 1.3.11. By Lemma 1.3.11, we have that $E_1^{a,b} = 0$ for $a + b = q - 1$ and $0 < a \leq q - 1$, except possibly for $(a, b) = (q - 1, 0)$ and, if $p = q - 1$, also $(a, b) = (0, q - 1)$. Thus, the only possible non-zero incoming differentials to $E_*^{q,0}$ are $d_1^{q-1,0}$ and, in case $p = q - 1$, also $d_q^{0,q-1}$. Also, there are no outgoing differentials out of $E_*^{q,0}$ by reason of degree.

If $p = q-1$ and $q \neq 1$, then as $p+q = n+r-1$, we have that $2p = n+r-2 \geq 2r$ and so $p \geq r$. Therefore by Lemma 1.3.14, all outgoing differentials of $E_q^{0,q-1}$ are zero. If $q = 1$, then $E_1^{0,q-1} = E_1^{0,0}$ and so it coincides with $E_1^{q-1,0}$. Either way, we therefore have that the only possibly nonzero incoming differential comes from $E_1^{q-1,0}$, and so $E_\infty^{q,0}$ is equal to $E_2^{q,0}$, i.e. $E_1^{q,0}$ modulo the image of $E_1^{q-1,0}$. Since all the differentials leaving $E_*^{q,0}$ are zero, we have the edge homomorphism

$$E_1^{q,0} = H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \to H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})$$

which is exactly the map δ. Since $E_2^{q,0} = E_\infty^{q,0}$, this gives an exact sequence

$$0 \to H^0(\mathcal{X}, i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) \to H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \xrightarrow{\delta} H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}).$$

Note that the map δ is surjective in this case since only $E_*^{q,0}$ can contribute. This completes the proof in case $p \neq q$.
If $p = q$ then the possibly nonzero term $E_1^{0,p}$ also contributes to $H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})$. There are no incoming differentials to $E_*^{0,p}$ and since $p = q = n + r - 1 - p$, we have $p \geq r$, so there no outgoing differentials by Lemma 1.3.14, and we have $E_1^{0,p} = E_\infty^{0,p}$. Thus, the edge homomorphism

$$\beta : H^p(\mathcal{X}, \Omega^p_{\mathcal{X}}) \to E_1^{0,p} = H^p(\mathcal{X}, i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n})$$

is surjective, and we find an exact sequence

$$\begin{aligned}0 \to H^0(\mathcal{X}, i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) \to H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \xrightarrow{\delta} H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})\\ \xrightarrow{\beta} H^p(\mathcal{X}, i^*\Omega^{p+1}_{\mathbb{P}^r\times\mathbb{P}^n}) \to 0\end{aligned}$$

This means that the image of δ is equal to the kernel of β, which by Lemma 1.3.13 is the kernel of i_*, which is by definition the primitive cohomology. □

1.3.2 Proof of Proposition 1.3.8

Proof of Proposition 1.3.8. We first note that $H^1(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) = 0$. Indeed, it follows from Proposition 1.2.10 that

$$\begin{aligned}H^1(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) = \bigoplus_{i+j=n+r} H^0(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(q)) \otimes H^1(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(mq))\\ \oplus H^1(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(q)) \otimes H^0(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(mq))\end{aligned}$$

Using Theorem 1.2.9 we observe that:

- If $q > 0$ then $H^1(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(mq)) = 0$ provided that $mq \geq j - n$. But if $mq < j - n$ we have that $j > n$ and so $H^1(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(mq))$ is always equal to zero. For similar reasons, $H^1(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(q))$ is always equal to zero and so $H^1(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) = 0$ in this case.

- If $q = 0$ then $H^1(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}) = 0$ provided that $j \neq 1$. However, in case $j = 1$ we have that $i = n+r-1 > 0$ as $n \geq 2$. Now from Theorem 1.2.9 it follows that $H^0(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}) = 0$. Similarly the second factor always vanishes.

This verifies the claim.
This means that the exact sequence (1.5) gives rise to an exact sequence

$$\begin{aligned}0 \to H^0(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) \to H^0(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X}))\\ \to H^0(\mathcal{X}, i^*\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X})) \to 0.\end{aligned}$$

We now consider res : $H^0(\mathbb{P}^r\times\mathbb{P}^n,\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X}))\to H^0(\mathcal{X},\Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X}))$. Noting that the kernel of res is $H^0(\mathbb{P}^r\times\mathbb{P}^n,\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X}))$, we see that res descends to a map

$$\widetilde{\mathrm{res}}: H^0(\mathcal{X},i^*\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X}))\to H^0(\mathcal{X},\Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X}))$$

with the same image as res. Furthermore, we observe that the map $\tilde{\psi}_{q+1}$ now gives rise to an isomorphism

$$k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]/(F)^{q-r,(q+1)m-(n+1)}\xrightarrow{\bar{\psi}_{q+1}} H^0(\mathcal{X},i^*\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X})). \tag{1.8}$$

We now consider the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{X},i^*\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X})) & \xleftarrow{\bar{\psi}_{q+1}} & k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]/(F)^{\alpha} \\ {\scriptstyle dF/F\wedge(-)}\uparrow & \searrow^{\widetilde{\mathrm{res}}} & \downarrow{\scriptstyle f} \\ H^0(\mathcal{X},i^*\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n}(q\mathcal{X})) & \xrightarrow{\pi_q} & H^0(\mathcal{X},\Omega^{n+r-1}_{\mathcal{X}}(q\mathcal{X})) \end{array} \tag{1.9}$$

where $\alpha=((q+1)-(r+1),(q+1)m-(n+1))$ is the bidegree from the statement and $f=\widetilde{\mathrm{res}}\circ\bar{\psi}_{q+1}$. For the rest of the proof, we will show that the image of the Jacobian ideal under f is the same as the image of π_q, i.e. the kernel of δ. This shows that the Jacobian ideal is precisely the kernel of the composition $\delta\circ f$. Note that the image of $\delta\circ f$ is the image of δ as res is surjective and $\bar{\psi}_{q+1}$ is an isomorphism. The image of δ however, is precisely the primitive cohomology. Therefore, noting that F is in the Jacobian ideal, we find an isomorphism

$$k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]/(F_0,\cdots,F_r,\bar{F}_0,\cdots,\bar{F}_n)^{\alpha}\to H^p(\mathcal{X},\Omega^p_{\mathcal{X}})_{prim}.$$

This will complete the proof.

We start by noting that the image of ideal $\left(F_0,\cdots,F_r,\bar{F}_0,\cdots,\bar{F}_n\right)/F$ under the isomorphism (1.8) is generated by elements of the form $F_i\omega\wedge\bar{\omega}$ and $\bar{F}_j\omega\wedge\bar{\omega}$. Following [50], we note that the sheaf $\Omega^r_{\mathbb{P}^r}(r+1)\otimes\Omega^{n-1}_{\mathbb{P}^n}(n)$ has global sections generated by the sections $\omega\wedge\tau_i$ for $i\in\{0,\cdots,n\}$ where

$$\tau_i=\sum_{j<i}(-1)^jX_jdX^{j,i}+\sum_{i<j}(-1)^{j+1}X_jdX^{i,j}.$$

So a section of this sheaf with a pole along $\mathcal{X}$ of order at most q is of the form $\sum_{i=0}^n\frac{B_i\omega\wedge\tau_i}{F^q}$ where B_i has bidegree $(q-(r+1),qm-n)$. We note that $dF=\sum_{i=0}^rF_idY_i+\sum_{j=0}^n\bar{F}_jdX_j$ and we also note that for $i,j\in\{0,\cdots,n\}$, we have that

$$dX_j\wedge\tau_i=\begin{cases}X_jdX^i & \text{if } i\neq j\\ \sum_{j<i}(-1)^{j+i-1}X_jdX^j+\sum_{j>i}(-1)^{j+i+1}X_jdX^j & \text{otherwise}\end{cases}$$

So

$$\left(\sum_{j=0}^n\bar{F}_jdX_j\right)\wedge\tau_i=\sum_{j\neq i}\bar{F}_jX_jdX^i+(-1)^{i-1}\bar{F}_i\sum_{j\neq i}(-1)^jX_jdX^j$$

$$= (-1)^{i-1}\bar{F}_i\bar{\omega} + mFdX^i.$$

It follows that modulo F, we have that

$$\frac{dF}{F} \wedge \sum_{i=0}^{n} \frac{B_i\omega \wedge \tau_i}{F^q} = \sum_{i=0}^{n}(-1)^{i-1}\frac{B_i\bar{F}_i\omega \wedge \bar{\omega}}{F^{q+1}}$$

We can do a similar thing for a section of $\Omega^{r-1}_{\mathbb{P}^r}(r) \otimes \Omega^n_{\mathbb{P}^n}(n+1)$.
Using [23, Exercise II.8.3] we see that $\Omega^{n+r-1}_{\mathbb{P}^r\times\mathbb{P}^n} = \bigoplus_{i+j=n+r-1} \pi_r^*\Omega^i_{\mathbb{P}^r} \otimes \pi_n^*\Omega^j_{\mathbb{P}^n}$. As the map $dF/F\wedge(-)$ increases the bidegrees of the corresponding elements of J by $(1, m-1)$, the only forms that will end up in the bidegree α when applying $dF/F\wedge(-)$ are either coming from $\Omega^r_{\mathbb{P}^r}\otimes\Omega^{n-1}_{\mathbb{P}^n}$ or from $\Omega^{r-1}_{\mathbb{P}^r}\otimes\Omega^n_{\mathbb{P}^n}$. Using the fact that $H^0(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X}))$ surjects onto $H^0(\mathcal{X}, i^*\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}((q+1)\mathcal{X}))$ the above computation shows that the image of the map $dF/F \wedge (-)$ in the diagram (1.9) is the same as the image of $\widetilde{\psi}_{q+1}$, and so the images remain the same after applying $\widetilde{\text{res}}$. This shows that f and π_q indeed have the same image, which completes the proof. □

1.4 Comparing the two products

The next goal is to compare the cup product on cohomology to the ring multiplication on the Jacobian ring. We keep all notation from the previous two sections.

Notation 1.4.1. Let $\mathcal{U} = \{U_0, \cdots, U_r, \bar{U}_0, \cdots, \bar{U}_n\}$ be the open cover of $\mathbb{P}^r\times\mathbb{P}^n$ where $U_i = \{F_i \neq 0\}$ for $i \in \{0, \cdots, r\}$ and $\bar{U}_j = \{\bar{F}_j \neq 0\}$ for $j \in \{0, \cdots, n\}$. Note that this is an open cover of $\mathbb{P}^r\times\mathbb{P}^n$ because $\mathcal{X}$ is smooth. We have that $\mathcal{U}$ restricts to a cover of $\mathcal{X}$, and we will use the same notation for both. Note that the U_i are not affine and that the F_i's and $\bar{F}_j$'s do not have the same bidegrees. We define an order on these open subsets as follows:

$$U_0 < U_1 < \cdots < U_r < \bar{U}_0 < \cdots < \bar{U}_n.$$

Notation 1.4.2. For $i \in \{0, \cdots, r\}$, we let K_i be inner multiplication with $\frac{\partial}{\partial Y_i}$, i.e. for $i \in \{0, \cdots, r\}$ we have that

$$K_i(dY_{i_1} \wedge \cdots \wedge dY_{i_l} \wedge dX_{j_1} \wedge \cdots \wedge dX_{j_k})$$
$$= \begin{cases} (-1)^{a-1}dY_{i_1} \wedge \cdots d\hat{Y}_{i_a} \cdots \wedge dY_{i_l} \wedge dX_{j_1} \wedge \cdots \wedge dX_{j_k} & \text{if } i = i_a \\ 0 & \text{if } i \notin \{i_1, \cdots, i_l\} \end{cases}$$

Similarly, for $j \in \{0, \cdots, n\}$, we let $\tilde{K}_j$ be inner multiplication with $\frac{\partial}{\partial X_j}$, i.e. for $j \in \{0, \cdots, n\}$, we have that

$$\bar{K}_j(dY_{i_1} \wedge \cdots \wedge dY_{i_l} \wedge dX_{j_1} \wedge \cdots \wedge dX_{j_k})$$
$$= \begin{cases} (-1)^{l+b-1}dY_{i_1} \wedge \cdots \wedge dY_{i_l} \wedge dX_{j_1} \wedge \cdots d\hat{X}_{j_b} \cdots \wedge dX_{j_k} & \text{if } j = j_b \\ 0 & \text{if } j \notin \{j_1, \cdots, j_k\} \end{cases}$$

We have that $K_i(\alpha \wedge \beta) = K_i(\alpha) \wedge \beta + (-1)^k \alpha \wedge K_i(\beta)$, if α is a k-form, and a similar formula holds for the $\bar{K}_j$'s.

Notation 1.4.3. For subsets $I = i_1, \cdots, i_l \subset \{0, \cdots, r\}$ and $J \subset \{0, \cdots, n\}$, we set:

- $U_{I,J} = \bigcap_{i \in I} U_i \cap \bigcap_{j \in J} \bar{U}_j$.
- $\Omega_I = (\prod_{i \in I} K_i)(\omega) = (K_{i_l} \circ K_{i_{l-1}} \circ \cdots \circ K_{i_1})(\omega)$ and $\bar{\Omega}_J = (\prod_{j \in J} \bar{K}_j)(\bar{\omega})$.
- $F_I = \prod_{i \in I} F_i$ and $\bar{F}_J = \prod_{j \in J} \bar{F}_j$.

Notation 1.4.4. For a sheaf $\mathcal{F}$ on $\mathcal{X}$ or on $\mathbb{P}^r \times \mathbb{P}^n$, we have the group $C^i(\mathcal{U}, \mathcal{F})$ consisting of all families $\{s_{I,J} \in \mathcal{F}(U_{I,J})\}_{I,J}$ where $\#I + \#J = i + 1$. These form the Čech complex

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \to \cdots \to C^{n+r+1}(\mathcal{U}, \mathcal{F}) \to 0$$

with differentials $\delta : C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$ given by

$$\delta(\{s_{I,J}\}_{I,J}) = \Big\{ \sum_{k=0}^{i+1} (-1)^k s_{(I,J)\setminus\{(I,J)_k\}}|_{U_{I,J}} \Big\}_{I,J}.$$

Here, $(I, J)_k$ denotes the k'th element of the ordered set (I, J). The cohomology groups of the above complex are denoted by $\check{H}^a(\mathcal{U}, \mathcal{F})$. Note that there are natural maps $\check{H}^a(\mathcal{U}, \mathcal{F}) \to H^a(\mathcal{X}, \mathcal{F})$ or $\check{H}^a(\mathcal{U}, \mathcal{F}) \to H^a(\mathbb{P}^r \times \mathbb{P}^n, \mathcal{F})$, by [23, Lemma II.4.4].

Notation 1.4.5. This notation is taken from [50, page 222] with a small adaptation, see the remark below. Fix $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = n + r - 1$. Consider the bidegree

$$\rho = (n - r - 1, (n + r + 1)m - 2(n + 1)).$$

For subsets $I \subset \{0, \cdots, r\}$ and $J \subset \{0, \cdots, n\}$ such that $\#I + \#J = n + r$, we define an element $\Omega(I, J) \in H^0(\mathcal{X} \cap U_{I,J}, i^*\Omega^{n+r-1}_{\mathbb{P}^r \times \mathbb{P}^n}(-\rho))$ as follows:

- If we are not in the situation where $\#I = r$ and $\#J = n$, then $\Omega(I, J) = 0$.
- If $I = \{i_0, \cdots, i_{r-1}\} \subset \{0, \cdots, r\}$ and $J = \{j_0, \cdots, j_{n-1}\} \subset \{0, \cdots, n\}$, then:
  - If $q \leq r - 1$, we write

$$I' = \begin{cases} \{i_0, \cdots, i_{q-1}\} & \text{if } q > 0 \\ \emptyset & \text{if } q = 0 \end{cases}$$

and

$$I'' = \begin{cases} \{i_{q+1}, \cdots, i_{r-1}\} & \text{if } q < r-1 \\ \emptyset & \text{if } q = r-1 \end{cases}$$

We define

$$\Omega(I,J) = \frac{(-1)^{nq}\Omega_{I',i_q} \wedge \bar{\omega} \wedge \Omega_{i_q,I''} \wedge \bar{\Omega}_J}{F_{I'}F_{i_q}^2 F_{I''}\bar{F}_J}.$$

– If $q \geq r$, then we write

$$J' = \begin{cases} \{j_0, \cdots, j_{q-r-1}\} & \text{if } q > r \\ \emptyset & \text{if } q = r \end{cases}$$

and

$$J'' = \begin{cases} \{j_{q-r+1}, \cdots, j_{n-1}\} & \text{if } q < n+r-1 \\ \emptyset & \text{if } q = n+r-1 \end{cases}$$

We define

$$\Omega(I,J) = \frac{(-1)^{r(n+q+r)}\Omega_I \wedge \bar{\Omega}_{J',j_{q-r}} \wedge \omega \wedge \bar{\Omega}_{j_{q-r},J''}}{F_I \bar{F}_{J'} \bar{F}_{j_{q-r}}^2 \bar{F}_{J''}}.$$

Remark 1.4.6. Note that $\Omega(I,J)$ can also be defined without distinguishing the case $\#I = r$ and $\#J = n$, which is the definition used in [50]. We then still have that $\Omega(I,J) = 0$ unless $\#I = r$ and $\#J = n$. Namely, $\Omega_{\{0,\cdots,r\}} = 0$ and $\bar{\Omega}_{\{0,\cdots,n\}} = 0$, which implies that $\Omega(I,J) = 0$ for $\#I \geq r+1$ or $\#J \geq n+1$. As $\#I + \#J = n+r$, this means that $\Omega(I,J) = 0$ unless $\#I = r$ and $\#J = n$.

In subsection 1.4.1, we will prove the following statement, which is a generalization of [50, Proposition 2.8] to other fields than $\mathbb{C}$.

Proposition 1.4.7. *For $A \in J^{q-r,(q+1)m-(n+1)}$ and $B \in J^{p-r,(p+1)m-(n+1)}$, write*

$$\omega_A = \psi_q(A) \in H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}) \text{ and } \omega_B = \psi_p(B) \in H^p(\mathcal{X}, \Omega^q_{\mathcal{X}}).$$

Then the cup product $\omega_A \cup \omega_B \in H^{n+r-1}(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}})$ is represented by the Čech cochain

$$\{\pi(AB\Omega(I,J))\}_{I,J} \in C^{n+r-1}(\mathcal{U}, \Omega^{n+r-1}_{\mathcal{X}}).$$

The argument is taken almost directly from [50], combined with some elements of the arguments in [33].

Remark 1.4.8 (Remark 1.3.9 continued)**.** Let $V(F_0) \subset \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial $F_0 \in k[X_0, \cdots, X_n]$ of degree m. If we set $r = 0$ then for $J = \{j_0, \cdots, j_{n-1}\} \subset \{0, \cdots, n\}$ and $I = \emptyset$, we have

$$\Omega(I,J) = \frac{\bar{\Omega}_{j_0,\cdots,j_q} \wedge \bar{\Omega}_{j_q,\cdots,j_{n-1}}}{Y_0^n \frac{\partial F_0}{\partial X_q} \prod_{j=0}^n \frac{\partial F_0}{\partial X_j}}.$$

One can check that Proposition 1.4.7 then becomes [33, Proposition 3.6(1)].

We can compute the element $i_*(\omega_A \cup \omega_B)$ as follows, generalizing [33, Proposition 3.7(2)] to a product of projective spaces.

Proposition 1.4.9. *We have that $i_*(\omega_A \cup \omega_B) \in H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$ is represented by the cochain in $\{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\} \in C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$ given by*

$$s_v = \frac{(-1)^{v+r+1} mABY_v F_v \omega \wedge \bar{\omega}}{\prod_{i=0}^r F_i \prod_{j=0}^n \bar{F}_j}$$

for $v \in \{0, \cdots, r\}$ corresponding to the intersection of all opens except for U_v and

$$\bar{s}_w = \frac{(-1)^{w+1} ABX_w \bar{F}_w \omega \wedge \bar{\omega}}{\prod_{i=0}^r F_i \prod_{j=0}^n \bar{F}_j}$$

for $w \in \{0, \cdots, n\}$ corresponding to the intersection of all opens except for $\bar{U}_w$.

It will have the following consequence, which can be viewed as a generalization of [50, Corollary 2.9] to other fields than $\mathbb{C}$.

Corollary 1.4.10. *Consider the morphism*

$$\tilde{\phi} : k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^\rho \to C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}), D \mapsto \{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\}$$

where

$$s_v = \frac{(-1)^{v+r+1} mDY_v F_v \omega \wedge \bar{\omega}}{\prod_{i=0}^r F_i \prod_{j=0}^n \bar{F}_j} \text{ and } \bar{s}_w = \frac{(-1)^{w+1} DX_w \bar{F}_w \omega \wedge \bar{\omega}}{\prod_{i=0}^r F_i \prod_{j=0}^n \bar{F}_j}$$

for $v \in \{0, \cdots, r\}$ and $w \in \{0, \cdots, n\}$. This gives rise to a surjective morphism $\phi : J^\rho \to H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) \cong k$, such that the diagram

$$\begin{array}{ccc} H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim} \otimes H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})_{prim} & \xrightarrow{i_* \circ \cup} & H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) \\ \uparrow{\scriptstyle \psi_p \otimes \psi_q} & & \uparrow{\scriptstyle \phi} \\ J^{q-r,(q+1)m-(n+1)} \otimes J^{p-r,(p+1)m-(n+1)} & \longrightarrow & J^\rho \end{array}$$

commutes.

Then in subsection 1.4.3, we use a slight variation on an argument from [27] to find the following.

Corollary 1.4.11. *The map ϕ is an isomorphism, except possibly when n is odd, $r = 1$ and $m = 2$.*

Remark 1.4.12. Note that if n is odd and $r = 1$, we have that $n + r - 1 = n$ is odd, so that $\mathcal{X}$ has odd dimension. By the Motivic Gauss Bonnet Theorem, see Theorem 1.1.16, we have that $\chi(\mathcal{X}/k)$ is hyperbolic in this case. Therefore, the one exception is not a problem for our purposes. Still, it is a good question why a complete intersection of two quadrics in an odd dimensional projective space is an exception. We do not have an explanation for this.

We also introduce a variant of the Jacobian ring, namely, the ring

$$\tilde{J} = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]/(Y_0F_0, \cdots, Y_rF_r, X_0\bar{F}_0, \cdots, X_n\bar{F}_n)$$

and show the following statement.

Proposition 1.4.13. *$\tilde{J}^{\rho+(r+1,n+1)}$ is a one dimensional vector space over k.*

1.4.1 Proof of Proposition 1.4.7

In order to prove Proposition 1.4.7, we first prove two lemmas and set up some notation.

Construction 1.4.14. Over an open U_i for $i \in \{0, \cdots, r\}$, one can define a splitting of the inclusion $dF/F \wedge (-) : \mathcal{O}_{\mathcal{X}}(-\mathcal{X}) \to i^*\Omega_{\mathbb{P}^r\times\mathbb{P}^n}$ by

$$H_i\left(\sum_{i=0}^{r} a_i dY_i + \sum_{k=0}^{n} b_k dX_k\right) = a_i F \cdot F_i^{-1}$$

extending to a map $H_i : i^*\Omega^a_{\mathbb{P}^r\times\mathbb{P}^n}(b\mathcal{X}) \to i^*\Omega^{a-1}_{\mathbb{P}^r\times\mathbb{P}^n}((b-1)\mathcal{X})$. Similarly, over $\bar{U}_j$ for $j \in \{0, \cdots, n\}$ one can define the splitting

$$\bar{H}_j\left(\sum_{i=0}^{r} a_i dY_i + \sum_{k=0}^{n} b_k dX_k\right) = b_j F \cdot \bar{F}_j^{-1}$$

extending to a map $\bar{H}_j : i^*\Omega^a_{\mathbb{P}^r\times\mathbb{P}^n}(b\mathcal{X}) \to i^*\Omega^{a-1}_{\mathbb{P}^r\times\mathbb{P}^n}((b-1)\mathcal{X})$. We get a map on Čech cochains

$$H : C^q(\mathcal{U}, i^*\Omega^a_{\mathbb{P}^r\times\mathbb{P}^n}(b\mathcal{X})) \to C^q(\mathcal{U}, i^*\Omega^{a-1}_{\mathbb{P}^r\times\mathbb{P}^n}((b-1)\mathcal{X}))$$

defined by

$$H(\{s_{I,J}\}_{I,J}) = \begin{cases} \{H_{(I,J)_0}(s_{I,J})\}_{I,J} & \text{if } I \neq \emptyset \\ \{\bar{H}_{(I,J)_0}(s_{I,J})\}_{I,J} & \text{otherwise} \end{cases}$$

where $(I, J)_0$ denotes the first element in an ordered index (I, J). We note that $H_i = F/F_i \cdot K_i$ for $i \in \{0, \cdots, r\}$ and $\bar{H}_j = F/\bar{F}_j \cdot \bar{K}_j$ for $j \in \{0, \cdots, n\}$.

The following statement is a generalization of [33, Lemma 3.4] to products of projective spaces.

Lemma 1.4.15. *Let $A \in k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]$ be a polynomial of bidegree $(b-(r+1), bm-(n+1))$ for some b. Then:*

1. *$res(A\omega \wedge \bar{\omega}/F^{b+1}) \in H^0(\mathcal{X}, \Omega^{n+r-1}_{\mathcal{X}}(b\mathcal{X}))$ is represented by the element $\{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\} \in C^0(\mathcal{U}, \Omega^{n+r-1}_{\mathcal{X}}(b\mathcal{X}))$ given by*

$$s_i = \frac{A\Omega_i \wedge \bar{\omega}}{F_i F^b} \in \Omega^{n+r-1}_{\mathcal{X}}(b\mathcal{X})(U_i)$$

for $i \in \{0, \cdots, r\}$ and

$$\bar{s}_j = \frac{(-1)^r A\omega \wedge \bar{\Omega}_j}{\bar{F}_j F^b} \in \Omega^{n+r-1}_{\mathcal{X}}(b\mathcal{X})(\bar{U}_j)$$

for $j \in \{0, \cdots, n\}$.

2. *For an element $\{\frac{A}{F_I \bar{F}_J F^b}\Omega_I \wedge \bar{\Omega}_J\}_{I,J} \in C^i(\mathcal{U}, i^*\Omega^{a-1}_{\mathbb{P}^r \times \mathbb{P}^n}(b-1)\mathcal{X})$ we have that*

$$H\left(\left\{(dF/F) \wedge (\frac{A}{F_I \bar{F}_J F^b}\Omega_I \wedge \bar{\Omega}_J)\right\}_{I,J}\right) = \left\{\frac{A}{F_I \bar{F}_J F^b}\Omega_I \wedge \bar{\Omega}_J\right\}_{I,J}.$$

3. *Applying the Čech differential δ to*

$$\alpha = \left\{\frac{(-1)^{(r+\#I)\#J}A}{F_I \bar{F}_J F^b}\Omega_I \wedge \bar{\Omega}_J\right\}_{\#I+\#J=i+1} \in C^i(\mathcal{U}, \Omega^a_{\mathcal{X}}(b\mathcal{X}))$$

we have that

$$\delta(\alpha) = \left\{(dF/F) \wedge \left((-1)^{i+1}\frac{(-1)^{(r+\#I')\#J'}A}{F_I \bar{F}_J F^{b-1}}\Omega_{I'} \wedge \bar{\Omega}_{J'}\right)\right\}_{\#I'+\#J'=i+2}$$

in $C^{i+1}(\mathcal{U}, \Omega^a_{\mathcal{X}}(b\mathcal{X}))$.

4. *Let $\pi : i^*\Omega^{a-1}_{\mathbb{P}^r \times \mathbb{P}^n}((b-1)\mathcal{X}) \to \Omega^{a-1}_{\mathcal{X}}((b-1)\mathcal{X})$ be the canonical projection. We have that $\pi \circ H$ is a splitting to*

$$dF/F \wedge (-) : C^i(\mathcal{U}, \Omega^{a-1}_{\mathcal{X}}((b-1)\mathcal{X})) \to C^i(\mathcal{U}, i^*\Omega^a_{\mathbb{P}^r \times \mathbb{P}^n}(b\mathcal{X})).$$

Proof. This follows the method of [11] directly. Note that to check an identity on a sheaf of p-forms on some open subset U of $\mathcal{X}$, it suffices to check on $f^{-1}(U)$ for $f : \mathcal{X}' \to \mathcal{X}$ any smooth morphism with $f^{-1}(U)$ nonempty. We will take f to be the restriction of $(\mathbb{A}^{r+1} \setminus \{0\}) \times (\mathbb{A}^{n+1} \setminus \{0\}) \to \mathbb{P}^r \times \mathbb{P}^n$ to $\mathcal{X}$. The point is that one does not need to assume that all the terms involved in the computation arise as forms on $\mathcal{X}$: the individual terms need not satisfy the Euler equations. We start by proving (1). Write $E_n = \sum_{i=0}^n X_i \partial/\partial X_i$ and $E_r = \sum_{i=0}^r Y_i \partial/\partial Y_i$, and let $dV_n = dX_0 \wedge \cdots \wedge dX_n$ and $dV_r = dY_0 \wedge \cdots \wedge dY_r$. Note that interior multiplication $\iota(E_n)$ with E_n gives $\iota(E_n)(dV_n) = \bar{\omega}$ and similarly we have that $\iota(E_r)(dV_r) = \omega$. We note that $dF \wedge dV_r \wedge dV_n = 0$, so

$$\begin{aligned} 0 &= \iota(E_n)\iota(E_r)(dF \wedge dV_r \wedge dV_n) \\ &= \iota(E_n)\left(F dV_r \wedge dV_n - dF \wedge \omega \wedge dV_n\right) \\ &= (-1)^{r+1} F dV_r \wedge \bar{\omega} - mF\omega \wedge dV_n + (-1)^r dF \wedge \omega \wedge \bar{\omega} \end{aligned}$$

Restricting to the affine cone over $\mathcal{X}$, we have that $F = 0$ and so we find that

$$dF \wedge \omega \wedge \bar{\omega} = 0. \tag{1.10}$$

Note that for $i \in \{0, \cdots, r\}$, we have that

$$K_i dF = K_i \left(\sum_{k=0}^{r} F_k dY_k + \sum_{l=0}^{n} \bar{F}_l dX_l \right) = F_i.$$

Applying K_i to (1.10) therefore yields that $F_i \omega \wedge \bar{\omega} = dF \wedge \Omega_i \wedge \bar{\omega}$. Similarly, applying $\bar{K}_j$ gives that $\bar{F}_j \omega \wedge \bar{\omega} = (-1)^r dF \wedge \omega \wedge \bar{\Omega}_j$ for $j \in \{0, \cdots, n\}$. We see from this that

$$\frac{A\omega \wedge \bar{\omega}}{F^{b+1}} = \begin{cases} \frac{A\Omega_i \wedge \bar{\omega} \wedge dF/F}{F_i F^b} & \text{for } i \in \{0, \cdots, r\} \\ \frac{(-1)^r A\omega \wedge \bar{\Omega}_j \wedge dF/F}{\bar{F}_j F^b} & \text{for } j \in \{0, \cdots, n\} \end{cases}$$

Applying the residue map to the left hand side, recalling the diagram (1.9) from the proof of Proposition 1.3.8, is the same as applying π to $\frac{A\Omega_i \wedge \bar{\omega}}{F_i F^b}$ or $\frac{(-1)^r A\omega \wedge \bar{\Omega}_j}{\bar{F}_j F^b}$. We find the result as desired.
To prove (2), let $I = \{i_0, \cdots, i_k\} \subset \{0, \cdots, r\}$ and $J = \{j_0, \cdots, j_l\} \subset \{0, \cdots, n\}$ be such that $\#I + \#J = i + 1$. If I is nonempty, we have that $K_{i_0} \Omega_I \wedge \bar{\Omega}_J = 0$ as $K_{i_0} K_I \omega = 0$ (one removes X_{i_0} from ω twice), and so

$$\begin{aligned} H_{i_0} \left(dF/F \wedge \left(\frac{A}{F_I \bar{F}_J F^b} \Omega_I \wedge \bar{\Omega}_J \right) \right) &= \frac{F}{F_{i_0}} K_{i_0} \left(dF/F \wedge \left(\frac{A}{F_I \bar{F}_J F^b} \Omega_I \wedge \bar{\Omega}_J \right) \right) \\ &= \frac{A}{F_I \bar{F}_J F^b} \Omega_I \wedge \bar{\Omega}_J \end{aligned}$$

as desired. If I is empty, one replaces H_{i_0} by $\bar{H}_{j_0}$, F_{i_0} by $\bar{F}_{j_0}$ and K_{i_0} by $\bar{K}_{j_0}$ and the proof works in the exact same way.
To prove statement (3), we consider two subsets $I' = \{i_0, \cdots, i_k\} \subset \{0, \cdots, r\}$ and $J' = \{j_0, \cdots, j_{i-k}\} \subset \{0, \cdots, n\}$ such that $\#I' + \#J' = i + 2$. We apply $K_{I'}$ to the identity (1.10), so that we find that

$$\begin{aligned} 0 &= (K_{i_k} \circ \cdots \circ K_{i_0})(dF \wedge \omega \wedge \bar{\omega}) \\ &= (K_{i_k} \circ \cdots \circ K_{i_1}) (F_{i_0} \omega \wedge \bar{\omega} - dF \wedge \Omega_{i_0} \wedge \bar{\omega}) \\ &= (K_{i_k} \circ \cdots \circ K_{i_2}) (F_{i_0} \Omega_{i_1} \wedge \bar{\omega} - F_{i_1} \Omega_{i_0} \wedge \bar{\omega} + dF \wedge \Omega_{i_0 i_1} \wedge \bar{\omega}) \\ &= \cdots \\ &= \sum_{l=0}^{k} (-1)^l F_{i_l} \Omega_{I' \setminus \{i_l\}} \wedge \bar{\omega} + (-1)^{k+1} dF \wedge \Omega_{I'} \wedge \bar{\omega} \end{aligned}$$

Applying $\bar{K}_{J'}$ gives

$$\begin{aligned} 0 &= (\bar{K}_{j_{i-k}} \circ \cdots \circ \bar{K}_{j_0}) \left(\sum_{l=0}^{k} (-1)^l F_{i_l} \Omega_{I' \setminus \{i_l\}} \wedge \bar{\omega} + (-1)^{k+1} dF \wedge \Omega_{I'} \wedge \bar{\omega} \right) \\ &= (\bar{K}_{j_{i-k}} \circ \cdots \circ \bar{K}_{j_1}) (\sum_{l=0}^{k} (-1)^{l+r-k} F_{i_l} \Omega_{I' \setminus \{i_l\}} \wedge \bar{\Omega}_{j_0} \end{aligned}$$

$$
\begin{aligned}
&+ (-1)^{k+1}\bar{F}_{j_0}\Omega_{I'} \wedge \bar{\omega} + (-1)^{k+1+r-k} dF \wedge \Omega_{I'} \wedge \bar{\Omega}_{j_0})\\
&= (\bar{K}_{j_{i-k}} \circ \cdots \circ \bar{K}_{j_2})(\sum_{l=0}^{k}(-1)^{l+2(r-k)} F_{i_l}\Omega_{I'\setminus\{i_l\}} \wedge \bar{\Omega}_{j_0 j_1}\\
&+ (-1)^{k+1+r-k-1}\bar{F}_{j_0}\Omega_{I'} \wedge \bar{\Omega}_{j_1} + (-1)^{k+1+r-k}\bar{F}_{j_1}\Omega_{I'} \wedge \bar{\Omega}_{j_0}\\
&+ (-1)^{k+1+2(r-k)} dF \wedge \Omega_{I'} \wedge \bar{\Omega}_{j_0 j_1})\\
&= \cdots\\
&= \sum_{l=0}^{k}(-1)^{l+(i-k+1)(r-k)} F_{i_l}\Omega_{I'\setminus\{i_l\}} \wedge \bar{\Omega}_{J'}\\
&+ (-1)^{k+1}\sum_{l=0}^{i-k}(-1)^{l+(i-k)(r-k-1)}\bar{F}_{j_l}\Omega_{I'} \wedge \bar{\Omega}_{J'\setminus\{j_l\}}\\
&+ (-1)^{k+1+(i-k+1)(r-k)} dF \wedge \Omega_{I'} \wedge \bar{\Omega}_{J'}
\end{aligned}
$$

so

$$
\sum_{l=0}^{k}(-1)^{l} F_{i_l}\Omega_{I'\setminus\{i_l\}} \wedge \bar{\Omega}_{J'} + (-1)^{k+1+r-i}\sum_{l=0}^{i-k}(-1)^{l}\bar{F}_{j_l}\Omega_{I'} \wedge \bar{\Omega}_{J'\setminus\{j_l\}} = (-1)^{k} dF \wedge \Omega_{I'} \wedge \bar{\Omega}_{J'}
$$

We see that

$$
\begin{aligned}
&\left(\delta\left(\left\{\frac{(-1)^{(r+\#I)\#J}A}{F_I\bar{F}_J F^b}\Omega_I \wedge \bar{\Omega}_J\right\}_{I,J}\right)\right)_{I',J'}\\
&= \frac{A}{F^b}\sum_{l=0}^{k}(-1)^{l+(r+k)(i-k+1)}\frac{F_{i_l}\Omega_{I'\setminus\{i_l\}} \wedge \bar{\Omega}_{J'}}{F_{I'}\bar{F}_{J'}}\\
&\quad + (-1)^{k+1}\frac{A}{F^b}\sum_{l=0}^{i-k}(-1)^{l+(r+k+1)(i-k)}\frac{\bar{F}_{j_l}\Omega_{I'} \wedge \bar{\Omega}_{J'\setminus\{j_l\}}}{F_{I'}\bar{F}_{J'}}\\
&= (-1)^{i+1}\frac{(-1)^{(r+k+1)(i-k-1)}A}{F_{I'}\bar{F}_{J'}F^b} dF \wedge \Omega_{I'} \wedge \bar{\Omega}_{J'}
\end{aligned}
$$

which proves the claim.
Finally, for (4), we note that as the original maps H_j are splittings, the composition $\pi \circ H$ is one too, which completes the proof. □

We can use this to prove the following result, which is a generalization of [33, Proposition 3.6] to products of projective spaces.

Lemma 1.4.16. *Let $A \in J^{q-r,(q+1)m-(n+1)}$, then $\psi_q(A) \in H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})$ is represented by the Čech cochain $\{\pi((-1)^{(r+\#I)\#J}A\Omega_I \wedge \bar{\Omega}_J/F_I\bar{F}_J)\}_{I,J}$ in $C^q(\mathcal{U}, \Omega^p_{\mathcal{X}})$.*

Proof. Recall that for $j \in \{0, \cdots, q-1\}$, there is the exact sequence (1.3) given by

$$
0 \to \Omega^{n+r-j-2}_{\mathcal{X}}((q-j-1)\mathcal{X}) \to i^*\Omega^{n+r-j-1}_{\mathbb{P}^r\times\mathbb{P}^n}((q-j)\mathcal{X}) \to \Omega^{n+r-j-1}_{\mathcal{X}}((q-j)\mathcal{X}) \to 0.
$$

We find coboundary maps

$$\delta_j : H^j(\mathcal{X}, \Omega_{\mathcal{X}}^{n+r-j-1}((q-j)\mathcal{X}) \to H^{j+1}(\mathcal{X}, \Omega_{\mathcal{X}}^{n+r-j-2}((q-j-1)\mathcal{X}).$$

We will show by induction on j that $\delta_{j-1} \circ \cdots \circ \delta_0(\text{res}(A\omega \wedge \bar{\omega}/F^{q+1}))$ is represented by the Čech cocycle

$$\{\pi((-1)^{j(j-1)/2+(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J / F_I \bar{F}_J F^{q-j})\}_{I,J} \in C^j(\mathcal{U}, \Omega_{\mathcal{X}}^{n+r-j-1}(q-j)\mathcal{X}).$$

Then taking $j = q$ will give the desired result: using [33, Remark 2.3], we have that the element above represents $\psi_q(A)$ for $j = q$ up to a factor $(-1)^{q(q-1)/2}$. First of all, note that the case where $j = 0$ is precisely Lemma 1.4.15, part (1). Now assume that $\delta_{j-1} \circ \cdots \circ \delta_0(\text{res}(A\omega \wedge \bar{\omega}/F^{q+1}))$ is represented by the Čech cocycle $\{\pi((-1)^{j(j-1)/2}(-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J / F_I \bar{F}_J F^{q-j})\}_{I,J}$ for some j. Then $\delta_j \circ \cdots \circ \delta_0(\text{res}(A\omega \wedge \bar{\omega}/F^{q+1}))$ is represented by the coboundary of

$$\{\pi((-1)^{j(j-1)/2}(-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J / F_I \bar{F}_J F^{q-j})\}_{I,J}.$$

Using [33, Remark 2.2], this is defined by lifting to the cochain

$$\{(-1)^{j(j-1)/2+(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J / F_I \bar{F}_J F^{q-j}\}_{I,J} \in C^j(\mathcal{U}, i^*\Omega_{\mathbb{P}^r \times \mathbb{P}^n}^{n+r-j-1}((q-j)\mathcal{X})$$

and applying the negative of the Čech coboundary operator δ, and then viewing this as an element coming from $C^{j+1}(\mathcal{U}, \Omega_{\mathcal{X}}^{n+r-j-2}((q-j-1)\mathcal{X}))$ of which the inclusion is induced by the map $dF/F \wedge (-)$. Using Lemma 1.4.15, part (3) we have that

$$\begin{aligned} -\delta &\left(\left\{ \frac{(-1)^{j(j-1)/2}(-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J}{F_I \bar{F}_J F^{q-j}} \right\}_{I,J} \right) \\ &= \left\{ (dF/F) \wedge \left(\frac{(-1)^{j(j+1)/2}(-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J}{F_I \bar{F}_J F^{q-j-1}} \right) \right\}_{I,J} \end{aligned}$$

By part (4), $\pi \circ H$ provides a splitting to $dF/F \wedge (-)$ implying by part (2) that the desired element is

$$\begin{aligned} (\pi \circ H) &\left(\left\{ \frac{dF}{F} \wedge \left(\frac{(-1)^{j(j+1)/2}(-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J}{F_I \bar{F}_J F^{q-j-1}} \right) \right\}_{I,J} \right) \\ &= \left\{ \pi \left((-1)^{j(j+1)/2} \frac{(-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J}{F_I \bar{F}_J F^{q-j-1}} \right) \right\}_{I,J} \end{aligned}$$

completing the induction. □

Proof of Proposition 1.4.7. We know from Lemma 1.4.16 that ω_A and ω_B are represented by the cochains

$$\{\pi((-1)^{(r+\#I)\#J} A\Omega_I \wedge \bar{\Omega}_J / F_I \bar{F}_J)\}_{I,J} \in C^q(\mathcal{U}, \Omega_{\mathcal{X}}^{n+r-1-q})$$

and

$$\{\pi((-1)^{(r+\#I)\#J} B\Omega_I \wedge \bar{\Omega}_J / F_I \bar{F}_J)\}_{I,J} \in C^p(\mathcal{U}, \Omega_{\mathcal{X}}^{n+r-1-p}).$$

Now let $I = \{i_0, \cdots, i_{r-1}\} \subset \{0, \cdots, r\}$ and $J = \{j_0, \cdots, j_{n-1}\} \subset \{0, \cdots, n\}$. First assume that $q \leq r-1$. Then by definition of the cup product on Čech cochains, we have that

$$\begin{aligned}(\omega_A \cup \omega_B)_{I,J} &= \pi\left((-1)^{(2r-q)n} \frac{A\Omega_{i_0,\cdots,i_q} \wedge \bar{\omega}}{F_{i_0,\cdots,i_q}} \cdot \frac{B\Omega_{i_q,\cdots,i_{r-1}} \wedge \bar{\Omega}_{j_0,\cdots,j_{n-1}}}{F_{i_q,\cdots,i_{r-1}}\bar{F}_{j_0,\cdots,j_{n-1}}}\right) \\ &= \pi\left(AB\Omega(I,J)\right)\end{aligned}$$

Similarly, if $q \geq r$, we have that

$$\begin{aligned}&(\omega_A \cup \omega_B)_{I,J} \\ &= \pi\left((-1)^{2r(q-r+1)+r(n-q-r)} \frac{A\Omega_{i_0,\cdots,i_{r-1}} \wedge \bar{\Omega}_{j_0,\cdots,j_{q-r}}}{F_{i_0,\cdots,i_{r-1}}\bar{F}_{j_0,\cdots,j_{q-r}}} \cdot \frac{B\omega \wedge \bar{\Omega}_{j_{q-r},\cdots,j_{n-1}}}{\bar{F}_{j_{q-r},\cdots,j_{n-1}}}\right) \\ &= \pi\left(AB\Omega(I,J)\right)\end{aligned}$$

which proves the statement. □

1.4.2 Proof of Proposition 1.4.9 and Corollary 1.4.10

In order to prove Proposition 1.4.9, we first show the following lemma.

Lemma 1.4.17. *Consider subsets*

$$I = \{i_0, \cdots, i_{r-1}\} \subset \{0, \cdots, r\} \text{ and } J = \{j_0, \cdots, j_{n-1}\} \subset \{0, \cdots, n\}$$

such that $I = \{0, \cdots, r\} \setminus \{v\}$ *and* $J = \{0, \cdots, n\} \setminus \{w\}$. *We have that*

$$dF \wedge \Omega(I,J) = \frac{(-1)^{v+w} Y_v F_v X_w \bar{F}_w \omega \wedge \bar{\omega}}{\prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_j}.$$

Proof. If $q \leq r-1$, we have by [11, Lemma on page 14] that

$$\left(\sum_{i=0}^{r} F_i dY_i\right) \wedge \Omega_{i_0,\cdots,i_q} \wedge \Omega_{i_q,\cdots,i_{r-1}} = (-1)^v Y_v F_{i_q} \wedge \omega.$$

Now noting that $\bar{\Omega}_{j_0,\cdots,j_{n-1}} = (-1)^w X_w$ and $dX_j \wedge \bar{\omega} = 0$ for all $j \in \{0, \cdots, n\}$, we see that

$$\begin{aligned}dF \wedge \Omega(I,J) &= \left(\sum_{i=0}^{r} F_i dY_i\right) \wedge \frac{(-1)^{nq} F_v \bar{F}_w \Omega_{i_0,\cdots,i_q} \wedge \bar{\omega} \wedge \Omega_{i_q,\cdots,i_{r-1}} \wedge \bar{\Omega}_{j_0,\cdots,j_{n-1}}}{F_{i_q} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_n} \\ &\quad + \left(\sum_{j=0}^{n} \bar{F}_j dX_j\right) \wedge \frac{(-1)^{nq} F_v \bar{F}_w \Omega_{i_0,\cdots,i_q} \wedge \bar{\omega} \wedge \Omega_{i_q,\cdots,i_{r-1}} \wedge \bar{\Omega}_{j_0,\cdots,j_{n-1}}}{F_{i_q} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_n} \\ &= \left(\sum_{i=0}^{r} F_i dY_i\right) \wedge \frac{(-1)^{w} X_w F_v \bar{F}_w \Omega_{i_0,\cdots,i_q} \wedge \Omega_{i_q,\cdots,i_{r-1}} \wedge \bar{\omega}}{F_{i_q} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_n}\end{aligned}$$

$$+ \left(\sum_{j=0}^{n} \bar{F}_j dX_j\right) \wedge \frac{(-1)^{n(r-1)+w} X_w F_v \bar{F}_w \bar{\omega} \wedge \Omega_{i_0,\cdots,i_q} \wedge \Omega_{i_q,\cdots,i_{r-1}}}{F_{i_q} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_n}$$
$$= \frac{(-1)^{v+w} Y_v F_v X_w \bar{F}_w \omega \wedge \bar{\omega}}{\prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_n}$$

If $q \geq r$, we have by [11, Lemma on page 14] that

$$\left(\sum_{j=0}^{n} \bar{F}_j dX_j\right) \wedge \bar{\Omega}_{j_0,\cdots,j_{q-r}} \wedge \bar{\Omega}_{j_{q-r},\cdots,j_{n-1}} = (-1)^w X_w \bar{F}_{j_{q-r}} \bar{\omega}.$$

Now as $\Omega_{i_0,\cdots,i_{r-1}} = (-1)^v Y_v$ and $dY_i \wedge \omega = 0$ for any $i \in \{0, \cdots, r\}$, we see that

$$dF \wedge \Omega(I, J)$$
$$= \left(\sum_{i=0}^{r} F_i dY_i\right) \wedge \frac{(-1)^{r(n+q+r)} F_v \bar{F}_w \Omega_{i_0,\cdots,i_{r-1}} \wedge \bar{\Omega}_{j_0,\cdots,j_{q-r}} \wedge \omega \wedge \bar{\Omega}_{j_{q-r},\cdots,j_{n-1}}}{\bar{F}_{j_{q-r}} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_j}$$
$$+ \left(\sum_{j=0}^{n} \bar{F}_j dX_j\right) \wedge \frac{(-1)^{r(n+q+r)} F_v \bar{F}_w \Omega_{i_0,\cdots,i_{r-1}} \wedge \bar{\Omega}_{j_0,\cdots,j_{q-r}} \wedge \omega \wedge \bar{\Omega}_{j_{q-r},\cdots,j_{n-1}}}{\bar{F}_{j_{q-r}} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_j}$$
$$= \left(\sum_{i=0}^{r} F_i dY_i\right) \wedge \frac{(-1)^{v+r} Y_v F_v \bar{F}_w \omega \wedge \bar{\Omega}_{j_0,\cdots,j_{q-r}} \wedge \bar{\Omega}_{j_{q-r},\cdots,j_{n-1}}}{\bar{F}_{j_{q-r}} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_j}$$
$$+ \left(\sum_{j=0}^{n} \bar{F}_j dX_j\right) \wedge \frac{(-1)^{v+rn} Y_v F_v \bar{F}_w \bar{\Omega}_{j_0,\cdots,j_{q-r}} \wedge \bar{\Omega}_{j_{q-r},\cdots,j_{n-1}} \wedge \omega}{\bar{F}_{j_{q-r}} \prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_j}$$
$$= \frac{(-1)^{v+w} Y_v F_v X_w \bar{F}_w \omega \wedge \bar{\omega}}{\prod_{i=0}^{r} F_i \prod_{j=0}^{n} \bar{F}_n}$$

which completes the proof. □

Proof of Proposition 1.4.9. Using [33, Remark 2.2] together with Lemma 1.2.7, we can represent $i_*(\omega_A \cup \omega_B))$ by lifting to the section $AB\Omega(I, J) \wedge dF/F$ of $\Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}(\log(\mathcal{X}))$ and then taking the negative of the Čech coboundary. Note that we use the diagram from the proof of Proposition 1.3.8 again to see that this is really the lift.
Now note that $C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$ has indices (I', J') where either:

- $I' = \{0, \cdots, r\}$ and $J' = \{0, \cdots, n\} \setminus \{w\}$ for a certain w
- $I' = \{0, \cdots, r\} \setminus \{v\}$ and $J' = \{0, \cdots, n\}$ for a certain v.

In the first case, we have using Lemma 1.4.17 that

$$\delta(\{ABdF/F \wedge \Omega(I, J)\}_{I,J})_{I',J'}$$

$$= AB\sum_{v=0}^{r}(-1)^{v}(-1)^{v+w}\frac{Y_vF_vX_w\bar{F}_w\omega\wedge\bar{\omega}}{F\prod_{i=0}^{r}F_i\prod_{j=0}^{n}\bar{F}_j}$$
$$= (-1)^{w}AB\frac{X_w\bar{F}_w\omega\wedge\bar{\omega}}{\prod_{i=0}^{r}F_i\prod_{j=0}^{n}\bar{F}_j}$$

If the index (I',J') is of the second form, we similarly find that

$$\delta(\{AB(dF/F)\wedge\Omega(I,J)\}_{I,J})_{I',J'}$$
$$= AB\sum_{w=0}^{n}(-1)^{v+w}(-1)^{w+r}\frac{Y_vF_vX_w\bar{F}_w\omega\wedge\bar{\omega}}{F\prod_{i=0}^{r}F_i\prod_{j=0}^{n}\bar{F}_j}$$
$$= (-1)^{v+r}mAB\frac{Y_vF_v\omega\wedge\bar{\omega}}{\prod_{i=0}^{r}F_i\prod_{j=0}^{n}\bar{F}_j}$$

This completes the proof. □

Proof of Corollary 1.4.10. Let $D\in k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]^{\rho}$. Note that $\tilde{\phi}(D)$ is a Čech cocycle, as we have that

$$\delta(\tilde{\phi}(D)) = \frac{D\left(m\sum_{v=0}^{r}(-1)^{2v+r+1}Y_vF_v\omega\wedge\bar{\omega}+\sum_{w=0}^{n}(-1)^{2w+r+2}X_w\bar{F}_w\omega\wedge\bar{\omega}\right)}{\prod_{i=0}^{r}F_i\prod_{j=0}^{n}\bar{F}_j}$$
$$= \frac{(-1)^{r+1}D(mF-mF)\omega\wedge\bar{\omega}}{\prod_{i=0}^{r}F_i\prod_{j=0}^{n}\bar{F}_j}$$
$$= 0$$

This means that $\tilde{\phi}$ induces a map

$$\bar{\phi}: k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]^{\rho}\to H^{n+r}(\mathbb{P}^r\times\mathbb{P}^n,\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}).$$

Using Proposition 1.4.9, we find the following commutative diagram

$$\begin{array}{ccc} H^q(\mathcal{X},\Omega^p_{\mathcal{X}})_{prim}\otimes H^p(\mathcal{X},\Omega^q_{\mathcal{X}})_{prim} & \xrightarrow{i_*\circ\cup} & H^{n+r}(\mathbb{P}^r\times\mathbb{P}^n,\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}) \\ \uparrow{\scriptstyle\tilde{\psi}_p\otimes\tilde{\psi}_q} & & \uparrow{\scriptstyle\bar{\phi}} \\ k[Y,X]^{q-r,(q+1)m-(n+1)}\otimes k[Y,X]^{p-r,(p+1)m-(n+1)} & \longrightarrow & k[Y,X]^{\rho} \end{array}$$

where we denote $k[Y,X]=k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]$. By Proposition 1.3.8, we have that $\tilde{\psi}_q$ descends to an isomorphism

$$J^{q-r,(q+1)m-n-1}\to H^q(\mathcal{X},\Omega^p_{\mathcal{X}})_{prim}$$

and similarly for $\tilde{\psi}_p$.
We note that $\bar{\phi}$ maps the Jacobian ideal to zero. To see this, suppose that $D\in k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]^{\rho}$ is a multiple of F_i for some $i\in\{0,\cdots,r\}$.

Write $\tilde{\phi}(D) = \{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\}$. Then $\{\xi_{I,J}\}_{I,J} \in C^{n+r-1}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n})$ given by

$$\xi_{I,J} = \begin{cases} (-1)^{i-1}s_v & \text{if } I = \{0,\cdots,r\}\setminus\{v,i\}, J = \{0,\cdots,n\} \text{ and } v < i \\ (-1)^{i}s_v & \text{if } I = \{0,\cdots,r\}\setminus\{v,i\}, J = \{0,\cdots,n\} \text{ and } v > i \\ (-1)^{i}\bar{s}_w & \text{if } I = \{0,\cdots,r\}\setminus\{i\}, J = \{0,\cdots,n\}\setminus\{w\} \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\delta(\xi_{I,J}) = \{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\}$. Indeed, for $I' = \{0,\cdots,r\}\setminus\{v\}$ with $v < i$ and $J' = \{0,\cdots,n\}$, we have that $\delta(\xi_{I,J})_{I',J'} = (-1)^{i-1}(-1)^{i-1}s_v = s_v$, and similarly for $v > i$. For $I' = \{0,\cdots,r\}$ and $J' = \{0,\cdots,n\}\setminus\{w\}$, we have that $\delta(\xi_{I,J})_{I',J'} = (-1)^i(-1)^i\bar{s}_w = \bar{s}_w$. Finally, for $I' = \{0,\cdots,r\}\setminus\{i\}$ and $J' = \{0,\cdots,n\}$, we have that

$$\delta(\xi_{I,J})_{I',J'} = \sum_{v\neq i}(-1)^{v+i-1}s_v + \sum_{w=0}^{r}(-1)^{i+w+r}\bar{s}_w = -(-1)^{i-1}(-1)^i s_i = s_i.$$

Therefore, $\tilde{\phi}(D)$ is a coboundary. Similarly, if D is a multiple of $\bar{F}_j$ for some $j \in \{0,\cdots,n\}$, then the element $\{\xi_{I,J}\}_{I,J} \in C^{n+r-1}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n})$ given by

$$\xi_{I,J} = \begin{cases} (-1)^{j+r}s_v & \text{if } I = \{0,\cdots,r\}\setminus\{v\}, J = \{0,\cdots,n\}\setminus\{j\} \\ (-1)^{j+r}\bar{s}_w & \text{if } I = \{0,\cdots,r\}, J = \{0,\cdots,n\}\setminus\{w,j\} \text{ and } w < j \\ (-1)^{j+r+1}\bar{s}_w & \text{if } I = \{0,\cdots,r\}, J = \{0,\cdots,n\}\setminus\{w,j\} \text{ and } w > j \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\delta(\xi_{I,J}) = \{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\}$.
So $\bar{\phi}$ descends to a map $\phi : J^\rho \to H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n})$ which makes the diagram commute. As the cup product is non-degenerate, we have that ϕ is surjective. □

1.4.3 One dimensionality of J^ρ and $\tilde{J}^{\rho+(r+1,n+1)}$

In this section, we will prove the following statement.

Proposition 1.4.18. *J^ρ is a one dimensional vector space over k, except possibly if n is odd, $r = 1$ and $m = 2$.*

Over $\mathbb{C}$, this is a special case of [27, Lemma 6.3] and the argument is partially the same. The idea to use the bundle $\Sigma_{\mathcal{L}}$ (see below) to give a description of the Jacobian ring and study its duality properties goes back to [20, Section 2]. It will follow that the map from Corollary 1.4.10 is an isomorphism whenever we are not in the situation where n is odd, $r = 1$ and $m = 2$.

Notation 1.4.19. Write $\mathcal{L} = \mathcal{O}(1, m)$ and let $\Sigma_{\mathcal{L}}$ be the bundle as defined in [27, Section 2.1]. There is a global presentation of $\Sigma_{\mathcal{L}}$ given by

$0 \to e_1 \cdot \mathcal{O}_{\mathbb{P}^r} \oplus e_2 \cdot \mathcal{O}_{\mathbb{P}^n} \to$

$$\mathrm{id}_{\mathcal{L}} \cdot \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n} \oplus \bigoplus_{i=0}^{r} \frac{\partial}{\partial Y_i} \cdot \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n}(1,0) \oplus \bigoplus_{j=0}^{n} \frac{\partial}{\partial X_j} \cdot \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n}(0,1) \to \Sigma_{\mathcal{L}} \to 0$$

where the first map is given by

$$e_1 \mapsto -\mathrm{id}_{\mathcal{L}} + \sum_{i=0}^{r} Y_i \frac{\partial}{\partial Y_i}, e_2 \mapsto -m\mathrm{id}_{\mathcal{L}} + \sum_{j=0}^{n} X_j \frac{\partial}{\partial X_j}.$$

We can map the above sequence into the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^n} \to \bigoplus_{i=0}^{r} \mathcal{O}(1,0) \oplus \bigoplus_{j=0}^{n} \mathcal{O}(0,1) \to T_{\mathbb{P}^r\times\mathbb{P}^n} \to 0$$

by sending $\mathrm{id}_{\mathcal{L}}$ to zero. This yields the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n} \to \Sigma_{\mathcal{L}} \to T_{\mathbb{P}^r\times\mathbb{P}^n} \to 0. \tag{1.11}$$

Consider the morphism $\Sigma_{\mathcal{L}} \to \mathcal{L}$ which sends a local section

$$a\mathrm{id}_{\mathcal{L}} + \sum_{i=0}^{r} b_i \frac{\partial}{\partial Y_i} + \sum_{j=0}^{n} c_j \frac{\partial}{\partial X_j}$$

to

$$aF + \sum_{i=0}^{r} b_i F_i + \sum_{j=0}^{n} c_j \bar{F}_j.$$

This gives rise to a surjective morphism $\Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1} \to \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n}$. Form the associated Koszul complex

$$0 \to \Lambda^{n+r+1}\Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-n-r-1} \to \cdots \to \Lambda^2\Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-2} \to \Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1} \to \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^n} \to 0. \tag{1.12}$$

Now applying the functor $\mathrm{Hom}(-, \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r})$ we find the exact sequence,

$$0 \to \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r} \to \Sigma_{\mathcal{L}}^{\vee} \otimes \mathcal{L} \otimes \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r} \to \cdots \to \Lambda^{n+r+1}\Sigma_{\mathcal{L}}^{\vee} \otimes \mathcal{L}^{n+r+1} \otimes \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r} \to 0 \tag{1.13}$$

defining a left resolution of $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}$. Note that

$$\Lambda^{n+r+1}\Sigma_{\mathcal{L}}^{\vee} \otimes \mathcal{L}^{n+r+1} \otimes \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r} = \mathcal{O}(\rho)$$

as $\det(\Sigma_{\mathcal{L}}) = \det(T_{\mathbb{P}^r\times\mathbb{P}^n}) = \mathcal{O}(r+1, n+1)$ using the exact sequence (1.11), $\mathcal{L}^{n+r+1} = \mathcal{O}(n+r+1, m(n+r+1))$ and $\Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r} = \mathcal{O}(-r-1, -n-1)$. Consider the hypercohomology spectral sequence

$$E_1^{p,q} = H^q(\mathbb{P}^r \times \mathbb{P}^n, \Lambda^{p+1}\Sigma_{\mathcal{L}}^{\vee} \otimes \mathcal{L}^{p+1} \otimes \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r}) \implies H^{p+q}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^n\times\mathbb{P}^r}).$$

Lemma 1.4.20. *We have that $E_1^{p,q} = 0$ for $q > 0$ and either $p+q = n+r$ or $p+q = n+r-1$, except in the case when n is odd, $r = 1$ and $m = 2$.*

Proof of Lemma 1.4.20. This proof is partially taken from [27]. Note that the sequence (1.11) gives rise to exact sequences

$$0 \to \Omega^j_{\mathbb{P}^n \times \mathbb{P}^r} \to \Lambda^j \Sigma^\vee_{\mathcal{L}} \to \Omega^{j-1}_{\mathbb{P}^n \times \mathbb{P}^r} \to 0$$

so that it is enough to prove that for $1 \leq s \leq n+r-1$, we have that

$$H^{n+r-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^s_{\mathbb{P}^r \times \mathbb{P}^n} \otimes \mathcal{L}^s \otimes \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0$$

and

$$H^{n+r-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{s-1}_{\mathbb{P}^r \times \mathbb{P}^n} \otimes \mathcal{L}^s \otimes \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0$$

and for $1 \leq s \leq n+r$, we have that

$$H^{n+r+1-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^s_{\mathbb{P}^r \times \mathbb{P}^n} \otimes \mathcal{L}^s \otimes \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0$$

and

$$H^{n+r+1-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{s-1}_{\mathbb{P}^r \times \mathbb{P}^n} \otimes \mathcal{L}^s \otimes \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0.$$

We show that the first condition holds except in the case when n is odd, $r = 1$ and $m = 2$; the others are similar. That is, we will show that

$$H^{n+r-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^s_{\mathbb{P}^r \times \mathbb{P}^n}(s-r-1, ms-n-1)) = 0 \tag{1.14}$$

for $1 \leq s \leq n+r-1$ except in the case when n is odd, $r = 1$ and $m = 2$. Using Proposition 1.2.10, we have that

$$\begin{aligned} &H^{n+r-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^s_{\mathbb{P}^r \times \mathbb{P}^n}(s-r-1, ms-n-1)) \\ &\quad = \bigoplus_{i+j=s} \bigoplus_{k+l=n+r-s} H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(s-r-1)) \otimes H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(ms-n-1)) \end{aligned}$$

Note that by Theorem 1.2.9 we have that $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(s-r-1)) = 0$ except possibly if we are in one of the following situations:

1. $k > 0, j = 0$ and $s \neq r+1$. Note that $H^l(\mathbb{P}^n, \mathcal{O}(ms-n-1))$ is zero except possibly for:
   - $l = 0$. Then $k = n+r-s$. If $k > r$, then $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(s-r-1)) = 0$. Otherwise, $n+r-s \leq r$ and so $s \geq n$ which implies that $i \geq n > r$ and so $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(s-r-1)) = 0$ in this case as well.
   - $l = n$. Then as $ms-n-1 > -n-1$, we have that

     $$H^n(\mathbb{P}^n, \mathcal{O}(ms-n-1)) = \left(\frac{1}{X_0, \cdots, X_n} k[X_0^{-1}, \cdots, X_n^{-1}]\right)_{ms-n-1} = 0.$$

2. $s = r+1$ and $i = k$. We note that:
   - If $l = 0$ then $k = n-1 > r$ and so $H^k(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(s-r-1)) = 0$.

- If $l > 0$ and $ms \neq n+1$ then $ms - r - 1 \geq j - n$ as
$$ms = m(i+j) \geq 2j \geq j+1. \tag{1.15}$$
It follows that $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(ms-n-1)) = 0$.
- If $l > 0$ and $ms = n+1$, then $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}) = 0$ provided that $j \neq l$. Therefore, $H^{n+r-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^s_{\mathbb{P}^r \times \mathbb{P}^n}(s-r-1, ms-n-1))$ is possibly nonzero if $l = j$, that is, $r = i+j-1 = k+l-1 = n-2$. In this case, it follows from $m(r+1) = ms = n+1 = r+3$ that $r = 1$ and $m = 2$, so $n = 3$.

3. $k = 0$ and $j > r+1$. Then $l = n+r-s > 0$ and:
   - If $ms \neq n+1$, we have that $ms - n - 1 \geq j - n$ by (1.15). This implies that $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(ms-n-1)) = 0$.
   - If $ms = n+1$, then $H^l(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}) = 0$ unless $l = j$. But if $l = j$ then $s = i+j \geq j = n+r-s$ implies that $2s \geq n+r \geq n+1$ while on the other hand $2s \leq ms = n+1$. We see from this that $H^{n+r-s}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^s_{\mathbb{P}^r \times \mathbb{P}^n}(s-r-1, ms-n-1))$ is possibly nonzero if $m = 2, r = 1$ and n is odd.

We conclude that the statement holds. □

Proof of Proposition 1.4.18. We have that $H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r}) \cong k$ is the final cohomology group of the sequence (1.13), by Lemma 1.4.20. Note that

$$H^0(\mathbb{P}^r \times \mathbb{P}^n, \mathcal{O}(\rho)) = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^\rho.$$

The image of the map

$$\Lambda^{n+r}\Sigma^\vee_{\mathcal{L}} \otimes \mathcal{L}^{n+r} \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r} \to \Lambda^{n+r+1}\Sigma^\vee_{\mathcal{L}} \otimes \mathcal{L}^{n+r+1} \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r}$$

is the ideal $(F_0, \cdots, F_r, \bar{F}_0, \cdots, \bar{F}_n)$ so $H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r}) \cong J^\rho$. In particular, J^ρ is one dimensional. □

Remark 1.4.21. As a consequence of this statement, the map ϕ from Corollary 1.4.10 is an isomorphism. This proves Corollary 1.4.11.

Remark 1.4.22. An interesting question is whether ϕ coincides with the map $J^\rho \to H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}$ which we get from the above argument. We have not been able to give a full answer, but suspect it might be true. A possible way to make the map from the above argument explicit might be as follows. Take a Čech resolution of all terms in the resolution (1.13). If we start with an element $\xi \in \mathcal{C}^{n+r-1}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$, then using that $H^{n+r-1}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) = 0$, the element comes from some $\xi' \in \mathcal{C}^{n+r-2}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$. Computing ξ' and applying the horizontal map, we find an element of $\mathcal{C}^{n+r-2}(\mathcal{U}, \Sigma^\vee_{\mathcal{L}} \otimes \mathcal{L}^{-n-r-1} \otimes \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$. Repeating the above procedure and keeping track of everything, we end up with an element in $\mathcal{C}^0(\mathcal{U}, \mathcal{O}(\rho))$. After that, we can check whether the resulting morphism is the same as ϕ.

We now prove Proposition 1.4.13.

Notation 1.4.23. We write $\mathcal{E} = \bigoplus_{i=1}^{n+r+1} \mathcal{O}(-1, -m)$.

There is a surjective morphism $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^n}$ sending local generators g_i for $i \in \{1, \cdots, r\}$ to $g_i Y_i F_i$ and $g_i \in \{r+1, \cdots, n+r+1\}$ to $g_i X_{i-r-1} \bar{F}_{i-r-1}$. This gives rise to an exact Koszul complex

$$0 \to \Lambda^{n+r+1}\mathcal{E} \to \cdots \to \Lambda^2 \mathcal{E} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^n} \to 0. \tag{1.16}$$

and applying $\mathrm{Hom}(-, \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r})$ we find

$$0 \to \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r} \to \mathcal{E}^\vee \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r} \to \cdots \to \Lambda^{n+r+1}\mathcal{E}^\vee \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r} \to 0. \tag{1.17}$$

Note that $\Lambda^{n+r+1}\mathcal{E}^\vee \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r} = \mathcal{O}(\rho - (r+1, n+1))$.

Lemma 1.4.24. *The complex (1.17) defines an acyclic resolution of* $\Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r}$.

Proof. We have that $\Lambda^i \mathcal{E}^\vee$ is a direct sum of terms $\mathcal{O}(i-r-1, mi-n-1)$. For $q > 0$ we have that

$$\begin{aligned} &H^q(\mathbb{P}^r \times \mathbb{P}^n, \mathcal{O}(i-r-1, mi-n-1)) \\ &\qquad = \bigoplus_{a+b=q} H^a(\mathbb{P}^r, \mathcal{O}(i-r-1)) \otimes H^b(\mathbb{P}^n, \mathcal{O}(mi-n-1)). \end{aligned} \tag{1.18}$$

Note that $H^a(\mathbb{P}^r, \mathcal{O}(i-r-1)) = 0$ unless possibly if $a \in \{0, r\}$. If $a = 0$, we have that $H^b(\mathbb{P}^n, \mathcal{O}(mi-n-1)) = 0$ unless possibly if $b = n$. But in that case

$$H^n(\mathbb{P}^n, \mathcal{O}(mi-n-1)) = \left(\frac{1}{X_0 \cdots X_n} k \left[\frac{1}{X_0}, \cdots, \frac{1}{X_n} \right] \right)^{mi-n-1} = 0$$

because $mi - n - 1 \geq -n$ (as $mi - n - 1 < -n$ would imply $mi < 1$). If $a = r$ then we similarly see that $H^r(\mathbb{P}^r, \mathcal{O}(i-r-1)) = 0$ as $i - r - 1 \geq -r$. ☐

Proof of Proposition 1.4.13. By Lemma 1.4.24 we have that $H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r})$ is the final cohomology group of the sequence (1.17). The image of the map

$$\Lambda^{n+r}\mathcal{E}^\vee \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r} \to \Lambda^{n+r+1}\mathcal{E}^\vee \otimes \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r}$$

is the ideal $(Y_1 F_1, \cdots, Y_r F_r, X_0 \bar{F}_0, \cdots, X_n \bar{F}_n)$. Because of the Euler relations, this is equal to the ideal $(Y_0 F_0, \cdots, Y_r F_r, X_0 \bar{F}_0, \cdots, X_n \bar{F}_n)$ so that we find that $k \cong H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^n \times \mathbb{P}^r}) \cong \tilde{J}^{\rho + (r+1, n+1)}$. ☐

1.5 Computing the trace map

Again, we keep the notation which was set up in the previous sections. For $A \in J^{p-r,(p+1)m-n-1}$ and $B \in J^{q-r,(q+1)m-n-1}$, consider their images

$$\omega_A = \psi_p(A) \in H^p(\mathcal{X}, \Omega^q_{\mathcal{X}}) \text{ and } \omega_B = \psi_q(B) \in H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})$$

under the isomorphism from Proposition 1.3.8. We note that as the trace map is compatible with pushforwards, we have that

$$\mathrm{Tr}_{\mathcal{X}}(\omega_A \cup \omega_B) = \mathrm{Tr}_{\mathbb{P}^r \times \mathbb{P}^n}(i_*(\omega_A \cup \omega_B)).$$

Therefore, we can compute $\mathrm{Tr}(\omega_A \cup \omega_B)$ by representing $i_*(\omega_A \cup \omega_B)$ on the open cover $\mathcal{U}$ using Proposition 1.4.9 and comparing it with a representation of $c_1(\mathcal{O}(1,m))^{n+r}$. We will do so on a refinement of $\mathcal{U}$, which we now first construct.

Notation 1.5.1. In this section we will make the following three extra assumptions:

- $m+1$ is invertible in k.
- $V(F_0), \cdots, V(F_r)$ are smooth hypersurfaces in $\mathbb{P}^n$ and they intersect transversally: $V(F_{i_0}, \cdots, F_{i_s})$ is a smooth closed subscheme of $\mathbb{P}^n$ which is of codimension $s+1$ for all $\{i_0, \cdots, i_s\} \subset \{0, \cdots, r\}$.
- The first assumption remains true after setting any proper subset of the X_i's equal to zero and replacing $\mathbb{P}^n$ with the linear subspace defined by the vanishing of the chosen X_i's.

Now let

$$G_0 = Y_0 F_0, \cdots, G_r = Y_r F_r, G_{r+1} = X_0 \bar{F}_0, \cdots, G_{n+r+1} = X_n \bar{F}_n.$$

The following is true because of the extra assumptions made in Notation 1.5.1.

Lemma 1.5.2. *The set of opens $\mathcal{V} = \{V_0, \cdots, V_{n+r+1}\}$ where $V_i = \{G_i \neq 0\}$ is an open cover of $\mathbb{P}^r \times \mathbb{P}^n$.*

Proof. Suppose that $x \in \mathbb{P}^r \times \mathbb{P}^n$ is not in $V_0 \cup \cdots \cup V_r$. Then we have that $Y_0 F_0 = \cdots = Y_r F_r = 0$ at x. As not all Y_i can be zero at x, there is some F_i which is zero. As $V(F_i)$ is smooth, we have that $\partial F_i / \partial X_j$ is nonzero at x for some j. So $\bar{F}_j = \sum_{i=0}^r Y_i \partial F_i / \partial X_j$ is nonzero at x. If X_j is nonzero at x, then $x \in V_{j+r+1}$. Otherwise, as $V(F_i)$ remains smooth after intersecting with $X_j = 0$, we have that $\partial F_i / \partial X_{j'}$ is nonzero at x for some $j' \neq j$. We now repeat the above argument until we find some open in $\mathcal{V}$ containing x. □

Proposition 1.5.3. *If we remove V_j for any $j \in \{0, \cdots, n+r+1\}$, this still results in an open cover.*

Proof. We have the two Euler equations $\sum_{i=0}^r Y_i F_i = F$ and $\sum_{i=0}^n X_i \bar{F}_i = mF$ and so $m \sum_{i=0}^r G_i - \sum_{i=r+1}^{n+r+1} G_i = 0$. □

Notation 1.5.4. We write $\mathcal{W}$ for the cover $\mathcal{V}$ with V_0 removed, i.e.

$$\mathcal{W} = \{V_1, \cdots, V_{n+r+1}\}.$$

Remark 1.5.5. Note that $\mathcal{V}$ is a refinement of $\mathcal{U}$ via the identity map on the index sets. The inclusion $\{1, \cdots, n+r+1\} \to \{0, \cdots, n+r+1\}$ makes $\mathcal{W}$ into a refinement of $\mathcal{V}$. This yields a composition of refinement maps

$$C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) \to C^{n+r}(\mathcal{V}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) \to C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$$

sending a cocycle $\{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\} \in C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$ - where s_i lives on the intersection of all opens except for U_i and $\bar{s}_j$ on the intersection of all opens except for $\bar{U}_j$ - to $\{s_0\} \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$.

Notation 1.5.6. Consider the matrix M given by

$$\begin{pmatrix} F_0 & 0 & \cdots & 0 & Y_0 \frac{\partial F_0}{\partial X_0} & Y_0 \frac{\partial F_0}{\partial X_1} & \cdots & Y_0 \frac{\partial F_0}{\partial X_n} \\ 0 & F_1 & \cdots & 0 & Y_1 \frac{\partial F_1}{\partial X_0} & Y_1 \frac{\partial F_1}{\partial X_1} & \cdots & Y_1 \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_r & Y_r \frac{\partial F_r}{\partial X_0} & Y_r \frac{\partial F_r}{\partial X_1} & \cdots & Y_r \frac{\partial F_r}{\partial X_n} \\ X_0 \frac{\partial F_0}{\partial X_0} & X_0 \frac{\partial F_1}{\partial X_0} & \cdots & X_0 \frac{\partial F_r}{\partial X_0} & \bar{F}_0 + X_0 \frac{\partial \bar{F}_0}{\partial X_0} & X_0 \frac{\partial \bar{F}_0}{\partial X_1} & \cdots & X_0 \frac{\partial \bar{F}_0}{\partial X_n} \\ X_1 \frac{\partial F_0}{\partial X_1} & X_1 \frac{\partial F_1}{\partial X_1} & \cdots & X_1 \frac{\partial F_r}{\partial X_1} & X_1 \frac{\partial \bar{F}_1}{\partial X_0} & \bar{F}_1 + X_1 \frac{\partial \bar{F}_1}{\partial X_1} & \cdots & X_1 \frac{\partial \bar{F}_1}{\partial X_n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_n \frac{\partial F_0}{\partial X_n} & X_n \frac{\partial F_1}{\partial X_n} & \cdots & X_n \frac{\partial F_r}{\partial X_n} & X_n \frac{\partial \bar{F}_n}{\partial X_0} & X_n \frac{\partial \bar{F}_n}{\partial X_1} & \cdots & \bar{F}_n + X_n \frac{\partial \bar{F}_n}{\partial X_n} \end{pmatrix} \tag{1.19}$$

Let $M_{i|j}$ be the minor with the i'th row and j'th column left out. Note that Y_i divides $\det(M_{0|i})$ for $i > 0$ and X_j divides $\det(M_{0|j})$.

We will prove the following statement in Section 1.5.1.

Lemma 1.5.7. *There exists a unique* $\tilde{C} \in k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^{\rho+(r+1,n+1)}$ *such that*

$$(m+1) Y_i X_j \tilde{C} = (-1)^j \det(M_{0|j+r+1}) Y_i + (-1)^{r+i} \det(M_{0|i}) X_j$$

for $i \in \{0, \cdots, r\}$ *and* $j \in \{0, \cdots, n\}$. *We have that* $c_1(\mathcal{O}(1, m))^{n+r}$ *is represented by*

$$\frac{\tilde{C} \omega \wedge \bar{\omega}}{\prod_{i=1}^r Y_i F_i \prod_{j=0}^n X_j \bar{F}_j} \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}).$$

Remark 1.5.8. One way to view the situation: we can embed $\mathbb{P}^r \times \mathbb{P}^n$ into $\mathbb{P}^{n+r+1}$ using coordinates $Y_0 F_0, \cdots, Y_r F_r, X_0 \bar{F}_0, \cdots, X_n \bar{F}_n$. This then lands in a hyperplane H. Computing the first Chern class boils down to pulling back the generator ω_H of Ω^{n+r}_H to $\mathbb{P}^r \times \mathbb{P}^n$.

We will then prove the following statement in Section 1.5.2.

Theorem 1.5.9. *Assume that we are not in the situation that* $\dim(\mathcal{X})$ *is odd,* $r = 1$ *and* $m = 2$. *Then the map*

$$\psi : J^\rho \to \tilde{J}^{\rho+(r+1,n+1)}, D \mapsto D \prod_{i=0}^r Y_i \prod_{j=0}^n X_j$$

is an isomorphism. Therefore, for the element $\tilde{C}$ from Lemma 1.5.7, we have that $\tilde{C} = \psi(C)$ for a unique $C \in J^\rho$. Write $AB = \lambda C$ in J^ρ for some $\lambda \in k^*$. Then

$$Tr(\omega_A \cup \omega_B) = (-1)^{r+1} m^{n+1} \binom{n+r}{r} \lambda.$$

We can then find an analogue of the Scheja-Storch generator from the classical case.

Definition 1.5.10. Suppose that $\binom{n+r}{r}$ is invertible in k. Then the trace one element $e_F = \frac{C}{m^n \binom{n+r}{r}} \in J^\rho$ is called the *Scheja-Storch generator* of X.

Remark 1.5.11. We conjecture that the assumption that $\binom{n+r}{r}$ is invertible in k is not necessary, i.e. that one can find a similar construction of the Scheja-Storch generator as in Construction 1.1.19.

Remark 1.5.12 (Continuation of Remark 1.3.9 and Remark 1.4.8)**.** If we take $r = 0$, the formula from Theorem 1.5.9 becomes

$$\mathrm{Tr}(\omega_A \cup \omega_B) = -m\lambda$$

for $A, B \in J$ such that $AB = \lambda e_F \in J^\rho$ for some $\lambda \in k^*$. This is in accordance with [33, Theorem 3.8].

In characteristic zero, one has an explicit formula for the Scheja-Storch generator in the case where $r = 0$, used in [33]. In Section 1.5.3, we show that under the map $Y_0 \mapsto 1$ from J to the classical Jacobian ring, the Scheja-Storch element from Definition 1.5.10 maps to this Scheja-Storch element up to a factor $\frac{m^{n+1}}{m+1}$.

1.5.1 Proof of Lemma 1.5.7

Before proving Lemma 1.5.7, we first prove a lemma in a slightly more general setting. Write $Z_0 = Y_0, \cdots, Z_r = Y_r, Z_{r+1} = X_0, \cdots, Z_{n+r+1} = X_n$ and let $G_0, \cdots, G_{n+r+1}$ be homogeneous polynomials in the Z_i of the same total degree $m+1$. Consider the matrix $M = (m_{ij})_{ij}$ with $m_{ij} = \frac{\partial G_i}{\partial Z_j}$.

Lemma 1.5.13. *Let $i, j \in \{0, \cdots, n+r+1\}$ be two distinct elements. We have that*

$$dG^{i,j} = \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l}$$

where $M_{ij|kl}$ is the minor of M with the i'th and j'th row and the k'th and l'th column removed.

Proof. We have that

$$dG^{i,j} = \bigwedge_{k \neq i,j} \left(\sum_{l=0}^{n+r+1} \frac{\partial G_k}{\partial Z_l} dZ_l \right)$$

$$= \sum_{(j_1,\cdots j_{n+r})} \prod_{k\neq i,j} m_{kj_k} dZ_{j_1} \wedge \cdots \wedge dZ_{j_{n+r}}$$
$$= \sum_{k<l} \sum_{\sigma \in S_{n+r}} \mathrm{sign}(\sigma) \prod_{p\neq i,j} m_{p\sigma(p)} dZ^{k,l}$$
$$= \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l}$$

where S_{n+r} is the symmetric group on $n+r$ elements and we used the Leibniz definition of a determinant. □

Now for $k \in \{0, \cdots, n+r+1\}$ we write

$$\tau_k = \sum_{i<k} (-1)^i Z_i dZ^{i,k} + \sum_{i>k} (-1)^{i+1} Z_i dZ^{k,i}.$$

Note that this definition is similar to the definition of generators τ_i of $\Omega^{l-1}_{\mathbb{P}^l}$ used in [50], which was also used in the argument of Proposition 1.3.8.

Lemma 1.5.14. *Let $i \in \{0, \cdots, n+r+1\}$. Then on the intersection of the opens $\{G_j \neq 0\}$ for $j \neq i$, we have that*

$$d\log\left(\frac{G_1}{G_0}\right) \wedge \cdots \wedge d\log\left(\frac{G_{i+1}}{G_{i-1}}\right) \wedge \cdots \wedge d\log\left(\frac{G_{n+r+1}}{G_{n+r}}\right)$$
$$= \frac{G_i}{(m+1)\prod_{j=0}^{n+r+1} G_j} \sum_{k=0}^{n+r+1} \det(M_{i|k}) \tau_k$$

Proof. We can compute that

$$d\log\left(\frac{G_1}{G_0}\right) \wedge \cdots \wedge d\log\left(\frac{G_{i+1}}{G_{i-1}}\right) \wedge \cdots \wedge d\log\left(\frac{G_{n+r+1}}{G_{n+r}}\right)$$
$$= \left(\frac{dG_{i+1}}{G_{i+1}} - \frac{dG_{i-1}}{G_{i-1}}\right) \prod_{j\neq i,i-1} \left(\frac{dG_{j+1}}{G_{j+1}} - \frac{dG_j}{G_j}\right)$$
$$= \frac{G_i \left(\sum_{j<i} (-1)^j G_j dG^{j,i} + \sum_{j>i} (-1)^{j+1} G_j dG^{i,j}\right)}{\prod_{p=0}^{n+r+1} G_p}$$
$$= \frac{G_i \sum_{j<i} (-1)^j G_j \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l}}{\prod_{p=0}^{n+r+1} G_p}$$
$$+ \frac{G_i \sum_{j>i} (-1)^{j+1} G_j \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l}}{\prod_{p=0}^{n+r+1} G_p}$$

Now recall the Euler equation $(m+1)G_j = \sum_{p=0}^{n+r+1} Z_p \frac{\partial G_j}{\partial Z_p} = \sum_{p=0}^{n+r+1} Z_p m_{jp}$. We see that

$$(m+1)\left(\sum_{j<i} (-1)^j G_j \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l} + \sum_{j>i} (-1)^{j+1} G_j \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l}\right)$$

$$
\begin{aligned}
&= \sum_{j<i} (-1)^j \sum_{p=0}^{n+r+1} Z_p m_{jp} \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l} \\
&\quad + \sum_{j>i} (-1)^{j+1} \sum_{p=0}^{n+r+1} Z_p m_{jp} \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l} \\
&= \sum_{p} (-1)^p Z_p \big(\sum_{j<i} (-1)^{j+p} m_{jp} + \sum_{j>i} (-1)^{j+p+1} m_{jp} \big) \sum_{k<l} \det(M_{ij|kl}) dZ^{k,l} \\
&= \sum_{p} (-1)^p Z_p \Big(\sum_{k<p} \big(\sum_{j<i} (-1)^{j+p} m_{jp} + \sum_{j>i} (-1)^{j+p+1} m_{jp} \big) \det(M_{ij|kp}) dZ^{k,p} \\
&\quad + \sum_{p<l} \big(\sum_{j<i} (-1)^{j+p} m_{jp} + \sum_{j>i} (-1)^{j+p+1} m_{jp} \big) \det(M_{ij|pl}) dZ^{p,l} \\
&\quad + \sum_{\substack{k<l \\ k,l \neq p}} \big(\sum_{j<i} (-1)^{j+p} m_{jp} + \sum_{j>i} (-1)^{j+p+1} m_{jp} \big) \det(M_{ij|kl}) dZ^{k,l} \Big) \\
&= \sum_{p} (-1)^p Z_p \big(\sum_{k<p} \det(M_{i|k}) dZ^{k,p} - \sum_{p>l} \det(M_{i|l}) dZ^{p,l} \big) \\
&= \sum_{k=0}^{n+r+1} \det(M_{i|k}) \big(\sum_{p<k} (-1)^p Z_p dZ^{p,k} + \sum_{p>k} (-1)^{p+1} Z_p dZ^{k,p} \big) \\
&= \sum_{k} \tau_k \det(M_{i|k})
\end{aligned}
$$

as desired. Note that the seventh line is zero, as this is the determinant of $M_{i|l}$ with the k'th row removed and replaced by the p'th one. Also, note that the sixth line picks up an extra minus as we have to jump over an extra column when computing the determinant. □

Now let

$$G_0 = Y_0 F_0, \cdots, G_r = Y_r F_r, G_{r+1} = X_0 \bar{F}_0, \cdots, G_{n+r+1} = X_n \bar{F}_n$$

as in the introduction of this section. Then those are all of bidegree $(1, m)$ and the total degree is $m+1$. The matrix M is exactly given by (1.19).

Proof of Lemma 1.5.7. We can represent $c_1(\mathcal{O}(1,m)) \in H^1(\mathbb{P}^r \times \mathbb{P}^n, \Omega^1_{\mathbb{P}^r \times \mathbb{P}^n})$ on the cover $\mathcal{W}$ by

$$\Big\{ d \log \Big(\frac{G_j}{G_i} \Big) \Big\}_{i,j} \in C^1(\mathcal{W}, \Omega^1_{\mathbb{P}^r \times \mathbb{P}^n}).$$

Taking the cup product $n+r$ times following the rules for a cup product on Čech cohomology, we see that $c_1(\mathcal{O}(1,m))^{n+r}$ is represented by

$$d \log \Big(\frac{G_2}{G_1} \Big) \wedge \cdots \wedge d \log \Big(\frac{G_{n+r+1}}{G_{n+r}} \Big) \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}).$$

By Lemma 1.5.14 this is equal to

$$\frac{\sum_{k=0}^{n+r+1}\det(M_{0|k})\tau_k}{(m+1)\prod_{i=1}^r Y_iF_i\prod_{j=0}^n X_j\bar{F}_j}.$$

The numerator $\sum_{k=0}^{n+r+1}\det(M_{0|k})\tau_k$ is a global section of the twisted sheaf $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(n+r+1, m(n+r+1))$. Because $\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}(r+1,n+1)$ is a trivial line bundle which has $\omega\wedge\bar{\omega}$ as a global generator, there exists a unique rational function $\tilde{C}\in k[Y_0,\cdots,Y_r,X_0,\cdots,X_n]^{n,(n+r+1)m-n-1}$ such that

$$\frac{\sum_{k=0}^{n+r+1}\det(M_{0|k})\tau_k}{(m+1)\prod_{i=1}^r Y_iF_i\prod_{j=0}^n X_j\bar{F}_j}=\frac{\tilde{C}\omega\wedge\bar{\omega}}{\prod_{i=1}^r Y_iF_i\prod_{j=0}^n X_j\bar{F}_j}.$$

In order to find $\tilde{C}$, consider the affine patch $\{Y_i\neq 0\},\{X_j\neq 0\}$ with coordinates $y_k=\frac{Y_k}{Y_i}, x_k=\frac{X_k}{X_j}$. We see that

$$\tau_k=(-1)^r dY^k\wedge\bar{\omega}=\begin{cases}0 & \text{if } k\neq i\\ (-1)^{r+i}\omega\wedge\bar{\omega} & \text{if } k=i\end{cases}$$

for $k\in\{0,\cdots,r\}$ and

$$\tau_k=\omega\wedge dX^{k-r-1}=\begin{cases}0 & \text{if } k\neq j+r+1\\ (-1)^{j}\omega\wedge\bar{\omega} & \text{if } k=j+r+1\end{cases}$$

for $k\in\{r+1,\cdots,n+r+1\}$. Therefore, $\sum_{k=0}^{n+r+1}\det(M_{0|k})\tau_k$ reduces to $(-1)^j\det(M_{0|j+r+1})+(-1)^{r+i}\det(M_{0|i})\omega\wedge\bar{\omega}$. We have that $\tilde{C}$ is of bidegree $\rho+(r+1,n+1)$. Homogenizing again and comparing coefficients of τ_k in $\omega\wedge\bar{\omega}$, we get

$$(m+1)Y_iX_j\tilde{C}=(-1)^j\det(M_{0|j+r+1})Y_i+(-1)^{r+i}\det(M_{0|i})X_j$$

as desired. □

1.5.2 Proof of Theorem 1.5.9

Proof of Theorem 1.5.9. Note that ψ is well defined, because an element of the Jacobian ideal $(F_0,\cdots,F_r,\bar{F}_0,\cdots,\bar{F}_n)$ will be mapped to zero.
Let $\phi: J^\rho\to H^{n+r}(\mathbb{P}^r\times\mathbb{P}^n,\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n})$ be the map from Corollary 1.4.10. Composing the map $J^\rho\to\check{H}^{n+r}(\mathcal{U},\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n})$ that gives rise to ρ with the refinement map from Remark 1.5.5, we find the morphism

$$\psi_J: J^\rho\to\check{H}^{n+r}(\mathcal{W},\Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}), D\mapsto\frac{(-1)^{r+1}mDY_0F_0\omega\wedge\bar{\omega}}{\prod_{i=0}^r F_i\prod_{j=0}^n\bar{F}_j}.$$

Because $\mathcal{W}$ is affine, the Čech cohomology of this cover computes the usual cohomology. This implies that ψ_J is surjective. Using Proposition 1.4.18, we

conclude that ψ_J is an isomorphism.
Now consider the morphism

$$k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^{\rho+(r+1,n+1)} \to C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}),$$
$$D \mapsto \frac{(-1)^{r+1} m D\omega \wedge \bar{\omega}}{\prod_{i=1}^r Y_i F_i \prod_{j=0}^n X_j \bar{F}_j}.$$

Note that coboundaries on $\mathcal{W}$ are precisely coming from the ideal generated by the G_i. We therefore find an induced morphism

$$\psi_{\tilde{J}} : \tilde{J}^{\rho+(r+1,n+1)} \to \check{H}^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}).$$

By Lemma 1.5.7, we have that $\tilde{C}$ maps to the nonzero element $c_1(\mathcal{O}(1,m))^{n+r}$, meaning that $\psi_{\tilde{J}}$ is surjective. Using Proposition 1.4.13, we see that $\psi_{\tilde{J}}$ is an isomorphism.
We now have the commutative diagram

$$\begin{array}{ccc} J^\rho & \xrightarrow{\psi} & \tilde{J}^{\rho+(r+1,n+1)} \\ & {\scriptstyle \psi_J} \searrow \quad \swarrow {\scriptstyle \psi_{\tilde{J}}} & \\ & \check{H}^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) & \end{array}$$

From this, we see that ψ has to be an isomorphism, which proves the first part of the statement.
Now using Proposition 1.4.9 and applying the refinement morphisms, we have that $i_*(\omega_A \cup \omega_B)$ is represented by

$$\frac{(-1)^{r+1} ABmY_0\omega \wedge \bar{\omega}}{\prod_{i=1}^r F_i \prod_{j=0}^n \bar{F}_j} \in C^{n+1}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}).$$

By Lemma 1.5.7, we have that $c_1(\mathcal{O}(1,m))^{n+r}$ is represented by

$$\frac{\tilde{C}\omega \wedge \bar{\omega}}{\prod_{i=1}^r Y_i F_i \prod_{j=0}^n X_j \bar{F}_j} \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}).$$

As ψ is an isomorphism, there exists a $C \in J^\rho$ such that $\tilde{C} = \psi(C)$, from which we see that C maps to $c_1(\mathcal{O}(1,m))^{n+r} \in H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$. Now using Proposition 1.4.18, we have that $AB = \lambda C$ for some $\lambda \in k$. Using that the trace of $c_1(\mathcal{O}(1,m))^{n+r}$ is equal to $\binom{n+r}{r} m^n$ we obtain the desired result. □

1.5.3 The Scheja-Storch generator in characteristic zero for $r = 0$

Notation 1.5.15. Assume in this section that $\mathrm{char}(k) = 0$.

Notation 1.5.16. As in Remark 1.3.9, let $X = V(F) \subset \mathbb{P}^n$ be a smooth hypersurface, defined by a homogeneous polynomial $F \in k[X_0, \cdots, X_n]$ of degree m. Form the hypersurface $\mathcal{X} = V(Y_0 F) \subset \mathbb{P}^0 \times \mathbb{P}^n$ and let $F_i = \frac{\partial F}{\partial X_i}$ and $F_{ij} = \frac{\partial F_i}{\partial X_j}$ for $i, j \in \{0, \cdots, n\}$. Note that we have Euler equations

$$(m-1)F_i = \sum_{j=0}^{n} X_j F_{ij}. \tag{1.20}$$

The Jacobian ring of $\mathcal{X}$ is the bigraded ring

$$J = k[Y_0, X_0, \cdots, X_n]/(F, Y_0F_0, \cdots, Y_0F_n).$$

Furthermore, we set

$$\tilde{J} = k[Y_0, X_0, \cdots, X_n]/(Y_0F, Y_0X_0F_0, \cdots, Y_0X_nF_n).$$

Let

$$J_X = k[X_0, \cdots, X_n]/(F_0, \cdots, F_n)$$

be the Jacobian ring of X as defined in [33], which is a usual graded ring. We have the map $f : J^{a,b} \to J_X^b, Y_0 \mapsto 1, X_j \mapsto X_j$ for all $a, b \in \mathbb{Z}$. Let

$$e_F = \frac{\det(\mathrm{Hess}(F))}{(m-1)^{n+1}} \in J_X^{(n+1)(m+2)}$$

be the classical Scheja-Storch element of X, used in the proof of [33, Lemma 3.7]. In this section we will prove the following result.

Proposition 1.5.17. *We have that* $f(\frac{(m+1)C}{m^{n+1}}) = e_F$ *in* $J_X^{(n+1)(m+2)}$.

Applying Theorem 1.5.9, we find the $(n+2) \times (n+2)$ matrix

$$M = \begin{pmatrix} F & Y_0F_0 & Y_0F_1 & \cdots & Y_0F_n \\ X_0F_0 & Y_0(F_0 + X_0F_{00}) & Y_0X_0F_{01} & \cdots & Y_0X_0F_{0n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_nF_n & Y_0X_nF_{n0} & Y_0X_nF_{n1} & \cdots & Y_0(F_n + X_nF_{nn}) \end{pmatrix}$$

and we note that this matrix has rank $n+1$, as the sum of the last $n+1$ rows is m times the first row.

Lemma 1.5.18. *We have that* $\det(M_{0|0}) = Y_0^{n+1} \frac{m^{n+1}}{(m-1)^{n+1}} \det(\mathit{Hess}(F)) \prod_{i=0}^{n} X_i$.

Proof. Let $\det(\mathrm{Hess}(F))_{j_0, \cdots, j_r}$ be the minor where the rows and columns $(j_0, \cdots, j_r)$ have been removed. We claim that

$$Y_0^{n+1} \det(\mathrm{Hess}(F))_{j_0, \cdots, j_r} \prod_{i \notin (j_0, \cdots, j_r)} X_i \prod_{i \in (j_0, \cdots, j_r)} F_i = Y_0^{n+1} \frac{\det(\mathrm{Hess}(F))}{(m-1)^{j+1}} \prod_{i=0}^{n} X_i.$$

Without loss of generality, we can assume that $(j_0, \cdots, j_r) = (k+1, \cdots, n)$ for some k. The proof of the claim proceeds by induction on k. For $k = n$, the result is clear. Now suppose that

$$Y_0^{n+1} \det(\mathrm{Hess}(F))_{k+1,\cdots,n} \cdot \prod_{i=0}^{k} X_i \cdot \prod_{i=k+1}^{n} F_i = Y_0^{n+1} \frac{\det(\mathrm{Hess}(F))}{(m-1)^{n-k}} \prod_{i=0}^{n} X_i$$

for some k. Denote $H = \mathrm{Hess}(F)_{k+1,\cdots,n}$ and write $H_{i,j}$ for the minor of H with the i'th row and j'th column removed. Note that

$$0 = (m-1)Y_0 X_j F_j = Y_0 \sum_{l=0}^{n} X_j X_l F_{lj}$$

in $\tilde{J}$ and so

$$\begin{aligned}
&((-1)^{k+j} X_k \det(H_{j,k}) - X_j \det(H_{k,k}))Y_0 \prod_{i=0}^{k-1} X_i \\
&\qquad = \left(-\sum_{i=0}^{k} (-1)^{i+j} (X_k F_{ki} + X_j F_{ji}) \det(H_{kj,ki}) \right) Y_0 \prod_{i=0}^{k-1} X_i \\
&\qquad = \left(-\sum_{i=0}^{k} (-1)^{i+j} (X_k F_{ki} - X_k F_{ki} - \sum_{l \neq j,k} X_l F_{li}) \det(H_{kj,ki}) \right) Y_0 \prod_{i=0}^{k-1} X_i \\
&\qquad = \left(\sum_{i=0}^{k} (-1)^{i+j} \sum_{l \neq j,k} X_l F_{li} \det(H_{kj,ki}) \right) Y_0 \prod_{i=0}^{k-1} X_i \\
&\qquad = 0
\end{aligned}$$

as the second sum on the fourth line is the determinant of $H_{k,k}$ with the j'th row replaced by the i'th row, which is zero.
We now have that

$$\begin{aligned}
Y_0^{n+1} \frac{\det(\mathrm{Hess}(F))}{(m-1)^{n-k}} \prod_{i=0}^{n} X_i &= \det(\mathrm{Hess}(F))_{k+1,\cdots,n} \prod_{i=0}^{k} X_i \prod_{i=k+1}^{n} F_i \\
&= Y_0^{n+1} \left(\sum_{j=0}^{k} (-1)^{k+j} F_{kj} \det(H_{k,j}) \right) \prod_{i=0}^{k} X_i \prod_{i=k+1}^{n} F_i \\
&= Y_0^{n+1} \left(\sum_{j=0}^{k} X_j F_{kj} \det(H_{k,k}) \right) \prod_{i=0}^{k-1} X_i \prod_{i=k+1}^{n} F_i \\
&= (m-1) Y_0^{n+1} \det(\mathrm{Hess}(F))_{k,\cdots,n} \prod_{i=0}^{k-1} X_i \prod_{i=k}^{n} F_i
\end{aligned}$$

This completes the proof of the claim.
Let $\tilde{M}_i$ be the matrix given by

$$\begin{pmatrix} X_0F_{00} & \cdots & X_0F_{0,i-1} & X_0F_{0,i+1} & \cdots & X_0F_{0n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{i-1}F_{i-1,i-1} & \cdots & X_{i-1}F_{i-1,i-1} & X_{i-1}F_{i-1,i+1} & \cdots & X_{i-1}F_{i-1,n} \\ X_{i+1}F_{i+1,0} & \cdots & X_{i+1}F_{i+1,i-1} & F_{i+1}+X_{i+1}F_{i+1,i+1} & \cdots & X_nF_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_nF_{n0} & \cdots & X_nF_{n,i-1} & X_nF_{n,i+1} & \cdots & F_n+X_nF_{nn} \end{pmatrix}$$

We find that

$$\begin{aligned} \det(M_{0|0}) = Y_0^{n+1}\det\begin{pmatrix} X_0F_{00} & \cdots & X_0F_{0n} \\ \vdots & \ddots & \vdots \\ X_nF_{n0} & \cdots & X_nF_{nn}\end{pmatrix} \\ + Y_0^{n+1}\sum_{i=0}^{n} F_i\det(\tilde{M}_i) \\ = Y_0^{n+1}\det(\mathrm{Hess}(F))\prod_{j=0}^{n}X_j \\ + Y_0^{n+1}\sum_{(j_0,\cdots,j_r)}\det(\mathrm{Hess}(F))_{j_0,\cdots,j_r}\prod_{i\notin(j_0,\cdots,j_r)}X_i\prod_{i\in(j_0,\cdots,j_r)}F_i \\ = Y_0^{n+1}\left(\frac{1}{(m-1)^{n+1}}\sum_{j=0}^{n+1}\binom{n+1}{j}(m-1)^{n+1-j}\right)\det(\mathrm{Hess}(F))\prod_{i=0}^{n}X_i \\ = Y_0^{n+1}\frac{m^{n+1}}{(m-1)^{n+1}}\det(\mathrm{Hess}(F))\prod_{i=0}^{n}X_i \end{aligned}$$

as desired. □

Proof of Proposition 1.5.17. We find by expanding to the first column that $\det(M_{0|n+2}) = 0$, as $Y_0X_iF_i = 0$ in $\tilde{J}$. Therefore, using Lemma 1.5.18 and Lemma 1.5.7 we have that

$$(m+1)\tilde{C} = Y_0^{n+1}\frac{m^{n+1}}{(m-1)^{n+1}}\prod_{i=0}^{n}X_i\det(\mathrm{Hess}(F))$$

and so using Theorem 1.5.9, we find that

$$C = Y_0^{n}\frac{m^{n+1}}{(m+1)(m-1)^{n+1}}\det(\mathrm{Hess}(F)) \in J^{\rho}.$$

This implies that $f(C) = \frac{m^{n+1}}{m+1}e_F$ in $J_X^{(n+1)(m+2)}$. □

1.6 Example: intersecting two generalized Fermat hypersurfaces of the same degree

To see an application of Theorem 1.5.9, we compute the quadratic Euler characteristic of a complete intersection of two generalized Fermat hypersurfaces of the same degree.

Notation 1.6.1. Let $m \geq 2$ be coprime to $\mathrm{char}(k)$, assume $m+1$ is invertible in k and let $F_0 = \sum_{i=0}^n a_i X_i^m$ and $F_1 = \sum_{i=0}^n b_i X_i^m$. Let $X = V(F_0, F_1)$ be their complete intersection. Furthermore, assume that $a_i b_j - a_j b_i \neq 0$ for all $j \neq i$. Write $L_i = (a_i Y_0 + b_i Y_1)$. Then $V(F_0)$ and $V(F_1)$ are both smooth, and so is X, and these conditions still hold when we set any subset of the X_i equal to zero. We have that

$$F = Y_0 F_0 + Y_1 F_1 = \sum_{i=0}^{n} (a_i Y_0 + b_i Y_1) X_i^m = \sum_{i=0}^{n} L_i X_i^m.$$

Again, we write $\mathcal{X} = V(F)$.

We will prove the following result.

Proposition 1.6.2. *Define*

$$A_{n,m} = \begin{cases} \frac{1}{2}\deg(c_n(T_{\mathcal{X}})) & \text{if } n \text{ or } m \text{ odd} \\ \frac{1}{2}\deg(c_n(T_{\mathcal{X}})) - n - 1 & \text{if } n, m \text{ even} \end{cases}$$

The quadratic Euler characteristic of $\mathcal{X}$ is equal to

$$\chi(\mathcal{X}/k) = \begin{cases} A_{n,m} H & \text{if } n \text{ or } m \text{ odd} \\ A_{n,m} H + \sum_{k=0}^{n} \langle \prod_{i=0, i\neq k}^{n} (a_k b_i - a_i b_k) \rangle & \text{if } n, m \text{ even} \end{cases}$$

This will imply the following.

Corollary 1.6.3. *Define*

$$B_{n,m} = \begin{cases} \frac{1}{2}\deg(c_{n-2}(T_X)) & \text{if } n \text{ odd} \\ \frac{1}{2}\deg(c_{n-2}(T_X)) - 1 & \text{if } n \text{ even, } m \text{ odd} \\ \frac{1}{2}\deg(c_{n-2}(T_X)) - n - 1 & \text{if } n, m \text{ even} \end{cases}$$

The quadratic Euler characteristic of X is equal to

$$\chi(X/k) = \begin{cases} B_{n,m} H & \text{if } n \text{ odd} \\ B_{n,m} H + \langle 1 \rangle & \text{if } n \text{ even, } m \text{ odd} \\ B_{n,m} H + \langle 1 \rangle + \sum_{k=0}^{n} \langle \prod_{i=0, i\neq k}^{n} (a_k b_i - a_i b_k) \rangle & \text{if } n, m \text{ even} \end{cases}$$

1.6.1 The case where $n = 2$

The case where $n = 2$ is special, so we treat that argument here first. In this case, $X = V(F_0, F_1)$ is the intersection of two Fermat curves $V(F_0)$ and $V(F_1)$ in $\mathbb{P}^2$ with

$$F_0 = a_0 X_0^m + a_1 X_1^m + a_2 X_2^m$$

and

$$F_1 = b_0 X_0^m + b_1 X_1^m + b_2 X_2^m$$

where the $a_i, b_i \in k^*$ satisfy $a_i b_j - a_j b_i \neq 0$ for all $i \neq j$. In order to calculate the corresponding quadratic Euler characteristic, we will need that for a separable field extension $k \subset L$, the natural map $\pi : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ induces a morphism $\pi_* : \mathrm{GW}(L) \to \mathrm{GW}(k)$ where for a form $\langle u \rangle \in \mathrm{GW}(L)$, we have that $\pi_* \langle u \rangle$ is given by the composition

$$L \times L \xrightarrow{\langle u \rangle} L \xrightarrow{\mathrm{Tr}_{L/k}} k.$$

By [24, Theorem 1.9] we have that $\chi(\mathrm{Spec}(L)/k) = \pi_*(\langle 1 \rangle)$. The following is a standard fact about quadratic forms, but we include a proof for the sake of completeness.

Lemma 1.6.4. *Let K be a perfect field of characteristic coprime to $2m$ and let $a \in K^*$. Consider the field extension $K(\alpha) = K[X]/(X^m + a)$ of K and let $u \in K(\alpha)^*$ be a unit. Then*

$$Tr_{K(\alpha)/K}(\langle u \rangle) = \begin{cases} \frac{m-1}{2} H + \langle um \rangle & \text{if } m \text{ is odd} \\ \frac{m-2}{2} H + \langle um \rangle + \langle -aum \rangle & \text{if } m \text{ is even} \end{cases}$$

Proof. Note that $K(\alpha)$ has the basis $1, \alpha, \alpha^2, \cdots, \alpha^{m-1}$ over K. We have that

$$\mathrm{Tr}_{K(\alpha)/K}(u\alpha^{i+j}) = \begin{cases} um & \text{if } i = j = 0 \\ -aum & \text{if } i + j = m \\ 0 & \text{otherwise} \end{cases}$$

Namely, if $i + j = m$, the multiplication by $u\alpha^m = -au$ corresponds to the diagonal matrix with $-au$ as its entries, and this has trace $-aum$. If $i = j = 0$, multiplication by u is the diagonal matrix with u on the diagonal, which has trace um. If i or j is not zero and $i + j \neq m$, we are taking the trace of the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -au & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -au & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -au \\ u & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & u & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which has trace zero.
Therefore, the quadratic form $\mathrm{Tr}_{K(\alpha)/K}(\langle u\rangle)$ corresponds to the symmetric bilinear form with matrix

$$\begin{pmatrix} um & 0 & 0 & \cdots & 0 & -uma \\ 0 & 0 & 0 & \cdots & -uma & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -uma & 0 & \cdots & 0 & 0 \\ -uma & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

which gives the form from the statement. □

Proposition 1.6.5. *The quadratic Euler characteristic of X equals*

$$\chi(X/k) = \begin{cases} \frac{(m+1)(m-1)}{2}H + \langle 1\rangle & \text{if } m \text{ is odd} \\ \frac{(m+2)(m-2)}{2}H + \langle 1\rangle + \sum_{i=0}^{2}\langle \prod_{j\neq i}(a_ib_j - a_jb_i)\rangle & \text{if } m \text{ is even} \end{cases}$$

Proof. Without loss of generality, we can assume that $X = V(F_0, F_1)$ lies inside the affine patch where $X_2 \neq 0$; otherwise, we change coordinates. Choosing coordinates $x = \frac{X_0}{X_2}$ and $y = \frac{X_1}{X_2}$ on $\mathbb{A}^2$, we have that X is the zero set of the ideal

$$(a_0x^m + a_1y^m + a_2, b_0x^m + b_1y^m + b_2).$$

Let K be the residue field of X, that is:

$$K = k[x,y]/(a_0x^m + a_1y^m + a_2, b_0x^m + b_1y^m + b_2).$$

Define

$$e = \frac{a_0b_2 - a_2b_0}{a_1b_0 - a_0b_1} \text{ and } f = \frac{a_1b_2 - a_2b_1}{a_0b_1 - a_1b_0}.$$

Then we can view the extension $k \subset K$ as one which takes place in two steps:

$$k \subset k(\alpha) = k[t]/(t^m + e) \subset K = k(\alpha)[s]/(s^m + f).$$

Indeed, the system of equations

$$a_0x^m + a_1y^m + a_2 = 0 \text{ and } b_0x^m + b_1y^m + b_2 = 0$$

implies that

$$(a_1b_0 - a_0b_1)y^m + a_2b_0 - a_0b_2 = 0 \text{ and } (a_0b_1 - a_1b_0)x^m + a_2b_1 - a_1b_2 = 0.$$

We see from Lemma 1.6.4 that for odd m, we have that

$$\begin{aligned} \mathrm{Tr}_{K/k}(\langle 1\rangle) &= \mathrm{Tr}_{k(\alpha)/k}(\mathrm{Tr}_{K/k(\alpha)}\langle 1\rangle) \\ &= \mathrm{Tr}_{k(\alpha)/k}\left(\frac{m-1}{2}H + \langle m\rangle\right) \\ &= \frac{m(m-1)}{2}H + \frac{m-1}{2}H + \langle m^2\rangle \end{aligned}$$

$$= \frac{(m+1)(m-1)}{2} H + \langle 1 \rangle$$

and for even m we compute

$$\begin{aligned}
\mathrm{Tr}_{K/k}(\langle 1 \rangle) &= \mathrm{Tr}_{k(\alpha)/k}(\mathrm{Tr}_{K/k(\alpha)} \langle 1 \rangle) \\
&= \mathrm{Tr}_{k(\alpha)/k}\left(\frac{m-2}{2} H + \langle m \rangle + \langle -fm \rangle \right) \\
&= \frac{m(m-2)}{2} H + \langle m^2 \rangle + \langle -m^2 e \rangle + \langle -m^2 f \rangle + \langle m^2 ef \rangle + (m-2)H \\
&= \frac{(m+2)(m-2)}{2} H + \langle 1 \rangle + \langle -e \rangle + \langle -f \rangle + \langle ef \rangle \\
&= \frac{(m+2)(m-2)}{2} H + \langle 1 \rangle + \langle (a_0 b_1 - a_1 b_0)(a_0 b_2 - a_2 b_0) \rangle \\
&\quad + \langle (a_1 b_0 - a_0 b_1)(a_1 b_2 - a_2 b_1) \rangle + \langle (a_2 b_0 - a_0 b_2)(a_2 b_1 - a_1 b_2) \rangle
\end{aligned}$$

which is the desired result. □

1.6.2 The Jacobian ring

In this situation, we can give a very explicit proof of the one dimensionality of the bidegree ρ part of the Jacobian ring

$$J = k[Y_0, Y_1, X_0, \cdots, X_n]/(F_0, F_1, mL_0 X_0^{m-1}, \cdots, mL_n X_n^{m-1})$$

and also give generators and understand their relations. This is following [50, Section 4 and Section 5.1].

Proposition 1.6.6. *Let $i, j, k \in \{0, \cdots, n\}$ be distinct. We can write L_i as a linear combination of L_j and L_k, more precisely, we have that*

$$L_i = \frac{a_k b_i - a_i b_k}{a_k b_j - a_j b_k} L_j + \frac{a_i b_j - b_i a_j}{a_k b_j - a_j b_k} L_k.$$

Proof. The expression $L_i = aL_j + bL_k$ leads to the system of equations

$$\begin{aligned}
a a_j + b a_k &= a_i \\
a b_j + b b_k &= b_i
\end{aligned}$$

These imply that $b a_k b_j - b b_k a_j = a_i b_j - b_i a_j$ and so $b = \frac{a_i b_j - b_i a_j}{a_k b_j - a_j b_k}$ implying that

$$a = a_j^{-1}(a_i - a_k b) = \frac{a_i a_k b_j - a_j a_i b_k - a_k a_i b_j + a_k b_i a_j}{a_j(a_k b_j - a_j b_k)} = \frac{a_k b_i - a_i b_k}{a_k b_j - a_j b_k}.$$

This proves the statement. □

Notation 1.6.7. Let $k, l \in \{0, \cdots, n\}$ be distinct.

Proposition 1.6.8. *The graded piece J^ρ is generated by the elements*

$$A_j = X_j^m \cdot X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq j,k,l} (a_i Y_0 + b_i Y_1)$$

for $j \in \{0, \cdots, n\} \setminus \{k, l\}$.

The statement will follow from two lemmas.

Lemma 1.6.9. *Consider a term*

$$A = X_0^{i_0} X_1^{i_1} \cdots X_n^{i_n} (a_{j_1} Y_0 + b_{j_1} Y_1) \cdots (a_{j_{n-2}} Y_0 + b_{j_{n-2}} Y_1)$$

where $i_0 + \cdots + i_n = (n+1)(m-2) + m$. If $i_k, i_l \geq m-1$ for $k, l \in \{0, \cdots, n\}$ distinct, then $A = 0$.

Proof. Assume without loss of generality that $i_0, i_1 \geq m - 1$. By Proposition 1.6.6, we can write any L_i for $i \geq 2$ as a linear combination of L_0 and L_1. This implies that A can be written as a linear combination of terms of the form $cX_0^{i_0} X_1^{i_1} \cdots X_n^{i_n} L_0^p L_1^q$ where $p + q = n - 2$ and $c \in k$ is some constant. But

$$(a_0 Y_0 + b_0 Y_1) X_0^{m-1} = 0 \text{ and } (a_1 Y_0 + b_1 Y_1) X_1^{m-1} = 0$$

in J and so $A = 0$. □

Lemma 1.6.10. *Let A be as in Lemma 1.6.9. Then $\max_{k=0,\cdots n} i_k \leq 2m - 2$.*

Proof. Suppose, without loss of generality, that $i_0 > 2m - 2$. We have that $X_0^m = -\frac{1}{a_0}(a_1 X_1^m + \cdots + a_n X_n^m)$ and so

$$A = -\frac{1}{a_0}(a_1 X_0^{i_0 - m} X_1^{i_1 + m} \cdots X_n^{i_n} (a_{j_1} Y_0 + b_{j_1} Y_1) \cdots (a_{j_{n-2}} Y_0 + b_{j_{n-2}} Y_1) + \\ \cdots + a_n X_0^{i_0 - m} X_1^{i_1} \cdots X_n^{i_n + m} (a_{j_1} Y_0 + b_{j_1} Y_1) \cdots (a_{j_{n-2}} Y_0 + b_{j_{n-2}} Y_1))$$

Note that $i_0 - m \geq 2m - 1 - m = m - 1$ and $i_k + m \geq m - 1$ for all $k \in \{1, \cdots, n\}$. By Lemma 1.6.9 this implies that $A = 0$. □

Proof of Proposition 1.6.8. First, note that J^ρ is generated by all the terms A as in Lemma 1.6.9. Now take such a term A and assume that it is nonzero. Then by Lemma 1.6.9, there can only be one $j \in \{0, \cdots, n\}$ such that $i_j \geq m - 1$, and by Lemma 1.6.10, all i_l are smaller than $2m - 2$. But as $i_0 + \cdots + i_n = (n+1)(m-2)+m$, the only possible way in which this can happen is if $i_j = 2m-2$ and all other i_l are equal to $m - 2$. Therefore, we can generate J^ρ by all terms of the form

$$X_j^m \cdot X_0^{m-2} \cdots X_n^{m-2} L_{j_1} \cdots L_{j_{n-2}}.$$

By Proposition 1.6.6, we can choose the generators so that $j_i \notin \{j, k, l\}$ for $i \in \{1, \cdots, n-2\}$. Furthermore, we can choose all j_i to be distinct as the total degree has to be $n - 2$. Finally, we can exclude the terms A_k and A_l using the relations $F_0 = 0$ and $F_1 = 0$. □

Lemma 1.6.11 (See [50], Lemma 4.9)**.** *Let* $p, q, r \in \{0, \cdots, n\}$ *be distinct. Then*

$$X_q^m \prod_{i \neq p,q,r} L_i = -\frac{a_p b_r - a_r b_p}{a_p b_q - b_p a_q} X_r^m \prod_{i \neq p,q,r} L_i$$

in J*.*

Proof. We first note that

$$\sum_{i=0}^{n} (a_p b_i - a_i b_p) X_i^m = a_p F_1 - b_p F_0 = 0$$

and so multiplying by $\prod_{i \neq p,q,r} L_i$ we see that

$$(a_p b_q - a_q b_p) X_q^m \prod_{i \neq p,q,r} L_i + (a_p b_r - a_r b_p) X_r^m \prod_{i \neq p,q,r} L_i = 0$$

as desired. □

Corollary 1.6.12. *Let* $j, j' \in \{1, \cdots, n-1\}$ *be distinct. We have that*

$$A_j = \frac{(a_{j'} b_k - a_k b_{j'})(a_l b_{j'} - a_{j'} b_l)}{(a_j b_k - a_k b_j)(a_l b_j - a_j b_l)} A_{j'}$$

in J^ρ*. In particular,* J^ρ *is one dimensional.*

Proof. Using Proposition 1.6.6 we see that

$$L_{j'} = \frac{a_{j'} b_j - a_j b_{j'}}{a_j b_k - a_k b_j} L_k + \frac{a_k b_{j'} - a_{j'} b_k}{a_j b_k - a_k b_j} L_j$$

so using Lemma 1.6.11 we have that

$$\begin{aligned}
A_j &= X_j^m \cdot X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq j,k,l} L_i \\
&= \frac{a_{j'} b_j - a_j b_{j'}}{a_j b_k - a_k b_j} X_j^m \cdot X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq j',j,l} L_i \\
&= -\frac{(a_{j'} b_j - a_j b_{j'})(a_l b_{j'} - a_{j'} b_l)}{(a_j b_k - a_k b_j)(a_l b_j - a_j b_l)} X_{j'}^m \cdot X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq j',j,l} L_i \\
&= \frac{(a_{j'} b_k - a_k b_{j'})(a_l b_{j'} - a_{j'} b_l)}{(a_j b_k - a_k b_j)(a_l b_j - a_j b_l)} X_{j'}^m \cdot X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq j',k,l} L_i
\end{aligned}$$

as desired. □

1.6.3 Computing the trace of multiples of the generators

Moving to the setting of Theorem 1.5.9, we note that in this case, we have that

$$M = \begin{pmatrix} F_0 & 0 & ma_0Y_0X_0^{m-1} & ma_1Y_0X_1^{m-1} & \cdots & ma_nY_0X_n^{m-1} \\ 0 & F_1 & mb_0Y_1X_0^{m-1} & mb_1Y_1X_1^{m-1} & \cdots & mb_nY_1X_n^{m-1} \\ ma_0X_0^m & mb_0X_0^m & m^2L_0X_0^{m-1} & 0 & \cdots & 0 \\ ma_1X_1^m & mb_1X_1^m & 0 & m^2L_1X_1^{m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ ma_nX_n^m & mb_nX_n^m & 0 & 0 & \cdots & m^2L_nX_n^{m-1} \end{pmatrix}$$

As in Section 1.6.2, let $k, l \in \{0, \cdots, n\}$ be distinct.

Lemma 1.6.13. *Let $A_j = X_j^m \cdot X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq j,k,l}(a_iY_0 + b_iY_1)$ be a generator as in Proposition 1.6.8 and assume that n is even. Let $A, B \in J$. If $AB = \lambda A_j$ in J^ρ for some $\lambda \in k^*$, we have that*

$$Tr(\omega_A \cup \omega_B) = m^{3n+2}(n+1)^2(a_jb_k - a_kb_j)(a_jb_l - a_lb_j)\lambda.$$

Proof. One can show that

$$\det(M_{0|1}) = -m^{2(n+1)}Y_1 \left(\prod_{i=0}^{n} L_iX_i^{m-1}\right) \sum_{i=0}^{n} \frac{a_ib_iX_i^m}{L_i}$$

and check that

$$\det(M_{0|n+2}) = (-1)^{n+1}m^{2n+1}X_n \left(\prod_{i=0}^{n} L_iX_i^{m-1}\right) \sum_{i=0}^{n} \frac{a_ib_iX_i^m}{L_i}.$$

It follows that

$$(m+1)\tilde{C} = -(m+1)m^{2n+1} \left(\prod_{i=0}^{n} L_iX_i^{m-1}\right) \sum_{i=0}^{n} \frac{a_ib_iX_i^m}{L_i}.$$

We note that for all i we have

$$\begin{aligned} a_ib_iX_i^mX_0^{m-2}\cdots X_n^{m-2}\prod_{p\neq i} L_p \\ &= A_i \cdot a_ib_iL_kL_l \\ &= A_i \cdot a_ib_i(a_ka_lY_0^2 + (a_kb_l + a_lb_k)Y_0Y_1 + b_kb_lY_1^2) \\ &= A_i \cdot (a_ia_ka_lb_iY_0^2 + (-(a_ib_k - a_kb_i)(a_ib_l - a_lb_i) \\ &\quad + a_i^2b_kb_l + b_i^2a_ka_l)Y_0Y_1 + a_ib_kb_lb_iY_1^2) \\ &= A_i \cdot (-(a_ib_k - a_kb_i)(a_ib_l - a_lb_i)Y_0Y_1 + a_ia_ka_lb_iY_0^2 \\ &\quad + (a_i^2b_kb_l + b_i^2a_ka_l)Y_0Y_1 + a_ib_kb_lb_iY_1^2) \end{aligned}$$

and we have that

$$\begin{aligned}(a_i a_k a_l b_i Y_0^2 &+ (a_i^2 b_k b_l + b_i^2 a_k a_l) Y_0 Y_1 + a_i b_k b_l b_i Y_1^2) X_i^m \\ &= (a_i a_k a_l b_i Y_0^2 - a_i b_i b_k b_l Y_1^2 - a_i a_k a_l b_i Y_0^2 + a_i b_k b_l b_i Y_1^2) X_i^m \\ &= 0\end{aligned}$$

in $\tilde{J}^{\rho+(r+1,n+1)}$. From this, we see that

$$\begin{aligned}\tilde{C} = m^{2n+1} Y_0 Y_1 X_0 \cdots X_n \Big(&\sum_{i \neq k,l} (a_i b_k - a_k b_i)(a_i b_l - a_l b_i) A_i \\ &+ (a_k b_{j'} - a_{j'} b_k)(a_k b_l - a_l b_k) \tilde{A}_k + (a_l b_{j'} - a_k b_{j'})(a_l b_k - a_k b_l) \tilde{A}_l \Big)\end{aligned}$$

for some $j' \notin \{j, k, l\}$, where

$$\tilde{A}_k = X_k^m X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq k,l,j'} L_i \text{ and } \tilde{A}_l = X_l^m X_0^{m-2} \cdots X_n^{m-2} \prod_{i \neq k,l,j'} L_i.$$

We note that

$$\begin{aligned}(a_l b_i - a_i b_l)&(a_k b_i - a_i b_k) A_i \\ &= (a_l b_i - a_i b_l)(a_k b_i - a_i b_k) \frac{(a_l b_j - a_j b_l)(a_k b_j - a_j b_k)}{(a_l b_i - a_i b_l)(a_k b_i - a_i b_k)} A_j \\ &= (a_j b_l - a_l b_j)(a_j b_k - a_k b_j) A_j\end{aligned}$$

for $i \neq j$ and that

$$\begin{aligned}(a_k b_{j'} - a_{j'} b_k)&(a_k b_l - a_l b_k) \tilde{A}_k \\ &= (a_k b_{j'} - a_{j'} b_k)(a_k b_l - a_l b_k) \frac{(a_j b_{j'} - a_{j'} b_j)(a_j b_l - a_l b_j)}{(a_k b_{j'} - a_{j'} b_k)(a_k b_l - a_l b_k)} \tilde{A}_j \\ &= \frac{(a_j b_k - a_k b_j)(a_j b_{j'} - a_{j'} b_j)(a_j b_l - a_l b_j)}{a_j b_{j'} - a_{j'} b_j} A_j \\ &= (a_j b_k - a_k b_j)(a_j b_l - a_l b_j) A_j\end{aligned}$$

and similarly, we have that

$$(a_l b_{j'} - a_k b_{j'})(a_l b_k - a_k b_l) \tilde{A}_l = (a_j b_k - a_k b_j)(a_j b_l - a_l b_j) A_j.$$

Putting this together, we see that

$$C = m^{2n+1}(n+1)(a_j b_k - a_k b_j)(a_j b_l - a_l b_j) A_j.$$

By Theorem 1.5.9, we have that

$$\mathrm{Tr}(\omega_A \cup \omega_B) = m^{3n+2}(n+1)^2 (a_j b_k - a_k b_j)(a_j b_l - a_l b_j) \lambda$$

as desired. □

1.6.4 The quadratic Euler characteristic

Notation 1.6.14. Assume that $n = 2p$ is even.

In order to prove Proposition 1.6.2, we will need to compute the form Q from Theorem 1.1.16, i.e. [35, Corollary 8.7], given by

$$H^p(\mathcal{X}, \Omega^p_{\mathcal{X}}) \times H^p(\mathcal{X}, \Omega^p_{\mathcal{X}}) \xrightarrow{\cup} H^n(\mathcal{X}, \Omega^n_{\mathcal{X}}) \xrightarrow{\mathrm{Tr}} k.$$

The result from the previous section will allow us to do so on primitive cohomology, but we will also need to understand the form Q on the complement.

Construction 1.6.15. Using Proposition 1.2.10, we have that $H^p(\mathbb{P}^1 \times \mathbb{P}^n, \Omega^p_{\mathbb{P}^1\times\mathbb{P}^n})$ has rank two over k. Generators are given by

$$\alpha = c_1(\mathcal{O}(1,0)) \cup c_1(\mathcal{O}(0,1))^{p-1} \text{ and } \beta = c_1(\mathcal{O}(0,1))^p.$$

Also, one can show that $H^{p+1}(\mathbb{P}^1 \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^1\times\mathbb{P}^n})$ has rank two. Generators are given by

$$\alpha' = c_1(\mathcal{O}(1,0)) \cup c_1(\mathcal{O}(0,1))^p \text{ and } \beta' = c_1(\mathcal{O}(0,1))^{p+1}.$$

Lemma 1.6.16. *The complement to $k \cdot i^*\alpha \oplus k \cdot i^*\beta$ inside $H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})$ under the trace pairing is precisely $H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim}$.*

Proof. Let $\gamma \in H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})$ be an arbitrary element. Using the projection formula, we note that

$$\begin{aligned}\mathrm{Tr}(i^*\alpha \cup \gamma) &= \mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(i_*(i^*\alpha \cup \gamma))\\ &= \mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\alpha \cup i_*\gamma).\end{aligned}$$

If $\gamma \in \ker(i_*)$, this implies that $\mathrm{Tr}(i^*\alpha \cup \gamma) = 0$, and a similar argument shows that $\mathrm{Tr}(i^*\beta \cup \gamma) = 0$, and so $\ker(i_*) = H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim}$ is contained in the complement of $k \cdot i^*\alpha \oplus k \cdot i^*\beta$.
We now show that the other inclusion holds. As α' and β' are generators of $H^{p+1}(\mathbb{P}^1 \times \mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^1\times\mathbb{P}^n})$, we have that $i_*\gamma = a\alpha' + b\beta'$ for certain $a, b \in k$. Note that $\alpha \cup \alpha' = 0 = \beta \cup \beta'$, while $\alpha \cup \beta'$ and $\beta \cup \alpha'$ give the generator of $H^{n+1}(\mathbb{P}^1 \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^1\times\mathbb{P}^n})$. It follows that

$$\mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\alpha \cup i_*\gamma) = b\mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\alpha \cup \beta') = b$$

and similarly

$$\mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\beta \cup i_*\gamma) = a\mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\beta \cup \alpha') = a.$$

If these are both zero then $a = b = 0$, i.e. we have $\gamma \in \ker(i_*) = H^p(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim}$. □

Proposition 1.6.17. *We have that $Tr(i^*\alpha \cup i^*\alpha) = 0$ and $Tr(i^*\beta \cup i^*\beta) = 1$.*

Proof. We have that

$$\begin{aligned}\mathrm{Tr}(i^*\alpha \cup i^*\alpha) &= \mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\alpha \cup i_*i^*\alpha)\\ &= \mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\alpha^2 \cup c_1(\mathcal{O}(1,m)))\\ &= 0\end{aligned}$$

as $\alpha^2 = 0$ and $i_*i^*\alpha = \alpha \cup c_1(\mathcal{O}(1,m))$. Similarly

$$\begin{aligned}\mathrm{Tr}(i^*\beta \cup i^*\beta) &= \mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(\beta^2 \cup c_1(\mathcal{O}(1,m)))\\ &= \mathrm{Tr}_{\mathbb{P}^1\times\mathbb{P}^n}(c_1(\mathcal{O}(0,1))^n \cup c_1(\mathcal{O}(1,m)))\\ &= 1\end{aligned}$$

as desired. □

Proof of Proposition 1.6.2. Using Theorem 1.1.16, we know that $\chi(\mathcal{X}/k)$ is hyperbolic for n odd. For n even, the quadratic Euler characteristic is equal to a hyperbolic form plus the trace form. Therefore, assume from now on that n is even.

In order to compute the trace form, we evaluate it on basis elements of J. Choose a generator $A_j = X_j^m X_0^{m-2} \cdots X_n^{m-2} \prod_{i\neq j,k,l} L_i$ of J^ρ as in Proposition 1.6.8. Note that if $AB = \lambda A_j$ for some $\lambda \in k^*$ and two distinct basis elements A, B, then $BA = \lambda A_j$ and one can check that this yields a hyperbolic form. If m is odd, there are no basis elements that square to a nonzero multiple of A_j. If $m = 2q$ is even, then $\rho = (n-2, 2q(n+2) - 2(n+1))$ is divisible by 2. For each subset $\{i_0, \cdots, i_{\frac{n-2}{2}}\} \subset \{0, \cdots, n\} \setminus \{j, k, l\}$, we find the element

$$A_{i_0,\cdots,i_{\frac{n-2}{2}}} = X_j^q X_0^{q-1} \cdots X_n^{q-1} \prod_{i\in\{i_0,\cdots,i_{\frac{n-2}{2}}\}} L_i$$

of $J^{\frac{\rho}{2}}$ such that, using Lemma 1.6.6 again, we have that

$$\begin{aligned}A^2_{i_0,\cdots,i_{\frac{n-2}{2}}} &= X_j^m X_0^{m-2} \cdots X_n^{m-2} \prod_{i\in\{i_0,\cdots,i_{\frac{n-2}{2}}\}} L_i^2\\ &= \left(\frac{\prod_{i\in\{i_0,\cdots,i_{\frac{n-2}{2}}\}}(a_ib_j - a_jb_i)}{\prod_{i\notin\{j,k,l,i_0,\cdots,i_{\frac{n-2}{2}}\}}(a_ib_j - a_jb_i)}\right) A_j\end{aligned}$$

We note that all such $A_{i_0,\cdots,i_{\frac{n-2}{2}}}$ are multiples of each other in $J^{\frac{\rho}{2}}$, because X_j has degree $2q - 1 = m - 1$, so we can use the same argument as for Proposition 1.6.8. Also, if $j, j' \in \{0, \cdots, n\}$ are distinct and $A, A' \in J^{\frac{\rho}{2}}$ are such that $A^2 = \lambda A_j$ and $(A')^2 = \lambda' A_{j'}$ for some $\lambda, \lambda' \in k^*$, then A and A' are distinct elements of $J^{\frac{\rho}{2}}$. Therefore, for each j there is exactly one basis element that squares to a nonzero multiple of A_j, and we choose this basis element to be such that j, k and l lie in the complement of the i_j.

Using Lemma 1.6.13, this gives rise to the term $\sum_{j=0}^n \langle \prod_{i\neq j}(a_jb_i - a_ib_j)\rangle$. Also, we note that by Proposition 1.6.17, the contribution coming from primitive cohomology is the form with matrix

$$\begin{pmatrix} 0 & m \\ m & 1 \end{pmatrix}$$

which is hyperbolic. Finally, as the rank of $\chi(\mathcal{X}/k)$ is equal to $\deg(c_n(T_{\mathcal{X}}))$ by [35, Theorem 5.3], we see that the coefficient of H is equal to $A_{n,m}$ as desired (also in the case where n is odd). $\square$

We can now also deduce Corollary 1.6.3.

Proof of Corollary 1.6.3. Using Proposition 1.1.8, we have that

$$\langle -1\rangle \chi(X/k) = \chi(\mathcal{X}/k) - \chi(\mathbb{P}^n/k)$$

and $\chi(\mathbb{P}^n/k) = \sum_{i=0}^n \langle -1\rangle^i$ as we saw in Example 1.1.10, i.e. [29, Proposition 1.4(4)]. So for odd n, we have that $\chi(\mathbb{P}^n/k)$ is hyperbolic, and for even n we get an extra $\langle 1\rangle$-term. This gives the desired statement. $\square$

1.6.5 Checking the answer using the quadratic Riemann-Hurwitz formula

There is another way to compute $\chi(\mathcal{X}/k)$: using the quadratic Riemann-Hurwitz formula from [29]. We will now do this and see that we recover Proposition 1.6.2.

Notation 1.6.18. Because we know that $\chi(\mathcal{X}/k)$ is hyperbolic if n is odd by Theorem 1.1.16, we assume throughout that n is even.

Note that the natural projection map $\mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^1$ yields a projective morphism $f : \mathcal{X} \to \mathbb{P}^1$. The fiber of f over a point $y \in \mathbb{P}^1$ is isomorphic to the zero locus of $\sum_{i=0}^n L_i(y)X_i^m$. This is smooth if $L_i(y) \neq 0$ for all $i \in \{0, \cdots, n\}$. If there is a $j \in \{0, \cdots, n\}$ such that $L_j(y) = 0$, we note that by Lemma 1.6.6, we must have that $L_i(y) \neq 0$ for all $i \neq j$. The fiber of f over y is now the cone over the zero locus of $\sum_{i\neq j} L_i(y)X_i^m$ with vertex $[e_j]$ given by the unit vector that has a one on the j'th spot and zero's everywhere else. The vertex $[e_j]$ is the only singular point. In particular, this fiber is smooth inside the $\mathbb{P}^{n-1}$ defined by setting $X_j = 0$.
To apply the quadratic Riemann-Hurwitz formula, the first thing which we need to do is to identify the set $\mathfrak{c}(f)$ of critical points of f, i.e. the locus of points of $\mathcal{X}$ where $df = 0$.

Proposition 1.6.19. *The critical locus $\mathfrak{c}(f)$ of f has $n + 1$ elements, and consists of those points satisfying $L_j = 0$ for some $j \in \{0, \cdots, n\}$ and $X_i = 0$ for all $i \neq j$.*

Proof. Let $j \in \{0, \cdots n\}$ and consider the affine patch of $\mathbb{P}^1 \times \mathbb{P}^n$ given by $X_j \neq 0$ and $Y_0 \neq 0$, with coordinates $y = \frac{Y_1}{Y_0}$ and $x_i = \frac{X_i}{X_j}$ for $i \neq j$. Here, $\mathcal{X}$ is given by the equation

$$\sum_{i\neq j} a_i x_i^m + y \sum_{i\neq j} b_i x_i^m + a_j + y b_j = 0.$$

This implies in particular that

$$m\sum_{i\neq j}(a_i+yb_i)x_i^{m-1}dx_i+\left(\sum_{i\neq j}b_ix_i^m+b_j\right)dy=0.$$

We have that f is given by $f(y,x_0,\cdots,x_{j-1},x_{j+1},\cdots x_n)=y$ and so

$$df=dy=-\frac{m}{\sum_{i\neq j}b_ix_i^m+b_j}\sum_{i\neq j}(a_i+yb_i)x_i^{m-1}dx_i.$$

This implies that $\sum_{i\neq j}L_i(y)x_i^{m-1}dx_i=0$ for those points. This gives us two possibilities for a critical point.
First, we can have that $L_i(y)\neq 0$ for $i\neq j$, so that we must have that $x_i=0$ for all $i\neq j$. We need in addition that $a_j+yb_j=L_j(y)=0$, as the critical point also has to lie on $\mathcal{X}$.
Secondly, we can have that there is some $k\neq j$ such that $L_k(y)=0$. In this case, we have that $x_i=0$ for all $i\neq k$. But for such a point to lie on $\mathcal{X}$, we need the condition that $L_j(y)=0$ again as well, which yields a contradiction, as $a_kb_j-a_jb_k\neq 0$.
Repeating this construction for other choices of j, we deduce the desired statement. □

Remark 1.6.20. All critical points may not lie in the same affine patch, but all critical values (so all $y\in\mathbb{P}^1$ such that $y=f(p)$ for p a critical point) do lie in the same affine patch of $\mathbb{P}^1$. Namely, if there would be a critical value with $Y_0=0$ then $L_j=0$ would imply that $Y_1=0$ (as all $a_i,b_i\in k^*$).

Notation 1.6.21. Let $y\subset\mathcal{X}$ be the subscheme of critical points of f. Consider the closed point y' of y given by $L_j=0$. Consider the affine patch of $\mathbb{P}^1\times\mathbb{P}^n$ given by $Y_0\neq 0$ and $X_j\neq 0$ as before. We have that $\mathcal{O}_{\mathcal{X},y'}$ is a regular local ring and we choose parameters $x_0,\cdots,x_{j-1},x_{j+1},\cdots,x_n$ which generate the maximal ideal $\mathfrak{m}_{y'}$ (we will not need L_j as an additional generator because we already work on $\mathcal{X}$). Furthermore, let x be the subscheme of $\mathbb{P}^1$ defined by $L_j=a_j+b_jy=0$. By [29, Remark 10.9] we have that

$$t_x=\frac{a_j+yb_j}{b_j}$$

is a normalized parameter. Define $s_i=-\frac{m}{\sum_{k\neq j}b_kx_k^m+b_j}(a_i+yb_i)x_i^{m-1}$ for $i\neq j$ and let

$$[B_{y'}]:=[B_{s_*,x_*}]\in\mathrm{GW}(k(y'))$$

be the corresponding Scheja-Storch form (see [29, Theorem 4.1 (3)]).

By the quadratic Riemann-Hurwitz formula, see [29, Corollary 10.6], we have that

$$\chi(\mathcal{X}/k)=\sum_{y'\in\mathfrak{c}(f)}\mathrm{Tr}_{k(y')/k}([B_{y'}])-D(f)\cdot H$$

where $D(f) \in \mathbb{Z}$ (see [29, Theorem 10.2]). Also, note that for all $y' \in \mathfrak{c}(f)$, we have that $k(y') = k$ and so the trace doesn't have any effect. It therefore remains to compute $[B_{y'}]$ for all $y' \in \mathfrak{c}(f)$.

Proposition 1.6.22. *We have that*

$$[B_{y'}] = \begin{cases} \frac{1}{2}(m-1)^n \cdot H & \textit{if } m \textit{ is odd} \\ (\frac{1}{2}((m-1)^n - 1) \cdot H + \langle \prod_{i \neq j}(a_i b_j - a_j b_i)\rangle) & \textit{if } m \textit{ is even} \end{cases}$$

Proof. Let y' be a critical point again. We note that $x_0, \cdots, x_n$ is a local framing for $(\det(\mathfrak{m}_{y'}/\mathfrak{m}_{y'}^2)^\vee)^{\otimes 2}$. The section we have is not diagonalizable, but we note that if we set

$$\lambda = -\frac{m}{\sum_{k \neq j} b_k x_k^m + b_j}$$

and do a change of coordinates where we switch dx_i with λdx_i for all i, it is, and it will only change the determinant by λ^n which will be a square as n is assumed to be even. We can therefore apply [29, Example 4.5] combined with [29, Corollary 4.3] to see that

$$[B_{y'}] = \langle (-\frac{m}{\sum_{i \neq j} b_i x_i^m + b_j})^n \prod_{i \neq j}(a_i + y b_i)\rangle \sum_{i=0}^{(m-1)^n - 1} \langle -1 \rangle^i.$$

We note that for m odd, $(m-1)^n$ is even and so

$$\sum_{i=0}^{(m-1)^n - 1} \langle -1 \rangle^i = \frac{1}{2}(m-1)^n \cdot H.$$

For even m, we have that

$$\sum_{i=0}^{(m-1)^n - 1} \langle -1 \rangle^i = \frac{1}{2}((m-1)^n - 1) \cdot H + \langle 1 \rangle.$$

Furthermore, the term $(-\frac{m}{\sum_{i \neq j} b_i x_i^m + b_j})^n$ is a square as we assumed that n is even.
Finally, as $L_j = 0$ we have that $y = -\frac{a_j}{b_j}$ and so

$$\langle \prod_{i \neq j}(a_i + y b_i)\rangle = \langle \frac{1}{b_j^n} \prod_{i \neq j}(a_i b_j - a_j b_i)\rangle = \langle \prod_{i \neq j}(a_i b_j - a_j b_i)\rangle.$$

So

$$[B_{y'}] = \begin{cases} \frac{1}{2}(m-1)^n \cdot H \langle \prod_{i \neq j}(a_i b_j - a_j b_i)\rangle & \text{if } m \text{ is odd} \\ (\frac{1}{2}(((m-1)^n - 1) \cdot H + \langle 1 \rangle)\langle \prod_{i \neq j}(a_i b_j - a_j b_i)\rangle & \text{if } m \text{ is even} \end{cases}$$

which proves the statement. □

Applying the quadratic Riemann-Hurwitz formula, we see from this that

$$\chi(\mathcal{X}/k) = \begin{cases} A_{n,m} \cdot H & \text{if } m \text{ is odd} \\ A_{n,m} \cdot H + \sum_{j=0}^{n} \langle \prod_{i \neq j} (a_i b_j - a_j b_i) \rangle & \text{if } m \text{ is even} \end{cases}$$

which coincides with the result of Proposition 1.6.2. This therefore gives the same quadratic Euler characteristic of X as we had before.

Chapter 2

Motivic Donaldson-Thomas invariants of $\mathbb{P}^3$

2.1 Classifying spaces and the Witt cohomology of BN_S

Notation 2.1.1. Inside SL_2 over a field k, there is the torus

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in k^* \right\}$$

Its normalizer N_S is generated by T and the "switching" element

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We have that GL_2 contains the torus

$$T_G = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} : t_1, t_2 \in k^* \right\}$$

and its normalizer N_G is generated by T_G and σ. Note that $N_S \subset N_G$ and that N_S is precisely the kernel of the restriction of the determinant character on N_G.

In this section, we recall the definition of a classifying space, together with some basic properties. We then give a summary of some results from the paper [28] by Levine, in which the Witt cohomology of the classifying space BN_S of N_S is computed, together with the Euler classes of some canonical rank two vector bundles on BN_S.

2.1.1 Classifying spaces

For algebraic groups G which are subgroups of GL_n, the corresponding classifying spaces BG have been constructed by Totaro in [53, Remark 1.4] and developed further by Edidin and Graham in [17]. We use the same model as [28], which comes from the model used in the paper [44, Section 4.2.2] of Morel and Voevodsky. We give a brief sketch of the construction.

Construction 2.1.2. Let $U_{n,n+j}$ denote the open subscheme of $\mathbb{A}^{n(n+j)}$ of $n \times n+j$ matrices which are of maximal rank n. We can form the sequence of inclusions

$$\cdots \to U_{n,n+j} \to U_{n,n+j+1} \to \cdots$$

where the maps insert a zero row in the last column. One defines the Indscheme $E\mathrm{GL}_n$ as the colimit of the above sequence. Note that $U_{n,n+j}$ has a free GL_n-action on it. Viewing $U_{n,n+j}$ as a subset of $\mathrm{Hom}(\mathbb{A}^{n+j}, \mathbb{A}^n)$, this action is induced by the one on $\mathbb{A}^n$.

Now let $G \subset \mathrm{GL}_n$ be an algebraic subgroup. We can restrict the action of GL_n on $U_{n,n+j}$ to obtain an action of G on $U_{n,n+j}$.

Notation 2.1.3. Denote EG for the colimit of the resulting sequence of inclusions. Write $B_jG = G \setminus U_{n,n+j}$ for the quotient scheme.

Definition 2.1.4. If $G \subset \mathrm{GL}_n$ is a smooth algebraic subgroup, we define its *classifying space* to be $BG = \mathrm{colim}_j B_jG$.

Remark 2.1.5. We have that B_jG is generally not equal to the presheaf quotient of $U_{n,n+j}$ by G, but it is the quotient as étale sheaves. The same holds for BG being the quotient of EG by G. Also, BG as defined above is independent up to $\mathbb{A}^1$-equivalence of the choice of embedding into GL_n. See [44, Proposition 4.2.6 and Remark 4.2.7] and the surrounding results for more on this. For special groups, BG is the quotient of EG by G as Zariski sheaves because of [17, Proposition 23]. In particular, this holds for SL_2 and GL_2.

Construction 2.1.6. For $n, m \in \mathbb{Z}_{\geq 1}$ with $m \geq n$, consider the map

$$U_{n,m} \to \mathbb{A}^{N+1}, M \mapsto (\det(M_I))_I$$

for $N = \binom{m}{n} - 1$, where the M_I are the $n \times n$-submatrices which one can form out of the columns of M (without changing their order). This induces an embedding

$$\mathrm{GL}_n \setminus U_{n,m} \to \mathbb{P}^N$$

because applying the GL_n-action multiplies the corresponding result by the determinant. One can prove that this map identifies the quotient $\mathrm{GL}_n \setminus U_{n,m}$ with the Grassmanian $\mathrm{Gr}(n, m)$, embedded in $\mathbb{P}^N$ via the classical Plücker embedding. In this way, one can realize $B\mathrm{GL}_n$ as $\mathrm{Gr}(n, \infty) \subset \mathbb{P}^\infty$. See [44, Proposition 3.7] for more details.

Remark 2.1.7. In particular, we have that $B\mathbb{G}_m \cong \mathbb{P}^\infty$ up to $\mathbb{A}^1$-equivalence.

Remark 2.1.8. In topology, classifying spaces classify principal G-bundles. For the construction above, it is however, not true in general that for any G-torsor $V \to X$ on a scheme, there is a classifying map $f : X \to BG$ such that

$V = f^*EG$. It does work for GL_n, however, whenever we have a bundle with a set of generating sections, for instance, for X affine. This is because those sections give an element of the Grassmanian, defining the desired map. See the papers [3] and [4] by Asok, Wendt and Hoyois for more on this topic.

Another fact about algebraic groups which we will need is the following one. This statement is well known, see for example [1, Example 6.1.11] for a reference.

Proposition 2.1.9. *Let $G \subset GL_n$ be an affine algebraic group. Then the Picard group of BG is in bijection with the character group of G.*

The idea of the proof is that a character $G \to \mathbb{G}_m$ gives rise to a morphism $BG \to B\mathbb{G}_m = \mathbb{P}^\infty$, and the pull-back of $\mathcal{O}(1)$ defines an element of the Picard group of BG. One can then construct an inverse operation to this, relying on the fact that $\mathrm{Pic}(U_{n,m}) = 0$.

2.1.2 Witt cohomology of BN_S and Euler classes of canonical rank two bundles

We now study [28, Proposition 5.5], which is a computation of the cohomology $H^*(BN_S, \mathcal{W})$ of BN_S where $\mathcal{W}$ is the sheaf of Witt rings, and [28, Theorem 7.1], which is a computation of the Euler classes of canonical rank two bundles on BN_S. Both of those results have been proven by Levine. For details on how to define the sheaf $\mathcal{W}$, see Morel's paper [42, Chapter 2]. For more details on how to take cohomology of $\mathcal{W}$ and how to define Euler classes in this theory, see [29, Section 2] or [35].

Construction 2.1.10. Consider the isomorphism $\mathbb{G}_m^2 \setminus \mathrm{GL}_2 \to (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ([a:b],[c:d]).$$

Here, the $\mathbb{G}_m^2$ action on GL_2 is given by

$$(\lambda,\mu) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \nu c & \nu d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and if we add the element σ, this gives the action of N_G on GL_2. We find that

$$N_G \setminus \mathrm{GL}_2 \cong S_2 \setminus ((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta)$$

where S_2 is the cyclic group of order 2. Note that $S_2 \setminus (\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Sym}^2(\mathbb{P}^1)$. Moreover, the morphism

$$\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2, ([X_0 : X_1],[Y_0 : Y_1]) \mapsto [X_0Y_0 : X_0Y_1 + X_1Y_0 : X_1Y_1]$$

induces an isomorphism $\mathrm{Sym}^2(\mathbb{P}^1) \cong \mathbb{P}^2$. The image of Δ is the set of all points of the shape $[X_0^2 : 2X_0X_1 : X_1^2]$ which form the curve $C = V(X_1^2 - 4X_0X_2)$. Therefore, we find that

$$N_S \setminus \mathrm{SL}_2 \cong N_G \setminus \mathrm{GL}_2 \cong \mathbb{P}^2 \setminus C.$$

Notation 2.1.11. We let $(N_S \setminus \mathrm{SL}_2) \times^{\mathrm{SL}_2} E\mathrm{SL}_2$ denote all pairs (a, b) inside $(N_S \setminus \mathrm{SL}_2) \times E\mathrm{SL}_2$ up to the equivalence relation $(a \cdot g, b) \sim (a, g \cdot b)$ for $g \in \mathrm{SL}_2$. With this notation, we have that

$$BN_S = N_S \setminus E\mathrm{SL}_2 \cong (N_S \setminus \mathrm{SL}_2) \times^{\mathrm{SL}_2} E\mathrm{SL}_2 \cong (\mathbb{P}^2 \setminus C) \times^{\mathrm{SL}_2} E\mathrm{SL}_2.$$

Proposition 2.1.12. *The polynomial $Q = X_1^2 - 4X_0X_2$ gives rise to a nowhere vanishing section of $\mathcal{O}_{\mathbb{P}^2 \setminus C}(2)$ which is SL_2-invariant.*

Proof. First, note that $[x : y : z] \mapsto Q([x : y : z])$ is a well defined nowhere vanishing section of $\mathcal{O}_{\mathbb{P}^2 \setminus C}(2)$, because Q is homogeneous and nonzero on $\mathbb{P}^2 \setminus C$. We now study the GL_2-action on $\mathbb{P}^2$ induced by the above chain of isomorphisms. Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2.$$

The GL_2-action on GL_2 that we start out with is given by right multiplication, so if we let A act on X we find

$$XA = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}.$$

Under the map $\mathbb{G}_m^2 \setminus \mathrm{GL}_2 \to (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$, the class of XA is mapped to the point $([a\alpha + b\gamma : a\beta + b\delta], [c\alpha + d\gamma : c\beta + d\delta])$. Now if we apply the map to $\mathbb{P}^2$, this point is sent to

$$[x\alpha^2 + y\alpha\gamma + z\gamma^2 : 2(x\alpha\beta + z\gamma\delta) + y(\alpha\delta + \beta\gamma) : x\beta^2 + y\beta\delta + z\delta^2]$$

where $x = ac, y = (ad + bc)$ and $z = bd$. Recall that X itself gets mapped to $[ac : ad+bc : bd]$ from which we see that A acts on a general point $[x : y : z] \in \mathbb{P}^2$ by sending it to the point above.
Having figured out the action, we now compute that

$$\begin{aligned}
Q([x : y : z] \cdot A) &= (2x\alpha\beta + y(\alpha\delta + \beta\gamma) + 2z\gamma\delta)^2 \\
&\quad - 4(x\alpha^2 + y\alpha\gamma + z\gamma^2)(x\beta^2 + y\beta\delta + z\delta^2) \\
&= x^2(4\alpha^2\beta^2 - 4\alpha^2\beta^2) \\
&\quad + xy(4\alpha\beta(\alpha\delta + \beta\gamma) - 4(\alpha^2\beta\delta + \alpha\beta^2\gamma)) \\
&\quad + y^2((\alpha\delta + \beta\gamma)^2 - 4\alpha\beta\gamma\delta) \\
&\quad + yz(4\gamma\delta(\alpha\delta + \beta\gamma) - 4(\beta\gamma^2\delta + \alpha\gamma\delta^2)) \\
&\quad + z^2(4\gamma^2\delta^2 - 4\gamma^2\delta^2) \\
&\quad + xz(8\alpha\beta\gamma\delta - 4(\alpha^2\delta^2 + \beta^2\gamma^2)) \\
&= y^2(\alpha\delta - \beta\gamma)^2 - 4xz(\alpha\delta - \beta\gamma)^2 \\
&= \det(A)^2 Q([x : y : z])
\end{aligned}$$

and so in $\mathcal{O}^*_{\mathbb{P}^2 \setminus C}$ we have that Q is SL_2-invariant under the induced action as desired. □

Construction 2.1.13. We consider the representation $\rho^- : N_S \to \mathbb{G}_m$ which sends σ to -1 and the diagonal matrices to 1. Pulling back the canonical bundle $\mathcal{O}(1)$ on $B\mathbb{G}_m$ via the induced map $BN_S \to B\mathbb{G}_m$ defines a line bundle γ_S on BN_S.

Remark 2.1.14. Note that $\mathcal{O}_{\mathbb{P}^2}(1)$ has a canonical GL_2-linearization. Therefore, we find an invertible sheaf γ_n on $N_S \setminus \mathrm{SL}_2 \times^{\mathrm{SL}_2} U_{2,n}$ for every n. The sheaf defined by those is exactly γ_S.

Notation 2.1.15. Similar to the above remark, $\mathcal{O}_{\mathbb{P}^2}(2)$ induces γ_S^2 on BN_S. Now by Proposition 2.1.12, we see that Q gives rise to a nowhere vanishing section of γ_S^2. Therefore, Q defines a quadratic form $\langle \bar{q} \rangle : \gamma_S^{-1} \to \mathcal{O}_{BN_S}$. We consider $\langle \bar{q} \rangle$ as a global section of the Witt sheaf on BN_S.

Remark 2.1.16. Using Proposition 2.1.9, we see that the Picard group of BN_S is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and it is generated by γ_S. Namely, a character of the torus is of the form

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{a-b}$$

for some $a, b \in \mathbb{Z}$ and from the relation

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \sigma = \sigma \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{2.1}$$

we see that for a character $\rho : N_S \to \mathbb{G}_m$ we have that $t^{a-b}\rho(\sigma) = \rho(\sigma)t^{b-a}$ so that $a = b$. As $\sigma^2 = -\mathrm{Id}$ we have that $\rho(\sigma^2) = (-1)^0 = 1$ and so a character is either the map that sends everything to 1 or ρ^-.

Notation 2.1.17. Note that the structure morphism $\mathbb{P}^2 \setminus C \to \mathrm{Spec}(k)$ induces a map

$$p : BN_S = (\mathbb{P}^2 \setminus C) \times^{\mathrm{SL}_2} E\mathrm{SL}_2 \to \mathrm{Spec}(k) \times^{\mathrm{SL}_2} E\mathrm{SL}_2 = B\mathrm{SL}_2.$$

Let $\mathcal{T}$ denote the tangent bundle of BN_S over $B\mathrm{SL}_2$. Let e be the Euler class of the canonical rank two bundle $\mathbb{A}^2 \times^{\mathrm{SL}2} B\mathrm{SL}_2$ on $B\mathrm{SL}_2$, where SL_2 acts on $\mathbb{A}^2$ by right matrix multiplication.

The following result was proven by Levine in [28].

Proposition 2.1.18 ([28], Proposition 5.5)**.** *Let k be a perfect field and let $W(k)[x_0, x_2]$ be the graded polynomial algebra over $W(k)$ on the generators x_0 of degree zero and x_2 of degree 2. Then $x_0 \mapsto \langle \bar{q} \rangle, x_2 \mapsto p^*e$ defines a $W(k)$-algebra isomorphism*

$$\psi : W(k)[x_0, x_2]/(x_0^2 - 1, (1 + x_0)x_2) \to H^*(BN_S, \mathcal{W}).$$

Moreover, $H^{*\geq 2}(BN_S, \mathcal{W}(\gamma_S))$ is the quotient of the free $H^*(BN_S, \mathcal{W})$-module on the generator $e(\mathcal{T})$ modulo the relation $(1 + \langle q \rangle)e(\mathcal{T}) = 0$.

In [28], there is also a computation of the Euler classes of rank two vector bundles on BN_S in terms of the above description. Following [28, Section 6], for $m \geq 1$, consider the representation $\rho_m : N_S \to \mathrm{GL}_2(\mathbb{A}^2)$ given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}$$
$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix}$$

and let ρ_m^- be given by $\rho_m^-(\sigma) = -\rho_m(\sigma)$. Finally, let ρ_0 be the trivial representation and let $\rho_0^- = \rho^-$. The representations ρ_m give rise to the rank two vector bundles

$$\tilde{O}(m) = \mathbb{A}^2 \times^{N_S} E\mathrm{SL}_2 \to BN_S$$

and the representations ρ_m^- similarly give rise to rank two vector bundles $\tilde{O}^-(m)$. By [28, Lemma 6.1], $\tilde{O}(2)$ is isomorphic to $\mathcal{T}$. Levine has proven the following result.

Theorem 2.1.19 ([28], Theorem 7.1)**.** *Suppose that k is a field of characteristic zero or of characteristic $p > 2$ such that p and m are coprime. Then*

$$e(\tilde{O}(m)) = \begin{cases} m \cdot p^*e \in H^2(BN_S, \mathcal{W}) & \text{if } m \equiv 1 \text{ mod } 4 \\ -m \cdot p^*e \in H^2(BN_S, \mathcal{W}) & \text{if } m \equiv 3 \text{ mod } 4 \\ \frac{m}{2} \cdot e(\mathcal{T}) \in H^2(BN_S, \mathcal{W}(\gamma_S)) & \text{if } m \equiv 2 \text{ mod } 4 \\ -\frac{m}{2} \cdot e(\mathcal{T}) \in H^2(BN_S, \mathcal{W}(\gamma_S)) & \text{if } m \equiv 0 \text{ mod } 4 \end{cases}$$

Furthermore, $e(\tilde{O}^-(m)) = -e(\tilde{O}(m))$.

Remark 2.1.20. Note that the above theorem gives a complete computation of Euler classes of bundles on BN_S. We have considered all possible twists as we saw in Remark 2.1.16. Also, let $\rho : N_S \to \mathrm{GL}(V)$ be a representation of N_S. Then we can restrict this to a representation of the torus inside of SL_2, and as representations of the torus T are semisimple, this will be diagonalizable. Adding the element σ and using the relation (2.1), we see that ρ is a direct sum of representations ρ_m for $m \geq 0$ or ρ^- (noting that any trivial subrepresentation is a ρ_0 and that ρ_m and ρ_m^- are isomorphic as representations). Note that this also proves that N_S-representations are semi-simple.

Remark 2.1.21. The Euler class $e(\tilde{O}^{\pm}(m))$ depends on a choice of isomorphism $\det(e(\tilde{O}^{\pm}(m))) \to \mathcal{O}_{BN_S}$ if m is odd, or $\det(e(\tilde{O}^{\pm}(m))) \to \gamma_S$ if m is even. As the isomorphism of representations $\tilde{O}^-(m) \to \tilde{O}(m)$ has determinant -1, one finds the minus sign in the formula for $e(\tilde{O}^-(m))$.

2.2 Summary of some results in [39] and [40] and an example

In this section, we give a summary of some results in the papers [39] and [40] by Maulik, Nekrasov, Okounkov and Pandharipande. We compute the Donaldson-

Thomas invariants for ideal sheaves of length 1 and 2 on $\mathbb{P}^3$ as an example.

2.2.1 Summary of some results in [39] and [40]

Notation 2.2.1. Let X be a smooth projective threefold over a field k. Suppose that $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. Let $\mathrm{Hilb}^n(X)$ be the moduli space of ideal sheaves of length n on X.

Remark 2.2.2. Hilbert schemes go back to Grothendieck, see [22], and see for instance Hartshorne's book [23] for more details.

Construction 2.2.3. There is a *perfect obstruction theory* on $\mathrm{Hilb}^n(X)$. This means that one can define a perfect complex $E_\bullet$ supported in cohomological degrees 0 and -1, together with a morphism $E_\bullet \to \mathbb{L}_{\mathrm{Hilb}^n(X)}$. Here $\mathbb{L}_{\mathrm{Hilb}^n(X)}$ is the cotangent complex of $\mathrm{Hilb}^n(X)$ over k. This morphism defines an isomorphism on cohomology groups in degree 0, and a surjection on cohomology groups in degree -1. Moreover, because of the condition that $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$, for all classes of ideal sheaves $[\mathcal{I}] \in \mathrm{Hilb}^n(X)$, there are isomorphisms

$$H^0(E_\bullet^\vee \otimes_{\mathcal{O}_{\mathrm{Hilb}^n(X)}} k([\mathcal{I}])) \to \mathrm{Ext}^1(\mathcal{I}, \mathcal{I})$$

and

$$H^{-1}(E_\bullet^\vee \otimes_{\mathcal{O}_{\mathrm{Hilb}^n(X)}} k([\mathcal{I}])) \to \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}).$$

Here $k([\mathcal{I}])$ denotes the residue field at $[\mathcal{I}]$.

Notation 2.2.4. From now on, assume that $k = \mathbb{C}$.

Remark 2.2.5. The above perfect obstruction theory on $\mathrm{Hilb}^n(X)$ for X a Calabi-Yau or Fano variety was constructed by Thomas in [51, Section 3], assuming a certain "tracelessness condition". It is shown in [39, Section 2.2] that one can extend this construction to smooth projective threefolds satisfying the condition that $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. In [9, Section 1.5], there is a general construction for integral proper 3-dimensional Gorenstein Deligne-Mumford stacks.

One can use the perfect obstruction theory to define the virtual fundamental class $[\mathrm{Hilb}^n(X)]^{vir} \in \mathrm{CH}_0(\mathrm{Hilb}^n(X))$.

Definition 2.2.6. The degree $I_n \in \mathrm{CH}^0(\mathrm{Spec}(\mathbb{C})) \cong \mathbb{Z}$ of $[\mathrm{Hilb}^n(X)]^{vir}$ is called the *Donaldson-Thomas invariant.*

Remark 2.2.7. See Fulton's book [18] for more details about the classical theory of Chow rings. For more details about virtual fundamental classes, see for instance [8] by Behrend and Fantechi, or [6], by Battistella, Carocci and Manolache. The theory of Donaldson-Thomas invariants was constructed and defined by Donaldson and Thomas, in [16] and [51].

Notation 2.2.8. Assume from now on that X is *toric*, i.e. there exists an embedding $\mathbb{T} \to X$ of the torus $\mathbb{T} = \mathbb{G}_m^3$ into X such that the image of $\mathbb{T}$ is a dense open in the Zariski topology and the action of $\mathbb{T}$ on itself extends to an action on X. We assume in addition that the very ample invertible sheaf $\mathcal{O}_X(1)$ on X defining the projective embedding of X has a $\mathbb{T}$-linearization.

In this situation, the action of $\mathbb{T}$ on X extends to an action of $\mathbb{T}$ on $\mathrm{Hilb}^n(X)$, and gives a $\mathbb{T}$-linearization on $E_\bullet$. Also, the perfect obstruction theory $E_\bullet$ gives rise to an equivariant perfect obstruction theory; see Levine's paper [32, Theorem 6.4] or [9, Proposition 2.4] for details. One can then define the equivariant virtual fundamental class $[\mathrm{Hilb}^n(X)]^{vir}_{\mathbb{T}} \in \mathrm{CH}_0^{\mathbb{T}}(\mathrm{Hilb}^n(X))$.

Notation 2.2.9. Let $I_n^{\mathbb{T}} \in \mathrm{CH}^0(B\mathbb{T})$ be the degree of $[\mathrm{Hilb}^n(X)]^{vir}_{\mathbb{T}}$.

See Totaro's paper [53, Section 1] for the definition of the Chow ring of a classifying space.

Remark 2.2.10. If we let $p : B\mathbb{T} \to \mathrm{Spec}(\mathbb{C})$ be the structure morphism, we have that $I_n^{\mathbb{T}} = p^* I_n$.

Definition 2.2.11. The *MacMahon function* is the function given by

$$M(q) = \prod_{n \geq 1} \frac{1}{(1 - q^n)^n}.$$

Remark 2.2.12. This function was first defined and conjectured to be the generating series for 3-dimensional partitions by Percy A. MacMahon in [38, Article 43]. A full statement and proof were later given by Stanley, see [49, Corollary 7.20.3].

In the papers [39] and [40], there is a proof of the following statement.

Theorem 2.2.13 ([40], Theorem 2)**.** *Let X be a smooth projective toric 3-fold. Then*

$$\sum_{n \geq 0} I_n q^n = M(-q)^{\deg_X c_3(T_X \otimes K_X)}.$$

Remark 2.2.14. This formula was later extended to all Calabi-Yau threefolds by Li, see [37], and then by Levine-Pandharipande, see [34].

The ingredients of the proof of this statement are the virtual localization formula from the paper [19] by Graber and Pandharipande, and the Bott residue formula, see Bott's paper [10], together with a computation of the virtual equivariant Euler class of the virtual tangent space of each fixed ideal. We now summarize how this computation works.
It is shown in [39, Sections 4.4-4.5] that the virtual localization formula now takes the following shape:

$$I_n = \sum_{[\mathcal{I}] \in \mathrm{Hilb}^n(X)^{\mathbb{T}}} \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathcal{I}, \mathcal{I}))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}))}.$$

Remark 2.2.15. In fact, the virtual localization formula computes $I_n^{\mathbb{T}}$ rather than I_n. Using Remark 2.2.10, one can then compute I_n from $I_n^{\mathbb{T}}$. But in our situation, we note that $\mathrm{CH}^0(B\mathbb{T}) \cong \mathbb{Z}$, so that all $I_n^{\mathbb{T}}$ are integers.

This means that for a fixed point $[\mathcal{I}]$, one needs the equivariant virtual Euler class of the virtual tangent space

$$\mathcal{T}_{[\mathcal{I}]} = \mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}).$$

Remark 2.2.16. Note that the virtual Euler class is not well defined in the category $K_0(\mathbb{T} - \mathrm{Reps})$ of virtual $\mathbb{T}$-representations. One would like to define $e^{\mathbb{T}}(V_0 - V_1) = \frac{e^{\mathbb{T}}(V_0)}{e^{\mathbb{T}}(V_1)}$ and then use that $e^{\mathbb{T}}(V_0 \oplus V_1) = e^{\mathbb{T}}(V_0)e^{\mathbb{T}}(V_1)$ to show that this is well defined. However, as the equivariant Euler class of a trivial representation is zero, this does not work. Instead, one uses that $\mathbb{T}$-representations are semi-simple and restricts to the subgroup of $K_0(\mathbb{T} - \mathrm{Reps})$ generated by all irreducible nontrivial representations.

Note that every representation of $\mathbb{T}$ has an associated trace character, which sends an element of $\mathbb{T}$ to the corresponding matrix and then to its trace in $\mathbb{C}$. We will find the Euler class of $\mathcal{T}_{[\mathcal{I}]}$ by computing its trace.

Notation 2.2.17. As explained in [39, Section 4.1], for each fixed ideal sheaf $[\mathcal{J}] \in \mathrm{Hilb}^n(X)$, we can choose a canonical $\mathbb{T}$-stable affine open

$$U_{\mathcal{J}} \cong \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3])$$

centered at the support of $\mathcal{J}$, and the $U_{\mathcal{J}}$ cover X. On such a chart $U_{\mathcal{J}}$, we may choose coordinates x_1, x_2, x_3 such that the $\mathbb{T}$-action is given by

$$(t_1, t_2, t_3)(x_1, x_2, x_3) = (t_1x_1, t_2x_2, t_3x_3).$$

As explained in [39, Section 4.2], the fixed ideal sheaves correspond to subschemes $Y \subset X$ supported on the fixed points. This implies that, for the fixed point $[\mathcal{I}]$, on an open $U_{\mathcal{J}}$ we have that $\mathcal{I}|_{U_{\mathcal{J}}} \subset \mathbb{C}[x, y, z]$ is a monomial ideal. Therefore, we find corresponding partitions

$$\pi_{\mathcal{J}} = \{(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geq 0} : x^{k_1}y^{k_2}z^{k_3} \notin \mathcal{I}|_{U_{\mathcal{J}}}\}.$$

One can prove (see [39, Equation (9)]) that

$$\begin{aligned}&\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}) \\ &\qquad = \sum_{[\mathcal{J}] \in \mathrm{Hilb}^n(X)^{\mathbb{T}}} \left(H^0(U_{\mathcal{J}}, \mathcal{O}_X) - \sum_{j=0}^{3} (-1)^j H^0(U_{\mathcal{J}}, \mathrm{Ext}^j(\mathcal{I}, \mathcal{I})) \right).\end{aligned}$$

Notation 2.2.18. Let $I \subset \mathbb{C}[x, y, z]$ be the restriction of $\mathcal{I}$ to $U_{\mathcal{J}}$. We consider the trace

$$Q_{\mathcal{J}}(t_1, t_2, t_3) = \mathrm{tr}_{\mathbb{C}[x,y,z]/I}(t_1, t_2, t_3) = \sum_{(k_1, k_2, k_3) \in \pi_{\mathcal{J}}} t_1^{k_1} t_2^{k_2} t_3^{k_3}.$$

One now defines

$$V_{\mathcal{J}} = Q_{\mathcal{J}}(t_1,t_2,t_3) - \frac{Q_{\mathcal{J}}(t_1^{-1},t_2^{-1},t_3^{-1})}{t_1t_2t_3} + Q_{\mathcal{J}}(t_1,t_2,t_3)Q_{\mathcal{J}}(t_1^{-1},t_2^{-1},t_3^{-1})\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}.$$

Theorem 2.2.19 ([39], Theorem 1)**.** *The $\mathbb{T}$-character of $\mathcal{T}_{[\mathcal{I}]}$ is given by*

$$tr_{\mathcal{T}_{[\mathcal{I}]}}(t_1,t_2,t_3) = \sum_{[\mathcal{J}]\in Hilb^n(X)^{\mathbb{T}}} V_{\mathcal{J}}.$$

Construction 2.2.20. As $\mathbb{T} \cong \mathbb{G}_m^3$, we have that $B\mathbb{T} \cong (\mathbb{P}^\infty)^3$. We have that $\mathrm{CH}^*(B\mathbb{T}) \cong \mathbb{Z}[e_1,e_2,e_3]$ where

$$e_1 = c_1(\mathcal{O}(1,0,0)), e_2 = c_1(\mathcal{O}(0,1,0)) \text{ and } e_3 = c_1(\mathcal{O}(0,0,1)).$$

See [53, Section 15] for more details.

Assuming the representation is diagonalized, each summand in $\mathrm{tr}_{\mathcal{T}_{[\mathcal{I}]}}(t_1,t_2,t_3)$ corresponds to one of the characters building up the representation, i.e. to line bundles on $B\mathbb{T}$ (see Proposition 2.1.9). Therefore, we have that each of these is a tensor product of the e_i's, e.g.

$$\begin{aligned} e^{\mathbb{T}}(t_1^a t_2^b t_3^c) &= c_1(\mathcal{O}(a,b,c)) \\ &= c_1(\mathcal{O}(a,0,0)) + c_1(\mathcal{O}(0,b,0)) + c_1(\mathcal{O}(0,0,c)) \\ &= ae_1 + be_2 + ce_3 \end{aligned}$$

Noting that the Euler class of a sum is the product of the Euler classes, if

$$\mathrm{tr}_{\mathcal{T}_{[\mathcal{I}]}}(t_1,t_2,t_3) = \sum_{(k_1,k_2,k_3)\in\mathbb{Z}^3} v_{k_1,k_2,k_3}\cdot t_1^{k_1}t_2^{k_2}t_3^{k_3}$$

we find the Euler class

$$e^{\mathbb{T}}(\mathcal{T}_{[\mathcal{I}]})^{-1} = \prod_{k\in\mathbb{Z}^3}(s_1k_1+s_2k_2+s_3k_3)^{-v_{k_1,k_2,k_3}}.$$

Here, s_1, s_2 and s_3 are the "tangent weights" of the action on $U_{\mathcal{J}}$, i.e. we have that $(t_1,t_2,t_3)\cdot(x,y,z) = (t_1^{s_1}x, t_2^{s_2}y, t_3^{s_3}z)$ on $U_{\mathcal{J}}$. Adding these contributions yields the Donaldson-Thomas invariant I_n.

2.2.2 An example

We apply the above strategy to the situation where $X = \mathbb{P}^3$, and we will show that $I_1 = 20$ and $I_2 = 150$.

Notation 2.2.21. Choose coordinates X_0, X_1, X_2, X_3 on $\mathbb{P}^3$, and consider the standard action of the three dimensional torus $\mathbb{T} \cong \mathbb{G}_m^3$ on $\mathbb{P}^3$ given by

$$(t_1, t_2, t_3) \cdot [X_0 : X_1 : X_2 : X_3] = [X_0 : t_1 X_1 : t_2 X_2 : t_3 X_3].$$

Write $U_i = \{X_i \neq 0\} \subset \mathbb{P}^3$.

The fixed points of this action are supported on the points

$$[1:0:0:0], [0:1:0:0], [0:0:1:0] \text{ and } [0:0:0:1].$$

Remark 2.2.22. Note that

$$\begin{aligned}
[X_0 : t_1 X_1 : t_2 X_2 : t_3 X_3] &= [t_1^{-1} X_0 : X_1 : t_1^{-1} t_2 X_2 : t_1^{-1} t_3 X_3] \\
&= [t_2^{-1} X_0 : t_1 t_2^{-1} X_1 : X_2 : t_2^{-1} t_3 X_3] \\
&= [t_3^{-1} X_0 : t_1 t_3^{-1} X_1 : t_2 t_3^{-1} X_2 : X_3]
\end{aligned}$$

We see from this that the tangent weights on U_0 are $((1,0,0),(0,1,0),(0,0,1))$, the tangent weights on U_1 are $((-1,0,0),(-1,1,0),(-1,0,1))$, the tangent weights on U_2 are $((0,-1,0),(1,-1,0),(0,-1,1))$ and the tangent weights on U_3 are $((0,0,-1),(1,0,-1),(0,1,-1))$.

To compute the invariant I_1, we need to consider the fixed points of which the ideal has length one. These are precisely the ideal sheaves supported on one of the above points, which are locally given by the ideal $(x,y,z) \subset \mathbb{C}[x,y,z]$. On U_0, we find $Q_0(t_1,t_2,t_3) = 1$ and so

$$V_0(t_1,t_2,t_3) = t_1^{-1} + t_2^{-1} + t_3^{-1} - t_1^{-1} t_2^{-1} - t_1^{-1} t_3^{-1} - t_2^{-1} t_3^{-1}.$$

We see from this that the Euler class of the virtual tangent space corresponding to the ideal sheaf of length 1 which is given by $(x,y,z) \subset \mathbb{C}[x,y,z]$ on U_0 is

$$\frac{(e_1+e_2)(e_1+e_3)(e_2+e_3)}{e_1 e_2 e_3}.$$

From

$$V(t_1^{-1}, t_1^{-1} t_2, t_1^{-1} t_3) = t_1 + t_1 t_2^{-1} + t_1 t_3^{-1} - t_1^2 t_2^{-1} - t_1^2 t_3^{-1} - t_1^2 t_2^{-1} t_3^{-1}$$

we find that the Euler class at $[0:1:0:0]$ is

$$\frac{(2e_1 - e_2 - e_3)(2e_1 - e_2)(2e_1 - e_3)}{e_1(e_1 - e_2)(e_1 - e_3)}.$$

Similarly, the class at $[0:0:1:0]$ is

$$\frac{(2e_2 - e_1 - e_3)(2e_2 - e_1)(2e_2 - e_3)}{e_2(e_2 - e_1)(e_2 - e_3)}$$

and at $[0:0:0:1]$ we find

$$\frac{(2e_3 - e_1 - e_2)(2e_3 - e_1)(2e_3 - e_2)}{e_3(e_3 - e_1)(e_3 - e_2)}.$$

Adding these yields $I_1 = 20$.
There are two categories of fixed points of length 2:

- Two points with both multiplicity one, corresponding to all possible products of the classes for $n = 1$.
- One point with multiplicity two in one tangent direction, corresponding to the ideals $(x^2, y, z), (x, y^2, z), (x, y, z^2)$ locally.

For the computation, we use some SAGE code, see here. We find that $I_2 = 150$.

Remark 2.2.23. This agrees with what one gets if one uses Theorem 2.2.13. Namely, we have that $\deg_{\mathbb{P}_3} c_3(T_{\mathbb{P}^3} \otimes K_{\mathbb{P}^3}) = -20$, because tensoring the Euler exact sequence with $K_{\mathbb{P}^3} \cong \mathcal{O}(-4)$ gives an exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-3)^4 \to T_{\mathbb{P}^3} \otimes K_{\mathbb{P}^3} \to 0$$

so we can compute the total Chern class of $T_{\mathbb{P}^3} \otimes K_{\mathbb{P}^3}$ in $\mathrm{CH}^*(\mathbb{P}^3) \cong \mathbb{Z}[t]/t^4$ as

$$\begin{aligned} c(T_{\mathbb{P}^3} \otimes K_{\mathbb{P}^3}) &= c(\mathcal{O}(-3))^4 / c(\mathcal{O}(-4)) \\ &= \frac{(1-3t)^4}{1-4t} \\ &= (1 - 12t + 54t^2 - 108t^3)(1 + 4t + 16t^2 + 64t^3) \end{aligned}$$

The coefficient of t^3 is $64 - 192 + 216 - 108 = -20$. Now we note that

$$M(-q)^{-20} = 1 + 20q + 150q^2 + \cdots$$

2.3 Motivic Donaldson-Thomas invariants of $\mathbb{P}^3$

Let $\mathcal{E}$ be an SL-oriented motivic ring spectrum, see for instance Ananyevskiy's paper [2] for a definition and more details. In Levine's paper [30], it is discussed how one can define a motivic analogue of a virtual fundamental class for a perfect obstruction theory $E_\bullet$ on a quasi-projective scheme Z over a perfect field k. This class is an element of the Borel-Moore homology $\mathcal{E}^{B.M.}(Z, \mathbb{V}(E_\bullet))$ where $\mathbb{V}(E_\bullet) = \mathrm{Spec}(\mathrm{Sym}^* E_0) - \mathrm{Spec}(\mathrm{Sym}^* E_1)$ is the virtual vector bundle associated to $E_\bullet$. If $E_\bullet$ has virtual rank zero and there is an isomorphism from the determinant of the obstruction theory to the square of a line bundle, a choice of such an isomorphism (called an *orientation* of $E_\bullet$) defines a degree of this class as an element of $\mathcal{E}^{0,0}(k)$ (see [30, Section 8.1]). In Levine's paper [31], this definition is extended to an equivariant setting if the scheme has an action on it of a smooth closed subgroup of GL_n, and [31] provides a proof of a virtual localization formula for this situation.

Notation 2.3.1. In this section and in the next, we work over the base field $\mathbb{R}$ unless specified otherwise. The reason for this is that in order to use the virtual localization formula for equivariant Witt cohomology as proven by Levine, see [31, Theorem 6.7], one needs to invert a certain integer. Also, the degrees of motivic Donaldson-Thomas invariants will land in the Witt ring of the ground field. As $\mathcal{W}(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$, we will not get much information out of the computations in this situation, especially if we need to invert 2. For $\mathbb{F}$ a finite field of odd characteristic, one has $4\mathcal{W}(\mathbb{F}) = 0$ (see [45, Corollary 3.11]), so in this situation, inverting 2 will also mean we do not get much out of the computation. On the other hand, $\mathcal{W}(\mathbb{R}) \cong \mathbb{Z}$, which is much more convenient. The arguments in this chapter do, however, mostly work over more general fields, in particular for any field which has an embedding into $\mathbb{R}$. For example, the map $\mathcal{W}(\mathbb{Q}) \to \mathcal{W}(\mathbb{R})$ is surjective with 2-primary kernel.

In this section and the next, we will study the motivic virtual fundamental classes corresponding to the following situation.

Notation 2.3.2. Let $a, b \in \mathbb{Z}$ be odd and such that

$$a, b, 3a - b, 3b - a, 3a + b, 3b + a, a - b \text{ and } a + b$$

are nonzero (the denominators in the computations in the next section are the reason for this assumption). Furthermore, assume that $a > 5b$ (so that we can decide for all the terms we will see in the next section whether they are positive or negative). Consider the action of N_S on $\mathbb{P}^3$ given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot [X_0 : X_1 : X_2 : X_3] = [t^a X_0 : t^{-a} X_1 : t^b X_2 : t^{-b} X_3]$$
$$\sigma \cdot [X_0 : X_1 : X_2 : X_3] = [-X_1 : X_0 : -X_3 : X_2].$$

This action does not have fixed points, but there are the two fixed couples $\{[1:0:0:0], [0:1:0:0]\}$ and $\{[0:0:1:0], [0:0:0:1]\}$.

Notation 2.3.3. We denote $U_i = \{X_i \neq 0\} \subset \mathbb{P}^3$ for $i \in \{0, \cdots, 3\}$.

Remark 2.3.4. We note that for even a or b, the above action would not be well defined, as the map $f : N_S \to \mathrm{SL}_2$ given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} t^a & 0 \\ 0 & t^{-a} \end{pmatrix}$$

and $\sigma \to \sigma$ is not a morphism. Indeed, if f were a morphism, we would have $f(\sigma^2) = f(\sigma)^2 = \sigma^2 = -\mathrm{Id}$. But $\sigma^2 = -\mathrm{Id}$, so that $f(\sigma^2) = f(-\mathrm{Id}) = \mathrm{Id}$, which is a contradiction. We would therefore have to send σ to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

but this has determinant -1 and is therefore not in SL_2.

As remarked before, we need that the determinant of the perfect obstruction theory (which is the same as in the previous section) is a square. One can construct an orientation on $\mathrm{Hilb}^n(X)$ for X a smooth projective threefold with an isomorphism $K_X \cong L^{\otimes 2}$. In the case of a Calabi-Yau threefold, this is due to Y. Toda [52, Proposition 3.1]. Following email correspondence with Toda describing his result and method of proof, M. Levine handled the case of arbitrary X in the paper [32]. We will discuss this orientation and its influence on signs in Section 2.3.1.
With a given orientation, one can make the following definition.

Definition 2.3.5. For $n \geq 0$, we let $\tilde{I}_n \in \mathcal{W}(\mathbb{R}) \cong \mathbb{Z}$ be the degree of the motivic virtual fundamental class of $\mathrm{Hilb}^n(\mathbb{P}^3)$.

One can show that the perfect obstruction theory $E_\bullet$ is N_S-equivariant, giving rise to an equivariant motivic virtual fundamental class of $\mathrm{Hilb}^n(\mathbb{P}^3)$.

Notation 2.3.6. Let $\tilde{I}_n^{N_S} \in H^*(BN_S, \mathcal{W})$ be the degree of the above equivariant virtual fundamental class.

Using [31, Theorem 6.7], if the torus T in N_S acts with isolated fixed points, we have that

$$\tilde{I}_n^{N_S} = \sum_{[\mathcal{I}] \in \mathrm{Hilb}^n(\mathbb{P}^3)^{N_S}} \frac{e(\mathrm{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}))}. \tag{2.2}$$

Again, one can find $\tilde{I}_n$ from this. We therefore need to compute the trace of the virtual representation

$$\mathrm{Ext}^2(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^1(\mathcal{I}, \mathcal{I})$$

of N_S for all ideal sheaves $\mathcal{I}$ that are isolated fixed points of the N_S-action; if the N_S-fixed locus on $\mathrm{Hilb}^n(X)$ has non-isolated fixed points, the method requires more work, and will not be discussed further here. We can compute the equivariant Euler classes from the trace using Proposition 2.1.19, because any N_S-representation can be decomposed as a sum of the representations $\rho_m^{\pm}$, see Remark 2.1.20. In order to find the trace, we use the strategy of [39] and [40]. The proofs are almost the same as in [39, Section 4], but included here for the reader's convenience. We first show the following.

Lemma 2.3.7. *For a fixed ideal sheaf $[\mathcal{I}] \in \mathrm{Hilb}^n(\mathbb{P}^3)$ under the N_S-action, we have that*

$$\mathit{Ext}^1(\mathcal{I}, \mathcal{I}) - \mathit{Ext}^2(\mathcal{I}, \mathcal{I}) = \sum_{i=0}^{3} \left(H^0(U_i, \mathcal{O}_{\mathbb{P}^3}) - \sum_{j=0}^{3} (-1)^j H^0(U_i, \mathcal{E}xt^j(\mathcal{I}, \mathcal{I})) \right)$$

as virtual representations of N_S.

Remark 2.3.8. Note that one can view the above as an N_S-representation by considering it as the sum of

$$\left(H^0(U_0, \mathcal{O}_{\mathbb{P}^3}) - \sum_{j=0}^{3}(-1)^j H^0(U_0, \operatorname{Ext}^j(\mathcal{I},\mathcal{I}))\right)$$
$$\oplus \left(H^0(U_1, \mathcal{O}_{\mathbb{P}^3}) - \sum_{j=0}^{3}(-1)^j H^0(U_1, \operatorname{Ext}^j(\mathcal{I},\mathcal{I}))\right)$$

and the corresponding term for U_2 and U_3.

We will then show the following.

Proposition 2.3.9. *Let $\mathcal{I}$ be an N_S-fixed ideal sheaf. For $i \in \{0, \cdots, 3\}$, write $R = \mathbb{R}[x,y,z] \cong H^0(U_i, \mathcal{O}_{\mathbb{P}^3})$ and let I be the image of the ideal sheaf $\mathcal{I}$. Let $\pi_I = \{(i,j,k) : x^i y^j z^k \notin I\}$ and suppose that s_1, s_2, s_3 are the tangent weights on U_i, i.e. $t \cdot (x,y,z) = (s_1 x, s_2 y, s_3 z)$. Set*

$$Q_i(t) = tr_{R/I}(t) = \sum_{(i,j,k)\in\pi_I} s_1^i s_2^j s_3^k.$$

We have that

$$tr_{Ext^1(I,I)-Ext^2(I,I)}(t) = \frac{s_1 s_2 s_3 Q(t) - Q(t^{-1}) + Q(t)Q(t^{-1})(1-s_1)(1-s_2)(1-s_3)}{s_1 s_2 s_3}.$$

Adding the traces on different U_i and filling in the correct tangent weights gives the trace of the virtual tangent space of $\mathcal{I}$, from which one can compute the Euler classes as before. In the next section, we will apply this to compute $\tilde{I}_n$ for $n \leq 6$.

2.3.1 The orientation and its influence on signs

Notation 2.3.10. Let X be a smooth projective threefold over a field k together with an N_S-action. Let $n \geq 1$ and let $\operatorname{Hilb}^n(X)$ be the Hilbert scheme of ideal sheaves of length n on X.

The following statement was proven for more general groups by Levine in the paper [32].

Proposition 2.3.11 ([32], Theorem 6.3)**.** *Suppose that there is an N_S-linearized very ample line bundle L on X and that we are given an N_S-linearized isomorphism $\omega_{X/k} \to L^{\otimes 2}$. Let $\mathcal{I}$ be the ideal sheaf corresponding to the universal subscheme $i : Z \to Hilb^n(X) \times X$ of $Hilb^n(X) \times X$. Let $p_Z : Z \to Hilb^n(X)$ be the natural projection of Z and let $p_2 : Hilb^n(X) \times X \to X$ be the projection to X. Then $E_\bullet$ has virtual rank zero and there exists a canonical N_S-equivariant isomorphism*

$$\rho : \det(E_\bullet) \to (\det(p_{Z,*}\mathcal{O}_Z) \otimes N_{Z/Hilb^n(X)}(i^* p_2^* L))^{\otimes -2}.$$

In particular, the determinant of the perfect obstruction theory is a square if $\omega_{X/k}$ is a square.

Here, $N_{Z/\mathrm{Hilb}^n(X)}$ is the norm of Z over $\mathrm{Hilb}^n(X)$, see [13, Section 7]. In the case where $X = \mathbb{P}^3$, we have that $\omega_{\mathbb{P}^3/k} = \mathcal{O}(-4)$ and so the above statement tells us that there is an orientation, so that the corresponding motivic Donaldson-Thomas invariants are well defined.
We now prove the following statement.

Proposition 2.3.12. *The orientation on $Hilb^n(\mathbb{P}^3)\times\mathbb{P}^3$ from Proposition 2.3.11 gives rise to an oriented basis of $Ext^2(\mathcal{I},\mathcal{I}) - Ext^1(\mathcal{I},\mathcal{I})$ for each N_S-fixed ideal sheaf $\mathcal{I}$. Every even negative weight in the trace induces a minus sign to the corresponding Euler class.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^3}$ be an ideal sheaf corresponding to an N_S-fixed point. Let I be the image of $\mathcal{I}$ on U_0. Let r be the rank of $\mathrm{Ext}^1(I,I)$. Because the perfect obstruction theory has virtual rank zero, we have that $\mathrm{Ext}^1(I,I)$ and $\mathrm{Ext}^2(I,I)$ have the same rank. Choose an oriented basis $e_1, \cdots, e_r, f_1, \cdots, f_r$ for $\mathrm{Ext}^2(I,I) - \mathrm{Ext}^1(I,I)$, i.e. a basis such that

$$\rho(\det(\mathrm{Ext}^2(I,I) - \mathrm{Ext}^1(I,I))) = \rho((e_1 \wedge \cdots \wedge e_r)\cdot(f_1 \wedge \cdots \wedge f_r)^{-1}) = 1.$$

By [32, Theorem 6.4], ρ is N_S-equivariant and so we have that

$$(-\sigma)(e_1), \cdots, (-\sigma)(e_r), (-\sigma)(f_1), \cdots, (-\sigma)(f_r)$$

is an oriented basis at $\sigma(I)$. This gives a basis

$$e_1, (-\sigma)(e_1), \cdots, e_r, (-\sigma)(e_r), f_1, (-\sigma)(f_1), \cdots, f_r, (-\sigma)(f_r)$$

for $\mathrm{Ext}^2(\mathcal{I},\mathcal{I}) - \mathrm{Ext}^1(\mathcal{I},\mathcal{I})$ on U_0 and U_1. This is compatible with the relative orientation as given by Proposition 2.3.11. In order to use Proposition 2.1.19 to compute Euler classes, we need to follow Remark 2.1.21 and use a basis which is compatible with the choice described there. This means: for each e_i with negative weight, one has to switch e_i and $(-\sigma)(e_i)$ in the above basis, because the positive weight always has to be the first. The same holds for the f_i. If the weight of a negative e_i or f_i is odd, these switches do not contribute a sign change to the Euler class, as $(-\sigma) = -\sigma$ in this case. If the weight of a negative e_i or f_i is even, this contributes a sign to the Euler class.
One can repeat this construction on U_2 and U_3. This proves the desired statement. □

2.3.2 Proof of Proposition 2.3.9

In this section, we prove Lemma 2.3.7 and Proposition 2.3.9. The proofs are almost exactly the same as in [39, Section 4], but included here for the reader's convenience. We start by finding a helpful expression for $\mathrm{Ext}^2(\mathcal{I},\mathcal{I})-\mathrm{Ext}^1(\mathcal{I},\mathcal{I})$ in terms of equivariant Euler characteristics.

Construction 2.3.13. For two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^3 \times \mathrm{Hilb}^n(\mathbb{P}^3)$, we have the derived Hom-set $R\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ in the bounded derived category. Its Euler characteristic is given by

$$\tilde{\chi}(\mathcal{F}, \mathcal{G}) = \sum_{i=0}^{3} (-1)^i \dim(\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})) \in \mathbb{Z}.$$

For two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^3 \times \mathrm{Hilb}^n(\mathbb{P}^3)$, the action of N_S extends to one on $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$, which is therefore an N_S-representation. This yields a refined Euler characteristic.

Definition 2.3.14. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $\mathbb{P}^3$. The *Euler characteristic* $\chi(\mathcal{F}, \mathcal{G})$ is given by the alternating sum of virtual N_S-representations

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_{i=0}^{3} (-1)^i \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \in K_0(N_S - \mathrm{Reps}).$$

We now make the following observation.

Lemma 2.3.15. *Let $\mathcal{I}$ be an ideal sheaf on $\mathbb{P}^3$ which is fixed by the N_S-action. We have that*

$$\chi(\mathcal{I}, \mathcal{I}) - \chi(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) = Ext^2(\mathcal{I}, \mathcal{I}) - Ext^1(\mathcal{I}, \mathcal{I}).$$

Proof. Writing out gives

$$\begin{aligned}
\chi(\mathcal{I}, \mathcal{I}) - \chi(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) &= \sum_{i=0}^{3} (-1)^i \mathrm{Ext}^i(\mathcal{I}, \mathcal{I}) - \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) \\
&= -\mathrm{Ext}^3(\mathcal{I}, \mathcal{I}) + \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) \\
&\quad + \mathrm{Hom}(\mathcal{I}, \mathcal{I}) - \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) \\
&= \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^1(\mathcal{I}, \mathcal{I})
\end{aligned}$$

because $\mathrm{Ext}^3(\mathcal{I}, \mathcal{I}) = 0$ by [39, Lemma 2] and $\mathrm{Hom}(\mathcal{I}, \mathcal{I}) = \mathcal{O}_{\mathbb{P}^3}$. ☐

Proof of Lemma 2.3.7. Consider the local-to-global spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}^3, \mathcal{E}xt^q(\mathcal{I}, \mathcal{I})) \implies \mathrm{Ext}^{p+q}(\mathcal{I}, \mathcal{I}).$$

Note that $E_2^{p,q} = 0$ if $p > 3$ or $q > 3$. Note that

$$\begin{aligned}
\sum_{p,q} (-1)^{p+q} E_s^{p,q} &= \sum_{p,q} (-1)^{p+q} (\ker(d_s^{p,q}) + \mathrm{im}(d_s^{p,q})) \\
&= \sum_{p,q} (-1)^{p+q} E_{s+1}^{p,q}
\end{aligned}$$

for all $s \geq 2$. In particular, this sums the infinity pages with the right signs, so that we find

$$\chi(\mathcal{I}, \mathcal{I}) = \sum_{i,j=0}^{3} (-1)^{i+j} H^i(\mathbb{P}^3, \mathcal{E}xt^j(\mathcal{I}, \mathcal{I})).$$

Using the U_i as a cover of $\mathbb{P}^3$, we can compute the above cohomology groups by computing Čech cohomology. Note that as $\mathcal{I}$ is only supported on points, we have that $\mathcal{I} = \mathcal{O}_{\mathbb{P}^3}$ on the intersection of two or more U_i. Therefore, the Čech complex is

$$0 \to \prod_{i=0}^{3} H^0(U_i, \chi(\mathcal{I}, \mathcal{I}) - \chi(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})) \to 0.$$

By Lemma 2.3.15, this gives the desired statement. □

In the remainder of this section, we will prove Proposition 2.3.9.

Proof of Proposition 2.3.9. We have that

$$\begin{aligned} \mathrm{tr}_R(t) &= \sum_{i,j,k} s_1^i s_2^j s_3^k \\ &= \left(\sum_i s_1^i\right)\left(\sum_j s_2^j\right)\left(\sum_k s_3^k\right) \\ &= \frac{1}{(1-s_1)(1-s_2)(1-s_3)} \end{aligned}$$

Consider a resolution of the ideal I given by

$$0 \to F_r \to \cdots \to F_1 \to I \to 0 \tag{2.3}$$

such that each term is of the form $F_i = \bigoplus_j R(d_{ij})$ for $d_{ij} = (d_{ij}^1, d_{ij}^2, d_{ij}^3) \in \mathbb{Z}^3$. One can always form a resolution like this if the ideal I is homogeneous: if $I = (f_0, \cdots, f_m)$ where f_i has degree d_i then we can take $F_1 = \bigoplus_{i=0}^m R(d_i)$, define F_2 based on the relations between the f_i and continue like that until we have a resolution. Consider the corresponding Poincaré polynomial

$$P(t) = \sum_{i,j} (-1)^i s_1^{d_{ij}^1} s_2^{d_{ij}^2} s_3^{d_{ij}^3}.$$

Note that for any $d = (d_1, d_2, d_3) \in \mathbb{Z}^3$, we have that

$$\mathrm{tr}_{R(d)}(t) = \sum_{i,j,k} t \cdot x^{i+d_1} y^{j+d_2} z^{k+d_3} = s_1^{d_1} s_2^{d_2} s_3^{d_3} \mathrm{tr}_R(t).$$

This implies that

$$\mathrm{tr}_{F_*}(t) = \sum_{i,j} (-1)^i \mathrm{tr}_{R(d_{ij})}(t)$$

$$= \operatorname{tr}_R(t) \sum_{i,j} (-1)^i s_1^{d_{ij}^1} s_2^{d_{ij}^2} s_3^{d_{ij}^3}$$
$$= \frac{P(t)}{(1-s_1)(1-s_2)(1-s_3)}$$

We have the exact sequence

$$0 \to F_* \to R \to R/I \to 0$$

and so $\operatorname{tr}_R(t) = \operatorname{tr}_{F_*}(t) + \operatorname{tr}_{R/I}(t)$. This implies that

$$Q(t) = \frac{1-P(t)}{(1-s_1)(1-s_2)(1-s_3)}.$$

We use the resolution F_* to compute that

$$\chi(I,I) = \sum_{i,j,k,l} (-1)^{i+k} \operatorname{Hom}(R(d_{ij}), R(d_{kl}))$$
$$= \sum_{i,j,k,l} (-1)^{i+k} R(d_{kl} - d_{ij})$$

This tells us that the trace is

$$\operatorname{tr}_{\chi(I,I)}(t) = \sum_{i,j,k,l} (-1)^{i+k} s_1^{d_{kl}^1 - d_{ij}^1} s_2^{d_{kl}^2 - d_{ij}^2} s_3^{d_{kl}^3 - d_{ij}^3} \operatorname{tr}_R(t)$$
$$= \frac{P(t^{-1})P(t)}{(1-s_1)(1-s_2)(1-s_3)}$$

and so

$$\operatorname{tr}_{R-\chi(I,I)}(t) = \frac{1-P(t^{-1})P(t)}{(1-s_1)(1-s_2)(1-s_3)}.$$

We have that $P(t) = -(1-s_1)(1-s_2)(1-s_3)Q(t) + 1$ and

$$P(t^{-1}) = -(1-s_1^{-1})(1-s_2^{-1})(1-s_3^{-1})Q(t^{-1}) + 1$$
$$= s_1 s_2 s_3 (1-s_1)(1-s_2)(1-s_3)Q(t^{-1}) + 1$$

and so we can rewrite this as

$$\operatorname{tr}_{R-\chi(I,I)}(t) = \frac{s_1 s_2 s_3 Q(t) - Q(t^{-1}) + Q(t)Q(t^{-1})(1-s_1)(1-s_2)(1-s_3)}{s_1 s_2 s_3}.$$

as desired. □

2.4 Computations of $\tilde{I}_n$ for $n \leq 6$

Notation 2.4.1. In this section, the base field is again assumed to be $\mathbb{R}$. See Notation 2.3.1.

Consider the action defined in Notation 2.3.2 again. In this section, we use the virtual localization formula (2.2) together with Proposition 2.1.19, Lemma 2.3.7 and Proposition 2.3.9 to compute the motivic Donaldson-Thomas invariants $\tilde{I}_n \in H^*(\mathrm{Spec}(\mathbb{R}), \mathcal{W}) \cong \mathbb{Z}$ for $n \leq 6$. Note that $\tilde{I}_n = 0$ whenever n is odd, because any fixed ideal sheaf needs to be invariant under the σ-action. We will see that $\tilde{I}_2 = 10, \tilde{I}_4 = 25$ and $\tilde{I}_6 = -50$. We have that

$$M(-q^2)^{-10} = 1 + 10q^2 + 25q^4 - 50q^6 - 240q^8 + \cdots$$

which might mean that this function predicts the next numbers. We therefore make the following conjecture.

Conjecture 2.4.2. *For $n \geq 0$, let $\tilde{I}_n$ be the degree of the motivic virtual fundamental class associated to $Hilb^n(\mathbb{P}^3)$. Then we have that*

$$\sum_{n \geq 0} \tilde{I}_n q^n = M(-q^2)^{\widetilde{\deg}(e(V))}.$$

where V is a certain locally free sheaf on $Hilb^2(\mathbb{P}^3)$ with quadratic Euler class $e(V)$ and $\widetilde{\deg}$ is the quadratic degree map.

We will make use of the following coordinates and tangent weights throughout.

Construction 2.4.3. Throughout, we use coordinates

- $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}, z = \frac{X_3}{X_0}$ on U_0.
- $u = \frac{X_0}{X_1}, v = \frac{X_2}{X_1}, w = \frac{X_3}{X_1}$ on U_1.
- $x' = \frac{X_0}{X_2}, y' = \frac{X_1}{X_2}, z' = \frac{X_3}{X_2}$ on U_2.
- $u' = \frac{X_0}{X_3}, v' = \frac{X_1}{X_3}, w' = \frac{X_2}{X_3}$ on U_3.

Construction 2.4.4. Choose coordinates $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}$ and $z = \frac{X_3}{X_0}$ on U_0. As $[t^a X_0 : t^{-a} X_1 : t^b X_2 : t^{-b} X_3] = [X_0 : t^{-2a} X_1 : t^{b-a} X_2 : t^{-a-b} X_3]$, we have that on U_0

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot (x, y, z) = (s_1 x, s_2 y, s_3 z)$$

where $s_1 = t^{-2a}, s_2 = t^{b-a}, s_3 = t^{-a-b}$. Repeating this for U_1, U_2 and U_3, we see that the "tangent weights" of the action are

$$\begin{aligned}
U_0: &\quad s_1 = t^{-2a}, s_2 = t^{b-a}, s_3 = t^{-a-b} \\
U_1: &\quad s_1 = t^{2a}, s_2 = t^{b+a}, s_3 = t^{a-b} \\
U_2: &\quad s_1 = t^{a-b}, s_2 = t^{-b-a}, s_3 = t^{-2b} \\
U_3: &\quad s_1 = t^{a+b}, s_2 = t^{b-a}, s_3 = t^{2b}
\end{aligned}$$

Note the symmetries between those tangent weights, which imply that for a fixed ideal sheaf $\mathcal{I}$ of even degree n supported on $\{[1:0:0:0],[0:1:0:0]\}$, we can compute the trace on U_1 as $V(t^{-1})$, where $V(t)$ is the trace on U_0. From the sum of those traces, we can find the Euler class $e(a,b)$. Furthermore, the Euler class of the corresponding fixed point supported on $\{[0:0:1:0],[0:0:0:1]\}$ is given by $e(b,a)$.

Remark 2.4.5. We note that for any $\lambda \in \mathbb{R}^*$, the ideal sheaf which is locally given by $(x+\lambda yz, y^2, z^2)$ is of length 4. These are nonisolated fixed points, defined by nonmonomial ideals. The method used for the other ideal sheaves does therefore not work for those ideal sheaves. Therefore, we have not been able to compute the invariant $\tilde{I}_n$ for $n \geq 8$.

Remark 2.4.6. Throughout, we make use of the SAGE code which one can find here. Note that for checking the signs coming from the use of Proposition 2.1.19, it is assumed in this code that a and b are both congruent to 1 modulo 4. One can check that the signs of the Euler classes do not change for other possible congruences of a, b modulo 4.

Remark 2.4.7. By Proposition 2.3.12, each even negative weight in the trace induces a minus sign to the corresponding Euler class. For all Euler classes computed in this section, the signs introduced by these even negative weights have been taken into account in the computation, and will not be mentioned further.

2.4.1 The computation for $n = 2$

Note that there are two ideal sheaves of length two. First, we have the subscheme supported on $\{[1:0:0:0],[0:1:0:0]\}$ of which the ideal sheaf $\mathcal{I}$ is given by $(x,y,z) \subset \mathbb{R}[x,y,z]$ on U_0 and by $(u,v,w) \subset \mathbb{R}[u,v,w]$ on U_1. The other one corresponds to the subscheme supported on $\{[0:0:1:0],[0:0:0:1]\}$, again with the ideal given by $(x',y',z') \subset \mathbb{R}[x',y',z']$ locally on U_2 and by $(u',v',w') \subset \mathbb{R}[u',v',w']$ on U_3.

Proposition 2.4.8. *The Euler class corresponding to the first point is*

$$e_{11} = \frac{(3a-b)(3a+b)}{(a-b)(a+b)}$$

and the class corresponding to the second point is

$$e_{12} = \frac{(3b-a)(3b+a)}{(b-a)(a+b)}.$$

We have that $\tilde{I}_2 = e_{11} + e_{12} = 10$.

Proof. We start with the first point. Plugging $Q = 1$ into the formula from Proposition 2.3.9 yields that the trace of $\mathrm{Ext}^1(\mathcal{I},\mathcal{I}) - \mathrm{Ext}^2(\mathcal{I},\mathcal{I})$ is given by

$$V_2 = s_1^{-1} + s_2^{-1} + s_3^{-1} - s_1^{-1}s_2^{-1} - s_1^{-1}s_3^{-1} - s_2^{-1}s_3^{-1}.$$

Filling in the tangent weights for U_0 and U_1 and adding up gives

$$(t^{a-b}+t^{b-a})+(t^{a+b}+t^{-a-b})-((t^{3a-b}+t^{-3a+b})+(t^{3a+b}+t^{-3a-b}).$$

Now using Proposition 2.1.19, we see that this leads to an Euler class

$$e_{11}=\frac{(3a-b)(3a+b)}{(a-b)(a+b)}.$$

We note that for a, b both congruent to 1 modulo four, we have mod 4 that $3b-a=2, 3b+a=0, b-a=0, a+b=2$ so that the use of Proposition 2.1.19 gives a plus. One can check that the sign does not change with all other possible congruences of a, b modulo 4.
Repeating the process for the second fixed point boils down to switching a and b, and gives us the Euler class

$$e_{12}=\frac{(3b-a)(3b+a)}{(b-a)(a+b)}$$

and adding those we get $e_{11}+e_{12}=10$. □

2.4.2 The computation for $n=4$

Note that between U_0 and U_1, the N_S-action will send $x \mapsto -u, y \mapsto w, z \mapsto -v$. From this we see that to be fixed under the N_S-action, an ideal sheaf supported on $\{[1:0:0:0],[0:1:0:0]\}$ of length four must correspond to one of the following:

1. (x^2, y, z) on U_0 and (u^2, v, w) on U_1.
2. (x, y^2, z) on U_0 and (u, v, w^2) on U_1.
3. (x, y, z^2) on U_0 and (u, v^2, w) on U_1.

One can find similar classes for fixed ideal sheaves of length four supported on $\{[0:0:1:0],[0:0:0:1]\}$. For $n=4$ there are thus seven fixed points if we add the point supported on all four points. We will prove the following result.

Proposition 2.4.9. *The fixed point supported on all four points has Euler class*

$$e_{21}=-\frac{(3a-b)(3a+b)(3b-a)(3b+a)}{(a-b)^2(a+b)^2}.$$

The points of type (1) have Euler class zero, and the points in type (2) give rise to Euler classes

$$e_{22}=\frac{(3a-b)^2(3a+b)(2a-b)}{b(a-b)^2(a+b)} \text{ and } e_{23}=\frac{(3b-a)^2(3b+a)(2b-a)}{a(b-a)^2(a+b)}.$$

The points of type (3) have Euler classes

$$e_{24}=-\frac{(3a-b)(3a+b)^2(2a+b)}{b(a-b)(a+b)^2} \text{ and } e_{25}=-\frac{(3b-a)(3b+a)^2(2b+a)}{a(b-a)(a+b)^2}.$$

We have that $\tilde{I}_4=e_{21}+e_{22}+e_{23}+e_{24}+e_{25}=25$.

Proof. First, there is the subscheme which consists of all four points. The Euler class is

$$e_{21} = e_{11}e_{12} = -\frac{(3a-b)(3a+b)(3b-a)(3b+a)}{(a-b)^2(a+b)^2}.$$

For the fixed point of type (1) supported on $\{[1:0:0:0],[0:1:0:0]\}$, plugging $Q = 1 + s_1$ into the formula from Proposition 2.3.9 gives

$$\begin{aligned} V_4 = {} & s_1^{-1} + s_2^{-1} + s_3^{-1} - s_1^{-1}s_2^{-1} - s_1^{-1}s_3^{-1} - s_2^{-1}s_3^{-1} \\ & + s_1^{-2} + s_1s_3^{-1} + s_1s_2^{-1} - s_1s_2^{-1}s_3^{-1} - s_1^{-2}s_2^{-1} - s_1^{-2}s_3^{-1}. \end{aligned} \tag{2.4}$$

Filling in the tangent weights on U_0 gives

$$\begin{aligned} & t^{2a} + t^{a-b} + t^{a+b} - t^{3a-b} - t^{2a} - t^{3a+b} + \\ & t^{4a} + t^{b-a} + t^{-b-a} - 1 - t^{5a-b} - t^{5a+b}. \end{aligned}$$

and filling in the weights of U_1 gives the same but with t and t^{-1} swapped. This yields an Euler class which is zero, because of the term 1 which represents a trivial subrepresentation of $\mathrm{Ext}^2(\mathcal{I},\mathcal{I})$, which contributes a zero to the numerator. Similarly, the fixed point supported on $\{[0:0:1:0],[0:0:0:1]\}$ of type (1) has Euler class equal to zero.
Now consider the fixed point supported on $\{[1:0:0:0],[0:1:0:0]\}$ of type (2). Filling in V_4 with the tangent weights of U_0 where we switch s_1 with s_2 gives

$$\begin{aligned} & t^{a-b} + t^{2a} + t^{a+b} - t^{3a-b} - t^{2a} - t^{3a+b} + \\ & t^{2a-2b} + t^{2b} + t^{b+a} - t^{2a+2b} - t^{3a-b} - t^{4a-2b}. \end{aligned}$$

If we fill in V_4 with the tangents weights of U_1 where we switch s_1 with s_3, this gives the same result with t and t^{-1} switched. This yields the Euler class e_{22}. Similarly, the fixed point supported on $\{[0:0:1:0],[0:0:0:1]\}$ of type (2) has Euler class e_{23}.
Finally, for the fixed point supported on $\{[1:0:0:0],[0:1:0:0]\}$ of type (3), filling in V_4 with the tangent weights of U_0 where we switch s_1 with s_3 gives

$$\begin{aligned} & t^{a+b} + t^{a-b} + t^{2a} - t^{3a+b} - t^{2a} - t^{3a-b} + \\ & t^{2a+2b} + t^{a-b} + t^{-2b} - t^{2a-2b} - t^{4a+2b} - t^{3a+b}. \end{aligned}$$

Again, filling in V_4 with the weights of U_1 and s_1 and s_2 switched boils down to switching t and t^{-1}. This yields the Euler class e_{24}, so that the fixed point supported on $\{[0:0:1:0],[0:0:0:1]\}$ of type (3) has Euler class e_{25} as desired.
One can check that for all possible congruence classes of a, b modulo 4, no signs are introduced because of the use of Proposition 2.1.19. □

2.4.3 The computation for $n = 6$

We will now prove the following.

Proposition 2.4.10. *We have that* $\tilde{I}_6 = -50$.

There are three types of fixed ideal sheaves of length six. First, from the lower degrees, we have the Euler classes

$$\begin{aligned} e_{31} &= e_{11} \cdot e_{23} \\ e_{32} &= e_{11} \cdot e_{25} \\ e_{33} &= e_{12} \cdot e_{22} \\ e_{34} &= e_{12} \cdot e_{24} \end{aligned}$$

Another class of ideal sheaves are the complete intersections, i.e. those supported on $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$ which correspond to either:

1. (x^3, y, z) on U_0 and (u^3, v, w) on U_1.
2. (x, y^3, z) on U_0 and (u, v, w^3) on U_1.
3. (x, y, z^3) on U_0 and (u, v^3, w) on U_1.

Again, each of those have a corresponding fixed point supported on $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$.

Lemma 2.4.11. *The fixed points of type (1) have Euler class equal to zero. The points of type (2) yield the Euler classes*

$$e_{35} = \frac{(3a-b)^2(3a+b)(a+3b)(5a-3b)(2a-b)^2}{3b^2(a-b)^3(a+b)(3b-a)}$$

and

$$e_{36} = \frac{(3b-a)^2(3b+a)(b+3a)(5b-3a)(2b-a)^2}{3a^2(b-a)^3(a+b)(3a-b)}.$$

The points of type (3) have Euler classes

$$e_{37} = -\frac{(3a-b)(3a+b)^2(a-3b)(5a+3b)(2a+b)^2}{3b^2(a-b)(a+b)^3(a+3b)}$$

and

$$e_{38} = -\frac{(3b-a)(3b+a)^2(b-3a)(5b+3a)(2b+a)^2}{3a^2(b-a)(a+b)^3(b+3a)}.$$

Proof. Consider the ideal of type (1). Plugging $Q = 1 + s_1 + s_1^2$ into the formula from Proposition 2.3.9 gives the trace

$$\begin{aligned} V_{61} = {} & s_1^{-1} + s_2^{-1} + s_3^{-1} - s_1^{-1}s_2^{-1} - s_1^{-1}s_3^{-1} - s_2^{-1}s_3^{-1} \\ & + s_1^{-2} + s_1 s_2^{-1} + s_1 s_3^{-1} - s_1 s_2^{-1} s_3^{-1} - s_1^{-2} s_2^{-1} - s_1^{-2} s_3^{-1} \\ & + s_1^{-3} + s_1^2 s_2^{-1} + s_1^2 s_3^{-1} - s_1^2 s_2^{-1} s_3^{-1} - s_1^{-3} s_2^{-1} - s_1^{-3} s_3^{-1}. \end{aligned}$$

Note that the first two lines are precisely the trace (2.4) we had in the $n = 4$ case. Filling in the tangent weights of U_0 in the last line of V_{61} gives

$$t^{6a} + t^{-3a+b} + t^{-3a-b} - t^{-2a} - t^{7a+b} - t^{7a-b}.$$

There is nothing here which cancels the trivial subrepresentation coming from the second line, implying that the corresponding Euler class is zero. Similarly, the Euler class of the corresponding fixed point supported on $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$ is zero.
For the fixed point supported on $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$ of type (2), filling in the tangent weights of U_0 in the last line of V_{61} and switching s_1 and s_2 gives

$$t^{3a-3b} + t^{-a+3b} + t^{2b} - t^{3b+a} - t^{5a-3b} - t^{4a-2b}$$

so that we find the Euler classes e_{35} and e_{36} in the statement.
For the fixed point supported on $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$ of type (3), filling in the tangent weights of U_0 in the last line of V_{61} and switching s_1 and s_3 gives

$$t^{3a+3b} + t^{-a-3b} + t^{2b} - t^{-3b+a} - t^{4a+2b} - t^{5a+3b}.$$

This gives the Euler classes e_{37} and e_{38} from the statement.
Again, one can check that for all possible congruence classes of a, b modulo 4, no signs are introduced because of the use of Proposition 2.1.19. □

The final class of ideal sheaves are those of which the ideal locally looks like a maximal ideal together with a variable, i.e. they are supported on $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$, and of one of the following types:

1. $((x, y)^2, z)$ on U_0 and $((u, w)^2, v)$ on U_1.
2. $((y, z)^2, x)$ on U_0 and $((v, w)^2, u)$ on U_1.
3. $((x, z)^2, y)$ on U_0 and $((u, v)^2, w)$ on U_1.

Again, each of those have a corresponding fixed point supported on $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$.

Lemma 2.4.12. *The ideals of type (1) yield Euler classes*

$$e_{39} = \frac{(3a + b)(3a - b)(5a - b)(2a - b)(2a + b)}{b^2(a - b)^2(a + b)}$$

and

$$e_{310} = \frac{(3b + a)(3b - a)(5b - a)(2b - a)(2b + a)}{a^2(a - b)^2(a + b)}.$$

The ideals of type (2) correspond to the Euler classes

$$e_{311} = \frac{9(3a + b)^3(3a - b)^3}{(a + b)^2(a - b)^2(a + 3b)(a - 3b)}$$

and

$$e_{312} = \frac{9(3b+a)^3(3b-a)^3}{(a+b)^2(a-b)^2(b+3a)(b-3a)}.$$

The ideals of type (3) give rise to the Euler classes

$$e_{313} = \frac{(3a-b)(3a+b)(5a+b)(2a+b)(2a-b)}{b^2(a-b)(a+b)^2}$$

and

$$e_{314} = \frac{(3b-a)(3b+a)(5b+a)(2b+a)(2b-a)}{a^2(b-a)(a+b)^2}.$$

Proof. For $((x,y)^2, z)$ on U_0, plugging $Q = 1 + s_1 + s_2$ into the formula from Proposition 2.3.9 yields

$$\begin{aligned} V_{62} = & 2s_1^{-1} + 2s_2^{-1} + s_3^{-1} - s_1^{-1}s_2^{-1} - 2s_1^{-1}s_3^{-1} - 2s_2^{-1}s_3^{-1} \\ & + s_1s_3^{-1} + s_2s_3^{-1} - s_1^{-2}s_2^{-1} - s_1^{-1}s_2^{-2} \\ & + s_1^{-2}s_2 + s_1s_2^{-2} - s_1^{-2}s_2s_3^{-1} - s_1s_2^{-2}s_3^{-1}. \end{aligned}$$

For the fixed point of type (1), filling in the tangent weights of U_0 in V_{62} gives

$$\begin{aligned} & 2t^{2a} + 2t^{a-b} + t^{a+b} - 2t^{2a} - 2t^{3a+b} - t^{3a-b} \\ & + t^{b-a} + t^{2b} - t^{5a-b} - t^{4a-2b} \\ & + t^{3a+b} + t^{-2b} - t^{4a+2b} - t^{a-b} \end{aligned}$$

which yields the Euler classes e_{39} and e_{310} as in the statement. For the fixed point of type (2), plugging the tangent weights of U_0 into V_{62} and switching s_2 and s_3 gives

$$\begin{aligned} & t^{a-b} + 2t^{2a} + 2t^{a+b} - 2t^{3a-b} - 2t^{2a} - t^{3a+b} \\ & + t^{-a-b} + t^{-2b} - t^{5a+b} - t^{4a+2b} \\ & - t^{4a-2b} - t^{a+b} + t^{3a-b} + t^{2b} \end{aligned}$$

which gives us the Euler classes e_{311} and e_{312} from the statement. Finally, for the fixed point of type (3), plugging the tangent weights of U_0 into V_{62} and switching s_1 and s_3 gives

$$\begin{aligned} & t^{2a} + 2t^{a+b} + 2t^{a-b} - 2t^{3a+b} - 2t^{3a-b} - t^{2a} \\ & + t^{a-b} + t^{a+b} - t^{3a+b} - t^{3a-b} \\ & - t^{3a+3b} - t^{3a-3b} + t^{a+3b} + t^{a-3b} \end{aligned}$$

which yields the Euler classes e_{313} and e_{314} as in the statement.
One can check that for all possible congruence classes of a, b modulo 4, no signs are introduced because of the use of Proposition 2.1.19. □

Adding the Euler classes together proves Proposition 2.4.10.

2.5 Cohomology of Witt sheaves on BN_G

It would be interesting to study the results of the previous sections for other actions than the one by N_S. For example, one could take the action by the normalizer N_G of the torus inside GL_2 on $\mathbb{P}^3$ given by

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot [X_0 : X_1 : X_2 : X_3] = [t_1^a X_0 : t_2^a X_1 : t_1^b X_2 : t_2^b X_3]$$
$$\sigma \cdot [X_0 : X_1 : X_2 : X_3] = [-X_1 : X_0 : -X_3 : X_2]$$

for $a, b \in \mathbb{Z}$ odd and nonzero.
For this situation, the construction of an orientation as described in Section 2.3.1 does not seem to work. Therefore, we have not been able to show that one can define motivic Donaldson-Thomas invariants in this setting. However, there are some things which we can do.
In this section, we will formulate and prove an analogue of Proposition 2.1.18, i.e. [28, Proposition 5.5], for BN_G. In order to do so, we first prove an analogue of [28, Theorem 4.1] to compare the cohomology on BN_S with that of BN_G, which we will then use to deduce the desired result from Proposition 2.1.18.

Notation 2.5.1. Throughout, we work over a perfect field k which is not of characteristic 2. We let $\pi : B\mathrm{SL}_n \to B\mathrm{GL}_n$ be the canonical map induced by the inclusion $\mathrm{SL}_n \to \mathrm{GL}_n$. Let $E_n = \mathbb{A}^n \times^{\mathrm{GL}_n} B\mathrm{GL}_n$ be the canonical rank n bundle, where GL_n acts on $\mathbb{A}^n$ by matrix multiplication. Similarly, let $E_2^{N_G} = \mathbb{A}^2 \times^{\mathrm{GL}_2} BN_G$ be the canonical rank 2 bundle on BN_G.

Remark 2.5.2. Let $a, b \in \mathbb{Z}$. Consider the N_G-representation $\rho_{a,b} : N_G \to \mathrm{GL}_2$ given by

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \begin{pmatrix} t_1^a t_2^b & 0 \\ 0 & t_1^b t_2^a \end{pmatrix}, \sigma \mapsto \begin{pmatrix} 0 & 1 \\ (-1)^{a-b} & 0 \end{pmatrix}$$

This gives rise to the rank two bundle $\tilde{O}(a, b)$ on BN_G. In combination with Proposition 2.1.19 and using that $\pi^*(\tilde{O}(a, b)) = \tilde{O}^{\pm}(|a - b|)$, one can compute the corresponding Euler classes.

2.5.1 Comparing cohomology for BN_S and BN_G

We start by considering the possible twists of cohomology on BN_G.

Construction 2.5.3. Similar to the case of N_S, one can define a representation $\rho_G^- : N_G \to \mathbb{G}_m$ which sends σ to -1 and the diagonal matrices to 1. We find a map $BN_G \to B\mathbb{G}_m$, and the pull-back of the canonical bundle on $B\mathbb{G}_m$ defines a line bundle γ_G on BN_G. Using Proposition 2.1.9 again, the Picard group of BN_G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, generated by γ_G and the determinant character det (sending σ to -1 and diagonal matrices to their determinant). This means that this time, we will have to consider the twists by γ_G, the determinant character det and $\det + \gamma_G$, because twists by the square of a line bundle do not change the cohomology.

Remark 2.5.4. Note that

$$\gamma_G = \rho_G^{-,*}\mathcal{O}(1) = \pi^*\rho^{-,*}\mathcal{O}(1) = \pi^*\gamma_S.$$

Let $\mathcal{E} \in \mathrm{SH}(k)$ be an SL-oriented motivic spectrum. Following [42, Section 6] and [2], the algebraic Hopf map

$$\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1, (x, y) \mapsto [x : y]$$

induces a map $\Sigma_{\mathbb{G}_m}\mathbb{S}_k \to \mathbb{S}_k$, i.e. an element $\eta_{\mathcal{E}} \in \mathcal{E}^{-1,-1}(k)$.

Definition 2.5.5. If $\eta_{\mathcal{E}}$ is invertible on $\mathcal{E}^{*,*}(k)$, we say that $\mathcal{E}$ is an *η-invertible theory.*

If $\mathcal{E}$ is η-invertible, for arbitrary vector bundles $E \to X$ there are Pontryagin classes $p_m(E) \in \mathcal{E}^{4m,2m}(X)$, see [2, Definition 19]. The following was proven by Ananyevskiy.

Theorem 2.5.6 (See Theorem 10 in [2])**.** *For $\mathcal{E}$ an η-invertible SL-oriented theory, one has*

$$\mathcal{E}^{*,*}(BSL_n) = \begin{cases} \mathcal{E}^{*,*}(k)[p_1, \cdots, p_{m-1}, e] & \text{for } n=2m \\ \mathcal{E}^{*,*}(k)[p_1, \cdots, p_m] & \text{for } n=2m+1 \end{cases}$$

where $p_i = p_i(E_n), e = e(E_n)$. If $n = 2m$, we have that $p_m(E_{2m}) = e(E_{2m})^2$.

In [28], there is the following induced result for GL_n. For $\mathcal{E}$ a SL-oriented motivic ring spectrum, there is the pull-back map

$$\pi^* : \mathcal{E}^{*,*}(B\mathrm{GL}_n) \to \mathcal{E}^{*,*}(B\mathrm{SL}_n)$$

and using the canonical trivialization $\theta : \pi^* \det^{-1}(E_n) \to \mathcal{O}_{B\mathrm{SL}_n}$ we also get a map

$$\pi^* : \mathcal{E}^{*,*}(B\mathrm{GL}_n, \det(E_n)^{-1}) \to \mathcal{E}^{*,*}(B\mathrm{SL}_n).$$

The following has been proven by Levine.

Theorem 2.5.7 ([28], Theorem 4.1)**.** *Let $\mathcal{E} \in SH(k)$ be an SL-oriented and η-invertible motivic ring spectrum. Suppose that either $\mathcal{E}^{0,0}$ is a Zariski sheaf or that the unit map makes it a $\mathcal{W}$-module. Then:*

- *For $n = 2m$, the map π^* induces an isomorphism*

  $$\mathcal{E}^{*,*}(BGL_n) \oplus \mathcal{E}^{*,*}(BGL_n; \det(E_n)) \to \mathcal{E}^{*,*}(BSL_n)$$

  where

  $$\mathcal{E}^{*,*}(BGL_n) = \mathcal{E}^{*,*}(k)[p_1, \cdots, p_{m-1}, e^2]$$

  and

  $$\mathcal{E}^{*,*}(BGL_n; \det(E_n)) = e \cdot \mathcal{E}^{*,*}(k)[p_1, \cdots, p_{m-1}, e^2]$$

- *For $n = 2m+1$, we have that $\mathcal{E}^{*,*}(BGL_n; \det(E_n)) = 0$ and the pull-back $\pi^* : \mathcal{E}^{*,*}(BGL_n) \to \mathcal{E}^{*,*}(BSL_n)$ is an isomorphism.*

We will prove a similar statement for the cohomology of BN_G and BN_S under an η-invertible motivic ring spectrum which is a $\mathcal{W}$-module by the unit map. The proof is almost exactly the same as the proof of [28, Theorem 4.1], but included here for the reader's convenience.

Notation 2.5.8. Let $L = \det(E_2^{N_G})$. Let

$$\mathrm{Th}(L) = L/(L \setminus 0_L)$$

be the Thom space of L.

Remark 2.5.9. Note that L does not have a canonical trivialization so that it does not have a Thom class. However, because of the canonical trivialization $L \oplus L^{-1} \to \mathcal{O}_{BN_G}$, one has the Thom class $\mathrm{th}(L \oplus L^{-1}) \in \mathcal{E}^{4,2}(\mathrm{Th}(L \oplus L^{-1}))$. By definition, we have that $\mathcal{E}^{4,2}(\mathrm{Th}(L \oplus L^{-1})) = \mathcal{E}^{2,1}(\mathrm{Th}(L); L^{-1})$. One defines the canonical Thom class

$$\mathrm{th}(L) \in \mathcal{E}^{2,1}(\mathrm{Th}(L); L^{-1})$$

to be the image of $\mathrm{th}(L \oplus L^{-1})$ under this identification. Let $s_{\mathrm{Th}(L)}$ be the map $\mathrm{Th}(L) \to BN_G$ induced by the zero section of L. One defines the Euler class $e(L) \in \mathcal{E}^{2,1}(BN_G; L^{-1})$ of L as the pull-back $(s_{\mathrm{Th}(L)})^*\mathrm{th}(L)$. See e.g. [35, Definition 3.11, Remark 5.2], where the treatment is inspired by that of [2], for more details.

Proposition 2.5.10. *Let $\mathcal{E} \in SH(k)$ be an SL-oriented and η-invertible motivic ring spectrum. Suppose that the unit map makes it a $\mathcal{W}$-module. Then the map π^* induces an isomorphism*

$$\mathcal{E}^{*,*}(BN_G) \oplus \mathcal{E}^{*,*}(BN_G; \det) \to \mathcal{E}^{*,*}(BN_S).$$

Similarly, π^* induces an isomorphism

$$\mathcal{E}^{*,*}(BN_G; \gamma_G) \oplus \mathcal{E}^{*,*}(BN_G; \gamma_G + \det) \to \mathcal{E}^{*,*}(BN_S; \gamma_S).$$

Proof. As $BN_S \to BN_G$ is a $\mathbb{G}_m$-torsor, we have that BN_S is a pull-back of $E\mathbb{G}_m$ under a classifying map $BN_G \to B\mathbb{G}_m$. Note that $E\mathbb{G}_m$ is equal to $\mathcal{O}(1)\setminus 0_{\mathcal{O}(1)}$ under the identification of $B\mathbb{G}_m$ with $\mathbb{P}^\infty$. As the Plücker embedding is given by the determinant, we have that $\mathcal{O}(1)$ pulls back to L, so that we realize BN_S as $L \setminus 0_L$. Note that the determinant character is also exactly given by the pull-back of $\mathcal{O}(1)$ under this classifying map, so that $L = \det$.
This gives us the open subspace $L \setminus 0_L$ and its closed complement 0_L giving rise to the localization sequence (see [35, Definition 6.2, A3 and Example 6.5])

$$\cdots \to \mathcal{E}^{a,b}(BN_G) \to \mathcal{E}^{a,b}(BN_S) \to \mathcal{E}^{a-1,b-1}(BN_G; L) \to \cdots \qquad (2.5)$$

where we used that $\mathcal{E}^{a,b}(L) = \mathcal{E}^{a,b}(BN_G)$ because of homotopy invariance and that $\mathcal{E}^{a-1,b-1}(BN_G; L) = \mathcal{E}^{a+1,b}(\mathrm{Th}(L))$ by definition.
We note that the maps $\mathcal{E}^{a-1,b-1}(BN_G; L) \to \mathcal{E}^{a+1,b}(BN_G)$ are by definition the composition

$$\mathcal{E}^{a-1,b-1}(BN_G; L) = \mathcal{E}^{a+1,b}(\mathrm{Th}(L)) \xrightarrow{q^*} \mathcal{E}^{a+1,b}(L) \xrightarrow{s^*} \mathcal{E}^{a+1,b}(BN_G)$$

where $s : BN_G \to L$ is the zero section and $q : L \to \mathrm{Th}(L)$ is the quotient map. Note that under the Thom isomorphism

$$(-) \cup \mathrm{th}(L \oplus L^{-1}) : \mathcal{E}^{a+1,b}(BN_G) \to \mathcal{E}^{a+5,b+2}(\mathrm{Th}(L \oplus L^{-1}))$$

these are precisely the cup product by the Euler class $e(L)$ of L. We can apply [28, Lemma 4.3] to see that $e(L) = 0$. Therefore, the localization sequence (2.5) breaks up into short exact sequences

$$0 \to \mathcal{E}^{a,b}(BN_G) \to \mathcal{E}^{a,b}(BN_S) \xrightarrow{\delta} \mathcal{E}^{a-1,b-1}(BN_G; L) \to 0.$$

We now define a splitting to the boundary map δ. Consider the pull-back

$$\begin{array}{ccc} \bar{\pi}^*L & & L \\ \big\downarrow & & \big\downarrow{\scriptstyle \bar{\pi}} \\ L & \xrightarrow{\bar{\pi}} & BN_G \end{array}$$

Note that L is locally $\mathbb{A}^1 \setminus \{0\} \cong \mathrm{Spec}(k[t])$. Let $t \in H^0(L, \bar{\pi}^*L)$ be the tautological section. Note that this is nowhere zero, so we find a section over $L \setminus 0_L$. Let $t^\vee \in H^0(L \setminus 0_L, \pi^*L^{-1})$ be the dual section. We find the quadratic form $\langle t^\vee \rangle \in \mathcal{W}(\pi^*L^{-1})$.
Now set $\langle t^\vee \rangle_{\mathcal{E}} = \langle t^\vee \rangle \cdot 1_{\mathcal{E}} \in \mathcal{E}^{0,0}(L \setminus 0_L, \pi^*L^{-1})$. Then one can do a local computation which shows that

$$\delta(\langle t^\vee \rangle_{\mathcal{E}}) = \eta \cdot 1_{\mathcal{E}} \in \mathcal{E}^{-1,-1}(BN_G).$$

This is by definition of the boundary locally, as it is defined and described in [43, Section 2.2]. Multiplication by $t^\vee$ defines a trivialization $\pi^*L \to \mathcal{O}_{BN_S}$ which gives a map $\langle t^\vee \rangle_{\mathcal{E}} \cdot (-) : \mathcal{E}^{a-2,b-1}(BN_S; \pi^*L) \to \mathcal{E}^{a,b}(BN_S)$. Now we can consider the composition

$$\mathcal{E}^{a-2,b-1}(BN_G; L) \to \mathcal{E}^{a-2,b-1}(BN_S; \pi^*L) \xrightarrow{\langle t^\vee \rangle_{\mathcal{E}} \cdot (-)} \mathcal{E}^{a,b}(BN_S)$$

which is invertible by η-invertibility. This provides an inverse for δ, which gives the desired result.
The second part of the statement can be proven in exactly the same way. □

2.5.2 Cohomology of Witt sheaves on BN_G

Notation 2.5.11. As in Section 2.1.2, let e be the Euler class of the canonical rank 2 bundle $\mathbb{A}^2 \times^{\mathrm{SL}_2} B\mathrm{SL}_2$ on $B\mathrm{SL}_2$. Let $\langle \bar{q} \rangle$ be the quadratic form corresponding to the curve C which was constructed in Section 2.1.2. Let $\mathcal{T}$ be the tangent bundle of BN_S over $B\mathrm{SL}_2$ and let $p : BN_S \to B\mathrm{SL}_2$ be the canonical map.

Notation 2.5.12. Note that $BN_G = (N_G \setminus \mathrm{GL}_2) \times^{\mathrm{GL}_2} E\mathrm{GL}_2$ and recall from Construction 2.1.10 that $N_G \setminus \mathrm{GL}_2 \cong \mathbb{P}^2 \setminus C$. Let $\mathcal{T}_G$ be the tangent bundle of BN_G over $B\mathrm{GL}_2$.

We can now prove an analogue of Proposition 2.1.18. Recall that from Proposition 2.1.18, we have that

$$H^*(BN_S, \mathcal{W}) \to W(k)[x_0, x_2]/(x_0^2 - 1, (1 + x_0)x_2), x_0 \mapsto \langle \bar{q} \rangle, x_2 \mapsto p^*e$$

defines a $\mathcal{W}(k)$-algebra isomorphism. Furthermore, $H^{*\geq 2}(BN_S, \mathcal{W}(\gamma_S))$ is the quotient of the free $H^*(BN_S, \mathcal{W})$-module on the generator $e(\mathcal{T})$ modulo the relation $(1 + \langle q \rangle)e(\mathcal{T}) = 0$.

Proposition 2.5.13. *Under the isomorphism*

$$H^*(BN_G, \mathcal{W}) \oplus H^*(BN_G; \mathcal{W}(\det)) \to H^*(BN_S, \mathcal{W}).$$

from Proposition 2.5.10, we have that $H^*(BN_G, \mathcal{W})$ corresponds to the subring generated by x_0 and x_2^2 of $H^*(BN_S, \mathcal{W}) \cong W(k)[x_0, x_2]/(x_0^2 - 1, (1 + x_0)x_2)$. Furthermore, $H^*(BN_G; \mathcal{W}(\det))$ corresponds to the ideal $x_2 H^*(BN_S, \mathcal{W})$.
Also, under the isomorphism

$$H^*(BN_G, \mathcal{W}(\gamma_G)) \oplus H^*(BN_G, \mathcal{W}(\gamma_G + \det)) \to H^*(BN_S, \mathcal{W}(\gamma_S))$$

we have that $H^{*\geq 2}(BN_G, \mathcal{W}(\gamma_G))$ is the submodule generated by $e(\mathcal{T}_G)$ and $H^{*\geq 2}(BN_G, \mathcal{W}(\gamma_G + \det))$ is the submodule generated by $x_2 \cdot e(\mathcal{T}_G)$.

Proof. We have that the Pontryagin class $p_1(E_2^{N_G}) \in H^*(BN_G, \mathcal{W})$ pulls back to

$$\pi^*(p_1(E_2^{N_G})) = p_1(p^*E_2) = p^*(e^2) \in H^*(BN_S, \mathcal{W}).$$

Furthermore, the Euler class $e(E_2^{N_G}) \in H^*(BN_G; \mathcal{W}(\det(E_2^{N_G})))$ pulls back to the Euler class p^*e. As $H^*(BN_G; \mathcal{W}(\det(E_2^{N_G})))$ is a $H^*(BN_G; \mathcal{W})$-module, this shows that $H^*(BN_G; \mathcal{W}(\det))$ contains

$$x_2 \cdot W(k)[x_0, x_2]/(x_0^2 - 1, (1 + x_0)x_2^2)$$

which concludes the proof of the first statement.
For the second statement, note that

$$\pi^* e(\mathcal{T}_G) = e(\pi^*\mathcal{T}_G) = e(\mathcal{T}).$$

Now consider the diagram

$$\begin{array}{ccc} H^{*\geq 2}(BN_S, \mathcal{W}) & \xleftarrow{\pi^*} & H^{*\geq 2}(BN_G, \mathcal{W}) \oplus H^{*\geq 2}(BN_G; \mathcal{W}(\det)) \\ \downarrow \phi & & \downarrow \psi \\ H^{*\geq 2}(BN_S, \mathcal{W}(\gamma_S)) & \xleftarrow{\pi^*} & H^{*\geq 2}(BN_G, \mathcal{W}(\gamma_G)) \oplus H^{*\geq 2}(BN_G; \mathcal{W}(\det + \gamma_G)) \end{array}$$

where the horizontal arrows are the isomorphisms from Proposition 2.5.10 and ψ is defined by sending 1 to $e(\mathcal{T}_G)$. We know that ϕ makes $H^*(BN_S, \mathcal{W}(\gamma_S))$ into a free $H^*(BN_S, \mathcal{W})$-module of rank 1 with generator $e(\mathcal{T})$. Moreover, ψ has to be injective as its composition with π^* is. Now as $\psi(x_2) = x_2 \cdot e(\mathcal{T}_G)$, the desired result follows. $\square$

Computation over C

Anna M. Viergever

```
R.<t1 ,t2, t3> = LaurentPolynomialRing(QQ)
C.<x, y, z> = PolynomialRing(QQ); D = C.fraction_field()

#A function which computes the Euler class of the ideal I with \
   tangent weights s1, s2, s3
def compute_euler_class(I, s1, s2, s3):
     basis = I.normal_basis()
     partition = [i.degrees() for i in basis]
     partitionlist = [list(i) for i in partition]
    Q = sum(t1^(a[0])*t2^(a[1])*t3^(a[2]) for a in partitionlist)
    Qbar = sum(t1^(-a[0])*t2^(-a[1])*t3^(-a[2]) for a in \
   partitionlist)
    V = Q - Qbar/(t1*t2*t3) + (Q*Qbar*(1 - t1)*(1 - t2)*(1 - t3 ))/(\
   t1*t2*t3)
     coefficients = V.coefficients()
     exponents = V.exponents()
     exponentslist = [list(i) for i in exponents]; length = len(\
   exponents)
     innerproducts = [b[0]*(s1[0]*x + s1[1]*y + s1[2]*z) + b[1]*(s2\
   [0]*x + s2[1]*y + s2[2]*z) + b[2]*(s3[0]*x + s3[1]*y + s3[2]*z) \
   for b in exponentslist]
     euler = prod(innerproducts[i]^(-coefficients[i]) for i in range\
   (0,length))
     return euler

#Tangent weights on different opens
w1 = [1, 0, 0]; w2 = [0, 1, 0]; w3 = [0, 0, 1]
v1 = [-1, 0, 0]; v2 = [-1, 1, 0]; v3 = [-1, 0, 1]
n1 = [1, -1, 0]; n2 = [0, -1, 0]; n3 = [0, -1, 1]
m1 = [1, 0, -1]; m2 = [0, 1, -1] ; m3 = [0, 0, -1]

#A function which computes the Euler classes of an ideal sheaf with \
   respect to the different weights and adds them together in a list\
    named "degrees"
def eulerclassesofideal(I, degrees):
```

```
    degrees.append(compute_euler_class(I, w1, w2, w3))
    degrees.append(compute_euler_class(I, v1, v2, v3))
    degrees.append(compute_euler_class(I, n1, n2, n3))
    degrees.append(compute_euler_class(I, m1, m2, m3))
    return
```

```
#Computing I_1
DegreeOne = []
eulerclassesofideal(ideal(x,y,z),DegreeOne)
sum(a for a in DegreeOne)
```

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```
#Computing I_2
DegreeTwo=[]
eulerclassesofideal(ideal(x^2,y,z),DegreeTwo)
eulerclassesofideal(ideal(x,y^2,z),DegreeTwo)
eulerclassesofideal(ideal(x,y,z^2),DegreeTwo)
products = []
for i in range (0, len(DegreeOne)):
    for j in range(0, len(DegreeOne)):
        if i < j:
            products.append(DegreeOne[i]*DegreeOne[j])
sum(a for a in DegreeTwo) + sum(a for a in products)
```

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Motivic computation

Anna M. Viergever

```
R.<s1,s2,s3> = LaurentPolynomialRing(ZZ)
R = PolynomialRing(QQ, 2, 'ab'); R
a,b = R.gens()
F = R.fraction_field(); F
C.<x,y,z> = PolynomialRing(QQ);
```

Multivariate Polynomial Ring in a, b over Rational Field
Fraction Field of Multivariate Polynomial Ring in a, b over Rational Field

```
#Function to compute the Euler class of the ideal I and add it to \
   the list degrees, together with the version with a and b swapped
def compute_euler_class(I, degrees):
     basis = I.normal_basis()
     partition = [i.degrees() for i in basis]
     partitionlist = [list(i) for i in partition]
    Q = sum(s1^(a[0])*s2^(a[1])*s3^(a[2]) for a in partitionlist)
     Qbar = sum(s1^(-a[0])*s2^(-a[1])*s3^(-a[2]) for a in \
   partitionlist)
    V = Q - Qbar/(s1*s2*s3) + (Q*Qbar*(1-s1)*(1-s2)*(1-s3))/(s1*s2*\
   s3); print("For the ideal"); pretty_print(I); print("the trace is\
   "); pretty_print(V)
     exponents = V.exponents()
     length = len(exponents)
     euler = 1
     print('We find factors')
     for i in range(0,length):
          monomialexponent = exponents[i]
          c1 = monomialexponent[0]
          c2 = monomialexponent[1]
          c3 = monomialexponent[2]
          coefficientpolynomial = V.coefficient((s1^c1)*(s2^c2)*(s3^c3\
   ))
          coefficient = coefficientpolynomial.constant_coefficient()
          factorwithoutcoeff = c1*(-2*a) + c2*(-a+b) + c3*(-a-b)
          if factorwithoutcoeff(1,1) % 4 == 0 or factorwithoutcoeff\
   (1,1) % 4 == 3:
```

```
            factorwithoutcoeff = -factorwithoutcoeff
            print('sign change')
        newfactor = (factorwithoutcoeff)^(-coefficient)
        print(newfactor)
        euler = euler*newfactor
    degrees.append(euler); degrees.append(euler(b,a))
    print("We find Euler classes"); pretty_print(euler.factor()); \
   print("and"); pretty_print(euler(b,a).factor())
    return
```

```
#Function to compute the products of Euler classes which correspond \
   to ideals of the same length
def compute_products_samelength(degrees):
    products = []
    for i in range(0, len(degrees)):
        for j in range(i, len(degrees)):
            if Mod(i,2) != Mod(j,2):
                products.append(degrees[i]*degrees[j])
    return products
```

```
#Function to compute the products of Euler classes which correspond \
   to ideals of different length
def compute_products(degrees1,degrees2):
    products = []
    for i in range(0, len(degrees1)):
        for j in range(0, len(degrees2)):
            if Mod(i,2) != Mod(j,2):
                products.append(degrees1[i]*degrees2[j])
    return products
```

```
#Computing \tilde{I}_2
DegreeOne=[]
compute_euler_class(ideal(x,y,z),DegreeOne)
print("Their sum is"); sum(a for a in DegreeOne)
```

For the ideal

$(x, y, z)\,\mathbb{Q}[x, y, z]$

the trace is

$s_3^{-1} + s_2^{-1} + s_1^{-1} - s_2^{-1}s_3^{-1} - s_1^{-1}s_3^{-1} - s_1^{-1}s_2^{-1}$

We find factors
1/(a + b)
sign change
1/(-a + b)
1/(2*a)
2*a
sign change
-3*a - b
3*a - b

We find Euler classes
$(a-b)^{-1} \cdot (a+b)^{-1} \cdot (3a-b) \cdot (3a+b)$
and
$(a-b)^{-1} \cdot (a+b)^{-1} \cdot (a-3b) \cdot (a+3b)$
Their sum is
10

```
#Computing \tilde{I}_4
DegreeTwo = []
compute_euler_class(ideal(x,y^2,z),DegreeTwo)
compute_euler_class(ideal(x,y,z^2),DegreeTwo)
print("Their sum is"); sum(a for a in DegreeTwo) + sum(a for a in \
    compute_products_samelength(DegreeOne))
```

For the ideal
$\left(x, y^2, z\right) \mathbb{Q}[x,y,z]$
the trace is
$s_2 s_3^{-1} + s_1^{-1} s_2 + s_3^{-1} - s_1^{-1} s_2 s_3^{-1} + s_2^{-1} + s_1^{-1} - s_2^{-1} s_3^{-1} - s_1^{-1} s_3^{-1} + s_2^{-2} - s_1^{-1} s_2^{-1} - s_2^{-2} s_3^{-1} - s_1^{-1} s_2^{-2}$
We find factors
1/(2*b)
1/(a + b)
1/(a + b)
sign change
-2*a - 2*b
sign change
1/(-a + b)
1/(2*a)
2*a
sign change
-3*a - b
sign change
1/(-2*a + 2*b)
3*a - b
3*a - b
4*a - 2*b
We find Euler classes
$(a-b)^{-2} \cdot b^{-1} \cdot (a+b)^{-1} \cdot (2a-b) \cdot (3a+b) \cdot (3a-b)^2$
and
$(-1) \cdot (-a+b)^{-2} \cdot a^{-1} \cdot (a+b)^{-1} \cdot (a-2b) \cdot (a+3b) \cdot (a-3b)^2$
For the ideal
$\left(x, y, z^2\right) \mathbb{Q}[x,y,z]$
the trace is
$s_2^{-1} s_3 + s_1^{-1} s_3 + s_3^{-1} + s_2^{-1} + s_1^{-1} - s_1^{-1} s_2^{-1} s_3 + s_3^{-2} - s_2^{-1} s_3^{-1} - s_1^{-1} s_3^{-1} - s_1^{-1} s_2^{-1} - s_2^{-1} s_3^{-2} - s_1^{-1} s_3^{-2}$
We find factors
1/(-2*b)
sign change
1/(-a + b)
1/(a + b)

sign change
1/(-a + b)
1/(2*a)
sign change
-2*a + 2*b
sign change
1/(-2*a - 2*b)
2*a
sign change
-3*a - b
3*a - b
sign change
-3*a - b
4*a + 2*b
We find Euler classes

$$(-1)\cdot(a+b)^{-2}\cdot b^{-1}\cdot(a-b)^{-1}\cdot(2a+b)\cdot(3a-b)\cdot(3a+b)^2$$

and

$$(a+b)^{-2}\cdot(-a+b)^{-1}\cdot a^{-1}\cdot(a-3b)\cdot(a+2b)\cdot(a+3b)^2$$

Their sum is
25

```
#Computing \tilde{I}_6
DegreeThree = []
compute_euler_class(ideal(x,y^3,z),DegreeThree)
compute_euler_class(ideal(x,y,z^3),DegreeThree)
compute_euler_class(ideal(x^2,x*y,y^2,z),DegreeThree)
compute_euler_class(ideal(y^2,y*z,z^2,x),DegreeThree)
compute_euler_class(ideal(x^2,x*z,z^2,y),DegreeThree)
print("Their sum is"); sum(a for a in DegreeThree) + sum(a for a in \
    compute_products(DegreeTwo,DegreeOne))
```

For the ideal

$$\left(x, y^3, z\right)\mathbb{Q}[x,y,z]$$

the trace is

$$s_2^2 s_3^{-1} + s_1^{-1}s_2^2 + s_2 s_3^{-1} - s_1^{-1}s_2^2 s_3^{-1} + s_1^{-1}s_2 + s_3^{-1} - s_1^{-1}s_2 s_3^{-1} + s_2^{-1} + s_1^{-1} - s_2^{-1}s_3^{-1} - s_1^{-1}s_3^{-1} + s_2^{-2} - s_1^{-1}s_2^{-1} - s_2^{-2}s_3^{-1} + s_2^{-3} - s_1^{-1}s_2^{-2} - s_2^{-3}s_3^{-1} - s_1^{-1}s_2^{-3}$$

We find factors
1/(-a + 3*b)
1/(2*b)
1/(2*b)
sign change
-a - 3*b
1/(a + b)
1/(a + b)
sign change
-2*a - 2*b
sign change
1/(-a + b)

1/(2*a)
2*a
sign change
-3*a - b
sign change
1/(-2*a + 2*b)
3*a - b
3*a - b
sign change
1/(-3*a + 3*b)
4*a - 2*b
4*a - 2*b
5*a - 3*b
We find Euler classes

$$\left(-\frac{1}{3}\right) \cdot (a-b)^{-3} \cdot b^{-2} \cdot (a-3b)^{-1} \cdot (a+b)^{-1} \cdot (a+3b) \cdot (3a+b) \cdot (5a-3b) \cdot (2a-b)^{2} \cdot (3a-b)^{2}$$

and

$$\left(\frac{1}{3}\right) \cdot (-a+b)^{-3} \cdot a^{-2} \cdot (-3a+b)^{-1} \cdot (a+b)^{-1} \cdot (a+3b) \cdot (3a-5b) \cdot (3a+b) \cdot (a-3b)^{2} \cdot (a-2b)^{2}$$

For the ideal

$$\left(x, y, z^{3}\right) \mathbb{Q}[x, y, z]$$

the trace is

$$s_2^{-1}s_3^{2} + s_1^{-1}s_3^{2} + s_2^{-1}s_3 + s_1^{-1}s_3 - s_1^{-1}s_2^{-1}s_3^{2} + s_3^{-1} + s_2^{-1} + s_1^{-1} - s_1^{-1}s_2^{-1}s_3 + s_3^{-2} - s_2^{-1}s_3^{-1} - s_1^{-1}s_3^{-1} - s_1^{-1}s_2^{-1} + s_3^{-3} - s_2^{-1}s_3^{-2} - s_1^{-1}s_3^{-2} - s_2^{-1}s_3^{-3} - s_1^{-1}s_3^{-3}$$

We find factors
sign change
1/(a + 3*b)
1/(-2*b)
1/(-2*b)
sign change
1/(-a + b)
a - 3*b
1/(a + b)
sign change
1/(-a + b)
1/(2*a)
sign change
-2*a + 2*b
sign change
1/(-2*a - 2*b)
2*a
sign change
-3*a - b
3*a - b
1/(3*a + 3*b)
sign change
-3*a - b
4*a + 2*b

4*a + 2*b
sign change
-5*a - 3*b
We find Euler classes

$$\left(-\frac{1}{3}\right) \cdot (a+b)^{-3} \cdot b^{-2} \cdot (a-b)^{-1} \cdot (a+3b)^{-1} \cdot (a-3b) \cdot (3a-b) \cdot (5a+3b) \cdot (2a+b)^2 \cdot (3a+b)^2$$

and

$$\left(-\frac{1}{3}\right) \cdot (a+b)^{-3} \cdot a^{-2} \cdot (-a+b)^{-1} \cdot (3a+b)^{-1} \cdot (a-3b) \cdot (3a-b) \cdot (3a+5b) \cdot (a+2b)^2 \cdot (a+3b)^2$$

For the ideal

$$\left(x^2, xy, y^2, z\right) \mathbb{Q}[x, y, z]$$

the trace is

$$s_1 s_3^{-1} + s_2 s_3^{-1} + s_3^{-1} + s_1 s_2^{-2} + 2s_2^{-1} + 2s_1^{-1} + s_1^{-2} s_2 - s_1 s_2^{-2} s_3^{-1} - 2s_2^{-1} s_3^{-1} - 2s_1^{-1} s_3^{-1} - s_1^{-2} s_2 s_3^{-1} - s_1^{-1} s_2^{-1} - s_1^{-1} s_2^{-2} - s_1^{-2} s_2^{-1}$$

We find factors
sign change
1/(a - b)
1/(2*b)
1/(a + b)
1/(-2*b)
sign change
1/(a^2 - 2*a*b + b^2)
1/(4*a^2)
sign change
1/(-3*a - b)
sign change
-a + b
4*a^2
sign change
9*a^2 + 6*a*b + b^2
4*a + 2*b
3*a - b
4*a - 2*b
sign change
-5*a + b
We find Euler classes

$$b^{-2} \cdot (a-b)^{-2} \cdot (a+b)^{-1} \cdot (2a-b) \cdot (2a+b) \cdot (3a-b) \cdot (3a+b) \cdot (5a-b)$$

and

$$(-1) \cdot (-a+b)^{-2} \cdot a^{-2} \cdot (a+b)^{-1} \cdot (a-5b) \cdot (a-3b) \cdot (a-2b) \cdot (a+2b) \cdot (a+3b)$$

For the ideal

$$\left(y^2, yz, z^2, x\right) \mathbb{Q}[x, y, z]$$

the trace is

$$s_1^{-1} s_2 + s_1^{-1} s_3 + s_2 s_3^{-2} + 2s_3^{-1} + 2s_2^{-1} + s_1^{-1} + s_2^{-2} s_3 - s_1^{-1} s_2 s_3^{-2} - s_2^{-1} s_3^{-1} - 2s_1^{-1} s_3^{-1} - 2s_1^{-1} s_2^{-1} - s_1^{-1} s_2^{-2} s_3 - s_2^{-1} s_3^{-2} - s_2^{-2} s_3^{-1}$$

We find factors
1/(a + b)
sign change
1/(-a + b)

sign change
1/(-a - 3*b)
1/(a^2 + 2*a*b + b^2)
sign change
1/(a^2 - 2*a*b + b^2)
1/(2*a)
1/(a - 3*b)
3*a + 3*b
2*a
sign change
9*a^2 + 6*a*b + b^2
9*a^2 - 6*a*b + b^2
sign change
-3*a + 3*b
sign change
-3*a - b
3*a - b
We find Euler classes

$$(9) \cdot (a-b)^{-2} \cdot (a+b)^{-2} \cdot (a-3b)^{-1} \cdot (a+3b)^{-1} \cdot (3a-b)^{3} \cdot (3a+b)^{3}$$

and

$$(9) \cdot (a-b)^{-2} \cdot (a+b)^{-2} \cdot (3a-b)^{-1} \cdot (3a+b)^{-1} \cdot (a-3b)^{3} \cdot (a+3b)^{3}$$

For the ideal

$$\left(x^2, xz, z^2, y\right) \mathbb{Q}[x,y,z]$$

the trace is

$$s_1 s_2^{-1} + s_2^{-1} s_3 + s_1 s_3^{-2} + 2s_3^{-1} + s_2^{-1} + 2s_1^{-1} + s_1^{-2} s_3 - s_1 s_2^{-1} s_3^{-2} - 2s_2^{-1} s_3^{-1} - s_1^{-1} s_3^{-1} - 2s_1^{-1} s_2^{-1} - s_1^{-2} s_2^{-1} s_3 - s_1^{-1} s_3^{-2} - s_1^{-2} s_3^{-1}$$

We find factors
1/(-a - b)
1/(-2*b)
1/(2*b)
1/(a^2 + 2*a*b + b^2)
sign change
1/(-a + b)
1/(4*a^2)
1/(3*a - b)
a + b
4*a^2
sign change
-3*a - b
9*a^2 - 6*a*b + b^2
4*a - 2*b
4*a + 2*b
5*a + b
We find Euler classes

$$b^{-2} \cdot (a+b)^{-2} \cdot (a-b)^{-1} \cdot (2a-b) \cdot (2a+b) \cdot (3a-b) \cdot (3a+b) \cdot (5a+b)$$

and

$$a^{-2} \cdot (a+b)^{-2} \cdot (-a+b)^{-1} \cdot (a-3b) \cdot (a-2b) \cdot (a+2b) \cdot (a+3b) \cdot (a+5b)$$

Their sum is
-50

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