Essays on Interest Rate Risk Management for Participating Life Insurance Products

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Contents

List of Abbreviations

ESTR Euro short-term rate

ALM Asset liability management

BaFin Bundesanstalt für Finanzdienstleistungsaufsicht

BEL Best estimate of liabilities

BSCR Basic solvency capital requirement

CE Certainty equivalent

CEIOPS Committee of European Insurance and Occupational Pensions Supervisors

CIR Cox, Ingersoll and Ross

CPT Cumulative prospect theory

CRRA Constant relative risk aversion

CVaR Conditional value at risk

DAV Deutsche Aktuarvereinigung

EIA Equity indexed annuity

EIOPA European Insurance and Occupational Pensions Authority

EONIA Euro OverNight Index Average

ES Expected shortfall

GDV Gesamtverband der Deutschen Versicherungswirtschaft

GSIB Global systematically important banks

MCPT Multi cumulative prospect theory

MLE Maximum likelihood estimation

MRRG Minimum return rate guarantee

RORAC Return on Risk-adjusted Capital

SCR Solvency capital requirement

SFP Shortfall probability

VaR Value at risk

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Chapter 1

General introduction

Insurance companies are currently facing many challenges. One of these challenges is the current low-interest phase, which has already lasted for several years. But the high fluctuation of interest rates also represents a fundamental financial risk for insurance companies. In general interest rate risk has a major impact on the solvency of insurance companies. In particular, the actuarial reserve of life insurance companies, whose portfolios consist of policies with long maturities that promise a relatively high guaranteed return, is affected by interest rate risk (cf. Berdin and Gründl (2015), Hieber et al. (2015)).

The subject of this work are life insurance companies who provide participating life insurance contracts. Especially German life insurer have a huge amount of long term contracts in their portfolio, such that they are exposed to potential risks on the financial stability, in particular interest rate risk i.e. losses due to interest rate fluctuations. Life insurance contracts with (innovative) guarantees, which are of particular relevance for life insurance companies due to the ongoing low-interest phase and the high fluctuations on the financial markets as well as the regulatory requirements from Solvency II (cf. Schmeiser and Wagner (2015)), are investigated in this thesis. Those participating life insurance contracts are contracts where the insured participates in the return of the insurers asset portfolio with her initially paid premium. In addition, a minimum return on her premium is often guaranteed. A distinction is made between terminal and cliquet-style guarantees, whereby terminal guarantees only take into account the return at the contract maturity, while cliquet-style guarantees do it regularly at specific intervals (e.g. annually).

In the German life insurance market, participating life insurance policies still account for a large proportion of all current insurance policies, although the number decreased in recent years (c.f. GDV (2022)). One risk affecting such policies is that the return on the insurer's investment portfolio may be lower than the guaranteed minimum return promised to the policyholders. In this case, the insurer is liable. For this reason, the policyholder pays a risk premium to the insurer. This risk threatens the solvency of the insurers and must therefore be adequately managed. Moreover, due to this, the minimum guaranteed return of such policies are regulated by local authorities. In Germany, an upper limit for these minimum guaranteed returns is set by the German Federal Ministry of Finance based on the recommendations of the German Association of Actuaries (DAV) and the German Federal Financial Supervisory Authority (BaFin). In order to earn the guaranteed returns on older policies, insurers must make investments that promise a higher return than the risk-free rate. Since a higher return is associated with more risk, insurers must therefore take more risks in order to avoid insolvency.

German government bond 30 years yield vs. upper bound on minimum guaranteed return from 01/2010 to 08/2022

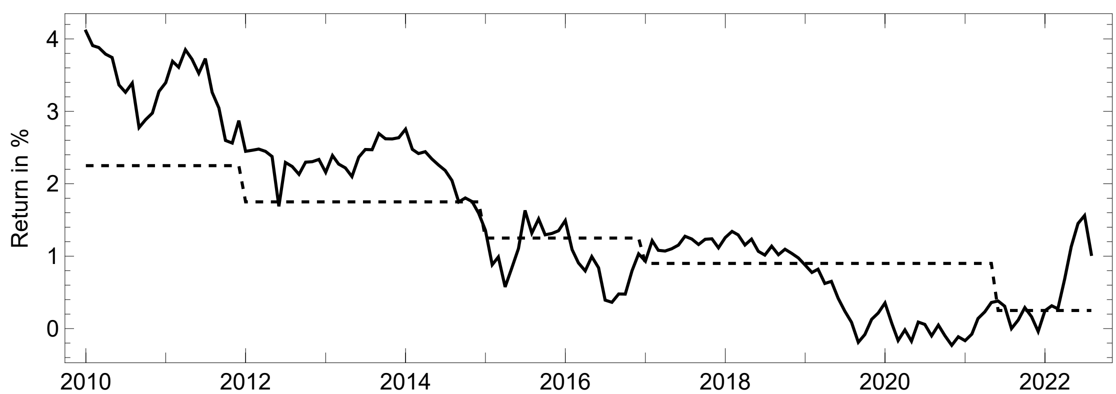

Figure 1.1: A comparison between the development of the monthly yield of German government bonds with a maturity of 30 years (solid line) and the upper bound on the minimum guaranteed return set by the German Federal Ministry of Finance (dashed line) in the period from January 2010 to August 2022. The data of the government bonds originates from Refinitiv Workspace while the data one the upper bound comes from the German Association of Actuaries (DAV).

Figure 1.1 shows the development of this upper limit for these minimum guaranteed returns (dashed line). Currently, the upper limit is 0.25%. Since this upper limit only applies to new policies, many insurers still have old policies in force that provide minimum guaranteed returns of over 2%. The upper limit for these minimum guaranteed returns was much higher before 2010 with values up to 4% from 1995 to 2000. In comparison, figure 1.1 also shows the development of the yield on German government bonds with a term of

30 years (solid line), which can be regarded as a representative risk-free interest rate due to its high rating. In fact, Figure 1.1 shows that the current risk-free interest rate is well below the current upper bound for minimum guaranteed returns. It can also be observed that participating life insurance contracts currently offered by German life insurers meet this upper limit.

The aim of this thesis is to examine participating life insurance products with regard to their exposure to interest rate risk. In addition to the question of the optimal design of these products, this thesis also deals with the question of how an appropriate choice of different products can hedge against interest rate risk. Furthermore, the regulatory basis for hedging interest rate risks in the EU given by Solvency II will be examined.

Chapter 2 analyzes minimum return rate guarantees (MRRGs) including fixed guarantee rates prevailing for the whole contract horizon as well as floating guarantee rates which are linked to the interest rate evolution. In a complete arbitrage free market where the asset and bond price dynamics are given by Gaussian processes, we obtain closed form pricing solutions for both guarantee schemes. Differences in the guarantee costs are then explained by the difference of the arbitrage free values of the fix and floating rate guarantees and the difference between cumulated volatilities resulting from forward and simple volatilities. We then consider the perspective of the asset liability management, i.e. we analyze the sensitivities of the asset and liability side against changes in the interest rate by determining the duration and convexity. We show that a combination of fix price and floating strike guarantees enables a natural hedge against changes in the interest rate. Furthermore we analyze the derived hedging strategy of fix and floating strike guarantees regarding the risk management of insurance companies by considering different risk measures like the value at risk and the conditional value at risk.

Chapter 3 analyzes the design of participating life insurance contracts with minimum return rate guarantees. The default risk, i.e. the risk that the value of the asset is lower than the intended guaranteed payout at the end of the term, is considered. In this context, a simple contract design of participating life insurance contracts with a minimum return guarantee (MRRG) is presented. Without default risk, the insured receives the maximum of a guaranteed rate and a participation in the investment returns. With default risk, the payoff is modified by a default put implying a compound option. Under regulatory guidelines for this probability of default, the optimal design of the MRRG is then determined in a simple Black Scholes model setup and under the assumption of a constant mix strategy. We represent the yearly returns of the liabilities by a portfolio of plain vanilla options. A closed form of the probability of default as a function of the equity ratio is derived. In a Black and Scholes model, the optimal payoff constrained by a maximal shortfall probability can be stated in closed form. Due to the completeness of the market, it can be implemented for any equity to debt ratio.

Chapter 4 analyzes the Solvency II capital requirements under the interest rate risk submodule. The aim of this chapter is to compile the results of these studies and to provide an overview of the strengths and weaknesses of the SCR with regard to protection against interest rate risk. At the end, the question whether the SCR under the current Solvency II regulations represents an adequate capital reserve for insurance companies regarding interest rate risk will be discussed on the basis of the available scientific literature. It begins by a brief introduction to Solvency II regulation. It then introduces the interest rate risk submodule and discusses the minimum capital requirements under the standard formula. Finally it summarizes the results from a collection of research papers that relate to capital requirements under the interest rate submodule.

Chapter 5 concludes this thesis. In addition, the appendix contains an analysis of the Vasicek model with respect to the derivation of the formulas, the simulation, and the estimation of the parameters.

Chapter 2

Natural hedging with fix and floating strike guarantees

2.1 Introduction

Low interest rate scenarios and (downward) changes in the term structure of interest rates may deteriorate the solvency situation of a life insurance company. In this chapter, we focus on the possibilities to build a natural hedge against interest rate risk by offering a suitable product mix of different minimum return rate guarantee schemes (MRRGs). In particular, we consider both, immunization of the interest rate risk on the liability side as well as on the asset and liability side simultaneously.

MRRGs are embedded in traditional and innovative participating life insurance products which are e.g. popular in German speaking countries.[1] Here, the savings account of the insured grows at least according to a guaranteed rate, but the insured also participates in the excess returns of the investment strategy conducted by the insurer, i.e. if there is any.

In view of the long-term horizons which are common in life insurance products, the guarantees (long term put options) can cause a substantial risk exposure to the provider. Recently, writing (traditional) long term terminal guarantees is often considered as too expensive by the insurance industry, in particular in view of low interest rate regimes and

[1] For example, reference is made to the products KonfortDynamik, InvestFlex and IndexSelect of Allianz, PrivatRente Performance of R+V and TwoTrust Vario of HDI.

the capital requirements posed by the Solvency II regulation, cf. Berdin and Gründl (2015) and Mirza and Wagner (2018). In consequence, there are innovative insurance products discussed and placed on the market linking the guaranteed rate to the interest rate evolution and/or the insurer's investment results which are considered as less expensive and riskily if compared to traditional guarantees, cf. Eling and Holder (2013) and Reuß et al. (2016).

In a dynamic version, a natural hedge (on the liability side) implies that the value of the product mix does not change if there is a change in the term structure of interest rate, i.e. if the interest rate sensitivities of both products (each weighted by the number of products) coincide. In the first instance, we analyze the possibilities of building a natural hedge. Regarding the products, we compare fixed guarantee rates prevailing for the whole contract horizon and floating guarantee rates linked to the interest rate evolution.

W.r.t. the numerical illustrations, we place ourselves in a complete and arbitrage free financial market model setup where the asset and bond price dynamics are given in terms of Gaussian processes. We assume a competitive insurance and financial market (as well as competition between the insurance and financial market) such that the guarantee contracts under consideration are priced under the no arbitrage condition. Using well known pricing results concerning the option to exchange two (lognormal) assets, we obtain closed form pricing solutions for the benchmark contract/guarantee designs. A contract is called fair (competitive) if the arbitrage free value of the liabilities to the insured are equal to her (up front) contribution. For a given initial contribution and guarantee rate, it is basically possible to determine the participation (in the excess returns) such that the contract is fair.[2] In particular, higher guarantee costs are associated with a lower participation fraction. We compare the participation rates of fixed rate and (interest rate linked) floating strike guarantees. The Gaussian model setup is especially convenient to obtain an intuitive explanation for the differences in the guarantee costs of fix and floating rate guarantees. Both guarantee costs are exclusively specified by the arbitrage free values of the (fix, floating) guaranteed amount and the difference between cumulated volatilities resulting from forward and simple volatilities of the asset side which is determined by an investment strategy in assets traded on the financial market. We then consider the price sensitivities of the asset (investment strategy) and liability side (consisting of fix and floating strike guarantees) towards changes in the interest rates and the correlation

[2] Alternatively, one can determine the fair guarantee rate for a given participation fraction.

between the assets and the interest rate. We show that fix and floating strike guarantees give rise to opposing effects such that a suitable combination of fix and floating interest rates can build a natural hedge against interest rate and correlation risk. Illustrations are given in terms of a Black and Scholes (1973)/extended Vasicek model (c.f. Hull and White (1990)). We further analyze the meaning of fix and floating strike guarantees for the risk management of insurance companies.

The contributions of the chapter can be summarized as follows. While natural hedging possibilities are for example discussed in the context of life insurance and annuities (cf. Gatzert and Wesker (2012)), we are, to the best of our knowledge, the first who discuss the possibility to build up a natural hedge by means of a suitable product mix of different guarantee schemes. Compared to an immunization of the buffer (difference of assets and liabilities) by adjusting the duration of the asset side, the natural hedge has in addition the following advantage. Reducing the (stochastic) duration of the asset side may be difficult because it is not necessarily possible to trade in (liquid) bonds which are consistent with the long maturities of life insurance contracts. In addition, any modification of the investment strategy (asset side) also impacts the value of the guarantees (liability side). In contrast, a natural ALM hedge based on a product mix on the liability side has no impact on the asset side.

This chapter is related to several strands of the literature including the ones on (i) pricing and hedging embedded guarantees/options, (ii) portfolio planning, (iii) innovative guarantee contracts, and (iv) natural hedging by means of life insurance products. Without postulating completeness, we only refer to the most related literature and hint at the additional literature given within the mentioned papers.

Pricing embedded options by no arbitrage already dates back to Brennan and Schwartz (1976). For a fair valuation of participating life insurance contracts, we refer, among others, to Briys and De Varenne (1994), Tanskanen and Lukkarinen (2003), Ballotta (2005), and Bauer et al. (2006), Eckert et al. (2016), Orozco-Garcia and Schmeiser (2019) and Bacinello et al. (2021). Risk management and hedging aspects are e.g. discussed in Mahayni and Schlögl (2008), Klusik and Palmowski (2011) and in the context of variable annuities in Feng and Yi (2019).

Literature on portfolio planning with a main focus on insurance contracts with guarantees includes e.g. Huang et al. (2008), Milevsky and Kyrychenko (2008), Boyle and Tian (2008), Branger et al. (2010), and Mahayni and Schneider (2016). Portfolio planning itself dates back to Merton (1975) who, amongst other results, solves the portfolio planning problem for a CRRA investor. For alternative or innovative guarantee contracts, we refer to Eling and Holder (2013), Mahayni and Muck (2017) and Ruß and Schelling (2018).

Hedging of mortality risk in life insurance contracts is related to our basic idea since contracts with opposing sensitivities to mortality risk are combined. Cox and Lin (2007) e.g. analyze natural hedging of life insurance and annuity liabilities. Wang et al. (2010) show that natural hedging can Gatzert and Wesker (2012) lower the sensitivity of an insurance portfolio with respect to mortality risk. They use simulations to select portfolios of insurance contracts which immunize the insurer's solvency against chances in mortality. While Wang et al. (2010) only concentrate on the liability side and do not consider the asset side, Gatzert and Wesker (2012) take into account both, asset and liabilities, as well as their interaction. Luciano et al. (2017) extend the previous literature by introducing hedges within a single generation and across generations in the presence of both, longevity and interest-rate risks. Wong et al. (2017) further analyze the effect of life insurance product design on natural hedging by using a variety of standard products. A natural hedge based on a product mix is also subject to the work of Bernard and Boyle (2011). The authors focus on equity indexed annuities and consider a standard equity linked contract and a so-called Monthly Sum Cap EIA, both accounting for a fix strike guarantee. However, we consider a product mix of a fix strike guarantee and a floating strike guarantee which is linked to the interest rate evolution in a participating life insurance contract where the insured also participates in the insurer's excess returns of the driven investment strategy.

The rest of the chapter is organized as follows. Section 3.2 introduces fix strike and floating strike guarantees. We state the model setup for the asset and interest rate evolution and derive the pricing formulas for the different benchmark guarantee schemes. In Section 3, we model the asset side of the insurer by means of a strategy conducted at the financial market. We then analyze the sensitivities of the asset and liability side w.r.t. changes in the interest rate. We then discuss the possibilities to obtain a natural hedge on the liability side which immunizes the buffer (assets minus liabilities) against changes in the interest rate. In Section 4, the results are illustrated in a Black and Scholes/Vasicek model setup. Section 2.6 concludes the paper.

2.2 Preliminaries

Throughout the following, we analyze two versions of MRRGs which are meaningful in the context of participating life insurance contracts. One version is implied by a guaranteed rate which terminal value is once determined at the contract inception. We refer to it as fix strike guarantee. The other version is a (stochastic) guarantee rate which is implied by the interest rate accumulation over the contract horizon. We call it floating strike guarantee. These two versions are understood as corner cases of more general contract designs.

Since the insured receives the maximum of the guarantee and a fraction of the asset result, we need assumptions on the evolution of the asset side and the interest rate dynamics. We use a Gaussian model setup which gives rise to closed-form (market consistent) values of the liabilities of the insurance company and allows to obtain first insights about the (basic) differences of the guarantee costs associated with fix strike and (interest rate) floating strikes. A detailed analysis of the interactions of asset and liabilities which are closely linked to the investment strategy conducted by the insurance company is dedicated to the subsequent section.

2.2.1 Contract design, model setup and pricing

The contribution of the insured consist of a single premium P_0 at the inception $t = 0$ of the contract.[3] Her payoff prevails at $T > 0$ and is given in terms of a participation on positive investment results/returns and includes a return guarantee. To simplify the expositions, we restrict ourselves to terminal guarantees and, as mentioned above, we distinguish between two corner cases. One is implied by a guarantee which is determined at the contract initialization. The other benchmark case is given by a guarantee which is proportional to a (stochastic) money account growing with the interest rates $(r_t)_{t\in[0,T]}$. Formally, the benchmark guarantee schemes (contracts, respectively) are described as follows. Let $(A_t)_{t\in[0,T]}$ denote the value process of the insurer's investment portfolio (asset side), then the payoff to the insured with a terminal guarantee is of the form

$$P_T = P_0 \max\left\{K, \alpha \frac{A_T}{A_0}\right\} = P_0 \left(\alpha \frac{A_T}{A_0} + \left(K - \alpha \frac{A_T}{A_0}\right)^+\right) \tag{2.1}$$

$$\text{where } K = f(I(0,T)) \text{ and } I(t,T) := \int_t^T r_u \, du$$

and α denotes the participation fraction of the investment return $\frac{A_T}{A_0}$. K represents the terminal guaranteed return which we refer to as strike guarantee, since the payoff can be

[3] Since we abstract from mortality or surrender risk, there is no loss of generality due to a single premium compared to more flexible premium payments.

represented in the form of a put option on insurer's assets with strike K. In the following we consider two types of guarantees:

(i) The fix strike guarantee is given by a constant strike $K = K_{\text{fix}}$.

(ii) The floating strike guarantee is given in terms of a strike which is proportional to the money market account, i.e. $K_{\text{fl.}} = \tilde{\alpha} e^{I(0,T)}$ where $\tilde{\alpha}$ denotes the fraction of the interest rate accumulation.

Assuming a competitive insurance (and financial) market, the participation fraction α (or the strike K of the guarantee option) is determined such that the initial contribution P_0 matches the (arbitrage-free) value of the contract payoff. Thus, we have to pose assumptions on the asset and interest rate dynamics. We assume a complete and arbitrage-free financial market model under interest rate risk where the dynamic of the price process $(A_t)_{t\in[0,T]}$ as well as the dynamics of the zero coupon bonds $B(\cdot, \bar{t})$ paying one monetary unit at maturity $\bar{t} \in [0, T]$ are lognormal.[4]

Thus, the index dynamic is modeled along the lines of Black and Scholes (1973), the interest rate dynamic is given by a Gaussian Markov Heath, Jarrow and Morton (1992) model. In particular, there exist a uniquely defined martingale measure P^* such that

$$dA_t = A_t \left(r_t \, dt + \sigma_A(t) \, dW_t^* \right) \tag{2.2}$$

$$dB(t, \bar{t}) = B(t, \bar{t}) \left(r_t \, dt + \sigma_{\bar{t}}(t) \, dW_t^* \right) \tag{2.3}$$

where W^* denotes a d-dimensional Brownian motion with respect to P^*, and σ_A and $\sigma_{\bar{t}}$ satisfy the usual regularity conditions.

To simplify the notation further, we introduce the cumulated volatility (of the process Z) from time t up to time T by $v(t, T)$, i.e.

$$v(t, T) := \sqrt{\int_t^T \|\sigma_Z(s)\|^2 \, ds} \tag{2.4}$$

where $\|\cdot\|$ depicts the euclidean norm. The subsequent pricing results can be traced back to the pricing of an option to exchange two assets. The pricing formula for a European

[4] We call a stochastic process $(Z_t)_{0\leq t\leq T}$ lognormal iff it is a solution of

$$dZ_t = Z_t(\mu_t \, dt + \sigma_Z(t) \, dW_t)$$

with deterministic dispersion coefficient $\sigma_Z : [0, T[\to R_+^d$

option to exchange two lognormal assets dates back to Margrabe (1978) and is summarized in the following lemma.

Lemma 2.2.1 (Pricing formula for exchange option) *Let X and Y denote two lognormal assets.*[5] *Then, the t–value of the European option to exchange the asset Y for the asset X at maturity T with payoff $[X_T - Y_T]^+$ is given by*

$$C(t, X_t, Y_t) = X_t \mathcal{N}(d_1(t, Z_t)) - Y_t \mathcal{N}(d_2(t, Z_t)) \tag{2.5}$$

where $Z := \frac{X}{Y}$ and

$$d_1(t, Z) := \frac{\ln(Z) + \frac{1}{2}v^2(t,T)}{v(t,T)}, \quad d_2(t,Z) = d_1(t,Z) - v(t,T),$$

$$\text{and } v(t,T) := \sqrt{\int_t^T \|\sigma_Z(s)\|^2 \, ds}.$$

Using the above pricing formula for the option to exchange two lognormal assets X and Y immediately implies closed-form pricing formulas for the insurance contracts with fixed and floating strikes. To simplify, from now on we concentrate on the case where $P_0 = A_0 = 1$, i.e. where the initial single premium paid by the insured as well as the initial value of the insurer's investment portfolio is normalized.

(i) First, consider the fix strike guarantee ($K = K_{\text{fix}}$). Let $Y_t = \alpha A_t$, i.e. Y is (along the lines of Eqn. (2.2)) a lognormal process. In addition, observe that the guarantee payoff is $[X_T - Y_T]^+$ where $X_T = K_{\text{fix}}$. Notice that the (arbitrage free) value of the payoff X_T at time t is given by the expectation (under the pricing measure P^* and given the information prevailing at t) of the discounted payoff, i.e.

$$X_t = E_* \left[e^{-I(t,T)} K_{fix} | \mathcal{F}_t \right] = K_{fix} E_* [e^{-I(t,T)} | \mathcal{F}_t] = B(t,T) K_{fix}.$$

In addition, we need the cumulated volatility $v(t,T)$ of the quotient process $Z_t = \frac{X_t}{Y_t}$. It is given by

$$v(t,T) = \sqrt{\int_t^T \|\sigma_Z(s)\|^2 \, ds} = \sqrt{\int_t^T \|\sigma_A(s) - \sigma_T(s)\|^2 \, ds}.$$

[5] A lognormal (synthetic) asset in fact means that its dynamic is lognormal and that under the risk neutral pricing measure the local drift coefficient is equal to r_t.

(ii) Now, consider the floating strike guarantee ($K = \tilde{\alpha} e^{I(0,T)}$). Again, let $Y_t = \alpha A_t$. Now it holds $X_T = \tilde{\alpha} e^{I(0,T)}$ such that the t-value of the asset X now is given by

$$X_t = E_* \left[e^{-I(t,T)} \tilde{\alpha} e^{I(0,T)} | \mathcal{F}_t \right] = \tilde{\alpha} E_* [e^{-I(t,T)+I(0,T)} | \mathcal{F}_t] = \tilde{\alpha} e^{I(0,t)}.$$

Notice that

$$dX_t = \, d \left(\tilde{\alpha} e^{\int_0^t r_u \, du} \right) = \tilde{\alpha} r_t e^{I(0,t)} \, dt = X_t r_t \, dt.$$

such that the cumulated volatility $v(t,T)$ of the quotient process $Z_t = \frac{X_t}{Y_t}$ is given by

$$v(t,T) = \sqrt{\int_t^T \|\sigma_Z(s)\|^2 \, ds} = \sqrt{\int_t^T \|\sigma_A(s)\|^2 \, ds}.$$

Thus, using Lemma 2.2.1 together with the payoff definition of fix and floating strike guarantees gives the following pricing results for the guarantee contracts.

Proposition 2.2.2 (Contract Pricing) *Let $P_0 = 1$ and let $C(t, X_t, Y_t)$ be defined by Equation 2.5. Then the arbitrage-free prices $C_w(t, K_t^w, \alpha A_t)$ ($w \in \{$fix, fl.$\}$) of the guarantee payoffs are given by*

$$C_w \left(t, K_t^w, \alpha A_t\right) = \alpha A_t + C \left(t, K_t^w, \alpha A_t\right)$$

(i) for $w =$ fix (fix strike guarantee) it holds

$$K_t^{fix} = B(t,T) K_{fix} \text{ and } v_{fix}(t,T) = \sqrt{\int_t^T \|\sigma_A(s) - \sigma_T(s)\|^2 \, ds}.$$

(ii) for $w =$ fl. (floating strike guarantee) it holds

$$K_t^{fl.} = \tilde{\alpha} e^{I(0,t)} \text{ and } v_{fl.}(t,T) = \sqrt{\int_t^T \|\sigma_A(s)\|^2 \, ds}.$$

Recall that $C(t, K, \alpha A_t)$ denotes the option price to exchange the strike K for the insured asset fraction αA_t. In its interpretation, this option is a put on the insurer's assets, i.e. a option to sell the assets back at strike K, which is granted in addition to the asset participation. The contract price of a fix strike (floating strike) guarantee contract is thus obviously increasing in K_{fix} (in $\tilde{\alpha}$). An important observation concerns the pricing impact of the correlation between the asset price and interest rate.

2.2.2 Fair pricing

Before further analyzing the effects of interest rate changes on the guarantee values, we first express the guarantee costs by means of the participation fraction α which ensures that the contract value is equal to the initial contribution of the insured.[6] We call the associated participation fraction the *fair* participation fraction. In fact, the participation fraction is valid if one assumes a competitive market.

Formally, let $C(t, X_t, Y_t)$ be defined by equation (2.5). For $w \in \{\text{fix, fl.}\}$ and for given guarantee strikes (K_{fix} or accumulation factor $\tilde{\alpha}$, respectively), the participation rate α^w is called fair iff

$$\alpha^w A_0 + C\left(0, K_0^w, \alpha^w A_0\right) = P_0$$

where K_t^w is given as in Proposition 2.2.2. Note that the guarantee costs can be resembled by the fair participation fraction α_{fair}, i.e. the lower the α_{fair}, the higher the guarantee costs. In particular, since we assume $P_0 = A_0 = 1$ the actual guarantee costs are

$$C\left(0, K_0^w, \alpha^w\right) = 1 - \alpha^w.$$

We want to shed a light on the question whether the costs of a floating strike guarantee are lower than the ones of a fix strike guarantee. The answer is simple if one assumes that the present value of the guarantee is equal, i.e.

$$\tilde{\alpha} = B(0,T)K_{fix}.$$

Recall that the option price which defines the guarantee costs is increasing in the cumulated volatility of the quotient process Z. Thus, for $\tilde{\alpha} = B(0,T)K_{fix}$, the difference of the floating strike guarantee and fix strike guarantee is determined by the difference of the cumulated volatilities. It immediately follows that for $\tilde{\alpha} = B(0,T)K_{fix}$ the guarantee costs of the floating strike guarantee are lower than the ones of the fix strike guarantee (i.e. $\alpha_{\text{fair}}^{\text{fix}} < \alpha_{\text{fair}}^{\text{fl.}}$), iff

$$\int_0^T \|\sigma_A(s) - \sigma_T(s)\|^2 \, ds > \int_0^T \|\sigma_A(s)\|^2 \, ds.$$

Remember that at time s the local forward volatility $\sqrt{\|\sigma_A(s) - \sigma_T(s)\|^2}$ which defines the guarantee costs of the fix strike guarantee is decreasing (increasing) in the local asset

[6] Notice that analogous reasoning can be conducted along the lines of the strikes, i.e. one can also fix the participation fraction α and determine the fair strike K_{fix} (accumulation factor $\tilde{\alpha}$).

bond (interest rate) correlation. In contrast, the correlation between the interest rate (bond prices) and asset value does not affect the guarantee costs of the benchmark floating strike guarantee. Thus, the difference of fix and floating strike guarantee costs is decreasing (increasing) in the asset bond (interest rate) correlation. Assuming a constant correlation between asset prices and interest rates e.g. implies that there exists a critical level ρ_{crit} of the correlation such that the fair participation fractions of fix and floating strike guarantees are equal.

In addition, it is worth mentioning that, ceteris paribus, the price of the floating strike guarantee does not depend on the initial interest rate term structure (zero bond curve, respectively). In contrast, falling interest rates (rising bond prices) positively impact the value of a fix strike guarantee contract. In addition, it is necessary to consider the impact of the term structure of interest rates which is introduced by its impact on the asset value A_t. The value of both guarantee versions, fix and floating, are given in terms of the prices of options written on the asset value of the insurance company. Thus, a change in the term structure of the interest rate effects the asset value and thus also the liability value. For further clarification, we are going to model the asset dynamics of the insurance company by means of the value process of an investment strategy with financial market instruments in the next section.

2.3 Financial market model, investment strategies, and risk management

Recall that in both cases, fix and floating strike guarantees, the contract value at time $t \in [0, T]$ depends on the dynamics of the insurer's asset side $(A_t)_{t \in [0,T]}$. In the previous section we assumed that the asset dynamics is described by a lognormal process (cf. Eqn. 2.2) such that the contract values are explicitly given in terms of the current value of the insurer's assets A_t and the riskiness of the asset side measured by the cumulated volatility $v(0, T)$ (cf. Proposition 2.2.2). In the following section we model the asset side as the result of an investment strategy conducted by the insurance company on the financial market. In the first instance we pose assumptions on the investment strategies and the investment opportunity set which are consistent with the closed-form pricing formulas derived in the previous subsection. Thus, the contract prices can still be expressed in closed-form. The same is true for the sensitivities of all risk factors.

2.3.1 Financial market model and investment strategies

Throughout the following, we assume that the insurer can invest in one risky asset S and (risky) bonds $B(\cdot, \bar{t})$ with maturities $\bar{t} \in [0, T]$. We assume that the bond dynamics (under the pricing measure P^*) are given as defined by Eqn. 2.3. The stock dynamic is modeled by

$$dS_t = S_t \left(r_t \, dt + \sigma_S \, dW_t^* \right) \tag{2.6}$$

where W^* denotes a d-dimensional Brownian motion with respect to P^* and σ_S denotes the constant stock volatility. In particular, P^* is uniquely defined (the financial market model is complete in the stock and $d-1$ bonds with different maturities). In order to stick to a Gaussian model setup w.r.t. the insurer's asset dynamics A_t (and also w.r.t. realistic applications), we assume that the investment strategy conducted by the insurer is a constant mix strategy, i.e. a strategy which is defined by constant fractions of portfolio wealth invested into the traded assets. Let the fractions of wealth invested in S and the bonds with maturities $t_1 < t_2 < \dots t_{d-1}$ be denoted by $\pi = (\pi_S, \pi_1, \dots, \pi_{d-1})$ where $\pi_S + \sum_{i=1}^{d-1} \pi_i = 1$. The above assumptions imply that the dynamics of the asset side is given by a (d-dimensional) lognormal process, i.e.

$$\frac{dA_t}{A_t} = \pi_S \frac{dS_t}{S_t} + \sum_{i=1}^{d-1} \pi_i \frac{dB(t, t_i)}{B(t, t_i)}. \tag{2.7}$$

Alternatively, the above dynamics can be represented in terms of the numbers $\phi_t^{(S)} = \frac{\pi_S A_t}{S_t}$ and $\phi_t^{(i)} = \frac{\pi_i A_t}{B(t,t_i)}$, i.e.

$$dA_t = \phi_t^{(S)} dS_t + \sum_{i=1}^{d-1} \phi_t^{(i)} dB(t, t_i). \tag{2.8}$$

Since the strategy $\phi = \left(\phi_t^{(S)}, \phi_t^{(1)}, \dots, \phi_t^{(d-1)} \right)$ is self-financing, we also have

$$A_t = \phi_t^{(S)} S_t + \sum_{i=1}^{d-1} \phi_t^{(i)} B(t, t_i). \tag{2.9}$$

Thus, we still can rely on the closed-form pricing formula summarized in Proposition 2.2.2 and shed further light on the (joined) sensitivities of the asset and liability side of the insurance company.

2.3.2 Interest rate sensitivities and natural hedging

Proposition 2.2.2 implies that the sensitivities (Greeks) of both, fix and floating strike, guarantee contracts can be stated in closed-form.[7]

Our main focus is on the asset liability management of interest rate risk. Thus, we also focus on the sensitivities of the asset and liability side, i.e. on the sensitivity of the buffer $A_t - L_t$. Since we intend an immunization on the basis of the duration as well as the convexity, we consider in the following the first two derivations according to the interest rate. Along the lines of our assumptions, A_t is defined by Eqn. (2.7). We assume that the insurer issues one type contract with a fix strike guarantee and one type of contract with a floating strike guarantee. Let η_{fix} and $\eta_{\text{fl.}}$ denote the fractions of those fix and floating contracts issued by the insurer and $\eta_{\text{fix}} + \eta_{\text{fl.}} = 1$ such that the liabilities can be described by

$$\begin{aligned} L_t &= \eta_{\text{fix}} C_{\text{fix}} \left(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t\right) + \eta_{\text{fl.}} C_{\text{fl.}} \left(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t\right) \\ &= \left(\eta_{\text{fix}} \alpha^{\text{fix}} + \eta_{\text{fl.}} \alpha^{\text{fl.}}\right) A_t + \eta_{\text{fix}} C \left(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t\right) + \eta_{\text{fl.}} C \left(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t\right). \end{aligned} \tag{2.10}$$

Eqn. (2.10) shows that the value of the liabilities can be divided into three parts. One part refers to the assets, one to the guarantee costs of the fix strike guarantee and one of the floating strike guarantee. It follows, that the buffer value can be described by

$$\begin{aligned} & A_t - L_t = \\ & \quad \left(1 - \eta_{\text{fix}} \alpha^{\text{fix}} - \eta_{\text{fl.}} \alpha^{\text{fl.}}\right) A_t - \eta_{\text{fix}} C \left(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t\right) - \eta_{\text{fl.}} C \left(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t\right). \end{aligned} \tag{2.11}$$

A default occurs if the buffer value becomes negative, i.e. $A_t - L_t < 0$. Notice that the proposed bond dynamics are affine, i.e.

$$B(t,T) = e^{-\mathcal{B}(t,T) r_t + \mathcal{A}(t,T)}. \tag{2.12}$$

With Eqns. (2.9) and (2.12), the interest rate sensitivities of the asset side are given by

$$\frac{\partial A_t}{\partial r_t} = \sum_{i=1}^{d-1} \frac{\pi_i A_t}{B(t,t_i)} \left(-\mathcal{B}(t,t_i) B(t,t_i)\right) = -A_t \sum_{i=1}^{d-1} \pi_i \mathcal{B}(t,t_i) < 0 \tag{2.13}$$

$$\frac{\partial^2 A_t}{(\partial r_t)^2} = -\frac{\partial A_t}{\partial r_t} \sum_{i=1}^{d-1} \pi_i \mathcal{B}(t,t_i) = A_t \left(\sum_{i=1}^{d-1} \pi_i \mathcal{B}(t,t_i)\right)^2 > 0 \tag{2.14}$$

[7] Closed form solutions for the price sensitivities are especially convenient since numerical approximation of the derivatives of non-closed-form solutions are demanding.

where $\mathcal{B}(t,t_i)$ denotes the duration of the bond maturing at t_i, i.e. $\mathcal{B}(t,t_i)$ is an increasing function in t_i and $\sum_{i=1}^{d-1}\pi_i\mathcal{B}(t,t_i)$ is the duration of the portfolio.

Proposition 2.3.1 (Interest rate sensitivities of the buffer value) *The interest rate sensitivities of the buffer value are given by*

$$\frac{\partial^{(k)}(A_t-L_t)}{(\partial r_t)^{(k)}} =$$

$$\left(1-\eta_{\mathit{fix}}\alpha^{\mathit{fix}}-\eta_{\mathit{fl.}}\alpha^{\mathit{fl.}}\right)\frac{\partial^{(k)}A_t}{(\partial r_t)^{(k)}}-\eta_{\mathit{fix}}\frac{\partial^{(k)}C\left(t,K_t^{\mathit{fix}},\alpha^{\mathit{fix}}A_t\right)}{(\partial r_t)^{(k)}}-\eta_{\mathit{fl.}}\frac{\partial^{(k)}C\left(t,K_t^{\mathit{fl.}},\alpha^{\mathit{fl.}}A_t\right)}{(\partial r_t)^{(k)}}.$$

For $w\in\{\mathit{fix},\mathit{fl.}\}$ $C\left(t,K_t^w,\alpha^wA_t\right)$ denotes the price of the exchange option according to Eqn. (2.5). It holds

$$\frac{\partial C(t,K_t^w,\alpha^wA_t)}{\partial r_t}=\frac{\partial K_t^w}{\partial r_t}\mathcal{N}\left(d_1\left(t,\frac{K_t^w}{\alpha^wA_t}\right)\right)-\alpha^w\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t,\frac{K_t^w}{\alpha^wA_t}\right)\right)$$

In particular,

$$\frac{\partial K_t^{\mathit{fix}}}{\partial r_t}=-\mathcal{B}(t,T)B(t,T)K_{\mathit{fix}}<0, \tag{2.15}$$

$$\frac{\partial K_t^{\mathit{fl.}}}{\partial r_t}=r_tK_{\mathit{fl.}}>0. \tag{2.16}$$

For the convexity it holds

$$\begin{aligned}\frac{\partial^2C(t,K_t^w,\alpha^wA_t)}{(\partial r_t)^2}&=\frac{\partial^2K_t^w}{(\partial r_t)^2}\mathcal{N}\left(d_1\left(t,\frac{K_t^w}{\alpha^wA_t}\right)\right)-\alpha^w\frac{\partial^2A_t}{(\partial r_t)^2}\mathcal{N}\left(d_2\left(t,\frac{K_t^w}{\alpha^wA_t}\right)\right)\\&\quad+\mathcal{N}'\left(d_1\left(t,\frac{K_t^w}{\alpha^wA_t}\right)\right)\frac{1}{v(t,T)}\left(\frac{1}{\sqrt{K_t^w}}\frac{\partial K_t^w}{\partial r_t}+\frac{\sqrt{K_t^w}}{A_t}\frac{\partial A_t}{\partial r_t}\right)^2\end{aligned}$$

and

$$\frac{\partial^2K_t^{\mathit{fix}}}{(\partial r_t)^2}=\mathcal{B}(t,T)^2B(t,T)K_{\mathit{fix}}>0, \tag{2.17}$$

$$\frac{\partial^2K_t^{\mathit{fl.}}}{(\partial r_t)^2}=\left(1+r_t^2\right)K_{\mathit{fl.}}>0. \tag{2.18}$$

A proof for Proposition 2.3.1 can be found in the appendix.

Proposition 2.3.1 states the following. With regard to the general interest rate sensitivity of the buffer value, we can see three influencing factors. One factor is represented by the interest sensitivity of the asset portfolio, one by the interest sensitivity of the exchange option price with the fix strike and one with the floating strike guarantee. The exchange option price sensitivity depends on two factors, one regarding the interest rate sensitivity of the strike and on regarding the interest rate sensitivity of the asset portfolio. For the fix strike one can see, that the interest rate sensitivity is negative (Eqn (2.15)). For the fix strike option it holds

$$\frac{\partial C(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t)}{\partial r_t} = \underbrace{\frac{\partial K_t^{\text{fix}}}{\partial r_t}}_{<0} \underbrace{\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}} A_t}\right)\right)}_{>0} - \alpha^{\text{fix}} \underbrace{\frac{\partial A_t}{\partial r_t}}_{<0} \underbrace{\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}} A_t}\right)\right)}_{>0}$$

In general the exchange option price with a fix strike is decreasing with increasing interest rate, if it holds

$$\frac{\partial K_t^{\text{fix}}}{\partial r_t} \mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}} A_t}\right)\right) < \alpha^{\text{fix}} \frac{\partial A_t}{\partial r_t} \mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}} A_t}\right)\right)$$

Thus, depending on the investment strategy (and the option's moneyness), we may observe a option price which is increasing, decreasing, or immune against interest rate changes. Formally, there exists a critical level l^* such that, for $-\frac{\partial A_t}{\partial r_t} < l^*$ (sufficient low interest rate duration of the asset side) it holds

$$\frac{\partial C(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t)}{\partial r_t} > 0.$$

The critical level is defined by

$$l^* = -\frac{\partial K_t^{\text{fix}}}{\partial r_t} \frac{\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}} A_t}\right)\right)}{\alpha^{\text{fix}} \mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}} A_t}\right)\right)} \tag{2.19}$$

Contrary to this the exchange option price with a floating strike is always increasing with increasing interest rate, i.e.

$$\frac{\partial C(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t)}{\partial r_t} = \underbrace{\frac{\partial K_t^{\text{fl.}}}{\partial r_t}}_{>0} \underbrace{\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}} A_t}\right)\right)}_{>0} - \alpha^{\text{fl.}} \underbrace{\frac{\partial A_t}{\partial r_t}}_{<0} \underbrace{\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}} A_t}\right)\right)}_{>0} > 0$$

Thus, if the directional effects are opposing each other, a suitable number of fix and floating strike guarantee products can (locally) immunize the interest rate sensitivity on the liability side. However, from the perspective of the asset liability management, it is merely important to immunize the difference of the asset and the liability value against interest rate changes. We call this a natural ALM hedge in fix and floating strike guarantee products.

Such a natural hedge is meaningful if the duration of the asset side is low compared to the duration implied by a full investment in the bond maturing at the terminal contract date T. In particular, this is the situation faced by the life insurance industry. In particular, assuming that the interest rate duration of the asset side is rather low we can determine the optimal fraction η^*_{fix} of fix strike guarantees such that the buffer value is immunized against a change in the interest rate, i.e. such that

$$\frac{\partial(A_t - L_t)}{\partial r_t} = 0.$$

There are a few comments worth mentioning. Basically, one can also obtain an immunization of the buffer by adjusting the duration of the asset side. Normally, the duration of the asset side of a life insurer is much lower than the one of the liability side. However, this may turn out difficult because of at least two reasons. For one, it is not necessarily possible to trade in (liquid) bonds which are consistent with the long maturities of life insurance contracts. In addition, any modification of the investment strategy (asset side) also impacts the value of the guarantees (liability side). In contrast, a natural ALM hedge which is constructed by a product mix (on the liability side) has no impact on the asset side.

In the following section, we illustrate and quantify the results by means of a two factor model.

2.4 Illustration – two factor model

The main focus of the following subsection is an illustration of the possibilities to obtain an interest rate immunization by introducing floating strike guarantees in addition to fix strike guarantees.

2.4.1 Two factor model ($d = 2$)

For the numerical illustrations we assume that the asset price dynamics S is lognormal with a constant volatility σ, and the interest rate dynamics are given along the lines of the (extended) Vasicek model, i.e. we place ourselves in a two dimensional Gaussian setup along the lines of Eqn. (2.2) and (2.3) where

$$\sigma_S = \begin{pmatrix} \rho\sigma \\ \sqrt{1-\rho^2}\sigma \end{pmatrix}, \quad \sigma_T(t) = \begin{pmatrix} \overline{\sigma}_T(t) \\ 0 \end{pmatrix} \text{ where } \overline{\sigma}_T(t) = \frac{\sigma_r}{a}(1 - e^{-a(T-t)}).$$

Alternatively, we can formulate this model setup by means of the asset and Vasicek interest rate dynamics (and in terms of two independent Brownian motions) where under the risk neutral probability it holds

$$\frac{dS_t}{S_t} = r_t\,dt + \rho\sigma\,dW_t^{(1)} + \sqrt{1-\rho^2}\sigma\,dW_t^{(2)} \tag{2.20}$$

$$dr_t = a(\theta(t) - r_t)\,dt + \sigma_r\,dW_t^{(1)}. \tag{2.21}$$

$W^{(1)}$ and $W^{(2)}$ denote two independent Brownian motions under the risk neutral measure $\mathbb{P}^*$. In addition, the volatility σ of the stock is a constant, σ_r is also a constant. a denotes the *speed of mean reversion* of the Ornstein Uhlenbeck process driving the interest rate dynamics with mean reversion level $\theta(t)$ where $\theta(t)$ is a deterministic function of the extended Vasicek model. In the special case of the Vasicek model it is constant, i.e. $\theta(t) = b$. In particular, the above mentioned interest rate dynamics implies the following bond price dynamics

$$\frac{dB(t,\bar{t})}{B(t,\bar{t})} = r_t\,dt - \sigma_r\mathcal{B}(t,\bar{t})\,dW_t^{(1)} \text{ where } \mathcal{B}(t,\bar{t}) = \frac{1}{a}\left(1 - e^{-a(\bar{t}-t)}\right) \tag{2.22}$$

implying (cf. e.g. Brigo and Mercurio (2006))

$$B(t,\bar{t}) = e^{-\mathcal{B}(t,\bar{t})r_t + \mathcal{A}(t,\bar{t})} \tag{2.23}$$

$$\text{where } \mathcal{A}(t,\bar{t}) = (\mathcal{B}(t,\bar{t}) - (\bar{t} - t))\left(b - \frac{\sigma_r^2}{2a^2}\right) - \frac{\sigma_r^2}{4a}\mathcal{B}^2(t,\bar{t}). \tag{2.24}$$

Notice that the interest rate model is complete in two bonds. In particular, this implies that any duration of the asset side (cf. Eqn. (2.13)) can be resembled by an investment strategy which only refers to two bonds. Thus, we simplify the exposition by referring to investment strategies including the stock and two bonds with different maturities.

Let $\pi = (\pi_S, \pi_1, \pi_2)$ with $(\pi_S + \pi_1 + \pi_2 = 1)$ denotes the (constant) fractions of wealth invested in the asset S and the zero coupon bonds with maturities T_1 and T_2. This implies the insurer's asset process A given by

$$\frac{dA_t}{A_t} = \pi_S\frac{dS_t}{S_t} + \pi_1\frac{dB(t,T_1)}{B(t,T_1)} + (1 - \pi_S - \pi_1)\frac{dB(t,T_2)}{B(t,T_2)}. \tag{2.25}$$

Since all three asset dynamics are given by lognormal processes, the process A is also lognormal. For the pricing purpose, we only need the volatility structure of A and $B(.,T)$, i.e. (in terms of a two dimensional Brownian motion)

$$\sigma_A(t) = \begin{pmatrix} \rho\pi_S\sigma + \pi_1\overline{\sigma}_{T_1}(t) + (1-\pi_S-\pi_1)\overline{\sigma}_{T_2}(t) \\ \sqrt{1-\rho^2}\pi_S\sigma \end{pmatrix}, \quad \sigma_T(t) = \begin{pmatrix} \overline{\sigma}_T(t) \\ 0 \end{pmatrix}$$

where $\overline{\sigma}_T(t) = \frac{\sigma_r}{a}(1 - e^{-a(T-t)})$. W.l.o.g., we restrict ourselves to the case where $T_1 = T$ and a very short bond $T_2 \to t$ (i.e. a cash position) such that

$$\sigma_A(t) = \begin{pmatrix} \rho\pi_S\sigma + \pi_B\overline{\sigma}_T(t) \\ \sqrt{1-\rho^2}\pi_S\sigma \end{pmatrix}, \quad \sigma_T(t) = \begin{pmatrix} \overline{\sigma}_T(t) \\ 0 \end{pmatrix}$$

where $\pi_B = \pi_1$ is the fraction invested in the bond and $1 - \pi_S - \pi_B$ is the fraction invested in the cash position.

Table 2.1: Benchmark parameter setup

Contract			Black Scholes			Vasicek						Portfolio	
α	T	P_0	S_0	σ	μ_S	r_0	a	$\tilde{b}$	b	σ_r	ρ	π_S	π_B
0.9	5	1	1	0.2	0.07	0.0115	0.3	0.042	0.0305	0.015	0.15	0.15	0.4

For simulation we need to change to the real world measure $\mathbb{P}$. Assuming a constant market price of interest rate risk λ_r leads to asset and interest rate dynamics under the real world measure given by

$$\frac{dS_t}{S_t} = \mu_S\, dt + \rho\sigma\, d\tilde{W}_t^{(1)} + \sqrt{1-\rho^2}\sigma\, d\tilde{W}_t^{(2)} \tag{2.26}$$

$$dr_t = a(\tilde{b} - r_t)\, dt + \sigma_r\, d\tilde{W}_t^{(1)}, \tag{2.27}$$

where $\tilde{b} = b + \frac{\lambda_r\sigma_r}{a}$. If not otherwise mentioned, we use the model parameters summarized in Table 2.1. For the market price of interest rate risk we set $\lambda_r = -0.23$.[8]

2.4.2 Illustration of correlation effects

Recall that at time t ($t \leq T$) the guarantee costs depend on the cumulated asset volatilities (respectively forward volatilities) $v(t, T)$. Depending on the investment fractions π_S (stock

[8] Our benchmark parameter setup is consistent with the one used in recent literature (e.g. Hieber et al (2019), Graf et al. (2011)).

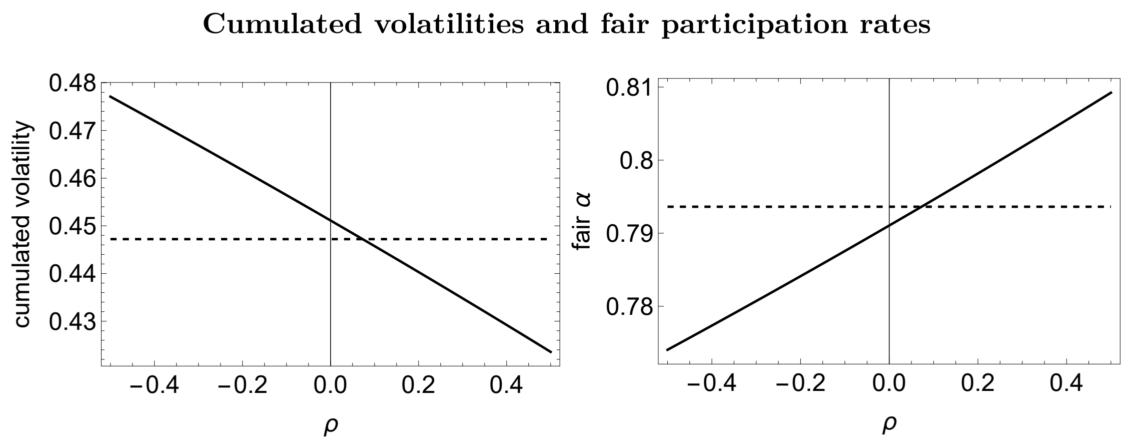

Figure 2.1: For varying asset bond correlation ρ, the figure on the left hand side depicts the cumulated volatility for fix (black line) and floating strike (dotted line) options. The figure on the right hand gives the corresponding fair participation rates. The underlying model parameters are given as summarized in Table 2.1. (We set $K_{\text{fix}} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{\text{fix}}$.

fraction) and π_B (wealth fraction invested in the bond with maturity T), it holds

$$\begin{aligned}
v_{\text{fix}}^2(t,T) &= \int_t^T \|\sigma_A(s) - \sigma_T(s)\|^2 \, ds \\
&= \pi_S^2 \sigma^2 (T-t) + \int_t^T \left(2\rho\pi_S(\pi_B - 1)\sigma\overline{\sigma}_T(s) + (\pi_B - 1)^2 \overline{\sigma}_T^2(s)\right) ds, \\
v_{\text{float.}}^2(t,T) &= \int_t^T \|\sigma_A(s)\|^2 \, ds = \pi_S^2 \sigma^2 (T-t) + \int_t^T \left(2\rho\pi_S\pi_B\sigma\overline{\sigma}_T(s) + \pi_B^2 \overline{\sigma}_T^2(s)\right) ds.
\end{aligned}$$

Assuming that the investment fractions are non-negative (no short positions), the above formulas implies that the directional effects of the bond position π_B on the volatilities (cumulated volatilities) of fix and floating strike guarantees are different, i.e. for a given asset fraction π_S. While the (cumulated) volatility σ_A needed for pricing the floating strike guarantee contract is increasing in the bond fraction π_B, this is not necessarily true for the forward volatility $\sigma_{A,T}$ needed for pricing the fix strike guarantee. Notice that the difference of fix and floating (cumulated) volatilities is determined by

$$\begin{aligned}
&\int_t^T \|\sigma_A(s) - \sigma_T(s)\|^2 \, ds - \int_t^T \|\sigma_A(s)\|^2 \, ds \\
&= -\, 2\rho\pi_s\sigma \int_t^T \overline{\sigma}_T(s) \, ds + (1 - 2\pi_B)\sigma \int_t^T \overline{\sigma}_T^2(s) \, ds,
\end{aligned}$$

i.e. for a sufficiently high asset bond correlation, the fix strike cumulated volatility is lower than the floating strike cumulated volatility.

Based on the benchmark parameters summarized in Table 2.1, Figure 2.1 illustrates the cumulated volatilities (and fair participation rates) of fix and floating strike guarantees in the special case that $\pi_S = 1$ and $\pi_B = 0$. Here, the cumulated volatility of the floating strike guarantee is constant while the one of the fix strike guarantee is decreasing.[9]

Cumulated volatilities for varying bond fraction

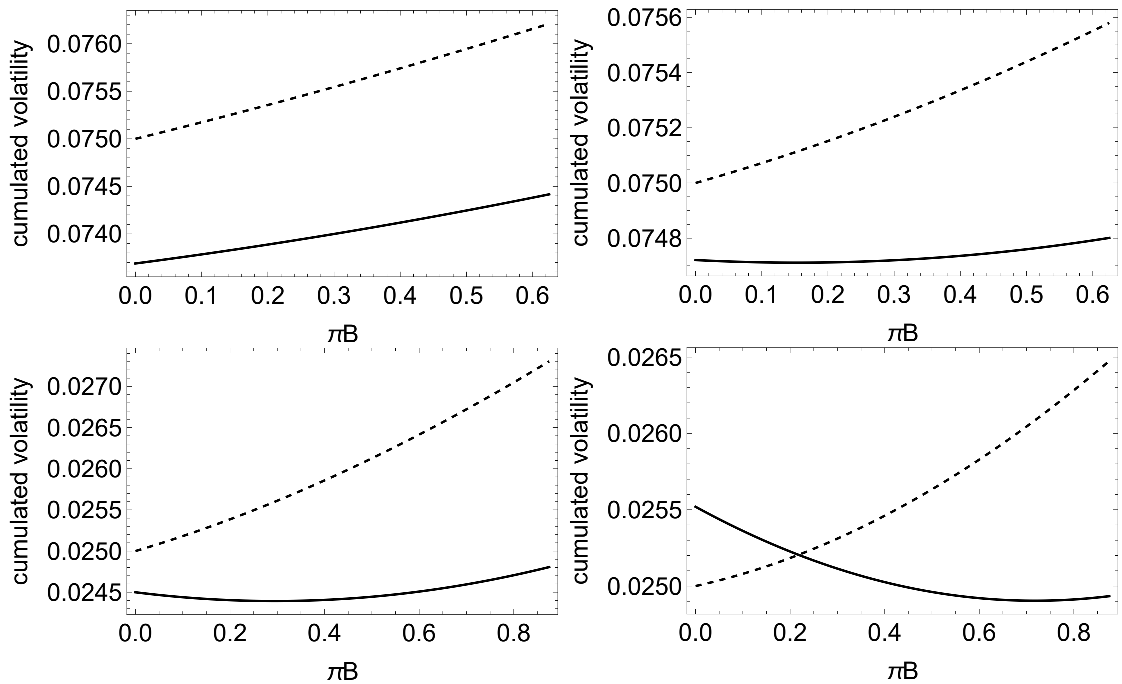

Figure 2.2: For varying bond fractions π_1, the figures depict the cumulated volatility for fix (black lines) and floating strike (dotted lines) guarantees. The upper (lower) figures refer to a stock fraction $\pi_S = 0.375$ ($\pi_S = 0.125$). The figures on the left (right) hand side are based on an asset bond correlation of $\rho = 0.25$ ($\rho = 0.1$). The underlying model parameters are given as summarized in Table 2.1. Furthermore we set $T = 1$.

[9] In particular, in the special case $\pi_S = 1$ and $\pi_B = 0$ (pure stock investment) it holds (cf. Appendix B.2)

$$\begin{aligned} v^2_{\text{fix}}(t,T) &= \int_t^T \|\sigma_A(s) - \sigma_T(s)\|^2 \, ds \\ &= \left(\sigma^2 - 2\rho\sigma\frac{\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right)(T-t) + 2\left(\rho\sigma\frac{\sigma_r}{a^2} - \frac{\sigma_r^2}{a^3}\right)\left(1 - e^{-a(T-t)}\right) + \frac{\sigma_r^2}{2a^3}\left(1 - e^{-2a(T-t)}\right), \\ v^2_{\text{float.}}(t,T) &= \int_t^T \|\sigma_A(s)\|^2 \, ds = \sigma^2(T-t). \end{aligned}$$

An illustration of the (cumulated option) volatility effects w.r.t. the asset fraction π_S, π_B and the asset bond correlation ρ is given by Figure 2.2. Along the lines of the above reasoning, for a given asset fraction, the cumulated volatilities embedded in the pricing of the floating strike guarantee options are increasing in the investment fraction for the long bond (cf. dashed lines). However, the effect may be reversed in the case of the fix strike guarantee options.

Cumulated volatilities depending on various parameters

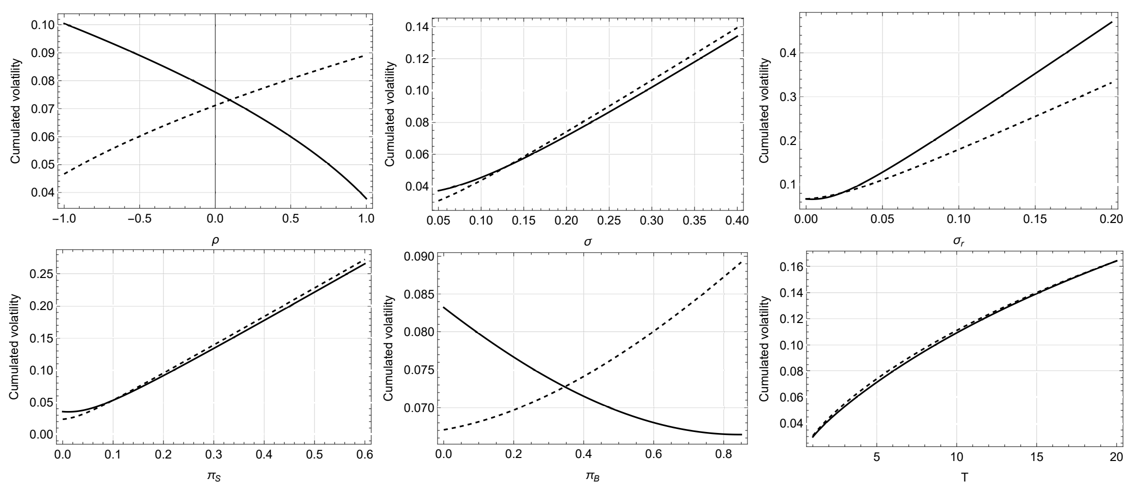

Figure 2.3: This Illustration shows the effect of different parameters on the cumulated volatility of fix (solid lines) and floating (dotted lines) strike guarantees. The upper row refers to the correlation ρ (left), stock volatility σ (mid) and interest rate volatility σ_r (right), while the lower row refers to the stock fraction π_S (left), bond fraction π_B (mid) and time to maturity T (right). The underlying model parameters are given as summarized in Table 2.1. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

A further analysis of the cumulated volatility is given in Figure 2.3, which illustrates the cumulated volatility of fix and floating strike guarantees in dependence of the correlation ρ, stock volatility σ, interest rate volatility σ_r, stock fraction π_S, bond fraction π_B and time to maturity T. It shows that the cumulated volatilities have opposing affects on the correlation ρ and the bond fraction π_B and otherwise same effects.

2.4.3 Illustration price and strikes

As mentioned above, the guarantee costs depend on the cumulated asset volatilities. After taking a further look on the cumulated volatilities we now take deeper look at the guarantee prices and examine the influence of the various parameters on them. Recall, that the contract prices are given in Proposition 2.2.2. Besides the cumulated volatility, the contract price also depends on the strike. Figure 2.4 illustrates the $K_{\text{fix}}^{\text{fair}}$ and $\tilde{\alpha}^{\text{fair}}$ for varying correlation ρ, interest rate volatility σ_r and time to maturity T.

At first glance, one can see that the correlation ρ have the same effect on both strikes, where $K_{\text{fix}}^{\text{fair}}$ is greater then $\tilde{\alpha}^{\text{fair}}$. For low interest rate volatility the $K_{\text{fix}}^{\text{fair}}$ is always greater then $\tilde{\alpha}^{\text{fair}}$ and for higher interest rate volatility otherwise. Moreover, Figure 2.4 shows that $K_{\text{fix}}^{\text{fair}}$ Gatzert and Wesker (2012) increases with a higher time to maturity T, while $\tilde{\alpha}^{\text{fair}}$ slightly decreases. An explanation for this distinction is given by the following.

K_{fix} is the terminal value of the fix strike, i.e. the value of the strike process

$$K_t^{fix} = E[e^{-I(t,T)}K_{fix}|\mathcal{F}_t] = E[e^{-I(t,T)}|\mathcal{F}_t]K_{fix} = B(t,T)K_{fix}$$

at time T, where $B(t,T)$ is the value of a Zero-Coupon-Bond with maturity T. At inception the strike value is $K_0^{fix} = B(0,T)K_{fix}$.

K_{float} is the terminal value of the float strike, i.e. the value of the strike process

$$K_t^{float} = E[e^{-I(t,T)}\tilde{\alpha}e^{I(0,T)}|\mathcal{F}_t] = E[e^{I(0,t)}|\mathcal{F}_t]\tilde{\alpha} = e^{I(0,t)}\tilde{\alpha}$$

at time T. At inception the strike value is $K_0^{float} = \tilde{\alpha}$. So the floating strike can be interpreted as investing the amount $\tilde{\alpha}$ on a bank account with continuously compounded interest rate $\frac{I(0,T)}{T}$.

To sum up $K_{fix} = K_T^{fix}$ and $\tilde{\alpha} = K_0^{float}$ so the difference is that for the fix strike guarantee the terminal guarantee is known at inception while for the floating strike guarantee the terminal guarantee depends on the interest rate dynamics.

Now we can finally examine the contract prices. To do so, Figure 2.5 illustrates the impact of the various parameters on the contract prices. Under investigation is the correlation ρ, interest rate volatility σ_r, stock fraction π_S, bond fraction π_B, the associated strikes and the time to maturity T.

First, it shows that the correlation has opposing effects, where the the fix strike guarantee is decreasing and the floating strike guarantee is increasing with increasing correlation. Both prices increasing with the interest rate volatility, whereby the fixed strike guarantee increases more steeply.

Furthermore the illustration shows, that the floating strike guarantee is higher than the fix strike guarantee, when the same strike is assumed. In addition, a lower floating

Fair strikes depending on various parameters

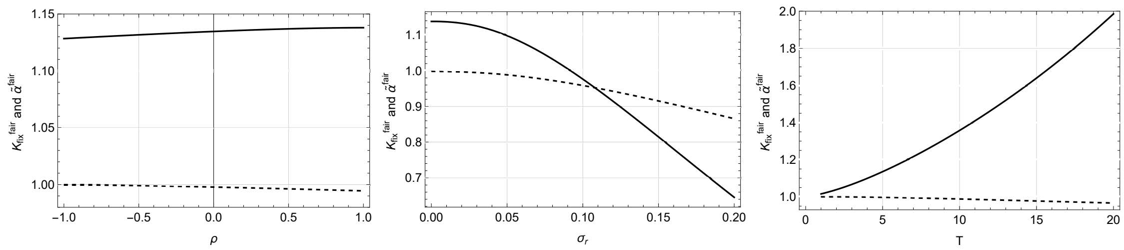

Figure 2.4: This Illustration shows the effect of different parameters on $K_{\text{fix}}^{\text{fair}}$ (solid lines) and $\tilde{\alpha}^{\text{fair}}$ (dotted lines). Figures refers to the correlation ρ (left), interest rate volatility σ_r (mid) and time to maturity T (left). The underlying model parameters are given as summarized in Table 2.1. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

strike is necessary for the guarantee price to be fair, then for the fix strike, such that the fair floating strike is lower than the fair fix strike, as stated above.

Moreover Figure 2.5 illustrates the effect of a increasing maturity on the guarantee prices. The guarantee prices shows an opposing behavior on the time to maturity. The floating strike guarantee is increasing with maturity while the fix strike guarantee is decreasing. The price of a floating strike guarantee is higher for longer maturities with $T > 5$ and lower for short maturities $T < 5$ than the price of a fix strike guarantee.

2.4.4 Interest rate sensitivity and convexity

In the following section we are going to investigate the interest rate sensitivity and convexity of the exchange option of the fix and floating strike guarantee as well as of the asset portfolio. Recall that the interest rate sensitivities of the asset portfolio are derived in Section 2.3.2 and given by

$$\frac{\partial A_t}{\partial r_t} = -A_t \pi_B \mathcal{B}(t,T)$$
$$\frac{\partial^2 A_t}{(\partial r_t)^2} = A_t \left(\pi_B \mathcal{B}(t,T)\right)^2 .$$

Moreover recall, that the interest rate sensitivities of the exchange option are stated in Proposition 2.3.1. As stated above, a natural hedge against interest rate risk is possible, if interest rate duration of the asset side is sufficient low, i.e. if $-\frac{\partial A_t}{\partial r_t} < l^*$ where the critical level is defined by Eqn. (2.19). We are now going to investigate if this condition is fulfilled for a model setup given by Table 2.1. Therefore, Figure 2.6 illustrates the relationship

Guarantee prices depending on various parameters

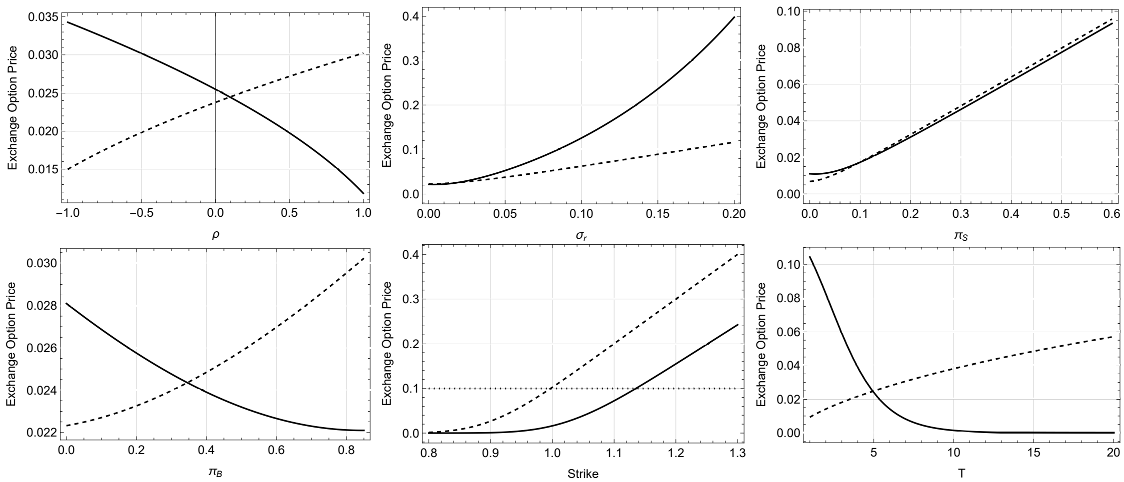

Figure 2.5: This Illustration shows the effect of different parameters on the fix (solid lines) and floating (dotted lines) strike guarantee prices. The upper row refers to the correlation ρ (left), interest rate volatility σ_r (mid) and stock fraction π_S (right) while the lower row refers to the bond fraction π_B (left), the strikes (mid) and time to maturity T (right). The underlying model parameters are given as summarized in Table 2.1. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

between the duration of the asset side given by $-\frac{\partial A_t}{\partial r_t}$ and the critical level l^*. The Figure shows that the condition is always fulfilled for different levels of stock fractions π_S, bond fractions π_B and time to maturities T. Moreover, the figure shows that the duration of the asset portfolio is independent of the stock fraction, which already follows from the above stated formula, as the stocks are independent of the interest rate. In addition, the duration of the asset portfolio increases with the bond fraction and the time to maturity.

Guarantee prices for various parameters

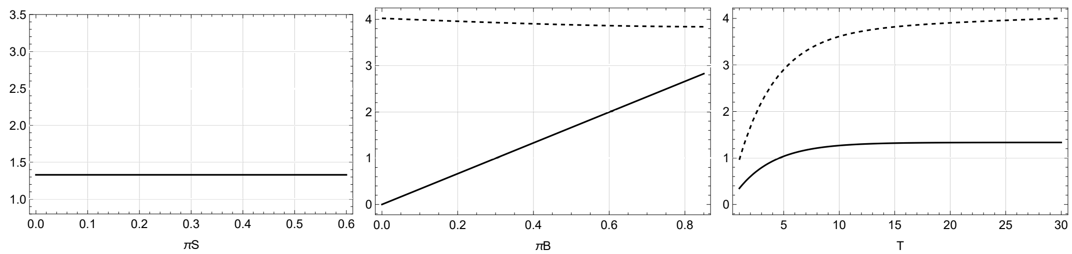

Figure 2.6: This Illustration shows the relationship between the duration of the asset side given by $-\frac{\partial A_t}{\partial r_t}$ (solid line) and the critical level l^* given by (2.19) (dashed line) for various stock fractions π_S, bond fractions π_B and time to maturities T. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

Interest rate sensitivity of asset portfolio and exchange options

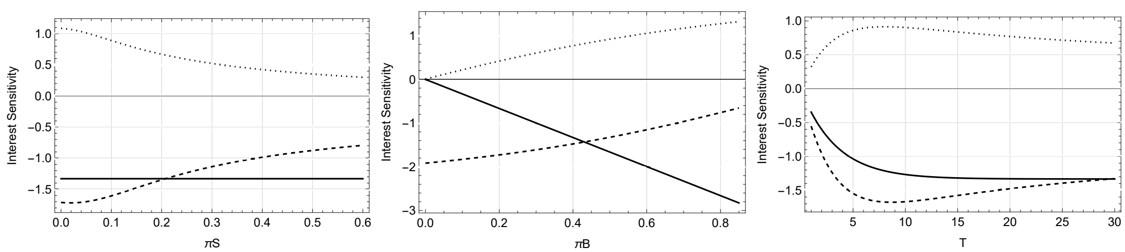

Figure 2.7: This Illustration shows the relationship between the duration of the asset side given by $-\frac{\partial A_t}{\partial r_t}$ (solid line) and the critical level l^* given by (2.19) (dashed line) for various stock fractions π_S, bond fractions π_B and time to maturities T. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

Next, we look at the interest rate sensitivity of the exchange options. Figure 2.7 therefore illustrates the interest rate sensitivity of the asset portfolio (solid line), the exchange option of the fix strike guarantee (dashed line) and of the floating strike guarantee (dotted line) for varying stock fraction π_S, bond fraction π_B and time to maturity T. First of all, it becomes apparent that the interest rate sensitivity of the floating strike is positive while for the fix strike and the asset portfolio is negative. It is also seen that the interest rate sensitivity of both exchange options increases with bond fraction, while it decreases for the asset portfolio.

To hedge against interest rate risk, the interest rate sensitivity of the asset side must coincide with that of the liability side. For this reason, we now look at the interest rate sensitivity of the buffer value. Recall that the interest rate sensitivities of the buffer value

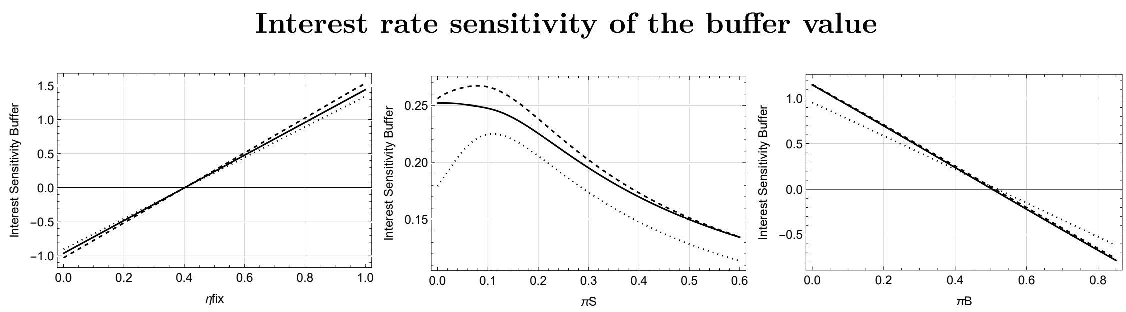

Figure 2.8: This Illustration shows the relationship between the duration of the asset side given by $-\frac{\partial A_t}{\partial r_t}$ (solid line) and the critical level l^* given by (2.19) (dashed line) for various stock fractions π_S, bond fractions π_B and time to maturities T. The solid line refers to short term $T = 5$, the dashed line to mid term $T = 10$ and the dotted line to long term $T = 20$ contracts. For the last two figures, it is assumed that an equal number of fixed strike and floating strike guarantees are sold, i.e. $\eta_{\text{fix}} = \eta_{\text{fl.}} = 0.5$. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

$A_t - L_t$ is also stated in Proposition 2.3.1. Figure 2.8 presents the interest rate sensitivities of the buffer value for varying fraction of fix strike guarantees sold η_{fix}, stock fraction π_S and bond fraction π_B. The first illustration shows that a natural hedge can be achieved for all maturities with $\eta_{\text{fix}} = 0.4$. The last figure shows that a natural hedge can also be achieved by a suitable bond fraction.

As already mentioned in the second chapter of this thesis, the first derivation only provides a hedge against small changes in the interest rate. To obtain a better hedge against interest rate risk, it is worthwhile to take into account the curvature represented by the second derivative. For this reason, we now also consider the convexity, which is represented by the second derivative according to the interest rate. The convexity of each exchange option and the buffer value is also stated in Proposition 2.3.1. Figure 2.9 presents the convexity of the asset portfolio (solid line), the exchange option of the fix strike guarantee (dashed line) and of the floating strike guarantee (dotted line) for varying stock fraction π_S, bond fraction π_B and time to maturity T. The figures show that the convexity of the fixed strike guarantee is Gatzert and Wesker (2012) larger than that of the floating strike and the asset portfolio.

Finally, we consider the convexity of the buffer value. Figure 2.10 presents the convexity of the buffer value for varying fraction of fix strike guarantees sold η_{fix}, stock fraction π_S and bond fraction π_B. The figures show that a natural hedge due to a convexity matching is not possible. In particular, it is not possible to hedge against interest rate risk simultaneously using interest rate sensitivity and convexity. It is not possible to find a

Convexity of asset portfolio and exchange options

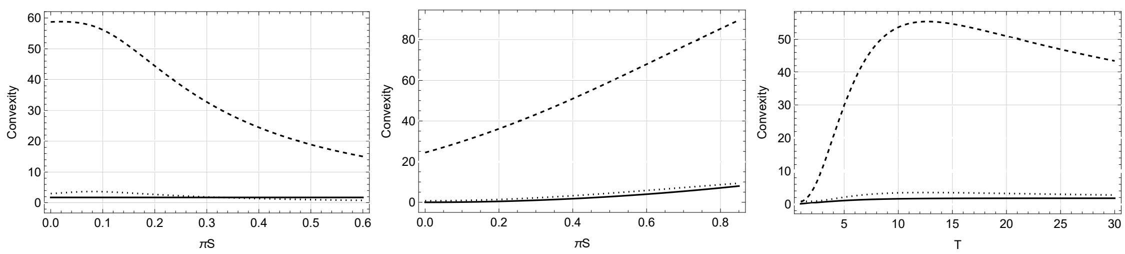

Figure 2.9: This Illustration shows the relationship between the duration of the asset side given by $-\frac{\partial A_t}{\partial r_t}$ (solid line) and the critical level l^* given by (2.19) (dashed line) for various stock fractions π_S, bond fractions π_B and time to maturities T. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

Convexity of the buffer

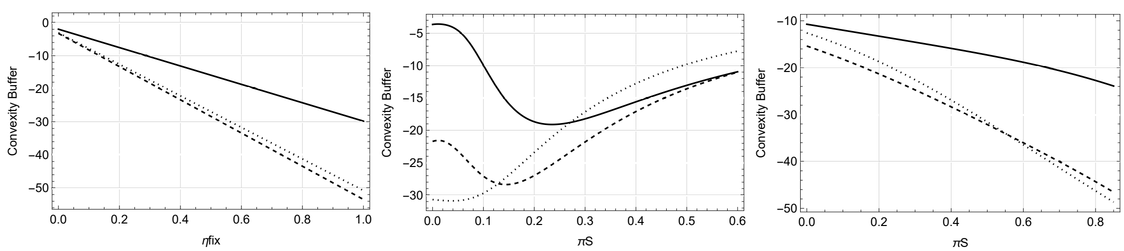

Figure 2.10: This Illustration shows the relationship between the duration of the asset side given by $-\frac{\partial A_t}{\partial r_t}$ (solid line) and the critical level l^* given by (2.19) (dashed line) for various stock fractions π_S, bond fractions π_B and time to maturities T. The solid line refers to short term $T = 5$, the dashed line to mid term $T = 10$ and the dotted line to long term $T = 20$ contracts. For the last two figures, it is assumed that an equal number of fixed strike and floating strike guarantees are sold, i.e. $\eta_{\text{fix}} = \eta_{\text{fl.}} = 0.5$. Furthermore we set $K_{fix} = e^{0.02}$ and $\tilde{\alpha} = B(0,T)K_{fix}$.

suitable combination of fixed and floating strike guarantees so that both the interest rate sensitivity and the convexity of the buffer value are zero. In the following section, we will restrict ourselves to hedging with the help of interest rate sensitivity. We will determine a suitable combination of fixed and floating strike guarantees so that the interest rate sensitivity of the buffer value is zero and then specify the associated convexity.

2.4.5 Illustration natural hedging

We now illustrate the perspective of the asset liability management. The main focus is on the possibility to obtain a natural hedge by means of a suitable mix of fix and floating strike

Table 2.2: Natural hedging

T	α	π_S	π_B	$K_{\text{fix}}^{\text{fair}}$	$\tilde{\alpha}^{\text{fair}}$	$\frac{\partial A_t}{\partial r_t}$	$\frac{\partial C_{\text{fix}}}{\partial r_t}$	$\frac{\partial C_{\text{float.}}}{\partial r_t}$	η_{fix}^*	$\frac{\partial A_t - L_t}{\partial r_t^2}$
5	0.9	0.15	0.	1.1331	0.9983	0.	-2.3088	0.0108	0.0047	-1.0330
5	0.9	0.15	0.2	1.1345	0.9980	-0.1757	-2.2194	0.1572	0.0736	-2.4691
5	0.9	0.15	0.4	1.1354	0.9973	-0.3515	-2.1123	0.2988	0.1385	-3.9745
5	0.9	0.15	0.6	1.1359	0.9963	-0.5272	-1.9882	0.4329	0.2006	-5.5759
5	0.9	0.15	0.8	1.1362	0.9949	-0.7030	-1.8492	0.5576	0.2609	-7.4074
10	0.9	0.15	0.	1.3425	0.9930	0.	-2.3179	0.0098	0.0042	-0.978
10	0.9	0.15	0.2	1.3511	0.9915	-0.1457	-2.3510	0.1170	0.0533	-2.5973
10	0.9	0.15	0.4	1.3574	0.9883	-0.2914	-2.3621	0.2149	0.0947	-4.2869
10	0.9	0.15	0.6	1.3614	0.9835	-0.4371	-2.3388	0.3004	0.1304	-5.9642
10	0.9	0.15	0.8	1.3632	0.9769	-0.5828	-2.2714	0.3728	0.1630	-7.6736
20	0.9	0.15	0.	1.9247	0.9799	0.	-1.9174	0.0085	0.0044	-0.8488
20	0.9	0.15	0.2	1.9585	0.9754	-0.0974	-2.0256	0.0683	0.0373	-1.7645
20	0.9	0.15	0.4	1.9852	0.9665	-0.1947	-2.1281	0.1195	0.0618	-2.7044
20	0.9	0.15	0.6	2.0036	0.9535	-0.2921	-2.2033	0.1604	0.0802	-3.6047
20	0.9	0.15	0.8	2.0128	0.9370	-0.3895	-2.2207	0.1923	0.0958	-4.4428

guarantees. To simplify the exposition, we normalize the current asset value to one, i.e. we set $A_t = 1$. In reality, the duration (minus times the interest rate sensitivity) of the asset side of an insurance company is lower than the duration of the liability side. Basically, the insurance company can obtain a higher duration of the asset side by changing its investment strategy towards investments in bonds with higher time to maturity. However, any modification of the investment strategy has also an impact on the liability side.

Recall that, in the two factor model setup, the interest rate duration can be resembled by the duration of an investment in two bonds only, in particular a long-term bond (maturing at the guarantee contract horizon T) and a very short term bond (i.e. cash position). In addition, we consider an investment in stocks. Notice that (for a given fraction of stock investment π_S), an increase of the investment fraction π_B of the long-term bond is the same as a reduction of the cash position $1 - \pi_S - \pi_B$. Intuitively, it is clear that the value of a fix (floating) strike guarantee is increasing (decreasing) in the fraction π_B of the long-term bond. While the fix strike guarantee can be honored almost surely for a sufficiently large investment in the long-term bond (in this case, the guarantee put is worthless), a higher cash position (accumulation according to the short term interest rate) decreases the value of the floating strike guarantee put. Thus, the directional effect

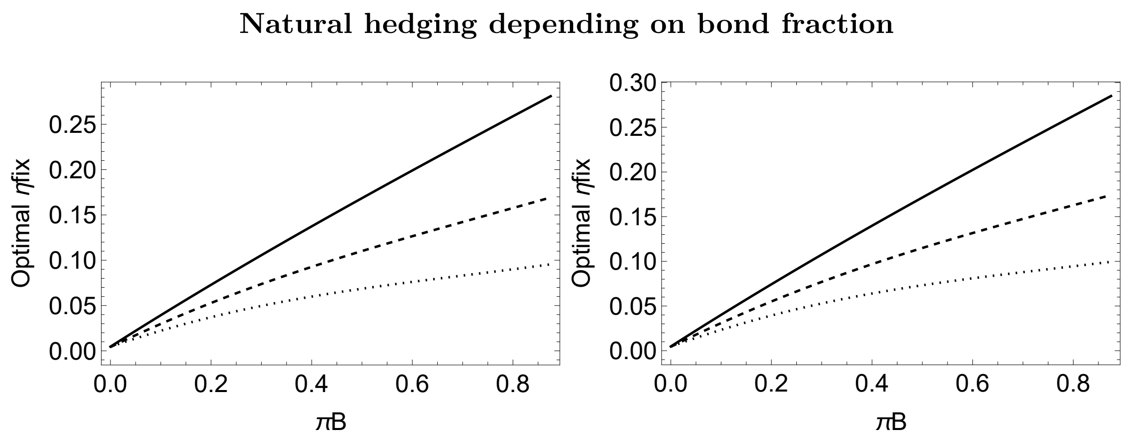

Figure 2.11: For varying bond fractions π_B, the figures depict the number of floating strike guarantees per fix strike guarantee. The black lines refer to a time to maturity $T = 5$, the dashed black lines to $T = 10$, and the dotted gray lines to $T = 20$. The stock fraction is equal to $\pi_S = 0.15$. The figure on the left (right) hand side are based on an asset bond correlation of $\rho = 0.25$ ($\rho = 0.1$). Otherwise, the underlying model parameters are given as summarized in Table 2.1.

of varying the investment fraction π_B has an opposing effect w.r.t. fix and floating strike guarantees.

In order to make the guarantee contracts comparable, we consider only contracts which are equal in their value. In Table 2.2, we fix the varying long-term bond investment fractions π_B and the participation rate $\alpha = 0.9$ and determine the fix strike K_{fix} (column five) and the floating strike accumulation factor $\tilde{\alpha}$ (column six) such that the contract value is equal to one. Along the lines of the above reasoning, observe that K_{fix}^{fair} ($\tilde{\alpha}^{fair}$) is increasing (decreasing) in the long-term investment fraction π_B, i.e. increasing π_B reduces (increases) the value of a fix (floating) strike guarantee which is compensated by increasing (decreasing) the strike. We then summarize the interest rate sensitivities of the asset side (column seven), the fair fix strike guarantee (column eight), and the floating strike guarantee (column nine) for different investment strategies (in terms of π_S and π_B). Observe that the interest rate duration (minus times the sensitivity) of the asset side is increasing in the long-term investment fraction π_B. The same is true for the (fairly priced) fix and floating strike guarantee contract. However, as long as the duration of the asset side is not too large, the sign of the interest rate duration of the floating strike contract is negative while the one of the fix strike guarantee is positive, i.e., ceteris paribus, writing floating strike guarantee contracts can reduce the duration of the liability side (and thus immunize the duration of the asset side). Then, the fraction of fix strike guarantees is stated in the penultimate column. A graphical illustration of this number is given in Figure 2.11. Finally, the last column states the convexity of the buffer immunized by the interest rate sensitivity.

To sum up, natural hedging with floating strike guarantees is possible if the duration of the asset side is rather low (consistent with the situation of participating life insurance). While switching to a higher duration on the asset side (increasing the money spent in long-term bonds, if possible) has an impact on the liability side at the same time, the reduction of the duration of the liability side can be obtained without impact on the asset duration.

2.5 Value at Risk

We now assess the natural hedging strategies using the 1 year value at risk in line with Solvency II. Therefore we generate 100,000 paths of short term interest rates using Equation (2.21) and investment portfolios using Equation (2.25) simultaneously based on the benchmark parameters summarized in Table 2.1. Prices of zero-coupon bonds are determined for each path using Equation (2.23). For given investment fractions π_S and π_B and participation rate $\alpha = 0.9$ the fix strike K_{fix} and the floating strike accumulation factor $\tilde{\alpha}$ are determined such that the contract value is fair. The values of the guarantees at time $t = 1$ are calculated as given in Proposition 2.2.2. For all simulation paths we then determine the buffer value after one year $B_1 = A_1 - L_1$ using Equation (2.11). In addition to the 1 year value at risk of the buffer value, we also calculate the conditional value at risk and the variance of the buffer value at time $t = 1$. Solvency II requires insurers to hold sufficient capital such that the probability of default is equal to 0.5%. Default occurs if the buffer value becomes negative, such the the 1 year value at risk and the conditional value at risk with confident level β is given by

$$
\begin{aligned}
VaR_\beta^{1yr} &= \underset{x}{\operatorname{argmin}} \left\{ \mathbb{P}\left(B_1 < x\right) = 1 - \beta \right\} \\
CVaR_\beta^{1yr} &= \mathbb{E}\left[B_1 | B_1 < VaR_\beta\right]
\end{aligned}
$$

To meet the capital requirements provided by Solvency II we consider a 1 year value at risk with confidence level 0.995. Furthermore, we choose a confident level of 0.985 for the conditional value at risk. First, we examine the impact of the distribution of fixed and floating strike guarantees on the risk measures. Therefore Figure 2.12 presents the different risk measures for varying levels of η_{fix}. Moreover a comparison of the 1 year value and the value at maturity T of each risk measure is made. The Figure shows, that

Risk measures for varying fraction of fix strike guarantees

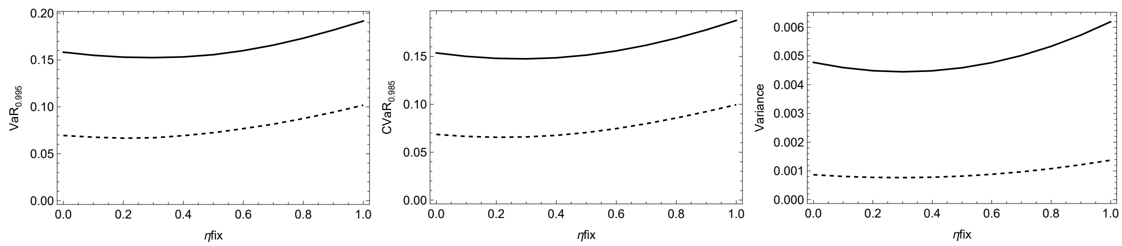

Figure 2.12: This Figure shows the Value ar Risk with confident level 0.995, the conditional Value at Risk with confident level 0.985 and the Variance of the Buffer for varying fraction of fix strike guarantees η_{fix}. The solid line refers to the value at maturity $T = 5$ and the dashed line to the value at time $t = 1$.

the 1 year value is always below the T year value. Furthermore the figure shows that a suitable combination of fix and floating strike guarantees can minimize the corresponding risk measure. The exact value as well as the values for different maturities can be found in the Appendix of this thesis.

Table 2.3 shows the results of the simulation by supplementing the previous results by the 1 year value at risk for a level of 0.995, the 1 year conditional value at risk with confidence level 0.985 and the 1 year variance of the buffer. We use a lower confidence level (98.5%) for 1 year conditional value at risk compared to the 1 year value at risk (99.5%) so that both values are comparable.

Table 2.3: Risk measures induced by natural hedging

T	α	π_S	π_B	$K_{\text{fix}}^{\text{fair}}$	$\tilde{\alpha}^{\text{fair}}$	η_{fix}^*	$\frac{\partial A_t - L_t}{\partial r_t^2}$	$VaR_{0.995}^{1yr}$	$CVaR_{0.985}^{1yr}$	$Variance$
5	0.9	0.15	0.	1.1331	0.9983	0.0047	-1.0330	0.0708	0.0693	0.00089
5	0.9	0.15	0.2	1.1345	0.9980	0.0736	-2.4691	0.0690	0.0674	0.00084
5	0.9	0.15	0.4	1.1354	0.9973	0.1385	-3.9745	0.0675	0.0659	0.00079
5	0.9	0.15	0.6	1.1359	0.9963	0.2006	-5.5759	0.0664	0.0648	0.00075
5	0.9	0.15	0.8	1.1362	0.9949	0.2609	-7.4074	0.0652	0.0640	0.00073
10	0.9	0.15	0.	1.3425	0.9930	0.0042	-0.9780	0.0692	0.0681	0.00080
10	0.9	0.15	0.2	1.3511	0.9915	0.0533	-2.5973	0.0661	0.0650	0.00072
10	0.9	0.15	0.4	1.3574	0.9883	0.0947	-4.2869	0.0631	0.0617	0.00065
10	0.9	0.15	0.6	1.3614	0.9835	0.1304	-5.9642	0.0598	0.0583	0.00058
10	0.9	0.15	0.8	1.3632	0.9769	0.1630	-7.6736	0.0564	0.0549	0.00052
20	0.9	0.15	0.	1.9247	0.9799	0.0044	-0.8488	0.0637	0.0623	0.00062
20	0.9	0.15	0.2	1.9585	0.9754	0.0373	-1.7645	0.0602	0.0586	0.00056
20	0.9	0.15	0.4	1.9852	0.9665	0.0618	-2.7044	0.0556	0.0542	0.00048
20	0.9	0.15	0.6	2.0036	0.9535	0.0802	-3.6047	0.0508	0.0495	0.00041
20	0.9	0.15	0.8	2.0128	0.9370	0.0958	-4.4428	0.0463	0.0451	0.00035

A graphical illustration of this result is given in Figure 2.13. For varying investment fractions in the corresponding long-term bond ($\pi_B = 0, 0.2, 0.4, 0.6, 0.8$), the fair initial strike of a fixed strike $K_{\text{fix}}^{\text{fair}}$ guarantee, the fair accumulation rate of a floating strike option $\tilde{\alpha}^{\text{fair}}$, the fraction of fix strike guarantee contracts which immunizes the buffer $A_t - L_t$ against changes in the interest rate and the resulting convexity are adopted from Table 2.2. Additionally the 1 year value at risk with confidence level 0.995, the 1 year conditional value at risk with confidence level 0.985 and the 1 year variance of the buffer are stated in the last three columns. Observe that all three risk measures are decreasing in the long term investment fraction π_B. The Figure illustrates the reduction in the risk for different maturities. The results can also contribute to the debate on the comparison of value at risk and conditional value at risk. As noticeable the conditional value at risk with confident level 0.985 is slightly below the value at risk with a higher confident level of 0.995. Furthermore, the results show that variance leads to a significant underestimation of risk compared to the other risk measures.

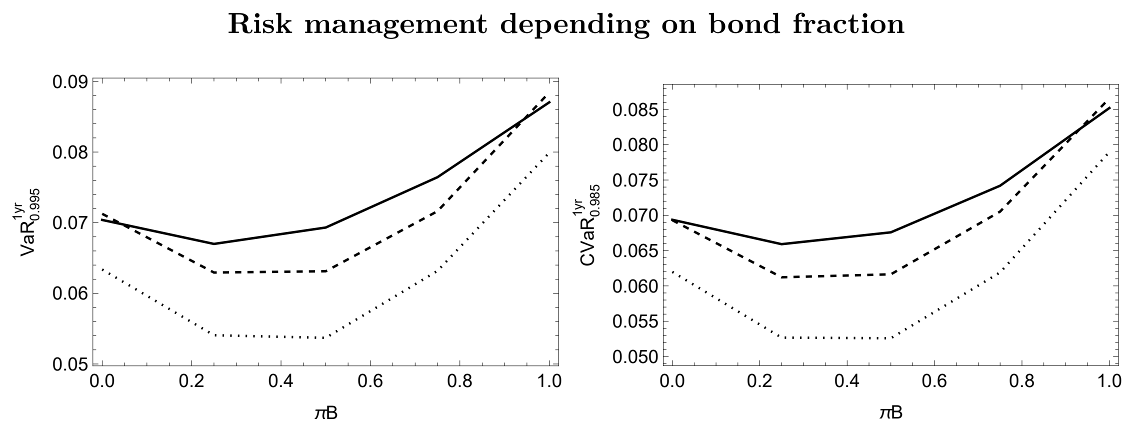

Figure 2.13: For varying bond fractions π_B, the figures depict the 1 year value at risk with confidence level 0.995 on the left and the 1 year conditional value at risk with confidence level 0.985 on the right. The black lines refer to a time to maturity $T = 5$, the dashed black lines to $T = 10$, and the dotted gray lines to $T = 20$. The stock fraction is equal to $\pi_S = 0.15$. Otherwise, the underlying model parameters are given as summarized in Table 2.1.

2.6 Conclusion

Low interest rate scenarios and (downward) changes in the term structure of interest rates may deteriorate the solvency situation of a life insurance company. In this paper, we focus on the possibilities to build a natural hedge against interest rate risk by offering a suitable product mix of different minimum return rate guarantee schemes (MRRGs).

We analyze two versions of MRRGs which are meaningful in the context of participating life insurance contracts. One version is implied by a guaranteed rate which is once determined at the contract inception (fix strike guarantee). The other version is a (stochastic) guarantee rate which is implied by the interest rate accumulation over the contract horizon (floating strike guarantee). We use a Gaussian model setup which gives rise to closed-form (market consistent) values of the liabilities. This is especially convenient to obtain insights in the price sensitivities.

We then propose a natural hedge against changes in the term structure of interest rates. The natural hedge is based on the coexistence of fix and floating strike guarantee products. Normally, the duration of the asset side of a life insurer is much lower than the one of the liability side. In this case, we show that selling floating strike guarantee products reduces the (stochastic) duration of the liability side such that it is possible to obtain an immunization.

Compared to an immunization of the buffer (difference of assets and liabilities) by adjusting the duration of the asset side, the natural hedge has the following advantage. Reducing the duration of the asset side may be difficult because it is not necessarily possible to trade in (liquid) bonds which are consistent with the long maturities of life insurance contracts. In addition, any modification of the investment strategy (asset side) also impacts the value of the guarantees (liability side). In contrast, a natural ALM hedge which is based on a product mix on the liability side has no impact on the asset side. Finally, we illustrate the results by means of a two factor model. Additionally we analyzed the risk profile by considering different risk measures.

Chapter 3

Participating life insurance contracts with minimum return rate guarantees under default risk

3.1 Introduction

This chapter analyzes the optimal design of participating life insurance contracts with minimum return rate guarantees (MRRGs) under default risk and is based on Mahayni et al. (2021).[1] The benefits to the insured are linked to an investment strategy which is conducted by the insurer on the financial market as e.g. observed in participating life insurance contracts. Unless there is a default event, the insured receives the maximum of a guaranteed rate and a participation in the investment returns. An optimal contract design implies the highest expected utility to the insured. The focus is on MRRGs which are fairly priced (pricing by no arbitrage condition) and satisfy regulatory requirements posed on the probability that the guarantees are violated (quantile MRRGs).

It is worth mentioning that we merely focus on a savings plan which is motivated by participating life insurance contracts. In reality, these contracts are much more complicated. They also include a term life insurance component and possess several premium payment options to policyholders. It is often criticized, that the underlying of this kind of

[1] In particular, we refer to annual return rate guarantees which are common in German-speaking countries.

life insurance product is in reality typically based on book values and not market values like it is suggested in most research papers. However, the main effect is, that the underlying possesses a lower volatility (via "smoothing") and - ceteris paribus - the value of the embedded options is lower. In any case, one can in principle account for this effect via choosing the "appropriate" volatility in the GBM - whenever her model is adjusted to empirical data via time series data. For a detailed description of participating life insurance contracts, we refer e.g., to Grosen and Jørgensen (2000) and Grosen and Jørgensen (2002). Additionally to these facts, the insurance companies even smoothen their asset and liability sides in reality to overcome bad financial years with the surplus of good years.[2] Furthermore, we also define the default event exclusively in terms of the investment returns and do not consider that the insurance company may itself default.

Considering the possibility that the liabilities (guarantees) can not be honored impedes the basic idea of a guarantee. However, in reality there is no guarantee prevailing with probability one. Any guarantee may fail in times of extremely negative market conditions, i.e. guarantees are only valid under sufficiently good scenarios. Thus, one may soften the term guarantee and imagine it as honored with a high probability (quantile guarantee). In the context of participating life insurance contracts the guarantee is secured by regulatory requirements on the maximal shortfall probability. For example, Solvency II contains the condition that the shortfall probability w.r.t. a time horizon of one year is limited to 0.5%. Intuitively, it is clear that the value of a guarantee is decreasing in the shortfall probability. Default risk mitigates the guarantee component (it is less often binding and thus the guarantee is cheaper than without default risk). In contrast, control of the shortfall probability makes the guarantee more binding. In summary, the pricing effects due to the impact of default risk are rather obvious. The impacts on the utility to the insured is more ambivalent, unless the insurer implements an optimal investment strategy. Therefore, our main focus is on the optimal contract design in the presence of an upper probability bound on the shortfall probability posed by the regulator, i.e. the optimal design of quantile MRRGs.

We proceed as follows. In the absence of mortality and surrender risk, we discuss the modification of the (return) payoff which arise by introducing default risk referred to a

[2] A paper on this topic is for example Maurer et al. (2016), where a stylized model with payout smoothing is provided and a literature overview of this topic is given. In addition, Kling et al. (2007) shows an example how smoothing can be modeled when analyzing some question related to participating products.

strictly binding guarantee. In a stylized manner, we model the asset side of the insurance company (the contract provider) by means of the value process of an admissible financial market investment strategy, i.e. a self-financing strategy where the initial value is given by the sum of equity and the contributions of the insureds. The liability side, i.e. the benefits to the insured, depends on the guarantee promise as well as on the question how the surpluses, if any, are distributed between the shareholders and the insured. This is modeled by a participation fraction on the investment returns. Considering default risk, the return payoff to the insured also depends on the amount of equity backing up the guarantee. If the intended payoff which is paid in a default free version is not obtained by the investment strategy, the remaining amount is provided by reducing the equity, i.e. unless the equity amount drops to zero.

In summary, the impact of the default risk on the contract pricing is captured by a short position in a default put option. In financial terms, the default put is a compound option (option on an option). The inner option is introduced by the guarantee option of the insured, i.e. arising from the (intended) guarantee. The outer option is implied by the default possibility, i.e. the intended payoff is only honored if the asset/investment performance is sufficiently good. We show that, w.r.t. each annual return payoff, the (return) payoff of the compound option can (for a suitable distinction of the equity to debt ratio compared to a function of the guarantee and participation fraction) be disentangled into a piecewise linear payoff function (of the investment return), i.e. the payoff can be stated in terms of plain vanilla options. The same is true for the liabilities to the insured (Proposition 3.2.5). Closed form solutions for pricing the default put and the insurance contract itself are possible in any financial market model setup which provides closed form solutions for plain vanilla options. Closed-form solutions for the return payoff in the context of no default risk but with mortality risk can be found e.g. in Bacinello (2001).

The Cliquet-style contracts can then be solved in closed form in any model and investment setup which implies independent and identically distributed return increments, at least if one assumes a constant or deterministic equity to debt fraction. Some general implications of considering (i) default risk and (ii) regulatory requirements on the shortfall probability can already be derived in a model free manner such that the results are valid in any arbitrage free model setup. We illustrate and quantify the results in a Black and Scholes model setup. This simple model setup in combination with the assumption that the insured is described by a constant relative risk aversion (CRRA) gives further insights on the utility effects from the perspective of the insured.

Due to the completeness of the model setup and the exclusion of mortality and surrender risk, we can even solve the resulting pure portfolio optimization problem and state the expected utility maximizing return payoff under the quantile condition posed by the regulator, i.e. the upper bound on the shortfall probability (Proposition 3.3.1).[3] In particular, the derivation of the optimal quantile contract is tractable because of the complete market assumption. W.l.o.g., one can analyze the relevant optimization problem without considering equity, i.e. by means of setting the equity to debt fraction to zero. Once the optimal return distribution is computed without equity, the same return payoff distribution can be implemented in the presence of any equity amount held by the insurance company. We compare the optimal quantile MRRG with the unrestricted solution (no shortfall condition posed by the regulator) as well as with solutions which are based on restrictions on the investment strategy implemented by the insurance company. For example, we consider the case that the insurer is restricted to constant mix strategies. Intuitively it is clear that the upper bound on the shortfall probability (if binding) affords some kind of quantile hedge. The resulting optimal payoff is not attainable without some (synthetic) option positions and can not be contained by a fixed sharing rule between equity and debt. We show that the utility loss to the insured arising if the insurer implements a suboptimal investment strategy can be significant.

The contributions of the paper can be summarized as follows. Based on the distinction between a high and a low equity to debt ratio (compared to the combination of guarantee and participation fraction), we state the return payoff to the insured (Proposition 3.2.5) by means of piecewise linear functions of the return of the insurers asset returns. On the one hand, this simplifies the pricing problem under default risk to the pricing of standard call (put) options. On the other hand, this already gives model independent insights, i.e. insights which are true w.r.t. any arbitrage free financial market model setup. For example, a low (high) equity to debt ratio implies a concave (piecewise concave and convex) payoff.[4] Thus, for a low equity to debt ratio, the value of the liabilities is decreasing in the riskiness of the insurer's assets. Consequently, the default risk dominates the guarantee option which contradicts the guarantee concept, i.e. if the admissible asset distributions are not restricted by an upper bound on the shortfall probability (on the

[3] Notice, that in general Solvency requirements and Solvency II in particular lead in fact to restrictions when it comes to optimal asset allocation settings.

[4] In our setup, a low equity to debt ratio is always implied if there is a return guarantee which gives a return accumulation higher (or equal) one.

guarantee). A further contribution is then given by deriving the optimal return payoff distribution to the insured (Proposition 3.3.1). Because of the market completeness, the optimal (return) payoff to the insured can be implemented for any equity to debt ratio. Finally it is important to point out that there are utility losses to the insured (and there is too much equity involved) if the insurer implements a suboptimal investment strategy.

Our paper is related to several strands of the literature including the ones on (i) pricing and hedging embedded guarantees/options, (ii) the impact of default risk (emphasizing on participating life insurance contracts), (iii) utility losses caused by guarantees and/or suboptimal investment decisions (conducted by insurance companies or pension funds), (iv) portfolio planning, (v) quantile hedging, and (vi) the analysis of piecewise convex and concave contingent payoffs. Without postulating completeness we only refer to the most related literature and hint at the additional literature given within the mentioned papers.

Pricing embedded options by no arbitrage already dates back to Brennan and Schwartz (1976). A more recent paper is Nielsen et al. (2011). Risk management and hedging aspects are discussed in Coleman et al. (2006), Coleman et al. (2007), and Mahayni and Schlögl (2008). An early paper which already provides tools to determine closed-form solutions for the solvency restriction based on a shortfall concept under certain distribution assumptions (normal and log normal case) is given by Winkler et al. (1972) using partial moments. Non-linear optimization problems under shortfall constrains have already been solved in the past, c.f.McCabe and Witt (1980) who calculated the optimal chance-constrained expected profit of a non-life insurer.

Considering default is, in the context of participating life insurance contracts, firstly analyzed in Briys and De Varenne (1997) and Grosen and Jørgensen (2002). More recent papers are Schmeiser and Wagner (2015) and Hieber et al. (2019). Other papers on participating life insurance contracts excluding default risk are e.g. Bacinello (2001) who discusses amongst other results how a minimum interest rate guarantee "technical rate") has to be set, such that the contracts are fairly priced and Gatzert et al. (2012) where the customer value of the policyholder is maximized.

Papers on utility losses caused by (suboptimal) investment strategies include Jensen and Sørensen (2001), Jensen and Nielsen (2016) and Chen et al. (2019).[5] Chen et al. (2019) consider a general utility maximization under fair-pricing and budget constraints

[5] In particular, Jensen and Sørensen (2001) analyze wealth losses for pension funds and emphasize that the individual investor can substantially suffer from the investment strategy conducted by the sponsor.

in a complete, arbitrage-free Black and Scholes model setup for an CRRA Investor. The payoff function is chosen such that it also includes default risk. They apply their results on equity-liked life insurances using a constant mix strategy and examine the effect of taxation.

Literature on portfolio planning with a main focus on insurance contracts with guarantees includes Huang et al. (2008), Milevsky and Kyrychenko (2008), Boyle and Tian (2008) and Mahayni and Schneider (2016). The general idea of maximizing the expected utility of the insured by choosing optimal parameter settings which fulfill fair pricing conditions has been provided in the literature before. The paper of Branger et al. (2010) analyzes different forms of point-to-point guarantees. Cliquet-style options are analyzed in Gatzert et al. (2012) and Schmeiser and Wagner (2015). In contrast to these articles we add the portfolio composition as a decision variable in the optimization problem to determine the overall expected utility maximizing payoff of the insured in quasi-closed form.

Portfolio planning itself dates back to Merton (1975) who, amongst other results, solves the portfolio planning problem for a CRRA investor. The solution for investors who must also manage market-risk exposure using the Value-at-Risk (VaR) is firstly mentioned in Basak and Shapiro (2001). Yiu (2004) solves the problem where the VaR constraint is posed for the entire investment horizon. More recently, Gao et al. (2016) derive the solution for an investor with a dynamic mean-variance-CVaR and a dynamic mean-variance-safety-first constraint. A joint (terminal) VaR and portfolio insurance constraint is considered in Chen et al. (2018a). Multiple VaR constraints are analyzed in Chen et al. (2018b).

With respect to European and American guarantees, we also refer to El Karoui et al. (2005). Quantile hedging already dates back to Föllmer and Leukert (1999). For an analysis of retail products with investment caps (piecewise convex and concave payoffs) we e.g. refer to Bernard and Li (2013), Bernard and Li (2013), Mahayni and Schneider (2016).

Literature on the insurance demand dates back to Leland (1980) and Benninga and Blume (1985) who show that in a complete financial market setup with risky and risk-free asset investments and a utility function with constant risk aversion the investor will never buy portfolio insurance, instead buys the asset itself directly. Ebert et al. (2012) confirm the result for guarantee contracts, i.e. for CRRA Investors with reasonable risk aversion parameter Cumulative Prospect Theory (CPT) can not explain the demand for complex guarantee contracts. Ruß and Schelling (2018) introduce the concept of Multi

Cumulative Prospect Theory (MCPT) which does not only consider the terminal value of the investment but also the annual value change. Under the MCPT the demand for complex guarantee products can be explained.

There is a great body of literature that discusses the duration of insurance liability. Briys and De Varenne (1997) present a model for pricing a terminal participating life insurance policy. Moreover, the effective duration and Macaulay duration is examined. Briys and De Varenne (1997) state, that the effective duration is greater than the Macaulay duration for maturity below five year. Otherwise the effective duration is less then the Macaulay duration.

Kim (2005) investigates the impact of surrender rates on the value and interest rate sensitivity of interest indexed annuities. In summary, Kim (2005) shows that surrender behavior has a large impact on the duration and convexity of interest indexed annuities. In addition, Kim (2005) shows that duration and convexity also depend on the choice of interest rate model. It should be noted that single-premium contracts are less common than level-premium contracts and that the duration of such products can differ significantly.

Tsai (2009) examines the duration of the policy reserves for an endowment life insurance. Tsai (2009) shows that the duration of the cover reserve can be negative and/or have a high value. Policies with abnormal high or negative duration have longer maturities and therefore smaller reserves than policies with normal figures, such that they have no big impact on the duration of the aggregated reserves. Tsai (2012) complements the findings of Tsai (2009) by adding an empirical surrender rate model analyzing the impact of surrender options on the durations of insurance policy reserves. In general, Tsai (2012) showed that surrender options reduce the duration of policy reserves.

Charupat et al. (2016) study the impact of the interest rate risk on the prices of life annuities. Charupat et al. (2016) show that insurer do not instantaneously adjust annuity prices to changes of interest rates. Furthermore Charupat et al. (2016) shows that the duration is higher when interest rates are increasing than decreasing. With increasing interest rates the annuity prices are decreasing more quickly and in larger magnitude than when interest rates are decreasing.

Finally, Lin and Tsai (2020) investigates natural hedging strategies for mortality and interest rate risk. For this purpose, Lin and Tsai (2020) derive closed-form solutions for mortality-interest rate duration and convexity of the net single premiums of whole life insurance and deferred whole life annuity products. Moreover Lin and Tsai (2020) compute a suitable portfolio allocation of both products such that the portfolio is immune against mortality and interest rate risk.

The rest of the paper is organized as follows. Section 3.2 describes the contract design. In particular, it is based on a combination of the contract parameters and the equity fraction such that the contract design gives no rise to any arbitrage opportunity. In addition, the contract design must meet some regulatory requirements regarding an upper bound on the shortfall probability. Along the ways, we give some convenient representations of the payoff profiles. We illustrate the contract design and some important properties in a Black and Scholes model setup. In Section 3.3, we derive the optimal contract design (return payoff, respectively) of a quantile minimum return guarantee (MRRG), i.e. a return guarantee which satisfies the fair pricing condition and an upper bound on the shortfall probability, and in view of an insured whose preferences are characterized by a constant relative risk aversion. We illustrate the utility loss to the insured which is caused if the insurer implements a suboptimal investment strategy. We further investigated on the interest rate risk of those quantile guarantees. We therefore calculated the effective duration and convexity for suitable combinations of the minimum return rate guarantee and the corresponding fair equity ratio. Section 3.5 concludes the paper.

3.2 Participating life insurance contracts with minimum return rate guarantees

3.2.1 Preliminaries

In the following section participating life insurance with minimum return rate guarantees are introduced and analyzed. Mortality risk and surrender risk are not taken into account. The policyholder pays a single premium at the inception of the insurance contract denoted by P_0. This premium is thus called upfront premium. Together with its existing equity amount, the contract provider invests this premium on the financial market. The existing equity amount of the contract provider is denoted by E_0 and the value of the investment portfolio at inception is given by $A_0 = P_0 + E_0$. Throughout the following we set $P_0 = 1$, so we normalize the policyholders premium, and we set $E_0 = \alpha^E$, where $\alpha^E \in [0, 1]$ represents the equity ratio (resp. equity to debt ratio). The terminal value of the investment portfolio is denoted by A_T, where T defines the maturity of the insurance contract. For simplification we set $T = 1$ fo this section, i.e. a one period setting.[6] This assumption implies, that the policyholder has no other premium payment option than the upfront premium.

At the contract maturity, the policyholder receives the premium paid, compounded at either a guaranteed interest rate $g \geq -1$ or a participation on the return of the investment portfolio whichever is higher. So g is referred to a the minimum return rate guarantee (MRRG). The terminal payoff of the insurance contract is then given by

$$P_1 = 1 + \max\left\{g, \alpha\left(\frac{A_1}{A_0} - 1\right)\right\} \tag{3.1}$$

where $\alpha \in [0, 1]$ denotes the participation fraction on the investment portfolio. The special case $g = -1$ represents a contract without guarantee. This leads to the following lemma.

Lemma 3.2.1 (Intended payoff representation)
The intended payoff of a participating life insurance contract with maturity $T = 1$ and initial premium paid by the policyholder $P_0 = 1$ is given by

$$P_1^I = 1 + g + \alpha\left(\frac{A_1}{A_0} - K\right)^+, \ \textit{with}\ K = 1 + \frac{g}{\alpha}, \tag{3.2}$$

where $\alpha \in [0, 1]$ denotes the participation fraction on the investment portfolio, $g \geq -1$ is the minimum return rate guarantee, $A_0 = 1 + \alpha^E$ is the value of the investment portfolio at inception stated with the equity ratio $\alpha^E \in [0, 1]$ and finally A_1 denotes the terminal value of the investment portfolio. Moreover $\left(\frac{A_1}{A_0} - K\right)^+ = max\left\{\frac{A_1}{A_0} - K, 0\right\}$.

The payoff presented in Lemma 3.2.1 is called intended payoff, because it may differ from the actual payoff L_1 in case of a default event, which we will introduce later in this section. Lemma 3.2.1 shows, that the intended payoff can be replicated by (i) a long position in $e^{-r}(1 + g)$ zero bonds with maturity $T = 1$ (where r denotes the continuously compounded bond yield), and (ii) a long position in $\frac{\alpha}{A_0}$ call options on the synthetic asset A with maturity $T = 1$ and strike $K = A_0\left(1 + \frac{g}{\alpha}\right)$. The effect of parameters α, g and α^E on the intended payoff P_1^I is illustrated in Figure 3.1.

[6] The assumption of a maturity $T = 1$ gives us the possibility to state the payoff of the insured in closed-form. For a maturity $T > 1$ this is not possible anymore. See for example the comment in the paper of Schmeiser and Wagner (2015) on page 669.

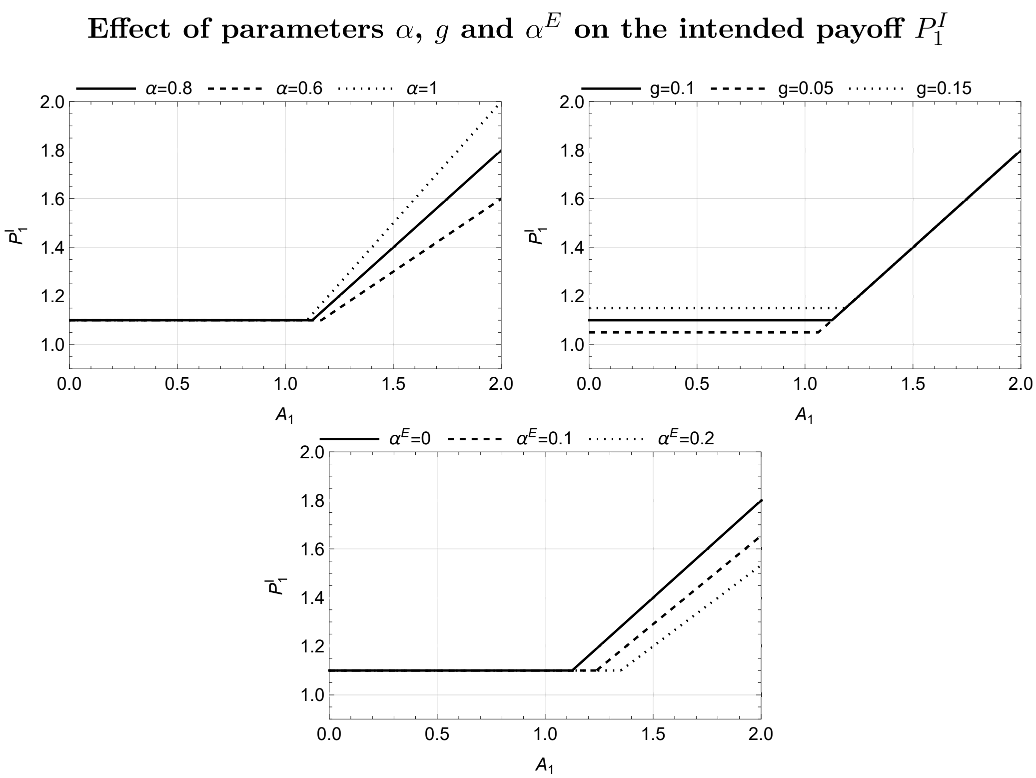

Figure 3.1: This figures illustrate the effect of parameters α, g and α^E on the intended payoff P_1^I. The figures show the intended payoff P_1^I in dependence of the terminal value of the investment portfolio A_1. By standard the parameters are $\alpha = 0.8$, $g = 0.1$ and $\alpha^E = 0$. The adjusted parameters are given in the legend above the corresponding illustration.

The standard case is given by the black solid line and refers to the parameter setup $\alpha = 0.8$, $g = 0.1$ and $\alpha^E = 0$. While the terminal value of the investment portfolio A_1 is below the strike $K = A_0 \left(1 + \frac{g}{\alpha}\right) = 1.125$, the payoff considers only the guaranteed part given by $1 + g = 1.1$. If the terminal value of the investment portfolio exceeds the strike, the payoff is increased by a participation fraction on the investment return by $0.8A_1 - 0.9$.

The upper left illustration shows the impact of the participation fraction α on the intended payoff P_1^I. It reflects the extent to which the policyholder participates in the investment return. A higher α increases this participation, which is shown by a steeper rise of the graph. In addition, the participation fraction also influences the strike and thus also the extent to which the policyholder participates in the investment return. As α increases, the strike decreases, which means that the investment return must be greater for the policyholder to participate.

The upper right illustration shows the impact of the minimum return rate guarantee g on the intended payoff P_1^I. It influences the guaranteed return that the policyholder receives. A higher g increases this guaranteed return. Furthermore, the minimum return rate guarantee also affects the strike. A falling g lowers the guaranteed return and thus leads to an earlier participation in investment return and vice versa.

Finally, the lower illustration shows the impact of the equity ratio α^E on the intended payoff P_1^I. The equity ratio mainly influences the initial value of the investment portfolio A_0. With a higher equity ratio, the initial value of the investment portfolio is higher. Thus, the final value of the investment portfolio must also be higher in order for the policyholder to participate in the investment return.

3.2.2 Consideration of default risk

In the following we first introduce a default event. A default occurs, if the terminal value of the investment portfolio A_1 is higher than the intended payoff P_1^I. Therefor the policyholder only receives the intended payoff P_1^I if the terminal value of the investment portfolio A_1 is sufficiently high. When default occurs and the terminal value of the investment portfolio A_1 is not sufficiently high the policyholder only receives the terminal value of the investment portfolio A_1 as compensation. The actual payoff L_1 to the policyholder where default risk is considered is thus given by

$$L_1 = P_1^I - \left(P_1^I - A_1\right)^+, \text{ where}$$
$$\left(P_1^I - A_1\right)^+ = \left(1 + g + \alpha\left(\frac{A_1}{A_0} - K\right)^+ - (1+\alpha^E)\frac{A_1}{A_0}\right)^+ \text{ with } K = 1 + \frac{g}{\alpha}$$

can be interpreted as a default put option of the contract provider. The terminal value of the default put option is the difference between the default free (intended) payoff and the payoff in presence of a default. Although the default put option is given in terms of a nested version of the max operator (a compound option feature), it is possible to disentangle the payoff in terms of the payoffs of plain vanilla options, only. To disentangle the nested payoff, we make a case distinction where we consider the inner call option $\left(\frac{A_1}{A_0} - K\right)^+$ given from the intended payoff and the (outer) default put option $\left(P_1^I - A_1\right)^+$ given from the actual payoff, respectively.

1. Case: $\frac{A_1}{A_0} \leq K_1 = 1 + \frac{g}{\alpha}$

At first we consider the case where the inner call option is out the money, i.e. $\frac{A_1}{A_0} \leq K_1 =$

$1+\frac{g}{\alpha}$, so the intended payoff is $P_1^I = 1+g$, so the intended payoff only considers the guaranteed return. Then the default put option is given by

$$\left(P_1^I - A_1\right)^+ = \left(1+g-(1+\alpha^E)\frac{A_1}{A_0}\right)^+ = (1+\alpha^E)\left(K_2 - \frac{A_1}{A_0}\right)^+, \tag{3.3}$$

$$\text{where } K_2 = \frac{1+g}{1+\alpha^E}. \tag{3.4}$$

Thus the default put option can be stated in terms of a put option on the terminal value of the investment portfolio A_1 with strike $\left(1+\alpha^E\right)K_2 = 1+g$.

2. Case: $\frac{A_1}{A_0} \geq K_1 = 1+\frac{g}{\alpha}$

Secondly we consider the case where the inner call option is in the money, i.e. $\frac{A_1}{A_0} \geq K_1 = 1+\frac{g}{\alpha}$, so the intended payoff is $P_1^I = 1+\alpha\left(\frac{A_1}{A_0}-1\right)$. Then the default put option is given by

$$\left(P_1^I - A_1\right)^+ = \left(1+\alpha\left(\frac{A_1}{A_0}-1\right)-(1+\alpha^E)\frac{A_1}{A_0}\right)^+ = (\alpha - A_0)\left(K_3 - \frac{A_1}{A_0}\right)^+, \tag{3.5}$$

$$\text{where } K_3 = \frac{1-\alpha}{1+\alpha^E-\alpha}. \tag{3.6}$$

Thus the default put option can be stated in terms of a put option on the terminal value of the investment portfolio A_1 with strike $\left(1+\alpha^E\right)K_3 = \left(1+\alpha^E\right)\frac{1-\alpha}{1+\alpha^E-\alpha}$.

In consequence, we can express the payoff of the default put option by means of piecewise linear functions as follows:

$$\begin{aligned}\left(P_1^I - A_1\right)^+ &= \begin{cases}\left(K_2-(1+\alpha^E)\frac{A_1}{A_0}\right) & \text{if } \frac{A_1}{A_0}\leq K_1 \text{ and } \frac{A_1}{A_0}\leq K_2\\ \frac{\alpha-1-\alpha^E}{1+\alpha^E}\left(K_3-(1+\alpha^E)\frac{A_1}{A_0}\right) & \text{if } \frac{A_1}{A_0}\geq K_1 \text{ and } \frac{A_1}{A_0}\leq K_3\\ 0 & \text{else}\end{cases}\\ &= \left(K_2-(1+\alpha^E)\frac{A_1}{A_0}\right)\mathbb{1}_{\{A_1\leq\min\{K_1,K_2\}\}}\\ &\quad+\frac{\alpha-1-\alpha^E}{1+\alpha^E}\left(K_3-(1+\alpha^E)\frac{A_1}{A_0}\right)\mathbb{1}_{\{K_1\leq A_1\leq K_3\}},\end{aligned}$$

where

$$K_1 = 1+\frac{g}{\alpha},\quad K_2 = \frac{1+g}{1+\alpha^E},\quad K_3 = \frac{1-\alpha}{1+\alpha^E-\alpha}. \tag{3.7}$$

A crucial distinction is given by a different ranking order of the strikes K_1, K_2 and K_3. However, the relation between the strikes is given by comparing the equity ratio α^E to the minimum return rate guarantee g and the participation fraction α. The result is summarized in the following lemma.

Lemma 3.2.2
Let K_1, K_2 and K_3 be defined as in Equation (3.7)*, then the following relations hold*

(i) $K_1 = K_2 = K_3 \Leftrightarrow \alpha^E = \frac{g(\alpha-1)}{g+\alpha}$

(ii) $K_1 \geq K_2 \geq K_3 \Leftrightarrow \alpha^E \geq \frac{g(\alpha-1)}{g+\alpha}$

(iii) $K_1 \leq K_2 \leq K_3 \Leftrightarrow \alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$

In particular it holds

$$\alpha^E = \frac{g(\alpha-1)}{g+\alpha} \Leftrightarrow g = \frac{\alpha\alpha^E}{\alpha-1-\alpha^E}$$

A visualization of the relation between the strikes and the minimum return rate guarantee g and the equity fraction α^E is given in Figure 3.2 for fix $\alpha = 0.8$.

First we discuss the dependence on the minimum return rate guarantee g, i.e. the upper row. The left hand side is based on $\alpha^E = 0$ and shows that the ordering of the strike changes at $g = 0$. The right hand side is based on $\alpha^E = 0.1$ and shows that the ordering of the strike changes at $g \leq 0$. In particular the strikes coincide at $g = \frac{\alpha\alpha^E}{\alpha-1-\alpha^E}$.

Nest we discuss the dependence on the equity fraction $\alpha^E = 0$, i.e. the lower row. The left hand side is based on $g \geq 0$ and shows that the ordering of the strike does not change at all. The right hand side is based on $g \leq 0$ and shows that the ordering of the strike changes at $\alpha^E = \frac{g(\alpha-1)}{g+\alpha}$

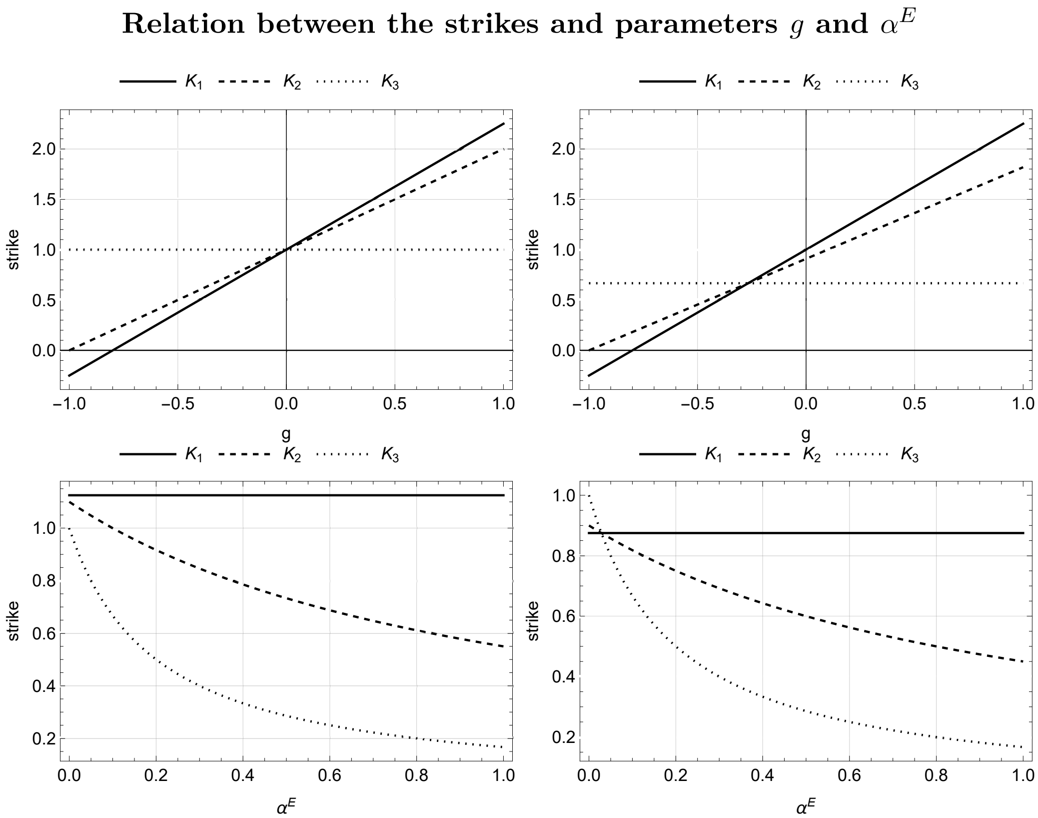

Figure 3.2: This figures illustrate the strikes K_1, K_2 and K_3 depending on the minimum return rate guarantee g in the upper row and the equity fraction α^E in the lower row. We fix $\alpha = 0.8$. The upper left hand figure is based on $\alpha^E = 0$ while the upper right hand figures is based on $\alpha^E = 0.1$. The lower left hand figure is based on $g = 0.1$ while the lower right hand figures is based on $g = -0.1$.

Let us now take a closer look on the term $\frac{g(\alpha-1)}{g+\alpha}$. At first one should recognize that $g \geq 0 \Rightarrow \frac{g(\alpha-1)}{g+\alpha} \leq 0$ for every $\alpha \in [0,1]$. Thus $g \geq 0$ implies $\alpha^E \geq \frac{g(\alpha-1)}{g+\alpha}$ as the equity ratio $\alpha^E \in [0,1]$ is nonnegative by definition. This is also shown in the illustration at the bottom left of Figure 3.2.

In summary, the payoff (return) of the default put can be represented as follows.

Proposition 3.2.3 (Payoff representation of the default put option)
The payoff of the default put option can be stated in terms of a piecewise linear function

in the asset increments $\frac{A_1}{A_0}$*, i.e.*

$$\left(P_1^I - A_1\right)^+ = \begin{cases} \left(1+\alpha^E\right)\left(K_2 - \frac{A_1}{A_0}\right)\mathbb{1}_{\left\{\frac{A_1}{A_0}\leq K_1\right\}} & \\ +\left(\alpha - 1 + \alpha^E\right)\left(K_3 - \frac{A_1}{A_0}\right)\mathbb{1}_{\left\{K_1\leq\frac{A_1}{A_0}\leq K_3\right\}} & \textit{for } \alpha^E \leq \frac{g(\alpha-1)}{g+\alpha} \\ \left(1+\alpha^E\right)\left(K_2 - \frac{A_1}{A_0}\right)^+ & \textit{for } \alpha^E > \frac{g(\alpha-1)}{g+\alpha} \end{cases}. \tag{3.8}$$

WE now take a look at the actual payoff L_1 to the policyholder. An intuitive way to understand the liability side under default risk is analogously given by stating the payoff L_1 depending on the asset increment $\frac{A_1}{A_0}$. First recall that, without default risk, the call option of the policyholder (cf. Lemma 3.2.1) is in the money if $\frac{A_1}{A_0} > K_1 = 1+\frac{g}{\alpha}$. Otherwise the intended payoff is $1+g$. In this case, under default risk, the policyholder only receives $1+g$ if this is possible, i.e. if $A_1 > 1+g$, or equivalent if $\frac{A_1}{A_0} > \frac{1+g}{1+\alpha^E} = K_2$. So for $\frac{A_1}{A_0} \leq K_1$, the policyholder only receives the minimum of $1+g$ and A_1.

Now, consider the case that $\frac{A_1}{A_0} > K_1$, i.e. $P_1^I = 1+\alpha\left(\frac{A_1}{A_0}-1\right)$. Again, under default risk, the policyholder nevertheless only receives the lower of $1+\alpha\left(\frac{A_1}{A_0}-1\right)$ and A_1, which is defined by the benchmark $K_3 = \frac{1-\alpha}{1+\alpha^E-\alpha}$. In summary, we obtain

$$L_1 = \begin{cases} (1+\alpha^E)\frac{A_1}{A_0} & \text{for } \frac{A_1}{A_0} < \min\{K_1, K_2\} \\ 1+g & \text{for } \min\{K_1, K_2\} \leq \frac{A_1}{A_0} < K_1 \\ (1+\alpha^E)\frac{A_1}{A_0} & \text{for } K_1 \leq \frac{A_1}{A_0} < \max\{K_1, K_3\} \\ 1+\alpha\left(\frac{A_1}{A_0}-1\right) & \text{for } \frac{A_1}{A_0} \geq \max\{K_1, K_2, K_3\}. \end{cases}$$

Before we state the actual payoff as a function of the equity ratio, we need a brief remark.

Remark 3.2.4

It follows from Lemma 3.2.2

$$\min\{K_1, K_2\} = \begin{cases} K_1 & \textit{for } \alpha^E \leq \frac{g(\alpha-1)}{g+\alpha} \\ K_2 & \textit{for } \alpha^E > \frac{g(\alpha-1)}{g+\alpha} \end{cases},$$

$$\max\{K_1, K_3\} = \begin{cases} K_3 & \textit{for } \alpha^E \leq \frac{g(\alpha-1)}{g+\alpha} \\ K_1 & \textit{for } \alpha^E > \frac{g(\alpha-1)}{g+\alpha} \end{cases},$$

$$\max\{K_1, K_2, K_3\} = \begin{cases} K_3 & \textit{for } \alpha^E \leq \frac{g(\alpha-1)}{g+\alpha} \\ K_1 & \textit{for } \alpha^E > \frac{g(\alpha-1)}{g+\alpha} \end{cases}.$$

Now we can specify the actual payoff as a function of the equity ratio.

Proposition 3.2.5 (Actual payoff representation)
Let K_1, K_2 and K_3 be defined as in Equation (3.7)*. Then it holds*

(i) ***Low equity ratio:*** *For $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$, the payoff (return) to the insured is given by*

$$L_1 = \begin{cases} (1+\alpha^E)\frac{A_1}{A_0} & \text{for } \frac{A_1}{A_0} < K_3 \\ 1+\alpha\left(\frac{A_1}{A_0}-1\right) & \text{for } \frac{A_1}{A_0} \geq K_3, \end{cases}$$

$$\text{i.e. } L_1 = (1+\alpha^E)\frac{A_1}{A_0} - (1-\alpha+\alpha^E)\left(\frac{A_1}{A_0}-K_3\right)^+. \tag{3.9}$$

(ii) ***High equity ratio:*** *For $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$ it holds*

$$L_1 = \begin{cases} (1+\alpha^E)\frac{A_1}{A_0} & \text{for } \frac{A_1}{A_0} < K_2 \\ 1+g & \text{for } K_2 \leq \frac{A_1}{A_0} < K_1 \\ 1+\alpha\left(\frac{A_1}{A_0}-1\right) & \text{for } \frac{A_1}{A_0} \geq K_1, \end{cases}$$

$$\text{i.e. } L_1 = (1+\alpha^E)\frac{A_1}{A_0} - \left(1+\alpha^E\right)\left(\frac{A_1}{A_0}-K_2\right)^+ + \alpha\left(\frac{A_1}{A_0}-K_1\right)^+. \tag{3.10}$$

For a low equity ratio (*Case (i)*), the above Proposition states that the liabilities of the insured are given by the payoff of

(i) one long position in the insurer's assets A and

(ii) $\frac{1-\alpha+\alpha^E}{1+\alpha^E}$ short calls on A with strike $\left(1+\alpha^E\right)K_3 = \left(1+\alpha^E\right)\frac{1-\alpha}{1-\alpha+\alpha^E}$.

For a high equity ratio (*Case (ii)*), the above Proposition states that the liabilities of the insured are given by the payoff of

(i) one long position in the insurer's assets A,

(ii) $\left(1+\alpha^E\right)$ short position in a call on A with strike $\left(1+\alpha^E\right)K_2 = 1+g$ and

(iii) α long calls with strike $\left(1+\alpha^E\right)K_1 = \left(1+\alpha^E\right)(1+\frac{g}{\alpha})$.

In addition, the above Proposition immediately implies the following important properties of the liability payoffs.

Corrolary 3.2.6 (Properties of the liability payoff)
Let L_1 be the liability payoff stated in Proposition 3.2.5, then it holds

(i) *L_1 is increasing in g and α^E. For $g > 0$, L_1 is increasing in α.*

(ii) *For $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$, L_1 is concave in $\frac{A_1}{A_0}$.*

(iii) *For $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$, L_1 is piecewise concave and piecewise convex in $\frac{A_1}{A_0}$.*

An illustration of L_1 is given in Figure 3.3. The left hand figure is based on $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$ (*low equity fraction*) while the right hand figure is based on the case $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$ (*high equity fraction*). In particular, the payoffs on the left hand side are concave while the payoffs on the right hand side are piecewise concave and convex while. Intuitively, it is clear that a higher amount of equity means that the real degree of guarantee is, ceteris paribus, higher than for a lower amount of equity. This is resembled in the payoff profiles, i.e. a higher amount of equity gives more convexity to the payoff profile (implying a more valuable guarantee).

Illustration of the contract payoff under default risk

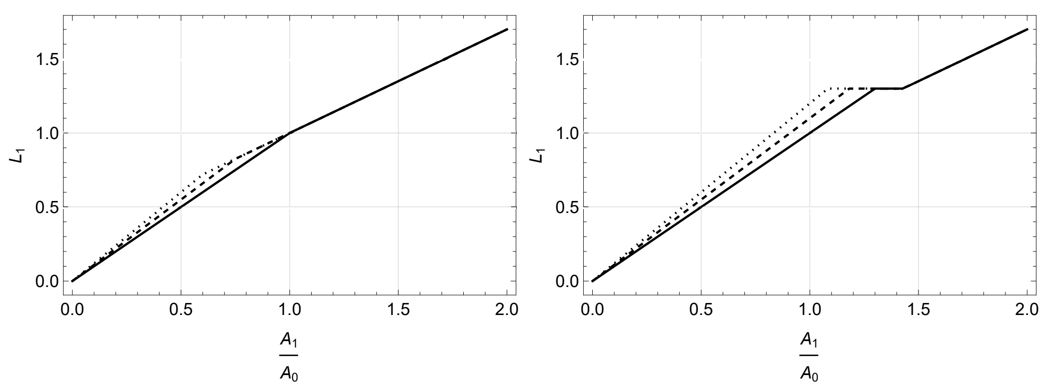

Figure 3.3: For varying asset increments $\frac{A_1}{A_0}$, the figures illustrate the actual contract payoff L_1. It holds $0 = \alpha_1^E < \alpha_2^E < \alpha_3^E$. The black solid line refer to $\alpha^E = 0$, the black dashed line to $\alpha^E = \alpha_2^E = 0.1$, and the dotted line to $\alpha^E = \alpha_3 = 0.2$. The left hand figure is based on $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$ (*low equity fraction*) while the right hand figure is based on $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$ (*high equity fraction*). Moreover we set $\alpha = 0.7$ and the minimum return rate guarantee $g = -0.3$ for the left figure and $g = 0.3$ for the right figure.

3.2.3 Fair pricing and regulatory requirements

Throughout the following analysis, we make some assumptions on the contract design (and the model setup for the financial market). We assume that the financial market model is arbitrage free. Furthermore, we assume that, because of competition, the contracts are fairly priced such that no arbitrage is introduced (among the insurers and between the insurance products and the financial market products):

Assumption 1 (No arbitrage)
We assume that the financial market model is arbitrage free. Thus, the fundamental theorem of asset pricing implies the existence of an equivalent pricing measure $\mathbb{P}^*$ such that the price of any traded asset X with payoff X_T at $T > 0$ is given by the expected discounted payoff under $\mathbb{P}^*$,i.e.

$$X_0 = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\int_0^T \tilde{r}_u\,du} X_T\right], \tag{3.11}$$

where $\tilde{r}_u$ denotes the forward rate, such that $\int_0^T \tilde{r}_u\,du$ is the continuously compounded interest rate prevailing at time T.

Assumption 2 (Fair pricing)
We assume competition between the insurance companies (and with the opportunity to invest in the financial market). In particular, we thus assume that the insurance contracts are fairly priced, i.e. depending on the investment decisions which are carried out by the insurer on the financial market, the contract prices are given by the arbitrage free (financial market) prices.[7]

Assumption 3 (Stakeholders)
The policyholders are not able to participate at the arbitrage free financial market, such that they cannot replicate future cash-flows. They just have the possibility to invest in the asset side of the insurance company. The insurer itself, resp. its shareholders, of course have this access to the market.[8]

In addition, we assume later that an admissible contract design must honor regulatory requirements as e.g. posed by an upper bound on the shortfall probability. First, we consider the assumption on the contract pricing and the implications of postulating an arbitrage

[7] It should be mentioned that in practice it would not be possible to e.g. make sure that all these contracts are initially fair: Rather, in practice, cross-subsidizing effects are unavoidable (cf. e.g. Hieber et al. (2015)).

[8] This assumption is reasonable and has often been used in other literature dealing with this topic, e.g. Schmeiser and Wagner (2015) or Briys and De Varenne (1997).

free financial model setup. Subsequently, we introduce the regulatory requirement and represent the shortfall probability in terms of the strikes introduced above.

Along the lines of Proposition 3.2.5, the arbitrage free value of the liabilities (and the default put, respectively) is given by the (arbitrage free) value of the corresponding portfolio of plain vanilla options. To simplify the exposition, we refer to a one year horizon, i.e. the call (or put) options have a maturity of $T = 1$. The (arbitrage free) value of a call (put) option (with maturity $T = 1$) and strike K is denoted by $Call(K)$ ($Put(K)$). To be more precise, $Call(K)$ ($Put(K)$) denotes the $t = 0$ value of the $T = 1$ payoff $(A_1 - K)^+$ ($(K - A_1)^+$, respectively).

Proposition 3.2.7 (Fair pricing conditions)
Assume that the asset A can be synthesized by a financial market strategy, i.e. the $t = 0$ price of the payoff A_1 is A_0 (A is an asset paying no dividends). In addition, assume that the financial market is arbitrage free. Then, the fair pricing condition is given by the condition that the market consistent price of the payoff L_1 is equal to $P_0 = 1$. In particular, depending on the equity ratio α^E, the minimum return rate guarantee g, and the participation fraction α, the following pricing conditions hold:

(i) ***Low equity ratio:*** *For $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$, it holds*

$$1 = \left(1 + \alpha^E\right) - (1 - \alpha + \alpha^E)Call(K_3). \tag{3.12}$$

(ii) ***High equity ratio:*** *For $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$, it holds*

$$1 = \left(1 + \alpha^E\right) - \left(1 + \alpha^E\right) Call(K_2) + \alpha Call(K_1). \tag{3.13}$$

where the strikes K_1, K_2 and K_3 are defined as in Equation (3.7).

Corrolary 3.2.8 (Properties of fair contracts under default risk)
The fair pricing conditions imply the following properties

(i) For $\alpha^E = 0$, a fair contract implies $\alpha^{fair} = 1$.

(ii) In the special case that $g = -1$ (no guarantee) it also holds $\alpha^{fair} = 1$.

The proof is straightforward and the results are intuitive: Part (i) states that without equity, the insured face the whole risk of the asset investments, i.e. the (fair) liabilities are given by $L_1 = A_1$. In particular, without further restrictions on the distribution of $\frac{A_1}{A_0}$, i.e. restrictions on the riskiness of the investment strategy, there is no guarantee without equity. The interpretation of part (ii) is analogous. Since there is no guarantee if $g = -1$, a fair contract must imply $L_1 = A_1$.

Now consider the condition that there is a regulatory requirement on the shortfall probability. Assume that the regulator requires an upper bound ϵ for the probability that the intended *guaranteed* accumulation P_1^I is not honored because the asset value A_1 is lower, i.e.

$$\mathbb{P}\left(A_1 < P_1^I\right) \leq \epsilon. \tag{3.14}$$

The event $\left\{A_1 < P_1^I\right\}$ can be represented in terms of the strikes given by Equation (3.7): K_1 defines the level of $\frac{A_1}{A_0}$ such that the inner option is in the money, i.e. where the intended payoff P_1^I pays out $1 + \alpha\left(\frac{A_1}{A_0} - 1\right)$ instead of $1 + g$. The strike K_2 defines the level of $\frac{A_1}{A_0}$ such that the put option is in the money, i.e. the intended Payoff P_1^I is equal to $1 + g$, but the asset side A_1 is lower. K_3 defines the level of $\frac{A_1}{A_0}$ where the liabilities can not be satisfied if the inner option is in the money, i.e.

$$\left\{A_1 < P_1^I\right\} = \left\{\frac{A_1}{A_0} \leq K_1; \frac{A_1}{A_0} < K_2\right\} \cup \left\{\frac{A_1}{A_0} > K_1; \frac{A_1}{A_0} < K_3\right\}. \tag{3.15}$$

With Lemma 3.2.2 and the representation of the shortfall event in Equation (3.15), we immediately obtain the following Proposition.

Proposition 3.2.9 (Shortfall probability)
The shortfall probability $\mathbb{P}\left(A_1 < P_1^I\right)$ *is given by*

$$\begin{aligned}\mathbb{P}\left(A_1 < P_1^I\right) &= \mathbb{P}\left(\frac{A_1}{A_0} < \min\{K_1, K_2\}\right) + \mathbb{P}\left(K_1 \leq \frac{A_1}{A_0} \leq \max\{K_1, K_3\}\right) \\ &= \mathbb{P}\left(\frac{A_1}{A_0} < K_3\right) \mathit{1}_{\left\{\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}\right\}} + \mathbb{P}\left(\frac{A_1}{A_0} \leq K_2\right) \mathit{1}_{\left\{\alpha^E > \frac{g(\alpha-1)}{g+\alpha}\right\}}.\end{aligned} \tag{3.16}$$

It is worth to emphasize that, e.g. in the context of Solvency II, the upper bound on the shortfall probability determines the amount of equity which is needed to assure the solvency to a high degree, i.e. to honor the liabilities to the insured. Obviously, the lower the strike is, the lower is the probability that the value of a given investment strategy drops below the strike. Since the above strikes are decreasing in the equity fraction α^E, a higher equity fraction is able to reduce the shortfall probability.[9]

[9] However, if one assumes a complete financial market model, any reduction in the shortfall probability can also be implemented by a change in the asset distribution by means of a suitable investment strategy.

3.2.4 Black and Scholes model setup and illustration

Along the lines of the previous subsections, the contracts can be fairly priced in closed form in any arbitrage free model setup which allows closed form solutions of plain vanilla options. For the sake of simplicity, we place ourselves in a Black and Scholes model setup to give some illustrations. The financial market model over the filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is given by the Black and Scholes model, i.e. there are two investment possibilities, a risky asset S and a risk-free asset B which accumulates according to a constant interest rate r. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by the standard Brownian motion $(W_t)_{t\in[0,T]}$. Because of the completeness of the Black and Scholes model, there exists a uniquely determined equivalent martingale measure $\mathbb{P}^*$ under which the process $(W_t^*)_{t\in[0,T]}$ defines a standard Brownian motion. In particular, the risky asset $(S_t)_{t\in[0,T]}$ and risk free bond dynamics $(B_t)_{t\in[0,T]}$ are given by

$$
\begin{aligned}
dS_t &= S_t\left(\mu\,dt + \sigma\,dW_t\right) = S_t\left(r\,dt + \sigma\,dW_t^*\right), \quad S_0 = s \\
dB_t &= B_t r\,dt, \quad B_0 = b.
\end{aligned}
$$

Under the real world probability measure $\mathbb{P}$, the asset price follows a geometric Brownian motion with constant drift μ ($\mu > r$) and constant volatility σ ($\sigma > 0$). Under the uniquely defined equivalent martingale measure (pricing measure) $\mathbb{P}^*$, the asset price follows a geometric Brownian motion with constant drift r and constant volatility σ ($\sigma > 0$). The risk free bond B grows at a constant interest rate r.

Constant mix strategies

Assuming that the insurer decides to implement an investment strategy which is described by a constant fraction of wealth $m^{(A)}$ invested in the risky asset (and the remaining fraction $1 - m^{(A)}$ is invested in the risk free bond) implies that the asset process is also given by a lognormal process, i.e.

$$
dA_t = A_t\left(m^{(A)}\frac{dS_t}{S_t} + (1 - m^{(A)})r\,dt\right).
$$

Thus, w.r.t. an investment horizon of $T = 1$, it holds

$$
A_1 = A_0 e^{\mu_A^{(RW)} - \frac{1}{2}\sigma_A^2 + \sigma_A W_1} = A_0 e^{r - \frac{1}{2}\sigma_A^2 + \sigma_A W_1^*}
$$

where $\mu_A^{(RW)} = m^{(A)}\mu + (1 - m^{(A)})r$ and $\sigma_A = m^{(A)}\sigma$.

$\mu^{(RW)}$ denotes the drift of the asset dynamics under the real word measure $\mathbb{P}$. Under the pricing measure $\mathbb{P}^*$, the drift is equal to r. In particular, let $N(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2 and $\Phi(\cdot)$ the cumulative distribution function of the standard normal distribution. Then it holds

$$\ln \frac{A_1}{A_0} \sim N\left(\mu_A - \frac{1}{2}\sigma_A^2, \sigma_A^2\right) \text{ under } \mathbb{P},$$
$$\ln \frac{A_1}{A_0} \sim N\left(r - \frac{1}{2}\sigma_A^2, \sigma_A^2\right) \text{ under } \mathbb{P}^*.$$

In consequence, the arbitrage free (competitive) price of the liabilities L_1 (the default put, respectively) can be derived by means of Proposition 3.2.7 where the call price formula $Call(K) = Call^{(BS)}(K, \sigma_A)$ is given by the Black and Scholes pricing formula (w.r.t. the returns), i.e.

$$Call^{(BS)}(K, \sigma_A) = \Phi(d_1(K, \sigma_A)) - e^{-r} K \Phi(d_2(K, \sigma_A)), \tag{3.17}$$
$$\text{where } d_1(K, \sigma_A) = \frac{-\ln K + r + \frac{1}{2}\sigma_A^2}{\sigma_A} \text{ and } d_2(K, \sigma_A) = d_1(K, \sigma_A) - \sigma_A.$$

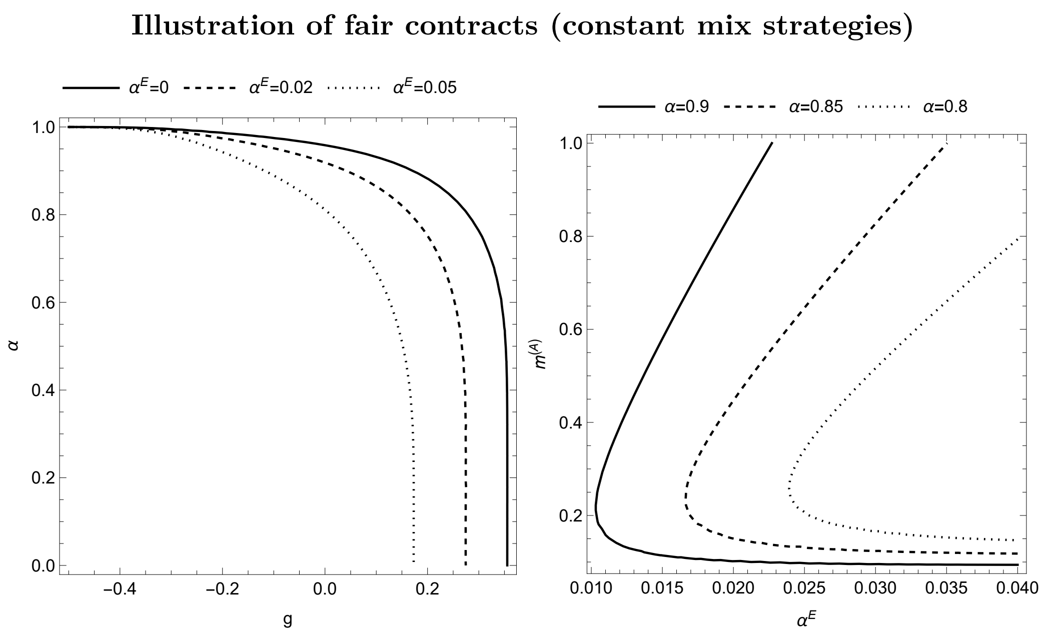

Figure 3.4: The contract and model parameters are given as in Table 3.1. The left figures illustrate fair tuples of the contract parameter (α, g). The black line refers to $\alpha_1^E = 0.01$, the black dashed line to $\alpha_2^E = 0.02$, and the dotted line to $\alpha_3^E = 0.05$. The figure on the right hand side (the black line, respectively) depicts fair contracts for the benchmark case in terms of fair combinations of the equity fraction α^E and the investment fraction $m^{(A)}$ (defining the volatility of the assets, i.e. $\sigma_A = m^{(A)}\sigma$). The solid line refers to $\alpha = 0.9$, the dashed line refers to a lower participation fraction $\alpha = 0.85$ and the dotted line refers to $\alpha = 0.8$.

Figure 3.4 gives an illustration of fair contract designs. The left figure illustrates fair tuples of the contract parameter (α, g). Along the lines of the model free results, the (return) payoff of the MRRG under default risk is increasing in α and g. Thus, in order to stay on a fair contract design, an increasing guarantee g must be compensated by decreasing the participation rate α. In addition, the fair (α, g) combinations are lower for higher equity fractions, i.e. the black line refers to $\alpha_1^E = 0.01$, the black dashed line to $\alpha_2^E = 0.02$, and the dotted line to $\alpha_3^E = 0.05$. This result is straightforward and can, for example, be found in Grosen and Jørgensen (2000). An interesting effect arises in view of the piecewise concave and piecewise convex payoff structures (implied by $g > 0$ and $\alpha^E > 0$, cf. Corollary 3.2.6). Although the contract value is increasing in the equity fraction α^E, this is not necessarily true with respect to the riskiness of the investments, i.e. w.r.t. $m^{(A)}$ (the volatility $\sigma_A = m^{(A)}\sigma$, respectively). Thus, for a fixed equity fraction α^E, there may be two investment fractions $m^{(A,1)}$ and $m^{(A,2)}$ such that the contract is fairly priced. This is illustrated in the right hand plot of Figure 3.4 which depicts fair contracts for the benchmark case in terms of fair combinations of the equity fraction α^E and the investment fraction $m^{(A)}$ (defining the volatility of the assets, i.e. $\sigma_A = m^{(A)}\sigma$). The solid line refers to $\alpha = 0.9$, the dashed line refers to a lower participation fraction $\alpha = 0.85$ and the dotted line refers to $\alpha = 0.8$. For the shortfall probability given in Proposition 3.2.9, the Black and Scholes model setup immediately implies

$$\mathbb{P}\left(A_1 < P_1^I\right) = \Phi\left(\overline{d}_0(K_3)\right) \mathbb{1}_{\left\{\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}\right\}} + \Phi\left(\overline{d}_0(K_2)\right) \mathbb{1}_{\left\{\alpha^E > \frac{g(\alpha-1)}{g+\alpha}\right\}}, \tag{3.18}$$
$$\text{where } \overline{d}_0(K) := \frac{\ln K - (\mu_A - \frac{1}{2}\sigma_A^2)}{\sigma_A}.$$

Again, notice that, e.g. in the context of Solvency II, the upper bound on the shortfall probability is posed to determine the amount of equity which is needed to assure the solvency to a high degree, i.e. to honor the liabilities to the insured. Obviously, the lower the strike is, the lower is the probability of a constant mix strategy that its terminal value drops below the strike. Since the above strikes are decreasing in the equity fraction α^E, a higher equity fraction is able to reduce the shortfall probability, cf. Figure 3.5 for an illustration.

It is worth noticing that any reduction of the shortfall probability can also be obtained by suitably adjusting the investment strategy, i.e. the distribution of A_1.

Table 3.1: Benchmark parameter settings

Model parameter			Contract parameter				Upper bound on $\mathbb{P}\left(A_1 < P_1^I\right)$
r	μ	σ	P_0	A_0	α	g	ϵ
0.03	0.07	0.2	1	$1+\alpha^E$	0.9	0.0175	0.005

3.3 Optimal design of quantile guarantees

The following section discusses, from the perspective of the insured, the optimal design of a MRRG under default risk and an upper bound on the shortfall probability. A fair contract design which provides a higher (expected) utility to the insured is also beneficial to the insurance company. The contract provider competes with other insurers and the financial market. Choosing among different contracts, the insured selects the contract which provides herself the highest (expected) utility. Throughout the following, we assume

Fair contracts honoring the upper bound on the shortfall probability

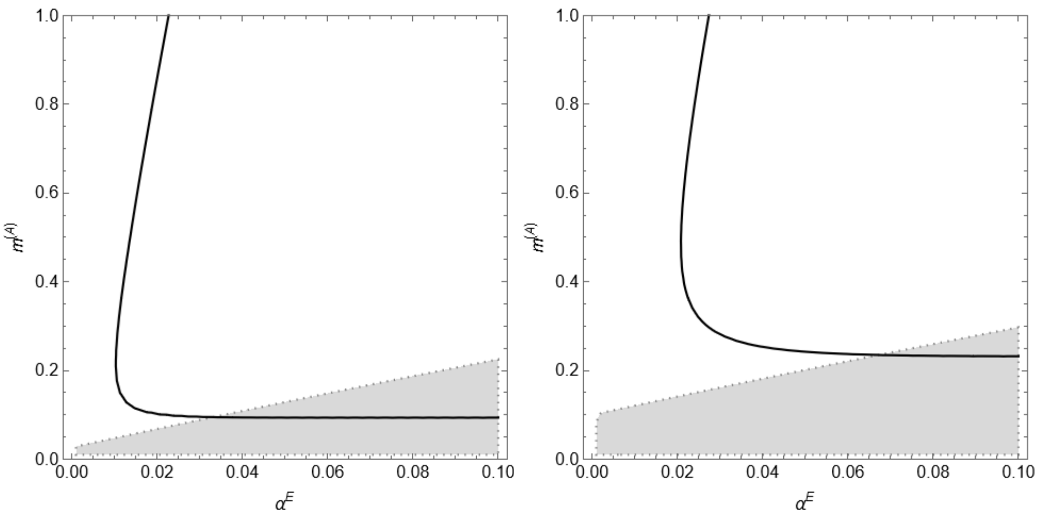

Figure 3.5: If not otherwise mentioned, the contract and model parameters are given as in Table 3.1. The black lines depict the fair contracts in terms of fair combinations of the equity fraction α^E and the investment fraction $m^{(A)}$ (defining the volatility of the assets, i.e. $\sigma_A = m^{(A)}\sigma$). The shaded region is the region where the upper bound on the shortfall probability ($\epsilon = 0.005$) is honored. While the figure on the left hand side refers to the benchmark guarantee $g = 0.0175$, the right hand side is implied by $g = -0.0175$.

that the preferences of the insured are described by a utility function $u = u^{(CRRA)}$ implying a constant relative risk aversion (CRRA) denoted by γ, i.e. $u^{(CRRA)}(x) = \frac{x^{1-\gamma}}{1-\gamma}$ ($\gamma > 1$) and $u^{(CRRA)}(x) = \ln x$ ($\gamma = 1$). Assuming CRRA preferences has its merits. There are empirical investigations which justify CRRA preference, cf. e.g. Chiappori and Paiella (2011). In addition, CRRA utility allows that the analysis is based on returns.[10] The relevant optimization problem is posed by maximizing the expected utility of the insured under constraints posed by a competitive market (fair pricing) and the restrictions posed by the regulator.[11] In the first instance, we formulate the optimization problem without

[10] It is worth mentioning that CRRA preferences can not explain the existence of (quantile) guarantees using, cf. Leland (1980). However one can understand that policy makers provide tax advantages for products with downside protection for old-age provision to reduce the risk of poverty among the elderly and possible implications for tax payers - even if downside protection reduces utility on the individual level for CRRA-type policyholders. For the effect of taxation on equity-linked life insurance we refer to Chen et al. (2019)

[11] The optimization procedure with a value at risk restriction can be referred to as a chance-constrained approach. It is transferable in a non-linear (deterministic) optimization program of normal of log normal

stating the optimization arguments, i.e.

$$\max \mathbb{E}_{\mathbb{P}}\left[u(L_1)\right] \text{ s.t. } \mathbb{P}\left(A_1 < P_1^I\right) \leq \epsilon,\ \mathbb{E}_{\mathbb{P}^*}\left[e^{-r}A_1\right] = 1 + \alpha^E \text{ and } \mathbb{E}_{\mathbb{P}^*}\left[e^{-r}L_1\right] = 1. \tag{3.19}$$

The first condition states the regulatory requirement on the upper bound on the shortfall of the intended payoff (guarantee) P_1^I. The second condition ensures that the asset value A_1 is obtainable by a self-financing investment strategy with initial investment $A_0 = 1 + \alpha^E$, and the third part captures the fair pricing of the liabilities, i.e. the $t = 0$ value of the liability payout at time $t = 1$ is equal to the contribution $P_0 = 1$. To shed further light on the (overall) optimal design of quantile guarantees, we discuss and compare (in the Black and Scholes model setup) different approaches concerning the arguments which are optimally chosen in the maximization problem (3.19) in order to maximize the utility which is provided to the insured. As a benchmark, we consider the optimal unconstrained strategy (no upper bound on the shortfall probability). For $\alpha^E = 0$, this is the classic Merton problem (cf. Merton (1975)). The solution implies the highest possible utility and thus provides an upper bound of the expected utility of all contract designs.

We also comment on an approach suggested in Schmeiser and Wagner (2015) who assume that the insurer implements a constant mix strategy, but can decide on the fraction of asset wealth which is invested riskily. The insurer simultaneously determines the equity fraction α^E and the investment fraction $m^{(A)}$ such that the pricing and shortfall constraints are satisfied for a given guarantee g. The utility to the insured is then maximized by selecting the guarantee g which gives the highest expected utility.

Finally, we consider the optimal solution under the pricing and shortfall constraints (without restricting the insurer's investment strategy to constant mix strategies).

3.3.1 The Merton solution as a benchmark

Assume that the insured is not committed to select among MRRG contracts, only. Instead, assume that she can, without transaction costs, dynamically trade on the financial market. In terms of the MRRG contracts, this is the special case that $\alpha^E = 0$ (the insured owns the

returns are assumed (cf. McCabe and Witt (1980)). Basically, we also consider log normal payoffs for $t = 1, 2, \ldots$ under a Geometric Brownian Motion (GBM) assumption. However we have added the assumption that the insured is described by a constant relative risk aversion (CRRA) which gives further insights on the utility effects from the perspective of the insured.

asset side herself) and a vanishing shortfall probability bound $\epsilon = 1$ (she is not restricted by the regulator). The optimization problem (3.19) then boils down to

$$\max_{A_1} \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{A_1}{A_0}\right)\right] \text{ s.t. } \mathbb{E}_{\mathbb{P}^*}\left[e^{-r}\frac{A_1}{A_0}\right] = 1,$$

i.e. the investor chooses the optimal payoff $L_1 = A_1$ (return, respectively, $A_0 = P_0 = 1$).[12] Assuming a Black and Scholes model setup to describe the financial market model, gives the classic Merton problem. The solution is firstly stated in Merton (1975). Under the real world measure $\mathbb{P}$, the optimal payoff $L_1^* = \frac{A_1^*}{A_0}$ is given by

$$\frac{A_1^*}{A_0} = e^{\mu_A^{(RW)} - \frac{1}{2}\sigma_A^2 + \sigma_A W_1}, \tag{3.20}$$

where $\mu_A^{(RW)} = m^{(A)}\mu + (1 - m^{(A)})r$, $\sigma_A = m^{(A)}\sigma$ and $m^{(A)} = \dfrac{\mu - r}{\gamma\sigma^2} =: m^{(Mer)}$.

In the optimum, the investor uses a constant mix strategy where the fraction $m^{(A)}$ of portfolio wealth which is invested riskily is given by the quotient of the (local) excess return $(\mu - r)$ and the squared asset volatility scaled by the parameter of relative risk aversion $\gamma\sigma^2$. The certainty equivalent wealth/return CE which makes the investor indifferent to the Merton payoff is defined by the condition $u(CE) = \mathbb{E}_{\mathbb{P}}[u(A_1)]$, i.e. $CE = u^{-1}(\mathbb{E}_{\mathbb{P}}[u(A_1)])$. Straightforward calculations imply

$$CE^* = e^{r + \frac{(\mu - r)^2}{2\gamma\sigma^2}} =: CE^{(Mer)} \text{ and } y^{CE*} = \ln CE^* = r + \frac{(\mu - r)^2}{2\gamma\sigma^2}, \tag{3.21}$$

where y^{CE^*} denotes the (optimal Merton) savings rate. Notice that the above CE^* defines an upper bound to all certainty equivalents which are implied by (admissible) MRRG contracts and refer to the upper bound by $CE^{(Mer)}$. Analogously, we refer to the optimal Merton payoff (fraction) by $A_1^{(Mer)}$ $(m^{(Mer)})$.

3.3.2 Upper bound on shortfall probability and restriction to constant mix strategies

Schmeiser and Wagner (2015) consider the optimization problem under a shortfall probability condition but assume that the insurer implements a constant mix strategy. In consequence, the insurer does not consider a quantile hedge to honor the guarantee. To ensure the shortfall probability condition for a given guarantee, the insurer is restricted to

[12] Recall that $\alpha^E = 0$ implies $\alpha = 1$, cf. Corollary 3.2.8 . With $A_0 = 1$ it follows $L_1 = A_1$.

suitable combinations of investment fractions and equity capital. Amongst other results, Schmeiser and Wagner (2015) consider the optimization problem

$$\max_{g \in \mathcal{G}} \mathbb{E}_{\mathbb{P}}[u(L_1)],$$

where $\mathcal{G}$ denotes the set of admissible guarantee rates and where the equity fraction α^E and the investment fraction of the asset side $m^{(A)}$ are determined simultaneously by the conditions[13]

$$\mathbb{P}\left(A_1 < P_1^I\right) \leq \epsilon \text{ and } \mathbb{E}_{\mathbb{P}^*}\left[e^{-r} L_1\right] = 1.$$

Notice that $\mathbb{P}\left(A_1 < P_1^I\right)$ is analytically given by Equation (3.18). The liability value $\mathbb{E}_{\mathbb{P}^*}\left[e^{-r} L_1\right]$ is stated in Proposition 3.2.7 in combination with Equation (3.17).[14] A few comments are worth mentioning here: Schmeiser and Wagner (2015) consider the exact fulfillment of the shortfall probability corresponding to the minimum safety requirement where the ruin probability $\mathbb{P}\left(A_1 < P_1^I\right)$ is equal to the upper bound ϵ. Intuitively, this is meaningful if the shortfall constraint is binding in the case without equity capital, i.e. if the upper bound on the shortfall probability ϵ is sufficiently low compared to the lowest guarantee contained in the set $\mathcal{G}$.

[13] Notice that the condition $\mathbb{E}_{\mathbb{P}^*}\left[e^{-r} A_1\right] = 1 + \alpha^E$ is ensured since the insurer implements a constant mix strategy with initial investment $1 + \alpha^E$.

[14] Once the equity fraction α^E and the investment fraction of the asset side $m^{(A)}$ are determined, the expected utility (and CE) can be stated in quasi closed form. Schmeiser and Wagner (2015) determine the solution by Monte Carlo simulations.

Table 3.2: Certainty equivalents of quantile MRRGs under the additional restriction to constant mix strategies ($\epsilon = 0.005$)

g	α^E	$m^{(A)}$	L_0	SFP	$CE^{\gamma=2}$	$CE^{\gamma=3.56}$	$CE^{\gamma=5.94}$
					$m^{(Mer)} = 0.5$	$m^{(Mer)} = 0.28$	$m^{(Mer)} = \mathbf{0.169}$
					$CE^{(Mer)} = 1.0408$	$CE^{(Mer)} = 1.0363$	$CE^{(Mer)} = 1.0339$
-0.100	0.1285	0.5277	1	0.005	**1.0405**	1.0341	1.0247
					(1.0406)		
-0.095	0.1250	0.5101	1	0.005	1.0405	1.0345	1.0257
-0.090	0.1211	0.4921	1	0.005	1.0404	1.0348	1.0266
-0.085	0.1175	0.4745	1	0.005	1.0403	1.0351	1.0275
-0.080	0.1140	0.4571	1	0.005	1.0401	1.0353	1.0283
-0.075	0.1105	0.4397	1	0.005	1.0400	1.0355	1.0290
-0.070	0.1073	0.4229	1	0.005	1.0398	1.0357	1.0297
-0.065	0.1034	0.4047	1	0.005	1.0396	1.0359	1.0304
-0.060	0.1000	0.3876	1	0.005	1.0394	1.0360	1.0310
-0.055	0.0970	0.3710	1	0.005	1.0392	1.0361	1.0315
-0.050	0.0925	0.3521	1	0.005	1.0389	1.0361	1.0320
-0.045	0.0890	0.3347	1	0.005	1.0386	**1.0361**	1.0324
						(1.0362)	
-0.040	0.0850	0.3165	1	0.005	1.0383	1.0361	1.0328
-0.035	0.0812	0.2987	1	0.005	1.0380	1.0360	1.0331
-0.030	0.0775	**0.2811**	1	0.005	1.0377	1.0359	1.0334
-0.025	0.0738	0.2634	1	0.005	1.0373	1.0358	1.0336
-0.020	0.0694	0.2443	1	0.005	1.0369	1.0356	1.0337
-0.015	0.0653	0.2259	1	0.005	1.0365	1.0354	1.0338
-0.010	0.0611	0.2074	1	0.005	1.0360	1.0351	**1.0338**
							(1.0338)
-0.005	0.0569	0.1887	1	0.005	1.0356	1.0349	1.0338
0.000	0.0519	**0.1684**	1	0.005	1.0350	1.0345	1.0336
0.005	0.0471	0.1485	1	0.005	1.0345	1.0341	1.0334
0.010	0.0419	0.1278	1	0.005	1.0339	1.0336	1.0331
0.015	0.0362	0.1063	1	0.005	1.0333	1.0331	1.0328
0.020	0.0299	0.0833	1	0.005	1.0326	1.0325	1.0323
0.025	0.0219	0.0567	1	0.005	1.0318	1.0317	1.0317

In addition, we consider an exogenously given participation fraction α (e.g. $\alpha = 0.9$ as implied by German legislation). However, α ($1-\alpha$, respectively) implicitly defines a guarantee fee, i.e. the insured gives up some upside participation for downside protection. In particular, if α is already sufficiently low (compared to g), there does not exist an equity fraction $\alpha^E \geq 0$ such that the (fair) pricing condition can be satisfied, cf. Figure 3.4 and the results in Schmeiser and Wagner (2015).

As a numerical example, we refer to the benchmark parameter setting summarized in Table 3.1 and consider the above optimization problem for the guarantees g, taking the values $g \in \mathcal{G} = \{-0.1, -0.095, \ldots, 0.02, 0.025\}$ and a shortfall probability bound given by $\epsilon = 0.005$. For each $g \in \mathcal{G}$, Table 3.2 summarizes the combination of equity fraction α^E and investment fraction $m^{(A)}$ (implying that the SFP is exactly met and the contract is fairly priced) as well as the certainty equivalent contract wealths CEs of insureds which are described by three different levels of relative risk aversion ($\gamma = 2$, 3.56, and 5.94). In addition, the Merton solution is summarized in the upper line. For each level of relative risk aversion, the highest certainty equivalent (CE) is marked which implies the optimal guarantee rate. Observe that the CEs obtained by the (optimal) contracts are close to (but below) the Merton solution. In addition, the corresponding investment fractions $m^{(A)}$ are close to (but above) the Merton fractions. Intuitively, this is explained by the participation fraction α which is (along the lines of the benchmark parametrization) equal to $\alpha = 0.9$, i.e. the investor gives up 10% of the upside returns.

3.3.3 Optimal quantile payoff

As mentioned above, the Black and Scholes model is complete such that any state dependent payoff is attainable, i.e. it can be synthesized by a self-financing strategy in the asset S and the risk free investment opportunity B. In addition with the assumption that the contracts are fairly priced, we can obtain the utility maximizing quantile guarantee payoff L_1 with an initial investment of $P_0 = 1$, i.e. the optimal payoff is independent of the equity fraction α^E. Thus, w.l.o.g. we can set $\alpha^E = 0$. Recall from Corollary 3.2.8 that for $\alpha^E = 0$, a fair contract implies $\alpha = 1$, i.e. $L_1 = A_1 = \frac{A_1}{A_0}$ (since $P_0 = 1$ and $A_0 = 1 + \alpha^E = 1$), such that the optimization problem (3.19) simplifies to

$$\max_{A_1} \mathbb{E}_{\mathbb{P}}\left[u(A_1)\right] \text{ s.t. } \mathbb{P}\left(A_1 < 1+g\right) \leq \epsilon \text{ and } \mathbb{E}_{\mathbb{P}^*}\left[e^{-r}A_1\right] = 1. \tag{3.22}$$

The solution to this problem can already fully be traced back to Basak and Shapiro (2001) who state the optimal payoff (in dependence of the state prices) under a terminal VaR constraint.[15]

Proposition 3.3.1 (Optimal quantile return payoff)
If the shortfall probability is not binding, i.e. if $\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq 1+g\right) \leq \epsilon$, *the optimal solution coincides with the Merton solution. If the shortfall probability is binding, i.e. if* $\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq 1+g\right) > \epsilon$, *the optimal return payoff w.r.t. the optimization problem (3.22) is given as follows*

$$\frac{A_1^*}{A_0} = \beta\frac{A_1^{(Mer)}}{A_0} + \left(1+g-\beta\frac{A_1^{(Mer)}}{A_0}\right) 1_{\left\{\underline{K}<\beta\frac{A_1^{(Mer)}}{A_0}\leq\overline{K}\right\}},$$

where $0 \leq \underline{K} \leq \overline{K} := 1+g$. $\underline{K}$ *is determined by the SFP bound* ϵ *and* β *by the pricing condition, i.e.*

$$\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq \frac{\underline{K}}{\beta}\right) = \epsilon \text{ and } 1-\beta = e^{-r}\mathbb{E}_{\mathbb{P}^*}\left[\left(1+g-\beta\frac{A_1^{(Mer)}}{A_0}\right) 1_{\left\{\underline{K}<\beta\frac{A_1^{(Mer)}}{A_0}\leq\overline{K}\right\}}\right].$$

In the limiting cases $\epsilon \to 1$ *(no constraint on the shortfall probability) and* $\epsilon \to 0$ *(full guarantee) it holds*

(i) For $\epsilon \to 1$ *(and/or* $\mathbb{P}\left(\frac{A_1^{(Mer)}}{A_0} \leq 1+g\right) \leq \epsilon$*), it holds* $\beta = 1$, *and* $\underline{K} = \overline{K}$, *i.e. the optimal (return) payoff is given by the Merton solution* $\left(\frac{A_1^*}{A_0} = \frac{A_1^{(Mer)}}{A_0}\right)$.

(ii) For $\epsilon \to 0$, *it holds* $\underline{K} = 0$ *(and* $\overline{K} = 1+g$*) such that*

$$\frac{A_1^*}{A_0} = (1+g) + \left(\beta\frac{A_1^{(Mer)}}{A_0} - (1+g)\right)^+,$$

where β *solves*

$$1 = e^{-r}(1+g) + \beta Call^{(BS)}\left(\frac{1+g}{\beta}, \sigma_A^{(Mer)}\right)$$

and $Call^{(BS)}$ *is given by Equation (3.17).*[16]

[15] Basak and Shapiro (2001) state the optimal solution in dependence of the state prices for a general class of utility functions in a dynamic complete market setup where the investor can choose between one risk-less bond and several risky stocks.

[16] Notice that the pricing condition is, by means of the put call parity, now given in terms of the call price.

Instead of explicitly stating the adoption to our setup, it is worth to comment on the intuition behind the result. Obviously, if the quantile constraint is not binding, the optimal solution is given by the Merton solution. W.r.t. the other limiting case where the return payoff is constrained by a shortfall probability of zero ($\epsilon \to 0$), we also refer to El Karoui et al. (2005). The optimal unconstrained payoff is a modification of the Merton solution (unconstrained solution).[17] Intuitively, it is clear that a full hedge of the guarantee features a put option. Notice that

$$(1+g) + \left(\beta \frac{A_1^{(Mer)}}{A_0} - (1+g)\right)^+ = \beta \frac{A_1^{(Mer)}}{A_0} + \left((1+g) - \beta \frac{A_1^{(Mer)}}{A_0}\right)^+ ,$$

i.e. the return of the Merton solution is backed up by a put option with strike $K = 1+g$. The put payoff gives the tightest (and thus cheapest) possibility to obtain a full hedge of the guarantee. Thus, it enables the investor to obtain the tightest modification of the unconstrained optimal payoff.

To honor the pricing condition, i.e. the value of the payoff must be equal to one, the investor can no longer obtain the full Merton return but only a fraction β of it. In particular, while the value of $\frac{A_1^{(Mer)}}{A_0}$ is equal to one, the investor now receives only a fraction of the return, i.e. in the presence of a (non vanishing) guarantee, her investment amount which is not needed to finance the put is only a fraction β $(0 < \beta < 1)$.

In summary, the fraction β is determined by a fix point problem which is due to the condition that the value of the put on the return $\beta \frac{A_1^{(Mer)}}{A_0}$ must be equal to the reduction of the initial investment $1-\beta$ (i.e. both sides depend on β). Intuitively it is now clear that any deviation from a perfect guarantee ($\epsilon \to 0$), an admissible shortfall probability which is higher than zero gives rise to lower hedging costs than the solution characterized above. While in the case of a zero shortfall probability the optimal payoff is given by

$$\beta \frac{A_1^{(Mer)}}{A_0} + \left((1+g) - \beta \frac{A_1^{(Mer)}}{A_0}\right) 1_{\left\{\underline{K} < \beta \frac{A_1^{(Mer)}}{A_0} \leq \overline{K}\right\}},$$

where $\underline{K} = 0$ and $\overline{K} = 1+g$, the investor is now allowed to implement a smaller *guarantee interval* $[\underline{K}, \overline{K}]$ where $0 \leq \underline{K} < \overline{K} \leq 1+g$. Notice that the upper bound on the shortfall probability implies that fixing either $\underline{K}$ or $\overline{K}$ implies the other strike such that β is determined by the resulting fix point problem.

[17] In fact, the result does not depend on the Black and Scholes model which implies the Merton solution.

However, the cheapest way to do so is by setting $\overline{K} = 1 + g$, i.e. starting with the high asset prices (Merton returns, respectively) which are linked to the cheapest states (to be hedged). In summary, the optimal quantile hedge is a scaled version of the Merton solution overlaid by the (cheapest) quantile hedge which honors the SFP bound.[18] In order to illustrate the improvement obtained by the optimal quantile hedge, we add in Table 3.2 the CEs associated with the optimal quantile guarantees, cf. italic numbers in brackets below the bold faced numbers referring to the optimal values under the restriction to constant mix strategies (and choosing the guarantee). Again, it is worth to emphasize that the optimal quantile payoff can be implemented for any equity fraction α^E of the insurer.

3.4 Interest rate risk management

Interest rate risk plays an important role for life insurance companies. In particular when assessing life insurance liabilities, interest rate risk should be considered. In the following section we shed a light on the interest rate risk of the above stated guarantee scheme. In order to evaluate the interest rate risk, we first recall the arbitrage-free price at time $t = 0$ of the payoff L_1 in a Black and Scholes model setup.

Proposition 3.4.1 (arbitrage-free price at time t of the payoff L) *The arbitrage-free price at time t of the payoff L is given by the following:*

(i) ***Low equity ratio:*** *For $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$, it holds*

$$L_t = A_t - (1 - \alpha + \alpha^E) Call(K_3). \tag{3.23}$$

(ii) ***High equity ratio:*** *For $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$, it holds*

$$L_t = A_t - \left(1 + \alpha^E\right) Call(K_2) + \alpha Call(K_1). \tag{3.24}$$

where the strikes K_1, K_2 and K_3 are defined as in Equation (3.7) and further $Call(K) = Call^{(BS)}(K, \sigma_A, t)$ is given by the Black and Scholes pricing formula (w.r.t. the returns),

[18] W.r.t. quantile hedges, the interested reader is referred to Föllmer and Leukert (1999) who show how to obtain the highest success probability when hedging a claim with a lower initial investment than the one needed for a full hedge (or the other way round).

i.e.

$$Call^{(BS)}(K,\sigma_A,t) = A_t\mathcal{N}(d_1(K,\sigma_A)) - e^{-r(T-t)}K\mathcal{N}(d_2(K,\sigma_A)), \tag{3.25}$$

$$\textit{where } d_1(K,\sigma_A) = \frac{\ln\left(\frac{A_t}{K}\right) + \left(r+\frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}} \textit{ and } d_2(K,\sigma_A) = d_1(K,\sigma_A) - \sigma_A\sqrt{T-t}.$$

Next we define the interest rate sensitivity of the liability L_t.

Proposition 3.4.2 (Interest rate sensitivities) *The interest rate sensitivity of the liability L_t is given by following:*

(i) ***Low equity ratio:*** *For $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$, it holds*

$$\frac{\partial L_t}{\partial r} = (1-m^{(A)})A_t - (1-\alpha+\alpha^E)\frac{\partial Call(K_3)}{\partial r}. \tag{3.26}$$

(ii) ***High equity ratio:*** *For $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$, it holds*

$$\frac{\partial L_t}{\partial r} = (1-m^{(A)})A_t - \left(1+\alpha^E\right)\frac{\partial Call(K_2)}{\partial r} + \alpha\frac{\partial Call(K_1)}{\partial r}. \tag{3.27}$$

where the strikes K_1, K_2 and K_3 are defined as in Equation (3.7) *and the interest rate sensitivity of the call option $Call(K) = Call^{(BS)}(K,\sigma_A,t)$ is given by*

$$\begin{aligned}
\frac{\partial Call^{(BS)}(K,\sigma_A,t)}{\partial r} =& (1-m^{(A)})A_t\mathcal{N}(d_1(K,\sigma_A)) + A_t\mathcal{N}'(d_1(K,\sigma_A))\frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}} \\
&+ (T-t)e^{-r(T-t)}K\mathcal{N}(d_2(K,\sigma_A)) \\
&- e^{-r(T-t)}K\mathcal{N}'(d_2(K,\sigma_A))\frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}} \\
\textit{where } d_1(K,\sigma_A) =& \frac{\ln\left(\frac{A_t}{K}\right) + \left(r+\frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}} \\
\textit{and } d_2(K,\sigma_A) =& d_1(K,\sigma_A) - \sigma_A\sqrt{T-t}.
\end{aligned}$$

Proof: The proof is given in Appendix C. □

The interest rate sensitivity of the asset side is given by

$$\frac{\partial A_t}{\partial r} = (1-m^{(A)})A_t$$

and the interest rate sensitivity of the buffer is then defined by $\frac{\partial A_t - L_t}{\partial r}$.

Interest rate sensitivities for different levels of g, α and α^E

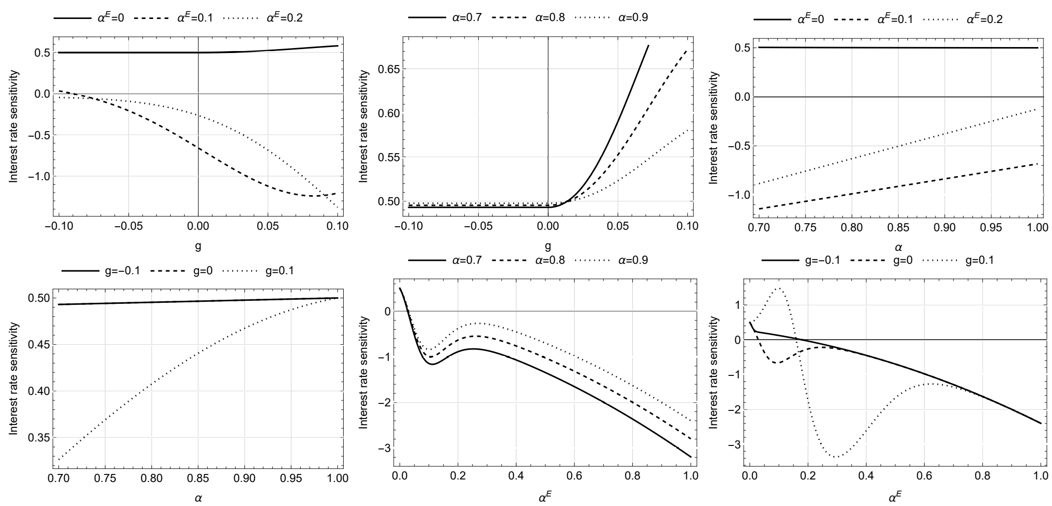

Figure 3.6: This Illustration show the interest rate sensitivity for different levels of g, α^E and α. If not stated otherwise the parameters are given by Table 3.1 and $\alpha^E = 0$.

First we discuss the interest rate sensitivity in more detail. Therefore, Figure 3.6 illustrates the interest rate sensitivity of the price of the liability L. For the parameter setup given by Table 3.1 and $\alpha^E = 0$ the illustration shows the interest rate sensitivity for different levels of g, α^E and α. For a rather low equity ratio the interest rate sensitivity is increasing in the guaranteed rate g. Furthermore the interest rate sensitivity is increasing with increasing participation fraction α. Moreover, Figure 3.6 shows that the equity ratio α^E does not have a monotonic effect on the interest rate sensitivity. Only in the case of a negative guarantee rate g it can be seen that the interest rate sensitivity falls monotonically as the equity ratio α^E rises.

We then examine interest rate sensitivity for quantile MRRGs. For this purpose, we add the corresponding values of the interest rate sensitivity to the Table 3.2. The results are given in Table 3.3. We also added the interest rate sensitivities of the asset portfolio as well as of the buffer. On the asset side, the influence of the interest rate is only represented by the investment in the risk-free bond. For this reason, the interest rate sensitivity of the asset portfolio increases as the guarantee rate g rises, as this is also accompanied by an increase in the share of the risk-free bond given by $(1 - m^{(A)})$. Table 3.3 shows that the interest rate sensitivity of the liabilities is first increasing with g until it hits its maximum in $g = -0.035$. Then it decreases with increasing g. For $g \leq -0.1$ and $g \geq 0$ the interest rate sensitivity of the liabilities is negative. Furthermore Table 3.3 shows that the asset portfolio is more sensitive to changes in the interest rate than the liabilities. The lowest interest rate sensitivity of the buffer value is therefore given by the smallest guarantee rate considered in this observation interval $g = -0.1$, because this also leads to the lowest bond fraction $(1 - m^{(A)})$. A visualization of these results is given in Figure 3.7.

Table 3.3: Interest rate sensitivity of quantile MRRGs

g	α^E	$m^{(A)}$	L_0	SFP	$\frac{\partial L_t}{\partial r}$	$\frac{\partial A_t}{\partial r}$	$\frac{\partial A_t - L_t}{\partial r}$
-0.100	0.1284	0.5274	1	0.005	-0.0090	0.5333	**0.5422**
-0.095	0.1252	0.5104	1	0.005	0.0055	0.5509	0.5454
-0.090	0.1213	0.4924	1	0.005	0.0210	0.5692	0.5482
-0.085	0.1178	0.4749	1	0.005	0.0353	0.5870	0.5517
-0.080	0.1142	0.4574	1	0.005	0.0489	0.6046	0.5557
-0.075	0.1107	0.4400	1	0.005	0.0617	0.6220	0.5603
-0.070	0.1073	0.4228	1	0.005	0.0735	0.6391	0.5656
-0.065	0.1037	0.4054	1	0.005	0.0845	0.6563	0.5717
-0.060	0.1001	0.3879	1	0.005	0.0948	0.6734	0.5786
-0.055	0.0962	0.3698	1	0.005	0.1045	0.6908	0.5863
-0.050	0.0928	0.3526	1	0.005	0.1118	0.7075	0.5956
-0.045	0.0891	0.3349	1	0.005	0.1180	0.7244	0.6064
-0.040	0.0853	0.3171	1	0.005	0.1219	0.7412	0.6192
-0.035	0.0815	0.2993	1	0.005	**0.1233**	0.7578	0.6345
-0.030	0.0774	0.2809	1	0.005	0.1221	0.7748	0.6527
-0.025	0.0735	0.2628	1	0.005	0.1169	0.7914	0.6745
-0.020	0.0695	0.2445	1	0.005	0.1068	0.8080	0.7012
-0.015	0.0653	0.2260	1	0.005	0.0889	0.8245	0.7357
-0.010	0.0611	0.2072	1	0.005	0.0641	0.8412	0.7772
-0.005	0.0567	0.1882	1	0.005	0.0247	0.8578	0.8331
0.000	0.0520	0.1687	1	0.005	-0.0359	0.8745	0.9104
0.005	0.0472	0.1487	1	0.005	-0.1224	0.8915	1.0139
0.010	0.0420	0.1280	1	0.005	-0.2632	0.9086	1.1719
0.015	0.0363	0.1063	1	0.005	-0.5088	0.9261	1.435
0.020	0.0298	0.0830	1	0.005	-1.0132	0.9443	1.9575
0.025	0.0220	0.0568	1	0.005	-2.4190	0.9640	3.3829

Interest rate sensitivity for different levels of g

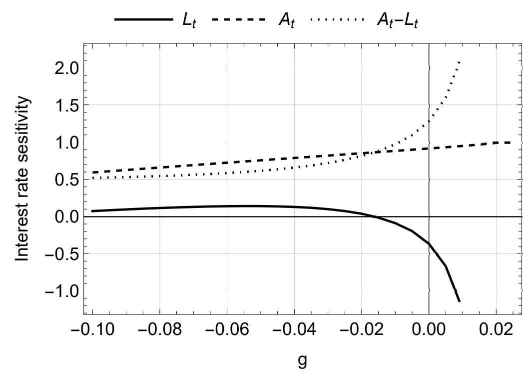

Figure 3.7: This Illustration show the interest rate sensitivity of quantile MRRGs L_t, the asset portfolio A_t and the Buffer $A_t - L_t$.

3.5 Conclusion

This chapter analyzes the optimal design of participating life insurance contracts with minimum return rate guarantees under default risk. The benefits to the insured depend on the performance of an investment strategy which is conducted by the insurer. This strategy is initialized by an amount given by the sum of equity and the contributions of the insured. Unless there is a default event, the insured receives the maximum of a guaranteed rate and a participation in the returns. Considering default risk modifies the payoff of the insured by means of a default put implying a compound option feature (nested maximum). Based on yearly returns, we show that,in spite of the compound option feature, the (yearly return) payoff of the default put (and the liabilities to the insured) can be represented by piecewise linear functions of the investment return, i.e. the payoff of a portfolio of plain vanilla options. Thus, the liabilities are easily priced in any model setup which gives closed form solutions for standard options.

In a complete market setup, we then derive the optimal (expected utility maximizing) quantile guarantee payoff of an investor/insured with constant relative risk aversion. Because of the completeness assumption, the return payoff can be implemented by the insurance company for any equity to debt ratio. We illustrate the utility loss which arise if the insurer implements a suboptimal investment strategy.

We further investigated on the interest rate risk of those quantile guarantees. We therefore calculated the effective duration and convexity for suitable combinations of the minimum return rate guarantee g and the corresponding fair equity ratio α^E.

For future work, it would be useful to extend the model to $T > 1$ and additionally consider, for example, periodic contributions. It could also be of interest to consider further interest rate models in order to ensure a more accurate assessment of interest rate risk. Furthermore, the consideration of mortality risks and surrender options would be a useful addition.

Chapter 4

The interest rate risk submodule under Solvency II: An Overview

4.1 Introduction

The crises of the past few years, as well as those that are still ongoing, have led to a rethinking on the part of financial regulators. In this process, regulation has become more comprehensive and demanding. In the area of banking regulation in Europe, a revision of Basel III regulation has been published in the form of the Review of the Trading Book. In the insurance sector, the area of risk measurement has been revised within Solvency II. In addition, the rapid development of financial products is accompanied by increasing complexity, which makes valuation more demanding. Furthermore, the ongoing low-interest phase continues to cause difficulties for insurance companies. Life insurance companies in particular are experiencing difficulties in fulfilling their guaranteed returns. Moreover, the range of innovative contracts is increasing compared to traditional life insurance policies, which also involve riskier investments. At the same time, the attractiveness for customers must be ensured.

In this respect, quantitative requirements for insurance companies are becoming increasingly important. Solvency II provides the regulatory basis for this. In order to maintain stability, insurance companies are required to form capital reserves. Too low capital reserves represents an insufficient hedge against various risks, while too high capital reserves impairs insurers' ability to do business by lowering their own funds. These capital reserves form the Solvency Capital Requirements (SCR) under Solvency II.

Solvency II is a supervisory system for the European insurance market which entered into force in 2016. Under the Solvency II regulation interest rate risk is being addressed by imposing the SCR based on the 99.5% Value at Risk. This SCR is basically determined either by a standard formula or an internal model. The standard formula is specified by the regulator with implementing standards, the internal model is developed by the insurer itself and is certified by the supervisory authorities. Most insurers to which this regulation applies use a standard formula to calculate the SCR. The standard formula is based on a bottom-up approach, where the basic SCR (BSCR) is being calculated based on six risk modules. In this paper we refer to the market risk model and more precisely to the interest rate risk sub module which is addressed in the delegated regulation (EUR-LEX (2015) Art. 165 ff.).

There is already a large body of literature that examines the SCR with regard to various aspects. The aim of this paper is to compile the results of these studies and to provide an overview of the strengths and weaknesses of the SCR with regard to protection against interest rate risk. This brief literature review shall establish an understanding of the Solvency II interest rate risk submodule. At the end, the question whether the SCR under the current Solvency II regulations represents an adequate capital reserve for insurance companies regarding interest rate risk will be discussed on the basis of the available scientific literature.

An overview on internal risk models in the context of Solvency II is given by Liebwein (2006). An analysis of the mathematical aspects of the standard formula is given by Scherer and Stahl (2021). Möhlmann (2021) investigates the sources of interest rate risk for life insurers.

The rest of the chapter is organized as follow. The following section begins with a brief introduction to Solvency II regulation. It then introduces the interest rate risk submodule and discusses the minimum capital requirements under the standard formula. The fourth section summarizes the results from a collection of research papers that relate to capital requirements under the interest rate submodule. The final section concludes.

4.2 Solvency II regulation and standard approach

The main objective of the supervision through the Solvency II regulation is the protection of policyholders. In this context, Solvency II regulation follows a three-pillar approach. The first pillar deals with quantitative requirements for capital adequacy, the second pillar with qualitative requirements for risk management and corporate governance, and the third pillar with market transparency and reporting requirements. The capital adequacy requirements under the first pillar are intended to ensure that insurance companies are protected against possible unexpected losses arising from the risks to which they are exposed. These capital requirements are determined in two steps. First, a solvency balance sheet is to be prepared containing all assets and values of all liabilities. The difference between the assets and values of all liabilities represents the basic own funds. In the second step, the SCR is determined on the basis of the various risk modules. Sufficient solvency is ensured if the basic own funds plus additional own funds exceed these SCR. The SCR should be high enough to ensure that insolvency occurs at most every 200 years (i.e. the one-year probability of default is less than or equal to 0.5%). In order to ensure this, the value at risk at the confidence level 0.995 of the distribution of own funds at the end of the year shall be used to determine the SCR in general. In order to determine the SCR, one can either use the standard formula, resort to an internal model, or use a partial model as a hybrid of both approaches. The standard formula is specified by regulation and provides a simplified representation and measurement of risks. The internal model, on the other hand, is developed by the insurer itself and provides a more detailed representation and measurement of the risks. Compared to the standard formula, the internal model is more complex, but offers more flexibility.

The standard formula is based on a modular structure. A distinction is made between actuarial risks (health, life and non-life risks), market risks, counterparty default risks and intangible asset risks, with some of these modules consisting of further submodules. The SCR using the standard formula are determined using a "bottom-up" basic principle. First, the SCR of the individual submodules are determined. Then, taking diversification effects into account, the SCR of the individual submodules are aggregated to form the basic SCR (BSCR). The final SCR is then composed of this BSCR, the SCR for operational risks and an adjustment for the risk-reducing effect of the surplus participation and deferred taxes.

The market risk module takes into account the risk arising from the level and volatility of the market prices of financial instruments that affect the measurement of assets and liabilities. The interest rate risk submodule is part of the market risk module. It represents the change in basic equity based on a change in the risk-free nominal interest rate. In the interest rate risk sub module, the capital requirements Mkt_{int} are determined based on the present value of interest rate risk exposures. The present value PV of all interest rate sensitive exposures is given by discounting the respective aggregated cash flows $CF(t) = \sum_{j=1} CF_j(t)$ of all single cash flows at time t using the risk-free rate r_t at time

t, i.e.

$$PV = \sum_{t=1}^{T} \frac{CF(t)}{(1+r_t)^t}$$

with $T = \max\{t|CF(t) \neq 0\}$. The interest rate sub module is based on the consideration of two different stress scenarios, a upward and a downward shift of the interest rate. Thus stressed present values are being determined adding a upward shock given by $s^{up}(t)$ at time t and a downward shock $s^{down}(t)$ at time t on the risk free rate r_t. The corresponding present value is then given by

$$PV^k = \sum_{t=1}^{T} \frac{CF(t)}{(1 + r_t \cdot (1 + s^k(t)))^t}, k \in \{\text{up, down}\}.$$

To obtain the Mkt_{int} the impact of the stress scenarios need to be calculated by determined the difference between the present value and the stressed present value

$$Mkt^k = PV - PK^k, k \in \{\text{up, down}\}.$$

The capital requirements for the interest rate sub module are then given by

$$Mkt_{int} = \max\{Mkt^{up}, Mkt^{down}\}.$$

4.3 Selected aspects of regulatory requirements for interest rate risk according the standard model of Solvency II addressed by the literature

4.3.1 Comparison between standard model and partial internal model

Gatzert and Martin (2012) compare the standard formula for the market risk module with a partial internal model by looking at a asset portfolio consisting of fixed income bonds and stocks. The SCR in the internal modal is calculated by the change of the market value of the asset portfolio over one year applying the Value at Risk with confidence level 99.5% which is in line with Solvency II. In the internal model it is assumed that the stocks follow a geometric Brownian motion and the short rate is described by the CIR short rate model. The capital requirements are then derived by

$$SCR^{IM} = MV(0) - VaR_{0.005}\left(e^{-\int_0^1 r_t dt} MV(1)\right)$$

where $MV(t)$ is the market value of the asset portfolio at time t. The market value of the asset portfolio at time $t = 1$ is discounted with the risk free short rate r_t. To asses credit risk, Gatzert and Martin (2012) uses the Jarrow-Turnbull model which quantifies credit risk by credit ratings and possible changes in those ratings. To take into account model risk Gatzert and Martin (2012) replace the stock, interest rate and credit risk models. Furthermore, diversification effects between stocks and bonds are also investigated in two ways. The first level diversification effect describes the diversification effects of pure stock or bond portfolios and is defined by

$$d_1(K) = \frac{SCR_K}{\sum_{k \in K} SCR_k} - 1, K \in \{S, B\}$$

where SCR_k is the capital requirement corresponding to a individual asset k of the asset class of stocks S or bonds B and SCR_K denotes the capital requirements for a portfolio consisting only of stocks or bonds. The second level diversification effect considers portfolio of both stocks and bonds and is defined by

$$d_2(S + B) = \frac{SCR_{S+B}}{SCR_S + SCR_B} - 1$$

where SCR_{S+B} denotes the capital requirements for a portfolio including stocks and bonds.

Gatzert and Martin (2012) demonstrate that the SCR calculated by the standard formula is inappropriate comparing to their internal model, especially for bond investments. They show that the predefined scenario in the standard model both over- and underestimates the actual risk belonging to investments, such that the risk situation is not being sufficiently reflected. Regarding diversification effects, Gatzert and Martin (2012) show that although the SCR for individual stocks is higher in case of the internal model, the SCR for a portfolio of those stocks in lower. So diversification effects implied a significant reduction of the SCR in the internal model compared to the standard model. Furthermore Gatzert and Martin (2012) show that model risk can have a strong impact on the internal model. The capital requirements are substantially higher if the stock dynamics are modeled by the Heston model instead of a geometric Brownian motion. Regarding interest rate models the SCR is higher using the Vasicek model than CIR model for high rated bonds like AA or AAA. For B-rated bonds this difference almost vanish. Gatzert and Martin (2012) explain this effect with the diminishing importance of interest rate risk for lower rated bonds, where credit risk is becoming the major risk driver in the SCR.

Asadi and Al Janabi (2020) compare the standard model under Solvency II with another innovative internal model, with regard to sufficient capital requirements in times of extreme events. Equity risk, interest rate risk and spread risk were taken into account for different stock and bond portfolios. With regard to the internal model, the marginal distribution of the individual stocks is modeled using a hybrid Glosten-Jagannathan-Runkle Extreme Value Theory (GJR-EVT) model. Subsequently, the students t-copula function is used to model the dependence structures of the individual stocks on each other. Finally, the SCR is determined using the value at risk. To assess the risks for bond portfolios, an approach based on Lando and Mortensen (2003) is used. The Cox-Ingersoll-Ross model is used to model the short rate.

On the one hand, Asadi and Al Janabi (2020) show that the internal model requires a higher SCR than the standard model for the equity risk for stock portfolios. For bonds with high rating quality and long maturity, the results between the internal model and the standard model are similar. This is also in line with the findings of Gatzert and Martin (2012). However, bonds with low rating quality and short maturity are overvalued according to the standard model. In this case, the interest rate risk is underestimated and the spread risk overestimated. This result contradicts the findings of Gatzert and Martin (2012), using their internal model approach.

4.3.2 Impact on asset allocation

Höring (2013) examines the impact of the Solvency II standard model capital requirement for the market risk module on the asset allocation of European insurers. In doing so, Höring (2013) examines whether the capital requirements for market risk under Solvency II lead to a mandatory portfolio rebalancing for European insurers. To test this, a fictitious and representative European life insurer is assumed. The capital that the life insurer must hold for market risk is determined and compared using the standard model under Solvency II and the Standard & Poor's (S&P) rating model. S&P evaluates insurance companies with regard to their creditworthiness on the basis of quantitative and qualitative criteria. In the process, certain security levels are defined for different rating levels. The target rating of "A" requires risk-based capital requirements based on a confidence level of 99.4 percent, which corresponds to the confidence level of 99.5 percent required by Solvency II for calibration. For a rating of "BBB", the confidence level is 97.2 percent, "AA" is based on a level of 99.7 percent and "AAA" is based on 99.9 percent. The diversification effects between shares, bonds and real estate are also taken into account at S&P. The market risk portfolio of the fictitious insurer consists of 82 percent debt instruments, of which 49 percent is sovereign debt and 36 percent is corporate debt. The remainder are covered bonds. The duration of the liabilities is 8.9 years and the duration gap between assets and liabilities is 2.1 years.

Höring (2013) shows that, taking into account the risk-mitigating effect of provisions and deferred taxes of Solvency II, insurance companies have to hold 68 percent more capital for net market risk with a target rating of "A" compared to Solvency II. If only the interest rate risk is considered, the capital requirements under Solvency II are 14 percent higher. Even for a target rating of "BBB", the capital requirements under S&P are 27 percent higher than under the Solvency II standard model. Höring (2013) concludes from these findings that insurers with a good credit rating and good regulatory solvency standing are not expected to adjust their asset allocation due to the introduction of the Solvency II regulation.

Nevertheless, the study presented is also subject to limitations. Firstly, when considering the fictitious insurer, no precise assumptions are made about the liability portfolio in terms of policies issued. Furthermore, a comparison of capital requirements is made between the standard and the rating model, which assumes that the insurers only maintain the regulatory minimum ratio. Also, the insurance company may use internal models or partial internal models instead of the standard formula.

Braun et al. (2018) examine the impact of Solvency II on risk-adjusted performance measures of life insurers through adjustments to asset allocations. For this purpose, a stylized insurance company is considered. A dynamic single-period model is used to model the assets and liabilities. The initial investment portfolio is divided into equity and the premium payments of the policyholders. The initial liabilities are given by the present value of the expected future payments to its policyholders. The asset return and the liability growth rate are assumed to be normally distributed. Furthermore, a diversification index is introduced, which is based on the Herfindahl index and reflects the level of concentration within the asset portfolios. The return on risk-adjusted capital (RORAC) is considered as a performance measure. Here, the expected change in equity is set in relation to the SCR according to the market risk standard model. Five typical investment types of a European life insurer are considered, consisting of stocks, government bonds, real estate, corporate bonds and money market instruments. Two scenarios are considered. For the first scenario, historical data between 2000 and 2015 is used to calibrate these investments and the investment in money market instruments is not considered. The second scenario takes into account data between 2011 and 2015 and includes the investment in money market instruments. This distinction is intended to represent different interest rate environments, business cycles and other macroeconomic effects.

The results from Braun et al. (2018) show that there is a negative correlation between the diversification index and the asset risk represented by the standard deviation of the asset portfolio. This means that a lower-risk portfolio implies greater diversification. Moreover, the diversification index in the first scenario is more dispersed than in the second scenario. Analyzing the expected profit for each portfolio, it can be seen that the average profit for the first scenario is 389 million euros and for the second scenario 287 million euros. Looking at the Solvency II capital requirements, the average capital requirement for the first scenario is 679 million euros and for the second scenario 724 million euros. Two conclusions can be drawn from these observations. First, asset portfolios with a higher risk achieve a higher expected profit. On the other hand, lower-risk and thus broadly diversified portfolios are subject to a higher capital requirement under Solvency II.

When analyzing the RORAC values, the reverse effect can be observed for the capital requirements. For the first scenario, the average RORAC values are 61.1% and for the second scenario 41.8%. Consequently, the low-risk and thus well-diversified portfolios are associated with low RORAC values. This is the exact opposite for the risky asset compositions. This shows that the RORAC values are more strongly influenced by the Solvency II capital requirements than the expected profits. This is the exact opposite for the risky asset portfolios. It shows that the RORAC values are more strongly influenced by the Solvency II capital requirements than the expected profits. To the extent that an insurance company uses RORAC as a key performance indicator, it will seek to reduce the cost of capital rather than maximize expected profits.

In summary, Braun et al. (2018) shows that the RORAC essentially depends on capital requirements. Expected profits, on the other hand, play a minor role. In addition, the standard formula requires a low capital charge for low-diversified portfolios with a high asset risk. This leads to high RORAC values. Lower-risk portfolios that are broadly diversified require a higher capital deposit. This results in low RORAC values. If a life insurer manages its performance measurement on the basis of RORAC, this can have a negative impact, especially for its stakeholders.

In addition, there is further literature that has examined the potential impact of Solvency II on asset allocation. Van Bragt et al. (2010) investigate the impact of Solvency II on the risk and return trade off for life insurance companies. They show that asset allocation and asset duration can have a large impact on the Solvency II capital requirements.

Reducing the interest rate risk by matching the duration of the asset and liabilities might decrease the long term expected return and thus also the capital requirements.

The actual changes in the asset allocation of German life insurers from 2005 to 2021 is illustrated in Figure 4.1 based on GDV (2014) and GDV (2022). The Solvency II directive was published in 2009 and has been in force since January 2016. Figure 4.1 shows that since publication, the proportion of stocks has increased annually from 25,4 % in 2009 to 41,4 % in 2010 and the proportion of bonds has decreased from 58,5 % in 2009 to 41,2 % in 2021.[1] Within the bond shares, it is clear that the share of registered bonds, promissory notes and loans is falling while the share of bearer bonds and other fixed-income securities is rising since publication.

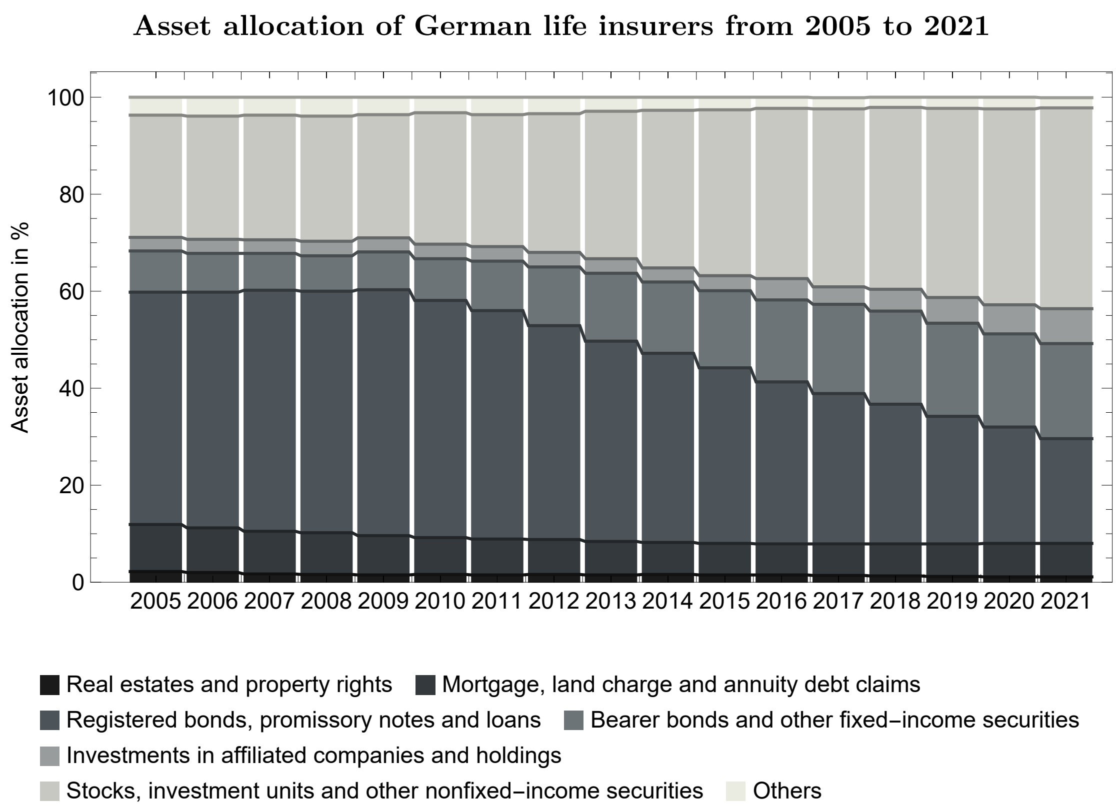

Figure 4.1: This Figure shows the changes in the asset allocation of German life insurers from 2005 to 2021 based on data from GDV (2014) and GDV (2022).

[1] Bond shares include registered bonds, promissory notes and loans as well as bearer bonds and other fixed-income securities.

4.3.3 Impact of model risk

Martin (2013) ties in with Gatzert and Martin (2012) and compares the SCR for interest rate risk and credit risk derived from the standard model with the one derived by the partial internal model introduced by Gatzert and Martin (2012) by assessing model risk. In doing so, the author refers to the choice of model for interest rates and examines what impact the choice of model has on the SCR with respect to bond investments. Three interest rate models in continuous time are considered, the Merton model, the Vasicek model and the CIR model. These models are considered because they provide bond prices with an affine term structure, which allow for analytical tractability. Moreover, similar to Gatzert and Martin (2012), credit ratings are adapted using the Jarrow Lando Turnbull model. So bonds are assigned with credit ratings whose dynamics are represented by Markov chains. Both corporate and government bonds are considered. For individual bonds as well as for bond portfolios, the SCR is calculated regarding interest rate risk, credit risk and for both risks, taking into account diversification benefits, using the internal model based on the three interest rate models and the Solvency II standard formula.

First, single corporate bonds were considered. For corporate bonds with a high rating, the SCR is affected by the choice of the interest model, as the interest risk drives a bigger role for the SCR compared to the credit rating. Here the Merton model lead to the highest SCR in the internal model, while CIR model leads to the lowest SCR. Nevertheless, the standard formula provides the highest SCR taking both risks into account. For corporate bonds with a low rating, the SCR with regard to the credit risk increases significantly for all approaches. It is clear that the standard formula significantly underestimates the credit risk compared to the internal model. With regard to the internal model, the SCR is similar for all interest rate models. For corporate bonds with a long maturity, the interest rate risk increases. The Merton model provides the highest SCR for interest rate risk, while the CIR model gives the lowest SCR. For government bonds, the results are the similar to those for corporate bonds.

The SCR for the bond portfolio is then considered, with the bond portfolio consisting largely of highly rated government and corporate bonds. The results for the interest rate risk as well as for the credit risk are the similar to the single bond exposure. In addition, the impact of the interest rate model parameters on the SCR in the internal model is examined. For this purpose, both the initial interest rates r_0 and the volatility σ_r of each interest rate process is separately increased and decreased by 20 percent. These shocks show a great influence on the interest rate risk. For example, a 20 percent decrease in the initial interest rate leads to a 17.5 percent decrease in the SCR in the CIR model, and a 20 percent increase in volatility leads to a 15.3 percent increase in the SCR in the Merton model. This identifies model parameter calibration as an important source of risk when using internal models.

Subsequently, the difference between the SCR determined by the standard model and the internal model is examined, considering all three interest rate models. It becomes clear that the SCR for the standard model differs significantly from the SCR for the internal models. The difference is smallest when the Vasicek model is used. The lowest SCR level is provided by the CIR model.

Overall, Martin (2013) shows that model risk has a major impact on the SCR for government and corporate bonds in several aspects. On the one hand, the choice of interest rate model in the internal model is an important factor. Furthermore, the model parameter calibration is also an important risk factor. Moreover, Martin (2013) shows that diversification has a large impact on the SCR of bond exposures.

4.3.4 Default probability

Braun et al. (2015) address the default probability implied by the standard formula in respect of the market risk submodule. In addition, the standard model is compared with an own partial internal model. Braun et al. (2015) consider a stylized balance sheet approach to capture the asset portfolio structure of life insurers. For this purpose, a large number of mean-variance efficient portfolios based on six different asset classes are constructed and then examined with regard to their SCR. The investment limits imposed by the German regulator are taken into account as well as short sale constraints in the construction of those portfolios. In total about 130,000 portfolios without investment limits and 75,000 portfolios with investment limits are considered. First, the capital requirements are calculated according to the standard model. It shows that portfolios with a very high proportion of investments in the money market and in government bonds are admissible under Solvency II. Admissible under Solvency II means that the following condition must be met:

$$SCR_{Mkt} \leq BOF_0$$

i.e., that the capital requirements under the Solvency II standard model regarding the market risk module are covered by the current basic own fund of the insurer. Subsequently, the actual default probability of the individual portfolios was determined using the partial internal model. This shows that some of the portfolios admissible under Solvency II have a significantly higher default probability than the 0.5 percent targeted by Solvency II. A sensitivity analysis shows that an increase in equity capital increases the number of admissible portfolios, but also increases the number of admissible portfolios with an actual default probability of more than 0.5 percent. This increases the likelihood of selecting a portfolio allocation with an inappropriately high default probability.

Braun et al. (2015) conclude from their findings that the current framework and the calibration of the Solvency II standard model for market risk are insufficient. Furthermore, they consider the assessment by the standard model using only stress scenarios to be incompatible with the insurance industry, as this approach is designed to avoid risks rather than to offset them.

Fischer and Schlütter (2015) shows that reducing the investment risk by increasing the stock risk parameters in the calibration of the standard formula does not necessarily lead to a reduction in the default probability.

4.3.5 Stress scenario calibration

EIOPA (2016) investigates whether the Solvency II interest rate risk calibration is still adequate in the current low interest situation. The standard formula for the interest rate sub module was firstly released in 2009 where negative interest rates were not yet foreseeable. Since then, no adjustments have been made to the shock scenarios with regard to negative interest rates. EIOPA (2016) states that one drawback of the standard formula is, that the shock scenarios are given in terms of percentage on the actual short rate, so the absolute shock is decreasing with decreasing interest rates and it is zero for negative rates. In the standard formula there is a minimum upward shock of 1% and no minimum downward shock. In addition, negative interest rates are not stressed downwards. EIOPA (2016) therefore concludes that the current shock calibration is not appropriate. Analogous to the upward show, introducing a minimum downward shock of 1%, as proposed in CEIOPS (2009), would result in only a small improvement. Furthermore, it would also only bring a smaller improvement if negative interest rates were also stressed downward. EIOPA (2016) proposes alternative methods to incorporate the stress scenarios into the risk-free curve. One approach is given by an additive stress where the stressed short rate is of the form

$$r_t^k = a^k r_t + b^k, k \in \{\text{up, down}\}$$

where a^k and b^k depend on the scenario and the maturity. To sum up, EIOPA (2016) show that the stress scenarios significantly underestimates interest rate risk in the current interest rate environment.

Fischer and Schlütter (2015) examine the impact of the standard formula calibration on insurer's capital and investment strategy. They show that the insurer decreases the fraction invested in risky stocks, when the stock risk parameter increases in the calibration, which is intuitive. A low equity risk stress factor in the calibration can lead to higher investment risk without impacting capital requirements.

4.3.6 Comparison between Value at Risk and Expected Shortfall

In accordance with Solvency II, the Value at Risk (VaR) with a level of 99.5 percent on a 1-year period is used to calibrate the stress scenarios of the standard model. Boonen (2017) investigates the impact on the SCR of using the Expected Shortfall (ES) instead of the VaR. Boonen (2017) also examines the impact on the SCR of empirically determining the stress scenarios. The market risk submodules equity risk, interest risk and the life submodule longevity risk are considered in this context. For this purpose, a fictitious life annuity insurer is considered, whose liability portfolio consists of 100,000 male policyholders aged between 21 and 79, with an average age of 50. The considered policy is a (deferred) single-life annuity that is paid at the end of each year starting at age 65. It is assumed that the insurer has no basic own funds. The insurers asset portfolio is composed of 25 percent global equity and 75 percent 5- and 30-year bonds.

A first result shows that the confidence level of the ES must be equal to 98.5 percent if the SCR under the VaR with confidence interval 99.5 percent should match to the SCR under the ES. If the VaR is considered at the level of 98.5 percent, a level of 97 percent must be considered for the ES. The stress scenarios are then determined using the VaR and the ES. The difference is small when the confidence level of 99.5 percent is considered for the VaR, with the VaR providing a slightly larger stress rate. If 98.5 percent is taken as the confidence level, the difference is greater. The downward shock is greater than the upward shock. Applying these stress scenarios to the determination of the SCR, the reduced total SCR is 23.24 percent of the best estimate of the liabilities (BEL), which represents the present value of the expected future liability payments. When applying these stress scenarios to the determination of the SCR, it can be seen that the SCR under VaR is always lower than under ES, if the same confidence level is considered for both risk measures. In addition, Boonen (2017) examines how proportion of the interest rate SCR to the reduced total SCR changes when the ES is used instead of the VaR in the calibration. It becomes clear that the change is small when a level of 99.5 percent is chosen for the VaR. When the level is lowered to 98.5 percent, the change becomes more noticeable, with the interest rate SCR increasing by 2.27 percent. Boonen (2017) then examines how the share of interest rate SCR changes when stress scenarios are determined empirically. For this purpose, it is assumed that the distribution of past returns corresponds to the expected future returns. The empirical VaR and ES are then derived in order to compute the stress scenarios. The interest rate SCR decreases by 13.09 percent compared to the SCR using the VaR with a level of 99.5 percent. The proportion of the interest rate SCR changes slightly with empirical calibration, both for a level of 99.5 percent and for 98.5 percent. However, the empirical calibration has a large impact on the other risk modules considered, so that overall it can be said that the differences in SCR allocations by using the ES instead of the VaR are significantly larger when the calibration of the stress scenario is done empirically. In a subsequent sensitivity analysis, in which the shares of the insurer's asset portfolio and the average age in the liability portfolio was varied, the above results can be confirmed.

4.3.7 Compersion between Solvency II and Basel III

Laas and Siegel (2017) compare the market risk and credit risk assessments in the Solvency II and Basel III standard model. In doing so, they examine both approaches both theoretically and quantitatively for their consistency. With regard to banking regulation, both the current Basel III regulation and the forthcoming adjustment of the market risk and credit risk frameworks named Basel III* are considered.

In the theoretical investigation, the mechanics of the standard models are opposed to each other. Significant discrepancies between the two approaches are becoming apparent. On the one hand, it is clear that under Basel III, capital requirements are only based on the asset side and the liability risk is not considered. In contrast, Solvency II takes both assets and liabilities into account. In Basel III, the 99 percent value at risk is used to calibrate parameters, while Solvency II requires the 99.5 percent value at risk. In the adjusted Basel III* regulation, the 97.5 percent expected shortfall is used. There are conceptual differences, particularly in the interest rate risk sub-model. On the one hand, Basel III distinguishes between specific and general capital charges. Under Basel III, capital requirements for assets are determined using predefined yield changes, their modified duration and their respective market value. Solvency II, on the other hand, uses interest rate shocks, as described above, to determine capital requirements. In addition, under Solvency II, liabilities are also taken into account and total requirements are determined using stress scenarios. Finally, interest rate risk for bonds in the banking book is not taken into account under Basel III. The adjustment of Basel IIII* also changes the assessment of interest rate risk. Under Basel III*, the risk-free yield curve is shocked at each maturity point separately, whereas under Solvency II the yield curve is stressed for all maturity points simultaneously. Thus, under Basel III*, this means that a number of changes in the present value is considered and aggregated.

Subsequently, a numerical investigation of the standard approaches is carried out. For this purpose, a stylized balance sheet approach is adopted to capture the asset portfolio structure of European life insurers, as in Braun et al. (2015). The asset portfolio is composed as follows: 35 percent government bonds, 38 percent bonds, 38 percent corporate bonds, 9 percent stocks, 4 percent property, 6 percent cash at bank, 6 percent residential mortgage loans, and 2 percent alternative investments. The liability side is structured in such a way that 87 percent represent life insurance liabilities and the remaining 13 percent are basic own funds. The asset portfolio in this stylized version is used to determine the capital requirements under both Solvency II and Basel III/III*. Taking into account the risk-mitigating effect of future discretionary participation and the loss-absorbing capacity of deferred taxes under Solvency II, the final SCR is lower than the capital requirements under Basel III and Basel III*. According to the banking regulations, a distinction is made between global systemically important banks (GSIB) and non-global systemically important banks (non-GSIB). GSIBs must take a larger buffer into account, so that GSIBs must meet significantly higher capital reserves overall. The numerical analysis shows that under the adjusted Basel III* regulation, non-GSIBs already have to set aside twice as high capital reserves as insurers under Solvency II. The capital requirements for GSIBs even exceed those of insurers by 139 percent.

Subsequently, the individual portfolio weights of the various assets were varied as part of a sensitivity analysis. The analysis shows that the Basel III* capital requirements exceed the Solvency II requirements in all adjusted portfolios.

4.3.8 A scenario-based approach

Schlütter (2021) presents a scenario-based approach for measuring interest rate risk that is intended to address the disadvantage of the current standard formula presented by the Solvency II regulation outlined by Gatzert and Martin (2012) and by EIOPA (2016). The value at risk is approximated using a principal component analysis to consider various stress scenarios. For this purpose, the interest rate is approximated using the Nelson-Siegel model as well as an adjusted version of the Nelson-Siegel model, which takes into account lower limits for the interest rates. The calculated value at risks approximations are backtested based on historical return curves and stochastic models. For this purpose, 1000 hypothetical asset-liability portfolios with associated deterministic cash flow patterns are considered. For each portfolio, the accuracy of the value at risk approximations is measured. The backtest shows that the results of the traditional Nelson-Siegel model are similar to the adjusted ones. This shows that by adjusting the Nelson-Siegel model with a lower bound on the interest rate, one does not get a deterioration in accuracy. Such a lower bound is particularly important in a low interest rate environment to avoid unjustifiably high negative interest rates. Furthermore, Schlütter (2021) shows that the approximation of the scenario-based approach can be improved by taking correlation parameters into account when aggregating the results of the individual scenarios.

4.4 Conclusion

Within these literature reviews it has been examined whether the SCR under the current Solvency II regulations represents an adequate capital reserve for insurance companies with regard to interest rate risk. The literature presented on this chapter has demonstrated the advantages and disadvantages of the standard formula in the assessment of interest rate risks.

First, the insurers asset allocation has a large impact on the SCR. In more detail, predefined scenario in the standard model both over- and underestimates the actual risk belonging to investments, such that the risk situation is not being sufficiently reflected.

Conversely, it was shown that insurers with a good credit rating and good regulatory solvency standing are not expected to adjust their asset allocation due to the introduction of the Solvency II regulation. This is due to the fact that the requirements of rating agencies are higher than under Solvency II in terms of the default probabilities of asset investments, so that the capital requirements seems not to be a binding constraint.

Furthermore, model risk is also a major factor influencing the determination of capital requirements under Solvency II. The choice of interest rate model also plays a significant role in the calculation of minimum capital requirements. The SCR for the standard model differs significantly from the SCR for the internal models. The difference is smallest when the Vasicek model is used.

When considering the probability of default, the current design and calibration of the Solvency II standard model appears to provide insufficient protection against interest rate risk. In particular, the assessment by the standard model using only stress scenarios to be incompatible with the insurance industry, as this approach is designed to avoid risks rather than to offset them.

A common point of criticism in the Solvency II calculation is the use of value at risk as the underlying risk measure. Since value at risk is not sub-additive, diversification is not rewarded. Furthermore, the value at risk does not take into account tail risks outside the confidence interval. As a result, worst case scenarios are not taken into consideration. A risk measure that does not have these disadvantages is the expected shortfall. It has been shown that for the calibration of the stress scenarios, the value at risk and the expected shortfall provide similar results. In order to generate the same results when calculating the minimum capital requirements, a confidence level of 98.8% must be considered in the case of the expected shortfall.

Comparison with the Basel III banking regulations reveals substantial discrepancies in the level of minimum capital requirements. The capital requirements under Basel II exceed those under Solvency II.

Chapter 5

General conclusion

This thesis can be placed within the literature in interest rate risk for participating life insurance products. In particular, we discussed the optimal design of participating life insurance contracts with minimum return rate guarantees under interest rate risk. In addition we focus on interest rate sensitivity of those products. Moreover we investigate on hedging strategy on interest rates for a portfolio of different participating guarantee schemes. The results are of interest to both practitioners and academics in the research field of liability insurance and risk management.

In the second chapter we focus on ways to build a natural hedge against interest rate risk by offering an appropriate product mix of different minimum return guarantees (MRRGs). In particular, we analyze two versions of MRRGs that are relevant in the context of participating life insurance contracts. One version implies a guaranteed rate that is fixed once at the inception of the contract (fix strike guarantee). The other version is a (stochastic) guarantee rate which is implied by the interest rate accumulation over the contract horizon (floating strike guarantee). Furthermore, we propose a natural hedge against changes in the term structure of interest rates. The natural hedge is based on the coexistence of fix and floating strike guarantee products. The duration of the asset side of a life insurer is usually much lower than that of the liability side. In this chapter, we show that selling floating strike guarantee products shortens the duration of the liability situation such that it is possible to achieve immunization.

In the third chapter we analyzes the optimal design of participating life insurance contracts with minimum return rate guarantees under default risk. Unless there is a default event, the insured receives the maximum of a guaranteed rate and a participation in the returns. Considering default risk modifies the payoff of the insured by means of a default put implying a compound option feature (nested maximum). Based on yearly returns, we show that, in spite of the compound option feature, the (yearly return) payoff of the default put (and the liabilities to the insured) can be represented by piecewise linear functions of the investment return, i.e. the payoff of a portfolio of plain vanilla options. Thus, the liabilities are easily priced in any model setup which gives closed form solutions for standard options. Furthermore we derive the optimal (expected utility maximizing) quantile guarantee payoff of an investor/insured with constant relative risk aversion in a complete market setup. We further investigated on the interest rate risk of those quantile guarantees. We therefore calculated the effective duration and convexity for suitable combinations of the minimum return rate guarantee and the corresponding fair equity ratio.

In the fourth chapter we analyzes the Solvency II capital requirements under the interest rate risk submodule. First we give a brief introduction to the Solvency II regulation. We then introduces the interest rate risk submodule and discusses the minimum capital requirements under the standard formula. Finally we summarizes the results from a collection of research papers that relate to capital requirements under the interest rate submodule.

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Appendix A

Interest rate modeling with the Vasicek model

In the following chapter two frequently used one factor short rate models are introduced. The first one, the Vasicek model, is discussed intensively while for the Cox Ingersoll Ross model only the most important information is addressed. The following elaborations are mostly based on Brigo and Mercurio (2006).

A.1 Vasicek Model

In the following section we introduce the Vasicek model which was first mentioned in Vasicek (1977). In the Vasicek model it is assumed that the short rate follows an Ornstein-Uhlenbeck process with constant coefficients such that the short rate dynamic is given by

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{A.1}$$

where r_0, a, b and σ are positive constants. r_0 is the initial process value at $t = 0$. The parameter b is called mean revision level. If the short rate r_t is higher than b, the drift term $(b - r_t)$ becomes negative and the drift will tend to decrease the process to the level of b. Otherwise, if the short rate r_t is below b, the drift term is positive and therefore it will tend to increase the process to the level of b. A process with this property is called mean-reverting. The parameter a is called mean revision speed and it defines the force of the drift term on the process. With higher a the force increases such that the process tends stronger to the mean revision level b. Last the parameter σ is called volatility and it depicts the impact of the randomness defined by the standard Brownian motion W_t.

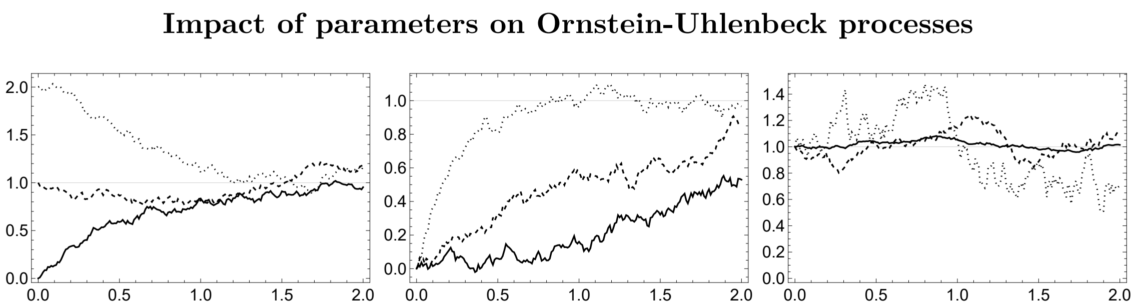

Figure A.1: The figures illustrates the impact of the parameter in the Ornstein-Uhlenbeck process. All figures shows three simulated paths where one parameter is varying and the others are fixed at a certain level. Each path consist of 200 simulated values. The left figure shows the impact of the mean revision level b by varying the initial value r_0, where $b = 1$, $a = 1$ and $\sigma = 0.2$ are fixed. The solid lines shows $r_0 = 0$, the dashed line $r_0 = 1$ and the dotted line $r_0 = 2$. The center figure shows the impact of the mean revision speed a, where $r_0 = 0$, $b = 1$ and $\sigma = 0.2$ are fixed. The solid lines shows $a = 0.3$, the dashed line $a = 1$ and the dotted line $a = 5$. The right figure illustrates the impact of the volatility σ, where $r_0 = 1$, $b = 1$ and $a = 1$ are fixed. The solid lines shows $\sigma = 0.05$, the dashed line $\sigma = 0.2$ and the dotted line $\sigma = 0.6$.

The impact of those parameters are illustrated in Figure A.1, showing the mean-reverting property. The left figure illustrates the impact of the mean revision level b by varying the initial value r_0. When the initial value is below the mean revision level b (solid line), the process tends to increase in time, while when it is above (dotted line), the process tends to decrease in time. When the process starts at the mean revision level (dashed line), the process fluctuates around the mean revision level. The center figure illustrates the impact of the mean revision speed a. With a rather low speed (solid line) the process tends to reach the mean revision level much slower than with a rather high speed (dotted line). At last, the right figure illustrates the impact of the volatility σ. This figure shows, that with increasing σ the process fluctuates more whereby for a low σ (solid line) the process seems to follow closer the mean revision level than for a high σ (dotted line), which shows more breakouts around the mean revision level. In the following the solution of the stochastic differential equation given by (A.1) is derived.

Theorem A.1.1 *The solution for the stochastic differential equation given by* (A.1) *for* $s \leq t$ *is*

$$r_t = r_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) + \sigma \int_s^t e^{-a(t-u)} dW_u.$$

Proof: First we rewrite the dynamics (A.1) to

$$dr_t = (ab - ar_t)dt + \sigma dW_t. \tag{A.2}$$

Then by using the chain rule it follows that

$$d\left(\frac{1}{a}e^{at}(ab - ar_t)\right) = -e^{at}dr_t + e^{at}(ab - ar_t)dt. \tag{A.3}$$

Multiplying both sides of (A.2) by $-e^{at}$ leads to

$$-e^{at}dr_t = -e^{at}(ab - ar_t)dt - e^{at}\sigma dW_t.$$

Inserting this to (A.3) gives

$$d\left(\frac{1}{a}e^{at}(ab - ar_t)\right) = -e^{at}(ab - ar_t)dt - e^{at}\sigma dW_t + e^{at}(ab - ar_t)dt = -e^{at}\sigma dW_t.$$

Integrating both sides gives

$$\begin{aligned}\frac{1}{a}e^{at}(ab - ar_t) - \frac{1}{a}e^{as}(ab - ar_s) &= -\sigma\int_s^t e^{au}dW_u \\ \Leftrightarrow \frac{1}{a}e^{at}(ab - ar_t) &= \frac{1}{a}e^{as}(ab - ar_s) - \sigma\int_s^t e^{au}dW_u.\end{aligned}$$

Finally multiplying both sides with ae^{-at} leads to

$$\begin{aligned}ab - ar_t &= e^{-a(t-s)}(ab - ar_s) - a\sigma\int_s^t e^{-a(t-u)}dW_u \\ \Leftrightarrow -ar_t &= -ab + e^{-a(t-s)}(ab - ar_s) - a\sigma\int_s^t e^{-a(t-u)}dW_u \\ \Leftrightarrow r_t &= b - e^{-a(t-s)}(b - r_s) + \sigma\int_s^t e^{-a(t-u)}dW_u \\ \Leftrightarrow r_t &= r_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) + \sigma\int_s^t e^{-a(t-u)}dW_u.\end{aligned}$$

□

Remark A.1.2 *Theorem A.1.1 implies that the short rate r_t is normally distributed with mean $\mathbb{E}[r_t|\mathcal{F}_s]$ and variance $Var[r_t|\mathcal{F}_s]$, i.e. $r_t \sim \mathcal{N}(\mathbb{E}[r_t|\mathcal{F}_s], Var[r_t|\mathcal{F}_s])$, where*

$$\begin{aligned}\mathbb{E}[r_t|\mathcal{F}_s] &= r_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) = r_s e^{-a(t-s)} + b - be^{-a(t-s)} \\ &= (r_s - b)\,e^{-a(t-s)} + b \\ Var[r_t|\mathcal{F}_s] &= \int_s^t \sigma^2 e^{-2a(t-u)}du = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right)\end{aligned}$$

This implies that for all t the short rate r_t can be negative with probability

$$\Phi\left(\frac{-\mathbb{E}\left[r_t|\mathcal{F}_s\right]}{\sqrt{Var\left[r_t|\mathcal{F}_s\right]}}\right) > 0$$

where Φ describes the cumulative distribution function of the standard normal distribution. The past development of the historical short rate has shown that a negative short rate is very much possible. Since $\lim_{t\to\infty} \mathbb{E}\left[r_t|\mathcal{F}_s\right] = b$ one can confirm the mean-reverting property of the process, as it shows, that the expected short rate tends to the mean revision level b for t going to infinity. So the mean revision level b is also referred to as a long term average rate. To derive the price of a zero coupon bond it is necessary to specify the distribution of the interest rate integral $I(s,t) := \int_s^t r_u du$.

Lemma A.1.3 *The interest rate integral $I(s,t) := \int_s^t r_u du$ is given by*

$$I(s,t) = \mathcal{B}(s,t)(r_s - b) + b(t-s) + \frac{\sigma}{a}\int_s^t \left(1 - e^{-a(t-u)}\right) dW_u$$

where $\mathcal{B}(s,t) := \frac{1}{a}\left(1 - e^{-a(t-s)}\right)$.

Proof:

$$\begin{aligned}
I(s,t) &= \int_s^t r_u du \\
&= \int_s^t \left(r_s e^{-a(u-s)} + b\left(1 - e^{-a(u-s)}\right) + \sigma \int_s^u e^{-a(u-v)} dW_v\right) du \\
&= r_s \int_s^t e^{-a(u-s)} du + b\int_s^t du - b\int_s^t e^{-a(u-s)} du + \sigma \int_s^t \int_s^u e^{-a(u-v)} dW_v du \\
&= r_s \mathcal{B}(s,t) + b(t-s) - b\mathcal{B}(s,t) + \sigma \int_s^t \int_v^t e^{-a(u-v)} du dW_v \\
&= r_s \mathcal{B}(s,t) + b(t-s) - b\mathcal{B}(s,t) + \frac{\sigma}{a}\int_s^t (1 - e^{-a(t-v)}) dW_v \\
&= \mathcal{B}(s,t)(r_s - b) + b(t-s) + \frac{\sigma}{a}\int_s^t (1 - e^{-a(t-v)}) dW_v
\end{aligned}$$

□

It is now possible to specify the distribution of $I(s,t)$.

Remark A.1.4 *The interest rate integral $I(s,t) := \int_s^t r_u du$ is normally distributed with mean $\mathbb{E}\left[I(s,t)|\mathcal{F}_s\right]$ and variance $Var\left[I(s,t)|\mathcal{F}_s\right]$, i.e.*

$$I(s,t) \sim \mathcal{N}\left(\mathbb{E}\left[I(s,t)|\mathcal{F}_s\right], Var\left[I(s,t)|\mathcal{F}_s\right]\right),$$

where

$$\begin{aligned}
\mathbb{E}\left[I(s,t)|\mathcal{F}_s\right] &= \mathcal{B}(s,t)(r_s - b) + b(t-s) \\
Var\left[I(s,t)|\mathcal{F}_s\right] &= \frac{\sigma^2}{a^2}\int_s^t (1-e^{-a(t-u)})^2 du = \frac{\sigma^2}{a^2}\int_s^t (1-2e^{-a(t-u)} + e^{-2a(t-u)})du \\
&= \frac{\sigma^2}{a^2}\left((t-s) - 2\mathcal{B}(s,t) + \frac{1}{2a}\left(1-e^{-2a(t-s)}\right)\right)
\end{aligned}$$

with $\mathcal{B}(s,t) := \frac{1}{a}\left(1-e^{-a(t-s)}\right)$.

Now it is possible to derive the price of a zero coupon bond.

Theorem A.1.5 *The arbitrage-free price at time s of a zero coupon bond with maturity t is given by*

$$B(s,t) = e^{\mathcal{A}(s,t) - \mathcal{B}(s,t)r_s}$$

where

$$\begin{aligned}
\mathcal{A}(s,t) &= \left(\frac{\sigma^2}{2a^2} - b\right)((t-s) - \mathcal{B}(s,t)) - \frac{\sigma^2}{4a}\mathcal{B}(s,t)^2 \\
\mathcal{B}(s,t) &= \frac{1}{a}\left(1-e^{-a(t-s)}\right)
\end{aligned}$$

The yield is then described by

$$y(s,t) = \frac{\mathcal{A}(s,t)}{t-s} - \frac{\mathcal{B}(s,t)}{t-s}r_s$$

Proof: A zero coupon bond pays one unit at maturity t. The arbitrage-free price is then given by the expectation of the discounted payoff under the risk neutral measure, i.e.

$$B(s,t) = \mathbb{E}\left[e^{-I(s,t)}|\mathcal{F}_s\right]$$

Since $I(s,t)$ is normally distributed with mean $\mathbb{E}\left[I(s,t)|\mathcal{F}_s\right]$ and variance $Var\left[I(s,t)|\mathcal{F}_s\right]$, it follows that $e^{-I(s,t)}$ is log-normally distributed with parameters $\mathbb{E}\left[-I(s,t)|\mathcal{F}_s\right]$ and $Var\left[-I(s,t)|\mathcal{F}_s\right]$. The expected value of $e^{-I(s,t)}$ is then given by

$$\mathbb{E}\left[e^{-I(s,t)}|\mathcal{F}_s\right] = e^{\mathbb{E}[-I(s,t)|\mathcal{F}_s] + \frac{1}{2}Var[-I(s,t)|\mathcal{F}_s]}$$

For the exponent it follows

$$
\begin{aligned}
&\mathbb{E}\left[-I(s,t)|\mathcal{F}_s\right] + \frac{1}{2}Var\left[-I(s,t)|\mathcal{F}_s\right]\\
&= -\mathcal{B}(s,t)(r_s - b) - b(t-s) + \frac{1}{2}\frac{\sigma^2}{a^2}\left((t-s) - 2\mathcal{B}(s,t) + \frac{1}{2a}\left(1 - e^{-2a(t-s)}\right)\right)\\
&= -\mathcal{B}(s,t)r_s + \mathcal{B}(s,t)b - b(t-s) + \frac{\sigma^2}{2a^2}(t-s) - \frac{\sigma^2}{a^2}\mathcal{B}(s,t) + \frac{\sigma^2}{2a^2}\left(\frac{1}{2a}\left(1 - e^{-2a(t-s)}\right)\right)
\end{aligned}
$$

In the next step the last term $\frac{1}{2a}\left(1 - e^{-2a(t-s)}\right)$ is unformed in the following way

$$
\begin{aligned}
-\frac{a}{2}\mathcal{B}(s,t)^2 + \mathcal{B}(s,t) &= -\frac{a}{2}\left(\frac{1}{a^2}\left(1 - 2e^{-a(t-s)} + e^{-2a(t-s)}\right)\right) + \frac{1}{a}\left(1 - e^{-a(t-s)}\right)\\
&= -\frac{1}{2a} - \frac{1}{a}e^{-a(t-s)} - \frac{1}{2a}e^{-2a(t-s)} + \frac{1}{a}1 - \frac{1}{a}e^{-a(t-s)}\\
&= \frac{1}{2a} - \frac{1}{2a}e^{-2a(t-s)}\\
&= \frac{1}{2a}\left(1 - e^{-2a(t-s)}\right)
\end{aligned}
$$

So it follows

$$
\begin{aligned}
&-\mathcal{B}(s,t)r_s + \mathcal{B}(s,t)b - b(t-s) + \frac{\sigma^2}{2a^2}(t-s) - \frac{\sigma^2}{a^2}\mathcal{B}(s,t) + \frac{\sigma^2}{2a^2}\left(\frac{1}{2a}\left(1 - e^{-2a(t-s)}\right)\right)\\
=&-\mathcal{B}(s,t)r_s - b\left((t-s) - \mathcal{B}(s,t)\right) + \frac{\sigma^2}{2a^2}(t-s) - \frac{\sigma^2}{a^2}\mathcal{B}(s,t) + \frac{\sigma^2}{2a^2}\left(-\frac{a}{2}\mathcal{B}(s,t)^2 + \mathcal{B}(s,t)\right)\\
=&-\mathcal{B}(s,t)r_s - b\left((t-s) - \mathcal{B}(s,t)\right) + \frac{\sigma^2}{2a^2}(t-s) - \frac{\sigma^2}{a^2}\mathcal{B}(s,t) - \frac{\sigma^2}{4a}\mathcal{B}(s,t)^2 + \frac{\sigma^2}{2a^2}\mathcal{B}(s,t)\\
=&-\mathcal{B}(s,t)r_s - b\left((t-s) - \mathcal{B}(s,t)\right) + \frac{\sigma^2}{2a^2}(t-s) - \frac{\sigma^2}{2a^2}\mathcal{B}(s,t) - \frac{\sigma^2}{4a}\mathcal{B}(s,t)^2\\
=&-\mathcal{B}(s,t)r_s - b\left((t-s) - \mathcal{B}(s,t)\right) + \frac{\sigma^2}{2a^2}\left((t-s) - \mathcal{B}(s,t)\right) - \frac{\sigma^2}{4a}\mathcal{B}(s,t)^2\\
=&-\mathcal{B}(s,t)r_s + \underbrace{\left(\frac{\sigma^2}{2a^2} - b\right)\left((t-s) - \mathcal{B}(s,t)\right) - \frac{\sigma^2}{4a}\mathcal{B}(s,t)^2}_{=:\mathcal{A}(s,t)}
\end{aligned}
$$

□

Theorem A.1.5 shows that the continuously-compounded spot rate is an affine function of the form

$$\mathcal{A}(s,t) - \mathcal{B}(s,t)r_s.$$

Models with this property are referred to as affine term-structure models.

A.1.1 Real World Measure

Under the real world measure $\mathbb{P}$ the interest rate dynamics can be described by

$$\begin{aligned} dr_t &= a\left(b + \frac{\lambda\sigma}{a} - r_t\right)dt + \sigma dW_t^{\mathbb{P}} \\ &= a\left(\tilde{b} - r_t\right)dt + \sigma dW_t^{\mathbb{P}} \end{aligned} \tag{A.4}$$

where λ is referred to as the market price of risk. It is usually assumed that the market price of risk is constant in the Vasicek model, such that the Novikov condition is satisfied, i.e. the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^t \lambda^2 r_s ds + \int_0^t \lambda r_s dW_s^{\mathbb{P}}}$$

is a martingale. By Girsanov theorem it follows that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and that

$$dW_t^{\mathbb{P}} = \lambda dt + dW_t$$

is a Brownian motion under $\mathbb{P}$.

A.1.2 Maximum Likelihood Estimation

In the following section we derive the maximum likelihood estimators for the Vasicek model. The estimation is made on the basis of $n+1$ historical short rates $r = (r_0, r_1, \ldots, r_n)$ with equidistant time partition dt. Since we use real world data as a basis of the estimation the estimation parameters are given by the short rate dynamics under the real world measure $\mathbb{P}$ given by (A.4). The mean and variance for each $r_i, i = 1, 2, \ldots, n$ is given by

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left[r_i\right] &= r_{i-1}e^{-adt} + \tilde{b}(1 - e^{-adt}) \\ Var_{\mathbb{P}}\left[r_i\right] &= \frac{\sigma^2}{2a}(1 - e^{-2adt}). \end{aligned}$$

For a sample $r = (r_0, r_1, \ldots, r_n)$ the probability function is therefore given by

$$f(r_i; a, \tilde{b}, \sigma^2) = \frac{1}{\sqrt{2\pi Var_{\mathbb{P}}\left[r_i\right]}} e^{-\frac{(r_i - \mathbb{E}_{\mathbb{P}}[r_i])^2}{2Var_{\mathbb{P}}[r_i]}}.$$

The likelihood function is then defined by

$$l(a, \tilde{b}, \sigma^2) = \prod_{i=1}^{n} f(r_i; a, \tilde{b}, \sigma^2) = \left(\frac{1}{\sqrt{2\pi Var_{\mathbb{P}}\left[r_i\right]}}\right)^n \prod_{i=1}^{n} e^{-\frac{(r_i - \mathbb{E}_{\mathbb{P}}[r_i])^2}{2Var_{\mathbb{P}}[r_i]}}.$$

The log-likelihood function is therefore given by

$$\begin{aligned} L(a,\tilde{b},\sigma^2) = \ln\left(l(a,\tilde{b},\sigma^2)\right) &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(Var_{\mathbb{P}}[r_i]\right) - \sum_{i=1}^{n}\frac{(r_i - \mathbb{E}_{\mathbb{P}}[r_i])^2}{2Var_{\mathbb{P}}[r_i]} \\ &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\frac{\sigma^2}{2a}(1-e^{-2adt})\right) - \sum_{i=1}^{n}\frac{(r_i - r_{i-1}e^{-adt} - \tilde{b}(1-e^{-adt}))^2}{\frac{\sigma^2}{a}(1-e^{-2adt})} \end{aligned}$$

The maximum likelihood estimators $\hat{a}$, $\hat{\tilde{b}}$ and $\hat{\sigma}^2$ are then given by maximizing the log-likelihood function, i.e.

$$\max_{(a,\tilde{b},\sigma^2)} L(a,\tilde{b},\sigma^2) = L(\hat{a},\hat{\tilde{b}},\hat{\sigma}^2).$$

It the next step we are going to derive closed form solutions for the maximum likelihood estimators $\hat{a}$, $\hat{\tilde{b}}$ and $\hat{\sigma}^2$.

Theorem A.1.6 *The maximum likelihood estimators $\hat{a}$, $\hat{\tilde{b}}$ and $\hat{\sigma}^2$ are given by*

$$\begin{aligned} \hat{a} &= -\frac{1}{dt}\ln\left(\frac{n\sum_{i=1}^{n} r_{i-1}r_i - \sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i}{n\sum_{i=1}^{n} r_{i-1}^2 - \left(\sum_{j=1}^{n} r_{j-1}\right)^2}\right) \\ \hat{\tilde{b}} &= \frac{\sum_{i=1}^{n}(r_i - r_{i-1}e^{-\hat{a}dt})}{n(1-e^{-\hat{a}dt})} \\ \hat{\sigma}^2 &= \frac{2\hat{a}}{n(1-e^{-2\hat{a}dt})}\sum_{i=1}^{n}(r_i - r_{i-1}e^{-\hat{a}dt} - \hat{\tilde{b}}(1-e^{-\hat{a}dt}))^2 \end{aligned}$$

Proof: At first we define

$$\begin{aligned} A &= e^{-adt} \\ B &= \frac{\sigma^2}{2a}(1-e^{-2adt}) \end{aligned}$$

such that the log-likelihood function can be rewritten as

$$L(a,\tilde{b},\sigma) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(B) - \sum_{i=1}^{n}\frac{(r_i - r_{i-1}A - \tilde{b}(1-A))^2}{2B}$$

Proof for $\hat{\tilde{b}}$: To find the maximum we need to derive the log-likelihood function regarding to $\tilde{b}$, i.e.

$$\frac{\partial L}{\partial \tilde{b}} = \frac{1-A}{B}\sum_{i=1}^{n}(r_i - r_{i-1}A - \tilde{b}(1-A)).$$

Setting the right side equal to zero and assuming $a \neq 0$ leads to

$$\begin{aligned}
&\sum_{i=1}^{n}(r_i - r_{i-1}A - \tilde{b}(1-A)) = 0 \\
\Leftrightarrow & n\tilde{b}(1-A) = \sum_{i=1}^{n}(r_i - r_{i-1}A) \\
\Leftrightarrow & \tilde{b} = \frac{\sum_{i=1}^{n}(r_i - r_{i-1}A)}{n(1-A)}
\end{aligned}$$

So the maximum likelihood estimator for $\tilde{b}$ is given by

$$\hat{\tilde{b}} = \frac{\sum_{i=1}^{n}(r_i - r_{i-1}e^{-adt})}{n(1-e^{-adt})}. \tag{A.5}$$

Proof for $\hat{\sigma}^2$: To find the maximum we first need to derive the log-likelihood function regarding to B, i.e.

$$\frac{\partial L}{\partial B} = -\frac{n}{2B} + \frac{1}{2B^2}\sum_{i=1}^{n}(r_i - r_{i-1}A - \tilde{b}(1-A))^2.$$

Setting the right side equal to zero and multiplying both sides by B leads to

$$\begin{aligned}
&-\frac{n}{2} + \frac{1}{2B}\sum_{i=1}^{n}(r_i - r_{i-1}A - \tilde{b}(1-A))^2 = 0 \\
\Leftrightarrow & \frac{1}{2B}\sum_{i=1}^{n}(r_i - r_{i-1}A - \tilde{b}(1-A))^2 = \frac{n}{2} \\
\Leftrightarrow & B = \frac{1}{n}\sum_{i=1}^{n}(r_i - r_{i-1}A - \tilde{b}(1-A))^2
\end{aligned}$$

Substituting $B = \frac{\sigma^2}{2a}(1-e^{-2adt})$ and $A = e^{-adt}$ gives finally the maximum likelihood estimator for σ^2

$$\hat{\sigma}^2 = \frac{2a}{n(1-e^{-2adt})}\sum_{i=1}^{n}(r_i - r_{i-1}e^{-adt} - \tilde{b}(1-e^{-adt}))^2$$

Proof for $\hat{a}$: To finally derive the maximum likelihood estimator for a we first need to rewrite the log-likelihood function $L(a, \tilde{b}, \sigma^2)$ by substituting $(1-A)\tilde{b}^* = \frac{1}{n}\sum_{j=1}^{n}(r_j - r_{j-1}A)$ from (A.5), i.e.

$$\begin{aligned}
L(a, \tilde{b}, \sigma^2) &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(B) - \frac{1}{2B}\sum_{i=1}^{n}(r_i - r_{i-1}A - \frac{1}{n}\sum_{j=1}^{n}(r_j - r_{j-1}A))^2 \\
&= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(B) - \frac{1}{2B}\sum_{i=1}^{n}\left(r_i - r_{i-1}A - \frac{1}{n}\left(\sum_{j=1}^{n} r_j\right) + A\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)^2
\end{aligned}$$

Now we need to derive this function by A in order to find maximum likelihood estimator for a, i.e.

$$\begin{aligned}
\frac{\partial L}{\partial A} =& \frac{1}{B}\sum_{i=1}^{n}\left[\left(r_{i-1}+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)\left(r_i-\left(r_{i-1}-\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)A-\frac{1}{n}\left(\sum_{j=1}^{n} r_j\right)\right)\right] \\
=& \frac{1}{B}\sum_{i=1}^{n}\left[\left(r_{i-1}+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)r_i-\left(r_{i-1}-\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)^2 A\right. \\
& \left.-\left(r_{i-1}+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)\left(\sum_{j=1}^{n} r_j\right)\right] \\
=& \frac{1}{B}\left(\sum_{i=1}^{n}\left[\left(r_{i-1}+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)r_i\right]-\sum_{i=1}^{n}\left[\left(r_{i-1}-\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)^2 A\right]\right. \\
& \left.-\sum_{i=1}^{n}\left[\left(r_{i-1}+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)\right)\frac{1}{n}\left(\sum_{j=1}^{n} r_j\right)\right]\right) \\
=& \frac{1}{B}\left(\sum_{i=1}^{n}\left[r_{i-1}r_i+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)r_i\right]\right. \\
& -\sum_{i=1}^{n}\left[\left(r_{i-1}^2-2r_{i-1}\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)+\frac{1}{n^2}\left(\sum_{j=1}^{n} r_{j-1}\right)^2\right)A\right] \\
& \left.-\sum_{i=1}^{n}\left[r_{i-1}\frac{1}{n}\left(\sum_{j=1}^{n} r_j\right)+\frac{1}{n^2}\left(\sum_{j=1}^{n} r_{j-1}\right)\left(\sum_{j=1}^{n} r_j\right)\right]\right) \\
=& \frac{1}{B}\left(\sum_{i=1}^{n}(r_{i-1}r_i)+\frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i\right. \\
& -\left(\sum_{i=1}^{n} r_{i-1}^2-2\frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_{i-1}+\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)^2\right)A \\
& \left.-\frac{1}{n}\sum_{i=1}^{n} r_{i-1}\sum_{j=1}^{n} r_j-\frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{j=1}^{n} r_j\right) \\
=& \frac{1}{B}\left(\sum_{i=1}^{n}(r_{i-1}r_i)-\frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i-\left(\sum_{i=1}^{n} r_{i-1}^2-\frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)^2\right)A\right)
\end{aligned}$$

Setting the last term equal to zero leads to

$$
\begin{aligned}
&\frac{1}{B}\left(\sum_{i=1}^{n}(r_{i-1}r_i) - \frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i - \left(\sum_{i=1}^{n} r_{i-1}^2 - \frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)^2\right)A\right) = 0 \\
\Leftrightarrow &\sum_{i=1}^{n}(r_{i-1}r_i) - \frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i - \left(\sum_{i=1}^{n} r_{i-1}^2 - \frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)^2\right)A = 0 \\
\Leftrightarrow &\left(\sum_{i=1}^{n} r_{i-1}^2 - \frac{1}{n}\left(\sum_{j=1}^{n} r_{j-1}\right)^2\right)A = \sum_{i=1}^{n}(r_{i-1}r_i) - \frac{1}{n}\sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i \\
\Leftrightarrow &A = \frac{n\sum_{i=1}^{n} r_{i-1}r_i - \sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i}{n\sum_{i=1}^{n} r_{i-1}^2 - \left(\sum_{j=1}^{n} r_{j-1}\right)^2}
\end{aligned}
$$

Finally the maximum likelihood estimator for a can be derived by substituting $A = e^{-adt}$, i.e.

$$
\begin{aligned}
e^{-adt} &= \frac{n\sum_{i=1}^{n} r_{i-1}r_i - \sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i}{n\sum_{i=1}^{n} r_{i-1}^2 - \left(\sum_{j=1}^{n} r_{j-1}\right)^2} \\
\Leftrightarrow \hat{a} &= -\frac{1}{dt}\ln\left(\frac{n\sum_{i=1}^{n} r_{i-1}r_i - \sum_{j=1}^{n} r_{j-1}\sum_{i=1}^{n} r_i}{n\sum_{i=1}^{n} r_{i-1}^2 - \left(\sum_{j=1}^{n} r_{j-1}\right)^2}\right)
\end{aligned}
$$

□

A.1.3 Parameter bias

In this section we investigate whether the maximum likelihood estimators are biased. The discrete time version of the Vasicek model is a autoregressive model of order 1 with autoregressive coefficient $\theta = e^{-adt}$, i.e. $AR(1)$. Marriott and Pope (1954) and Kendall (1954) showed, that

$$\mathbb{E}[\hat{\theta}] = \theta - \frac{1+3\theta}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

where n is the sample size and $\hat{\theta}$ is the maximum likelihood estimator of the autoregressive coefficient θ.

To illustrate that bias of the parameter a we use parameters regularly used in the literature, i.e. Hieber et al. (2019) and Graf et al. (2011). We chose $r_0 = 0.0115$, $\tilde{b} = 0.0305$, and $\sigma = 0.015$. For 100 different values of $a \in [-0.5, 0.5]$ we generate 500 paths of monthly short rates with a total sample size of 240 per path, using the above stated dynamics. For each path we then derive the maximum likelihood estimator $\hat{a}$.

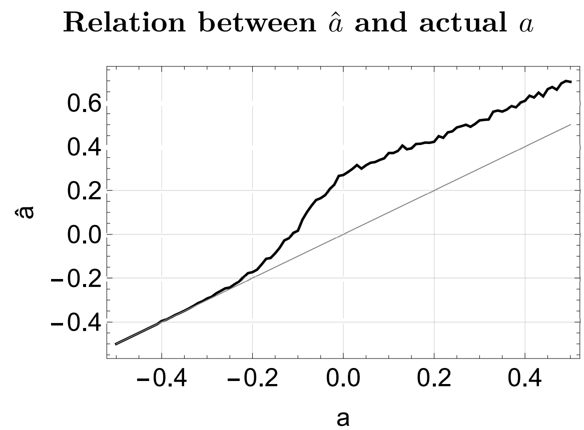

Figure A.2: Relation between $\hat{a}$ and actual a.

Figure A.2 illustrates the bias of the maximum likelihood estimator $\hat{a}$ and the actual a. The black line represents the relationship between the actual value a and its estimator $\hat{a}$. The Figure shows that for small values of $a < -0.2$ the distortion is rather small as it follows the gray line. For values between -0.2 to 0 the bias is increasing. For values above 0 the bias is present, but not increasing with a. There are several methods mentioned in the literature to correct this bias. One is a jackknife method proposed by Phillips and Yu (2005). At first, the sample, consisting of n observations is divided into m consecutive subsamples each with l observations, such that $n = ml$. Then the jackknife bias corrected estimator is given by

$$\hat{\theta}_{JK} = \frac{m}{m-1}\hat{\theta} - \frac{\sum_{i=1}^{m}\hat{\theta}_i}{m^2 - m}$$

where $\hat{\theta}$ is the maximum likelihood estimator of $\theta = e^{-adt}$ for the whole sample and $\hat{\theta}_i$ are the maximum likelihood estimators for each of the m different subsamples. Phillips and Yu (2005) suggest to use $m = 4$ which gives the best trade-off between bias reduction and variance inflation. The result of the jack bias corrected estimator is illustrated in Figure A.3.

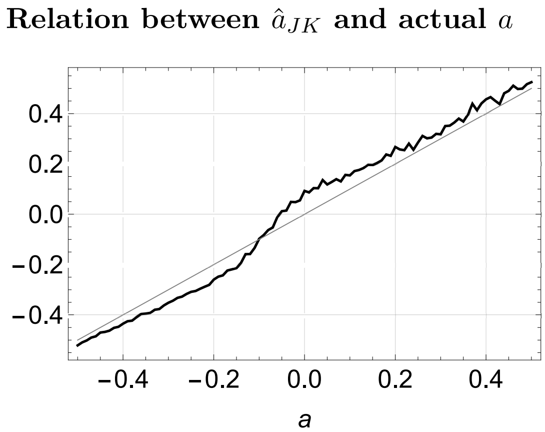

Figure A.3: Relation between $\hat{a}_{JK}$ and actual a.

A.1.4 Estimation

In the following section we are going to talk about the parameter estimation based of real world interest rates used as reference rates. This data is collected in the real world, so the statistical properties of those reference rates characterize the distribution of the short rate process r_t under the real world measure $\mathbb{P}$. Therefore the model parameters derived by estimation using real world references rates are given by the real world measure dynamics. Later we are going to talk about calibrating the short rate by using observed prices of derivatives and other financial products. Those prices are calculated using the risk neutral measure $\mathbb{Q}$, so the parameters derived by those prices are under the risk neutral dynamics.

Estimation on EONIA rates

In the following section we are going to calibrate the Vasicek model using the above derived maximum likelihood estimators. We use daily Euro OverNight Index Average (EONIA) rates from its inception, which was 04.01.1999 to 08.11.2021 as our basis for the estimation. The development of the EONIA rate is illustrated in Figure A.4.

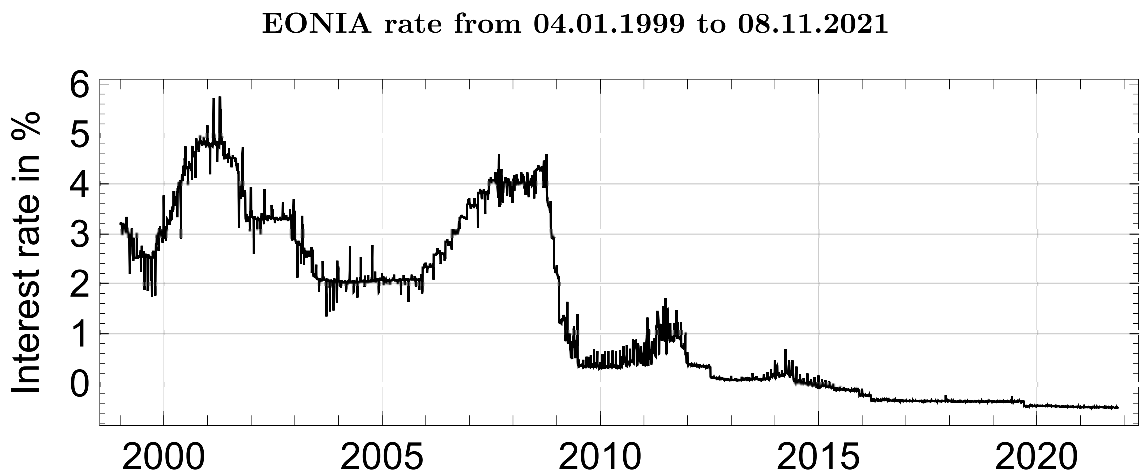

Figure A.4: The Figure shows the development of the EONIA rate from 04.01.1999 to 08.11.2021.

The EONIA rates are given on a daily basis, so we set $dt = \frac{1}{260}$. In total we have $n = 5961$ interest rates given. The maximum likelihood estimators are calculated using the formulas given in Theorem A.1.6. The results are stated in Table A.1.

Table A.1: MLE results based on EONIA

$\hat{a}$	$\hat{\hat{b}}$	$\hat{\sigma}$
0.385955	0.00965424	0.0178384

Estimation on Euro short-term rate

The Euro short-term rate (ESTR) is a new reference rate calculated by the European Central Bank based on European money market data. It is meant to extend the EONIA rates by also taking into account lending to money market funds, insurance companies and other financial corporations while the EONIA only refer to inter-banking lending, which gives the ESTR bigger relation to insurance companies. The ESTR is being published as of the 1. October 2019 and is calculated by using overnight unsecured fixed rate deposit transactions over 1 million euro for each TARGET2 business day. The development of the ESTR rate from 01.10.2019 to 09.11.2021 is illustrated in Figure A.5.

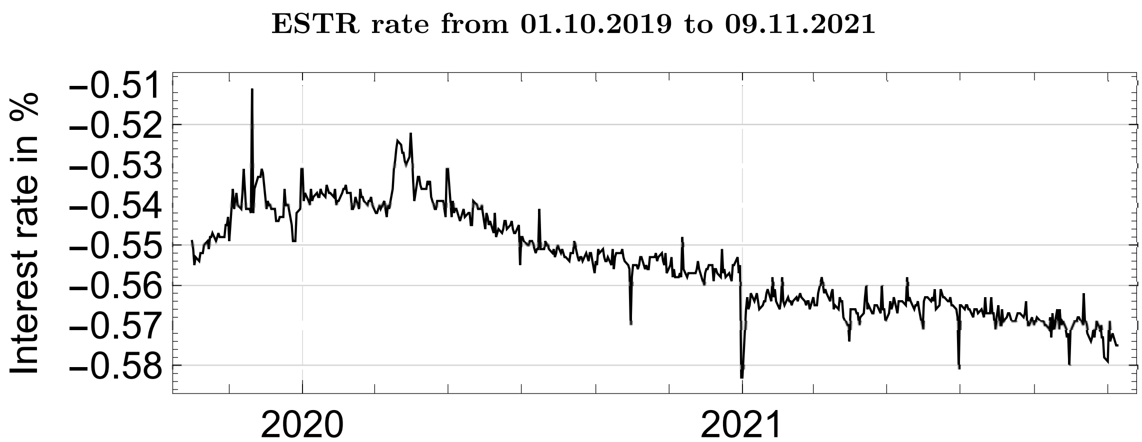

Figure A.5: The Figure shows the development of the ESTR rate from 01.10.2019 to 09.11.2021.

The ESTR is also given on a daily basis, so we set $dt = \frac{1}{260}$. In total we have $n = 551$ interest rates given. The maximum likelihood estimators are calculated using the formulas given in Theorem A.1.6. The results are stated in Table A.2.

Table A.2: MLE results based on ESTR

$\hat{a}$	$\hat{\hat{b}}$	$\hat{\sigma}$
2.68049	−0.00557684	0.021722

A.1.5 Calibration

In this section we calibrate the Vasicek model by using prices of financial products. As already mention, those prices are calculated under the risk neutral measure $\mathbb{Q}$, so the parameters derived by those prices are under the risk neutral dynamics. One possibility is by using zero coupon bonds. The arbitrage-free price at time 0 of a zero coupon bond with maturity T is given by $B(0,T)$ which was derived in Theorem A.1.5. The observed price of zero coupon bonds with maturity T on the market are given by $B^M(0,T)$. One way of calibration is by choosing the parameters such that the aggregated quadratic difference between the observed price and the calculated price for different maturities $T = T_1, T_2, \ldots, T_N$ is minimized, i.e.

$$\min_{a,b,\sigma,r_0} \sum_{i=1}^{N} \left(B^M(0,T_i) - B(0,T_i)\right)^2 .$$

This problem can simply be rewritten in terms of the corresponding zero bond yields, i.e.

$$\min_{a,b,\sigma,r_0} \sum_{i=1}^{N} \Big(y^M(0,T_i) - y(0,T_i) \Big)^2 . \tag{A.6}$$

For our calibration we use German treasury notes and zero coupon bonds. The yields are

Table A.3: German treasury notes as of 22.11.2021

Bond	Maturity	Coupon	Time to maturity	Yield in %
2019 IV Schatz	10.12.2021	0.00	0.05	-0.1491
2020 Schatz	11.03.2022	0.00	0.30	-0.7653
2020 II Schatz	10.06.2022	0.00	0.55	-0.7925
2020 III Schatz	16.09.2022	0.00	0.82	$-0,7511$
2020 IV Schatz	15.12.2022	0.00	1.06	$-0,7868$
2021 Schatz	10.03.2023	0.00	1.30	$-0,8257$
2021 II Schatz	16.06.2023	0.00	1.56	$-0,8431$
2021 III Schatz	15.09.2023	0.00	1.81	$-0,8204$
2021 IV. (2023) Schatz	15.12.2023	0.00	2.06	$-0,7746$

Table A.4: German zero coupon bond as of 22.11.2021

Bond	Maturity	Coupon	Time to maturity	Yield
2016 (2026) Bund	15.08.2026	0.00	$4,73$	$-0,6777$
2021 (2028) Bund	15.11.2028	0.00	$6,99$	$-0,5123$
2020 (2027) Bund	15.11.2027	0.00	$5,98$	$-0,5781$
2019 (2029) Bund	15.08.2029	0.00	$7,73$	$-0,4788$
2020 (2030) Bund	15.08.2030	0.00	$8,24$	$-0,4662$
2020 (2030) II Bund	15.02.2030	0.00	$8,73$	$-0,4429$
2021 (2031) Bund	15.02.2031	0.00	9.24	$-0,3904$
2021 (2031) II Bund	15.08.2031	0.00	9.73	$-0,3378$
2020 (2035) Bund	15.05.2035	0.00	$13,48$	$-0,2114$
2021 (2036) Bund	15.05.2036	0.00	14.49	$-0,1641$
2019 (2050) Bund	15.08.2050	0.00	28.75	$-0,0383$
2021 (2052) Bund	15.08.2052	0.00	30.75	$-0,0065$

The results of the calibration by solving Eq. (A.6) using the yields given in Table A.3 and Table A.4 are stated in Table A.5.

Table A.5: Calibration results based on German yield curve

a	b	σ	r_0
0.324746	0.0208519	0.0711287	0.00631842

Yield curve of german government bonds as of 22.11.2021

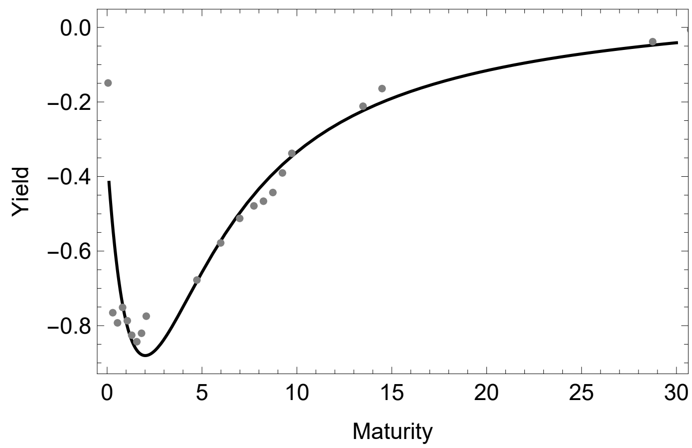

Figure A.6: This figure shows the Yield curve of german government bonds as of 22.11.2021.

Appendix B

Appendix to Chapter 2

B.1 Derivation Margrabe formula

B.1.1 Model free pricing formula

We consider a time horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We assume two non dividend paying assets, whose price process is defined by $(S_t^1)_{t\in[0,T]}$ and $(S_t^2)_{t\in[0,T]}$. The terminal payoff of a European option to exchange the asset S^1 for the asset S^2 at time T is given by $[S_T^2 - S_T^1]^+ = \max[S_T^2 - S_T^1, 0]$. We assume a arbitrage-free market, so there exist a martingale measure $\tilde{\mathbb{P}}_{S^1}$ and $\tilde{\mathbb{P}}_{S^2}$, which represent risk-neutral measures. Let $C(t, S_t^1, S_t^2)$ denotes the price at time t of a European option to exchange the asset S^1 for the asset S^2 at time T, i.e. $C(T, S_T^1, S_T^2) = [S_T^2 - S_T^1]^+$. Under the risk-neutral measure its holds

$$\begin{aligned}
C(t,S_t^1,S_t^2) &= S_t^1\tilde{\mathbb{E}}_{S^1}\left[\frac{C(T,S_T^1,S_T^2)}{S_T^1}|\mathcal{F}_t\right]\\
&= S_t^1\tilde{\mathbb{E}}_{S^1}\left[\frac{(S_T^2-S_T^1)^+}{S_T^1}|\mathcal{F}_t\right]\\
&= S_t^1\tilde{\mathbb{E}}_{S^1}\left[\left(\frac{S_T^2}{S_T^1}-1\right)^+|\mathcal{F}_t\right]\\
&= S_t^1\tilde{\mathbb{E}}_{S^1}\left[\left(\frac{S_T^2}{S_T^1}-1\right)\mathbb{1}_{\{S_T^2>S_T^1\}}|\mathcal{F}_t\right]\\
&= S_t^1\tilde{\mathbb{E}}_{S^1}\left[\frac{S_T^2}{S_T^1}\mathbb{1}_{\{S_T^2>S_T^1\}}|\mathcal{F}_t\right]-S_t^1\tilde{\mathbb{E}}_{S^1}\left[\mathbb{1}_{\{S_T^2>S_T^1\}}|\mathcal{F}_t\right]\text{(Change numèraire } S^1 \text{ to } S^2)\\
&= S_t^2\tilde{\mathbb{E}}_{S^2}\left[\mathbb{1}_{\{S_T^2>S_T^1\}}|\mathcal{F}_t\right]-S_t^1\tilde{\mathbb{P}}_{S^1}\left(S_T^2>S_T^1\right)\\
&= S_t^2\tilde{\mathbb{P}}_{S^2}\left(S_T^2>S_T^1\right)-S_t^1\tilde{\mathbb{P}}_{S^1}\left(S_T^2>S_T^1\right)
\end{aligned}$$

B.1.2 General model setup

Now we specify the model by assuming the following dynamics for the two assets S^1 and S^2 with Brownian motions W_t^1 and W_t^2.

$$\begin{aligned}
dS_t^1 &= S_t^2\left(\mu_{S^1}(t)dt+\sigma_{1,S^1}(t)dW_t^1+\sigma_{2,S^1}(t)dW_t^2\right)\\
dS_t^2 &= S_t^2\left(\mu_{S^2}(t)dt+\sigma_{S^2}(t)dW_t^1\right)
\end{aligned}$$

where $\mu_{S^1}(t),\mu_{S^2}(t),\sigma_{S^1}(t),\sigma_{S^2}(t)$ are deterministic functions and $\sigma_{S^1}(t),\sigma_{S^2}(t)\geq 0$. The solutions of above dynamics are given by geometric Brownian motions

$$\begin{aligned}
S_t^1 &= S_0^1 e^{\int_0^t\left(\mu_{S^1}(u)-\frac{1}{2}\sigma^1(u,S_u^1)\right)du+\int_0^t\sigma_{S^1}(u)\rho dW_u^1+\int_0^t\sigma_{S^1}(u)\sqrt{1-\rho^2}dW_t^2}\\
S_t^2 &= S_0^2 e^{\int_0^t\left(\mu_{S^2}(u)-\frac{1}{2}\sigma^2(u,S_u^2)\right)du+\int_0^t\sigma_{S^2}(u)dW_u^1}
\end{aligned}$$

The dynamics of the fraction $\frac{S_t^2}{S_t^1}$ can then be derived by Ito's lemma

$$\begin{aligned}
d\frac{S_t^2}{S_t^1} =& \frac{1}{S_t^1}dS_t^2 - \frac{S_t^2}{(S_t^1)^2}dS_t^1 - \frac{1}{(S_t^1)^2}dS_t^2 dS_t^1 + \frac{S_t^2}{(S_t^1)^3}dS_t^1 dS_t^1 \\
=& \frac{S_t^2}{S_t^1}\left(\mu_{S^2}(t)dt + \sigma_{S^2}(t)dW_t^1\right) - \frac{S_t^2}{S_t^1}\left(\mu_{S^1}(t)dt + \sigma_{1,S^1}(t)dW_t^1 + \sigma_{2,S^1}(t)dW_t^2\right) \\
&- \frac{S_t^2}{S_t^1}\left(\mu_{S^2}(t)dt + \sigma_{S^2}(t)dW_t^1\right)\left(\mu_{S^1}(t)dt + \sigma_{1,S^1}(t)dW_t^1 + \sigma_{2,S^1}(t)dW_t^2\right) \\
&+ \frac{S_t^2}{S_t^1}\left(\mu_{S^1}(t)dt + \sigma_{1,S^1}(t)dW_t^1 + \sigma_{2,S^1}(t)dW_t^2\right)\left(\mu_{S^1}(t)dt + \sigma_{1,S^1}(t)dW_t^1 + \sigma_{2,S^1}(t)dW_t^2\right) \\
=& \frac{S_t^2}{S_t^1}\left[\mu_{S^2}(t)dt + \sigma_{S^2}(t)dW_t^1 - \mu_{S^1}(t)dt - \sigma_{1,S^1}(t)dW_t^1 - \sigma_{2,S^1}(t)dW_t^2\right. \\
&\left. -\sigma_{1,S^1}(t)\sigma_{S^2}(t)dt + \sigma_{1,S^1}(t)^2 dt + \sigma_{2,S^1}(t)^2 dt\right] \\
=& \frac{S_t^2}{S_t^1}\left[\left(\mu_{S^2}(t) - \mu_{S^1}(t) - \sigma_{S^2}(t)\sigma_{1,S^1}(t) + \sigma_{1,S^1}(t)^2 + \sigma_{2,S^1}(t)^2\right)dt\right. \\
&\left. +(\sigma_{S^2}(t) - \sigma_{1,S^1}(t))dW_t^1 - \sigma_{2,S^1}(t)dW_t^2\right] \\
=& \frac{S_t^2}{S_t^1}\left[\left(\mu_{S^2}(t) - \mu_{S^1}(t) - \sigma_{S^2}(t)\sigma_{1,S^1}(t) + \sigma_{1,S^1}(t)^2 + \sigma_{2,S^1}(t)^2\right)dt\right. \\
&\left. +\sqrt{(\sigma_{S^2}(t) - \sigma_{1,S^1}(t))^2 + \sigma_{2,S^1}(t)^2}dW_t^3\right] \\
=& \frac{S_t^2}{S_t^1}\left[\underbrace{\left(\mu_{S^2}(t) - \mu_{S^1}(t) - \sigma_{S^2}(t)\sigma_{1,S^1}(t) + \sigma_{1,S^1}(t)^2 + \sigma_{2,S^1}(t)^2\right)}_{:=\mu_{\frac{S^2}{S^1}}(t)}dt\right. \\
&\left. + \underbrace{\sqrt{(\sigma_{S^2}(t) - \sigma_{1,S^1}(t))^2 + \sigma_{2,S^1}(t)^2}}_{:=\sigma_{\frac{S^2}{S^1}}(t)}dW_t^3\right] \\
=& \frac{S_t^2}{S_t^1}\left[\mu_{\frac{S^2}{S^1}}(t)dt + \sigma_{\frac{S^2}{S^1}}(t)dW_t^3\right]
\end{aligned}$$

We can now define the market price of risk by

$$\theta(t) = \frac{\mu_{\frac{S^2}{S^1}}(t)}{\sigma_{\frac{S^2}{S^1}}(t)}$$

We define the Radon-Nikodým derivative by

$$\frac{d\tilde{\mathbb{P}}_{S^1}}{d\mathbb{P}} = e^{\int_0^t(-\theta(u))dW_u^4 - \frac{1}{2}\int_0^t(-\theta(u))^2 du}$$

Then by Girsanov theorem it follows that

$$dW_t^4 = -\theta(t)dt + d\tilde{W}_t$$

and $\tilde{W}_t$ is a Brownian motion under the risk neutral measure. So the dynamics of the fraction $\frac{S_t^2}{S_t^1}$ can be rewritten as

$$
\begin{aligned}
d\frac{S_t^2}{S_t^1} =& \frac{S_t^2}{S_t^1}\left[\sigma_{\frac{S^2}{S^1}}(t)\theta(t)dt + \sigma_{\frac{S^2}{S^1}}(t)\left(-\theta(t)dt + d\tilde{W}_t\right)\right] \\
=& \frac{S_t^2}{S_t^1}\sigma_{\frac{S^2}{S^1}}(t)d\tilde{W}_t
\end{aligned}
$$

By applying Ito's lemma the solution of this is given by

$$
\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1}e^{-\int_0^t \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2du + \int_0^t \sigma_{\frac{S^2}{S^1}}(u)d\tilde{W}_u}
$$

This shows the distribution of the fraction $\frac{S_t^2}{S_t^1}$ under the risk neutral measure $\tilde{\mathbb{P}}_{S^1}$. Now we do a numéraire change to the risk neutral measure $\tilde{\mathbb{P}}_{S^2}$. The Radon-Nikodým derivative is given by

$$
\frac{d\tilde{\mathbb{P}}_{S^2}}{d\tilde{\mathbb{P}}_{S^1}} = \frac{\frac{S_t^2}{S_t^1}}{\frac{S_0^2}{S_0^1}} = e^{\int_0^t \sigma_{\frac{S^2}{S^1}}(u)d\tilde{W}_u - \int_0^t \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2du}
$$

where the market price of risk is now given by $-\sigma_{\frac{S^2}{S^1}}(u)$. Then by Girsanov theorem it follows that

$$
d\tilde{W}_t = -\left(-\sigma_{\frac{S^2}{S^1}}(u)\right)dt + d\overline{W}_t
$$

and $\overline{W}_t$ is a Brownian motion. So the dynamics of the fraction $\frac{S_t^2}{S_t^1}$ can be rewritten as

$$
\begin{aligned}
d\frac{S_t^2}{S_t^1} =& \frac{S_t^2}{S_t^1}\sigma_{\frac{S^2}{S^1}}(t)\left(\sigma_{\frac{S^2}{S^1}}(u)dt + d\overline{W}_t\right) \\
=& \frac{S_t^2}{S_t^1}\left(\sigma_{\frac{S^2}{S^1}}(u)^2dt + \sigma_{\frac{S^2}{S^1}}(t)d\overline{W}_t\right)
\end{aligned}
$$

By applying Ito's lemma the solution of this is given by

$$
\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1}e^{\int_0^t \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2du + \int_0^t \sigma_{\frac{S^2}{S^1}}(u)d\overline{W}_u}
$$

This shows the distribution of the fraction $\frac{S_t^2}{S_t^1}$ under the risk neutral measure $\tilde{\mathbb{P}}_{S^2}$. Now we derive the Probabilities $\tilde{\mathbb{P}}_{S^1}\left(S_T^2 > S_T^1\right)$ and $\tilde{\mathbb{P}}_{S^2}\left(S_T^2 > S_T^1\right)$.

$$\begin{aligned}
\tilde{\mathbb{P}}_{S^1}\left(S_T^2 > S_T^1\right) &= \tilde{\mathbb{P}}_{S_t^1}\left(\frac{S_T^2}{S_T^1} > 1\right) \\
&= \tilde{\mathbb{P}}_{S^1}\left(\frac{S_t^2}{S_t^1} e^{-\int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du + \int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\tilde{W}_u} > 1\right) \\
&= \tilde{\mathbb{P}}_{S^1}\left(e^{-\int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du + \int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\tilde{W}_u} > \frac{S_t^1}{S_t^2}\right) \\
&= \tilde{\mathbb{P}}_{S^1}\left(-\int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du + \int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\tilde{W}_u > \ln\left(\frac{S_t^1}{S_t^2}\right)\right) \\
&= \tilde{\mathbb{P}}_{S^1}\left(\int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\tilde{W}_u > \ln\left(\frac{S_t^1}{S_t^2}\right) + \int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du\right) \\
&= \tilde{\mathbb{P}}_{S^1}\left(-\int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\tilde{W}_u \leq \ln\left(\frac{S_t^2}{S_t^1}\right) - \int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du\right) \\
&= \Phi\left(\underbrace{\frac{\ln\left(\frac{S_t^2}{S_t^1}\right) - \frac{1}{2}\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}{\sqrt{\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}}}_{:=d_2(t)}\right) = \Phi\left(d_2(t)\right)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbb{P}}_{S^2}\left(S_T^2 > S_T^1\right) &= \tilde{\mathbb{P}}_{S^2}\left(\frac{S_T^2}{S_T^1} > 1\right) \\
&= \tilde{\mathbb{P}}_{S^2}\left(\frac{S_t^2}{S_t^1} e^{\int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du + \int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\overline{W}_u} > 1\right) \\
&= \tilde{\mathbb{P}}_{S^2}\left(e^{\int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du + \int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\overline{W}_u} > \frac{S_t^1}{S_t^2}\right) \\
&= \tilde{\mathbb{P}}_{S^2}\left(\int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du + \int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\overline{W}_u > \ln\left(\frac{S_t^1}{S_t^2}\right)\right) \\
&= \tilde{\mathbb{P}}_{S^2}\left(\int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\overline{W}_u > \ln\left(\frac{S_t^1}{S_t^2}\right) - \int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du\right) \\
&= \tilde{\mathbb{P}}_{S^2}\left(-\int_t^T \sigma_{\frac{S^2}{S^1}}(u) d\overline{W}_u \leq \ln\left(\frac{S_t^2}{S_t^1}\right) + \int_t^T \frac{1}{2}\sigma_{\frac{S^2}{S^1}}(u)^2 du\right) \\
&= \Phi\left(\underbrace{\frac{\ln\left(\frac{S_t^2}{S_t^1}\right) + \frac{1}{2}\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}{\sqrt{\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}}}_{:=d_1(t)}\right) = \Phi\left(d_1(t)\right)
\end{aligned}$$

To sum up the price at time t of a European option to exchange the asset S^1 for the asset S^2 at time T

$$C(t, S_t^1, S_t^2) = S_t^2 \Phi\left(d_1(t)\right) - S_t^1 \Phi\left(d_2(t)\right)$$

where

$$d_1(t) := \frac{\ln\left(\frac{S_t^2}{S_t^1}\right) + \frac{1}{2}\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}{\sqrt{\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}}, \quad d_2(t) = d_1(t) - \sqrt{\int_t^T \sigma_{\frac{S^2}{S^1}}(u)^2 du}$$

and $\sigma_{\frac{S^2}{S^1}}(t)^2 = (\sigma_{S^2}(t) - \sigma_{1,S^1}(t))^2 + \sigma_{2,S^1}(t)^2$

B.1.3 Black Scholes model and Heath Jarrow Morton model

We assume a complete and arbitrage-free financial market model under interest rate risk where the dynamic of the price process $(A_t)_{t\in[0,T]}$ as well as the dynamics of the zero coupon bonds $B(\cdot, T)$ paying one monetary unit at maturity $T > 0$ are lognormal. Thus, the index dynamic is modeled along the lines of Black and Scholes (1973), the interest rate dynamic is given by a Gaussian Markov Heath, Jarrow and Morton model introduced by Heath et al. (1992). In particular, there exist a uniquely defined martingale measure P^* such that

$$\begin{aligned} A_t &= A_t \left(r_t dt + \sigma_{1,A}(t) dW_t^1 + \sigma_{2,A}(t) dW_t^2 \right) \\ B(t,T) &= B(t,T) \left(r_t dt + \sigma_T(t) dW_t^1 \right) \end{aligned}$$

where W^* denotes a d-dimensional Brownian motion with respect to P^*, and σ_A and σ_T satisfy the usual regularity conditions. The first asset from the above model S_t^1 is then represented by the fraction of the investment return of the asset A_t, i.e. $S_t^1 = \alpha \frac{A_t}{A_0}$. The dynamic can then be derived by Ito‘s lemma and is given by

$$d\left(\alpha \frac{A_t}{A_0} \right) = \alpha \frac{A_t}{A_0} \left(r_t dt + \sigma_{1,A}(t) dW_t^1 + \sigma_{2,A}(t) dW_t^2 \right)$$

Fix strike

For the second asset S_t^2 we distinguish between two cases. The first case is the fix strike guarantee. Then the asset is given by $S_t^2 = B(t,T)K_{\text{fix}}$, where K_{fix} is constant. Then the dynamic can be derived either by Ito‘s Lemma and is given by

$$d(B(t,T)K_{\text{fix}}) = B(t,T)K_{\text{fix}} \left(r_t dt + \sigma_T(t) dW_t^1 \right)$$

The price at time t of a European option to exchange the asset S^1 for the asset S^2 at time T

$$C\left(t, \alpha \frac{A_t}{A_0}, B(t,T)K_{\text{fix}} \right) = B(t,T)K_{\text{fix}} \Phi\left(d_1(t) \right) - \alpha \frac{A_t}{A_0} \Phi\left(d_2(t) \right)$$

where

$$d_1(t) := \frac{\ln\left(\frac{B(t,T)K_{\text{fix}}}{\alpha \frac{A_t}{A_0}} \right) + \frac{1}{2} \int_t^T \sigma(u)^2 du}{\sqrt{\int_t^T \sigma(u)^2 du}}, \quad d_2(t) = d_1(t) - \sqrt{\int_t^T \sigma(u)^2 du}$$

and $\sigma(t)^2 = (\sigma_T(t) - \sigma_{1,A}(t))^2 + \sigma_{2,A}(t)^2$

Floating strike

The second is the floating strike guarantee. Then the asset is given by $S_t^2 = \tilde{\alpha} e^{I(0,t)}$, where $I(t,T) := \int_t^T r_u du$ and $(r_t)_{t \in [0,T]}$ is the instantaneous spot rate. The dynamics are the given by

$$d\left(\tilde{\alpha} e^{I(0,t)} \right) = \tilde{\alpha} e^{I(0,t)} r_t dt$$

The price at time t of a European option to exchange the asset S^1 for the asset S^2 at time T

$$C\left(t, \alpha \frac{A_t}{A_0}, \tilde{\alpha} e^{I(0,t)} \right) = \tilde{\alpha} e^{I(0,t)} \Phi\left(d_1(t) \right) - \alpha \frac{A_t}{A_0} \Phi\left(d_2(t) \right)$$

where

$$d_1(t) := \frac{\ln\left(\frac{\tilde{\alpha}e^{I(0,t)}}{\alpha\frac{A_t}{A_0}}\right) + \frac{1}{2}\int_t^T \sigma(u)^2 du}{\sqrt{\int_t^T \sigma(u)^2 du}}, \quad d_2(t) = d_1(t) - \sqrt{\int_t^T \sigma(u)^2 du}$$

and $\sigma(t)^2 = \sigma_{1,A}(t)^2 + \sigma_{2,A}(t)^2$

B.2 Cumulated volatilities

Lemma B.2.1 *For*

$$\sigma_S = \begin{pmatrix} \rho\sigma \\ \sqrt{1-\rho^2}\sigma \end{pmatrix}, \quad \sigma_T(t) = \begin{pmatrix} \frac{\sigma_r}{a}(1-e^{-a(T-t)}) \\ 0 \end{pmatrix}$$

and $\sigma_Z(t) = \sigma_{S,T}(t) = \sigma_S - \sigma_T(t)$ *it holds*

$$\begin{aligned} &\int_t^T \|\sigma_Z(s)\|^2 ds \\ &= \left(\sigma^2 - 2\rho\sigma\frac{\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right)(T-t) + 2\left(\rho\sigma\frac{\sigma_r}{a^2} - \frac{\sigma_r^2}{a^3}\right)\left(1-e^{-a(T-t)}\right) + \frac{\sigma_r^2}{2a^3}\left(1-e^{-2a(T-t)}\right). \end{aligned}$$

Proof:

$$\begin{aligned} \|\sigma_Z(t)\|^2 &= \left(\rho\sigma - \frac{\sigma_r}{a}(1-e^{-a(T-t)})\right)^2 + (1-\rho^2)\sigma^2 \\ &= \underbrace{\left(\sigma^2 - 2\rho\sigma\frac{\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right)}_{A_1} + \underbrace{2\left(\rho\sigma\frac{\sigma_r}{a} - \frac{\sigma_r^2}{a^2}\right)}_{A_2} e^{-a(T-t)} + \underbrace{\frac{\sigma_r^2}{a^2}}_{A_3} e^{-2a(T-t)} \end{aligned}$$

It follows

$$\begin{aligned} &\int_t^T \|\sigma_Z(s)\|^2 ds = A_1(T-t) + A_2\frac{1}{a}e^{-a(T-t)} + A_3\frac{1}{2a}e^{-2a(T-t)} \\ &= \left(\sigma^2 - 2\rho\sigma\frac{\sigma_r}{a} + \frac{\sigma_r^2}{a^2}\right)(T-t) + 2\left(\rho\sigma\frac{\sigma_r}{a^2} - \frac{\sigma_r^2}{a^3}\right)\left(1-e^{-a(T-t)}\right) + \frac{\sigma_r^2}{2a^3}\left(1-e^{-2a(T-t)}\right). \end{aligned}$$

B.3 Derivatives of the exchange option price $C(t, K_t^w, \alpha^w A_t)$

$$C(t, K_t^w, \alpha^w A_t) := K_t^w \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w A_t \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)$$

First derivatives of d_1 and d_2

$$\begin{aligned} d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right) &:= \frac{\ln \frac{K_t^w}{\alpha^w A_t} + \frac{1}{2} v^2(t,T)}{v(t,T)}, \\ d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right) &:= d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right) - v(t,T) \end{aligned}$$

$$\begin{aligned} d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right) &:= \frac{\ln(K_t^w)}{v(t,T)} - \frac{\ln(\alpha^w A_t)}{v(t,T)} + \frac{1}{2} v(t,T), \\ d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right) &:= \frac{\ln(K_t^w)}{v(t,T)} - \frac{\ln(\alpha^w A_t)}{v(t,T)} - \frac{1}{2} v(t,T) \end{aligned}$$

First derivative of d_1 and d_2 regarding A_t

$$\frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial A_t} = \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial A_t} := -\frac{1}{A_t v(t,T)}$$

First derivative of d_1 and d_2 regarding K_t^w

$$\frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial K_t^w} = \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial K_t^w} := \frac{1}{K_t^w v(t,T)}$$

First derivative of d_1 and d_2 regarding r_t

$$\begin{aligned} \frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} = \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} &= \frac{1}{v(t,T)} \frac{\partial \ln(K_t^w)}{\partial r_t} - \frac{1}{v(t,T)} \frac{\partial \ln(\alpha^w A_t)}{\partial r_t} \\ &= \frac{1}{v(t,T) K_t^w} \frac{\partial K_t^w}{\partial r_t} - \frac{1}{v(t,T) A_t} \frac{\partial A_t}{\partial r_t} \end{aligned}$$

First derivative of d_1 and d_2 regarding cumulated volatility $v(t,T)$

$$\begin{aligned} \frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial v(t,T)} &= -\frac{\ln \frac{K_t^w}{\alpha^w A_t}}{v^2(t,T)} + \frac{1}{2}, \\ \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial v(t,T)} &= -\frac{\ln \frac{K_t^w}{\alpha^w A_t}}{v^2(t,T)} - \frac{1}{2} \end{aligned}$$

First derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding A_t

$$
\begin{aligned}
\frac{\partial C(t, K_t^w, \alpha^w A_t)}{\partial A_t} =& K_t^w \underbrace{\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial A_t}}_{(a)} - \alpha^w \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& - \alpha^w A_t \underbrace{\frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial A_t}}_{(b)} \\
=& - \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{A_t v(t,T)} - \alpha^w \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{A_t v(t,T)} \\
=& -\alpha^w \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)
\end{aligned}
$$

Derivative (a)

$$
\begin{aligned}
\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial A_t} &= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial A_t} \\
&= - \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{A_t v(t,T)}
\end{aligned}
$$

Derivative (b)

$$
\begin{aligned}
\frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial A_t} &= \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial A_t} \\
&= - \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{A_t v(t,T)} \\
&\underset{(*)}{=} - \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{\alpha^w A_t} \frac{1}{A_t v(t,T)} \\
&= - \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{\alpha^w A_t^2 v(t,T)}
\end{aligned}
$$

Derivative $(*)$

$$
\begin{aligned}
\frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} &= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right) - v(t,T)\right)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)^2 - 2d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right) v(t,T) + v(t,T)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)^2}{2}} e^{d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right) v(t,T)} e^{-\frac{v(t,T)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)^2}{2}} e^{\ln \frac{K_t^w}{\alpha^w A_t} + \frac{v(t,T)^2}{2}} e^{-\frac{v(t,T)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)^2}{2}} \frac{K_t^w}{\alpha^w A_t} e^{\frac{v(t,T)^2}{2}} e^{-\frac{v(t,T)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)^2}{2}} \frac{K_t^w}{\alpha^w A_t} \\
&= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{\alpha^w A_t}
\end{aligned}
$$

First derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding K_t^w

$$
\begin{aligned}
\frac{\partial C(t, K_t^w, \alpha^w A_t)}{\partial K_t^w} =& \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) + K_t^w \underbrace{\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial K_t^w}}_{(c)} \\
& - \alpha^w A_t \underbrace{\frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial K_t^w}}_{(d)} \\
=& \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) + \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{v(t,T)} \\
& - \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{v(t,T)} \\
=& \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)
\end{aligned}
$$

Derivative (c)

$$
\begin{aligned}
\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial K_t^w} &= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial K_t^w} \\
&= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{K_t^w v(t,T)}
\end{aligned}
$$

Derivative (d)

$$
\begin{aligned}
\frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial K_t^w} &= \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial K_t^w} \\
&= \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{K_t^w v(t,T)} \\
&\underset{(*)}{=} \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{\alpha^w A_t} \frac{1}{K_t^w v(t,T)} \\
&= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{\alpha^w A_t v(t,T)}
\end{aligned}
$$

Second derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding A_t

$$\begin{aligned}
\frac{\partial^2 C(t, K_t^w, \alpha^w A_t)}{(\partial A_t)^2} &= -\alpha^w \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial A_t} \\
&\underset{(b)}{=} \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{K_t^w}{A_t^2 v(t,T)} \\
&= \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \frac{K_t^w}{A_t^2 v(t,T)}
\end{aligned}$$

Second derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding K_t^w

$$\begin{aligned}
\frac{\partial^2 C(t, K_t^w, \alpha^w A_t)}{(\partial K_t^w)^2} &= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{K_t^w} \\
&\underset{(c)}{=} \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{K_t^w v(t,T)} \\
&= \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \frac{1}{K_t^w v(t,T)}
\end{aligned}$$

Second derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding A_t and K_t^w

$$\begin{aligned}
\frac{\partial \frac{\partial C(t, K_t^w, \alpha^w A_t)}{\partial A_t}}{\partial K_t^w} &= -\alpha^w \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial K_t^w} \\
&\underset{(c)}{=} -\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{A_t v(t,T)} \\
&= -\mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \frac{1}{A_t v(t,T)}
\end{aligned}$$

Second derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding K_t^w and A_t

$$\begin{aligned}
\frac{\partial \frac{\partial C(t, K_t^w, \alpha^w A_t)}{\partial K_t^w}}{\partial A_t} &= -\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial A_t} \\
&\underset{(a)}{=} -\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)} \frac{1}{A_t v(t,T)} \\
&= -\mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \frac{1}{A_t v(t,T)}
\end{aligned}$$

First derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding r_t

$$
\begin{aligned}
\frac{\partial C(t, K_t^w, \alpha^w A_t)}{\partial r_t} =& \frac{\partial K_t^w}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) + K_t^w \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial r_t} \\
& - \alpha^w \frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w A_t \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial r_t} \\
\underset{(s.b.)}{=}& \frac{\partial K_t^w}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) + K_t^w \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
& - \alpha^w \frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w A_t \mathcal{N}'\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
=& \frac{\partial K_t^w}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + K_t^w \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\left(\frac{1}{v(t,T)K_t^w}\frac{\partial K_t^w}{\partial r_t} - \frac{1}{v(t,T)A_t}\frac{\partial A_t}{\partial r_t}\right) \\
& - \alpha^w A_t \mathcal{N}'\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\left(\frac{1}{v(t,T)K_t^w}\frac{\partial K_t^w}{\partial r_t} - \frac{1}{v(t,T)A_t}\frac{\partial A_t}{\partial r_t}\right) \\
\underset{(*)}{=}& \frac{\partial K_t^w}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{1}{v(t,T)}\frac{\partial K_t^w}{\partial r_t} - \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{K_t^w}{v(t,T)A_t}\frac{\partial A_t}{\partial r_t} \\
& - \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{1}{v(t,T)}\frac{\partial K_t^w}{\partial r_t} + \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{K_t^w}{v(t,T)A_t}\frac{\partial A_t}{\partial r_t} \\
=& \frac{\partial K_t^w}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)
\end{aligned}
$$

Derivative $\mathcal{N}(\cdot)$ regarding r_t

$$
\begin{aligned}
\frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial r_t} &= \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}\frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
&= \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
\frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial r_t} &= \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}\frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
&= \mathcal{N}'\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)\frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t}
\end{aligned}
$$

Second derivative of $C(t, K_t^w, \alpha^w A_t)$ regarding r_t

$$
\begin{aligned}
\frac{\partial^2 C(t, K_t^w, \alpha^w A_t)}{(\partial r_t)^2} =& \frac{\partial^2 K_t^w}{(\partial r_t)^2} \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) + \frac{\partial K_t^w}{\partial r_t} \frac{\partial \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial r_t} \\
& - \alpha^w \frac{\partial^2 A_t}{(\partial r_t)^2} \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial A_t}{\partial r_t} \frac{\partial \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right)}{\partial r_t} \\
\underset{(s.o.)}{=} & \frac{\partial^2 K_t^w}{(\partial r_t)^2} \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial^2 A_t}{(\partial r_t)^2} \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \frac{\partial K_t^w}{\partial r_t} \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \frac{\partial d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
& - \alpha^w \frac{\partial A_t}{\partial r_t} \mathcal{N}'\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \frac{\partial d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)}{\partial r_t} \\
\underset{(*)}{=} & \frac{\partial^2 K_t^w}{(\partial r_t)^2} \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial^2 A_t}{(\partial r_t)^2} \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \frac{\partial K_t^w}{\partial r_t} \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \left(\frac{1}{v(t,T) K_t^w} \frac{\partial K_t^w}{\partial r_t} - \frac{1}{v(t,T) A_t} \frac{\partial A_t}{\partial r_t}\right) \\
& - \frac{\partial A_t}{\partial r_t} \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \left(\frac{1}{v(t,T) A_t} \frac{\partial K_t^w}{\partial r_t} - \frac{K_t^w}{v(t,T) A_t^2} \frac{\partial A_t}{\partial r_t}\right) \\
=& \frac{\partial^2 K_t^w}{(\partial r_t)^2} \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial^2 A_t}{(\partial r_t)^2} \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \left(\frac{1}{v(t,T) K_t^w} \left(\frac{\partial K_t^w}{\partial r_t}\right)^2 - \frac{1}{v(t,T) A_t} \frac{\partial K_t^w}{\partial r_t} \frac{\partial A_t}{\partial r_t}\right) \\
& - \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \left(\frac{1}{v(t,T) A_t} \frac{\partial K_t^w}{\partial r_t} \frac{\partial A_t}{\partial r_t} - \frac{K_t^w}{v(t,T) A_t^2} \left(\frac{\partial A_t}{\partial r_t}\right)^2\right) \\
=& \frac{\partial^2 K_t^w}{(\partial r_t)^2} \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial^2 A_t}{(\partial r_t)^2} \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& \quad \left(\frac{1}{v(t,T) K_t^w} \left(\frac{\partial K_t^w}{\partial r_t}\right)^2 - 2 \frac{1}{v(t,T) A_t} \frac{\partial K_t^w}{\partial r_t} \frac{\partial A_t}{\partial r_t} + \frac{K_t^w}{v(t,T) A_t^2} \left(\frac{\partial A_t}{\partial r_t}\right)^2\right) \\
=& \frac{\partial^2 K_t^w}{(\partial r_t)^2} \mathcal{N}\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) - \alpha^w \frac{\partial^2 A_t}{(\partial r_t)^2} \mathcal{N}\left(d_2\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \\
& + \mathcal{N}'\left(d_1\left(t, \frac{K_t^w}{\alpha^w A_t}\right)\right) \left(\frac{1}{\sqrt{v(t,T) K_t^w}} \frac{\partial K_t^w}{\partial r_t} + \frac{\sqrt{K_t^w}}{\sqrt{v(t,T)} A_t} \frac{\partial A_t}{\partial r_t}\right)^2
\end{aligned}
$$

B.4 Derivatives of the asset portfolio A_t

$$A_t = \phi_t^{(S)} S_t + \sum_{i=1}^{d-1} \phi_t^{(i)} B(t, t_i), \qquad \phi_t^{(S)} = \frac{\pi_S A_t}{S_t},\ \phi_t^{(i)} = \frac{\pi_i A_t}{B(t, t_i)}$$

Bondprice $B(s, t)$

$$\begin{aligned}
B(s,t) &= e^{-\mathcal{B}(s,t) r_s + \mathcal{A}(s,t)} \\
\mathcal{B}(s,t) &= \frac{1}{a}(1 - e^{-a(t-s)}) \\
\mathcal{A}(s,t) &= (\mathcal{B}(s,t) - (t-s)) \left(b - \frac{\sigma_r^2}{2a^2} \right) - \frac{\sigma_r^2}{4a} \mathcal{B}^2(s,t).
\end{aligned}$$

First derivative of $B(s, t)$ regarding r_t

$$\frac{\partial B(s,t)}{\partial r_s} = -\mathcal{B}(s,t) B(s,t)$$

Second derivative of $B(s, t)$ regarding r_t

$$\begin{aligned}
\frac{\partial^2 B(s,t)}{\partial r_s^2} &= -\mathcal{B}(s,t) \frac{\partial B(s,t)}{\partial r_s} \\
&= \mathcal{B}(s,t)^2 B(s,t)
\end{aligned}$$

First derivative of A_t regarding r_t

$$\begin{aligned}
\frac{\partial A_t}{\partial r_t} &= \sum_{i=1}^{d-1} \phi_t^{(i)} \frac{\partial B(t, t_i)}{\partial r_t} \\
&= -\sum_{i=1}^{d-1} \frac{\pi_i A_t}{B(t, t_i)} \mathcal{B}(t, t_i) B(t, t_i) \\
&= -A_t \sum_{i=1}^{d-1} \pi_i \mathcal{B}(t, t_i)
\end{aligned}$$

Second derivative of A_t regarding r_t

$$
\begin{aligned}
\frac{\partial^2 A_t}{(\partial r_t)^2} &= -\frac{\partial A_t}{\partial r_t} \sum_{i=1}^{d-1} \pi_i \mathcal{B}(t, t_i) \\
&= A_t \left(\sum_{i=1}^{d-1} \pi_i \mathcal{B}(t, t_i) \right)^2
\end{aligned}
$$

B.5 Derivatives of strikes $K_t^{\textbf{fix}}$ and $K_t^{\textbf{fl.}}$

Fix Strike $K_t^{\textbf{fix}}$

$$K_t^{\text{fix}} := B(t,T)K_{\text{fix}}$$

First derivative $K_t^{\textbf{fix}}$ regarding r_t

$$\begin{aligned}\frac{\partial K_t^{\text{fix}}}{\partial r_t} &= K_{\text{fix}}\frac{\partial B(t,T)}{\partial r_t}\\ &= -K_{\text{fix}}\mathcal{B}(t,T)B(t,T)\end{aligned}$$

Second derivative $K_t^{\textbf{fix}}$ regarding r_t

$$\begin{aligned}\frac{\partial^2 K_t^{\text{fix}}}{(\partial r_t)^2} &= -K_{\text{fix}}\mathcal{B}(t,T)\frac{\partial B(t,T)}{\partial r_t}\\ &= K_{\text{fix}}\mathcal{B}(t,T)^2B(t,T)\end{aligned}$$

Floating Strike $K_t^{\textbf{fl.}}$

$$K_t^{\text{fl.}} := \tilde{\alpha}e^{\int_0^t r_s ds}$$

First derivative $K_t^{\textbf{fl.}}$ regarding r_t

$$\begin{aligned}\frac{\partial K_t^{\text{fl.}}}{\partial r_t} &= \tilde{\alpha}\frac{\partial e^{\int_0^t r_s ds}}{\partial r_t}\\ &= \tilde{\alpha}r_t e^{\int_0^t r_s ds}\end{aligned}$$

Second derivative $K_t^{\textbf{fl.}}$ regarding r_t

$$\begin{aligned}\frac{\partial^2 K_t^{\text{fl.}}}{(\partial r_t)^2} &= \tilde{\alpha}e^{\int_0^t r_s ds} + \tilde{\alpha}r_t\frac{\partial e^{\int_0^t r_s ds}}{\partial r_t}\\ &= \tilde{\alpha}e^{\int_0^t r_s ds} + \tilde{\alpha}r_t^2 e^{\int_0^t r_s ds}\\ &= \tilde{\alpha}e^{\int_0^t r_s ds}(1+r_t^2)\end{aligned}$$

B.6 Derivatives of the Buffer $A_t - L_t$

Liabilities

$$\begin{aligned}
L_t &= \eta^{\text{fix}} C_{\text{fix}}(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t) + \eta^{\text{fl.}} C_{\text{fl.}}(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t) \\
&= \eta^{\text{fix}} \left(\alpha^{\text{fix}} A_t + C(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t)\right) + \eta^{\text{fl.}} \left(\alpha^{\text{fl.}} A_t + C(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t)\right) \\
&= \left(\eta^{\text{fix}} \alpha^{\text{fix}} + \eta^{\text{fl.}} \alpha^{\text{fl.}}\right) A_t + \eta^{\text{fix}} C(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t) + \eta^{\text{fl.}} C(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t)
\end{aligned}$$

Buffer $A_t - L_t$

$$A_t - L_t = \left(1 - \eta^{\text{fix}} \alpha^{\text{fix}} - \eta^{\text{fl.}} \alpha^{\text{fl.}}\right) A_t - \eta^{\text{fix}} C(t, K_t^{\text{fix}}, \alpha^{\text{fix}} A_t) - \eta^{\text{fl.}} C(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}} A_t)$$

First derivative of $A_t - L_t$ regarding r_t

$$\begin{aligned}
\frac{\partial A_t - L_t}{\partial r_t} &= \left(1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}}\right)\frac{\partial A_t}{\partial r_t} - \eta^{\text{fix}}\frac{\partial C(t, K_t^{\text{fix}}, \alpha^{\text{fix}}A_t)}{\partial r_t} - \eta^{\text{fl.}}\frac{\partial C(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}}A_t)}{\partial r_t} \\
&= \left(1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}}\right)\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\left[\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) - \alpha^{\text{fix}}\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right)\right] \\
&\quad - \eta^{\text{fl.}}\left[\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) - \alpha^{\text{fl.}}\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)\right] \\
&= \left(1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}}\right)\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) + \eta^{\text{fix}}\alpha^{\text{fix}}\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) \\
&\quad - \eta^{\text{fl.}}\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) + \eta^{\text{fl.}}\alpha^{\text{fl.}}\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) \\
&= \left(1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}}\right)\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) - \eta^{\text{fl.}}\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) \\
&\quad + \eta^{\text{fix}}\alpha^{\text{fix}}\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) + \eta^{\text{fl.}}\alpha^{\text{fl.}}\frac{\partial A_t}{\partial r_t}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) \\
&= \left(1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}}\right)\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) - \eta^{\text{fl.}}\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) \\
&\quad + \left[\eta^{\text{fix}}\alpha^{\text{fix}}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) + \eta^{\text{fl.}}\alpha^{\text{fl.}}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)\right]\frac{\partial A_t}{\partial r_t} \\
&= \Bigg[1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}} + \eta^{\text{fix}}\alpha^{\text{fix}}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) \\
&\qquad + \eta^{\text{fl.}}\alpha^{\text{fl.}}\mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)\Bigg]\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) - \eta^{\text{fl.}}\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg[1 + \eta^{\text{fix}}\alpha^{\text{fix}}\left(1 - \mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right)\right) \\
&\qquad + \eta^{\text{fl.}}\alpha^{\text{fl.}}\left(1 - \mathcal{N}\left(d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)\right)\Bigg]\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) - \eta^{\text{fl.}}\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right) \\
&= \Bigg[1 + \eta^{\text{fix}}\alpha^{\text{fix}}\mathcal{N}\left(-d_2\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) + \eta^{\text{fl.}}\alpha^{\text{fl.}}\mathcal{N}\left(-d_2\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)\Bigg]\frac{\partial A_t}{\partial r_t} \\
&\quad - \eta^{\text{fix}}\frac{\partial K_t^{\text{fix}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fix}}}{\alpha^{\text{fix}}A_t}\right)\right) - \eta^{\text{fl.}}\frac{\partial K_t^{\text{fl.}}}{\partial r_t}\mathcal{N}\left(d_1\left(t, \frac{K_t^{\text{fl.}}}{\alpha^{\text{fl.}}A_t}\right)\right)
\end{aligned}
$$

Second derivative of $A_t - L_t$ regarding r_t

$$
\frac{\partial^2 A_t - L_t}{(\partial r_t)^2} = \left(1 - \eta^{\text{fix}}\alpha^{\text{fix}} - \eta^{\text{fl.}}\alpha^{\text{fl.}}\right)\frac{\partial^2 A_t}{(\partial r_t)^2} - \eta^{\text{fix}}\frac{\partial^2 C(t, K_t^{\text{fix}}, \alpha^{\text{fix}}A_t)}{(\partial r_t)^2} - \eta^{\text{fl.}}\frac{\partial^2 C(t, K_t^{\text{fl.}}, \alpha^{\text{fl.}}A_t)}{(\partial r_t)^2}
$$

B.7 Simulation

In this section we present the simulation used to calculate the risk measures.

B.7.1 Product design

Pay-Off:

$$P_T^{\text{fix}} = P_0 \max\left\{K_{\text{fix}}, \alpha\frac{A_T}{A_0}\right\}$$
$$P_T^{\text{fl.}} = P_0 \max\left\{\tilde{\alpha}e^{I(0,T)}, \alpha\frac{A_T}{A_0}\right\}$$

Assume $P_0 = 1$. Price at time t:

$$C^{\text{fix}}(t, K_{\text{fix}}) = \alpha\frac{A_t}{A_0} + C\left(t, \alpha\frac{A_t}{A_0}, B(t,T)K_{\text{fix}}\right)$$
$$C^{\text{fl.}}(t, \tilde{\alpha}) = \alpha\frac{A_t}{A_0} + C\left(t, \alpha\frac{A_t}{A_0}, \tilde{\alpha}e^{I(0,t)}\right)$$

Fair Pricing:

$$C^{\text{fix}}(0, K_{\text{fix}}) = \alpha\frac{A_0}{A_0} + C\left(0, \alpha\frac{A_0}{A_0}, B(0,T)K_{\text{fix}}\right) = P_0 = 1 \Rightarrow C\left(0, \alpha, B(0,T)K_{\text{fix}}\right) = 1 - \alpha$$
$$C^{\text{fl.}}(0, \tilde{\alpha}) = \alpha\frac{A_0}{A_0} + C\left(0, \alpha\frac{A_0}{A_0}, \tilde{\alpha}\right) = P_0 = 1 \Rightarrow C\left(0, \alpha, \tilde{\alpha}\right) = 1 - \alpha$$

Choose $K_{\text{fix}}^{\text{fair}}$ and $\tilde{\alpha}^{\text{fair}}$ such that

$$C\left(0, \alpha, B(0,T)K_{\text{fix}}^{\text{fair}}\right) = 1 - \alpha$$
$$C\left(0, \alpha, \tilde{\alpha}^{\text{fair}}\right) = 1 - \alpha$$

B.7.2 Model specification (Risk neutral measure)

Stock:

$$\frac{dS_t}{S_t} = r_t dt + \rho\sigma_S dW_t^1 + \sqrt{1-\rho^2}\sigma_S dW_t^2$$

Interest Rate:

$$dr_t = a(b - r_t)dt + \sigma_r dW_t^1$$

Bond:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt - \sigma_r \frac{1}{a}\left(1 - e^{-a(T-t)}\right) dW_t^1$$

Bank account:

$$\frac{d(e^{I(0,t)})}{e^{I(0,t)}} = r_t dt$$

Investment Portfolio:

$$\begin{aligned}\frac{dA_t}{A_t} =& \pi_S \frac{dS_t}{S_t} + \pi_B \frac{dB(t,T)}{B(t,T)} + (1-\pi_S-\pi_B)\frac{d(e^{I(0,t)})}{e^{I(0,t)}} \\ =& \pi_S \left(r_t dt + \rho\sigma_S dW_t^1 + \sqrt{1-\rho^2}\sigma_S dW_t^2 \right) + \pi_B \left(r_t dt - \sigma_r \frac{1}{a}\left(1-e^{-a(T-t)}\right) dW_t^1 \right) \\ &+ (1-\pi_S-\pi_B) r_t dt \\ =& r_t dt + \underbrace{\left(\pi_S\rho\sigma_S - \pi_B\sigma_r \frac{1}{a}\left(1-e^{-a(T-t)}\right)\right)}_{=:\sigma_{1,A}(t)} dW_t^1 + \underbrace{\pi_S\sqrt{1-\rho^2}\sigma_S}_{=:\sigma_{2,A}(t)} dW_t^2\end{aligned}$$

B.7.3 Model specification (Real world measure)

The dynamic of the interest rate in the Vasicek model are given under the risk neutral measure $\mathbb{P}^*$

$$dr_t^{\mathbb{P}^*} = a(b^* - r_t^{\mathbb{P}^*})dt + \sigma_r dW_t^{1,\mathbb{P}^*}.$$

For simulation we need to change to the real world measure $\mathbb{P}$. Assuming a constant market price of interest rate risk λ leads to $\mathbb{P}$ dynamics

$$dr_t^{\mathbb{P}} = a(b - r_t^{\mathbb{P}})dt + \sigma_r dW_t^{1,\mathbb{P}}$$

where $b = b^* + \lambda\frac{\sigma_r}{a}$. Furthermore under the real world measure $\mathbb{P}$ the stock dynamic is given by

$$\frac{dS_t}{S_t}^{\mathbb{P}} = \mu_S dt + \rho\sigma_S dW_t^{1,\mathbb{P}} + \sqrt{1-\rho^2}\sigma_S dW_t^{2,\mathbb{P}}.$$

The Investment Portfolio dynamics are then given by

$$\frac{dA_t}{A_t}^{\mathbb{P}} = (\pi_S\mu_S + (1-\pi_S)r_t^{\mathbb{P}})dt + \|\sigma_A(t)\| dW_t^{3,\mathbb{P}}$$

$$A_0 = \pi_S \frac{S_0}{S_0} + \pi_B \frac{B(0,T)}{B(0,T)} + (1-\pi_S-\pi_B) = 1$$

Liabilities:

$$L_t = \alpha A_t + \eta_{\text{fix}} C\left(t, \alpha A_t, B(t,T) K_{\text{fix}}^{\text{fair}}\right) + (1-\eta_{\text{fix}}) C\left(t, \alpha A_t, \tilde{\alpha}^{\text{fair}} e^{I(0,t)}\right)$$

$$\begin{aligned}L_0 &= \alpha + \eta_{\text{fix}} C\left(0, \alpha, B(0,T) K_{\text{fix}}^{\text{fair}}\right) + (1-\eta_{\text{fix}}) C\left(0, \alpha, \tilde{\alpha}^{\text{fair}}\right) \\ &= \alpha + \eta_{\text{fix}}(1-\alpha) + (1-\eta_{\text{fix}})(1-\alpha) = 1\end{aligned}$$

$$\begin{aligned}L_T &= \eta_{\text{fix}} P_T^{\text{fix}} + (1-\eta_{\text{fix}}) P_T^{\text{fl.}} \\ &= \eta_{\text{fix}} \max\left\{K_{\text{fix}}^{\text{fair}}, \alpha A_T\right\} + (1-\eta_{\text{fix}}) \max\left\{\tilde{\alpha}^{\text{fair}} e^{I(0,T)}, \alpha A_T\right\}\end{aligned}$$

B.7.4 Distribution

If

$$\frac{dS_t}{S_t} = r_t dt + \rho\sigma_S dW_t^{(1)} + \sqrt{1-\rho^2}\sigma_S dW_t^{(2)}$$
$$dr_t = a(b - r_t)dt + \sigma_r dW_t^{(1)}$$

it holds

$$\left[\begin{array}{l|l} r_t & r_s \\ ln\frac{S_t}{S_s} & \frac{S_s}{S_0} \\ \int_s^t r_u du & \int_0^s r_u du \end{array}\right] \sim N\left(\begin{array}{l|l} (r_s - b)\, e^{-a(t-s)} + b & \\ \mathcal{B}(s,t)(r_s - b) + \left(b - \frac{1}{2}\sigma_S^2\right)(t-s) & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \\ \mathcal{B}(s,t)(r_s - b) + b(t-s) & \end{array}\right)$$

where

$$c_{11} = \frac{\sigma_r^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$
$$c_{12} = c_{21} = \left(\frac{\sigma_r}{a} + \rho\sigma_S\right)\sigma_r\mathcal{B}(s,t) - \frac{\sigma_r^2}{2a^2}(1 - e^{-2a(t-s)})$$
$$c_{13} = c_{31} = \frac{\sigma_r^2}{a}\mathcal{B}(s,t) - \frac{\sigma_r^2}{2a^2}(1 - e^{-2a(t-s)})$$
$$c_{22} = 2\frac{\sigma_r}{a}\left(-\rho\sigma_S - \frac{\sigma_r}{a}\right)\mathcal{B}(s,t) + \frac{\sigma_r^2}{2a^3}(1 - e^{-2a(t-s)}) + \left(\frac{\sigma_r^2}{a^2} + 2\rho\sigma_S\frac{\sigma_r}{a} + \sigma_S^2\right)(t-s)$$
$$c_{33} = \frac{\sigma_r^2}{a^2}\left((t-s) - 2\mathcal{B}(s,t) + \frac{1}{2a}\left(1 - e^{-2a(t-s)}\right)\right)$$
$$c_{23} = c_{32} = c_{33} + \rho\sigma_S\frac{\sigma_r}{a}\left((t-s) - \mathcal{B}(s,t)\right)$$

and

$$\mathcal{B}(s,t) := \frac{1}{a}(1 - e^{-a(t-s)}).$$

Proof:

$$r_t = r_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) + \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}$$

$$\mathbb{E}\left[r_t|\mathcal{F}_s\right] = r_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) = r_s e^{-a(t-s)} + b - be^{-a(t-s)} = (r_s - b)\, e^{-a(t-s)} + b$$
$$Var\left[r_t|\mathcal{F}_s\right] = \int_s^t \sigma_r^2 e^{-2a(t-u)}du = \frac{\sigma_r^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$

$$I(s,t) := \int_s^t r_u du$$

$$
\begin{aligned}
I(s,t) &= \int_s^t \left(r_s e^{-a(u-s)} + b\left(1 - e^{-a(u-s)}\right) + \sigma_r \int_s^u e^{-a(u-v)} dW_v^{(1)} \right) du \\
&= r_s \int_s^t e^{-a(u-s)} du + b \int_s^t 1 du - b \int_s^t e^{-a(u-s)} du + \sigma_r \int_s^t \int_s^u e^{-a(u-v)} dW_v^{(1)} du \\
&= r_s \mathcal{B}(s,t) + b(t-s) - b\mathcal{B}(s,t) + \sigma_r \int_s^t \int_v^t e^{-a(u-v)} du dW_v^{(1)} \text{(Fubini for stoch. integrals)} \\
&= r_s \mathcal{B}(s,t) + b(t-s) - b\mathcal{B}(s,t) + \frac{\sigma_r}{a} \int_s^t (1 - e^{-a(t-u)}) dW_u^{(1)} \\
&= \mathcal{B}(s,t)(r_s - b) + b(t-s) + \frac{\sigma_r}{a} \int_s^t (1 - e^{-a(t-u)}) dW_u^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[I(s,t)|\mathcal{F}_s\right] &= \mathcal{B}(s,t)(r_s - b) + b(t-s) \\
Var\left[I(s,t)|\mathcal{F}_s\right] &= \frac{\sigma_r^2}{a^2} \int_s^t (1 - e^{-a(t-u)})^2 du = \frac{\sigma_r^2}{a^2} \int_s^t (1 - 2e^{-a(t-u)} + e^{-2a(t-u)}) du \\
&= \frac{\sigma_r^2}{a^2} \left((t-s) - \frac{2}{a}\left(1 - e^{-a(t-s)}\right) + \frac{1}{2a}\left(1 - e^{-2a(t-s)}\right) \right)
\end{aligned}
$$

$$
\begin{aligned}
ln\left(\frac{S_t}{S_s}\right) &= \int_s^t r_u du - \frac{1}{2}\sigma_S^2(t-s) + \int_s^t \rho\sigma_S dW_u^{(1)} + \int_s^t \sqrt{1-\rho^2}\sigma_S dW_u^{(2)} \\
&= \mathcal{B}(s,t)(r_s - b) + b(t-s) + \frac{\sigma_r}{a} \int_s^t (1 - e^{-a(t-u)}) dW_u^{(1)} - \frac{1}{2}\sigma_S^2(t-s) + \int_s^t \rho\sigma_S dW_u^{(1)} \\
&\quad + \int_s^t \sqrt{1-\rho^2}\sigma_S dW_u^{(2)} \\
&= \mathcal{B}(s,t)(r_s - b) + \left(b - \frac{1}{2}\sigma_S^2\right)(t-s) + \int_s^t \left(\frac{\sigma_r}{a}(1 - e^{-a(t-u)}) + \rho\sigma_S\right) dW_u^{(1)} \\
&\quad + \int_s^t \sqrt{1-\rho^2}\sigma_S dW_u^{(2)}
\end{aligned}
$$

$$\begin{aligned}
\mathbb{E}\left[ln\left(\frac{S_t}{S_s}\right)\middle|\mathcal{F}_s\right] =& \mathcal{B}(s,t)(r_s-b)+\left(b-\frac{1}{2}\sigma_S^2\right)(t-s)\\
Var\left[ln\left(\frac{S_t}{S_s}\right)\middle|\mathcal{F}_s\right] =& \int_s^t \left(\frac{\sigma_r}{a}(1-e^{-a(t-u)})+\rho\sigma_S\right)^2 du + \int_s^t (1-\rho^2)\sigma_S^2 du\\
=& \int_s^t \left(\frac{\sigma_r^2}{a^2}(1-e^{-a(t-u)})^2 + 2\rho\sigma_S\frac{\sigma_r}{a}(1-e^{-a(t-u)})+\rho^2\sigma_S^2\right) du\\
&+(1-\rho^2)\sigma_S^2(t-s)\\
=& \int_s^t \left(\frac{\sigma_r^2}{a^2}(1-e^{-a(t-u)})^2 + 2\rho\sigma_S\frac{\sigma_r}{a} - 2\rho\sigma_S\frac{\sigma_r}{a}e^{-a(t-u)}\right) du\\
&+\rho^2\sigma_S^2(t-s)+(1-\rho^2)\sigma_S^2(t-s)\\
=& \int_s^t \left(\frac{\sigma_r^2}{a^2}(1-e^{-a(t-u)})^2 - 2\rho\sigma_S\frac{\sigma_r}{a}e^{-a(t-u)}\right) du\\
&+2\rho\sigma_S\frac{\sigma_r}{a}(t-s)+\sigma_S^2(t-s)\\
=& \int_s^t \left(\frac{\sigma_r^2}{a^2}(1-2e^{-a(t-u)}+e^{-2a(t-u)}) - 2\rho\sigma_S\frac{\sigma_r}{a}e^{-a(t-u)}\right) du\\
&+\left(2\rho\sigma_S\frac{\sigma_r}{a}+\sigma_S^2\right)(t-s)\\
=& \int_s^t \left(\frac{\sigma_r^2}{a^2} - 2\frac{\sigma_r^2}{a^2}e^{-a(t-u)}+\frac{\sigma_r^2}{a^2}e^{-2a(t-u)} - 2\rho\sigma_S\frac{\sigma_r}{a}e^{-a(t-u)}\right) du\\
&+\left(2\rho\sigma_S\frac{\sigma_r}{a}+\sigma_S^2\right)(t-s)\\
=& \int_s^t \left(2\frac{\sigma_r}{a}\left(-\rho\sigma_S-\frac{\sigma_r}{a}\right)e^{-a(t-u)}+\frac{\sigma_r^2}{a^2}e^{-2a(t-u)}\right) du\\
&+\frac{\sigma_r^2}{a^2}(t-s)+\left(2\rho\sigma_S\frac{\sigma_r}{a}+\sigma_S^2\right)(t-s)\\
=& 2\frac{\sigma_r}{a}\left(-\rho\sigma_S-\frac{\sigma_r}{a}\right)\int_s^t e^{-a(t-u)}du+\frac{\sigma_r^2}{a^2}\int_s^t e^{-2a(t-u)}du\\
&+\left(\frac{\sigma_r^2}{a^2}+2\rho\sigma_S\frac{\sigma_r}{a}+\sigma_S^2\right)(t-s)\\
=& 2\frac{\sigma_r}{a^2}\left(-\rho\sigma_S-\frac{\sigma_r}{a}\right)(1-e^{-a(t-s)})+\frac{\sigma_r^2}{2a^3}(1-e^{-2a(t-s)})\\
&+\left(\frac{\sigma_r^2}{a^2}+2\rho\sigma_S\frac{\sigma_r}{a}+\sigma_S^2\right)(t-s)
\end{aligned}$$

$$
\begin{aligned}
Cov\left[ln\left(\frac{S_t}{S_s}\right), I(s,t)\middle|\mathcal{F}_s\right] =& Cov\left[\int_s^t \left(\frac{\sigma_r}{a}(1-e^{-a(t-u)}) + \rho\sigma_S\right)dW_u^{(1)},\right.\\
&\left.\frac{\sigma_r}{a}\int_s^t (1-e^{-a(t-u)})dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& Cov\left[\int_s^t \frac{\sigma_r}{a}(1-e^{-a(t-u)})dW_u^{(1)}, \frac{\sigma_r}{a}\int_s^t (1-e^{-a(t-u)})dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&+ Cov\left[\int_s^t \rho\sigma_S dW_u^{(1)}, \frac{\sigma_r}{a}\int_s^t (1-e^{-a(t-u)})dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& Var\left[\int_s^t \frac{\sigma_r}{a}(1-e^{-a(t-u)})dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&+ \rho\sigma_S\frac{\sigma_r}{a}Cov\left[\int_s^t dW_u^{(1)}, \int_s^t (1-e^{-a(t-u)})dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& Var\left[I(s,t)|\mathcal{F}_s\right] + \rho\sigma_S\frac{\sigma_r}{a}\int_s^t (1-e^{-a(t-u)})du\\
=& Var\left[I(s,t)|\mathcal{F}_s\right] + \rho\sigma_S\frac{\sigma_r}{a}\left((t-s) - \frac{1}{a}(1-e^{-a(t-s)})\right)
\end{aligned}
$$

$$
\begin{aligned}
Cov\left[ln\left(\frac{S_t}{S_s}\right), r_t\middle|\mathcal{F}_s\right] =& Cov\left[\int_s^t \left(\frac{\sigma_r}{a}(1-e^{-a(t-u)}) + \rho\sigma_S\right)dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& Cov\left[\int_s^t \frac{\sigma_r}{a}(1-e^{-a(t-u)})dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&+ Cov\left[\int_s^t \rho\sigma_S dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& Cov\left[\int_s^t \frac{\sigma_r}{a}dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&- Cov\left[\int_s^t \frac{\sigma_r}{a}e^{-a(t-u)}dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&+ Cov\left[\int_s^t \rho\sigma_S dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& \frac{\sigma_r^2}{a}Cov\left[\int_s^t dW_u^{(1)}, \int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&- \frac{\sigma_r^2}{a}Var\left[\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
&+ \rho\sigma_S\sigma_r Cov\left[\int_s^t dW_u^{(1)}, \int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\\
=& \frac{\sigma_r^2}{a}\int_s^t e^{-a(t-u)}du - \frac{\sigma_r^2}{a}\int_s^t e^{-2a(t-u)}du + \rho\sigma_S\sigma_r\int_s^t e^{-a(t-u)}du\\
=& \left(\frac{\sigma_r}{a} + \rho\sigma_S\right)\sigma_r\int_s^t e^{-a(t-u)}du - \frac{\sigma_r^2}{a}\int_s^t e^{-2a(t-u)}du\\
=& \left(\frac{\sigma_r}{a} + \rho\sigma_S\right)\frac{\sigma_r}{a}(1-e^{-a(t-s)}) - \frac{\sigma_r^2}{2a^2}(1-e^{-2a(t-s)})
\end{aligned}
$$

$$\begin{aligned}
Cov\left[I(s,t), r_t|\mathcal{F}_s\right] &= Cov\left[\frac{\sigma_r}{a}\int_s^t(1-e^{-a(t-u)})dW_u^{(1)}, \sigma_r\int_s^t e^{-a(t-u)}dW_u^{(1)}|\mathcal{F}_s\right] \\
&= \frac{\sigma_r^2}{a}\left(Cov\left[\int_s^t dW_u^{(1)}, \int_s^t e^{-a(t-u)}dW_u^{(1)}|\mathcal{F}_s\right] - Var\left[\int_s^t e^{-a(t-u)}dW_u^{(1)}\middle|\mathcal{F}_s\right]\right) \\
&= \frac{\sigma_r^2}{a}\int_s^t e^{-a(t-u)}du - \frac{\sigma_r^2}{a}\int_s^t e^{-2a(t-u)}du \\
&= \frac{\sigma_r^2}{a^2}(1-e^{-a(t-s)}) - \frac{\sigma_r^2}{2a^2}(1-e^{-2a(t-s)})
\end{aligned}$$

In particular, if

$$\frac{dS_t}{S_t} = (r_t + \sigma_S\lambda_t)dt + \rho\sigma_S dW_t^{(1)} + \sqrt{1-\rho^2}\sigma_S dW_t^{(2)}$$

where $\lambda_t = \frac{\mu_S - r_t}{\sigma_S}$ it holds

$$\mathbb{E}\left[ln\left(\frac{S_t}{S_s}\right)\middle|\mathcal{F}_s\right] = \sigma_S\int_s^t \lambda_u du + \mathcal{B}(s,t)(r_s - b) + \left(b - \frac{1}{2}\sigma_S^2\right)(t-s).$$

B.7.5 Parameters

The parameters are chosen to be in line with recent literature (e.g. Hieber et al (2019), Graf et al. (2011)).

Contract parameters:

α	T	P_0
0.9	5	1

Black Scholes model parameters:

S_0	σ_S	μ_S
1	0.2	0.07

Vasicek model parameters:

r_0	a	b^* (Risk neutral)	λ	b (Real world)	σ_r	ρ
0.0115	0.3	0.042	−0.23	0.0305	0.015	0.15

Calculated fair prices:

$$\Rightarrow K_{\text{fix}}^{\text{fair}} = 1.1287$$

$$\Rightarrow \tilde{\alpha}^{\text{fair}} = 0.9982$$

Investment Parameter:

π_S	π_B
0.15	0.4

$\Delta t = \frac{1}{365} \Rightarrow 365 \cdot 5 + 1 = 1826$ values per Path

B.7.6 Simulation

Simulation of one path of interest rate, asset portfolio and interest rate integral

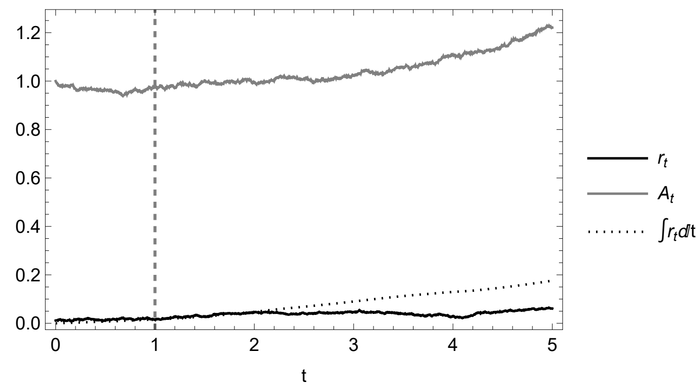

Figure B.1: This Figure illustrates a simulated path of the interest rate, asset portfolio and interest rate integral under the real world measure.

B.7.7 First simulation results

Simulating A_T, $P_T^{\mathbf{fix}}$ and $P_T^{\mathbf{fl.}}$ for $T = 5$

Table B.1: Summery distribution and risk for $T = 5$

	Mean	Median	SD	Variance	$VaR_{0.995}$	$CVaR_{0.985}$
A_T	1.1510	1.1475	0.0875	0.0076	0.9439	0.9478
${P_T}^{\text{fix}}$	1.1400	1.1354	0.0181	0.0003	1.1354	1.1312
${P_T}^{\text{float}}$	1.1144	1.1119	0.0652	0.0043	0.9577	0.9615

Resulted histograms for $T = 5$

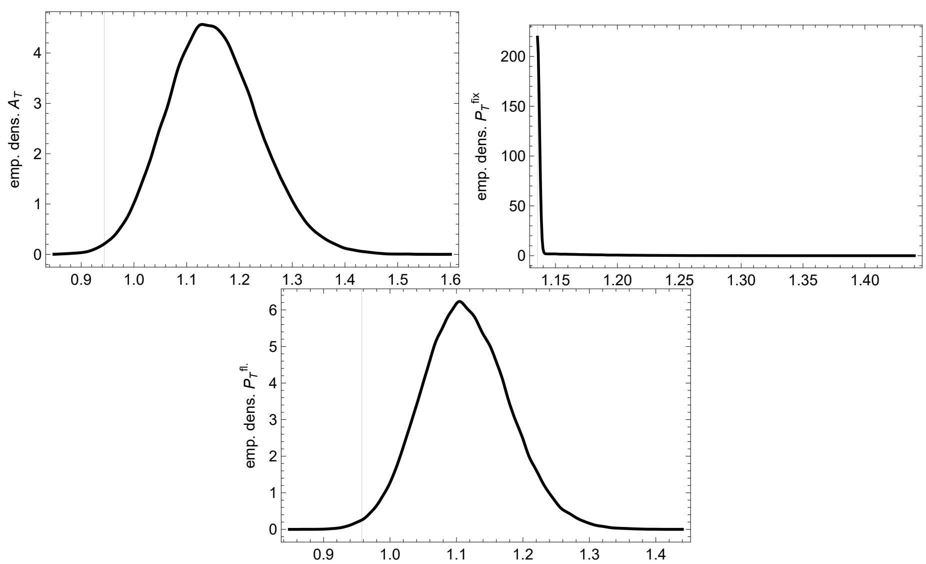

Figure B.2: This figure shows the histogram of the simulated terminal value of the asset portfolio (top left), fix strike guarantee (top right) and floating strike guarantee (bottom) with maturity $T = 5$.

To P_T^{fix}: Remember that $K_{\text{fix}}^{\text{fair}} = 1.1287$. In most of the cases the fraction of the investment portfolio is smaller then the strike, i.e. $\alpha A_T < K_{\text{fix}}^{\text{fair}}$, so the exchange option is out-of-the-money ($\alpha = 0.9$). To $P_T^{\text{fl.}}$: In the Vasicek model negative interest rates are possible, so the bank account can be decreasing. So the initial investment $\tilde{\alpha}^{\text{fair}} = 0.9982$ can decrease either.

Simulating A_T, P_T^{fix} and $P_T^{\text{fl.}}$ for $T = 10$

Table B.2: Summery distribution and risk for $T = 10$

	Mean	Median	SD	Variance	$VaR_{0.995}$	$CVaR_{0.985}$
A_T	1.3693	1.3603	0.1573	0.0247	1.0094	1.0165
${P_T}^{\text{fix}}$	1.3738	1.3574	0.0484	0.0023	1.3574	1.3465
${P_T}^{\text{float}}$	1.3068	1.2983	0.1407	0.0198	0.9892	0.9950

Resulted histograms for $T = 10$

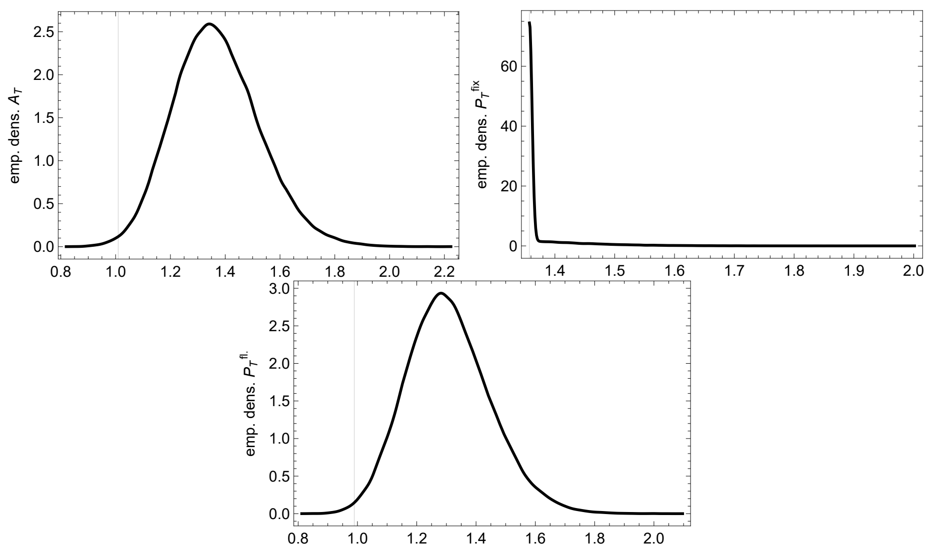

Figure B.3: This figure shows the histogram of the simulated terminal value of the asset portfolio (top left), fix strike guarantee (top right) and floating strike guarantee (bottom) with maturity $T = 10$.

Simulating A_T, $P_T^{\textbf{fix}}$ and $P_T^{\textbf{fl.}}$ for $T = 20$

Table B.3: Summery distribution and risk for $T = 20$

	Mean	Median	SD	Variance	$VaR_{0.995}$	$CVaR_{0.985}$
A_T	1.9731	1.9450	0.3380	0.1142	1.2529	1.1900
${P_T}^{\text{fix}}$	2.0343	1.9852	0.1273	0.0162	1.9852	1.9522
${P_T}^{\text{float}}$	1.8533	1.8261	0.3160	0.0998	1.1854	1.1263

Resulted histograms for $T = 20$

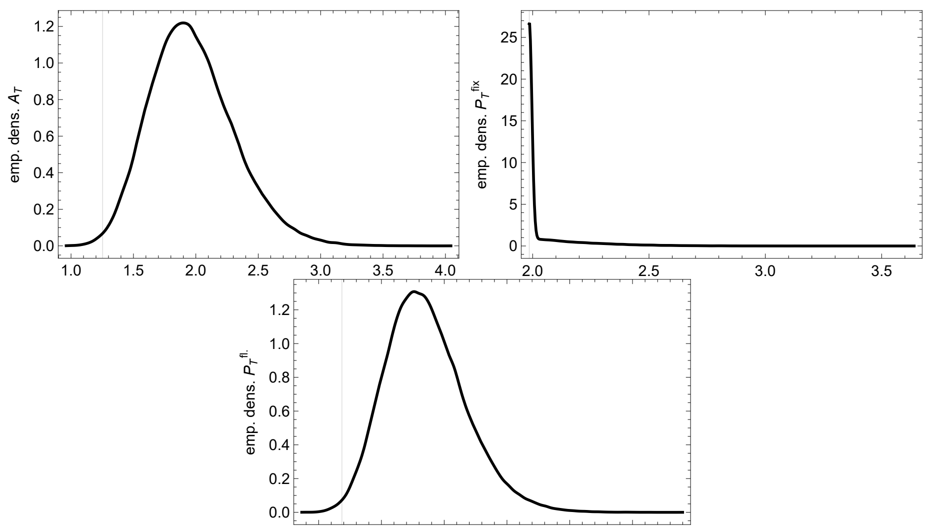

Figure B.4: This figure shows the histogram of the simulated terminal value of the asset portfolio (top left), fix strike guarantee (top right) and floating strike guarantee (bottom) with maturity $T = 20$.

B.7.8 Simulating buffer at time $t = T$ and $t = 1$

Table B.4: Risk measures for varying fractions of fix strike guarantees with $T = 5$

	$t = T$			$t = 1$		
η_{fix}	$VaR_{0.995}$	$CVaR_{0.985}$	Variance	$VaR_{0.995}$	$CVaR_{0.985}$	Variance
0	0.158366	0.153661	0.00477675	0.0695071	0.0685395	0.000867232
0.1	0.155018	0.150083	0.00459635	0.0676866	0.0664799	0.000809602
0.2	0.153047	0.147995	0.00448746	**0.0666358**	**0.06559**	0.000776033
0.3	**0.152547**	**0.147496**	**0.00445008**	0.0670639	0.0659501	**0.000766526**
0.4	0.153289	0.148596	0.0044842	0.0693217	0.0675758	0.000781079
0.5	0.155649	0.151379	0.00458983	0.0724549	0.0705056	0.000819694
0.6	0.160002	0.155794	0.00476696	0.0768014	0.0746235	0.00088237
0.7	0.165866	0.161707	0.0050156	0.0815773	0.0797514	0.000969107
0.8	0.173245	0.169065	0.00533574	0.0876077	0.0857038	0.00107991
0.9	0.181964	0.177767	0.00572739	0.0942533	0.0923372	0.00121477
1	0.191561	0.187596	0.00619055	0.101819	0.0995766	0.00137369

Risk measures for varying fractions of fix strike guarantees with $T = 5$

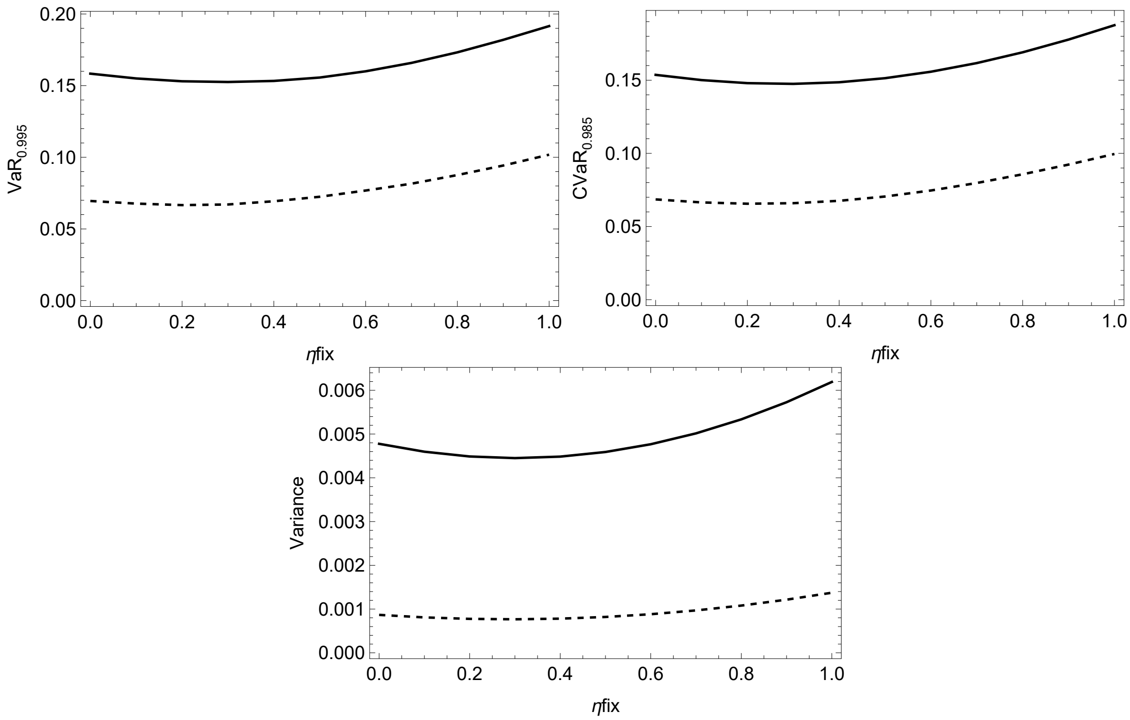

Figure B.5: This figure shows the different risk measures of the buffer for varying frictions of fix strike guarantees η_{fix}. The solid line refers to the buffer value at maturity $T = 5$ and the dashed line refers to the buffer value at $t = 1$.

Table B.5: Risk measures for varying fractions of fix strike guarantees with $T = 10$

	$t = T$			$t = 1$		
η_{fix}	$VaR_{0.995}$	$CVaR_{0.985}$	Variance	$VaR_{0.995}$	$CVaR_{0.985}$	Variance
0	0.27178	0.2635	0.00959667	0.0680055	0.0659657	0.000726966
0.1	0.258107	0.250383	0.00901034	0.0638115	0.0623843	0.000648894
0.2	0.247923	0.240807	0.00871245	0.0619253	0.0601993	0.000601434
0.3	0.242179	0.235614	**0.008703**	**0.0612037**	**0.0598364**	**0.000584585**
0.4	**0.241614**	**0.235473**	0.008982	0.0631142	0.0616242	0.000598349
0.5	0.247745	0.241224	0.00954944	0.0671364	0.0655056	0.000642724
0.6	0.261498	0.252229	0.0104053	0.0721997	0.0711717	0.000717712
0.7	0.277029	0.268522	0.0115496	0.0799544	0.0782453	0.000823311
0.8	0.296498	0.289293	0.0129824	0.0882434	0.0864377	0.000959522
0.9	0.321584	0.313722	0.0147036	0.0979583	0.095453	0.00112635
1	0.348025	0.34093	0.0167133	0.107906	0.105128	0.00132378

Risk measures for varying fractions of fix strike guarantees with $T = 10$

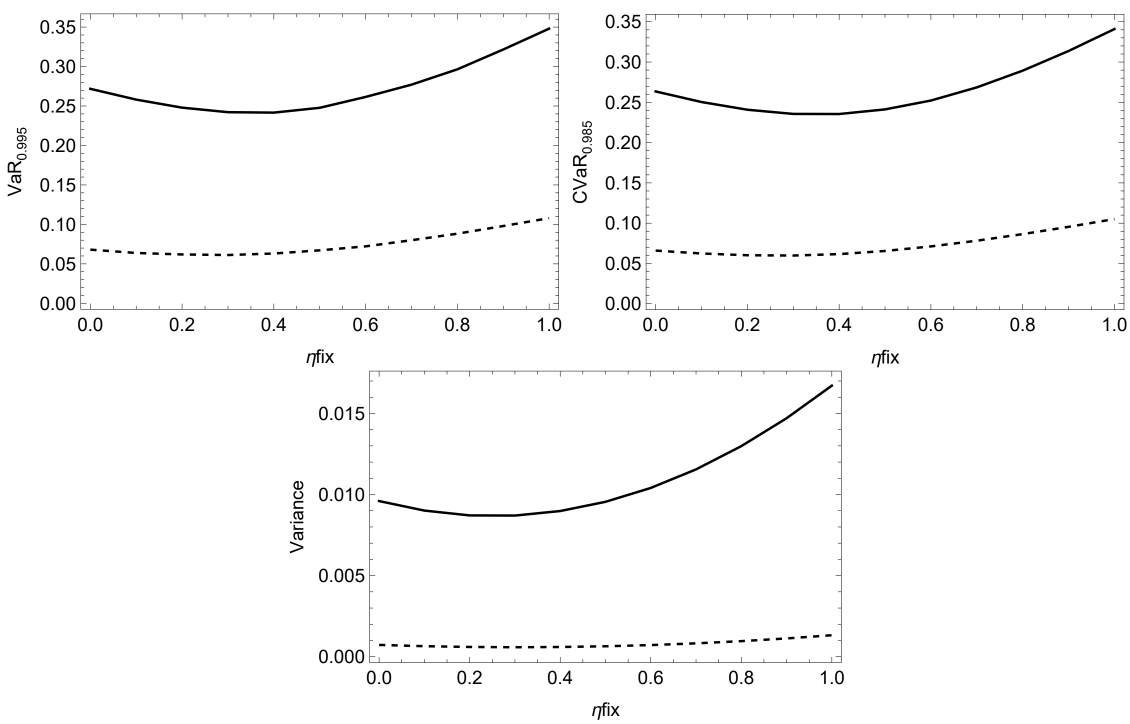

Figure B.6: This figure shows the different risk measures of the buffer for varying frictions of fix strike guarantees η_{fix}. The solid line refers to the buffer value at maturity $T = 10$ and the dashed line refers to the buffer value at $t = 1$.

Table B.6: Risk measures for varying fractions of fix strike guarantees with $T = 20$

	$t = T$			$t = 1$		
η_{fix}	$VaR_{0.995}$	$CVaR_{0.985}$	Variance	$VaR_{0.995}$	$CVaR_{0.985}$	Variance
0	0.52785	0.510567	0.0231846	0.0574121	0.0561246	0.000521507
0.1	0.490174	0.475111	0.0219246	0.0541114	0.05286	0.00045967
0.2	0.462045	0.447466	**0.0219002**	0.0522175	0.0509258	0.000421634
0.3	0.44294	0.43029	0.0231115	**0.051976**	**0.0507322**	**0.000407402**
0.4	**0.439257**	**0.427366**	0.0255585	0.0536088	0.0525241	0.000416972
0.5	0.450904	0.441922	0.0292411	0.0571585	0.0563008	0.000450344
0.6	0.4851	0.474488	0.0341593	0.0630621	0.0618151	0.00050752
0.7	0.532326	0.522464	0.0403132	0.0697324	0.0686174	0.000588497
0.8	0.592056	0.581094	0.0477028	0.0771709	0.0762803	0.000693277
0.9	0.659804	0.647062	0.056328	0.0856038	0.0845477	0.00082186
1	0.732273	0.717069	0.0661889	0.0946566	0.0933164	0.000974245

Risk measures for varying fractions of fix strike guarantees with $T = 20$

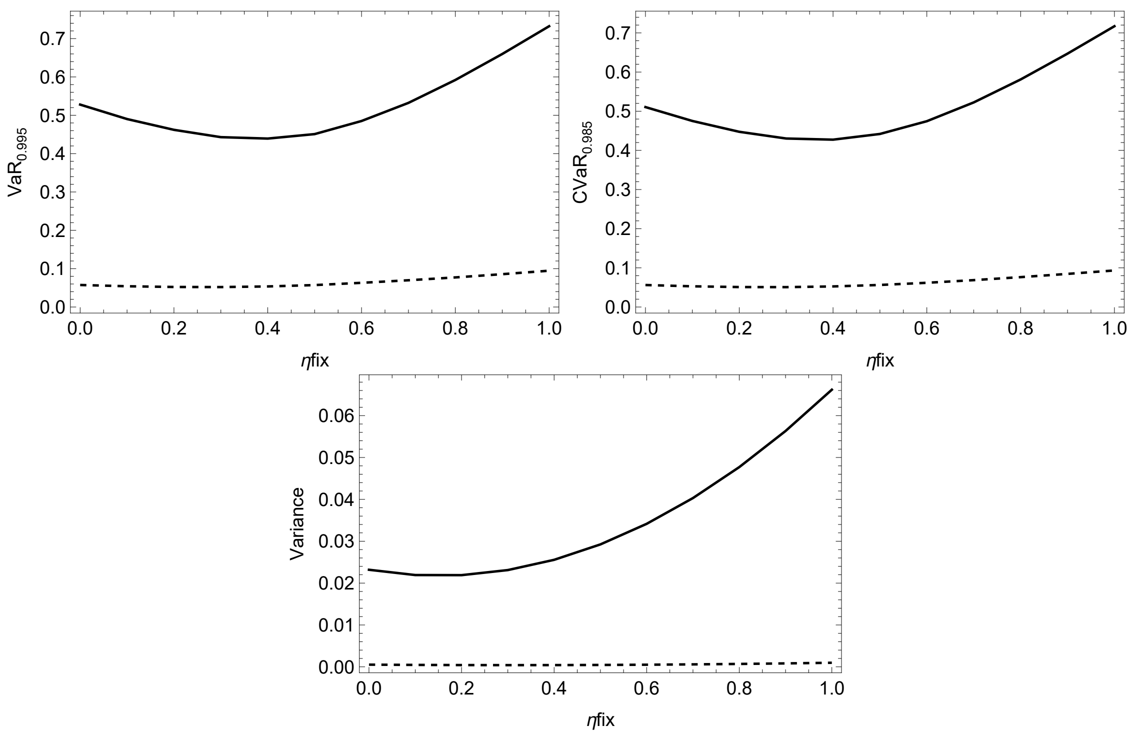

Figure B.7: This figure shows the different risk measures of the buffer for varying frictions of fix strike guarantees η_{fix}. The solid line refers to the buffer value at maturity $T = 20$ and the dashed line refers to the buffer value at $t = 1$.

B.8 Natural Hedge

B.8.1 Natural hedge for varying π_B

Table B.7: Natural hedge and corresponding risk measures for varying π_B

T	α	π_S	π_B	$K_{\text{fix}}^{\text{fair}}$	$\tilde{\alpha}^{\text{fair}}$	$\frac{\partial A_t}{\partial r_t}$	$\frac{\partial C_{\text{fix}}}{\partial r_t}$	$\frac{\partial C_{\text{float.}}}{\partial r_t}$	optimal η_{fix}	$\frac{\partial A_t - L_t}{\partial r_t^2}$	$VaR_{0.995}^{1yr}$	$CVaR_{0.985}^{1yr}$	$Variance$
5	0.9	0.15	0.	1.1331	0.9983	0.	-2.3088	0.0108	0.0047	-1.033	0.0703888	0.0693662	0.000893831
5	0.9	0.15	0.2	1.1345	0.998	-0.5179	-1.9415	0.4425	0.2073	-6.1761	0.0669932	0.0659117	0.000785225
5	0.9	0.15	0.4	1.1354	0.9973	-1.0358	-1.5456	0.8601	0.4006	-13.1202	0.0693139	0.0675887	0.00078123
5	0.9	0.15	0.6	1.1359	0.9963	-1.5537	-1.1275	1.2557	0.5921	-21.6301	0.0764413	0.074194	0.000876458
5	0.9	0.15	0.8	1.1362	0.9949	-2.0717	-0.6955	1.6235	0.7894	-32.0565	0.0870402	0.0852003	0.00107602
10	0.9	0.15	0.	1.3425	0.993	0.	-2.3179	0.0098	0.0042	-0.978	0.0712563	0.0693063	0.000804759
10	0.9	0.15	0.2	1.3511	0.9915	-0.6335	-2.0226	0.4764	0.216	-10.0161	0.0629426	0.0612202	0.000623105
10	0.9	0.15	0.4	1.3574	0.9883	-1.267	-1.6683	0.9034	0.4006	-23.3638	0.0631254	0.0616404	0.000598511
10	0.9	0.15	0.6	1.3614	0.9835	-1.9004	-1.2551	1.2763	0.5793	-40.7303	0.0716256	0.0705287	0.00071341
10	0.9	0.15	0.8	1.3632	0.9769	-2.5339	-0.7958	1.593	0.7729	-62.9031	0.0884368	0.0865697	0.000981401
20	0.9	0.15	0.	1.9247	0.9799	0.	-1.9174	0.0085	0.0044	-0.8488	0.0633762	0.061984	0.000625943
20	0.9	0.15	0.2	1.9585	0.9754	-0.665	-1.7278	0.4185	0.226	-9.1943	0.0540688	0.0526731	0.000448071
20	0.9	0.15	0.4	1.9852	0.9665	-1.33	-1.4769	0.7707	0.4021	-22.2354	0.0536981	0.052583	0.000417423
20	0.9	0.15	0.6	2.0036	0.9535	-1.995	-1.1529	1.0537	0.5679	-40.6657	0.0631775	0.0619123	0.000511566
20	0.9	0.15	0.8	2.0128	0.937	-2.6601	-0.7602	1.2746	0.7571	-65.9711	0.0799075	0.0789697	0.000747863

Optimal fraction of fix strike guarantees for varying π_B and T with fixed $\pi_S = 0.15$

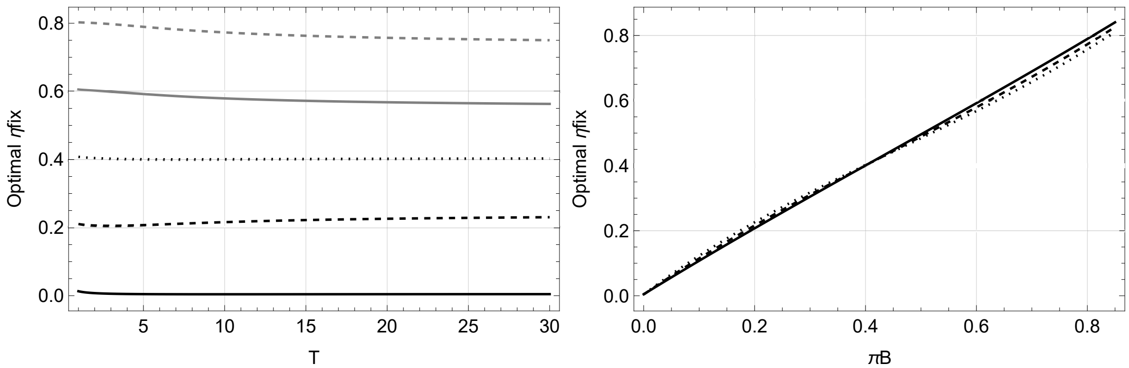

Figure B.8: This figure presents the optimal fraction of fix strike guarantees η_{fix} depending on the stock fraction π_B and the time to maturity T. The left figure shows the optimal fraction depending on T for different levels of π_B and the right figure shows the optimal fraction depending on π_B for different maturities T. For the left line, the black solid line refers to $\pi_B = 0$, the black dashed line to $\pi_B = 0.2$, the dotted line to $\pi_B = 0.4$, the gray solid line to $\pi_B = 0.6$ and finally the gray dashed line to $\pi_B = 0.8$. For the right figure, the solid line refers to $T = 5$, the dashed line to $T = 10$ and the dotted line to $T = 20$.

Convexity depending on the maturity T for different levels of π_B with fixed $\pi_S = 0.15$

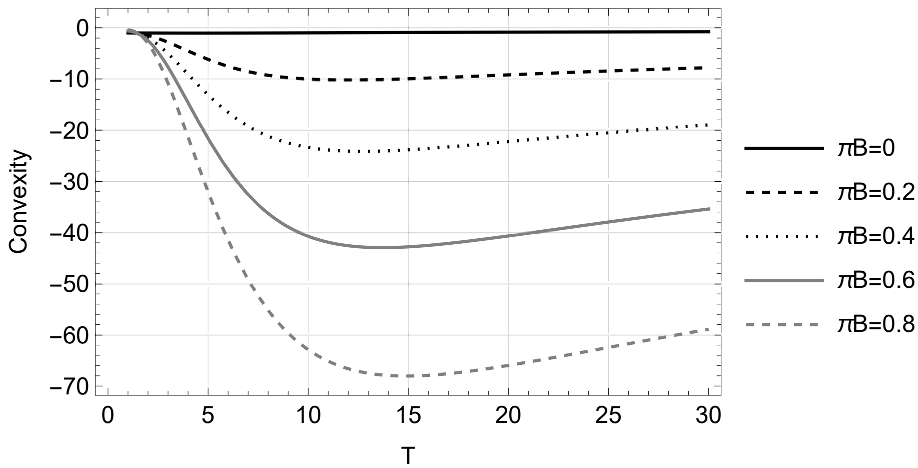

Figure B.9: This figure shows the convexity of the buffer depending on the maturity for varying levels of π_B. η_{fix} is chosen such that the interest rate sensitivity of the buffer is zero. The black solid line refers to $\pi_B = 0$, the black dashed line to $\pi_B = 0.2$, the dotted line to $\pi_B = 0.4$, the gray solid line to $\pi_B = 0.6$ and finally the gray dashed line to $\pi_B = 0.8$.

1 year risk measures for varying bond fraction

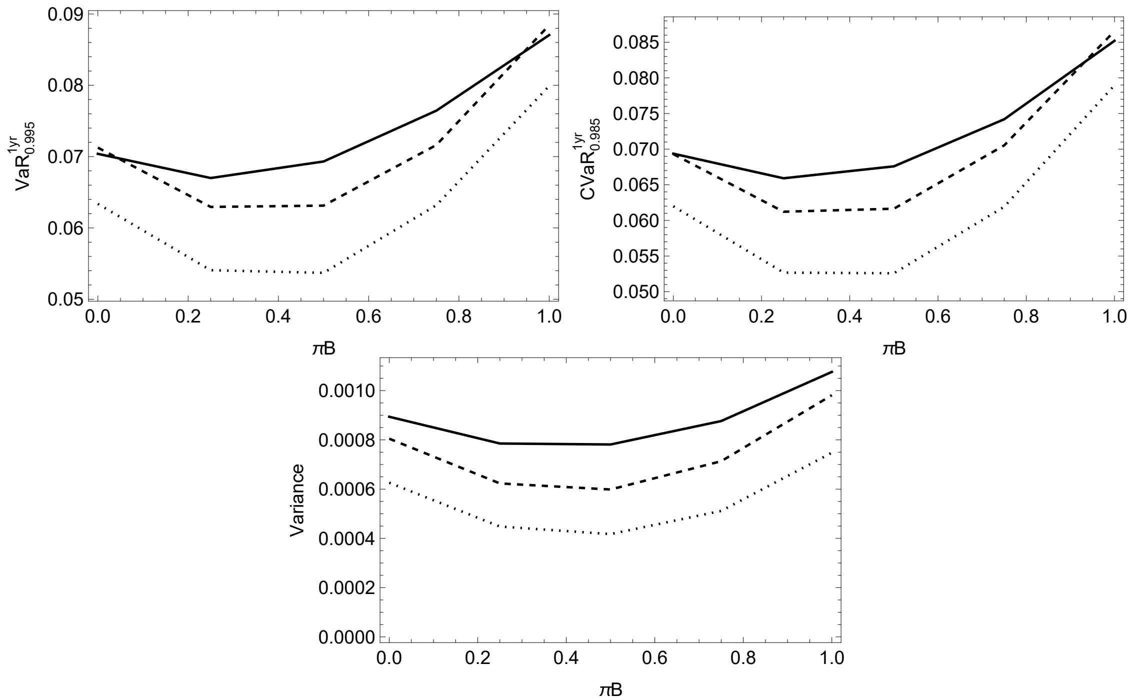

Figure B.10: This Figure presents the 1 year risk measures of the buffer for varying stock fraction π_B. The upper left figure illustrates the $VaR_{0.995}$, the upper right figure shows the $CVaR_{0.985}$ and the lower figure presents the $Variance$. The solid line refers to a time to maturity $T = 5$, the dashed line to $T = 10$ and the dotted line to $T = 20$.

Contour plot of the optimal fraction of fix strike guarantees depending on T and pi_B

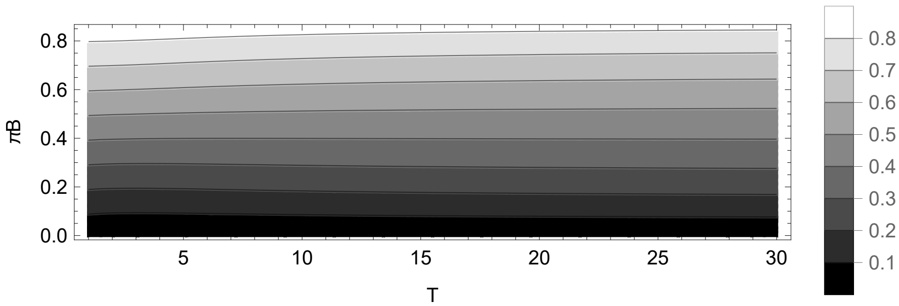

Figure B.11: This Figure presents a contour plot of the optimal fraction of fix strike guarantees η_{fix} depending on the maturity T and the bond fraction π_B. η_{fix} is choose such that the interest rate sensitivity of the buffer is zero. The brighter the surface, the higher is the η_{fix}.

Contour plot of the convexity of the buffer depending on T and pi_B

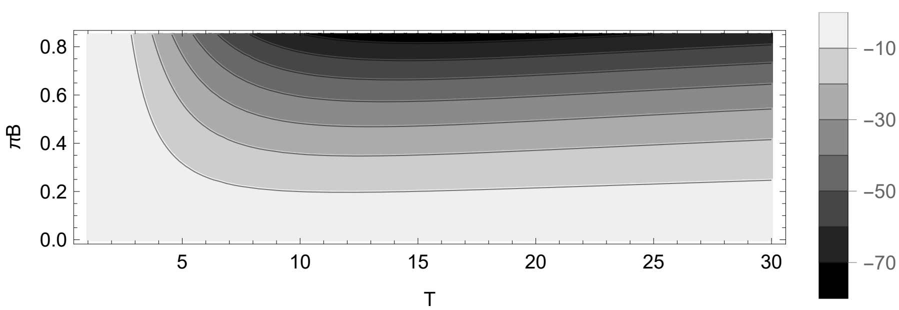

Figure B.12: This Figure presents a contour plot of the convexity of the buffer depending on the maturity T and the stock fraction π_B. η_{fix} is chosen such that the interest rate sensitivity of the buffer is zero. The brighter the surface, the closer the convexity is to zero.

B.8.2 Natural hedge for varying π_S

Table B.8: Natural hedge and corresponding risk measures for varying π_S

T	α	π_S	π_B	$K_{\text{fix}}^{\text{fair}}$	$\tilde{\alpha}^{\text{fair}}$	$\frac{\partial A_t}{\partial r_t}$	$\frac{\partial C_{\text{fix}}}{\partial r_t}$	$\frac{\partial C_{\text{float.}}}{\partial r_t}$	optimal η_{fix}	$\frac{\partial A_t - L_t}{\partial r_t^2}$	$VaR_{0.995}^{1yr}$	$CVaR_{0.985}^{1yr}$	$Variance$
5	0.9	0.	0.4	1.138	1.	-1.0358	-1.6551	0.9437	0.403	-2.9232	0.0717837	0.070077	0.000920237
5	0.9	0.2	0.4	1.131	0.9929	-1.0358	-1.4476	0.7894	0.3992	-15.3858	0.0655133	0.063883	0.000686269
5	0.9	0.4	0.4	1.0923	0.9567	-1.0358	-1.1091	0.5622	0.3984	-12.821	0.0458108	0.0445687	0.000389606
5	0.9	0.6	0.4	1.034	0.9044	-1.0358	-0.9041	0.428	0.399	-9.0116	0.0307331	0.0297722	0.000242982
10	0.9	0.	0.4	1.37	0.9998	-1.267	-1.9034	1.1379	0.4158	-18.1193	0.0718628	0.0701619	0.00083868
10	0.9	0.2	0.4	1.3444	0.9773	-1.267	-1.5331	0.803	0.398	-22.1078	0.0572594	0.0558675	0.000496728
10	0.9	0.4	0.4	1.2585	0.9101	-1.267	-1.1357	0.5355	0.3963	-13.9289	0.0374609	0.0364531	0.000254014
10	0.9	0.6	0.4	1.1478	0.8277	-1.267	-0.9165	0.3935	0.3971	-8.9761	0.0257325	0.0249462	0.000154622
20	0.9	0.	0.4	2.0266	0.9968	-1.33	-1.7115	1.0873	0.436	-27.1311	0.067489	0.0660922	0.000663031
20	0.9	0.2	0.4	1.9484	0.9451	-1.33	-1.348	0.6703	0.398	-19.1583	0.047452	0.0464717	0.000335673
20	0.9	0.4	0.4	1.7398	0.8357	-1.33	-0.9856	0.4238	0.395	-10.3429	0.0302377	0.0295826	0.000162658
20	0.9	0.6	0.4	1.505	0.7195	-1.33	-0.794	0.3004	0.396	-6.2858	0.0213304	0.0208317	0.0000978783

Optimal fraction of fix strike guarantees for varying π_S and T with fixed $\pi_B = 0.4$

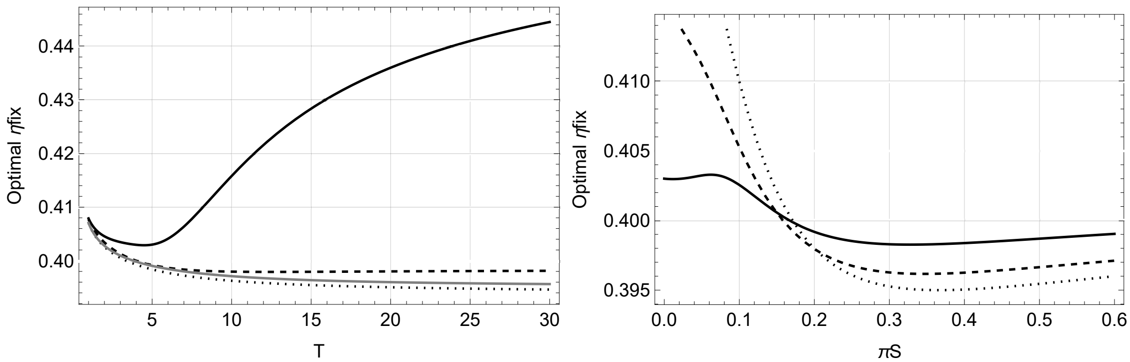

Figure B.13: This figure presents the optimal fraction of fix strike guarantees η_{fix} depending on the stock fraction π_S and the time to maturity T. The left figure shows the optimal fraction depending on T for different levels of π_S and the right figure shows the optimal fraction depending on π_S for different maturities T. For the left line, the black solid line refers to $\pi_S = 0$, the dashed line to $\pi_S = 0.2$, the dotted line to $\pi_S = 0.4$ and finally the gray solid line to $\pi_S = 0.6$. For the right figure, the solid line refers to $T = 5$, the dashed line to $T = 10$ and the dotted line to $T = 20$.

Convexity depending on the maturity T for different levels of π_S with fixed $\pi_B = 0.4$

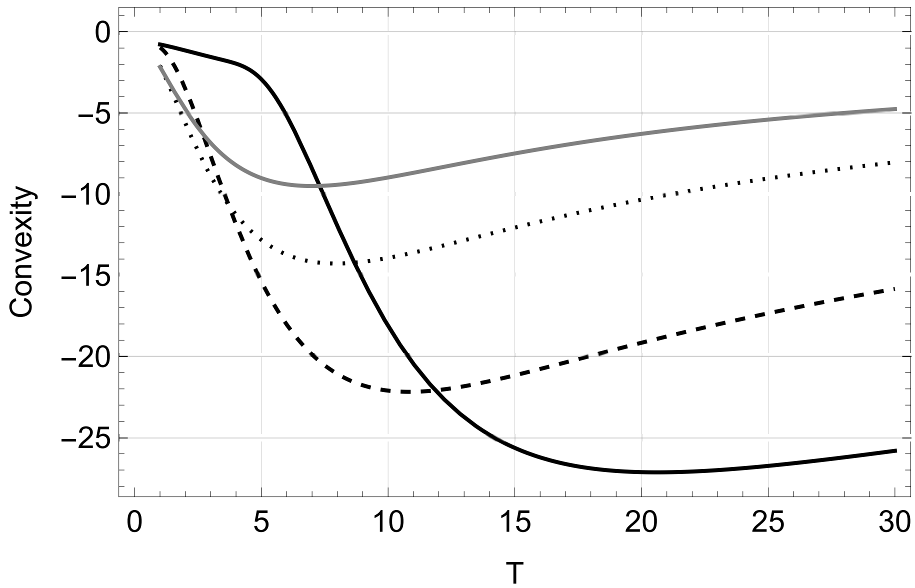

Figure B.14: This figure shows the convexity of the buffer depending on the maturity for varying levels of π_S. η_{fix} is chosen such that the interest rate sensitivity of the buffer is zero. The black solid line refers to $\pi_S = 0$, the dashed line to $\pi_S = 0.2$, the dotted line to $\pi_S = 0.4$ and finally the gray solid line to $\pi_S = 0.6$.

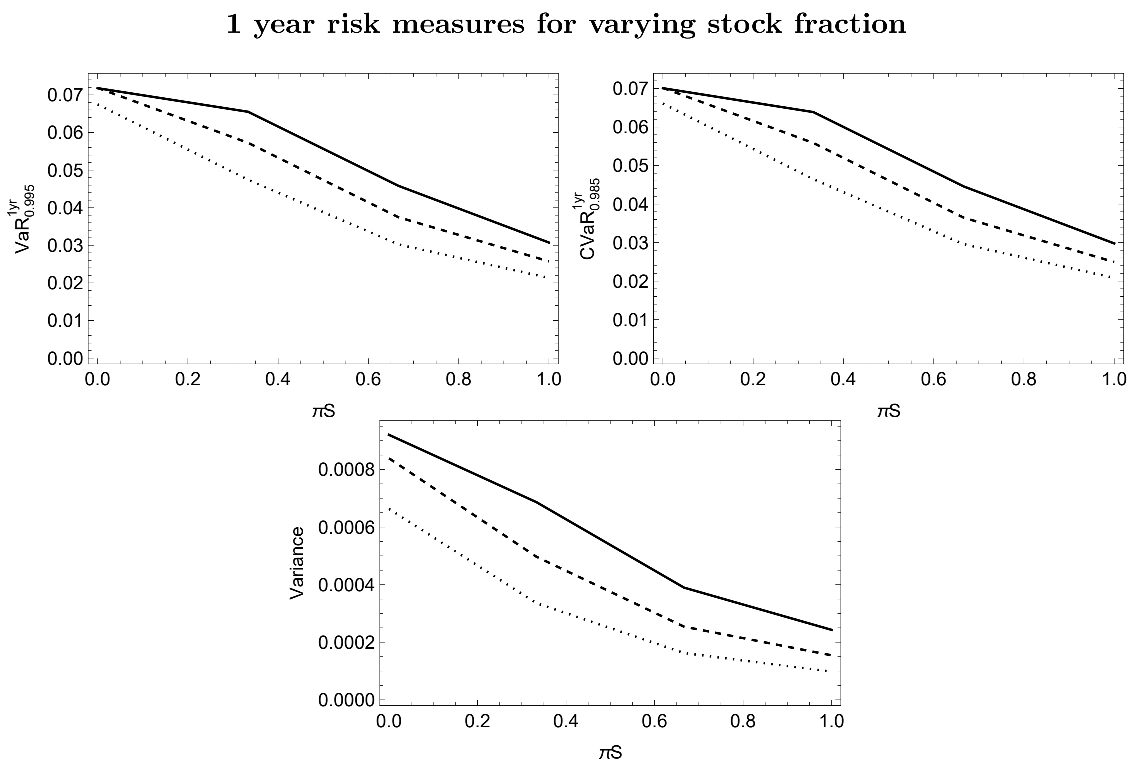

Figure B.15: This Figure presents the 1 year risk measures of the buffer for varying stock fraction π_S. The upper left figure illustrates the $VaR_{0.995}$, the upper right figure shows the $CVaR_{0.985}$ and the lower figure presents the $Variance$. The solid line refers to a time to maturity $T = 5$, the dashed line to $T = 10$ and the dotted line to $T = 20$.

Contour plot of the optimal fraction of fix strike guarantees depending on T and pi_S

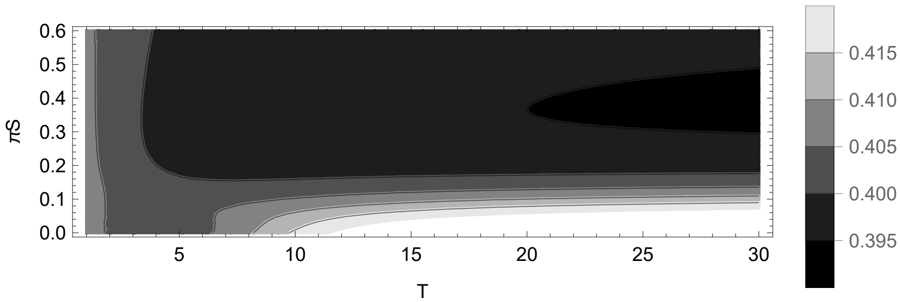

Figure B.16: This Figure presents a contour plot of the optimal fraction of fix strike guarantees η_{fix} depending on the maturity T and the stock fraction π_S. η_{fix} is choose such that the interest rate sensitivity of the buffer is zero. The brighter the surface, the higher is the η_{fix}.

Contour plot of the convexity of the buffer depending on T and pi_S

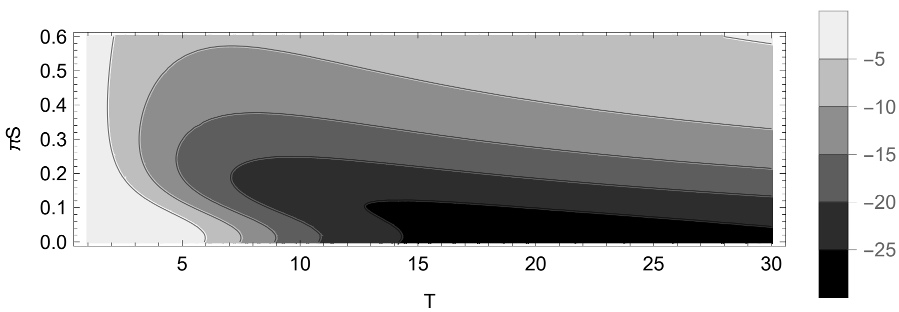

Figure B.17: This Figure presents a contour plot of the convexity of the buffer depending on the maturity T and the stock fraction π_S. η_{fix} is chosen such that the interest rate sensitivity of the buffer is zero. The brighter the surface, the closer the convexity is to zero.

Appendix C

Appendix to Chapter 3

C.1 Proof of Proposition 3.4.2

Recall that the arbitrage-free price at time t of the payoff L is given by the following:

(i) **Low equity ratio:** For $\alpha^E \leq \frac{g(\alpha-1)}{g+\alpha}$, it holds

$$L_t = A_t - (1 - \alpha + \alpha^E)Call(K_3). \tag{C.1}$$

(ii) **High equity ratio:** For $\alpha^E > \frac{g(\alpha-1)}{g+\alpha}$, it holds

$$L_t = A_t - \left(1 + \alpha^E\right) Call(K_2) + \alpha Call(K_1). \tag{C.2}$$

where the strikes K_1, K_2 and K_3 are defined as in Equation (3.7) and $Call(K) = Call^{(BS)}(K, \sigma_A, t)$ is given by the Black and Scholes pricing formula (w.r.t. the returns), i.e.

$$Call^{(BS)}(K, \sigma_A, t) = A_t \mathcal{N}(d_1(K, \sigma_A)) - e^{-r(T-t)} K \mathcal{N}(d_2(K, \sigma_A)), \tag{C.3}$$

$$\text{where } d_1(K, \sigma_A) = \frac{\ln\left(\frac{A_t}{K}\right) + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A \sqrt{T-t}} \text{ and } d_2(K, \sigma_A) = d_1(K, \sigma_A) - \sigma_A \sqrt{T-t}.$$

The interest rate sensitivity of the asset portfolio is then given by

$$\begin{aligned} A_t &= A_0 e^{m^{(A)}\mu + (1-m^{(A)})r - \frac{1}{2}(m^{(A)}\sigma)^2 + m^{(A)}\sigma W_t} \\ \frac{\partial A_t}{\partial r} &= (1 - m^{(A)}) A_0 e^{m^{(A)}\mu + (1-m^{(A)})r - \frac{1}{2}(m^{(A)}\sigma)^2 + m^{(A)}\sigma W_t} = (1 - m^{(A)}) A_t. \end{aligned}$$

$$
\begin{aligned}
d_1(K,\sigma_A) &= \frac{\ln\left(\frac{A_t}{K}\right) + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}} = \frac{\ln A_t - \ln K + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}} \\
&= \frac{\ln A_t}{\sigma_A\sqrt{T-t}} - \frac{\ln K}{\sigma_A\sqrt{T-t}} + \frac{r\sqrt{T-t}}{\sigma_A} + \frac{\frac{1}{2}\sigma_A^2\sqrt{T-t}}{\sigma_A} \\
\frac{\partial d_1(K,\sigma_A)}{\partial r} &= \frac{\frac{\partial \ln A_t}{\partial r}}{\sigma_A\sqrt{T-t}} + \frac{\sqrt{T-t}}{\sigma_A} = \frac{\frac{1}{A_t}\frac{\partial A_t}{\partial r}}{\sigma_A\sqrt{T-t}} + \frac{\sqrt{T-t}}{\sigma_A} = \frac{1-m^{(A)}}{\sigma_A\sqrt{T-t}} + \frac{\sqrt{T-t}}{\sigma_A} \\
&= \frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}} = \frac{\partial d_2(K,\sigma_A)}{\partial r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{N}(d_1(K,\sigma_A))}{\partial r} &= \frac{\partial \mathcal{N}(d_1(K,\sigma_A))}{\partial d_1(K,\sigma_A)}\frac{\partial d_1(K,\sigma_A)}{\partial r} = \mathcal{N}'(d_1(K,\sigma_A))\frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}} \\
\frac{\partial \mathcal{N}(d_2(K,\sigma_A))}{\partial r} &= \frac{\partial \mathcal{N}(d_2(K,\sigma_A))}{\partial d_2(K,\sigma_A)}\frac{\partial d_2(K,\sigma_A)}{\partial r} = \mathcal{N}'(d_2(K,\sigma_A))\frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Call^{(BS)}(K,\sigma_A,t)}{\partial r} =& \frac{\partial A_t}{\partial r}\mathcal{N}(d_1(K,\sigma_A)) + A_t\frac{\partial \mathcal{N}(d_1(K,\sigma_A))}{\partial r} \\
& - \frac{\partial\left(e^{-r(T-t)}K\right)}{\partial r}\mathcal{N}(d_2(K,\sigma_A)) - e^{-r(T-t)}K\frac{\partial \mathcal{N}(d_2(K,\sigma_A))}{\partial r} \\
=& \frac{\partial A_t}{\partial r}\mathcal{N}(d_1(K,\sigma_A)) + A_t\frac{\partial \mathcal{N}(d_1(K,\sigma_A))}{\partial r} \\
& + (T-t)e^{-r(T-t)}K\mathcal{N}(d_2(K,\sigma_A)) - e^{-r(T-t)}K\frac{\partial \mathcal{N}(d_2(K,\sigma_A))}{\partial r} \\
=& (1-m^{(A)})A_t\mathcal{N}(d_1(K,\sigma_A)) + A_t\mathcal{N}'(d_1(K,\sigma_A))\frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}} \\
& + (T-t)e^{-r(T-t)}K\mathcal{N}(d_2(K,\sigma_A)) \\
& - e^{-r(T-t)}K\mathcal{N}'(d_2(K,\sigma_A))\frac{(1-m^{(A)})(T-t)}{\sigma_A\sqrt{T-t}}
\end{aligned}
$$

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