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Advances in the Analysis, Numerics and Optimization of Maxwell Variational Inequalities

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To my father, who never got to see this journey.

-In loving memory-

Preface

The present dissertation has been developed over the past three years during my doctoral studies at the University of Duisburg-Essen and is the result of my research on obstacle problems in electromagnetism. During these three years, I have gained an immense amount of new experiences, not only in the field of mathematics.

First of all, I would like to express my gratitude to both of my parents for their unwavering support. I would like to extend my particular gratitude to my father, from whom I have learned so much and who is unfortunately not with us anymore.

A huge amount of appreciation is devoted to my partner Johanna, during the last eight years, I felt utterly supported in good as well as in bad time periods.

I am also grateful to all of my todays friends, many of whom I have got to know during my undergraduate studies in Essen. I very much value the way we approached mathematics as a subject, and I will always cherish the numerous hours that we spend together discussing it in the LuDi.

Especially, I want to mention Malte and Luis who I shared an office with. I am sure that our daily mathematical exchange greatly contributed to this thesis. Moreover, I thank Gabriele from the University of Trento for our fruitful collaboration resulting from his visit in Essen in the summer of 2021.

Last but not least, I would like to express my sincere appreciation to my advisor Irwin Yousept. I am very grateful for the opportunity to work on this topic and have greatly enjoyed the journey. Starting my studies during a time period that was soon overshadowed by lockdowns and limited mathematical interaction, it was only through your support that I had the opportunity to meet - and present our research to - a big part of the community at a variety of conferences. I am very thankful for the countless discussions that we had and particularly for the encouragement that I received at times that felt not so successful to me.

Maurice Hensel

Abstract

This thesis aims to contribute to a more thorough understanding of the analysis, numerics and optimization of Maxwell-structured variational inequalities. Motivated by applications that require a certain shielding of electric or magnetic fields, we investigate the evolutionary Maxwell obstacle problem first introduced by Duvaut and Lions and certain variants thereof. To begin with, we analyze the mathematical modeling of the famous eddy current approximation in the Maxwell obstacle problem. Through the usage of an implicit Euler scheme in time and its rigorous convergence analysis, we are able to prove a well-posedness result for the eddy current model. We present uniform a priori estimates that provide information on the quantitative precision of the eddy current approximation. The numerical experiments corresponding to the Maxwell obstacle problem suggest that a combination of the implicit Euler method with a mixed finite element method is too computationally costly. Based on this observation, we introduce an alternative time-discretization by the so called leapfrog stepping. The resulting fully discrete scheme turns out to be indeed way more efficient since it completely eliminates the variational inequality character. We prove the stability and convergence of the fully discrete scheme, which requires us to construct a novel constraint preserving mollification operator for vector fields that admit a weak curl. Thereafter, we study a quasilinear first kind variational inequality with a bilateral differential constraint. We propose a tailored regularization approach by the use of which we examine both the well-posedness and the optimal control of our problem. In particular, invoking curl-projection and cut-off type arguments, we are able to derive a set of necessary optimality conditions for the optimal control problem. Finally, we turn our attention to the construction of an efficient solver for a quasi-variational inequality with applications in superconductivity. Here, we utilize once again a time-discretization by the leapfrog stepping with the aim of eliminating the present quasi-variational character. Exploiting the explicit choice of our nonlinearity, we are able to prove the stability and convergence of the scheme for source and temperature data of merely bounded variation in time. The thesis is concluded with an outlook on the modeling of magnetic levitation phenomena accompanied by numerical experiments.

Zusammenfassung

Ziel dieser Arbeit ist es, zu einem tieferen Verständnis der Analysis, Numerik und Optimierung von Variationsungleichungen mit einer Maxwell-Struktur beizutragen. Motiviert durch Anwendungen, die eine gewisse Abschirmung von elektrischen und magnetischen Feldern erfordern, untersuchen wir das, zuerst durch Duvaut und Lions eingeführte, Maxwell-Hindernisproblem. Zunächst analysieren wir die mathematische Modellierung der Wirbelstrom-Approximation für das Maxwell-Hindernisproblem. Durch die Verwendung eines impliziten Euler-Verfahrens in der Zeit und dessen rigorose Konvergenzanalyse sind wir in der Lage die Wohlgestelltheit des Wirbelstrom-Problems zu zeigen. Wir stellen A-priori-Abschätzungen vor, die Aufschluss über die Genauigkeit der Wirbelstrom-Approximation geben. Die numerischen Experimente zum Maxwell-Hindernisproblem legen nahe, dass eine Kombination des impliziten Euler-Verfahrens mit einer gemischten Finite-Elemente-Methode zu rechenaufwändig ist. Basierend auf dieser Beobachtung führen wir eine alternative Zeitdiskretisierung durch das so genannte Leapfrog-Verfahren ein. Das daraus resultierende volldiskrete Verfahren erweist sich in der Tat als wesentlich effizienter, da es den Charakter der Variationsungleichung vollständig eliminiert. Wir beweisen die Stabilität und Konvergenz des volldiskreten Schemas und konstruieren dafür einen neuartigen, schrankeneinhaltenden Glättungsoperators für Vektorfelder, die eine schwache Rotation besitzen. Danach untersuchen wir eine quasilineare Variationsungleichung erster Art mit einer bilateralen Beschränkung für die Rotation. Wir schlagen einen Regularisierungsansatz vor, mit dem wir sowohl die Wohlgestelltheit als auch die optimale Steuerung unseres Problems untersuchen. Insbesondere sind wir in der Lage, unter Verwendung von Projektions- und Abschneideargumenten eine Reihe von notwendigen Optimalitätsbedingungen für das optimale Steuerungsproblem abzuleiten. Schließlich widmen wir uns der Konstruktion eines effizienten Lösungsalgorithmus für eine Quasi-Variationsungleichung mit Anwendungen in der Supraleitung. Hier verwenden wir erneut eine Zeitdiskretisierung durch das Leapfrog-Verfahrem mit dem Ziel, den Quasi-Variationsungleichungs-Charakter zu eliminieren. Unter Ausnutzung der expliziten Wahl unserer Nichtlinearität sind wir in der Lage, die Stabilität und Konvergenz des Schemas für Quell- und Temperaturdaten von lediglich beschränkter Variation zu beweisen. Die Arbeit schließt mit einem Ausblick auf die Modellierung von Phänomenen im Bereich der magnetischen Levitation, begleitet von numerischen Experimenten.

CONTENTS

CHAPTER 1

INTRODUCTION

> From a long view of the history of mankind, seen from, say, ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.
>
> Richard Feynman, 1964

It is uncertain whether Richard Feynman's statement is accurate, but certainly James Clerk Maxwell is considered one of the most important physicists in history. In the middle of the 19th century, precisely in 1861, Maxwell's paper *On Physical Lines of Force* [96] was published. It includes a set of twenty equations that explain the behavior and interaction of electric and magnetic fields. In the form of partial differential equations (PDEs), these twenty equations first appeared in their fully developed form in the book *A Treatise on Electricity and Magnetism* [97] in 1873. They were later unified by Oliver Heaviside into four PDEs, up to today known as Maxwell's equations, which unify the previously separate theories of electricity, magnetism, and light. In their classical formulation, Maxwell's equations have a long history in the modeling of phenomena related to electromagnetism. Specifically, they are used to describe the time evolution of electromagnetic waves in a given medium.

In this thesis, we are particularly interested in the dynamics and propagation of electromagnetic fields under the influence of constraints. Here, one may think of the case where certain magnetic or conducting materials serve as a barrier to redirect or block electromagnetic fields in a specific domain of interest (cf. [124]). Mathematically speaking, this leads to variational inequalities with a Maxwell structure.

The notion of variational inequalities goes back to Antonio Signorini. It was 1959 when Signorini posed the problem (cf. [126] and the previous foundational note [125]) of determining the displacement in a heavy, linearly elastic body on a rigid and frictionless horizontal plane. The model that was presented (nowadays called Signorini problem) seems to be the first appearance of a problem carrying the structure of what is up to today known as a variational inequality. The main difficulty of this problem is that the contact set between the elastic body and the plane can be very complicated and the existence and uniqueness of solutions to the Signorini problem were therefore unknown for several years. Roughly four years later, Gaetano Fichera, a student of Signorini, was able to present a full proof (cf. [54]) showing the existence and uniqueness of a solution. His techniques relied on characterizing the position field for the Signorini problem as the minimizer of a certain potential energy over a convex set (cf. [15]). In 1965, the french mathematician Georges Duvaut was part of the audience when Fichera presented his results at a conference in Italy. After Duvaut's return to France, he had several graduate students work on this particular topic which led to a strong french activity in this area. After all, this period of time could be considered the birth of the field of variational inequalities.

The very first contribution to variational inequalities (VIs) in electromagnetism was made by Duvaut and Lions [48], who explored and analyzed the electromagnetic wave propagation in a polarizable medium through a Maxwell obstacle problem: In the free region, electromagnetic fields satisfy Maxwell's equations, while in the shielded area (unilateral or bilateral) constraints are imposed on the fields. In their seminal work, Duvaut and Lions considered a bilateral electric constraint of the type

$$|\boldsymbol{E}(t,x)| \leq d(x) \quad \text{a.e. in } (0,T)\times\Omega$$

for some obstacle $d\colon \Omega \to [0,\infty]$, where $\boldsymbol{E}$ denotes the electric field. More recently, building on [143], Yousept [144] refined the developed theory by Duvaut and Lions to allow a more general constraint structure, namely that the variational inequality character is also present in Faraday's law, leading to a very general problem with simultaneous shielding in both electric and magnetic field (see the problem formulation (2.40)). From a mathematical perspective, such a problem comes with a number of issues. First of all, if there is no additional structural assumption on the electromagnetic obstacle set, uniqueness of a solution is not available. Moreover, the spatial regularity for solutions of (2.40) is generally very low (see Theorem 2.13). For this reasons, throughout this thesis, we will focus on the individual presence of an electric or magnetic obstacle.

1.1 Contribution

The main goal of this dissertation is to make advances within the research area of obstacle problems with applications in both electric and magnetic shielding. The present study addresses four individual topics that are interconnected through their focus on Maxwell-structured variational inequalities.

(a) In Chapter 3 we analyze the mathematical modeling of the famous eddy current approximation in the Maxwell obstacle problem. Here, the medium is assumed to be solely open, containing conducting and non-conducting materials with certain properties of anisotropy and non-smoothness. The proposed evolutionary PDE-model (P_{ec}) preserves the Faraday law and excludes the displacement current from the governing Ampère-Maxwell variational inequality in (P). Our study strives to justify this model and delivers two main results: global well-posedness (Theorem 3.6) of the model and its quantitative precision by uniform a priori estimates (Theorem 3.17). Here, the proof of the well-posedness result is non-standard due to the low regularity assumption on the initial value (see (3.5)). We overcome this issue by introducing certain correction terms (see (3.10)) which need to be handled in the underlying stability analysis. The uniform a priori estimates yield an explicit bound for the smallness condition on the ratio between the electric permittivity and the electric conductivity in the region where the displacement current is disregarded. Below this threshold, the eddy current solution provides the desired reasonable approximation and justifies the proposed model. We also carry out a numerical test which confirms the convergence rate (3.72) as the electric permittivity ϵ decreases.

(b) The employed Rothe method used for the well-posedness result of the eddy current problem (P_{ec}) (in particular the problem (P)) suggests its combination with a mixed finite element method in space (as briefly carried out in Section 3.4). However, to numerically solve the involved elliptic **curl**-**curl** variational inequality in (P_N), an iterative solver such as the semi-smooth Newton method or the primal dual active set strategy would be required. In order not to rely on the mentioned iterative solver, in Chapter 4 we propose and examine a different finite element method (FEM) for the Maxwell obstacle problem (P). Based on the leapfrog time-stepping and the Nédélec edge elements, we set up a fully discrete FEM $(\mathrm{P}_{N,h})$ where the obstacle is discretized in such a way that no additional nonlinear solver is required for the computation of the discrete VI. Our construction allows us to find the explicit unique analytical solution to our discrete VI (see Theorem 4.3). While the $\boldsymbol{L}^2$-stability is achieved for the discrete solutions and the associated difference quotients, the scheme only guarantees the $\boldsymbol{L}^1$-stability for the discrete magnetic **curl**-field in the obstacle region (see Proposition 4.9). The lack of the global $\boldsymbol{L}^2$-stability for the magnetic rotational field is justified by the low regularity issue in Maxwell obstacle problems (see Theorem 2.13) and turns out to be the main challenge in the convergence analysis. Our convergence proof consists of two main stages. First, exploiting the $\boldsymbol{L}^1$-stability in the obstacle region, we derive a convergence result towards a weaker system involving smooth feasible test functions. In the second step, we recover the original system by enlarging the feasible test function set through a specific constraint preserving mollification process in the spirit of Ern and Guermond [50]. Here, using techniques from geometrical analysis, we construct a mollification operator for $\boldsymbol{H}(\mathbf{curl})$-fields (see Theorem 4.14) which is able to preserve certain constraints that appear in the obstacle set of the variational inequality. We present 3D numerical results of the proposed FEM confirming the theoretical convergence result and in particular the Faraday shielding effect.

(c) Having covered both the eddy current approximation and numerical analysis of the electric shielding problem (P), in Chapter 5 we turn our attention to the shielding of magnetic fields by ferromagnetic materials. The resulting model is given by an $\boldsymbol{H}(\mathbf{curl})$-quasilinear first kind variational inequality with a bilateral vector **curl**-constraint. For this (time-independent) VI, we examine both the well-posedness and the optimal control. We propose a tailored regularization approach based on the Helmholtz decomposition and a reduction of the first-order constraint to the zeroth-order one in combination with a smoothed Yosida penalization. In this way, a suitable family of approximating quasilinear variational equalities is obtained. The corresponding limiting analysis not only leads to a well-posedness result for the VI but also reveals its dual formulation (see Theorem 5.5). The last part is devoted to the analysis of the corresponding optimal control problem, which is mainly complicated by the involvement of the $\boldsymbol{H}(\mathbf{curl})$-quasilinearity, the bilateral vector **curl**-constraint, and the non-smoothness. On the basis of the proposed regularization, as the final novelty, we derive necessary optimality conditions, including a characterization of the limiting dual multiplier through **curl**-projection and cut-off type arguments (see Theorem 5.11).

(d) Differently from Chapters 3 to 5, in Chapter 6 we are concerned with the derivation and numerical analysis of an efficient solver for a quasi-variational inequality (QVI) of the second kind with applications in superconductivity. To obtain a fully discrete scheme, we employ again a time-discretization by the leapfrog stepping in combination with a mixed finite element method in a way such that we are able to completely eliminate the QVI character and replace it with an $\boldsymbol{L}^2$-structured VI for which an explicit analytical solution is available (Theorem 6.6). Compared with known numerical algorithms for QVIs, no fixed-point type iteration is needed, leading to exceptionally low computational effort. The convergence analysis of the discrete scheme $(\mathrm{QVI}_{N,h})$ involves similar ideas to the convergence analysis in Chapter 4 but requires a careful stability analysis as we work with general source and temperature data of merely bounded variation in time (Assumption 6.2 and Lemma 6.9). We close the chapter by presenting numerical tests for different configurations of a specific nonlinearity known from the physics literature.

Motivated by the numerical experiments in Chapter 6, the thesis is concluded with an outlook on the modeling of magnetic levitation phenomena. In Chapter 6 we tackle the dependence of the nonlinearity on the magnetic field strength to obtain a more realistic model. As proposed in the physics literature [35, 130], the interaction force between a permanent magnet and a superconductor is given by a term that involves both the current density and the magnetic field (see (7.1)). We invoke this force term to compute the displacement of the superconductor at any given time step, resulting in the model (QVI_{lev}) given by a quasi-variational inequality involving a complicated nonlinearity. In particular, to the best of the authors knowledge, many numerical experiments using the force term (7.1) have been carried out by approximating the QVI-character by a certain power law, but a full QVI model has not yet been approached. The (numerical) analysis of (QVI_{lev}) appears to be very challenging, however, Chapter 7 offers first numerical results, justifying a future investigation of the model.

The remainder of this dissertation is organized as follows. We use the upcoming chapter to give a background on Maxwell's equations, variational inequalities and their combination. Thereafter following, Chapters 3 to 6 directly relate to the descriptions from (a)-(d) and in particular they are in large parts similar to one of the following publications, listed in the exact same order. Consequently, we will not explicitly highlight any quotations from these papers individually.

Publications

[64] M. Hensel and I. Yousept. Eddy Current Approximation in Maxwell Obstacle Problems. *Interfaces and Free Boundaries*, 10.4171/ifb/486, 2022

[65] M. Hensel and I. Yousept. Numerical Analysis for Maxwell Obstacle Problems in Electric Shielding. *SIAM J. Numer. Anal.*, 60(3):1083-1110, 2022

[38] G. Caselli, M. Hensel and I. Yousept. Quasilinear Variational Inequalities in Ferromagnetic Shielding: Well-posedness, Regularity, and Optimal Control. To be published, *SIAM J. Control Optim.*, 2023

[66] M. Hensel, M. Winckler and I. Yousept. Numerical Analysis for Maxwell Quasi-variational Inequalities in Superconductivity. *Preprint*, 2023

CHAPTER 2

BACKGROUND

Let us use this chapter to provide a background on the topics that are relevant to any of the chapters following later: Maxwell's equations and variational inequalities. After giving a brief historical background on the basic equations of electromagnetism, we introduce the underlying Sobolev-type function spaces $\boldsymbol{H}(\mathbf{curl})$ and $\boldsymbol{H}(\mathrm{div})$ which are pivotal for a treatment by means of a modern functional analytic approach. We state the key properties of these spaces and particularly discuss their missing compactness properties. Afterwards, we turn our attention to the basic concepts of variational inequalities. In this context, we look into the basic Poisson-type obstacle problem and discuss differentiability properties of its solution mapping. We close the chapter by combining the two mentioned topics, resulting in *variational inequalities in electromagnetism*. We recall the classical result by Duvaut and Lions [48] and present recent results obtained in [143, 144] and how they relate to the content of this thesis.

2.1 Maxwell's Equations

The first part of this section is concerned with a historical introduction of Maxwell's equations, as those are, together with the variational inequality character, at the very heart of every problem formulation in this thesis. For a more in-depth perspective on the topic, we refer the reader to [48, Chapter 7, Section 2], [7, Section 1.1], as well as the classical works [62, 77]. To begin with, let us introduce the two physical quantities

$$q\colon [0,T]\times\mathbb{R}^3 \to \mathbb{R} \qquad \text{(the electric charge)}$$
$$\boldsymbol{J}\colon [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{(the current density).}$$

For the sake of simplicity and since we are primarily interested in a formal derivation of Maxwell's equations, let us assume that both $q, \boldsymbol{J}$ and all following physical quantities are smooth in space and time. Now, let a physical medium be represented by a compact set $D \subset \mathbb{R}^3$ with smooth boundary ∂D. The first part of the derivation of Maxwell's equations is based on a fundamental observation by the two scientists William Watson and Benjamin Franklin in the middle of the 17th century, namely that electric charge is conserved (cf. [55]). More precisely, their discovery entailed that, as long as there is no external addition of charges, the change in time of the total electric charge contained in the interior of the medium D equals the flux of charges through the boundary ∂D. In mathematical terms, this conservation law is known in the integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D q\,\mathrm{d}x = -\int_{\partial D} \boldsymbol{J}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{2.1}$$

where $\boldsymbol{n}\colon \partial D \to \mathbb{R}^3$ denotes the outer unit normal of D. Now, (2.1) may also be rewritten in a differential form. To obtain such differential form we apply the famous divergence theorem by Gauss and gain advantage upon the fact that the integration in (2.1) is carried out on an arbitrary compact set $D \subset \mathbb{R}^3$ with smooth boundary so that we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} q + \operatorname{div} \boldsymbol{J} = 0. \tag{2.2}$$

Now, we introduce a new vector field

$$\boldsymbol{D} \colon [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{(the electric induction)},$$

as the potential of charge, so that

$$q = \operatorname{div} \boldsymbol{D}. \tag{2.3}$$

Then, applying (2.3) to (2.2) entails

$$\operatorname{div} \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{D} + \boldsymbol{J} \right) = 0,$$

which is why, by the classical Helmholtz theorem, there must exist another vector field

$$\boldsymbol{H} \colon [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{(the magnetic field)},$$

with the property

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{D} + \boldsymbol{J} - \mathbf{curl}\, \boldsymbol{H} = 0. \tag{2.4}$$

The latter equation is called the differential form of the Ampère-Maxwell equation, an extension of the original circuital law by Ampère in which conservation of charge was not yet considered. Of course, (2.4) may also be accompanied by some contribution of an external source $\boldsymbol{f} \colon [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$, so that a generalized form of the Ampère-Maxwell equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{D} - \mathbf{curl}\, \boldsymbol{H} + \boldsymbol{J} = \boldsymbol{f}. \tag{2.5}$$

The next relation was found by Faraday, who by means of physical experiments observed around the year 1830 that a change of a magnetic field in time induces an electric field (cf. [53]). More precisely, he stated that the derivative with respect to time of the flux of

$$\boldsymbol{B} \colon [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{(the magnetic induction)}$$

through some (smooth and orientable) surface $S \subset \partial D \subset \mathbb{R}^3$ equals the opposite of the circulation of

$$\boldsymbol{E} \colon [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{(the electric field)}$$

along its contour $\partial S \subset \mathbb{R}^3$. In mathematical terms, the Faraday law may be written in the integral form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}S = - \int_{\partial S} \boldsymbol{E} \cdot \boldsymbol{\tau} \, \mathrm{d}s, \tag{2.6}$$

where $\boldsymbol{\tau} \colon \partial S \to \mathbb{R}^3$ denotes the tangent vector on the curve ∂S oriented counterclockwise to the normal $\boldsymbol{n}_{|S}$. Now, applying the Stokes theorem allows us to rewrite the right-hand side of the previous equation as

$$- \int_{\partial S} \boldsymbol{E} \cdot \boldsymbol{\tau} \, \mathrm{d}s = - \int_S \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{2.7}$$

and since the surface S and the medium D was arbitrary, it follows from (2.6) and (2.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} = -\,\mathbf{curl}\,\boldsymbol{E}, \tag{2.8}$$

the differential form of the Faraday Law. Collecting (2.3), (2.5) and (2.8), this results in the basic laws of electromagnetism, later coined Maxwell's equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D} - \mathbf{curl}\,\boldsymbol{H} + \boldsymbol{J} = \boldsymbol{f} & \text{in } (0,T)\times\mathbb{R}^3 \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} + \mathbf{curl}\,\boldsymbol{E} = 0 & \text{in } (0,T)\times\mathbb{R}^3 \\ \operatorname{div}\boldsymbol{D} = q & \text{in } (0,T)\times\mathbb{R}^3 \\ \operatorname{div}\boldsymbol{B} = 0 & \text{in } (0,T)\times\mathbb{R}^3. \end{cases} \tag{2.9}$$

Here, the property $\operatorname{div}\boldsymbol{B} = 0$ is based on the second equation in (2.9) and the assumption that $\operatorname{div}\boldsymbol{B}(t) = 0$ for at least one time instant $t \in (0,T)$.

The system (2.9) includes both the electric (resp. magnetic) induction and field as unknown vector-valued variables. At a microscopic level the electric (resp. magnetic) induction and field coincide up to a constant. At the macroscopic level, i.e., on the whole spatial domain, it is therefore reasonable to assume that they also coincide up to some (potentially matrix-valued) material parameter. We say that the electric (resp. magnetic) induction and field are related through constitutive equations. To be more specific, the electric permittivity $\epsilon\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ relates the electric induction and the electric field by

$$\boldsymbol{D} = \epsilon\boldsymbol{E}. \tag{2.10}$$

In a similar way, the magnetic permeability $\mu\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ relates the magnetic induction and the magnetic field by

$$\boldsymbol{B} = \mu\boldsymbol{H}. \tag{2.11}$$

We want to mention that we made a serious simplification here, namely that the dependence between the quantities in (2.10) and (2.11) is linear. There also exist materials for which it is necessary to work with nonlinear constitutive relations. An application where such materials arise is investigated in Chapter 5. Assuming that the medium is sufficiently stable in the sense that it possesses an electric resistivity ρ not depending on electromagnetic quantities, Ohm's law holds true. In this case, the conductivity $\sigma = 1/\rho\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ relates the current density and the electric field by

$$\boldsymbol{J} = \sigma\boldsymbol{E}. \tag{2.12}$$

Now, applying the constitutive relations (2.10), (2.11) and (2.12) to (2.9), we obtain the reduced Maxwell system

$$\begin{cases} \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E} - \mathbf{curl}\,\boldsymbol{H} + \sigma\boldsymbol{E} = \boldsymbol{f} & \text{in } (0,T)\times\mathbb{R}^3 \\ \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0 & \text{in } (0,T)\times\mathbb{R}^3 \\ \operatorname{div}\epsilon\boldsymbol{E} = q & \text{in } (0,T)\times\mathbb{R}^3 \\ \operatorname{div}\mu\boldsymbol{H} = 0 & \text{in } (0,T)\times\mathbb{R}^3. \end{cases} \tag{2.13}$$

Of course, one could for instance also add a source term for the Faraday law in (2.13). Finally, in view of rigorous mathematical analysis, the system (2.13) should be accompanied by appropriate boundary and initial conditions which will be introduced at a later point.

2.1.1 The Underlying Function Spaces

Let us use this section to introduce the required functional analytic framework for the treatment of Maxwell's equations. In what follows, given a real Banach space V, we denote its topological dual space with V^* and we indicate its norm with $\|\cdot\|_V$. In the case that V is a real Hilbert space, we represent with $(\cdot,\cdot)_V$ the associated scalar product. If $V=\mathbb{R}^d$ for some $d\in\mathbb{N}$, we simply write a dot and $|\cdot|$ for the Euclidean scalar product and norm. Discussing problems of Maxwell-type, there naturally arise function spaces of $\mathbb{R}^3$-valued functions. We will therefore use a bold typeface to indicate them.

As we have already seen in the previous section, Maxwell's equations involve the differential operators **curl** and div. In a natural way, these operators can be defined on Hilbert spaces involving fewer regularity than the well-known Sobolev space H^1. To begin with, let $\mathcal{O}\subset\mathbb{R}^3$ be an open set. We denote by $L^2(\mathcal{O})$ (resp. $\boldsymbol{L}^2(\mathcal{O})$) the space of all (equivalence classes of) $\mathbb{R}$-valued (resp. $\mathbb{R}^3$-valued) Lebesgue square-integrable functions. Moreover, we write $\mathcal{C}_0^\infty(\mathcal{O})$ (resp. $\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O})$) for the space of infinitely differentiable $\mathbb{R}$-valued (resp. $\mathbb{R}^3$-valued) functions with compact support in $\mathcal{O}$. For convenience, let us recall the definition of a weak gradient. Let $u\in L^2(\mathcal{O})$ and $\boldsymbol{v}\in\boldsymbol{L}^2(\mathcal{O})$ be given. Then, $\boldsymbol{v}$ is called the weak gradient of u if and only if

$$\int_{\mathcal{O}} u\,\mathrm{div}\,\boldsymbol{\phi}\,\mathrm{d}x=-\int_{\mathcal{O}}\boldsymbol{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x\quad\forall\boldsymbol{\phi}\in\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O}).$$

One usually writes $\nabla u:=\boldsymbol{v}$. Along the same principle, we can now introduce the notions of a weak **curl** and a weak divergence. To this aim, let $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{L}^2(\mathcal{O})$ be given. Then, $\boldsymbol{v}$ is called the weak **curl** of $\boldsymbol{u}$ (we write $\mathbf{curl}\,\boldsymbol{u}:=\boldsymbol{v}$) if and only if

$$\int_{\mathcal{O}}\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\phi}\,\mathrm{d}x=\int_{\mathcal{O}}\boldsymbol{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x\quad\forall\boldsymbol{\phi}\in\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O}).\tag{2.14}$$

In a similar way, if $\boldsymbol{u}\in\boldsymbol{L}^2(\mathcal{O})$ and $v\in L^2(\mathcal{O})$, the function v is called the weak divergence of $\boldsymbol{u}$ (we write $\mathrm{div}\,\boldsymbol{u}:=v$) if and only if

$$\int_{\mathcal{O}}\boldsymbol{u}\cdot\nabla\phi\,\mathrm{d}x=-\int_{\mathcal{O}}v\phi\,\mathrm{d}x\quad\forall\phi\in\mathcal{C}_0^\infty(\mathcal{O}).\tag{2.15}$$

In contrast to the weak gradient, the existence of the weak **curl** or the weak divergence is not equivalent to the existence of the occurring weak partial derivatives in the classical definition of **curl** or div. This makes these two concepts indeed more general. Similar to the classical definition in the H^1-case, that is,

$$H^1(\mathcal{O}):=\{u\in L^2(\mathcal{O})\mid\nabla u\in\boldsymbol{L}^2(\mathcal{O})\},$$

we are interested in the spaces of vector fields where the weak **curl** (resp. the weak divergence) exists. Therefore, we introduce the Hilbert spaces

$$\begin{aligned}\boldsymbol{H}(\mathbf{curl},\mathcal{O})&:=\{\boldsymbol{u}\in\boldsymbol{L}^2(\mathcal{O})\mid\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{L}^2(\mathcal{O})\}\\ \boldsymbol{H}(\mathrm{div},\mathcal{O})&:=\{\boldsymbol{u}\in\boldsymbol{L}^2(\mathcal{O})\mid\mathrm{div}\,\boldsymbol{u}\in L^2(\mathcal{O})\},\end{aligned}$$

endowed with their natural graph norms

$$\begin{aligned}\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\mathcal{O})}&:=\left(\|\cdot\|^2_{\boldsymbol{L}^2(\mathcal{O})}+\|\,\mathbf{curl}\cdot\|^2_{\boldsymbol{L}^2(\mathcal{O})}\right)^{\frac{1}{2}}\\ \|\cdot\|_{\boldsymbol{H}(\mathrm{div},\mathcal{O})}&:=\left(\|\cdot\|^2_{\boldsymbol{L}^2(\mathcal{O})}+\|\,\mathrm{div}\cdot\|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}.\end{aligned}$$

Here the **curl** and div operators are to be understood in the sense of (2.14) and (2.15). We define the subspaces $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O})$ and $\boldsymbol{H}_0(\mathrm{div}, \mathcal{O})$ as the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O})$ with respect to the corresponding topology, i.e.,

$$\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \coloneqq \overline{\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O})}^{\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\mathcal{O})}} \quad \text{and} \quad \boldsymbol{H}_0(\mathrm{div}, \mathcal{O}) \coloneqq \overline{\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O})}^{\|\cdot\|_{\boldsymbol{H}(\mathrm{div},\mathcal{O})}}. \tag{2.16}$$

Assuming that $\mathcal{O}$ is a bounded Lipschitz domain, the spaces in (2.16) can be written (similarly to the H^1-case) as the kernel of appropriate trace mappings. To expand on this, let $\tau\colon H^1(\mathcal{O}) \to L^2(\partial\mathcal{O})$ indicate the standard trace mapping. Then, we denote by

$$H^{1/2}(\partial\mathcal{O}) \coloneqq \tau(H^1(\mathcal{O}))$$

the fractional Sobolev space to the exponent 1/2 endowed with the quotient norm

$$\|g\|_{H^{1/2}(\partial\mathcal{O})} \coloneqq \inf_{\substack{v\in H^1(\mathcal{O})\\ \tau(v)=g}} \|v\|_{H^1(\mathcal{O})} \quad \forall g \in H^{1/2}(\partial\mathcal{O}). \tag{2.17}$$

There are several other representations for the fractional Sobolev space $H^{1/2}(\partial\mathcal{O})$, as can be found in [47] or [60]. Now, denoting with $\boldsymbol{n}\colon \partial\mathcal{O} \to \mathbb{R}^3$ the outward unit normal of $\mathcal{O}$ and under the mentioned assumptions on $\mathcal{O}$, there exist tangential and normal trace maps (cf. [107])

$$\tau_t\colon \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \to \boldsymbol{H}^{-1/2}(\partial\mathcal{O}) \quad \text{and} \quad \tau_n\colon \boldsymbol{H}(\mathrm{div}, \mathcal{O}) \to H^{-1/2}(\partial\mathcal{O}),$$

generalizing the boundary evaluations $\boldsymbol{u}\times\boldsymbol{n}_{|\partial\mathcal{O}}$ and $\boldsymbol{u}\cdot\boldsymbol{n}_{|\partial\mathcal{O}}$ defined for continuous fields $\boldsymbol{u}\colon \overline{\mathcal{O}} \to \mathbb{R}^3$, so that

$$\begin{aligned} \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \mid \tau_t(\boldsymbol{u}) = 0\} \\ \boldsymbol{H}_0(\mathrm{div}, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \mathcal{O}) \mid \tau_n(\boldsymbol{u}) = 0\}. \end{aligned} \tag{2.18}$$

Here, $H^{-1/2}(\partial\mathcal{O})$ (resp. $\boldsymbol{H}^{-1/2}(\partial\mathcal{O})$) stands for the topological dual space of $H^{1/2}(\partial\mathcal{O})$ (resp. $\boldsymbol{H}^{1/2}(\partial\mathcal{O})$). For further information on trace spaces, we refer the reader to [5] and [36]. As pointed out, the characterization in (2.18) relies on additional regularity assumptions on the domain. For this reason, we try to solely depend on another type of representation not requiring additional regularity. For the convenience of the reader, we provide an elementary proof for the following result

Lemma 2.1. *Let $\mathcal{O} \subset \mathbb{R}^3$ be open. Then, it holds that*

$$\begin{aligned} \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) &= \{\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \mid (\boldsymbol{z}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\mathcal{O})} = (\mathbf{curl}\,\boldsymbol{z}, \boldsymbol{v})_{\boldsymbol{L}^2(\mathcal{O})}\ \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \mathcal{O})\} \\ \boldsymbol{H}_0(\mathrm{div}, \mathcal{O}) &= \{\boldsymbol{z} \in \boldsymbol{H}(\mathrm{div}, \mathcal{O}) \mid (\boldsymbol{z}, \nabla v)_{\boldsymbol{L}^2(\mathcal{O})} = -\,(\mathrm{div}\,\boldsymbol{z}, v)_{L^2(\mathcal{O})}\ \forall v \in H^1(\mathcal{O})\}. \end{aligned} \tag{2.19}$$

Proof. We show the characterization for $\boldsymbol{H}_0(\mathrm{div}, \mathcal{O})$. The other case follows an analogous argumentation. For simplicity, we denote

$$\boldsymbol{Z} \coloneqq \{\boldsymbol{z} \in \boldsymbol{H}(\mathrm{div}, \mathcal{O}) \mid (\boldsymbol{z}, \nabla v)_{\boldsymbol{L}^2(\mathcal{O})} = -\,(\mathrm{div}\,\boldsymbol{z}, v)_{L^2(\mathcal{O})}\ \forall v \in H^1(\mathcal{O})\}.$$

By definition, the adjoint of the (unbounded) operator $\nabla\colon H^1(\mathcal{O}) \subset L^2(\mathcal{O}) \to \boldsymbol{L}^2(\mathcal{O})$ is characterized by

$$\forall \boldsymbol{z} \in D(\nabla^*): \quad (\nabla v, \boldsymbol{z})_{\boldsymbol{L}^2(\mathcal{O})} = (v, \nabla^*\boldsymbol{z})_{L^2(\mathcal{O})} \quad \forall v \in H^1(\mathcal{O}).$$

Thus, by the definition of $\boldsymbol{Z}$, this shows that

$$\nabla^* = -\,\mathrm{div}_{|\boldsymbol{Z}}\colon \boldsymbol{Z} \subset \boldsymbol{L}^2(\mathcal{O}) \to L^2(\mathcal{O}).$$

On the other hand, comparing with (2.15), it holds that

$$\left(-\operatorname{div}_{|\boldsymbol{H}_0(\operatorname{div},\mathcal{O})}\right)^* = \nabla\colon H^1(\mathcal{O}) \subset L^2(\mathcal{O}) \to \boldsymbol{L}^2(\mathcal{O}).$$

We conclude that

$$-\operatorname{div}_{|\boldsymbol{H}_0(\operatorname{div},\mathcal{O})} = \overline{-\operatorname{div}_{|\boldsymbol{H}_0(\operatorname{div},\mathcal{O})}} = \left(-\operatorname{div}_{|\boldsymbol{H}_0(\operatorname{div},\mathcal{O})}\right)^{**} = \nabla^* = -\operatorname{div}_{|\boldsymbol{Z}},$$

which ultimately yields $\boldsymbol{H}_0(\operatorname{div},\mathcal{O}) = \boldsymbol{Z}$. □

Let us now introduce the kernels of the divergence and the **curl** in their respective spaces, i.e.,

$$\begin{aligned}
\boldsymbol{H}(\operatorname{div}{=}0,\mathcal{O}) &:= \{\boldsymbol{z} \in \boldsymbol{H}(\operatorname{div},\mathcal{O}) \mid \operatorname{div}\boldsymbol{z} = 0\} \\
\boldsymbol{H}_0(\operatorname{div}{=}0,\mathcal{O}) &:= \{\boldsymbol{z} \in \boldsymbol{H}_0(\operatorname{div},\mathcal{O}) \mid \operatorname{div}\boldsymbol{z} = 0\} \\
\boldsymbol{H}(\mathbf{curl}{=}0,\mathcal{O}) &:= \{\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl},\mathcal{O}) \mid \mathbf{curl}\,\boldsymbol{z} = 0\} \\
\boldsymbol{H}_0(\mathbf{curl}{=}0,\mathcal{O}) &:= \{\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \mid \mathbf{curl}\,\boldsymbol{z} = 0\}
\end{aligned}$$

which are henceforth endowed with the $\boldsymbol{L}^2(\mathcal{O})$-topology. In the case that given from the context it is clear which domain we are talking about, we skip indicating the domain, for example, we solely write $\boldsymbol{H}(\mathbf{curl})$ or $\boldsymbol{H}(\operatorname{div})$. By the constitutive relations (2.10), (2.11) and (2.12) we have seen that certain (possibly matrix-valued) material parameters are involved in the Maxwell system (2.13). A rigorous treatment of this Maxwell system requires certain regularity assumptions on the material parameters. In that context, we denote by $L^\infty_{\mathrm{sym}}(\mathcal{O})^{3\times 3}$ the space of all (equivalence classes of) symmetric $\mathbb{R}^{3\times 3}$-valued Lebesgue measurable and essentially bounded functions with respect to the spectral norm, i.e.,

$$\|\alpha\|_{L^\infty(\mathcal{O})^{3\times 3}} := \operatorname*{ess\,sup}_{x\in\mathcal{O}} \max_{|\xi|\le 1} |\alpha(x)\xi| < \infty \quad \forall \alpha \in L^\infty_{\mathrm{sym}}(\mathcal{O})^{3\times 3}. \tag{2.20}$$

For a given uniformly positive definite matrix-valued function $\alpha \in L^\infty_{\mathrm{sym}}(\mathcal{O})^{3\times 3}$, that is, there exists a constant $\underline{\alpha} > 0$ such that

$$\alpha(x)\xi\cdot\xi \ge \underline{\alpha}|\xi|^2 \quad \text{for a.e. } x \in \mathcal{O} \text{ and all } \xi \in \mathbb{R}^3,$$

we denote by $\boldsymbol{L}^2_\alpha(\mathcal{O})$ the space $\boldsymbol{L}^2(\mathcal{O})$ equipped with the weighted scalar product $(\alpha\cdot,\cdot)_{\boldsymbol{L}^2(\mathcal{O})}$. In the case that the weight function α is scalar-valued, i.e., $\alpha \in L^\infty(\mathcal{O})$ such that there exists a constant $\underline{\alpha} > 0$ with

$$\alpha(x) \ge \underline{\alpha} \quad \text{for a.e. } x \in \mathcal{O},$$

we identify α with its product with the identity matrix in $\mathbb{R}^{3\times 3}$, so that our notation of $\boldsymbol{L}^2_\alpha(\mathcal{O})$ withstands. Note that the weighted space $\boldsymbol{L}^2_\alpha(\mathcal{O})$ is merely for convenience since, under the assumptions on the weight function α, we have the equality $\boldsymbol{L}^2_\alpha(\mathcal{O}) = \boldsymbol{L}^2(\mathcal{O})$ as sets and an isomorphism $\boldsymbol{L}^2_\alpha(\mathcal{O}) \cong \boldsymbol{L}^2(\mathcal{O})$ as vector spaces.

2.1.2 A Lack of Compactness

The previous functional analytic framework, building upon the Hilbert spaces $\boldsymbol{H}(\mathbf{curl},\mathcal{O})$ and $\boldsymbol{H}(\operatorname{div},\mathcal{O})$, allows us to formulate the arguably biggest challenge within the treatment of Maxwell-structured problems: the omnipresent lack of compactness. In the famous H^1-setting, it is the well-known Rellich-Kondrachov theorem which states (under reasonable assumptions on the domain $\mathcal{O}$) that the embedding $H^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ is compact. Contrary, the embeddings

$\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ and $\boldsymbol{H}(\mathrm{div},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ are not compact. Even restricting to the subspaces $\boldsymbol{H}_0(\mathbf{curl},\mathcal{O})$, $\boldsymbol{H}_0(\mathrm{div},\mathcal{O})$ or $\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap \boldsymbol{H}(\mathrm{div},\mathcal{O})$, the corresponding embeddings fail to be compact. Loosely speaking, this is due to the fact that the kernel of both (unbounded) operators

$$\mathbf{curl}\colon \boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \subset \boldsymbol{L}^2(\mathcal{O}) \to \boldsymbol{L}^2(\mathcal{O}) \quad \text{and} \quad \mathrm{div}\colon \boldsymbol{H}_0(\mathrm{div},\mathcal{O}) \subset \boldsymbol{L}^2(\mathcal{O}) \to L^2(\mathcal{O}) \tag{2.21}$$

is too large, namely, we have the inclusions

$$N(\mathbf{curl}) \supset \overline{\nabla H_0^1(\mathcal{O})} \qquad \text{and} \qquad N(\mathrm{div}) \supset \overline{\mathbf{curl}\,\boldsymbol{H}_0(\mathbf{curl})}.$$

Here, $N(\mathbf{curl})$ and $N(\mathrm{div})$ denote the kernels of the operators in (2.21). The situation for the unbounded operator $\nabla\colon H_0^1(\mathcal{O}) \subset L^2(\mathcal{O}) \to \boldsymbol{L}^2(\mathcal{O})$ is different, since here it holds that $N(\nabla) = \{0\}$. These observations are formalized within the following Proposition. The first statement and its proof can also be found in [14, Proposition 2.7].

Proposition 2.2. *Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, the following two assertions hold true.*

(i) The inclusion $\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap \boldsymbol{H}(\mathrm{div},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ fails to be compact.

(ii) Both the inclusions $\boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ and $\boldsymbol{H}_0(\mathrm{div},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ are not compact.

Proof. Ad (i): First, $H^{1/2}(\partial\mathcal{O})$ endowed with the norm in (2.17) is a separable Hilbert space (cf. [60, Theorem 1.5]). For this reason, there exists an orthonormal basis $\{g_k\}_{k=1}^\infty \subset H^{1/2}(\partial\mathcal{O})$. Now, for $k \in \mathbb{N}$, we consider the inhomogeneous Dirichlet problems

$$\begin{cases} -\Delta u_k = 0 & \text{in } \mathcal{O} \\ \quad u_k = g_k & \text{in } \partial\mathcal{O}. \end{cases} \tag{2.22}$$

By the solution theory for inhomogeneous Dirichlet problems (cf. [60, Proposition 1.1]), it follows that there exists a unique solution $u_k \in H^1(\mathcal{O})$ to (2.22) which depends continuously on the boundary data. In particular, this implies the boundedness of the sequence $\{u_k\}_{k=1}^\infty$ in $H^1(\mathcal{O})$. For $k \in \mathbb{N}$, we define $\boldsymbol{v}_k \coloneqq \nabla u_k$. Then, since by construction it holds that

$$\mathbf{curl}\,\boldsymbol{v}_k = 0 \quad \text{and} \quad \mathrm{div}\,\boldsymbol{v}_k = 0 \quad \text{a.e. in } \mathcal{O},$$

the sequence $\{\boldsymbol{v}_k\}_{k=1}^\infty$ is bounded in $\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap \boldsymbol{H}(\mathrm{div},\mathcal{O})$. It remains to show that there exists no subsequence of $\{\boldsymbol{v}_k\}_{k=1}^\infty$ which converges strongly in $\boldsymbol{L}^2(\mathcal{O})$. To this aim, let us assume that indeed there exists some $\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{O})$ and a subsequence, not denoted any different, such that $\boldsymbol{v}_k \to \boldsymbol{v}$ strongly in $\boldsymbol{L}^2(\mathcal{O})$ as $k \to \infty$. Since $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\mathcal{O})$, the Rellich-Kondrachov theorem (see [1, Theorem 6.3]) implies that there exists $u \in L^2(\mathcal{O})$ such that, up to another subsequence, it holds that $u_k \to u$ strongly in $L^2(\mathcal{O})$ as $k \to \infty$. We conclude that $\boldsymbol{v} = \nabla u$ so that up to our particular choice of a subsequence, $\{u_k\}_{k=1}^\infty$ converges strongly to u in $H^1(\mathcal{O})$. By the construction of the norm in (2.17), the trace map with image in $H^{1/2}(\partial\mathcal{O})$, i.e., $\tau\colon H^1(\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$, is bounded. Consequently, it follows that

$$g_k = \tau(u_k) \to \tau(u) \quad \text{strongly in } H^{1/2}(\partial\mathcal{O}) \quad \text{as } k \to \infty.$$

Since $\{g_k\}_{k=1}^\infty$ is an orthonormal basis for $H^{1/2}(\partial\mathcal{O})$, it converges weakly to 0 in $H^{1/2}(\partial\mathcal{O})$, so that $\tau(u) = 0$. Finally, this implies that the sequence $\{g_k\}_{k=1}^\infty$ converges strongly to 0 in $H^{1/2}(\partial\mathcal{O})$, which cannot happen due to $\|g_k\|_{H^{1/2}(\partial\mathcal{O})} = 1$ for every $k \in \mathbb{N}$. This completes the proof.

Ad (ii): We show that the embedding $\boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ fails to be compact. First of all, by the Rellich-Kondrachov theorem (implying the standard Poincaré inequality for $H_0^1(\mathcal{O})$), the space $\nabla H_0^1(\mathcal{O}) \subset \boldsymbol{L}^2(\mathcal{O})$ is closed and it is, therefore, a separable Hilbert space when endowed with the $\boldsymbol{L}^2(\mathcal{O})$-norm. Now, for $k \in \mathbb{N}$, we choose $u_k \in H_0^1(\mathcal{O})$ such that the elements $\boldsymbol{v}_k := \nabla u_k$ form an orthonormal basis of $\nabla H_0^1(\mathcal{O})$. Since, for every $k \in \mathbb{N}$, the function u_k satisfies the Dirichlet boundary condition and since $\mathbf{curl}\,\boldsymbol{v}_k = \mathbf{curl}\,\nabla u_k = 0$, it follows that the sequence $\{\boldsymbol{v}_k\}_{k=1}^{\infty} = \{\nabla u_k\}_{k=1}^{\infty}$ is contained in $\boldsymbol{H}_0(\mathbf{curl}{=}0,\mathcal{O})$. Thus, the sequence $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$ converges weakly to 0 in $\boldsymbol{H}_0(\mathbf{curl},\mathcal{O})$ but not strongly to 0 in $\boldsymbol{L}^2(\mathcal{O})$, again due to orthonormality. The arguments for the non-compactness of the inclusion $\boldsymbol{H}_0(\operatorname{div},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$ are similar. Here, one uses the fact that $\overline{\mathbf{curl}\,\boldsymbol{H}_0(\mathbf{curl},\mathcal{O})}$ is a separable Hilbert space when endowed with the $\boldsymbol{L}^2(\mathcal{O})$-norm satisfying $\overline{\mathbf{curl}\,\boldsymbol{H}_0(\mathbf{curl},\mathcal{O})} \subset \boldsymbol{H}_0(\operatorname{div}{=}0,\mathcal{O})$. This concludes the proof. □

As a remedy for Proposition 2.2, a compact embedding can be obtained when adding a boundary condition to one of the spaces within the intersection $\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap \boldsymbol{H}(\operatorname{div},\mathcal{O})$. For the following result we refer to [20, 114, 134, 135]:

Theorem 2.3. *Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, both inclusions*

$$\boldsymbol{X}_N(\mathcal{O}) := \boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \cap \boldsymbol{H}(\operatorname{div},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$$
$$\boldsymbol{X}_T(\mathcal{O}) := \boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap \boldsymbol{H}_0(\operatorname{div},\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O})$$

are compact.

In particular, the respective kernels, also called the spaces of magnetic and electric harmonic fields,

$$\begin{aligned}\mathcal{H}(e,\mathcal{O}) &:= \boldsymbol{H}_0(\mathbf{curl}{=}0,\mathcal{O}) \cap \boldsymbol{H}(\operatorname{div}{=}0,\mathcal{O}) = \{\boldsymbol{v} \in \boldsymbol{X}_N(\mathcal{O}) \mid \mathbf{curl}\,\boldsymbol{v} = 0 \text{ and } \operatorname{div}\boldsymbol{v} = 0\}\\ \mathcal{H}(m,\mathcal{O}) &:= \boldsymbol{H}(\mathbf{curl}{=}0,\mathcal{O}) \cap \boldsymbol{H}_0(\operatorname{div}{=}0,\mathcal{O}) = \{\boldsymbol{v} \in \boldsymbol{X}_T(\mathcal{O}) \mid \mathbf{curl}\,\boldsymbol{v} = 0 \text{ and } \operatorname{div}\boldsymbol{v} = 0\}\end{aligned} \quad (2.23)$$

are finite-dimensional. If we assume that the domain $\mathcal{O}$ has a connected boundary, the space $H(e,\mathcal{O})$ is trivial. On the other hand, if we assume that $\mathcal{O}$ is simply connected, the space $H(m,\mathcal{O})$ is trivial. We refer to [7, Appendix A.4] or [14] for a more detailed description of the spaces of magnetic and electric harmonic fields. From the compactness in Theorem 2.3 we obtain the following Poincaré-Friedrichs-type inequality.

Corollary 2.4. *Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary. Then, there exists a constant $C_{\mathrm{p}} > 0$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\mathcal{O})} \le C_{\mathrm{p}}\Big(\|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\mathcal{O})} + \|\operatorname{div}\boldsymbol{v}\|_{L^2(\mathcal{O})}\Big) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N(\mathcal{O}). \quad (2.24)$$

The inequality in (2.24) is of particular relevance to us within the space

$$\boldsymbol{X}_{N,0}(\mathcal{O}) := \{\boldsymbol{v} \in \boldsymbol{X}_N(\mathcal{O}) \mid \operatorname{div}\boldsymbol{v} = 0\} = \boldsymbol{X}_N(\mathcal{O}) \cap \boldsymbol{H}(\operatorname{div}{=}0,\mathcal{O}).$$

Then, (2.24) simplifies to

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\mathcal{O})} \le C_{\mathrm{p}}\|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\mathcal{O})} \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}(\mathcal{O}). \quad (2.25)$$

We end this section by recalling some orthogonal decomposition results for which we provide an elementary proof.

Lemma 2.5. *Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, we have the basic orthogonal decompositions*

$$\boldsymbol{L}^2(\mathcal{O}) = \nabla H^1(\mathcal{O}) \oplus \boldsymbol{H}_0(\mathrm{div}{=}0, \mathcal{O}) \tag{2.26}$$
$$\boldsymbol{L}^2(\mathcal{O}) = \nabla H_0^1(\mathcal{O}) \oplus \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O}). \tag{2.27}$$

If $\mathcal{O}$ is additionally simply connected, it holds that

$$\boldsymbol{L}^2(\mathcal{O}) = \nabla H^1(\mathcal{O}) \oplus \mathbf{curl}\left(\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})\right). \tag{2.28}$$

If the boundary $\partial\mathcal{O}$ is connected, it holds that

$$\boldsymbol{L}^2(\mathcal{O}) = \nabla H_0^1(\mathcal{O}) \oplus \mathbf{curl}\left(\boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}_0(\mathrm{div}{=}0, \mathcal{O})\right). \tag{2.29}$$

Proof. Using the projection theorem in Hilbert spaces, we obtain a decomposition of the form

$$H_2 = \overline{R(A)} \oplus \overline{R(A)}^{\perp} = \overline{R(A)} \oplus N(A^*), \tag{2.30}$$

where $A\colon D(A) \subset H_1 \to H_2$ denotes some (unbounded) linear operator with range $R(A) \subset H_2$. Now, the decomposition in (2.30) may be applied to the operator $\nabla\colon H^1(\mathcal{O}) \subset L^2(\mathcal{O}) \to \boldsymbol{L}^2(\mathcal{O})$ to obtain

$$\boldsymbol{L}^2(\mathcal{O}) = \overline{\nabla H^1(\mathcal{O})} \oplus \boldsymbol{H}_0(\mathrm{div}{=}0, \mathcal{O}).$$

Using the regularity assumption on $\mathcal{O}$ we can use the compact embedding $H^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ to see that $\nabla H^1(\mathcal{O})$ is already closed (cf. [113, Corollary 2.6] or alternatively [87, Theorem 12.23]). This shows the decomposition (2.26). The second decomposition (2.27) is shown in the same way. Towards the refined decomposition in (2.28), we apply (2.30) to the operator $\mathbf{curl}\colon \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \subset \boldsymbol{L}^2(\mathcal{O}) \to \boldsymbol{L}^2(\mathcal{O})$ to obtain

$$\boldsymbol{L}^2(\mathcal{O}) = \overline{\mathbf{curl}\, \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O})} \oplus \boldsymbol{H}(\mathbf{curl}{=}0, \mathcal{O}). \tag{2.31}$$

Considering the decomposition (2.27), we intersect both sides in (2.27) with $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O})$ to find

$$\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) = \nabla H_0^1(\mathcal{O}) \oplus \left(\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})\right). \tag{2.32}$$

Applying (2.32) to (2.31) yields

$$\begin{aligned}
\boldsymbol{L}^2(\mathcal{O}) &= \overline{\mathbf{curl}\left(\nabla H_0^1(\mathcal{O}) \oplus (\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O}))\right)} \oplus \boldsymbol{H}(\mathbf{curl}{=}0, \mathcal{O}) \\
&= \overline{\mathbf{curl}\left(\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})\right)} \oplus \boldsymbol{H}(\mathbf{curl}{=}0, \mathcal{O}) \\
&= \mathbf{curl}\left(\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})\right) \oplus \boldsymbol{H}(\mathbf{curl}{=}0, \mathcal{O}),
\end{aligned} \tag{2.33}$$

where we used the compact embedding from Theorem 2.3 to obtain that $\mathbf{curl}(\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})) \subset \boldsymbol{L}^2(\mathcal{O})$ is closed (cf. [113, Corollary 2.6]). Now, taking another intersection with the space $\boldsymbol{H}_0(\mathrm{div}{=}0, \mathcal{O})$ in (2.33) implies

$$\boldsymbol{H}_0(\mathrm{div}{=}0, \mathcal{O}) = \mathbf{curl}\left(\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap \boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})\right) \oplus \mathcal{H}(m, \mathcal{O}). \tag{2.34}$$

Finally, applying (2.34) to the decomposition (2.26) and using that $\mathcal{O}$ is simply connected, we conclude (2.28). The decomposition in (2.29) is shown analogously. This completes the proof. □

2.2 Elliptic Variational Inequalities

This section is devoted to the very basic concepts from the theory of elliptic variational inequalities accompanied by the standard example given by the Poisson-type obstacle problem in the classical Sobolev space H^1. In particular, we introduce known techniques to study the sensitivity analysis of the Poisson-type obstacle problem and discuss the application of these techniques to electromagnetic obstacle problems.

Let H be a Hilbert space and let $a\colon H \times H \to \mathbb{R}$ be a bounded and coercive bilinear form, so that there exist constants $\overline{c}, \underline{c} > 0$, such that the bilinear form a satisfies

$$\begin{aligned} |a(u,v)| &\leq \overline{c}\|u\|_H\|v\|_H \quad && \forall u,v \in H \\ a(u,u) &\geq \underline{c}\|u\|_H^2 && \forall u \in H. \end{aligned}$$

Further, let $L\colon H \to \mathbb{R}$ be a linear and bounded form on H, i.e., $L \in H^*$. Now, given a convex, sequentially lower semi-continuous and proper functional $j\colon H \to \mathbb{R} \cup \{\infty\}$, the problem

$$\begin{cases} \text{Find } u \in H, \text{ s.t.} \\ a(u, v-u) + j(v) - j(u) \geq L(v-u) \quad \forall v \in H \end{cases} \tag{EVI}$$

is called a variational inequality. Historically, it has been convenient to distinguish two cases. In the case of j being equal to the indicator function of a non-empty, closed and convex set $K \subset H$, i.e.,

$$j = I_K\colon H \to \mathbb{R} \cup \{\infty\}, \quad v \mapsto \begin{cases} 0, & \text{if } v \in K \\ \infty, & \text{if } v \notin K, \end{cases}$$

the problem (EVI) is called a variational inequality of the first kind. Otherwise, (EVI) is called a variational inequality of the second kind. Given the mentioned properties of the set $K \subset H$, it is easily shown that the functional I_K is indeed convex, sequentially lower semi-continuous and proper. Moreover, for $j = I_K$, (EVI) can be rewritten as

$$\begin{cases} \text{Find } u \in K, \text{ s.t.} \\ a(u, v-u) \geq L(v-u) \quad \forall v \in K. \end{cases} \tag{2.35}$$

It goes almost without saying that if the nonlinear character of (EVI) is not present, that is $j = 0$, the problem (EVI) reduces to the standard variational problem of finding $u \in H$ such that

$$a(u,v) = L(v) \quad \forall v \in H.$$

The following result summarizes the well-posedness theory for both first and second kind elliptic variational inequalities and is known for many decades. It goes back to Lions and Stampacchia [91, Theorem 2.1 & Theorem 2.2] and can alternatively be found in [61, Theorem 3.1 & Theorem 4.1].

Theorem 2.6. *Let H be a Hilbert space. Further, let $a\colon H \times H \to \mathbb{R}$ be a bounded and coercive bilinear form and let $L \in H^*$ be given. Moreover, let $j\colon H \to \mathbb{R} \cup \{\infty\}$ be convex, sequentially lower semi-continuous and proper. Then, there exists a unique solution $u \in H$ to* (EVI)*. Moreover, if $K \subset H$ is a closed, convex, non-empty set and $j = I_K$ (i.e., we are in the setting of* (2.35)*), then it additionally holds that $u \in K$.*

In the case that the bilinear form is additionally symmetric, the solution to (EVI) is characterized as the unique minimizer of a certain energy functional. We quickly recall the result for variational inequalities of the first kind.

Proposition 2.7. *In the setting of Theorem 2.6 assume that the bilinear form a is additionally symmetric. Then, there is a unique minimizer $u \in K$ to*

$$\min_{v\in K} \frac{1}{2}a(v,v) - L(v). \tag{2.36}$$

Moreover, $u \in K$ is the minimizer of (2.36) *if and only if $u \in K$ is the unique solution to* (2.35).

2.2.1 The Obstacle Problem

In this section, we want to discuss the issue that comes with the non-smooth character of problems such as (2.35). In particular, we want to present the main ideas regarding the sensitivity analysis of (2.35) by considering the puristic H^1-obstacle problem, which serves as one of the prime examples for variational inequalities of the first kind. In this way, we hope to give the reader a better understanding on why the variational inequality in Chapter 5 needs a particular treatment in opposition to using the known results presented in this section.

Given a natural number $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ denote an open and bounded subset. Further, let $f \in L^2(\Omega)$ be given. The problem of focus reads

$$\begin{cases} \text{Find } u \in K, \text{ s.t.} \\ \int_\Omega \nabla u \cdot \nabla(v-u)\,\mathrm{d}x \geq \int_\Omega f(v-u)\,\mathrm{d}x \quad \forall v \in K, \end{cases} \tag{2.37}$$

where the obstacle set is given by

$$K \coloneqq \{v \in H_0^1(\Omega) \mid \psi_- \leq v \leq \psi_+ \text{ a.e. in } \Omega\}$$

with obstacles $\psi_-, \psi_+ \colon \Omega \to [-\infty, \infty]$ so that K is non-empty . Here, the right-hand side may more generally be replaced by a linear functional $L \in H^{-1}(\Omega) = H_0^1(\Omega)^*$. Now, well-posedness of the problem in (2.37) is guaranteed by the general result in Theorem 2.6, so that for every $f \in L^2(\Omega)$ there exists a unique solution $u \in K$ to (2.37) which gives rise to the solution mapping

$$S \colon L^2(\Omega) \to H_0^1(\Omega), \quad f \mapsto u.$$

For what concerns the differentiability of the solution mapping S, the situation is rather poor: even in the present case of (2.37), only Hadamard-directional (Gâteaux) differentiability can be achieved. In the seminal work by Mignot [99], the sensitivity analysis hinges on the notions of regular Dirichlet spaces and polyhedricity as the main ingredients. We also refer to [41] for a more in-depth presentation of the general sensitivity analysis for both first and second kind VIs. The following definition can also be found in [42].

Definition 2.8 (Regular L^2-Dirichlet space)**.** Let M denote a locally compact and separable metric space. Denoting by $\mathcal{B}(M)$ the Borel σ-algebra of M, we assume that $\mu \colon \mathcal{B}(M) \to [0,\infty]$ is a measure that is strictly positive on non-empty open sets and finite on compact sets. Then, we call $(V, \mathfrak{b})$ a regular L^2-Dirichlet space when V is a subspace of $L^2(M,\mu)$ and $\mathfrak{b} \colon V \times V \to \mathbb{R}$ is a symmetric, positive semi-definite bilinear form such that $(V, \mathfrak{b})$ satisfies

(i) V is dense in $L^2(M,\mu)$ w.r.t. the norm $\|\cdot\|_{L^2(M,\mu)}$,

(ii) V is a Hilbert space equipped with the scalar product $(\cdot,\cdot)_V \coloneqq (\cdot,\cdot)_{L^2(M,\mu)} + \mathfrak{b}(\cdot,\cdot)$,

(iii) for every $v \in V$, it holds that $u \coloneqq \min(1, \max(0, v)) \in V$ and $\mathfrak{b}(u,u) \leq \mathfrak{b}(v,v)$,

(iv) the intersection $V \cap \mathcal{C}_0(M)$ is dense in V w.r.t. the norm $\|\cdot\|_V$,

(v) the intersection $V \cap \mathcal{C}_0(M)$ is dense in $\mathcal{C}_0(M)$ w.r.t. the norm $\|\cdot\|_{L^\infty(M,\mu)}$.

Here, $\mathcal{C}_0(M)$ denotes the space of continuous functions on M with compact support in M.

For a rather abstract introduction to Dirichlet spaces we refer to [26, 27] and [57]. As a result of Stampacchia's lemma (cf. [82, Chapter II, Theorem A.1]), given some $v \in H_0^1(\Omega)$, the required cut-off type mapping in condition (iii) of Definition 2.8 for $V = H_0^1(\Omega)$ is again an element of $H_0^1(\Omega)$. For this reason, the Sobolev space $H_0^1(\Omega)$ is a standard example for a regular L^2-Dirichlet space when endowed with the form

$$\mathfrak{b}\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}, \quad (v, w) \mapsto \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x. \tag{2.38}$$

The upcoming notion of polyhedricity requires basic concepts from convex analysis. For this reason, let us recall certain types of cones in Banach spaces. These concepts and more information on the topic can be found in [29, Section 2.2.4].

Definition 2.9. Let X be a Banach space and let $K \subset X$ be a convex and non-empty subset. Then, for $x \in K$, we define

$$\begin{aligned} \mathcal{T}_K^{\mathrm{rad}}(x) &\coloneqq \mathbb{R}^+(K - x), && \text{the radial cone at } x \in K,\\ \mathcal{T}_K(x) &\coloneqq \overline{\mathcal{T}_K^{\mathrm{rad}}(x)}, && \text{the tangent cone at } x \in K,\\ \mathcal{N}_K(x) &\coloneqq \{x^* \in X^* \mid x^*(y) \le 0 \quad \forall y \in \mathcal{T}_K(x)\}, && \text{the normal cone at } x \in K. \end{aligned}$$

Based on these different concepts of cones in Banach spaces we can introduce the concept of polyhedricity in Banach spaces. Proposition 2.7 shows that (2.37) can be interpreted as an optimization problem. Already from finite-dimensional optimization, it is known that polyhedric sets play an important role. As for example discussed in [63], polyhedricity in a general Hilbert space provides a notion of curvature which can be very beneficial when investigating differentiability properties of projections onto non-empty, closed and convex sets.

Definition 2.10. Let X be a Banach space and let $K \subset X$ be a closed, convex and non-empty subset. Then, K is said to be polyhedric at $x \in K$ for $x^* \in \mathcal{N}_K(x)$ if

$$\mathcal{T}_K(x) \cap N(x^*) = \overline{\mathcal{T}_K^{\mathrm{rad}}(x) \cap N(x^*)}.$$

In the case that K is polyhedric at $x \in K$ for every $x^* \in \mathcal{N}_K(x)$, we say that K is polyhedric at $x \in K$.

There is actually a distinction to be made between polyhedric sets in the sense of the last definition and sets which are the intersection of finitely many half-spaces. Those two notions only coincide in finite dimensions. We refer the reader to [132] for a thorough overview of the topic of polyhedricity in infinite dimensions and to [42, 131] for limitations of the concept of polyhedricity in regular L^2-Dirichlet spaces. The following result is central to the sensitivity analysis of variational inequalities and was first proven by Mignot in [99].

Theorem 2.11. *Assume that $(V, \mathfrak{b})$ is a regular L^2-Dirichlet space on the underlying locally compact and separable metric space M. Let $\psi_-, \psi_+ \colon M \to [-\infty, \infty]$ be Borel-measurable functions such that the set*

$$K \coloneqq \{v \in V \mid \psi_- \le v \le \psi_+ \ \mu\text{-a.e. in } M\}$$

is non-empty. Then, K is polyhedric at every point.

With the polyhedricity at hand, let us state the key result regarding the differentiability of the solution mapping associated with (2.37). As mentioned, the result was already a part of the seminal work [99]. The developed techniques and certain extensions can, for instance, be found in [41, Chapter 2]. In particular, the following result is a special case of [41, Corollary 3.4.3] and is obtained by combining [41, Theorem 3.3.5] with Theorem 2.11 and the fact that $H_0^1(\Omega)$ endowed with the form in (2.38) is a regular L^2-Dirichlet space.

Corollary 2.12. *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be non-empty and open. Then, the solution mapping $S\colon L^2(\Omega) \to H_0^1(\Omega), f \mapsto u$ associated with the obstacle problem in (2.37) is Hadamard-directionally (in particular Gâteaux) differentiable. The directional derivative $d := S'(f,g)$ in $f \in L^2(\Omega)$ into a direction $g \in L^2(\Omega)$ is uniquely characterized by the variational inequality*

$$\begin{cases} \text{Find } d \in \mathcal{T}_K(u) \cap N(f+\Delta u), \ \text{s.t.} \\ \displaystyle\int_\Omega \nabla d \cdot \nabla (z-d)\,\mathrm{d}x \geq \int_\Omega g(z-d)\,\mathrm{d}x \quad \forall z \in \mathcal{T}_K(u) \cap N(f+\Delta u). \end{cases}$$

Here, the element $f + \Delta u$ is to be interpreted as an element of $H^{-1}(\Omega)$.

We want to underline that the employed techniques for Corollary 2.12, i.e., polyhedricity in regular L^2-Dirichlet spaces, very specifically use

1. the structure of the Sobolev space $H_0^1(\Omega)$ in the sense of Stampacchia's lemma
2. the structure of the underlying obstacle set $K \subset H_0^1(\Omega)$.

In this thesis, we are primarily concerned with Maxwell-structured problems based on the space $\boldsymbol{H}(\mathbf{curl})$ containing vector-valued functions. Here, it is not clear how the concept of regular L^2-Dirichlet spaces generalizes to the vector context. In particular, the minimum or maximum operation as in condition (iii) of Definition 2.8 would have to be replaced. Besides difficulties within the notion of L^2-Dirichlet spaces in this case, problems also arise in terms of polyhedricity in the vector-valued case, even more so when combined with an obstacle set featuring a first-order differential constraint. As shown in [41, Theorem 3.47], if $n \geq 3$ and if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, the set

$$K_\nabla := \{v \in H_0^1(\Omega) \mid |\nabla v(x)|_\infty \leq 1 \text{ a.e. on } \Omega\}$$

is not polyhedric in $H_0^1(\Omega)$. In fact, using that $\nabla\colon H_0^1(\Omega) \to \nabla H_0^1(\Omega)$ defines an isomorphism, the non-polyhedricity of K_∇ in $H_0^1(\Omega)$ is equivalent (see [41, Theorem 3.48]) to the non-polyhedricity of

$$\nabla K_\nabla = \{w \in \nabla H_0^1(\Omega) \mid |w(x)|_\infty \leq 1 \text{ a.e. on } \Omega\}$$

in $L^2(\Omega)^n$. We may conclude that, in the vector-valued case, even when considering a constraint with respect to the maximums norm $|\cdot|_\infty$, polyhedricity can in general not be expected.

2.3 Variational Inequalities in Electromagnetism

The focus of this section is to combine the previously introduced variational inequality character with Maxwell's equations. As a result, one obtains a class of variational inequalities with applications in electromagnetism.

Years after the first investigation of Maxwell variational inequalities (MVIs), the study of MVIs has gained more and more attention due to their paramount applications in superconductivity (see [19, 31, 78, 86, 120, 121, 141]). Miranda et al. [105] established a general framework for the well-posedness of parabolic MVIs and Maxwell quasi-variational inequalities. While the aforementioned contributions are primarily devoted to the well-posedness analysis, numerical methods for MVIs were proposed and analyzed in [49, 136, 137].

Due to the involved complexity in MVIs, a unified treatment of both first and second kind VIs seems to have its limits (cf. [143]), so that it is worth studying the two cases separately. Since, throughout this thesis, we will mostly address VIs of the first kind with an obstacle-type structure, let us use this section to give a basic introduction to the topic and recall some of the important results that are available.

2.3.1 Maxwell VIs of the First Kind

The main application of MVIs of the first kind lies in the shielding of electromagnetic (EM) fields. Electromagnetic shielding is a physical process of redirecting or reducing electromagnetic fields by conductive or magnetic materials. For instance, obstacles made out of conductive materials can be used to block or redirect electric fields. This physical phenomenon was discovered in 1836 by Michael Faraday, who experimentally verified that a conductive enclosure is able to eliminate the effect of an external electric field by charge cancelation on the boundary and leaving a zero field inside. Such an effect is also known under the term Faraday cage. Faraday cage effects can also be treated by homogenization techniques (see [39,95] for electrostatic Faraday cage models), which are however not the focus here. Further, ferromagnetic materials with high magnetic permeability (cobalt, nickel, etc.) can realize magnetic shielding by diverting the magnetic flux to another path. The mentioned examples belong either to electric or magnetic shielding but not to both at the same time. We shall address later that, mathematically speaking, the problem of blocking or redirecting both electric and magnetic field at the same time is rather delicate.

Nowadays, EM shielding is indispensable in many technological and daily applications, including microwave ovens, mobile phones, aircraft, magnetic resonance imaging, circuits, semiconductor chips, and many other electronic devices. In fact, EM shielding is utterly required in every application demanding the reduction of undesired electromagnetic interference.

The main focus of this thesis is on Maxwell VIs of the first kind. Here, the PDE-models under consideration are:

1. The evolutionary obstacle problem, that is, we arrive at a hyperbolic variational inequality of the first kind with a general electric obstacle set. This case particularly includes bilateral constraints on the electric field.

2. An $\boldsymbol{H}(\mathbf{curl})$-quasilinear first kind variational inequality with a bilateral differential **curl**-constraint.

As pointed out in the introduction, the very first contribution to this research direction was made by Duvaut and Lions, who explored and analyzed the electromagnetic wave propagation in a polarizable medium through a Maxwell obstacle problem involving an electric constraint of the type

$$|\boldsymbol{E}(x,t)| \leq d(x) \quad \text{a.e. in } \Omega \times (0,T)$$

for some obstacle $d\colon \Omega \to [0,\infty]$. This leads to the very specific underlying (electric) feasible set

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{v}(x)| \leq d(x) \quad \text{for a.e. } x \in \Omega\}.$$

To formulate the hyperbolic Maxwell obstacle problem introduced by Duvaut and Lions, suppose that $\Omega \subset \mathbb{R}^3$ is an open set representing an anisotropic medium in which the electric field

$\boldsymbol{E}\colon (0,T)\times\Omega \to \mathbb{R}^3$ and the magnetic field $\boldsymbol{H}\colon (0,T)\times\Omega \to \mathbb{R}^3$ are acting in a finite time interval $(0,T)$. Given initial data $(\boldsymbol{E}_0,\boldsymbol{H}_0)\in(\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl}))\times\boldsymbol{H}(\mathbf{curl})$ and an applied current source $\boldsymbol{f}\in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega))$, find a unique solution

$$(\boldsymbol{E},\boldsymbol{H})\in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega))\cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega))$$

such that

$$\begin{cases}\displaystyle\int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))+\sigma\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))-\boldsymbol{H}(t)\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x\\ \displaystyle\geq\int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x\quad\forall\boldsymbol{v}\in\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl})\text{ for a.e. }t\in(0,T)\\ \displaystyle\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t)+\mathbf{curl}\,\boldsymbol{E}(t)=0\quad\text{for a.e. }t\in(0,T)\\ \boldsymbol{E}(t)\in\boldsymbol{K}\quad\text{for a.e. }t\in(0,T)\\ (\boldsymbol{E},\boldsymbol{H})(0)=(\boldsymbol{E}_0,\boldsymbol{H}_0).\end{cases}\tag{2.39}$$

Here, $\epsilon,\mu,\sigma\colon\Omega\to\mathbb{R}^{3\times3}$ denote, respectively, the electric permittivity, the magnetic permeability, and the electric conductivity. All these coefficients are allowed to be non-smooth. Moreover, as the medium Ω may contain different conducting and non-conducting materials, the electric conductivity σ is assumed to be merely positive semi-definite.

Note that (2.39) preserves the Faraday law but modifies the Ampère-Maxwell equation into a variational inequality of the first kind. Based on the method of vanishing **curl**-**curl**-viscosity and constraint penalization and under rather strong assumptions on the initial value, Duvaut and Lions proved a well-posedness result [48, Chapter 7, Theorem 8.1] which was modified some years later by Milani [101, 102] to the case of a time dependent obstacle $d = d(x,t)$. For the problem (2.39) under the weaker assumptions from Assumption 3.1, a well-posedness result is available in [144, Theorems 1 and 2].[1] As we shall see later, the well-posedness of (2.39) is also a special case of one of our results from the next chapter (see Theorem 3.6). More recently, building on [143], Yousept [144] generalized the result by Duvaut and Lions [48] to allow a more general constraint structure $\boldsymbol{K}\subset\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)$, that is, the variational inequality character is also present in Faraday's law, leading to the problem

$$\begin{cases}\displaystyle\int_0^T\int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))+\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t)\cdot(\boldsymbol{w}-\boldsymbol{H}(t))-\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{v}+\boldsymbol{E}(t)\cdot\mathbf{curl}\,\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t\\ \displaystyle\geq\int_0^T\int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))+\boldsymbol{g}(t)\cdot(\boldsymbol{w}-\boldsymbol{H}(t))\,\mathrm{d}x\,\mathrm{d}t\\ \text{for all }(\boldsymbol{v},\boldsymbol{w})\in L^2((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl}))\text{ with }(\boldsymbol{v},\boldsymbol{w})(t)\in\boldsymbol{K}\text{ for a.e. }t\in(0,T)\\ (\boldsymbol{E},\boldsymbol{H})(t)\in\boldsymbol{K}\quad\text{for a.e. }t\in(0,T)\\ (\boldsymbol{E},\boldsymbol{H})(0)=(\boldsymbol{E}_0,\boldsymbol{H}_0).\end{cases}\tag{2.40}$$

Note that this VI formulation is weaker as it involves time dependent test functions and an averaging of the involved inequality in time. In particular, as we will see in the next theorem, this generalized structure, possibly assuming a simultaneous obstacle for both electric and magnetic field, leads to a severe loss of regularity. Nevertheless, assuming that the domain contains free electric and magnetic regions, partial regularity can be recovered. In this context, open subsets

[1] At least for the case $\sigma = 0$. However, the extension to the case of a non-vanishing conductivity σ which is positive semi-definite is not an issue.

$\Omega_E, \Omega_H \subset \Omega$ are electric (resp. magnetic) free regions, if it holds that

$$(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K} \quad \Rightarrow \quad (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}) \in \boldsymbol{K} \quad \forall \tilde{\boldsymbol{v}} = \begin{cases} \boldsymbol{v}_E & \text{in } \Omega_E \\ \boldsymbol{v} & \text{in } \Omega \setminus \Omega_E \end{cases} \quad \forall \tilde{\boldsymbol{w}} = \begin{cases} \boldsymbol{w}_H & \text{in } \Omega_H \\ \boldsymbol{w} & \text{in } \Omega \setminus \Omega_H \end{cases} \tag{2.41}$$

for any $(\boldsymbol{v}_E, \boldsymbol{v}_H) \in \boldsymbol{L}^2(\Omega_E) \times \boldsymbol{L}^2(\Omega_H)$. For the next theorem we refer to [144, Theorem 1].

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^3$ be an open set and let $\boldsymbol{K} \subset \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ be a closed and convex set containing $(0,0)$. Moreover, let the material parameters $\epsilon, \mu \in L^\infty_{\text{sym}}(\mathcal{O})^{3\times 3}$ be uniformly positive definite. Then, for every right-hand side $(\boldsymbol{f}, \boldsymbol{g}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))$ and every initial value $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})) \cap \boldsymbol{K}$, the general Maxwell obstacle problem* (2.40) *admits a (not necessarily unique) solution $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))$. If the obstacle set $\boldsymbol{K} \subset \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ satisfies* (2.41) *for open subsets $\Omega_E, \Omega_H \subset \Omega$, then every solution to* (2.40) *satisfies*

$$\boldsymbol{E}_{|\Omega_H} \in L^\infty((0,T), \boldsymbol{H}(\mathbf{curl}, \Omega_H)) \quad \text{and} \quad \boldsymbol{H}_{|\Omega_E} \in L^\infty((0,T), \boldsymbol{H}(\mathbf{curl}, \Omega_E))$$

and is a solution to the classical Ampère-Maxwell equation in Ω_E and the Faraday equation in Ω_H, that is,

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{H} = \boldsymbol{f} & \text{a.e. in } (0,T) \times \Omega_E \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = \boldsymbol{g} & \text{a.e. in } (0,T) \times \Omega_H. \end{cases}$$

If additionally no obstacle is present for the magnetic field $\boldsymbol{H}$, i.e., it holds that $\Omega_H = \Omega$, then $\boldsymbol{E} \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}))$.

Note that the previous theorem generalizes the well-known result by Duvaut and Lions in the sense that (2.40) coincides with (2.39) when there is no obstacle for the magnetic field $\boldsymbol{H}$. In this circumstance, the identical regularity for the electric field is achieved, namely that $\boldsymbol{E} \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}))$. As it is pointed out in [144], the general case leads to a loss of both the electric boundary condition and any additional spatial regularity. In particular, classical energy arguments as typically used to show uniqueness are not applicable here. As a remedy, hinging on a specific splitting assumption on the respective free regions Ω_E and Ω_H, the author in [144] shows that a uniqueness result can be achieved. Especially, uniqueness holds in the simpler setting $\Omega_H = \Omega$ or $\Omega_E = \Omega$.

For the issues mentioned, throughout this thesis, the remainder of this thesis will be concerned with problems in electric *or* magnetic shielding. We will see in Chapter 4 that already for the case of shielding in the electric field, that is $\Omega_H = \Omega$, the loss of $\boldsymbol{H}(\mathbf{curl})$-regularity of the magnetic field makes the required techniques for the numerical analysis genuinely non-standard.

CHAPTER 3

EDDY CURRENT APPROXIMATION IN MAXWELL OBSTACLE PROBLEMS

In this chapter, we aim to explore the eddy current (magneto-quasistatic) approximation and its justification for the hyperbolic Maxwell obstacle problem with shielding for the electric field. For the convenience of the reader, let us recall the problem formulation of the Maxwell obstacle problem. It reads to find a unique solution

$$(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega))$$

to the system

$$\begin{cases} \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + \sigma\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl}) \text{ for a.e. } t\in(0,T) \\ \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t\in(0,T) \\ \boldsymbol{E}(t)\in\boldsymbol{K} \quad \text{for a.e. } t\in(0,T) \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{P}$$

Here, $\boldsymbol{K}\subset\boldsymbol{L}^2(\Omega)$ represents a general obstacle set, which does not necessarily depend on a specific obstacle function $d\colon\Omega\to[0,\infty]$. The precise mathematical assumptions for all data involved in (P) (and more generally its upcoming eddy current approximation (P_{ec})) are specified in Assumption 3.1.

Our analysis is mainly motivated by the profound role of eddy current modeling in electrical engineering applications and low-frequency physics. Generally speaking, the eddy current model approximates the full Maxwell system by excluding the displacement current $\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}$ but still preserving the Faraday law. Such approximations are widely used in the engineering community and particularly reasonable if the electric permittivity is significantly smaller than the electric conductivity, and the corresponding wavelength is much larger than the diameter of Ω. From among many other contributions to the eddy current model, we refer to the monographs by Alonso and Valli [7], Bossavit [30,32], and the papers [3,4,13,16,46,71,73,103,110,128,129,139]. While the mathematical and numerical analysis for the eddy current equations seems to have reached an advanced stage of development, so far, we are not aware of any previous study regarding the justification of eddy current modeling for (P).

We focus on an eddy current model allowing the displacement current to be disregarded in an open conducting subregion $\Omega_\sigma\subset\Omega$. More precisely, we look for a unique solution

$$(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega))$$

to

$$\begin{cases} \int_{\Omega\setminus\Omega_\sigma} \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x + \int_\Omega \sigma\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl}) \text{ for a.e. } t\in(0,T) \\ \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t\in(0,T) \\ \boldsymbol{E}(t)\in\boldsymbol{K} \quad \text{for a.e. } t\in(0,T) \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0) \quad \text{a.e. in } (\Omega\setminus\Omega_\sigma)\times\Omega. \end{cases} \tag{$\mathrm{P_{ec}}$}$$

To justify the eddy current model ($\mathrm{P_{ec}}$), there are two open mathematical questions to be rigorously addressed and answered. First, the model itself ($\mathrm{P_{ec}}$) has to be reasonable in the sense that there exists a unique solution to ($\mathrm{P_{ec}}$). Second, under a suitable condition, its solution must provide a good estimation for the original problem (P). In particular, inspired from the time-harmonic case [7], ($\mathrm{P_{ec}}$) should serve as a reasonable approximation if the quantity $\underline{\sigma}^{-1}\,\|\epsilon\|_{L^\infty(\Omega_\sigma)^{3\times3}}$ is small enough with $\underline{\sigma}>0$ denoting a uniform lower bound for the lowest eigenvalues of $\sigma(x)$ for almost all $x\in\Omega_\sigma$.

We develop three novelties delivering positive answers to the issues mentioned above. The first novelty concerns the a priori analysis for the time-discrete approximation (P_N) of ($\mathrm{P_{ec}}$) based on the Rothe method. Here, our analysis hinges on the mild compatibility assumption (3.5) for the initial data $(\boldsymbol{E}_0,\boldsymbol{H}_0)$ in the subset Ω_σ. With the compatibility condition, we prove the stability of (P_N) (see Theorem 3.5) through the use of special correction terms developed using the variational inequality structure of ($\mathrm{P_{ec}}$). Then, the analysis for the time-discrete scheme (P_N) allows us to establish a well-posedness result for ($\mathrm{P_{ec}}$) as the second novelty (Theorem 3.6). To be more precise, applying the stability result to a specific interpolation of (P_N) and passing to the limit in the time-discretization, the weak-star limit of the interpolation turns out to satisfy ($\mathrm{P_{ec}}$) leading to an existence result for ($\mathrm{P_{ec}}$). We note that the standard technique of passing to the weak-star limit of the piecewise linear interpolations fails to work, as (P_N) does not admit sufficient stability of its solutions in Ω_σ. This difficulty is overcome by considering the weak-star limit of the piecewise constant interpolations. Let us point out that ($\mathrm{P_{ec}}$) does not exclude the displacement current $\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}$ in $\Omega\setminus\Omega_\sigma$. In Section 3.2.1, we address the case ($\mathrm{P^0_{ec}}$) where the displacement current is entirely neglected both in the conducting and nonconducting regions (Ω_σ and $\Omega\setminus\Omega_\sigma$). It turns out that the proposed techniques for ($\mathrm{P_{ec}}$) can be extended to ($\mathrm{P^0_{ec}}$), leading to a well-posedness result for ($\mathrm{P^0_{ec}}$) (see Theorem 3.12) under additional assumptions (Assumption 3.8). Our final result is the justification for ($\mathrm{P_{ec}}$) (see Theorem 3.17): If $|\Omega\setminus\Omega_\sigma|\neq0$, then the solution $(\boldsymbol{E}_{\mathrm{ec}},\boldsymbol{H}_{\mathrm{ec}})$ of ($\mathrm{P_{ec}}$) approximates the solution $(\boldsymbol{E},\boldsymbol{H})$ to (P) through the following uniform a priori estimate:

$$\begin{aligned} &\|(\boldsymbol{E},\boldsymbol{H})-(\boldsymbol{E}_{\mathrm{ec}},\boldsymbol{H}_{\mathrm{ec}})\|_{\mathcal{C}([0,T],\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2_\mu(\Omega))} + \|\boldsymbol{E}-\boldsymbol{E}_{\mathrm{ec}}\|_{L^2((0,T),\boldsymbol{L}^2_\sigma(\Omega_\sigma))} \\ &\quad\leq 2\left(\frac{L(\Omega_\sigma)^2T}{\underline{\sigma}} + \frac{2L(\Omega\setminus\Omega_\sigma)T}{\sqrt{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}}\sqrt{\frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}+\frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}}\right)^{1/2}\left\|\frac{\epsilon}{\underline{\sigma}}\right\|_{L^\infty(\Omega_\sigma)^{3\times3}}, \end{aligned} \tag{3.1}$$

where $L(\Omega_\sigma)>0$ (resp. $L(\Omega\setminus\Omega_\sigma)>0$) stands for the Lipschitz constant of $\boldsymbol{f}_{|\Omega_\sigma}$ (resp. $\boldsymbol{f}_{|\Omega\setminus\Omega_\sigma}$), and $\underline{\epsilon}(\Omega\setminus\Omega_\sigma)>0$ denotes a uniform lower bound for the lowest eigenvalues of $\epsilon(x)$ for almost all $x\in\Omega\setminus\Omega_\sigma$. If $\Omega_\sigma=\Omega$, i.e., if the displacement current is completely removed in the conducting medium Ω, then the following precision is obtained for the eddy current approximation:

$$\|\boldsymbol{H}-\boldsymbol{H}_{\mathrm{ec}}\|_{\mathcal{C}([0,T],\boldsymbol{L}^2_\mu(\Omega))} + \|\boldsymbol{E}-\boldsymbol{E}_{\mathrm{ec}}\|_{L^2((0,T),\boldsymbol{L}^2_\sigma(\Omega))} \leq \frac{2L(\Omega)\sqrt{T}}{\sqrt{\underline{\sigma}}}\left\|\frac{\epsilon}{\underline{\sigma}}\right\|_{L^\infty(\Omega)^{3\times3}}. \tag{3.2}$$

We emphasize that in many electromagnetic applications (cf. [7, 93]), the ratio $\|\epsilon/\underline{\sigma}\|_{L^\infty(\Omega_\sigma)^{3\times 3}}$ is often negligibly small. For instance, stainless steel and copper admits, respectively, the value $6.14\cdot 10^{-18}$ and $1.56\cdot 10^{-19}$ for the corresponding ratio. This property is in particular satisfied by every good conductor Ω_σ (see [93]) as the electric permittivity ϵ is in this case very close to the one in vacuum $\approx 8.85\cdot 10^{-12}$, and the electric conductivity σ is in the order of 10^6-10^7. Therefore, the achieved estimation reveals the desired approximation by the eddy current solution with a specific bound for the smallness condition on the quantity $\|\epsilon/\underline{\sigma}\|_{L^\infty(\Omega_\sigma)^{3\times 3}}$. At the same time, it guarantees the strong convergence of (P_{ec}) towards (P) with a linear convergence rate in terms of $\|\epsilon\|_{L^\infty(\Omega_\sigma)^{3\times 3}}$. Last but not least, all theoretical results, in particular the uniform estimates (3.1) and (3.2), apply as well to the classical Maxwell equations by simply considering $\boldsymbol{K} = \boldsymbol{L}^2(\Omega)$.

Another important application of the eddy current model arises in the context of type-II superconductivity. The corresponding model leads to parabolic obstacle problems with first-order gradient or **curl**-constraints. We refer to [31, 49, 104, 120, 136, 141, 142] for contributions in this research direction. More recently, a unified analysis for nonlinear parabolic obstacle problems, including those with **curl**-type constraints, has been recently developed by Miranda et al. [105].

The remainder of this chapter is organized as follows. After presenting the required assumptions for our analysis, we introduce the formulation of the time-discrete scheme together with its associated a priori stability analysis. Section 3.2 is devoted to the existence and uniqueness analysis for (P_{ec}). Thereafter, in Section 3.3, we prove Theorem 3.17 for the justification of the eddy current model, and the final section features a numerical test verifying the a priori estimate and the predicted convergence rate from Theorem 3.17.

Let us now present the mathematical assumptions for our analysis.

Assumption 3.1.

(A3.1) Suppose that $\Omega \subset \mathbb{R}^3$ is open and contains a given (possibly empty) open subset $\Omega_\sigma \subset \Omega$.

(A3.2) The electric permittivity $\epsilon\colon \Omega \to \mathbb{R}^{3\times 3}$ and the magnetic permeability $\mu\colon \Omega \to \mathbb{R}^{3\times 3}$ are of class $L^\infty_{\mathrm{sym}}(\Omega)^{3\times 3}$ and uniformly positive definite, i.e., there exist constants $\underline{\epsilon}, \underline{\mu} > 0$ such that

$$\epsilon(x)\xi\cdot\xi \geq \underline{\epsilon}|\xi|^2 \quad \text{and} \quad \mu(x)\xi\cdot\xi \geq \underline{\mu}|\xi|^2 \quad \text{for a.e. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^3. \tag{3.3}$$

The electric conductivity $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}$ is of class $L^\infty_{\mathrm{sym}}(\Omega)^{3\times 3}$ and positive semi-definite. Furthermore, it is uniformly positive definite on Ω_σ, i.e., there exists a constant $\underline{\sigma} > 0$ such that

$$\sigma(x)\xi\cdot\xi \geq \underline{\sigma}|\xi|^2 \quad \text{for a.e. } x\in\Omega_\sigma \text{ and all } \xi\in\mathbb{R}^3. \tag{3.4}$$

(A3.3) The obstacle set $\boldsymbol{K} \subset \boldsymbol{L}^2(\Omega)$ is assumed to be closed and convex containing 0.

(A3.4) The applied current source fulfills $\boldsymbol{f} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$ with Lipschitz constant $L \geq 0$.

(A3.5) The initial value satisfies $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ and

$$\int_{\Omega_\sigma} (\sigma\boldsymbol{E}_0 - \mathbf{curl}\,\boldsymbol{H}_0)\cdot(\boldsymbol{v}-\boldsymbol{E}_0)\,\mathrm{d}x \geq \int_{\Omega_\sigma} \boldsymbol{f}(0)\cdot(\boldsymbol{v}-\boldsymbol{E}_0)\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl}). \tag{3.5}$$

Remark 3.2. The condition (3.5) is obviously satisfied if

$$\sigma\boldsymbol{E}_0 - \mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{f}(0) \quad \text{a.e. in } \Omega_\sigma. \tag{3.6}$$

If (3.6) fails to hold, then (3.5) may still be valid. For instance, if the feasible set $\boldsymbol{K}$ fulfills

$$\boldsymbol{e} \in \boldsymbol{K} \quad \Rightarrow \quad \boldsymbol{e}(x) = 0 \quad \text{for a.e. } x \in \Omega_\sigma, \tag{3.7}$$

then the condition (3.5) is satisfied for all $\boldsymbol{f}(0) \in \boldsymbol{L}^2(\Omega)$ and all $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$. Note that (3.7) is highly relevant to the physical phenomenon of electric shielding such as the Faraday cage to block the effects of external electric fields in the material Ω_σ. Another example is the case where $\boldsymbol{K}$ only permits a certain feasible direction of the electric field in Ω_σ such as

$$\boldsymbol{e} \in \boldsymbol{K} \quad \Rightarrow \quad e_1(x) \geq 0 \quad \text{and} \quad e_2(x) = e_3(x) = 0 \quad \text{for a.e. } x \in \Omega_\sigma. \tag{3.8}$$

If $\boldsymbol{E}_0(x) = 0$ holds for a.e. $x \in \Omega_\sigma$, and the first components of $\mathbf{curl}\,\boldsymbol{H}_0$ and $\boldsymbol{f}(0)$ are non-positive a.e. in Ω_σ, then the condition (3.5) is satisfied in the case of (3.8).

Lastly, we note that in the case (P), that is to say $\Omega_\sigma = \emptyset$, (3.5) is always satisfied since all integrals over Ω_σ vanish. In particular, for (P) there is no restriction on the initial value other than $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$.

3.1 Analysis of the Time-discrete Approximation to (P_{ec})

This section is devoted to the analysis of the time-discrete approximation to (P_{ec}) based on the Rothe method. Let us begin by introducing an equidistant partition of the time interval $[0, T]$ as follows: Given $N \in \mathbb{N}$, we set

$$\tau := \frac{T}{N}, \quad 0 = t_0 < t_1 < \ldots < t_N = T \quad \text{with} \quad t_n := n\tau, \quad n \in \{0, \ldots, N\}.$$

Furthermore, we introduce the backward Euler difference quotients

$$\delta \boldsymbol{E}_n := \frac{\boldsymbol{E}_n - \boldsymbol{E}_{n-1}}{\tau}, \quad \delta \boldsymbol{H}_n := \frac{\boldsymbol{H}_n - \boldsymbol{H}_{n-1}}{\tau} \quad \forall n \in \{1, \ldots, N\} \tag{3.9}$$

and set $\boldsymbol{f}_n := \boldsymbol{f}(t_n) \in \boldsymbol{L}^2(\Omega)$ for all $n \in \{0, \ldots, N\}$. Invoking these quantities, the time-discrete (Euler) approximation to (P_{ec}) reads as follows: Find $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ such that

$$\begin{cases} \displaystyle\int_{\Omega \setminus \Omega_\sigma} \epsilon \delta \boldsymbol{E}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x + \int_\Omega \sigma \boldsymbol{E}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n) - \boldsymbol{H}_n \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x \\ \displaystyle\geq \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \quad \forall n \in \{1, \ldots, N\} \\ \mu \delta \boldsymbol{H}_n + \mathbf{curl}\,\boldsymbol{E}_n = 0 \quad \forall n \in \{1, \ldots, N\}. \end{cases} \tag{P_N}$$

To derive an existence and uniqueness result for (P_N), let us consider a bounded and coercive bilinear form

$$a\colon \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}, \quad (\boldsymbol{E}, \boldsymbol{v}) \mapsto \int_{\Omega \setminus \Omega_\sigma} \epsilon \boldsymbol{E} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega \tau \sigma \boldsymbol{E} \cdot \boldsymbol{v} + \tau^2 \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x$$

and bounded linear forms

$$F_n\colon \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}, \quad \boldsymbol{v} \mapsto \int_\Omega \tau \boldsymbol{f}_n \cdot \boldsymbol{v} + \tau \boldsymbol{H}_{n-1} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} \epsilon \boldsymbol{E}_{n-1} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall n \in \{1, \ldots, N\}.$$

In view of (3.9), (P_N) is equivalent to the problem of successively finding elements

$$(\boldsymbol{E}_1, \boldsymbol{H}_1), \ldots, (\boldsymbol{E}_N, \boldsymbol{H}_N) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$$

such that

$$a(\boldsymbol{E}_n, \boldsymbol{v} - \boldsymbol{E}_n) \geq F_n(\boldsymbol{v} - \boldsymbol{E}_n) \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \quad \text{and} \quad \boldsymbol{H}_n = -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_n + \boldsymbol{H}_{n-1}.$$

The well-posedness of (P_N) therefore follows Theorem 2.6, which we summarize in the following lemma:

Lemma 3.3. *Let Assumption 3.1 hold. Then, for every $N \in \mathbb{N}$, the time-discrete problem (P_N) admits a unique solution $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$.*

In the upcoming theorem, we prove our first main result on the stability for (P_N). For the convenience of the reader, we recall Gronwall's lemma in its discrete version in the following auxiliary lemma (see [44, p. 280]).

Lemma 3.4. *Let $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ be sequences of nonnegative real numbers satisfying*

$$a_n \leq c + \sum_{k=0}^{n-1} a_k b_k \quad \forall n \in \mathbb{N}$$

for some constant $c > 0$. Then, it holds that

$$a_n \leq c \exp\left(\sum_{k=0}^{n-1} b_k\right) \quad \forall n \in \mathbb{N}.$$

As pointed out in the introduction, our upcoming stability proof is based on the use of certain correction terms for the initial data $\boldsymbol{z} \in \boldsymbol{L}^2(\Omega \setminus \Omega_\sigma)$ and $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ as follows:

$$\boldsymbol{z} := \epsilon\boldsymbol{E}_0 + \sigma\boldsymbol{E}_0 - \mathbf{curl}\,\boldsymbol{H}_0 - \boldsymbol{f}_0 \quad \text{a.e. on } \Omega \setminus \Omega_\sigma \quad \text{and} \quad \boldsymbol{w} := \mu\boldsymbol{H}_0 + \mathbf{curl}\,\boldsymbol{E}_0 \quad \text{a.e. on } \Omega. \tag{3.10}$$

Theorem 3.5. *Let Assumption 3.1 hold. Then, there exists a positive real constant C, depending only on $T, \epsilon, \mu, \sigma, \boldsymbol{f}, \boldsymbol{E}_0, \boldsymbol{H}_0$, such that, for every $N \in \mathbb{N}$ the unique solution $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ of (P_N) satisfies*

$$\max_{1 \leq n \leq N} \Big[\|\boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)} + \|\delta\boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} + \|\delta\boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega)} \Big] \leq C. \tag{3.11}$$

Proof. Let $N \in \mathbb{N}$ be arbitrarily fixed, and let $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ denote the unique solution to (P_N). Let now $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ be arbitrarily fixed. Multiplying the above equation for $\boldsymbol{z}$ with $\boldsymbol{v} - \boldsymbol{E}_0$ and integrating the resulting equality over $\Omega \setminus \Omega_\sigma$, we obtain that

$$\int_{\Omega\setminus\Omega_\sigma} \epsilon\boldsymbol{E}_0 \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\sigma} (\sigma\boldsymbol{E}_0 - \mathbf{curl}\,\boldsymbol{H}_0) \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x = \int_{\Omega\setminus\Omega_\sigma} (\boldsymbol{f}_0 + \boldsymbol{z}) \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x.$$

Then, combining the above equality with (3.5), it follows that

$$\int_{\Omega\setminus\Omega_\sigma} \epsilon\boldsymbol{E}_0 \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x + \int_{\Omega} (\sigma\boldsymbol{E}_0 - \mathbf{curl}\,\boldsymbol{H}_0) \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x \geq \int_{\Omega} \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\sigma} \boldsymbol{z} \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x,$$

and consequently applying the characterization from Lemma 2.1, it follows that the initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ satisfy

$$\begin{cases} \displaystyle\int_{\Omega\setminus\Omega_\sigma} \epsilon\boldsymbol{E}_0 \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x + \int_{\Omega} \sigma\boldsymbol{E}_0 \cdot (\boldsymbol{v} - \boldsymbol{E}_0) - \boldsymbol{H}_0 \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x \\ \displaystyle\geq \int_{\Omega} \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\sigma} \boldsymbol{z} \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \\ \mu\boldsymbol{H}_0 + \mathbf{curl}\,\boldsymbol{E}_0 = \boldsymbol{w}. \end{cases} \tag{3.12}$$

Now, the underlying system (3.12) allows us to incorporate the initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0)$ to the time-discrete scheme (P_N) and preserve its pivotal structure for our stability analysis. To realize this, we employ the quantities

$$\delta \boldsymbol{E}_0 \coloneqq \boldsymbol{E}_0, \qquad \delta \boldsymbol{H}_0 \coloneqq \boldsymbol{H}_0, \tag{3.13}$$
$$\boldsymbol{z}_n^N \coloneqq \begin{cases} \boldsymbol{z} & \text{if } n = 0 \\ 0 & \text{if } n \in \{1, \dots, N\}, \end{cases} \qquad \boldsymbol{w}_n^N \coloneqq \begin{cases} \boldsymbol{w} & \text{if } n = 0 \\ 0 & \text{if } n \in \{1, \dots, N\}, \end{cases}$$

and deduce from (3.12) that the unique solution to (P_N) fulfills

$$\begin{cases} \int_{\Omega \setminus \Omega_\sigma} \epsilon \delta \boldsymbol{E}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n) \,\mathrm{d}x + \int_\Omega \sigma \boldsymbol{E}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n) - \boldsymbol{H}_n \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}_n) \,\mathrm{d}x \\ \geq \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n) \,\mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} \boldsymbol{z}_n^N \cdot (\boldsymbol{v} - \boldsymbol{E}_n) \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \\ \forall n \in \{0, \dots, N\} \\ \mu \delta \boldsymbol{H}_n + \mathbf{curl}\, \boldsymbol{E}_n = \boldsymbol{w}_n^N \quad \forall n \in \{0, \dots, N\}. \end{cases} \tag{3.14}$$

For every $n \in \{1, \dots, N\}$, setting $\boldsymbol{v} = \boldsymbol{E}_{n-1}$ (resp. $\boldsymbol{v} = \boldsymbol{E}_n$) in the n-th inequality of (3.14) (resp. the $(n-1)$-th inequality of (3.14)) and then dividing the resulting inequalities by $-\tau$, we obtain

$$\int_{\Omega \setminus \Omega_\sigma} \epsilon \delta \boldsymbol{E}_n \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x + \int_\Omega \sigma \boldsymbol{E}_n \cdot \delta \boldsymbol{E}_n - \boldsymbol{H}_n \cdot \mathbf{curl}\, \delta \boldsymbol{E}_n \,\mathrm{d}x \leq \int_\Omega \boldsymbol{f}_n \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} \boldsymbol{z}_n^N \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x \tag{3.15}$$

and

$$-\int_{\Omega \setminus \Omega_\sigma} \epsilon \delta \boldsymbol{E}_{n-1} \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x - \int_\Omega \sigma \boldsymbol{E}_{n-1} \cdot \delta \boldsymbol{E}_n - \boldsymbol{H}_{n-1} \cdot \mathbf{curl}\, \delta \boldsymbol{E}_n \,\mathrm{d}x$$
$$\leq -\int_\Omega \boldsymbol{f}_{n-1} \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x - \int_{\Omega \setminus \Omega_\sigma} \boldsymbol{z}_{n-1}^N \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x. \tag{3.16}$$

On the other hand, the second equation in (3.14) yields that

$$\mathbf{curl}\, \delta \boldsymbol{E}_n = -\tau^{-1} \mu (\delta \boldsymbol{H}_n - \delta \boldsymbol{H}_{n-1}) + \tau^{-1} (\boldsymbol{w}_n^N - \boldsymbol{w}_{n-1}^N) \quad \forall n \in \{1, \dots, N\}. \tag{3.17}$$

Adding (3.15) and (3.16) together and utilizing (3.17) as well as the positive semi-definiteness of σ, we get

$$\int_{\Omega \setminus \Omega_\sigma} \epsilon (\delta \boldsymbol{E}_n - \delta \boldsymbol{E}_{n-1}) \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x + \int_{\Omega_\sigma} \sigma (\boldsymbol{E}_n - \boldsymbol{E}_{n-1}) \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x + \int_\Omega \mu (\delta \boldsymbol{H}_n - \delta \boldsymbol{H}_{n-1}) \cdot \delta \boldsymbol{H}_n \,\mathrm{d}x$$
$$\leq \int_\Omega (\boldsymbol{f}_n - \boldsymbol{f}_{n-1}) \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x + \int_{\Omega \setminus \Omega_\sigma} (\boldsymbol{z}_n^N - \boldsymbol{z}_{n-1}^N) \cdot \delta \boldsymbol{E}_n \,\mathrm{d}x + \int_\Omega (\boldsymbol{w}_n^N - \boldsymbol{w}_{n-1}^N) \cdot \delta \boldsymbol{H}_n \,\mathrm{d}x \tag{3.18}$$

for all $n \in \{1, \dots, N\}$. Note that given a Hilbert space H and $a_0, \dots, a_{n_0} \in H$ for $n_0 \in \mathbb{N}$, the binomial type formula

$$\sum_{n=1}^{n_0} (a_n - a_{n-1}, a_n)_H = \frac{1}{2} \left(\|a_{n_0}\|_H^2 - \|a_0\|_H^2 + \sum_{n=1}^{n_0} \|a_n - a_{n-1}\|_H^2 \right) \tag{3.19}$$

holds true. Let now $n_0 \in \{1, \dots, N\}$ be arbitrarily fixed, and we sum up the inequality (3.18) over $\{1, \dots, n_0\}$. Then, applying the binomial formula (3.19) along with Hölder's inequality and (3.4), it follows that

$$\frac{1}{2} \left(\|\delta \boldsymbol{E}_{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega \setminus \Omega_\sigma)} - \|\delta \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega \setminus \Omega_\sigma)} + \sum_{n=1}^{n_0} \|\delta \boldsymbol{E}_n - \delta \boldsymbol{E}_{n-1}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega \setminus \Omega_\sigma)} \right) \tag{3.20}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\|\delta \boldsymbol{H}_{n_0}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} - \|\delta \boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \sum_{n=1}^{n_0}\|\delta \boldsymbol{H}_n - \delta \boldsymbol{H}_{n-1}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right) + \sum_{n=1}^{n_0}\tau\underline{\sigma}\,\|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\\
&\le \sum_{n=1}^{n_0}\|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} + \sum_{n=1}^{n_0}\|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|_{\boldsymbol{L}^2(\Omega_\sigma)}\|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega_\sigma)}\\
&+\sum_{n=1}^{n_0}\left\|\boldsymbol{z}_n^N - \boldsymbol{z}_{n-1}^N\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} + \sum_{n=1}^{n_0}\left\|\boldsymbol{w}_n^N - \boldsymbol{w}_{n-1}^N\right\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)}\,.
\end{aligned}
$$

Using Young's inequality together with an estimate of the type $\underline{\alpha}\|\cdot\|^2_{\boldsymbol{L}^2(\Omega)} \le \|\cdot\|^2_{\boldsymbol{L}^2_\alpha(\Omega)}$ and the Lipschitz property of $\boldsymbol{f}$, the first and second terms in the right-hand side of (3.20) can be estimated by

$$
\begin{aligned}
&\sum_{n=1}^{n_0}\|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \qquad (3.21)\\
\le\ & \sum_{n=1}^{n_0}\left(\frac{N}{\underline{\epsilon}}\|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} + \frac{1}{4N}\|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\right)\\
\le\ & \sum_{n=1}^{n_0}\left(\frac{N}{\underline{\epsilon}}L^2\tau^2 + \frac{1}{4N}\|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\right)\\
\underbrace{\le}_{\tau=\frac{T}{N}}\ & \frac{L^2T^2}{\underline{\epsilon}} + \frac{1}{4}\|\delta \boldsymbol{E}_{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \sum_{n=1}^{n_0-1}\frac{1}{4N}\|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=1}^{n_0}\|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|_{\boldsymbol{L}^2(\Omega_\sigma)}\|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega_\sigma)} &\le \sum_{n=1}^{n_0}\left(\frac{1}{4\tau\underline{\sigma}}\|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} + \tau\underline{\sigma}\,\|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\right) \qquad (3.22)\\
&\le \frac{L^2T}{4\underline{\sigma}} + \sum_{n=1}^{n_0}\tau\underline{\sigma}\,\|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\,.
\end{aligned}
$$

For the remaining terms on the right-hand side of (3.20), we find by Young's inequality and the triangle inequality that

$$
\begin{aligned}
&\sum_{n=1}^{n_0}\left\|\boldsymbol{z}_n^N - \boldsymbol{z}_{n-1}^N\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \underbrace{=}_{(3.13)} \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \qquad (3.23)\\
&\le \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_1 - \delta \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} + \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\delta \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\\
&\le \frac{2}{\underline{\epsilon}}\|\boldsymbol{z}\|^2_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} + \frac{1}{4}\|\delta \boldsymbol{E}_1 - \delta \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{4}\|\delta \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\,,
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&\sum_{n=1}^{n_0}\left\|\boldsymbol{w}_n^N - \boldsymbol{w}_{n-1}^N\right\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)} \qquad (3.24)\\
\le\ & \frac{2}{\underline{\mu}}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{4}\|\delta \boldsymbol{H}_1 - \delta \boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{1}{4}\|\delta \boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\,.
\end{aligned}
$$

Applying (3.21)-(3.24) to (3.20) along with $\delta \boldsymbol{E}_0 = \boldsymbol{E}_0$ and $\delta \boldsymbol{H}_0 = \boldsymbol{H}_0$, it follows after some rearrangement that

$$
\frac{1}{4}\|\delta \boldsymbol{E}_{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{2}\|\delta \boldsymbol{H}_{n_0}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le \frac{L^2T^2}{\underline{\epsilon}} + \frac{L^2T}{4\underline{\sigma}} + \frac{3}{4}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \frac{3}{4}\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)}
$$

$$+\frac{2}{\underline{\epsilon}}\|\boldsymbol{z}\|^2_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\frac{2}{\underline{\mu}}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+\sum_{n=1}^{n_0-1}\frac{1}{4N}\|\delta\boldsymbol{E}_n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}.$$

By virtue of Lemma 3.4, we eventually deduce that

$$\|\delta\boldsymbol{E}_{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\|\delta\boldsymbol{H}_{n_0}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\leq C\exp\left(\sum_{n=1}^{n_0-1}\frac{1}{N}\right)\leq C\exp(1),$$

with a generic constant $C>0$, depending only on $T,L,\epsilon,\mu,\ \sigma,\boldsymbol{E}_0,\boldsymbol{H}_0$. In particular, it holds that

$$\|\delta\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\|\delta\boldsymbol{H}_{n_0}\|_{\boldsymbol{L}^2(\Omega)}\leq C. \tag{3.25}$$

From (3.25) and the reversed triangle inequality, it follows by the definition of the difference quotients (3.9) that

$$\begin{aligned}\|\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}_{n_0}\|_{\boldsymbol{L}^2(\Omega)}&\leq\tau C+\|\boldsymbol{E}_{n_0-1}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}_{n_0-1}\|_{\boldsymbol{L}^2(\Omega)}\\&\leq\cdots\leq n_0\tau C+\|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}\\&\leq TC+\|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{3.26}$$

Furthermore, the estimate

$$\|\mathbf{curl}\,\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega)}\leq C \tag{3.27}$$

immediately results from (3.25) along with the discrete Faraday law in (P_N). We are left with showing the estimate for $\boldsymbol{E}_{n_0}$ in $\boldsymbol{L}^2(\Omega_\sigma)$. To do so, we test with $\boldsymbol{v}=0$ in (P_N) and use the positive semi-definiteness of σ to obtain

$$\begin{aligned}&\int_{\Omega\setminus\Omega_\sigma}\epsilon\delta\boldsymbol{E}_{n_0}\cdot\boldsymbol{E}_{n_0}\,\mathrm{d}x+\int_{\Omega_\sigma}\sigma\boldsymbol{E}_{n_0}\cdot\boldsymbol{E}_{n_0}\,\mathrm{d}x-\int_\Omega\boldsymbol{H}_{n_0}\cdot\mathbf{curl}\,\boldsymbol{E}_{n_0}\,\mathrm{d}x\\&\leq\int_{\Omega\setminus\Omega_\sigma}\boldsymbol{f}_{n_0}\cdot\boldsymbol{E}_{n_0}\,\mathrm{d}x+\int_{\Omega_\sigma}\boldsymbol{f}_{n_0}\cdot\boldsymbol{E}_{n_0}\,\mathrm{d}x.\end{aligned} \tag{3.28}$$

Applying the estimate

$$\int_{\Omega_\sigma}\boldsymbol{f}_{n_0}\cdot\boldsymbol{E}_{n_0}\,\mathrm{d}x\leq\frac{1}{2\underline{\sigma}}\|\boldsymbol{f}_{n_0}\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}+\frac{\underline{\sigma}}{2}\|\boldsymbol{E}_{n_0}\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}$$

to (3.28) together with (3.4) and Hölder's inequality, we end up with

$$\begin{aligned}\frac{\underline{\sigma}}{2}\|\boldsymbol{E}_{n_0}\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\leq&\|\boldsymbol{f}_{n_0}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}+\frac{1}{2\underline{\sigma}}\|\boldsymbol{f}_{n_0}\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\\&+\|\delta\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\|\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}_{n_0}\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega)},\end{aligned} \tag{3.29}$$

where all the terms on the right-hand side are bounded due to the stability shown before. Since $n_0\in\{1,\ldots,N\}$ was chosen arbitrarily, (3.25), (3.26), (3.27) and (3.29) imply that the a priori estimate (3.11) is valid. □

3.2 Well-posedness

This section is devoted to the well-posedness analysis for the eddy current obstacle problem (P_{ec}) based on the time-discrete approximation (P_N). As a preparation, for every $N\in\mathbb{N}$, we set up piecewise linear and piecewise constant (in time) interpolations out of the solution $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ of (P_N) as follows:

$$\boldsymbol{E}_N\colon [0,T] \to \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}), \quad t \mapsto \begin{cases} \boldsymbol{E}_0 & \text{if } t = 0 \\ \boldsymbol{E}_{n-1} + (t - t_{n-1})\delta \boldsymbol{E}_n & \text{if } t \in (t_{n-1}, t_n] \end{cases} \tag{3.30}$$

$$\boldsymbol{H}_N\colon [0,T] \to \boldsymbol{L}^2(\Omega), \quad t \mapsto \begin{cases} \boldsymbol{H}_0 & \text{if } t = 0 \\ \boldsymbol{H}_{n-1} + (t - t_{n-1})\delta \boldsymbol{H}_n & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

and

$$\overline{\boldsymbol{E}}_N\colon [0,T] \to \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}), \quad t \mapsto \begin{cases} \boldsymbol{E}_0 & \text{if } t = 0 \\ \boldsymbol{E}_n & \text{if } t \in (t_{n-1}, t_n] \end{cases} \tag{3.31}$$

$$\overline{\boldsymbol{H}}_N\colon [0,T] \to \boldsymbol{L}^2(\Omega), \quad t \mapsto \begin{cases} \boldsymbol{H}_0 & \text{if } t = 0 \\ \boldsymbol{H}_n & \text{if } t \in (t_{n-1}, t_n] \end{cases}$$

$$\overline{\boldsymbol{f}}_N\colon [0,T] \to \boldsymbol{L}^2(\Omega), \quad t \mapsto \begin{cases} \boldsymbol{f}_0 & \text{if } t = 0 \\ \boldsymbol{f}_n & \text{if } t \in (t_{n-1}, t_n]. \end{cases}$$

In view of (P_N), it follows immediately that the above interpolations satisfy

$$\begin{cases} \displaystyle\int_{\Omega\setminus\Omega_\sigma} \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_N(t) \cdot (\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t))\,\mathrm{d}x \\ \displaystyle + \int_\Omega \sigma \overline{\boldsymbol{E}}_N(t) \cdot (\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t)) - \overline{\boldsymbol{H}}_N(t) \cdot \mathbf{curl}(\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t))\,\mathrm{d}x \\ \displaystyle \geq \int_\Omega \overline{\boldsymbol{f}}_N(t) \cdot (\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t))\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \text{ for all } t \in (0,T] \\ \displaystyle \mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t) + \mathbf{curl}\,\overline{\boldsymbol{E}}_N(t) = 0 \quad \text{for all } t \in (0,T] \\ \overline{\boldsymbol{E}}_N(t) \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \quad \text{for all } t \in [0,T]. \end{cases} \tag{$\widetilde{\mathrm{P}}_N$}$$

Theorem 3.6. *Let Assumption 3.1 hold. Then, the eddy current obstacle problem* $(\mathrm{P}_{\mathrm{ec}})$ *admits a unique solution* $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega \setminus \Omega_\sigma) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega))$.

Proof. Existence of a solution. By our construction (3.30) and (3.31), Theorem 3.5 yields the existence of a subsequence of $\{(\boldsymbol{E}_N, \boldsymbol{H}_N)\}_{N=1}^\infty$, denoted again by the same symbol, such that

$$\begin{aligned} (\boldsymbol{E}_N, \boldsymbol{H}_N) &\overset{*}{\rightharpoonup} (\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \text{ as } N \to \infty \\ (\overline{\boldsymbol{E}}_N, \overline{\boldsymbol{H}}_N) &\overset{*}{\rightharpoonup} (\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}}) \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \text{ as } N \to \infty \\ \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_N, \boldsymbol{H}_N) &\overset{*}{\rightharpoonup} (\boldsymbol{\xi}, \boldsymbol{\zeta}) \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega \setminus \Omega_\sigma) \times \boldsymbol{L}^2(\Omega)) \text{ as } N \to \infty \end{aligned} \tag{3.32}$$

for some $(\boldsymbol{E}, \boldsymbol{H}), (\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}}) \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega))$ and $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in L^\infty((0,T), \boldsymbol{L}^2(\Omega \setminus \Omega_\sigma) \times \boldsymbol{L}^2(\Omega))$. Furthermore, (3.30) and (3.31) also imply

$$\left\| \boldsymbol{E}_N(t) - \overline{\boldsymbol{E}}_N(t) \right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \leq \tau \max_{1 \leq n \leq N} \|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \underbrace{\leq}_{(3.11)} \frac{TC}{N} \quad \forall t \in [0,T] \tag{3.33}$$

$$\left\| \boldsymbol{H}_N(t) - \overline{\boldsymbol{H}}_N(t) \right\|_{\boldsymbol{L}^2(\Omega)} \leq \tau \max_{1 \leq n \leq N} \|\delta \boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)} \underbrace{\leq}_{(3.11)} \frac{TC}{N} \quad \forall t \in [0,T],$$

and consequently

$$\lim_{N\to\infty} \left\| \boldsymbol{E}_N - \overline{\boldsymbol{E}}_N \right\|_{L^\infty((0,T),\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma))} = \lim_{N\to\infty} \left\| \boldsymbol{H}_N - \overline{\boldsymbol{H}}_N \right\|_{L^\infty((0,T),\boldsymbol{L}^2(\Omega))} = 0. \tag{3.34}$$

By the above convergence properties together with (3.32), we obtain that

$$\boldsymbol{E} = \overline{\boldsymbol{E}} \quad \text{a.e. in } (0,T)\times(\Omega\setminus\Omega_\sigma) \quad \text{and} \quad \boldsymbol{H} = \overline{\boldsymbol{H}} \quad \text{a.e. in } (0,T)\times\Omega. \tag{3.35}$$

Let us now verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}} = \boldsymbol{\xi} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{H}} = \boldsymbol{\zeta}. \tag{3.36}$$

Indeed, the definition of the weak time derivative implies that

$$\begin{aligned}
&\int_0^T (\boldsymbol{\xi}(t), \boldsymbol{\phi}(t))_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\, \mathrm{d}t \underbrace{=}_{(3.32)} \lim_{N\to\infty} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_N(t), \boldsymbol{\phi}(t)\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \mathrm{d}t \\
&= \lim_{N\to\infty} -\int_0^T \left(\boldsymbol{E}_N(t), \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(t)\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \mathrm{d}t \underbrace{=}_{(3.32)\&(3.35)} -\int_0^T \left(\overline{\boldsymbol{E}}(t), \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(t)\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \mathrm{d}t \\
&\quad \forall \boldsymbol{\phi} \in \mathcal{C}_0^\infty((0,T), \boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)),
\end{aligned}$$

and hence $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}} = \boldsymbol{\xi}$. Analogous arguments are also valid for $\overline{\boldsymbol{H}}$ which concludes (3.36). Altogether, the weak star limit $(\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}})$ enjoys the regularity property

$$(\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega)). \tag{3.37}$$

As the next step, we verify Faraday's law for $(\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}})$. According to $(\widetilde{\mathrm{P}}_N)$, it holds that

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t) + \mathbf{curl}\,\overline{\boldsymbol{E}}_N(t) = 0 \quad \forall t \in (0,T],$$

from which it follows that

$$\begin{aligned}
&\int_0^T \left(\mu\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{H}}(t) + \mathbf{curl}\,\overline{\boldsymbol{E}}(t), \boldsymbol{\phi}(t)\right)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t \\
&\qquad \underbrace{=}_{(3.32)\&(3.36)} \lim_{N\to\infty} \int_0^T \left(\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t) + \mathbf{curl}\,\overline{\boldsymbol{E}}_N(t), \boldsymbol{\phi}(t)\right)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t = 0
\end{aligned}$$

for all $\boldsymbol{\phi} \in \mathcal{C}_0^\infty((0,T), \boldsymbol{L}^2(\Omega))$. As a consequence, by the fundamental theorem of variational calculus, we obtain

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{H}}(t) + \mathbf{curl}\,\overline{\boldsymbol{E}}(t) = 0 \quad \text{for a.e. } t \in (0,T). \tag{3.38}$$

Let us now prove the pointwise weak convergence

$$(\boldsymbol{E}_N, \boldsymbol{H}_N)(t) \rightharpoonup (\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}})(t) \quad \text{weakly in } \boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2(\Omega) \text{ as } N\to\infty \text{ for all } t\in[0,T]. \tag{3.39}$$

To this aim, let $t \in (0,T], \boldsymbol{w} \in \boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)$, and $\phi \in \mathcal{C}^1([0,t])$ be arbitrarily fixed. Integration by parts yields

$$\begin{aligned}
&\left(\overline{\boldsymbol{E}}(t), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \phi(t) - \left(\overline{\boldsymbol{E}}(0), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \phi(0) \qquad (3.40)\\
= \quad &\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\overline{\boldsymbol{E}}(s), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \phi(s) + \left(\overline{\boldsymbol{E}}(s), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \frac{\mathrm{d}}{\mathrm{d}s}\phi(s)\,\mathrm{d}s
\end{aligned}$$

$$\underbrace{=}_{(3.32),(3.35),(3.36)} \lim_{N\to\infty} \left(\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{E}_N(s), \boldsymbol{w} \right)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \phi(s) + (\boldsymbol{E}_N(s), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \frac{\mathrm{d}}{\mathrm{d}s}\phi(s)\, \mathrm{d}s \right)$$
$$= \lim_{N\to\infty} \Big((\boldsymbol{E}_N(t), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\, \phi(t) - (\boldsymbol{E}_N(0), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\, \phi(0) \Big).$$

Choosing $\phi(t) \neq 0$ and $\phi(0) = 0$ (resp. $\phi(t) = 0$ and $\phi(0) \neq 0$) yields $\boldsymbol{E}_N(t) \rightharpoonup \overline{\boldsymbol{E}}(t)$ weakly in $\boldsymbol{L}^2(\Omega \setminus \Omega_\sigma)$ as $N \to \infty$ for all $t \in [0,T]$. By the same argumentation, we derive the pointwise weak convergence for $\{\boldsymbol{H}_N\}_{N=1}^\infty$. In conclusion, (3.39) is valid. As a direct consequence of (3.39) and $(\boldsymbol{E}_N, \boldsymbol{H}_N)(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0)$ for all $N \in \mathbb{N}$, we have

$$\begin{aligned} \overline{\boldsymbol{E}}(0) &= \boldsymbol{E}_0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma \\ \overline{\boldsymbol{H}}(0) &= \boldsymbol{H}_0 \quad \text{a.e. in } \Omega, \end{aligned} \tag{3.41}$$

which is exactly the initial value condition in $(\mathrm{P}_{\mathrm{ec}})$. Let us now introduce the subset

$$\widehat{\boldsymbol{K}} := \{\boldsymbol{w} \in L^2((0,T), \boldsymbol{L}^2(\Omega)) \mid \boldsymbol{w}(t) \in \boldsymbol{K} \text{ for a.e. } t \in (0,T)\}.$$

By definition, since $\boldsymbol{K}$ is a closed and convex subset of $\boldsymbol{L}^2(\Omega)$, the subset $\widehat{\boldsymbol{K}} \subset L^2((0,T), \boldsymbol{L}^2(\Omega))$ is closed and convex. Therefore, since $\overline{\boldsymbol{E}}_N \in \widehat{\boldsymbol{K}}$ for all $N \in \mathbb{N}$, the convergence property (3.32) implies that

$$\overline{\boldsymbol{E}} \in \widehat{\boldsymbol{K}} \quad \Rightarrow \quad \overline{\boldsymbol{E}}(t) \in \boldsymbol{K} \quad \text{for a.e. } t \in (0,T). \tag{3.42}$$

By virtue of (3.37), (3.38), (3.41), and (3.42), the weak limit $(\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}})$ is a solution to $(\mathrm{P}_{\mathrm{ec}})$ once we are able to show that it satisfies the variational inequality in $(\mathrm{P}_{\mathrm{ec}})$. In view of (3.31) and the Lipschitz regularity $\boldsymbol{f} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$, it holds that

$$\lim_{N\to\infty} \overline{\boldsymbol{f}}_N = \boldsymbol{f} \quad \text{in } L^2((0,T), \boldsymbol{L}^2(\Omega)). \tag{3.43}$$

Let now $\boldsymbol{v} \in L^2((0,T), \boldsymbol{H}_0(\mathbf{curl}))$ be arbitrarily fixed and satisfy $\boldsymbol{v}(t) \in \boldsymbol{K}$ for a.e. $t \in (0,T)$. By standard properties of the limit superior, we deduce that

$$\int_0^T \Big(\boldsymbol{f}(t), \boldsymbol{v}(t) - \overline{\boldsymbol{E}}(t) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t \underbrace{=}_{(3.43)} \lim_{N\to\infty} \int_0^T \Big(\overline{\boldsymbol{f}}_N(t), \boldsymbol{v}(t) - \overline{\boldsymbol{E}}_N(t) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t \tag{3.44}$$
$$\underbrace{\leq}_{(\widetilde{\mathrm{P}}_N)} \limsup_{N\to\infty} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E}_N(t), \boldsymbol{v}(t) - \overline{\boldsymbol{E}}_N(t) \right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \Big(\sigma \overline{\boldsymbol{E}}_N(t), \boldsymbol{v}(t) - \overline{\boldsymbol{E}}_N(t) \Big)_{\boldsymbol{L}^2(\Omega)}$$
$$- \Big(\overline{\boldsymbol{H}}_N(t), \mathbf{curl}(\boldsymbol{v}(t) - \overline{\boldsymbol{E}}_N(t)) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t$$
$$\underbrace{\leq}_{(3.32)\&(3.36)} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t} \overline{\boldsymbol{E}}(t), \boldsymbol{v}(t) \right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} \mathrm{d}t - \liminf_{N\to\infty} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E}_N(t), \overline{\boldsymbol{E}}_N(t) \right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} \mathrm{d}t$$
$$+ \int_0^T \Big(\sigma \overline{\boldsymbol{E}}(t), \boldsymbol{v}(t) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t - \liminf_{N\to\infty} \int_0^T \Big(\sigma \overline{\boldsymbol{E}}_N(t), \overline{\boldsymbol{E}}_N(t) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t$$
$$- \int_0^T \Big(\overline{\boldsymbol{H}}(t), \mathbf{curl}\, \boldsymbol{v}(t) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t + \limsup_{N\to\infty} \int_0^T \Big(\overline{\boldsymbol{H}}_N(t), \mathbf{curl}\, \overline{\boldsymbol{E}}_N(t) \Big)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t.$$

Our next step is to estimate the remaining terms on the right-hand side of (3.44). First of all, by the weak sequential lower semi-continuity of the squared norm, we infer that

$$\liminf_{N\to\infty}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_N(t),\boldsymbol{E}_N(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}t=\liminf_{N\to\infty}\frac{1}{2}\left(\|\boldsymbol{E}_N(T)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}-\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\right)$$
$$\underbrace{\geq}_{(3.39)}\frac{1}{2}\left(\left\|\overline{\boldsymbol{E}}(T)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}-\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\right)\underbrace{=}_{(3.41)}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}}(t),\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}t,\quad(3.45)$$

and consequently

$$\liminf_{N\to\infty}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_N(t),\overline{\boldsymbol{E}}_N(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}t\geq\liminf_{N\to\infty}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_N(t),\overline{\boldsymbol{E}}_N(t)-\boldsymbol{E}_N(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}t$$
$$+\liminf_{N\to\infty}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_N(t),\boldsymbol{E}_N(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}t\underbrace{\geq}_{(3.34)\&(3.45)}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}}(t),\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}t.\quad(3.46)$$

Furthermore, the positive semi-definiteness of σ implies

$$\begin{aligned}&\liminf_{N\to\infty}\int_0^T\left(\sigma\overline{\boldsymbol{E}}_N(t),\overline{\boldsymbol{E}}_N(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\qquad(3.47)\\ =\;&\liminf_{N\to\infty}\int_0^T\left(\sigma(\overline{\boldsymbol{E}}_N(t)-\overline{\boldsymbol{E}}(t)),\overline{\boldsymbol{E}}_N(t)-\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2(\Omega)}\\ &\qquad+\left(\sigma(\overline{\boldsymbol{E}}_N(t)-\overline{\boldsymbol{E}}(t)),\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2(\Omega)}+\left(\sigma\overline{\boldsymbol{E}}(t),\overline{\boldsymbol{E}}_N(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\\ \geq\;&\liminf_{N\to\infty}\int_0^T\left(\sigma(\overline{\boldsymbol{E}}_N(t)-\overline{\boldsymbol{E}}(t)),\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2(\Omega)}+\left(\sigma\overline{\boldsymbol{E}}(t),\overline{\boldsymbol{E}}_N(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\\ \underbrace{=}_{(3.32)}&\int_0^T\left(\sigma\overline{\boldsymbol{E}}(t),\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Using once again the weak sequential lower semi-continuity of the squared norm, we find that

$$\limsup_{N\to\infty}-\int_0^T\left(\boldsymbol{H}_N(t),\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t=\limsup_{N\to\infty}\frac{1}{2}\left(\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\|\boldsymbol{H}_N(T)\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right)\quad(3.48)$$
$$\underbrace{\leq}_{(3.39)}\frac{1}{2}\left(\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\left\|\overline{\boldsymbol{H}}(T)\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right)\underbrace{=}_{(3.41)}-\int_0^T\left(\overline{\boldsymbol{H}}(t),\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{H}}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t$$
$$\underbrace{=}_{(3.38)}\int_0^T\left(\overline{\boldsymbol{H}}(t),\mathbf{curl}\,\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t,$$

and therefore

$$\begin{aligned}&\limsup_{N\to\infty}\int_0^T\left(\overline{\boldsymbol{H}}_N(t),\mathbf{curl}\,\overline{\boldsymbol{E}}_N(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\qquad(3.49)\\ \underbrace{=}_{(\widetilde{\mathrm{P}}_N)}&\limsup_{N\to\infty}-\int_0^T\left(\overline{\boldsymbol{H}}_N(t),\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t\\ \leq\;&\limsup_{N\to\infty}-\int_0^T\left(\overline{\boldsymbol{H}}_N(t)-\boldsymbol{H}_N(t),\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t\\ &+\limsup_{N\to\infty}-\int_0^T\left(\boldsymbol{H}_N(t),\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_N(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t\end{aligned}$$

$$\underbrace{\leq}_{(3.34)\&(3.48)} \int_0^T \left(\overline{\boldsymbol{H}}(t), \mathbf{curl}\,\overline{\boldsymbol{E}}(t)\right)_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t.$$

Applying (3.46), (3.47), and (3.49) to (3.44) results in

$$\begin{aligned} &\int_0^T \int_{\Omega\setminus\Omega_\sigma} \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}}(t)\cdot(\boldsymbol{v}(t)-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x \\ &+\int_\Omega \sigma\overline{\boldsymbol{E}}(t)\cdot(\boldsymbol{v}(t)-\overline{\boldsymbol{E}}(t))-\overline{\boldsymbol{H}}(t)\cdot\mathbf{curl}(\boldsymbol{v}(t)-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x\,\mathrm{d}t \\ \geq &\int_0^T\int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}(t)-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x\,\mathrm{d}t \\ &\forall \boldsymbol{v}\in L^2((0,T),\boldsymbol{H}_0(\mathbf{curl})) \text{ with } \boldsymbol{v}(t)\in\boldsymbol{K} \text{ for a.e. } t\in(0,T). \end{aligned} \tag{3.50}$$

Finally, to show that the variational inequality in ($\mathrm{P_{ec}}$) holds, let us assume the contrary, i.e.,

$$\exists \boldsymbol{q}\in\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl})\ \exists M\subset(0,T) \text{ with } |M|>0 \text{ s.t. } \int_{\Omega\setminus\Omega_\sigma}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}}(t)\cdot(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x$$
$$+\int_\Omega \sigma\overline{\boldsymbol{E}}(t)\cdot(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))-\overline{\boldsymbol{H}}(t)\cdot\mathbf{curl}(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x<\int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x \quad \text{for a.e. } t\in M,$$

which implies

$$\begin{aligned} \int_M\int_{\Omega\setminus\Omega_\sigma}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{E}}(t)\cdot(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x+\int_\Omega\sigma\overline{\boldsymbol{E}}(t)\cdot(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))-\overline{\boldsymbol{H}}(t)\cdot\mathbf{curl}(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x\,\mathrm{d}t \\ <\int_M\int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{q}-\overline{\boldsymbol{E}}(t))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.51}$$

Inserting $\boldsymbol{v}:=\chi_M\boldsymbol{q}+\chi_{(0,T)\setminus M}\overline{\boldsymbol{E}}$ into (3.50) immediately contradicts (3.51). In conclusion, $(\overline{\boldsymbol{E}},\overline{\boldsymbol{H}})$ satisfies the variational inequality in ($\mathrm{P_{ec}}$). This completes the existence proof.

Uniqueness and Lipschitz stability. Let $(\boldsymbol{E}_1,\boldsymbol{H}_1)$ and $(\boldsymbol{E}_2,\boldsymbol{H}_2)$ denote, respectively, solutions to ($\mathrm{P_{ec}}$) associated with the initial data $(\boldsymbol{E}_0^1,\boldsymbol{H}_0^1),(\boldsymbol{E}_0^2,\boldsymbol{H}_0^2)$ and the right-hand sides $\boldsymbol{f}_1,\boldsymbol{f}_2$ satisfying Assumption 3.1. Setting $\boldsymbol{v}=\boldsymbol{E}_2(s)$ in ($\mathrm{P_{ec}}$) for $\boldsymbol{E}=\boldsymbol{E}_1$ (resp. $\boldsymbol{v}=\boldsymbol{E}_1(s)$ in ($\mathrm{P_{ec}}$) for $\boldsymbol{E}=\boldsymbol{E}_2$) and multiplying with -1, we have

$$\begin{aligned} \int_{\Omega\setminus\Omega_\sigma}\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}_1(s)\cdot(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))\,\mathrm{d}x+\int_\Omega\sigma\boldsymbol{E}_1(s)\cdot(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))-\boldsymbol{H}_1(s)\cdot\mathbf{curl}(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))\,\mathrm{d}x \\ \leq\int_\Omega \boldsymbol{f}_1(s)\cdot(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))\,\mathrm{d}x \quad \text{for a.e. } s\in(0,T) \end{aligned} \tag{3.52}$$

and

$$\begin{aligned} &-\int_{\Omega\setminus\Omega_\sigma}\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}_2(s)\cdot(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))\,\mathrm{d}x \\ &-\int_\Omega\sigma\boldsymbol{E}_2(s)\cdot(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))-\boldsymbol{H}_2(s)\cdot\mathbf{curl}(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))\,\mathrm{d}x \\ \leq &-\int_\Omega \boldsymbol{f}_2(s)\cdot(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))\,\mathrm{d}x \quad \text{for a.e. } s\in(0,T). \end{aligned} \tag{3.53}$$

In addition, by the Faraday law for $(\boldsymbol{E}_1,\boldsymbol{H}_1)$ and $(\boldsymbol{E}_2,\boldsymbol{H}_2)$, it holds that

$$\mathbf{curl}(\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s))=-\mu\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{H}_1(s)-\boldsymbol{H}_2(s)) \quad \text{for a.e. } s\in(0,T). \tag{3.54}$$

Adding (3.52) and (3.53) together and then applying (3.54) to the resulting inequality, we obtain by using the properties of σ as well as Hölder's and Young's inequalities that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\underline{\sigma}\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{H}_1(s)-\boldsymbol{H}_2(s)\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\
&\le\|\boldsymbol{f}_1(s)-\boldsymbol{f}_2(s)\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\\
&\quad+\|\boldsymbol{f}_1(s)-\boldsymbol{f}_2(s)\|_{\boldsymbol{L}^2(\Omega_\sigma)}\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|_{\boldsymbol{L}^2(\Omega_\sigma)}\\
&\le\left(\frac{1}{2\underline{\epsilon}}+\frac{1}{2\underline{\sigma}}\right)\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^2_{\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))}+\frac{1}{2}\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\frac{\underline{\sigma}}{2}\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}
\end{aligned}$$

for a.e. $s\in(0,T)$. By integration over the time interval $(0,t)$ and rearrangement, it follows that

$$\begin{aligned}
&\|\boldsymbol{E}_1(t)-\boldsymbol{E}_2(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\underline{\sigma}\int_0^t\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\,\mathrm{d}s+\|\boldsymbol{H}_1(t)-\boldsymbol{H}_2(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\
&\le\left\|\boldsymbol{E}_0^1-\boldsymbol{E}_0^2\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\left\|\boldsymbol{H}_0^1-\boldsymbol{H}_0^2\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\left(\frac{t}{\underline{\epsilon}}+\frac{t}{\underline{\sigma}}\right)\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^2_{\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))}\\
&\quad+\int_0^t\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}s\\
&\le\max\left\{1,\frac{t}{\underline{\epsilon}}+\frac{t}{\underline{\sigma}}\right\}\left(\left\|(\boldsymbol{E}_0^1,\boldsymbol{H}_0^1)-(\boldsymbol{E}_0^2,\boldsymbol{H}_0^2)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2_\mu(\Omega)}+\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^2_{\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))}\right)\\
&\quad+\int_0^t\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\,\mathrm{d}s\quad\forall t\in[0,T].
\end{aligned}$$

Employing the Gronwall lemma, we then arrive at

$$\begin{aligned}
&\|(\boldsymbol{E}_1,\boldsymbol{H}_1)(t)-(\boldsymbol{E}_2,\boldsymbol{H}_2)(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2_\mu(\Omega)}+\underline{\sigma}\int_0^t\|\boldsymbol{E}_1(s)-\boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\,\mathrm{d}s\\
&\le\mathrm{e}^t\max\left\{1,\frac{t}{\underline{\epsilon}}+\frac{t}{\underline{\sigma}}\right\}\left(\left\|(\boldsymbol{E}_0^1,\boldsymbol{H}_0^1)-(\boldsymbol{E}_0^2,\boldsymbol{H}_0^2)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2_\mu(\Omega)}+\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^2_{\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))}\right)
\end{aligned}\tag{3.55}$$

for all $t\in[0,T]$. In view of (3.55), we conclude that $(\mathrm{P_{ec}})$ admits at most one solution. □

Remark 3.7. Introducing the subset

$$\begin{aligned}
\mathcal{U}:=\Big\{&(\boldsymbol{f},\boldsymbol{E}_0,\boldsymbol{H}_0)\in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega))\times(\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl}))\times\boldsymbol{H}(\mathbf{curl})\;\Big|\\
&\int_{\Omega_\sigma}(\sigma\boldsymbol{E}_0-\mathbf{curl}\,\boldsymbol{H}_0)\cdot(\boldsymbol{v}-\boldsymbol{E}_0)\,\mathrm{d}x\ge\int_{\Omega_\sigma}\boldsymbol{f}(0)\cdot(\boldsymbol{v}-\boldsymbol{E}_0)\,\mathrm{d}x\quad\forall\boldsymbol{v}\in\boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl})\Big\}
\end{aligned}$$

of $\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))\times\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)$, the solution operator associated to $(\mathrm{P_{ec}})$,

$$\Phi\colon\mathcal{U}\to\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2(\Omega))\cap L^2((0,T),\boldsymbol{L}^2(\Omega_\sigma)\times\boldsymbol{L}^2(\Omega)),\quad(\boldsymbol{f},\boldsymbol{E}_0,\boldsymbol{H}_0)\mapsto(\boldsymbol{E},\boldsymbol{H}),$$

is Lipschitz continuous as a consequence of (3.55).

3.2.1 The Full Eddy Current Case in the Presence of a Nonconducting Region

Up to this point, the displacement current $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}$ was only neglected in the region where σ is uniformly positive definite. In this section, we suppose that $\Omega\setminus\Omega_\sigma$ is of nonzero Lebesgue measure and represents an insulating region, i.e.,

$$\sigma=0\quad\text{a.e. in }\Omega\setminus\Omega_\sigma.$$

Our focus lies on the full eddy current case where the displacement current is completely removed in the whole domain containing the insulating region $\Omega\setminus\Omega_\sigma$. Here, the previously developed analysis serves as the foundation to cover this case with some additional assumptions as follows:

Assumption 3.8.

(B3.1) It holds that $|\Omega \setminus \Omega_\sigma| \neq 0$ and

$$\begin{aligned} \sigma &= 0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma \\ \boldsymbol{f} &= 0 \quad \text{a.e. in } (0,T) \times (\Omega \setminus \Omega_\sigma) \\ \mathbf{curl}\, \boldsymbol{H}_0 &= 0 \quad \text{a.e. in } \Omega \setminus \Omega_\sigma. \end{aligned} \tag{3.56}$$

(B3.2) The obstacle set $\boldsymbol{K}$ satisfies one of the following conditions:

$$\text{(i) } \exists C > 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K} \,:\, \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \leq C. \tag{3.57a}$$

(ii) $\boldsymbol{K} \subset \boldsymbol{X}_\epsilon(\Omega)$ and Ω is a bounded Lipschitz domain with a connected boundary, (3.57b)

where $\boldsymbol{X}_\epsilon(\Omega) \coloneqq \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid (\epsilon \boldsymbol{v}, \nabla\phi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \phi \in H^1_0(\Omega)\}$.

Remark 3.9.

(i) As $\Omega \setminus \Omega_\sigma$ represents an insulating region such as air, it is physically reasonable to assume that no current source is present in the insulator. The assumption (3.56) on the vanishing source and vanishing initial rotational magnetic field in the insulator is indeed common in the study of the eddy current problems (see, e.g., [129, p. 42] or [7, p. 239]).

(ii) The condition (3.57a) is obviously satisfied if the obstacle set $\boldsymbol{K}$ is bounded in $\boldsymbol{L}^2(\Omega)$. A prominent example is the set $\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{v}(x)| \leq d(x) \text{ for a.e. } x \in \Omega\}$ for some electric obstacle $d \in L^2(\Omega)$. On the other hand, the condition (3.57b) describes a physical medium with vanishing charge density, i.e., the case where the electric field satisfies $\operatorname{div}(\epsilon \boldsymbol{E}) \equiv 0$.

Let us now state the full eddy current problem we focus on in this section:

$$\begin{cases} \displaystyle\int_{\Omega_\sigma} \sigma \boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\,\mathrm{d}x - \int_\Omega \boldsymbol{H}(t) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t))\,\mathrm{d}x \\ \displaystyle\geq \int_{\Omega_\sigma} \boldsymbol{f}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \text{ for a.e. } t \in (0,T) \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}(t) + \mathbf{curl}\, \boldsymbol{E}(t) = 0 \quad \text{for a.e. } t \in (0,T) \\ \boldsymbol{E}(t) \in \boldsymbol{K} \quad \text{for a.e. } t \in (0,T), \quad \boldsymbol{H}(0) = \boldsymbol{H}_0 \quad \text{a.e. in } \Omega. \end{cases} \tag{$\mathrm{P}^0_{\mathrm{ec}}$}$$

Note that in contrast to (P_{ec}), the problem ($\mathrm{P}^0_{\mathrm{ec}}$) comprises an elliptic VI for the electric field $\boldsymbol{E}$ and an evolutionary equation for the magnetic field $\boldsymbol{H}$ which is why we do not impose any initial condition for $\boldsymbol{E}$ (cf. [103] for the case of the full eddy current equations with a constant and scalar conductivity $\sigma > 0$). The time-discrete approximation for ($\mathrm{P}^0_{\mathrm{ec}}$) reads as finding $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ such that

$$\begin{cases} \displaystyle\int_{\Omega_\sigma} \sigma \boldsymbol{E}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x - \int_\Omega \boldsymbol{H}_n \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x \\ \displaystyle\geq \int_{\Omega_\sigma} \boldsymbol{f}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \quad \forall n \in \{1, \ldots, N\} \\ \mu \delta \boldsymbol{H}_n + \mathbf{curl}\, \boldsymbol{E}_n = 0 \quad \forall n \in \{1, \ldots, N\}. \end{cases} \tag{P^0_N}$$

To prove the well-posedness of (P^0_N), we reformulate it as a minimization problem in a Hilbert space as follows:

Lemma 3.10. *Let Assumption 3.1 and Assumption 3.8 be satisfied. Then, the time-discrete problem* (P_N^0) *admits a solution* $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$*. If* (3.57b) *holds true, then the solution to* (P_N^0) *is unique.*

Proof. First, using the discrete Faraday law, we rewrite the problem (P_N^0) as

$$\begin{aligned}&\int_{\Omega_\sigma} \sigma \boldsymbol{E}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x + \tau \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_n \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x \\ \geq &\int_{\Omega_\sigma} \boldsymbol{f}_n \cdot (\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x + \int_\Omega \boldsymbol{H}_{n-1} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}_n)\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \quad \forall n \in \{1, \dots, N\},\end{aligned}$$

which is equivalent to the minimization problem

$$\begin{aligned}\min_{\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})} &\Big(\frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)} + \frac{\tau}{2}\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2_{\mu^{-1}}(\Omega)} \\ &- \int_{\Omega_\sigma} \boldsymbol{f}_n \cdot \boldsymbol{v}\,\mathrm{d}x - \int_\Omega \boldsymbol{H}_{n-1} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x\Big) \quad \forall n \in \{1, \dots, N\}.\end{aligned} \tag{3.58}$$

Next, let $n \in \{1, \dots, N\}$ be arbitrarily fixed. For any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$, it holds that

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)} + \frac{\tau}{2}\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2_{\mu^{-1}}(\Omega)} - \int_{\Omega_\sigma} \boldsymbol{f}_n \cdot \boldsymbol{v}\,\mathrm{d}x - \int_\Omega \boldsymbol{H}_{n-1} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ \geq &\frac{1}{4}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)} + \frac{\tau}{4}\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2_{\mu^{-1}}(\Omega)} - \frac{1}{\underline{\sigma}}\|\boldsymbol{f}_n\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} - \frac{\overline{\mu}}{\tau}\|\boldsymbol{H}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} \\ \geq &- \frac{1}{\underline{\sigma}}\|\boldsymbol{f}_n\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} - \frac{\overline{\mu}}{\tau}\|\boldsymbol{H}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{3.59}$$

This shows that the objective functional associated with (3.58) is bounded from below. Therefore, since $\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ is non-empty, there exists an infimal sequence $\{\boldsymbol{v}_k^n\}_{k=1}^\infty \subset \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ for the minimization problem (3.58). Thanks to (3.59), the infimal sequence $\{\boldsymbol{v}_k^n\}_{k=1}^\infty \subset \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ satisfies

$$\|\boldsymbol{v}_k^n\|_{\boldsymbol{L}^2(\Omega_\sigma)} + \|\,\mathbf{curl}\,\boldsymbol{v}_k^n\|_{\boldsymbol{L}^2(\Omega)} \leq C, \quad k \in \mathbb{N}, \tag{3.60}$$

for some constant $C > 0$, independent of k. Now, if (3.57a) is satisfied, then in view of (3.60) it follows that the infimal sequence $\{\boldsymbol{v}_k^n\}_{k=1}^\infty$ is bounded in $\boldsymbol{H}_0(\mathbf{curl})$. On the other hand, if (3.57b) is satisfied, then it implies the Poincaré-Friedrichs-type inequality [6, Lemma 3.1] (compare also with Corollary 2.4 for the version without ϵ)

$$\exists C_{\mathrm{p}} > 0 \quad \forall \boldsymbol{v} \in \boldsymbol{X}_\epsilon(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}) \,:\, \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C_{\mathrm{p}}\|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.61}$$

which yields due to (3.60) the boundedness of $\{\boldsymbol{v}_k^n\}_{k=1}^\infty$ in $\boldsymbol{H}_0(\mathbf{curl})$. In conclusion, for every $n \in \{1, \dots, N\}$, the existence of a minimizer to (3.58) follows by standard arguments as in the proof of the direct method of variational calculus. Finally, if (3.57b) holds true, then due to (3.61) the objective functional associated with (3.58) is strictly convex, and so the minimization problem (3.58) admits a unique solution. □

Lemma 3.11. *Let Assumption 3.1 and Assumption 3.8 hold. Then, there exists a positive real constant* C*, depending only on* $T, \mu, \sigma, \boldsymbol{f}, \boldsymbol{H}_0$ *such that, for every* $N \in \mathbb{N}$*, every solution* $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ *of* (P_N^0) *satisfies*

$$\max_{1 \leq n \leq N} \Big[\|\boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega)}\Big] \leq C. \tag{3.62}$$

Proof. Let $N \in \mathbb{N}$ be arbitrarily fixed and let $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^N \subset (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ denote a solution to (P_N^0). Further, let $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ and $n_0 \in \{1, \dots, N\}$ be arbitrarily fixed. The lines of argumentation are similar to Theorem 3.5 where we simply set $\boldsymbol{z}$ to be zero thanks to (3.56) and since ϵ does not appear in (P_N^0). Then, together with the fact that $\boldsymbol{f} = 0$ a.e. in $(0,T) \times (\Omega \setminus \Omega_\sigma)$, by an analogous argumentation to the proof of Theorem 3.5, it follows that

$$\begin{aligned}&\frac{1}{2}\left(\|\delta \boldsymbol{H}_{n_0}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} - \|\delta \boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \sum_{n=1}^{n_0} \|\delta \boldsymbol{H}_n - \delta \boldsymbol{H}_{n-1}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right) + \sum_{n=1}^{n_0} \tau \underline{\sigma} \|\delta \boldsymbol{E}_n\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} \qquad (3.63)\\ &\le \sum_{n=1}^{n_0} \|\boldsymbol{f}_n - \boldsymbol{f}_{n-1}\|_{\boldsymbol{L}^2(\Omega_\sigma)} \|\delta \boldsymbol{E}_n\|_{\boldsymbol{L}^2(\Omega_\sigma)} + \sum_{n=1}^{n_0} \left\|\boldsymbol{w}_n^N - \boldsymbol{w}_{n-1}^N\right\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{H}_n\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

In turn, this implies the estimate

$$\frac{1}{2} \|\delta \boldsymbol{H}_{n_0}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le \frac{L^2 T}{4\underline{\sigma}} + \frac{3}{4} \|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{2}{\underline{\mu}} \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

In view of the above estimate we obtain $\|\boldsymbol{H}_{n_0}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le C$ and $\|\mathbf{curl}\, \boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega)} \le C$ for a constant $C > 0$ due to (3.26) and the discrete Faraday law in (P_N^0). The bound on $\|\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega_\sigma)}$ is obtained by testing with $\boldsymbol{v} = 0$ in (P_N^0) and proceeding as in (3.28) and (3.29). The bound on $\|\boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega \setminus \Omega_\sigma)}$ is an immediate result of (3.57a) or (3.57b) along with the Poincaré-Friedrichs-type inequality (3.61) and the estimate $\|\mathbf{curl}\, \boldsymbol{E}_{n_0}\|_{\boldsymbol{L}^2(\Omega)} \le C$. □

In view of (P_N^0), invoking again the constructions (3.30) and (3.31) it follows that the interpolations satisfy

$$\begin{cases} \displaystyle\int_{\Omega_\sigma} \sigma \overline{\boldsymbol{E}}_N(t) \cdot (\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t))\, \mathrm{d}x - \int_\Omega \overline{\boldsymbol{H}}_N(t) \cdot \mathbf{curl}(\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t))\, \mathrm{d}x \\ \displaystyle\ge \int_{\Omega_\sigma} \overline{\boldsymbol{f}}_N(t) \cdot (\boldsymbol{v} - \overline{\boldsymbol{E}}_N(t))\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \text{ for all } t \in (0,T] \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}_N(t) + \mathbf{curl}\, \overline{\boldsymbol{E}}_N(t) = 0 \quad \text{for all } t \in (0,T] \\ \overline{\boldsymbol{E}}_N(t) \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \quad \text{for all } t \in [0,T]. \end{cases} \qquad (\widetilde{\mathrm{P}}_N^0)$$

Theorem 3.12. *Let Assumption 3.1 and Assumption 3.8 hold. Then, the eddy current obstacle problem* $(\mathrm{P}_{\mathrm{ec}}^0)$ *admits a solution* $(\boldsymbol{E}, \boldsymbol{H}) \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl})) \times W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$. *If* (3.57b) *holds true, then the solution to* $(\mathrm{P}_{\mathrm{ec}}^0)$ *is unique.*

Proof. First, as in the proof of Theorem 3.6, the a priori estimate from Lemma 3.11 yields the existence of a subsequence of $\{(\boldsymbol{E}_N, \boldsymbol{H}_N)\}_{N=1}^\infty$, denoted again by the same symbol, such that

$$\begin{aligned}&(\overline{\boldsymbol{E}}_N, \boldsymbol{H}_N, \overline{\boldsymbol{H}}_N, \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}_N) \overset{*}{\rightharpoonup} (\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}}, \overline{\boldsymbol{H}}, \frac{\mathrm{d}}{\mathrm{d}t} \overline{\boldsymbol{H}})\\ &\text{weakly-* in } L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))\end{aligned}$$

as $N \to \infty$ for some $(\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}}) \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl})) \times W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$. Passing to the limit in the discrete Faraday law as in (3.38), we then obtain

$$\mu \frac{\mathrm{d}}{\mathrm{d}t} \overline{\boldsymbol{H}}(t) + \mathbf{curl}\, \overline{\boldsymbol{E}}(t) = 0 \quad \text{for a.e. } t \in (0,T).$$

Analogously to (3.40), we obtain the pointwise weak convergence

$$\boldsymbol{H}_N(t) \rightharpoonup \overline{\boldsymbol{H}}(t) \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \text{ as } N \to \infty \text{ for all } t \in [0,T],$$

which implies the initial condition $\overline{\boldsymbol{H}}(0) = \boldsymbol{H}_0$ a.e. in Ω. Also, as in the proof of Theorem 3.6, the previous weak-star convergence yields the feasibility $\overline{\boldsymbol{E}}(t) \in \boldsymbol{K}$ for a.e. $t \in (0,T)$. Ultimately, the final passage to the limit in $(\widetilde{\mathrm{P}}^0_N)$ follows again the same arguments as in the proof of Theorem 3.6. In conclusion, the weak-star limit $(\overline{\boldsymbol{E}}, \overline{\boldsymbol{H}})$ satisfies $(\mathrm{P}^0_{\mathrm{ec}})$. Let us now assume that (3.57b) is valid and let $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, and $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ denote, respectively, solutions to $(\mathrm{P}^0_{\mathrm{ec}})$. Setting $\boldsymbol{v} = \boldsymbol{E}_2(s)$ in $(\mathrm{P}^0_{\mathrm{ec}})$ for $\boldsymbol{E} = \boldsymbol{E}_1$ (resp. $\boldsymbol{v} = \boldsymbol{E}_1(s)$ in $(\mathrm{P}^0_{\mathrm{ec}})$ for $\boldsymbol{E} = \boldsymbol{E}_2$) we can proceed as in (3.52), (3.53) and (3.54) to obtain the estimate

$$\|\boldsymbol{E}_1(s) - \boldsymbol{E}_2(s)\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{H}_1(s) - \boldsymbol{H}_2(s)\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \leq 0 \quad \text{for a.e. } s \in (0,T).$$

As $\boldsymbol{H}_1(0) = \boldsymbol{H}_0 = \boldsymbol{H}_2(0)$, the above inequality implies that $\boldsymbol{H}_1 = \boldsymbol{H}_2$, which yields due to the Faraday law in $(\mathrm{P}^0_{\mathrm{ec}})$ that $\mathbf{curl}(\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0$. As a result of the Poincaré-Friedrichs-type inequality (3.61) it then follows that $\boldsymbol{E}_1 = \boldsymbol{E}_2$. This completes the proof. □

Remark 3.13. We want to mention that even in the case of the eddy current equations, without assuming something similar to (3.57b), uniqueness of the solution can in general not be expected.

Remark 3.14. The analysis in this section with respect to (3.57b) reveals that a local Poincaré-Friedrichs-type inequality in the insulator $\Omega \setminus \Omega_\sigma$ is sufficient to obtain an existence and uniqueness result for $(\mathrm{P}^0_{\mathrm{ec}})$. This allows us to work with another obstacle set $\boldsymbol{K}$ as follows: Suppose again that Ω is a bounded Lipschitz domain such that $\overline{\Omega_\sigma} \subset \Omega$ and $\Omega \setminus \Omega_\sigma$ is connected. Then, for the obstacle set $\boldsymbol{K}$, an alternative assumption to (3.57b) reads

$$\begin{aligned}\boldsymbol{K} \subset \widetilde{\boldsymbol{X}}_\epsilon(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid (\epsilon\boldsymbol{v}, \nabla\phi)_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} = 0 \quad \forall \phi \in H^1(\Omega \setminus \Omega_\sigma),\\ (\epsilon\boldsymbol{v}, \boldsymbol{h})_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} = 0 \quad \forall \boldsymbol{h} \in \boldsymbol{\mathcal{H}}\},\end{aligned}$$

where $\boldsymbol{\mathcal{H}}$ denotes the finite-dimensional vector space of Neumann fields related to topological quantities of the physical domain Ω and the insulating region $\Omega \setminus \Omega_\sigma$ (see [7, Page 13] for its definition and the simplified version in (2.23)). As proven in [7, Lemma 2.2], the following Poincaré-Friedrichs-type inequality holds true:

$$\exists C_{\mathrm{p}} > 0 \quad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{X}}_\epsilon(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}) : \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \leq C_{\mathrm{p}} \Big(\|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega_\sigma)}\Big). \tag{3.64}$$

With (3.64) at hand, the existence of a unique solution to $(\mathrm{P}^0_{\mathrm{ec}})$ is obtained under minor changes of this section.

3.3 Justification of the Eddy Current Model

Theorem 3.6 implies that both the Maxwell obstacle problem (P) (by choosing $\Omega_\sigma = \emptyset$) and the eddy current model $(\mathrm{P}_{\mathrm{ec}})$ admit unique solutions, which we denote in the following, respectively, by

$$(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega \setminus \Omega_\sigma) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \tag{3.65}$$

and

$$(\boldsymbol{E}_{\mathrm{ec}}, \boldsymbol{H}_{\mathrm{ec}}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega \setminus \Omega_\sigma) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)). \tag{3.66}$$

Our goal now is to justify the eddy current model ($\mathrm{P_{ec}}$) in the sense that its unique solution $(\boldsymbol{E}_{\mathrm{ec}}, \boldsymbol{H}_{\mathrm{ec}})$ is close to $(\boldsymbol{E}, \boldsymbol{H})$ under a reasonable smallness condition on $\|\epsilon/\underline{\sigma}\|_{L^\infty(\Omega_\sigma)^{3\times 3}}$. By proposing an additional assumption on the initial data (see Assumption 3.15), we are able to not only justify the eddy current model but also prove an a priori error estimate for the eddy current approximation with a linear convergence rate in terms of $\|\epsilon\|_{L^\infty(\Omega_\sigma)^{3\times 3}}$.

Assumption 3.15. The initial value $(\boldsymbol{E}_0, \boldsymbol{H}_0)$ satisfies

$$\int_{\Omega\setminus\Omega_\sigma} (\sigma \boldsymbol{E}_0 - \mathbf{curl}\, \boldsymbol{H}_0)\cdot(\boldsymbol{v} - \boldsymbol{E}_0)\,\mathrm{d}x \geq \int_{\Omega\setminus\Omega_\sigma} \boldsymbol{f}(0)\cdot(\boldsymbol{v}-\boldsymbol{E}_0)\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}\cap \boldsymbol{H}_0(\mathbf{curl}) \tag{3.67}$$

and

$$\mathbf{curl}\,\boldsymbol{E}_0 = 0 \quad \text{a.e. on } \Omega. \tag{3.68}$$

Remark 3.16. Assumptions (3.67) is of technical importance and is trivially satisfied if $\sigma\boldsymbol{E}_0 - \mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{f}(0)$ a.e. in $\Omega\setminus\Omega_\sigma$. Note that in real applications $\Omega\setminus\Omega_\sigma$ typically represents a nonconducting medium such that the conductivity $\sigma_{|\Omega\setminus\Omega_\sigma}$ is zero. In this case (3.67) is satisfied if $-\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{f}(0)$ a.e. in $\Omega\setminus\Omega_\sigma$.

In the following, if $|\Omega\setminus\Omega_\sigma|\neq 0$, the constant $\underline{\epsilon}(\Omega\setminus\Omega_\sigma)$ denotes a uniform lower bound for the lowest eigenvalues of ϵ in $\Omega\setminus\Omega_\sigma$, i.e., it satisfies

$$\epsilon(x)\xi\cdot\xi \geq \underline{\epsilon}(\Omega\setminus\Omega_\sigma)|\xi|^2 \quad \text{for a.e. } x\in\Omega\setminus\Omega_\sigma \text{ and all } \xi\in\mathbb{R}^3. \tag{3.69}$$

Furthermore, let $L(\Omega_\sigma)$ and $L(\Omega\setminus\Omega_\sigma)$ denote, respectively, the Lipschitz constants of $\boldsymbol{f}$ in Ω_σ and $\Omega\setminus\Omega_\sigma$, i.e.,

$$\begin{aligned} \|\boldsymbol{f}(t_1)-\boldsymbol{f}(t_2)\|_{\boldsymbol{L}^2(\Omega_\sigma)} &\leq L(\Omega_\sigma)|t_1-t_2| \quad \forall t_1,t_2\in[0,T] \\ \|\boldsymbol{f}(t_1)-\boldsymbol{f}(t_2)\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} &\leq L(\Omega\setminus\Omega_\sigma)|t_1-t_2| \quad \forall t_1,t_2\in[0,T]. \end{aligned} \tag{3.70}$$

Theorem 3.17. *Let Assumption 3.1 and Assumption 3.15 be satisfied. If $|\Omega\setminus\Omega_\sigma|\neq 0$, then it holds that*

$$\begin{aligned} &\|(\boldsymbol{E},\boldsymbol{H}) - (\boldsymbol{E}_{\mathrm{ec}},\boldsymbol{H}_{\mathrm{ec}})\|_{\mathcal{C}([0,T],\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)\times\boldsymbol{L}^2_\mu(\Omega))} + \|\boldsymbol{E}-\boldsymbol{E}_{\mathrm{ec}}\|_{L^2((0,T),\boldsymbol{L}^2_\sigma(\Omega_\sigma))} \\ &\quad\leq 2\left(\frac{L(\Omega_\sigma)^2T}{\underline{\sigma}} + \frac{2L(\Omega\setminus\Omega_\sigma)T}{\sqrt{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}}\sqrt{\frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}}\right)^{1/2}\left\|\frac{\epsilon}{\underline{\sigma}}\right\|_{L^\infty(\Omega_\sigma)^{3\times3}}. \end{aligned} \tag{3.71}$$

If $\Omega_\sigma = \Omega$, then

$$\|\boldsymbol{H}-\boldsymbol{H}_{\mathrm{ec}}\|_{\mathcal{C}([0,T],\boldsymbol{L}^2_\mu(\Omega))} + \|\boldsymbol{E}-\boldsymbol{E}_{\mathrm{ec}}\|_{L^2((0,T),\boldsymbol{L}^2_\sigma(\Omega))} \leq 2\frac{L\sqrt{T}}{\sqrt{\underline{\sigma}}}\left\|\frac{\epsilon}{\underline{\sigma}}\right\|_{L^\infty(\Omega)^{3\times3}}. \tag{3.72}$$

Remark 3.18. If the applied current source $\boldsymbol{f}$ is only acting in the conducting region Ω_σ, we have $L(\Omega\setminus\Omega_\sigma)=0$ so that the upper bound for (3.71) precisely coincides with the one in (3.72) given by $2L\sqrt{T}/\sqrt{\underline{\sigma}}\,\|\epsilon/\underline{\sigma}\|_{L^\infty(\Omega)^{3\times3}}$.

Proof. We split the proof into three parts.
Step 1: Boundedness of $t\mapsto \int_0^t \left\|\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s$ with an upper bound being independent of $\epsilon_{|\Omega_\sigma}$. Setting $\boldsymbol{v} = \boldsymbol{E}(s+h)$ (resp. $\boldsymbol{v}=\boldsymbol{E}(s)$) in (P) for $t=s$ (resp. $t=s+h$) and then

adding the resulting inequalities, we obtain (similarly to the uniqueness proof for Theorem 3.6) by employing the Faraday law, the properties of σ, and Hölder's inequality that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\|\boldsymbol{E}(s+h)-\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\|\boldsymbol{E}(s+h)-\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)} \\ &\quad+\underline{\sigma}\left\|\boldsymbol{E}(s+h)-\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\|\boldsymbol{H}(s+h)-\boldsymbol{H}(s)\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \\ &\leq\left\|\boldsymbol{f}(s+h)-\boldsymbol{f}(s)\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\left\|\boldsymbol{E}(s+h)-\boldsymbol{E}(s)\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \\ &\quad+\left\|\boldsymbol{f}(s+h)-\boldsymbol{f}(s)\right\|_{\boldsymbol{L}^2(\Omega_\sigma)}\left\|\boldsymbol{E}(s+h)-\boldsymbol{E}(s)\right\|_{\boldsymbol{L}^2(\Omega_\sigma)} \end{aligned}$$

for a.e. $s\in(0,T)$ and a.e. $h\in(0,T-s)$. Integrating the above inequality over $(0,t)$ and dividing by h^2, we obtain that

$$\begin{aligned} &\frac{1}{2}\left\|\frac{\boldsymbol{E}(t+h)-\boldsymbol{E}(t)}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}-\frac{1}{2}\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\frac{1}{2}\left\|\frac{\boldsymbol{E}(t+h)-\boldsymbol{E}(t)}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)} \\ &-\frac{1}{2}\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)}+\underline{\sigma}\int_0^t\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \\ &+\frac{1}{2}\left\|\frac{\boldsymbol{H}(t+h)-\boldsymbol{H}(t)}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\frac{1}{2}\left\|\frac{\boldsymbol{H}(h)-\boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \\ \leq\ &\int_0^t\left\|\frac{\boldsymbol{f}(s+h)-\boldsymbol{f}(s)}{h}\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\mathrm{d}s \\ &+\int_0^t\left\|\frac{\boldsymbol{f}(s+h)-\boldsymbol{f}(s)}{h}\right\|_{\boldsymbol{L}^2(\Omega_\sigma)}\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \\ \underbrace{\leq}_{(3.69)}\ &\frac{1}{\sqrt{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}}\int_0^t\left\|\frac{\boldsymbol{f}(s+h)-\boldsymbol{f}(s)}{h}\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}s \\ &+\int_0^t\frac{1}{2\underline{\sigma}}\left\|\frac{\boldsymbol{f}(s+h)-\boldsymbol{f}(s)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s+\int_0^t\frac{\underline{\sigma}}{2}\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \\ &\forall t\in(0,T),\ h\in(0,T-t). \end{aligned} \tag{3.73}$$

Note that if $|\Omega\setminus\Omega_\sigma|=0$ then all integrals over $\Omega\setminus\Omega_\sigma$ vanish, and we may simply set $\underline{\epsilon}(\Omega\setminus\Omega_\sigma)=1$ in the case of $|\Omega\setminus\Omega_\sigma|=0$. Now, by the Lipschitz property (3.70) and the regularity property $\boldsymbol{E}\in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega))$, it follows that

$$\begin{aligned} &\left\|\frac{\boldsymbol{E}(t+h)-\boldsymbol{E}(t)}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\underline{\sigma}\int_0^t\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \\ \leq&\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)}+\left\|\frac{\boldsymbol{H}(h)-\boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \\ &+\frac{2L(\Omega\setminus\Omega_\sigma)t}{\sqrt{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}\right\|_{L^\infty((0,t),\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma))}+\frac{L(\Omega_\sigma)^2t}{\underline{\sigma}}\quad\forall t\in(0,T),\ h\in(0,T-t). \end{aligned} \tag{3.74}$$

Our goal now is to show the boundedness of the difference quotients at the point 0 appearing on the right-hand side of (3.74). Setting $\boldsymbol{v}=\boldsymbol{E}_0$ in (P) yields

$$\begin{aligned} &\int_\Omega\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\cdot(\boldsymbol{E}(s)-\boldsymbol{E}_0)+\sigma\boldsymbol{E}(s)\cdot(\boldsymbol{E}(s)-\boldsymbol{E}_0)-\boldsymbol{H}(s)\cdot\mathbf{curl}(\boldsymbol{E}(s)-\boldsymbol{E}_0)\,\mathrm{d}x \\ &\qquad\leq\int_\Omega\boldsymbol{f}(s)\cdot(\boldsymbol{E}(s)-\boldsymbol{E}_0)\,\mathrm{d}x \end{aligned} \tag{3.75}$$

for a.e. $s \in (0,T)$. On the other hand, since $\boldsymbol{E}(s) \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ holds for a.e. $s \in (0,T)$, a combination of (3.5) and (3.67) ensures that

$$\int_\Omega -\epsilon \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}_0}_{=0} \cdot (\boldsymbol{E}(s) - \boldsymbol{E}_0) - \sigma \boldsymbol{E}_0 \cdot (\boldsymbol{E}(s) - \boldsymbol{E}_0) + \boldsymbol{H}_0 \cdot \mathbf{curl}(\boldsymbol{E}(s) - \boldsymbol{E}_0)\,\mathrm{d}x$$
$$\leq \int_\Omega -\boldsymbol{f}(0) \cdot (\boldsymbol{E}(s) - \boldsymbol{E}_0)\,\mathrm{d}x \quad \text{for a.e. } s \in (0,T). \quad (3.76)$$

Therefore, adding (3.75) and (3.76) together results in

$$\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{E}(s)-\boldsymbol{E}_0)\cdot(\boldsymbol{E}(s)-\boldsymbol{E}_0)+\sigma(\boldsymbol{E}(s)-\boldsymbol{E}_0)\cdot(\boldsymbol{E}(s)-\boldsymbol{E}_0)-(\boldsymbol{H}(s)-\boldsymbol{H}_0)\cdot\mathbf{curl}(\boldsymbol{E}(s)-\boldsymbol{E}_0)\,\mathrm{d}x$$
$$\leq \int_\Omega (\boldsymbol{f}(s) - \boldsymbol{f}(0)) \cdot (\boldsymbol{E}(s) - \boldsymbol{E}_0)\,\mathrm{d}x \quad \text{for a.e. } s \in (0,T). \quad (3.77)$$

In addition, the Faraday law in (P) along with (3.68) yields

$$\mathbf{curl}(\boldsymbol{E}(s) - \boldsymbol{E}_0) = -\mu \frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{H}(s) - \boldsymbol{H}_0) \quad \text{for a.e. } s \in (0,T). \quad (3.78)$$

Applying (3.78) to (3.77), integrating the resulting inequality over $(0,h)$ and dividing by h^2, we follow the same argumentation as before to deduce by Hölder's and Young's inequalities as well as the properties of σ that

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \frac{1}{2}\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)} + \underline{\sigma}\int_0^h \left\|\frac{\boldsymbol{E}(s)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \\
&+ \frac{1}{2}\left\|\frac{\boldsymbol{H}(h)-\boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \\
\leq{}& \int_0^h \left\|\frac{\boldsymbol{f}(s)-\boldsymbol{f}(0)}{h}\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)} \left\|\frac{\boldsymbol{E}(s)-\boldsymbol{E}_0}{h}\right\|_{\boldsymbol{L}^2(\Omega\setminus\Omega_\sigma)}\mathrm{d}s \\
&+ \int_0^h \left\|\frac{\boldsymbol{f}(s)-\boldsymbol{f}(0)}{h}\right\|_{\boldsymbol{L}^2(\Omega_\sigma)} \left\|\frac{\boldsymbol{E}(s)-\boldsymbol{E}_0}{h}\right\|_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \\
\leq{}& \int_0^h \left(\frac{1}{2\underline{\epsilon}(\Omega\setminus\Omega_\sigma)} + \frac{1}{2\underline{\sigma}}\right)\left\|\frac{\boldsymbol{f}(s)-\boldsymbol{f}(0)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{d}s + \frac{1}{2}\int_0^h \left\|\frac{\boldsymbol{E}(s)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}s \\
&+ \frac{\underline{\sigma}}{2}\int_0^h \left\|\frac{\boldsymbol{E}(s)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s \quad \forall h \in (0,T),
\end{aligned}$$

and consequently, by the Lipschitz continuity of $\boldsymbol{f}$ as well as rearrangement, we arrive at

$$\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)} + \left\|\frac{\boldsymbol{H}(h)-\boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}$$
$$\leq \frac{1}{3}hL^2\left(\frac{1}{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)} + \frac{1}{\underline{\sigma}}\right) + \int_0^h \left\|\frac{\boldsymbol{E}(s)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}\mathrm{d}s \quad \forall h \in (0,T).$$

In conclusion, Gronwall's lemma delivers

$$\left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \left\|\frac{\boldsymbol{E}(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_\sigma)} + \left\|\frac{\boldsymbol{H}(h)-\boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}$$

$$\leq \frac{1}{3} h L^2 \left(\frac{1}{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)} + \frac{1}{\underline{\sigma}} \right) \mathrm{e}^h \tag{3.79}$$

for all $h \in (0, T)$. Going back to (3.74) and on account of (3.79), we attain

$$\begin{aligned} &\left\| \frac{\boldsymbol{E}(t+h) - \boldsymbol{E}(t)}{h} \right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} + \underline{\sigma} \int_0^t \left\| \frac{\boldsymbol{E}(s+h) - \boldsymbol{E}(s)}{h} \right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} \mathrm{d}s \leq \frac{1}{3} h L^2 \left(\frac{1}{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)} + \frac{1}{\underline{\sigma}} \right) \mathrm{e}^h \\ &+ \frac{2L(\Omega \setminus \Omega_\sigma) t}{\sqrt{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)}} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E} \right\|_{L^\infty((0,t), \boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma))} + \frac{L(\Omega_\sigma)^2 t}{\underline{\sigma}} \quad \forall t \in (0,T), h \in (0, T-t). \end{aligned} \tag{3.80}$$

By passing to the limit $h \to 0$ in the first term of the left-hand side of (3.80), we obtain that

$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E}(t) \right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)} &\leq \frac{2L(\Omega \setminus \Omega_\sigma) T}{\sqrt{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)}} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E} \right\|_{L^\infty((0,T), \boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma))} + \frac{L(\Omega_\sigma)^2 T}{\underline{\sigma}} \\ &\leq \frac{2L(\Omega \setminus \Omega_\sigma)^2 T^2}{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)} + \frac{1}{2} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E} \right\|^2_{L^\infty((0,T), \boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma))} + \frac{L(\Omega_\sigma)^2 T}{\underline{\sigma}} \end{aligned}$$

for a.e. $t \in (0, T)$, from which it follows that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E} \right\|^2_{L^\infty((0,T), \boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma))} \leq \frac{4L(\Omega \setminus \Omega_\sigma)^2 T^2}{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2 T}{\underline{\sigma}}. \tag{3.81}$$

Finally, using again the regularity property $\boldsymbol{E} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$, Fatou's lemma yields

$$\begin{aligned} &\int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{E}(s) \right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} \mathrm{d}s \leq \liminf_{h \to 0} \int_0^t \left\| \frac{\boldsymbol{E}(s+h) - \boldsymbol{E}(s)}{h} \right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)} \mathrm{d}s \\ &\underbrace{\leq}_{(3.80)\&(3.81)} \frac{L(\Omega_\sigma)^2 T}{\underline{\sigma}^2} + \frac{2L(\Omega \setminus \Omega_\sigma) T}{\underline{\sigma} \sqrt{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)}} \sqrt{\frac{4L(\Omega \setminus \Omega_\sigma)^2 T^2}{\underline{\epsilon}(\Omega \setminus \Omega_\sigma)} + \frac{2L(\Omega_\sigma)^2 T}{\underline{\sigma}}} \quad \forall t \in (0,T). \end{aligned} \tag{3.82}$$

Step 2: The proof of (3.71) *for* $|\Omega \setminus \Omega_\sigma| \neq 0$. We start by inserting $\boldsymbol{v} = \boldsymbol{E}(s)$ in ($\mathrm{P_{ec}}$) and $\boldsymbol{v} = \boldsymbol{E}_{\mathrm{ec}}(s)$ in (P) to obtain that

$$\begin{aligned} &\int_{\Omega\setminus\Omega_\sigma} \epsilon \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{E}_{\mathrm{ec}}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x \\ &+ \int_\Omega \sigma \boldsymbol{E}_{\mathrm{ec}}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) - \boldsymbol{H}_{\mathrm{ec}}(s) \cdot \mathbf{curl}(\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x \\ \leq &\int_\Omega \boldsymbol{f}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x \quad \text{for a.e. } s \in (0,T) \end{aligned} \tag{3.83}$$

and

$$\begin{aligned} -\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{E}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x - \int_\Omega \sigma \boldsymbol{E}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) - \boldsymbol{H}(s) \cdot \mathbf{curl}(\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x \\ \leq -\int_\Omega \boldsymbol{f}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x \quad \text{for a.e. } s \in (0,T). \end{aligned} \tag{3.84}$$

Adding the inequalities (3.83) and (3.84) together results in

$$\int_{\Omega\setminus\Omega_\sigma} \epsilon \frac{\mathrm{d}}{\mathrm{d}s} (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x - \int_{\Omega_\sigma} \epsilon \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{E}(s) \cdot (\boldsymbol{E}_{\mathrm{ec}}(s) - \boldsymbol{E}(s)) \,\mathrm{d}x \tag{3.85}$$

$$+\int_\Omega \sigma(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\cdot(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))-(\boldsymbol{H}_{\text{ec}}(s)-\boldsymbol{H}(s))\cdot\mathbf{curl}(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\,\mathrm{d}x\leq 0$$

for a.e. $s\in(0,T)$. By the Faraday law for the solutions of (P_{ec}) and (P), we have that

$$\mathbf{curl}(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))=-\mu\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{H}_{\text{ec}}(s)-\boldsymbol{H}(s))\quad\text{for a.e. } s\in(0,T), \tag{3.86}$$

and thus applying (3.86) to (3.85) leads to

$$\begin{aligned}&\int_{\Omega\setminus\Omega_\sigma}\epsilon\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\cdot(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\,\mathrm{d}x\\&+\int_\Omega\mu\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{H}_{\text{ec}}(s)-\boldsymbol{H}(s))\cdot(\boldsymbol{H}_{\text{ec}}(s)-\boldsymbol{H}(s))\,\mathrm{d}x\\&+\int_\Omega\sigma(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\cdot(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\,\mathrm{d}x\leq\int_{\Omega_\sigma}\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\cdot(\boldsymbol{E}_{\text{ec}}(s)-\boldsymbol{E}(s))\,\mathrm{d}x\end{aligned} \tag{3.87}$$

for a.e. $s\in(0,T)$. Since $\boldsymbol{E}(0)=\boldsymbol{E}_{\text{ec}}(0)=\boldsymbol{E}_0$ in $\Omega\setminus\Omega_\sigma$ and $\boldsymbol{H}(0)=\boldsymbol{H}_{\text{ec}}(0)=\boldsymbol{H}_0$ in Ω, we find after integrating (3.87) over $(0,t)$ that

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{E}(t)-\boldsymbol{E}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\frac{1}{2}\|\boldsymbol{H}(t)-\boldsymbol{H}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\int_0^t\|\boldsymbol{E}(s)-\boldsymbol{E}_{\text{ec}}(s)\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)}\,\mathrm{d}s\\&\leq\int_0^t\left\|\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\right\|_{\boldsymbol{L}^2(\Omega_\sigma)}\|\boldsymbol{E}(s)-\boldsymbol{E}_{\text{ec}}(s)\|_{\boldsymbol{L}^2(\Omega_\sigma)}\,\mathrm{d}s\\&\leq\frac{1}{2\underline{\sigma}}\int_0^t\left\|\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s+\frac{1}{2}\int_0^t\|\boldsymbol{E}(s)-\boldsymbol{E}_{\text{ec}}(s)\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)}\,\mathrm{d}s\quad\forall t\in(0,T),\end{aligned} \tag{3.88}$$

and consequently

$$\begin{aligned}&\|\boldsymbol{E}(t)-\boldsymbol{E}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}(t)-\boldsymbol{H}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\int_0^t\|\boldsymbol{E}(s)-\boldsymbol{E}_{\text{ec}}(s)\|^2_{\boldsymbol{L}^2_\sigma(\Omega_\sigma)}\,\mathrm{d}s\\&\leq\frac{1}{\underline{\sigma}}\int_0^t\left\|\epsilon\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s\underbrace{\leq}_{(2.20)}\frac{1}{\underline{\sigma}}\|\epsilon\|^2_{L^\infty(\Omega_\sigma)^{3\times3}}\int_0^t\left\|\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2(\Omega_\sigma)}\mathrm{d}s\quad\forall t\in(0,T).\end{aligned} \tag{3.89}$$

Eventually, applying (3.82) to (3.89) yields

$$\begin{aligned}&\|\boldsymbol{E}(t)-\boldsymbol{E}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega\setminus\Omega_\sigma)}+\|\boldsymbol{H}(t)-\boldsymbol{H}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\|\boldsymbol{E}-\boldsymbol{E}_{\text{ec}}\|^2_{L^2((0,t),\boldsymbol{L}^2_\sigma(\Omega_\sigma))}\\&\leq\left(\frac{L(\Omega_\sigma)^2T}{\underline{\sigma}}+\frac{2L(\Omega\setminus\Omega_\sigma)T}{\sqrt{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}}\sqrt{\frac{4L(\Omega\setminus\Omega_\sigma)^2T^2}{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}+\frac{2L(\Omega_\sigma)^2T}{\underline{\sigma}}}\right)\left\|\frac{\epsilon}{\underline{\sigma}}\right\|^2_{L^\infty(\Omega_\sigma)^{3\times3}}\quad\forall t\in(0,T).\end{aligned}$$

In view of the regularity properties (3.65) and (3.66), the above pointwise estimate leads immediately to the uniform estimate (3.71).

Step 3: The proof of (3.72) *for* $\Omega_\sigma=\Omega$. In this case, the inequality (3.80) turns out to be

$$\underline{\sigma}\int_0^t\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{d}s\leq\frac{1}{3}hL^2\left(\frac{1}{\underline{\epsilon}(\Omega\setminus\Omega_\sigma)}+\frac{1}{\underline{\sigma}}\right)\mathrm{e}^h+\frac{L^2t}{\underline{\sigma}},$$

and so by Fatou's lemma

$$\int_0^t\left\|\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{E}(s)\right\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{d}s\leq\liminf_{h\to0}\int_0^t\left\|\frac{\boldsymbol{E}(s+h)-\boldsymbol{E}(s)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{d}s\leq\frac{L^2t}{\underline{\sigma}^2}\quad\forall t\in(0,T). \tag{3.90}$$

Now, in the case of $\Omega_\sigma = \Omega$, the inequality (3.89) reads as

$$\|\boldsymbol{H}(t) - \boldsymbol{H}_{\text{ec}}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \|\boldsymbol{E} - \boldsymbol{E}_{\text{ec}}\|^2_{L^2((0,t),\boldsymbol{L}^2_\sigma(\Omega))} \leq \frac{1}{\underline{\sigma}} \|\epsilon\|^2_{L^\infty(\Omega)^{3\times 3}} \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{E}(s) \right\|^2_{\boldsymbol{L}^2(\Omega)} \mathrm{d}s \quad \forall t \in (0,T). \tag{3.91}$$

The final claim (3.72) follows therefore by applying (3.90) to (3.91). □

3.4 Numerical Verification

We close this chapter with a brief numerical verification of our theoretical findings. In particular, our numerical test confirms the linear convergence rate with respect to $\epsilon_{|\Omega_\sigma}$ for the eddy current approximation (Theorem 3.17). Note that the following example is of merely academic nature as the conducting domain is chosen to be equal to the whole domain. So, for the test, we consider $\Omega = (-1,1)^3$, $T = 1$, $\mu \equiv 1$, $\sigma \equiv 1$, $\Omega_\sigma = \Omega$ and $(0,0)$ as an initial value. For the applied current source, we choose

$$\boldsymbol{f}\colon [0,1] \times \Omega \to \mathbb{R}^3, \quad \boldsymbol{f}(t,x_1,x_2,x_3) := \begin{cases} \left(0, \dfrac{-tx_3}{\sqrt{x_2^2+x_3^2}}, \dfrac{tx_2}{\sqrt{x_2^2+x_3^2}}\right) & \text{if } (x_1,x_2,x_3) \in P \\ 0 & \text{if } (x_1,x_2,x_3) \notin P, \end{cases}$$

where $P = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid 0 \leq x_1 \leq 0.5, 0.3 \leq \sqrt{x_2^2+x_3^2} \leq 0.5\}$ models a cylindrical pipe coil. Furthermore, the feasible set is set to be

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{v}(x)|_\infty \leq 5 \cdot 10^{-4} \text{ for a.e. } x \in \omega\},$$

with the obstacle region

$$\omega := \{(x_1,x_2,x_3) \in \Omega \mid -0.25 \leq x_1 \leq -0.125, |x_2| \leq 0.5, |x_3| \leq 0.5\}.$$

Note that the choice of the bound $5 \cdot 10^{-4}$ in the obstacle set $\boldsymbol{K}$ is of no particular importance. With the choice of our bound, we strive to model the effects of electric shielding. Our numerical computation is based on the time-discrete (implicit Euler) scheme (P_N) along with the space discretization consisting of Nédélec's edge elements (cf. [108] or (4.5)) for $\boldsymbol{E}$ and piecewise constant elements for $\boldsymbol{H}$ (cf. (4.6)). The corresponding finite element approximations of the time-discrete problems in (P_N) (with roughly 829.000 degrees of freedom) were solved by the primal dual active set algorithm (see [75,76]) implemented on the open-source platform FEniCS [92]. The visualizations were done with ParaView. We note that the primal dual method approximates the elliptic variational inequalities in (P_N) by equalities on the corresponding active and inactive sets that are iteratively updated. To verify the convergence of the eddy current approximation, we use the quantity

$$\text{Error}_k = \|(\boldsymbol{E},\boldsymbol{H})_k - (\boldsymbol{E}_{\text{ec}},\boldsymbol{H}_{\text{ec}})\|_{L^2((0,T),\boldsymbol{L}^2(\Omega)) \times \mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))}$$

with $(\boldsymbol{E},\boldsymbol{H})_k$ being the numerical solution to (P) for $\epsilon = \frac{1}{2^k}$ and $(\boldsymbol{E}_{\text{ec}},\boldsymbol{H}_{\text{ec}})$ being the numerical solution to the eddy current model (P_{ec}). Furthermore, to check the experimental order of convergence with respect to ϵ, we make use of the following quantity:

$$\text{EOC}_k = \frac{\log(\text{Error}_{k+1}) - \log(\text{Error}_k)}{\log(2^{-(k+1)}) - \log(2^{-k})}.$$

Table 3.1 depicts the computed error and experimental order of convergence for $k = 4, \dots, 14$. In agreement with our theoretical finding (Theorem 3.17), we observe that the eddy current approximation $(\mathrm{P}_{\mathrm{ec}})$ becomes closer and closer to (P) as ϵ decreases. More importantly, the experimental order of convergence is readily very close to 1, which exactly confirms the linear convergence rate in the a priori error estimate (3.72).

Table 3.1: Convergence behavior of the eddy current approximation.

$\frac{\epsilon}{\sigma}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	$\frac{1}{2^6}$	$\frac{1}{2^7}$	$\frac{1}{2^8}$	$\frac{1}{2^9}$	$\frac{1}{2^{10}}$	$\frac{1}{2^{11}}$	$\frac{1}{2^{12}}$	$\frac{1}{2^{13}}$	$\frac{1}{2^{14}}$
Error_k	$1.9 \cdot 10^{-3}$	$9.9 \cdot 10^{-4}$	$5.1 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$6.7 \cdot 10^{-5}$	$3.4 \cdot 10^{-5}$	$1.7 \cdot 10^{-5}$	$8.5 \cdot 10^{-6}$	$4.2 \cdot 10^{-6}$	$2.1 \cdot 10^{-6}$
EOC_k	0.9487	0.9512	0.9547	0.9749	0.9866	0.9933	0.9972	0.9984	0.9985	0.9991	0.9999

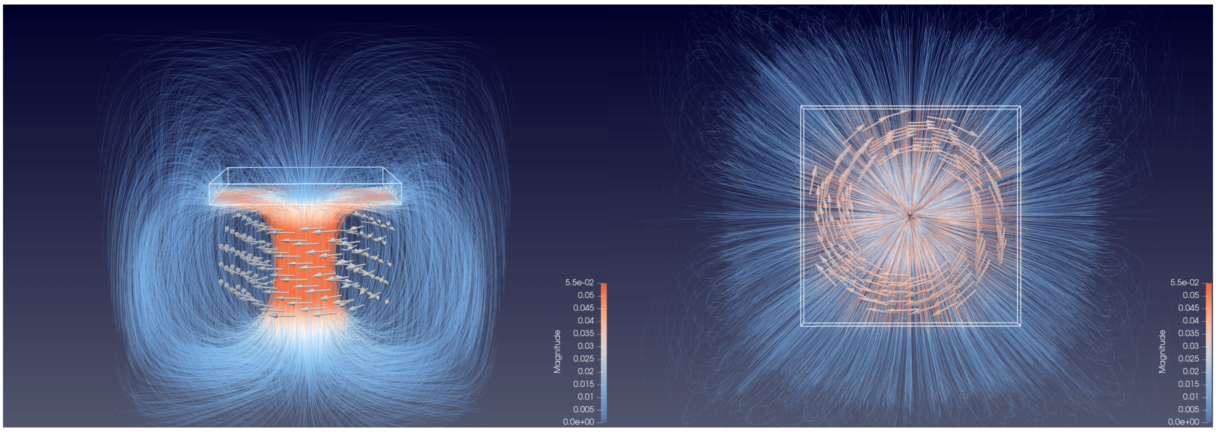

Figure 3.1: Computed magnetic field from two different views at the last time step together with the applied circular current and the outlined obstacle.

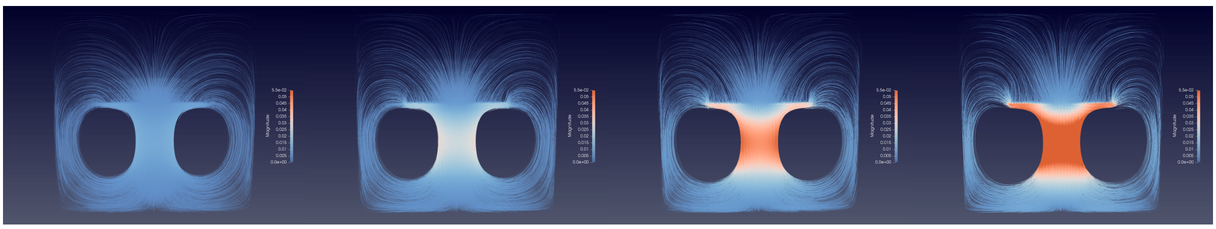

Figure 3.2: Evolution of the magnetic field at the time steps $t_n = \frac{n}{4}$ with $n \in \{1, 2, 3, 4\}$.

CHAPTER 4

NUMERICAL ANALYSIS FOR MAXWELL OBSTACLE PROBLEMS IN ELECTRIC SHIELDING

The previous chapter established a well-posedness result for the eddy current problem (P_{ec}) and in particular the problem (P) which was mainly based on the Rothe method, i.e., a discretization in time. In view of obtaining a fully discrete scheme for (P), it would now be natural to combine the time-discrete problems (P_N) with a mixed finite element methods in space. In fact, as described in Section 3.4, we have done precisely that for our numerical test. However, choosing the time-discrete problems (P_N) as a baseline for further spatial discretization is problematic since the numerical resolution of the involved elliptic **curl-curl** variational inequality requires the use of an iterative solver such as the semi-smooth Newton method or the primal dual active set strategy. Since the usage of such an iterative solver in every time step leads to extremely high computational costs, we use this chapter to propose and examine a different fully discrete approximation for the Maxwell obstacle problem (P).

Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral Lipschitz domain representing the hold-all domain. For our upcoming numerical analysis, we consider a slightly simplified and more specific structure for the electric obstacle set than in Chapter 3. Namely, we consider a polyhedral Lipschitz domain ω satisfying $\overline{\omega} \subset \Omega$. The subset ω stands for the obstacle region representing the area shielded by a closed conductive enclosure. Thus, a pointwise constraint is applied to the electric field in ω leading to the following feasible electric set:

$$\boldsymbol{K} \coloneqq \{\boldsymbol{e} \in \boldsymbol{L}^2(\Omega) \;\mid\; |\boldsymbol{e}(x)| \leq d \quad \text{for a.e. } x \in \omega\} \tag{4.1}$$

for some fixed upper bound $d \in [0, \infty)$. Then, given initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ and a source field $\boldsymbol{f} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$, the electric obstacle problem is the same as in Chapter 3 with the specific choice of (4.1) for the obstacle set, i.e.,

$$\begin{cases} \displaystyle\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) + \sigma \boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t))\,\mathrm{d}x \\ \displaystyle\geq \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \text{ for a.e. } t \in (0,T) \\ \displaystyle\mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t \in (0,T) \\ (\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \\ \boldsymbol{E}(t) \in \boldsymbol{K} \text{ for all } t \in [0,T] \text{ and } (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases} \tag{P}$$

The existence of a unique solution $(\boldsymbol{E}, \boldsymbol{H})$ to (P) follows from our previous result Theorem 3.6 by considering the case $\Omega_\sigma = \emptyset$. As apparent from Theorem 2.13, in the free region $\Omega \setminus \overline{\omega}$, the unique solution of (P) satisfies the Ampère-Maxwell equation and the local magnetic $\boldsymbol{L}^2$-regularity property $\mathbf{curl}\,\boldsymbol{H} \in L^\infty((0,T),\boldsymbol{L}^2(\Omega\setminus\overline{\omega}))$. However, in the obstacle region ω, the $\boldsymbol{L}^2$-regularity of $\mathbf{curl}\,\boldsymbol{H}$ is not a priori guaranteed.

In the following, we aim to construct and analyze an efficient finite element method for (P). We are not aware of any previous work on the numerical analysis of (P). In particular, our discretization does not employ the implicit Euler method used in Chapter 3. To describe our method, let us begin by introducing a partition of the time interval $[0,T]$ as follows: Given $N\in\mathbb{N}$, we set

$$\tau := \frac{T}{N},\quad 0=t_0<t_{\frac12}<t_1<\ldots<t_{N-\frac12}<t_N=T \quad\text{with}\quad t_n:=n\tau\quad \forall n\in\{0,\ldots,N\}$$

and intermediate time steps

$$t_{n-\frac12} := \frac{t_n+t_{n-1}}{2} = t_n-\frac{\tau}{2}\quad \forall n\in\{1,\ldots,N\}.$$

Motivated by the leapfrog (Yee) time-stepping [138] (cf. Li et al. [88,90] and Monk et al. [45,106]), we consider

- the Ampère-Maxwell variational inequality in (P) at the intermediate time steps $t_{n-\frac12}$
- the Faraday equation in (P) at the time steps t_n

and make use of the following central difference approximations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t_{n-\frac12}) \approx \frac{\boldsymbol{E}(t_n)-\boldsymbol{E}(t_{n-1})}{\tau},\quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t_n)\approx\frac{\boldsymbol{H}(t_{n+\frac12})-\boldsymbol{H}(t_{n-\frac12})}{\tau},$$

and mean value approximations

$$\boldsymbol{E}(t_{n-\frac12})\approx\frac{\boldsymbol{E}(t_n)+\boldsymbol{E}(t_{n-1})}{2}.$$

Then, invoking the piecewise constant finite element space $\mathbf{DG}_h$ (see (4.6)) and the lowest-order Nédélec finite element space $\mathbf{ND}_h$ (see (4.5)) for the spatial discretization of the electric and magnetic fields, respectively, we arrive at

$$\begin{cases}\displaystyle\int_\Omega \epsilon\delta\boldsymbol{E}_h^n\cdot(\boldsymbol{v}_h-\boldsymbol{E}_h^{n-\frac12})+\sigma\boldsymbol{E}_h^{n-\frac12}\cdot(\boldsymbol{v}_h-\boldsymbol{E}_h^{n-\frac12})-\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac12}\cdot(\boldsymbol{v}_h-\boldsymbol{E}_h^{n-\frac12})\,\mathrm{d}x\\ \displaystyle\geq\int_\Omega \boldsymbol{f}_h^{n-\frac12}\cdot(\boldsymbol{v}_h-\boldsymbol{E}_h^{n-\frac12})\,\mathrm{d}x\quad\forall\boldsymbol{v}_h\in\boldsymbol{K}\cap\mathbf{DG}_h\quad\forall n\in\{1,\ldots,N\}\\ \displaystyle\int_\Omega\mu\delta\boldsymbol{H}_h^{n+\frac12}\cdot\boldsymbol{w}_h+\boldsymbol{E}_h^n\cdot\mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x=0\quad\forall\boldsymbol{w}_h\in\mathbf{ND}_h\quad\forall n\in\{1,\ldots,N\},\end{cases}\tag{LF$_{N,h}$}$$

where

$$\delta\boldsymbol{E}_h^n:=\frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau},\quad \delta\boldsymbol{H}_h^{n+\frac12}:=\frac{\boldsymbol{H}_h^{n+\frac12}-\boldsymbol{H}_h^{n-\frac12}}{\tau},\quad \boldsymbol{E}_h^{n-\frac12}:=\frac{\boldsymbol{E}_h^n+\boldsymbol{E}_h^{n-1}}{2}\tag{4.2}$$

for all $n\in\{1,\ldots,N\}$. Furthermore, $\boldsymbol{E}_h^0\in\mathbf{DG}_h$, $\boldsymbol{H}_h^{\frac12}\in\mathbf{ND}_h$, and $\boldsymbol{f}_h^{n-\frac12}\in\mathbf{DG}_h$ are given proper finite element approximations specified as in (4.12). Now, to complete the discrete scheme, we have to properly include the obstacle structure $\boldsymbol{K}$ in the discrete system. We propose to apply the pointwise electric constraint at *the intermediate time steps* $t_{n-\frac12}$ (instead of at the time steps t_n), i.e.,

$$\boldsymbol{E}_h^{n-\frac12}\in\boldsymbol{K}\cap\mathbf{DG}_h\quad\forall n\in\{1,\ldots,N\}.\tag{4.3}$$

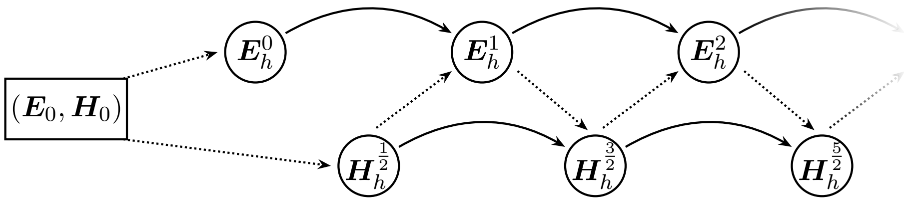

Figure 4.1: Schematic drawing of the leapfrog stepping.

The choice (4.3) is of paramount importance to obtain an efficiently computable explicit formula for the discrete electric field (Theorem 4.3). Thus, differently from the implicit Euler method, the numerical realization of our discretization does not require an additional nonlinear solver for solving the underlying VI. Altogether, utilizing

$$\delta \boldsymbol{E}_h^n = \frac{2}{\tau}\left(\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1}\right) \quad \forall n \in \{1, \ldots, N\} \tag{4.4}$$

and (4.2)-(4.3) in ($\mathrm{LF}_{N,h}$), we finally end up with the following fully discrete FEM:

$$\begin{cases} \text{Find } \{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (\boldsymbol{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ such that} \\ \int_\Omega \left(\frac{2\epsilon}{\tau} + \sigma\right) \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \geq \int_\Omega \left(\boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}\right) \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\} \\ \boldsymbol{E}_h^n = 2\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1} \\ \int_\Omega \frac{\mu}{\tau}\boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h + \boldsymbol{E}_h^n \cdot \mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x = \int_\Omega \frac{\mu}{\tau}\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x \\ \forall \boldsymbol{w}_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\}. \end{cases} \tag{$\mathrm{P}_{N,h}$}$$

The basic idea of the time-discretization by leapfrog stepping is visualized in Figure 4.1. We emphasize that, in contrast to the finite element approximation carried out in Section 3.4 where we used the Nédélec finite elements for the electric field and the piecewise constant finite elements for the magnetic field, we now use the Nédélec finite elements for the magnetic field and the piecewise constant finite elements for the electric field.

In the upcoming sections, we analyze the proposed FEM ($\mathrm{P}_{N,h}$) and deliver three main novelties: Well-posedness, stability, and convergence. The well-posedness of ($\mathrm{P}_{N,h}$) is obtained by Theorem 2.6 due to our particular choice (4.3), which leads to a computable explicit formula for the (exact) discrete electric field (see Theorem 4.3). The stability analysis relies on an additional $\boldsymbol{H}^1(\Omega)$-regularity assumption for the initial electric field $\boldsymbol{E}_0$. Along with a linear CFL-condition (4.19), it allows us to prove $\boldsymbol{L}^2$-stability for the discrete solutions and the associated difference quotients (4.2) (see Proposition 4.6 and Corollary 4.8). Based on Proposition 4.6, our analysis reveals local $\boldsymbol{L}^2$-stability for $\{\mathbf{curl}\,\boldsymbol{H}_h^{n+1/2}\}$ in the free region $\Omega \setminus \overline{\omega}$, while only $\boldsymbol{L}^1$-stability for $\{\mathbf{curl}\,\boldsymbol{H}_h^{n+1/2}\}$ is achieved in the obstacle region ω (see Proposition 4.9). This result is somehow justified by the low regularity issue in (P) pointed out earlier: In the free region $\Omega \setminus \overline{\omega}$, we have $\mathbf{curl}\,\boldsymbol{H} \in L^\infty((0,T), \boldsymbol{L}^2(\Omega \setminus \overline{\omega}))$, but there is no a priori knowledge on the $\boldsymbol{L}^2$-regularity of $\mathbf{curl}\,\boldsymbol{H}$ in the obstacle region ω (cf. Theorem 2.13). The lack of the global $\boldsymbol{L}^2$-stability for the rotational field makes the convergence analysis of ($\mathrm{P}_{N,h}$) rather challenging. Our strategy to prove a convergence result (Theorem 4.15) comprises two main stages. First, exploiting the $\boldsymbol{L}^2$-stability estimates (Proposition 4.6 and Corollary 4.8) and the $\boldsymbol{L}^1$-stability result (Proposition 4.9), we derive a convergence result for ($\mathrm{P}_{N,h}$) towards a weaker system (4.77) involving smooth test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. The second step is to recover the original system (P) from (4.77) by enlarging the feasible smooth test function set to $\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$.

We realize this part through a mollification process, which requires us to construct a mollification operator for $\boldsymbol{H}(\mathbf{curl})$-fields (see Theorem 4.14) which is able to preserve the specific constraint appearing in the obstacle set (4.1). The existence of a mollification operator for $\boldsymbol{H}(\mathbf{curl})$-fields is ensured by a recent result from Ern and Guermond [50]. The generalization to constraint preserving mollification, carried out in Section 4.3.1, uses tools from geometrical analysis and calls for a modification of the techniques presented in [50].

Let us now present the basic assumption for our analysis.

Assumption 4.1. There exist a family of Lipschitz polyhedral domains $\{\Omega_j\}_{j=1}^{j_0}$ in Ω and a subfamily $\{\Omega_j^\omega\}_{j=1}^{l_0} \subset \{\Omega_j\}_{j=1}^{j_0}$ such that

$$\Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j \in \{1, \dots, j_0\}, \quad \overline{\Omega} = \bigcup_{j=1}^{j_0} \overline{\Omega}_j, \quad \overline{\omega} = \bigcup_{j=1}^{l_0} \overline{\Omega_j^\omega}.$$

All material parameters are assumed to be piecewise constants, i.e., there exist real constants $c_j^\epsilon, c_j^\mu > 0$ and $c_j^\sigma \geq 0$ such that

$$\epsilon(x) = c_j^\epsilon, \quad \mu(x) = c_j^\mu, \quad \sigma(x) = c_j^\sigma \quad \text{for a.e. } x \in \Omega_j \text{ and every } j \in \{1, \dots, j_0\}.$$

Furthermore, we denote the lower bounds for ϵ and μ, respectively, by $\underline{\epsilon}, \underline{\mu} \in (0, \infty)$, i.e., $\epsilon(x) \geq \underline{\epsilon}$ and $\mu(x) \geq \underline{\mu}$ hold for a.e. $x \in \Omega$.

4.1 Well-posedness

In all what follows, let Assumption 4.1 be satisfied. Let $\{\mathcal{T}_h\}_{h>0}$ denote a quasi-uniform family of triangulations of Ω with $h > 0$ standing for the largest diameter of $T \in \mathcal{T}_h$. In particular, it holds that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \quad \forall h > 0.$$

The triangulation is chosen such that, for every $h > 0$, there exists a subfamily $\mathcal{T}_h^\omega$ of $\mathcal{T}_h$ with the property

$$\overline{\omega} = \bigcup_{T \in \mathcal{T}_h^\omega} T,$$

and that ϵ, μ and σ are constant in every $T \in \mathcal{T}_h$. We denote the Nédélec finite element space of the first family [108] (cf. [32]) by

$$\mathbf{ND}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathbf{curl}) \mid \boldsymbol{v}_{h|T} = a_T + b_T \times \cdot \ \text{ for some } a_T, b_T \in \mathbb{R}^3 \ \forall T \in \mathcal{T}_h\}, \tag{4.5}$$

and the piecewise constant finite element space by

$$\mathbf{DG}_h := \{\boldsymbol{w}_h \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{w}_{h|T} = a_T \ \text{ for some } a_T \in \mathbb{R}^3 \ \forall T \in \mathcal{T}_h\}. \tag{4.6}$$

Let us now introduce the standard $\boldsymbol{L}^2(\Omega)$-orthogonal projector onto $\mathbf{DG}_h$ by $\boldsymbol{Q}_h\colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ defined by

$$\boldsymbol{Q}_h \boldsymbol{v} = \sum_{T\in\mathcal{T}_h} \chi_T \frac{1}{|T|}\int_T \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \tag{4.7}$$

where $\chi_T : \mathbb{R}^3 \to \{0,1\}$ denotes the characteristic function of T. For every $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, it is well-known that $\boldsymbol{Q}_h\boldsymbol{v} \to \boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$ as $h \to 0$. Especially, if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, we obtain convergence with a linear rate, i.e., there exists a constant $C > 0$, s.t.

$$\|\boldsymbol{Q}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le Ch\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \quad \forall h > 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega). \tag{4.8}$$

Further important properties of $\boldsymbol{Q}_h\colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ are summarized in the following lemma:

Lemma 4.2. *Let Assumption 4.1 hold. Then,* $\boldsymbol{Q}_h\colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ *satisfies*

$$\begin{aligned} \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \quad &\Rightarrow \quad \|\boldsymbol{Q}_h\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall h > 0 &(4.9)\\ \boldsymbol{v} \in \boldsymbol{K} \quad &\Rightarrow \quad \boldsymbol{Q}_h\boldsymbol{v} \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall h > 0 &(4.10)\\ \boldsymbol{v} \in \boldsymbol{\mathcal{C}}^{0,1}(\overline{\Omega}) \quad &\Rightarrow \quad \|\boldsymbol{Q}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} \le \mathrm{Lip}(\boldsymbol{v})h \quad \forall h > 0, &(4.11) \end{aligned}$$

where $\mathrm{Lip}(\boldsymbol{v}) > 0$ *denotes the Lipschitz constant of* $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^{0,1}(\overline{\Omega})$.

Proof. The first property (4.9) is an immediate consequence of the definition. Suppose that $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ satisfies $|\boldsymbol{v}(x)| \le d$ for a.e. $x \in \omega$. Then, for almost every $y \in \omega$, it follows that

$$|\boldsymbol{Q}_h\boldsymbol{v}(y)| = \left|\sum_{T\in\mathcal{T}_h} \chi_T(y)\frac{1}{|T|}\int_T \boldsymbol{v}\,\mathrm{d}x\right| \le \sum_{T\in\mathcal{T}_h^\omega} \chi_T(y)\frac{1}{|T|}\int_T |\boldsymbol{v}|\,\mathrm{d}x \le d \sum_{T\in\mathcal{T}_h^\omega} \chi_T(y) = d.$$

In conclusion, (4.10) is valid. Now, suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^{0,1}(\overline{\Omega})$. Then,

$$\begin{aligned} \|\boldsymbol{Q}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} &= \operatorname*{ess\,sup}_{y\in\Omega} \left|\sum_{T\in\mathcal{T}_h} \chi_T(y)\frac{1}{|T|}\int_T \boldsymbol{v}(x)\,\mathrm{d}x - \boldsymbol{v}(y)\right| \\ &\le \operatorname*{ess\,sup}_{y\in\Omega} \sum_{T\in\mathcal{T}_h} \chi_T(y)\frac{1}{|T|}\int_T |\boldsymbol{v}(x) - \boldsymbol{v}(y)|\,\mathrm{d}x \le \mathrm{Lip}(\boldsymbol{v})h. \end{aligned}$$

This completes the proof. □

Let us now state the initial discrete values and right-hand side data involved in $(\mathrm{P}_{N,h})$:

$$\boldsymbol{f}_h^{n-\frac{1}{2}} := \boldsymbol{Q}_h\boldsymbol{f}(t_{n-\frac{1}{2}}), \quad \boldsymbol{E}_h^0 = \boldsymbol{Q}_h\boldsymbol{E}_0, \quad \boldsymbol{H}_h^{\frac{1}{2}} = \boldsymbol{\Pi}_h\boldsymbol{H}_0 \quad \forall n \in \{1,\dots,N\} \quad \forall h > 0, \tag{4.12}$$

where $\boldsymbol{\Pi}_h\colon \boldsymbol{H}(\mathbf{curl}) \to \mathbf{ND}_h$ denotes the classical Hilbert projection.

Theorem 4.3. *Let Assumption 4.1 hold. Then, for every* $N \in \mathbb{N}$ *and* $h > 0$, *the fully discrete problem* $(\mathrm{P}_{N,h})$ *admits a unique solution* $\{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (\boldsymbol{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h$. *In particular,* $\boldsymbol{E}_h^{n-\frac{1}{2}}$ *explicitly comes as*

$$\boldsymbol{E}_h^{n-\frac{1}{2}} = \begin{cases} \dfrac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} & \text{on } \mathcal{M}_h^{n-\frac{1}{2}} \\ \left(\dfrac{2\epsilon}{\tau} + \sigma\right)^{-1} \boldsymbol{g}_h^{n-\frac{1}{2}} & \text{on } \Omega \setminus \mathcal{M}_h^{n-\frac{1}{2}}, \end{cases} \tag{4.13}$$

with right-hand sides and strict superlevel sets

$$\boldsymbol{g}_h^{n-\frac{1}{2}} := \boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1} \quad \text{and} \quad \mathcal{M}_h^{n-\frac{1}{2}} := \left\{ x \in \omega \;\middle|\; \left(\frac{2\epsilon}{\tau} + \sigma\right)^{-1} |\boldsymbol{g}_h^{n-\frac{1}{2}}(x)| > d \right\}.$$

Proof. Let $n \in \{1, \dots, N\}$ be arbitrarily fixed. We assume that $(\boldsymbol{E}_h^{n-1}, \boldsymbol{H}_h^{n-\frac{1}{2}})$ is already computed in agreement with $(\mathrm{P}_{N,h})$. By virtue of Theorem 2.6, we obtain the existence of a unique solution $\boldsymbol{E}_h^{n-\frac{1}{2}} \in \boldsymbol{K} \cap \mathbf{DG}_h$ to the $\boldsymbol{L}^2(\Omega)$-elliptic variational inequality

$$\int_\Omega \left(\frac{2\epsilon}{\tau} + \sigma\right) \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\, \mathrm{d}x \geq \int_\Omega \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h\,. \tag{4.14}$$

The discrete magnetic field $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \mathbf{ND}_h$ is then obtained by the Lax-Milgram lemma since $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}$ are equivalent norms in the finite-dimensional space $\mathbf{ND}_h$. Let us now verify the explicit formula (4.13). Let $\boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h$. First, it holds that

$$\begin{aligned} &\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left| \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right| \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \right) \mathrm{d}x \\ = &\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left| \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right| \underbrace{\boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \boldsymbol{v}_h}_{\leq d|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \mathrm{d}x - \int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left| \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right| d|\boldsymbol{g}_h^{n-\frac{1}{2}}|\, \mathrm{d}x \leq 0. \end{aligned} \tag{4.15}$$

Since

$$\frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 = \frac{d}{\left(\frac{2\epsilon}{\tau} + \sigma\right)^{-1} |\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 < 0 \quad \text{on } \mathcal{M}_h^{n-\frac{1}{2}},$$

multiplying (4.15) by a sign implies

$$\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left(\frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right) \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \right) \mathrm{d}x \geq 0. \tag{4.16}$$

Now, rearrangement in (4.16) yields

$$\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left(\frac{2\epsilon}{\tau} + \sigma\right) \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \right) \mathrm{d}x \geq \int_{\mathcal{M}_h^{n-\frac{1}{2}}} \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \right) \mathrm{d}x. \tag{4.17}$$

By construction, for the set $\Omega \setminus \mathcal{M}_h^{n-\frac{1}{2}}$ there is nothing to show. As a conclusion, $\boldsymbol{E}_h^{n-\frac{1}{2}}$ as stated in (4.13), is the unique solution to (4.14). □

4.2 Stability

From the classical inverse estimate for finite-dimensional subspaces of $H^1(\Omega)$ (see [2, Theorem 1.3]) and the continuous embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{H}(\mathbf{curl})$, we obtain an inverse estimate for the space $\mathbf{ND}_h$. To be specific, there exists a constant $C_{\text{inv}} > 0$ such that

$$\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_{\text{inv}}}{h} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \mathbf{ND}_h\,. \tag{4.18}$$

Assumption 4.4. We require the following growth condition on the time-discretization parameter and regularity of the initial data .

(i) The linear CFL-condition

$$\tau \leq \frac{1}{2c_\nu C_{\text{inv}}} h \tag{4.19}$$

holds true. Here, $c_\nu := 1/\sqrt{\underline{\epsilon}\underline{\mu}}$ denotes the uniform lower bound for the wave propagation speed in Ω.

(ii) The initial electromagnetic field $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ is assumed to additionally satisfy $\boldsymbol{E}_0 \in \boldsymbol{H}^1(\Omega)$.

Both the CFL-condition (4.19) (see also [115]) and $\boldsymbol{E}_0 \in \boldsymbol{H}^1(\Omega)$ serve as the fundaments for our stability analysis. In view of (4.8), the corresponding discrete initial value $\boldsymbol{E}_h^0 = \boldsymbol{Q}_h \boldsymbol{E}_0$ satisfies

$$\|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} \leq Ch\|\boldsymbol{E}_0\|_{\boldsymbol{H}^1(\Omega)} \quad \forall h > 0. \tag{4.20}$$

In the sequel, we mainly take advantage of the structure ($\text{LF}_{N,h}$), which is by the construction automatically satisfied by the unique solution to ($\text{P}_{N,h}$).

Lemma 4.5. *Let Assumption 4.1 and Assumption 4.4 hold. Then, there exists a constant $C > 0$ such that for all $h > 0$ and $N \in \mathbb{N}$ the unique solution to* ($\text{P}_{N,h}$) *satisfies*

$$\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C. \tag{4.21}$$

Proof. Let $h > 0$ and $N \in \mathbb{N}$ be arbitrarily fixed. We start by setting $\boldsymbol{v}_h = 0$ in ($\text{LF}_{N,h}$) to obtain that

$$\int_\Omega \epsilon \delta \boldsymbol{E}_h^1 \cdot \boldsymbol{E}_h^{\frac{1}{2}} + \sigma \boldsymbol{E}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} - \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x \leq \int_\Omega \boldsymbol{f}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x. \tag{4.22}$$

Multiplying the above inequality by τ, applying (4.4), and using that σ is nonnegative implies,

$$\int_\Omega 2\epsilon(\boldsymbol{E}_h^{\frac{1}{2}} - \boldsymbol{E}_h^0) \cdot \boldsymbol{E}_h^{\frac{1}{2}} - \tau\, \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x \leq \int_\Omega \tau \boldsymbol{f}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x,$$

from which we deduce that

$$2\underline{\epsilon}\|\boldsymbol{E}_h^{\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\tau \boldsymbol{f}_h^{\frac{1}{2}} + 2\epsilon \boldsymbol{E}_h^0 + \tau\, \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{E}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.23}$$

Now, by construction of $\boldsymbol{f}_h^{\frac{1}{2}}, \boldsymbol{H}_h^{\frac{1}{2}}$ and $\boldsymbol{E}_h^0$, the first norm on the right-hand side of (4.23) is uniformly bounded. As a consequence, it follows

$$\|\boldsymbol{E}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C \tag{4.24}$$

with $C > 0$, independent of N and h. Let us mention that according to Assumption 4.4 along with (4.10) and (4.12), the field $\boldsymbol{E}_h^0$ is admissible, i.e., $\boldsymbol{E}_h^0 \in \boldsymbol{K} \cap \mathbf{DG}_h$. Thus, we may set $\boldsymbol{v}_h = \boldsymbol{E}_h^0$ in ($\text{LF}_{N,h}$) to conclude that $\delta \boldsymbol{E}_h^1$ admits a uniform bound in $\boldsymbol{L}^2(\Omega)$. This way we receive after multiplication with $-\frac{2}{\tau}$ together with applying (4.4) that

$$\int_\Omega \epsilon \delta \boldsymbol{E}_h^1 \cdot \delta \boldsymbol{E}_h^1 + \sigma \boldsymbol{E}_h^{\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^1 - \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^1 \,\mathrm{d}x \leq \int_\Omega \boldsymbol{f}_h^{\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^1 \,\mathrm{d}x$$

and therefore

$$\underline{\epsilon}\|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{f}_h^{\frac{1}{2}} - \sigma \boldsymbol{E}_h^{\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)}.$$

Utilizing (4.24) then leads to

$$\|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} \le C \tag{4.25}$$

with $C > 0$, independent of N and h. Let us now prove that $\delta\boldsymbol{H}_h^{\frac{3}{2}}$ admits a uniform bound in $\boldsymbol{L}^2(\Omega)$. We start by setting $\boldsymbol{w}_h = \delta\boldsymbol{H}_h^{\frac{3}{2}}$ in $(\mathrm{LF}_{N,h})$ to derive

$$\int_\Omega \mu\delta\boldsymbol{H}_h^{\frac{3}{2}} \cdot \delta\boldsymbol{H}_h^{\frac{3}{2}} + \boldsymbol{E}_h^1 \cdot \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x = 0. \tag{4.26}$$

To complete the proof, we employ (4.26) to estimate

$$\begin{aligned}
&\underline{\mu}\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} \le \left|\int_\Omega \boldsymbol{E}_h^1 \cdot \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x\right| \\
\underbrace{\le}_{(2.19)}\ & \left|\int_\Omega (\boldsymbol{E}_h^1 - \boldsymbol{E}_h^0)\cdot \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x\right| + \left|\int_\Omega (\boldsymbol{E}_h^0 - \boldsymbol{E}_0)\cdot \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x\right| + \left|\int_\Omega \mathbf{curl}\,\boldsymbol{E}_0 \cdot \delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x\right| \\
\le\ & \tau\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)}\|\,\mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\|\,\mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
& + \|\,\mathbf{curl}\,\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{\le}_{(4.18)}\ & \left(\tau\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)}\frac{C_{\mathrm{inv}}}{h} + \|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\frac{C_{\mathrm{inv}}}{h} + \|\,\mathbf{curl}\,\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\right)\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{\le}_{(4.19),(4.20)}\ & \left(\frac{\sqrt{\epsilon}\sqrt{\underline{\mu}}}{2}\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)} + CC_{\mathrm{inv}}\|\boldsymbol{E}_0\|_{\boldsymbol{H}^1(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\right)\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{\le}_{(4.25)}\ & C\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}
\end{aligned}$$

with a constant $C > 0$, independent of N and h. □

Proposition 4.6. *Let Assumption 4.1 and Assumption 4.4 be satisfied. Then, there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \ge 2$ and $h > 0$ the unique solution to* $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n\in\{2,\dots,N\}}\left[\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}\right] \le C. \tag{4.27}$$

Proof. Let $N \in \mathbb{N}$ with $N \ge 2$ and $h > 0$ be arbitrarily fixed. We choose $n_0 \in \{2,\dots,N\}$ and $n \in \{2,\dots,n_0\}$. Let us first note that it holds

$$\boldsymbol{E}_h^{n-\frac{3}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}} = \frac{\boldsymbol{E}_h^{n-1} + \boldsymbol{E}_h^{n-2}}{2} - \frac{\boldsymbol{E}_h^{n} + \boldsymbol{E}_h^{n-1}}{2} = -\frac{\tau}{2}(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1}). \tag{4.28}$$

By construction, both the fields $\boldsymbol{E}_h^{n-\frac{1}{2}}$ and $\boldsymbol{E}_h^{n-\frac{3}{2}}$ are admissible, i.e., they belong to $\boldsymbol{K}\cap\mathbf{DG}_h$. Hence, we are able to test with $\boldsymbol{E}_h^{n-\frac{3}{2}}$ (resp. with $\boldsymbol{E}_h^{n-\frac{1}{2}}$) in the n-th inequality of $(\mathrm{LF}_{N,h})$ (resp. the $(n-1)$-th inequality of $(\mathrm{LF}_{N,h})$) and thus obtain by multiplication with $-\frac{2}{\tau}$ together with (4.28) that

$$\begin{aligned}
\int_\Omega \epsilon\delta\boldsymbol{E}_h^n \cdot (\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1}) + \sigma\boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1}) - \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x& \\
\le \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x&
\end{aligned} \tag{4.29}$$

and

$$-\int_\Omega \epsilon\delta\boldsymbol{E}_h^{n-1}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})+\sigma\boldsymbol{E}_h^{n-\frac{3}{2}}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})-\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{3}{2}}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \leq -\int_\Omega \boldsymbol{f}_h^{n-\frac{3}{2}}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x. \quad (4.30)$$

Adding together (4.29) and (4.30) as well as using σ being nonnegative yields

$$\int_\Omega \epsilon(\delta\boldsymbol{E}_h^n-\delta\boldsymbol{E}_h^{n-1})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})-\mathbf{curl}(\boldsymbol{H}_h^{n-\frac{1}{2}}-\boldsymbol{H}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \leq \int_\Omega(\boldsymbol{f}_h^{n-\frac{1}{2}}-\boldsymbol{f}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x. \quad (4.31)$$

We sum up the inequality (4.31) over $\{2,\ldots,n_0\}$:

$$\sum_{n=2}^{n_0}\int_\Omega \epsilon(\delta\boldsymbol{E}_h^n-\delta\boldsymbol{E}_h^{n-1})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x-\sum_{n=2}^{n_0}\int_\Omega\mathbf{curl}(\boldsymbol{H}_h^{n-\frac{1}{2}}-\boldsymbol{H}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \leq \sum_{n=2}^{n_0}\int_\Omega(\boldsymbol{f}_h^{n-\frac{1}{2}}-\boldsymbol{f}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x. \quad (4.32)$$

For the left-hand side of (4.32), we have

$$\begin{aligned}\sum_{n=2}^{n_0}\int_\Omega \epsilon(\delta\boldsymbol{E}_h^n-\delta\boldsymbol{E}_h^{n-1})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x &= \sum_{n=2}^{n_0}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\|\delta\boldsymbol{E}_h^{n-1}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \quad (4.33)\\ &= \|\delta\boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\|\delta\boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\end{aligned}$$

and

$$\begin{aligned}&-\sum_{n=2}^{n_0}\int_\Omega\mathbf{curl}(\boldsymbol{H}_h^{n-\frac{1}{2}}-\boldsymbol{H}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \quad (4.34)\\ &=-\tau\sum_{n=2}^{n_0}\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x\\ &=-\tau\sum_{n=2}^{n_0-1}\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x-\tau\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n_0}\,\mathrm{d}x\\ &\quad-\tau\sum_{n=3}^{n_0}\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n-1}\,\mathrm{d}x-\tau\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\cdot\delta\boldsymbol{E}_h^1\,\mathrm{d}x\\ &=-\tau\sum_{n=2}^{n_0-1}\int_\Omega\mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x-\tau\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n_0}\,\mathrm{d}x\\ &\quad-\tau\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\cdot\delta\boldsymbol{E}_h^1\,\mathrm{d}x.\end{aligned}$$

For the first summand on the right-hand side of (4.34),

$$\begin{aligned}&-\tau\sum_{n=2}^{n_0-1}\int_\Omega\mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x \quad (4.35)\\ &=-\sum_{n=2}^{n_0-1}\int_\Omega\mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\boldsymbol{E}_h^n\,\mathrm{d}x+\sum_{n=2}^{n_0-1}\int_\Omega\mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\boldsymbol{E}_h^{n-1}\,\mathrm{d}x=:R.\end{aligned}$$

Testing with $\boldsymbol{w}_h = \delta \boldsymbol{H}_h^{n+\frac{1}{2}} + \delta \boldsymbol{H}_h^{n-\frac{1}{2}}$ in the n-th (resp. the $(n-1)$-th) equality of $(\mathrm{LF}_{N,h})$, we continue with

$$\begin{aligned} R &= \sum_{n=2}^{n_0-1} \int_\Omega \mu \delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot (\delta \boldsymbol{H}_h^{n+\frac{1}{2}} + \delta \boldsymbol{H}_h^{n-\frac{1}{2}}) \,\mathrm{d}x - \sum_{n=2}^{n_0-1} \int_\Omega \mu \delta \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\delta \boldsymbol{H}_h^{n+\frac{1}{2}} + \delta \boldsymbol{H}_h^{n-\frac{1}{2}}) \,\mathrm{d}x \\ &= \sum_{n=2}^{n_0-1} \int_\Omega \mu (\delta \boldsymbol{H}_h^{n+\frac{1}{2}} - \delta \boldsymbol{H}_h^{n-\frac{1}{2}}) \cdot (\delta \boldsymbol{H}_h^{n+\frac{1}{2}} + \delta \boldsymbol{H}_h^{n-\frac{1}{2}}) \,\mathrm{d}x \\ &= \sum_{n=2}^{n_0-1} \|\delta \boldsymbol{H}_h^{n+\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} - \|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} = \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} - \|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}. \end{aligned} \tag{4.36}$$

Let us now consider the inverse estimate (4.18) and the imposed CFL-condition (4.19) to obtain that

$$\begin{aligned} &\tau \int_\Omega \mathbf{curl}\, \delta \boldsymbol{H}_h^{n_0-\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^{n_0} \,\mathrm{d}x \le \tau \|\mathbf{curl}\, \delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2(\Omega)} \\ \le& \frac{\tau}{h} \frac{C_{\mathrm{inv}}}{\sqrt{\underline{\epsilon}}\sqrt{\underline{\mu}}} \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2_\mu(\Omega)} \|\delta \boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \frac{1}{2} \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2_\mu(\Omega)} \|\delta \boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\ \le& \frac{1}{4} \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{1}{4} \|\delta \boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \end{aligned} \tag{4.37}$$

By an analogous argumentation, we infer that

$$\tau \int_\Omega \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \cdot \delta \boldsymbol{E}_h^1 \,\mathrm{d}x \le \frac{1}{4} \|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{1}{4} \|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \tag{4.38}$$

Finally let us estimate the right-hand side of (4.32) as follows:

$$\begin{aligned} &\sum_{n=2}^{n_0} \int_\Omega (\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) \,\mathrm{d}x \le \sum_{n=2}^{n_0} \|\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \\ \le\ & \sum_{n=2}^{n_0} \frac{4N}{\underline{\epsilon}} \|\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} + \sum_{n=2}^{n_0} \frac{\underline{\epsilon}}{16N} \left(\|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \right)^2 \\ \underbrace{\le}_{(4.9)}\ & \frac{4L^2T^2}{\underline{\epsilon}} + \sum_{n=2}^{n_0} \frac{\underline{\epsilon}}{8N} \left(\|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{E}_h^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} \right) \\ \le\ & \frac{4L^2T^2}{\underline{\epsilon}} + \sum_{n=1}^{n_0} \frac{1}{4N} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \frac{4L^2T^2}{\underline{\epsilon}} + \frac{1}{4} \|\delta \boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \sum_{n=1}^{n_0-1} \frac{1}{4N} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}, \end{aligned} \tag{4.39}$$

where $L > 0$ denotes the Lipschitz constant of $\boldsymbol{f} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega))$. Applying (4.33)-(4.39) to (4.32) now yields

$$\begin{aligned} &\frac{1}{2} \|\delta \boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{3}{4} \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \\ \le& \frac{4L^2T^2}{\underline{\epsilon}} + \frac{5}{4} \|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{5}{4} \|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \sum_{n=1}^{n_0-1} \frac{1}{4N} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \end{aligned}$$

The discrete version of the Gronwall lemma (see Lemma 3.4) together with Lemma 4.5 then leads to

$$\|\delta \boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le C \exp\left(\sum_{n=1}^{n_0-1} \frac{1}{N} \right) \le C \exp(1),$$

or equivalently

$$\|\delta \boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C$$

with a generic constant $C > 0$, independent of N and h. This completes the proof. □

Remark 4.7. We underline that (4.27) does not guarantee the stability of $\|\delta \boldsymbol{H}_h^{N+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}$. The stability of this term can be obtained by performing one additional step $N+1$ in $(\mathrm{LF}_{N,h})$ under an appropriate choice for $\boldsymbol{f}_h^{N+\frac{1}{2}}$. However, as we shall see in the upcoming section, the estimate (4.27) is readily sufficient for proving the convergence of the proposed scheme $(\mathrm{LF}_{N,h})$, i.e., without performing one additional step $N+1$ in $(\mathrm{LF}_{N,h})$.

Corollary 4.8. *Let Assumption 4.1 and Assumption 4.4 be satisfied. Then, there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$ the unique solution to* $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n \in \{1,\dots,N\}} \left[\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \right] \leq C. \tag{4.40}$$

Proof. Using the reversed triangle inequality, it follows by the definition of the difference quotients (4.2) together with Proposition 4.6 and Lemma 4.5 that

$$\frac{1}{\tau}\left(\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} - \|\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\right) \leq \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \leq C \quad \forall n \in \{1,\dots,N\},$$

which implies

$$\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \leq \tau C + \|\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall n \in \{1,\dots,N\}.$$

Using the same argumentation for $\|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}$, we derive iteratively that

$$\begin{aligned} \|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} &\leq n\tau C + \|\boldsymbol{E}_h^0\|_{\boldsymbol{L}^2(\Omega)} + (n-1)\tau C + \|\boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq 2TC + \|\boldsymbol{E}_h^0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C \quad \forall n \in \{1,\dots,N\} \end{aligned} \tag{4.41}$$

with a generic constant $C > 0$, independent of N and h. □

Proposition 4.9. *Let Assumption 4.1 and Assumption 4.4 be satisfied. Then, there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$ the unique solution to* $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n \in \{1,\dots,N-1\}} \left[\|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^1(\omega)} + \|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega\setminus\omega)} \right] \leq C. \tag{4.42}$$

Proof. Let $n \in \{1,\dots,N-1\}$ be arbitrarily fixed. We define

$$\boldsymbol{z}_h^{n-\frac{1}{2}}(x) := \begin{cases} \dfrac{d \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}(x)}{|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}(x)|} & \text{if } \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}(x) \neq 0 \\ 0 & \text{if } \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}(x) = 0, \end{cases} \tag{4.43}$$

$$\boldsymbol{z}_{h,\omega}^{n-\frac{1}{2}} := \begin{cases} \boldsymbol{z}_h^{n-\frac{1}{2}} & \text{on } \omega \\ \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{on } \Omega\setminus\omega. \end{cases}$$

Obviously, $\boldsymbol{z}_{h,\omega}^{n-\frac{1}{2}}$ is an element of the set $\boldsymbol{K} \cap \mathbf{DG}_h$, and we can therefore set $\boldsymbol{v}_h = \boldsymbol{z}_{h,\omega}^{n-\frac{1}{2}}$ in $(\mathrm{LF}_{N,h})$ to obtain

$$\int_\omega \epsilon\delta \boldsymbol{E}_h^n \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}}) + \sigma \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}}) - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \geq \int_\omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x. \quad (4.44)$$

Altogether, in view of Proposition 4.6 and Corollary 4.8 as well as for a generic constant $C > 0$, independent of N and h, we obtain that

$$\begin{aligned} d\|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^1(\omega)} &\underbrace{=}_{(4.43)} \int_\omega \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{z}_h^{n-\frac{1}{2}}\,\mathrm{d}x \\ &\underbrace{\leq}_{(4.44)} \int_\omega \left(\epsilon\delta\boldsymbol{E}_h^n + \sigma\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{1}{2}}\right) \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x + \int_\omega \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n-\frac{1}{2}}\,\mathrm{d}x \\ &\leq C + \int_\omega \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n-\frac{1}{2}}\,\mathrm{d}x \\ &= C + \frac{1}{2}\int_\omega \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n}\,\mathrm{d}x + \frac{1}{2}\int_\omega \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n-1}\,\mathrm{d}x \\ &\underbrace{=}_{(\mathrm{LF}_{N,h})} C - \frac{1}{2}\int_\omega \mu\delta\boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{H}_h^{n-\frac{1}{2}}\,\mathrm{d}x - \frac{1}{2}\int_\omega \mu\delta\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{H}_h^{n-\frac{1}{2}}\,\mathrm{d}x \leq C. \end{aligned}$$

Now, to obtain a bound for the term $\|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega\setminus\omega)}$, we define

$$\boldsymbol{z}_{h,\Omega\setminus\omega}^{n-\frac{1}{2}} := \begin{cases} \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{on } \omega \\ \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{on } \Omega\setminus\omega. \end{cases}$$

Then, $\boldsymbol{z}_{h,\Omega\setminus\omega}^{n-\frac{1}{2}}$ is also an element of the set $\boldsymbol{K}\cap\mathbf{DG}_h$, and so using it as a test function in $(\mathrm{LF}_{N,h})$ leads to

$$\begin{aligned} \|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega\setminus\omega)} &\leq \int_{\Omega\setminus\omega} \left(\epsilon\delta\boldsymbol{E}_h^n + \sigma\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{1}{2}}\right) \cdot \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\,\mathrm{d}x \\ &\leq C\|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega\setminus\omega)}, \end{aligned}$$

again for a constant $C > 0$, independent of N and h, where we have used Proposition 4.6 and Corollary 4.8 for the last inequality. This completes the proof. □

4.3 Convergence

Given $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$, we consider the solution $\{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N$ to $(\mathrm{P}_{N,h})$. Invoking those finite element solutions, we set up linear and piecewise constant interpolations

$$\begin{aligned} \boldsymbol{E}_{N,h}, \overline{\boldsymbol{E}}_{N,h}, \widehat{\boldsymbol{f}}_{N,h} &\colon [0,T] \to \mathbf{DG}_h \\ \widehat{\boldsymbol{E}}_{N,h} &\colon [0,T] \to \boldsymbol{K}\cap\mathbf{DG}_h \\ \boldsymbol{H}_{N,h}, \widehat{\boldsymbol{H}}_{N,h} &\colon [0,T] \to \mathbf{ND}_h, \end{aligned}$$

which, for $t \in [0,T]$, are defined by

$$\boldsymbol{E}_{N,h}(t) = \begin{cases} \boldsymbol{E}_h^0 & \text{if } t = 0 \\ \boldsymbol{E}_h^{n-1} + (t - t_{n-1})\delta \boldsymbol{E}_h^n & \text{if } t \in (t_{n-1}, t_n], \end{cases} \tag{4.45}$$

$$\boldsymbol{H}_{N,h}(t) = \begin{cases} \boldsymbol{H}_h^{\frac{1}{2}} & \text{if } t = 0 \\ \boldsymbol{H}_h^{n-\frac{1}{2}} + (t - t_{n-1})\delta \boldsymbol{H}_h^{n+\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n \in \{1, \dots, N-1\} \\ \boldsymbol{H}_h^{N-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n = N, \end{cases}$$

and

$$\widehat{\boldsymbol{E}}_{N,h}(t) = \begin{cases} \boldsymbol{E}_h^0 & \text{if } t = 0 \\ \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n], \end{cases} \tag{4.46}$$

$$\widehat{\boldsymbol{H}}_{N,h}(t) = \begin{cases} \boldsymbol{H}_h^{\frac{1}{2}} & \text{if } t = 0 \\ \boldsymbol{H}_h^{n-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n \in \{1, \dots, N-1\} \\ \boldsymbol{H}_h^{N-\frac{3}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n = N, \end{cases}$$

$$\overline{\boldsymbol{E}}_{N,h}(t) = \begin{cases} \boldsymbol{E}_h^0 & \text{if } t = 0 \\ \boldsymbol{E}_h^n & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

$$\widehat{\boldsymbol{f}}_{N,h}(t) = \begin{cases} \boldsymbol{f}_h^{\frac{1}{2}} & \text{if } t = 0 \\ \boldsymbol{f}_h^{n-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n]. \end{cases}$$

Note that, since the pointwise electric constraint is applied at the intermediate time steps $t_{n-\frac{1}{2}}$ instead of at the time steps t_n, only the range of the piecewise constant interpolation $\widehat{\boldsymbol{E}}_{N,h}$ is contained in the obstacle set $\boldsymbol{K}$. By the above construction and in view of $(\mathrm{LF}_{N,h})$ as well as (4.3), it then follows that

$$\begin{cases} \displaystyle\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t)) + \sigma \widehat{\boldsymbol{E}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t)) \\ \qquad - \mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t))\, \mathrm{d}x \\ \geq \displaystyle\int_\Omega \widehat{\boldsymbol{f}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t))\, \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall t \in \left(0, T - \frac{T}{N}\right] \\ \displaystyle\int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t) \cdot \boldsymbol{w}_h + \overline{\boldsymbol{E}}_{N,h}(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h \, \mathrm{d}x = 0 \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h \quad \forall t \in \left(0, T - \frac{T}{N}\right] \\ \widehat{\boldsymbol{E}}_{N,h}(t) \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall t \in [0,T]. \end{cases} \tag{$\widetilde{\mathrm{P}}_{N,h}$}$$

The convergence analysis of the scheme $(\widetilde{\mathrm{P}}_{N,h})$ turns out to be challenging due to the lack of $L^\infty((0,T), \boldsymbol{L}^2(\omega))$-boundedness of $\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}$. Provided the weaker boundedness in the space $L^\infty((0,T), \boldsymbol{L}^1(\omega))$ (see Proposition 4.9), our first step consists of bypassing the missing boundedness by exploiting $\boldsymbol{Q}_h \boldsymbol{v}$ for functions $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. In this way, we are able to derive a convergence result towards a solution of a time integrated version of the variational inequality in (P) with test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. The final step is to enlarge the test function set to $\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$, which requires the construction of a constraint preserving mollification operator.

4.3.1 Constraint Preserving Mollification

Recently, Ern and Guermond [50] established novel mollification operators with pivotal commuting and convergence properties (cf. also [40] and [51]). Their construction is based on the

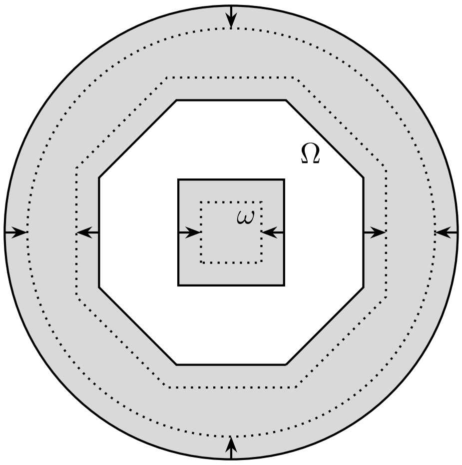

Figure 4.2: Schematic drawing of $\mathcal{O}$ (gray) and its inwardly transversal vector field.

use of a transversal vector field [72] along with a cut-off strategy in a careful combination with mollification techniques. Our goal is to extend [50] to constraint preserving mollification operators in the sense that the mollification of a function in $\boldsymbol{K}$ lies as well in $\boldsymbol{K}$. The extension is mainly complicated due to the fact that there is no a priori knowledge of how the obstacle region ω behaves under the expansion as in [50]. We tackle this issue by modifying the mollification operator in [50] with a certain scaling and by choosing the transversal vector field in a way that the obstacle set boundary $\partial\omega$ is transported inwardly (cf. Figure 4.2). Given $\boldsymbol{v} \in \boldsymbol{L}^1(\Omega)$ we denote its zero-extension to the whole space $\mathbb{R}^3$ by $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^1(\mathbb{R}^3)$. Furthermore, let

$$\rho\colon \mathbb{R}^3 \to \mathbb{R}, \quad \rho(x) = \begin{cases} \eta \exp\left(-\dfrac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $\eta > 0$ is chosen such that

$$\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = \int_{B(0,1)} \rho(x)\,\mathrm{d}x = 1. \tag{4.47}$$

At first, since Ω is bounded, there exist some $x_\Omega \in \mathbb{R}^3$ and a radius $r_\Omega > 0$ such that $\overline{\Omega} \subset B(x_\Omega, r_\Omega)$. Then

$$\mathcal{O} \coloneqq B(x_\Omega, r_\Omega) \setminus \left(\overline{\Omega} \setminus \omega\right) = \left(B(x_\Omega, r_\Omega) \setminus \overline{\Omega}\right) \cup \omega$$

represents a bounded and open set with Lipschitz boundary. Let us now introduce the notion of a transversal vector field. Here, we present the definition in its full generality, in particular, we do not restrict to Lipschitz domains but use sets of locally finite perimeter instead. A Lebesgue-measurable set $\mathcal{U} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is said to have locally finite perimeter, if

$$\chi_{\mathcal{U}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d). \tag{4.48}$$

From the theory of functions with bounded variation (cf. [52, Chapter 5]), specifically the Gauss-Green Theorem [52, Theorem 5.16 and Theorem 5.23], we can characterize (4.48) equivalently as

$$|\partial_\star \mathcal{U} \cap K| < \infty \quad \text{for every compact set } K \subset \mathbb{R}^d,$$

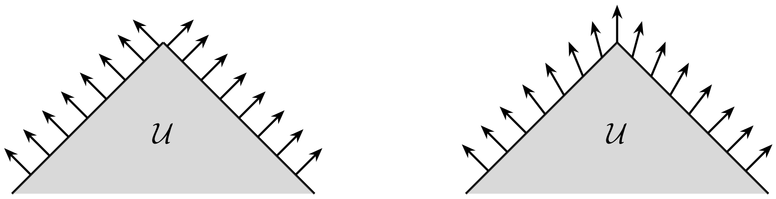

Figure 4.3: Part of a domain $\mathcal{U}$ with its discontinuous outward unit normal field (left) and with an example of a (normalized and smooth) outwardly transversal vector field (right).

where the measure-theoretic boundary $\partial_\star\mathcal{U}$ of $\mathcal{U}$ is defined by

$$\partial_\star\mathcal{U} := \left\{ x \in \partial\mathcal{U} \mid \limsup_{r\to 0^+} r^{-d} \min\left\{ |\mathcal{U}\cap B(x,r)|, |\mathcal{U}^c \cap B(x,r)| \right\} > 0 \right\}.$$

Definition 4.10. Let $d \in \mathbb{N}$ be given and let $\mathcal{U} \subset \mathbb{R}^d$ be an open set of locally finite perimeter, with outward unit normal $\boldsymbol{n}$ and surface measure σ.

(i) Given $r > 0$ and $x_0 \in \partial\mathcal{U}$, we say that a continuous vector field $\boldsymbol{v}\colon B(x_0,r)\cap\partial\mathcal{U} \to \mathbb{R}^d$ is (outwardly) transversal to $\partial\mathcal{U}$ near x_0, if there exists $\kappa > 0$ such that

$$\boldsymbol{n}(x)\cdot\boldsymbol{v}(x) \geq \kappa \quad \text{for } \sigma\text{-a.e. } x \in B(x_0,r)\cap\partial\mathcal{U}. \tag{4.49}$$

(ii) We say that $\mathcal{U}$ has continuous (outwardly) transversal vector fields provided that for every $x_0 \in \mathcal{U}$ there exists a radius $r > 0$ and a continuous vector field $\boldsymbol{v}\colon B(x_0,r)\cap\partial\mathcal{U} \to \mathbb{R}^d$ which is (outwardly) transversal to $\partial\mathcal{U}$ near x_0.

(iii) The set $\mathcal{U}$ is said to have a continuous (outwardly) globally transversal vector field, if there exist a continuous vector field $\boldsymbol{V}\colon \partial\mathcal{U} \to \mathbb{R}^d$ and a constant $\kappa > 0$, the transversality constant of $\boldsymbol{V}$, such that

$$\boldsymbol{n}(x)\cdot\boldsymbol{V}(x) \geq \kappa \quad \text{for } \sigma\text{-a.e. } x \in \partial\mathcal{U}. \tag{4.50}$$

The notion of inward transversality is obtained by replacing the outward unit normal $\boldsymbol{n}$ by the inward unit normal $-\boldsymbol{n}$, so that (4.49) and (4.50) are, respectively, replaced by

$$\begin{aligned} \boldsymbol{n}(x)\cdot\boldsymbol{v}(x) &\leq -\kappa \quad \text{for } \sigma\text{-a.e. } x \in B(x_0,r)\cap\partial\mathcal{U} \\ \boldsymbol{n}(x)\cdot\boldsymbol{V}(x) &\leq -\kappa \quad \text{for } \sigma\text{-a.e. } x \in \partial\mathcal{U}. \end{aligned}$$

We will soon see that it is relevant for us to obtain existence of globally transversal fields which are already defined on the whole space. The following result (cf. [72, Proposition 2.3 (iv)]) gives a sufficient condition for the existence of such fields.

Proposition 4.11. *Let $d \in \mathbb{N}$ be given and let $\mathcal{U} \subset \mathbb{R}^d$ be an open set of locally finite perimeter with compact boundary. We assume further that $\mathcal{U}$ has locally transversal vector fields and that*

$$|\partial_\star\mathcal{U}\cap B(x,r)| > 0 \quad \forall x \in \partial\mathcal{U} \quad \forall r > 0. \tag{4.51}$$

Then, there exists $\boldsymbol{X} \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^d)$ whose restriction to $\partial\mathcal{U}$ is globally transversal to $\partial\mathcal{U}$ with the property $|\boldsymbol{X}(y)| = 1$ for every $y \in \partial\mathcal{U}$.

We refer to Figure 4.3 for a geometric comparison of the outward unit normal and a specific choice of a (normalized and smooth) outwardly transversal vector field.

In particular, as stated in [72, Corollary 2.13], Lipschitz regularity ensures the existence of a continuous globally transversal vector field:

Proposition 4.12. *Let $d \in \mathbb{N}$ be given and let $\mathcal{U} \subset \mathbb{R}^d$ be non-empty, open, and bounded. Then, if the boundary $\partial\mathcal{U}$ is Lipschitz, it holds that $\mathcal{U}$ is of finite perimeter and $\mathcal{U}$ has continuous outwardly (resp. inwardly) globally transversal vector fields $\pm\boldsymbol{V}\colon \partial\mathcal{U} \to \mathbb{R}^d$.*

Therefore, as a result of Proposition 4.12, the set $\mathcal{O}$ admits a continuous inwardly globally transversal vector field, i.e., there exist a vector field $\widehat{\boldsymbol{k}} \in \boldsymbol{\mathcal{C}}(\partial\mathcal{O})$ and a transversality constant $\kappa > 0$ with the property $\widehat{\boldsymbol{k}}(x) \cdot \boldsymbol{n}(x) \leq -\kappa$ for a.e. $x \in \partial\mathcal{O}$. Here, $\boldsymbol{n}$ denotes the unit normal vector field pointing outward on $\mathcal{O}$. Now, by the piecewise smoothness of $\partial\mathcal{O}$ in combination with [146, Lemma 5.9.5] and [94, Remark 15.1], the measure-theoretic boundary $\partial_\star\mathcal{O}$ coincides with $\partial\mathcal{O}$ up to a set of (surface-)measure zero, as a result of which we can deduce that

$$|\partial_\star\mathcal{O} \cap B(x,r)| = |\partial\mathcal{O} \cap B(x,r)| > 0 \quad \forall x \in \partial\mathcal{O} \quad \forall r > 0, \tag{4.52}$$

that is, the condition in (4.51) is satisfied. Together with the boundary $\partial\mathcal{O}$ being compact, we are able to apply Proposition 4.11 which implies that there exists a vector field $\boldsymbol{k} \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^3)$ whose restriction to $\partial\mathcal{O}$ is inwardly globally transversal to $\partial\mathcal{O}$ with $|\boldsymbol{k}(y)| = 1$ for every $y \in \partial\mathcal{O}$. By the use of this special vector field, for every $\delta > 0$, we introduce the mapping

$$\boldsymbol{\theta}_\delta\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad y \mapsto y + \delta\boldsymbol{k}(y). \tag{4.53}$$

Lemma 4.13. *There exist $\delta_0 > 0$ and $\zeta > 0$ such that*

$$\boldsymbol{\theta}_\delta(\mathcal{O}) + B(0, \delta\zeta) \subset \mathcal{O} \quad \forall \delta \in (0, \delta_0).$$

Proof. As shown in the proof of [72, Proposition 4.15], there exists some $\delta_{0,1} > 0$ such that

$$\partial\boldsymbol{\theta}_\delta(\mathcal{O}) = \{y + \delta\boldsymbol{k}(y) \mid y \in \partial\mathcal{O}\} \quad \forall \delta \in (0, \delta_{0,1}).$$

As obtained from the proof of Lemma 4.16 in [72], there exists some $\delta_{0,2} > 0$ such that

$$H\colon \partial\mathcal{O} \times (-\delta_{0,2}, \delta_{0,2}) \to \mathbb{R}^3, \quad (x, \delta) \mapsto y + \delta\boldsymbol{k}(y)$$

is a bi-Lipschitz mapping. In particular, with L_H denoting the Lipschitz constant of H, it holds that

$$|H(y,\delta) - H(z,\rho)| \geq \frac{1}{L_H}|(y,\delta) - (z,\rho)| \quad \forall (y,\delta), (z,\rho) \in \partial\mathcal{O} \times (-\delta_{0,2}, \delta_{0,2}).$$

Let now $\delta_0 \coloneqq \min\{\delta_{0,1}, \delta_{0,2}\}$ and $\delta \in (0, \delta_0)$ be arbitrarily fixed. Given $y \in \partial\mathcal{O}$, $y + \delta\boldsymbol{k}(y) \in \partial\boldsymbol{\theta}_\delta(\mathcal{O})$, and $z \in \partial\mathcal{O}$, we have

$$|y + \delta\boldsymbol{k}(y) - z| = |H(y,\delta) - H(z,0)| \geq \frac{1}{L_H}|(y,\delta) - (z,0)| = \frac{1}{L_H}\sqrt{|y-z|^2 + \delta^2} \geq \frac{1}{L_H}\delta.$$

Therefore, $\operatorname{dist}(\partial\boldsymbol{\theta}_\delta(\mathcal{O}), \partial\mathcal{O}) \geq \frac{1}{L_H}\delta$ holds for every $\delta \in (0, \delta_0)$. Now, [72, Proposition 4.15] yields that $\overline{\boldsymbol{\theta}_\delta(\mathcal{O})} \subset \mathcal{O}$, and therefore the claim follows for $\zeta \coloneqq \frac{1}{L_H}$ and δ_0 as above. ∎

By $\mathbb{K}_\delta$ we denote the Jacobian mapping $\mathrm{D}\boldsymbol{\theta}_\delta\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. It is known (see [50, p. 59-60]) that there exists a constant $c_\theta > 0$ such that for $\delta > 0$ it holds

$$\sup_{y\in\Omega} \|\mathbb{K}_\delta(y) - \mathbb{I}\|_{\mathbb{R}^{3\times 3}} \leq c_\theta \delta. \tag{4.54}$$

We now introduce the following mollification operator:

$$\boldsymbol{\mathcal{K}}_\delta\colon \boldsymbol{L}^1(\Omega) \to \boldsymbol{L}^1(\Omega), \quad \boldsymbol{v} \mapsto \frac{1}{1+c_\theta\delta} \int_{B(0,1)} \rho(x)\mathbb{K}_\delta^{\mathrm{T}}(\cdot)\widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(\cdot) + \delta\zeta x)\,\mathrm{d}x, \tag{4.55}$$

where $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^1(\mathbb{R}^3)$ is the zero-extension of $\boldsymbol{v} \in \boldsymbol{L}^1(\Omega)$. In the following theorem, we prove the main constraint preserving property of the mollification (4.55) relying on the use of the following positive constants:

$$c_{\boldsymbol{k}} \coloneqq \max_{p\in\overline{\omega}} |\boldsymbol{k}(p)| \quad \text{and} \quad \lambda \coloneqq \operatorname{dist}(\omega, \mathbb{R}^3 \setminus \Omega).$$

Note that $\lambda > 0$ holds true due to $\overline{\omega} \subset \Omega$.

Theorem 4.14. *For $\delta \in (0, \delta_0)$, it holds that*

$$\boldsymbol{v} \in \boldsymbol{L}^1(\Omega) \;\Rightarrow\; \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega) \quad \textit{and} \quad \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \;\Rightarrow\; \lim_{\delta\to 0} \|\boldsymbol{K}_\delta \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} = 0. \tag{4.56}$$

If $0 < \delta < \min\left\{\delta_0, \frac{\lambda}{c_{\boldsymbol{k}}+\zeta}\right\}$, then $\boldsymbol{\mathcal{K}}_\delta$ satisfies

$$\boldsymbol{v} \in \boldsymbol{K} \quad \Rightarrow \quad \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{K}. \tag{4.57}$$

Proof. The vector field $\boldsymbol{k}$ is particularly inwardly (globally) transversal for $\partial(B(x_\Omega, r_\Omega)\setminus\overline{\Omega})$. The proof for (4.56) therefore follows the same arguments as in [50, Lemma 4.1] and [50, Theorem 4.4] together with the fact that $\frac{1}{1+c_\theta\delta} \to 1$ for $\delta \to 0$. Let now $0 < \delta < \min\left\{\delta_0, \frac{\lambda}{c_{\boldsymbol{k}}+\zeta}\right\}$. Due to Lemma 4.13 we know that

$$\boldsymbol{\theta}_\delta(\omega) + B(0, \delta\zeta) \subset \mathcal{O} = \left(B(x_\Omega, r_\Omega) \setminus \overline{\Omega}\right) \cup \omega. \tag{4.58}$$

Let us now prove that (4.58) can be refined as

$$\boldsymbol{\theta}_\delta(\omega) + B(0, \delta\zeta) \subset \omega. \tag{4.59}$$

To this aim, we assume the contrary: There exist $y \in \omega$ and $x \in B(0, \delta\zeta)$ such that

$$\boldsymbol{\theta}_\delta(y) + x \in B(x_\Omega, r_\Omega) \setminus \overline{\Omega}. \tag{4.60}$$

Then, (4.60) leads to a contradiction as follows:

$$\lambda = \operatorname{dist}(\omega, \mathbb{R}^3 \setminus \Omega) \leq \operatorname{dist}(\omega, \boldsymbol{\theta}_\delta(y) + x) = \inf_{z\in\omega} |\boldsymbol{\theta}_\delta(y) + x - z| \leq |\boldsymbol{\theta}_\delta(y) + x - y| \underbrace{=}_{(4.53)} |\delta\boldsymbol{k}(y) + x|$$
$$\leq \delta \max_{p\in\overline{\omega}} |\boldsymbol{k}(p)| + \delta\zeta = \delta(c_{\boldsymbol{k}} + \zeta) < \lambda,$$

where the last inequality follows from our particular choice of δ. This concludes (4.59). Let now $\boldsymbol{v} \in \boldsymbol{K}$ be given. In view of (4.55) and (4.59), it holds for a.e. $y \in \omega$ that

$$|\boldsymbol{K}_\delta \boldsymbol{v}(y)| = \left| \frac{1}{1+c_\theta\delta} \int_{B(0,1)} \rho(x)\mathbb{K}_\delta^{\mathrm{T}}(y)\widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(y) + \delta\zeta x)\,\mathrm{d}x \right|$$

$$
\begin{aligned}
&\leq \frac{1}{1+c_\theta\delta}\int_{B(0,1)}\left|\rho(x)\left(\mathbb{K}_\delta^{\mathrm{T}}(y)-\mathbb{I}\right)\widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(y)+\delta\zeta x)\right|\,\mathrm{d}x\\
&\quad+\frac{1}{1+c_\theta\delta}\int_{B(0,1)}|\rho(x)\widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(y)+\delta\zeta x)|\,\mathrm{d}x\\
&\underbrace{=}_{(4.59)} \frac{1}{1+c_\theta\delta}\int_{B(0,1)}\left|\rho(x)\left(\mathbb{K}_\delta^{\mathrm{T}}(y)-\mathbb{I}\right)\boldsymbol{v}(\underbrace{\boldsymbol{\theta}_\delta(y)+\delta\zeta x)}_{\in\omega}\right|\,\mathrm{d}x\\
&\quad+\frac{1}{1+c_\theta\delta}\int_{B(0,1)}\left|\rho(x)\boldsymbol{v}(\underbrace{\boldsymbol{\theta}_\delta(y)+\delta\zeta x)}_{\in\omega}\right|\,\mathrm{d}x\\
&\underbrace{\leq}_{\boldsymbol{v}\in\boldsymbol{K}} \frac{d}{1+c_\theta\delta}\int_{B(0,1)}\rho(x)\left\|\mathbb{K}_\delta^{\mathrm{T}}(y)-\mathbb{I}\right\|_{\mathbb{R}^{3\times3}}\,\mathrm{d}x+\frac{d}{1+c_\theta\delta}\int_{B(0,1)}\rho(x)\,\mathrm{d}x\\
&\underbrace{\leq}_{(4.47),(4.54)} d\left(\frac{c_\theta\delta}{1+c_\theta\delta}+\frac{1}{1+c_\theta\delta}\right)=d.
\end{aligned}
$$

In conclusion, (4.57) is valid. □

4.3.2 Convergence Result

In the following, let $N = N(h) \in \mathbb{N}$ denote a natural number depending on $h > 0$ with the property $N(h) \to \infty$ as $h \to 0$ maintaining the linear CFL-condition (4.19).

Theorem 4.15. *Let Assumption 4.1 and Assumption 4.4 hold. Then*

$$
\begin{aligned}
(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) &\overset{*}{\rightharpoonup} (\boldsymbol{E},\boldsymbol{H}) \quad \textit{weakly-* in } L^\infty((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \textit{ as } h\to 0\\
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) &\overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H}) \quad \textit{weakly-* in } L^\infty((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \textit{ as } h\to 0,
\end{aligned}
$$

where $(\boldsymbol{E},\boldsymbol{H})$ *is the unique solution to* (P). *If additionally*

$$
\boldsymbol{H}\in L^1((0,T),\boldsymbol{H}(\mathbf{curl})) \quad \textit{and} \quad \{\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}\}_{h>0} \textit{ is bounded in } L^p((0,T),\boldsymbol{L}^2(\omega)) \tag{4.61}
$$

for some $p > 1$, *then*

$$
(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) \to (\boldsymbol{E},\boldsymbol{H}) \quad \textit{in } \mathcal{C}([0,T],\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \textit{ as } h\to 0.
$$

Proof. The proof is divided into four parts:
Step 1: Preparation. Proposition 4.6, Corollary 4.8, and Proposition 4.9 yield the existence of subsequences, denoted w.l.o.g. by the same symbol, such that

$$
\begin{aligned}
(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) &\overset{*}{\rightharpoonup} (\boldsymbol{E},\boldsymbol{H}) \quad \text{weakly-* in } L^\infty((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \text{ as } h\to 0 \qquad (4.62)\\
(\widehat{\boldsymbol{E}}_{N,h},\widehat{\boldsymbol{H}}_{N,h}) &\overset{*}{\rightharpoonup} (\widehat{\boldsymbol{E}},\widehat{\boldsymbol{H}}) \quad \text{weakly-* in } L^\infty((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \text{ as } h\to 0\\
\overline{\boldsymbol{E}}_{N,h} &\overset{*}{\rightharpoonup} \overline{\boldsymbol{E}} \quad \text{weakly-* in } L^\infty((0,T),\boldsymbol{L}^2(\Omega)) \text{ as } h\to 0\\
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) &\overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H}) \quad \text{weakly-* in } L^\infty((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \text{ as } h\to 0.
\end{aligned}
$$

Similarly to the proof of Theorem 3.6, the constructions (4.45) and (4.46) imply

$$
\|\boldsymbol{E}_{N,h}(t)-\overline{\boldsymbol{E}}_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} \leq \max_{n\in\{1,\dots,N\}} \tau\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{TC}{N} \quad \forall t\in[0,T] \tag{4.63}
$$

$$\|\overline{\boldsymbol{E}}_{N,h}(t) - \widehat{\boldsymbol{E}}_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} \leq \max_{n\in\{1,\dots,N\}} \frac{\tau}{2}\|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{TC}{2N} \quad \forall t \in [0,T]$$

$$\|\boldsymbol{H}_{N,h}(t) - \widehat{\boldsymbol{H}}_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} \leq \max_{n\in\{1,\dots,N-1\}} \tau\|\delta \boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{TC}{N} \quad \forall t \in [0,T]$$

from which we conclude that $\boldsymbol{E} = \widehat{\boldsymbol{E}} = \overline{\boldsymbol{E}}$ and $\boldsymbol{H} = \widehat{\boldsymbol{H}}$. By standard arguments (cf. again the proof of Theorem 3.6), the first and last convergence properties in (4.62) lead to the following pointwise weak convergence:

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})(t) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H})(t) \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \quad \text{as } h \to 0 \quad \forall t \in [0,T]. \tag{4.64}$$

Let us now verify the Faraday law for the weak limit $(\boldsymbol{E}, \boldsymbol{H})$. Given $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$, there exists a sequence $\{\boldsymbol{w}_h\}_{h>0} \subset \boldsymbol{H}(\mathbf{curl})$ with $\boldsymbol{w}_h \in \mathbf{ND}_h$ for all $h > 0$ such that $\boldsymbol{w}_h \to \boldsymbol{w}$ in $\boldsymbol{H}(\mathbf{curl})$ as $h \to 0$. Using this converging sequence and (4.62), we deduce that

$$\begin{aligned}
&\int_0^T \left(\left(\mu \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}(t), \boldsymbol{w} \right)_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{E}(t), \mathbf{curl}\, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \right) \phi(t) \,\mathrm{d}t \\
\underbrace{=}_{(4.62)} \;& \lim_{h\to 0} \int_0^T \left(\left(\mu \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}_{N,h}(t), \boldsymbol{w}_h \right)_{\boldsymbol{L}^2(\Omega)} + \left(\overline{\boldsymbol{E}}_{N,h}(t), \mathbf{curl}\, \boldsymbol{w}_h \right)_{\boldsymbol{L}^2(\Omega)} \right) \phi(t) \,\mathrm{d}t \\
\underbrace{=}_{(\widetilde{\mathrm{P}}_{N,h})} \;& \lim_{h\to 0} \int_{T-\frac{T}{N}}^T \left(\left(\mu \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}_{N,h}(t), \boldsymbol{w}_h \right)_{\boldsymbol{L}^2(\Omega)} + \left(\overline{\boldsymbol{E}}_{N,h}(t), \mathbf{curl}\, \boldsymbol{w}_h \right)_{\boldsymbol{L}^2(\Omega)} \right) \phi(t) \,\mathrm{d}t \\
= \;& 0 \quad \forall \phi \in \mathcal{C}_0^\infty(0,T),
\end{aligned}$$

where the last equality holds true since the integrand is uniformly bounded in time (Proposition 4.6 and Corollary 4.8). Thus, by the fundamental theorem of variational calculus and since $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$ was chosen arbitrarily, it follows from the above identity and Lemma 2.1 that

$$\boldsymbol{E}(t) \in \boldsymbol{H}_0(\mathbf{curl}) \quad \text{with} \quad \mathbf{curl}\, \boldsymbol{E}(t) = -\mu \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H}(t) \quad \text{for a.e. } t \in (0,T), \tag{4.65}$$

which particularly implies that

$$\boldsymbol{E} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl})). \tag{4.66}$$

Since $(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})(0) = (\boldsymbol{E}_h^0, \boldsymbol{H}_h^{\frac{1}{2}}) \to (\boldsymbol{E}_0, \boldsymbol{H}_0)$ holds, we obtain, thanks to (4.64), that

$$(\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \tag{4.67}$$

By the construction (4.46), it holds that $\widehat{\boldsymbol{E}}_{N,h}(t) \in \boldsymbol{K} \cap \mathbf{DG}_h$ for all $t \in [0,T]$, and so (4.63) and (4.64) imply that

$$\boldsymbol{E}(t) \in \boldsymbol{K} \quad \forall t \in [0,T] \tag{4.68}$$

since $\boldsymbol{K}$ is weakly closed in $\boldsymbol{L}^2(\Omega)$.

Step 2: Derivation of the weak system (4.77) *for* $(\boldsymbol{E}, \boldsymbol{H})$*.* Let $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ be arbitrarily fixed. In view of (4.10), $\boldsymbol{Q}_h \boldsymbol{v} \in \boldsymbol{K} \cap \mathbf{DG}_h$ such that we may insert $\boldsymbol{v}_h = \boldsymbol{Q}_h \boldsymbol{v}$ in $(\widetilde{\mathrm{P}}_{N,h})$ to deduce that

$$\int_0^T (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \tag{4.69}$$

$$
\begin{aligned}
&\underbrace{=}_{(4.9),(4.11),(4.62)} \limsup_{h\to 0} \int_0^{T-\frac{T}{N}} (\widehat{\boldsymbol{f}}_{N,h}(t), \boldsymbol{Q}_h \boldsymbol{v} - \widehat{\boldsymbol{E}}_{N,h}(t))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \\
&\underbrace{\leq}_{(\widetilde{\mathrm{P}}_{N,h}),(4.62)} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v}\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t - \liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t \\
&\qquad + \int_0^T (\sigma \boldsymbol{E}(t), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t - \liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\sigma \widehat{\boldsymbol{E}}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t \\
&\qquad - \liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}(t), \boldsymbol{Q}_h \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t \\
&\qquad + \limsup_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t.
\end{aligned}
$$

Let us now proceed by estimating the individual parts appearing on the right-hand side of (4.69). At first, for $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, we estimate

$$
\begin{aligned}
&\left(\boldsymbol{E}_{N,h}\left(T - \frac{T}{N}\right) - \boldsymbol{E}(T), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \\
&= \left(\boldsymbol{E}_{N,h}\left(T - \frac{T}{N}\right) - \boldsymbol{E}_{N,h}(T), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{E}_{N,h}(T) - \boldsymbol{E}(T), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \\
&= -\int_{T-\frac{T}{N}}^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t + (\boldsymbol{E}_{N,h}(T) - \boldsymbol{E}(T), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \to 0 \text{ as } h \to 0,
\end{aligned}
$$

where we have used the boundedness of $\{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}\}_{h>0}$ and (4.64) for the above convergence. Using the same argumentation for the discrete magnetic fields, we obtain that

$$
(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})\left(T - \frac{T}{N}\right) \rightharpoonup (\boldsymbol{E}(T), \boldsymbol{H}(T)) \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \text{ as } h \to 0. \tag{4.70}
$$

Now, by the weak sequential lower semi-continuity of the squared norm, we infer

$$
\begin{aligned}
&\liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \boldsymbol{E}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t \\
&= \liminf_{h\to 0} \frac{1}{2}\left(\left\|\boldsymbol{E}_{N,h}\left(T - \frac{T}{N}\right)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \|\boldsymbol{E}^0_h\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right) \\
&\underbrace{\geq}_{(4.70)} \frac{1}{2}\left(\|\boldsymbol{E}(T)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right) = \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t,
\end{aligned}
\tag{4.71}
$$

and therefore

$$
\begin{aligned}
&\liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t \\
&= \liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t \\
&\qquad + \liminf_{h\to 0} \int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \boldsymbol{E}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t \\
&\underbrace{\geq}_{(4.63),(4.71)} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} \mathrm{d}t.
\end{aligned}
\tag{4.72}
$$

Furthermore,

$$\begin{aligned}
&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\sigma\widehat{\boldsymbol{E}}_{N,h}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t \qquad (4.73)\\
=\;&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)),\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)\Big)_{\boldsymbol{L}^2(\Omega)}+\Big(\sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)),\boldsymbol{E}(t)\Big)_{\boldsymbol{L}^2(\Omega)}\\
&\qquad+\Big(\sigma\boldsymbol{E}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
\geq\;&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)),\boldsymbol{E}(t)\Big)_{\boldsymbol{L}^2(\Omega)}+\Big(\sigma\boldsymbol{E}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
\underbrace{=}_{(4.62)}\;&\int_0^T(\sigma\boldsymbol{E}(t),\boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

By the same argumentation as in (4.71) and (4.72), we infer that

$$\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t),\widehat{\boldsymbol{H}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t\geq\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t),\boldsymbol{H}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t,$$

from which it follows that

$$\begin{aligned}
&\limsup_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t \qquad (4.74)\\
\underbrace{=}_{(4.63)}\;&\limsup_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\overline{\boldsymbol{E}}_{N,h}(t)\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
\underbrace{=}_{(\widetilde{\mathrm{P}}_{N,h})}\;&-\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t),\widehat{\boldsymbol{H}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t \qquad (4.75)\\
\leq\;&-\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t),\boldsymbol{H}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\mathrm{d}t\\
\underbrace{=}_{(4.65)}\;&\int_0^T(\mathbf{curl}\,\boldsymbol{E}(t),\boldsymbol{H}(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Due to Proposition 4.9 and (4.11), it holds that

$$\left|\int_0^{T-\frac{T}{N}}\Big(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{Q}_h\boldsymbol{v}-\boldsymbol{v}\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\right|\leq\|\,\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}\|_{L^1((0,T),\boldsymbol{L}^1(\Omega))}\|\boldsymbol{Q}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}\to 0,$$

and consequently

$$\begin{aligned}
&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{Q}_h\boldsymbol{v}\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t \qquad (4.76)\\
\geq&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{Q}_h\boldsymbol{v}-\boldsymbol{v}\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t+\int_0^{T-\frac{T}{N}}\Big(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{v}\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
=&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\Big(\widehat{\boldsymbol{H}}_{N,h}(t),\mathbf{curl}\,\boldsymbol{v}\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\underbrace{=}_{(4.62)}\int_0^T(\boldsymbol{H}(t),\mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Applying (4.72)-(4.76) to (4.69) and taking (4.65)-(4.68) into account, we conclude that the weak-star limit $(\boldsymbol{E}, \boldsymbol{H})$ satisfies

$$\begin{cases} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ \quad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \\ \geq \int_0^T (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega) \\ \mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\, \boldsymbol{E}(t) = 0 \quad \text{for a.e. } t \in (0,T) \\ (\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \\ \boldsymbol{E}(t) \in \boldsymbol{K} \text{ for all } t \in [0,T] \text{ and } (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases} \tag{4.77}$$

Step 3: (4.77) $\Rightarrow$ (P) *through a mollification process.* Let $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ and $0 < \delta < \min\left\{\delta_0, \frac{\lambda}{c_{\boldsymbol{k}} + \zeta}\right\}$. In view of Theorem 4.14, $\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ is a feasible test function for (4.77). For this reason,

$$\int_0^T (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \underbrace{=}_{(4.56)} \lim_{\delta \to 0} \int_0^T (\boldsymbol{f}(t), \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \tag{4.78}$$

$$\begin{aligned} &\underbrace{\leq}_{(4.77)} \lim_{\delta \to 0} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ &\qquad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \\ &\underbrace{=}_{(4.56)} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t. \end{aligned}$$

Since simple (in time) functions with values in $\boldsymbol{H}_0(\mathbf{curl})$ are dense in $L^2((0,T), \boldsymbol{H}_0(\mathbf{curl}))$, it follows that

$$\begin{aligned} &\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v}(t) - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{v}(t) - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ &\qquad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{v}(t) - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \\ &\geq \int_0^T (\boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \\ &\forall \boldsymbol{v} \in L^2((0,T), \boldsymbol{H}_0(\mathbf{curl})) \text{ with } \boldsymbol{v}(t) \in \boldsymbol{K} \text{ for a.e. } t \in (0,T). \end{aligned} \tag{4.79}$$

Finally, to prove that the Ampère-Maxwell VI in (P) is satisfied, we follow the same argumentation as in the proof of Theorem 3.6 by assuming the contrary:

$$\begin{aligned} &\exists \boldsymbol{z} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})\ \exists M \subset (0,T) \text{ with } |M| > 0 \text{ s.t. } \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) \,\mathrm{d}x \\ &+ \int_\Omega \sigma \boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}(\boldsymbol{z} - \boldsymbol{E}(t)) \,\mathrm{d}x < \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) \,\mathrm{d}x \quad \text{for a.e. } t \in M, \end{aligned}$$

which implies

$$\int_M \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) \,\mathrm{d}x + \int_\Omega \sigma \boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}(\boldsymbol{z} - \boldsymbol{E}(t)) \,\mathrm{d}x \,\mathrm{d}t$$

$$< \int_M \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t. \quad (4.80)$$

Inserting $\boldsymbol{v} := \chi_M \boldsymbol{z} + \chi_{(0,T)\setminus M} \boldsymbol{E}$ into (4.79) immediately contradicts (4.80). In conclusion, we have shown that $(\boldsymbol{E}, \boldsymbol{H})$ is the unique solution to (P).

Step 4: Uniform convergence. Suppose that $\boldsymbol{H} \in L^1((0,T), \boldsymbol{H}(\mathbf{curl}))$ and $\{\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}\}_{h>0}$ is bounded in $L^p((0,T), \boldsymbol{L}^2(\omega))$ for some $p > 1$. Let $\boldsymbol{v} \in \boldsymbol{K}$. As shown in Theorem 4.14, it holds that $\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{K}$. Now, [50, Theorem 4.4] additionally reveals that $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ is sufficient to obtain $\|\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \to 0$ as $\delta \to 0$. Thus, together with $\boldsymbol{H} \in L^1((0,T), \boldsymbol{H}(\mathbf{curl}))$, we obtain

$$\int_\Omega \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E}(t) + \sigma \boldsymbol{E}(t) - \mathbf{curl}\, \boldsymbol{H}(t) \right) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\, \mathrm{d}x \\ \geq \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \text{ for a.e. } t \in (0,T). \quad (4.81)$$

Now, $(\widetilde{\mathrm{P}}_{N,h})$ implies that

$$\int_\Omega \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E}_{N,h}(t) + \sigma \widehat{\boldsymbol{E}}_{N,h}(t) - \mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}(t) \right) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t))\, \mathrm{d}x \\ \geq \int_\Omega \widehat{\boldsymbol{f}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t))\, \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall t \in \left(0, T - \frac{T}{N}\right]. \quad (4.82)$$

For a.e. $t \in (0, T - T/N]$, the inequalities (4.81) and (4.82) allow for testing with $\boldsymbol{v} = \widehat{\boldsymbol{E}}_{N,h}(t)$ and $\boldsymbol{v}_h = \boldsymbol{Q}_h \boldsymbol{E}(t)$. Let $\rho \in (0,T)$ be arbitrarily fixed. Adding the resulting inequalities and integrating over the time interval $(0, \rho)$ then yields

$$\begin{aligned} &\int_0^\rho \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \qquad (4.83) \\ &+ \int_0^\rho \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E}_{N,h}(t) \cdot (\boldsymbol{E}(t) - \boldsymbol{Q}_h \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \\ &+ \int_0^\rho \int_\Omega \sigma (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t + \int_0^\rho \int_\Omega \sigma \widehat{\boldsymbol{E}}_{N,h}(t) \cdot (\boldsymbol{E}(t) - \boldsymbol{Q}_h \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \\ &- \int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \\ &- \int_0^\rho \int_\Omega \mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}(t) \cdot (\boldsymbol{E}(t) - \boldsymbol{Q}_h \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \\ \leq &\int_0^\rho \int_\Omega (\widehat{\boldsymbol{f}}_{N,h}(t) - \boldsymbol{f}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t + \int_0^\rho \int_\Omega \widehat{\boldsymbol{f}}_{N,h}(t) \cdot (\boldsymbol{E}(t) - \boldsymbol{Q}_h \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \end{aligned}$$

for sufficiently small $h > 0$. The first term on the left-hand side of the last inequality satisfies

$$\begin{aligned} &\limsup_{h\to 0} \int_0^\rho \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \qquad (4.84) \\ = &\limsup_{h\to 0} \int_0^\rho \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}_{N,h}(t))\, \mathrm{d}x\, \mathrm{d}t \\ &+ \frac{1}{2} \|\boldsymbol{E}_{N,h}(\rho) - \boldsymbol{E}(\rho)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \frac{1}{2} \|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \\ \underbrace{=}_{(4.63)} &\limsup_{h\to 0} \|\boldsymbol{E}_{N,h}(\rho) - \boldsymbol{E}(\rho)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \end{aligned}$$

The remaining pointwise norm can be extracted as follows:

$$\limsup_{h\to 0} - \int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \qquad (4.85)$$

$$
\begin{aligned}
\underbrace{=}_{(4.61),(4.63)} \quad & \limsup_{h\to 0} -\int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\overline{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \\
= \quad & \limsup_{h\to 0} -\int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{\Pi}_h \boldsymbol{H}(t)) \cdot (\overline{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\, \mathrm{d}x\, \mathrm{d}t \\
\underbrace{=}_{(2.19),(\mathrm{P}),(\widetilde{\mathrm{P}}_{N,h})} \quad & \limsup_{h\to 0} \int_0^\rho \int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{\Pi}_h \boldsymbol{H}(t))\, \mathrm{d}x\, \mathrm{d}t \\
= \quad & \limsup_{h\to 0} \int_0^\rho \int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t))\, \mathrm{d}x\, \mathrm{d}t \\
= \quad & \limsup_{h\to 0} \|\boldsymbol{H}_{N,h}(\rho) - \boldsymbol{H}(\rho)\|^2_{\boldsymbol{L}^2_\mu(\Omega)},
\end{aligned}
$$

where the same argument as in (4.84) was used for the last equality. Let us recall that, due to Lemma 4.5, Proposition 4.6, and Corollary 4.8, there exists a constant $C > 0$, independent of h, such that

$$
\|(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})\|_{W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times \boldsymbol{L}^2(\Omega))} + \|\widehat{\boldsymbol{E}}_{N,h}\|_{L^\infty((0,T),\boldsymbol{L}^2(\Omega))} \leq C \quad \forall h > 0. \tag{4.86}
$$

On the other hand, by the convergence property of $\boldsymbol{Q}_h$ along with $\boldsymbol{E} \in L^\infty((0,T), \boldsymbol{L}^2(\Omega))$ and (4.9), the Lebesgue dominated convergence theorem implies that

$$
\|\boldsymbol{E} - \boldsymbol{Q}_h \boldsymbol{E}\|_{L^s((0,T),\boldsymbol{L}^2(\Omega))} \to 0 \quad \text{as } h \to 0 \quad \forall 1 \leq s < \infty. \tag{4.87}
$$

Indeed, given any $1 \leq s < \infty$, the necessary bound for the Lebesgue dominated convergence theorem is obtained as follows:

$$
\begin{aligned}
\|\boldsymbol{E}(t) - \boldsymbol{Q}_h \boldsymbol{E}(t)\|^s_{\boldsymbol{L}^2(\Omega)} \leq \; & \Big(\|\boldsymbol{E}(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{Q}_h \boldsymbol{E}(t)\|_{\boldsymbol{L}^2(\Omega)}\Big)^s \\
\underbrace{\leq}_{(4.9)} \; & 2^s \|\boldsymbol{E}(t)\|^s_{\boldsymbol{L}^2(\Omega)} \qquad \text{for a.e. } t \in (0,T)
\end{aligned}
$$

for the right-hand side being of class $L^\infty(0,T) \hookrightarrow L^1(0,T)$. Thus, by (4.86) and the positive semi-definiteness of σ, applying (4.84)–(4.87) to (4.83), using the assumed boundedness of $\{\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}\}_{h>0}$ in $L^p((0,T), \boldsymbol{L}^2(\omega))$ with $p > 1$ together with the shown boundedness in $L^\infty((0,T), \boldsymbol{L}^2(\Omega \setminus \omega))$ from Proposition 4.9, we find that

$$
\lim_{h\to 0} \|(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})(\rho) - (\boldsymbol{E}, \boldsymbol{H})(\rho)\|_{\boldsymbol{L}^2(\Omega)\times \boldsymbol{L}^2(\Omega)} \to 0 \quad \text{as } h \to 0 \quad \forall \rho \in (0,T).
$$

Finally, utilizing once again the boundedness in (4.86), the proof is finished by the use of Arzelà-Ascoli's theorem for Banach space-valued functions (see [85, Chapter III, Theorem 3.1]). □

4.4 Numerical Tests

To close this chapter, we carry out numerical tests of the proposed FEM $(\mathrm{P}_{N,h})$. We consider a numerical simulation with $\Omega = (-1,1)^3$, $T = 1$, $\epsilon, \mu = 1$, $\sigma = 0$, $(\boldsymbol{E}_0, \boldsymbol{H}_0) = (0,0)$, and

$$
\boldsymbol{f}\colon [0,1] \times \Omega \to \mathbb{R}^3, \quad \boldsymbol{f}(t,\cdot) = (0, 2 + 10t, 0).
$$

The obstacle set is defined by

$$
\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{v}(x)| \leq 0.05 \text{ for a.e. } x \in \omega\}, \quad \omega = (-0.25, 0.25) \times (-0.5, 0.5)^2. \tag{4.88}
$$

Note that $d = 0.05$ for the electric obstacle in $\boldsymbol{K}$ is just an arbitrary choice. We may as well set $d = 0$ or any nonnegative real number for the upper bound d.

As stated in the introduction, thanks to Theorem 4.3, there is no need to invoke an additional nonlinear solver for the computation of the VI in $(\mathrm{P}_{N,h})$. Its exact solution is given by (4.13), which makes the numerical realization of $(\mathrm{P}_{N,h})$ particularly efficient and superior to the implicit Euler time-stepping. As to numerical precision, we went with 320 time steps and roughly 1.800.000 degrees of freedom (DoF) for $\mathbf{ND}_h$ as well as roughly 4.700.000 degrees of freedom for $\mathbf{DG}_h$. Figure 4.4 depicts two computed electric fields at the final time step $t = T$. The left figure depicts the computed electric field of the classical Maxwell equations in the absence of an electric obstacle, i.e, $\boldsymbol{K} = \boldsymbol{L}^2(\Omega)$, whereas the second one is the computed solution based on $(\mathrm{P}_{N,h})$ with the given obstacle (4.88). See also Figure 4.5 for the evolution of the electric field at $t = 1/4, 1/2, 3/4, 1$. Evidently, our numerical method is able to confirm the Faraday shielding effect in the obstacle region ω.

Finally, to test the convergence behavior of $(\mathrm{P}_{N,h})$, we fix the above-mentioned computed solution as a reference solution $(\boldsymbol{E}_{\mathrm{ref}}, \boldsymbol{H}_{\mathrm{ref}})$ since the true solution is unknown. Thereafter, we consider four different numerical solutions at coarser grids (maintaining a linear CFL-condition) and compute their error to the reference solution based on

$$\begin{aligned}\mathrm{Err}_{N,h}(\boldsymbol{E}) &:= \max_{n\in\{0,\dots,N\}} \|\boldsymbol{E}_{N,h}(t_n) - \boldsymbol{E}_{\mathrm{ref}}(t_n)\|_{\boldsymbol{L}^2(\Omega)}\\ \mathrm{Err}_{N,h}(\boldsymbol{H}) &:= \max_{n\in\{0,\dots,N\}} \|\boldsymbol{H}_{N,h}(t_n) - \boldsymbol{H}_{\mathrm{ref}}(t_n)\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

We should point out that, based on our numerical tests, the error quantities $\mathrm{Err}_{N,h}(\boldsymbol{E})$ and $\mathrm{Err}_{N,h}(\boldsymbol{H})$ coincide with the corresponding errors at the final time $t_N = T$, i.e., $\mathrm{Err}_{N,h}(\boldsymbol{E}) = \|\boldsymbol{E}_{N,h}(T) - \boldsymbol{E}_{\mathrm{ref}}(T)\|_{\boldsymbol{L}^2(\Omega)}$ and $\mathrm{Err}_{N,h}(\boldsymbol{H}) = \|\boldsymbol{H}_{N,h}(T) - \boldsymbol{H}_{\mathrm{ref}}(T)\|_{\boldsymbol{L}^2(\Omega)}$. From Table 4.1, we can clearly monitor a convergence behavior as the discretization becomes finer and finer, which serves as well as numerical evidence of our convergence result (Theorem 4.15).

Table 4.1: Convergence behavior of the scheme.

N	$5 \cdot 2^2$	$5 \cdot 2^3$	$5 \cdot 2^4$	$5 \cdot 2^5$	$5 \cdot 2^6$
h	$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$
$\mathrm{DoF}(\mathbf{DG}_h)$	1.152	9.216	31.024	589.824	4.718.592
$\mathrm{DoF}(\mathbf{ND}_h)$	604	4.184	73.728	238.688	1.872.064
$\mathrm{Err}_{N,h}(\boldsymbol{E})$	3.2415	1.2647	0.9207	0.5267	–
$\mathrm{Err}_{N,h}(\boldsymbol{H})$	3.1408	1.4920	0.8186	0.4352	–

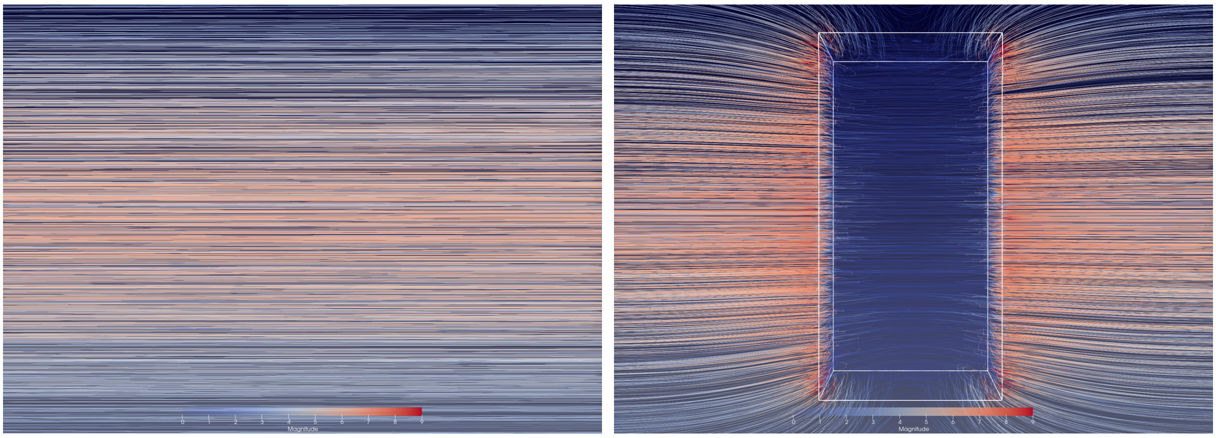

Figure 4.4: Electric field without obstacle (left) and with obstacle (right).

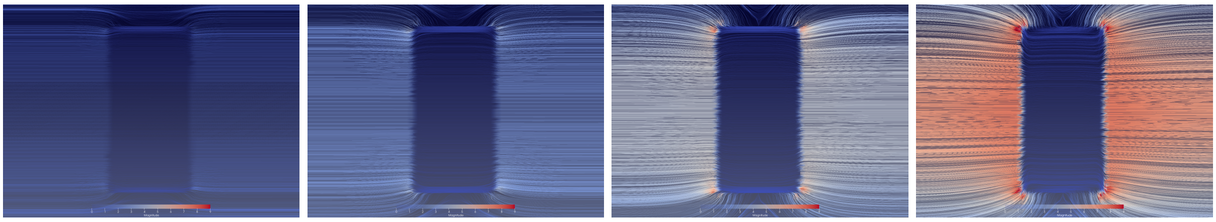

Figure 4.5: Evolution of the shielded electric field (2D slice) at $t = 1/4, 1/2, 3/4, 1$.

CHAPTER 5

QUASILINEAR VARIATIONAL INEQUALITIES IN FERROMAGNETIC SHIELDING: WELL-POSEDNESS, REGULARITY, AND OPTIMAL CONTROL

Having covered both the eddy current approximation and the numerical analysis of the hyperbolic electric shielding problem (P), in this chapter, we turn our attention to the shielding of magnetic fields by ferromagnetic materials in the static regime through the magnetic vector potential formulation. The resulting model is given by an $\boldsymbol{H}(\mathbf{curl})$-quasilinear first kind variational inequality with a bilateral vector **curl**-constraint. To begin with, let us recall that, in the free region, as a particular case of Maxwell's equations, magnetostatic equations read as

$$\begin{cases} \mathbf{curl}(\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}) + \nabla\phi = \boldsymbol{J} & \text{in } \Omega \\ \operatorname{div}\boldsymbol{A} = 0 & \text{in } \Omega \\ \boldsymbol{A}\times\boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Here, $\Omega \subset \mathbb{R}^3$ represents a bounded Lipschitz domain with a connected boundary, $\boldsymbol{A}\colon \Omega \to \mathbb{R}^3$ denotes the magnetic vector potential, $\boldsymbol{J}\colon \Omega \to \mathbb{R}^3$ the current density, $\phi\colon \Omega \to \mathbb{R}$ the Lagrange multiplier, and $\boldsymbol{n}$ the unit outward normal to $\partial\Omega$. Furthermore, $\nu\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ describes the nonlinear magnetic reluctivity modeling the physical dependency of ferromagnetic materials on the magnetic induction $\mathbf{curl}\,\boldsymbol{A}$. From among many works on (5.1), we refer the reader to [17, 79, 140]. In the present chapter, we consider a variational inequality of the first kind for (5.1), in which the magnetic induction strength $|\,\mathbf{curl}\,\boldsymbol{A}|$ is constrained to lie underneath a certain level leading to the following feasible set:

$$\boldsymbol{K} \coloneqq \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \mid |\,\mathbf{curl}\,\boldsymbol{v}| \leq d \ \text{ a.e. in } \Omega\} \quad \text{for a given nonnegative } d \in L^2(\Omega). \tag{5.2}$$

Following the theory of variational inequalities (cf. Section 2.2 and [119]), we formulate the first kind variational inequality for the quasilinear magnetostatic field equations (5.1) and (5.2) as follows:

$$\begin{cases} \text{Find } (\boldsymbol{A},\phi) \in \boldsymbol{K} \times H_0^1(\Omega), \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x + \int_\Omega \nabla\phi\cdot\boldsymbol{v}\,\mathrm{d}x \geq \int_\Omega \boldsymbol{J}\cdot(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{K} \\ \displaystyle\int_\Omega \boldsymbol{A}\cdot\nabla\psi\,\mathrm{d}x = 0 \quad \forall\psi\in H_0^1(\Omega). \end{cases} \tag{VI}$$

Motivated by the technological applications of ferromagnetic shielding, we make the first attempt to analyze (VI) and the corresponding optimal control problem (5.3). Due to the involved

$\boldsymbol{H}(\mathbf{curl})$-quasilinearity and the non-smooth character in (VI), the analysis is genuinely nonstandard and challenging. In particular, it requires a substantial extension of developed techniques from the existing literature. First, we develop a regularization approach (VE_γ) for (VI) by means of the Helmholtz decomposition and a reformulation of the first-order constraint (5.2) through the zeroth-order one (5.7) in combination with a smoothed Yosida penalization. By the limiting analysis of (VE_γ), we establish the well-posedness of (VI) and its dual formulation (Theorem 5.5).

After analyzing both the well-posedness for (VI), the second part of this chapter is devoted to the optimal control problem. Our aim is to find an optimal current source in the ferromagnetic shielding process (VI) which minimizes the $\boldsymbol{L}^2$-distance between the induced magnetic induction and the desired one. This leads to the following minimization problem:

$$\begin{cases} \min \displaystyle\int_\Omega |\,\mathbf{curl}\,\boldsymbol{A} - \boldsymbol{B}_\mathrm{d}|^2\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\boldsymbol{J}|^2\,\mathrm{d}x \\ \text{s.t. (VI).} \end{cases} \tag{5.3}$$

In the setting of (5.3), the vector field $\boldsymbol{B}_\mathrm{d} \in \boldsymbol{L}^2(\Omega)$ denotes the desired magnetic induction, and $\lambda > 0$ the control cost parameter. Let us emphasize that the primary difficulty of (5.3) lies not only in the $\boldsymbol{H}(\mathbf{curl})$-quasilinearity and the bilateral vector **curl**-constraint (5.2) but also in the lack of differentiability. Even for the simpler H^1-case, the directional differentiability of the solution mapping of the corresponding variational inequality in the presence of bilateral or gradient constraints cannot be expected (see the considerations in Section 2.2.1). All these aspects together make the analysis of (5.3) particularly delicate. While the mathematical analysis for the optimal control of $H^1(\Omega)$-type variational inequalities seems to have reached an advanced stage of development (cf. [18, 24, 28, 56, 69, 70, 74, 75, 99, 100]), the present work is the first to address (5.3). In fact, we are not aware of any previous contributions towards optimal control of Maxwell variational inequalities. The final novelty is therefore the derivation of necessary optimality conditions for the non-smooth optimal control problem (5.3) (see Theorem 5.11). In particular, our proof extends established Maxwell techniques for optimal control [111, 128, 129, 140, 142] and develops new ideas to cope with the aforementioned complexity involved in (5.3). We note that the results of this chapter are not restricted to the objective functional (5.3) involving only the first-order term $\mathbf{curl}\,\boldsymbol{A}$. Following [140, Theorem 3.8 and Remark 3.9], we obtain comparable results for objective functionals involving the zeroth-order term $\|\boldsymbol{A} - \boldsymbol{A}_\mathrm{d}\|^2_{\boldsymbol{L}^2(\Omega)}$ with a given $\boldsymbol{A}_\mathrm{d} \in \boldsymbol{L}^2(\Omega)$.

Let us now present the basic (physical) assumptions for our analysis. We assume the magnetic reluctivity $\nu\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ to be a Carathéodory function: For every $s \in \mathbb{R}_0^+$, the function $\nu(\cdot, s)$ is measurable, and, for almost every $x \in \Omega$, the function $\nu(x, \cdot)$ is continuous. By $\nu_0 > 0$ we denote the magnetic reluctivity in a vacuum. Further conditions on the nonlinearity (cf. [17, 79]) are collected in the following assumption which we assume to be valid throughout the whole document.

Assumption 5.1. There exist constants $\underline{\nu}, \overline{\nu} \in (0, \nu_0)$ such that

$$\begin{aligned} \underline{\nu} \le \nu(x,s) &\le \nu_0 && \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}_0^+ \\ (\nu(x,s)s - \nu(x,\hat{s})\hat{s})(s - \hat{s}) &\ge \underline{\nu}(s - \hat{s})^2 && \forall s, \hat{s} \in \mathbb{R}_0^+ \\ |\nu(x,s)s - \nu(x,\hat{s})\hat{s}| &\le \overline{\nu}|s - \hat{s}| && \forall s, \hat{s} \in \mathbb{R}_0^+ \end{aligned}$$

holds true.

Under Assumption 5.1, it holds for a.e. $x \in \Omega$ that

$$\begin{aligned} (\nu(x,|s|)s - \nu(x,|\hat{s}|)\hat{s}) \cdot (s-\hat{s}) &\geq \underline{\nu}|s-\hat{s}|^2 \quad \forall s,\hat{s} \in \mathbb{R}^3 \\ |\nu(x,|s|)s - \nu(x,|\hat{s}|)\hat{s}| &\leq L|s-\hat{s}| \quad \forall s,\hat{s} \in \mathbb{R}^3, \end{aligned} \tag{5.4}$$

where $L = 2\nu_0 + \overline{\nu}$. A proof for (5.4) can be found in [140, Lemma 2.2].

5.1 Regularization of (VI)

We propose a regularization approach for (VI) based on three main steps:

1. Reduction of (VI) to the lower level problem ($\mathrm{VI}_{\mathrm{sol}}$) with a divergence-free source term.
2. Reformulation of the first-order constraint (5.2) by the zeroth-order one (5.7) and the application of the Yosida regularization to the subdifferential of the indicator function for the zeroth-order obstacle set.
3. Smoothing of the maximum function (5.8).

For the first step, let us consider a solenoidal source term $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$ and test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$ in (VI). In this particular case, since

$$\int_\Omega \boldsymbol{v} \cdot \nabla\psi \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}{=}0) \quad \forall \psi \in H_0^1(\Omega), \tag{5.5}$$

(VI) leads to the following problem

$$\begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0), \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x \geq \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0). \end{cases} \tag{$\mathrm{VI}_{\mathrm{sol}}$}$$

Let us remark that the auxiliary problem ($\mathrm{VI}_{\mathrm{sol}}$) is indeed helpful for our investigation and serves as a basis for our well-posedness result for (VI). More precisely, applying the Helmholtz decomposition (2.27) to the source term

$$\boldsymbol{L}^2(\Omega) \ni \boldsymbol{J} = \boldsymbol{J}_{\mathrm{sol}} + \nabla\phi_{\boldsymbol{J}} \in \boldsymbol{H}(\mathrm{div}{=}0) \oplus \nabla H_0^1(\Omega), \tag{5.6}$$

we show in Theorem 5.5 that the solution $\boldsymbol{A}$ to ($\mathrm{VI}_{\mathrm{sol}}$) for $\boldsymbol{J}_{\mathrm{sol}}$ as in (5.6) turns out to be the unique solution to (VI) with the corresponding (unique) multiplier given by $\phi_{\boldsymbol{J}}$ from (5.6). For the second step, we introduce the zeroth-order obstacle set

$$\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)} \coloneqq \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{v}| \leq d \text{ a.e. on } \Omega\},$$

with which we can reformulate our first-order constraint as

$$\boldsymbol{A} \in \boldsymbol{K} \quad \Leftrightarrow \quad \mathbf{curl}\,\boldsymbol{A} \in \boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}. \tag{5.7}$$

Based on the proposed reformulation, we invoke the Yosida regularization of the subdifferential of the indicator function $I_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}}$ which is given by $\gamma(\mathrm{Id} - \mathbb{P}_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}})$ (cf. [122, pp. 137] and [21, Corollary 12.30]) with $\gamma > 0$ being the regularization parameter. Here, $\mathbb{P}_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}}$ denotes the Hilbert projection onto the non-empty, closed, and convex set $\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)} \subset \boldsymbol{L}^2(\Omega)$. The simplified $\boldsymbol{L}^2(\Omega)$ structure of $\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}$ now allows us to find an explicit expression (cf. [68, Example 4.2]) for the associated Yosida approximation as follows:

$$\gamma(\mathrm{Id} - \mathbb{P}_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}})(\boldsymbol{v}) = \gamma\boldsymbol{\theta}(\cdot, \boldsymbol{v}(\cdot)),$$

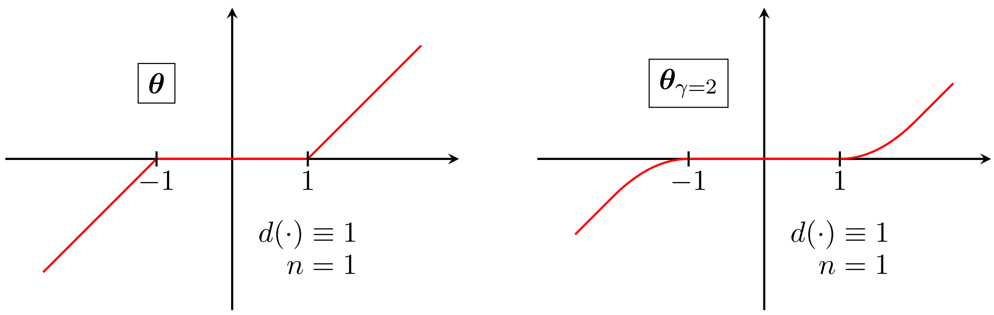

Figure 5.1: Illustration of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_\gamma$ for $\gamma = 2$ in the one-dimensional case $n = 1$ with the obstacle choice $d(\cdot) \equiv 1$.

with

$$\boldsymbol{\theta}\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\theta}(x, s) := \begin{cases} \max(|s| - d(x), 0)\dfrac{s}{|s|} & \text{if } s \neq 0 \\ 0 & \text{if } s = 0. \end{cases} \tag{5.8}$$

Note that for a.e. $x \in \Omega$, the function $\boldsymbol{\theta}(x, \cdot)$ is continuous but not differentiable. For our final step, we therefore regularize the non-smooth function $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta}_\gamma\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, s) \mapsto \begin{cases} \max_\gamma(|s| - d(x), 0)\dfrac{s}{|s|} & \text{if } s \neq 0 \\ 0 & \text{if } s = 0, \end{cases}$$

where

$$\max_\gamma(\cdot, 0)\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} x - \gamma^{-1} & \text{if } x \geq 2\gamma^{-1} \\ \dfrac{\gamma}{4}x^2 & \text{if } x \in (0, 2\gamma^{-1}) \\ 0 & \text{if } x \leq 0. \end{cases}$$

Geometrically speaking, the function $\max_\gamma(\cdot, 0)$ is a continuously differentiable regularization of $\max(\cdot, 0)$, which approximates the kink at 0 by a quadratic function in the interval $(0, 2\gamma^{-1})$. We also refer to Figure 5.1 for an illustration of $\boldsymbol{\theta}$ and its regularization $\boldsymbol{\theta}_\gamma$.

Altogether, for every $\gamma > 0$, we consider the following regularized problem:

$$\begin{cases} \text{Find } \boldsymbol{A}_\gamma \in \boldsymbol{X}_{N,0}, \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma|) \operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} \,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma) \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} \,\mathrm{d}x \\ \displaystyle = \int_\Omega \boldsymbol{J}_{\text{sol}} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{cases} \tag{VE$_\gamma$}$$

We shall see later in Section 5.3 that (VE$_\gamma$) serves as the state equation for the regularized optimal control problem (P$_\gamma$).

Lemma 5.2. *Let $\gamma > 0$. Then, the mapping $\boldsymbol{\theta}_\gamma$ is continuously differentiable with respect to the* *second variable, with derivative* $\mathrm{D}_s\boldsymbol{\theta}_\gamma\colon \Omega\times\mathbb{R}^3\to\mathbb{R}^{3\times 3}$ *given by*

$$(x,s)\mapsto\begin{cases}\dfrac{s\otimes s}{|s|^2}+\dfrac{|s|-d(x)-\gamma^{-1}}{|s|}\left(\mathrm{Id}-\dfrac{s\otimes s}{|s|^2}\right) & \text{if } |s|\geq d(x)+2\gamma^{-1}\\ \dfrac{\gamma}{2}(|s|-d(x))\dfrac{s\otimes s}{|s|^2}+\dfrac{\gamma(|s|-d(x))^2}{4|s|}\left(\mathrm{Id}-\dfrac{s\otimes s}{|s|^2}\right) & \text{if } |s|\in(d(x),d(x)+2\gamma^{-1})\\ 0 & \text{if } |s|\leq d(x).\end{cases} \tag{5.9}$$

For all $s\in\mathbb{R}^3$ *and almost every* $x\in\Omega$*, the matrix* $\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s)\in\mathbb{R}^{3\times 3}$ *is symmetric and positive semi-definite. Moreover,* $\mathrm{D}_s\boldsymbol{\theta}_\gamma\colon\Omega\times\mathbb{R}^3\to\mathbb{R}^{3\times3}$ *is uniformly bounded. Finally, for almost every* $x\in\Omega$*,* $\boldsymbol{\theta}_\gamma(x,\cdot)\colon\mathbb{R}^3\to\mathbb{R}^{3\times 3}$ *is monotone, Lipschitz-continuous, and it holds that*

$$|\boldsymbol{\theta}_\gamma(x,s)-\boldsymbol{\theta}(x,s)|\leq\frac{3}{\gamma}\quad\text{for a.e. } x\in\Omega \text{ and all } s\in\mathbb{R}^3. \tag{5.10}$$

Proof. It is apparent that $\boldsymbol{\theta}_\gamma$ is continuously differentiable at $s\neq 0$ since it is the product and composition of $\mathcal{C}^1$-mappings and the same is true if $s=0$ and $d(x)>0$. If s and $d(x)$ are both zero, it suffices to check that $s\mapsto s|s|$ is continuously differentiable at the origin, which is easily verified. A direct computation shows that the Jacobian is given by (5.9). Note that for each $s\in\mathbb{R}^3$, the matrices

$$\frac{s\otimes s}{|s|^2}\quad\text{and}\quad\mathrm{Id}-\frac{s\otimes s}{|s|^2}$$

are (symmetric) projection matrices. Thus, they have spectrum $\{0,1\}$ so that $\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s)$ is positive semi-definite for each $s\in\mathbb{R}^3$ and almost every $x\in\Omega$. Now, we can apply [118, Theorem 12.3] to conclude that $\boldsymbol{\theta}_\gamma$ is monotone w.r.t. the second variable. For the uniform boundedness of $\mathrm{D}_s\boldsymbol{\theta}_\gamma$, we observe that

$$\frac{|s\otimes s|_{\mathbb{R}^{3\times3}}}{|s|^2}=1\quad\text{and}\quad\frac{|s|-d(x)-\gamma^{-1}}{|s|}\leq 1,$$

where $|\cdot|_{\mathbb{R}^{3\times3}}$ denotes the spectral norm. Therefore, there exists a constant $C>0$ such that $|\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s)|\leq C$ for almost all $x\in\Omega$ and all $s\in\mathbb{R}^3$. Combining [118, Theorem 9.2] and [118, Theorem 9.7], this also implies the Lipschitz continuity of $\boldsymbol{\theta}_\gamma$. To finish the proof we calculate

$$|\boldsymbol{\theta}_\gamma(x,s)-\boldsymbol{\theta}(x,s)|\leq\begin{cases}\Big||s|-d(x)-\gamma^{-1}-(|s|-d(x))\Big| & \text{if } |s|\geq d(x)+2\gamma^{-1}\\ \Big|\dfrac{\gamma}{4}(|s|-d(x))^2-(|s|-d(x))\Big| & \text{if } |s|\in(d(x),d(x)+2\gamma^{-1}),\end{cases}$$

which yields the desired estimate (5.10). □

Lemma 5.3. *For every* $\boldsymbol{J}_{\mathrm{sol}}\in\boldsymbol{H}(\mathrm{div}{=}0)$*, the regularized problem* (VE$_\gamma$) *admits a unique solution* $\boldsymbol{A}_\gamma\in\boldsymbol{X}_{N,0}$*.*

Proof. In view of the Browder-Minty theorem, we define an operator $\mathcal{M}_\gamma\colon\boldsymbol{X}_{N,0}\to\boldsymbol{X}_{N,0}^*$ by

$$\begin{aligned}\langle\mathcal{M}_\gamma\boldsymbol{A},\boldsymbol{v}\rangle_{\boldsymbol{X}_{N,0}^*,\boldsymbol{X}_{N,0}}:=&\int_\Omega\nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x\\&+\gamma\int_\Omega\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A})\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x\quad\forall\boldsymbol{A},\boldsymbol{v}\in\boldsymbol{X}_{N,0},\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\boldsymbol{X}_{N,0}^*,\boldsymbol{X}_{N,0}}$ denotes the duality pairing between $\boldsymbol{X}_{N,0}$ and $\boldsymbol{X}_{N,0}^*$. A combination of Assumption 5.1, the Poincaré-Friedrichs inequality (2.25), and Lemma 5.2 (the monotonicity and continuity of $\boldsymbol{\theta}_\gamma$) implies that $\mathcal{M}_\gamma$ is strongly monotone and hemicontinuous. Indeed, for any $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \boldsymbol{X}_{N,0}$, it holds that

$$\begin{aligned}
&\langle \mathcal{M}_\gamma \boldsymbol{A}_1 - \mathcal{M}_\gamma \boldsymbol{A}_2, \boldsymbol{A}_1 - \boldsymbol{A}_2 \rangle_{\boldsymbol{X}^*_{N,0}, \boldsymbol{X}_{N,0}} \qquad (5.11)\\
&= \int_\Omega (\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_1|)\,\mathbf{curl}\,\boldsymbol{A}_1 - \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_2|)\,\mathbf{curl}\,\boldsymbol{A}_2) \cdot \mathbf{curl}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\,\mathrm{d}x \\
&\quad + \gamma \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_1) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_2)) \cdot \mathbf{curl}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\,\mathrm{d}x \\
&\geq \int_\Omega (\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_1|)\,\mathbf{curl}\,\boldsymbol{A}_1 - \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_2|)\,\mathbf{curl}\,\boldsymbol{A}_2) \cdot \mathbf{curl}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\,\mathrm{d}x \\
&\underbrace{\geq}_{(5.4)} \underline{\nu} \|\,\mathbf{curl}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\|^2_{\boldsymbol{L}^2(\Omega)} \underbrace{\geq}_{(2.25)} \frac{\underline{\nu}\min\{1, C_\mathrm{p}^{-2}\}}{2} \|\boldsymbol{A}_1 - \boldsymbol{A}_2\|^2_{\boldsymbol{X}_{N,0}},
\end{aligned}$$

which implies the strong monotonicity of $\mathcal{M}_\gamma$. The hemicontinuity of $\mathcal{M}_\gamma$ follows immediately from the continuity properties of the nonlinearities ν and $\boldsymbol{\theta}_\gamma$ in combination with Lebesgue's dominated convergence theorem. Since the right-hand side in (VE$_\gamma$) induces a functional in $\boldsymbol{X}^*_{N,0}$, the usage of the Browder-Minty theorem completes the proof. □

Lemma 5.4. *For every $\boldsymbol{J}_\mathrm{sol} \in \boldsymbol{H}(\mathrm{div}{=}0)$, the problem* (VI$_\mathrm{sol}$) *admits a unique solution $\boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$. Furthermore, the unique solution $\boldsymbol{A}_\gamma \in \boldsymbol{X}_{N,0}$ of* (VE$_\gamma$) *converges strongly in $\boldsymbol{X}_{N,0}$ to the unique solution $\boldsymbol{A}$ of* (VI$_\mathrm{sol}$) *as $\gamma \to \infty$.*

Proof. Let $\gamma > 0$ be given. Testing (VE$_\gamma$) with $\boldsymbol{v} = \boldsymbol{A}_\gamma$ leads to

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{A}_\gamma\,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{A}_\gamma\,\mathrm{d}x = \int_\Omega \boldsymbol{J}_\mathrm{sol} \cdot \boldsymbol{A}_\gamma\,\mathrm{d}x. \quad (5.12)$$

Utilizing Assumption 5.1, the Poincaré-Friedrichs inequality (2.25), and Lemma 5.2 (the monotonicity of $\boldsymbol{\theta}_\gamma$), a straightforward computation in the fashion of (5.11) in combination with Hölder's and Young's inequalities shows that the sequence $\{\boldsymbol{A}_\gamma\}_{\gamma>0} \subset \boldsymbol{X}_{N,0}$ is bounded, and consequently there exists a subsequence, still denoted in the same way, and $\boldsymbol{A} \in \boldsymbol{X}_{N,0}$, such that

$$\boldsymbol{A}_\gamma \rightharpoonup \boldsymbol{A} \quad \text{weakly in } \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty. \quad (5.13)$$

By the compactness of the embedding Theorem 2.3, we also obtain that $\boldsymbol{A}_\gamma \to \boldsymbol{A}$ strongly in $\boldsymbol{L}^2(\Omega)$ as $\gamma \to \infty$. Next we shall prove that $\boldsymbol{A} \in \boldsymbol{K}$, i.e., $|\,\mathbf{curl}\,\boldsymbol{A}| \leq d$ a.e. in Ω. Dividing the equation in (5.12) by γ and due to the boundedness of $\{\boldsymbol{A}_\gamma\}_{\gamma>0}$ in $\boldsymbol{X}_{N,0}$ implying the boundedness of $\{\mathbf{curl}\,\boldsymbol{A}_\gamma\}_{\gamma>0}$ in $\boldsymbol{L}^2(\Omega)$, we get

$$\int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{A}_\gamma\,\mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty, \quad (5.14)$$

while from (VE$_\gamma$) it also follows that

$$\int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \quad (5.15)$$

As $\boldsymbol{\theta}_\gamma$ is monotone in the second variable, it holds for all $\boldsymbol{v} \in \boldsymbol{X}_{N,0}$ that

$$\begin{aligned}
0 &\leq \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v})\,\mathrm{d}x \\
&= \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v})\,\mathrm{d}x + \int_\Omega (\boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{v}) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v})\,\mathrm{d}x \\
&\quad - \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{v}) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v})\,\mathrm{d}x.
\end{aligned}$$

Thanks to (5.14), (5.15), and the strong $\boldsymbol{L}^2(\Omega)$-convergence of $\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{v})$ towards $\boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{v})$ (see (5.10)), we obtain after passing to the limit $\gamma \to \infty$ in the previous inequality that

$$\int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{v}) \cdot \mathbf{curl}(\boldsymbol{A} - \boldsymbol{v})\, \mathrm{d}x \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \tag{5.16}$$

Now we take $s \in (0,1)$, $\widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}$ arbitrarily fixed and set $\boldsymbol{v} = \boldsymbol{A} + s\widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}$ in (5.16) to deduce that

$$\int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{v}}\, \mathrm{d}x \geq 0 \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}. \tag{5.17}$$

By the continuity of $\boldsymbol{\theta}$ with respect to the second variable, it follows that $\boldsymbol{\theta}(\cdot, \mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})) \to \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A})$ a.e. in Ω as $s \to 0$. Moreover, we have

$$\begin{aligned} |\boldsymbol{\theta}(\cdot, \mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}}))| &\underbrace{=}_{(5.8)} |\max(|\mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})| - d, 0)| \\ &\leq ||\mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})| - d| \leq |\,\mathbf{curl}\, \boldsymbol{A}| + |\,\mathbf{curl}\, \widetilde{\boldsymbol{v}}| + d \quad \text{for all } s \in (0,1) \text{ and a.e. in } \Omega, \end{aligned}$$

and therefore we apply Lebesgue's dominated convergence theorem to pass to the limit $s \to 0$ in (5.17). This implies

$$\int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A}) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{v}}\, \mathrm{d}x \geq 0 \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0} \quad \Rightarrow \quad \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A}) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{v}}\, \mathrm{d}x = 0 \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0},$$

and setting $\widetilde{\boldsymbol{v}} = \boldsymbol{A}$ in the last equation finally yields

$$0 = \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A}) \cdot \mathbf{curl}\, \boldsymbol{A}\, \mathrm{d}x = \int_\Omega \underbrace{\max(|\,\mathbf{curl}\, \boldsymbol{A}| - d, 0)|\,\mathbf{curl}\, \boldsymbol{A}|}_{\geq 0}\, \mathrm{d}x,$$

which implies

$$\max(|\,\mathbf{curl}\, \boldsymbol{A}(x)| - d(x), 0)|\,\mathbf{curl}\, \boldsymbol{A}(x)| = 0 \quad \text{for a.e. } x \in \Omega \quad \Rightarrow \quad \boldsymbol{A} \in \boldsymbol{K}.$$

Let us now show that the weak convergence (5.13) is strong. To this end, first we test (VE$_\gamma$) with $\boldsymbol{v} = \boldsymbol{A}_\gamma - \boldsymbol{A} \in \boldsymbol{X}_{N,0}$ to obtain

$$\begin{aligned} \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x. \end{aligned} \tag{5.18}$$

In view of Assumption 5.1 and (2.25), there exists a constant $C_\nu > 0$ such that

$$\begin{aligned} &\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ &\qquad \geq C_\nu \|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{X}_{N,0}} + \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} &\int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ &\qquad = \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A})) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \geq 0, \end{aligned} \tag{5.20}$$

where for the latter equality we used $\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}) = 0$ since $|\,\mathbf{curl}\, \boldsymbol{A}(x)| \leq d(x)$ for a.e. $x \in \Omega$. Applying (5.19) and (5.20) to (5.18) leads to

$$C_\nu \|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{X}_{N,0}} \leq \int_\Omega \boldsymbol{J}_{\text{sol}} \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x - \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x.$$

Since, by the convergence (5.13), the right-hand side of the above inequality tends to 0 as $\gamma \to \infty$, it follows that

$$\boldsymbol{A}_\gamma \to \boldsymbol{A} \quad \text{strongly in } \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty. \tag{5.21}$$

We are left to show that $\boldsymbol{A}$ is a solution to (VI_{sol}). To this end, let $\boldsymbol{v} \in \boldsymbol{K}$. Testing ($\text{VE}_\gamma$) with $\boldsymbol{v} - \boldsymbol{A}_\gamma$ yields that

$$\begin{aligned}
&\int_\Omega \boldsymbol{J}_{\text{sol}} \cdot (\boldsymbol{v} - \boldsymbol{A}_\gamma)\, \mathrm{d}x \qquad (5.22)\\
&= \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\, \mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\, \mathrm{d}x\\
&= \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\, \mathrm{d}x\\
&\quad + \gamma \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) - \underbrace{\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{v})}_{=0}) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\, \mathrm{d}x\\
&\leq \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\, \mathrm{d}x,
\end{aligned}$$

where we exploited the fact that $\boldsymbol{v} \in \boldsymbol{K}$ and that $\boldsymbol{\theta}_\gamma$ is monotone. In view of (5.21), after passing to the limit $\gamma \to \infty$ in (5.22), we obtain that $\boldsymbol{A}$ is a solution to (VI_{sol}). Uniqueness is obtained by a standard energy argument exploiting once again the monotonicity of $\boldsymbol{\theta}_\gamma$. This concludes the proof. □

5.2 Well-posedness

Theorem 5.5. *Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ be given with the associated Helmholtz decomposition*

$$\boldsymbol{J} = \boldsymbol{J}_{\text{sol}} + \nabla \phi_{\boldsymbol{J}} \in \boldsymbol{H}(\text{div=0}) \oplus \nabla H^1_0(\Omega). \tag{5.23}$$

Furthermore, let $\boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div=0})$ denote the unique solution to (VI_{sol}) *for $\boldsymbol{J}_{\text{sol}} \in \boldsymbol{H}(\text{div=0})$ given by* (5.23)*. Then, $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ is the unique solution to* (VI)*, and there exists a unique $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$, the so called dual multiplier, such that*

$$\begin{cases}
\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x + \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v} + \nabla \phi_{\boldsymbol{J}} \cdot \boldsymbol{v}\, \mathrm{d}x\\
= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})\\
\int_\Omega \boldsymbol{A} \cdot \nabla \psi\, \mathrm{d}x = 0 \quad \forall \psi \in H^1_0(\Omega)\\
\int_\Omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}) \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}.
\end{cases} \tag{5.24}$$

Proof. First, the unique solution $\boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div=0})$ of (VI_{sol}) satisfies

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v}_{\text{sol}} - \boldsymbol{A})\, \mathrm{d}x \geq \int_\Omega \boldsymbol{J}_{\text{sol}} \cdot (\boldsymbol{v}_{\text{sol}} - \boldsymbol{A})\, \mathrm{d}x \quad \forall \boldsymbol{v}_{\text{sol}} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div=0}). \tag{5.25}$$

Recalling from the Helmholtz decomposition (2.27), it holds that

$$\forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \;\; \exists(\boldsymbol{v}_{\text{sol}}, \phi_{\boldsymbol{v}}) \in \boldsymbol{H}(\text{div}{=}0) \times H_0^1(\Omega) \,:\, \boldsymbol{v} = \boldsymbol{v}_{\text{sol}} + \nabla \phi_{\boldsymbol{v}}.$$

If $\boldsymbol{v} \in \boldsymbol{K}$ (see (5.2) for its definition), then we obtain from the above decomposition that $\boldsymbol{v}_{\text{sol}} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)$ since $|\,\mathbf{curl}\, \boldsymbol{v}_{\text{sol}}| = |\,\mathbf{curl}(\boldsymbol{v}_{\text{sol}} + \nabla\phi_{\boldsymbol{v}})| = |\,\mathbf{curl}\, \boldsymbol{v}| \leq d$ a.e. in Ω. For this reason,

$$\forall \boldsymbol{v} \in \boldsymbol{K} \;\; \exists(\boldsymbol{v}_{\text{sol}}, \phi_{\boldsymbol{v}}) \in (\boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)) \times H_0^1(\Omega) \,:\, \boldsymbol{v} = \boldsymbol{v}_{\text{sol}} + \nabla \phi_{\boldsymbol{v}}. \tag{5.26}$$

As $\boldsymbol{J}_{\text{sol}} \in \boldsymbol{H}(\text{div}{=}0)$ and $\mathbf{curl}\, \nabla \equiv 0$, it follows by applying (5.26) to (5.25) that

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x \geq \int_\Omega \boldsymbol{J}_{\text{sol}} \cdot (\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{5.27}$$

Applying the decomposition (5.23) to (5.27) and taking $\boldsymbol{A} \in \boldsymbol{H}(\text{div}{=}0)$ into account, we obtain that

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x + \int_\Omega \nabla \phi_{\boldsymbol{J}} \cdot \boldsymbol{v}\, \mathrm{d}x \geq \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Hence, $(\boldsymbol{A}, \phi_{\boldsymbol{J}}) \in (\boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)) \times H_0^1(\Omega)$ is a solution to (VI). Towards uniqueness, let $(\widetilde{\boldsymbol{A}}, \widetilde{\phi}) \in (\boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)) \times H_0^1(\Omega)$ be another solution to (VI). Considering only test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)$ in (VI), we obtain due to (5.23) and (5.5) that $\widetilde{\boldsymbol{A}}$ is a solution to (VI$_{\text{sol}}$), which by the uniqueness of the solution to (VI$_{\text{sol}}$) implies that $\widetilde{\boldsymbol{A}} = \boldsymbol{A}$. Next, for any $\varphi \in H_0^1(\Omega)$, we have that $\boldsymbol{A} + \nabla\varphi \in \boldsymbol{K}$ since $|\,\mathbf{curl}(\boldsymbol{A} + \nabla\varphi)| = |\,\mathbf{curl}\, \boldsymbol{A}| \leq d$ a.e. in Ω. Thus, testing the variational inequality for the solution $(\boldsymbol{A}, \widetilde{\phi})$ to (VI) with $\boldsymbol{v} = \boldsymbol{A} + \nabla\varphi \in \boldsymbol{K}$, we obtain due to $\mathbf{curl}\, \nabla \equiv 0$ that

$$\int_\Omega \nabla \widetilde{\phi} \cdot \nabla \varphi\, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \nabla \varphi\, \mathrm{d}x \quad \forall \varphi \in H_0^1(\Omega). \tag{5.28}$$

Applying (5.23) to (5.28), we end up with

$$\int_\Omega \nabla(\widetilde{\phi} - \phi_{\boldsymbol{J}}) \cdot \nabla\varphi\, \mathrm{d}x = 0 \quad \forall \varphi \in H_0^1(\Omega) \quad \Rightarrow \quad \widetilde{\phi} = \phi_{\boldsymbol{J}}.$$

In conclusion, $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ is the unique solution to (VI).

Let us now prove that $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ satisfies the dual characterization (5.24). In view of Lemma 5.4, $\{\boldsymbol{A}_\gamma\}_{\gamma>0} \subset \boldsymbol{H}_0(\mathbf{curl})$ is bounded, and hence it follows from (VE$_\gamma$) that $\{\gamma \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma)\}_{\gamma>0}$ is bounded in $[\mathbf{curl}\, \boldsymbol{X}_{N,0}]^*$. Therefore, we find $\boldsymbol{\Psi} \in [\mathbf{curl}\, \boldsymbol{X}_{N,0}]^*$ such that after selecting a subsequence

$$\gamma \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \rightharpoonup \boldsymbol{\Psi} \quad \text{weakly in } [\mathbf{curl}\, \boldsymbol{X}_{N,0}]^* \quad \text{as } \gamma \to \infty. \tag{5.29}$$

At the same time, since $\mathbf{curl}\, \boldsymbol{X}_{N,0} \subset \boldsymbol{L}^2(\Omega)$ is closed (cf. (2.25)), Riesz's representation theorem implies the existence of $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$ such that

$$\boldsymbol{\Psi}(\mathbf{curl}\, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \tag{5.30}$$

Combining (5.29) with (5.30), it follows that

$$\gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \to \int_\Omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty. \tag{5.31}$$

Due to (5.31) and the strong convergence $\boldsymbol{A}_\gamma \to \boldsymbol{A}$ in $\boldsymbol{X}_{N,0}$ as $\gamma \to \infty$ (see Lemma 5.4), we obtain after passing to the limit $\gamma \to \infty$ in (VE$_\gamma$) that

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x = \int_\Omega \boldsymbol{J}_{\text{sol}} \cdot \boldsymbol{v}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \tag{5.32}$$

As a result of (2.25), (5.32) implies that $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$ is unique. Indeed, assuming that there exist $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \boldsymbol{X}_{N,0}$ satisfying (5.32), it follows that

$$\int_\Omega \mathbf{curl}(\boldsymbol{m}_1 - \boldsymbol{m}_2) \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}.$$

Then, inserting $\boldsymbol{v} = \boldsymbol{m}_1 - \boldsymbol{m}_2 \in \boldsymbol{X}_{N,0}$ and taking into account (2.25), we obtain that $\boldsymbol{m}_1 = \boldsymbol{m}_2$. Now, since $\mathbf{curl}\, \nabla \equiv 0$ and $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$, it follows from the Helmholtz decomposition $\boldsymbol{H}_0(\mathbf{curl}) = \boldsymbol{X}_{N,0} \oplus \nabla H_0^1(\Omega)$ that the variational equality (5.32) is valid for all test functions in $\boldsymbol{H}_0(\mathbf{curl})$, i.e., it holds for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$ that

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{v} \, \mathrm{d}x \underbrace{=}_{(5.23)} \int_\Omega (\boldsymbol{J} - \nabla \phi_{\boldsymbol{J}}) \cdot \boldsymbol{v} \, \mathrm{d}x. \tag{5.33}$$

For the last part in (5.24) we take $\boldsymbol{v} \in \boldsymbol{K} \subset \boldsymbol{H}_0(\mathbf{curl})$ and test equation (5.33) with $\boldsymbol{v} - \boldsymbol{A}$ to deduce that

$$\begin{aligned} &\int_\Omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}) \, \mathrm{d}x \\ &\quad = \int_\Omega (\boldsymbol{J} - \nabla \phi_{\boldsymbol{J}}) \cdot (\boldsymbol{v} - \boldsymbol{A}) \, \mathrm{d}x - \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}) \, \mathrm{d}x \underbrace{\leq}_{(\mathrm{VI})} 0. \end{aligned}$$

To summarize, we have proven that there is a unique $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$ such that the unique solution $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ of (VI) satisfies the dual characterization (5.24). This completes the proof. □

5.3 Optimal Control

In what follows, we denote the control-to-state mapping for (VI) by

$$\boldsymbol{G} \colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_{N,0}, \quad \boldsymbol{J} \mapsto \boldsymbol{A}.$$

In view of Theorem 5.5, the restriction of $\boldsymbol{G}$ onto the subspace $\boldsymbol{H}(\mathrm{div}{=}0)$, i.e., $\boldsymbol{G} \colon \boldsymbol{H}(\mathrm{div}{=}0) \to \boldsymbol{X}_{N,0}$ serves as the control-to-state mapping for ($\mathrm{VI}_{\mathrm{sol}}$). Invoking $\boldsymbol{G}$, we reformulate the optimal control problem (5.3) as

$$\min_{\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)} F(\boldsymbol{J}) \coloneqq \frac{1}{2} \|\, \mathbf{curl}\, \boldsymbol{G}(\boldsymbol{J}) - \boldsymbol{B}_{\mathrm{d}} \|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2} \|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{P}$$

Lemma 5.6. *The optimal control problem* (P) *admits an optimal solution. Every optimal solution to* (P) *enjoys a higher regularity property in* $\boldsymbol{H}(\mathrm{div}{=}0)$.

Proof. By an application of standard techniques (cf. [140, Proposition 3.2] or [37, Lemma 2.1.7] for details), the control-to-state mapping $\boldsymbol{G} \colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_{N,0}$ is weak-strong continuous. Thus, the existence of an optimal solution to (P) follows from classical arguments (cf. [127]). Let $\boldsymbol{J}^\star \in \boldsymbol{L}^2(\Omega)$ be an optimal solution to (P). Our goal now is to prove the higher regularity property $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$. In view of (2.27), $\boldsymbol{J}^\star$ admits the following orthogonal decomposition

$$\begin{aligned} &\boldsymbol{J}^\star = \boldsymbol{J}^\star_{\mathrm{sol}} + \nabla \phi_{\boldsymbol{J}^\star} \in \boldsymbol{H}(\mathrm{div}{=}0) \oplus \nabla H_0^1(\Omega) \\ &\qquad \Rightarrow \quad \|\boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{J}^\star_{\mathrm{sol}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{5.34}$$

Let us now consider the optimal control problem

$$\min_{\boldsymbol{J}\in \boldsymbol{H}(\text{div}=0)} F(\boldsymbol{J}). \tag{5.35}$$

Since $\boldsymbol{G}_{|\boldsymbol{H}(\text{div}=0)}$ is as well weak-strong continuous, there exists a minimizer $\widetilde{\boldsymbol{J}}^\star_{\text{sol}} \in \boldsymbol{H}(\text{div}=0)$ for (5.35), i.e.,

$$F(\widetilde{\boldsymbol{J}}^\star_{\text{sol}}) \leq F(\boldsymbol{J}) \qquad \forall \boldsymbol{J} \in \boldsymbol{H}(\text{div}=0).$$

It then follows that

$$\begin{aligned} F(\widetilde{\boldsymbol{J}}^\star_{\text{sol}}) \leq F(\boldsymbol{J}^\star_{\text{sol}}) &= \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J}^\star_{\text{sol}}) - \boldsymbol{B}_{\text{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}^\star_{\text{sol}}\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\underbrace{=}_{(5.34)} \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J}^\star_{\text{sol}}) - \boldsymbol{B}_{\text{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{\lambda}{2}\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)} \\ &= F(\boldsymbol{J}^\star) - \frac{\lambda}{2}\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)} \leq F(\widetilde{\boldsymbol{J}}^\star_{\text{sol}}) - \frac{\lambda}{2}\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

where for the last inequality we used the fact that $\boldsymbol{J}^\star$ is optimal for (P). The above inequalities yield that $\nabla\phi_{\boldsymbol{J}^\star} = 0$, which in turn implies $\phi_{\boldsymbol{J}^\star} = 0$ due to $\phi_{\boldsymbol{J}^\star} \in H^1_0(\Omega)$. Thus, by the decomposition in (5.34), we come to the conclusion that $\boldsymbol{J}^\star = \boldsymbol{J}^\star_{\text{sol}} \in \boldsymbol{H}(\text{div}=0)$. This completes the proof. □

Remark 5.7. Lemma 5.6 implies that any optimal solution $\boldsymbol{J}^\star \in \boldsymbol{L}^2(\Omega)$ of (P) is also an optimal solution of the $\boldsymbol{H}(\text{div}=0)$-reduced problem

$$\min_{\boldsymbol{J}\in \boldsymbol{H}(\text{div}=0)} \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J}) - \boldsymbol{B}_{\text{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{5.36}$$

On the other hand, given an optimal solution $\boldsymbol{J}^\star_{\text{sol}} \in \boldsymbol{H}(\text{div}=0)$ for (5.36), it is straightforward to verify that $\boldsymbol{J}^\star_{\text{sol}}$ is as well an optimal solution of (P). In that sense, both problems (P) and (5.36) are equivalent, and it is, therefore, sufficient to focus on the derivation of optimality conditions for (5.36).

5.3.1 Necessary Optimality Conditions for (P)

This section is devoted to the establishment of an optimality system for (P). To overcome the underlying non-smoothness, we consider a smoothed version of (5.36) built upon the approximation (VE_γ) in the spirit of Barbu [18]: Given an arbitrarily fixed optimal solution $\boldsymbol{J}^\star \in \boldsymbol{H}(\text{div}=0)$ to (P), we consider

$$\min_{\boldsymbol{J}_\gamma\in \boldsymbol{H}(\text{div}=0)} F_\gamma(\boldsymbol{J}_\gamma) := \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{G}_\gamma(\boldsymbol{J}_\gamma) - \boldsymbol{B}_{\text{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}_\gamma\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}_\gamma - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)}, \tag{P_γ}$$

where $\boldsymbol{G}_\gamma \colon \boldsymbol{H}(\text{div}=0) \to \boldsymbol{X}_{N,0}$ denotes the reduced control-to-state mapping for (VE_γ) based on Lemma 5.3. Note that, as a consequence of Lemma 5.4, standard arguments yield that

$$\begin{aligned} &\boldsymbol{J}_\gamma \rightharpoonup \boldsymbol{J} \quad \text{weakly in } \boldsymbol{H}(\text{div}=0) \quad \text{as } \gamma \to \infty \\ &\qquad\qquad \Rightarrow \qquad \boldsymbol{G}_\gamma(\boldsymbol{J}_\gamma) \to \boldsymbol{G}(\boldsymbol{J}) \quad \text{strongly in } \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty. \end{aligned} \tag{5.37}$$

Lemma 5.8. *Let $\gamma > 0$. Then, there exists an optimal solution $\boldsymbol{J}^\star_\gamma \in \boldsymbol{H}(\text{div}=0)$ to the problem* (P_γ).

Proof. As before, by well-known techniques, the mapping $\boldsymbol{G}_\gamma\colon \boldsymbol{H}(\mathrm{div}{=}0) \to \boldsymbol{X}_{N,0}$ is weak-strong continuous. Therefore, the existence of an optimal solution $\boldsymbol{J}_\gamma^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ of (P_γ) is guaranteed. □

We note that in general the uniqueness of optimal solutions to (P_γ) cannot be guaranteed since $\boldsymbol{G}_\gamma$ is nonlinear.

Next, for ease of notation, we introduce a vector version of the nonlinearity ν by means of the mapping

$$\boldsymbol{\mathcal{F}}\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,s) \mapsto \nu(x,|s|)s,$$

for which we require the following regularity assumption to hold:

Assumption 5.9. For almost every $x \in \Omega$, both the mappings $\nu(x,\cdot)\colon (0,\infty) \to \mathbb{R}$ and $\boldsymbol{\mathcal{F}}(x,\cdot)\colon \mathbb{R}^3 \to \mathbb{R}^3$ are continuously differentiable. Moreover, there is a constant $C > 0$, such that

$$\left|\frac{\partial \boldsymbol{\mathcal{F}}_{\boldsymbol{i}}}{\partial s_j}(x,s)\right| \leq C \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}^3$$

for all $i,j \in \{1,2,3\}$.

Assumption 5.9 is obviously satisfied for $\nu \equiv 1$, i.e., for the case where the quasilinearity is not present. A non-trivial example for a choice of ν satisfying both Assumption 5.1 and Assumption 5.9 can be found in [140, Example 3.5].

Now, let $\gamma > 0$ be arbitrarily fixed. Further let $\overline{\boldsymbol{J}}, \boldsymbol{J} \in \boldsymbol{H}(\mathrm{div}{=}0)$ and let $\overline{\boldsymbol{A}}_\gamma = \boldsymbol{G}_\gamma(\overline{\boldsymbol{J}})$ be the corresponding state. To obtain differentiability properties of $\boldsymbol{G}_\gamma$, we introduce an auxiliary linear problem, it reads

$$\begin{cases} \text{Find } \mathfrak{A}_\gamma \in \boldsymbol{X}_{N,0}, \text{ s.t.} \\ \displaystyle\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\,\mathbf{curl}\,\mathfrak{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \gamma \int_\Omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\,\mathbf{curl}\,\mathfrak{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{cases} \tag{5.38}$$

Now, [140, Proposition 3.7] provides us with

$$\mathrm{D}_s\boldsymbol{\mathcal{F}}(x,s)y \cdot y \geq \underline{\nu}|y| \quad \text{for a.e. } x \in \Omega \text{ and all } s,y \in \mathbb{R}^3. \tag{5.39}$$

Furthermore, as a consequence of Lemma 5.2, it holds that

$$\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s)y \cdot y \geq 0 \quad \text{for a.e. } x \in \Omega \text{ and all } s,y \in \mathbb{R}^3. \tag{5.40}$$

Since $\overline{\boldsymbol{A}}_\gamma$ is fixed, the left-hand side of (5.38) induces a bilinear form. According to the properties (5.39) and (5.40), this bilinear form is coercive. Thanks to the uniform boundedness of $\mathrm{D}_s\boldsymbol{\mathcal{F}}$ from Assumption 5.9 and the uniform boundedness of $\mathrm{D}_s\boldsymbol{\theta}_\gamma$ from Lemma 5.2, the resulting bilinear form is also bounded. Hence, by the Lax-Milgram theorem, (5.38) admits a unique solution $\mathfrak{A}_\gamma \in \boldsymbol{X}_{N,0}$. Taking into account Assumption 5.9, it is readily straightforward to verify the weak Gâteaux differentiability of $\boldsymbol{G}_\gamma\colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_{N,0}$ (see the proof of [140, Proposition 3.7] or [37, Lemma 2.2.7]). The Gâteaux derivative $\boldsymbol{G}_\gamma'(\overline{\boldsymbol{J}})\boldsymbol{J}$ is given by the unique solution $\mathfrak{A}_\gamma$ to (5.38).

As a consequence of the weak Gâteaux differentiability, standard adjoint techniques (cf. [127] or [37, Theorem 2.2.8]) imply necessary optimality conditions for (P_γ) which are collected in the following lemma:

Lemma 5.10. *Let $\gamma > 0$ and let $\boldsymbol{J}^\star_\gamma \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal control for* (P_γ). *Then, there exists a tuple $(\boldsymbol{A}^\star_\gamma, \boldsymbol{Q}^\star_\gamma) \in \boldsymbol{X}_{N,0} \times \boldsymbol{X}_{N,0}$ such that*

$$\begin{cases} \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad = \int_\Omega \boldsymbol{J}^\star_\gamma \cdot \boldsymbol{v}\,\mathrm{d}x \\ \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0} \end{cases}$$

$$\begin{cases} \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \gamma \int_\Omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ = \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0} \end{cases}$$

$$\boldsymbol{J}^\star_\gamma = -\frac{1}{2}\lambda^{-1}\boldsymbol{Q}^\star_\gamma + \frac{1}{2}\boldsymbol{J}^\star. \tag{5.41}$$

In all what follows, for every $\gamma > 0$, let $\boldsymbol{J}^\star_\gamma \in \boldsymbol{H}(\mathrm{div}{=}0)$ denote an optimal solution to (P_γ) with the associated state and adjoint state $(\boldsymbol{A}^\star_\gamma, \boldsymbol{Q}^\star_\gamma) \in \boldsymbol{X}_{N,0} \times \boldsymbol{X}_{N,0}$ satisfying (5.41). Our final goal is to establish necessary optimality conditions for (P) by means of a limit passage in the necessary optimality systems (5.41). Generally speaking, as part of necessary optimality conditions, one would expect a certain orthogonal relation between the dual multiplier and the optimal state (cf. [98, 100]). In the case of (P), difficulties arise due to the involved quasilinearity and especially the first-order bilateral vector **curl**-constraint in the underlying $\boldsymbol{H}(\mathbf{curl})$-structured variational inequality. Consequently, we can only prove the boundedness of $\{\gamma\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\}_{\gamma>0}$ and $\{\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in $[\mathbf{curl}\,\boldsymbol{X}_{N,0}]^*$ and not in $\boldsymbol{L}^2(\Omega)$. We tackle this difficulty by employing the Hilbert projector into the space $\mathbf{curl}\,\boldsymbol{X}_{N,0}$ given by

$$\mathbb{P}_{\mathbf{curl}} \coloneqq \mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}_{N,0}} \colon \boldsymbol{L}^2(\Omega) \to \mathbf{curl}\,\boldsymbol{X}_{N,0}, \tag{5.42}$$

taking into account that $\mathbf{curl}\,\boldsymbol{X}_{N,0} \subset \boldsymbol{L}^2(\Omega)$ is closed, and the following tailored cut-off type function:

$$\boldsymbol{\varrho} \colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\varrho}(x,s) \coloneqq s \min\left(1, \frac{d(x)}{|s|}\right) = \begin{cases} s & \text{if } |s| \le d(x) \\ d(x)\dfrac{s}{|s|} & \text{if } |s| > d(x). \end{cases}$$

Theorem 5.11. *Let $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal solution of* (P). *Then, there exist an optimal state $\boldsymbol{A}^\star \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$, an adjoint state $\boldsymbol{Q}^\star \in \boldsymbol{X}_{N,0}$, a state multiplier $\boldsymbol{m}^\star \in \boldsymbol{X}_{N,0}$, and a triple of adjoint multipliers $(\boldsymbol{n}^\star, \boldsymbol{\sigma}^\star_{d_+}, \boldsymbol{\sigma}^\star_{d_-}) \in \boldsymbol{X}_{N,0} \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ such that*

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}^\star|)\,\mathbf{curl}\,\boldsymbol{A}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{m}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{J}^\star \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0} \tag{5.43}$$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{m}^\star \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}^\star)\,\mathrm{d}x \le 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K} \tag{5.44}$$

$$\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0} \tag{5.45}$$

$$\boldsymbol{J}^\star = -\lambda^{-1}\boldsymbol{Q}^\star \tag{5.46}$$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{Q}^\star\,\mathrm{d}x \ge 0 \tag{5.47}$$

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d\frac{\mathbf{curl}\,\boldsymbol{A}^\star}{|\,\mathbf{curl}\,\boldsymbol{A}^\star|} - \mathbf{curl}\,\boldsymbol{A}^\star \right) \mathrm{d}x = 0, \quad \mathbf{curl}\,\boldsymbol{n}^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}. \tag{5.48}$$

Moreover, after selection of a subsequence, the triplet of adjoint multipliers $(\boldsymbol{n}^\star, \boldsymbol{\sigma}^\star_{d_+}, \boldsymbol{\sigma}^\star_{d_-})$ *is characterized by*

$$\begin{aligned}
\gamma\mathbb{P}_{\mathbf{curl}}\left(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\right) &\rightharpoonup \mathbf{curl}\,\boldsymbol{n}^\star && \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty \\
\gamma\mathbb{P}_{\mathbf{curl}}\left(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\right)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|>d\}} &\rightharpoonup \boldsymbol{\sigma}^\star_{d_+} && \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty \\
\gamma\mathbb{P}_{\mathbf{curl}}\left(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\right)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|\le d\}} &\rightharpoonup \boldsymbol{\sigma}^\star_{d_-} && \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty.
\end{aligned}$$

Proof. The proof is divided into three steps.

Step 1 (Limiting process). Let $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal solution of (5.36). Combining Lemma 5.4 with standard arguments from [18] taking into account the penalty term $\frac{\lambda}{2}\|\boldsymbol{J}_\gamma - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)}$ in (P_γ), there exists a sequence $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0} \subset \boldsymbol{H}(\mathrm{div}{=}0)$ of optimal solutions to (P_γ) such that

$$\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star \quad \text{strongly in } \boldsymbol{H}(\mathrm{div}{=}0) \quad \text{as } \gamma \to \infty.$$

Combining Lemma 5.4 with Theorem 5.5 and (5.37) implies that

$$\begin{aligned}
\boldsymbol{A}^\star_\gamma &\to \boldsymbol{A}^\star && \text{strongly in } \boldsymbol{X}_{N,0} && \text{as } \gamma \to \infty \\
\gamma\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma) &\rightharpoonup \mathbf{curl}\,\boldsymbol{m}^\star && \text{weakly in } [\mathbf{curl}\,\boldsymbol{X}_{N,0}]^* && \text{as } \gamma \to \infty,
\end{aligned} \tag{5.49}$$

where $(\boldsymbol{A}^\star, \boldsymbol{m}^\star) \in (\boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)) \times \boldsymbol{X}_{N,0}$ is the unique solution to the dual formulation (5.24) with right-hand side $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ and $\phi = 0$, i.e., (5.43) and (5.44). Let us now invoke the necessary optimality conditions (5.41) for the optimal control $\boldsymbol{J}^\star_\gamma$ of the regularized problem (P_γ). Inserting $\boldsymbol{v} = \boldsymbol{Q}^\star_\gamma$ in (5.41) and taking (5.39) as well as the adjoint equation in (5.40) into account, we obtain the boundedness of $\{\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in $\boldsymbol{X}_{N,0}$. Furthermore, in view of (5.41), the boundedness of $\{\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ and $\{\boldsymbol{A}_\gamma\}_{\gamma>0}$ as well as Assumption 5.9 yield the boundedness of $\{\gamma\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in the dual space $[\mathbf{curl}\,\boldsymbol{X}_{N,0}]^*$. Altogether, there exist $\boldsymbol{Q}^\star \in \boldsymbol{X}_{N,0}$ and $\boldsymbol{n}^\star_0 \in \boldsymbol{X}_{N,0}$ such that, after selection of a subsequence, we obtain

$$\begin{aligned}
\boldsymbol{Q}^\star_\gamma &\rightharpoonup \boldsymbol{Q}^\star && \text{weakly in } \boldsymbol{X}_{N,0} && \text{as } \gamma \to \infty \\
\gamma\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma &\rightharpoonup \mathbf{curl}\,\boldsymbol{n}^\star_0 && \text{weakly in } [\mathbf{curl}\,\boldsymbol{X}_{N,0}]^* && \text{as } \gamma \to \infty.
\end{aligned} \tag{5.50}$$

Taking into account (5.49) and Assumption 5.9, we apply Lebesgue's dominated convergence theorem to deduce

$$\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{v} \to \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)\,\mathbf{curl}\,\boldsymbol{v} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty \tag{5.51}$$

for every $\boldsymbol{v} \in \boldsymbol{X}_{N,0}$. It now follows from (5.49), (5.50), and (5.51) that $(\boldsymbol{Q}^\star, \boldsymbol{n}^\star_0)$ satisfies the adjoint equation (5.45). Moreover, passing to the limit $\gamma \to \infty$ in the representation for the optimal control in (5.41), we conclude that (5.46) is valid.

Step 2 (Orthogonality condition). For every $\gamma > 0$, employing (5.42), we decompose the field $\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma$ as

$$\begin{aligned}
\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma &= \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma) + \boldsymbol{z}_\gamma, \\
\text{with} \quad \boldsymbol{z}_\gamma &:= (I - \mathbb{P}_{\mathbf{curl}})(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma) \in (\mathbf{curl}\,\boldsymbol{X}_{N,0})^\perp.
\end{aligned} \tag{5.52}$$

By definition, for every $\gamma > 0$, there exists $\boldsymbol{g}_\gamma \in \boldsymbol{X}_{N,0}$, so that $\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma) = \mathbf{curl}\,\boldsymbol{g}_\gamma$. Inserting $\boldsymbol{v} = \boldsymbol{g}_\gamma$ in the adjoint equation in (5.41) yields

$$\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{g}_\gamma \,\mathrm{d}x + \gamma\int_\Omega (\mathbf{curl}\,\boldsymbol{g}_\gamma + \boldsymbol{z}_\gamma)\cdot \mathbf{curl}\,\boldsymbol{g}_\gamma \,\mathrm{d}x$$

$$= \int_\Omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\, \boldsymbol{g}_\gamma \,\mathrm{d}x.$$

Since $\boldsymbol{z}_\gamma \in (\mathbf{curl}\, \boldsymbol{X}_{N,0})^\perp$, the $\boldsymbol{L}^2(\Omega)$-inner product between $\boldsymbol{z}_\gamma$ and $\mathbf{curl}\, \boldsymbol{g}_\gamma$ vanishes such that

$$\begin{aligned} &\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^\mathrm{T}\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \cdot \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \,\mathrm{d}x + \gamma^2 \|\, \mathbf{curl}\, \boldsymbol{g}_\gamma \|^2_{\boldsymbol{L}^2(\Omega)} \\ = &\int_\Omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \boldsymbol{B}_\mathrm{d}) \cdot \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \,\mathrm{d}x. \end{aligned}$$

In view of Assumption 5.9, the sequences $\{\mathbf{curl}\, \boldsymbol{A}_\gamma^\star\}_{\gamma>0}$ and $\{\mathbf{curl}\, \boldsymbol{Q}_\gamma^\star\}_{\gamma>0}$ are bounded in $\boldsymbol{L}^2(\Omega)$. Thus, an application of the Hölder and Young inequalities implies that $\{\gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma\}_{\gamma>0} \subset \boldsymbol{L}^2(\Omega)$ is bounded. As a consequence, there exists $\boldsymbol{n}^\star \in \boldsymbol{X}_{N,0}$, such that, after selecting a subsequence, it holds that

$$\gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star) = \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \rightharpoonup \mathbf{curl}\, \boldsymbol{n}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty, \tag{5.53}$$

where we used the fact that $\mathbf{curl}\, \boldsymbol{X}_{N,0} \subset \boldsymbol{L}^2(\Omega)$ is closed. Since, according to (5.50) and (5.52), it also holds that

$$\int_\Omega \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \to \int_\Omega \mathbf{curl}\, \boldsymbol{n}_0^\star \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \quad \text{as } \gamma \to \infty \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0},$$

we infer that

$$\int_\Omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \mathbf{curl}\, \boldsymbol{n}_0^\star \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}.$$

Next, for every $\gamma > 0$, we set

$$\begin{aligned} \boldsymbol{\sigma}_{\gamma,d_+} &\coloneqq \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star) \chi_{\{|\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star| > d\}} = \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \chi_{\{|\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star| > d\}} \\ \boldsymbol{\sigma}_{\gamma,d_-} &\coloneqq \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star) \chi_{\{|\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star| \le d\}} = \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \chi_{\{|\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star| \le d\}}. \end{aligned}$$

By definition, it holds that

$$\begin{aligned} |\boldsymbol{\sigma}_{\gamma,d_+}| &\le |\gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma| \quad \text{a.e. in } \Omega \\ |\boldsymbol{\sigma}_{\gamma,d_-}| &\le |\gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma| \quad \text{a.e. in } \Omega. \end{aligned}$$

Consequently, as $\{\gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma\}_{\gamma>0} \subset \boldsymbol{L}^2(\Omega)$ is bounded, the sequences $\{\boldsymbol{\sigma}_{\gamma,d_+}\}_{\gamma>0}$ and $\{\boldsymbol{\sigma}_{\gamma,d_-}\}_{\gamma>0}$ are also bounded in $\boldsymbol{L}^2(\Omega)$. For this reason, there exist $\boldsymbol{\sigma}_{d_+}^\star, \boldsymbol{\sigma}_{d_+}^\star \in \boldsymbol{L}^2(\Omega)$ such that, after extracting a subsequence, we obtain that

$$\begin{aligned} \boldsymbol{\sigma}_{\gamma,d_+} &\rightharpoonup \boldsymbol{\sigma}_{d_+}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty \\ \boldsymbol{\sigma}_{\gamma,d_-} &\rightharpoonup \boldsymbol{\sigma}_{d_-}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty. \end{aligned} \tag{5.54}$$

Finally, due to

$$\boldsymbol{\sigma}_{\gamma,d_+} + \boldsymbol{\sigma}_{\gamma,d_-} = \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma \quad \forall \gamma > 0$$

and the weak convergence (5.53), we come to the conclusion that $\mathbf{curl}\, \boldsymbol{n}^\star = \boldsymbol{\sigma}_{d_+}^\star + \boldsymbol{\sigma}_{d_-}^\star$. Let us now make use of the cut-off type function

$$(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\le d}(x) \coloneqq \boldsymbol{\varrho}(x, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star) = \begin{cases} \mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x) & \text{if } |\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)| \le d(x) \\ d(x) \dfrac{\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)}{|\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)|} & \text{if } |\, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)| > d(x) \end{cases}$$

for a.e. $x \in \Omega$. Since, for a.e. $x \in \Omega$, the mapping $\boldsymbol{\varrho}(x,\cdot)$ is Lipschitz continuous with Lipschitz constant 1 (cf. [52, Theorem 4.5]), we obtain that

$$\int_\Omega |(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\le d} - \mathbf{curl}\,\boldsymbol{A}^\star|^2\,\mathrm{d}x \underbrace{=}_{|\,\mathbf{curl}\,\boldsymbol{A}^\star|\le d} \int_\Omega |(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\le d} - (\mathbf{curl}\,\boldsymbol{A}^\star)^{\le d}|^2\,\mathrm{d}x$$
$$= \int_\Omega |\boldsymbol{\varrho}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma) - \boldsymbol{\varrho}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star)|^2\,\mathrm{d}x \le \int_\Omega |\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \mathbf{curl}\,\boldsymbol{A}^\star|^2\,\mathrm{d}x \underbrace{\to}_{(5.49)} 0 \quad \text{as } \gamma\to\infty.$$

Therefore, it holds that

$$(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\le d} \to \mathbf{curl}\,\boldsymbol{A}^\star \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma\to\infty. \tag{5.55}$$

Combining (5.55) with (5.53) and the fact that $\mathbf{curl}\,\boldsymbol{n}^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}$ holds, we obtain

$$\int_\Omega \gamma\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma)\cdot(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\le d}\mathrm{d}x$$
$$\to \int_\Omega \boldsymbol{\sigma}^\star_{d_+}\cdot\mathbf{curl}\,\boldsymbol{A}^\star\,\mathrm{d}x + \int_\Omega \boldsymbol{\sigma}^\star_{d_-}\cdot\mathbf{curl}\,\boldsymbol{A}^\star\,\mathrm{d}x \tag{5.56}$$

as $\gamma\to\infty$. On the other hand, in view of the definition of the cut-off mapping $\boldsymbol{\varrho}$, the left-hand side of the latter equation can be rewritten as

$$\int_\Omega \gamma\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma)\cdot(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\le d}\,\mathrm{d}x \tag{5.57}$$
$$= \int_\Omega \gamma\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|>d\}}\cdot d\frac{\mathbf{curl}\,\boldsymbol{A}^\star_\gamma}{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|}\,\mathrm{d}x$$
$$+ \int_\Omega \gamma\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|\le d\}}\cdot\mathbf{curl}\,\boldsymbol{A}^\star_\gamma\,\mathrm{d}x$$
$$= \int_\Omega \boldsymbol{\sigma}_{\gamma,d_+}\cdot d\frac{\mathbf{curl}\,\boldsymbol{A}^\star_\gamma}{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|}\,\mathrm{d}x + \int_\Omega \boldsymbol{\sigma}_{\gamma,d_-}\cdot\mathbf{curl}\,\boldsymbol{A}^\star_\gamma\,\mathrm{d}x$$
$$\to \int_\Omega \boldsymbol{\sigma}^\star_{d_+}\cdot d\frac{\mathbf{curl}\,\boldsymbol{A}^\star}{|\,\mathbf{curl}\,\boldsymbol{A}^\star|}\,\mathrm{d}x + \int_\Omega \boldsymbol{\sigma}^\star_{d_-}\cdot\mathbf{curl}\,\boldsymbol{A}^\star\,\mathrm{d}x \quad \text{as } \gamma\to\infty,$$

where the last convergence follows again from (5.49) and (5.54). Comparing (5.57) and (5.56) concludes the proof for the orthogonality condition (5.48).

Step 3 (Sign condition). Finally, let us prove the sign condition (5.47). For this last step, we test the adjoint equation in (5.41) with $\boldsymbol{v} = \boldsymbol{Q}^\star_\gamma$ to obtain

$$\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\cdot\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\,\mathrm{d}x - \int_\Omega(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}})\cdot\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\,\mathrm{d}x$$
$$= -\gamma\int_\Omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\cdot\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\,\mathrm{d}x \le 0. \tag{5.58}$$

Next, let us estimate

$$\liminf_{\gamma\to\infty}\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\cdot\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\,\mathrm{d}x \tag{5.59}$$
$$= \liminf_{\gamma\to\infty}\Big[\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}(\boldsymbol{Q}^\star_\gamma - \boldsymbol{Q}^\star)\cdot\mathbf{curl}(\boldsymbol{Q}^\star_\gamma - \boldsymbol{Q}^\star)\,\mathrm{d}x$$
$$+ 2\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\cdot\mathbf{curl}\,\boldsymbol{Q}^\star\,\mathrm{d}x$$

$$
\begin{aligned}
&\qquad - \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \Big] \\
&\underbrace{\geq}_{(5.39)} 2 \liminf_{\gamma \to \infty} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
&\qquad - \limsup_{\gamma \to \infty} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x. \\
&\underbrace{\geq}_{(5.51)} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x.
\end{aligned}
$$

Using the limiting adjoint equation (5.45), we ultimately find that

$$
\begin{aligned}
&\qquad - \int_\Omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
&\underbrace{=}_{(5.45)} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x - \int_\Omega (\mathbf{curl}\, \boldsymbol{A}^\star - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
&\underbrace{\leq}_{(5.59)} \liminf_{\gamma \to \infty} \left(\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \,\mathrm{d}x - \int_\Omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \,\mathrm{d}x \right) \\
&\underbrace{\leq}_{(5.58)} 0.
\end{aligned}
$$

This completes the proof. □

CHAPTER 6

NUMERICAL ANALYSIS FOR MAXWELL QUASI-VARIATIONAL INEQUALITIES IN SUPERCONDUCTIVITY

Depending on the properties of the given medium, different configurations and variations of Maxwell's equations need to be considered. For instance, as briefly reported in Section 2.1, if the medium is isotropic with good conducting properties, it is well-known that Ohm's law holds true. In that case, the current density can be related with a multiple of the electric field.

Nowadays, many modern technologies make use of some sort of superconducting material. Generally speaking, superconductivity comprises physical properties of certain materials which cause it to lose its electrical resistance. By the Meissner effect, the loss of electrical resistance causes any magnetic field to be expelled. In this chapter, we are particularly interested in high-temperature type-II superconductors. Those being superconductors in which

- the superconducting state occurs below $-195.8°$C, the boiling point of liquid nitrogen (high-temperature),

- the transition between the superconducting and non-superconducting state is not abrupt (type-II).

See Figure 6.1 for a visualization of the non-abrupt transition of type-II superconductors. In the presence of a superconductor, Ohm's law needs to be replaced by a nonlinear and non-smooth constitutive relation between the electric field $\boldsymbol{E}$ and the current density $\boldsymbol{J}$. A prominent model was proposed by Bean (cf. [22, 23]), his critical state model postulates that

(B1) the current density strength $|\boldsymbol{J}|$ cannot exceed some critical value $j_c \in \mathbb{R}^+$,

(B2) the electric field $\boldsymbol{E}$ vanishes if $|\boldsymbol{J}| < j_c$,

(B3) the electric field $\boldsymbol{E}$ is parallel to $\boldsymbol{J}$.

Including (B1)-(B3) into the classical evolutionary Maxwell's equations, one ends up with a hyperbolic variational inequality of the second kind without temperature and magnetic field dependence. The resulting model was studied for the first time in [141] and later generalized in [143]. Dependence on the temperature, i.e., the handling of a nonlinearity $j_c = j_c(\cdot, \theta)$, was first analyzed in [136]. Neglecting the dependence on the magnetic field is physically reasonable as long as the magnetic field strength is not so strong. However, with the strength of the magnetic field exceeding a certain level, as found by Kim et al. in [81], an accurate model needs to account for the dependence not only on the temperature θ but also on the magnetic field $\boldsymbol{H}$.

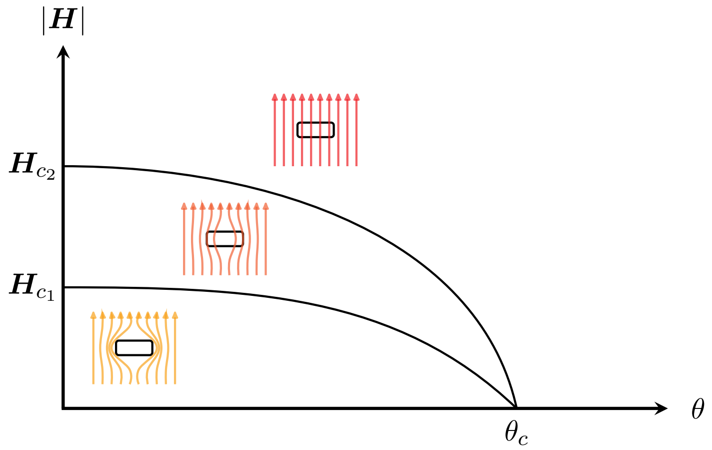

Figure 6.1: Expulsion of magnetic field lines by type-II superconductors depending on critical values of both temperature and magnetic field.

For this reason, we consider a nonnegative function

$$j_c \colon \Omega \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R},$$

allowing for the more general dependence $j_c = j_c(\cdot, \theta, \boldsymbol{H})$. Here, $\Omega \subset \mathbb{R}^3$ represents our computational medium, which we assume to be a bounded polyhedral Lipschitz domain. Given a time horizon $T \in \mathbb{R}^+$, we are able to include the Bean-Kim model into Maxwell's equations so that we end up with the nonlinear and non-smooth system

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{H} + \boldsymbol{J} = \boldsymbol{f} & \text{in } (0,T) \times \Omega \\ \mu \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = 0 & \text{in } (0,T) \times \Omega \\ \boldsymbol{J}(x,t) \cdot \boldsymbol{E}(x,t) = j_c(x, \theta(x,t), \boldsymbol{H}(x,t)) |\boldsymbol{E}(x,t)| & \text{in } (0,T) \times \Omega \\ |\boldsymbol{J}(x,t)| \leq j_c(x, \theta(x,t), \boldsymbol{H}(x,t)) & \text{in } (0,T) \times \Omega \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{in } (0,T) \times \partial\Omega \\ (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0) & \text{in } \Omega. \end{cases} \tag{6.1}$$

Assumptions on the data involved in (6.1) can be found in Assumption 6.1. In particular, we rely on a local boundedness and local Lipschitz assumption of the nonlinearity j_c with respect to the temperature and a global boundedness and global Lipschitz assumption of j_c with respect to the magnetic field. We refer to [43] for a physical justification of these assumptions. As it turns out, a corresponding weak formulation of (6.1) does not lead to a hyperbolic VI but rather to a hyperbolic quasi-variational inequality, namely finding

$$(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}) \cap \mu^{-1} \boldsymbol{H}_0(\mathrm{div}{=}0)),$$

such that

$$\begin{cases} \displaystyle\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \mathbf{curl}\,\boldsymbol{H}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \qquad + j(\theta(t),\boldsymbol{H}(t),\boldsymbol{v}) - j(\theta(t),\boldsymbol{H}(t),\boldsymbol{E}(t)) \geq \displaystyle\int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \text{for a.e. } t\in(0,T) \text{ and all } \boldsymbol{v}\in \boldsymbol{L}^2(\Omega) \\ \mu\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t\in(0,T) \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{QVI}$$

Here, the L^1-type nonlinearity $j\colon L^2(\Omega)\times \boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)\to\mathbb{R}$ is defined by

$$j(\eta,\boldsymbol{h},\boldsymbol{v}) := \int_\Omega j_c(\cdot,\eta,\boldsymbol{h})|\boldsymbol{v}|\,\mathrm{d}x \quad \forall(\eta,\boldsymbol{h},\boldsymbol{v})\in L^2(\Omega)\times\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega).$$

Lately, there has been an increasing interest in QVIs of both first and second kind, in particular, their sensitivity analysis (cf. [8–11]). We also mention the publications [19, 84, 120] dealing with QVIs in the context of superconductivity. In contrast to the mentioned papers, our model involves no simplified $W^{1,p}$-structure but rather includes the natural spaces from the theory of Maxwell's equations. The well-posedness for our formulation (QVI) goes back to [145] and appears to be the first contribution towards quasi-variational inequalities with a Maxwell structure.

The present chapter is mainly concerned with the development of an efficient solver for (QVI) and its numerical analysis. Our novelties include the stability and strong convergence of the scheme in ($\text{QVI}_{N,h}$) towards the unique solution of (QVI). In particular, as a result of our tailored stability analysis and by making use of the specific structure of the nonlinearity, we are able to strengthen the well-posedness result from [145] in the sense of allowing for source and temperature data to be merely of bounded variation instead of H^1 in time.

While there are several contributions towards the numerical analysis of QVIs (cf. [109,112]), to the best of the authors knowledge, this work is the first one to deal with the numerical resolution of a QVI with a hyperbolic character. The main difficulty in the discretization of the present problem (QVI) lies in the fact that there is no general well-posedness theory for elliptic QVIs. This problem especially comes to light when employing standard discretization methods such as the implicit Euler method. Here, the nonlinearity j_c in the resulting elliptic QVI formulation for the electric field would even depend on its rotational field, making the existence and uniqueness of solutions especially delicate.

We overcome this complexity by employing a discretization by the leapfrog stepping in combination with a discontinuous Galerkin discretization for the electric field similar to Chapter 4. In this way, we are able to completely eliminate the QVI character in our fully discrete scheme ($\text{QVI}_{N,h}$) and replace it with an $\boldsymbol{L}^2$-structured VI for which we can explicitly compute its analytical solution in terms of the given data (see Theorem 6.6). Compared with other types of discretizations in possible combination with fixed-point type iterations, this leads to a highly efficient solve for the unique solution of (QVI). Supplementary to the references on the leapfrog stepping in Chapter 4, let us mention the contributions from the mathematical community [33, 34, 89, 117], but also from the engineering community [58, 59, 67, 83] in the context of wave equations, in particular electromagnetic problems. However, we are only aware of the previous contribution towards leapfrog discretization for variational inequalities in Chapter 4.

Let us now introduce the pivotal Hilbert space for this chapter by

$$\boldsymbol{X}^{(\mu)}(\Omega) := \boldsymbol{H}(\mathbf{curl}) \cap \mu^{-1}\boldsymbol{H}_0(\mathrm{div}{=}0) = \{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl}) : (\boldsymbol{v},\nabla\phi)_{\boldsymbol{L}^2_\mu(\Omega)} = 0 \quad \forall\phi\in H^1(\Omega)\}.$$

As we will work with source term and temperature of bounded variation, let us denote by $\mathcal{P}$ the set of all partitions of the interval $[0,T]$, i.e.,

$$\mathcal{P} = \{P = \{s_0, \dots, s_{n_P}\} \subset [0,T] \ : \ n_P \in \mathbb{N} \text{ and } s_i \leq s_{i+1} \text{ for every } 0 \leq i \leq n_{\mathcal{P}} - 1\}.$$

For a given Banach space V, we then denote

$$\mathrm{BV}([0,T],V) := \{g\colon [0,T] \to V \ : \ \mathrm{TV}(g) < \infty\},$$

where, for $g\colon [0,T] \to V$, the total variation is defined by

$$\mathrm{TV}(g) := \sup_{P \in \mathcal{P}} \sum_{n=0}^{n_P - 1} \|g(s_{n+1}) - g(s_n)\|_V. \tag{6.2}$$

Note that any $g \in \mathrm{BV}([0,T],V)$ is already contained in $L^\infty((0,T),V)$. In the stability analysis following later, we will use the following equation which holds for $\{\boldsymbol{a}_n\}_{n=1}^{i_0} \subset \mathbb{R}^d$ and $\{\boldsymbol{b}_n\}_{n=0}^{i_0} \subset \mathbb{R}^d$ with $d, i_0 \in \mathbb{N}$,

$$\sum_{n=1}^{i_0} \boldsymbol{a}_n \cdot (\boldsymbol{b}_n + \boldsymbol{b}_{n-1}) = \sum_{n=1}^{i_0 - 1} (\boldsymbol{a}_{n+1} + \boldsymbol{a}_n) \cdot \boldsymbol{b}_n + \boldsymbol{a}_1 \cdot \boldsymbol{b}_0 + \boldsymbol{a}_{i_0} \cdot \boldsymbol{b}_{i_0}. \tag{6.3}$$

We now summarize the mathematical assumptions for (QVI).

Assumption 6.1 (Regularity assumptions on the material parameters)**.**

(A6.1) The material parameters $\epsilon, \mu \in L^\infty(\Omega)$ are strictly positive, i.e., there exist positive constants $\underline{\epsilon}, \overline{\epsilon}, \underline{\mu}, \overline{\mu} > 0$ such that

$$\underline{\epsilon} \leq \epsilon(x) \leq \overline{\epsilon} \quad \text{and} \quad \underline{\mu} \leq \mu(x) \leq \overline{\mu} \quad \text{for a.e. } x \in \Omega.$$

(A6.2) For every $(y, \boldsymbol{z}) \in \mathbb{R} \times \mathbb{R}^3$, $j_c(\cdot, y, \boldsymbol{z})\colon \Omega \to \mathbb{R}$ is Lebesgue-measurable and nonnegative.

(A6.3) For every $M > 0$, there exists a constant $C(M) > 0$ such that

$$0 \leq j_c(x, y, \boldsymbol{z}) \leq C(M)$$

for a.e. $x \in \Omega$, every $y \in [-M, M]$, and every $\boldsymbol{z} \in \mathbb{R}^3$ satisfying $|\boldsymbol{z}| \leq M$.

(A6.4) For every $M > 0$, there exists a constant $L(M) > 0$ such that

$$|j_c(x, y_1, \boldsymbol{z}_1) - j_c(x, y_2, \boldsymbol{z}_2)| \leq L(M)\,(|y_1 - y_2| + |\boldsymbol{z}_1 - \boldsymbol{z}_2|)$$

for a.e. $x \in \Omega$, every $y_1, y_2 \in [-M, M]$ and $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^3$.

Moreover, we employ the following assumptions for the given data.

Assumption 6.2 (Regularity assumptions on the given data)**.**

(A6.5) Suppose that

$$\boldsymbol{f} \in \mathrm{BV}([0,T], \boldsymbol{L}^2(\Omega)) \quad \text{and} \quad \theta \in \mathrm{BV}([0,T], L^2(\Omega)) \cap \mathcal{C}([0,T], L^\infty(\Omega)).$$

(A6.6) The initial data fulfills the regularity property

$$(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{X}^{(\mu)}(\Omega).$$

Remark 6.3. The assumptions (A6.1)–(A6.4) are physically reasonable and for instance reported in [43].

6.1 Leapfrog Scheme

In this section, we want to introduce a fully discrete scheme to (QVI) in a similar fashion as in Chapter 4. Therefore, we choose again a family of quasi-uniform triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω with $h > 0$ standing for the largest diameter of $T \in \mathcal{T}_h$. Our mixed finite element method is based on the Nédélec finite element space $\mathbf{ND}_h$ and the finite element space of piecewise constant functions $\mathbf{DG}_h$. For their definition we refer to Chapter 4. In addition to these spaces, we introduce the space of continuous piecewise linear elements by

$$\Theta_h := \{\psi_h \in H^1(\Omega) \ : \ \psi_{h|T} = \boldsymbol{a}_T \cdot x + b_T \text{ with } \boldsymbol{a}_T \in \mathbb{R}^3, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h\}.$$

By means of the above space, the subspace $\boldsymbol{X}_h^{(\mu)} \subset \mathbf{ND}_h$ is defined by

$$\boldsymbol{X}_h^{(\mu)} := \{\boldsymbol{w}_h \in \mathbf{ND}_h \ : \ (\boldsymbol{w}_h, \nabla\psi_h)_{\boldsymbol{L}^2_\mu(\Omega)} = 0 \quad \forall \psi_h \in \Theta_h\},$$

i.e., it consists of all discrete μ-divergence-free edge element functions. The space $\boldsymbol{X}_h^{(\mu)}$ satisfies the following well-known discrete compactness result (cf. [80] for the original result with $\mu \equiv 1$).

Lemma 6.4. *Let $\{\boldsymbol{z}_h\}_{h>0} \subset \boldsymbol{H}(\mathbf{curl})$ be bounded and satisfy $\boldsymbol{z}_h \in \boldsymbol{X}_h^{(\mu)}$ for every $h > 0$. Then, there exists a subsequence $\{\boldsymbol{z}_{h_n}\}_{n=1}^\infty \subset \{\boldsymbol{z}_h\}_{h>0}$ with $h_n \to 0$ as $n \to \infty$ such that*

$$\begin{aligned} \boldsymbol{z}_{h_n} &\to \boldsymbol{z} && \text{strongly in } \boldsymbol{L}^2_\mu(\Omega) \text{ as } n \to \infty \\ \mathbf{curl}\, \boldsymbol{z}_{h_n} &\rightharpoonup \mathbf{curl}\, \boldsymbol{z} && \text{weakly in } \boldsymbol{L}^2(\Omega) \text{ as } n \to \infty \end{aligned}$$

for some $\boldsymbol{z} \in \boldsymbol{X}^{(\mu)}(\Omega)$.

Let us now introduce the operators that link the original function spaces with their respective finite element discretizations. At first, let us recall the $\boldsymbol{L}^2(\Omega)$-orthogonal projector onto $\mathbf{DG}_h$ (see Chapter 4) satisfying

$$\boldsymbol{Q}_h \boldsymbol{u} := \operatorname*{arg\,min}_{\boldsymbol{v}_h \in \mathbf{DG}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \Leftrightarrow \quad \int_\Omega (\boldsymbol{Q}_h \boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{v}_h \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,. \tag{6.4}$$

Thanks to the best approximation property, it satisfies

$$\begin{aligned} \|\boldsymbol{Q}_h \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} &\to 0 \qquad \text{as } h \to 0 \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \\ \|\boldsymbol{Q}_h \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} &\leq Ch \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \quad \forall h > 0, \end{aligned} \tag{6.5}$$

where the constant $C > 0$ is independent of $\boldsymbol{v}$ and h. Moreover, the following solution operator links $\boldsymbol{H}(\mathbf{curl})$ to $\mathbf{ND}_h$ and preserves the divergence properties for the discrete function (cf. [136, Definition 3.2] for the case $\mu = 1$).

Definition 6.5. For every $h > 0$ and $\boldsymbol{y} \in \boldsymbol{H}(\mathbf{curl})$, we denote the solution operator of the discrete variational mixed problem

$$\begin{cases} (\mathbf{curl}\, \boldsymbol{y}_h, \mathbf{curl}\, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} = (\mathbf{curl}\, \boldsymbol{y}, \mathbf{curl}\, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}_h \in \mathbf{ND}_h \\ (\boldsymbol{y}_h, \nabla\psi_h)_{\boldsymbol{L}^2_\mu(\Omega)} = (\boldsymbol{y}, \nabla\psi_h)_{\boldsymbol{L}^2_\mu(\Omega)} & \forall \psi_h \in \Theta_h \end{cases} \tag{6.6}$$

by $\boldsymbol{\Phi}_h \colon \boldsymbol{H}(\mathbf{curl}) \to \mathbf{ND}_h$ with $\boldsymbol{\Phi}_h \boldsymbol{y} := \boldsymbol{y}_h$.

Note that, $\mathbf{\Phi}_h$ satisfies

$$\begin{aligned} &\|\mathbf{\Phi}_h \boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} \leq C\|\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} \quad \forall h>0, \quad \forall \boldsymbol{y} \in \boldsymbol{H}(\mathbf{curl}) \\ &\lim_{h\to 0} \|\mathbf{\Phi}_h \boldsymbol{y} - \boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} = 0 \quad \forall \boldsymbol{y} \in \boldsymbol{H}(\mathbf{curl}), \end{aligned} \tag{6.7}$$

where the constant $C > 0$ is independent of h and $\boldsymbol{y}$. Let us also recall the pivotal inverse estimate (cf. (4.18)) for finite-element functions in $\mathbf{ND}_h$: There exists a constant $C_{\text{inv}} > 0$ that is independent of h such that

$$\|\mathbf{curl}\, \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_{\text{inv}}}{h} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \mathbf{ND}_h\,. \tag{6.8}$$

Following the construction in Chapter 4, we propose a time-discretization by the leapfrog scheme. To this aim, let us fix $N \in \mathbb{N}$. We define again an equidistant partition of $[0,T]$ as follows:

$$\tau := \frac{T}{N}, \quad 0 = t_0 < t_1 < \cdots < t_N = T \quad \text{with} \quad t_n := n\tau$$

for all $n \in \{0,\dots,N\}$. Moreover, we employ a growth restriction on τ. For some $0 < \alpha < 1$ it holds that

$$\frac{\tau}{h} \leq \sqrt{\frac{\underline{\epsilon\mu}}{2C_{\text{inv}}}(1-\alpha)} \quad \Leftrightarrow \quad \alpha \leq 1 - \frac{2C_{\text{inv}}^2 \tau^2}{\underline{\epsilon\mu} h^2}. \tag{6.9}$$

Moreover, we introduce the source and temperature time-discretizations

$$\boldsymbol{f}^{n-\frac{1}{2}} := \boldsymbol{f}(t_{n-\frac{1}{2}}) \quad \text{and} \quad \theta^{n-\frac{1}{2}} := \theta(t_{n-\frac{1}{2}}) \quad \forall n \in \{1,\dots,N\}.$$

Let us now propose the fully discrete scheme for (QVI):

$$\begin{cases} \text{For every } n \in \{1,\dots,N\} \text{ find } \boldsymbol{E}_h^n \in \mathbf{DG}_h \text{ such that} \\ \displaystyle\int_\Omega \epsilon\delta \boldsymbol{E}_h^n \cdot \left(\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}\right) - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}\right) \mathrm{d}x + j(\theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h) \\ \quad - j(\theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}, \boldsymbol{E}_h^{n-\frac{1}{2}}) \geq \displaystyle\int_\Omega \boldsymbol{f}^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}\right) \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h \\ \text{and for every } n \in \{1,\dots,N-1\} \text{ find } \boldsymbol{H}_h^{n+\frac{1}{2}} \in \mathbf{ND}_h \text{ such that} \\ \displaystyle\int_\Omega \mu\delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h \,\mathrm{d}x + \int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\, \boldsymbol{w}_h \,\mathrm{d}x = 0 \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h \\ \boldsymbol{E}_h^0 := \boldsymbol{Q}_h \boldsymbol{E}_0 \in \mathbf{DG}_h, \quad \boldsymbol{H}_h^{\frac{1}{2}} := \mathbf{\Phi}_h \boldsymbol{H}_0 \in \boldsymbol{X}_h^{(\mu)}, \end{cases} \tag{QVI$_{N,h}$}$$

where we use the same notation as in Chapter 4, namely,

$$\delta \boldsymbol{E}_h^n := \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau} \quad \text{and} \quad \boldsymbol{E}_h^{n-\frac{1}{2}} := \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2} \tag{6.10}$$

for all $n \in \{1,\dots,N\}$ as well as

$$\delta \boldsymbol{H}_h^{n+\frac{1}{2}} := \frac{\boldsymbol{H}_h^{n+\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{1}{2}}}{\tau}$$

for every $n \in \{1,\dots,N-1\}$. Furthermore, for the sake of a simpler notation, for every $n \in \{1,\dots,N\}$, we define the nonlinear function $\varphi_h^{n-\frac{1}{2}} \colon \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ by

$$\varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}) := j(\theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}, \boldsymbol{v}) = \int_\Omega j_{c,h}^{n-\frac{1}{2}} |\boldsymbol{v}| \,\mathrm{d}x \quad \text{with} \quad j_{c,h}^{n-\frac{1}{2}} := j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}). \tag{6.11}$$

The initial discrete magnetic field satisfies the discrete regularity $\boldsymbol{H}_h^{\frac{1}{2}} \in \boldsymbol{X}_h^{(\mu)}$ as $\boldsymbol{H}_0 \in \boldsymbol{X}^{(\mu)}(\Omega)$ (see (A6.6)) and the construction of $\boldsymbol{\Phi}_h$ in (6.6). Following Chapter 4, an important error estimate for the discrete initial electric field follows immediately from (A6.6) and (6.5). There exists a constant $C > 0$, independent of $h > 0$, such that

$$\|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq Ch \quad \forall h > 0. \tag{6.12}$$

The following theorem provides the well-posedness of $(\mathrm{QVI}_{N,h})$ and its proof is the foundation for our numerical computations.

Theorem 6.6. *Let Assumption 6.1 and Assumption 6.2 be satisfied. Then, for every $h > 0$ and $N \in \mathbb{N}$, the system $(\mathrm{QVI}_{N,h})$ admits a unique solution $(\boldsymbol{\mathcal{E}}_h^N, \boldsymbol{\mathcal{H}}_h^{N-1})$, where $\boldsymbol{\mathcal{E}}_h^N = \{\boldsymbol{E}_h^n\}_{n=1}^N$ and $\boldsymbol{\mathcal{H}}_h^{N-1} = \{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1}$. Moreover, $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \boldsymbol{X}_h^{(\mu)}$ for every $n \in \{1, \ldots, N-1\}$. If in addition the functions $\epsilon, j_{c,h}^{n-\frac{1}{2}}$, and $\boldsymbol{f}^{n-\frac{1}{2}}$ are piecewise constant for some $n \in \{1, \ldots, N\}$ in accordance with $\mathcal{T}_h$, then $\boldsymbol{E}_h^{n-\frac{1}{2}} \in \mathbf{DG}_h$ is explicitly given by*

$$\boldsymbol{E}_h^{n-\frac{1}{2}} = \frac{\tau\epsilon^{-1}}{2}\left(\boldsymbol{\omega}_h^n - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n\right), \quad \boldsymbol{\omega}_h^n := \boldsymbol{f}^{n-\frac{1}{2}} + \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}. \tag{6.13}$$

Here, $\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} : \mathbf{DG}_h \to \mathbf{DG}_h$ denotes the Hilbert projector onto the subdifferential $\partial\varphi_h^{n-\frac{1}{2}}(0) \subset \mathbf{DG}_h$ satisfying

$$\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{v}_h = \frac{j_{c,h}^{n-\frac{1}{2}}\boldsymbol{v}_h}{\max\left(|\boldsymbol{v}_h|, j_{c,h}^{n-\frac{1}{2}}\right)} \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,.$$

Proof. Let $N \in \mathbb{N}$, $h > 0$, $n \in \{1, \ldots, N\}$ and assume that $(\boldsymbol{E}_h^{n-1}, \boldsymbol{H}_h^{n-\frac{1}{2}}) \in \mathbf{DG}_h \times \mathbf{ND}_h$ is already computed. First of all, we note that

$$\delta\boldsymbol{E}_h^n = \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau} \underbrace{=}_{(6.10)} \frac{2}{\tau}\left(\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1}\right). \tag{6.14}$$

Therefore, we insert (6.14) into $(\mathrm{QVI}_{N,h})$ and obtain the discrete variational inequality to compute $\boldsymbol{E}_h^{n-\frac{1}{2}} \in \mathbf{DG}_h$:

$$\begin{aligned}\int_\Omega \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) \\ \geq \int_\Omega (\boldsymbol{f}^{n-\frac{1}{2}} + \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}) \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,.\end{aligned} \tag{6.15}$$

Thanks to its $\boldsymbol{L}^2$-structure, (6.15) admits a unique solution $\boldsymbol{E}_h^{n-\frac{1}{2}} \in \mathbf{DG}_h$ (see Theorem 2.6). In view of (6.14), it follows that $\boldsymbol{E}_h^n = 2\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1} \in \mathbf{DG}_h$ solves the discrete variational inequality in $(\mathrm{QVI}_{N,h})$. With $\boldsymbol{E}_h^n$ at hand, and if $n < N$, we compute $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \mathbf{ND}_h$ as the unique solution to the discrete linear equation

$$\int_\Omega \mu\boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x = \int_\Omega \mu\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x - \tau\int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h\,. \tag{6.16}$$

In this way we have shown that $(\mathrm{QVI}_{N,h})$ admits a unique solution $(\boldsymbol{\mathcal{E}}_h^N, \boldsymbol{\mathcal{H}}_h^{N-1})$. Finally, the fact that $\mathbf{curl}\,\nabla \equiv 0$ in combination with $\boldsymbol{H}_h^{\frac{1}{2}} \in \boldsymbol{X}_h^{(\mu)}$ and (6.16), yields the desired regularity property $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \boldsymbol{X}_h^{(\mu)}$ for every $n \in \{1, \dots, N-1\}$ by inductive reasoning.

Let now $n \in \{1, \dots, N\}$ such that $\epsilon, j_{c,h}^{n-\frac{1}{2}}$, and $\boldsymbol{f}^{n-\frac{1}{2}}$ are piecewise constant in accordance with $\mathcal{T}_h$. Then, first of all, according to the definition of the subdifferential, it holds that

$$\begin{aligned}\partial\varphi_h^{n-\frac{1}{2}}(0) &= \{\boldsymbol{v}_h \in \mathbf{DG}_h \;:\; (\boldsymbol{v}_h, \boldsymbol{p}_h)_{\boldsymbol{L}^2(\Omega)} \leq \varphi_h^{n-\frac{1}{2}}(\boldsymbol{p}_h) - \varphi_h^{n-\frac{1}{2}}(0) \quad \forall \boldsymbol{p}_h \in \mathbf{DG}_h\,\} \\ &\underbrace{=}_{(6.11)} \{\boldsymbol{v}_h \in \mathbf{DG}_h \;:\; (\boldsymbol{v}_h, \boldsymbol{p}_h)_{\boldsymbol{L}^2(\Omega)} \leq (j_{c,h}^{n-\frac{1}{2}}, |\boldsymbol{p}_h|)_{L^2(\Omega)} \quad \forall \boldsymbol{p}_h \in \mathbf{DG}_h\,\}.\end{aligned} \tag{6.17}$$

By the piecewise constant structure of $\mathbf{DG}_h$, setting

$$\boldsymbol{p}_h = \begin{cases} \boldsymbol{v}_h & \text{on } T \\ 0 & \text{on } \Omega \setminus T \end{cases}$$

for $T \in \mathcal{T}_h$ in (6.17) yields

$$\partial\varphi_h^{n-\frac{1}{2}}(0) = \{\boldsymbol{v}_h \in \mathbf{DG}_h \;:\; |\boldsymbol{v}_{h|T}| \leq j_{c,h}^{n-\frac{1}{2}}{}_{|T} \quad \text{for all } T \in \mathcal{T}_h\}. \tag{6.18}$$

Further, by definition of the Hilbert projector together with the piecewise constant structure of $\mathbf{DG}_h$, for every $\boldsymbol{v}_h \in \mathbf{DG}_h$, $\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} \boldsymbol{v}_h$ is given by the unique minimizer to

$$\begin{aligned}\min_{\boldsymbol{w}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \|\boldsymbol{v}_h - \boldsymbol{w}_h\|^2_{\boldsymbol{L}^2(\Omega)} &= \min_{\boldsymbol{w}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \sum_{T \in \mathcal{T}_h} \int_T |\boldsymbol{v}_h - \boldsymbol{w}_h|^2 \,\mathrm{d}x \\ &= \min_{\boldsymbol{w}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \sum_{T \in \mathcal{T}_h} |T| |\boldsymbol{v}_{h|T} - \boldsymbol{w}_{h|T}|^2.\end{aligned} \tag{6.19}$$

In view of (6.18), it follows that, for every $T \in \mathcal{T}_h$, $(\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} \boldsymbol{v}_h)_{|T} \in \mathbb{R}^3$ minimizes the problem

$$\min_{x \in \mathbb{R}^3} \left|\boldsymbol{v}_{h|T} - x\right|^2 \quad \text{s.t.} \quad |x| \leq j_{c,h}^{n-\frac{1}{2}}{}_{|T}. \tag{6.20}$$

The solution to the three-dimensional minimization problem (6.20) is exactly given by the projection of the vector $\boldsymbol{v}_{h|T} \in \mathbb{R}^3$ onto the euclidean ball with radius $j_{c,h}^{n-\frac{1}{2}}{}_{|T}$. In conclusion,

$$\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} \boldsymbol{v}_h = \frac{j_{c,h}^{n-\frac{1}{2}} \boldsymbol{v}_h}{\max\left(|\boldsymbol{v}_h|, j_{c,h}^{n-\frac{1}{2}}\right)} \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,.$$

Now, let us verify that $\boldsymbol{E}_h^{n-\frac{1}{2}}$ given by (6.13) solves (6.15). To this aim, we recall from the classical Hilbert projection theorem that $\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} \boldsymbol{\omega}_h^n$ is characterized by the solution to the variational inequality

$$\left(\boldsymbol{\omega}_h^n - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} \boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)} \boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \leq 0 \quad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0). \tag{6.21}$$

Using once again the piecewise constant structure, we set

$$\widetilde{\boldsymbol{v}_h} = \begin{Bmatrix} \boldsymbol{v}_h & \text{on } T \\ \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n & \text{on } \Omega\setminus T \end{Bmatrix} \in \partial\varphi_h^{n-\frac{1}{2}}(0)$$

in (6.21) to obtain its equivalence to

$$\left(\boldsymbol{\omega}_h^n - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(T)} \leq 0 \quad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0) \quad \forall T \in \mathcal{T}_h. \tag{6.22}$$

Using that ϵ is both positive and piecewise constant, we multiply the inequality in (6.22) with $\tau\epsilon^{-1}/2$, which yields

$$\left(\frac{\tau\epsilon^{-1}}{2}(\boldsymbol{\omega}_h^n - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n), \boldsymbol{v}_h - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(T)} \leq 0 \quad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0) \quad \forall T \in \mathcal{T}_h. \tag{6.23}$$

Utilizing (6.23), it follows that

$$\begin{aligned}(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n)_{\boldsymbol{L}^2(\Omega)} \\ \underbrace{=}_{(6.13)} \left(\frac{\tau\epsilon^{-1}}{2}(\boldsymbol{\omega}_h^n - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n), \boldsymbol{v}_h - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \leq 0 \quad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0),\end{aligned}$$

which implies

$$\begin{aligned}(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \leq (\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0) \\ \Rightarrow \max_{\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} (\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n)_{\boldsymbol{L}^2(\Omega)},\end{aligned} \tag{6.24}$$

since $\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n \in \partial\varphi_h^{n-\frac{1}{2}}(0)$. On the other hand, by setting $\boldsymbol{p}_h = \boldsymbol{E}_h^{n-\frac{1}{2}}$ in (6.17) we obtain

$$\max_{\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} (\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \leq \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) = \int_\Omega j_{c,h}^{n-\frac{1}{2}}|\boldsymbol{E}_h^{n-\frac{1}{2}}|\,\mathrm{d}x. \tag{6.25}$$

Introducing

$$\boldsymbol{q}_h^n(x) = \begin{cases} \dfrac{j_{c,h}^{n-\frac{1}{2}}(x)\boldsymbol{E}_h^{n-\frac{1}{2}}(x)}{|\boldsymbol{E}_h^{n-\frac{1}{2}}(x)|} & \text{if } \boldsymbol{E}_h^{n-\frac{1}{2}}(x) \neq 0 \\ 0 & \text{else,} \end{cases}$$

we have

$$\int_\Omega j_{c,h}^{n-\frac{1}{2}}|\boldsymbol{E}_h^{n-\frac{1}{2}}|\,\mathrm{d}x = (\boldsymbol{q}_h^n, \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} \leq \max_{\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} (\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)}, \tag{6.26}$$

since $\boldsymbol{q}_h^n \in \partial\varphi_h^{n-\frac{1}{2}}(0)$ according to (6.18). Hence, combining (6.24) with (6.25) and (6.26) yields

$$\varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) = (\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n, \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)}, \tag{6.27}$$

from which it follows that

$$\begin{aligned}
&\left(\frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}\right)_{\boldsymbol{L}^2(\Omega)} + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) \\
\underbrace{=}_{(6.13)}\ & (\boldsymbol{\omega}_h^n - \boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) \\
=\ & (\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n, \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} \\
& - (\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) \\
\underbrace{\geq}_{(6.17)}\ & (\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{P}_{\partial\varphi_h^{n-\frac{1}{2}}(0)}\boldsymbol{\omega}_h^n, \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) \\
\underbrace{=}_{(6.27)}\ & (\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,.
\end{aligned}$$

This finishes the proof. □

Remark 6.7. Although the explicit formula (6.13) is not necessary for the well-posedness analysis, in the same way as in Theorem 4.3, it is crucial for the numerical computation of the solution. By means of this explicit formula, the computation of the solution for one step in the iterative scheme ($\mathrm{QVI}_{N,h}$) is reduced to setting $\boldsymbol{E}_h^{n-\frac{1}{2}}$ according to (6.13) and solving the linear problem (6.16). In comparison with fully implicit schemes (e.g. the implicit Euler) and particularly fixed-point iterations that are commonly used in the context of QVIs, this should result in a significant decrease in computational costs.

In order to prove the zero- and first-order stability of our fully discrete scheme ($\mathrm{QVI}_{N,h}$), as in Chapter 4, we have to establish a first-order stability result for the initial leapfrog time step.

Lemma 6.8. *Under Assumption 6.1, Assumption 6.2 and* (6.9) *there exists a constant $C > 0$, independent of N and h, such that*

$$\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C.$$

Proof. We begin by testing ($\mathrm{QVI}_{N,h}$) for $n = 1$ with $\boldsymbol{v}_h = \boldsymbol{E}_h^{\frac{1}{2}} - \delta\boldsymbol{E}_h^1 \in \mathbf{DG}_h$ such that we obtain

$$\|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq \int_\Omega (\boldsymbol{f}^{\frac{1}{2}} + \mathbf{curl}\,\boldsymbol{H}_h^{\frac{1}{2}}) \cdot \delta\boldsymbol{E}_h^1 \,\mathrm{d}x + \int_\Omega j_c(\cdot, \theta^{\frac{1}{2}}, \boldsymbol{H}_h^{\frac{1}{2}})(|\boldsymbol{E}_h^{\frac{1}{2}} - \delta\boldsymbol{E}_h^1| - |\boldsymbol{E}_h^{\frac{1}{2}}|)\,\mathrm{d}x.$$

Thanks to the uniform boundedness of $\boldsymbol{H}_h^{\frac{1}{2}}$ with respect to $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}$ (see (6.7)), (A6.3), and (A6.6), we obtain

$$\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq \underline{\epsilon}^{-\frac{1}{2}}\big(\|\boldsymbol{f}^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|j_c(\cdot, \theta^{\frac{1}{2}}, \boldsymbol{H}_h^{\frac{1}{2}})\|_{L^2(\Omega)}\big) \leq C \tag{6.28}$$

for a constant $C > 0$, independent of N and h. Additionally, ($\mathrm{QVI}_{N,h}$) for $n = 1$ tested with $\boldsymbol{w}_h = \delta\boldsymbol{H}_h^{\frac{3}{2}}$ gives us after some rearrangements and integration by parts

$$\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} = \int_\Omega \mu\delta\boldsymbol{H}_h^{\frac{3}{2}} \cdot \delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x \underbrace{=}_{(\mathrm{QVI}_{N,h})} -\int_\Omega \boldsymbol{E}_h^1 \cdot \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x$$

$$= -\int_\Omega (\tau\delta\boldsymbol{E}_h^1 + \boldsymbol{E}_h^0 - \boldsymbol{E}_0)\cdot \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x - \int_\Omega \mathbf{curl}\,\boldsymbol{E}_0\cdot\delta\boldsymbol{H}_h^{\frac{3}{2}}\,\mathrm{d}x$$
$$\underbrace{\leq}_{(6.8)} \frac{C_{\text{inv}}}{\sqrt{\underline{\epsilon}\underline{\mu}}}\left(\frac{\tau}{h}\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{\|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2_\epsilon(\Omega)}}{h} + \|\mathbf{curl}\,\boldsymbol{E}_0\|_{\boldsymbol{L}^2_\epsilon(\Omega)}\right)\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2_\mu(\Omega)}.$$

Finally, the assertion follows from (6.9), (6.28), and (6.12). □

In the following two lemmas, we prove zero- and first-order stability estimates for the fully discrete solution to $(\mathrm{QVI}_{N,h})$.

Lemma 6.9. *Under Assumption 6.1, Assumption 6.2 and* (6.9)*, there exists a constant $C > 0$, independent of N and h, such that*

$$\max_{1\leq n\leq N}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \max_{1\leq n\leq N-1}\|\delta\boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C.$$

Proof. First, fix $n\in\{2,\dots,N\}$ and test the n-th inequality in $(\mathrm{QVI}_{N,h})$ with $\boldsymbol{v}_h = \boldsymbol{E}_h^{n-\frac{3}{2}}$ and the $(n-1)$-th inequality with $\boldsymbol{v}_h = \boldsymbol{E}_h^{n-\frac{1}{2}}$. Then, adding the resulting inequalities and using (4.28) implies

$$\int_\Omega \epsilon(\delta\boldsymbol{E}_h^n - \delta\boldsymbol{E}_h^{n-1})\cdot(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1}) - \tau\,\mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \tag{6.29}$$
$$-\frac{2}{\tau}\left(\varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}}) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) + \varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) - \varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}})\right)$$
$$\leq \int_\Omega (\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x.$$

After summing (6.29) up over $\{2,\dots,i_0\}$ for a fixed $i_0\in\{2,\dots,N\}$, we obtain

$$\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \tau\sum_{n=2}^{i_0}\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \tag{6.30}$$
$$\leq \|\delta\boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \sum_{n=2}^{i_0}\int_\Omega(\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x$$
$$+\frac{2}{\tau}\sum_{n=2}^{i_0}\left(\varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}}) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) + \varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) - \varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}})\right).$$

Let us now estimate the terms in (6.30) separately. For the first sum on the right-hand side of (6.30) it holds that

$$\sum_{n=2}^{i_0}\int_\Omega(\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n + \delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x \tag{6.31}$$
$$\leq \frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n=2}^{i_0}\|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n=2}^{i_0}\|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2_\epsilon(\Omega)}$$
$$\leq \frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n=2}^{i_0-1}\|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n=1}^{i_0-1}\|\boldsymbol{f}^{n+\frac{1}{2}} - \boldsymbol{f}^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)}$$
$$+\frac{2}{\underline{\epsilon}}\|\boldsymbol{f}^{i_0-\frac{1}{2}} - \boldsymbol{f}^{i_0-\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{8}\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.$$

Now, introducing the set

$$\mathcal{I}_{i_0} := \{n \in \{1,\dots,i_0-1\} \ : \ \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq 1\}$$

and recalling the definition (6.2) with $V = \boldsymbol{L}^2(\Omega)$, it holds that

$$\begin{aligned}
&\sum_{n=2}^{i_0-1} \|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
\leq& \sum_{n\in\mathcal{I}_{i_0}\setminus\{1\}} \|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \sum_{n\in\mathcal{I}^c_{i_0}\setminus\{1\}} \|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
\leq& \mathrm{TV}(\boldsymbol{f}) + \sum_{n\in\mathcal{I}^c_{i_0}\setminus\{1\}} \|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{6.32}$$

In an analogous way, we calculate

$$\begin{aligned}
&\sum_{n=1}^{i_0-1} \|\boldsymbol{f}^{n+\frac{1}{2}} - \boldsymbol{f}^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\qquad\qquad \leq \mathrm{TV}(\boldsymbol{f}) + \sum_{n\in\mathcal{I}^c_{i_0}} \|\boldsymbol{f}^{n+\frac{1}{2}} - \boldsymbol{f}^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{6.33}$$

Then, applying (6.32) and (6.33) to (6.31) leads to

$$\begin{aligned}
&\sum_{n=2}^{i_0} \int_\Omega (\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) \,\mathrm{d}x \\
\leq& \frac{1}{8} \|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{2}{\underline{\epsilon}} \mathrm{TV}(\boldsymbol{f})^2 + \frac{2}{\sqrt{\underline{\epsilon}}} \mathrm{TV}(\boldsymbol{f}) + \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n\in\mathcal{I}^c_{i_0}\setminus\{1\}} \|\boldsymbol{f}^{n-\frac{1}{2}} - \boldsymbol{f}^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&+ \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n\in\mathcal{I}^c_{i_0}} \|\boldsymbol{f}^{n+\frac{1}{2}} - \boldsymbol{f}^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{6.34}$$

Moreover, thanks to (A6.6), we make use of the local Lipschitz-properties of j_c (cf. (A6.4) and (A6.5)) in order to estimate the second sum on the right-hand side of (6.30)

$$\begin{aligned}
&\frac{2}{\tau} \sum_{n=2}^{i_0} \left(\varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}}) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) + \varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) - \varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}}) \right) \\
\underbrace{=}_{(6.11)}& \ 2 \sum_{n=2}^{i_0} \int_\Omega \left(j_c(\cdot, \theta^{n-\frac{3}{2}}, \boldsymbol{H}_h^{n-\frac{3}{2}}) - j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}) \right) \left(\frac{|\boldsymbol{E}_h^{n-\frac{1}{2}}| - |\boldsymbol{E}_h^{n-\frac{3}{2}}|}{\tau} \right) \mathrm{d}x \\
\underbrace{\leq}_{(A6.3)–(A6.5)}& \ \frac{L(\|\theta\|_{C([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}} \sum_{n=2}^{i_0} \|\theta^{n-\frac{1}{2}} - \theta^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&+ \frac{\tau L(\|\theta\|_{C([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}\underline{\mu}}} \sum_{n=2}^{i_0} \|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2_\mu(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
\leq& \ \frac{L(\|\theta\|_{C([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}} \sum_{n=2}^{i_0-1} \|\theta^{n-\frac{1}{2}} - \theta^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)}
\end{aligned} \tag{6.35}$$

$$+\frac{2L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}}\|\theta^{i_0-\frac{1}{2}}-\theta^{i_0-\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{1}{8}\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}$$
$$+\frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon\mu}}\sum_{n=2}^{i_0}\|\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{\tau}{8T}\sum_{n=2}^{i_0}\|\delta\boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.$$

Now, using again the definition (6.2) this time with $V=L^2(\Omega)$ and following the same argumentation as in (6.32), we obtain

$$\begin{aligned}&\sum_{n=2}^{i_0-1}\|\theta^{n-\frac{1}{2}}-\theta^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)}\\&\le\mathrm{TV}(\theta)+\sum_{n\in\mathcal{I}^c_{i_0}\setminus\{1\}}\|\theta^{n-\frac{1}{2}}-\theta^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\end{aligned}\tag{6.36}$$

and hence applying (6.36) to (6.35) results in

$$\begin{aligned}&\frac{2}{\tau}\sum_{n=2}^{i_0}\Big(\varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}})-\varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}})+\varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}})-\varphi^{n-\frac{3}{2}}(\boldsymbol{E}_h^{n-\frac{3}{2}})\Big)\\&\le\frac{L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}}\Big(\mathrm{TV}(\theta)+\sum_{n\in\mathcal{I}^c_{i_0}\setminus\{1\}}\|\theta^{n-\frac{1}{2}}-\theta^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\Big)\\&\quad+\frac{2L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}}\mathrm{TV}(\theta)^2+\frac{1}{4}\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\\&\quad+\frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon\mu}}\sum_{n=2}^{i_0}\|\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{\tau}{8T}\sum_{n=2}^{i_0-1}\|\delta\boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.\end{aligned}\tag{6.37}$$

The estimation of the second term on the left-hand side of (6.30) requires the following formula: for every $n\in\{2,\dots,N-1\}$ it holds that

$$-\tau\int_\Omega\mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x=\|\delta\boldsymbol{H}_h^{n+\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\|\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}.\tag{6.38}$$

In fact, substracting the n-th and $(n-1)$-th equations of $(\mathrm{QVI}_{N,h})$ results in

$$-\int_\Omega(\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1})\cdot\mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x=\int_\Omega\mu(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}-\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\boldsymbol{w}_h\,\mathrm{d}x\quad\forall\boldsymbol{w}_h\in\mathbf{ND}_h\,.$$

Hence, choosing $\boldsymbol{w}_h=\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\in\mathbf{ND}_h$ implies (6.38). Now, utilizing (6.3) and (6.8) we obtain an estimate for the second term on the left-hand side of (6.30)

$$\begin{aligned}&\tau\sum_{n=2}^{i_0}\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x\\&\underbrace{=}_{(6.3)}\tau\sum_{n=2}^{i_0-1}\int_\Omega\mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\delta\boldsymbol{E}_h^n+\mathbf{curl}\,\delta\boldsymbol{H}_h^{i_0-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{i_0}+\mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\cdot\delta\boldsymbol{E}_h^1\,\mathrm{d}x\\&\underbrace{\le}_{(6.38)}-\|\delta\boldsymbol{H}_h^{i_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{2C^2_{\mathrm{inv}}\tau^2}{\underline{\epsilon\mu}h^2}\|\delta\boldsymbol{H}_h^{i_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\&\quad+\frac{1}{8}\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\frac{2C^2_{\mathrm{inv}}\tau^2}{\underline{\epsilon\mu}h^2}\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{1}{8}\|\delta\boldsymbol{E}_h^{1}\|_{\boldsymbol{L}^2_\epsilon(\Omega)},\end{aligned}\tag{6.39}$$

where in the final step of the estimation we have also used the Hölder and Young inequalities as well as (6.8) to obtain the estimates

$$\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{i_0-\frac12}\cdot\delta\boldsymbol{E}_h^{i_0}\,\mathrm{d}x \le \frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\|\delta\boldsymbol{H}_h^{i_0-\frac12}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac18\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}$$

and

$$\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac32}\cdot\delta\boldsymbol{E}_h^{1}\,\mathrm{d}x \le \frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\|\delta\boldsymbol{H}_h^{\frac32}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac18\|\delta\boldsymbol{E}_h^{1}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.$$

In conclusion, we insert (6.34), (6.37) and (6.39) into (6.30) to obtain

$$\begin{aligned}
&\frac12\|\delta\boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \left(1-\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2} - \frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}\underline{\mu}}\right)\|\delta\boldsymbol{H}_h^{i_0-\frac12}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \qquad (6.40)\\
\le&\frac98\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \left(1+\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\right)\|\delta\boldsymbol{H}_h^{\frac32}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{2}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\boldsymbol{f}) + \frac{L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\theta)\\
&+\frac{2}{\underline{\epsilon}}\mathrm{TV}(\boldsymbol{f})^2 + \frac{2L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}}\mathrm{TV}(\theta)^2 + \frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n\in\mathcal{I}_{i_0}^c\setminus\{1\}}\|\boldsymbol{f}^{n-\frac12}-\boldsymbol{f}^{n-\frac32}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\\
&+\frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n\in\mathcal{I}_{i_0}^c}\|\boldsymbol{f}^{n+\frac12}-\boldsymbol{f}^{n-\frac12}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\\
&+\frac{L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}}\sum_{n\in\mathcal{I}_{i_0}^c\setminus\{1\}}\|\theta^{n-\frac12}-\theta^{n-\frac32}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\\
&+\frac{\tau}{8T}\sum_{n=2}^{i_0-1}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}\underline{\mu}}\sum_{n=2}^{i_0-1}\|\delta\boldsymbol{H}_h^{n-\frac12}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}.
\end{aligned}$$

Thanks to the growth condition (6.9), there exists a fixed $N_0\in\mathbb{N}$ such that for $\tau=T/N$ it holds that

$$1-\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2} - \frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}\underline{\mu}} \ge \alpha - \frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}\underline{\mu}} \ge \frac{\alpha}{2} \quad \forall N\ge N_0. \quad (6.41)$$

Now, by means of (6.9) and Lemma 6.8, the first two terms on the right-hand side of (6.40) are bounded independently of N and h, so that we can define

$$\begin{aligned}
\beta :=& \max_{(N,h)\in\mathbb{N}\times(0,\infty)}\left(\frac98\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \left(1+\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\right)\|\delta\boldsymbol{H}_h^{\frac32}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right) + \frac{2}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\boldsymbol{f}) \qquad (6.42)\\
&+\frac{L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\theta) + \frac{2}{\underline{\epsilon}}\mathrm{TV}(\boldsymbol{f})^2 + \frac{2L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}}\mathrm{TV}(\theta)^2.
\end{aligned}$$

To simplify the estimate (6.40), for $n\in\{1,\dots,i_0-1\}$, let us introduce the coefficient

$$h_n := \begin{cases}
\frac{1}{\sqrt{\underline{\epsilon}}}\|\boldsymbol{f}^{\frac32}-\boldsymbol{f}^{\frac12}\|_{\boldsymbol{L}^2(\Omega)} & \text{if } n=1\\[1ex]
\max\Big\{\frac{1}{\sqrt{\underline{\epsilon}}}\|\boldsymbol{f}^{n-\frac12}-\boldsymbol{f}^{n-\frac32}\|_{\boldsymbol{L}^2(\Omega)} + \frac{1}{\sqrt{\underline{\epsilon}}}\|\boldsymbol{f}^{n+\frac12}-\boldsymbol{f}^{n-\frac12}\|_{\boldsymbol{L}^2(\Omega)} & \\
\quad + \frac{L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}}\|\theta^{n-\frac12}-\theta^{n-\frac32}\|_{\boldsymbol{L}^2(\Omega)} + \frac{\tau}{8T}, & \\
\quad \frac{2\tau TL(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon}\underline{\mu}}\Big\} & \text{if } n\in\{2,\dots,i_0-1\},
\end{cases}$$

by the use which, in combination with (6.41) and (6.42), it follows that

$$\frac{\alpha}{2}\Big(\|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \|\delta \boldsymbol{H}_h^{i_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\Big) \le \beta + \sum_{n=1}^{i_0-1} h_n \Big(\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\Big).$$

Invoking the discrete version of Gronwall's inequality, this implies

$$\|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \|\delta \boldsymbol{H}_h^{i_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le \beta \exp\left(\sum_{n=1}^{i_0-1} h_n\right).$$

It remains to show that the term $\sum_{n=1}^{i_0-1} h_n$ is bounded. Indeed, it holds that

$$\sum_{n=1}^{i_0-1} h_n \le \max\left\{\frac{2}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\boldsymbol{f}) + \frac{L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\theta) + \frac{1}{8}, \frac{2T^2 L(\|\theta\|_{\mathcal{C}([0,T],L^\infty(\Omega))})^2}{\underline{\epsilon\mu}}\right\}.$$

As N_0 was fixed and $i_0 \in \{2, \dots, N\}$ chosen arbitrarily, we deduce with Lemma 6.8 that

$$\max_{1\le n\le N} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \max_{1\le n\le N-1} \|\delta \boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \le C \quad \forall N \in \mathbb{N} \tag{6.43}$$

for a constant $C > 0$, independent of N and h. This completes the proof. □

Lemma 6.10. *Let Assumption 6.1, Assumption 6.2 and* (6.9) *be satisfied. Then, there exists a constant $C > 0$ which is independent of N and h such that*

$$\max_{1\le n\le N} \|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \max_{1\le n\le N-1} \|\boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \max_{1\le n\le N-1} \|\mathbf{curl}\, \boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \le C.$$

Proof. Using the reversed triangle inequality, it follows by definition of the difference quotients together with Lemma 6.9 that

$$\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \tau C + \|\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \dots \le n\tau C + \|\boldsymbol{E}_h^0\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le TC + \|\boldsymbol{E}_h^0\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le C$$

for any $n \in \{1, \dots, N\}$. Using the same argumentation for the discrete magnetic fields, it follows that

$$\max_{1\le n\le N} \|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \max_{1\le n\le N-1} \|\boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \le C. \tag{6.44}$$

Now, we insert $\boldsymbol{v}_h = \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \boldsymbol{E}_h^{n-\frac{1}{2}} \in \mathbf{DG}_h$ for $n \in \{1, \dots, N\}$ into (QVI$_{N,h}$) such that we obtain

$$\begin{aligned}
&\|\mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\le \int_\Omega \big(\epsilon \delta \boldsymbol{E}_h^n - \boldsymbol{f}^{n-\frac{1}{2}}\big) \cdot \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \,\mathrm{d}x + \varphi_h^{n-\frac{1}{2}}(\mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \boldsymbol{E}_h^{n-\frac{1}{2}}) - \varphi_h^{n-\frac{1}{2}}(\boldsymbol{E}_h^{n-\frac{1}{2}}) \\
&\le \big(\|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})\|_{L^2(\Omega)}\big)\|\mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Ultimately, using (6.43), (6.44), and (A6.3), the assertion follows. □

6.2 Convergence

The previous stability estimates enable the establishment of a convergence result for ($\mathrm{QVI}_{N,h}$), which yields the well-posedness of (QVI) as a direct consequence. We will not present any details on the convergence here, as it is in large parts similar to Section 4.3. Invoking similar interpolation techniques, in particular, using the piecewise continuous and piecewise constant interpolations for the discrete electric and magnetic fields as in (4.45) and (4.46), the resulting convergence result reads as follows.

Theorem 6.11. *Let Assumption 6.1, Assumption 6.2 and* (6.9) *be satisfied. Then, there exists*

$$(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \times L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{X}^{(\mu)}(\Omega))$$

such that for $N = N(h)$ *with* $N(h) \to \infty$ *as* $h \to 0$ *it holds that*

$$\begin{aligned}\lim_{h\to 0}\|\boldsymbol{E}_{N,h}-\boldsymbol{E}\|_{\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))} &= \lim_{h\to 0}\|\overline{\boldsymbol{E}}_{N,h}-\boldsymbol{E}\|_{L^\infty((0,T),\boldsymbol{L}^2(\Omega))} = 0\\ \lim_{h\to 0}\|\boldsymbol{H}_{N,h}-\boldsymbol{H}\|_{\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega))} &= \lim_{h\to 0}\|\overline{\boldsymbol{H}}_{N,h}-\boldsymbol{H}\|_{L^\infty((0,T),\boldsymbol{L}^2(\Omega))} = 0.\end{aligned}$$

Moreover, $(\boldsymbol{E},\boldsymbol{H})$ *is the unique solution to* (QVI).

Comparing with the proof of Theorem 4.15, the constraint preserving mollification process presented in Section 4.3.1 is not needed. However, the convergence proof involves the usage of the compactness property in Lemma 6.4 to pass to the limit in the nonlinearity. Moreover, to be able to pass to the limit in the piecewise constant interpolation of the source term $\boldsymbol{f}$, one uses the fact that $\boldsymbol{f}$ is of bounded variation, and therefore continuous almost everywhere (cf. [12, Corollary 3.33]). A complete proof can be found in the corresponding preprint [66].

6.3 Numerical Experiments

In this final section, we present a numerical test for (QVI) based on the proposed discrete scheme ($\mathrm{QVI}_{N,h}$). As mentioned in the introduction, experiments in the physics literature report the dependence of the critical current density j_c not only on the temperature but also on the magnetic field $\boldsymbol{H}$. With our numerical test we strive to show the impact of the magnetic field dependence on the simulation. In particular, we replicate the findings in [43,81,133], where the authors agree upon the fact that for certain types of superconductors the critical current density $j_c = j_c(\theta,\boldsymbol{H})$ should vanish for high values of $|\boldsymbol{H}|$. Based on physical experiments, the findings in [43] suggest the Kim-like critical-state model in its power form, that is, the choice

$$j_c(\cdot,\theta,\boldsymbol{H}) \coloneqq \frac{c(1-\theta)^2}{1+|\boldsymbol{H}|^\beta}\chi_{\Omega_{\mathrm{sc}}}(\cdot), \quad \text{where } c>0 \text{ and } \beta \geq 0, \tag{6.45}$$

to model the magnetic field dependence for materials covered in specifically designed silver-sheathed tapes. Here, $\Omega_{\mathrm{sc}} \subset \Omega$ denotes a high temperature superconductor. For our numerical test we consider the computational domain $\Omega = (-1,1)^3$, the time horizon $T = 1$, the material parameters $\epsilon, \mu \equiv 1$, and $(\boldsymbol{E}_0,\boldsymbol{H}_0) = (0,0)$ as an initial value. For our superconductor Ω_{sc}, we use

$$\Omega_{\mathrm{sc}} = \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \sqrt{x_1^2+x_2^2+x_3^2} \leq 0.2\},$$

and along with the choice $c = 4\cdot 10^3$ and the temperature distribution $\theta(x,t) = t$, we consider three different configurations for the parameter β:

(i) $\beta = 0$: the case of no magnetic field dependence.

(ii) $\beta = 1$ and $\beta = 3$: the case of the magnetic field strength entering to the exponents 1 and 3.

Finally, for the applied current source, we choose $\boldsymbol{f} \colon [0,1] \times \Omega \to \mathbb{R}^3$, defined by

$$\boldsymbol{f}(t, x_1, x_2, x_3) \coloneqq \begin{cases} 10(1+4t^2)\left(0, \dfrac{-x_3}{\sqrt{x_2^2+x_3^2}}, \dfrac{x_2}{\sqrt{x_2^2+x_3^2}}\right) & \text{if } (x_1,x_2,x_3) \in P \\ 0 & \text{if } (x_1,x_2,x_3) \notin P, \end{cases}$$

where $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_1| \leq 0.5, 0.3 \leq \sqrt{x_2^2 + x_3^2} \leq 0.5\}$ models a cylindrical pipe coil. The mentioned setup and the resulting magnetic field at one time instant are visualized in Figure 6.2. To realize the proposed numerical resolution of (QVI), we implemented ($\mathrm{QVI}_{N,h}$) with time steps according to the CFL-condition (6.9) and roughly 3 million DoFs in the mixed finite element space, where we have in particular refined the underlying mesh around the superconductor Ω_{sc}. As observable in Figure 6.3, in the beginning, the superconducting effect is fully present for every configuration of β. This is in line with the choice of the initial value. With the evolution of time, the strength of the magnetic field enhances, which leads to Ω_{sc} gradually leaving its superconducting state. Specifically, we see that the breakdown of the superconducting state is hugely accelerated with the choice $\beta = 3$, whereas for $\beta = 1$, the solution is comparable to the solution for $\beta = 0$, i.e., the case of no magnetic field dependence. At the final time, the superconducting state is fully broken down independent of the configuration of β. We conclude that including the quasi-variational character is particularly important to the modeling of superconductors when strong magnetic fields are considered.

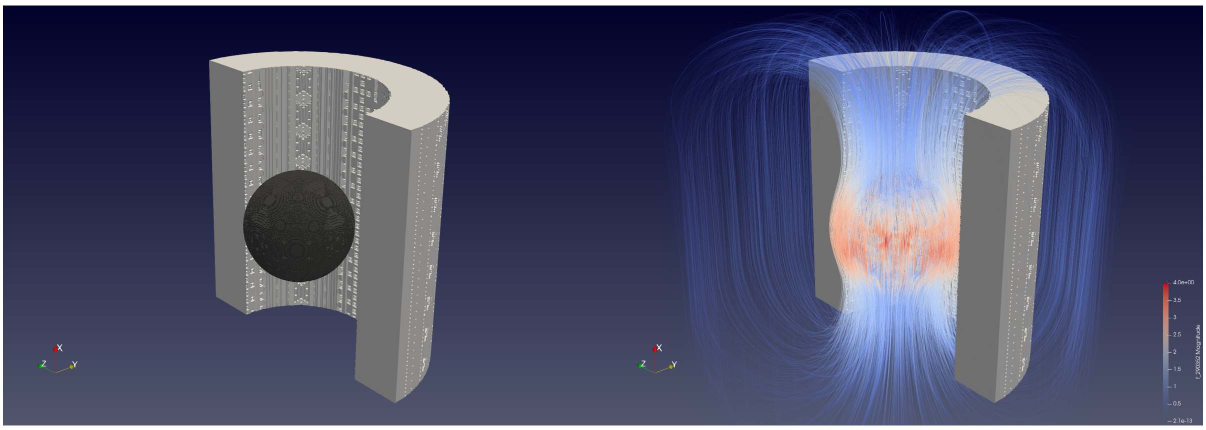

Figure 6.2: Clipped pipe coil with the superconductor in its center (left) and 3D visualization at a fixed time step (right).

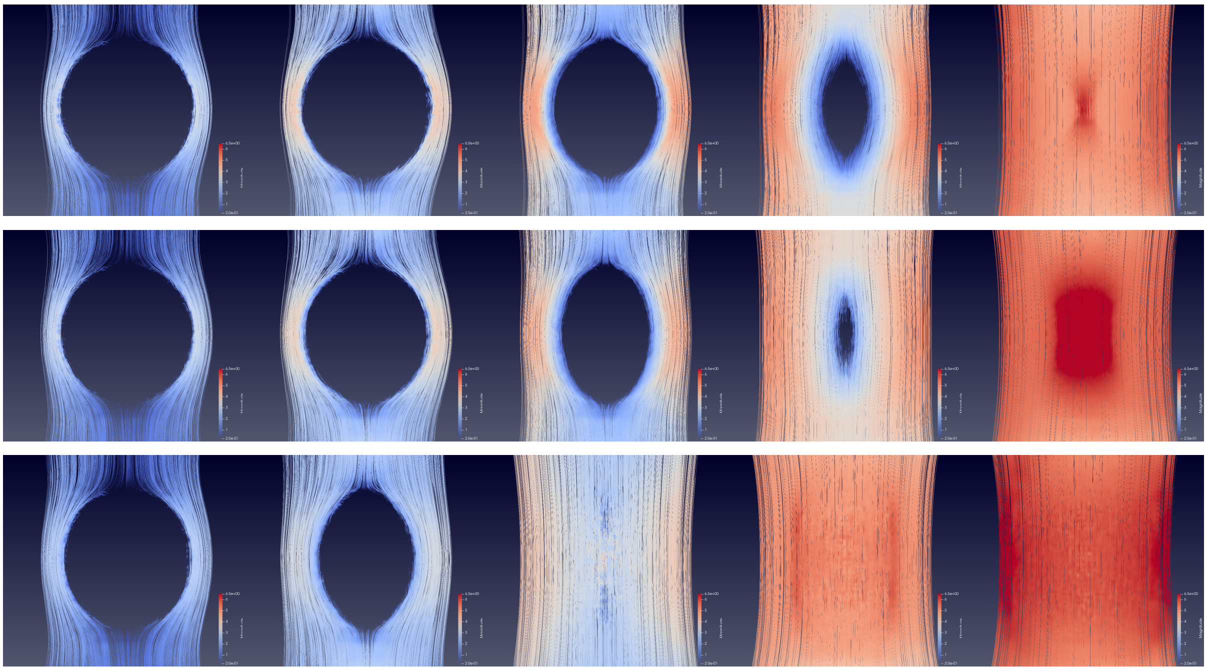

Figure 6.3: Time-evolution of the magnetic field for the three different configurations of β (2D slice). The first line shows the case $\beta = 0$, while the second and third line show the cases $\beta = 1$ and $\beta = 3$.

CHAPTER 7

OUTLOOK

In this thesis, we have covered a variety of problem statements within the realm of Maxwell variational inequalities. Motivated by the eddy current approximation of the hyperbolic Maxwell obstacle problem, we moved on to the construction of an efficient solver and its numerical analysis. Then, inspired by applications in ferromagnetic shielding, we covered the analysis and optimal control of a quasilinear variational inequality. Finally, with the help of the previous techniques, we investigated the numerical analysis of a second kind Maxwell quasi-variational inequality with applications in superconductivity.

Certainly, every single of these topics has its own interesting and challenging continuations. However, rather than going into detail in this direction, let us use this final chapter for a small preview on a very recent work in progress that is related to the last chapter.

7.1 Towards Modeling of Magnetic Levitation Phenomena

In the classic physical experiment of placing a superconductor over a permanent magnet, levitation of the superconductor can be observed assuming the superconductor is in its superconducting state. As described in [35, 130], this levitation is related to the Meissner effect and stable levitation mainly occurs due to hysterical forces resulting from the interaction of flux lines with defects in the material. The literature concerned with a macroscopical description of this phenomenon is quite rich. In particular, we refer to the paper [25] for a topical overview.

Given a superconductor $\Omega_{\mathrm{sc}} \subset \Omega$, physical experiments suggest that the interaction force between the permanent magnet and the superconductor Ω_{sc} can be described by the term

$$F(\Omega_{\mathrm{sc}}, \boldsymbol{J}, \boldsymbol{H}) := \int_{\Omega_{\mathrm{sc}}} \boldsymbol{J}(t,x) \times \boldsymbol{H}(t,x) \, \mathrm{d}x, \tag{7.1}$$

where $\boldsymbol{J}$ denotes the current density. Many numerical experiments using such quantity have already been realized by approximating the present QVI-character by a certain power law for the current density (cf. [116, 123]). However, to the best of the authors knowledge, a full QVI model in the context of magnetic levitation has not yet been considered.

In this last part, building upon (QVI), we strive to present a reasonable model that does not rely on a simplification by a power law for the current density. In particular, based on the interaction force term (7.1), we construct a specific nonlinearity j_c to model the displacement of the superconductor Ω_{sc} under the influence of a magnetic field. As a baseline, we modify the model from the previous chapter, resulting in another hyperbolic Maxwell quasi-variational inequality of the second kind with an L^1-type nonlinearity:

$$\begin{cases} \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \mathbf{curl}\,\boldsymbol{H}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \quad + \int_\Omega j_c(\cdot,t,\boldsymbol{H},\boldsymbol{J})|\boldsymbol{v}|\,\mathrm{d}x - \int_\Omega j_c(\cdot,t,\boldsymbol{H},\boldsymbol{J})|\boldsymbol{E}(t)|\,\mathrm{d}x \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \text{for a.e. } t\in(0,T) \text{ and all } \boldsymbol{v}\in \boldsymbol{L}^2(\Omega) \\ \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t\in(0,T) \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{QVI$_{\text{lev}}$}$$

Note that, compared to (QVI), the critical current in (QVI$_{\text{lev}}$) not only depends on the magnetic field $\boldsymbol{H}$, but also on the current density $\boldsymbol{J}$ and $t\in(0,T)$. For our specific choice of j_c, we start with an initial configuration of the superconductor $\Omega_{\text{sc}}(0)\subset\Omega$ having mass $m\in(0,\infty)$. Then, we define

$$\begin{aligned} j_c\colon \Omega\times(0,T)\times L^\infty((0,T),\boldsymbol{X}^{(\mu)}(\Omega))\times L^\infty((0,T),\boldsymbol{L}^2(\Omega)) &\to \mathbb{R} \\ (x,t,\boldsymbol{H},\boldsymbol{J}) &\mapsto c\chi_{\Omega_{\text{sc}}(t)}(x), \end{aligned} \tag{7.2}$$

where $c>0$ is a chosen constant and

$$\begin{aligned} \Omega_{\text{sc}}(t) &= \Omega_{\text{sc}}(0) + \int_0^t\int_0^s m^{-1}F(\Omega_{\text{sc}}(\tau),\boldsymbol{J}_{|(0,t)},\boldsymbol{H}_{|(0,t)})\,\mathrm{d}\tau\,\mathrm{d}s \in \mathbb{R}^3 \\ \boldsymbol{J} &= \mathbf{curl}\,\boldsymbol{H} - \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}. \end{aligned} \tag{7.3}$$

Here, based on Newton's second law of motion, the term $m^{-1}F(\Omega_{\text{sc}}(\tau),\boldsymbol{J},\boldsymbol{H})$ describes the acceleration of the superconductor $\Omega_{\text{sc}}(\tau)$ at time τ and its displacement is therefore obtained by double integration as performed in (7.3). For simplicity, we assume that the mass m is large enough so that the superconductor stays within the physical domain, i.e., $\Omega_{\text{sc}}(t)\subset\Omega$ for all times $t\in[0,T]$.

Since the critical current in (7.2) depends on the current density and therefore implicitly on $\mathbf{curl}\,\boldsymbol{H}$ and the time derivative of $\boldsymbol{E}$, the model (QVI$_{\text{lev}}$) becomes highly complicated and does not fit into the previous framework of (QVI). As a remedy, at least the dependence on the time derivative can be disregarded when considering the eddy current case, i.e., ϵ being negligibly small.

To numerically solve the system (QVI$_{\text{lev}}$) we choose once again the leapfrog stepping in the same way as in Chapter 4. The time integration in (7.3) is handled by the usage of a basic quadrature rule.

For our provisional computational results, we chose a similar applied current source as before, namely

$$\boldsymbol{f}\colon\Omega\to\mathbb{R}^3,\quad \boldsymbol{f}(x_1,x_2,x_3) := \begin{cases} \left(0, \frac{-x_3}{\sqrt{x_2^2+x_3^2}}, \frac{x_2}{\sqrt{x_2^2+x_3^2}}\right) & \text{if } (x_1,x_2,x_3)\in P \\ 0 & \text{if } (x_1,x_2,x_3)\notin P, \end{cases}$$

where $P=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid -0.8\le x_1\le 0, 0.3\le\sqrt{x_2^2+x_3^2}\le 0.5\}$ models a cylindrical pipe coil. For our setup, we further went with the disk-shaped initial superconductor

$$\Omega_{\text{sc}}(0) = \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid -0.3\le x_1\le -0.1, \sqrt{x_2^2+x_3^2}\le 0.25\}.$$

The superconductor is assumed to have mass $m = 5 \cdot 10^{-3}$. Further, we opt for the constant $c = 1000$ in (7.2) and $\epsilon = 1$, hence we do not consider the eddy current case. Figure 7.1 shows the pipe coil and the superconductor at the initial time as well as a two-dimensional slice of the magnetic field at a given time step in which we observe full expulsion of the magnetic field from the superconductor. In Figure 7.2 we can observe the time evolution of the magnetic field and the change in position of the superconductor. Starting from the initial position at time zero, the superconductor experiences an upward force, resulting in an acceleration in the upward direction, leading to an increase in height as the magnetic field gets stronger. Figure 7.3 shows that the speed of the superconductor increases steadily over time. With the superconductor leaving the strongest area of the magnetic field, as observable in Figure 7.3, its speed tends to be constant which means that there is no further acceleration. Note that the given model does not account for any gravitational effects, which is why we cannot expect that the position of the superconductor becomes stationary.

The computational results appear to be promising and justify a future investigation of the proposed model.

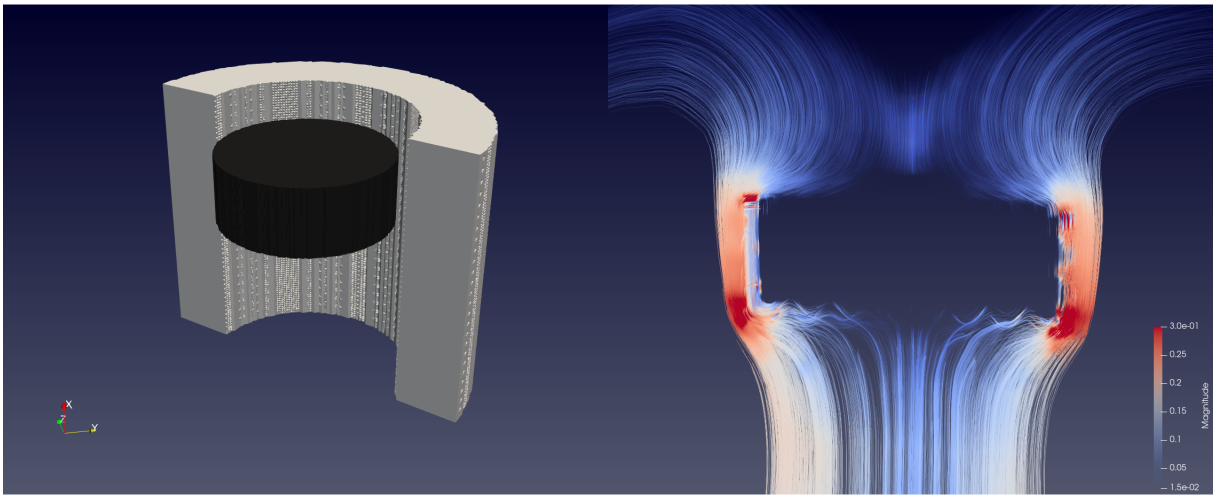

Figure 7.1: The pipe coil P in grey and the initial superconductor in black (left). A 2D slice of the magnetic field at a given time step (right).

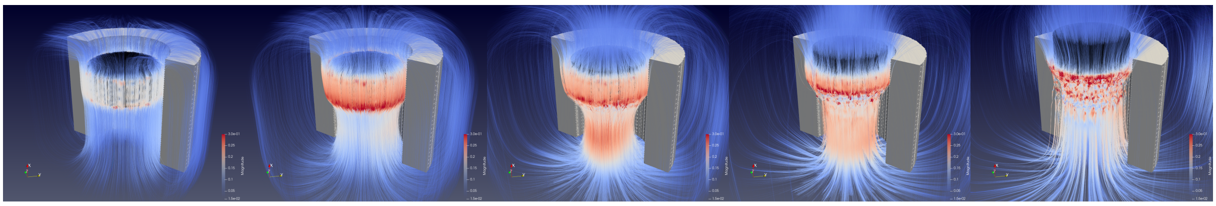

Figure 7.2: Evolution of the magnetic field and displacement of the superconductor.

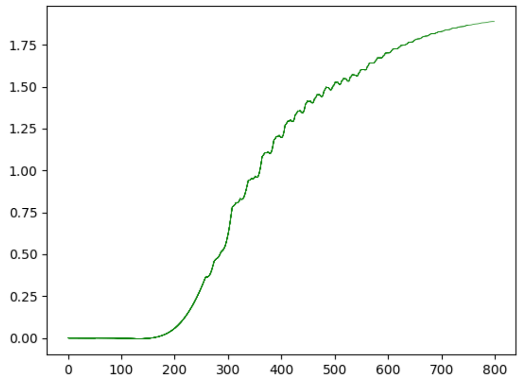

Figure 7.3: Graph of the superconductors speed over 800 time steps.

BIBLIOGRAPHY

[1] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces*. ISSN. Elsevier Science, 2003.

[2] M. Ainsworth and J. T. Oden. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000.

[3] A. Alonso Rodríguez. A mathematical justification of the low-frequency heterogeneous time-harmonic Maxwell equations. *Math. Models Methods Appl. Sci.*, 9(3):475–489, 1999.

[4] A. Alonso Rodríguez, R. Hiptmair, and A. Valli. A hybrid formulation of eddy current problems. *Numer. Methods Partial Differential Equations*, 21(4):742–763, 2005.

[5] A. Alonso Rodríguez and A. Valli. Some remarks on the characterization of the space of tangential traces of $H(\mathrm{rot};\Omega)$ and the construction of an extension operator. *Manuscripta Math.*, 89(2):159–178, 1996.

[6] A. Alonso Rodríguez and A. Valli. Unique solvability for high-frequency heterogeneous time-harmonic Maxwell equations via the Fredholm alternative theory. *Math. Methods Appl. Sci.*, 21(6):463–477, 1998.

[7] A. Alonso Rodríguez and A. Valli. *Eddy current approximation of Maxwell equations*, volume 4 of *MS&A. Modeling, Simulation and Applications*. Springer-Verlag Italia, Milan, 2010.

[8] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Directional differentiability for elliptic quasi-variational inequalities of obstacle type. *Calc. Var. Partial Differential Equations*, 58(1):Paper No. 39, 47, 2019.

[9] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Existence, iteration procedures and directional differentiability for parabolic QVIs. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 95, 53, 2020.

[10] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. On the differentiability of the minimal and maximal solution maps of elliptic quasi-variational inequalities. *Journal of Mathematical Analysis and Applications*, 507(1):125732, 2022.

[11] A. Alphonse, M. Hintermüller, and C. N. Rautenberg. Optimal control and directional differentiability for elliptic quasi-variational inequalities. *Set-Valued Var. Anal.*, 30(3):873–922, 2022.

[12] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Science Publications. Clarendon Press, 2000.

[13] H. Ammari, A. Buffa, and J.-C. Nédélec. A justification of eddy currents model for the Maxwell equations. *SIAM J. Appl. Math.*, 60(5):1805–1823, 2000.

[14] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998.

[15] S. S. Antman. The influence of elasticity on analysis: modern developments. *Bull. Amer. Math. Soc. (N.S.)*, 9(3):267–291, 1983.

[16] L. Arnold and B. Harrach. A unified variational formulation for the parabolic-elliptic eddy current equations. *SIAM J. Appl. Math.*, 72(2):558–576, 2012.

[17] F. Bachinger, U. Langer, and J. Schöberl. Numerical analysis of nonlinear multiharmonic eddy current problems. *Numer. Math.*, 100(4):593–616, 2005.

[18] V. Barbu. *Optimal control of variational inequalities*, volume 100 of *Research Notes in Mathematics.* Pitman, Boston, MA, 1984.

[19] J. W. Barrett and L. Prigozhin. A quasi-variational inequality problem in superconductivity. *Math. Models Methods Appl. Sci.*, 20(5):679–706, 2010.

[20] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, 48(4):2912–2943, 2016.

[21] H. H. Bauschke and P. L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011. With a foreword by Hédy Attouch.

[22] C. P. Bean. Magnetization of hard superconductors. *Phys. Rev. Lett.*, 8:250–253, 1962.

[23] C. P. Bean. Magnetization of high-field superconductors. *Rev. Mod. Phys.*, 36:31–39, 1964.

[24] M. Bergounioux. Optimal control of an obstacle problem. *Appl. Math. Optim.*, 36(2):147–172, 1997.

[25] P. Bernstein and J. Noudem. Superconducting magnetic levitation: principle, materials, physics and models. *Superconductor Science Technology*, 33(3):033001, March 2020.

[26] A. Beurling and J. Deny. Espaces de Dirichlet. I. Le cas élémentaire. *Acta Math.*, 99:203–224, 1958.

[27] A. Beurling and J. Deny. Dirichlet spaces. *Proc. Nat. Acad. Sci. U.S.A.*, 45:208–215, 1959.

[28] F. Bonnans and E. Casas. An extension of Pontryagin's principle for state-constrained optimal control of semilinear elliptic equations and variational inequalities. *SIAM J. Control Optim.*, 33(1):274–298, 1995.

[29] J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems.* Springer Series in Operations Research and Financial Engineering. Springer New York, 2000.

[30] A. Bossavit. *Électromagnétisme, en vue de la modélisation*, volume 14 of *Mathématiques & Applications (Berlin) [Mathematics & Applications].* Springer-Verlag, Paris, 1993.

[31] A. Bossavit. Numerical modelling of superconductors in three dimensions: A model and a finite element method. *Magnetics, IEEE Transactions on*, 30:3363 – 3366, 1994.

[32] A. Bossavit. *Computational electromagnetism.* Electromagnetism. Academic Press, Inc., San Diego, CA, 1998. Variational formulations, complementarity, edge elements.

[33] A. Bossavit. Discretization of electromagnetic problems: The Generalized Finite Differences approach. In *Numerical Methods in Electromagnetics*, volume 13 of *Handbook of Numerical Analysis*, pages 105–197. Elsevier, 2005.

[34] A. Bossavit and L. Kettunen. Yee-like schemes on a tetrahedral mesh, with diagonal lumping. *Int. J. Numer. Model. El.*, 12(1-2):129–142, 1999.

[35] E. H. Brandt. Rigid levitation and suspension of high-temperature superconductors by magnets. *American Journal of Physics*, 58(1):43–49, 1990.

[36] A. Buffa, M. Costabel, and D. Sheen. On traces for $\boldsymbol{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, 2002.

[37] G. Caselli. *Some optimization problems in electromagnetism.* Università degli studi di Trento, PhD thesis, 2022.

[38] G. Caselli, M. Hensel, and I. Yousept. Quasilinear Variational Inequalities in Ferromagnetic Shielding: Well-posedness, Regularity, and Optimal Control. To be published, *SIAM J. Control Optim.*, 2023.

[39] S. J. Chapman, D. P. Hewett, and L. N. Trefethen. Mathematics of the Faraday cage. *SIAM Rev.*, 57(3):398–417, 2015.

[40] S. H. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus. *Math. Comp.*, 77(262):813–829, 2008.

[41] C. Christof. *Sensitivity analysis of elliptic variational inequalities of the first and the second kind.* Technische Universität Dortmund, doctoral dissertation, 2018.

[42] C. Christof and G. Wachsmuth. On the non-polyhedricity of sets with upper and lower bounds in dual spaces. *GAMM-Mitt.*, 40(4):339–350, 2018.

[43] M. Ciszek, B. A. Glowacki, S. P. Ashworth, A. M. Campbell, W. Y. Liang, R. Flükiger, and R. E. Gladyshevskii. AC losses and critical currents in Ag/(Tl,Pb,Bi)-1223 tape. *Physica C Superconductivity*, 260(1):93–102, February 1996.

[44] D. S. Clark. Short proof of a discrete Gronwall inequality. *Discrete Appl. Math.*, 16(3):279–281, 1987.

[45] G. Cohen and P. Monk. Gauss point mass lumping schemes for Maxwell's equations. *Numer. Methods Partial Differential Equations*, 14(1):63–88, 1998.

[46] M. Costabel, M. Dauge, and S. Nicaise. Singularities of eddy current problems. *M2AN Math. Model. Numer. Anal.*, 37(5):807–831, 2003.

[47] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[48] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics.* Springer-Verlag, Berlin-New York, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[49] C. M. Elliott and Y. Kashima. A finite-element analysis of critical-state models for type-II superconductivity in 3D. *IMA J. Numer. Anal.*, 27(2):293–331, 2007.

[50] A. Ern and J.-L. Guermond. Mollification in strongly Lipschitz domains with application to continuous and discrete de Rham complexes. *Comput. Methods Appl. Math.*, 16(1):51–75, 2016.

[51] A. Ern and J.-L. Guermond. Abstract nonconforming error estimates and application to boundary penalty methods for diffusion equations and time-harmonic Maxwell's equations. *Comput. Methods Appl. Math.*, 18(3):451–475, 2018.

[52] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[53] M. Faraday. Experimental researches in electricity. *Philosophical Transactions of the Royal Society of London*, 122:125–162, 1832.

[54] G. Fichera. Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., VIII. Ser., Sez. I*, 7:91–140, 1964.

[55] B. Franklin. *Experiments and observations on electricity.* Printed and sold by E. Cave, at St. John's Gate, 1751.

[56] A. Friedman. Optimal control for variational inequalities. *SIAM J. Control Optim.*, 24(3):439–451, 1986.

[57] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, extended edition, 2011.

[58] A. Gansen, M. El Hachemi, S. Belouettar, O. Hassan, and K. Morgan. An effective 3d leapfrog scheme for electromagnetic modelling of arbitrary shaped dielectric objects using unstructured meshes. *Comput. Mech.*, 56:1023–1037, 2015.

[59] A. Gansen, M. El Hachemi, S. Belouettar, O. Hassan, and K. Morgan. A 3d unstructured mesh FDTD scheme for EM modelling. *Arch. Computat. Methods Eng.*, 2020.

[60] V. Girault and P.-A. Raviart. *Finite element methods for Navier-stokes equations.* Springer Series in Computational Mathematics. Springer, Berlin, Germany, 1986 edition, May 1986.

[61] R. Glowinski. *Numerical methods for nonlinear variational problems.* Scientific Computation. Springer-Verlag, Berlin, 2008. Reprint of the 1984 original.

[62] D. J. Griffiths. *Introduction to electrodynamics; 3rd ed.* Prentice-Hall, Upper Saddle River, NJ, 1999.

[63] A. Haraux. How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. *Journal of the Mathematical Society of Japan*, 29(4):615 – 631, 1977.

[64] M. Hensel and I. Yousept. Eddy Current Approximation in Maxwell Obstacle Problems. *Interfaces and Free Boundaries,* 10.4171/ifb/486, 2022.

[65] M. Hensel and I. Yousept. Numerical Analysis for Maxwell Obstacle Problems in Electric Shielding. *SIAM J. Numer. Anal.*, 60(3):1083–1110, 2022.

[66] M. Hensel, I. Yousept, and M. Winckler. Numerical Analysis for Maxwell Quasi-variational Inequalities in Superconductivity. Preprint, 2023.

[67] F. Hermeline, S. Layouni, and P. Omnes. A finite volume method for the approximation of Maxwell's equations in two space dimensions on arbitrary meshes. *J. Comput. Phys.*, 227(22):9365–9388, 2008.

[68] R. Herzog and C. Meyer. Optimal control of static plasticity with linear kinematic hardening. *ZAMM Z. Angew. Math. Mech.*, 91(10):777–794, 2011.

[69] M. Hintermüller. Inverse coefficient problems for variational inequalities: optimality conditions and numerical realization. *M2AN Math. Model. Numer. Anal.*, 35(1):129–152, 2001.

[70] M. Hintermüller. An active-set equality constrained Newton solver with feasibility restoration for inverse coefficient problems in elliptic variational inequalities. *Inverse Problems*, 24(3):034017, 23, 2008.

[71] M. Hintermüller, A. Laurain, and I. Yousept. Shape sensitivities for an inverse problem in magnetic induction tomography based on the eddy current model. *Inverse Problems*, 31(6):065006, 25, 2015.

[72] S. Hofmann, M. Mitrea, and M. Taylor. Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. *J. Geom. Anal.*, 17(4):593–647, 2007.

[73] R. H. W. Hoppe and J. Schöberl. Convergence of adaptive edge element methods for the 3D eddy currents equations. *J. Comput. Math.*, 27(5):657–676, 2009.

[74] K. Ito and K. Kunisch. Optimal control of elliptic variational inequalities. *Appl. Math. Optim.*, 41(3):343–364, 2000.

[75] K. Ito and K. Kunisch. Semi-smooth Newton methods for variational inequalities of the first kind. *M2AN Math. Model. Numer. Anal.*, 37(1):41–62, 2003.

[76] K. Ito and K. Kunisch. The primal-dual active set method for nonlinear optimal control problems with bilateral constraints. *SIAM J. Control Optim.*, 43(1):357–376, 2004.

[77] J. D. Jackson. *Classical electrodynamics.* Wiley, New York, 3rd edition, 1999.

[78] F. Jochmann. On a first-order hyperbolic system including Bean's model for superconductors with displacement current. *J. Differential Equations*, 246(6):2151–2191, 2009.

[79] B. Kaltenbacher, M. Kaltenbacher, and S. Reitzinger. Identification of nonlinear B-H curves based on magnetic field computations and multigrid methods for ill-posed problems. *European J. Appl. Math.*, 14(1):15–38, 2003.

[80] F. Kikuchi. On a discrete compactness property for the Nédélec finite elements. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(3):479–490, 1989.

[81] Y. B. Kim, C. F. Hempstead, and A. R. Strnad. Magnetization and critical supercurrents. *Phys. Rev.*, 129:528–535, Jan 1963.

[82] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and their Applications.* Academic Press, New York, 1980.

[83] H. Kono, M. Fujino, M. Yokoyama, K. Yonezawa, Y. Takahashi, C. Isokawa, and A. Tatematsu. Applying FDTD simulation to lightning surge route analysis in microwave relay stations. *J. Int. Counc. Electr. Eng.*, 1(1):60–66, 2011.

[84] M. Kunze and J. F. Rodrigues. An elliptic quasi-variational inequality with gradient constraints and some of its applications. *Math. Methods Appl. Sci.*, 23(10):897–908, 2000.

[85] S. Lang. *Real and Functional Analysis.* Springer-Verlag New York, 3 edition, 1993.

[86] A. Laurain, M. Winckler, and I. Yousept. Shape optimization for superconductors governed by $H(\mathrm{curl})$-elliptic variational inequalities. *SIAM J. Control Optim.*, 59(3):2247–2272, 2021.

[87] G. Leoni. *A first course in Sobolev spaces*, volume 105 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2009.

[88] J. Li. Unified analysis of leap-frog methods for solving time-domain Maxwell's equations in dispersive media. *J. Sci. Comput.*, 47(1):1–26, 2011.

[89] J. Li and S. Shields. Superconvergence analysis of Yee scheme for metamaterial Maxwell's equations on non-uniform rectangular meshes. *Numer. Math.*, 134(4):741–781, 2016.

[90] J. Li, J. Wang Waters, and E. A. Machorro. An implicit leap-frog discontinuous Galerkin method for the time-domain Maxwell's equations in metamaterials. *Comput. Methods Appl. Mech. Engrg.*, 223/224:43–54, 2012.

[91] J.-L. Lions and G. Stampacchia. Variational inequalities. *Comm. Pure Appl. Math.*, 20:493–519, 1967.

[92] A. Logg, K.-A. Mardal, and G. N. Wells, editors. *Automated solution of differential equations by the finite element method*, volume 84 of *Lecture Notes in Computational Science and Engineering.* Springer, Heidelberg, 2012. The FEniCS book.

[93] P. Lorrain and D.-R. Corson. *Electromagnetic Fields and Waves.* W. H. Freeman, 1970.

[94] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[95] P. A. Martin. On acoustic and electric Faraday cages. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 470(2171):20140344, 14, 2014.

[96] J. C. Maxwell. On physical lines of force. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 21(139):161–175, March 1861.

[97] J. C. Maxwell. *A Treatise on Electricity and Magnetism: 2.* Clarendon Press series. Clarendon Press, 1873.

[98] C. Meyer, A. Rademacher, and W. Wollner. Adaptive optimal control of the obstacle problem. *SIAM J. Sci. Comput.*, 37(2):A918–A945, 2015.

[99] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *J. Functional Analysis*, 22(2):130–185, 1976.

[100] F. Mignot and J.-P. Puel. Optimal control in some variational inequalities. *SIAM J. Control Optim.*, 22(3):466–476, 1984.

[101] A. Milani. On a variational inequality with time dependent convex constraints for the Maxwell equations. *Rend. Sem. Mat. Univ. Politec. Torino*, 36:389–401 (1979), 1977/78.

[102] A. Milani. On a variational inequality with time dependent convex constraint for the Maxwell equations. II. *Rend. Sem. Mat. Univ. Politec. Torino*, 43(1):171–183, 1985.

[103] A. Milani and R. Picard. Weak solution theory for Maxwell's equations in the semistatic limit case. *J. Math. Anal. Appl.*, 191(1):77–100, 1995.

[104] F. Miranda, J. F. Rodrigues, and L. Santos. On a p-curl system arising in electromagnetism. *Discrete Contin. Dyn. Syst. Ser. S*, 5(3):605–629, 2012.

[105] F. Miranda, J. F. Rodrigues, and L. Santos. Evolutionary quasi-variational and variational inequalities with constraints on the derivatives. *Adv. Nonlinear Anal.*, 9(1):250–277, 2020.

[106] P. Monk. A mixed method for approximating Maxwell's equations. *SIAM J. Numer. Anal.*, 28(6):1610–1634, 1991.

[107] P. Monk. *Finite element methods for Maxwell's equations.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[108] J.-C. Nédélec. Mixed finite elements in R^3. *Numer. Math.*, 35(3):315–341, 1980.

[109] Y. Nesterov and L. Scrimali. Solving strongly monotone variational and quasi-variational inequalities. *Discrete Contin. Dyn. Syst.*, 31(1078-0947-2011-4-1383):1383, 2011.

[110] S. Nicaise and F. Tröltzsch. A coupled Maxwell integrodifferential model for magnetization processes. *Math. Nachr.*, 287(4):432–452, 2014.

[111] S. Nicaise and F. Tröltzsch. Optimal control of some quasilinear Maxwell equations of parabolic type. *Discrete Contin. Dyn. Syst. Ser. S*, 10(6):1375–1391, 2017.

[112] M. A. Noor. An iterative scheme for a class of quasi variational inequalities. *J. Math. Anal. Appl.*, 110(2):463–468, 1985.

[113] D. Pauly. Solution theory, variational formulations, and functional a posteriori error estimates for general first order systems with applications to electro-magneto-statics and more. *Numer. Funct. Anal. Optim.*, 41(1):16–112, 2020.

[114] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Math. Z.*, 187(2):151–164, 1984.

[115] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1994.

[116] L. Queval, K. Liu, W. Yang, V. M. R. Zermeno, and G. Ma. Superconducting magnetic bearings simulation using an H-formulation finite element model. *Superconductor Science and Technology*, 31(8):084001, August 2018.

[117] J. Räbinä, L. Kettunen, S. Mönkölä, and T. Rossi. Generalized wave propagation problems and discrete exterior calculus. *ESAIM: M2AN*, 52(3):1195–1218, 2018.

[118] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.

[119] J. F. Rodrigues. *Obstacle problems in mathematical physics*, volume 134 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987.

[120] J. F. Rodrigues and L. Santos. A parabolic quasi-variational inequality arising in a superconductivity model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(1):153–169, 2000.

[121] J. F. Rodrigues and L. Santos. Quasivariational solutions for first order quasilinear equations with gradient constraint. *Arch. Ration. Mech. Anal.*, 205(2):493–514, 2012.

[122] T. Roubíček. *Nonlinear partial differential equations with applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser Verlag, Basel, 2005.

[123] F. Sass, G. G. Sotelo, R. de Andrade Junior, and F. Sirois. H-formulation for simulating levitation forces acting on HTS bulks and stacks of 2G coated conductors. *Superconductor Science Technology*, 28(12):125012, December 2015.

[124] V. Shukla. Review of electromagnetic interference shielding materials fabricated by iron ingredients. *Nanoscale Adv.*, 1:1640–1671, 2019.

[125] A. Signorini. Sopra alcune questioni di statica dei sistemi continui. *Ann. Scuola Norm. Super. Pisa Cl. Sci. (2)*, 2(2):231–251, 1933.

[126] A. Signorini. Questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. Appl., V. Ser.*, 18:95–139, 1959.

[127] F. Tröltzsch. *Optimal control of partial differential equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels.

[128] F. Tröltzsch and A. Valli. Optimal control of low-frequency electromagnetic fields in multiply connected conductors. *Optimization*, 65(9):1651–1673, 2016.

[129] F. Tröltzsch and A. Valli. Optimal voltage control of non-stationary eddy current problems. *Math. Control Relat. Fields*, 8(1):35–56, 2018.

[130] S. O. Valenzuela, G. A. Jorge, and E. Rodríguez. Measuring the interaction force between a high temperature superconductor and a permanent magnet. *American Journal of Physics*, 67(11):1001–1006, 1999.

[131] G. Wachsmuth. Pointwise constraints in vector-valued Sobolev spaces: with applications in optimal control. *Appl. Math. Optim.*, 77(3):463–497, 2018.

[132] G. Wachsmuth. A guided tour of polyhedric sets: basic properties, new results on intersections, and applications. *J. Convex Anal.*, 26(1):153–188, 2019.

[133] J. H. P. Watson. Magnetization of synthetic filamentary superconductors. b. the dependence of the critical current density on temperature and magnetic field. *Journal of Applied Physics*, 39(7):3406–3413, 1968.

[134] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.*, 2(1):12–25, 1980.

[135] N. Weck. Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries. *J. Math. Anal. Appl.*, 46:410–437, 1974.

[136] M. Winckler and I. Yousept. Fully discrete scheme for Bean's critical-state model with temperature effects in superconductivity. *SIAM J. Numer. Anal.*, 57(6):2685–2706, 2019.

[137] M. Winckler, I. Yousept, and J. Zou. Adaptive edge element approximation for $\boldsymbol{H}(\mathbf{curl})$ elliptic variational inequalities of second kind. *SIAM J. Numer. Anal.*, 58(3):1941–1964, 2020.

[138] K. Yee. Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Transactions on Antennas and Propagation*, 14(3):302–307, 1966.

[139] I. Yousept. Finite element analysis of an optimal control problem in the coefficients of time-harmonic eddy current equations. *J. Optim. Theory Appl.*, 154(3):879–903, 2012.

[140] I. Yousept. Optimal control of quasilinear H(curl)-elliptic partial differential equations in magnetostatic field problems. *SIAM J. Control Optim.*, 51(5):3624–3651, 2013.

[141] I. Yousept. Hyperbolic Maxwell variational inequalities for Bean's critical-state model in type-II superconductivity. *SIAM J. Numer. Anal.*, 55(5):2444–2464, 2017.

[142] I. Yousept. Optimal control of non-smooth hyperbolic evolution Maxwell equations in type-II superconductivity. *SIAM J. Control Optim.*, 55(4):2305–2332, 2017.

[143] I. Yousept. Hyperbolic Maxwell variational inequalities of the second kind. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 34, 2020.

[144] I. Yousept. Well-posedness theory for electromagnetic obstacle problems. *J. Differential Equations*, 269(10):8855–8881, 2020.

[145] I. Yousept. Maxwell quasi-variational inequalities in superconductivity. *ESAIM Math. Model. Numer. Anal.*, 55(4):1545–1568, 2021.

[146] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.