Faculty of Mathematics
University of Duisburg-Essen
Thea-Leymann-Straße 9, 45127 Essen, Germany

Gamma convergence and Cosserat curvy shell models

M.Sc. Maryam Mohammadi Saem
Karaj, Persia

Dissertation

to attain the academic degree
Doctor of Natural Sciences (Dr. rer. nat.)
at the Faculty of Mathematics,
University of Duisburg-Essen

DOI: 10.17185/duepublico/78204
URN: urn:nbn:de:hbz:465-20230419-083612-9

Day of defense: 02.12.2022

First reviewer: Prof. Dr. rer. nat. habil. Patrizio Neff
Faculty of Mathematics
University of Duisburg-Essen

Second reviewer: Prof. Dr. rer. nat. Ionel-Dumitrel Ghiba
Faculty of Mathematics
Alexandru Ioan Cuza University of Iasi

Acknowledgment

After four years study and research in the group of Nonlinear Analysis and Modeling, now it is the time to thank people who supported me in all this long journey until I reached the goal. After the help of God, it was my supervisor who helped me to come back to Germany and start another scientifically exciting life. I would like to appreciate by heart, my supervisor Prof. Dr. Patrizio Neff who was not only my supervisor in Mathematics but a patient supporter in my private life as well. He provided a nice and friendly environment for our group which helped me to get familiar with the subject of Gamma convergence and at the end we could have interesting results in this subject. Next, I would like to thank Prof. Ionel-Dumitrel Ghiba for his kind patronage and the pleasant mathematical discussions with him.

I had a chance to work in a very skillful group with talented colleges. Thanks go to Dr. Peter Lewintan, Dr. Jendrik Voss, Dr. Hassam Ahmed Khan and Dr. Robert J. Martin who always responded to my requests. I appreciate the continuous support of a very kind lady, Frau Anja Schulte who is always patient for help.

I am extremely thankful to my beloved parents whom I could not visit since several years. They always gave me good mood to continue my study with motivation. I wish for both health and happy life. I should not forget my kind sisters and their family. They also tolerated this distance between us with patience and were encouraging their youngest sister. I kiss all my nephews and nieces who missed me but tried not to show it to cheer me up.

Contents

Introduction

The Encyclopedia Britannica describes *elasticity* as the "ability of a deformed material body to return to its original shape and size when the forces causing the deformation are removed". The theory of (non-linear) elasticity provides a mathematical framework to model the response of different elastic materials under given loads. The elastic body is generally modeled as a continuum, thus the theory is concerned with the macroscopic effects rather than the underlying microscopic mechanism of atomic bonding that causes the elastic behavior.

To this day, there exists no mathematical model which correctly describes the entirety of elastic behavior. In fact, precise description of physical materials as ideally elastic is not even possible because of various interfering effects like thermodynamics, plasticity, and microscopic structures. Thus in a real-world scenario, there is no material response which is a purely elastic continuum behavior. Therefore, for the concept of elasticity, it is crucial to find a reasonable simplification that is capable of expressing an approximate description of the actual physical behavior. For an ideal theoretical elastic material, all other mentioned effects are left out and the act of deforming a body is reduced to its resulting configuration to omit time-dependency.

Shell theory is the basic of the first Part in this dissertation. Shells are formed from two curvy layers with a common inner surface in between with a very small thickness. We note that the thickness in one direction is much more smaller than the two other dimensions which are orthogonal to the direction of the thickness. The theory of shells is similar to the theory of plates with more specification like curvature. The thickness of the shell can play a role to effect the boundary and external loads.

In the first part of this dissertation we consider a three dimensional Cosserat model and derive a curvy shell model with small thickness. By applying the nonlinear scaling for both deformation field φ and the microrotation R, we obtain the homogenized membrane energy W_{mp}. The homogenized curvature energy W_{curv} is obtained separately in the second chapter of the first Part. A combination of these homogenized energies and applying the concept of Γ-convergence will lead us to find a minimizer for the sequence of energies as the thickness leads to zero.

In the second part, we discuss some properties of the scenes of minimal surfaces. Minimal surfaces are defined as surfaces with zero mean curvature, which a parametrized minimal surface satisfies in Lagrane's equation. For many years the only known complete, embedded minimal surfaces of finite topology were the catenoid and helicoid which we disscused about them in section 7.5. In this chapter we also assume that an in-plane drill rotation is the deformation mapping from a smooth regular shell surface to another which is parameterized on the same domain and we show that this is not possible unless all rotations at a portion of the boundary are fixed.

Publications

This dissertaion is based on the following papers which respectively are published, submitted and in preparation:

- M. Mohammadi Saem, P. Lewintan and P. Neff. *On in-plane drill rotations for Cosserat surfaces.* The Royal Society Publishing (2021) [78].
- M. Mohammadi Saem, I.D. Ghiba and P. Neff. *A geometrically nonlinear Cosserat (micropolar) curvy shell model via Gamma convergence.* arxive (2022) [102].
- I.D. Ghiba, M. Mohammadi Saem and P. Neff. *On the choice of third order Cosserat curvature tensors in the shell model.* in preparation (2022) [55].

Glossary

$\mathbb{R}_+ = (0, \infty)$	the set of positive real numbers
$\overline{\mathbb{R}} = [-\infty, \infty]$	the set of all real numbers including $-\infty, +\infty$
$\mathbb{R}^{n\times n}$	the set of all real $n \times n$ matrices
$\mathrm{GL}(n) = \{X \in \mathbb{R}^{n\times n} \mid \det X \neq 0\}$	class of invertible matrices
$\mathrm{O}(3) = \{X \in \mathbb{R}^{3\times 3} \mid X^T X = \mathbb{1}_3\}$	class of orthogonal matrices
$\mathrm{SO}(3) = \{X \in \mathbb{R}^{3\times 3} \mid X^T X = \mathbb{1}_3\, , \det(X) = 1\}$	class of special orthogonal matrices
$\mathfrak{so}(3) = \{X \in \mathbb{R}^{3\times 3} \mid X^T = -X\}$	class of skew-symmetric matrices
$\Omega \subset \mathbb{R}^n$	reference configuration
$\Omega_h \subset \mathbb{R}^n$	fictitious flat Cartesian configuration
$\varphi\colon \Omega \to \mathbb{R}^n$	deformation mapping
$F = \nabla\varphi = \big(\varphi_{i,x_j}\big) \in \mathrm{GL}^+(n)$	deformation gradient
$\operatorname{tr} X = \langle X, \mathbb{1}\rangle$	trace of X
$\langle X, Y\rangle = \operatorname{tr}(XY^T)$	the standard Euclidean scalar product
$\lVert X\rVert = \langle X, X\rangle^{\frac{1}{2}}$	the associated norm to the Euclidean scalar product
$\operatorname{dev} X = X - \frac{1}{n}\operatorname{tr}(X)\mathbb{1}$	deviatoric (trace-free) part of $X \in \mathbb{R}^{n\times n}$
$\operatorname{sym} X = \frac{1}{2}\big(X + X^T\big)$	symmetric part of X
$\operatorname{skew} X = \frac{1}{2}\big(X - X^T\big)$	skew-symmetric part of X
$\operatorname{Cof} F = \det F \cdot F^{-T}$	cofactor of $F \in \mathrm{GL}^+(n)$
$\operatorname{axl} A$	canonical identification of $A \in \mathfrak{so}(3)$ and $\operatorname{axl} A \in \mathbb{R}^3$
$\operatorname{Anti}(v)$	the inverse of axl on the vector v
μ, λ	elastic Lamé parameters, μ shear modulus
μ_c	Cosserat couple modulus
L_c	internal length
κ	bulk modulus, $\kappa = \frac{3\lambda+2\mu}{3}$
$h > 0$	the thickness of a thin shell
$\mathcal{H}$	harmonic mean
R, Q	rotation matrices
H	mean curvature
K	Gauss curvature
$\mathrm{D}X$	the Jacobian matrix of matrix X

Preliminaries

This section contains some of the required definitions and concepts which are used in this dissertation.

0.1. Vectors

Definition 0.1.1. Let $a, b \in \mathbb{R}^n$ be two vectors. We denote the *scalar product* on $\mathbb{R}^n$ with

$$\langle a, b\rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} a_i b_i \,, \quad \text{and} \quad \|a\|^2 = \langle a, a\rangle \,. \tag{0.1.1}$$

Assume that $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. The *cross product* of the two vectors a, b is expressed by the following determinant

$$a \times b = \det \begin{pmatrix} + & - & + \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \,, \tag{0.1.2}$$

which is a perpendicular vector to the plane which contains the vectors a, b. If the vectors a, b are parallel, then $a \times b = 0$. Moreover, the cross product is anti-commutative, that is $a \times b = -(b \times a)$. The following properties hold for vectors $a, b, c, d \in \mathbb{R}^3$:

- $\langle a, (b \times c)\rangle = \langle b, (c \times a)\rangle = \langle c, (a \times b)\rangle \,,$
- $\langle (a \times b), (c \times d)\rangle = \langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle \,,$
- $a \times (b \times c) = b\langle a, c\rangle - c\langle a, b\rangle \,,$ (0.1.3)
- $(a \times b) \times (a \times c) = \langle a, (b \times c)\rangle a \,.$

0.1.1. Some inequalities in vector spaces

Remark 1. *(Young's inequality) For every $a, b \geq 0$, it holds*

$$a\, b \leq \frac{a^p}{p} + \frac{b^q}{q} \,, \tag{0.1.4}$$

for $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Another version can be seen like

$$a\, b \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2 \,, \qquad \varepsilon > 0 \,. \tag{0.1.5}$$

The generalized Young inequality is

$$\prod_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{1}{p_i} (a_i)^{p_i} \,,$$

with $p_i > 1$, $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, for $i = 1, \cdots, n$.

Remark 2. *(Cauchy-Schwarz inequality) For vectors $x, y \in \mathbb{R}^n$, it holds*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \,. \tag{0.1.6}$$

0.2. Matrices and related spaces

Definition 0.2.1. Assume that $X \in \mathbb{R}^{3\times 3}$ is a quadratic matrix. We denote the *transpose* matrix of X by X^T and it is defined as follows

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \qquad X^T = \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}, \tag{0.2.1}$$

and $X^T \in \mathbb{R}^{n\times n}$. It can be seen that, $(X^T)^T = X$. Regarding to the transpose matrix of any matrix X, we have the two following definitions

$$\operatorname{sym} X := \frac{1}{2}(X + X^T), \qquad \operatorname{skew} X := \frac{1}{2}(X - X^T), \tag{0.2.2}$$

where $\operatorname{sym} X$ and $\operatorname{skew} X$ denote the *symmetric* and the *skew symmetric* part of X, respectively. Consequently, one has the following orthogonal decomposition for any matrix $X \in \mathbb{R}^{n\times n}$

$$X = \operatorname{sym} X + \operatorname{skew} X. \tag{0.2.3}$$

The *deviatoric part* of X is defined by

$$\operatorname{dev} X := X - \frac{1}{n}\operatorname{tr}(X)\,\mathbb{1}_n, \tag{0.2.4}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix in $\mathbb{R}^{n\times n}$. By definition it holds $\operatorname{tr}(\operatorname{dev} X) = 0$ and therefore the deviatoric part is known as *trace free* part of X. We have the following orthogonal decomposition for any matrix $X \in \mathbb{R}^{n\times n}$ as well

$$X = \operatorname{dev}\operatorname{sym} X + \operatorname{skew} X + \frac{1}{n}\operatorname{tr}(X)\,\mathbb{1}_n. \tag{0.2.5}$$

In the following we introduce some sets of matrices which will be used in this dissertation. The set of all *invertible* matrices is denoted by $\mathrm{GL}(n)$ and is defined

$$\mathrm{GL}(n) := \{X \in \mathbb{R}^{n\times n} \mid \det X \neq 0\}. \tag{0.2.6}$$

$\mathrm{GL}(n)^+$ denotes the set of all *invertible matrices with positive determinant.* The following sets are all subsets of $\mathrm{GL}(n)$ and defined

$$\mathrm{O}(n) := \{X \in \mathbb{R}^{n\times n} \mid X^T X = \mathbb{1}_n\},$$

the *orthogonal group*,

$$\mathrm{SO}(n) := \{X \in \mathbb{R}^{n\times n} \mid X^T X = \mathbb{1}_n, \det(X) = 1\}, \tag{0.2.7}$$

the *special orthogonal group* (where in linear transformation each element in this group acts as a rotation) and

$$\mathfrak{so}(n) := \{X \in \mathbb{R}^{n\times n} \mid X^T = -X\}, \tag{0.2.8}$$

the set of *skew symmetric* matrices. A quadratic $n \times n$ matrix X is *symmetric positive definite*, if it is symmetric and for all nonzero vectors $\xi \in \mathbb{R}^n$, we have

$$\langle X\xi, \xi\rangle > 0. \tag{0.2.9}$$

Definition 0.2.2. For two $n \times n$ matrices X, Y, the *standard Euclidean scalar product* is given by

$$\langle X, Y\rangle_{\mathbb{R}^{n\times n}} = \operatorname{tr}(XY^T), \tag{0.2.10}$$

and the associated (squared) norm is

$$\|X\|^2_{\mathbb{R}^{n\times n}} = \langle X, X\rangle_{\mathbb{R}^{n\times n}}. \tag{0.2.11}$$

Therefore, for any matrix $X \in \mathbb{R}^{n\times n}$, $\operatorname{tr}(X) = \langle X, \mathbb{1}_n\rangle_{\mathbb{R}^{n\times n}}$. By using the fact that $\operatorname{tr}(X) = \operatorname{tr}(X^T)$, we see

$$\langle X, Y\rangle = \operatorname{tr}(XY^T) = \langle XY^T, \mathbb{1}\rangle = \langle \mathbb{1}, X^T Y\rangle = \operatorname{tr}(X^T Y) = \langle X^T Y, \mathbb{1}\rangle = \langle Y, X\rangle. \tag{0.2.12}$$

Definition 0.2.3. The *canonical identification* of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl} X\colon \mathfrak{so}(3) \mapsto \mathbb{R}^3$. By using the standard vector product we define $(\operatorname{axl} X) \times \xi = X.\xi$ for all vectors $\xi \in \mathbb{R}^3$, such that

$$\operatorname{axl}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \operatorname{Anti}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \tag{0.2.13}$$

where the inverse of axl is denoted by $\operatorname{Anti}\colon \mathbb{R}^3 \to \mathfrak{so}(3)$.

0.2.1. Cofactor of a matrix

Definition 0.2.4. Let A be a $n \times n$ square matrix. The *minor* A_{ij} of the i-th row and j-th column is the determinant of the submatrix of A which is formed by deleting the i-th row and j-th column of the original matrix A.
The (i,j) *cofactor* is obtained by $\widetilde{A}_{ij} = (-1)^{i+j}\det(A_{ij})$, $i,j = 1,\dots,n$. By the cofactor matrix of A we mean the $n \times n$ matrix $\operatorname{Cof} A$, in which the (i,j) entry is $\widetilde{A}_{ij}$.
For example, in the case $n = 3$ we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Longrightarrow \operatorname{Cof} A = \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{32}a_{13} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}. \tag{0.2.14}$$

Lemma 0.2.5. *Assume that* $A \in \mathbb{R}^{n\times n}$, *then*

$$A.(\operatorname{Cof} A)^T = \det A \cdot \mathbb{1}_3\,. \tag{0.2.15}$$

Corollary 0.2.6. *Let* $A \in \mathrm{GL}(n)$. *Then*

$$A.(\operatorname{Cof} A)^T = \det A \cdot \mathbb{1} \quad\Longleftrightarrow\quad \operatorname{Cof} A \cdot A^T = \det A \cdot \mathbb{1} \quad\Longleftrightarrow\quad \operatorname{Cof} A = \det A \cdot A^{-T}\,. \tag{0.2.16}$$

Lemma 0.2.7. *(Nanson's formula) Let* $A \in \mathbb{R}^{3\times 3}$ *and* $a, b \in \mathbb{R}^3$. *Then,*

$$(Aa) \times (Ab) = (\operatorname{Cof} A)(a \times b)\,, \tag{0.2.17}$$

where $a \times b$ *is the cross product between two vectors* a, b.

We need to notice that for three column vectors $a, b, c \in \mathbb{R}^3$ we have

$$\begin{aligned} \det(a|b|c) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= c_1(a_2b_3 - b_2a_3) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1) \\ &= (a_2b_3 - b_2a_3)c_1 - (a_1b_3 - a_3b_1)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= \langle a \times b, c\rangle\,. \end{aligned}$$

Lemma 0.2.8. *Assume that* $X, Y \in \mathbb{R}^{n\times n}$ *and* $F \in \mathrm{GL}(n)$. *Then*

1. $(\operatorname{Cof} F)^{-1} = \operatorname{Cof}(F^{-1})$,
2. $(\operatorname{Cof} X)^T = \operatorname{Cof}(X^T)$,
3. $\operatorname{Cof}(XY) = \operatorname{Cof} X \operatorname{Cof} Y$.

Theorem 0.2.9. *(Piola-identity) Let* $\Omega \subset \mathbb{R}^3$ *be an open subset of* $\mathbb{R}^3$ *and let* $\varphi : \Omega \to \Omega' \subset \mathbb{R}^3$ *be a diffeomorphism. Then*

$$\operatorname{Div} \operatorname{Cof} \nabla\varphi = 0\,. \tag{0.2.18}$$

Proof. We have

$$\nabla\varphi = \begin{pmatrix} \varphi_{1,x_1} & \varphi_{1,x_2} & \varphi_{1,x_3} \\ \varphi_{2,x_1} & \varphi_{2,x_2} & \varphi_{2,x_3} \\ \varphi_{3,x_1} & \varphi_{3,x_2} & \varphi_{3,x_3} \end{pmatrix}. \tag{0.2.19}$$

Each element of $\operatorname{Cof}\nabla\varphi$ is written like

$$(\operatorname{Cof}\nabla\varphi)_{ij} = \partial_{j+1}\varphi_{i+1}\cdot\partial_{j+2}\varphi_{i+2} - \partial_{j+2}\varphi_{i+1}\cdot\partial_{i+1}\varphi_{i+2}\,, \tag{0.2.20}$$

where all indices are counted modulo 3. Now we have

$$\begin{aligned}(\operatorname{Div}\operatorname{Cof}\nabla\varphi)_{ij} &= \sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x_j}(\operatorname{Cof}\nabla\varphi)_{ij}\\ &= \sum_{j=1}^{3}(\underbrace{\partial_{j,j+1}\varphi_{i+1}\cdot\partial_{j+2}\varphi_{i+2}}_{=:a_j}+\underbrace{\partial_{j+1}\varphi_{i+1}\cdot\partial_{j,j+2}\varphi_{i+2}}_{:=b_j})\\ &\quad -\sum_{j=1}^{3}(\underbrace{\partial_{j,j+2}\varphi_{i+1}\cdot\partial_{j+1}\varphi_{i+2}}_{:=c_j}+\underbrace{\partial_{j+2}\varphi_{i+1}\cdot\partial_{j,j+1}\varphi_{i+2}}_{:=d_j}) = 0\,,\quad \forall i,j\in\{1,2,3\}\,,\end{aligned} \tag{0.2.21}$$

because $a_j - c_{j+1} = 0$ and $b_j - d_{j+2} = 0$. ■

Assume A is a $n\times n$ matrix with the eigenvalues $\lambda_1,\ldots,\lambda_n$. The *principal matrix invariants* are defined by

$$I_k(A) = \sum_{1\le j_1<\ldots<j_k\le n}\prod_{i=1}^{n}\lambda_{j_i}\,,\qquad \text{for}\quad 0\le k\le n\,. \tag{0.2.22}$$

We have

Lemma 0.2.10. *For $A\in\mathbb{R}^{3\times3}$,*

1. $I_1(A) = \operatorname{tr}(A)\,,$
2. $I_2(A) = \operatorname{tr}(\operatorname{Cof} A)\,,$
3. $I_3(A) = \det A\,.$

From the definition and for $n = 3$ we have

$$\begin{aligned}&I_1(A) = \operatorname{tr}(A) = \lambda_1+\lambda_2+\lambda_3\,,\quad I_2(A) = \operatorname{tr}(\operatorname{Cof} A) = \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1\,,\\ &I_3(A) = \det A = \lambda_1\lambda_2\lambda_3\,.\end{aligned} \tag{0.2.23}$$

For any matrix $A\in\mathbb{R}^{n\times n}$ the *characteristic polynomial* of A is defined by

$$\det(A-\lambda\mathbb{1}) = (-1)^n(\lambda^n + c_{n-1}\lambda^{n-1}+\ldots+c_1\lambda+c_0)\,. \tag{0.2.24}$$

The following theorem (*Cayley-Hamilton*) will state that every matrix with nonzero determinant is a root of its characteristic polynomial.

Theorem 0.2.11. *For any matrix $A\in\mathbb{R}^{n\times n}$*

$$A^n + c_{n-1}A^{n-1}+\ldots+c_1A+(-1)^n\det A.\mathbb{1} = 0\,. \tag{0.2.25}$$

The Cayley-Hamilton theorem can be rewritten by using the principal invariants as following

Theorem 0.2.12. *For any matrix $A\in\mathbb{R}^{n\times n}$,*

$$A^n - I_1(A)A^{n-1}+\ldots+(-1)^{n-1}I_{n-1}(A)A+(-1)^nI_n(A)\mathbb{1} = 0\,. \tag{0.2.26}$$

Proof. From the following characteristic polynomial

$$\det(A-\lambda\mathbb{1}) = (-1)^n\lambda^n+(-1)^{n-1}I_1(A)\lambda^{n-1}\pm\cdots-I_{n-1}(A)\lambda+I_n(A) = 0\,, \tag{0.2.27}$$

and eq. (0.2.16) we obtain

$$\det(A-\lambda\mathbb{1})\mathbb{1} = (A-\lambda\mathbb{1})\operatorname{Cof}(A-\lambda\mathbb{1})^T = (A-\lambda\mathbb{1})\operatorname{adj}(A-\lambda\mathbb{1})\,, \tag{0.2.28}$$

where have $\operatorname{adj}(A-\lambda\mathbb{1}) = \sum_{i=0}^{\infty} Y^i\lambda^i$, with $Y^i \in \mathbb{R}^{n\times n}$ are unknown coefficients. The entries in $\operatorname{adj}(\mathbb{1}-\lambda A)$ are polynomials in λ up to the maximal power on $n-1$. Therefore, for $k \geq n$, $Y^k = 0$, we have the following calculations

$$(A-\lambda\mathbb{1})\operatorname{adj}(A-\lambda\mathbb{1}) = \det(A-\lambda\mathbb{1}) \iff (A-\lambda\,\mathbb{1})\sum_{i=0}^{n-1} Y^i\lambda^i = \Big(\sum_{i=0}^{n-1} I_{n-1}(A)\lambda^i + (-1)^n\lambda^n\Big)\mathbb{1}\,, \tag{0.2.29}$$

and it is true if and only if

$$AY^0 + \sum_{i=1}^{n-1}\lambda^i(AY^i - Y^{i-1}) - Y^{n-1}\lambda^n = I_n(A)\mathbb{1} + \sum_{i=1}^{n-1}(-1)^i I_{n-1}(A)\lambda^i\mathbb{1} + (-1)^n\lambda^n\mathbb{1}\,, \quad \forall\lambda\in\mathbb{R}\,. \tag{0.2.30}$$

One may have

$$AY^0 = I_n(A)\mathbb{1}\,, \qquad AY^i - Y^{i-1} = (-1)^i I_{n-1}(A)\mathbb{1}\,, \quad \forall 1\leq i\leq n-1\,, \quad -Y^{n-1} = (-1)^n\mathbb{1}\,. \tag{0.2.31}$$

Multiplication above equations from the left side by A^i, we obtain

$$AY^0 + \sum_{i=1}^{n-1}(Ai+1Y^i - A^iY^{i-1}) - A^nY^{n-1} = I_n(A)\mathbb{1} - I_{n-1}(A)A + \cdots + (-1)^n A^n\,. \tag{0.2.32}$$

By noticing the telescoping sum in the left hand side, which is zero, we will obtain the conclusion. ■

So, for $n = 3$, the statement of the theorem reads

$$-A^3 + I_1(A)A^2 - I_2(A)A + I_3(A) = 0 \qquad\iff\qquad -A^3 + \operatorname{tr}(A)A^2 - \operatorname{tr}(\operatorname{Cof} A)A + \det A\cdot\mathbb{1} = 0\,. \tag{0.2.33}$$

Next we introduce the *right and left Cauchy-Green stretch tensors* with $C = F^TF = \mathbb{1}+2E$ and $B = FF^T$ respectively, where $E = \frac{1}{2}(F^TF-\mathbb{1})$ is the *Green-St Venant strain tensor*, for $F\in\mathrm{GL}^+(3)$. More details about strain tensors are available in Section 2.1. We have the following properties for the principal invariant of the strain tensors as follow

Lemma 0.2.13. *The following properties hold*

$$\begin{aligned}
I_1(C) &= \|F\|^2 = 3 + \operatorname{tr}(F^TF-\mathbb{1})\,,\\
&= \operatorname{tr}(\mathbb{1}+F^TF-\mathbb{1}) = \operatorname{tr}(\mathbb{1}) + \operatorname{tr}(F^TF-\mathbb{1}) = 3 + \operatorname{tr}(F^TF-\mathbb{1})\,,\\
I_2(C) &= \|\operatorname{Cof} F\|^2 = 3 + 2\operatorname{tr}(F^TF-\mathbb{1}) + \mathrm{h.o.t}(F)\,,\\
I_3(C) &= 1 + \operatorname{tr}(F^TF-\mathbb{1}) + \mathrm{h.o.t}(F)\,,
\end{aligned}$$

where h.o.t(F) *denotes the higher order terms dependent on F. The same hold for $B = FF^T$.*

Proof. We have

$$\begin{aligned}
I_1(C) &= \operatorname{tr}(C) = \langle C,\mathbb{1}\rangle = \langle F^TF,\mathbb{1}\rangle = \langle F,F\rangle = \|F\|^2\\
&= \operatorname{tr}(\mathbb{1}+2E) = \operatorname{tr}(\mathbb{1}) + 2\operatorname{tr}(E) = 3 + 2\operatorname{tr}(E)\,,
\end{aligned} \tag{0.2.34}$$

and

$$\begin{aligned}
I_2(C) &= \operatorname{tr}(\operatorname{Cof} C) = \langle \det C.C^{-T},\mathbb{1}\rangle = \det(F^TF)\langle (F^TF)^{-T},\mathbb{1}\rangle = (\det F)^2\langle F^{-T},F^{-T}\rangle\\
&= \langle \operatorname{Cof} F,\operatorname{Cof} F\rangle = \|\operatorname{Cof} F\|^2\\
&= \operatorname{tr}(\operatorname{Cof}(\mathbb{1}+2E)) = 3 + 4\operatorname{tr} E + \mathrm{h.o.t}(E)\,,
\end{aligned} \tag{0.2.35}$$

and finally,

$$I_3(C) = \det(F^TF) = (\det F)^2 = \det(\mathbb{1}+2E) = 1 + 2\operatorname{tr} E + \mathrm{h.o.t}(E)\,.$$ ■

0.2.2. Derivatives of functions on $\mathbb{R}^{n\times n}$

In the subject of dimensional reduction (direct approach) we will do some derivatives of some scalar valued functions on the set of matrices. The Taylor series expression will be used to reach this goal. So assume that $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a smooth scalar valued function. For every $H \in \mathbb{R}^{n\times n}$, the Taylor expression of f at point (matrix) $X \in \mathbb{R}^{n\times n}$ is as following

$$f(X+H) = f(X) + \langle \mathrm{D}f(X), H\rangle + O(H^2)\,. \tag{0.2.36}$$

If $g\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a matrix valued function, then for $X, H \in \mathbb{R}^{n\times n}$ we have

$$g(X+H) = g(X) + \mathrm{D}g(X).H + O(H^2)\,. \tag{0.2.37}$$

For some matrix valued functions we may have the following Lemma.

Lemma 0.2.14. *For matrices $X, H \in \mathbb{R}^{n\times n}$ we have:*

1. $D[X^T X].H = X^T H + H^T X$,
2. $D[\operatorname{dev} X].H = \operatorname{dev} H$,
3. $D[\operatorname{sym} X].H = \operatorname{sym} H$,
4. $D[\operatorname{skew} X].H = \operatorname{skew} H$,
5. $D[\operatorname{Cof} X].H = -\langle \operatorname{Cof} X, H\rangle X^{-T} H^T X^{-T}$,
6. $D[\det X].H = \langle \operatorname{Cof} X, H\rangle$.

Proof. Let us call $g_1(X) := X^T X$. For the matrix $(X+H)$ we have

$$g_1(X+H) = (X+H)^T(X+H) = (X^T+H^T)(X+H) = \underbrace{X^T X}_{=g_1(X)} + X^T H + H^T X + \underbrace{H^T H}_{\text{h.o.t}(H)}\ . \tag{0.2.38}$$

By comparing the right side of this relation and the Taylor expression of g_1, we obtain that $D[g_1(X)].H = X^T H + H^T X$. Now assume that $g_2(X) := \operatorname{dev} X$. Similarly, we have

$$\begin{aligned} g_2(X+H) &= \operatorname{dev}(X+H) = (X+H) - \frac{1}{n}\operatorname{tr}(X+H)\mathbb{1} \\ &= \underbrace{X - \frac{1}{n}\operatorname{tr}(X)\mathbb{1}}_{=g_2(X)} + H - \frac{1}{n}\operatorname{tr}(H)\mathbb{1}\,, \end{aligned} \tag{0.2.39}$$

and therefore, $D[g_2(X)]\cdot H = H - \frac{1}{n}\operatorname{tr}(H)\mathbb{1} = \operatorname{dev}(H)$. We call $g_3(X) = \operatorname{sym}(X)$. Hence

$$\begin{aligned} g_3(X+H) = \operatorname{sym}(X+H) &= \frac{1}{2}\big((X+H) + (X+H)^T\big) = \frac{1}{2}\big((X+X^T) + (H+H^T)\big) \\ &= \underbrace{\frac{1}{2}(X+X^T)}_{=g_3(X)} + \frac{1}{2}(H+H^T)\,, \end{aligned} \tag{0.2.40}$$

and finally $D[g_3(X)].H = \operatorname{sym} H$. The last function is $g_4(X) := \operatorname{skew} X$. We have

$$g_4(X+H) = \operatorname{skew}(X+H) = \frac{1}{2}\big((X+H) - (X+H)^T\big) = \underbrace{\frac{1}{2}(X-X^T)}_{=g_4(X)} + \frac{1}{2}(H-H^T)\,,$$

and hence $D[g_4(X)].H = \operatorname{skew} H$.
Denote $g_5(X) = \operatorname{Cof} X$. The, we have

$$\begin{aligned} \operatorname{Cof}(X+H) &= \det(X+H)(X+H)^{-T} = (\det X + \langle \operatorname{Cof} X, H\rangle + \text{h.o.t }(H)^2)(X^{-T} - X^{-T}H^T X^{-T} + \text{h.o.t }(H)) \\ &= \det X \cdot X^{-T} - \det X \cdot X^{-T} H^T X^{-T} + \langle \operatorname{Cof} X, H\rangle X^{-T} - \langle \operatorname{Cof} X, H\rangle X^{-T} H^T X^{-T} + \text{h.o.t }(H) \\ &= \operatorname{Cof} X - \langle \operatorname{Cof} X, H\rangle X^{-T} + \langle \operatorname{Cof} X, H\rangle X^{-T} - \langle \operatorname{Cof} X, H\rangle X^{-T} H^T X^{-T} + \text{h.o.t }(H) \\ &= \operatorname{Cof} X - \langle \operatorname{Cof} X, H\rangle X^{-T} H^T X^{-T} + \text{h.o.t }(H)\,. \end{aligned}$$

Hence,

$$D[g_5(X)].H = -\langle \operatorname{Cof} X, H\rangle X^{-T} H^T X^{-T}\,. \tag{0.2.41}$$

Denoting $g_6(X) = \det X$. Then, we have

$$\begin{aligned}\det(X+H) &= \det(X(\mathbb{1}+HX^{-1}H)) = \det X \det(\mathbb{1}+X^{-1}H) = \det X[1+\langle \mathbb{1}, X^{-1}H\rangle, \text{h.o.t}(H)]\\ &= \det X + \det X\langle X^{-T}, H\rangle + \text{h.o.t}(H) = \det X + \langle \operatorname{Cof} X, H\rangle + \text{h.o.t}(H)\,,\end{aligned} \tag{0.2.42}$$

which leads to

$$D[g_6(X)].H = \langle \operatorname{Cof} X, H\rangle\,.$$

∎

Lemma 0.2.15. *For every $X, H \in \mathbb{R}^{n\times n}$,*

1. $D[\|X\|].H = \frac{1}{\|X\|}\langle X, H\rangle\,,$
2. $D[\|X\|^2].H = 2\langle X, H\rangle\,.$

Proof. Assume that $k(X) = \|X\|$. The Taylor expression for k is

$$k(X+H) = k(X) + \langle D[k(X)], H\rangle + \text{h.o.t}(H)\,. \tag{0.2.43}$$

From that we obtain

$$\|X+H\|^2 = \big(\|X\| + \langle D[\|X\|], H\rangle + \text{h.o.t}(H)\big)^2\,, \tag{0.2.44}$$

this implies that

$$\|X\|^2 + \langle X, H\rangle + \langle H, X\rangle + \underbrace{\|H\|^2}_{\text{h.o.t}} = \|X\|^2 + 2\|X\|\langle D[\|X\|], H\rangle + (\langle D[\|X\|], H\rangle)^2 + \text{h.o.t}(H)\,. \tag{0.2.45}$$

Therefore,

$$D[\|X\|].H = \frac{1}{2\|X\|}\big(\langle X, H\rangle + \langle H, X\rangle\big) = \frac{1}{\|X\|}\langle X, H\rangle\,. \tag{0.2.46}$$

Similarly, assume that $t(X) = \|X\|^2$. Hence,

$$\|X\|^2 + \langle X, H\rangle + \langle H, X\rangle + \|H\|^2 = \|X+H\|^2 = \|X\|^2 + D[\|X\|^2].H + \text{h.o.t}(H)\,, \tag{0.2.47}$$

and then

$$\mathrm{D}[\|X\|^2].H = \langle X, H\rangle + \langle H, X\rangle = \operatorname{tr}(XH^T + HX^T) = 2\langle X, H\rangle. \tag{0.2.48}$$

∎

0.2.3. Minimum problems

In mathematical analysis that uses variations, the calculus of variations can be used in order to find the maxima and minima of functionals, which can be done by applying some small changes in functions and functionals. To this aim, the *Euler-Lagrange equations* is a system of second order ordinary differential equations whose solutions are stationary points of the given action functional. Actually this equation is useful for solving optimization problems, where we are looking for minimizing or maximizing the function. For example, we are searching to find the curve in the plane of the shortest length connecting two points. The solution is just the straight line between two points. Otherwise, the solution will not be so clear and it is possible we have more than one solution. For example, assume the disturbed curve $y(x)$ which varies between two fixed points $a, b \in \mathbb{R}^2$ in a plane. Assume that l denotes the length between $(a, A), (b, B) \in \mathbb{R}^2$. One can see that l can take the following form

$$l[y] = \int_a^b \sqrt{1 + y'(x)^2}\,\mathrm{d}x\,, \qquad y(a) = A\,, \quad y(b) = B\,. \tag{0.2.49}$$

For finding the shortest path between a and b, we should minimize the functional l, but regarding to the function $y(x)$. Assume that $h(x)$ is another differentiable function that satisfies $h(a) = h(b) = 0$. Then we can define a new path by

$$\widehat{y}(x,t) = y(x) + t\,h(x)\,, \quad \text{for} \quad t \in \mathbb{R}\,. \tag{0.2.50}$$

Because $\widehat{y}(a,t) = y(a) + th(a) = A$ and $\widehat{y}(b,t) = y(b) + th(b) = B$, we can consider $\widehat{y}$ as a path between the points (a, A) and (b, B). The length of the new path can be defined like

$$\ell[y + th] = \int_a^b \sqrt{1 + ((y + th)')^2}\,\mathrm{d}x = \int_a^b \sqrt{1 + (y' + th')^2}\,\mathrm{d}x\,.$$

We notice that $\ell[y + th]$ is a real valued function. Indeed, $\ell[y + th]$, for fixed y and h, takes real number t and gives the length of $y + t\,h$ as a real number. On the other hand, we assumed that y is the minimal path, which means $\ell[y + th]$ must take the minimum at $t = 0$ for every direction h. This means,

$$\frac{d}{dt}\ell[y + th]\Big|_{t=0} = 0\,, \tag{0.2.51}$$

which is equivalent to

$$\frac{d}{dt}\ell[y + th]\Big|_{t=0} = \Big(\frac{d}{dt}\int_a^b \sqrt{1 + (y' + th')^2}\mathrm{d}x\Big)\Big|_{t=0}\,. \tag{0.2.52}$$

Obviously, a, b are independent from t. By applying Leibniz's integral rule, we obtain

$$\begin{aligned}\frac{d}{dt}\ell[y + th]\Big|_{t=0} &= \int_a^b \frac{d}{dt}\sqrt{1 + (y' + th')^2}\Big|_{t=0}\mathrm{d}x \\ &= \int_a^b \frac{(y' + th')h'}{\sqrt{1 + (y' + th')^2}}\Big|_{t=0}\mathrm{d}x \\ &= \int_a^b \frac{y'h'}{\sqrt{1 + (y')^2}}\mathrm{d}x\,.\end{aligned} \tag{0.2.53}$$

We obtain

$$\int_a^b \frac{y'}{\sqrt{1 + (y')^2}}h'\mathrm{d}x = 0\,. \tag{0.2.54}$$

After integration by parts, we arrive at

$$\int_a^b \Big(\frac{y'}{\sqrt{1 + (y')^2}}\Big)'h\,\mathrm{d}x = 0\,. \tag{0.2.55}$$

We already assumed that h is an arbitrary function with zero boundary rules, therefore,

$$\Big(\frac{y'}{\sqrt{1 + (y')^2}}\Big)' = 0\,, \tag{0.2.56}$$

and by integrating both sides we have

$$\frac{y'}{\sqrt{1 + (y')^2}} = \alpha, \text{ for some constant } \alpha\,. \tag{0.2.57}$$

Generally, assume that $W : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ and assume that $u \in C^1(\Omega, \mathbb{R}^m)$. Now the question is: find a u that minimizes the following function

$$I(u) = \int_\Omega W(x, u(x), \nabla u(x))\,\mathrm{d}x\,. \tag{0.2.58}$$

There is a necessary condition for existence of an extermum for solving the above problem.

Theorem 0.2.16. *Let I be a function of the form*

$$I(u) = \int_\Omega W(x, u(x), \nabla u(x))\,dx\,, \tag{0.2.59}$$

such that $u \in C^1(\Omega, \mathbb{R}^m)$. Then the Euler-Lagrange equation (in the strong form) reads

$$\mathrm{D}_u W(x, u(x), \nabla u(x)) - \mathrm{Div}\,\Big(\mathrm{D}_{\nabla u} W(x, u(x), \nabla u(x))\Big) = 0\,. \tag{0.2.60}$$

0.2.4. Convexity and local minima

Assume that X, Y are two normed vector spaces, $L(X, Y)$ is the vector space of all continuous linear mappings from X to Y and $f\colon \Omega \subset X \to Y$, with Ω as an open subset of X.

Definition 0.2.17. The mapping f is *differentiable* at a point $x \in \Omega$ if there exists an element $\mathrm{D}f(x)$ of the space $L(X, Y)$ such that

$$f(x+h) = f(x) + \mathrm{D}f(x)h + o(h)\,, \tag{0.2.61}$$

where $o(h)$ denotes the higher order therms of h.

One of the applications of differentiability of functions, is the famous *Taylor series formula.* The expression of the Taylor formula for any function depends on the order of differentiability of the function. Therefore, we have the following theorem

Theorem 0.2.18. *Let X, Y be two normed vector spaces, Ω be an open subset of X, $[x, x+h]$ be a closed segment contained in Ω, $f\colon \Omega \subset X \to Y$ be a given mapping and let m be an integer such that $m \geq 1$. If f is $(m-1)$ times differentiable in Ω and m times differentiable at the point x, then*

$$f(x+h) = f(x) + \mathrm{D}f(x)h + \cdots + \frac{1}{m!}\,\mathrm{D}^{(m)}f(x)h^m + o(h^{m+1})\,. \tag{0.2.62}$$

For a Proof we refer to [33, Theorem 1.3-3].

Existence of a local minimizer and convexity of a function like f, are some other utilization of differentiable mappings.

Theorem 0.2.19. *Let Ω be an open subset of a normed space X and let $f\colon \Omega \subset X \to \mathbb{R}$ be a differentiable function in Ω. Assume that $a \in \Omega$ and $\mathrm{D}f(a) = 0$. If the function f is twice differentiable in Ω and if there exists an open neighborhood $V \subset \Omega$ of the point a such that*

$$\mathrm{D}^2 f(x)(h,h) \geq 0\,, \qquad \textit{for all} \quad x \in V\,, h \in X\,, \tag{0.2.63}$$

then the point a is a local minimum of the function f.

Definition 0.2.20. Let $f\colon \Omega \subset X \to \mathbb{R}$ be a function. The function f is called *convex* if for all $0 \leq t \leq 1$ and all $x_1, x_2 \in X$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)\,. \tag{0.2.64}$$

In the following, we can see the relation between convexity and derivatives

Theorem 0.2.21. ***(convexity and first derivative)*** *Assume $f\colon \Omega \subset X \to \mathbb{R}$ is a differentiable function in Ω. The function f is convex on Ω* ***if and only if***

$$f(x+h) \geq f(x) + \mathrm{D}f(x).h\,, \quad \textit{for all} \quad x, h \in X\,. \tag{0.2.65}$$

Theorem 0.2.22. ***(convexity and second derivative)*** *Assume $f\colon \Omega \subset X \to \mathbb{R}$ is a twice differentiable function in Ω. The function f is convex* ***if and only if***

$$\mathrm{D}^2 f(x)(h,h) \geq 0\,, \quad \textit{for all} \quad x, h \in X\,. \tag{0.2.66}$$

The next theorem will show the property of the minimum of a convex function,

Theorem 0.2.23. *[33, Theorem 4.7-8] Let $f\colon \Omega \subset X \to \mathbb{R}$ be a convex function defined on a convex subset Ω of a normed space X.*

- *Any local minimum of f on Ω is a minimum.*
- *If f is strictly convex, it has at most one minimum on Ω, and it is a strict minimum.*
- *Let g be differentiable at a point $x \in \Omega$. a point x is a minimium pf g on Ω if and only if*

$$\mathrm{D}g(x)(h) \geq 0\,, \qquad \textit{for all} \quad x, h \in \Omega. \tag{0.2.67}$$

- *If the set Ω is open, a point x is a minimum of g if and only if $\mathrm{D}g(x) = 0$.*

Part I.

Cosserat curvy shell model

1. Γ- limit and Γ-convergence

1.1. Lower semi-continuity

One of the questions which always may arise is how to determine sufficient conditions to ensure the convergence of sequences of minimum problems and their related minimizers. Γ-convergence is one the important tools for answering this question. In fact, Γ-convergence is able to answer the following family of minimum problems

$$\min\{I_j(u) \,\big|\, u \in X_j\}\,. \tag{1.1.1}$$

A way to describe the behavior of the solutions of above problem can be provided by replacing such a family by following problem

$$\min\{I(u) \,\big|\, u \in X\}\,, \tag{1.1.2}$$

which takes the related behavior of minimizers and for which a solution can be more easily obtained [25]. The most important feature of Γ-convergence is about the possibility of obtaining converging sequences (or subsequences) from minimizers of (1.1.1). One of the basic requirements is *compactness*:

"the idea of convergence of functions u_ϵ should be in a way that ensures the existence of a limit of minimizers of (1.1.1). The convergence of functions may give a negative answer just because the minimizers will not converge. Hence, a candidate space like X, can be one which is equipped with the compactness property for solving the limit problems".

We notice that the argument of Γ-convergence depends on compactness, therefore, it is more convenient to have a space with weaker topologies which guarantee the existence of a convergence sequence (or subsequence) [25].

Later on, we will see that the compactness property can be replaced with *coerciveness* property. Of course in the case that the functionals F_ϵ are energy functionals, by considering the *existence* of a minimizer for the homogenized energy, we may answer the compactness. The *fundamental theorem of* Γ*-convergence* can be summarized as

$$\Gamma\text{-convergence} + \text{equi-coerciveness (compactness)} \quad\Longrightarrow\quad \text{convergence of minimum problems.} \tag{1.1.3}$$

Of course, beside the study of asymptotic properties of minimum problems, Γ-convergence has some other kind of uses. One is the construction of suitable Γ-convergence functionals I_j to a given I_0. Moreover, Γ-convergence is also used in the "justification" of physical theories through a limit procedure. One example is the derivation of low-dimensional theories from three-dimensional elasticity, another one is the deduction of properties in Continuum Mechanics from atomistic potentials.

Definition 1.1.1. Assume that $f\colon X \to \overline{\mathbb{R}}$. The *lower limit* of f at point x is

$$\begin{aligned}\liminf_{y\to x} f(y) &= \inf\{\liminf_j f(x_j) \,\big|\, x_j \in X\,, x_j \to x\} \\ &= \inf\{\lim_j f(x_j) \,\big|\, x_j \in X\,, x_j \to x\,, \exists \lim_j f(x_j)\}\,.\end{aligned} \tag{1.1.4}$$

The *upper limit* of f at point x is

$$\begin{aligned}\limsup_{y\to x} f(y) &= \sup\{\limsup_j f(x_j) \,\big|\, x_j \in X\,, x_j \to x\} \\ &= \sup\{\lim_j f(x_j) \,\big|\, x_j \in X\,, x_j \to x\,, \exists \lim_j f(x_j)\}\,.\end{aligned} \tag{1.1.5}$$

The lower limit is written like " $\liminf$ " and the upper limit like " $\limsup$ ".

It can be seen that [25]

$$\liminf_{y\to x}(f(y)+g(y)) \geq \liminf_{y\to x} f(y) + \liminf_{y\to x} g(y)\,, \tag{1.1.6}$$

$$\liminf_{y\to x}(f(y)+g(y)) \leq \limsup_{y\to x} f(y) + \liminf_{y\to x} g(y)\,, \tag{1.1.7}$$

$$\liminf_{y\to x}(-f(y)) = -\limsup_{y\to x}(f(y))\,. \tag{1.1.8}$$

Lemma 1.1.2. *(**Fatou's Lemma**) Let (X,Σ,ν) be a measure space and $\{f_j\colon X\to[0,\infty]\}$ be a sequence of nonnegative measurable functions. Then the function $\liminf_{j\to\infty} f_j$ is measurable and*

$$\int_X \liminf_{j\to\infty} f_j\, d\nu \leq \liminf_{j\to\infty}\int_X f_j\, d\nu\,. \tag{1.1.9}$$

Definition 1.1.3. Assume that X is a metric space. The function $I\colon X\to\overline{\mathbb{R}}$ is said to be *lower semi-continuous* (*l.s.c*) at the point $x\in X$, if for every sequence x_j converging to x we have

$$I(x) \leq \liminf_{j} I(x_j)\,, \tag{1.1.10}$$

or,

$$I(x) = \min\{\liminf_{j} I(x_j)\ : x_j\to x\}\,. \tag{1.1.11}$$

If I is lower semi-continuous at all point $x\in X$, then I is lower semi-continuous on X.

Example 1.1.4. Assume the space $X=\mathbb{R}$ with the metric $d(x,y)=|x-y|$ for all $x,y\in\mathbb{R}$. Let $g\in C(\mathbb{R})$ and $a\in\mathbb{R}$. We define

$$f(x) := \begin{cases} g(x) & x\neq 0\\ a & x=0\end{cases} \tag{1.1.12}$$

The function f is lower semi-continuous at $x=0$ if and only if $a\leq g(0)$. □

Proof. Assume that $a\leq g(0)$. Then, according to the definition of semi-continuity we have

$$a\leq g(0) \qquad\Longrightarrow\qquad f(0)=a\leq g(0)=g(\lim_{x_j\to 0} x_j)=\lim_{x_j\to 0} g(x_j)=\lim_{0\neq x_j\to 0} f(x_j)\,, \tag{1.1.13}$$

which for $x=0$ gives

$$f(x) \leq \lim_{0\neq x_j\to 0} f(x_j)\,, \tag{1.1.14}$$

and shows that f is lower semi-continuous at $x=0$. Now assume that f is lower semi-continuous at x. Then, for every sequence (x_j) with $x_j\to x$, we have

$$f(x) \leq \lim_{x_j\to x} f(x_j)\,. \tag{1.1.15}$$

One may use the opposite direction of the relation (1.1.13). Since *l.s.c* happened for every $x_j\to x$, we may assume that $x_j\to 0$. Then, we obtain $a\leq g(0)$. ■

Remark 3. *The following conditions are equivalent:*

1. *$I\colon X\to\overline{\mathbb{R}}$ is lower semi-continuous,*
2. *for all $x\in X$, $I(x)=\liminf_{y\to x} I(y)$,*
3. *for all $\alpha\in\mathbb{R}$, the sublevel set $\{I\leq\alpha\}:=\{x\in X\colon I(x)\leq\alpha\}$ is closed.*

Remark 4. *(i) If I and J are lower semi-continuous, then $I+J$ is as well. The proof can be seen by using the property of* lim inf*.*
(ii) If $\{I_i\ :\ i\in\mathbb{N}\}$ is a family of lower semi-continuous functions, then the function $J(x)=\sup_i I_i(x)$ is lower semi-continuous too.

1.2. Gamma convergence and recovery sequence

Definition 1.2.1. Let us assume that X be a metric space. We call the sequence $I_j\colon X \to \overline{\mathbb{R}}$, Γ-*convergent* to $I_0\colon X \to \overline{\mathbb{R}}$ in X, if for all $x \in X$ we have the two following conditions

1. (**liminf inequality**) for every convergent sequence $x_j \to x$

$$I_0(x) \leq \liminf_j I_j(x_j)\,, \qquad (\liminf \text{ inequality})\,. \tag{1.2.1}$$

2. (**limsup inequality**) there exists a convergent sequence $x_j \to x$ such that

$$I_0(x) \geq \limsup_j I_j(x_j)\,, \qquad (\limsup \text{ inequality})\,. \tag{1.2.2}$$

In other words, I_0 is a *lower bound* for the sequence I_j, in the sense that $I_0(x) \leq I_j(x_j) + o(1)$ whenever $x_j \to x$.

Regarding to $\limsup$ inequality, we should notice that I_0, actually, is an upper bound for the sequence I_j and computing that, is related to an *ansatz* which is dependent on the construction of the sequence (x_j). Now the critical matter is selecting the right concept of convergence $x_j \to x$. It is remarkable that the sequence is not already chosen and should be selected in a way that can help the equi-coerciveness (Definition 1.2.5) of the family I_j.

The function I_0 is called Γ- *limit* of (I_j) and one can write $I_0 = \Gamma\text{-}\lim_j I_j$.

According to the definition of Γ-limit, we can obtain

$$I_0(x) \leq \liminf_j I_j(x_j) \leq \limsup_j I_j(x_j) \leq I_0(x)\,, \tag{1.2.3}$$

which means $I_0(x) = \lim_j I_j(x_j)$. Therefore, for the second condition, we can have the following alternative condition,

- (**existence of a suitable recovery sequence**) there exists a sequence (x_j) converging to x such that

$$I_0(x) = \lim_j I_j(x_j)\,. \tag{1.2.4}$$

As before mentioned, for solving the convergence problem, we may work on the spaces with weaker topology which always will guarantee the existence of a convergence subsequence. But this is not the only point which should be noticed through the subject of Γ-convergence. Another important issue is selecting the right *energy scaling.* As a matter of fact, there is a possibility that the given sequence (I_j) will not act properly to reach the equi-coercivity, even with the convergence sequence. But choosing the right scaling of the variables will be one of the important steps for solving the minimization problem.

From the Γ-convergence of functionals I_j to I_0, we do not immediatly deduce the convergence of minimum problems with Dirichlet boundary conditions. In fact, to do so we must prove the *compatibility* of the condition $u = \varphi$ on $\partial\Omega$; i.e., that the functionals

$$I_j^{\varphi}(u) = \begin{cases} I_j(u) = \int_\Omega f_j(x, Du)\,dx & \text{if } u = \varphi \text{ on } \partial\Omega \\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.2.5}$$

Γ-convergence to I_0^{φ} analogously defined.

Remark 5. *The Γ-convergence has the following properties:*

- *Γ-limit and continuous functions: If the family I_j, Γ-converges to I_0 and $J\colon X \to [-\infty, +\infty]$ is a continuous function, then $(I_j + J)$ Γ-converges to $I_0 + J$. Indeed, for all $x \in X$, with $x_j \to x$ we have*

$$I_j(x) + J(x) \leq \liminf_j I_j(x_j) + \lim_j J(x_j) = \liminf_j (I_j + J)(x_j)\,. \tag{1.2.6}$$

Since the $\limsup$ *condition holds as well, we get*

$$I_0(x) + J(x) = \lim_j I_j(x_j) + \lim_j J(x_j) = \lim_j (I_j + J)(x_j)\,, \tag{1.2.7}$$

which gives

$$\limsup_j (I_j + J)(x_j) \leq I_0(x) + J(x)\,. \tag{1.2.8}$$

Therefore, (x_j) is a recovery sequence for $I_0 + J$ and it shows that $I_j + J$ Γ-converges to $I_0 + J$.

- Γ*-limit of a constant sequence: Let $I_j = I$ be a sequence, for all $j \in \mathbb{N}$; that means I_j is a constant sequence. Assume that (I_j) Γ-convergences, hence for "hypothetical" Γ-limit, I_∞ should hold*

$$I_\infty(x) \leq \liminf_j I(x_j)\,, \qquad x_j \to x\,, \qquad x \in X\,. \tag{1.2.9}$$

*But if I is **not** lower semi-continues, then there exists $\overline{x}$ and a sequence $\overline{x}_j$ such that $\overline{x}_j \to \overline{x}$ and*

$$\liminf_j I(\overline{x}_j) \leq I(\overline{x})\,. \tag{1.2.10}$$

*Therefore, $I_\infty(\overline{x}) \neq I(\overline{x})$. This shows that Γ-convergence **does not** satisfy the requirement that a sequence $I_j = I$ convergences to I (unless I is lower semi-continuous). Therefore,*

$$\Gamma\text{-}\lim I_j(x) = I(x) \qquad \text{if and only if} \qquad I \text{ is lower semi-continuous}\,. \tag{1.2.11}$$

Definition 1.2.2. [Upper and lower Γ-limits] Assume $I_j \colon X \to \overline{R}$ and $x \in X$. The quantity

$$\Gamma\text{-}\liminf_j I_j(x) = \inf\{\liminf_j I_j(x_j) \,:\, x_j \to x\}\,, \tag{1.2.12}$$

is called the Γ-*lower limit* of the sequence (I_j) at the point x. Similarly

$$\Gamma\text{-}\limsup_j I_j(x) = \inf\{\limsup_j I_j(x_j) \,:\, x_j \to x\}\,, \tag{1.2.13}$$

is called the Γ-*upper limit* of the sequence (I_j) at x. If one has the following equality

$$\Gamma\text{-}\liminf_j I_j(x) = \Gamma_\infty = \Gamma\text{-}\limsup_j I_j(x)\,, \tag{1.2.14}$$

for some $\Gamma_\infty \in [-\infty, +\infty]$, then, we can write $\Gamma_\infty = \Gamma\text{-}\lim_j I_j(x)$ and we say that Γ_∞ is the Γ-limit of the sequence (I_j) at the point $x \in X$.

In [25], Proposition 1.28, we can find the following result.

Proposition 1. ***[Lower semi-continuity of* Γ*-limits]*** *The Γ-upper and lower limits of a sequence (I_j) are lower semi-continuous functions.*

Remark 6. *Γ-limit has the following properties:*

1. *If (I_{j_k}) is a subsequence of (I_j) then*

$$\Gamma\text{-}\liminf_j I_j \leq \Gamma\text{-}\liminf_k I_{j_k}\,, \qquad \Gamma\text{-}\limsup_k I_{j_k} \leq \Gamma\text{-}\limsup_j I_j\,. \tag{1.2.15}$$

If $I_\infty = \Gamma\text{-}\lim_j I_j$ exists, then for every increasing sequence of integers (j_k) we have $I_\infty = \Gamma\text{-}\lim_k f_{j_k}$.

2. *If J is a continuous function, then $I_\infty + J = \Gamma\text{-}\lim_j(I_j + J)$. Moreover, if $J_j \to J$ convergence uniformly and J is continuous, then $I_\infty + J = \Gamma\text{-}\lim_j(I_j + J_j)$. And if I_j converges uniformly to I on an open set U, then*

$$\Gamma\text{-}\lim_j I_j = \operatorname{sc} I\,, \tag{1.2.16}$$

where $\operatorname{sc} I$ is the lower semi-continues envelope of I which is the greatest lower semi-continues function not greater than I.

Proposition 2. *(Compactness of Γ-convergence) Let (X, d) be a seperable metric space with metric d and let $I_j \colon X \to \overline{\mathbb{R}}$ be a function, for all $j \in \mathbb{N}$. Then there exsits a subsequence (I_{j_k}) such that for all $x \in X$, the Γ-limit of the sequence (I_{j_k}) exists.*

Remark 7. *If (X,d) in above Proposition is not separable, then the result of this Proposition will fail. We may assume that $X = \{-1,1\}^N$ with the discrete topology. X is metrizable and here Γ-convergence on X is equivalent to pointwise convergence. Let us take the sequence $I_j\colon X \to \{-1,1\}$ defined by $I_j(\boldsymbol{x}) = x_j$, where $\boldsymbol{x} = (x_0, x_1, \ldots)$. Assume that (I_{j_k}) is a subsequence of (I_j) and assume $\boldsymbol{x}$ is defined by $x_{j_k} = (-1)^k$ and if $j \notin \{j_k : k \in \mathbb{N}\}$ then $x_j = 1$. Hence, $\lim_k I_{j_k}(x)$ does not exist and therefore no subsequence of (I_j) Γ-converges.*

Γ-convergence also has the following result regarding to subsequences

Proposition 3. ***[Urysohn property of Γ-convergence]*** *The Γ-limit for the sequence (I_j) exists, i.e., $I_\infty = \Gamma\text{-}\lim_j I_j$, if and only if for every subsequence (I_{j_k}) there exists a further subsequence which Γ-convergence to I_∞.*

Proof. See [25], Proposition 1.44. ■

Definition 1.2.3. We call that I_{h_j}, Γ-converges to I_0, if for all sequences $(h_j) \to 0$ we have $\Gamma\text{-}\lim_j I_{h_j} = I_0$.

Definition 1.2.4. The function $I\colon X \to \overline{\mathbb{R}}$ is *coercive* on X if the closure of the sublevel set $\{x \in X \,|\, I(x) \le \alpha\}$ is compact in X, for every $\alpha \in \mathbb{R}$.

Definition 1.2.5. The sequence of functionals $I_j\colon X \to \overline{\mathbb{R}}$ is *equi-coercive* if for each $K > 0$ there exists a compact set $K_c \subset X$ such that $\{x \in X \,|\, I_j(x) \le K\} \subset K_c$, independently of $j > 0$.

Note that, if the family I_j is equi-coercive, then lim inf-inequality condition immediately implies one inequality for the minimum problem: if (x_j) is a minimizing sequence and $x_j \to x_0$ then

$$\inf I_0 \le I_0(x_0) \le \liminf_{j\to 0} I_j(x_j) = \liminf_{j\to 0} I_j \,. \tag{1.2.17}$$

Theorem 1.2.6 (Characterization of equi-coerciveness)**.** *The sequence of functionals $I_{h_j}\colon X \to \mathbb{R}$ is equi-coercive if and only if there exists a lower semi-continuous coercive function $\Psi\colon X \to \overline{\mathbb{R}}$ such that $I_{h_j} \ge \Psi$ on X for every $h_j > 0$.*

Theorem 1.2.7. *(Coerciveness of Γ-limit) Suppose that the sequence of functionals $I_{h_j}\colon X \to \mathbb{R}$ is equi-coercive. Then the upper and lower Γ-limits are both coercive and*

$$\min_{x\in X}(\Gamma\text{-}\liminf_{h_j} I_{h_j})(x) = \liminf_{h_j} \inf_{x\in X} I_{h_j}(x) \,. \tag{1.2.18}$$

If in addition, the sequence of integral functionals $I_{h_j}\colon X \to \overline{\mathbb{R}}$, Γ-convergence to a functional $I_0\colon X \to \overline{\mathbb{R}}$, then I_0 itself is coercive and

$$\min_{x\in X} I_0(x) = \lim_{h_j} \inf_{x\in X} I_{h_j}(x) \,. \tag{1.2.19}$$

The variational nature of Γ-convergence is now evident. This theorem expresses the convergence of the minimum problems related to the I_j's to the corresponding one for I_0. Moreover, the existence of solutions to the latter problem is guaranteed.

1.3. Sobolev spaces

In order to study and answering the questions regarding to existence and uniqueness of a minimizer, it is necessary to define the suitable spaces which include these properties. *Sobolev spaces* are required tool for simplifying the study of linear and nonlinear partial differential equations and their problems. But before that we present the definition of *Lebesgue spaces.*

Definition 1.3.1. Let Ω be an open subset of $\mathbb{R}^n$ and p be a real number satisfying $p \in [1,\infty)$. The space of functions, which are *Lebesgue integrable* on Ω to the power of p is denoted by

$$L^p(\Omega) = \{\, f \;|\; \int_\Omega |f(x)|^p \,\mathrm{d}x < \infty \} \,, \tag{1.3.1}$$

with the following norm

$$\|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}} . \tag{1.3.2}$$

In the case $p = 2$, $L^2(\Omega)$ with the following inner product is a Hilbert space

$$\langle f, g\rangle = \int_\Omega f\, g\, \mathrm{d}x \tag{1.3.3}$$

For $p = \infty$ we have

$$L^\infty(\Omega) = \{\, f \;\big|\; |f(x)| < \infty \quad \text{almost everywhere in} \quad \Omega\} , \tag{1.3.4}$$

equipped with the norm

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| . \tag{1.3.5}$$

Remark 8. *(Hölder's inequality) Let $\frac{1}{p} + \frac{1}{q} = 1$, with $p, q \in [1, \infty]$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and it holds that*

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} . \tag{1.3.6}$$

If $p = q = 2$, then we will arrive at the Cauchy-Schwarz inequality

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} . \tag{1.3.7}$$

Definition 1.3.2. Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, then the *Sobolev space* $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \,\big|\, \mathrm{D}^\alpha u \in L^p(\Omega) , \quad \forall\alpha \quad \text{with} \quad |\alpha| \le k\} , \tag{1.3.8}$$

which with the following norm will be a Banach space

$$\|u\|_{W^{k,p}(\Omega)} := (\sum_{|\alpha|\le k} \|\mathrm{D}^\alpha u\|^p_{L^p(\Omega)})^{\frac{1}{p}} , \qquad 1 \le p < \infty . \tag{1.3.9}$$

For $p = 2$ we will have the notation $H^k(\Omega) := W^{k,2}(\Omega)$, which is a Hilbert space with the following inner product

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{|\alpha|\le k} \langle \mathrm{D}^\alpha u, \mathrm{D}^\alpha v\rangle_{L^2(\Omega)} . \tag{1.3.10}$$

As well, for $1 < p < \infty$, $W^{k,p}(\Omega)$ is reflexive. It can be seen that the space $W^{k,p}$ is contained in the space $L^p(\Omega)$.

Definition 1.3.3. Assume that X, Y are two normed vector spaces. We say that X is *embedded* in Y, and denote by $X \hookrightarrow Y$, if $X \subset Y$ and there is a constant c such that $\|v\|_Y \le c\,\|v\|_X$, for all $v \in X$.

Definition 1.3.4. A normed vector space X is *compactly embedded* in a normed vector space Y, if $X \hookrightarrow Y$ and the continous injection $j: X \to Y$ with $j(x) = x \in Y$ is a *compact* linear operator, that is, if j maps each bounded sequence (x^k) into a sequence $(j(x^k))$ that contains a subsequence converging to some limit in Y. We will denote this kind of embedding by $X \Subset Y$.

The following theorem is called *Rellich-Kondrachov embedding theorem*, which gives the properties of compact embedding spaces

Theorem 1.3.5. *[33, Theorem 6.1-5. p.278] Let Ω be a domain in $\mathbb{R}^n$, $k > 0$ be an integer, and let $p \in [1, \infty)$. Then the following compact embedding holds*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) , \quad \textit{for all } q \textit{ with } 1 \le q < \infty, \textit{ if } k = \frac{n}{p} . \tag{1.3.11}$$

The following compact embedding is one of the special case of the above theorem

$$H^1(\Omega) \Subset L^2(\Omega)\,, \tag{1.3.12}$$

which is independent of the dimension n.

Definition 1.3.6. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in a Hilbert space H and assume that $x \in H$. The sequence $(x_n)_{n\in\mathbb{N}}$ is *weakly convergent* to x, denoted by $x_n \rightharpoonup x$, if and only if for all $y \in H$

$$\lim_{n\to\infty} \langle y, x_n\rangle = \langle y, x\rangle\,, \qquad (\langle y, x_n\rangle \to \langle y, x\rangle)\,. \tag{1.3.13}$$

$W^{k,2}(\Omega)$ can be an example of a Hilbert space. We say that the sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ *converges weakly* to $u \in W^{1,2}(\Omega)$, $u_n \rightharpoonup u$, if for every $v \in W^{1,2}(\Omega)$ we have $\langle u_n, v\rangle_{W^{1,2}(\Omega)} \to \langle u, v\rangle_{W^{1,2}(\Omega)}$.

Theorem 1.3.7. *(a) In a Banach space, every weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ is bounded and for the limit x, we have*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|\,. \tag{1.3.14}$$

*(b) In a reflexive Banach space, a bounded sequence **contains** a weakly convergent subsequence.*

From the fact that $W^{1,2}(\Omega)$ is a Hilbert space and consequently a Banach space, one can obtain that (a) and (b) hold as well for the space $W^{1,2}(\Omega)$.

Theorem 1.3.8. *(Mazur's theorem) Assume that u_n is weakly convergent to u in the Hilbert space $W^{1,2}(\Omega)$. Then, there exist a sequence of convex combinations like $\tilde{u}_n = \sum_{j=i}^{m_i} a_{i,j} u_j$, with $a_{i,j} \ge 0$ and $\sum_{j=i}^{m_i} a_{i,j} = 1$, such that $\tilde{u}_n \to u$, i.e. $\tilde{u}_n$ converges strongly to u.*

2. Theory of elasticity

The *theory of elasticity* treats the relationship between forces applied to an elastic object and the resulting deformations. In physics, elasticity is defined as the resistant ability of a body to return to its original size and shape when the force or the influence is gone. The stress, strain and displacements in an elasticity problem can be obtained from a series of basic equations and boundary conditions. Throughout deriving such equations, we may consider all the factors, but the results can be so complicated which may give no solution. Hence, we can consider some basic assumptions in order to reach possible solutions. While we are considering these assumptions, we may neglect some other factors that have minor influence on the result. For example we may assume that the whole body is continuous and there is no empty place in the body. Therefore, the physical quantities of the body like stress, strain and displacement can be distributed continuously on the body. Also we can assume that the body is perfectly elastic. We need to have a *homogeneous*[1] body such that all elastic properties are the same through the body, i.e. the elastic constants can be independent from the position in the body. The other condition that we may assume for the body is isotropy, which means all the elastic properties are the same in all directions. Thus, as mentioned already, by applying these assumptions, we can make the problem easier to solve. We only consider quasi-static problems, i.e., the energy does not have a kinetic part.

One of the usage of elasticity is to analyse the stress and displacement of elements within the time of receiving the force on the body and then to check the strength, stiffness and stability of the body. The theory of elasticity contains equilibrium equations related to stress, kinematic equations related to displacement and strain, constitutive equations related to stress and strain, boundary conditions related to physical domain and uniqueness constraint related to the applicability of the solution.

The *infinitesimal strain theory* is a mathematical approach for describing the deformation of a solid body in which the *displacement* and *rotations* of the particles of material are assumed to be much smaller than any relevant dimension of the body. It means the geometry and fundamental properties of the material at each point of the body can be assumed to be unchanged by the deformation. In opposite, the *finite strain theory* deals with strains and rotations which are large enough to change the particles of the body.

The *deformation* of a body has two components: displacement of a *rigid body*[2] and a deformation. The displacement of a rigid body consists of translation and rotation. In this case there is no changes in the shape or size. These kind of changes is due to deformation which will happen to a body from the initial configuration to a deformed one. Indeed, the deformation occurs when the distance between any two particles in the undeformed configuration changes.

Definition 2.0.1. A *deformation* of the reference configuration Ω is a mapping

$$\varphi\colon \Omega \to \mathbb{R}^3 \,, \tag{2.0.1}$$

which is smooth enough, injective on $\partial\Omega$ and preserves the orientation.

By defining $\partial_i = \frac{\partial}{\partial x_i}$, at each point of Ω we have

$$\nabla\varphi := \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 & \partial_3\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 & \partial_3\varphi_2 \\ \partial_1\varphi_3 & \partial_2\varphi_3 & \partial_3\varphi_3 \end{pmatrix} . \tag{2.0.2}$$

The 3×3 matrix $\nabla\varphi$ is the *deformation gradient*, and by noticing the orientation-preserving property of the deformation, we will have the following *orientation preserving condition*

$$\det \nabla\varphi(x) > 0 \,, \tag{2.0.3}$$

for all $x \in \Omega$. This shows that the matrix $\nabla\varphi$ at any point $x \in \Omega$ is invertible as well.
The vector field $u\colon \Omega \to \mathbb{R}^3$, $u = (u_1, u_2, u_3)$, with the following definition is called the *displacement*

$$u(x) = \varphi(x) - x \,, \tag{2.0.4}$$

[1] A body is called *homogeneous*, when all the points have the same property.
[2] A rigid body is a solid body that the changes after the deformation are very small that can be omitted.

Similarly we have the gradient of the displacement like following

$$\nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix}, \tag{2.0.5}$$

and we will have the following relation between the gradients

$$\nabla \varphi = \mathbb{1} + \nabla u\,. \tag{2.0.6}$$

Therefore, for each point $x \in \Omega$, $\varphi(x)$ is the deformed point under the deformation φ which maps to the *deformed configuration* $\varphi(\Omega)$.

More details about the volume, surface and length of the reference configuration and the deformed configuration can be found in [30]. Let us assume that W represents the elastic energy of arbitrary

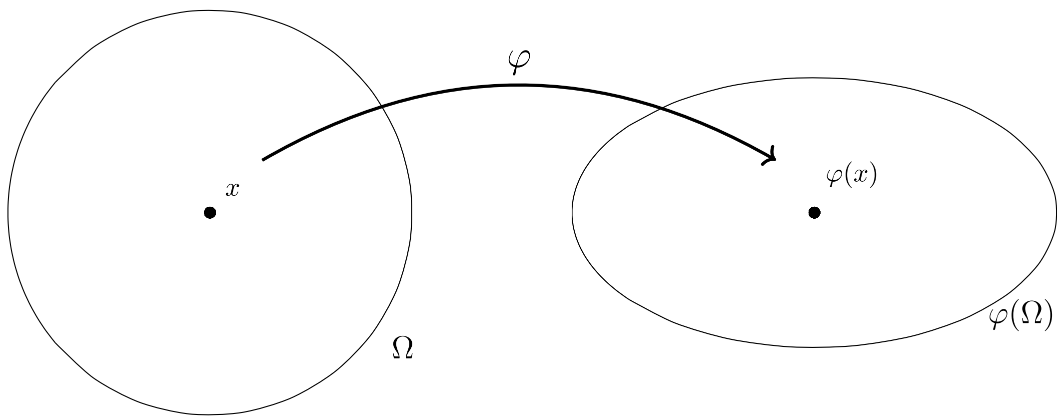


Figure 2.1.: A deformed point x from the reference configuration Ω to the point $\varphi(x)$ in the deformed configuration $\varphi(\Omega)$.

infinitesimal cubes instead of the energy of the whole body. Then the total energy $I(\varphi)$ will be concluded through the summation of the energy of all infinitesimal cubes. We assume that the whole body is homogeneous. It will be helpful because the energy of each cube does not depend on its location in the body. This is the first step to calculate the total energy. In the second step, we will approximate these infinitesimal squares: Assume that x_0 is the center of an infinitesimal square in the body Ω. The Taylor series expression for the deformation φ at point x_0 can be like

$$\varphi(x_0 + h) = \varphi(x_0) + \nabla\varphi(x_0)h + o(h^2)\,, \tag{2.0.7}$$

where $\varphi(x_0)$ is the translation of all infinitesimal squares, $\nabla\varphi(x_0)h$ is a linear term and $o(h^2)$ shows the higher order terms of which for infinitsimal squares, h will vanish for infinitesimal cubes ($h \to 0$).
Hence, it follows that for infinitesimal squares case, dependency of the energy on $\nabla\varphi(x_0)$ is also a good approximation. The total energy of Ω will be obtained from a summation on the energy of all infinitesimal squares

$$\sum_{i,j} W(\nabla\varphi(x_{0,ij}))\delta x_i\,, \qquad \delta x_i \to 0\,, \tag{2.0.8}$$

and

$$I(\varphi(x)) := \int_\Omega W(\nabla\varphi(x))\,dx\,, \tag{2.0.9}$$

for $I\colon C^2(\Omega, \mathbb{R}^3) \to \mathbb{R}$. We will call $I(\varphi)$ the *energy density* and $W(F)$ the *energy function*. We notice that $W\colon \operatorname{GL}^+(3) \to \mathbb{R}$ takes the element with dimension 9, while the domain of I is infinite.

2.1. Strain tensors

Assume that φ is a deformation which is differentiable at every point $x \in \Omega$. Hence, for every point $x + \delta x \in \Omega$ and by using the definition of differentiablitiy, we have

$$\varphi(x+\delta x) - \varphi(x) = \nabla\varphi(x)\delta x + O(\delta x)\,, \tag{2.1.1}$$

and

$$\|\varphi(x+\delta x) - \varphi(x)\|^2 = \langle \nabla\varphi(x)\delta x, \nabla\varphi(x)\delta x\rangle + O(\delta x^2)\,. \tag{2.1.2}$$

We already defined the symmetric right Cauchy-Green strain tensor

$$C := \nabla\varphi^T\nabla\varphi\,, \tag{2.1.3}$$

and the symmetric left Cauchy-Green strain tensor

$$B := \nabla\varphi\nabla\varphi^T\,. \tag{2.1.4}$$

One may define $F := \nabla\varphi$, therefore, $C = F^T F\,, \quad B = F F^T$, by which both have the same characteristic polynomial.
As it is seen, the tensor C is one of the good tools to measure the strain, by which we mean the "changes in form or size". The other class of deformations that they have no strain during their deformation are *rigid deformation*

Definition 2.1.1. The mapping φ is called a *rigid transformation* if it has the form

$$\varphi(x) = a + Q\,x\,, \tag{2.1.5}$$

for $a \in \mathbb{R}, Q \in \mathrm{SO}(3)$ and all $x \in \Omega$.

Indeed, the rigid deformation rotates the reference configuration around the origin and translates it by a vector a, without any strain. Now the question is that, how the deformation $\varphi\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ looks like that we can have $C = F^T F = \nabla\varphi^T\nabla\varphi = \mathbb{1}$. The answers can be

1. **Translation.** The simple case is $\varphi(x) = x + b$, for constant $b = (b_1, b_2, b_3) \in \mathbb{R}^3$

$$\varphi(x_1,x_2,x_3) = \begin{pmatrix} x_1+b_1\\ x_2+b_2\\ x_3+b_3\end{pmatrix}\,, \qquad F = \nabla\varphi = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \mathbb{1} \qquad \Rightarrow \qquad C = F^T F = \mathbb{1}\,. \tag{2.1.6}$$

2. **Linear mapping.** In this case for constant matrix $A \in \mathbb{R}^{3\times3}$, we define $\varphi(x) = A.x$. Then

$$\varphi(x_1,x_2,x_3) = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3\\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3\\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3\end{pmatrix} \quad \Rightarrow \quad F = \nabla\varphi = \begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix} = A\,, \tag{2.1.7}$$

   it means $C = A^T A$. Now the question is that under which condition for A we can have $A^T A = \mathbb{1}$. This situation can happen when $A \in \mathrm{O}(3)$ (an orthogonal matrix). The matrix A can be a rotation or a reflector. Since it is assumed that the matrix is orientation preserving, one may obtain that A is a rotation and $A \in \mathrm{SO}(3)$.

The opposite case can happen under some conditions,

Theorem 2.1.2. *[33, Theorem 1.8-1] Let Ω be an open connected subset of $\mathbb{R}^n$ and let there be given a mapping $\varphi \in C^1(\Omega, \mathbb{R}^n)$ that satisfies*

$$\nabla\varphi(x)^T\nabla\varphi(x) = \mathbb{1}\,, \quad \textit{for all} \quad x \in \Omega\,. \tag{2.1.8}$$

Then there exists a vector $a \in \mathbb{R}^n$ and one orthogonal matrix $Q \in O(n)$ such that

$$\varphi(x) = a + Q\,x\,, \quad \textit{for all} \quad x \in \Omega\,. \tag{2.1.9}$$

Proof. The goal is to show that for every $x_0 \in \Omega$ there exists an open convex subset $V \subset \Omega$ such that for all $x, y \in V$

$$\|\varphi(x) - \varphi(y)\| = \|x - y\| \,. \tag{2.1.10}$$

By applying the mean value theorem, for all $x, y \in V$ we obtain

$$\|\varphi(x) - \varphi(y)\| \leq \sup_{x \in U} |\nabla\varphi(z)| \|x - y\| \,, \tag{2.1.11}$$

where $U \subset V$ is the line between x and y and $|F| = \sup_{\|\xi\|=1} \|F\xi\|_{\mathbb{R}^2} = \lambda_{\max}$ denotes the operator norm of $\nabla\varphi$. The assumption $\nabla\varphi(x)^T \nabla\varphi(x) = \mathbb{1}$ shows that the largest singular value of $C = F^T F$ is 1 which guarantees that the largest singular value of $\nabla\varphi(z)$ is also 1 for all $z \in \Omega$. Hence,

$$\|\varphi(x) - \varphi(y)\| \leq \|x - y\| \,, \qquad \forall x, y \in V \,. \tag{2.1.12}$$

Using the inversion theorem shows that the mapping φ is locally invertible in Ω; it means that there exists an inverse mapping like $\Psi \colon W \to V$ which like φ is continuously differentiable. By using the fact that $\nabla_\xi \Psi^T(\xi) \nabla_\xi \Psi(\xi) = \mathbb{1}$, proves the opposite of the inequality (2.1.12).
Now we prove that $\nabla\varphi$ is constant on V. Let us assume the mapping $G \colon V \times V \to \mathbb{R}$ with

$$G(x, y) = \|\varphi(y) - \ \varphi(x)\|^2 - \|y - x\|^2 = \sum_{k=1}^{n} (\varphi_k(y) - \varphi_k(x))^2 - \sum_{k=1}^{n} (y_k - x_k)^2 \,. \tag{2.1.13}$$

This mapping is constant which means its derivative is zero and it holds

$$\begin{aligned} 0 = \frac{\partial}{\partial x_j} \frac{\partial G}{\partial y_i}(x, y) &= \frac{\partial}{\partial x_j} \Big[\sum_{k=1}^{n} 2(\varphi_k(y) - \varphi_k(x)) \frac{\partial \varphi_k}{\partial y_i}(y) - 2(y_i - x_i) \Big] \\ &= -2 \sum_{k=1}^{n} \frac{\partial \varphi_k}{\partial x_j}(x) \frac{\partial \varphi_k}{\partial y_i}(y) + 2\delta_{ij} \,, \end{aligned} \tag{2.1.14}$$

which for all $i, j \in \{1, \cdots, n\}$ leads to

$$\delta_{ij} = \sum_{k=1}^{n} \frac{\partial \varphi_k}{\partial x_j}(x) \frac{\partial \varphi_k}{\partial y_i}(y) = (\nabla\varphi(x)^T \nabla\varphi(y))_{ij} \iff \mathbb{1} = \nabla\varphi(x)^T \nabla\varphi(y) \iff \nabla\varphi(x) = \nabla\varphi(y) \,, \tag{2.1.15}$$

for all $x, y \in V$. We could show that $\nabla\varphi$ is constant which means $\nabla\varphi = M \in \mathbb{R}^{n \times n}$ is also constant on Ω, since Ω is connected. According to $\nabla\varphi(x)^T \nabla\varphi(x) = \mathbb{1}$, we obtain that the $M \in \mathrm{O}(n)$. Therefore, φ has the following form

$$\varphi(x) = Q\,x + b \,, \tag{2.1.16}$$

for all $x \in \Omega$, $Q \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$, which implies that the gradient of φ, $\nabla\varphi$, is a simple rotation. ■

Theorem 2.1.3. *Let Ω be an open connected subset of $\mathbb{R}^n$ and let there be given two mappings $\varphi, \Psi \in C^1(\Omega, \mathbb{R}^n)$ such that*

$$\nabla\varphi(x)^T \nabla\varphi(x) = \nabla\Psi(x)^T \nabla\Psi(x) \,, \tag{2.1.17}$$

with injective Ψ and $\det \nabla\Psi(x) \neq 0$, *for all $x \in \Omega$. Then there exists a vector $a \in \mathbb{R}^n$ and an orthogonal matrix $Q \in O(n)$ such that*

$$\varphi(x) = a + Q\,\Psi(x) \,, \quad \textit{for all} \quad x \in \Omega \,. \tag{2.1.18}$$

These theorems are significant to understand the importance of the tensor C. From the first theorem we define

$$E := \frac{1}{2}(C - \mathbb{1}) \,, \tag{2.1.19}$$

which is a measure of the deflection between a certain deformation and a rigid deformation. Because, the deformation is rigid if and only if $C = \mathbb{1}$. The tensor E is called *Green-St Venant strain tensor* and can be rewritten as

$$E = \frac{1}{2}(C - \mathbb{1}) = \frac{1}{2}(F^T F - \mathbb{1}) \,. \tag{2.1.20}$$

In the expression of the gradient of the displacement u, we have

$$C = \nabla\varphi^T\nabla\varphi = \mathbb{1} + \nabla u^T + \nabla u + \nabla u^T\nabla u = \mathbb{1} + 2E\,, \tag{2.1.21}$$

and respectively

$$E(u) := E = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T\nabla u)\,. \tag{2.1.22}$$

Theorem 2.1.4. *If $F = \nabla\varphi$ is invertible ($\det F > 0$), then C is symmetric positive definite ($C \in \mathrm{Sym}^+(3)$).*

2.2. Stress tensor

As mentioned already, the external force on a certain part of the body is called *stress.* Regarding to different type of strains, there are different types of stress tensors, which through certain formulas can be exchanged to each other. A tool for measuring the stress is *Cauchy stress* σ (true stress), which is a second order symmetric tensor and defines the value of the stress at each point of the deformed body. That is, if n^φ is an arbitrary direction in a cut on the deformed body, the Cauchy traction $\sigma.n^\varphi \in \mathbb{R}^3$ is the force that is important at the intersection to keep the separated material in its deformed shape. Cauchy's Theorem illustrates that the Cauchy stress is symmetric and the tensor field and the vector field are related by a *partial differential equation* (PDE) in the deformed configuration.
The Cauchy stress tensor is using the deformed coordinate system on behalf of using the initial coordinate system of the reference configuration. However, the deformation itself is not known. Therefore, this is one of the disadvantages of the Cauchy stress tensor. Hence, we may introduce another alternative stress tensor which is called *first Piola-Kirchhoff stress tensor* S_1, which describes the force of the deformed material per original area. The following formula shows how σ and S_1 are transformed into each other

$$S_1(F) = \sigma(F)\,\mathrm{Cof}\,F\,, \tag{2.2.1}$$

where $F = \nabla\varphi$ for deformation φ. Although the Cauchy stress tensor is symmetric, the first Piola-Kirchhoff stress tensor is **not** symmetric. Indeed,

$$S_1(F)^T = (\mathrm{Cof}\,F)^T\sigma(F) = \det F^T \cdot F^{-1} \cdot \frac{S_1(F)}{\det F} \cdot F^T \tag{2.2.2}$$

$$= F^{-1}S_1(F)F^T\,, \tag{2.2.3}$$

where we have used $\sigma(F) = S_1(F)(\mathrm{Cof}\,F)^{-1}$ and $(\mathrm{Cof}\,F)^T = \mathrm{Cof}(F^T) = \det F^T.F^{-1}$.
For the energy function W in the following minimum problem

$$\int_\Omega W(F)\,\mathrm{d}x \to \quad \min \quad F\,, \qquad \varphi|_{\partial\Omega} = \varphi_0\,, \qquad F = \nabla\varphi\,, \tag{2.2.4}$$

and the assumption of hyperelasticity, we can define the first Piola-Kirchhoff stress tensor as following

$$S_1(F) = \mathrm{D}_F W(F)\,. \tag{2.2.5}$$

The other type of stress tensor is the *second Piola-Kirchhoff stress tensor*

$$S_2(F) = F^{-1}S_1(F)\,. \tag{2.2.6}$$

Both Piola-Kirchhoff stress tensors are dependent on the deformation φ, first because of the Piola transform and second because of the Cauchy stress tensor which is also dependent on φ.
There is another type of stress tensor which is called *Biot stress tensor* and is shown by

$$T_{\mathrm{Biot}}(F) = R^T S_1(F)\,. \tag{2.2.7}$$

where R is the rotation in the polar decomposition $F = RU = VR$ [93].

A material is called *isotropic* if its properties remain the same in different directions. In opposite, a material is called *anisotropic* when its properties vary when measured in different directions. For example glass, metals and plastics are kind of isotropic materials while composites and woods tend to show anisotropic properties.

Definition 2.2.1. A constitutive equation $\sigma\colon \mathrm{GL}^+(3) \to \mathrm{Sym}(3)$ is called *isotropic* if for a rotation $Q \in \mathrm{SO}(3)$ we have $\sigma(FQ).n^\varphi = \sigma(F).n^\varphi$ for any arbitrary directions n^φ on the deformed configuration. Hence, a material is isotropic if and only if the Cauchy stress satisfies

$$\sigma(FQ) = \sigma(F)\,, \qquad \forall F \in \mathrm{GL}^+(3)\,, \quad Q \in \mathrm{O}(3)\,, \tag{2.2.8}$$

or equivalently

$$S_1(FQ) = \sigma(FQ)\,\mathrm{Cof}(QF) = \sigma(F)\,\mathrm{Cof}(F)\det(Q)Q^{-T} = S_1(F)\,Q\,. \tag{2.2.9}$$

Indeed, the rotation of the whole body Ω before the actual deformation has no impact on the stress inside the deformed material.

Definition 2.2.2. A constitutive equation $\sigma\colon \mathrm{GL}^+(3) \to \mathrm{Sym}(3)$ is called *objective* if for a rotation $Q \in \mathrm{SO}(3)$ we have $Q\sigma(F).n^\varphi = \sigma(QF).Qn^\varphi$ for any arbitrary directions n^φ on the deformed configuration. Therefore, a material is isotropic if and only if the Cauchy stress satisfies

$$\sigma(QF) = Q\sigma(F)Q^T\,, \qquad F \in \mathrm{GL}^+(3)\,, \quad Q \in \mathrm{SO}(3)\,. \tag{2.2.10}$$

Equivalently,

$$S_1(QF) = \sigma(QF)\,\mathrm{Cof}(QF) = Q\sigma(F)Q^T\det(Q)Q^{-T}\,\mathrm{Cof}(F) = QS_1(F)\,. \tag{2.2.11}$$

In fact, the rotation of the whole body Ω after the actual deformation has no impact on the stress inside the deformed material.

A combination of objectivity and isotropy, can give the following Corollary

Corollary 2.2.3. *For all $F \in \mathrm{GL}^+(3)$ and $Q \in \mathrm{SO}(3)$, on objective and isotropic material it holds*

$$\sigma(Q^TFQ) = Q^T\sigma(FQ)Q = Q^T\sigma(F)Q\,, \qquad S_1(Q^TFQ) = Q^TS_1(FQ) = Q^TS_1(F)Q\,. \tag{2.2.12}$$

Theorem 2.2.4 (Polar decomposition)**.** *Assume that $F \in \mathrm{GL}^+(3)$. There exist positive definite symmetric matrices U, V and $R \in \mathrm{SO}(n)$ such that we have the following unique representation*

$$F = RU = VR\,. \tag{2.2.13}$$

The above representation is called *polar decomposition* for matrix F. For the right Cauchy- Green strain tensor (2.1.3) we have

$$C = F^TF = (RU)^TRU = U^TR^TRU = U^TU = U^2 \quad \Longleftrightarrow \quad U = \sqrt{C} = \sqrt{F^TF}\,. \tag{2.2.14}$$

Simillarly, for the left Cauchy-green strain tensor we obtain

$$B = FF^T = VR(VR)^T = VRR^TV^T = VV^T = V^2 \quad \Longleftrightarrow \quad V = \sqrt{B} = \sqrt{FF^T}\,. \tag{2.2.15}$$

Definition 2.2.5. We call an energy function *isotropic*, if for all $Q \in \mathrm{SO}(3)$ and $F \in \mathrm{GL}^+(3)$ holds

$$W(FQ) = W(F)\,, \tag{2.2.16}$$

and *objective* if

$$W(QF) = W(F)\,. \tag{2.2.17}$$

In an objective energy, the energy value will not be changed by a rotation after the deformation of the body, but isotropy studies a rigid rotation earlier than the existent rotation.

In the nonlinear elasticity theory, we would like to predict the elastic behavior of a body under the force of the boundary and the strength of the body. With $C - \mathbb{1}$ we have found a first candidate for characterizing the distortion in the material. Now we want to know how the independency of distortion from the stored energy of the body looks like.

For the energy function, one of the candidates can be $\|C - \mathbb{1}\|^2$, which is meaningful when C is constant on Ω.

Theorem 2.2.6. *Assume that the energy function W is objective. Then there exists $\widehat{W}\colon \operatorname{Sym}^+(3)\to\mathbb{R}$ such that $W(F)=\widehat{W}(U)$, for all $U\in\operatorname{Sym}^+(3)$.* ■

By using the above Theorem for $C=F^TF$, we may write

$$W(F)=\|F^TF-\mathbb{1}\|^2=\widehat{W}(C)=\|C-\mathbb{1}\|^2\,, \tag{2.2.18}$$

where $C-\mathbb{1}\in\operatorname{Sym}^+(3)$. By $F=\mathbb{1}+\nabla u$, and using the Taylor expansion, the formula will be

$$\begin{aligned}\widehat{W}(C)&=\widehat{W}\big((\mathbb{1}+\nabla u)^T(\mathbb{1}+\nabla u)\big)=\widehat{W}\big(\mathbb{1}+2\operatorname{sym}\nabla u+\nabla u^T\nabla u\big)\\&=\widehat{W}(\mathbb{1})+\langle \mathrm{D}\widehat{W}(\mathbb{1}),2\operatorname{sym}\nabla u+\nabla u\nabla u^T\rangle\\&\quad+\frac{1}{2}\,\mathrm{D}^2\widehat{W}(\mathbb{1})[2\operatorname{sym}\nabla u+\nabla u\nabla u^T,2\operatorname{sym}\nabla u+\nabla u\nabla u^T]+\text{h.o.t}\\&=\widehat{W}(\mathbb{1})+\langle \mathrm{D}\widehat{W}(\mathbb{1}),C-\mathbb{1}\rangle+2\,\mathrm{D}^2\widehat{W}(\mathbb{1})[C-\mathbb{1},C-\mathbb{1}]+\text{h.o.t}\,.\end{aligned} \tag{2.2.19}$$

Obviously in the reference configuration there is no stress and strain. Therefore, $\varphi(x)=x$, which means $F=\mathbb{1}$. So, according to (2.2.18) we have $\widehat{W}(\mathbb{1})=0$. Consequently, $S_1(\mathbb{1})=0, S_2(\mathbb{1})=0$ and $\sigma(\mathbb{1})=0$. We notice that $S_1(C)=\mathrm{D}\widehat{W}(C)$. Hence, $\mathrm{D}\widehat{W}(\mathbb{1})=0$. Now, the relation (2.2.19) can be rewritten as

$$\widehat{W}(C)=2\,\mathrm{D}^2\widehat{W}(\mathbb{1})[C-\mathbb{1},C-\mathbb{1}]+\text{h.o.t}\,, \tag{2.2.20}$$

which for small strains the higher order terms can be omitted and hence

$$\widehat{W}(C)=2\,\mathrm{D}^2\widehat{W}(\mathbb{1})[C-\mathbb{1},C-\mathbb{1}]=\langle \mathbb{C}(C-\mathbb{1}),(C-\mathbb{1})\rangle\,. \tag{2.2.21}$$

where $\mathbb{C}$ is a forth-order tensor with 81 independent components. Since C is assumed to be symmetric, the number of independent components regarding to $C-\mathbb{1}$ will be reduced to 36. On the other hand, $\mathbb{C}$ is self adjont which gives us $\mathbb{C}\sim\mathbb{R}^{21}$. Now assume that W is isotropic. From $W(F)=W(\sqrt{C})=\widehat{W}(C)=:\widetilde{W}(C-\mathbb{1})$ we obtain

$$\begin{aligned}\widetilde{W}(Q(C-\mathbb{1})Q^T)&=\widetilde{W}(QF^T(Q^TQ)FQ^T-QQ^T)=\widetilde{W}((QFQ^T)^T(QFQ^T)-\mathbb{1})\\&=W(QFQ^T)=W(F)=\widetilde{W}(C-\mathbb{1})\,.\end{aligned} \tag{2.2.22}$$

The assumption of isotropy and above formula will help us to reduce the number of independent components further.

We already have seen that the deformation has the form $\varphi(x)=x+u(x)$, where $u(x)$ is the displacement field. We put $\varepsilon:=\operatorname{sym}\nabla u$, and we call it *infinitesimal strain tensor*. Therefore, the relation (2.2.19) can be seen as

$$\begin{aligned}\widehat{W}(C)&=2\,\mathrm{D}^2\widehat{W}(\mathbb{1})[C-\mathbb{1},C-\mathbb{1}]+\text{h.o.t}=2\,\mathrm{D}^2\widehat{W}(\mathbb{1})[\operatorname{sym}\nabla u,\operatorname{sym}\nabla u]+\text{h.o.t}\\&=4\langle\mathbb{C}.\varepsilon,\varepsilon\rangle+\text{h.o.t}=:W_{\text{lin}}(\varepsilon)\,,\end{aligned} \tag{2.2.23}$$

where h.o.t contains also the higher order terms of $\operatorname{sym}\nabla u$. Generally,

$$W_{\text{lin}}\colon \operatorname{Sym}(3)\cong\mathbb{R}^6\to\mathbb{R}\,,\qquad W_{\text{lin}}(\varepsilon)=4\langle\mathbb{C}\,\varepsilon,\varepsilon\rangle\quad\text{where}\quad \mathbb{C}\colon\mathbb{R}^6\to\mathbb{R}^6\,. \tag{2.2.24}$$

We notice that regarding to (2.2.22), when the energy function W is isotropic, the linearized energy W_{lin} will be also isotropic.

Remark 9. *The most important benefit from Theorem 2.2.6 can be reducing the dimension of the entries. We notice that $F\in\mathbb{R}^{3\times3}(\dim=9)$ which means the needed entries in matrix F are nine. Regarding to this theorem we will replace F with another matrix like $U\in\operatorname{Sym}^+(3)(\dim=9)$ which again the number of entries are nine but some of them are repeated, like*

$$U=\begin{pmatrix}a&b&c\\b&d&f\\c&f&e\end{pmatrix}\,. \tag{2.2.25}$$

Lemma 2.2.7. *Every quadratic isotropic function $W\colon\mathbb{R}^{3\times3}\to\mathbb{R}$ has the following unique representation*

$$W(X)=\alpha_1\|\operatorname{dev}\operatorname{sym}X\|^2+\alpha_2\|\operatorname{skew}X\|^2+\frac{\alpha_3}{2}(\operatorname{tr}X)^2\,,\quad \alpha_1,\alpha_2,\alpha_3\in\mathbb{R}\,, \tag{2.2.26}$$

for $X\in\mathbb{R}^{3\times3}$.

Proof. For $\alpha_1 = \alpha_2 = \alpha_3 = 1$ we know the decomposition

$$\|X\|^2 = \|\mathrm{dev}\,\mathrm{sym}\,X + \mathrm{skew}\,X + \frac{1}{3}\,\mathrm{tr}\,X \cdot \mathbb{1}\|^2 = \|\mathrm{dev}\,\mathrm{sym}\,X\|^2 + \|\mathrm{skew}\,X\|^2 + \frac{1}{3}(\mathrm{tr}\,X)^2 \tag{2.2.27}$$

from

$$\mathbb{R}^{n\times n} = (\mathrm{Sym}(n) \cap \mathfrak{sl}(n)) \oplus \mathfrak{so}(n) \oplus \mathbb{R}.\mathbb{1}. \tag{2.2.28}$$

More details about material symmetry [47, p.198] shows that for isotropic solids the number of independent constants for the elasticity tensor $\mathbb{C}$ is three. The relation (2.2.28) shows linearly in-dependency of the terms. Now it is sufficient to show that the decomposition (2.2.26) is invariant under $X \mapsto QXQ^T$ in the sense that

$$\begin{aligned}
\mathrm{tr}(QXQ^T) &= \langle QXQ^T, \mathbb{1}\rangle = \langle QXQ^T, QQ^T\rangle = \langle X, \mathbb{1}\rangle = \mathrm{tr}\,X\,,\\
\mathrm{sym}(QXQ^T) &= \frac{1}{2}\Big(QXQ^T + (QXQ^T)^T\Big) = \frac{1}{2}\left(QXQ^T + QX^TQ^T\right) = Q(\mathrm{sym}\,X)Q^T,\\
\mathrm{skew}(QXQ^T) &= \frac{1}{2}\Big(QXQ^T - (QXQ^T)^T\Big) = \frac{1}{2}\left(QXQ^T - QX^TQ^T\right) = Q(\mathrm{skew}\,X)Q^T,\\
\mathrm{dev}(QXQ^T) &= QXQ^T - \frac{1}{3}\,\mathrm{tr}(QXQ^T)\mathbb{1} = QXQ^T - \frac{1}{3}(\mathrm{tr}\,X)QQ^T = Q\Big(X - \frac{1}{3}\,\mathrm{tr}\,X\Big)Q^T = Q(\mathrm{dev}\,X)Q^T,\\
\|QXQ^T\|^2 &= \langle QXQ^T, QXQ^T\rangle = \langle Q^TQXQ^TQ, X\rangle = \langle X, X\rangle = \|X\|^2.
\end{aligned}$$

■

Now by using this lemma and knowing the fact that $\varepsilon \in \mathrm{Sym}(3)$, we may have the following representation for W_{lin}

$$W_{\mathrm{lin}}(\varepsilon) = \alpha_1\|\mathrm{dev}\,\mathrm{sym}\,\varepsilon\|^2 + \frac{\alpha_3}{2}(\mathrm{tr}\,\varepsilon)^2\,. \tag{2.2.29}$$

Theorem 2.2.8. *The general quadratic isotropic expression for* $\varepsilon = \mathrm{sym}\,\nabla u \in \mathrm{Sym}(3)$ *is*

$$W_{\mathrm{lin}}(\varepsilon) = \mu\|\mathrm{dev}\,\mathrm{sym}\,\varepsilon\|^2 + \frac{\kappa}{2}(\mathrm{tr}\,\varepsilon)^2 = \mu\|\varepsilon\|^2 + \frac{\lambda}{2}(\mathrm{tr}\,\varepsilon)^2\,, \tag{2.2.30}$$

with constants $\mu, \lambda \in \mathbb{R}$.

The constants μ, λ are called the Lamé constants. The parameter μ is also known as the *shear modulus* and κ is the infinitesimal bulk modulus.
Notice that in the three-dimensional case

$$\|\varepsilon\|^2 = \|\mathrm{dev}\,\varepsilon + \frac{1}{3}\,\mathrm{tr}(\varepsilon)\mathbb{1}\|^2 = \|\mathrm{dev}\,\varepsilon\|^2 + \frac{1}{9}\,\mathrm{tr}(\varepsilon)^2\langle\mathbb{1},\mathbb{1}\rangle = \|\mathrm{dev}\,\varepsilon\|^2 + \frac{1}{3}\,\mathrm{tr}(\varepsilon)^2\,, \tag{2.2.31}$$

which gives

$$\|\mathrm{dev}\,\varepsilon\|^2 = \|\varepsilon\|^2 - \frac{1}{3}\,\mathrm{tr}(\varepsilon)^2\,. \tag{2.2.32}$$

By inserting this formula in (2.2.30) we have

$$\begin{aligned}
W_{\mathrm{lin}}(\varepsilon) &= \mu\|\varepsilon\|^2 + \frac{\lambda}{2}\,\mathrm{tr}(\varepsilon)^2\\
&= \mu\|\mathrm{dev}\,\varepsilon\|^2 + \frac{\mu}{3}\,\mathrm{tr}(\varepsilon)^2 + \frac{\lambda}{2}\,\mathrm{tr}(\varepsilon)^2 = \mu\|\mathrm{dev}\,\varepsilon\|^2 + \frac{2\mu+3\lambda}{6}\,\mathrm{tr}(\varepsilon)^2\\
&= \mu\|\mathrm{dev}\,\varepsilon\|^2 + \frac{\kappa}{2}\,\mathrm{tr}(\varepsilon)^2\,,
\end{aligned} \tag{2.2.33}$$

where the bulk modulus is $\kappa = \frac{2\mu+3\lambda}{3}$. In fact, κ is a measure of the resistance of the body to compression. The Cauchy stress tensor related to the linear energy can be obtained from the formula

$$\sigma^{\mathrm{lin}}(\varepsilon) = \mathrm{D}_\varepsilon W_{\mathrm{lin}}(\varepsilon)\,. \tag{2.2.34}$$

Therefore,

$$\sigma(\varepsilon) = 2\mu\,\varepsilon + \lambda\,\mathrm{tr}(\varepsilon)\mathbb{1} = 2\mu\,\mathrm{dev}\,\varepsilon + \kappa\,\mathrm{tr}(\varepsilon)\mathbb{1}\,. \tag{2.2.35}$$

We have the following theorem from [32, Theorem 3.2-3]

Theorem 2.2.9 (Singular value decomposition of a matrix)**.** *Let F be an arbitrary real 3×3 matrix, with singular values $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+$. Then there exist orthogonal matrices $R_1, R_2 \in \mathrm{SO}(3)$ such that*

$$F = R_1 \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) R_2. \tag{2.2.36}$$

Lemma 2.2.10. *Assume that $W\colon \mathrm{GL}^+(3) \to \mathbb{R}$ is any isotropic and objective energy function. Then, there exist a unique mapping $g\colon \mathbb{R}^3_+ \to \mathbb{R}$ with $W(F) = g(\lambda_1, \lambda_2, \lambda_3)$ for all $F \in \mathrm{GL}^+(3)$ and for singular values $\lambda_1, \lambda_2, \lambda_3$. The mapping g is invariant under permutation of its arguments.*

Proof. We assumed that W is isotropic and objective. Therefore, for all $Q_1, Q_2 \in \mathrm{SO}(3)$, we may write

$$W(F) = W(Q_1 F Q_2) = W(Q_1 R_1 \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) R_2 Q_2)\,. \tag{2.2.37}$$

Let us select $Q_1 = R_1^T$ and $Q_2 = R_2^T$. Then

$$W(F) = W(R_1^T R_1 \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) R_2 R_2^T) = W(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)) := g(\lambda_1, \lambda_2, \lambda_3)\,, \tag{2.2.38}$$

where $g\colon \mathbb{R}^3 \to \mathbb{R}$. ■

One can see that this lemma is about reducing the dimension. Obviously, W applies on a space with 9 dimension, while g needs only to apply on 3 dimensional space.

Assume that F is an arbitrary matrix of order n and assume that λ_i, for $1 \le i \le n$, denotes the n eigenvalue of the symmetric and positive definite matrix $F^T F$. The n numbers

$$v_i(F) := \sqrt{\lambda_i(F^T F)}\,, \qquad 1 \le i \le n\,, \tag{2.2.39}$$

are called the *singular values* of the matrix F. For the case $n = 3$ one may write

$$v_1(F) = \sqrt{\lambda_1(F^T F)}\,, \quad v_2(F) = \sqrt{\lambda_2(F^T F)}\,, \qquad v_3(F) = \sqrt{\lambda_3(F^T F)}\,. \tag{2.2.40}$$

Theorem 2.2.11. *Let F be an arbitrary real square matrix with singular values $v_i(F)$. Then there exist orthogonal matrices P and Q depending on v_i such that*

$$F = P(\operatorname{diag} v_i(F)) Q^T\,. \tag{2.2.41}$$

An immediate consequence of this theorem shows that the two matrices $F^T F$ and $F F^T$ are always orthogonaly equivalent.

Corollary 2.2.12. *Let F be a square matrix with $\det F > 0$. Then there exists a singular value decomposition like*

$$F = R_1 \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) R_2\,, \quad \textit{with} \quad R_1, R_2 \in \mathrm{SO}(3)\,, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+\,. \tag{2.2.42}$$

3. Shell theory

The theory of shells is an important branch of the theory of deformable solids. By a *shell* we mean two outer curved surfaces which are somehow parallel to each other and they have a common inner surface in the thickness $\pm\frac{h}{2}$, where we assume that the thickness h is small and perpendicular to the curved surface. The inner surface is called *mid-surface*. The deformation of the shell can be recovered according to the deformation of the mid-surface. Also, the stress system of the shell can be referred to the stress of the mid-surface. If the shells are designed very suitable, they can tolerate large loads, for example, airplane's wings. In general, the shell and plate theories are purposed for the study of *thin bodies*, in other word, bodies in which the thickness in one direction is much smaller than the two other dimensions which are orthogonal to the direction of thickness. As simple examples of shells we may mention, vehicle bodies in automotive industry, cell walls, biological membranes, composite material and plates.

Indeed, a shell has all the specifications of a plate with an extra specification which is curvature. Note that the structure of a shell is more complicated than a plate and it is because of the relation between the bending and stretching. Shells can be divided in different types based on their curvature, for example cylindrical and spherical. However, shells are categorized in two different classes, thin shells which is our subject in this dissertation and thick shells. As we mentioned already, a shell is thin when the ratio of the thickness over the radius of curvature of the midsurface is small enough to neglect (max $\left(\frac{h}{R}\right) \leq \frac{1}{20}$, where R is the radius of the curvature of the midsurface).

3.1. Dimensional reduction

Dimension reduction is changing the data from a higher dimensional spaces to a lower one. The dimensional reduction of a given continuum-mechanical model is an old subject and it has seen many "solutions". One of the methods is the *derivation approach*, i.e., reducing a given three dimensional model via physically reasonable constitutive assumptions on the kinematics to a two dimensional model as opposed to either the *intrinsic approach* , which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the *asymptotic methods*, which tries to establish two-dimensional equations by formal expansion of the three dimensional solution in power series in terms of a small parameter. The intrinsic approach is closley related to the *direct approach* which takes the shell to be a two-dimensional directed medium in the sense of a *restricted Cosserat-surface.*
The philosophy behind the derivation approach is expressed by W. T. Koiter [72, p. 93]: "Any two-dimensional theory of thin shells is necessarily of an approximate character. An exact two-dimensional theory of shells can not exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional. ... Since the theory we have to deal with is approximate in character, we feel that extreme rigor in its development is hardly desirable. ... Flexible bodies like thin shells require a flexible approach." As well as in [65, p-58] it is mentioned: "The theory of Cosserat is exact, but shell theory derived from the three-dimensional equations is approximate. It may be a matter of taste, but we prefer to regard an exact theory as more fundamental. The Cosserat theory of shells (Cosserat surface) is on a comparable footing with any exact three-dimensional continuum theory."
The basic task of any shell theory is a consistent reduction of some presumably "exact" 3D-theory to 2D. Actually the minimization problem will be adopted from the original physical space to a "shell-like" theory, i.e, the new thin domain. A thorough mathematical analysis of linear, infinitesimal displacement shell theory, based on asymptotic methods, is found in [34]. A detailed presentation of the classical shell theories can be found in [79]. One of the most famous theories in the field of reduction is *Reissner-Mindlin plate theory* which is an extension of *Kirchhoff-Love* theory. The Kirchhoff-Love theory considers shear deformations through the thickness of the plate while the Reissner-Mindlin theory considers the deformations and stresses in a plate which its thickness is very smaller than the planar dimensions. They are intended for some thick plates in which their normal vector will remain straight but after deformation it is not necessarily perpendicular to the mid-surface.
Actually, by introducing a proper *ansatz*, one can reduce the dimension from 3 to 2 (engineering method). Indeed, concerning the boundary conditions in any energy model, we may define the ansatz, and by in-

serting the ansatz in the main formula, we will get a reduction in the dimension of the energy and have the homogenized energy in 2-dimension.

4. Linear and nonlinear scaling

As it is already mentioned, the goal of dimension reduction is to reformulate the original problem on a thin 3-D domain, as well as shell theory. Therefore, a problem which is defined on a physical space $\mathbb{E}$ including units of measurement will be adapted to a plate-like theory. This means we are given a three-dimensional *thin domain* $\Omega_h \subset \mathbb{R}^3$ with

$$\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}], \tag{4.0.1}$$

where ω is a bounded Lipschitz domain in $\mathbb{R}^2$ with Lipschitz boundary and $h > 0$ is the *non-dimensional relative characteristic thickness* with $h \ll 1$. Assume the following boundary conditions on $\partial\omega \times (-h, h)$

$$\begin{aligned} u_\alpha(h)(x) &= \zeta_{\alpha\beta} x_\beta\,, \\ u_3(h)(x) &= 0\,, \qquad\qquad x = (x_\gamma, x_3) \in \partial\omega \times (-h, h)\,, \end{aligned} \tag{4.0.2}$$

where u is the displacement field, the index 3 denotes the transverse direction and Greek indices, α, β, γ range between 1 and 2. By using this conditions, we may write the minimizing elastic energy like

$$\int_{\Omega_h} W(\nabla u)\, dx\,, \tag{4.0.3}$$

where u satisfies the boundary conditions and W is a homogeneous (x_3-independent) elastic energy density.
Scaling of independent and (or) dependent variables is the usual first step when we are dealing with the dimension reduction thin domain.
The type of the scaling is crucial for the application of the Γ-convergence. The most effect of choosing a right scaling can be seen finally in defining the recovery sequence for Γ-$\limsup$ condition, in the definition of Γ-convergence.
There are two different type of scalings: *nonlinear* or *natural scaling* and the other one is *linear elasticity scaling.*
The nonlinear or natural scaling for a vector field $z\colon \Omega_h \longrightarrow \mathbb{R}^3$ is $z^\natural\colon \Omega_1 \longrightarrow \mathbb{R}^3$, where only the independent variables will be scaled

$$\begin{aligned} &x_1 = \eta_1\,, \quad x_2 = \eta_2\,, \quad x_3 = h\eta_3\,, \\ &z^\natural\Big(x_1, x_2, \frac{1}{h}x_3\Big) := z(x_1, x_2, x_3)\,, \qquad \text{nonlinear scaling}\,. \end{aligned} \tag{4.0.4}$$

The gradient of z with respect to $x = (x_1, x_2, x_3)$ is

$$\begin{aligned} \nabla_x z(x_1, x_2, x_3) &= \Big(\partial_{\eta_1} z^\natural(\eta_1, \eta_2, \eta_3) \mid \partial_{\eta_2} z^\natural(\eta_1, \eta_2, \eta_3) \mid \frac{1}{h}\partial_{\eta_3} z^\natural(\eta_1, \eta_2, \eta_3)\Big) \\ &= \begin{pmatrix} \partial_{\eta_1} z_1^\natural(\eta) & \partial_{\eta_2} z_1^\natural(\eta) & \frac{1}{h}\partial_{\eta_3} z_1^\natural(\eta) \\ \partial_{\eta_1} z_2^\natural(\eta) & \partial_{\eta_2} z_2^\natural(\eta) & \frac{1}{h}\partial_{\eta_3} z_2^\natural(\eta) \\ \partial_{\eta_1} z_3^\natural(\eta) & \partial_{\eta_2} z_3^\natural(\eta) & \frac{1}{h}\partial_{\eta_3} z_3^\natural(\eta) \end{pmatrix} := \nabla_\eta^h z^\natural(\eta)\,. \end{aligned} \tag{4.0.5}$$

In *linear* scaling the behavior of the components is different. The in-plane components x_1, x_2 of the vector field are independent and the out-of-plane component x_3 will be dependent on the scaling. That is,

$$\begin{aligned} &x_1 = \eta_1, \qquad x_2 = \eta_2, \qquad x_3 = h\eta_3, \\ &\begin{pmatrix} z_1^\flat(x_1, x_2, \frac{1}{h}x_3) \\ z_2^\flat(x_1, x_2, \frac{1}{h}x_3) \\ z_3^\flat(x_1, x_2, \frac{1}{h}x_3) \end{pmatrix} := \begin{pmatrix} z_1(x_1, x_2, x_3) \\ z_2(x_1, x_2, x_3) \\ h z_3(x_1, x_2, x_3) \end{pmatrix}, \qquad \text{linear scaling}\,. \end{aligned} \tag{4.0.6}$$

The related gradient of z with respect to $x = (x_1, x_2, x_3)$ is as follows

$$\begin{aligned}\nabla_x z(x_1, x_2, x_3) &= \Big(\partial_{\eta_1} z^\flat(\eta_1, \eta_2, \eta_3) \mid \partial_{\eta_2} z^\flat(\eta_1, \eta_2, \eta_3) \mid \frac{1}{h}\partial_{\eta_3} z^\flat(\eta_1, \eta_2, \eta_3)\Big) \\ &= \begin{pmatrix} \partial_{\eta_1} z_1^\flat(\eta) & \partial_{\eta_2} z_1^\flat(\eta) & \frac{1}{h}\partial_{\eta_3} z_1^\flat(\eta) \\ \partial_{\eta_1} z_2^\flat(\eta) & \partial_{\eta_2} z_2^\flat(\eta) & \frac{1}{h}\partial_{\eta_3} z_2^\flat(\eta) \\ \frac{1}{h}\partial_{\eta_1} z_3^\flat(\eta) & \frac{1}{h}\partial_{\eta_2} z_3^\flat(\eta) & \frac{1}{h^2}\partial_{\eta_3} z_3^\flat(\eta) \end{pmatrix} := \nabla_\eta^h z^\flat(\eta). \end{aligned} \tag{4.0.7}$$

5. The three dimensional Cosserat model

If a three-dimensional elastic body is very thin in one direction, it has special load-bearing capacities. Due to the geometry, it is always tempting to try to come up with simplified equations for this situation. The ensuing theory is subsumed under the name shell theory. We speak of a flat shell problem if the reference configuration is flat, i.e., the undeformed configuration is given by $\Omega_h = \omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, with $\omega \subset \mathbb{R}^2$ and $h \ll 1$, and of a shell (or curvy shell) if the reference configuration is curvy, in the sense that the undeformed configuration is given by $\Omega_\xi = \Theta(\Omega_h)$, with Θ a C^1-diffeomorphism $\Theta\colon \mathbb{R}^3 \to \mathbb{R}^3$.

There are many different ways to mathematically describe the response of shells and of obtaining two-dimensional field equations. One method is called the *derivation approach*. The idea of this method is reducing the dimension of a given 3 dimensional model to 2 dimensions through physically reasonable constitution assumptions on the kinematics [72]. Neff has introduced this derivation procedure based on the geometrically nonlinear Cosserat model in his habilitation thesis [85, 80]. The other approach is the *intrinsic approach* which from the beginning views the shell as a two-dimensional surface and refers to methods from differential geometry [5, 7, 65]. The *asymptotic method* seeks, by using the formal expansion of the three-dimensional solution in power series in terms of a small thickness parameter to establish two-dimensional equations. Moreover, the *direct approach* [64] assumes that the shell is a two-dimensional medium which has additional extrinsic directors in the concept of a restricted Cosserat surface ([13, 35, 37, 36, 38, 46, 65, 101, 23, 24]). Of course, the intrinsic approach is related to the direct approach. More information regarding to this method can be found in [80, 86, 87, 88].

One of the most famous shell theories is the *Reissner-Mindlin membrane-bending model* which is an extension of the *Kirchhoff-Love* membrane-bending model [12] (the Koiter model [11]). The kinematic assumptions in this theory are that straight lines normal to the reference mid-surface remain straight and normal to the mid-surface after deformation. The Reissner-Mindlin theory can be applied for thick plates and it does not require the cross-section to be perpendicular to the axial axes after deformation, i.e. it includes transverse shear. A serious drawback of both these theories is that a geometrically nonlinear, physically linear membrane-bending model is typically not well-posed ([63]) and needs specific modifications [10, 11] to re-establish well-posedness.

There is another powerful tool that one can use to perform the dimensional reduction namely Γ- *convergence*. In this case, a given 3D model is dimensionally reduced via physically reasonable assumptions on the scaling of the energy.

In this regard, one of the first advances in finite elasticity was the derivation of a nonlinear membrane model (energy scaling with h) which is given in [75]. After that, the idea of Γ-convergence was developed in [50, 52, 51, 53], where different scalings on the applied forces are considered, see also [26, 104].

A notorious property of the Γ-limit model based on classical elasticity is its de-coupling of the limit into either a membrane-like (scaling with h) or bending-like problem (scaling with h^3), see e.g. [15, 68].

In this chapter we will use the idea of Γ-convergence to deduce our two-dimensional curvy shell model from a 3-dimensional geometrically nonlinear Cosserat model ([91]). This work is a challenging extension of the Cosserat membrane Γ-limit for flat shells, which was previously obtained by Neff and Chelminski in [90], to the situation of shells with initial curvature.

The Cosserat model was introduced in 1909 by the Cosserat brothers [38, 40, 39]. They imposed a principal of least action, combining the classical deformation $\varphi\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ and an independent triad of orthogonal directors, the microrotation $\overline{R}\colon \Omega \subset \mathbb{R}^3 \to \mathrm{SO}(3)$. Invariance of the energy under superposed rigid body motions (left-invariance under SO(3)) allowed them to conclude the suitable form of the energy as $W = W(\overline{R}^T \nabla\varphi, \overline{R}^T \partial_{x_1}\overline{R}, \overline{R}^T \partial_{x_2}\overline{R}, \overline{R}^T \partial_{x_3}\overline{R})$. The balance of force equation appears by taking variations w.r.t φ and balance of angular momentum follows from taking variations of $\overline{R} \in \mathrm{SO}(3)$. Here, as additional structural assumption we will assume material isotropy, i.e., right-invariance of the energy under SO(3). In addition we will only consider a physically linear version of the model (quadratic energy in suitable strains) which allows a complete and definite representation of the energy, see eq. (5.1.5).

In the geometric description of shells the normal to the midsurface and the tangent plane appear naturally and the Darboux-Frenet-frame can be used. The underlying Cosserat model immediately generalizes this concept in that the additional microrotation field R can replace the Darboux-Frenet frame. The third column of the microrotation matrix R generalizes the normal in a Kirchhoff-Love model and the director in a Reissner-Mindlin model. Note that the Cosserat model allows for global minimizers [85].

Concerning now the thin shell Γ-limit, we choose the nonlinear scaling and concentrate on a $O(h)$-model, i.e. the membrane response. Since, however, the 3-D Cosserat model already features curvature terms (derivatives of the microrotations), these terms "survive" the Γ-limit procedure and scale with h, while dedicated bending- like terms scaling with h^3 do not appear[1].

The major difficulty compared to the flat shell Γ-limit in [90] is therefore the incorporation of the curved reference configuration. This problem is solved by introducing a multiplicative decomposition of the appearing fields into elastic and (compatible) permanent parts. The permanent parts encode the geometry of the curved surface given by Θ. In this way, we are able to avoid completely the use of the intrinsic geometry of the curved shell.

The related Cosserat shell model in [56, 57] is obtained by the derivation approach. There, the 2-dimensional model depends on the deformation of the midsurface $m\colon \omega \to \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s}\colon \omega \to \mathrm{SO}(3)$ for $\omega \subset \mathbb{R}^2$, the same as here. The resulting reduced energy contains a membrane part, membrane-bending part and bending-curvature part, while the Cosserat Γ-limit model obtained in this chapter contains only the membrane energy and the curvature energy separately.

The membrane part is a combination of the shell energy and transverse shear energy and the curvature part includes the 2-dimensional Cosserat-curvature energy of the shell.

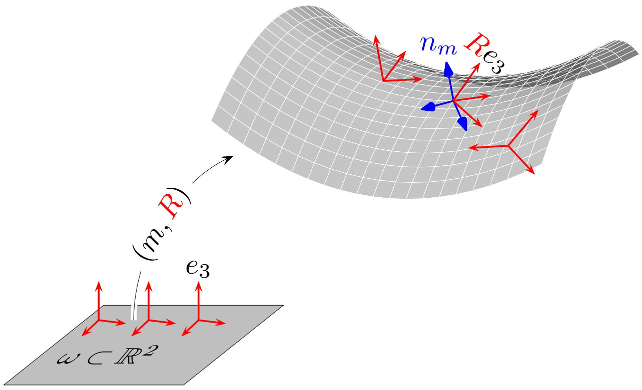


Figure 5.1.: The mapping $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ describes the deformation of a flat midsurface $\omega \subset \mathbb{R}^2$. The Frenet-Darboux frame (in blue, trièdre caché) is tangent to the midsurface m. The independent orthogonal frame mapped by $R \in \mathrm{SO}(3)$ is the trièdre mobile (in red, not necessary tangent to the midsurface). Both fields m and R are coupled in the variational problem. This picture describes the situation of a flat Cosserat shell.

5.1. The variational problem defined on the thin curved reference configuration

Let us consider an elastic material which in its reference configuration fills the three dimensional *shell-like thin* domain $\Omega_\xi \subset \mathbb{R}^3$, i.e., we assume that there exists a C^1-diffeomorphism $\Theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\Theta(x_1, x_2, x_3) := (\xi_1, \xi_2, \xi_3)$ such that $\Theta(\Omega_h) = \Omega_\xi$ and $\omega_\xi = \Theta(\omega \times \{0\})$, where $\Omega_h \subset \mathbb{R}^3$ with $\Omega_h = \omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, and $\omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary $\partial\omega$. The scalar $0 < h \ll 1$ is called *thickness* of the shell, while the domain Ω_h is called *fictitious flat Cartesian configuration* of the body. We consider the following diffeomorphism $\Theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ which describes the curved surface of the shell

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3\, n_0(x_1, x_2)\,, \tag{5.1.1}$$

[1]Observe that the surviving Cosserat curvature is not related to the change of curvature tensor, which measures the change of mean curvature and Gauß curvature of the surface, see Acharya [2], Anicic and Legér [12] as well as the recent work by Silhavy [105] and [58, 59, 60, 62]).

where $y_0\colon \omega \to \mathbb{R}^3$ is a $C^2(\omega)$-function and $n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}$ is the unit normal vector on ω_ξ. Remark that

$$\nabla_x \Theta(x_3) \;=\; (\nabla y_0 | n_0) + x_3 (\nabla n_0 | 0) \;\; \forall x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right), \quad \nabla_x \Theta(0) \;=\; (\nabla y_0 |\, n_0), \quad [\nabla_x \Theta(0)]^{-T} e_3 \;=\; n_0, \tag{5.1.2}$$

and $\det \nabla_x \Theta(0) = \det(\nabla y_0 | n_0) = \sqrt{\det[(\nabla y_0)^T \nabla y_0]}$ represents the surface element. In the following we identify the *W*eingarten map (or shape operator) on $y_0(\omega)$ with its associated matrix by $\mathrm{L}_{y_0} = \mathrm{I}_{y_0}^{-1} \mathrm{II}_{y_0}$, where $\mathrm{I}_{y_0} := [\nabla y_0]^T \nabla y_0 \in \mathbb{R}^{2\times 2}$ and $\mathrm{II}_{y_0} := -[\nabla y_0]^T \nabla n_0 \in \mathbb{R}^{2\times 2}$ are the matrix representations of the *f*irst fundamental form (metric) and the *s*econd fundamental form of the surface $y_0(\omega)$, respectively. Then, the *G*auß curvature K of the surface $y_0(\omega)$ is determined by $\mathrm{K} = \mathit{det}\, \mathrm{L}_{y_0}$ and the *m*ean curvature H through $2\,\mathrm{H} := \mathrm{tr}(\mathrm{L}_{y_0})$. We denote the principal curvatures of the surface by κ_1 and κ_2.

We note that $\det \nabla\Theta(x_3) = 1 - 2\,\mathrm{H}\, x_3 + \mathrm{K}\, x_3^2 = (1-\kappa_1 x_3)(1-\kappa_2 x_3) > 0$. Therefore, $1 - 2\,\mathrm{H}\, x_3 + \mathrm{K}\, x_3^2 > 0$, $\forall\, x_3 \in [-h/2, h/2]$ if and only if $1 > \kappa_1 x_3$ and $1 > \kappa_2 x_3$, for all $x_3 \in [-h/2, h/2]$. These conditions are equivalent with $|\kappa_1|\, \frac{h}{2} < 1$ and $|\kappa_2|\, \frac{h}{2} < 1$, i.e., equivalent with

$$h \max\{ \sup_{(x_1,x_2)\in\omega} |\kappa_1|, \sup_{(x_1,x_2)\in\omega} |\kappa_2| \} < 2\,. \tag{5.1.3}$$

We assume that after a deformation process given by the function $\varphi_\xi : \Omega_\xi \to \mathbb{R}^3$, the reference configuration Ω_ξ is mapped to the deformed configuration $\Omega_c = \varphi_\xi(\Omega_\xi)$. In the Cosserat theory, each point of the reference body is an endowed with three independent orthogonal directors, i.e., with a matrix $\overline{R}_\xi : \Omega_\xi \to \mathrm{SO}(3)$ called the *microrotation* tensor. Let us remark that while the tensor $\mathrm{polar}(\nabla_\xi \varphi_\xi) \in \mathrm{SO}(3)$ of the polar decomposition of $F_\xi := \nabla_\xi \varphi_\xi$ is not independent on φ_ξ, the tensor $\overline{R}_\xi$ of Cosserat theory is independent of $\nabla \varphi_\xi$. In other words, in general, $\overline{R}_\xi \neq \mathrm{polar}(\nabla_\xi \varphi_\xi)$.

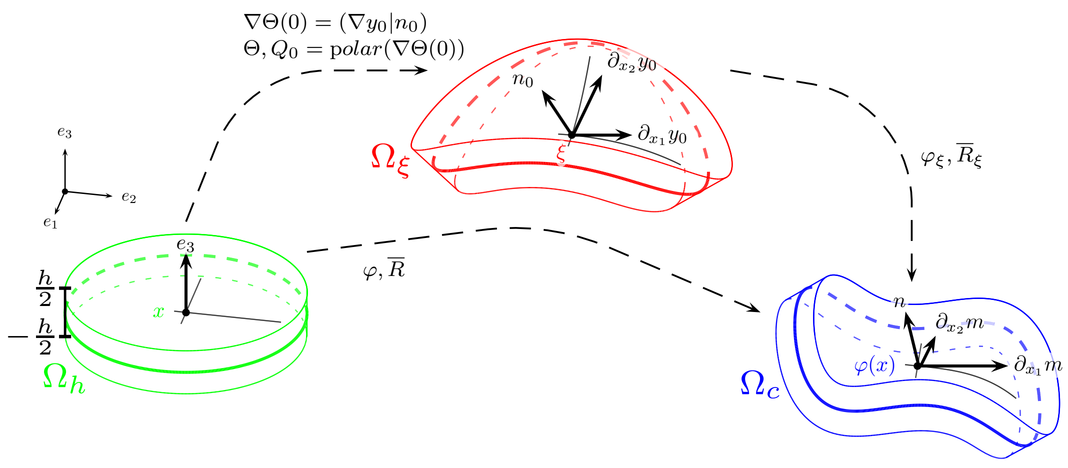


Figure 5.2.: Kinematics of the 3D-Cosserat model. In each point $\xi \in \Omega_\xi$ of the curvy reference configuration, there is the deformation $\varphi_\xi\colon \Omega_\xi \to \mathbb{R}^3$ and the microrotation $\overline{R}_\xi\colon \Omega_\xi \to \mathrm{SO}(3)$. We introduce a fictitious flat configuration Ω_h and refer all fields to that configuration. This introduces a multiplicative split of the total deformation $\varphi\colon \Omega_h \to \mathbb{R}^3$ and total rotation $\overline{R}\colon \Omega_h \to \mathrm{SO}(3)$ into "elastic" parts ($\varphi_\xi\colon \Omega_\xi \to \mathbb{R}^3$ and $\overline{R}_\xi\colon \Omega_\xi \to \mathrm{SO}(3)$) and compatible "plastic" parts (given by $\Theta : \Omega_h \to \Omega_\xi$ and $Q_0 : \Omega_h \to \mathrm{SO}(3)$). The "intermediate" configuration Ω_ξ is compatible by construction.

In a geometrical nonlinear Cosserat elastic 3D model, the deformation φ_ξ and the microrotation $\overline{R}_\xi$ are the solutions of the following *nonlinear minimization problem* on Ω_ξ:

$$I(\varphi_\xi, F_\xi, \overline{R}_\xi, \alpha_\xi) = \int_{\Omega_\xi} \Big[W_{\mathrm{mp}}(\overline{U}_\xi) + W_{\mathrm{curv}}(\alpha_\xi) \Big]\, dV_\xi - \Pi(\varphi_\xi, \overline{R}_\xi) \mapsto \min. \qquad \text{w.r.t} \quad (\varphi_\xi, \overline{R}_\xi)\,, \tag{5.1.4}$$

where

$$
\begin{aligned}
F_\xi &:= \nabla_\xi \varphi_\xi \in \mathbb{R}^{3\times 3} && \text{(the deformation gradient)},\\
\overline{U}_\xi &:= \overline{R}_\xi^T F_\xi \in \mathbb{R}^{3\times 3} && \text{(the non-symmetric Biot-type stretch tensor)},\\
\alpha_\xi &:= \overline{R}_\xi^T \operatorname{Curl}_\xi \overline{R}_\xi \in \mathbb{R}^{3\times 3} && \text{(the second order dislocation density tensor [82])}, \qquad (5.1.5)\\
W_{\rm mp}(\overline{U}_\xi) &:= \mu\, \|\operatorname{dev}\operatorname{sym}(\overline{U}_\xi - \mathbb{1}_3)\|^2 + \mu_c\, \|\operatorname{skew}(\overline{U}_\xi - \mathbb{1}_3)\|^2 + \frac{\kappa}{2}\,[tr(\operatorname{sym}(\overline{U}_\xi - \mathbb{1}_3))]^2 && \text{(physically linear)},\\
W_{\rm curv}(\alpha_\xi) &:= \mu\, L_c^2 \left(a_1\, \|\operatorname{dev}\operatorname{sym}\alpha_\xi\|^2 + a_2\, \|\operatorname{skew}\alpha_\xi\|^2 + a_3\,[\operatorname{tr}(\alpha_\xi)]^2\right) && \text{(quadratic curvature energy)},
\end{aligned}
$$

and $dV(\xi)$ denotes the volume element in the Ω_ξ-configuration. The total stored energy can be seen by $W = W_{\rm mp} + W_{\rm curv}$, with $W_{\rm mp}$ as membrane energy and $W_{\rm curv}$ as curvature energy. Clearly, W depends on the deformation gradient $F_\xi = \nabla_\xi \varphi_\xi$ and the microrotation $\overline{R}_\xi$. As before, the parameters μ and λ are the *Lamé constants* of classical isotropic elasticity, $\kappa = \dfrac{2\,\mu\, + 3\,\lambda}{3}$ is the *infinitesimal bulk modulus*, $\mu_c > 0$ is the *Cosserat couple modulus* and $L_c > 0$ is the *internal length* and responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples (Cosserat models [38]). If not stated otherwise, we assume that $\mu\, > 0$, $\kappa > 0$, $\mu_c > 0$. We also assume that $a_1 > 0, a_2 > 0$ and $a_3 > 0$, which assures the *coercivity* and *convexity of the curvature* energy [90].

The *external loading potential* denoted by $\Pi(\varphi_\xi, \overline{R}_\xi)$, is given by

$$
\Pi(\varphi_\xi, \overline{R}_\xi) = \Pi_f(\varphi_\xi) + \Pi_c(\overline{R}_\xi)\,,
$$

where

$$
\begin{aligned}
\Pi_f(\varphi_\xi) &:= \int_{\Omega_\xi} \langle f, u_\xi\rangle\, dV_\xi = \text{potential of external applied body forces } f\,,\\
\Pi_c(\overline{R}_\xi) &:= \int_{\Gamma_\xi} \langle c, \overline{R}_\xi\rangle\, dS_\xi = \text{potential of external applied boundary couple forces } c\,,
\end{aligned}
$$

with $u_\xi = \varphi_\xi - \xi$ the displacement vector. We will assume that the external loads satisfy in regularity condition:

$$
f \in {\rm L}^2(\Omega_\xi, \mathbb{R}^3)\,, \quad c \in {\rm L}^2(\Gamma_\xi, \mathbb{R}^3)\,, \quad \overline{R}_\xi \in {\rm L}^2(\Omega_\xi, \mathbb{R}^3)\,. \tag{5.1.6}
$$

For simplicity, we consider only Dirichlet-type boundary conditions on $\Gamma_\xi = \gamma_\xi \times \big[-\frac{h}{2}, \frac{h}{2}\big]$, $\gamma_\xi \subset \partial\omega_\xi$, i.e., we assume that

$$
\varphi_\xi = \varphi_\xi^d \qquad \text{on} \quad \Gamma_\xi, \tag{5.1.7}
$$

where φ_ξ^d is a given function on Γ_ξ.

In [80] existence of minimizers is shown for positive Cosserat couple modulus $\mu_c > 0$. The case $\mu_c = 0$ can be handled as well with a slight modification of the curvature energy. The form of the curvature energy $W_{\rm curv}$ is not that originally considered in [81]. Indeed, Neff [81] used a curvature energy expressed in terms of the *third order curvature tensor* $\mathfrak{A}_\xi = (\overline{R}_\xi^T \nabla(\overline{R}_\xi.e_1)\,|\,\overline{R}_\xi^T \nabla(\overline{R}_\xi.e_2)\,|\,\overline{R}_\xi^T \nabla(\overline{R}_\xi.e_3))$. The new form of the energy based on the *second order dislocation density tensor* α_ξ simplifies considerably the representation by allowing to use the orthogonal decomposition

$$
\overline{R}_\xi^T \ \operatorname{Curl}_\xi \ \overline{R}_\xi = \alpha_\xi = \operatorname{dev}\ \operatorname{sym}\ \alpha_\xi + \operatorname{skew}\ \alpha_\xi + \frac{1}{3}\ \operatorname{tr}(\alpha_\xi)\mathbb{1}_3. \tag{5.1.8}
$$

Moreover, it yields an equivalent control of spatial derivatives of rotations [82] and allows us to write the curvature energy in a fictitious Cartesian configuration in terms of the so-called *wryness tensor* [82, 44]

$$
\Gamma_\xi := \Big(\operatorname{axl}(\overline{R}_\xi^T\, \partial_{\xi_1} \overline{R}_\xi)\,|\,\operatorname{axl}(\overline{R}_\xi^T\, \partial_{\xi_2} \overline{R}_\xi)\,|\,\operatorname{axl}(\overline{R}_\xi^T\, \partial_{\xi_3} \overline{R}_\xi)\Big) \in \mathbb{R}^{3\times 3}, \tag{5.1.9}
$$

since (see [82]) the following close relationship between the *wryness tensor* and the *dislocation density tensor* holds

$$
\alpha_\xi \ = \ -\Gamma_\xi^T + \operatorname{tr}(\Gamma_\xi)\,\mathbb{1}_3\,, \qquad \text{or equivalently,} \qquad \Gamma_\xi \ = \ -\alpha_\xi^T + \frac{1}{2}\operatorname{tr}(\alpha_\xi)\,\mathbb{1}_3\,. \tag{5.1.10}
$$

For infinitesimal strains this formula is well-known under the name Nye's formula, and $-\Gamma$ is also called Nye's curvature tensor [95]. Our choice of the *second order dislocation density tensor* α_ξ has some further implications, e.g., the coupling between the membrane part, the membrane-bending part, the bending-curvature part and the curvature part of the energy of the shell model is transparent and will coincide with shell-bending curvature tensors elsewhere considered [45].

Within our assumptions on the constitutive coefficients, together with the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(3)$ and with the definition

$$\begin{aligned} W_{\rm mp}(X) &:= W^\infty_{\rm mp}(\text{sym } X) + \mu_{\rm c}\|\text{skew } X\|^2 \qquad \forall X \in \mathbb{R}^{3\times 3}, \\ W^\infty_{\rm mp}(S) &= \mu\,\|S\|^2 + \frac{\lambda}{2}[\text{tr}(S)]^2 \qquad\qquad \forall S \in \text{Sym}(3), \end{aligned} \tag{5.1.11}$$

it follows that there exist positive constants c_1^+, c_2^+, C_1^+ and C_2^+ such that for all $X \in \mathbb{R}^{3\times3}$ the following inequalities hold

$$\begin{aligned} C_1^+\|S\|^2 \geq\ & W^\infty_{\rm mp}(S) \geq\ c_1^+\|S\|^2 && \forall S \in \text{Sym}(3), \\ C_1^+\|\text{sym } X\|^2 + \mu_{\rm c}\,\|\text{skew } X\|^2 \geq\ & W_{\rm mp}(X) \geq\ c_1^+\|\text{sym } X\|^2 + \mu_{\rm c}\,\|\text{skew } X\|^2 && \forall X \in \mathbb{R}^{3\times3}, \\ C_2^+\|X\|^2 \geq\ & W_{\rm curv}(X) \geq\ c_2^+\|X\|^2 && \forall X \in \mathbb{R}^{3\times3}. \end{aligned} \tag{5.1.12}$$

Here, c_1^+ and C_1^+ denote respectively the smallest and the largest eigenvalues of the quadratic form $W^\infty_{\rm mp}(X)$. Hence, they are independent of $\mu_{\rm c}$. Both $W_{\rm mp}$ and $W_{\rm curv}$ are quadratic convex and coercive functions of $\overline{U}_\xi$ and α_ξ, respectively.

The regularity condition of the external loads allows us to conclude that

$$|\Pi_f(\varphi_\xi)| = |\int_{\Omega_\xi} \langle f, u_\xi\rangle\, dV_\xi| \leq \|f\|_{{\rm L}^2(\Omega_\xi)}\|u_\xi\|_{{\rm L}^2(\Omega_\xi)}\,, \tag{5.1.13}$$

which implies that

$$|\Pi_f(\varphi_\xi)| = |\int_{\Omega_\xi} \langle f, u_\xi\rangle dV_\xi| \leq \|f\|_{{\rm L}^2(\Omega_\xi)}\|u_\xi\|_{{\rm W}^{1,2}(\Omega_\xi)}\,. \tag{5.1.14}$$

Similarly we have

$$|\Pi_c(\overline{R}_\xi)| = |\int_{\Gamma_\xi} \langle c, \overline{R}_\xi\rangle dS_\xi| \leq \|c\|_{{\rm L}^2(\Gamma_\xi)}\|\overline{R}_\xi\|_{{\rm L}^2(\Gamma_\xi)}\,. \tag{5.1.15}$$

Note that $\|\overline{R}_\xi\|^2 = 3$. By using the fact that $\|\overline{R}_\xi\|^2_{{\rm L}^2(\Gamma_\xi)} = (3\,\text{area}\,\Gamma_\xi)$, we get

$$|\Pi(\varphi_\xi, \overline{R}_\xi)| \leq \|f\|_{{\rm L}^2(\Omega_\xi)}\|u_\xi\|_{{\rm W}^{1,2}(\Omega_\xi)} + \|c\|_{{\rm L}^2(\Gamma_\xi)}(3\,\text{area}\,\Gamma_\xi)^{\frac{1}{2}}\,. \tag{5.1.16}$$

This boundedness will be later used in the subject of Γ-convergence.

5.2. Transformation of the problem from Ω_ξ to the fictitious flat configuration Ω_h

The first step in a shell model is to transform the problem to a variational problem defined on the fictitious flat configuration $\Omega_h = \omega \times \big[-\frac{h}{2}, \frac{h}{2}\big]$. This process is going to be done with the help of the diffeomorphism Θ. To this aim, we define the mapping

$$\varphi\colon \Omega_h \to \Omega_c\,, \qquad \varphi(x_1,x_2,x_3) = \varphi_\xi(\Theta(x_1,x_2,x_3))\,.$$

The function φ maps Ω_h (fictitious flat Cartesian configuration) into Ω_c (deformed current configuration). Moreover, we consider the *elastic microrotation* $\overline{Q}_e\colon \Omega_h \to \mathrm{SO}(3)$ similarly defined by

$$\overline{Q}_e(x_1,x_2,x_3) := \overline{R}_\xi(\Theta(x_1,x_2,x_3))\,, \tag{5.2.1}$$

and the *elastic Biot-type stretch tensor* $\overline{U}_e\colon \Omega_h \to \mathrm{Sym}(3)$ is then given by

$$\overline{U}_e(x_1,x_2,x_3) := \overline{U}_\xi(\Theta(x_1,x_2,x_3))\,. \tag{5.2.2}$$

We also have the polar decomposition $\nabla_x\Theta = Q_0\,U_0$, where

$$Q_0 = \mathrm{polar}(\nabla_x\Theta) = \mathrm{polar}([\nabla_x\Theta]^{-T}) \in \mathrm{SO}(3) \quad \text{and} \quad U_0 \in \mathrm{Sym}^+(3)\,. \tag{5.2.3}$$

Now by using (5.2.1), we define the *total microrotation* tensor

$$\overline{R}\colon \Omega_h \to \mathrm{SO}(3), \qquad \overline{R}(x_1,x_2,x_3) = \overline{Q}_e(x_1,x_2,x_3)\,Q_0(x_1,x_2,x_3)\,. \tag{5.2.4}$$

By applying the chain rule for φ one obtains

$$\nabla_x\varphi(x_1,x_2,x_3) = \nabla_\xi\varphi_\xi(\Theta(x_1,x_2,x_3))\,\nabla_x\Theta(x_1,x_2,x_3)\,, \tag{5.2.5}$$

or equivalently

$$F_\xi(\Theta(x_1,x_2,x_3)) = F(x_1,x_2,x_3)\,[\nabla_x\Theta(x_1,x_2,x_3)]^{-1}\,. \tag{5.2.6}$$

Finally the *elastic non-symmetric stretch tensor* expressed on Ω_h is defined by

$$\overline{U}_e = \overline{Q}_e^T F[\nabla_x\Theta]^{-1} = Q_0\overline{R}^T F[\nabla_x\Theta]^{-1}\,. \tag{5.2.7}$$

Note that $\partial_{x_k}\overline{Q}_e = \sum_{i=1}^3 \partial_{\xi_i}\overline{R}_\xi\,\partial_{x_k}\xi_i$, $\partial_{\xi_k}\overline{R}_\xi = \sum_{i=1}^3 \partial_{x_i}\overline{Q}_e\,\partial_{\xi_k}x_i$ and

$$\begin{aligned}
\overline{R}_\xi^T\,\partial_{\xi_k}\overline{R}_\xi &= \sum_{i=1}^3(\overline{Q}_e^T\partial_{x_i}\overline{Q}_e)\,\partial_{\xi_k}x_i = \sum_{i=1}^3\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{ik}\,, \\
\mathrm{axl}\big(\overline{R}_\xi^T\,\partial_{\xi_k}\overline{R}_\xi\big) &= \sum_{i=1}^3 \mathrm{axl}\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{ik}.
\end{aligned} \tag{5.2.8}$$

Thus, we have from the chain rule

$$\begin{aligned}
\Gamma_\xi &= \Big(\sum_{i=1}^3 axl\,\Big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\Big)([\nabla_x\Theta]^{-1})_{i1}\,\Big|\,\sum_{i=1}^3 axl\,\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{i2}\,\Big|\,\sum_{i=1}^3 axl\,\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{i3}\Big) \\
&= \Big(\mathrm{axl}(\overline{Q}_e^T\,\partial_{x_1}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_e^T\,\partial_{x_2}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_e^T\,\partial_{x_3}\overline{Q}_e)\Big)[\nabla_x\Theta]^{-1}.
\end{aligned} \tag{5.2.9}$$

We recall again the Nye's formula

$$\alpha_\xi = -\Gamma_\xi^T + \mathrm{tr}(\Gamma_\xi)\mathbb{1}_3\,, \qquad \text{or} \qquad \Gamma_\xi = -\alpha_\xi^T + \frac{1}{2}\,\mathrm{tr}(\alpha_\xi)\mathbb{1}_3\,. \tag{5.2.10}$$

Define

$$\Gamma_e := \Big(\mathrm{axl}(\overline{Q}_e^T\,\partial_{x_1}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_e^T\,\partial_{x_2}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_e^T\,\partial_{x_3}\overline{Q}_e)\Big), \qquad \alpha_e := \overline{Q}_e^T\,\mathrm{Curl}_x\,\overline{Q}_e. \tag{5.2.11}$$

Using Nye's formula for α_e and Γ_e, we deduce (see [56])

$$\begin{aligned}\alpha_\xi &= [\nabla_x\Theta]^{-T}\alpha_e - \frac{1}{2}\,\mathrm{tr}(\alpha_e)\,[\nabla_x\Theta]^{-T} - \mathrm{tr}(\,[\nabla_x\Theta]^{-T}\alpha_e)\,\mathbb{1}_3 + \frac{1}{2}\,\mathrm{tr}(\alpha_e)\;\mathrm{tr}([\nabla_x\Theta]^{-1})\,\mathbb{1}_3\\ &= [\nabla_x\Theta]^{-T}\alpha_e - \mathrm{tr}(\alpha_e^T\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3 - \frac{1}{2}\,\mathrm{tr}(\alpha_e)\,\Big([\nabla_x\Theta]^{-T} - \mathrm{tr}([\nabla_x\Theta]^{-1})\,\mathbb{1}_3\Big).\end{aligned} \tag{5.2.12}$$

However, we will not use this formula to rewrite the curvature energy in the fictitious Cartesian configuration Ω_h, since it is easier to use (from (5.1.10))

$$\begin{aligned}\mathrm{sym}\;\alpha_\xi &= -\,\mathrm{sym}\;\Gamma_\xi + \mathrm{tr}(\Gamma_\xi)\,\mathbb{1}_3 \;=\; -\,\mathrm{sym}(\Gamma_e\,[\nabla_x\Theta]^{-1}) + \mathrm{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3,\\ \mathrm{dev\;sym}\;\alpha_\xi &= -\,\mathrm{dev\;sym}\;\Gamma_\xi \;=\; -\mathrm{dev\;sym}(\Gamma_e\,[\nabla_x\Theta]^{-1}),\\ \mathrm{skew}\,\alpha_\xi &= -\,\mathrm{skew}\;\Gamma_\xi \;=\; -\,\mathrm{skew}(\Gamma_e\,[\nabla_x\Theta]^{-1}),\\ \mathrm{tr}(\alpha_\xi) &= -\,\mathrm{tr}(\Gamma_\xi) + 3\;\mathrm{tr}(\Gamma_\xi) \;=\; 2\;\mathrm{tr}(\Gamma_\xi) \;=\; 2\;\mathrm{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1}),\end{aligned} \tag{5.2.13}$$

for expressing the curvature energy in terms of $\Gamma_e\,[\nabla_x\Theta]^{-1}$ as

$$W_{\mathrm{curv}}(\alpha_\xi) = \mu\,L_{\mathrm{c}}^2\,\big(a_1\,\|\mathrm{dev\;sym}(\Gamma_e\,[\nabla_x\Theta]^{-1})\|^2 + a_2\,\|\mathrm{skew}(\Gamma_e\,[\nabla_x\Theta]^{-1})\|^2 + 4\,a_3\,[tr(\Gamma_e\,[\nabla_x\Theta]^{-1})]^2\big)\,. \tag{5.2.14}$$

Note that using

$$\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e \;=\; Q_0\,\overline{R}^T\,\partial_{x_i}(\overline{R}\,Q_0^T) \;=\; Q_0(\overline{R}^T\,\partial_{x_i}\overline{R})\,Q_0^T - Q_0(Q_0^T\partial_{x_i}Q_0)\,Q_0^T, \quad i \;=\; 1,2,3\,, \tag{5.2.15}$$

and the invariance ([56], relation (3.12))

$$\mathrm{axl}(Q\,A\,Q^T) \;=\; Q\,\mathrm{axl}(A) \qquad \forall\, Q \in \mathrm{SO}(3) \quad \text{and} \quad \forall\, A \in \mathfrak{so}(3), \tag{5.2.16}$$

we obtain the following form of the wryness tensor

$$\begin{aligned}\Gamma(x_1,x_2,x_3) :&= \Gamma_\xi(\Theta(x_1,x_2,x_3)) \;=\; \Gamma_e\,[\nabla_x\Theta]^{-1}\\ &= Q_0\Big[\Big(\mathrm{axl}(\overline{R}^T\,\partial_{x_1}\overline{R})\,|\,\mathrm{axl}(\overline{R}^T\,\partial_{x_2}\overline{R})\,|\,\mathrm{axl}(\overline{R}^T\,\partial_{x_3}\overline{R})\Big)\\ &\qquad - \Big(\mathrm{axl}(Q_0^T\,\partial_{x_1}Q_0)\,|\,\mathrm{axl}(Q_0^T\,\partial_{x_2}Q_0)\,|\,\mathrm{axl}(Q_0^T\,\partial_{x_3}Q_0)\Big)\Big]\,[\nabla_x\Theta]^{-1}.\end{aligned} \tag{5.2.17}$$

Now the minimization problem on the curved reference configuration Ω_ξ is transformed to the fictitious flat Cartesian configuration Ω_h as follows

$$I = \int_{\Omega_h}\Big[W_{\mathrm{mp}}(\overline{U}_e) + \widetilde{W}_{\mathrm{curv}}(\Gamma)\Big]\det(\nabla_x\Theta)\;dV - \widetilde{\Pi}(\varphi,\overline{Q}_e) \;\mapsto \min. \qquad \text{w.r.t} \quad (\varphi,\overline{Q}_e)\,, \tag{5.2.18}$$

where

$$\begin{aligned}W_{\mathrm{mp}}(\overline{U}_e) &= \;\mu\,\|\mathrm{sym}(\overline{U}_e - \mathbb{1}_3)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U}_e - \mathbb{1}_3)\|^2 + \frac{\lambda}{2}[\mathrm{tr}(\mathrm{sym}(\overline{U}_e - \mathbb{1}_3))]^2\\ &= \mu\,\|\mathrm{dev\,sym}(\overline{U}_e - \mathbb{1}_3)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U}_e - \mathbb{1}_3)\|^2 + \frac{\kappa}{2}[\mathrm{tr}(\mathrm{sym}(\overline{U}_e - \mathbb{1}_3))]^2\,,\\ \widetilde{W}_{\mathrm{curv}}(\Gamma) &= \;\mu\,L_{\mathrm{c}}^2\,\big(a_1\,\|\mathrm{dev\;sym}\,\Gamma\|^2 + a_2\,\|\mathrm{skew}\,\Gamma\|^2 + 4\,a_3\,[tr(\Gamma)]^2\big)\\ &= \;\mu\,L_{\mathrm{c}}^2\,\big(b_1\,\|\mathrm{sym}\;\Gamma\|^2 + b_2\,\|\mathrm{skew}\;\Gamma\|^2 + \,b_3\,[\mathrm{tr}(\Gamma)]^2\big)\;,\end{aligned} \tag{5.2.19}$$

where $b_1 = a_1, b_2 = a_2$, $b_3 = \frac{12a_3 - a_1}{3}$ and $\widetilde{\Pi}(\varphi,\overline{Q}_e) = \widetilde{\Pi}_f(\varphi) + \widetilde{\Pi}_c(\overline{Q}_e)$, with the following forms

$$\begin{aligned}\widetilde{\Pi}_f(\varphi) := \Pi_f(\varphi_\xi) &= \int_{\Omega_\xi}\langle f, u_\xi\rangle\,dV_\xi = \int_{\Omega_h}\langle \widetilde{f},\widetilde{u}\rangle\,dV\,,\\ \widetilde{\Pi}_c(\overline{Q}_e) := \Pi_c(\overline{R}_\xi) &= \int_{\Gamma_\xi}\langle c, \overline{R}_\xi\rangle\,dS_\xi = \int_{\Gamma_h}\langle \widetilde{c},\overline{Q}_e\rangle\,dS\,,\end{aligned} \tag{5.2.20}$$

with $\widetilde{u}(x_i) = \varphi(x_i) - \Theta(x_i)$ the displacement vector, $\overline{R} = \overline{Q}_e\,Q_0$ the total microrotation, the vector fields $\widetilde{f}$ and $\widetilde{c}$ can be determined in terms of f and c, respectively, for instance (see [31, Theorem 1.3.-1])

$$\widetilde{f}(x) = f(\Theta(x))\det(\nabla_x\Theta), \qquad\qquad \widetilde{c}(x) = c(\Theta(x))\det(\nabla_x\Theta). \tag{5.2.21}$$

Note that regarding to the regularity condition (5.1.6), the following regularity conditions will hold as well

$$\widetilde{f} \in \mathrm{L}^2(\Omega_h,\mathbb{R}^3)\,, \quad \widetilde{c} \in \mathrm{L}^2(\Gamma_h,\mathbb{R}^3)\,, \quad \overline{Q}_e \in \mathrm{L}^2(\Gamma_h,\mathbb{R}^3)\,. \tag{5.2.22}$$

The Dirichlet-type boundary conditions (in the sense of the traces) on $\Gamma_\xi = \gamma_\xi \times \big[-\frac{h}{2},\frac{h}{2}\big]$, $\gamma_\xi \subset \partial\omega_\xi$, read on the boundary $\Gamma_h = \gamma \times \big[-\frac{h}{2},\frac{h}{2}\big]$, $\gamma = \Theta^{-1}(\gamma_\xi) \subset \partial\omega$, as $\varphi = \varphi_d^h$ on Γ_h, where $\varphi_d^h = \Theta^{-1}(\varphi_\xi^h)$.

5.3. Construction of the family of functionals I_{h_j}

5.3.1. Nonlinear scaling for the gradient of the deformation and the microrotation

In order to apply the methods of Γ-convergence, the first step is to transform our problem further from Ω_h to a *domain* with fixed thickness $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$. For this goal, scaling of the variables (dependent/independent) would be the first step. However, it is important to know which kind of scaling is suitable for our variables.

In a first step we will apply the nonlinear scaling to the deformation. For $\Omega_1 = \omega \times \Big[-\dfrac{1}{2}, \dfrac{1}{2}\Big] \subset \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$, we define the scaling transformations

$$\begin{aligned} \zeta\colon\ \eta \in \Omega_1 \mapsto \mathbb{R}^3\,, \qquad & \zeta(\eta_1,\eta_2,\eta_3) := (\eta_1,\eta_2,h\,\eta_3)\,,\\ \zeta^{-1}\colon\ x \in \Omega_h \mapsto \mathbb{R}^3\,, \qquad & \zeta^{-1}(x_1,x_2,x_3) := (x_1,x_2,\frac{x_3}{h})\,, \end{aligned} \tag{5.3.1}$$

with $\zeta(\Omega_1) = \Omega_h$. By using the relation (4.0.4) and above transformations we obtain the formula for the transformed deformation φ as

$$\varphi(x_1,x_2,x_3) = \varphi^\natural(\zeta^{-1}(x_1,x_2,x_3)) \quad \forall x \in \Omega_h; \quad \varphi^\natural(\eta) = \varphi(\zeta(\eta)) \quad \forall \eta \in \Omega_1\,,$$

$$\nabla_x \varphi(x_1,x_2,x_3) = \begin{pmatrix} \partial_{\eta_1}\varphi_1^\natural(\eta) & \partial_{\eta_2}\varphi_1^\natural(\eta) & \frac{1}{h}\partial_{\eta_3}\varphi_1^\natural(\eta) \\ \partial_{\eta_1}\varphi_2^\natural(\eta) & \partial_{\eta_2}\varphi_2^\natural(\eta) & \frac{1}{h}\partial_{\eta_3}\varphi_2^\natural(\eta) \\ \partial_{\eta_1}\varphi_3^\natural(\eta) & \partial_{\eta_2}\varphi_3^\natural(\eta) & \frac{1}{h}\partial_{\eta_3}\varphi_3^\natural(\eta) \end{pmatrix} = \nabla_\eta^h \varphi^\natural(\eta) = F_h^\natural. \tag{5.3.2}$$

Now we will do the same process for the microrotation tensor $\overline{Q}_e^\natural\colon \Omega_1 \to \mathrm{SO}(3)$

$$\overline{Q}_e(x_1,x_2,x_3) = \overline{Q}_e^\natural(\zeta^{-1}(x_1,x_2,x_3)) \quad \forall x \in \Omega_h\,; \ \ \overline{Q}_e^\natural(\eta) = \overline{Q}_e(\zeta(\eta))\,, \quad \forall \eta \in \Omega_1\,,$$

as well as for $\nabla_x\Theta(x)$, the matrices of its polar decomposition $\nabla_x\Theta(x) = Q_0(x)U_0(x)$, in the sense that

$$(\nabla_x\Theta)^\natural(\eta) = (\nabla_x\Theta)(\zeta(\eta)), \qquad Q_0^\natural(\eta) = Q_0(\zeta(\eta)), \qquad U_0^\natural(\eta) = U_0(\zeta(\eta))\,. \tag{5.3.3}$$

We also define $\overline{R}^\natural\colon \Omega_1 \to \mathrm{SO}(3)$

$$\overline{R}(x_1,x_2,x_3) = \overline{R}^\natural(\zeta^{-1}(x_1,x_2,x_3)) \quad \forall x \in \Omega_h\,; \qquad \overline{R}^\natural(\eta) = \overline{R}(\zeta(\eta))\,, \quad \forall \eta \in \Omega_1\,.$$

With this, the non-symmetric stretch tensor expressed in a point of Ω_1 is given by

$$\overline{U}_e^\natural = \overline{Q}_e^{\natural,T} F_h^\natural [(\nabla_x\Theta)^\natural]^{-1} = \overline{Q}_e^{\natural,T} \nabla_\eta^h \varphi^\natural(\eta)[(\nabla_x\Theta)^\natural]^{-1}\,. \tag{5.3.4}$$

Since for $\eta_3 = 0$ their values expressed in terms of $(\eta_1,\eta_2,0)$ and $(x_1,x_2,0)$ coincide, we will omit the sign $\cdot^\natural$ and we will understand from the context the variables into discussion, i.e.,

$$\begin{aligned} (\nabla_x\Theta)(0) := (\nabla y_0\,|n_0) = (\nabla_x\Theta)^\natural(\eta_1,\eta_2,0) \equiv (\nabla_x\Theta)(x_1,x_2,0),\\ Q_0(0) := Q_0^\natural(\eta_1,\eta_2,0) \equiv Q_0(x_1,x_2,0), \qquad U_0(0) := U_0^\natural(\eta_1,\eta_2,0) \equiv U_0(x_1,x_2,0). \end{aligned}$$

Therefore, we have

$$\overline{Q}_e^\natural(\eta) = \overline{R}^\natural(\eta)(Q_0^\natural(\eta))^T, \qquad \overline{U}_e^\natural(\eta) = \overline{Q}_e^{\natural,T}(\eta)F_h^\natural(\eta)[(\nabla_x\Theta)^\natural]^{-1} = Q_0^\natural(\eta)\overline{R}^{\natural,T}(\eta)F_h^\natural(\eta)[(\nabla_x\Theta)^\natural]^{-1}\,, \tag{5.3.5}$$

and

$$\Gamma_h^\natural = \Big(\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_1}\overline{Q}_{e,h}^\natural)\,|\,\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)\,|\,\frac{1}{h}\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_3}\overline{Q}_{e,h}^\natural)\Big)[(\nabla_x\Theta)^\natural]^{-1}. \tag{5.3.6}$$

5.3.2. Transformation of the problem from Ω_h to a fixed domain Ω_1

The next step, in order to apply the Γ-convergence technique, is to transform the minimization problem onto the *fixed domain* Ω_1, which is independent from the thickness h. According to the results from the previous subsection, we have the following minimization problem on Ω_1

$$
\begin{aligned}
I_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) &= \int_{\Omega_1} \Big(W_{\rm mp}(\overline{U}_h^\natural) + \widetilde{W}_{\rm curv}(\Gamma_h^\natural) \Big) \det(\nabla_\eta \zeta(\eta)) \det((\nabla_x \Theta)^\natural)\, dV_\eta - \Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural) \\
&= \underbrace{\int_{\Omega_1} h \Big[\Big(W_{\rm mp}(\overline{U}_h^\natural) + \widetilde{W}_{\rm curv}(\Gamma_h^\natural) \Big) \det((\nabla_x \Theta)^\natural) \Big]\, dV_\eta}_{:= J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural)} - \Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural) \mapsto \min \text{ w.r.t } (\varphi^\natural, \overline{Q}_e^\natural)\,, \qquad (5.3.7)
\end{aligned}
$$

where

$$
\begin{aligned}
W_{\rm mp}(\overline{U}_h^\natural) &= \mu\, \|{\rm sym}(\overline{U}_h^\natural - \mathbb{1}_3)\|^2 + \mu_c\, \|{\rm skew}(\overline{U}_h^\natural - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [{\rm tr}({\rm sym}(\overline{U}_h^\natural - \mathbb{1}_3))]^2\,, \\
\widetilde{W}_{\rm curv}(\Gamma_h^\natural) &= \mu\, L_c^2 \Big(a_1\, \|{\rm dev\ sym}\, \Gamma_h^\natural\|^2 + a_2\, \|{\rm skew}\, \Gamma_h^\natural\|^2 + a_3\, [tr(\Gamma_h^\natural)]^2 \Big)\,, \qquad (5.3.8)
\end{aligned}
$$

with $\Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural) = \Pi_f^\natural(\varphi^\natural) + \Pi_c^\natural(\overline{Q}_e^\natural)$,

$$
\begin{aligned}
\Pi_f^\natural(\varphi^\natural) &:= \widetilde{\Pi}_f(\varphi) = \int_{\Omega_h} \langle \widetilde{f}, \widetilde{u} \rangle\, dV = \int_{\Omega_1} \langle \widetilde{f}^\natural, \widetilde{u}^\natural \rangle\, \det(\nabla_\eta \zeta(\eta))\, dV_\eta = h \int_{\Omega_1} \langle \widetilde{f}^\natural, \widetilde{u}^\natural \rangle\, dV_\eta\,, \\
\Pi_c^\natural(\overline{Q}_e^\natural) &:= \widetilde{\Pi}_c(\overline{Q}_e) = \int_{\Gamma_h} \langle \widetilde{c}, \overline{Q}_e \rangle dS = \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_e^\natural \rangle\, \det(\nabla_\eta \zeta(\eta))\, dS_\eta = h \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_e^\natural \rangle\, dS_\eta\,, \qquad (5.3.9)
\end{aligned}
$$

with $\widetilde{f}^\natural(\eta) = \widetilde{f}(\zeta(\eta))$, $\widetilde{u}^\natural(\eta) = \widetilde{u}(\zeta(\eta))$, $\widetilde{c}^\natural(\eta) = \widetilde{c}(\zeta(\eta))$ and $\overline{Q}_e^\natural(\eta) = \overline{Q}_e(\zeta(\eta))$. Here we recall that regarding to the regularity condition (5.2.22), it holds

$$
\widetilde{f}^\natural \in {\rm L}^2(\Omega_1, \mathbb{R}^3)\,, \qquad \widetilde{c}^\natural \in {\rm L}^2(\Gamma_1, \mathbb{R}^3)\,, \qquad \overline{Q}^\natural \in {\rm L}^2(\Gamma_1, \mathbb{R}^3)\,. \qquad (5.3.10)
$$

Therefore, we may write

$$
\begin{aligned}
|\Pi_f^\natural(\varphi^\natural)| &= |h \int_{\Omega_1} \langle \widetilde{f}^\natural, \widetilde{u}^\natural \rangle dV_\eta| \le h \|\widetilde{f}^\natural\|_{{\rm L}^2(\Omega_1)} \|\widetilde{u}^\natural\|_{{\rm L}^2(\Omega_1)}\,, \\
|\Pi_c^\natural(\overline{Q}_e^\natural)| &= |h \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_e^\natural \rangle dS_\eta| \le h \|\widetilde{c}^\natural\|_{{\rm L}^2(\Gamma_1)} \|\overline{Q}_e^\natural\|_{{\rm L}^2(\Gamma_1)}\,, \qquad (5.3.11)
\end{aligned}
$$

and consequently

$$
|\Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural)| \le h \Big[\|\widetilde{f}^\natural\|_{{\rm L}^2(\Omega_1)} \|\widetilde{u}^\natural\|_{{\rm L}^2(\Omega_1)} + \|\widetilde{c}^\natural\|_{{\rm L}^2(\Gamma_1)} \|\overline{Q}_e^\natural\|_{{\rm L}^2(\Gamma_1)} \Big]\,. \qquad (5.3.12)
$$

The Dirichlet-type boundary conditions (in the sense of the trace) on $\Gamma_h = \gamma \times \big[-\frac{h}{2}, \frac{h}{2}\big]$, $\gamma = \Theta^{-1}(\gamma_\xi) \subset \partial\omega$, read on the boundary $\Gamma_1 = \gamma \times \big[-\frac{1}{2}, \frac{1}{2}\big]$ as $\varphi^\natural = \varphi_d^\natural$ on Γ_1, where $\varphi_d^\natural = \Theta^{-1}(\varphi_d^h)$.

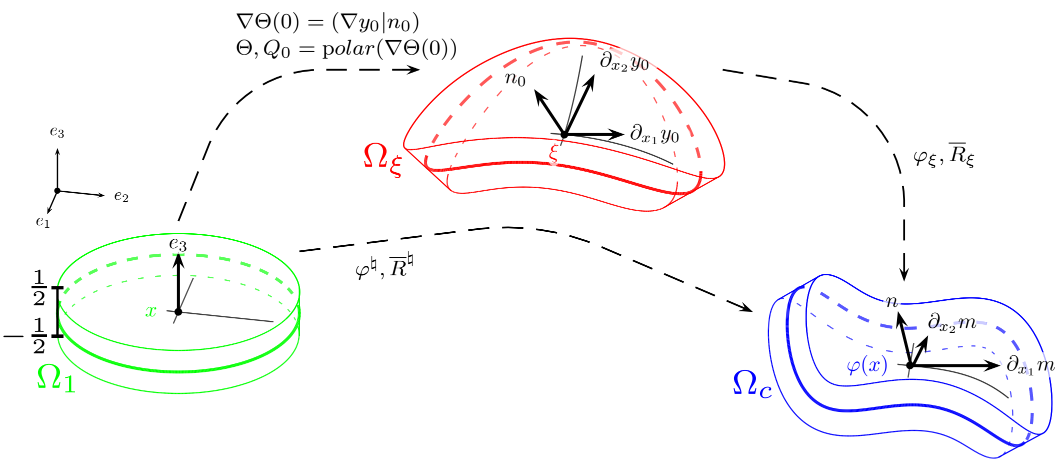


Figure 5.3.: The complete picture of the involved domains. Ω_1 is the fictitious flat domain with unit thickness, Ω_ξ denotes the curved reference configuration, Ω_c is the current deformed configuration. Again, the reference configuration Ω_ξ takes on the role of a compatible intermediate configuration in the multiplicative decomposition.

5.4. Equi-coercivity and compactness of the family of energy functionals

5.4.1. The set of admissible solutions

Due to the scaling, we have obtained a family of functionals

$$J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) = \int_{\Omega_1} h \left[\left(W_{\rm mp}(\overline{U}_h^\natural) + \widetilde{W}_{\rm curv}(\Gamma_h^\natural) \right) \det((\nabla_x \Theta)^\natural) \right] dV_\eta \,, \tag{5.4.1}$$

depending on the thickness h. The next step is to prepare a suitable space X on which the existence of Γ-convergence will be studied. As already mentioned, for applying the techniques of Γ-limit we need to assume that the space X is separable or metrizable. Since working in $\mathrm{H}^1(\Omega_1, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3))$ means to consider the weak topopolgy, which does not give rise to a metric space, we introduce the following spaces:

$$\begin{aligned} X &:= \{(\varphi^\natural, \overline{Q}_e^\natural) \in \mathrm{L}^2(\Omega_1, \mathbb{R}^3) \times \mathrm{L}^2(\Omega_1, \mathrm{SO}(3))\} \,, \\ X' &:= \{(\varphi^\natural, \overline{Q}_e^\natural) \in \mathrm{H}^1(\Omega_1, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3))\} \,, \\ X_\omega &:= \{(\varphi, \overline{Q}_e) \in \mathrm{L}^2(\omega, \mathbb{R}^3) \times \mathrm{L}^2(\omega, \mathrm{SO}(3))\} \,, \\ X'_\omega &:= \{(\varphi, \overline{Q}_e) \in \mathrm{H}^1(\omega, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\omega, \mathrm{SO}(3))\} \,. \end{aligned} \tag{5.4.2}$$

We also consider the following admissible sets

$$\begin{aligned} \mathcal{S}' &:= \{(\varphi, \overline{Q}_e) \in \mathrm{H}^1(\Omega_1, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3)) \,\big|\, \varphi|_{\Gamma_1}(\eta) = \varphi_d^\natural(\eta)\} \,, \\ \mathcal{S}'_\omega &:= \{(\varphi, \overline{Q}_e) \in \mathrm{H}^1(\omega, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\omega, \mathrm{SO}(3)) \,\big|\, \varphi|_{\partial\omega}(\eta_1, \eta_2) = \varphi_d^\natural(\eta_1, \eta_2, 0)\} \,. \end{aligned} \tag{5.4.3}$$

By the imbedding theorem ([32], Theorem 6.1-3), the imbedding $X' \subset X$ is true and clearly $X_\omega \subset X$, $X'_\omega \subset X'$ [2].

The functionals in our analysis are obtained by extending the functionals J_h (respectively I_h) to the entire space X and to take their averages over the thickness, through

$$\begin{aligned} \mathcal{I}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) &= \begin{cases} \dfrac{1}{h} \, I_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \\ &= \begin{cases} \dfrac{1}{h} \, J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) - \dfrac{1}{h} \Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \end{aligned} \tag{5.4.4}$$

[2] Since $\infty > \int_\omega |\varphi|^2 \, dx \, dy = \int_\omega \int_{-1/2}^{1/2} |\varphi|^2 \, dz \, dx \, dy = \int_{\Omega_1} |\varphi|^2 \, dV$, which means any element from X_ω, belongs to X as well.

The main aim of the current chapter is to find the Γ-limit of the family of functional $\mathcal{I}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural)$, i.e., to obtain an energy functional expressed only in terms of the weak limit of a subsequence of $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in X$, when h_j goes to zero. In other words, as we will see, to construct an energy function depending only on quantities defined on the midsurface of the shell-like domain, see Figure 5.4 . As a first step we consider the functionals

$$\mathcal{J}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) = \begin{cases} \dfrac{1}{h} J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \tag{5.4.5}$$

5.4.2. Equi-coercivity and compactness of the family $\mathcal{J}_h^\natural$

Theorem 5.4.1. *Assume that the initial configuration is defined by a continuous injective mapping* $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ *which admits an extension to* $\overline{\omega}$ *into* $C^2(\overline{\omega}; \mathbb{R}^3)$ *such that* $\det[\nabla_x \Theta(0)] \geq a_0 > 0$ *on* $\overline{\omega}$, *where* a_0 *is a positive constant, and assume that the boundary data satisfies the conditions*

$$\varphi_d^\natural = \varphi_d\big|_{\Gamma_1} \text{ (in the sense of traces) for } \varphi_d \in H^1(\Omega_1, \mathbb{R}^3). \tag{5.4.6}$$

Consider a sequence $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in X$, *such that the energy functionals* $\mathcal{J}_{h_j}^\natural(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural)$ *are bounded as* $h_j \to 0$. *Let the constitutive parameters satisfy*

$$\mu > 0, \qquad \kappa > 0, \qquad \mu_c > 0, \qquad a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0. \tag{5.4.7}$$

Then the sequence $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural)$ *admits a subsequence which is weakly convergent to* $(\varphi_0^\natural, \overline{Q}_{e,0}^\natural) \in X_\omega$.

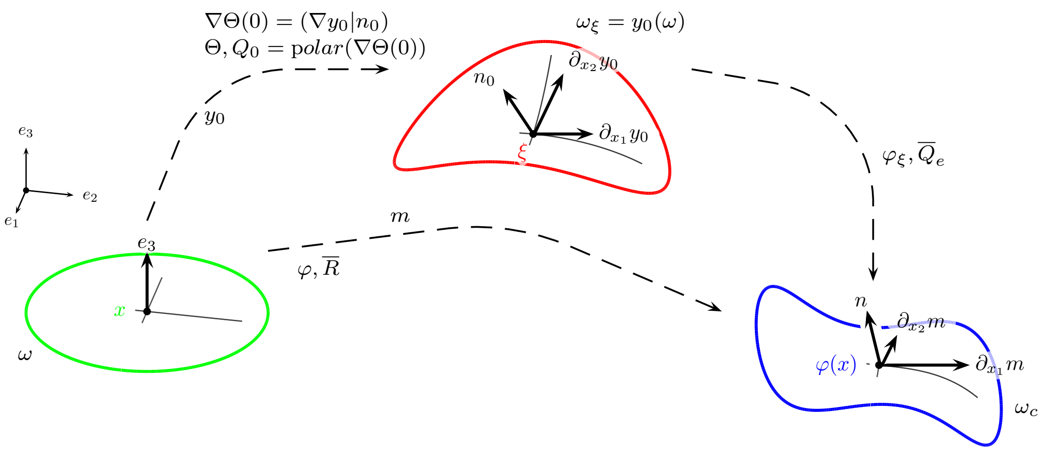


Figure 5.4.: Kinematics of the dimensionally reduced Cosserat shell model. All fields are referred to two-dimensional surfaces. The geometry of the curved surface ω_ξ is fully encoded by the map Θ. Instead of the elastic deformation starting from ω_ξ, the total deformation m from the fictitious flat midsurface ω is considered, likewise for the total rotation $\overline{R}$.

Proof. Consider the sequence $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in X$, such that the energy functionals $\mathcal{J}_{h_j}^\natural(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural)$ are bounded as $h_j \to 0$. Obviously this implies that $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in \mathcal{S}'$ for all h_j. We have

$$\begin{aligned} 2\Big(\|\overline{U}_{h_j}^\natural - \mathbb{1}_3\|^2 + \|\mathbb{1}_3\|^2\Big) &\geq (\|\overline{U}_{h_j}^\natural - \mathbb{1}_3\| + \|\mathbb{1}_3\|)^2 \geq \|\overline{U}_{h_j}^\natural\|^2 = \|\overline{Q}_e^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\|^2 \\ &= \langle \overline{Q}_e^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}, \overline{Q}_e^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1} \rangle \\ &= \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\|^2. \end{aligned} \tag{5.4.8}$$

Thus, we deduce with $(5.4.8)_1$

$$\|\overline{U}_{h_j}^\natural - \mathbb{1}_3\|^2 \geq \frac{1}{2} \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\|^2 - 3. \tag{5.4.9}$$

But

$$\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\| = \|\nabla_\eta^{h_j}\varphi^\natural_{h_j}[(\nabla_x\Theta)^\natural(\eta)]^{-1}[(\nabla_x\Theta)^\natural(\eta)]\| \le \|\nabla_\eta^{h_j}\varphi^\natural_{h_j}[(\nabla_x\Theta)^\natural(\eta)]^{-1}\|\cdot\|(\nabla_x\Theta)^\natural(\eta)]\|\,, \tag{5.4.10}$$

and we obtain

$$\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}[(\nabla_x\Theta)^\natural(\eta)]^{-1}\| \ge \|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|\frac{1}{\|(\nabla_x\Theta)^\natural(\eta)\|}. \tag{5.4.11}$$

From the formula $[(\nabla_x\Theta)^\natural(\eta)] = (\nabla y_0|n_0) + h_j\eta_3(\nabla n_0|0)$ we get

$$\begin{aligned}\|(\nabla_x\Theta)^\natural(\eta)\| &\le \|(\nabla y_0|n_0)\| + h_j\,|\eta_3|\,\|(\nabla n_0|0)\| \le \|(\nabla y_0|n_0)\| + h_j\,\|(\nabla n_0|0)\|\\ &< \|(\nabla y_0|n_0)\| + \|(\nabla n_0|0)\|\,,\end{aligned} \tag{5.4.12}$$

since $h_j \ll 1$. Thus

$$\frac{1}{\|(\nabla_x\Theta)^\natural(\eta)\|} \ge \frac{1}{\|(\nabla y_0|n_0)\| + \|(\nabla n_0|0)\|}. \tag{5.4.13}$$

Moreover, since $y_0 \in C^2(\overline{\omega};\mathbb{R}^3)$, it follows that for h_j small enough that there exists $c_1>0$ such that

$$\frac{1}{\|(\nabla_x\Theta)^\natural(\eta)\|} \ge c_1. \tag{5.4.14}$$

Therefore, from (5.4.9) and (5.4.11), we get that there exist $c_1, c_2 > 0$ such that

$$\|\overline{U}^\natural_{h_j} - \mathbb{1}_3\|^2 \ge \frac{c_1}{2}\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|^2 - c_2. \tag{5.4.15}$$

From the hypothesis we have

$$\begin{aligned}\infty > \mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j}) &\ge \int_{\Omega_1}\Big(W_{\rm mp}(\overline{U}^\natural_{h_j}) + \widetilde{W}_{\rm curv}(\Gamma^\natural_{h_j})\Big)\det((\nabla_x\Theta)^\natural)\,dV_\eta\\ &\ge \int_{\Omega_1} W_{\rm mp}(\overline{U}^\natural_{h_j})\det((\nabla_x\Theta)^\natural)\,dV_\eta \ge \min(c_1^+,\mu_{\rm c})\int_{\Omega_1}\|\overline{U}^\natural_{h_j}-\mathbb{1}_3\|^2\det((\nabla_x\Theta)^\natural)dV_\eta,\end{aligned} \tag{5.4.16}$$

where c_1^+ denotes the smallest eigenvalue of the quadratic form $W^\infty_{\rm mp}(X)$.

Let us recall that $\det(\nabla_x\Theta(x_3)) = \det(\nabla y_0|n_0)\Big[1-2\,x_3\,{\rm H} + x_3^2\,{\rm K}\Big] = \det(\nabla y_0|n_0)(1-\kappa_1\,x_3)(1-\kappa_2\,x_3)$, where H,K are the mean curvature and Gauß curvature, respectively. But $(1-\kappa_1\,x_3)(1-\kappa_2\,x_3) > 0$, $\forall\, x_3 \in [-h_j/2, h_j/2]$ if and only if h_j satisfies the hypothesis. Therefore, there exists a constant $c>0$ such that

$$\det(\nabla_x\Theta(x_3)) \ge c\det(\nabla y_0|n_0) \quad \forall\, x_3 \in [-h/2, h/2]\,. \tag{5.4.17}$$

Due to the hypothesis $\det[\nabla_x\Theta(0)] \ge a_0 > 0$ this implies that there exists a constant $c>0$ such that

$$\det(\nabla_x\Theta(x_3)) \ge c \quad \forall\, x_3 \in [-h_j/2, h_j/2]\,, \tag{5.4.18}$$

which means that $\det(\nabla_x\Theta(x_3)^\natural) \ge c \quad \forall\, x_3 \in [-1/2, 1/2]$.

Hence, from (5.4.16), (5.4.15) and (5.4.18), it follows that for small enough h_j there exist constants $c_1>0$ and $c_2>0$ such that

$$\begin{aligned}\infty > \mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j}) &\ge c_1\int_{\Omega_1}\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|^2 dV_\eta - c_2\\ &\ge c_1\int_{\Omega_1}\left(\|\partial_{\eta_1}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_2}\varphi^\natural_{h_j}\|^2 + \frac{1}{h_j^2}\|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2\right)dV_\eta - c_2.\end{aligned} \tag{5.4.19}$$

Furthermore, due to the hypothesis on h, it is clear that there exists $c>0$ such that

$$\|\partial_{\eta_1}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_2}\varphi^\natural_{h_j}\|^2 + \frac{1}{h_j^2}\|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2 \ge c\,\Big(\|\partial_{\eta_1}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_2}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2\Big)\,, \tag{5.4.20}$$

which implies the existence of $c_1, c_2 > 0$ such that

$$\infty > \mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \geq c_1 \int_{\Omega_1} \underbrace{\left(\|\partial_{\eta_1}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_2}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2\right)}_{=:\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|^2} dV_\eta - c_2. \tag{5.4.21}$$

We also obtain, applying the Poincaré–inequality [96], that there exists a constant $C > 0$ such that

$$\begin{aligned}\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|^2_{\mathrm{L}^2(\omega)} &= \|\nabla_\eta^{h_j}\varphi^\natural_{h_j} - \nabla_\eta^{h_j}\varphi_d + \nabla_\eta^{h_j}\varphi_d\|^2_{\mathrm{L}^2(\omega)}\\ &\geq (\|\nabla_\eta^{h_j}(\varphi^\natural_{h_j} - \varphi_d)\|_{\mathrm{L}^2(\omega)} - \|\nabla_\eta^{h_j}\varphi_d\|_{\mathrm{L}^2(\omega)})^2\\ &= \|\nabla_\eta^{h_j}(\varphi^\natural_{h_j} - \varphi_d)\|^2_{\mathrm{L}^2(\omega)} - 2\|\nabla_\eta^{h_j}(\varphi^\natural_{h_j} - \varphi_d)\|_{\mathrm{L}^2(\omega)}\|\nabla_\eta^{h_j}\varphi_d\|_{\mathrm{L}^2(\omega)} + \|\nabla_\eta^{h_j}\varphi_d\|^2_{\mathrm{L}^2(\omega)}\\ &\geq C\,\|\varphi^\natural_{h_j} - \varphi_d\|^2_{\mathrm{H}^1(\omega)} - 2\,\|\varphi^\natural_{h_j} - \varphi_d\|_{\mathrm{H}^1(\omega)}\|\nabla_\eta^{h_j}\varphi_d\|_{\mathrm{L}^2(\omega)} + \|\nabla_\eta^{h_j}\varphi_d\|^2_{\mathrm{L}^2(\omega)}\\ &\geq C\,\|\varphi^\natural_{h_j} - \varphi_d\|^2_{\mathrm{H}^1(\omega)} - \frac{1}{\varepsilon}\,\|\varphi^\natural_{h_j} - \varphi_d\|^2_{\mathrm{H}^1(\omega)} - \varepsilon\|\nabla_\eta^{h_j}\varphi_d\|^2_{\mathrm{L}^2(\omega)} + \|\nabla_\eta^{h_j}\varphi_d\|^2_{\mathrm{L}^2(\omega)} \qquad \forall\,\varepsilon > 0,\end{aligned} \tag{5.4.22}$$

where we have used Young's and Poincaré's inequality. Therefore, by choosing $\varepsilon > 0$ small enough, (5.4.22) ensures the existence of constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\begin{aligned}\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|^2_{\mathrm{L}^2(\omega)} &\geq c_1\|\varphi^\natural_{h_j} - \varphi_d\|^2_{\mathrm{H}^1(\omega)} - c_2 \geq \frac{c_1}{2}2\,(\|\varphi^\natural_{h_j}\|_{\mathrm{H}^1(\omega)} - \|\varphi_d\|_{\mathrm{H}^1(\omega)})^2 - c_2\\ &\geq \frac{c_1}{2}\|\varphi^\natural_{h_j}\|^2_{\mathrm{H}^1(\omega)} + \frac{c_1}{2}\|\varphi_d\|^2_{\mathrm{H}^1(\omega)} - c_2.\end{aligned} \tag{5.4.23}$$

Thus, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\|\nabla_\eta^{h_j}\varphi^\natural_{h_j}\|^2_{\mathrm{L}^2(\omega)} \geq \frac{c_1}{2}\|\varphi^\natural_{h_j}\|^2_{\mathrm{H}^1(\omega)} - c_2, \tag{5.4.24}$$

which implies the uniform bound for $\varphi^\natural_{h_j}$ in $\mathcal{S}'$. On the other hand, since

$$\|\partial_{\eta_1}\varphi^\natural_{h_j}\|^2 + \|\partial_{\eta_2}\varphi^\natural_{h_j}\|^2 + \frac{1}{h_j^2}\|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2 \geq \frac{1}{h_j^2}\|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2, \tag{5.4.25}$$

from (5.4.19) it results that $\frac{1}{h_j^2}\|\partial_{\eta_3}\varphi^\natural_{h_j}\|^2$ is bounded, i.e., there is $c > 0$, such that

$$\|\partial_{\eta_3}\varphi^\natural_{h_j}\|_{\mathrm{L}^2(\Omega)} \leq c\,h_j. \tag{5.4.26}$$

This means that $\partial_{\eta_3}\varphi^\natural_{h_j} \to 0$ strongly in $\mathrm{L}^2(\Omega)$, when $h_j \to 0$.

Hence, considering $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$, such that the energy functionals $\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j})$ are bounded, it follows that any limit point $\varphi^\natural_0$ of $\varphi^\natural_{h_j}$ for the weak topology of $\mathrm{L}^2(\Omega_1, \mathbb{R}^3)$ (which exists due to its uniform boundedness in $\mathrm{H}^1(\omega, \mathbb{R}^3)$) satisfies

$$\partial_{\eta_3}\varphi^\natural_0 = 0 \quad \Rightarrow \quad \varphi^\natural_0 \in \mathrm{H}^1(\omega, \mathbb{R}^3). \tag{5.4.27}$$

Similar arguments for the curvature energy implies that there exists $c > 0$ such that

$$\begin{aligned}\infty > \mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) &\geq \int_{\Omega_1} \widetilde{W}_{\mathrm{curv}}(\Gamma^\natural_{h_j})\det((\nabla_x\Theta)^\natural)\,dV_\eta \geq \int_{\Omega_1} c\;\|\Gamma^\natural_{h_j}\|^2\det((\nabla_x\Theta)^\natural)\,dV_\eta\\ &= c\int_{\Omega_1}\left\|\Big(\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h_j})\,\Big|\,\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h_j})\,\Big|\,\frac{1}{h_j}\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_3}\overline{Q}^\natural_{e,h_j})\Big)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\right\|^2\det((\nabla_x\Theta)^\natural)\,dV_\eta\,.\end{aligned} \tag{5.4.28}$$

In the next step, as in the deduction of (5.4.8)–(5.4.19), it will be shown that for $a_1, a_2, a_3 > 0$ there exists $c > 0$ such that

$$\begin{aligned}\infty &> c\int_{\Omega_1}\left(\|\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h_j})\|^2 + \|\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h_j})\|^2 + \frac{1}{h_j^2}\|\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_3}\overline{Q}^\natural_{e,h_j})\|^2\right)dV_\eta\\ &= c\int_{\Omega_1}\left(\|\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h_j}\|^2 + \|\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h_j}\|^2 + \frac{1}{h_j^2}\|\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_3}\overline{Q}^\natural_{e,h_j}\|^2\right)dV_\eta\\ &= c\int_{\Omega_1}\left(\|\partial_{\eta_1}\overline{Q}^\natural_{e,h_j}\|^2 + \|\partial_{\eta_2}\overline{Q}^\natural_{e,h_j}\|^2 + \frac{1}{h_j^2}\|\partial_{\eta_3}\overline{Q}^\natural_{e,h_j}\|^2\right)dV_\eta\,.\end{aligned} \tag{5.4.29}$$

With the same argument as the membrane part, we deduce

$$\infty > c \int_{\Omega_1} \left(\|\partial_{\eta_1} \overline{Q}^\natural_{e,h_j}\|^2 + \|\partial_{\eta_2} \overline{Q}^\natural_{e,h_j}\|^2 + \|\partial_{\eta_3} \overline{Q}^\natural_{e,h_j}\|^2 \right) dV_\eta \,, \tag{5.4.30}$$

where $c > 0$. Hence, it follows that $\partial_{\eta_i} \overline{Q}^\natural_{e,h_j}$ is bounded in $\mathrm{L}^2(\Omega_1, \mathbb{R}^{3\times 3})$, for $i = 1, 2, 3$. Since $\overline{Q}^\natural_{e,h_j} \in \mathrm{SO}(3)$, we have $\|\overline{Q}^\natural_{e,h_j}\|^2 = 3$ and therefore $\overline{Q}^\natural_{e,h_j}$ is bounded in $\mathrm{L}^2(\Omega_1, \mathbb{R}^{3\times 3})$. Hence, we can infer that the sequence $\overline{Q}^\natural_{e,h_j}$ is bounded in $\mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3))$, independently from h_j.

Therefore, there is a subsequence from $\overline{Q}^\natural_{e,h_j}$ which is weakly convergent (without relabeling) to $\overline{Q}^\natural_{e,0}$. That is

$$\overline{Q}^\natural_{e,h_j} \rightharpoonup \overline{Q}^\natural_{e,0} \quad \text{in} \quad \mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3))\,. \tag{5.4.31}$$

In addition, from (5.4.29), we also obtain that there exists $c > 0$ such that $c\, h_j > \|\partial_{\eta_3} \overline{Q}^\natural_{e,h_j}\|_{\mathrm{L}^2(\Omega_1, \mathrm{SO}(3))}$. This means that $\partial_{\eta_3} \overline{Q}^\natural_{e,h_j} \to 0$ strongly in $\mathrm{L}^2(\Omega_1, \mathrm{SO}(3))$, when $h_j \to 0$. Hence, considering $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$, such that the energy functional $\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j})$ are bounded, it follows that any limit point $\overline{Q}^\natural_{e,0}$ of $\overline{Q}^\natural_{e,h_j}$ for the weak topology of X satisfies

$$\partial_{\eta_3} \overline{Q}^\natural_{e,0} = 0 \quad \Rightarrow \quad \overline{Q}^\natural_{e,0} \in \mathrm{W}^{1,2}(\omega, \mathrm{SO}(3)). \tag{5.4.32}$$

From (5.4.27), (5.4.32) and due to the continuity of the trace operator we obtain that considering $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$, such that the energy functional $\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j})$ are bounded, it follows that any limit point $(\varphi^\natural_0, \overline{Q}^\natural_{e,0})$ for the weak topology of X belongs to $\mathcal{S}'_\omega$ (since actually, such a sequence belongs to $\mathcal{S}'$). ■

Since the embedding $X' \subset X$ is compact, it follows that the set of the sequence of energies due to the scaling is a subset of X', and hence, we have obtained that the family of energy functionals $J^\natural_h$ is equi-coercive with respect to X.

5.5. The construction of the Γ-limit J_0 of the rescaled energies

In this section we construct the Γ-limit of the rescaled energies

$$\mathcal{J}^\natural_h(\varphi^\natural, \nabla^h_\eta \varphi^\natural, \overline{Q}^\natural_e, \Gamma^\natural_h) = \begin{cases} \dfrac{1}{h} J^\natural_h(\varphi^\natural, \nabla^h_\eta \varphi^\natural, \overline{Q}^\natural_e, \Gamma^\natural_h) & \text{if } (\varphi^\natural, \overline{Q}^\natural_e) \in \mathcal{S}', \\ +\infty & \text{else in } X, \end{cases} \tag{5.5.1}$$

with

$$J^\natural_h(\varphi^\natural, \nabla^h_\eta \varphi^\natural, \overline{Q}^\natural_e, \Gamma^\natural_h) = \int_{\Omega_1} h \left[\left(W_{\mathrm{mp}}(\overline{U}^\natural_h) + \widetilde{W}_{\mathrm{curv}}(\Gamma^\natural_h) \right) \det((\nabla_x \Theta)^\natural) \right] dV_\eta. \tag{5.5.2}$$

5.5.1. Auxiliary optimization problem

For $\varphi^\natural : \Omega_1 \to \mathbb{R}^3$ and $\overline{Q}^\natural_e : \Omega_1 \to \mathrm{SO}(3)$ we associate the non fully dimensional reduced elastic shell stretch tensor

$$\overline{U}_{\varphi^\natural, \overline{Q}^\natural_e} := \overline{Q}^{\natural,T}_e (\nabla_{(\eta_1,\eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}\,, \tag{5.5.3}$$

and the non fully dimensional reduced elastic shell strain tensor

$$\mathcal{E}_{\varphi^\natural, \overline{Q}^\natural_e} := (\overline{Q}^{\natural,T}_e \nabla_{(\eta_1,\eta_2)} \varphi^\natural - (\nabla y_0)^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1} = \overline{U}_{\varphi^\natural, \overline{Q}^\natural_e} - ((\nabla y_0)^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}\,. \tag{5.5.4}$$

Here, "non-fully" means that the introduced quantities still depend on η_3 and h, because the elements $\nabla_{(\eta_1,\eta_2)} \varphi^\natural$ still depend on η_3 and $\overline{Q}^{\natural,T}$ depends on h.

For reaching our goal we need to solve the following optimization problem: for $\varphi^\natural : \Omega_1 \to \mathbb{R}^3$ and $\overline{Q}_e^\natural : \Omega_1 \to \mathrm{SO}(3)$, we determine a vector $d^* \in \mathbb{R}^3$ through

$$W_{\rm mp}^{{\rm hom},\natural}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}) = W_{\rm mp}\Big(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1}\Big) := \inf_{c\in\mathbb{R}^3} W_{\rm mp}\Big(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|c)[(\nabla_x\Theta)^\natural]^{-1}\Big). \tag{5.5.5}$$

The motivation for this optimization problem is to minimize the effect of the derivative in the η_3-direction in the local energy $W_{\rm mp}$. Due to the coercivity and continuity of the energy $W_{\rm mp}$, it is clear that this function is well defined and the infimum is attained. Note that $\varphi^\natural$ and $\overline{Q}_e^\natural$ depend on η_3 and h. Hence $W_{\rm mp}^{{\rm hom},\natural}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})$ depends on η_3 and h. While it is not immediately clear why $W_{\rm mp}\Big(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1}\Big)$ can be expressed as a function of $\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}$, this aspect will be clarified in the rest of this subsection.

Now we calculate the variation of the energy (5.5.5) at equilibrium to be minimized over $c \in \mathbb{R}^3$ in order to determine the minimizer d^*. For arbitrary increment $\delta d^* \in \mathbb{R}^3$, we have

$$\forall\ \delta d^* \in \mathbb{R}^3: \quad \big\langle {\rm D}W_{\rm mp}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1}), \overline{Q}_e^{\natural,T}(0|0|\delta d^*)[(\nabla_x\Theta)^\natural]^{-1}\big\rangle = 0. \tag{5.5.6}$$

We do some lengthy but straightforward calculations using the fact that $[\nabla_x\Theta]^{-T}e_3 = n_0$ and $[(\nabla_x\Theta)^\natural]^{-T}e_3 = n_0$, as well. By applying ${\rm D}W_{\rm mp}$ we obtain

$$\begin{aligned}
&\big\langle 2\,\mu\,\Big({\rm sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big), \overline{Q}_e^{\natural,T}(0|0|\delta d^*)[(\nabla_x\Theta)^\natural]^{-1}\big\rangle_{\mathbb{R}^{3\times3}}\\
&\quad + \big\langle 2\,\mu_c\,\Big({\rm skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1})\Big), \overline{Q}_e^{\natural,T}(0|0|\delta d^*)[(\nabla_x\Theta)^\natural]^{-1}\big\rangle_{\mathbb{R}^{3\times3}}\\
&\quad + \lambda\,{\rm tr}\Big({\rm sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big)\,\langle \mathbb{1}_3, \overline{Q}_e^{\natural,T}(0|0|\delta d^*)[(\nabla_x\Theta)^\natural]^{-1}\rangle_{\mathbb{R}^{3\times3}} = 0.
\end{aligned} \tag{5.5.7}$$

This is equivalent to

$$\begin{aligned}
&\big\langle 2\,\mu\,\overline{Q}_e^\natural\Big({\rm sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big)[(\nabla_x\Theta)^\natural]^{-T}e_3, \delta d^*\big\rangle_{\mathbb{R}^3}\\
&\quad + \big\langle 2\,\mu_c\,\overline{Q}_e^\natural\Big({\rm skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1})\Big)[(\nabla_x\Theta)^\natural]^{-T}e_3, \delta d^*\big\rangle_{\mathbb{R}^3}\\
&\quad + \lambda\,{\rm tr}\Big({\rm sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big)\,\langle \overline{Q}_e^\natural[(\nabla_x\Theta)^\natural]^{-T}e_3, \delta d^*\rangle_{\mathbb{R}^3} = 0\,,
\end{aligned} \tag{5.5.8}$$

and it gives

$$\begin{aligned}
&\big\langle 2\,\mu\,\overline{Q}_e^\natural\Big({\rm sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big)n_0, \delta d^*\big\rangle_{\mathbb{R}^3}\\
&\quad + \Big\langle 2\,\mu_c\,\overline{Q}_e^\natural\Big({\rm skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0, \delta d^*\Big\rangle_{\mathbb{R}^3}\\
&\quad + \lambda\,{\rm tr}\Big({\rm sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big)\langle \overline{Q}_e^\natural n_0, \delta d^*\rangle_{\mathbb{R}^3} = 0.
\end{aligned} \tag{5.5.9}$$

Recall that the *first Piola–Kirchhoff stress tensor* in the reference configuration Ω_ξ is given by $S_1(F_\xi, \overline{R}_\xi) := {\rm D}_{F_\xi}W_{\rm mp}(F_\xi, \overline{R}_\xi)$, while the *Biot-type stress tensor* is $T_{\rm Biot}(\overline{U}_\xi) := {\rm D}_{\overline{U}_\xi}W_{\rm mp}(\overline{U}_\xi)$. Since ${\rm D}_{F_\xi}\overline{U}_\xi.X = \overline{R}_\xi^T X$ and $\langle {\rm D}_{F_\xi}W_{\rm mp}(F_\xi, \overline{R}_\xi), X\rangle = \langle {\rm D}_{\overline{U}_\xi}W_{\rm mp}(\overline{U}_\xi), {\rm D}_{F_\xi}\overline{U}_\xi X\rangle$, $\forall X \in \mathbb{R}^{3\times3}$, we obtain

$${\rm D}_{F_\xi}W_{\rm mp}(F_\xi, \overline{R}_\xi) = \overline{R}_\xi\ {\rm D}_{\overline{U}_\xi}W_{\rm mp}(\overline{U}_\xi)\,. \tag{5.5.10}$$

Therefore, $S_1(F_\xi, \overline{R}_\xi) = \overline{R}_\xi\, T_{\rm Biot}(\overline{U}_\xi)$ and $T_{\rm Biot}(\overline{U}_\xi) = \overline{R}_\xi^T\, S_1(F_\xi, \overline{R}_\xi)$. Here, we have

$$T_{\rm Biot}(\overline{U}_\xi) = 2\,\mu\ {\rm sym}(\overline{U}_\xi - \mathbb{1}_3) + 2\,\mu_c\ {\rm skew}(\overline{U}_\xi - \mathbb{1}_3) + \lambda\,{\rm tr}({\rm sym}(\overline{U}_\xi - \mathbb{1}_3))\mathbb{1}_3\,, \tag{5.5.11}$$

where $\overline{U}_\xi(\Theta(x_1,x_2,x_3)) = \overline{U}_e(x_1,x_2,x_3)$. Thus, we can express the first Piola Kirchhoff stress tensor

$$S_1(F_\xi, \overline{R}_\xi) = \overline{R}_\xi\Big[2\,\mu\ {\rm sym}(\overline{R}_\xi^T F_\xi - \mathbb{1}_3) + 2\,\mu_c\ {\rm skew}(\overline{R}_\xi^T F_\xi - \mathbb{1}_3) + \lambda\,{\rm tr}({\rm sym}(\overline{R}_\xi^T F_\xi - \mathbb{1}_3))\mathbb{1}_3\Big], \tag{5.5.12}$$

with $\overline{R}_\xi(\Theta(x_1,x_2,x_3)) = \overline{Q}_e(x_1,x_2,x_3)$ for the elastic microrotation $\overline{Q}_e\colon \Omega_h \to \mathrm{SO}(3)$. Hence, we must have

$$\forall \delta d^* \in \mathbb{R}^3: \qquad \langle S_1((\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1}, \overline{Q}_e^\natural)n_0, \delta d^*\rangle_{\mathbb{R}^3} = 0, \tag{5.5.13}$$

implying

$$S_1((\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1},\overline{Q}_e^\natural)\,n_0=0 \qquad \forall\,\eta_3\in\left[-\frac{1}{2},\frac{1}{2}\right]. \tag{5.5.14}$$

In shell theories, the usual assumption is that the normal stress on the transverse boundaries are vanishing, that is

$$S_1(F_\xi,\overline{R}_\xi)\big|_{\omega_\xi^\pm}(\pm n_0)=0\,, \qquad \text{(normal stress on lower and upper faces is zero)}. \tag{5.5.15}$$

We notice that the condition (5.5.14) is for all $\eta_3\in\left[-\frac{1}{2},\frac{1}{2}\right]$, while the condition (5.5.15) is only for $\eta_3=\pm\frac{1}{2}$. Therefore, it is possible that the Cosserat-membrane type Γ-limit underestimates the real stresses (e.g., the transverse shear stresses). From the relation between the first Piola-Kirchhoff tensor and the Biot-stress tensor we obtain

$$T_{\rm Biot}\Big(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1}\Big)n_0=0\,, \qquad \forall\,\eta_3\in[-\frac{1}{2},\frac{1}{2}]\,, \tag{5.5.16}$$

or, equivalently,

$$T_{\rm Biot}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*})\,n_0=0, \tag{5.5.17}$$

where

$$T_{\rm Biot}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*})=2\,\mu\,{\rm sym}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*}-\mathbb{1}_3)+2\,\mu_c\,{\rm skew}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*}-\mathbb{1}_3)+\lambda\,{\rm tr}({\rm sym}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*}-\mathbb{1}_3))\mathbb{1}_3\,, \tag{5.5.18}$$

and we have introduced the notation $\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*}:=\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1}$. With the help of the following decomposition

$$\begin{aligned}\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*}-\mathbb{1}_3&=(\overline{Q}_e^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural-(\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}+(0|0|\overline{Q}_e^{\natural,T}d^*-n_0)[(\nabla_x\Theta)^\natural]^{-1}\\&=\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}+(0|0|\overline{Q}_e^{\natural,T}d^*-n_0)[(\nabla_x\Theta)^\natural]^{-1}\,,\end{aligned} \tag{5.5.19}$$

with $\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}=(\overline{Q}_e^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural-(\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}$, and relations (A.1.1)-(A.1.3), the relation (5.5.18) can be expressed as

$$\begin{aligned}T_{\rm Biot}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,d^*})n_0&=\mu\left(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(\overline{Q}_e^{\natural,T}d^*-n_0)+[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}d^*-n_0)^Tn_0\right)\\&\quad+\mu_c\left(-\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(\overline{Q}_e^{\natural,T}d^*-n_0)-[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}d^*-n_0)^Tn_0\right)\\&\quad+\lambda\Big(\langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\mathbb{1}_3\rangle n_0+(\overline{Q}_e^{\natural,T}d^*-n_0)n_0\otimes n_0\Big)\\&=(\mu+\mu_c)(\overline{Q}_e^{\natural,T}d^*-n_0)+(\mu-\mu_c)\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(\mu-\mu_c)((0|0|\overline{Q}_e^{\natural,T}d^*-n_0)[(\nabla_x\Theta)^\natural]^{-1})^Tn_0\\&\quad+\lambda\,{\rm tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0+\lambda(\overline{Q}_e^{\natural,T}d^*-n_0)n_0\otimes n_0,\end{aligned} \tag{5.5.20}$$

and the condition (5.5.17) on $T_{\rm Biot}$ reads

$$\begin{aligned}(\mu+\mu_c)(\overline{Q}_e^{\natural,T}d^*-n_0)+(\mu-\mu_c)(\overline{Q}_e^{\natural,T}d^*-n_0)n_0\otimes n_0+\lambda(\overline{Q}_e^{\natural,T}d^*-n_0)n_0\otimes n_0&\\=-\Big[(\mu-\mu_c)\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+\lambda\,{\rm tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\Big],&\end{aligned} \tag{5.5.21}$$

where $((0|0|\overline{Q}_e^{\natural,T}d^*-n_0)[(\nabla_x\Theta)^\natural]^{-1})^Tn_0=(\overline{Q}_e^{\natural,T}d^*-n_0)n_0\otimes n_0$. Before continuing the calculations, we introduce the tensor

$${\rm A}_{y_0}:=(\nabla y_0|0)\;[(\nabla_x\Theta)(0)]^{-1}=\mathbb{1}_3-n_0\otimes n_0\in{\rm Sym}(3), \tag{5.5.22}$$

and we notice that, identically as in the proof of Lemma 4.3 in [56], we can show that

$$\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}{\rm A}_{y_0}=\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural} \qquad\Longleftrightarrow\qquad \mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0=0. \tag{5.5.23}$$

Actually, for an arbitrary matrix $X = (*|*|0)\,[\nabla_x\Theta(0)]^{-1}$, since $\mathrm{A}_{y_0}^2 = \mathrm{A}_{y_0} \in \mathrm{Sym}(3)$ and $X\mathrm{A}_{y_0} = X$, we have

$$\big\langle (\mathbb{1}_3 - \mathrm{A}_{y_0})\,X, \mathrm{A}_{y_0}\,X\big\rangle = \big\langle (\mathrm{A}_{y_0} - \mathrm{A}_{y_0}^2)\,X,\,X\big\rangle = 0,$$

but also

$$(\mathbb{1}_3 - \mathrm{A}_{y_0})\,X^T = \big(X(\mathbb{1}_3 - \mathrm{A}_{y_0})\big)^T = \big(X - X\mathrm{A}_{y_0}\big)^T = 0, \tag{5.5.24}$$

and consequently

$$\big\langle X^T(\mathbb{1}_3 - \mathrm{A}_{y_0}), \mathrm{A}_{y_0}\,X\big\rangle = 0 \qquad \text{as well as} \qquad \big\langle X^T(\mathbb{1}_3 - \mathrm{A}_{y_0}), (\mathbb{1}_3 - \mathrm{A}_{y_0})\,X\big\rangle = 0.$$

In addition, since $\mathrm{A}_{y_0} = \mathbb{1}_3 - (0|0|n_0)\,(0|0|n_0)^T = \mathbb{1}_3 - n_0\otimes n_0$, the following equalities holds

$$\begin{aligned}\|(\mathbb{1}_3 - \mathrm{A}_{y_0})\,X\|^2 &= \big\langle\, X, (\mathbb{1}_3 - \mathrm{A}_{y_0})^2\,X\big\rangle = \big\langle\, X, (\mathbb{1}_3 - \mathrm{A}_{y_0})\,X\big\rangle = \big\langle\, X, (0|0|n_0)\,(0|0|n_0)^T\,X\big\rangle \\ &= \big\langle\,(0|0|n_0)^T X, (0|0|n_0)^T\,X\big\rangle = \|X\,(0|0|n_0)^T\|^2 = \|X^T\,(0|0|n_0)\|^2 = \|X^T\,n_0\|^2.\end{aligned} \tag{5.5.25}$$

We have the following decomposition

$$\begin{aligned}(\overline{Q}_e^{\natural,T} d^* - n_0) &= \mathbb{1}_3(\overline{Q}_e^{\natural,T} d^* - n_0) = (A_{y_0} + n_0\otimes n_0)(\overline{Q}_e^{\natural,T} d^* - n_0) \\ &= A_{y_0}(\overline{Q}_e^{\natural,T} d^* - n_0) + n_0\otimes n_0(\overline{Q}_e^{\natural,T} d^* - n_0).\end{aligned} \tag{5.5.26}$$

By using that

$$n_0\otimes n_0(\overline{Q}_e^{\natural,T} d^* - n_0) = n_0\langle n_0, (\overline{Q}_e^{\natural,T} d^* - n_0)\rangle = \langle(\overline{Q}_e^{\natural,T} d^* - n_0), n_0\rangle n_0 = (\overline{Q}_e^{\natural,T} d^* - n_0)n_0\otimes n_0, \tag{5.5.27}$$

and with (5.5.21), we get

$$\begin{aligned}(\mu+\mu_c)A_{y_0}(\overline{Q}_e^{\natural,T} d^* - n_0) + (\mu+\mu_c)n_0\otimes n_0(\overline{Q}_e^{\natural,T} d^* - n_0) + (\mu-\mu_c)n_0\otimes n_0(\overline{Q}_e^{\natural,T} d^* - n_0) \\ + \lambda\, n_0\otimes n_0(\overline{Q}_e^{\natural,T} d^* - n_0) = -\Big[(\mu-\mu_c)\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 + \lambda\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\Big].\end{aligned} \tag{5.5.28}$$

Therefore,

$$\Big((\mu+\mu_c)A_{y_0} + (2\mu+\lambda)n_0\otimes n_0\Big)(\overline{Q}_e^{\natural,T} d^* - n_0) = -\Big[(\mu-\mu_c)\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 + \lambda\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\Big]. \tag{5.5.29}$$

Direct calculation shows

$$\Big((\mu+\mu_c)A_{y_0} + (2\mu+\lambda)n_0\otimes n_0\Big)^{-1} := \Big(\frac{1}{\mu+\mu_c}A_{y_0} + \frac{1}{2\mu+\lambda}n_0\otimes n_0\Big). \tag{5.5.30}$$

Next, by using

$$\begin{aligned}A_{y_0}n_0 &= (\mathbb{1}_3 - n_0\otimes n_0)n_0 = n_0 - n_0\langle n_0, n_0\rangle = n_0 - n_0 = 0, \\ n_0\otimes n_0\,\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 &= (0|0|n_0)(0|0|n_0)^T\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 = (0|0|n_0)\Big((\overline{Q}_e^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural - (\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}(0|0|n_0)\Big)^T n_0 \\ &= (0|0|n_0)\Big((\overline{Q}_e^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural - (\nabla y_0)^\natural|0)(0|0|e_3)\Big)^T n_0 = 0,\end{aligned} \tag{5.5.31}$$

eq. (5.5.29) can be written as

$$\begin{aligned}\overline{Q}_e^{\natural,T} d^* - n_0 &= -\Big[\frac{1}{\mu+\mu_c}A_{y_0} + \frac{1}{2\mu+\lambda}n_0\otimes n_0\Big] \times \Big[(\mu-\mu_c)\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 + \lambda\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\Big] \qquad (5.5.32)\\ &= -\Big[\frac{\mu-\mu_c}{\mu+\mu_c}A_{y_0}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 + \frac{\mu-\mu_c}{2\mu+\lambda}\,n_0\otimes n_0\,\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 + \frac{\lambda}{\mu+\mu_c}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})A_{y_0}n_0 \\ &\quad + \frac{\lambda}{2\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})(n_0\otimes n_0)n_0\Big] = -\Big[\frac{\mu-\mu_c}{\mu+\mu_c}A_{y_0}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0 + \frac{\lambda}{2\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\Big].\end{aligned}$$

Simplifying (5.5.32) we obtain

$$d^* = \Big(1 - \frac{\lambda}{2\,\mu\, + \lambda}\langle \mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}, \mathbb{1}_3\rangle\Big)\overline{Q}_e^\natural n_0 + \frac{\mu_c\, - \mu}{\mu_c\, + \mu}\,\overline{Q}_e^\natural \mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0.$$

In terms of $\overline{Q}_e^\natural = \overline{R}^\natural Q_0^{\natural,T}$ we obtain the following expression for d^*

$$\begin{aligned} d^* =& \Big(1 - \frac{\lambda}{2\,\mu\, + \lambda}\langle (Q_0^\natural \overline{R}^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural - (\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}, \mathbb{1}_3\rangle\Big)\overline{R}^\natural Q_0^{\natural,T} n_0 \\ &+ \frac{\mu_c\, - \mu}{\mu_c\, + \mu}\,\overline{R}^\natural Q_0^{\natural,T}\Big((Q_0^\natural \overline{R}^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural - (\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}\Big)^T n_0. \end{aligned} \tag{5.5.33}$$

Now we insert d^* in the membrane energy

$$W_{\rm mp}(\overline{U}_h^\natural) = \mu\,\|{\rm sym}(\overline{U}_h^\natural - \mathbb{1}_3)\|^2 + \mu_c\,\|{\rm skew}(\overline{U}_h^\natural - \mathbb{1}_3)\|^2 + \frac{\lambda}{2}[{\rm tr}({\rm sym}(\overline{U}_h^\natural - \mathbb{1}_3))]^2\,.$$

Using (A.2.6), (A.2.11) and (A.2.12) in Appendix, we obtain the explicit form of the homogenized energy for the membrane part

$$\begin{aligned} W_{\rm mp}^{\rm hom,\natural}(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}) =& \mu\,\|{\rm sym}\,\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \frac{\mu}{2}\frac{(\mu_c\, - \mu\,)^2}{(\mu_c\, + \mu\,)^2}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 + \frac{\mu\,(\mu_c\, - \mu\,)}{(\mu_c\, + \mu\,)}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 \\ &+ \frac{\mu\,\lambda^2}{(2\,\mu\, + \lambda)^2}\,{\rm tr}(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural})^2 + \mu_c\,\|{\rm skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1})\|^2 \\ &+ \frac{\mu_c}{2}\frac{(\mu_c\, - \mu\,)^2}{(\mu_c\, + \mu\,)^2}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 - \frac{\mu_c\,(\mu_c\, - \mu\,)}{(\mu_c\, + \mu\,)}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 + \frac{2\,\mu^{\,2}\lambda}{(2\,\mu\, + \lambda)^2}\,{\rm tr}(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural})^2, \end{aligned} \tag{5.5.34}$$

and finally

$$W_{\rm mp}^{\rm hom,\natural}(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}) = W_{\rm shell}(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}) - \frac{(\mu_c\, - \mu\,)^2}{2(\mu_c\, + \mu\,)}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2, \tag{5.5.35}$$

where

$$W_{\rm shell}(X) = \mu\,\|{\rm sym}\,X\|^2 + \mu_c\,\|{\rm skew}\,X\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,\big[{\rm tr}\,X\,\big]^2.$$

Using the orthogonal decomposition in the tangential plane and in the normal direction, gives

$$X = X^\parallel + X^\perp, \qquad X^\parallel := {\rm A}_{y_0}\,X, \qquad X^\perp := (\mathbb{1}_3 - {\rm A}_{y_0})\,X, \tag{5.5.36}$$

we deduce that for all $X = (*|*|0)\cdot[\nabla_x\Theta(0)]^{-1}$ we have the following split in the expression of the considered quadratic forms

$$W_{\rm shell}(X) = \mu\,\|{\rm sym}\,X^\parallel\|^2 + \mu_c\,\|{\rm skew}\,X^\parallel\|^2 + \frac{\mu\, + \mu_c}{2}\,\|X^\perp\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,\big[{\rm tr}(X)\big]^2. \tag{5.5.37}$$

Moreover, using that for all $X = (*|*|0)\,[\nabla_x\Theta(0)]^{-1}$, it holds that

$${\rm tr}(X^\perp) = {\rm tr}\,\big((\mathbb{1}_3 - {\rm A}_{y_0})X\big) = {\rm tr}(X) - {\rm tr}({\rm A}_{y_0}X) = {\rm tr}(X) - {\rm tr}(X\,{\rm A}_{y_0}) = 0, \tag{5.5.38}$$

we obtain

$$\begin{aligned} W_{\rm shell}\big(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}\big) =& \mu\,\|{\rm sym}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \mu_c\,\|{\rm skew}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,[{\rm tr}(\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural})]^2 + \frac{\mu\, + \mu_c}{2}\,\|\mathcal{E}^\perp_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 \\ =& \mu\,\|{\rm sym}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \mu_c\,\|{\rm skew}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,[{\rm tr}(\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural})]^2 + \frac{\mu\, + \mu_c}{2}\,\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2. \end{aligned} \tag{5.5.39}$$

Therefore, the homogenized energy for the membrane part is

$$\begin{aligned} W_{\rm mp}^{\rm hom,\natural}(\mathcal{E}_{\varphi^\natural, \overline{Q}_e^\natural}) =& \mu\,\|{\rm sym}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \mu_c\,\|{\rm skew}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,\big[{\rm tr}(\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural})\big]^2 \\ &+ \frac{\mu\, + \mu_c}{2}\,\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 - \frac{(\mu_c\, - \mu\,)^2}{2(\mu_c\, + \mu\,)}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 \\ =& \mu\,\|{\rm sym}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \mu_c\,\|{\rm skew}\,\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,\big[{\rm tr}(\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural})\big]^2 + \frac{2\,\mu\,\mu_c}{\mu_c\, + \mu}\|\mathcal{E}^T_{\varphi^\natural, \overline{Q}_e^\natural} n_0\|^2 \\ =& W_{\rm shell}\big(\mathcal{E}^\parallel_{\varphi^\natural, \overline{Q}_e^\natural}\big) + \frac{2\,\mu\,\mu_c}{\mu_c\, + \mu}\|\mathcal{E}^\perp_{\varphi^\natural, \overline{Q}_e^\natural}\|^2. \end{aligned} \tag{5.5.40}$$

5.5.2. Homogenized membrane energy

Now, we will be able to propose the form of the homogenized membrane energy. To each pair $(m, \overline{Q}_{e,0})$, where $m : \omega \to \mathbb{R}^3$, $\overline{Q}_{e,0} : \omega \to \mathrm{SO}(3)$, we associate the *elastic shell strain tensor*

$$\mathcal{E}_{m,s} := (\overline{Q}_{e,0}^T \nabla m - \nabla y_0|0)[\nabla_x \Theta(0)]^{-1}\,, \tag{5.5.41}$$

and we define the homogenized energy

$$W_{\mathrm{mp}}^{\mathrm{hom}}(\mathcal{E}_{m,s}) := \inf_{\widetilde{d}\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\overline{Q}_{e,0}^T(\nabla m|\widetilde{d})[(\nabla_x\Theta)(0)]^{-1}\Big) = \inf_{\widetilde{d}\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\mathcal{E}_{m,s} - (0|0|\widetilde{d})[(\nabla_x\Theta)(0)]^{-1}\Big). \tag{5.5.42}$$

Direct calculations as in the previous subsection (5.5.1) show us that the infimum is attained for

$$\widetilde{d}^* = \Big(1 - \frac{\lambda}{2\,\mu + \lambda}\langle \mathcal{E}_{m,s}, \mathbb{1}_3\rangle\Big)\overline{Q}_{e,0} n_0 + \frac{\mu_c - \mu}{\mu_c + \mu}\,\overline{Q}_{e,0}\mathcal{E}_{m,s}^T n_0\,, \tag{5.5.43}$$

and

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{hom}}(\mathcal{E}_{m,s}) &= \mu\,\|\mathrm{sym}\;\mathcal{E}_{m,s}^{\|}\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\;\mathcal{E}_{m,s}^{\|}\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}_{m,s}^{\|})\big]^2 + \frac{2\,\mu\,\mu_{\mathrm{c}}}{\mu_c + \mu}\|\mathcal{E}_{m,s}^T n_0\|^2 \\ &= W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}^{\|}\big) + \frac{2\,\mu\,\mu_{\mathrm{c}}}{\mu_c + \mu}\|\mathcal{E}_{m,s}^{\perp}\|^2, \end{aligned} \tag{5.5.44}$$

where

$$W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}^{\|}\big) = \mu\,\|\mathrm{sym}\;\mathcal{E}_{m,s}^{\|}\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\;\mathcal{E}_{m,s}^{\|}\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}_{m,s}^{\|})\big]^2. \tag{5.5.45}$$

Note that $W_{\mathrm{mp}}^{\mathrm{hom},\natural}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})$ constructed in (5.5.40) depends on η_3 and h, while $W_{\mathrm{mp}}^{\mathrm{hom}}(\mathcal{E}_{m,s})$ in (5.5.44) does not depend on η_3 and h, since $\overline{Q}_{e,0}$ and $[(\nabla_x\Theta)(0)]$ do not depend on η_3 and h.

5.5.3. Homogenized curvature energy

We define the homogenized curvature energy as

$$\begin{aligned} \widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}(\mathcal{K}_e^\natural) :&= \widetilde{W}_{\mathrm{curv}}\Big(\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_1}\overline{Q}_e^\natural)\,|\,\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_2}\overline{Q}_e^\natural)\,|\;\mathrm{axl}(A^*)\,\Big)[(\nabla_x\Theta)^\natural]^{-1} \\ &= \inf_{A\in\mathfrak{so}(3)} \widetilde{W}_{\mathrm{curv}}\Big(\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_1}\overline{Q}_e^\natural)\,|\,\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_2}\overline{Q}_e^\natural)\,|\;\mathrm{axl}(A)\,\Big)[(\nabla_x\Theta)^\natural]^{-1}, \end{aligned} \tag{5.5.46}$$

where

$$\mathcal{K}_e^\natural := \Big(\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_1}\overline{Q}_e^\natural)\,|\,\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_2}\overline{Q}_e^\natural)\,|0\Big)[(\nabla_x\Theta)^\natural]^{-1}\,,$$

represents a not fully reduced elastic shell bending-curvature tensor, in the sense that it still depends on η_3 and h, since $\overline{Q}_e^\natural = \overline{Q}_e^\natural(\eta_1,\eta_2,\eta_3)$. Therefore, $\widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}(\mathcal{K}_e^\natural)$ given by the above definitions still depends on η_3 and h.

As in the case of the homogenized membrane part in (5.3.8), from which we obtained the unknown d^*, one can explicitly determine the infinitesimal microrotation $A^* \in \mathfrak{so}(3)$ as well. Ghiba et.al, in [55] obtained the homogenized quadratic curvature energy (see Chapter 6 for explicit calculations). However in this chapter, it is enough to see that $\widetilde{W}_{\mathrm{curv}}^{\mathrm{hom}}$ is uniquely defined and has the other requirements like remaining convex in its argument and having the same growth as $\widetilde{W}_{\mathrm{curv}}$. Therefore,

$$\widetilde{W}_{\mathrm{curv}}\Big(\Gamma_h^\natural\Big) \geq \widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}(\mathcal{K}_e^\natural), \tag{5.5.47}$$

i.e.,

$$\begin{aligned} &\widetilde{W}_{\mathrm{curv}}\Big(\big(\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_1}\overline{Q}_{e,h}^\natural)\,|\,\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)\,|\,\frac{1}{h}\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_3}\overline{Q}_{e,h}^\natural)\,\big)[(\nabla_x\Theta)^\natural]^{-1}\Big) \\ &\qquad \geq \widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}\Big(\big(\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_1}\overline{Q}_e^\natural)\,|\,\mathrm{axl}(\overline{Q}_e^{\natural,T}\,\partial_{\eta_2}\overline{Q}_e^\natural)\,|0\,\big)[(\nabla_x\Theta)^\natural]^{-1}\Big)\,, \end{aligned} \tag{5.5.48}$$

where this relation will help us in subsection 5.6.1 to show the lim inf condition for the curvature energy. In order to construct the Γ-limit, we have to define a homogenized curvature energy. This energy will be expressed in terms of the elastic shell bending-curvature tensor

$$\mathcal{K}_{e,s} := \Big(\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{x_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{x_2}\overline{Q}_{e,0})\,|0\Big)[\nabla_x\Theta(0)\,]^{-1}\not\in \mathrm{Sym}(3)\,,\ \text{elastic shell bending–curvature tensor}\,,$$

which will be defined for any $\overline{Q}_{e,0}:\omega\to \mathrm{SO}(3)$. For $\overline{Q}_{e,0}:\omega\to \mathrm{SO}(3)$, we set

$$\begin{aligned}\widetilde{W}_{\mathrm{curv}}^{\mathrm{hom}}(\mathcal{K}_{e,s}) :&= \widetilde{W}_{\mathrm{curv}}^{*}\Big(\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_2}\overline{Q}_{e,0})\,|\ \mathrm{axl}(A^*)\,\Big)[(\nabla_x\Theta)^\natural(0)]^{-1} \qquad (5.5.49)\\ &= \inf_{A\in\mathfrak{so}(3)}\widetilde{W}_{\mathrm{curv}}\Big(\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_2}\overline{Q}_{e,0})\,|\ \mathrm{axl}(A)\,\Big)[(\nabla_x\Theta)^\natural(0)]^{-1}\,.\end{aligned}$$

Again note that while $\widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}(\mathcal{K}_e^\natural)$ (previously constructed) depends on η_3 and h, $\widetilde{W}_{\mathrm{curv}}^{\mathrm{hom}}(\mathcal{K}_{e,s})$ does not depend on η_3 and h, since $\overline{Q}_{e,0}$ and $[(\nabla_x\Theta)(0)]$ do not depend on η_3 and h.

5.6. Γ-convergence of $\mathcal{J}_{h_j}$

We are now ready to formulate the main result of this chapter

Theorem 5.6.1. *Assume that the initial configuration of the curved shell is defined by a continuous injective mapping* $y_0:\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ *which admits an extension to* $\overline{\omega}$ *into* $C^2(\overline{\omega};\mathbb{R}^3)$ *such that for*

$$\Theta(x_1,x_2,x_3) = y_0(x_1,x_2) + x_3 n_0(x_1,x_2)$$

we have $\det[\nabla_x\Theta(0)]\geq a_0>0$ *on* $\overline{\omega}$, *where* a_0 *is a constant, and assume that the boundary data satisfy the conditions*

$$\varphi_d^\natural = \varphi_d\big|_{\Gamma_1}\ \textit{(in the sense of traces) for}\ \ \varphi_d\in H^1(\Omega_1;\mathbb{R}^3). \qquad (5.6.1)$$

Let the constitutive parameters satisfy

$$\mu>0,\qquad \kappa>0,\qquad \mu_c>0,\qquad a_1>0,\qquad a_2>0,\qquad a_3>0\,. \qquad (5.6.2)$$

Then, for any sequence $(\varphi_{h_j}^\natural,\overline{Q}_{e,h_j}^\natural)\in X$ *such that* $(\varphi_{h_j}^\natural,\overline{Q}_{e,h_j}^\natural)\to(\varphi_0,\overline{Q}_{e,0})$ *as* $h_j\to 0$, *the sequence of functionals* $\mathcal{J}_{h_j}:X\to\overline{\mathbb{R}}$ *Γ-converges to the limit energy functional* $\mathcal{J}_0: X\to\overline{\mathbb{R}}$ *defined by*

$$\mathcal{J}_0(m,\overline{Q}_{e,0}) = \begin{cases}\int_\omega[W_{mp}^{hom}(\mathcal{E}_{m,\overline{Q}_{e,0}}) + \widetilde{W}_{curv}^{hom}(\mathcal{K}_{e,s})]\det(\nabla y_0|n_0)\,d\omega & \textit{if}\quad (m,\overline{Q}_{e,0})\in\mathcal{S}'_\omega\,,\\ +\infty & \textit{else in } X,\end{cases} \qquad (5.6.3)$$

where

$$\begin{aligned} m(x_1,x_2) := \varphi_0(x_1,x_2) &= \lim_{h_j\to 0}\varphi_{h_j}^\natural(x_1,x_2,\frac{1}{h_j}x_3), \qquad \overline{Q}_{e,0}(x_1,x_2) = \lim_{h_j\to 0}\overline{Q}_{e,h_j}^\natural(x_1,x_2,\frac{1}{h_j}x_3),\\ \mathcal{E}_{m,\overline{Q}_{e,0}} &= (\overline{Q}_{e,0}^T\nabla m - \nabla y_0|0)[\nabla_x\Theta(0)]^{-1}, \qquad (5.6.4)\\ \mathcal{K}_{e,s} &= \Big(\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{x_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{x_2}\overline{Q}_{e,0})\,|0\Big)[\nabla_x\Theta(0)\,]^{-1}\not\in \mathit{Sym}(3)\,,\end{aligned}$$

and

$$\begin{aligned} W_{mp}^{hom}(\mathcal{E}_{m,\overline{Q}_{e,0}}) &= \mu\,\|\mathrm{sym}\,\mathcal{E}_{m,\overline{Q}_{e,0}}^{\|}\|^2 + \mu_c\,\|\mathrm{skew}\,\mathcal{E}_{m,\overline{Q}_{e,0}}^{\|}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}_{m,\overline{Q}_{e,0}}^{\|})\big]^2 + \frac{2\,\mu\,\mu_c}{\mu_c+\mu}\|\mathcal{E}_{m,\overline{Q}_{e,0}}^T n_0\|^2\\ &= W_{\mathrm{shell}}(\mathcal{E}_{m,\overline{Q}_{e,0}}^{\|}) + \frac{2\,\mu\,\mu_c}{\mu_c+\mu}\|\mathcal{E}_{m,\overline{Q}_{e,0}}^{\perp}\|^2, \qquad (5.6.5)\\ \widetilde{W}_{curv}^{hom}(\mathcal{K}_{e,s}) &= \inf_{A\in\mathfrak{so}(3)}\widetilde{W}_{curv}\Big(\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_2}\overline{Q}_{e,0})\,|\ \mathrm{axl}(A)\,\Big)[(\nabla_x\Theta)^\natural(0)]^{-1}\\ &= \mu L_c^2\Big(b_1\|\mathrm{sym}\,\mathcal{K}_{e,s}^{\|}\|^2 + b_2\|\mathrm{skew}\,\mathcal{K}_{e,s}^{\|}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\,\mathrm{tr}(\mathcal{K}_{e,s}^{\|})^2 + \frac{2\,b_1 b_2}{b_1+b_2}\|\mathcal{K}_{e,s}^{\perp}\|\Big)\,.\end{aligned}$$

Proof. The first part of the proof is represented by the proof of equi-coercivity and compactness of the family of energy functionals which are already done. The rest of the proof will be divided into two parts which make the subjects of the following two subsections. ■

5.6.1. Step 1 of the proof. The lim-inf condition

In this section we prove the following lemma

Lemma 5.6.2. *In the hypothesis of Theorem 5.6.1, for any sequence* $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$ *such that* $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \to (\varphi^\natural_0, \overline{Q}^\natural_{e,0})$ *for* $h_j \to 0$*, i.e.,*

$$\varphi^\natural_{h_j} \to \varphi^\natural_0 \qquad in \quad L^2(\Omega_1, \mathbb{R}^3), \qquad \overline{Q}^\natural_{e,h_j} \to \overline{Q}^\natural_{e,0} \qquad in \quad L^2(\Omega_1, SO(3)), \tag{5.6.6}$$

we have

$$\mathcal{J}_0(\varphi^\natural_0, \overline{Q}^\natural_{e,0}) \le \liminf_{h_j \to 0} \mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}). \tag{5.6.7}$$

Proof. It is clear that we may restrict our proof to the sequences $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in \mathcal{S}' \subset X'$, i.e., to sequences in which the functionals $\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j})$ are finite, since otherwise the statement is satisfied. In addition, any $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j})$ such that $\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) < \infty$ is uniformly bounded in X'. Therefore, there exists a subsequence (not relabeled) which is weakly convergent in X'. Due to the strong convergence of the original sequence, the considered subsequence is weakly convergent to $(\varphi^\natural, \overline{Q}^\natural_{e,0})$, i.e.,

$$\varphi^\natural_{h_j} \rightharpoonup \varphi^\natural_0 \qquad \text{in} \quad \mathrm{L}^2(\Omega_1, \mathbb{R}^3), \qquad \overline{Q}^\natural_{e,h_j} \rightharpoonup \overline{Q}^\natural_{e,0} \qquad \text{in} \quad \mathrm{L}^2(\Omega_1, \mathrm{SO}(3)). \tag{5.6.8}$$

Therefore, we have the weak convergence $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j})$ (without relabeling it) to $(\varphi^\natural_0, \overline{Q}^\natural_{e,0})$ in $\mathrm{H}^1(\omega, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\omega, \mathrm{SO}(3))$. For $\overline{U}^\natural_h = \overline{Q}^{\natural,T}_e \nabla^h_\eta \varphi^\natural [(\nabla_x \Theta)^\natural]^{-1}$ we have

$$W_{\mathrm{mp}}(\overline{U}^\natural_h) = \mu\, \|\mathrm{sym}(\overline{U}^\natural_h - \mathbb{1}_3)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U}^\natural_h - \mathbb{1}_3)\|^2 + \frac{\lambda}{2}[\mathrm{tr}(\mathrm{sym}(\overline{U}^\natural_h - \mathbb{1}_3))]^2\,, \tag{5.6.9}$$

while for $\mathcal{E}_{\varphi^\natural, \overline{Q}^\natural_e} = \mathcal{E}^{\parallel}_{\varphi^\natural, \overline{Q}^\natural_e} + \mathcal{E}^{\perp}_{\varphi^\natural, \overline{Q}^\natural_e}$ with $\mathcal{E}_{\varphi^\natural, \overline{Q}^\natural_e} = (\overline{Q}^{\natural,T}_e \nabla_{(\eta_1,\eta_2)} \varphi^\natural - [\nabla y_0]^\natural | 0)[(\nabla_x \Theta)^\natural]^{-1}$ we have

$$W^{\mathrm{hom},\natural}_{\mathrm{mp}}(\mathcal{E}_{\varphi^\natural, \overline{Q}^\natural_e}) = \mu\, \|\mathrm{sym}\, \mathcal{E}^{\parallel}_{\varphi^\natural, \overline{Q}^\natural_e}\|^2 + \mu_c\, \|\mathrm{skew}\, \mathcal{E}^{\parallel}_{\varphi^\natural, \overline{Q}^\natural_e}\|^2 + \frac{\lambda\, \mu}{\lambda + 2\,\mu}\, \big[\mathrm{tr}(\mathcal{E}^{\parallel}_{\varphi^\natural, \overline{Q}^\natural_e})\big]^2 + \frac{2\,\mu\, \mu_c}{\mu_c + \mu} \|\mathcal{E}^{\perp}_{\varphi^\natural, \overline{Q}^\natural_e}\|^2\,.$$

Hence, for the sequence $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in \mathrm{H}^1(\Omega_1, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3))$ where $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \to (\varphi^\natural_0, \overline{Q}^\natural_{e,0})$ with $J^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) < \infty$, we have

$$W_{\mathrm{mp}}(\overline{Q}^{\natural,T}_{e,h_j} \nabla^{h_j}_\eta \varphi^\natural_{h_j} [(\nabla_x \Theta)^\natural]^{-1}) = W_{\mathrm{mp}}\Big(\overline{Q}^{\natural,T}_{e,h_j} (\nabla_{(\eta_1,\eta_2)} \varphi^\natural_{h_j} | \frac{1}{h_j} \partial_{\eta_3} \varphi^\natural_{h_j})[(\nabla_x \Theta)^\natural]^{-1}\Big) \ge W^{\mathrm{hom},\natural}_{\mathrm{mp}}\Big(\mathcal{E}_{\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}}\Big), \tag{5.6.10}$$

where we recall that $\mathcal{E}_{\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}} := (\overline{Q}^{\natural,T}_{e,h_j} \nabla_{(\eta_1,\eta_2)} \varphi^\natural_{h_j} - (\nabla y_0)^\natural | 0)[(\nabla_x \Theta)^\natural]^{-1}\,.$

Then by taking the integral over Ω_1 on both sides and taking the lim inf for h_j, we obtain

$$\liminf_{h_j \to 0} \int_{\Omega_1} W_{\mathrm{mp}}(\overline{Q}^{\natural,T}_{e,h_j} \nabla^{h_j}_\eta \varphi^\natural_{h_j} [(\nabla_x \Theta)^\natural]^{-1})\, \det[\nabla_x \Theta]^\natural(\eta)\, dV_\eta \ge \liminf_{h_j \to 0} \int_{\Omega_1} W^{\mathrm{hom},\natural}_{\mathrm{mp}}\Big(\mathcal{E}_{\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}}\Big)\, \det[\nabla_x \Theta]^\natural(\eta)\, dV_\eta\,.$$

In the expression of $\mathcal{E}_{\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}}$, the quantity $[\nabla_x \Theta]^{-1}$ is evaluated in $(x_1, x_2, x_3) = (\eta_1, \eta_2, h\,\eta_3)$. Therefore, we have to study its behaviour for $h_j \to 0$. In addition, we recall the convergence results [43, Lemma 1]:

$$\begin{aligned}
\lim_{h_j \to 0} \det[\nabla_x \Theta]^\natural(\eta_1, \eta_2, \eta_3) &= \lim_{h_j \to 0} \det[\nabla_x \Theta]^\natural(x_1, x_2, \frac{1}{h_j} x_3) = \det[\nabla_x \Theta]^\natural(\eta_1, \eta_2, 0) \\
&= \det(\nabla y_0 | n_0) \quad \text{ in } \quad C^0(\overline{\Omega}), \\
\lim_{h_j \to 0} [(\nabla_x \Theta)^{-1}]^\natural(\eta_1, \eta_2, \eta_3) &= \lim_{h_j \to 0} [(\nabla_x \Theta)^{-1}]^\natural(x_1, x_2, \frac{1}{h_j} x_3) = [(\nabla_x \Theta)^{-1}]^\natural(\eta_1, \eta_2, 0) \\
&=: (\nabla_x \Theta)^{-1}(0) \text{ in } \quad C^0(\overline{\Omega}).
\end{aligned} \tag{5.6.11}$$

Due to (5.6.11), the weak convergence of the sequence $\varphi^\natural_{h_j}$ and the strong convergence of the sequence $\overline{Q}^\natural_{e,h_j}$, we have the weak convergence

$$\begin{aligned}\mathcal{E}_{\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j}} &= (\overline{Q}^{\natural,T}_{e,h_j}\nabla_{(\eta_1,\eta_2)}\varphi^\natural_{h_j} - (\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}\\ &\rightharpoonup (\overline{Q}^{\natural,T}_{e,0}\nabla_{(\eta_1,\eta_2)}\varphi^\natural_0 - (\nabla y_0)^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}(0) =: \mathcal{E}_{\varphi^\natural_0,\overline{Q}^\natural_{e,0}}. \end{aligned} \tag{5.6.12}$$

Using again (5.6.11), the convexity of the energy function $W^{\text{hom},\natural}_{\text{mp}}$ with respect to $\mathcal{E}_{\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j}}$, the Fatou's Lemma, the characterization of lim inf and the weak convergence (5.6.12) we get

$$\liminf_{h_j\to 0}\int_{\Omega_1} W^{\text{hom},\natural}_{\text{mp}}\Big(\mathcal{E}_{\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j}}\Big)\det[\nabla_x\Theta]^\natural(\eta)\,dV_\eta \ge \int_{\Omega_1} W^{\text{hom},\natural}_{\text{mp}}\Big(\mathcal{E}_{\varphi^\natural_0,\overline{Q}^\natural_{e,0}}\Big)\det(\nabla y_0|n_0)\,dV_\eta\,. \tag{5.6.13}$$

Since both $\varphi^\natural_0$ and $\overline{Q}^\natural_{e,0}$ are independent of the transerve variable η_3, we also obtain

$$\begin{aligned}\liminf_{h_j\to 0}\int_{\Omega_1} W_{\text{mp}}(\overline{Q}^{\natural,T}_{e,h_j}\nabla^{h_j}_\eta\varphi^\natural_{h_j}[(\nabla_x\Theta)^\natural]^{-1})\det[\nabla_x\Theta]^\natural(\eta)\,dV_\eta &\ge \int_{-\frac12}^{\frac12}\int_\omega W^{\text{hom},\natural}_{\text{mp}}\Big(\mathcal{E}_{\varphi^\natural_0,\overline{Q}^\natural_{e,0}}\Big)\det(\nabla y_0|n_0)\,dV_\eta\\ &= \int_\omega W^{\text{hom}}_{\text{mp}}\Big(\mathcal{E}_{m,\overline{Q}_{e,0}}\Big)\det(\nabla y_0|n_0)\,d\omega\,. \end{aligned} \tag{5.6.14}$$

We do the same process for the curvature energy, by using (5.5.47), the convexity of $\widetilde{W}^{\text{hom}}_{\text{curv}}$ in its argument and the weak convergence

$$\begin{aligned}&\Big(\text{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h_j})\,|\,\text{axl}(\overline{Q}^{\natural,T}_{e,h_j}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h_j})\,|0\Big)[(\nabla_x\Theta)^\natural(\eta)\,]^{-1}\\ &\qquad\rightharpoonup \Big(\text{axl}(\overline{Q}^{\natural,T}_{e,0}\,\partial_{\eta_1}\overline{Q}^\natural_{e,0})\,|\,\text{axl}(\overline{Q}^{\natural,T}_{e,0}\,\partial_{\eta_2}\overline{Q}^\natural_{e,0})\,|0\Big)[\nabla_x\Theta(0)\,]^{-1}\,. \end{aligned} \tag{5.6.15}$$

Using also (5.6.11), we arrive at

$$\begin{aligned}\liminf_{h_j}\int_{\Omega_1}\widetilde{W}_{\text{curv}}(\Gamma^\natural_h)\det[\nabla_x\Theta]^\natural(\eta)\,dV_\eta &\ge \liminf_{h_j}\int_{\Omega_1}\widetilde{W}^{\text{hom},\natural}_{\text{curv}}(\mathcal{K}^\natural_e)\det[\nabla_x\Theta]^\natural(\eta)\,dV_\eta\\ &\ge \liminf_{h_j}\int_{\Omega_1}\widetilde{W}^{\text{hom}}_{\text{curv}}(\mathcal{K}_{e,s})\det[\nabla_x\Theta]^\natural(\eta)\,dV_\eta \ge \int_{\Omega_1}\widetilde{W}^{\text{hom}}_{\text{curv}}(\mathcal{K}_{e,s})\det(\nabla y_0|n_0)\,dV_\eta\\ &= \int_{-\frac12}^{\frac12}\int_\omega \widetilde{W}^{\text{hom}}_{\text{curv}}(\mathcal{K}_{e,s})\det(\nabla y_0|n_0)\,dV_\eta = \int_\omega \widetilde{W}^{\text{hom}}_{\text{curv}}(\mathcal{K}_{e,s})\det(\nabla y_0|n_0)\,d\omega\,. \end{aligned} \tag{5.6.16}$$

Since, $W_{\text{mp}}(\overline{Q}^{\natural,T}_{e,h_j}\nabla^{h_j}_\eta\varphi^\natural_{h_j}[(\nabla_x\Theta)^\natural]^{-1}) > 0$ and $\widetilde{W}_{\text{curv}}(\Gamma^\natural_h) > 0$, by combining (5.6.14) and (5.6.16) we deduce

$$\begin{aligned}&\liminf_{h_j}\int_{\Omega_1}[W_{\text{mp}}(\overline{Q}^{\natural,T}_{e,h_j}\nabla^{h_j}_\eta\varphi^\natural_{h_j}[(\nabla_x\Theta)^\natural]^{-1}) + \widetilde{W}_{\text{curv}}(\Gamma^\natural_h)]\det[\nabla_x\Theta]^\natural(\eta)\,dV_\eta \qquad (5.6.17)\\ &\qquad\ge \int_\omega\Big(W^{\text{hom}}_{\text{mp}}(\mathcal{E}_{m,\overline{Q}_{e,0}}) + \widetilde{W}^{\text{hom}}_{\text{curv}}(\mathcal{K}_{e,s})\Big)\det(\nabla y_0|n_0)\,d\omega = \mathcal{J}_0(m,\overline{Q}_{e,0})\,,\end{aligned}$$

where we have used that $\overline{Q}^\natural_{e,0}\equiv\overline{Q}_{e,0}$ and $m=\varphi_0$. Hence, the lim-inf inequality, (5.6.7) is proven. ■

5.6.2. Step 2 of the proof: The lim-sup condition - recovery sequence

Now we show the following lemma

Lemma 5.6.3. *In the hypothesis of Theorem 5.6.1, for all* $(\varphi^\natural_0,\overline{Q}^\natural_{e,0})\in L^2(\Omega_1)\times L^2(\Omega_1,SO(3))$ *there exists* $(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j})\in L^2(\Omega_1)\times L^2(\Omega_1,SO(3))$ *with* $(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j})\to(\varphi^\natural_0,\overline{Q}^\natural_{e,0})$ *such that*

$$\mathcal{J}_0(\varphi^\natural_0,\overline{Q}^\natural_{e,0}) \ge \limsup_{h_j\to 0}\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j}). \tag{5.6.18}$$

Proof. Similar to the case of the lim-inf inequality, we can restrict our attention to sequences $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$ such that $\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) < \infty$. Therefore, the sequence $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$ has a weakly convergent subsequence in X', and we can focus on the space $\mathrm{H}^1(\Omega_1, \mathbb{R}^3) \times \mathrm{W}^{1,2}(\Omega_1, \mathrm{SO}(3))$.
One of the requirements for Γ-convergence, is the existence of a recovery sequence. Thus, the idea is to define an expansion for the deformation and the microrotation through the thickness. In reality, the minimizers of the energy model can be a good candidate for constructing the recovery sequence. To do so, we look at the first order Taylor expansion of the nonlinear deformation $\varphi^\natural_{h_j}$ in thickness direction η_3

$$\varphi^\natural_{h_j}(\eta_1, \eta_2, \eta_3) = \varphi^\natural_{h_j}(\eta_1, \eta_2, 0) + \eta_3\, \partial_{\eta_3} \varphi^\natural_{h_j}(\eta_1, \eta_2, 0)\,. \tag{5.6.19}$$

With the formula

$$d^* = \Big(1 - \frac{\lambda}{2\,\mu + \lambda} \langle \mathcal{E}_{m,s}, \mathbb{1}_3\rangle\Big) \overline{Q}^\natural_{e,0} n_0 + \frac{\mu_c - \mu}{\mu_c + \mu}\, \overline{Q}^\natural_{e,0} \mathcal{E}^T_{m,s} n_0\,, \tag{5.6.20}$$

and replacing $\frac{1}{h_j}\partial_{\eta_3}\varphi^\natural_{h_j}(\eta_1, \eta_2, 0)$ with $d^*(\eta_1, \eta_2)$, which means $\partial_{\eta_3}\varphi^\natural_{h_j}(\eta_1, \eta_2, 0) = h_j d^*(\eta_1, \eta_2)$, we make an ansatz for our recovery sequence as following

$$\varphi^\natural_{h_j}(\eta_1, \eta_2, \eta_3) := \varphi^\natural_0(\eta_1, \eta_2) + h_j\, \eta_3\, d^*(\eta_1, \eta_2). \tag{5.6.21}$$

Since $\nabla_{(\eta_1,\eta_2)}\varphi^\natural \in \mathrm{L}^2(\omega, \mathbb{R}^3)$ and $\overline{Q}_{e,0} \in \mathrm{SO}(3)$, we obtain that d^* belongs to $\mathrm{L}^2(\omega, \mathbb{R}^3)$ and by letting $h_j \to 0$, it can be seen that for this ansatz $\varphi^\natural_{h_j} \to \varphi_0$.
The reconstruction for the rotation $\overline{Q}_{e,0}$ is not obvious, since on the one hand we have to maintain the rotation constraint along the sequence and on the other hand we must approach the lower bound, which excludes the simple reconstruction $\overline{Q}^\natural_{e,h_j}(\eta_1, \eta_2, \eta_3) = \overline{Q}_{e,0}(\eta_1, \eta_2)$. In order to meet both requirements we consider therefore

$$\overline{Q}^\natural_{e,h_j}(\eta_1, \eta_2, \eta_3) := \overline{Q}_{e,0}(\eta_1, \eta_2) \cdot \exp(h_j\, \eta_3\, A^*(\eta_1, \eta_2)), \tag{5.6.22}$$

where $A^* \in \mathfrak{so}(3)$ is the term obtained in (5.5.46), depending on the given $\overline{Q}_{e,0}$, and we note that $A^* \in \mathrm{L}^2(\omega, \mathfrak{so}(3))$ by the coercivity of $\widetilde{W}_{\text{curv}}$. Since $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, we obtain that $\overline{Q}^\natural_{e,h_j} \in \mathrm{SO}(3)$ and for $h_j \to 0$, we have $\overline{Q}^\natural_{e,h_j} \to \overline{Q}_{e,0} \in \mathrm{L}^2(\Omega_1, \mathrm{SO}(3))$.
Since d^* need not to be differentiable, we should consider another modified recovery sequence. For fixed $\varepsilon > 0$, we select $d_\varepsilon \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^3)$ such that $\|d_\varepsilon - d^*\|_{\mathrm{L}^2(\omega,\mathbb{R}^3)} < \varepsilon$. Therefore, accordingly we define the final recovery sequence for the deformation as following

$$\varphi^\natural_{h_j,\varepsilon}(\eta_1, \eta_2, \eta_3) := \varphi^\natural_0(\eta_1, \eta_2) + h_j\, \eta_3\, d_\varepsilon(\eta_1, \eta_2). \tag{5.6.23}$$

The same argument holds for A^*: for fixed $\varepsilon > 0$ we may choose $A_\varepsilon \in \mathrm{W}^{1,2}(\omega, \mathfrak{so}(3))$ such that $\|A_\varepsilon - A^*\|_{\mathrm{L}^2(\omega,\mathfrak{so}(3))} < \varepsilon$. Hence, the final recovery sequence for the microrotation is like

$$\overline{Q}^\natural_{e,h_j,\varepsilon}(\eta_1, \eta_2, \eta_3) := \overline{Q}_{e,0}(\eta_1, \eta_2) \cdot \exp(h_j\, \eta_3\, A_\varepsilon(\eta_1, \eta_2)). \tag{5.6.24}$$

The gradient of the new recovery sequence of deformation is

$$\begin{aligned}
\nabla_\eta \varphi^\natural_{h_j,\varepsilon}(\eta_1, \eta_2, \eta_3) &= (\nabla_{(\eta_1,\eta_2)}\varphi^\natural_0(\eta_1, \eta_2)|0) + h_j(0|d_\varepsilon(\eta_1, \eta_2)) + h_j\eta_3(\nabla_{(\eta_1,\eta_2)} d_\varepsilon(\eta_1, \eta_2)|0) \\
&= (\nabla\varphi^\natural_0(\eta_1, \eta_2)|h_j d_\varepsilon(\eta_1, \eta_2)) + h_j\eta_3(\nabla d_\varepsilon(\eta_1, \eta_2)|0),
\end{aligned} \tag{5.6.25}$$

and the different terms in the curvature energy are

$$\begin{aligned}
\overline{Q}^{\natural,T}_{e,h_j,\varepsilon}\partial_{\eta_1}\overline{Q}^\natural_{e,h_j,\varepsilon} &= \exp(h_j\eta_3 A_\varepsilon)^T \overline{Q}^T_{e,0}[\partial_{\eta_1}\overline{Q}_{e,0}\exp(h_j\eta_3 A_\varepsilon) + \overline{Q}_{e,0} D\exp(h_j\eta_3 A_\varepsilon).[h_j\eta_3\partial_{\eta_1} A_\varepsilon]], \\
\overline{Q}^{\natural,T}_{e,h_j,\varepsilon}\partial_{\eta_2}\overline{Q}^\natural_{e,h_j,\varepsilon} &= \exp(h_j\eta_3 A_\varepsilon)^T \overline{Q}^T_{e,0}[\partial_{\eta_2}\overline{Q}_{e,0}\exp(h_j\eta_3 A_\varepsilon) + \overline{Q}_{e,0} D\exp(h_j\eta_3 A_\varepsilon).[h_j\eta_3\partial_{\eta_2} A_\varepsilon]], \\
\overline{Q}^{\natural,T}_{e,h_j,\varepsilon}\partial_{\eta_3}\overline{Q}^\natural_{e,h_j,\varepsilon} &= \exp(h_j\eta_3 A_\varepsilon)^T \overline{Q}^T_{e,0}[\partial_{\eta_3}\overline{Q}_{e,0}\exp(h_j\eta_3 A_\varepsilon) + \overline{Q}_{e,0} D\exp(h_j\eta_3 A_\varepsilon).[h_j A_\varepsilon]] \\
&= h_j\exp(h_j\eta_3 A_\varepsilon(\eta_1, \eta_2))^T D\exp(h_j\eta_3 A_\varepsilon(\eta_1, \eta_2)).[A_\varepsilon],
\end{aligned} \tag{5.6.26}$$

with $\partial_{\eta_i} A_\varepsilon \in \mathfrak{so}(3)$. Now we introduce the quantities

$$
\begin{aligned}
\widetilde{U}_0 &= \overline{Q}_{e,0}^T(\nabla\varphi_0(\eta_1,\eta_2)|d^*(\eta_1,\eta_2))[(\nabla_x\Theta)(0)]^{-1},\\
\widetilde{U}_{h_j}^\varepsilon &= \overline{Q}_{e,h_j,\varepsilon}^{\natural,T}\Big((\nabla\varphi_0(\eta_1,\eta_2)|d_\varepsilon(\eta_1,\eta_2)) + h_j\eta_3(\nabla d_\varepsilon(\eta_1,\eta_2)|0)\Big)[(\nabla_x\Theta)^\natural(\eta)]^{-1},\\
\widetilde{U}_0^\varepsilon &= \overline{Q}_{e,0}^T(\nabla\varphi_0(\eta_1,\eta_2)|d_\varepsilon(\eta_1,\eta_2))[(\nabla_x\Theta)(0)]^{-1}, \qquad (5.6.27)\\
\Gamma_{h_j,\varepsilon}^\natural &:= \Big(\underbrace{\operatorname{axl}\Big(\overline{Q}_{e,h_j,\varepsilon}^{\natural,T}\partial_{\eta_1}\overline{Q}_{e,h_j,\varepsilon}^\natural\Big)}_{:=\Gamma_{h_j,\varepsilon}^{1,\natural}} \,\Big|\, \underbrace{\operatorname{axl}\Big(\overline{Q}_{e,h_j,\varepsilon}^{\natural,T}\partial_{\eta_2}\overline{Q}_{e,h_j,\varepsilon}^\natural\Big)}_{:=\Gamma_{h_j,\varepsilon}^{2,\natural}} \,\Big|\, \underbrace{\frac{1}{h_j}\operatorname{axl}\Big(\overline{Q}_{e,h_j,\varepsilon}^{\natural,T}\partial_{\eta_3}\overline{Q}_{e,h_j,\varepsilon}^\natural\Big)}_{:=\Gamma_{h_j,\varepsilon}^{3,\natural}}\Big)[(\nabla_x\Theta)^\natural(\eta)]^{-1},\\
\Gamma_0 &:= \Big(\underbrace{\operatorname{axl}\Big(\overline{Q}_{e,0}^T\partial_{\eta_1}\overline{Q}_{e,0}\Big)}_{:=\Gamma_0^1} \,\Big|\, \underbrace{\operatorname{axl}\Big(\overline{Q}_{e,0}^T\partial_{\eta_2}\overline{Q}_{e,0}\Big)}_{:=\Gamma_0^2} \,\Big|\, 0\Big)\,[(\nabla_x\Theta)(0)]^{-1}\,.
\end{aligned}
$$

Note that

$$
\Gamma_{h_j,\varepsilon}^{3,\natural} := \operatorname{axl}\Big(\exp(h_j\eta_3 A_\varepsilon(\eta_1,\eta_2))^T\,\mathrm{Dexp}(h_j\eta_3 A_\varepsilon(\eta_1,\eta_2)).[A_\varepsilon]\Big). \qquad (5.6.28)
$$

It holds

$$
\begin{aligned}
&\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0^\varepsilon\| \to 0, \quad \text{as} \quad h_j \to 0, && \|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\| \to 0, \quad \text{as} \quad h_j \to 0, \varepsilon \to 0,\\
&\|\Gamma_{h_j,\varepsilon}^{i,\natural} - \Gamma_0^i\| \to 0, \quad \text{as} \quad h_j \to 0, \varepsilon \to 0, \quad i = 1,2\,, && \|\Gamma_{h_j,\varepsilon}^{3,\natural} - \operatorname{axl} A_\varepsilon\| \to 0, \quad \text{as} \quad h_j \to 0.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\|\widetilde{U}_0^\varepsilon - \widetilde{U}_0\|^2 &= \|\overline{Q}_{e,0}^T(\nabla\varphi_0|d_\varepsilon)[\nabla_x\Theta(0)]^{-1} - \overline{Q}_{e,0}^T(\nabla\varphi_0|d^*)[\nabla_x\Theta(0)]^{-1})\|^2\\
&= \|\overline{Q}_{e,0}^T(0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}\|^2\\
&= \langle\overline{Q}_{e,0}^T(0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}, \overline{Q}_{e,0}^T(0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}\rangle\\
&= \langle\overline{Q}_{e,0}\overline{Q}_{e,0}^T(0|0|d_\varepsilon - d^*), (0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}[\nabla_x\Theta(0)]^{-T}\rangle \qquad (5.6.29)\\
&= \langle(0|0|d_\varepsilon - d^*), (0|0|d_\varepsilon - d^*)(\widehat{\mathrm{I}}_{y_0})^{-1}\rangle = \langle(0|0|d_\varepsilon - d^*)^T(0|0|d_\varepsilon - d^*), (\widehat{\mathrm{I}}_{y_0})^{-1}\rangle\\
&= \langle(0|0|(d_\varepsilon - d^*)^T(d_\varepsilon - d^*)), (\widehat{\mathrm{I}}_{y_0})^{-1}\rangle = \langle d_\varepsilon - d^*, d_\varepsilon - d^*\rangle = \|d_\varepsilon - d^*\|^2 \;\to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

We may write

$$
\begin{aligned}
\mathcal{J}_{h_j}^{\natural,\mathrm{mp}}(\varphi_{h_j,\varepsilon}^\natural, \overline{Q}_{e,h_j,\varepsilon}^\natural) &:= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{U}_{h_j}^\varepsilon)\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta\\
&= \int_{\Omega_1}\Big[W_{\mathrm{mp}}(\widetilde{U}_{h_j}^\varepsilon) - W_{\mathrm{mp}}(\widetilde{U}_0) + W_{\mathrm{mp}}(\widetilde{U}_0)\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta \qquad (5.6.30)\\
&= \int_{\Omega_1}\Big[W_{\mathrm{mp}}(\widetilde{U}_{h_j}^\varepsilon + \widetilde{U}_0 - \widetilde{U}_0) - W_{\mathrm{mp}}(\widetilde{U}_0) + W_{\mathrm{mp}}(\widetilde{U}_0)\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta\\
&\le \int_{\Omega_1}\Big[|W_{\mathrm{mp}}(\widetilde{U}_{h_j}^\varepsilon + \widetilde{U}_0 - \widetilde{U}_0) - W_{\mathrm{mp}}(\widetilde{U}_0)| + W_{\mathrm{mp}}(\widetilde{U}_0)\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta,
\end{aligned}
$$

where we used that W_{mp} is positive. The exact quadratic expansion in the neighborhood of the point $\widetilde{U}_{h_j}^\varepsilon = \widetilde{U}_0 + \widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0$ for W_{mp} is given by

$$
W_{\mathrm{mp}}(\widetilde{U}_0 + \widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0) = W_{\mathrm{mp}}(\widetilde{U}_0) + \big\langle DW_{\mathrm{mp}}(\widetilde{U}_0), \widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\big\rangle + \frac{1}{2}D^2W_{\mathrm{mp}}(\widetilde{U}_0).(\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0, \widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0)\,.
$$

Therefore, with the assumption that $\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\| \le 1$, we have the following relations

$$
\begin{aligned}
&\mathcal{J}_{h_j}^{\natural,\mathrm{mp}}(\varphi_{h_j,\varepsilon}^\natural, \overline{Q}_{e,h_j,\varepsilon}^\natural)\\
&\quad\le \int_{\Omega_1}\Big[W_{\mathrm{mp}}(\widetilde{U}_0) + \|DW_{\mathrm{mp}}(\widetilde{U}_0)\|\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\| + \frac{1}{4}\|D^2W_{\mathrm{mp}}(\widetilde{U}_0)\|\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\|^2\,\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta\\
&\quad\le \int_{\Omega_1}\Big[W_{\mathrm{mp}}(\widetilde{U}_0) + C\|\widetilde{U}_0\|\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\| + C_1\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\|\,\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta\\
&\quad\le \int_{\Omega_1}\Big[W_{\mathrm{mp}}(\widetilde{U}_0) + (C\|\widetilde{U}_0\| + C_1)\|\widetilde{U}_{h_j}^\varepsilon - \widetilde{U}_0\|\,\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta, \qquad (5.6.31)
\end{aligned}
$$

where C and C_1 are upper bounds for $\|DW_{\rm mp}(\widetilde{U}_0)\|$ and $\|D^2W_{\rm mp}(\widetilde{U}_0)\|$, respectively. Now we consider the terms of $\widetilde{W}_{\rm curv}$

$$\begin{aligned}
\mathcal{J}^{\natural,\rm curv}_{h_j}(\Gamma^\natural_{h_j,\varepsilon}) &:= \int_{\Omega_1} \widetilde{W}_{\rm curv}\Big((\Gamma^{1,\natural}_{h_j,\varepsilon},\Gamma^{2,\natural}_{h_j,\varepsilon},\Gamma^{3,\natural}_{h_j,\varepsilon})[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta\\
&\le \int_{\Omega_1}\Big[\widetilde{W}_{\rm curv}\Big((\Gamma^{1,\natural}_{h_j,\varepsilon},\Gamma^{2,\natural}_{h_j,\varepsilon},\Gamma^{3,\natural}_{h_j,\varepsilon})[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)-\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A_\varepsilon)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\\
&\qquad+\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A_\varepsilon)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)-\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A^*)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\\
&\qquad+\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A^*)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta \qquad (5.6.32)\\
&\le \int_{\Omega_1}\Big[\Big|\widetilde{W}_{\rm curv}\Big((\Gamma^{1,\natural}_{h_j,\varepsilon},\Gamma^{2,\natural}_{h_j,\varepsilon},\Gamma^{3,\natural}_{h_j,\varepsilon})[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)-\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A_\varepsilon)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\Big|\\
&\qquad+\Big|\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A_\varepsilon)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)-\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A^*)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\Big|\\
&\qquad+\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A^*)[(\nabla_x\Theta)^\natural(\eta)]^{-1}\Big)\Big]\,\det((\nabla_x\Theta)^\natural)(\eta)\,dV_\eta,
\end{aligned}$$

where we have used the triangle inequality.

Note that beside the boundedness of $\det[\nabla_x\Theta]^\natural(0)$, due to the hypothesis that $\det[\nabla_x\Theta(0)]\ge a_0>0$, it follows that there exits a constant $C>0$ such that

$$\forall\, x\in\overline{\omega}:\qquad \|[\nabla_x\Theta(0)]^{-1}\|\le C. \tag{5.6.33}$$

We notice that both energies are positive and $\det[\nabla_x\Theta](0)$ is bounded. Also $\widetilde{W}_{\rm curv}$ is continuous and $\|A_\varepsilon-A^*\|_{{\rm L}^2(\omega,\mathfrak{so}(3))}<\varepsilon$. By using (5.5.46) and (5.6.11), and applying $\limsup_{h_j\to0}$ on both sides of (5.6.31) and (5.6.32) with $h_j\to0$ and $\varepsilon\to0$ we get

$$\begin{aligned}
\limsup_{h_j\to0}\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j,\varepsilon},\overline{Q}^\natural_{e,h_j,\varepsilon}) &\le \int_{\Omega_1}(W_{\rm mp}(\widetilde{U}_0)+\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,A^*)[(\nabla_x\Theta)^\natural(0)]^{-1}\Big)\,\det[\nabla_x\Theta](0)\,dV_\eta\\
&=\int_{\Omega_1}(W_{\rm mp}(\widetilde{U}_0)+\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,0)[(\nabla_x\Theta)(0)]^{-1}\Big)\,\det[\nabla_x\Theta](0)\,dV_\eta.
\end{aligned} \tag{5.6.34}$$

However, $W_{\rm mp}(\widetilde{U}_0)$ and $\widetilde{W}_{\rm curv}\Big((\Gamma^1_0,\Gamma^2_0,0)[(\nabla_x\Theta)(0)]^{-1}\Big)$ are already independent of the third variable η_3, hence we deduce

$$\limsup_{h_j\to0}\mathcal{J}^\natural_{h_j}(\varphi^\natural_{h_j,\varepsilon},\overline{Q}^\natural_{e,h_j,\varepsilon})\le\mathcal{J}_0(m,\overline{Q}_{e,0}),\qquad \varphi^\natural\equiv\varphi,\qquad \overline{Q}^\natural_{e,0}\equiv\overline{Q}_{e,0}\qquad\text{and}\qquad m=\varphi_0\,. \qquad \blacksquare$$

5.7. The Gamma-limit including loads

The main result of this chapter is the following theorem

Theorem 5.7.1. *Assume that the initial configuration is defined by a continuous injective mapping* $y_0:\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ *which admits an extension to* $\overline{\omega}$ *into* $C^2(\overline{\omega};\mathbb{R}^3)$ *such that* $\det[\nabla_x\Theta(0)]\ge a_0>0$ *on* $\overline{\omega}$ *where* a_0 *is a constant, and assume that the boundary data satisfy the conditions*

$$\varphi^\natural_d=\varphi_d\big|_{\Gamma_1}\text{(in the sense of traces) for }\ \varphi_d\in H^1(\Omega_1;\mathbb{R}^3). \tag{5.7.1}$$

Let the constitutive parameters satisfy

$$\mu>0,\qquad \kappa>0,\qquad \mu_c>0,\qquad a_1>0,\qquad a_2>0,\qquad a_3>0, \tag{5.7.2}$$

Then, for any sequence $(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j})\in X$ *such that* $(\varphi^\natural_{h_j},\overline{Q}^\natural_{e,h_j})\to(\varphi_0,\overline{Q}_{e,0})$ *as* $h_j\to0$*, the sequence of functionals* $\mathcal{I}_{h_j}:X\to\overline{\mathbb{R}}$

$$\mathcal{I}^\natural_{h_j}(\varphi^\natural,\nabla^{h_j}_\eta\varphi^\natural,\overline{Q}^\natural_e,\Gamma^\natural_{h_j})=\begin{cases}\dfrac{1}{h_j}J^\natural_{h_j}(\varphi^\natural,\nabla^{h_j}_\eta\varphi^\natural,\overline{Q}^\natural_e,\Gamma^\natural_{h_j})-\dfrac{1}{h}\Pi^\natural_{h_j}(\varphi^\natural,\overline{Q}^\natural_e) & \text{if } (\varphi^\natural,\overline{Q}^\natural_e)\in\mathcal{S}',\\ +\infty & \text{else in } X\,,\end{cases} \tag{5.7.3}$$

Γ-converges to the limit energy functional $\mathcal{I}_0 \colon X \to \overline{\mathbb{R}}$ defined by

$$\mathcal{I}_0(m, \overline{Q}_{e,0}) = \begin{cases} \mathcal{J}_0(m, \overline{Q}_{e,0}) - \Pi(m, \overline{Q}_{e,0}) & \text{if} \quad (m, \overline{Q}_{e,0}) \in \mathcal{S}'_\omega\,, \\ +\infty & \text{else in } X, \end{cases} \tag{5.7.4}$$

where

$$\mathcal{J}_0(m, \overline{Q}_{e,0}) = \begin{cases} \int_\omega [W^{hom}_{mp}(\mathcal{E}_{m,\overline{Q}_{e,0}}) + \widetilde{W}^{hom}_{curv}(\mathcal{K}_{e,s})] \det(\nabla y_0 | n_0)\, d\omega & \text{if} \quad (m, \overline{Q}_{e,0}) \in \mathcal{S}'_\omega\,, \\ +\infty & \text{else in } X, \end{cases} \tag{5.7.5}$$

and $\Pi(m, \overline{Q}_{e,0}) = \Pi_{\widetilde{f},\omega}(\widetilde{u}_0) + \Pi_{\widetilde{c},\gamma_1}(\overline{Q}_{e,0})$ as the external load.

Remark 10. *Before proving the above theorem, we will give the expression of the external loads potential in Ω_1. We have*

$$\Pi^\natural_h(\varphi^\natural, \overline{Q}^\natural_e) = \Pi^\natural_f(\varphi^\natural) + \Pi^\natural_c(\overline{Q}^\natural_e)\,, \quad \Pi^\natural_f(\varphi^\natural) = h \int_{\Omega_1} \langle \widetilde{f}^\natural, \widetilde{u}^\natural \rangle\, dV_\eta\,, \quad \Pi^\natural_c(\overline{Q}^\natural_e) = h \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}^\natural_e \rangle\, dS_\eta\,, \tag{5.7.6}$$

with $\widetilde{f}^\natural(\eta) = \widetilde{f}(\zeta(\eta))$, $\widetilde{u}^\natural(\eta) = \widetilde{u}(\zeta(\eta))$, $\widetilde{c}^\natural(\zeta) = \widetilde{c}(\zeta(\eta))$, $\overline{Q}^\natural_e(\eta) = \overline{Q}_e(\zeta(\eta))$ and $\widetilde{u}^\natural(\eta_i) = \varphi^\natural(\eta_i) - \Theta^\natural(\eta_i)$. We use the following expressions

$$\begin{aligned} &\Theta^\natural(\eta) = y^\natural_0(\eta_1, \eta_2) + h_j \eta_3 n_0(\eta_1, \eta_2)\,, \qquad \varphi^\natural_{h_j}(\eta) = \varphi^\natural_0(\eta_1, \eta_2) + h_j \eta_3 d^*(\eta_1, \eta_2)\,, \\ &\widetilde{u}^\natural(\eta_i) = \varphi^\natural(\eta_i) - \Theta^\natural(\eta_i) = \underbrace{\Big(\varphi^\natural_0(\eta_1, \eta_2) - y^\natural_0(\eta_1, \eta_2)\Big)}_{\widetilde{u}_0(\eta_1, \eta_2)} + h_j \eta_3 \Big(d^*(\eta_1, \eta_2) - n_0(\eta_1, \eta_2)\Big)\,. \end{aligned} \tag{5.7.7}$$

We calculate the loads separately. We have

$$\begin{aligned} \Pi^\natural_f(\varphi^\natural_{h_j}) &= h_j \int_{\Omega_1} \langle \widetilde{f}^\natural, \widetilde{u}^\natural \rangle dV_\eta = h_j \int_{\Omega_1} \langle \widetilde{f}^\natural, \widetilde{u}_0(\eta_1, \eta_2) \rangle dV_\eta + h_j^2 \eta_3 \int_{\Omega_1} \langle \widetilde{f}^\natural, (d^*(\eta_1, \eta_2) - n_0(\eta_1, \eta_2)) \rangle dV_\eta \\ &= h_j \int_\omega \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \widetilde{f}^\natural, \widetilde{u}_0(\eta_1, \eta_2) \rangle d\eta_3\, d\omega + h_j^2 \int_\omega \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3 \langle \widetilde{f}^\natural, (d^*(\eta_1, \eta_2) - n_0(\eta_1, \eta_2)) \rangle d\eta_3\, d\omega \\ &= h_j \int_\omega \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{f}^\natural\, d\eta_3, \widetilde{u}_0(\eta_1, \eta_2) \rangle\, d\omega + h_j^2 \int_\omega \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3 \widetilde{f}^\natural\, d\eta_3, (d^* - n_0)(\eta_1, \eta_2) \rangle\, d\omega := \Pi_{\widetilde{f},\omega}(\widetilde{u}_0)\,. \end{aligned} \tag{5.7.8}$$

For applying the same method for the potential of external applied boundary surface couple, we need to have an approximation for the exponential function which is already used in the expression of the recovery sequence for the microrotation $\overline{Q}^\natural_{e,h_j}$, i.e., $\exp(X) = \mathbb{1} + X + \frac{1}{2!}X^2 + \cdots$, which implies

$$\overline{Q}^\natural_{e,h_j} = \overline{Q}_{e,0} \cdot \exp(h_j \eta_3 A^*(\eta_1, \eta_2)) = \overline{Q}_{e,0} + \overline{Q}_{e,0} h_j \eta_3 A^*(\eta_1, \eta_2) + \frac{1}{2} \overline{Q}_{e,0} h_j^2 \eta_3^2 A^*(\eta_1, \eta_2)^2 + \cdots\,. \tag{5.7.9}$$

Hence,

$$\begin{aligned} \Pi^\natural_c(\overline{Q}^\natural_{e,h_j}) &= h_j \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} + \overline{Q}_{e,0} h_j \eta_3 A^*(\eta_1, \eta_2) + \frac{1}{2} \overline{Q}_{e,0} h_j^2 \eta_3^2 A^*(\eta_1, \eta_2)^2 + \cdots \rangle\, dS_\eta \\ &= h_j \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} \rangle\, dS_\eta + h_j^2 \eta_3 \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} A^*(\eta_1, \eta_2) \rangle\, dS_\eta \\ &\quad + \frac{1}{2} h_j^3 \eta_3^2 \int_{\Gamma_1} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} A^*(\eta_1, \eta_2)^2 \rangle\, dS_\eta + \cdots \\ &= h_j \int_{(\gamma_1 \times [-\frac{1}{2}, \frac{1}{2}])} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} \rangle\, dS_\eta + h_j^2 \eta_3 \int_{(\gamma_1 \times [-\frac{1}{2}, \frac{1}{2}])} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} A^*(\eta_1, \eta_2) \rangle\, dS_\eta + O(h_j^3) \\ &= h_j \int_{\gamma_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} \rangle\, d\eta_3\, ds + h_j^2 \eta_3 \int_{\gamma_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \widetilde{c}^\natural, \overline{Q}_{e,0} A^*(\eta_1, \eta_2) \rangle\, d\eta_3\, ds + O(h_j^3) \\ &= \underbrace{h_j \int_{\gamma_1} \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{c}^\natural\, d\eta_3, \overline{Q}_{e,0} \rangle\, ds + h_j^2 \int_{\gamma_1} \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3 \widetilde{c}^\natural\, d\eta_3, \overline{Q}_{e,0} A^*(\eta_1, \eta_2) \rangle\, ds}_{:= \Pi_{\widetilde{c},\gamma_1}(\overline{Q}_{e,0})} + O(h_j^3)\,. \end{aligned} \tag{5.7.10}$$

Therefore,

$$\Pi^{\natural}_{h_j}(\varphi^{\natural},\overline{Q}^{\natural}_e)=\Pi_{\widetilde{f},\omega}(\widetilde{u}_0)+\Pi_{\widetilde{c},\gamma_1}(\overline{Q}_{e,0})+O(h_j^3)=\Pi(m,\overline{Q}_{e,0})+O(h^3_{h_j})\,,\qquad \widetilde{u}_0=m-y_0\,, \tag{5.7.11}$$

which regularity condition confirm the boundedness and continuity of external loads.

Now we come back to the proof of Theorem 5.7.1.

Proof of Theorem 5.7.1. As a first step we have considered the functionals

$$\mathcal{J}^{\natural}_h(\varphi^{\natural},\nabla^h_\eta\varphi^{\natural},\overline{Q}^{\natural}_e,\Gamma^{\natural}_h)=\begin{cases}\dfrac{1}{h}\,J^{\natural}_h(\varphi^{\natural},\nabla^h_\eta\varphi^{\natural},\overline{Q}^{\natural}_e,\Gamma^{\natural}_h) & \text{if } (\varphi^{\natural},\overline{Q}^{\natural}_e)\in\mathcal{S}',\\ +\infty & \text{else in } X.\end{cases} \tag{5.7.12}$$

In subsections 5.6.1 and 5.6.2, we have shown that the following inequality holds

$$\limsup_{h_j\to 0}\mathcal{J}^{\natural}_{h_j}(\varphi^{\natural}_{h_j,\varepsilon},\overline{Q}^{\natural}_{e,h_j,\varepsilon})\le\mathcal{J}_0(\varphi_0,\overline{Q}_{e,0})\le\liminf_{h_j\to 0}\mathcal{J}^{\natural}_{h_j}(\varphi^{\natural}_{h_j,\varepsilon},\overline{Q}^{\natural}_{e,h_j,\varepsilon})\,, \tag{5.7.13}$$

which implies that $\mathcal{J}_0(\varphi_0,\overline{Q}_{e,0})$ is the Γ-lim of the sequence $\mathcal{J}^{\natural}_{h_j}(\varphi^{\natural}_{h_j,\varepsilon},\overline{Q}^{\natural}_{e,h_j,\varepsilon})$, i.e.,

$$\mathcal{J}_0(\varphi_0,\overline{Q}_{e,0})=\Gamma\text{-}\lim(\mathcal{J}^{\natural}_{h_j}(\varphi^{\natural}_{h_j,\varepsilon},\overline{Q}^{\natural}_{e,h_j,\varepsilon}))\,,\qquad m\equiv\varphi_0\,. \tag{5.7.14}$$

Remark 10, shows that the family $(\mathcal{J}^{\natural}_{h_j}(\varphi^{\natural},\overline{Q}^{\natural}_e)-\Pi^{\natural}_{h_j}(\varphi^{\natural},\overline{Q}^{\natural}_e))_j$ is Γ-convergent (because the external load potential is continuous). This guarantees the existence of Γ-convergence for the family $(\mathcal{I}^{\natural}_{h_j})_j$. Therefore, we may write

$$\mathcal{I}_0(m,\overline{Q}_{e,0})=\Gamma\text{-}\lim\mathcal{I}^{\natural}_{h_j}(\varphi^{\natural}_{h_j,\varepsilon},\overline{Q}^{\natural}_{e,h_j,\varepsilon})=\mathcal{J}_0(m,\overline{Q}_{e,0})-\Pi(\varphi_0,\overline{Q}_{e,0})\,,\qquad m\equiv\varphi_0\,, \tag{5.7.15}$$

which is the desired formula. ■

5.8. Consistency with related shell and plate models

5.8.1. A comparison to the Cosserat plate model derived using the Γ-convergence method

In this part we check whether our model is consistent with the Cosserat plate model obtained in [90]. In the case of the plate model (flat initial configuration) we can assume that $\Theta(x_1,x_2,x_3)=(x_1,x_2,x_3)$ which gives $\nabla_x\Theta=\mathbb{1}_3$ and $y_0(x_1,x_2)=(x_1,x_2):=\mathrm{id}(x_1,x_2)$. Also $Q_0=\mathbb{1}_3$, $n_0=e_3$ and $\overline{Q}_{e,0}(x_1,x_2)=\overline{R}(x_1,x_2)$.

The family of functionals [27, 28] coincide with that considered in the analysis of Γ- convergence for a flat referential configuration, while its Γ-limit is

$$\mathcal{J}_0(m,\overline{R})=\begin{cases}\frac{\mathrm{d}}{\mathrm{d}\int_\omega}[W^{\mathrm{hom}}_{\mathrm{mp}}(\mathcal{E}^{plate}_{m,s})+\overline{W}^{\mathrm{hom}}_{\mathrm{curv}}(\mathcal{K}^{plate}_{e,s})]\,d\omega & \text{if } \ (m,\overline{R})\in\mathcal{S}'_\omega\,,\\ +\infty & \text{else in } X,\end{cases} \tag{5.8.1}$$

where

$$\begin{aligned}\mathcal{E}^{plate}_{m,s}&=\overline{R}^T(\nabla m|0)-\mathbb{1}^\flat_2=\overline{R}^T(\nabla m|0)-\mathbb{1}_3+e_3\otimes e_3,\\ \mathcal{K}^{plate}_{e,s}&=\Big(\mathrm{axl}(\overline{Q}^T_{e,0}\,\partial_{x_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}^T_{e,0}\,\partial_{x_2}\overline{Q}_{e,0})\,|0\Big)\notin\mathrm{Sym}(3)\,,\end{aligned} \tag{5.8.2}$$

and

$$\begin{aligned}W^{\mathrm{hom}}_{\mathrm{mp}}(\mathcal{E}^{plate}_{m,s})&=\mu\,\|\mathrm{sym}\,[\mathcal{E}^{plate}_{m,s}]^{\parallel}\|^2+\mu_{\mathrm{c}}\,\|\mathrm{skew}\,[\mathcal{E}^{plate}_{m,s}]^{\parallel}\|^2+\frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}([\mathcal{E}^{plate}_{m,s}]^{\parallel})\big]^2+\frac{2\,\mu\,\mu_{\mathrm{c}}}{\mu_c\,+\mu}\|[\mathcal{E}^{plate}_{m,s}]^T\,e_3\|^2\\ &=W_{\mathrm{shell}}\big([\mathcal{E}^{plate}_{m,s}]^{\parallel}\big)+\frac{2\,\mu\,\mu_{\mathrm{c}}}{\mu_c\,+\mu}\|[\mathcal{E}^{plate}_{m,s}]^{\perp}\|^2,\\ \overline{W}^{\mathrm{hom}}_{\mathrm{curv}}(\mathcal{K}^{plate}_{e,s})&=\inf_{A\in\mathfrak{so}(3)}\overline{W}^*_{\mathrm{curv}}\Big(\mathrm{axl}(\overline{R}^T\,\partial_{\eta_1}\overline{R})\,|\,\mathrm{axl}(\overline{R}^T\,\partial_{\eta_2}\overline{R})\,|\,\mathrm{axl}(A)\Big),\end{aligned} \tag{5.8.3}$$

together with

$$[\mathcal{E}^{\rm plate}_{m,s}]^{\parallel} := (\mathbb{1}_3 - e_3 \otimes e_3)\, [\mathcal{E}^{\rm plate}_{m,s}], \qquad [\mathcal{E}^{\rm plate}_{m,s}]^{\perp} := (e_3 \otimes e_3)\, [\mathcal{E}^{\rm plate}_{m,s}]\,, \tag{5.8.4}$$

where $W_{\rm shell}(X) = \mu \|{\rm sym}\, X\|^2 + \mu_c \|{\rm skew}\, X\|^2 + \frac{\lambda\mu}{\lambda+2\mu}[{\rm tr}(X)]^2$. Let us denote by $\overline{R}_i$ the columns of the matrix $\overline{R}$, i.e., $\overline{R} = \big(\overline{R}_1\,|\,\overline{R}_2\,|\,\overline{R}_3\big)$, $\overline{R}_i = \overline{R}\, e_i$. Since $(\mathbb{1}_3 - e_3 \otimes e_3)\overline{R}^T = \big(\overline{R}_1\,|\,\overline{R}_2\,|\,0\big)^T$, it follows that $[\mathcal{E}^{\rm plate}_{m,s}]^{\parallel} = \big(\overline{R}_1\,|\,\overline{R}_2\,|\,0\big)^T (\nabla m|0) - \mathbb{1}_2^\flat = \big(\big(\overline{R}_1\,|\,\overline{R}_2\big)^T\, \nabla m\big)^\flat - \mathbb{1}_2^\flat$, while

$$[\mathcal{E}^{\rm plate}_{m,s}]^{\perp} = (0\,|\,0\,|\,\overline{R}_3)^T (\nabla m|0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \langle \overline{R}_3, \partial_{x_1} m\rangle & \langle \overline{R}_3, \partial_{x_2} m\rangle & 0 \end{pmatrix}. \tag{5.8.5}$$

Hence, in the Cosserat plate model we have

$$\begin{aligned} W^{\rm hom}_{\rm mp}(\mathcal{E}^{\rm plate}_{m,s}) &= \mu\, \|{\rm sym}\,\big(\big(\overline{R}_1\,|\,\overline{R}_2\big)^T\, \nabla m - \mathbb{1}_2\big)\|^2 + \mu_{\rm c} \|{\rm skew}\,\big(\big(\overline{R}_1\,|\,\overline{R}_2\big)^T\, \nabla m - \mathbb{1}_2\big)\|^2 \\ &\quad + \frac{\lambda\,\mu}{\lambda + 2\,\mu} [{\rm tr}\big(\big(\overline{R}_1\,|\,\overline{R}_2\big)^T\, \nabla m - \mathbb{1}_2\big)]^2 + \frac{2\,\mu\,\mu_{\rm c}}{\mu_c + \mu} (\langle \overline{R}_3, \partial_{x_1} m\rangle^2 + \langle \overline{R}_3, \partial_{x_2} m\rangle^2)\,, \end{aligned} \tag{5.8.6}$$

which agrees with the Γ-limit found in [90].

5.8.2. A comparison with the nonlinear Cosserat shell model obtained via the derivation approach

In [56], under assumptions (5.1.3) upon the thickness by using the derivation approach, the authors have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \to \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s} : \omega \to {\rm SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \overline{Q}_{e,s})$ the functional

$$I(m, \overline{Q}_{e,s}) = \int_\omega \Big[\, W_{\rm memb}\big(\mathcal{E}_{m,s}\big) + W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) + W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big)\Big]\, \underbrace{det(\nabla y_0|n_0)}_{det \nabla\Theta}\, d\omega\,, \tag{5.8.7}$$

where the membrane part $W_{\rm memb}\big(\mathcal{E}_{m,s}\big)$, the membrane–bending part $W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big)$ and the bending–curvature part $W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big)$ of the shell energy density are given by

$$\begin{aligned} W_{\rm memb}\big(\mathcal{E}_{m,s}\big) &= \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big), \\ W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) &= \Big(\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\, {\rm B}_{y_0} + {\rm C}_{y_0} \mathcal{K}_{e,s}\big) \\ &\quad - \frac{h^3}{3} {\rm H}\, \mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\, \mathcal{K}_{e,s}\big) \\ &\quad + \frac{h^3}{6}\, \mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\, \mathcal{K}_{e,s}){\rm B}_{y_0}\big) \\ &\quad + \frac{h^5}{80}\, W_{\rm mp}\big((\mathcal{E}_{m,s}\, {\rm B}_{y_0} + {\rm C}_{y_0} \mathcal{K}_{e,s}){\rm B}_{y_0}\,\big), \\ W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big) &= \Big(h - {\rm K}\,\frac{h^3}{12}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big) + \Big(\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\,\big) \\ &\quad + \frac{h^5}{80}\, W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}^2_{y_0}\big), \end{aligned} \tag{5.8.8}$$

where

$$\begin{aligned} W_{\rm shell}(X) &= \mu\, \|{\rm sym}\, X\|^2 + \mu_{\rm c} \|{\rm skew}\, X\|^2 + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\, \big[{\rm tr}(X)\big]^2\,, \\ &= \mu\, \|{\rm dev\, sym}\, X\|^2 + \mu_{\rm c} \|{\rm skew}\, X\|^2 + \frac{2\,\mu\,(2\,\lambda + \mu\,)}{3(\lambda + 2\,\mu\,)}\, [{\rm tr}(X)]^2\,, \end{aligned} \tag{5.8.9}$$

$$\begin{aligned} \mathcal{W}_{\rm shell}(X, Y) &= \mu\, \big\langle {\rm sym}\, X,\, {\rm sym}\, Y\big\rangle + \mu_{\rm c} \big\langle {\rm skew}\, X,\, {\rm skew}\, Y\big\rangle + \frac{\lambda\,\mu}{\lambda + 2\,\mu}\, {\rm tr}(X)\, {\rm tr}(Y), \\ W_{\rm mp}(X) &= \mu\, \|{\rm sym}\, X\|^2 + \mu_{\rm c} \|{\rm skew}\, X\|^2 + \frac{\lambda}{2}\, \big[\, {\rm tr}(X)\,\big]^2 = \mathcal{W}_{\rm shell}(X) + \frac{\lambda^2}{2\,(\lambda + 2\,\mu\,)}\, [{\rm tr}(X)]^2, \\ W_{\rm curv}(X) &= \mu\, L_{\rm c}^2\, \big(b_1\, \|{\rm dev\ sym}\, X\|^2 + b_2\, \|{\rm skew}\, X\|^2 + 4 b_3\, [{\rm tr}(X)]^2\big)\,, \quad \forall\, X, Y \in \mathbb{R}^{3\times 3}\,. \end{aligned}$$

In the formulation of the minimization problem, the Weingarten map (or shape operator) is defined by $\mathrm{L}_{y_0} = \mathrm{I}_{y_0}^{-1}\mathrm{II}_{y_0} \in \mathbb{R}^{2\times 2}$, where $\mathrm{I}_{y_0} := [\nabla y_0]^T\,\nabla y_0 \in \mathbb{R}^{2\times 2}$ and $\mathrm{II}_{y_0} := -[\nabla y_0]^T\,\nabla n_0 \in \mathbb{R}^{2\times 2}$ are the matrix representations of the $f$irst fundamental form (metric) and the $s$econd fundamental form of the surface, respectively. In that paper, the authors have also introduced the tensors defined by

$$\mathrm{A}_{y_0} := (\nabla y_0|0)\;[\nabla\Theta_x(0)\,]^{-1} \in \mathbb{R}^{3\times 3}, \qquad \mathrm{B}_{y_0} := -(\nabla n_0|0)\;[\nabla\Theta_x(0)\,]^{-1} \in \mathbb{R}^{3\times 3}, \tag{5.8.10}$$

and the so-called *alternator tensor* C_{y_0} of the surface [112]

$$\mathrm{C}_{y_0} := \det(\nabla\Theta_x(0)\,)\,[\nabla\Theta_x(0)\,]^{-T}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}[\nabla\Theta_x(0)\,]^{-1}. \tag{5.8.11}$$

Comparing with the Γ-limit obtained in the current chapter, the internal energy density obtained via the derivation approach depends also on

$$\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s} = -\,[\nabla\Theta_x(0)\,]^{-T}\left(\begin{array}{c|c} \mathcal{R} - \mathcal{G}\,\mathrm{L}_{y_0} & 0\\ \mathcal{T}\,\mathrm{L}_{y_0} & 0\end{array}\right)[\nabla\Theta_x(0)\,]^{-1}, \tag{5.8.12}$$

where the nonsymmetric quantity $\mathcal{R}-\mathcal{G}\,\mathrm{L}_{y_0}$ represents $t$he change of curvature tensor. The choice of this name is justified subsequently in the framework of the linearized theory, see [59]. Let us notice that the elastic shell bending–curvature tensor $\mathcal{K}_{e,s}$ is not capable to measure the change of curvature, see [58, 59], and that sometimes a confusion is made between bending and change of curvature measures, see also [2, 105, 12, 8, 9]

If we ignore the effect of the change of curvature tensor, there exists no coupling terms in $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$ and we obtain a particular form of the energy obtained via the derivation approach, i.e.,

$$W_{\mathrm{our}}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) = \Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\mathrm{shell}}(\mathcal{E}_{m,s}) + \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\mathrm{curv}}(\mathcal{K}_{e,s}), \tag{5.8.13}$$

where

$$\begin{aligned} W_{\mathrm{shell}}(\mathcal{E}_{m,s}) &= \mu\,\|\mathrm{sym}\;\mathcal{E}^{\|}_{m,s}\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\;\mathcal{E}^{\|}_{m,s}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}^{\|}_{m,s})\big]^2 + \frac{\mu+\mu_{\mathrm{c}}}{2}\,\|\mathcal{E}^{\perp}_{m,s}\|^2 \\ &= \mu\,\|\mathrm{sym}\;\mathcal{E}^{\|}_{m,s}\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\;\mathcal{E}^{\|}_{m,s}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}^{\|}_{m,s})\big]^2 + \frac{\mu+\mu_{\mathrm{c}}}{2}\,\|\mathcal{E}^{T}_{m,s}\,n_0\|^2, \end{aligned} \tag{5.8.14}$$

and

$$W_{\mathrm{curv}}(\mathcal{K}_{e,s}) = \mu\,L_{\mathrm{c}}^2\left(b_1\,\|\mathrm{sym}\,\mathcal{K}^{\|}_{e,s}\|^2 + b_2\,\|\mathrm{skew}\,\mathcal{K}^{\|}_{e,s}\|^2 + \frac{12\,b_3 - b_1}{3}\,[\mathrm{tr}(\mathcal{K}^{\|}_{e,s})]^2 + \frac{b_1+b_2}{2}\,\|\mathcal{K}^{\perp}_{e,s}\|^2\right). \tag{5.8.15}$$

Skipping now all bending related h^3-terms we note that there is only one difference between the membrane energy obtained via the derivation approach and the membrane energy obtained via Γ-convergence, i.e., the weight of the energy terms $\|\mathcal{E}^{T}_{m,s}\,n_0\|^2$:

- derivation approach: the algebraic mean of μ and μ_{c}, i.e., $\dfrac{\mu+\mu_{\mathrm{c}}}{2}$;
- Γ-convergence: the harmonic mean of μ and μ_{c}, i.e., $\dfrac{2\,\mu\,\mu_{\mathrm{c}}}{\mu+\mu_{\mathrm{c}}}$.

This difference has already been observed for the Cosserat plate [91].

We recall again the obtained curvature energy in [55] as

$$W^{\mathrm{hom}}_{\mathrm{curv}}(\mathcal{K}_{e,s}) = \mu L_c^2\Big(b_1\|\mathrm{sym}\,\mathcal{K}^{\|}_{e,s}\|^2 + b_2\|\mathrm{skew}\,\mathcal{K}^{\|}_{e,s}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\,\mathrm{tr}(\mathcal{K}^{\|}_{e,s})^2 + \frac{2\,b_1 b_2}{b_1+b_2}\|\mathcal{K}^{\perp}_{e,s}\|^2\Big). \tag{5.8.16}$$

A comparison between (5.8.15) and (5.8.16) shows that, like the case for the membrane part, the weight of the energy terms $\|\mathcal{K}^{\perp}_{e,s}\|^2 = \|\mathcal{K}^{T}_{e,s}n_0\|^2$ are different as following

- derivation approach: the algebraic mean of b_1 and b_2, i.e., $\dfrac{b_1+b_2}{2}$;
- Γ-convergence: the harmonic mean of b_1 and b_2, i.e., $\dfrac{2\,b_1 b_2}{b_1+b_2}$.

In the model obtained via the derivation approach [56], the constitutive coefficients in the shell model depend on both the Gauß curvature K and the mean curvature H. In the approach presented in the current chapter this does not occur. However, we will consider this aspect in forthcoming works, by considering the Γ-limit method in order to obtain higher order terms in terms of the thickness in the membrane energy, see [50, 52, 51, 53].

5.8.3. A comparison with the general 6-parameter shell model

In the resultant 6-parameter theory of shells, the strain energy density for isotropic shells has been presented in various forms. The simplest expression $W_{\rm P}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ has been proposed in the papers [27, 28] in the form

$$\begin{aligned}2\,W_{\rm P}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) = \quad & C\big[\,\nu\,(\mathrm{tr}\,\mathcal{E}^{\|}_{m,s})^2+(1-\nu)\,\mathrm{tr}((\mathcal{E}^{\|}_{m,s})^T\mathcal{E}^{\|}_{m,s})\big]+\alpha_s\,C(1-\nu)\,\|\mathcal{E}^{T}_{m,s}n_0\|^2\\ &+\,D\big[\,\nu\,(\mathrm{tr}\,\mathcal{K}^{\|}_{e,s})^2+(1-\nu)\,\mathrm{tr}((\mathcal{K}^{\|}_{e,s})^T\mathcal{K}^{\|}_{e,s})\big]+\alpha_t\,D(1-\nu)\,\|\mathcal{K}^{T}_{e,s}n_0\|^2\,, \qquad (5.8.17)\end{aligned}$$

with the Poisson ratio $\nu=\frac{\lambda}{2(\mu+\lambda)}$.

In [45], Eremeyev and Pietraszkiewicz have proposed a more general form of the strain energy density, namely

$$\begin{aligned}2\,W_{\mathrm{E}P}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) = \quad & \alpha_1\,\big(\mathrm{tr}\,\mathcal{E}^{\|}_{m,s}\big)^2+\alpha_2\,\mathrm{tr}\big(\mathcal{E}^{\|}_{m,s}\big)^2+\alpha_3\,\mathrm{tr}\big((\mathcal{E}^{\|}_{m,s})^T\mathcal{E}^{\|}_{m,s}\big)+\alpha_4\,\|\mathcal{E}^{T}_{m,s}n_0\|^2\\ &+\beta_1\,\big(\mathrm{tr}\,\mathcal{K}^{\|}_{e,s}\big)^2+\beta_2\,\mathrm{tr}\big(\mathcal{K}^{\|}_{e,s}\big)^2+\beta_3\,\mathrm{tr}\big((\mathcal{K}^{\|}_{e,s})^T\mathcal{K}^{\|}_{e,s}\big)+\beta_4\,\|\mathcal{K}^{T}_{e,s}n_0\|^2. \qquad (5.8.18)\end{aligned}$$

Already, note the absence of coupling terms involving $\mathcal{K}^{\|}_{e,s}$ and $\mathcal{E}^{\|}_{m,s}$. The eight coefficients α_k, β_k $(k=1,2,3,4)$ can depend in general on the structure of the curvature tensor

$$\mathcal{K}^0\;=\;Q_0(\,\mathrm{axl}(Q_0^T\partial_{x_1}Q_0)\,|\,\mathrm{axl}(Q_0^T\partial_{x_2}Q_0)\,|\,0\,)[\nabla\Theta(0)]^{-1}\,,$$

of the curved reference configuration. We can decompose the strain energy density (5.8.18) in the in-plane part $W_{\mathrm{plane}-EP}(\mathcal{E}_{m,s})$ and the curvature part $W_{\mathrm{curv}-EP}(\mathcal{K}_{e,s})$ and write their expressions in the form

$$\begin{aligned}W_{EP}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) &= W_{\mathrm{plane}-EP}(\mathcal{E}_{m,s})+W_{\mathrm{curv}-EP}(\mathcal{K}_{e,s})\,, \qquad (5.8.19)\\ 2\,W_{\mathrm{plane}-EP}(\mathcal{E}_{m,s}) &= (\alpha_2+\alpha_3)\,\|\mathrm{sym}\,\mathcal{E}^{\|}_{m,s}\|^2+(\alpha_3-\alpha_2)\,\|\mathrm{skew}\,\mathcal{E}^{\|}_{m,s}\|^2+\alpha_1\,\big(\mathrm{tr}(\mathcal{E}^{\|}_{m,s})\big)^2+\alpha_4\,\|\mathcal{E}^{T}_{m,s}n_0\|^2,\\ 2\,W_{\mathrm{curv}-EP}(\mathcal{K}_{e,s}) &= (\beta_2+\beta_3)\,\|\mathrm{sym}\,\mathcal{K}^{\|}_{e,s}\|^2+(\beta_3-\beta_2)\,\|\mathrm{skew}\,\mathcal{K}^{\|}_{e,s}\|^2+\beta_1\,\big(\mathrm{tr}(\mathcal{K}^{\|}_{e,s})\big)^2+\beta_4\,\|\mathcal{K}^{T}_{e,s}n_0\|^2.\end{aligned}$$

By comparing our membrane energy

$$\begin{aligned}W_{\rm mp}^{\rm hom}(\mathcal{E}_{m,s}) &= \mu\,\|\mathrm{sym}\;\mathcal{E}^{\|}_{m,s}\|^2+\mu_{\rm c}\,\|\mathrm{skew}\;\mathcal{E}^{\|}_{m,s}\|^2+\frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}^{\|}_{m,s})\big]^2+\frac{2\,\mu\,\mu_{\rm c}}{\mu_c\,+\mu}\|\mathcal{E}^{T}_{m,s}n_0\|^2 \qquad (5.8.20)\\ &= W_{\rm shell}\big(\mathcal{E}^{\|}_{m,s}\big)+\frac{4\,\mu\,\mu_{\rm c}}{\mu_c\,+\mu}\|\mathcal{E}^{\perp}_{m,s}\|^2,\end{aligned}$$

with $W_{\mathrm{E}P}\big(\mathcal{E}_{m,s},\mathcal{K}_{e,s}\big)$ we deduce the following identification of the constitutive coefficients $\alpha_1\,,...,\alpha_4$

$$\alpha_1=h\,\frac{2\,\mu\,\lambda}{2\,\mu\,+\lambda}\,,\qquad \alpha_2=h\,(\mu\,-\mu_{\rm c}),\qquad \alpha_3=h\,(\mu\,+\mu_{\rm c}),\qquad \alpha_4=h\frac{2\,\mu\;\mu_{\rm c}}{\mu\,+\mu_c}.$$

We observe that

$$\mu_{\rm c}^{\rm drill}\;:=\;\alpha_3-\alpha_2\;=\;2\,h\,\mu_{\rm c}\,, \qquad (5.8.21)$$

which means that the in-plane rotational couple modulus $\mu_{\rm c}^{\rm drill}$ of the Cosserat shell model is determined by the Cosserat couple modulus $\mu_{\rm c}$ of the 3D Cosserat material. An analogous conclusion is given in [6] where linear deformations are considered.

Now a comparison between our curvature energy

$$W_{\rm curv}^{\rm hom}(\mathcal{K}_{e,s})=\mu L_c^2\Big(b_1\|\mathrm{sym}\,\mathcal{K}^{\|}_{e,s}\|^2+b_2\|\mathrm{skew}\,\mathcal{K}^{\|}_{e,s}\|^2+\frac{b_1b_3}{(b_1+b_3)}\,\mathrm{tr}(\mathcal{K}^{\|}_{e,s})^2+\frac{2\,b_1b_2}{b_1+b_2}\|\mathcal{K}^{\perp}_{e,s}\|^2\Big)\,. \qquad (5.8.22)$$

and $W_{\mathrm{curv}-EP}(\mathcal{K}_{e,s})$, leads us to the identification of the constitutive coefficients $\beta_1,\cdots,\beta_4$

$$\beta_1=2\mu L_c^2\,\frac{b_1b_3}{b_1+b_3}\,,\qquad \beta_2=\mu L_c^2 b_1\,,\qquad \beta_3=\mu L_c^2(b_1+b_2)\,,\qquad \beta_4=4\mu L_c^2\frac{b_1b_2}{b_1+b_2}\,.$$

5.8.4. A comparison to another $O(h^5)$-Cosserat shell model

In [22], by using a method which extends the reduction procedure from classical elasticity to the case of Cosserat shells, Bîrsan has obtained a minimization problem, which for the particular case of a quadratic ansatz for the deformation map and skipping higher order terms is based on the following energy

$$I(m,\overline{Q}_{e,s})=\int_\omega \Big[W^{(quad)}_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) + W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big)\Big] \underbrace{det(\nabla y_0|n_0)}_{det\nabla\Theta}\, da\,, \tag{5.8.23}$$

with $W^{(quad)}_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) = h\, W_{\rm Coss}\big(\mathcal{E}_{m,s}\big)$ and $W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big) = h\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big)$, where

$$\begin{aligned}
W_{\rm Coss}(X) &= \mathcal{W}_{\rm Coss}(X,X) = \mu\, \lVert {\rm sym}\, X^{\parallel}\rVert^2 + \mu_{\rm c}\, \lVert {\rm skew}\, X^{\parallel}\rVert^2 + \frac{2\mu\,\mu_{\rm c}}{\mu\,+\mu_{\rm c}}\, \lVert X^{\perp}\rVert^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\, \big[{\rm tr}(X)\big]^2\,,\\
\mathcal{W}_{\rm Coss}(X,Y) &= \mu\, \big\langle {\rm sym}\, X^{\parallel}, {\rm sym} Y^{\parallel}\big\rangle + \mu_{\rm c}\, \big\langle {\rm skew}\, X^{\parallel}, {\rm skew} Y^{\parallel}\big\rangle + \frac{2\mu\,\mu_{\rm c}}{\mu\,+\mu_{\rm c}}\, \big\langle X^{\perp}, Y^{\perp}\big\rangle + \frac{\lambda\,\mu}{\lambda+2\,\mu}\, {\rm tr}(X)\, {\rm tr}(Y)\,,\\
W_{\rm mp}(X) &= \mu\, \lVert {\rm sym}\, X\rVert^2 + \mu_{\rm c} \lVert {\rm skew}\, X\rVert^2 + \frac{\lambda}{2}\, \big[{\rm tr}(X)\big]^2 = \mathcal{W}_{\rm shell}(X,X) + \frac{\lambda^2}{2\,(\lambda+2\,\mu)}\, [{\rm tr}(X)]^2,\\
W_{\rm curv}(X) &= \mu\, L_{\rm c}^2\, \big(b_1\, \lVert {\rm dev\ sym}\, X\rVert^2 + b_2\, \lVert {\rm skew}\, X\rVert^2 + 4 b_3\, [{\rm tr}(X)]^2\big)\,, \quad \forall\, X, Y\in\mathbb{R}^{3\times 3}\,.
\end{aligned}$$

As it can be seen, in the obtained model by Birsan, there are some coupled terms of stress tensor and bending-curvature tensor, too. This is not surprising, since Birsan has obtained the starting example from the model in [56]. The main difference, in comparison to the model obtained in [56] is that $W_{\rm coss}$ with the formula

$$W_{\rm Coss}(X) = \mathcal{W}_{\rm Coss}(X,X) = \mu\, \lVert {\rm sym}\, X^{\parallel}\rVert^2 + \mu_{\rm c}\, \lVert {\rm skew}\, X^{\parallel}\rVert^2 + \frac{2\mu\,\mu_{\rm c}}{\mu\,+\mu_{\rm c}}\, \lVert X^{\perp}\rVert^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\, \big[{\rm tr}(X)\big]^2\,,$$

from [56] is replaced by

$$\mathcal{W}_{\rm Coss}(X,Y) := W_{\rm shell}(X^{\parallel}, Y^{\parallel}) + \frac{2\,\mu\,\mu_c}{\mu+\mu_c}\big\langle X^{\perp}, Y^{\perp}\big\rangle\,, \tag{5.8.24}$$

for all tensors $X,\, Y\in\mathbb{R}^{3\times 3}$ of the form $(*|*|0)\cdot[\nabla_x\Theta(0)]^{-1}$. Note that

$$\mathcal{W}_{\rm shell}(X,Y) := W_{\rm shell}(X^{\parallel}, Y^{\parallel}) + \frac{\mu+\mu_c}{2}\big\langle X^{\perp}, Y^{\perp}\big\rangle\,, \tag{5.8.25}$$

holds true for all tensors $X,\, Y\in\mathbb{R}^{3\times 3}$ of the form $(*|*|0)\cdot[\nabla_x\Theta(0)]^{-1}$. Hence, for this type of tensors we have

$$\mathcal{W}_{\rm Coss}(X,Y) := \mathcal{W}_{\rm shell}(X,Y) - \frac{\mu+\mu_c}{2}\big\langle X^{\perp}, Y^{\perp}\big\rangle + \frac{2\,\mu\,\mu_c}{\mu+\mu_c}\big\langle X^{\perp}, Y^{\perp}\big\rangle\,. \tag{5.8.26}$$

The main point of the comparison presented in this subsection is that the membrane term of order $O(h)$ coincide with the homogenized membrane energy determined by us in this manuscript, i.e.,

$$W^{\rm hom}_{\rm mp}(\mathcal{E}_{m,s}) \equiv W_{\rm Coss}(\mathcal{E}_{m,s}). \tag{5.8.27}$$

With a small comparison between the obtained membrane energy via Γ-convergence and the one obtained via the derivation approach model by Birsan, obviously we see that for a $O(h)$-Cosserat shell theory, there is no difference between the coefficients, i.e.,

- special derivation approach: the harmonic mean of μ and μ_c; $\dfrac{2\mu\mu_c}{\mu+\mu_c}$,
- Γ-limit approach: the harmonic mean of μ and μ_c; $\dfrac{2\mu\mu_c}{\mu+\mu_c}$.

5.9. Linearisation of the Γ-limit Cosserat shell model

5.9.1. The linearised model

In this section we develop the linearization of the Γ-limit functional for the elastic Cosserat shell model, i.e., for situations of small midsurface deformations and small curvature. Let us consider

$$m(x_1, x_2) = y_0(x_1, x_2) + v(x_1, x_2), \tag{5.9.1}$$

where $v : \omega \to \mathbb{R}^3$ is the infinitesimal shell-midsurface displacement. For the rotation tensor $\overline{Q}_{e,0} \in \mathrm{SO}(3)$ there exists a skew-symmetric matrix

$$\overline{A}_\vartheta := \mathrm{Anti}(\vartheta_1, \vartheta_2, \vartheta_3) := \begin{pmatrix} 0 & -\vartheta_3 & \vartheta_2 \\ \vartheta_3 & 0 & -\vartheta_1 \\ -\vartheta_2 & \vartheta_1 & 0 \end{pmatrix} \in \mathfrak{so}(3), \qquad \mathrm{Anti} : \mathbb{R}^3 \to \mathfrak{so}(3), \tag{5.9.2}$$

where $\vartheta = \mathrm{a}xl(\overline{A}_\vartheta)$ denotes the axial vector of $\overline{A}_\vartheta$, such that $\overline{Q}_{e,0} := \exp(\overline{A}_\vartheta) = \sum_{k=0}^{\infty} \frac{1}{k!} \overline{A}_\vartheta^k = \mathbb{1}_3 + \overline{A}_\vartheta + \text{h.o.t.}$ The tensor field $\overline{A}_\vartheta$ is the infinitesimal microrotation. Here, "h.o.t" stands for terms of higher order than linear with respect to u and $\overline{A}_\vartheta$.

Using these linearisations of the kinematic variables, we find the linearisations of the strain tensors. Indeed, since

$$\overline{Q}_{e,0}^T \nabla m - \nabla y_0 = (\mathbb{1}_3 + \overline{A}_\vartheta^T + \text{h.o.t.})(\nabla v + \nabla y_0) - \nabla y_0 = \nabla v - \overline{A}_\vartheta \nabla y_0 + \text{h.o.t.}, \tag{5.9.3}$$

we get for the non-symmetric *shell strain tensor* (which characterises both the in-plane deformation and the transverse shear deformation)

$$\mathcal{E}_{m,s} = (\overline{Q}_{e,0}^T \nabla m - \nabla y_0 \mid 0)\, [\nabla\Theta\,]^{-1},$$

the linearization

$$\mathcal{E}_{m,s}^{\text{lin}} = (\nabla v - \overline{A}_\vartheta \nabla y_0 \mid 0)\, [\nabla\Theta\,]^{-1} = (\partial_{x_1} u - \vartheta \times a_1 \mid \partial_{x_2} u - \vartheta \times a_2 \mid 0)\, [\nabla\Theta\,]^{-1} \notin \mathrm{Sym}(3).$$

And for the *shell bending-curvature tensor*

$$\mathcal{K}_{e,s} := \Big(\mathrm{axl}(\overline{Q}_{e,0}^T \partial_{x_1} \overline{Q}_{e,0}) \,|\, \mathrm{axl}(\overline{Q}_{e,0}^T \partial_{x_2} \overline{Q}_{e,0}) \,|\, 0\Big)\, [\nabla\Theta\,]^{-1}, \tag{5.9.4}$$

we calculate

$$\overline{Q}_{e,0}^T \partial_{x_\alpha} \overline{Q}_{e,0} = (\mathbb{1}_3 - \overline{A}_\vartheta)\; \partial_{x_\alpha} \overline{A}_\vartheta + \text{h.o.t.} = \partial_{x_\alpha} \overline{A}_\vartheta + \text{h.o.t.} = \underbrace{\overline{A}_{\partial_{x_\alpha}\vartheta}}_{\equiv\, Anti\, \partial_{x_\alpha}\vartheta \,=\, \partial_{x_\alpha} Anti\, \vartheta} + \text{h.o.t.}\,, \tag{5.9.5}$$

i.e.,

$$\mathrm{axl}\big(\overline{Q}_{e,0}^T \partial_{x_\alpha} \overline{Q}_{e,0}\big) = \partial_{x_\alpha} \vartheta + \text{h.o.t.}\,, \tag{5.9.6}$$

and we deduce

$$\mathcal{K}_{e,s}^{\text{lin}} = (\mathrm{axl}\big(\partial_{x_1} \overline{A}_\vartheta\big) \,|\, \mathrm{axl}\big(\partial_{x_2} \overline{A}_\vartheta\big) \,|\, 0)\, [\nabla\Theta\,]^{-1}, \tag{5.9.7}$$

together with

$$\mathcal{K}_{e,s}^{\text{lin}} = (\partial_{x_1} \vartheta \,|\, \partial_{x_2} \vartheta \,|\, 0)\, [\nabla\Theta\,]^{-1} = (\nabla\vartheta \,|\, 0)\, [\nabla\Theta\,]^{-1}. \tag{5.9.8}$$

The form of the energy density remains unchanged upon linearization, since the model is physically linear. Thus, the linearization of the Γ-limits reads: for a midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the micro-rotation vector field $\vartheta : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$:

$$\mathcal{J}_0(m, \overline{Q}_{e,0}) = \int_\omega \Big[\overline{W}_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}^{\text{lin}}) + \overline{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}^{\text{lin}})\Big] det(\nabla y_0 | n_0) da - \overline{\Pi}^{lin}(u, \vartheta)\,,$$

where

$$\begin{aligned} \overline{W}_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}^{\text{lin}}) &= \mu\, \|\mathrm{sym}\, \mathcal{E}_{m,s}^{\text{lin},\|}\|^2 + \mu_{\text{c}}\, \|\mathrm{skew}\, \mathcal{E}_{m,s}^{\text{lin},\|}\|^2 + \frac{\lambda\, \mu}{\lambda + 2\, \mu}\, \big[\mathrm{tr}(\mathcal{E}_{m,s}^{\text{lin},\|})\big]^2 + \frac{2\, \mu\, \mu_{\text{c}}}{\mu_c + \mu} \|\mathcal{E}_{m,s}^{\text{lin},T} n_0\|^2 \\ &= W_{\text{shell}}\big(\mathcal{E}_{m,s}^{\text{lin},\|}\big) + \frac{2\, \mu\, \mu_{\text{c}}}{\mu_c + \mu} \|\mathcal{E}_{m,s}^{\text{lin},\perp}\|^2, \end{aligned} \tag{5.9.9}$$

$$\overline{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}^{\text{lin}}) = \inf_{A \in \mathfrak{so}(3)} \overline{W}_{\text{curv}}^{*}\Big(\mathrm{axl}(\overline{Q}_e^T\, \partial_{\eta_1} \overline{Q}_e) \,|\, \mathrm{axl}(\overline{Q}_e^T\, \partial_{\eta_2} \overline{Q}_e) \,|\, \mathrm{axl}(A)\Big)[(\nabla_x \Theta)^\natural]^{-1}\,,$$

and $\overline{\Pi}^{lin}(u, \vartheta)$ is the linearization of the continuous external loading potential $\overline{\Pi}$.

5.9.2. A comparison with the linear Reissner-Mindlin membrane-bending model

The following model

$$\begin{aligned}&\int_\omega h\Big(\mu\,\|\mathrm{sym}\,\nabla(v_1,v_2)\|^2+\frac{\kappa\mu}{2}\|\nabla v_3-\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\|^2+\frac{\mu\,\lambda}{2\,\mu\,+\lambda}\,\mathrm{tr}(\mathrm{sym}\,\nabla(v_1,v_2))^2\Big)\\ &\qquad+\frac{h^3}{12}\Big(\mu\,\|\mathrm{sym}\,\nabla(\theta_1,\theta_2)\|^2+\frac{\mu\,\lambda}{2\,\mu\,+\lambda}\,\mathrm{tr}(\nabla(\theta_1,\theta_2))^2\Big)d\omega\to\min\ \text{w.r.t.}(v,\theta)\,, \qquad (5.9.10)\\ &\qquad v|_{\gamma_0}=u^d(x,y,0)\,,\quad -\theta|_{\gamma_0}=(u^d_{1,z},u^d_{2,z},0)^T\,,\end{aligned}$$

is the linear Reissner-Mindlin membrane-bending model which has five degree of freedom, three from the midsurface displacement $v\colon\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ and the other two are from the out-of-plane rotation parameter $\theta\colon\omega\to\mathbb{R}^2$ that describes the infinitesimal increment of the director and $0<\kappa\leq 1$ is the so called *shear correction factor*. In this model the drill rotations (rotations about the normal) are absent.

As derived in [88], the Reissner-Mindlin membrane-bending model can be obtained as Γ-limit of the linear Cosserat elasticity model. Neff et al. in [91] applied the nonlinear scaling for deformation and linear scaling for the microrotation for the minimization problem with respect to $(u,\overline{A})$:

$$I(u,\overline{A})=\int_{\Omega_h}W_{\mathrm{mp}}(\overline{\varepsilon})+W_{\mathrm{curv}}(\nabla\,\mathrm{axl}\,\overline{A})\;dS\mapsto\min\quad \text{w.r.t}\quad(u,\overline{A})\,, \qquad (5.9.11)$$

where $\overline{\varepsilon}=\nabla u-\overline{A}$, and

$$\begin{aligned}W_{\mathrm{mp}}(\overline{\varepsilon})&=\mu\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2+\mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2+\frac{\lambda}{2}[\mathrm{tr}(\overline{\varepsilon})]^2\,,\\ W_{\mathrm{curv}}(\mathcal{A})&=\mu\,\frac{\widehat{L}_c^2(h)}{2}\Big(\alpha_1\|\mathrm{sym}\,\nabla\,\mathrm{axl}\,\overline{A}\|^2+\alpha_2\|\mathrm{skew}\,\nabla\,\mathrm{axl}\,\overline{A}\|^2+\frac{\alpha_3}{2}[\mathrm{tr}(\nabla\,\mathrm{axl}\,\overline{A})]^2\Big)\,, \qquad (5.9.12)\end{aligned}$$

for $\alpha_1,\alpha_2,\alpha_3\geq 0$. Then, they obtained the following minimization problem:

$$I^{\mathrm{hom}}(v,\overline{A})=\int_\omega W^{\mathrm{hom}}_{\mathrm{mp}}(\nabla v,\mathrm{axl}\,\overline{A})+W^{\mathrm{hom}}_{\mathrm{curv}}(\nabla\,\mathrm{axl}\,\overline{A})\;d\omega\,, \qquad (5.9.13)$$

with respect to (v,θ), where $v\colon\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ is the deformation of the midsurface and $\overline{A}\colon\omega\subset\mathbb{R}^2\to\mathfrak{so}(3)$ as the infinitesimal microrotation of the plate on ω with the boundary condition $v|_{\gamma_0}=u_d(x,y,0),\gamma_0\subset\partial\omega$ and

$$\begin{aligned}W^{\mathrm{hom}}_{\mathrm{mp}}(\nabla v,\theta)&:=\mu\,\|\mathrm{sym}\,\nabla_{(\eta_1,\eta_2)}(v_1,v_2)\|^2+2\frac{\mu\,\mu_c}{\mu\,+\mu_c}\|\nabla_{(\eta_1,\eta_2)}v_3-\begin{pmatrix}-\theta_2\\ \theta_1\end{pmatrix}\|^2+\frac{\mu\,\lambda}{2\,\mu\,+\lambda}\,\mathrm{tr}[\nabla_{(\eta_1,\eta_2)}(v_1,v_2)]^2\,,\\ W^{\mathrm{hom}}_{\mathrm{curv}}(\nabla\theta)&:=\mu\,\frac{\widehat{L}_c^2(h)}{2}\Big(\alpha_1\|\mathrm{sym}\,\nabla_{(\eta_1,\eta_2)}(\theta_1,\theta_2)\|^2+\frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\,\mathrm{tr}[\nabla_{(\eta_1,\eta_2)}(\theta_1,\theta_2)]^2\Big)\,, \qquad (5.9.14)\end{aligned}$$

and it can be seen that this formula is just the Reissner-Mindlin model which is obtained by Γ-convergence, upon selecting $\alpha_1=\mu\,,\alpha_3=\lambda$. In this formula one can recognize the harmonic mean $\mathcal{H}$

$$\frac{1}{2}\mathcal{H}(\mu\,,\frac{\lambda}{2})=\frac{\mu\,\lambda}{2\,\mu\,+\lambda}\,,\qquad \mathcal{H}(\mu\,,\mu_c\,)=\frac{2\,\mu\,\mu_c}{\mu\,+\mu_c}\,,\qquad \frac{1}{2}\mathcal{H}(\alpha_1,\frac{\alpha_3}{2})=\frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\,. \qquad (5.9.15)$$

In the current chapter we used the nonlinear scaling for both deformation and microrotation, while in [91], they applied linear scaling for microrotation and nonlinear scaling for deformation. The other comparison is regarding the th elastic shell strain tensor and elastic shell bending curvature tensor which in our model are not de-coupled, and in (5.9.14) the in-plane deflections v_1,v_2 are not decoupled from θ_3 as well.

5.9.3. Aganovic and Neff's model

Aganović et al.[4] proposed a linear Cosserat flat shell model based on asymptotic analysis of the linear isotropic micropolar Cosserat model. They used the nonlinear scaling for both the displacement and infinitesimal microrotations. Therefore, their minimization problem reads:

$$\begin{aligned}&\int_\omega h\Big(\mu\,\|\mathrm{sym}\,(\nabla(v_1,v_2)-\begin{pmatrix}0&-\theta_3\\ \theta_3&0\end{pmatrix})\|^2+\mu_c\,\|\mathrm{skew}\,(\nabla(v_1,v_2)-\begin{pmatrix}0&-\theta_3\\ \theta_3&0\end{pmatrix})\|^2+\frac{2\,\mu\,\mu_c}{\mu\,+\mu_c}\|\nabla v_3-\begin{pmatrix}-\theta_2\\ \theta_1\end{pmatrix}\|^2\\ &\quad+\frac{\mu\,\lambda}{2\,\mu\,+\lambda}\,\mathrm{tr}(\mathrm{sym}\,(\nabla(v_1,v_2)-\begin{pmatrix}0&-\theta_3\\ \theta_3&0\end{pmatrix})\,)^2\Big) \qquad (5.9.16)\\ &\quad+\mu\,\frac{h\,L_c^2}{2}\Big(\alpha_1\|\mathrm{sym}\,\nabla(\theta_1,\theta_2)\|^2+\alpha_2\|\mathrm{skew}\,\nabla(\theta_1,\theta_2)\|^2+\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\|\nabla\theta_3\|^2+\frac{\alpha_1\alpha_3}{2\alpha_1+\alpha_3}\,\mathrm{tr}(\nabla(\theta_1,\theta_2))^2\Big)\,d\omega\\ &\quad\to\min\ \text{w.r.t.}(v,\theta)\,,\end{aligned}$$

where it is assumed that $\alpha_2, \kappa > 0$, otherwise this model with the assumption $\alpha_2 = 0$ will give the Reissner-Mindlin model. This means that we can not ignore the in-plane drill component θ_3 here and in the case of $\alpha_2 > 0$ one does not obtain the Reissner-Mindlin model. The asymptotic model coincides with the assumptions of Neff et al. in [90], where their assumption was about scaling the nonlinear Cosserat plate model with nonlinear scaling for both deformation and microrotation. The membrane part of this energy coincides with the homogenized membrane energy of our model with the same coefficients.

6. Homogenized curvature energy

6.1. Homogenized quadratic flat curvature energy

In this section, we assume that we are working in the context of a three-dimensional isotropic Cosserat shell which has a thin flat reference configuration. The goal is to derive a dimensionally reduced flat shell model. We use Γ-convergence arguments (see [25]). The homogenized curvature energy turns out to be

$$W_{\mathrm{curv}}^{\mathrm{hom}}(\Gamma_{\square}^{\natural}) = \inf_{c^*\in\mathbb{R}^3} W_{\mathrm{curv}}((\Gamma_1|\Gamma_2|c^*)[(\nabla_x\Theta)^{\natural}]^{-1})\,. \tag{6.1.1}$$

Interesting, this calculus can be made explicit once the curvature energy is written in terms of Γ, where

$$\Gamma = \Big(\operatorname{axl}(\overline{Q}_e^T\partial_{x_1}\overline{Q}_e)|\operatorname{axl}(\overline{Q}_e^T\partial_{x_2}\overline{Q}_e)|\operatorname{axl}(\overline{Q}_e^T\partial_{x_3}\overline{Q}_e)\Big)[(\nabla_x\Theta)]^{-1}\,.$$

In this part we assume that the shell in its initial configuration is flat and the $3D$ Cosserat curvature energy assumes the form

$$W_{\mathrm{curv}}(\Gamma) = \mu L_c^2(a_1\|\operatorname{sym}\Gamma\|^2 + a_2\|\operatorname{skew}\Gamma\|^2 + a_3\operatorname{tr}(\Gamma)^2)\,, \tag{6.1.2}$$

where $a_1, a_2, a_3 > 0$. Assume that we have done the nonlinear scaling [91] for the matrix Γ. Therefore,

$$\Gamma_h^{\natural} = \begin{pmatrix} \Gamma_{11}^{\natural} & \Gamma_{12}^{\natural} & c_1 \\ \Gamma_{21}^{\natural} & \Gamma_{22}^{\natural} & c_2 \\ \Gamma_{31}^{\natural} & \Gamma_{32}^{\natural} & c_3 \end{pmatrix}, \tag{6.1.3}$$

and the homogenized curvature energy is given by

$$W_{\mathrm{curv}}^{\mathrm{hom}}(\Gamma_h^{\natural}) = W_{\mathrm{curv}}((\Gamma_1|\Gamma_2|c) = \inf_{c^*\in\mathbb{R}^3} W_{\mathrm{curv}}((\Gamma_1|\Gamma_2|c^*)[(\nabla_x\Theta)^{\natural}]^{-1})\,. \tag{6.1.4}$$

By using the relation (6.1.2), we start to do the calculations for symmetric, skew-symmetric and trace parts as

$$\operatorname{sym}\Gamma_h^{\natural} = \begin{pmatrix} \Gamma_{11}^{\natural} & \frac{\Gamma_{12}^{\natural}+\Gamma_{21}^{\natural}}{2} & \frac{c_1+\Gamma_{31}^{\natural}}{2} \\ \frac{\Gamma_{21}^{\natural}+\Gamma_{12}^{\natural}}{2} & \Gamma_{22}^{\natural} & \frac{c_2+\Gamma_{32}^{\natural}}{2} \\ \frac{\Gamma_{31}^{\natural}+c_1}{2} & \frac{\Gamma_{32}^{\natural}+c_2}{2} & c_3 \end{pmatrix}, \qquad \operatorname{skew}\Gamma_h^{\natural} = \begin{pmatrix} 0 & \frac{\Gamma_{12}^{\natural}-\Gamma_{21}^{\natural}}{2} & \frac{c_1-\Gamma_{31}^{\natural}}{2} \\ \frac{\Gamma_{21}^{\natural}-\Gamma_{12}^{\natural}}{2} & 0 & \frac{c_2-\Gamma_{32}^{\natural}}{2} \\ \frac{\Gamma_{31}^{\natural}-c_1}{2} & \frac{\Gamma_{32}^{\natural}-c_2}{2} & 0 \end{pmatrix}, \tag{6.1.5}$$

and

$$\operatorname{tr}(\Gamma_h^{\natural}) = (\Gamma_{11}^{\natural} + \Gamma_{22}^{\natural} + c_3)\,. \tag{6.1.6}$$

The idea is minimizing the energy model regarding to the available unknowns inside the curvature energy. So, let us assume that the total energy is as following

$$I^{\natural}(\varphi^{\natural}, \nabla_\eta^h\varphi^{\natural},\ \Gamma_h^{\natural}) = \int_{\Omega_h} W_{\mathrm{mp}}(U_h^{\natural}) + W_{\mathrm{curv}}(\Gamma_h^{\natural})\, dV_\eta\,. \tag{6.1.7}$$

We notice that the the membrane part and curvature part are completely decoupled.
We have

$$\begin{aligned} W_{\mathrm{curv}}(\Gamma_h^{\natural}) = \mu L_c^2\Big(& a_1\big(\Gamma_{11}^{\natural,2} + \frac{1}{2}(\Gamma_{12}^{\natural}+\Gamma_{21}^{\natural})^2 + \frac{1}{2}(c_1+\Gamma_{31}^{\natural})^2 + \Gamma_{22}^{\natural,2} + \frac{1}{2}(c_2+\Gamma_{32}^{\natural})^2 + c_3^2\big) \\ &+ a_2\big(\frac{1}{2}(\Gamma_{12}^{\natural}-\Gamma_{21}^{\natural})^2 + \frac{1}{2}(c_1-\Gamma_{31}^{\natural})^2 + \frac{1}{2}(c_2-\Gamma_{32}^{\natural})^2\big) \\ &+ a_3(\Gamma_{11}^{\natural}+\Gamma_{22}^{\natural}+c_3)^2\Big)\,. \end{aligned} \tag{6.1.8}$$

Now we apply the Euler-Lagrange equations to determine the unknowns. But first, we recall that

$$\begin{aligned} I_0(\varphi,\Gamma) &= \inf_{b^*\in\mathbb{R}^3, c^*\in\mathbb{R}^3} \int_{\Omega_1} W_{\rm mp}(\overline{Q}^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|b^*)) + W_{\rm curv}((\Gamma_1|\Gamma_2|c^*))\, dV_\eta \\ &= I(\varphi,\Gamma,b,c) = \inf_{b^*\in\mathbb{R}^3, c^*\in\mathbb{R}^3} I(\varphi,\Gamma,b^*,c^*)\,, \end{aligned} \tag{6.1.9}$$

where $c=(c_1,c_2,c_3)^T$. Notice that the two energies are decoupled. Hence, for solving the minimization problem we will have

$$\begin{aligned} 0 &= \frac{\partial I}{\partial c_1} = a_1(c_1+\Gamma^\natural_{31}) + a_2(c_1-\Gamma^\natural_{31}) = (a_1+a_2)c_1 + (a_1-a_2)\Gamma^\natural_{31} \quad \Rightarrow \quad c_1 = \frac{a_2-a_1}{a_1+a_2}\Gamma^\natural_{31}\,,\\ 0 &= \frac{\partial I}{\partial c_2} = a_1(c_2+\Gamma^\natural_{32}) + a_2(c_2-\Gamma^\natural_{32}) = (a_1+a_2)c_2 + (a_1-a_2)\Gamma^\natural_{32} \quad \Rightarrow \quad c_2 = \frac{a_2-a_1}{a_1+a_2}\Gamma^\natural_{32}\,,\\ 0 &= \frac{\partial I}{\partial c_3} = a_1 c_3 + a_3(\Gamma^\natural_{11}+\Gamma^\natural_{22}+c_3) \quad \Rightarrow \quad c_3 = \frac{-a_3}{a_1+a_3}(\Gamma^\natural_{11}+\Gamma^\natural_{22})\,. \end{aligned} \tag{6.1.10}$$

By inserting the unknowns inside $W_{\rm curv}$, we have

$$\begin{aligned} W_{\rm curv}(\Gamma^\natural_h) &= \mu L_c^2\Big(a_1\big(\Gamma^{\natural,2}_{11}+\Gamma^{\natural,2}_{22}+(\frac{-a_3}{a_1+a_3}(\Gamma^\natural_{11}+\Gamma^\natural_{22}))^2+\frac12(\Gamma^\natural_{21}+\Gamma^\natural_{12})^2+\frac12(\frac{a_2-a_1}{a_1+a_2}\Gamma^\natural_{31}+\Gamma^\natural_{31})^2\\ &\qquad +\frac12(\frac{a_2-a_1}{a_1+a_2}\Gamma^\natural_{32}+\Gamma^\natural_{32})^2\big)\\ &\qquad + a_2\big(\frac12(\Gamma^\natural_{12}-\Gamma^\natural_{21})^2+\frac12(\frac{a_2-a_1}{a_1+a_2}\Gamma^\natural_{31}-\Gamma^\natural_{31})^2+\frac12(\frac{a_2-a_1}{a_1+a_2}\Gamma^\natural_{32}-\Gamma^\natural_{32})^2\big)\\ &\qquad + a_3\big((\Gamma^\natural_{11}+\Gamma^\natural_{22})-\frac{a_3}{a_1+a_3}(\Gamma^\natural_{11}+\Gamma^\natural_{22})\big)^2\Big)\\ &= \mu L_c^2\Big(a_1\big(\Gamma^{\natural,2}_{11}+\Gamma^{\natural,2}_{22}+\frac{a_3^2}{(a_1+a_3)^2}(\Gamma^\natural_{11}+\Gamma^\natural_{22})^2+\frac12(\Gamma^\natural_{21}+\Gamma^\natural_{12})^2+2\frac{a_2^2}{(a_1+a_2)^2}\Gamma^{\natural,2}_{31}\\ &\qquad +2\frac{a_2^2}{(a_1+a_2)^2}\Gamma^{\natural,2}_{32}\big)\\ &\qquad + a_2\big(\frac12(\Gamma^\natural_{12}-\Gamma^\natural_{21})^2+2\frac{a_1^2}{(a_1+a_2)^2}\Gamma^{\natural,2}_{31}+2\frac{a_1^2}{(a_1+a_2)^2}\Gamma^{\natural,2}_{32}\big)\\ &\qquad + a_3\frac{a_1^2}{(a_1+a_3)^2}(\Gamma^\natural_{11}+\Gamma^\natural_{22})^2\Big)\\ &= \mu L_c^2\Big(a_1\big(\Gamma^{\natural,2}_{11}+\Gamma^{\natural,2}_{22}\big)+\frac{a_1a_3}{(a_1+a_3)}(\Gamma^\natural_{11}+\Gamma^\natural_{22})^2+\frac{a_1}{2}(\Gamma^\natural_{21}+\Gamma^\natural_{12})^2+2\frac{a_1a_2}{(a_1+a_2)}\Gamma^{\natural,2}_{31}\\ &\qquad +2\frac{a_1a_2}{(a_1+a_2)}\Gamma^{\natural,2}_{32}+\frac{a_2}{2}(\Gamma^\natural_{21}-\Gamma^\natural_{12})^2\Big)\\ &= \mu L_c^2\Big(a_1\|{\rm sym}\,\Gamma^\natural_\square\|^2+a_2\|{\rm skew}\,\Gamma^\natural_\square\|^2+\frac{a_1a_3}{(a_1+a_3)}\,{\rm tr}(\Gamma^\natural_\square)^2+\frac{2a_1a_2}{(a_1+a_2)}\|\begin{pmatrix}\Gamma^\natural_{31}\\ \Gamma^\natural_{32}\end{pmatrix}\|^2\Big)\,, \end{aligned} \tag{6.1.11}$$

where $\Gamma^\natural_\square = \begin{pmatrix}\Gamma^\natural_{11} & \Gamma^\natural_{12}\\ \Gamma^\natural_{21} & \Gamma^\natural_{22}\end{pmatrix}$. Therefore, the homogenized curvature energy in plat model is

$$W^{\rm hom}_{\rm curv}(\Gamma^\natural_h) = \mu L_c^2\Big(a_1\|{\rm sym}\,\Gamma^\natural_\square\|^2+a_2\|{\rm skew}\,\Gamma^\natural_\square\|^2+\frac{a_1a_3}{(a_1+a_3)}\,{\rm tr}(\Gamma^\natural_\square)^2+\frac{2a_1a_2}{(a_1+a_2)}\|\begin{pmatrix}\Gamma^\natural_{31}\\ \Gamma^\natural_{32}\end{pmatrix}\|^2\Big)\,. \tag{6.1.12}$$

6.2. Homogenized curvature energy for the curvy shell model

Let us consider an elastic material which in its reference configuration fills the three dimensional *shell-like thin* domain $\Omega_\xi\subset\mathbb{R}^3$, i.e., we assume that there exists a C^1-diffeomorphism $\Theta\colon\mathbb{R}^3\to\mathbb{R}^3$ with $\Theta(x_1,x_2,x_3):=(\xi_1,\xi_2,\xi_3)$ such that $\Theta(\Omega_h)=\Omega_\xi$ and $\omega_\xi=\Theta(\omega\times\{0\})$, where $\Omega_h\subset\mathbb{R}^3$ for $\Omega_h=\omega\times\big[-\frac{h}{2},\frac{h}{2}\big]$, with $\omega\subset\mathbb{R}^2$ a bounded domain with Lipschitz boundary $\partial\omega$. The scalar $0<h\ll1$ is called *thickness* of the shell, while the domain Ω_h is called *fictitious Cartesian configuration* of the body. In fact, in this chapter, we consider the following diffeomorphism $\Theta\colon\mathbb{R}^3\to\mathbb{R}^3$ which describes the curved surface of the shell

$$\Theta(x_1,x_2,x_3) = y_0(x_1,x_2)+x_3\,n_0(x_1,x_2)\,, \tag{6.2.1}$$

where $y_0\colon \omega \to \mathbb{R}^3$ is a $C^2(\omega)$-function and $n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}$ is the unit normal vector on ω_ξ. Remark that

$$\nabla_x\Theta(x_3) \;=\; (\nabla y_0|n_0) + x_3(\nabla n_0|0) \;\; \forall\, x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right), \quad \nabla_x\Theta(0) \;=\; (\nabla y_0|\, n_0), \quad [\nabla_x\Theta(0)]^{-T}\, e_3 \;=\; n_0, \tag{6.2.2}$$

and $\det \nabla_x\Theta(0) = \det(\nabla y_0|n_0) = \sqrt{\det[(\nabla y_0)^T\nabla y_0]}$ represents the surface element.

6.2.1. Euler-Lagrange equations

We have the following curvature energy formula for a curvy configuration

$$W_{\rm curv}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}) = \mu L_c^2\Big(b_1\|\operatorname{dev}\operatorname{sym}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 + b_2\,\|\operatorname{skew}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 + 4b_3\operatorname{tr}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1})^2\Big)\,, \tag{6.2.3}$$

which we can rewrite as

$$W_{\rm curv}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}) = \mu L_c^2\Big(a_1\|\operatorname{sym}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 + a_2\,\|\operatorname{skew}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 + a_3\operatorname{tr}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1})^2\Big)\,, \tag{6.2.4}$$

where $a_3 = \frac{12b_3 - b_1}{3}$.

We need to find

$$W_{\rm curv}^{\rm hom}(\Gamma^\natural_\square) = W_{\rm curv}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}) = \inf_{c^*\in\mathbb{R}^3} W_{\rm curv}((\Gamma_1|\Gamma_2|c^*)[(\nabla_x\Theta)^\natural]^{-1})\,. \tag{6.2.5}$$

The Euler-Lagrange equations appear from variations with respect to arbitrary $\delta c \in \mathbb{R}^3$.

$$\langle DW_{\rm curv}((\Gamma_1|\Gamma_2|c^*)[(\nabla_x\Theta)^\natural]^{-1}, (0|0|\delta c^*)[(\nabla_x\Theta)^\natural]^{-1})\rangle = 0\,. \tag{6.2.6}$$

We have

$$\begin{aligned}
&\langle 2a_1\operatorname{sym}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}), (0|0|\delta c)[(\nabla_x\Theta)^\natural]^{-1}\rangle + \langle 2a_2\operatorname{skew}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}), (0|0|\delta c)[(\nabla_x\Theta)^\natural]^{-1}\rangle\\
&\qquad + 2a_3\operatorname{tr}(\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1})\langle \mathbb{1}_3, (0|0|\delta c)[(\nabla_x\Theta)^\natural]^{-1}\rangle = 0\,, \qquad (6.2.7)\\
&\langle 2a_1\operatorname{sym}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})[(\nabla_x\Theta)^\natural]^{-T}, (0|0|\delta c)\rangle + \langle 2a_2\operatorname{skew}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})[(\nabla_x\Theta)^\natural]^{-T}, (0|0|\delta c)\rangle\\
&\qquad + 2a_3\operatorname{tr}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\langle [(\nabla_x\Theta)^\natural]^{-T}, (0|0|\delta c)\rangle = 0\,,\\
&\langle 2a_1\operatorname{sym}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\underbrace{[(\nabla_x\Theta)^\natural]^{-T}e_3}_{=n_0}, \delta c\rangle + \langle 2a_2\operatorname{skew}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})[(\nabla_x\Theta)^\natural]^{-T}e_3, \delta c\rangle\\
&\qquad + 2a_3\operatorname{tr}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\langle [(\nabla_x\Theta)^\natural]^{-T}e_3, \delta c\rangle = 0\,,\\
&\langle 2a_1\operatorname{sym}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})n_0, \delta c\rangle + \langle 2a_2\operatorname{skew}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})n_0, \delta c\rangle\\
&\qquad + 2a_3\operatorname{tr}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\langle n_0, \delta c\rangle = 0\,,\\
&\langle\Big(2a_1\operatorname{sym}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}) + 2a_2\operatorname{skew}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}) + 2a_3\operatorname{tr}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0, \delta c\rangle = 0\,.
\end{aligned}$$

Because this relation holds for arbitrary $\delta c \in \mathbb{R}^3$, we get

$$\Big(2a_1\operatorname{sym}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}) + 2a_2\operatorname{skew}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}) + 2a_3\operatorname{tr}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0 = 0\,. \tag{6.2.8}$$

We write

$$(\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1} = (\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1} + (0|0|c)[(\nabla_x\Theta)^\natural]^{-1}\,. \tag{6.2.9}$$

Therefore, we have

$$\begin{aligned}
2\operatorname{sym}\big((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}\big)n_0 &= 2\Big(\operatorname{sym}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}) + \operatorname{sym}((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0\\
&= (\Gamma_1|\Gamma_2|0)\underbrace{[(\nabla_x\Theta)^\natural]^{-1}n_0}_{=e_3} + ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + (0|0|c)[(\nabla_x\Theta)^\natural]^{-1}n_0\\
&\qquad + ((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0\\
&= ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + c + ((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0\,. \qquad (6.2.10)
\end{aligned}$$

A similar calculation shows that

$$2\,\text{skew}\,\big((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1}\big)n_0 = 2\Big(\,\text{skew}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}) + \text{skew}((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0$$
$$= -((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + c - ((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0\,. \quad (6.2.11)$$

The trace term can be calculated like

$$2a_3\,\text{tr}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})n_0 = 2a_3\Big(\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}) + \text{tr}((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0$$
$$= 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0 + 2a_3\langle(0|0|c)[(\nabla_x\Theta)^\natural]^{-1}, \mathbb{1}_3\rangle_{\mathbb{R}^{3\times3}}n_0$$
$$= 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0 + 2a_3\langle c, \underbrace{[(\nabla_x\Theta)^\natural]^{-T}e_3}_{=n_0}\rangle n_0$$
$$= 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0 + 2a_3\,c\,n_0\otimes n_0\,. \quad (6.2.12)$$

By using (6.2.8), we obtain

$$a_1((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + a_1 c + a_1((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 - a_2((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + a_2 c$$
$$- a_2((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0 + 2a_3\,c\,n_0\otimes n_0 = 0\,. \quad (6.2.13)$$

Gathering similar terms gives us

$$(a_1-a_2)((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + (a_1+a_2)c + (a_1-a_2)((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0$$
$$+ 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0 + 2a_3\,c\,n_0\otimes n_0 = 0\,. \quad (6.2.14)$$

We have

$$((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 = (c(0|0|e_3)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 = (cn_0)^T n_0 = n_0^T c^T n_0 = \langle n_0, c^T\rangle n_0 = n_0\langle n_0, c\rangle$$
$$= n_0\otimes n_0\,c = c n_0\otimes n_0\,. \quad (6.2.15)$$

Let us define the lifted quantity $\mathrm{I}^\flat_{y_0} = \begin{pmatrix} & I_{y_0} & 0\\ & & 0\\ 0 & 0 & 1\end{pmatrix}$. We introduce the symmetric tensor $A_{y_0} := (\nabla y_0|0)[\nabla_x\Theta(0)]^{-1} \in \text{SO}(3)$. Because

$$A_{y_0} = (\nabla y_0|0)[\nabla_x\Theta(0)]^{-1} = (\nabla y_0|n_0)\mathbb{1}_2^\flat[\nabla_x\Theta(0)]^{-1} \quad (6.2.16)$$
$$= [\nabla_x\Theta(0)]^{-T}[\nabla_x\Theta(0)]^T[\nabla_x\Theta(0)]\mathbb{1}_2^\flat[\nabla_x\Theta(0)]^{-1}$$
$$= [\nabla_x\Theta(0)]^{-T}\widehat{\mathrm{I}}_{y_0}\mathbb{1}_2^\flat[\nabla_x\Theta(0)]^{-1} = [\nabla_x\Theta(0)]^{-T}\mathrm{I}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1}\,,$$

where $\widehat{\mathrm{I}}_{y_0} = \mathrm{I}^\flat_{y_0} + \widehat{0}_3$, with $\widehat{0}_3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$.

By using the decomposition

$$\mathbb{1}_3 c = A_{y_0}c + n_0\otimes n_0 c\,, \quad (6.2.17)$$

we have

$$(a_1-a_2)((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 + (a_1+a_2)(A_{y_0}c + n_0\otimes n_0 c) + (a_1-a_2)n_0\otimes n_0 c$$
$$+ 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0 + 2a_3 n_0\otimes n_0 c = 0\,, \quad (6.2.18)$$

and

$$((a_1+a_2)A_{y_0} + 2(a_1+a_3)n_0\otimes n_0)c = -(a_1-a_2)((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0$$
$$- 2a_3\,\text{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0\,. \quad (6.2.19)$$

We introduce (see [22])

$$((a_1+a_2)A_{y_0} + 2(a_1+a_3)n_0\otimes n_0)^{-1} = \Big(\frac{1}{a_1+a_2}A_{y_0} + \frac{1}{2(a_1+a_3)}n_0\otimes n_0\Big)\,. \quad (6.2.20)$$

Therefore,

$$c = (a_2 - a_1)\Big[(\frac{1}{a_1+a_2}A_{y_0} + \frac{1}{2(a_1+a_3)}n_0\otimes n_0)((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0\Big]$$
$$- 2a_3\,\mathrm{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})\Big[(\frac{1}{a_1+a_2}A_{y_0} + \frac{1}{2(a_1+a_3)}n_0\otimes n_0)n_0\Big]\,, \quad (6.2.21)$$

which will be reduced to

$$c = \frac{(a_2-a_1)}{a_1+a_2}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T n_0 - \frac{2a_3}{2(a_1+a_3)}\,\mathrm{tr}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})n_0\,, \quad (6.2.22)$$

where we have used the fact that $A_{y_0}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T = ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T$. Because,

$$\begin{aligned}A_{y_0}((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T &= \mathbb{1}_3((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T - n_0\otimes n_0((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T\\
&= ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T - (0|0|n_0)(0|0|n_0)^T[(\nabla_x\Theta)^\natural]^{-T}(\Gamma_1|\Gamma_2|0)^T n_0\\
&= ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T - (0|0|n_0)(\underbrace{[(\nabla_x\Theta)^\natural]^{-1}(0|0|n_0)}_{(0|0|e_3)})^T(\Gamma_1|\Gamma_2|0)^T n_0\\
&= ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T - \underbrace{(0|0|n_0)(0|0|e_3)^T(\Gamma_1|\Gamma_2|0)^T}_{=0_{\mathbb{R}^{3\times3}}} n_0\\
&= ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T\,.\end{aligned}$$

6.2.2. Calculations for the homogenized curvature energy

In this part we insert the minimizer c in (6.2.22), in the energy (6.2.4). With the definition $\mathcal{K}_{e,s} := (\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}$, we have

$$\begin{aligned}&\|\mathrm{sym}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 = \|\mathrm{sym}((\Gamma_1|\Gamma_2|c)[(\nabla_x\Theta)^\natural]^{-1})\|^2 \qquad (6.2.23)\\
&\quad= \|\mathrm{sym}\left((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1} + (0|0|c)[(\nabla_x\Theta)^\natural]^{-1}\right)\|^2\\
&\quad= \|\mathrm{sym}\left((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}\right)\|^2 + \|\mathrm{sym}\left((0|0|c)[(\nabla_x\Theta)^\natural]^{-1}\right)\|^2\\
&\qquad+ 2\left\langle \mathrm{sym}\left((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}\right), \mathrm{sym}\left((0|0|c)[(\nabla_x\Theta)^\natural]^{-1}\right)\right\rangle\\
&\quad= \|\mathrm{sym}\,\mathcal{K}_{e,s}\|^2 + \|\mathrm{sym}\left(\frac{a_2-a_1}{a_1+a_2}\mathcal{K}_{e,s}^T(0|0|n_0)[(\nabla_x\Theta)^\natural]^{-1} - \frac{a_3}{(a_1+a_3)}\,\mathrm{tr}(\mathcal{K}_{e,s})(0|0|n_0)[(\nabla_x\Theta)^\natural]^{-1}\right)\|^2\\
&\qquad+ 2\left\langle \mathrm{sym}\,\mathcal{K}_{e,s}, \mathrm{sym}\left(\frac{a_2-a_1}{a_1+a_2}\mathcal{K}_{e,s}^T(0|0|n_0)[(\nabla_x\Theta)^\natural]^{-1} - \frac{a_3}{(a_1+a_3)}\,\mathrm{tr}(\mathcal{K}_{e,s})(0|0|n_0)[(\nabla_x\Theta)^\natural]^{-1}\right)\right\rangle\,,\end{aligned}$$

and

$$\begin{aligned}&\|\mathrm{sym}\left(\frac{a_2-a_1}{a_1+a_2}\mathcal{K}_{e,s}^T(0|0|n_0)[(\nabla_x\Theta)^\natural]^{-1} - \frac{a_3}{(a_1+a_3)}\,\mathrm{tr}(\mathcal{K}_{e,s})(0|0|n_0)[(\nabla_x\Theta)^\natural]^{-1}\right)\|^2\\
&= \frac{(a_2-a_1)^2}{(a_1+a_2)^2}\|\mathrm{sym}(\mathcal{K}_{e,s}^T n_0\otimes n_0)\|^2 + \frac{a_3^2}{(a_1+a_3)^2}\,\mathrm{tr}(\mathcal{K}_{e,s})^2\|n_0\otimes n_0\|^2\\
&\quad- 2\,\frac{a_2-a_1}{a_1+a_2}\,\frac{a_3}{(a_1+a_3)}\,\mathrm{tr}(\mathcal{K}_{e,s})\langle \mathrm{sym}(\mathcal{K}_{e,s}^T n_0\otimes n_0), n_0\otimes n_0\rangle\\
&= \frac{(a_2-a_1)^2}{(a_1+a_2)^2}\left\langle \mathrm{sym}(\mathcal{K}_{e,s}^T n_0\otimes n_0), \mathrm{sym}(\mathcal{K}_{e,s}^T n_0\otimes n_0)\right\rangle + \frac{a_3^2}{(a_1+a_3)^2}\,\mathrm{tr}(\mathcal{K}_{e,s})^2\\
&\quad- \frac{a_2-a_1}{a_1+a_2}\,\frac{a_3}{(a_1+a_3)}\,\mathrm{tr}(\mathcal{K}_{e,s})\langle \mathcal{K}_{e,s}^T n_0\otimes n_0, n_0\otimes n_0\rangle\\
&\quad- \frac{a_2-a_1}{a_1+a_2}\,\frac{a_3}{(a_1+a_3)}\,\mathrm{tr}(\mathcal{K}_{e,s})\langle n_0\otimes n_0\,\mathcal{K}_{e,s}, n_0\otimes n_0\rangle \qquad (6.2.24)\\
&= \frac{(a_2-a_1)^2}{4(a_1+a_2)^2}\langle \mathcal{K}_{e,s}^T n_0\otimes n_0, \mathcal{K}_{e,s}^T n_0\otimes n_0\rangle + \frac{(a_2-a_1)^2}{4(a_1+a_2)^2}\langle \mathcal{K}_{e,s}^T n_0\otimes n_0, n_0\otimes n_0\,\mathcal{K}_{e,s}\rangle\\
&\quad+ \frac{(a_2-a_1)^2}{4(a_1+a_2)^2}\langle n_0\otimes n_0\,\mathcal{K}_{e,s}, \mathcal{K}_{e,s}^T n_0\otimes n_0\rangle + \frac{(a_2-a_1)^2}{4(a_1+a_2)^2}\langle n_0\otimes n_0\,\mathcal{K}_{e,s}, n_0\otimes n_0\,\mathcal{K}_{e,s}\rangle\\
&\quad+ \frac{a_3^2}{(a_1+a_3)^2}\,\mathrm{tr}(\mathcal{K}_{e,s})^2\\
&= \frac{(a_2-a_1)^2}{2(a_1+a_2)^2}\|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{a_3^2}{(a_1+a_3)^2}\,\mathrm{tr}(\mathcal{K}_{e,s})^2.\end{aligned}$$

Note that

$$\begin{aligned}
\langle \mathcal{K}_{e,s}^T, n_0\otimes n_0\rangle &= \langle ((\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1})^T, n_0\otimes n_0\rangle = \langle (\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural(x_3)]^{-1}, (0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}\rangle\\
&= \left\langle (\Gamma_1|\Gamma_2|0), (0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}[(\nabla_x\Theta)^\natural(0)]^{-T}\begin{pmatrix} \mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T}\right\rangle\\
&= \left\langle (0|0|n_0)^T(\Gamma_1|\Gamma_2|0), \begin{pmatrix} \mathrm{I}_{y_0}^{-1} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}\begin{pmatrix} \mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T}\right\rangle\\
&= \left\langle \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ * & * & 0\end{pmatrix}, \begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{pmatrix}\right\rangle = 0\,. \qquad (6.2.25)
\end{aligned}$$

We also observe that

$$\begin{aligned}
n_0\otimes n_0[(\nabla_x\Theta)^\natural(\eta_3)]^{-T} &= (0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}[(\nabla_x\Theta)^\natural(\eta_3)]^{-T}\\
&= (0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}[(\nabla_x\Theta)^\natural(0)]^{-T}\begin{pmatrix} \mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T} \qquad (6.2.26)\\
&= (0|0|n_0)\begin{pmatrix} \mathrm{I}_{y_0}^{-1} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}\begin{pmatrix} \mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T} = (0|0|n_0)\begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{pmatrix} = (0|0|n_0)\,,
\end{aligned}$$

Since we have used the matrix expression $\mathcal{K}_{e,s} = (\Gamma_1|\Gamma_2|0)[(\nabla_x\Theta)^\natural]^{-1}$ and $n_0\otimes n_0 = (0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}$, we deduce

$$\langle \mathcal{K}_{e,s}^T n_0\otimes n_0, \mathcal{K}_{e,s}^T n_0\otimes n_0\rangle = \langle \mathcal{K}_{e,s}^T n_0, \mathcal{K}_{e,s}^T n_0\rangle = \|\mathcal{K}_{e,s}^T n_0\|^2\,,$$

because for every vector $\widehat{u}, v\in\mathbb{R}^3$ we have

$$\begin{aligned}
\langle \widehat{u}\otimes n_0, v\otimes n_0\rangle &= \langle (v\otimes n_0)^T\widehat{u}\otimes n_0, \mathbb{1}\rangle = \langle (n_0\otimes v)\widehat{u}\otimes n_0, \mathbb{1}\rangle = \langle n_0\otimes n_0\langle v,\widehat{u}\rangle, \mathbb{1}\rangle\\
&= \langle v,\widehat{u}\rangle\cdot\underbrace{\langle n_0, n_0\rangle}_{=1} = \langle v,\widehat{u}\rangle\,. \qquad (6.2.27)
\end{aligned}$$

On the other hand,

$$2\left\langle \operatorname{sym}\mathcal{K}_{e,s}, \operatorname{sym}\Big(\frac{a_2-a_1}{a_1+a_2}\mathcal{K}_{e,s}^T n_0\otimes n_0 - \frac{a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s}) n_0\otimes n_0\Big)\right\rangle = \frac{a_2-a_1}{a_1+a_2}\|\mathcal{K}_{e,s}^T n_0\|^2\,. \qquad (6.2.28)$$

Therefore, we see that

$$\|\operatorname{sym}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 = \|\operatorname{sym}\mathcal{K}_{e,s}\|^2 + \frac{(a_1-a_2)^2}{2(a_1+a_2)^2}\|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{a_3^2}{(a_1+a_3)^2}\operatorname{tr}(\mathcal{K}_{e,s})^2 + \frac{a_2-a_1}{a_1+a_2}\|\mathcal{K}_{e,s}^T n_0\|^2\,. \qquad (6.2.29)$$

Now we continue the calculations for the skew-symmetric part,

$$\|\operatorname{skew}\Gamma^\natural[(\nabla_x\Theta)^\natural]^{-1}\|^2 = \|\operatorname{skew}\mathcal{K}_{e,s}\|^2 + \|\operatorname{skew}((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})\|^2 + 2\langle \operatorname{skew}\mathcal{K}_{e,s}, \operatorname{skew}((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})\rangle. \qquad (6.2.30)$$

In a similar manner, we calculate the terms separately. Since $n_0\otimes n_0$ is symmetric, we obtain

$$\begin{aligned}
\|\operatorname{skew}((0|0|c)[(\nabla_x\Theta)^\natural]^{-1})\|^2 &= \|\operatorname{skew}\Big(\frac{a_2-a_1}{a_1+a_2}\mathcal{K}_{e,s}^T\, n_0\otimes n_0 - \frac{a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s})\, n_0\otimes n_0\Big)\|^2 \qquad (6.2.31)\\
&= \frac{(a_1-a_2)^2}{(a_1+a_2)^2}\|\operatorname{skew}(\mathcal{K}_{e,s}^T\, n_0\otimes n_0)\|^2.
\end{aligned}$$

But, we have

$$\begin{aligned}
\|\operatorname{skew}(\mathcal{K}_{e,s}^T n_0\otimes n_0)\|^2 &= \frac{1}{4}\left\langle \mathcal{K}_{e,s}^T n_0\otimes n_0, \mathcal{K}_{e,s}^T\, n_0\otimes n_0\right\rangle - \frac{1}{4}\left\langle \mathcal{K}_{e,s}^T n_0\otimes n_0, n_0\otimes n_0\,\mathcal{K}_{e,s}\right\rangle\\
&\quad - \frac{1}{4}\left\langle n_0\otimes n_0\,\mathcal{K}_{e,s}, \mathcal{K}_{e,s}^T\, n_0\otimes n_0\right\rangle + \frac{1}{4}\left\langle n_0\otimes n_0\,\mathcal{K}_{e,s}, n_0\otimes n_0\,\mathcal{K}_{e,s}\right\rangle \qquad (6.2.32)\\
&= \frac{1}{2}\|\mathcal{K}_{e,s}^T n_0\|^2\,,
\end{aligned}$$

where we used the fact that $(n_0 \otimes n_0)^2 = (n_0 \otimes n_0)$. We have as well

$$
\begin{aligned}
2\langle \operatorname{skew} \mathcal{K}_{e,s}, \operatorname{skew}((0|0|c)[(\nabla_x \Theta)^\natural]^{-1})\rangle &= 2\,\frac{(a_2-a_1)}{(a_1+a_2)}\,\big\langle \operatorname{skew} \mathcal{K}_{e,s}, \operatorname{skew}(\mathcal{K}_{e,s}^T n_0 \otimes n_0)\big\rangle \qquad (6.2.33)\\
&= \frac{(a_2-a_1)}{2(a_1+a_2)}\langle \mathcal{K}_{e,s}, \mathcal{K}_{e,s}^T n_0 \otimes n_0\rangle - \frac{(a_2-a_1)}{2(a_1+a_2)}\langle \mathcal{K}_{e,s}, n_0 \otimes n_0\, \mathcal{K}_{e,s}\rangle\\
&\quad - \frac{(a_2-a_1)}{2(a_1+a_2)}\langle \mathcal{K}_{e,s}^T, \mathcal{K}_{e,s}^T n_0 \otimes n_0\rangle + \frac{(a_2-a_1)}{2(a_1+a_2)}\langle \mathcal{K}_{e,s}^T, n_0 \otimes n_0\, \mathcal{K}_{e,s}\rangle\\
&= -\frac{(a_2-a_1)}{(a_1+a_2)}\|\mathcal{K}_{e,s}^T n_0\|^2\,,
\end{aligned}
$$

and we obtain

$$
\|\operatorname{skew} \Gamma^\natural[(\nabla_x \Theta)^\natural]^{-1}\|^2 = \|\operatorname{skew} \mathcal{K}_{e,s}\|^2 + \frac{(a_2-a_1)^2}{2(a_1+a_2)^2}\|\mathcal{K}_{e,s}^T n_0\|^2 - \frac{(a_2-a_1)}{(a_1+a_2)}\|\mathcal{K}_{e,s}^T n_0\|^2. \qquad (6.2.34)
$$

A further needed calculation is

$$
\begin{aligned}
\Big[\operatorname{tr}(\Gamma^\natural[(\nabla_x \Theta)^\natural]^{-1})\Big]^2 &= \Big(\operatorname{tr}(\mathcal{K}_{e,s}) + \operatorname{tr}\big((0|0|c)[(\nabla_x \Theta)^\natural]^{-1})\big)\Big)^2 \qquad (6.2.35)\\
&= \Big(\operatorname{tr}(\mathcal{K}_{e,s}) + \frac{(a_2-a_1)}{2(a_1+a_2)}\langle \mathcal{K}_{e,s}^T n_0 \otimes n_0, \mathbb{1}_3\rangle - \frac{a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s})\underbrace{\langle n_0 \otimes n_0, \mathbb{1}_3\rangle}_{\langle n_0, n_0\rangle = 1}\Big)^2\\
&= \frac{a_1^2}{(a_1+a_3)^2}\operatorname{tr}(\mathcal{K}_{e,s})^2.
\end{aligned}
$$

Now we apply the above calculations in the formula (6.2.4), and obtain

$$
\begin{aligned}
W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) = \mu L_c^2\Big(& a_1(\|\operatorname{sym} \mathcal{K}_{e,s}\|^2 + \frac{(a_1-a_2)^2}{2(a_1+a_2)^2}\|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{a_3^2}{(a_1+a_3)^2}\operatorname{tr}(\mathcal{K}_{e,s})^2 + \frac{a_2-a_1}{a_1+a_2}\|\mathcal{K}_{e,s}^T n_0\|^2)\\
&+ a_2(\|\operatorname{skew} \mathcal{K}_{e,s}\|^2 + \frac{(a_2-a_1)^2}{2(a_1-a_2)^2}\|\mathcal{K}_{e,s}^T n_0\|^2 - \frac{a_2-a_1}{a_1+a_2}\|\mathcal{K}_{e,s}^T n_0\|^2)\\
&+ a_3\frac{a_1^2}{(a_1+a_3)^2}\operatorname{tr}(\mathcal{K}_{e,s})^2\Big)\,, \qquad (6.2.36)
\end{aligned}
$$

which reduces to

$$
W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) = \mu L_c^2\Big(a_1\|\operatorname{sym} \mathcal{K}_{e,s}\|^2 + a_2\|\operatorname{skew} \mathcal{K}_{e,s}\|^2 - \frac{(a_1-a_2)^2}{2(a_1+a_2)}\|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{a_1 a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s})^2\Big)\,. \qquad (6.2.37)
$$

One may apply the orthogonal decomposition of a matrix X

$$
X = X^\parallel + X^\perp, \qquad X^\parallel := \mathrm{A}_{y_0}\, X, \qquad X^\perp := (\mathbb{1}_3 - \mathrm{A}_{y_0})\, X, \qquad (6.2.38)
$$

for the matrix $\mathcal{K}_{e,s}$, where $A_{y_0} = (\nabla y_0|0)[\nabla_x \Theta(0)]^{-1} \in \mathrm{SO}(3)$. After inserting the decomposition inside the homogenized curvature energy, we get

$$
\begin{aligned}
W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) &= \mu L_c^2\Big(a_1\|\operatorname{sym} \mathcal{K}_{e,s}\|^2 + a_2\|\operatorname{skew} \mathcal{K}_{e,s}\|^2 - \frac{(a_1-a_2)^2}{2(a_1+a_2)}\|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{a_1 a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s})^2\Big)\\
&= \mu L_c^2\Big(a_1\|\operatorname{sym} \mathcal{K}_{e,s}^\parallel\|^2 + a_2\|\operatorname{skew} \mathcal{K}_{e,s}^\parallel\|^2 - \frac{(a_1-a_2)^2}{2(a_1+a_2)}\|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{a_1 a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s}^\parallel)^2\\
&\qquad + \frac{a_1+a_2}{2}\|\mathcal{K}_{e,s}^T n_0\|\Big)\\
&= \mu L_c^2\Big(a_1\|\operatorname{sym} \mathcal{K}_{e,s}^\parallel\|^2 + a_2\|\operatorname{skew} \mathcal{K}_{e,s}^\parallel\|^2 + \frac{a_1 a_3}{(a_1+a_3)}\operatorname{tr}(\mathcal{K}_{e,s}^\parallel)^2 + \frac{2a_1 a_2}{a_1+a_2}\|\mathcal{K}_{e,s}^\perp\|\Big)\,. \qquad (6.2.39)
\end{aligned}
$$

6.2.3. Consistency check: obtaining the flat model from the curvy one

Now let us assume that in the homogenized energy which we obtained in (6.2.39) we have $\nabla\Theta = \mathbb{1}_3, \nabla y_0 = \mathbb{1}$. Also $n_0 = [\nabla_x \Theta(0)]e_3 = e_3$. Then for the matrix

$$
\mathcal{K}_{e,s} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\Gamma_\square} & 0\\ & & 0\\ \hline \Gamma_{31} & \Gamma_{32} & 0 \end{array}\right)[(\nabla_x \Theta)^\natural]^{-1}\,,
$$

with $\Gamma_\square = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$ we have

$$W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) = \mu L_c^2 \Big(a_1 \|\text{sym}\,\Gamma_\square\|^2 + a_2 \|\text{skew}\,\Gamma_\square\|^2 + \frac{a_1 a_3}{(a_1 + a_3)}\,\text{tr}(\Gamma_\square)^2 + \frac{2 a_1 a_2}{a_1 + a_2} \left\| \begin{pmatrix} \Gamma_{31} \\ \Gamma_{32} \end{pmatrix} \right\|^2 \Big) . \quad (6.2.40)$$

A comparison between (6.1.12) and (6.2.40), shows that the homogenized flat curvature energy can be obtained from the curvature one.

6.3. Conclusion

In this part we have considered the Γ-limit procedure in order to derive a Cosserat thin shell model having a curved reference configuration. The paper is based on the development in [90], where the Γ-limit was obtained for a flat reference configuration of the shell. Here, the major complication arises from the curvy shell reference configuration. By introducing suitable mappings, we can encode the "curvy" information on a fictitious flat reference configuration. There, we use the nonlinear scaling for both the nonlinear deformation and the microrotation. This leads to a Cosserat membrane model, in which the effect of Cosserat-curvature survive the Γ-limit procedure. The homogenized membrane and curvature energy expressions are made explicit after some lengthy technical calculations. This is only possible because we use a physically linear, isotropic Cosserat model. Since the limit equations are obtained by Γ-convergence, they are automatically well-posed. We finally compare the Cosserat membrane shell model with some other dimensionally reduced proposals and linearizations. The full regularity of weak solutions for this Cosserat shell model (for some choice of constitutive parameters) will be established in [54].

Part II.

Drill rotations for Cosserat surfaces

7. Rotations and Cosserat surfaces

In this part we consider the apparently novel question whether nontrivial pure in-plane drill rotations may appear in the deformation of shells if boundary conditions are prescribed that fix the Cosserat drill rotations at a portion of the boundary.

We show under some natural smoothness assumptions that pure in-plane drill rotations as deformation mappings of a C^2-smooth regular shell surface to another one parametrized over the same domain are impossible provided that the rotations are fixed at a portion of the boundary. Put otherwise, if the tangent vectors of the new surface are obtained locally by only rotating the given tangent vectors, and if these rotations have a rotation axis which coincides everywhere with the normal of the initial surface, then the two surfaces are equal provided they coincide at a portion of the boundary. In the language of differential geometry of surfaces we show that any isometry which leaves normals invariant and which coincides with the given surface at a portion of the boundary, is the identity mapping.

In this context, we prove the following main improved rigidity result for surfaces

Proposition 7.0.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\overline{\omega}, \mathbb{R}^3)$ are regular surfaces, $Q \in C^1(\overline{\omega}, \mathrm{SO}(3))$ and*

$$\begin{aligned} \mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x), \quad Q(x)\,n_0(x) = n_0(x)\,, \quad x \in \overline{\omega}\,, \\ m|_{\gamma_d} = y_0|_{\gamma_d}\,, \end{aligned} \tag{7.0.1}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$ and γ_d is a relatively open, non-empty subset of the boundary $\partial\omega$. Then $m \equiv y_0$.

The interest for this question is not coming from differential geometry per se, but is motivated from shell models with independent director fields, so called Cosserat-surfaces [38]. The additional field is a rotation vector field $Q \in \mathrm{SO}(3)$, necessitating additional balance equations and offering the possibility to introduce new (material) parameters into the model, coupling in-plane tangent vector fields and the rotation field Q by a stiffness $\mu_c > 0$. The question of how to determine the Cosserat couple modulus μ_c is largely open in the dedicated literature [18, 16, 17, 19, 21, 29, 70, 98, 99, 109, 107, 3, 20, 49, 69, 73, 108, 111]. We focus therefore on the effect, this Cosserat couple modulus μ_c may have and arrive at investigating *pure in-plane drill rotations* for arbitrary shell surfaces. In the course of this investigation (see section 7.1 below) we connect the initial question to more standard rigidity results [74, 94, 2, 30] for solid bodies and thin shells.

With the result of Proposition 7.0.1, in the end we are seeing that the stiffness μ_c in Cosserat shell models is arguably connected to a boundary condition and therefore, its status as material parameter is in doubt.

Proposition 7.0.1 can be seen in the language of classical differential geometry of surfaces as:

Corollary 7.0.2. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\overline{\omega}, \mathbb{R}^3)$ are two regular surfaces and*

$$\begin{aligned} \mathrm{I}_m(x) = [\mathrm{D}m(x)]^T\,\mathrm{D}m(x) = [\mathrm{D}y_0(x)]^T\,\mathrm{D}y_0(x) = \mathrm{I}_{y_0}(x)\,, \qquad n(x) = n_0(x) \qquad \forall x \in \overline{\omega}\,, \\ m|_{\gamma_d} = y_0|_{\gamma_d}\,, \end{aligned} \tag{7.0.2}$$

where $n = \frac{\partial_1 m \times \partial_2 m}{\|\partial_1 m \times \partial_2 m\|}$ and $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ are the respective normal fields and γ_d is a relatively open, non-empty subset of the boundary $\partial\omega$. Then $m \equiv y_0$.

The results of this Part should perhaps not come as a surprise to experts in the field of differential geometry. Indeed, except for minimal surfaces, the Gauss map n_0 already determines the surface essentially, cf. [66, Theorem 2.5]. On the contrary, minimal surfaces come with a family of 'associate surfaces', which have all the same Gauss map but are distinct to each other. Comparable results to Corollary 7.0.2 can be found in [1, 66, 48] where different methods of proof were used and any connection to applications were missing. However, the latter result does not belong to the standard textbook knowledge in differential geometry and it is completely unknown in the field of shell-theory. Our aim is to give a straight forward proof without the techniques coming from differential geometry. We use the Rodrigues representation formula for rotations with given axis as well as repeated properties of the cross-product.

In order to set the stage, we recall some of the better known rigidity and integrability theorems, for 3D−bulk materials and 2D−surfaces. For a warm up, we tackle first the small rotation problem which already discloses some of the necessary techniques. In the subsequent section 7.7 we give the proof of Proposition 7.0.1 for the large rotation problem.

The complementary problem

$$\mathrm{D}m(x) = U(x)\ \mathrm{D}y_0(x), \tag{7.0.3}$$

of finding all "compatible" in-plane stretches U, characterized by

$$U(x)\, n_0(x) = \kappa^+(x)\, n_0(x), \quad U(x) \in \mathrm{Sym}^+(3), \quad \kappa^+(x) > 0, \tag{7.0.4}$$

has been completely solved more than ten years ago by Szwabowicz [106].[1]

Moreover, we have done some results regarding to *minimal surfaces.* If we consider the definition of a minimal surface in mathematics aspect, we may consider surfaces that locally minimize their area; or equivalently, when the *mean curvature* is zero (H = 0).
A famous example of minimal surfaces is the surface formed by the soap solution. This can be achieved when we dip two wire rings inside a soap solution. Then what is produced is the boundary of this soap film (which is a minimal surface) is the frame of the wire.
There is another equaivalent definition for minimal surfaces as following:
A surface $X \subset \mathbb{R}^3$ is called a minimal surface if and only if X is a *least-area surface.* A least area surface is a surface which its area is less than or equal to the area of any other surface having the same boundary.
J. L. Lagrange (1768) was one of the first researchers who considered minimal surfaces by bringing up the following variational problem. The task is: find a least area surface which is stretched across a given closed contour. If we assume that the mentioned surface is $z = z(x, y)$, then Lagrange showed that the surface z should satisfies the Euler-Lagrange equation:

$$(1+q^2)\frac{\partial^2 z}{\partial^2 x} - 2pq\frac{\partial^2 z}{\partial x \partial y} + (1+p^2)\frac{\partial^2 z}{\partial^2 y} = 0\,, \tag{7.0.5}$$
$$p = \frac{\partial z}{\partial x}\,, \qquad q = \frac{\partial z}{\partial y}\,.$$

After a while, it was discovered by others that the minimality condition of a surface shows H = 0, and hence, a surface with zero mean curvature is called *minimal.*
There are some examples of minimal surfaces like *catenoid* and *helicoid* (see subsection 7.5). In reality, a catenoid is a form that is obtained by a soap film which is stretched over two discs of wire and are perpendicular to the line which connect their centers. A catenoid is a member of the family of surfaces which are obtained from the revolution of the curves $y = a \cosh x/b$ around the x-axis. Nevertheless, just the special case $a = b$ causes that the corresponding surface can be a minimal surface. The catenoid is locally isometric to the helicoid. Indeed, a helicoid is a ruled surface which can be assumed as a straight line that rotates at a constant angular rate around a fixed axis and at the same time step by step collapses at a constant rate k along this axis. In parametric form, one can see a helicoid as following

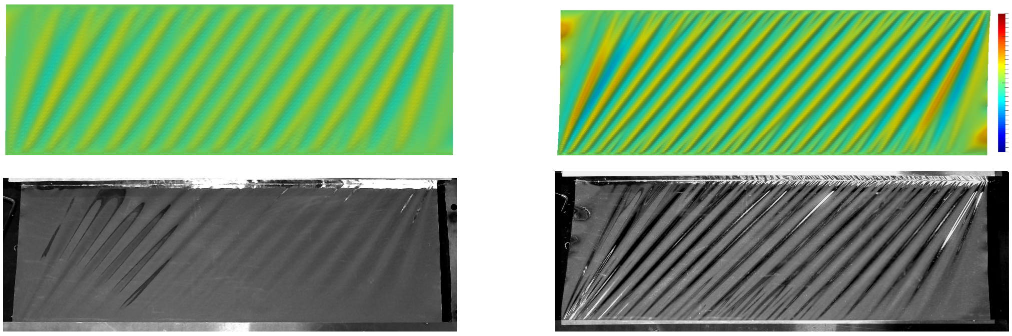
$$x = \rho \cos t\,, \qquad y = \rho \sin t\,, \qquad z = \rho \arctan \alpha + kt\,, \tag{7.0.6}$$

where α is the constant angle. We notice that for $\alpha = \frac{\pi}{2}$ the helicoid is called *straight or right*, otherwise is called *oblique.* Every straight helicoid is a minimal surface in the sense of zero mean curvature.
It is interesting to know that although a sphere is counted as a minimial surface in the aspect of minimization the surface area to volume ration, but it is not qualified as a minimial surface in the sense of mathematics definition.

7.1. Engineering motivation: Cosserat shell models

The elastic range of many engineering materials is restricted to small finite strains. Thin structures may typically undergo large rotations (by bending) but are accompanied by small elastic strains.

[1] Szwabowicz uses a different notation, but his stretch tensor is basically the stretch tensor U satisfying (7.0.4). Note that since $U \in \mathrm{Sym}^+(3)$ it can be orthogonally diagonalized, the stretch $U(x)$ satisfying (7.0.4) leaves the tangent plane $T_{y_0(x)}y_0(\omega)$ invariant. Therefore, the Gauss map is preserved as well.



In [84] the following geometrically nonlinear (but small elastic strain) isotropic planar shell model has been derived for such a situation: find the midsurface deformation $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the independent rotation field $\overline{R} : \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3)$ minimizing the elastic energy

$$\begin{aligned} I(m,\overline{R}) = \int_\omega h\Big[\mu \underbrace{\|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T \mathrm{D}m - \mathbb{1}_2)\|^2}_{\text{shear-stretch energy}} + \mu_c \underbrace{\|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T \mathrm{D}m)\|^2}_{\text{first order drill energy}} \\ + \frac{(\mu+\mu_c)}{2} \underbrace{\Big(\langle \overline{R}_3, \partial_1 m\rangle^2 + \langle \overline{R}_3, \partial_2 m\rangle^2\Big)}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda} \underbrace{\mathrm{tr}(\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T \mathrm{D}m - \mathbb{1}_2))^2}_{\text{stretch energy}}\Big] \\ + h\Big[\mu L_c^2 \|\mathcal{K}_s\|^2 + \mu L_c^{2+q}\|\mathcal{K}_s\|^{2+q}\Big] \qquad (7.1.1) \\ + \frac{h^3}{12}\Big[\mu\|\mathrm{sym}\,\mathcal{K}_b\|^2 + \mu_c\|\mathrm{skew}\,\mathcal{K}_b\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}[\mathcal{K}_b]^2\Big]\, d\omega \to \min \ \text{w.r.t}\ (m,\overline{R}), \end{aligned}$$

where the Cosserat curvature tensor is given by

$$\mathcal{K}_s = (\overline{R}^T(\mathrm{D}(\overline{R}.e_1)|0), \overline{R}^T(\mathrm{D}(\overline{R}.e_2)|0), \overline{R}^T(\mathrm{D}(\overline{R}.e_3)|0)), \qquad \mathcal{K}_b := \mathcal{K}_{s,3} = \overline{R}^T(\mathrm{D}(\overline{R}.e_3)|0), \qquad (7.1.2)$$

and the boundary condition of place for the midsurface deformation m on the Dirichlet part of the lateral boundary, $m|_{\gamma_d} = g_d(x, y, 0)$ is imposed. This shell model is derived by dimensional descent from a three-dimensional bulk Cosserat model [38, 84] and the appearing parameters are the isotropic shear modulus $\mu > 0$, the second Lamé parameter λ (with $2\mu + \lambda > 0$) and the so-called **Cosserat couple modulus** $\mu_c \geq 0$, while $h > 0$ is the thickness of the shell, $L_c \geq 0$ is a characteristic length and $q \geq 0$. This Cosserat shell model can be naturally related to the general six-parameter theory of shells [18, 16, 17, 19, 21, 29, 70, 98, 99, 109, 107], see also [3, 20, 49, 69, 73, 108, 111]. One of the typical energy terms in these models is connected to so-called *in-plane* drill rotations [19, 110]. These in-plane drill rotations describe local rotations of the shell midsurface with rotation axis given by the local shell normal n_0 of y_0. Typically, the constitutive coefficients which governs this deformation mode are difficult to establish (and the ubiquitous Cosserat couple modulus $\mu_c > 0$ appears prominently). Naghdi-type shell models with only one independent "Cosserat"-director do not have the drill-degree of freedom [77] but allow for transverse shear. Classical shell models neither have drill nor transverse shear [56, 45, 100]. On the contrary, rotations about in-plane axis describe bending and twist. Even though a classical shell model (with Kirchhoff-Love normality assumption) does not have this kinematic degree of freedom, numerical approaches may introduce artificially shell-elements that possess locally this degree of freedom. The question is then which amount of stiffness should be adopted. It is observed that higher artificial in-plane drill stiffness strongly affects the calculated solution. In this context it is also known that flat shell topologies allow for unconstrained drill rotations and we will observe this in this chapter as well: indeed unconstrained drill rotations may be observed not only for flat surfaces but for any minimal surface as well.

The extension of the planar shell model to initially curved shells has been recently given in [61, 56, 57]. The planar shell model (7.1.1) has been used to successfully predict the wrinkling behavior of very thin elastic sheets [103]. In these calculations, however, the Cosserat couple modulus μ_c has been set to zero throughout and $q = 2$ has been adopted. In this case, the term in (7.1.1) denoted by "first order drill energy" will drop out, while all other terms basically remain the same. It seems therefore mandatory to devote special attention to this in-plane drill term in order to understand it's physical and mathematical significance. This will be undertaken next.

7.2. On the physical concept of in-plane drill-linear torsional spring

Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ smooth and regular describing the mid-surface of a shell.

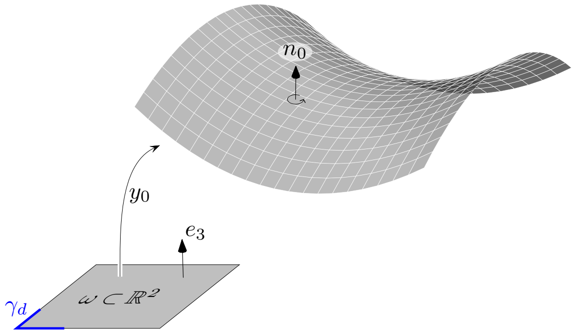


Figure 7.1: The midsurface $y_0 \in C^2(\overline{\omega}, \mathbb{R}^3)$ of a shell is visualized, together with a tangent plane $T_{y_0(x)}y_0$ spanned by $\partial_1 y_0(x), \partial_2 y_0(x)$ and with unit normal $n_0(x)$. Prescribed boundary conditions at γ_d fix the shell mid-surface in space.

Let us analyze the energy term corresponding to drill rotations shown in (7.1.1). In order to measure in-plane drill rotations of a shell in a continuum description one first needs to endow the shell with a given orthonormal frame, tangent to the surface y_0, against which in-plane rotations can be seen. The role of this frame will be taken here by $Q_0 \in \mathrm{SO}(3)$, defined by $Q_0 := \mathrm{polar}(\mathrm{D}y_0|n_0)$ (already used as such by Darboux, see [41]), where $\mathrm{polar}(F)$ denotes the orthogonal part in the polar-decomposition of $F \in \mathrm{GL}^+(3)$. First, it holds that

$$\mathrm{skew}(Q_0^T(\mathrm{D}y_0|n_0)) = 0 \qquad \text{for} \qquad Q_0 \;=\; \mathrm{polar}(\mathrm{D}y_0|n_0) \in \mathrm{SO}(3), \tag{7.2.1}$$

due to the properties of the polar decomposition [92]

$$\begin{aligned}(\mathrm{D}y_0|n_0) &= Q_0 \underbrace{\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\sqrt{[\mathrm{D}y_0]^T\,\mathrm{D}y_0}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 1\end{array}\right)}_{\in \mathrm{Sym}^+(3)} \\ \overset{Q_0 e_3 = n_0}{\iff} \quad & (Q_{0_1}, Q_{0_2})^T\,\mathrm{D}y_0 = \sqrt{[\mathrm{D}y_0]^T\,\mathrm{D}y_0} \in \mathrm{Sym}^+(2)\,. \end{aligned} \tag{7.2.2}$$

Here, it can be seen that $Q_0 : \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3)$ is an orthonormal frame whose third column coincides with the normal n_0 of the surface such that there is also *no* induced *transverse shear*. The three dimensional condition (7.2.1) can be expressed equivalently as (see also (7.2.2))

$$\mathrm{skew}(Q_0^T(\mathrm{D}y_0|n_0)) = \mathrm{skew}\left(\begin{array}{c|c} (Q_{0_1}, Q_{0_2})^T\,\mathrm{D}y_0 & 0 \\ \hline 0 & 1\end{array}\right) = 0 \iff \mathrm{skew}[(Q_{0_1}|Q_{0_2})^T\,\mathrm{D}y_0] = 0,$$

which is the "drill energy" argument from equation (7.1.1) and in this special situation we have an initially "drill-free" setting.

Now, what happens if we only **locally rotate (drill)** the given tangent vectors $\partial_1 y_0, \partial_2 y_0$ about the rotation axis n_0? For this, we take a drill rotation $Q(\alpha)\,n_0 = n_0$, $Q(\alpha) = Q(\alpha(x)) \in \mathrm{SO}(3)$, where $\alpha = \alpha(x)$ is the rotation angle and $n_0 = n_0(x)$ is the prescribed axis of rotation normal to the surface y_0 and we consider locally the mapping

$$\mathrm{D}y_0 \to Q(\alpha)\,\mathrm{D}y_0\,, \tag{7.2.3}$$

which leaves the first fundamental form $\mathrm{I}_{y_0} = \mathrm{D}y_0^T\,\mathrm{D}y_0 = \big(Q(\alpha)\,\mathrm{D}y_0\big)^T\big(Q(\alpha)\,\mathrm{D}y_0\big)$ invariant. This implies that the surface y_0 is locally changed isometrically. For simplicity, taking into account the subsequent representation (7.3.2) of rotations with given axis of rotations, we consider presently only small drill rotation angles α so that we can duly approximate

$$Q(\alpha) \approx \mathbb{1} + \alpha\ \mathrm{Anti}(n_0). \tag{7.2.4}$$

Inserting then $(\mathbb{1}+\alpha\ \mathrm{Anti}(n_0))\,\mathrm{D}y_0$ instead of $\mathrm{D}y_0$ into the drill term from (7.2.3) we obtain

$$\begin{aligned}\mathrm{skew}&\Big[(Q_{0_1}|Q_{0_2})^T(\mathbb{1}+\alpha\ \mathrm{Anti}(n_0))\Big(\partial_1 y_0|\partial_2 y_0\Big)\Big]\\ &=\underbrace{\mathrm{skew}((Q_{0_1}|Q_{0_2})^T\,\mathrm{D}y_0)}_{=0}+\mathrm{skew}\,\Big[(Q_{0_1}|Q_{0_2})^T\,\alpha\,\Big(n_0\times\partial_1 y_0\Big|n_0\times\partial_2 y_0\Big)\Big] \qquad (7.2.5)\\ &=\frac{\alpha}{2}\begin{pmatrix}0 & -[\langle Q_{0_2},n_0\times\partial_1 y_0\rangle-\langle Q_{0_1},n_0\times\partial_2 y_0\rangle]\\ [\langle Q_{0_2},n_0\times\partial_1 y_0\rangle-\langle Q_{0_1},n_0\times\partial_2 y_0\rangle] & 0\end{pmatrix}.\end{aligned}$$

We will now show that for any non-zero small angle of rotation α, expression (7.2.5) is non-zero implying that the related drill energy term $h\,\mu_c\|\mathrm{skew}((Q_{0_1}|Q_{0_2})^T\,\mathrm{D}y_0)\|^2$ serves to introduce a **linear torsional spring stiffness** against superposed in-plane rotations (with spring constant $h\mu_c$, where $\mu_c\geq 0$ is the Cosserat couple modulus).

Note that at present, the discussion is purely local: at no place did we require that $Q(\alpha(x))\,\mathrm{D}y_0(x)$ can be determined as the gradient of a mapping. The global question whether $Q(\alpha)\ \mathrm{D}y_0$ can be the gradient of a regular embedding $m:\omega\to\mathbb{R}^3$ with $\omega\subset\mathbb{R}^2$ will be considered next.

7.2.1. Setting of the differential geometric problem

Consider a given initial curved shell surface parametrized locally by $y_0:\overline{\omega}\subset\mathbb{R}^2\to\mathbb{R}^3$, where we assume that y_0 is sufficiently smooth and regular ($\mathrm{rank}(\mathrm{D}y_0)=2$). Let $m:\overline{\omega}\subset\mathbb{R}^2\to\mathbb{R}^3$ be any smooth deformation of the given surface y_0 parametrized over the same domain and consider a smooth in-plane drill rotation field

$$Q:\overline{\omega}\subset\mathbb{R}^2\to\mathrm{SO}(3),\qquad Q(x)\,n_0(x)=n_0(x)\,, \qquad (7.2.6)$$

where $n_0=\dfrac{\partial_1 y_0\times\partial_2 y_0}{\|\partial_1 y_0\times\partial_2 y_0\|}$ is the unit normal vector field on the initial surface y_0. We will assume further on that

$$Q|_{\gamma_d}=\mathbb{1}, \qquad (7.2.7)$$

where γ_d is a relatively open, non-empty subset of the boundary $\partial\omega$. The motivation for this boundary condition will be given in Lemma 7.4.1.

Problem 1. Let $\omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and assume that $m,y_0\in C^2(\overline{\omega},\mathbb{R}^3)$ are regular surfaces. Does there exist a nontrivial in-plane drill rotation field $Q\in C^1(\overline{\omega},\mathrm{SO}(3))$ such that

$$\begin{aligned}&\mathrm{D}m(x)=Q(x)\,\mathrm{D}y_0(x),\qquad Q(x)n_0(x)=n_0(x),\qquad x\in\overline{\omega},\\ &Q|_{\gamma_d}=\mathbb{1}.\end{aligned} \qquad (7.2.8)$$

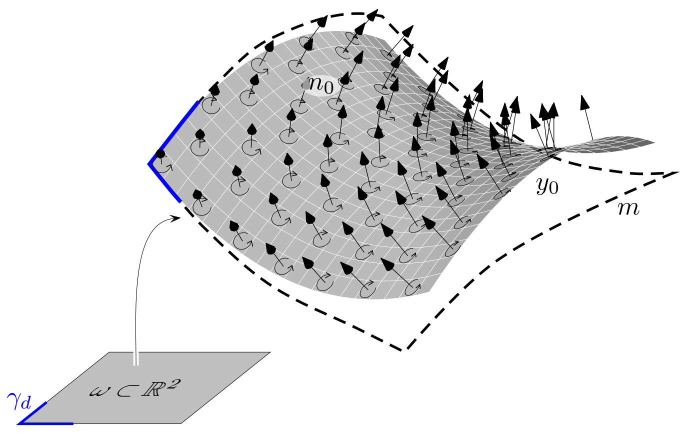


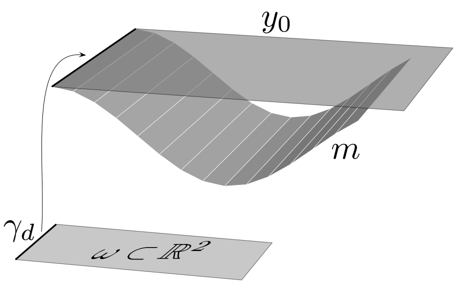
Figure 7.2: At any point of the surface y_0 tangent planes are rotated in their own plane leaving the orientation of the surface invariant to obtain a new surface m. How can m look like if at a part of the boundary $\gamma_d\subset\partial\omega$ the surfaces m and y_0 coincide?

Remark 7.2.1. For a given shell surface y_0 any pure bending (flexure) deformation m satisfies locally

$$\mathrm{D}m(x)=Q(x)\,\mathrm{D}y_0(x),\qquad Q(x)\in\mathrm{SO}(3), \qquad (7.2.9)$$

such that the first fundamental forms coincide $\mathrm{I}_m = [\mathrm{D}m]^T\,\mathrm{D}m = [\mathrm{D}y_0]^T\,\mathrm{D}y_0 = \mathrm{I}_{y_0}$. Considering a flat piece of paper assumed to be made of unstretchable material, the rotations can even be fixed at one side of the paper still allowing for nontrivial bending deformations of the paper. However, the appearing local rotation field $Q(x) \in \mathrm{SO}(3)$ in this case is not of in-plane drill type, i.e., the rotation axis of Q is not everywhere given by n_0. □

Figure 7.3: A pure bending deformation of a surface leaves length invariant, the first fundamental form is unchanged, but there is local rotation. We have $\mathrm{D}m = Q(x)\,\mathrm{D}y_0$ for some non-constant $Q \in \mathrm{SO}(3)$, but the rotation in this example does not have an in-plane rotation axis.



Remark 7.2.2 (Rigidity in 3D)**.** Assume that $M : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ and $Y_0 : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ are two diffeomorphisms, the formally similar to (7.2.8) looking condition

$$\mathrm{D}M(x) = Q(x)\,\mathrm{D}Y_0(x)\,, \qquad Q(x) \in \mathrm{SO}(3), \qquad x \in \Omega\,, \tag{7.2.10}$$

implies that $Q(x) \equiv \mathrm{const}$ by rigidity [94, 2], as can easily be seen as follows.

By the chain rule we have

$$\mathrm{D}(M(Y_0^{-1}(\xi))) = \mathrm{D}M(Y_0^{-1}(\xi))\ \mathrm{D}[Y_0^{-1}(\xi)]\,, \quad \text{where} \quad \mathrm{D}[Y_0^{-1}(\xi)] = [\mathrm{D}Y_0(x)]^{-1},$$

therefore,

$$\begin{aligned} &\mathrm{D}(M(Y_0^{-1}(\xi))) = Q(x)[\mathrm{D}Y_0(x)]\ [\mathrm{D}Y_0(x)]^{-1}, \\ &\Rightarrow \quad \mathrm{D}(M(Y_0^{-1}(\xi))) = Q(x) \in \mathrm{SO}(3) \overset{\text{rigidity}}{\Longrightarrow} Q \equiv \mathrm{const}. \end{aligned} \tag{7.2.11}$$

Note that the smoothness of $Q : \Omega \to \mathrm{SO}(3)$ can be a priori controlled by the smoothness of M and Y_0. □

Remark 7.2.3 (Rigidity in 3D)**.** In the 3D−case we have another condition which turns out to yield homogeneous rotations as well. Assume again that $M, Y_0 : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ are two diffeomorphisms. Then

$$\forall x \in \Omega : \quad [\mathrm{D}M(x)]^T\,\mathrm{D}M(x) = [\mathrm{D}Y_0(x)]^T\,\mathrm{D}Y_0(x) \quad \Longleftrightarrow \quad M(x) = \overline{Q}\,Y_0(x)\,, \tag{7.2.12}$$

where $\overline{Q} \in \mathrm{SO}(3)$ is a constant rotation, as is shown, e.g. in [30]. However, as already seen, 2D−structures are much more flexible in the sense that for $m, y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ (smooth embeddings)

$$\mathrm{I}_m = [\mathrm{D}m(x)]^T\,\mathrm{D}m(x) = [\mathrm{D}y_0(x)]^T\,\mathrm{D}y_0(x) = \mathrm{I}_{y_0} \quad \not\Longrightarrow \quad m(x) = \overline{Q}\,y_0(x)\,, \tag{7.2.13}$$

as any pure bending deformation shows. □

Example 7.2.4 (Flat case)**.** Consider $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ with $y_0(x) = (x_1, x_2, 0)^T$. Then $\mathrm{D}y_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $n_0 \equiv e_3$. Thus, the conditions $Q(x)n_0 = n_0$, $m \in C^1(\overline{\omega}, \mathbb{R}^3)$ and $\mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x)$ for $Q(x) \in \mathrm{SO}(3)$, together with $Q|_{\gamma_d} = \mathbb{1}_3$, imply

$$\mathrm{D}m(x) = Q(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\widehat{Q}(x)$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{Q}(x) \\ \hline 0 \end{pmatrix}, \tag{7.2.14}$$

with $\widehat{Q}(x) \in \mathrm{SO}(2)$. Hence, $\mathrm{D}m_3(x) \equiv 0$ and

$$\mathrm{D}\begin{pmatrix} m_1(x) \\ m_2(x) \end{pmatrix} = \widehat{Q}(x) \in \mathrm{SO}(2). \tag{7.2.15}$$

Then, again by rigidity [94] we obtain that $\widehat{Q} \equiv \mathrm{const}$. Applying the boundary conditions we have $\widehat{Q}|_{\gamma_d} = \mathbb{1}_2$ and finally $Q \equiv \mathbb{1}_3$. Thus $m - y_0 \equiv \mathrm{const}$. □

7.3. Preliminaries on rotations in $\mathrm{SO}(3)$ and the Euler-Rodrigues formula

We need to consider the matrix exponential function

$$\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3), \qquad \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \quad A \in \mathfrak{so}(3). \tag{7.3.1}$$

According to Euler, any rotation can be realized by a rotation around one axis with a certain rotation angle. However, any rotation $Q \in \mathrm{SO}(3)$ with prescribed rotation axis n_0 and angle of rotation α can be written with the Euler-Rodrigues representation in matrix notation as [97]

$$\begin{aligned} Q(\alpha) &= \exp(\mathrm{Anti}(\alpha\, n_0)) = \mathbb{1} + \sin\alpha\ \mathrm{Anti}(n_0) + (1-\cos\alpha)\,(\mathrm{Anti}(n_0))^2 \\ &= (1-\cos\alpha)\, n_0 \otimes n_0 + \cos\alpha\, \mathbb{1} + \sin\alpha\ \mathrm{Anti}(n_0)\,. \end{aligned} \tag{7.3.2}$$

For small rotation angle $|\alpha| \ll 1$ the above representation (7.3.2) will be approximated by

$$Q \approx \mathbb{1} + \alpha\ \mathrm{Anti}(n_0), \tag{7.3.3}$$

since $\sin\alpha \to \alpha$ and $1-\cos\alpha \to 0$ as $|\alpha| \to 0$. For later use, note that any nontrivial rotation $\mathbb{1} \neq Q \in \mathrm{SO}(3)$ has (only) one rotation axis $\eta \in \mathbb{R}^3$ such that $Q\eta = \eta$ where η is the eigenvector to the real eigenvalue 1.

Taking the trace in (7.3.2), we also see that

$$\mathrm{tr}(Q(\alpha(x))) = 2\cos\alpha(x) + 1 \qquad \Longleftrightarrow \qquad \cos(\alpha(x)) = \frac{\mathrm{tr}(Q(\alpha(x))) - 1}{2}\,. \tag{7.3.4}$$

The inverse cosine is a multivalued function and each branch is differentiable only on $(-1,1)$. Thus, for $Q = \mathbb{1}$ or $Q = \mathrm{diag}(-1,-1,1)$ we have $\frac{\mathrm{tr}(Q)-1}{2} \in \{-1,1\}$, i.e., in the neighborhood of both these rotations the simple formula (7.3.4) is not meaningful for extracting a smooth rotation angle. In order to solve this problem for small rotation angle α (for Q near to $\mathbb{1}$) we proceed as follows (the simple idea is taken from [76]). Multiplying (7.3.2) on both sides with $\mathrm{Anti}(n_0)$ from the left gives

$$\mathrm{Anti}(n_0)Q(\alpha) = \underbrace{\mathrm{Anti}(n_0)}_{\in\mathfrak{so}(3)} + \sin\alpha(\mathrm{Anti}(n_0))^2 + (1-\cos\alpha)\underbrace{(\mathrm{Anti}(n_0))^3}_{\in\mathfrak{so}(3)}\,. \tag{7.3.5}$$

Tanking the trace gives

$$\mathrm{tr}\left(\mathrm{Anti}(n_0)Q(\alpha)\right) = -\sin\alpha\, \|\mathrm{Anti}(n_0)\|^2 = -2\sin\alpha\,, \tag{7.3.6}$$

since $\mathrm{Anti}(n_0) \in \mathfrak{so}(3)$ and $\|n_0\| = 1$. Thus with (7.3.4) we arrive at

$$\sin\alpha = -\frac{\mathrm{tr}\left(\mathrm{Anti}(n_0)Q\right)}{2}, \qquad \cos\alpha = \frac{\mathrm{tr}(Q)-1}{2} \quad \overset{\mathrm{tr}(Q)\neq 1}{\Longrightarrow} \quad \tan\alpha = -\frac{\mathrm{tr}\left(\mathrm{Anti}(n_0)Q\right)}{\mathrm{tr}(Q)-1}, \tag{7.3.7}$$

whereby any branch of the inverse tangent is smooth on $\mathbb{R}$. This shows that for $\mathrm{tr}(Q) - 1 > 0$, i.e., in a large neighborhood of $Q = \mathbb{1}$, the extraction of the rotation angle α from the rotation Q is as smooth as Q and the surface allows.

Lemma 7.3.1. *Assume $y \in C^2(\overline{\omega}, \mathbb{R}^3)$ is a regular surface and let $Q \in C^1(\overline{\omega}, \mathrm{SO}(3))$ be given. Assume that for a point $x_0 \in \omega$ and $\alpha_0 \in \mathbb{R}$ it holds*

$$Q(x_0) = (1-\cos\alpha_0)\, n_0(x_0) \otimes n_0(x_0) + \cos\alpha_0\, \mathbb{1}_3 + \sin\alpha_0\ \mathrm{Anti}(n_0(x_0))\,, \tag{7.3.8}$$

where n_0 is the normal field on y_0. Then there exists a neighborhood $\mathcal{U}(x_0) \subset \omega$ and a continuously-differentiable function

$$\alpha\colon \mathcal{U}(x_0) \to \mathbb{R} \quad \textit{satisfying} \quad \alpha(x_0) = \alpha_0\,, \tag{7.3.9}$$

such that for all $x \in \mathcal{U}(x_0)$

$$Q(x) = (1-\cos\alpha(x))\, n_0(x) \otimes n_0(x) + \cos\alpha(x)\, \mathbb{1}_3 + \sin\alpha(x)\ \mathrm{Anti}(n_0(x))\,. \tag{7.3.10}$$

Proof. The C^1-regularity of α in a sufficiently small neighborhood of x_0 follows from one of the expressions contained in (7.3.7). Indeed, for $\mathrm{tr}(Q(x_0)) \neq 1$ consider in $(7.3.7)_3$ the branch of the inverse tangent which contains α_0. Otherwise, for $\mathrm{tr}(Q(x_0)) = 1$ take in $(7.3.7)_2$ the branch of the inverse cosine which contains α_0, cf. Figure 7.4. ■

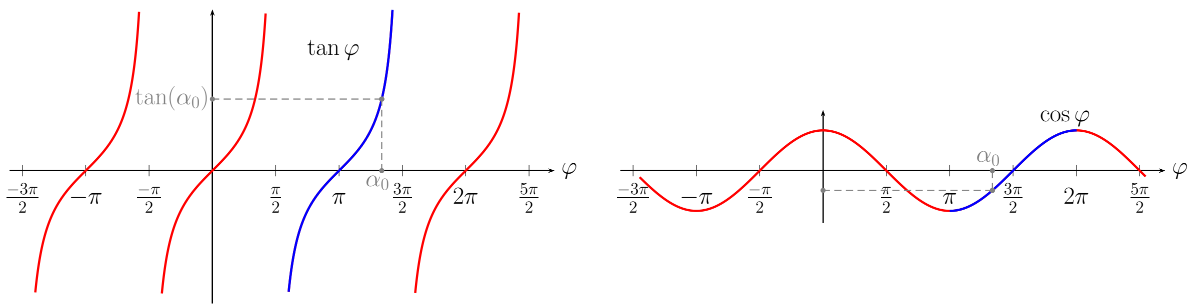


Figure 7.4.: The range for the corresponding branch of the inverse trigonometric functions are indicated in blue.

7.4. Boundary conditions

Lemma 7.4.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^1(\overline{\omega}, \mathbb{R}^3)$ are regular surfaces, $Q \in C^0(\overline{\omega}, \mathrm{SO}(3))$ and*

$$\mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x)\,, \qquad x \in \overline{\omega}\,, \qquad m|_{\gamma_d} = y_0|_{\gamma_d}\,, \tag{7.4.1}$$

where γ_d is a relatively open, non-empty subset of the boundary $\partial\omega$. If for all $x \in \gamma_d$ we have $Q(x)n_0(x) = n_0(x)$, where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$, then $Q|_{\gamma_d} \equiv \mathbb{1}$.

Remark 7.4.2. For the conclusion of this lemma we only need the assumptions on γ_d. However, the required conditions in the interior of ω are those which will be considered later. □

Proof of Lemma 7.4.1. Consider a $C^{0,1}$-parametrization $\gamma : (0,1) \to \mathbb{R}^2$, $\gamma(0,1) \subset \gamma_d \subset \partial\omega$. Then, $m(\gamma(s)) = y_0(\gamma(s))$ on $(0,1)$ implies for $\dot\gamma(s) \in \mathbb{R}^2$

$$\frac{d}{ds} m(\gamma(s)) = \frac{d}{ds} y_0(\gamma(s)) \;\Rightarrow\; \mathrm{D}m(\gamma(s))\dot\gamma(s) = \mathrm{D}y_0(\gamma(s))\dot\gamma(s) \;\in \mathbb{R}^3 \qquad \text{a.e. on } (0,1). \tag{7.4.2}$$

Hence,

$$\mathrm{D}m(\gamma(s))\,\dot\gamma(s) \overset{(7.4.1)}{=} Q(\gamma(s))\,\mathrm{D}y_0(\gamma(s))\,\dot\gamma(s) \overset{(7.4.2)}{=} Q(\gamma(s))\,\underbrace{\mathrm{D}m(\gamma(s))\,\dot\gamma(s)}_{=:q(s)\in\mathbb{R}^3} \qquad \text{a.e. on } (0,1). \tag{7.4.3}$$

Thus $q(s) = Q(\gamma(s))q(s)$, for almost all $s \in (0,1)$. Moreover, the vector $q(s)$ is a tangent vector to y_0 at $y_0(\gamma(s))$. Together with the assumption $Q(\gamma(s))\,n_0(\gamma(s)) = n_0(\gamma(s))$ it follows that $Q(\gamma(s))$ has two linear independent eigenvectors $q(s)$ and $n_0(\gamma(s))$. Since the axis of rotation is unique for any nontrivial rotation, it follows $Q(\gamma(\cdot)) = \mathbb{1}$ a.e. on $(0,1)$ and by continuity $Q|_{\gamma_d} \equiv \mathbb{1}$. ■

We repeat a similar reasoning for the small rotation case.

Lemma 7.4.3. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^1(\overline{\omega}, \mathbb{R}^3)$ are two regular surfaces, $A \in C(\overline{\omega}, \mathfrak{so}(3))$ and*

$$\mathrm{D}m(x) = (\mathbb{1} + A(x))\,\mathrm{D}y_0(x)\,, \qquad x \in \overline{\omega}\,, \qquad m|_{\gamma_d} = y_0|_{\gamma_d}\,, \tag{7.4.4}$$

where γ_d is a relatively open, non-empty subset of the boundary $\partial\omega$. If for all $x \in \gamma_d \subset \partial\omega$ we have $A(x)n_0(x) = 0$, where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$, then $A|_{\gamma_d} \equiv 0$.

Proof. We have again

$$\mathrm{D}m(\gamma(s))\dot\gamma(s) = \mathrm{D}y_0(\gamma(s))\dot\gamma(s) \;\in \mathbb{R}^3 \qquad \text{a.e. on } (0,1)\,, \tag{7.4.5}$$

for a $C^{0,1}$-parametrization $\gamma : (0,1) \to \mathbb{R}^2$, $\gamma(0,1) \subset \gamma_d \subset \partial\omega$. Hence, we obtain a.e. on $(0,1)$

$$\begin{aligned} \mathrm{D}y_0(\gamma(s))\dot\gamma(s) &\overset{(7.4.5)}{=} \mathrm{D}m(\gamma(s))\,\dot\gamma(s) \overset{(7.4.4)}{=} \mathrm{D}y_0(\gamma(s))\,\dot\gamma(s) + A(\gamma(s))\,\mathrm{D}y_0(\gamma(s))\,\dot\gamma(s) \\ \iff \qquad & 0 = A(\gamma(s))\,\underbrace{\mathrm{D}y_0(\gamma(s))\,\dot\gamma(s)}_{=:q(s)} = A(\gamma(s))\,q(s)\,, \quad \text{where} \quad q(s) \perp n_0(\gamma(s))\,. \end{aligned} \tag{7.4.6}$$

From the assumption we moreover have $A(\gamma(s))\,n_0(\gamma(s)) = 0$ so that with $A(\gamma(s))\,q(s) = 0$ along γ_d we obtain $A(\gamma(s)) = 0$, since any non-zero skew-symmetric 3×3 matrix A has rank two. ■

7.5. Family of minimal surfaces

One of the family of minimal surfaces is *catenoid*. It is shown that catenoid is a minimal surface. Actually this name comes from the rotating a certain *catenary* about some axis. Let us assume that the axis for rotation is x_3-axis. Then all catenoids are generated by rotating the catenaries

$$x_1 = \alpha \cosh(\frac{x_3 - x_{3_0}}{\alpha}), \tag{7.5.1}$$

where x_{3_0} and α are arbitrary constants with $\alpha \neq 0$.
By choosing $x_{3_0} = 0$, we can see a parametrized representation for a catenoid like $X\colon \mathbb{C} \to \mathbb{R}^3$

$$X(x_1, x_2) = \begin{pmatrix} \alpha \cosh x_1 \cos x_2 \\ -\alpha \cosh x_1 \sin x_2 \\ \alpha x_1 \end{pmatrix}, \quad \text{for} \quad -\infty < x_1 < \infty\,, \; 0 \leq x_2 < 2\pi\,, \tag{7.5.2}$$

and $\mathbf{x} = (x_1, x_2) \in \mathbb{C}$, that is, $\mathbf{x} = x_1 + ix_2$. If the mapping $f\colon \mathbb{C} \to \mathbb{C}^3$ denotes the isotropic curve with

$$f(x_1, x_2) = (\alpha \cosh \mathbf{x}, \alpha i \sinh \mathbf{x}, \alpha \mathbf{x}), \tag{7.5.3}$$

we may write

$$X(\mathbf{x}) = \operatorname{Re} f(\mathbf{x}). \tag{7.5.4}$$

Definition 7.5.1. *An associated family of a minimal surface* is a one-parameter family of surfaces which share the same displacement of the vectors between two corresponding points in the range. It means, all of the members of an associate family have the same domain, Gauss map and metric.

For the catenoid (7.5.2), the adjoint surface of (7.5.4) can be

$$X^*(\mathbf{x}) := \operatorname{Im} f(\mathbf{x}), \tag{7.5.5}$$

and the adjoint of the catanoid is the matrix

$$\begin{pmatrix} \alpha \sinh x_1 \sin x_2 \\ \alpha \sinh x_1 \cos x_2 \\ \alpha x_2 \end{pmatrix}. \tag{7.5.6}$$

In other word, one may write

$$X^* = \alpha Y(x_2) + \sinh x_1 Z(x_2), \tag{7.5.7}$$

where

$$Y(x_2) = (0, 0, x_2), \qquad Z(x_2) = (\sin x_2, \cos x_2, 0). \tag{7.5.8}$$

Hence, for every $x_2 \in R$ the curve $X^*(., x_2)$ is a straight line which meets the x_3-axis perpendicularly. Generally, we see that X^* is generated ba y screw motion of some straight lines which meet the x_3-axis perpendicularly. Therefore, X^* is called *helicoid* or *screw surface*. So, the helicoid X^*, which is the adjoint of the catenoid X, is a ruled surface with the x_3-axis as its directrix.

Remark 11. *Assume that $Z(w, \theta)$ is the associate surfeces. The coordinates of*

$$Z(w, \theta) = \operatorname{Re}\{e^{-i\theta} f(w)\}, \quad \theta \in \mathbb{R}, \tag{7.5.9}$$

to the catenoid $X(\boldsymbol{x})$, also to the helicoid $X^*(\boldsymbol{x})$ are

$$\begin{pmatrix} \alpha \cosh x_1 \cos x_2 \cos\theta + \alpha \sinh x_1 \sin x_2 \sin\theta \\ -\alpha \cosh x_1 \sin x_2 \cos\theta + \alpha \sinh x_1 \cos x_2 \sin\theta \\ \alpha x_1 \cos\theta + \alpha x_1 \sin\theta \end{pmatrix}. \tag{7.5.10}$$

Generally, a catenoid can be written as

$$X^{cat}(\boldsymbol{x}) = (\cosh x_1 \cos x_2, -\cosh x_1 \sin x_2, x_1), \tag{7.5.11}$$

and the helicoid as

$$X^{hel}(\boldsymbol{x}) = (\sinh x_1 \sin x_2, \sinh x_1 \cos x_2, x_2)\,. \tag{7.5.12}$$

So we can write the curve like following

$$X(\boldsymbol{x}) = \alpha X^{cat}(\boldsymbol{x}) + \beta X^{hel}(\boldsymbol{x})\,. \tag{7.5.13}$$

By choosing

$$\alpha = c\cos\theta\,, \quad \beta = c\sin\theta \qquad \textit{with} \quad c = \sqrt{\alpha^2+\beta^2}\,, \tag{7.5.14}$$

and hence,

$$X(\boldsymbol{x}) = c[\cos\theta X^{cat}(\boldsymbol{x}) + \sin\theta X^{hel}(\boldsymbol{x})]\,. \tag{7.5.15}$$

In the special case for $\alpha = cos\theta$ and $\beta = \sin\theta$, gives

$$X(\mathbf{x}) = \cos\theta X^{\text{cat}}(\mathbf{x}) + \sin\theta X^{\text{hel}}(\mathbf{x})\,. \tag{7.5.16}$$

Their partial derivatives fulfill

$$\partial_1 X^\theta = \cos\theta\cdot\partial_1 X^{\text{cat}} - \sin\theta\cdot\partial_2 X^{\text{cat}} \quad \text{and} \quad \partial_2 X^\theta = \sin\theta\cdot\partial_1 X^{\text{cat}} + \cos\theta\cdot\partial_2 X^{\text{cat}}\,, \tag{7.5.17}$$

so that, the surface normals remain unchanged

$$n_{X^\theta}(x_1,x_2) = n_{X^{\text{cat}}}(x_1,x_2) = n_{X^{\text{hel}}}(x_1,x_2) \qquad \text{for all } (x_1,x_2)\in\omega. \tag{7.5.18}$$

Further properties of the members of this associate family X^θ can be found in [71] and [42, Chapter 3], as well as the references cited therein.

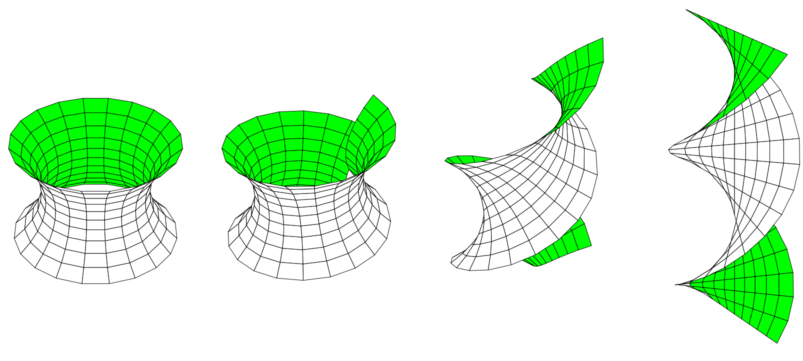


Figure 7.5: Four consecutive steps of an isometric deformation of a catenoid (left) into a helicoid (right). Such a transformation exists, since both are members of the same associate family X^θ. Note that every member of the deformation family has vanishing mean curvature, i.e., is a minimal surface.

(a) $\theta = 0$ (b) $\theta = \frac{\pi}{25}$ (c) $\theta = \frac{\pi}{4}$ (d) $\theta = \frac{\pi}{2}$

To see (7.5.17) and (7.5.18) we take the partial derivatives of (7.5.16):

$$\partial_j X^\theta = \cos\theta\cdot\partial_j X^{\text{cat}} + \sin\theta\cdot\partial_j X^{\text{hel}} \qquad \text{for } j = 1,2. \tag{7.5.19}$$

Since the partial derivatives of the catenoid and the helicoid satisfy the Cauchy-Riemann equations

$$\partial_1 X^{\text{cat}} = \partial_2 X^{\text{hel}} \quad \text{and} \quad \partial_2 X^{\text{cat}} = -\partial_1 X^{\text{hel}}\,, \tag{7.5.20}$$

we obtain (7.5.17), which in matrix notation reads

$$\mathrm{D}X^\theta = \mathrm{D}X^{\text{cat}}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \tag{7.5.21}$$

Moreover,

$$\begin{aligned}\partial_1 X^\theta \times \partial_2 X^\theta &\overset{(7.5.17)}{=} \cos^2\theta \cdot \partial_1 X^{\rm cat} \times \partial_2 X^{\rm cat} - \sin^2\theta \cdot \partial_2 X^{\rm cat} \times \partial_1 X^{\rm cat} \\ &= \partial_1 X^{\rm cat} \times \partial_2 X^{\rm cat}\,, \end{aligned} \tag{7.5.22}$$

which gives (7.5.18). Furthermore,

$$\partial_1 X^{\rm cat} = \begin{pmatrix} \cos x_2 \sinh x_1 \\ \sin x_2 \sinh x_1 \\ 1 \end{pmatrix} \quad \text{and} \quad \partial_2 X^{\rm cat} = \begin{pmatrix} -\sin x_2 \cosh x_1 \\ \cos x_2 \cosh x_1 \\ 0 \end{pmatrix}, \tag{7.5.23}$$

so that

$$\|\partial_1 X^{\rm cat}\|^2 = \sinh^2 x_1 + 1 = \cosh^2 x_1 = \|\partial_2 X^{\rm cat}\|^2 \quad \text{and} \quad \langle \partial_1 X^{\rm cat}, \partial_2 X^{\rm cat}\rangle = 0. \tag{7.5.24}$$

In regard with (7.5.17) we obtain

$$\|\partial_1 X^\theta\|^2 \overset{(7.5.24)_2}{=} \cos^2\theta \|\partial_1 X^{\rm cat}\|^2 + \sin^2\theta \|\partial_2 X^{\rm cat}\|^2 \overset{(7.5.24)_1}{=} \cosh^2 x_1\,, \tag{7.5.25a}$$

$$\|\partial_2 X^\theta\|^2 \overset{(7.5.24)_2}{=} \sin^2\theta \|\partial_1 X^{\rm cat}\|^2 + \cos^2\theta \|\partial_2 X^{\rm cat}\|^2 \overset{(7.5.24)_1}{=} \cosh^2 x_1 = \|\partial_1 X^\theta\|^2\,, \tag{7.5.25b}$$

$$\langle \partial_1 X^\theta, \partial_2 X^\theta \rangle \overset{(7.5.24)_2}{=} \cos\theta \sin\theta \|\partial_1 X^{\rm cat}\|^2 - \sin\theta\cos\theta \|\partial_2 X^{\rm cat}\|^2 \overset{(7.5.24)_1}{=} 0. \tag{7.5.25c}$$

In other words, the first fundamental form of all members of the associate family remains unchanged and is given by

$$\mathrm{I}_{X^\theta}(x_1,x_2) = [\mathrm{D}X^\theta]^T \,\mathrm{D}X^\theta = \begin{pmatrix} \|\partial_1 X^\theta\|^2 & \langle \partial_1 X^\theta, \partial_2 X^\theta\rangle \\ \langle \partial_1 X^\theta, \partial_2 X^\theta\rangle & \|\partial_2 X^\theta\|^2 \end{pmatrix} = \cosh^2 x_1 \cdot \mathbb{1}_2. \tag{7.5.26}$$

Thus,

$$[\mathrm{D}X^{\rm cat}]^T \,\mathrm{D}X^\theta \overset{(7.5.21)}{=} [\mathrm{D}X^{\rm cat}]^T \,\mathrm{D}X^{\rm cat} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \overset{(7.5.26)}{=} \cosh^2 x_1 \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{7.5.27}$$

and $[\mathrm{D}X^{\rm cat}]^T \,\mathrm{D}X^\theta \notin \mathrm{Sym}^+(2)$ is not a pure in-plane stretch.
Moreover, as in (7.7.12), there exists an in-plane drill rotation $Q^\theta(x) \in \mathrm{SO}(3)$ which fulfills

$$\mathrm{D}X^\theta(x) = Q^\theta(x)\,\mathrm{D}X^{\rm cat}(x) \quad \text{and} \quad Q^\theta(x)\, n_0(x) = n_0(x), \quad \text{where } n_0(x) \coloneqq n_{X^\theta}(x) = n_{X^{\rm cat}}(x). \tag{7.5.28}$$

Next we show, that the (constant) rotation angle, cf. Lemma 7.7.1, extracted from $Q^\theta(x)$ is already given by $-\theta$ (or differs from it by an integer multiple of 2π), so that we have the representation

$$\begin{aligned} Q^\theta(x) &= (1-\cos(-\theta))\, n_0(x) \otimes n_0(x) + \cos(-\theta)\,\mathbb{1} + \sin(-\theta)\,\mathrm{Anti}(n_0(x)) \\ &= (1-\cos\theta)\, n_0(x)\otimes n_0(x) + \cos\theta\,\mathbb{1} - \sin\theta\,\mathrm{Anti}(n_0(x)). \end{aligned} \tag{7.5.29}$$

For that purpose, note that

$$\partial_1 X^{\rm cat} \times \partial_2 X^{\rm cat} = \begin{pmatrix} -\cos x_2 \cosh x_1 \\ -\sin x_2 \cosh x_1 \\ \sinh x_1 \cosh x_1 \end{pmatrix} \quad \Rightarrow \quad \|\partial_1 X^{\rm cat} \times \partial_2 X^{\rm cat}\| = \cosh^2 x_1, \tag{7.5.30}$$

so that

$$n_0 = \frac{1}{\cosh x_1} \begin{pmatrix} -\cos x_2 \\ -\sin x_2 \\ \sinh x_1 \end{pmatrix}, \quad \text{and} \quad n_0 \times \partial_1 X^{\rm cat} = \partial_2 X^{\rm cat}, \quad n_0 \times \partial_2 X^{\rm cat} = -\partial_1 X^{\rm cat}. \tag{7.5.31}$$

Hence, with $n_0 \otimes n_0\, \partial_j X^{\rm cat} = n_0\, \langle n_0, \partial_j X^{\rm cat}\rangle \equiv 0$, we obtain

$$Q^\theta \partial_1 X^{\rm cat} \overset{(7.5.29)}{=} \cos\theta\, \partial_1 X^{\rm cat} - \sin\theta\, \mathrm{Anti}(n_0) \partial_1 X^{\rm cat} \overset{(7.5.31)_2}{=} \cos\theta\, \partial_1 X^{\rm cat} - \sin\theta\, \partial_2 X^{\rm cat} \overset{(7.5.17)_1}{=} \partial_1 X^\theta\,, \tag{7.5.32}$$

as well as

$$Q^\theta \partial_2 X^{\rm cat} \overset{(7.5.29)}{=} \cos\theta\, \partial_2 X^{\rm cat} - \sin\theta\, \mathrm{Anti}(n_0)\partial_2 X^{\rm cat} \overset{(7.5.31)_3}{=} \cos\theta\, \partial_2 X^{\rm cat} + \sin\theta\, \partial_1 X^{\rm cat} \overset{(7.5.17)_2}{=} \partial_2 X^\theta\,, \tag{7.5.33}$$

and we have shown that

$$Q^\theta\, \mathrm{D} X^{\rm cat} = \mathrm{D} X^\theta\,, \tag{7.5.34}$$

where Q^θ has the expression (7.5.29) and its columns read with the representation of the normal $(7.5.31)_1$

$$Q^\theta\, e_1 = \begin{pmatrix} -\frac{\left(\cos^2 x_2 - \cosh^2 x_1\right)\cos\theta - \cos^2 x_2}{\cosh^2 x_1} \\ -\frac{\cosh x_1 \sinh x_1 \sin\theta + \cos x_2 \sin x_2 \cos\theta - \cos x_2 \sin x_2}{\cosh^2 x_1} \\ -\frac{\cosh x_1 \sin x_2 \sin\theta - \sinh x_1 \cos x_2 \cos\theta + \sinh x_1 \cos x_2}{\cosh^2 x_1} \end{pmatrix}, \tag{7.5.35a}$$

$$Q^\theta\, e_2 = \begin{pmatrix} \frac{\cosh x_1 \sinh x_1 \sin\theta - \cos x_2 \sin x_2 \cos\theta + \cos x_2 \sin x_2}{\cosh^2 x_1} \\ -\frac{\left(\sin^2 x_2 - \cosh^2 x_1\right)\cos\theta - \sin^2 x_2}{\cosh^2 x_1} \\ \frac{\cosh x_1 \cos x_2 \sin\theta + \sinh x_1 \sin x_2 \cos\theta - \sinh x_1 \sin x_2}{\cosh^2 x_1} \end{pmatrix}, \tag{7.5.35b}$$

$$Q^\theta\, e_3 = \begin{pmatrix} \frac{\cosh x_1 \sin x_2 \sin\theta + \sinh x_1 \cos x_2 \cos\theta - \sinh x_1 \cos x_2}{\cosh^2 x_1} \\ -\frac{\cosh x_1 \cos x_2 \sin\theta - \sinh x_1 \sin x_2 \cos\theta + \sinh x_1 \sin x_2}{\cosh^2 x_1} \\ \frac{\cos\theta + \cosh^2 x_1 - 1}{\cosh^2 x_1} \end{pmatrix}. \tag{7.5.35c}$$

Recall, that if a surface X is parametrized conformally, i.e., it holds $\|\partial_1 X\| = \|\partial_2 X\|$ and $\langle \partial_1 X, \partial_2 X\rangle = 0$, then it is a minimal surface (i.e. has vanishing mean curvature everywhere) if and only if $\Delta X \equiv 0$ holds, cf. [42, p.72]. Thus, in regard with (7.5.26), to check that all members of the associate family X^θ are, indeed, minimal surfaces, we compute

$$\Delta X^\theta \overset{(7.5.17)}{=} \cos\theta \cdot \Delta X^{\rm cat} \overset{(7.5.23)}{\equiv} 0. \tag{7.5.36}$$

Furthermore, if a minimal surface is parametrized conformally, then the same holds for its corresponding Gauss map, cf. [42, p.74]. Indeed, all members of the associate family X^θ are minimal surfaces and for their Gauss maps (which all coincide) $n_0 : \omega \to \mathbb{S}^2$ it holds

$$\mathrm{I}_{n_0} = [\mathrm{D} n_0]^T\, \mathrm{D} n_0 \overset{(7.5.31)_1}{=} \frac{1}{\cosh^2 x_1} \cdot \mathbb{1}_2\,, \tag{7.5.37}$$

which shows that n_0 is also parametrized conformally.

Let us mention, that the constancy of the rotation angle can also be achieved without applying Lemma 7.7.1. For that purpose, let us here call the in-plane drill rotation $\widehat{Q}(x) \in \mathrm{SO}(3)$ which fulfills

$$\mathrm{D} X^\theta(x) = \widehat{Q}(x)\, \mathrm{D} X^{\rm cat}(x) \quad \text{and} \quad \widehat{Q}(x)\, n_0(x) = n_0(x), \quad \text{where } n_0(x) := n_{X^\theta}(x) = n_{X^{\rm cat}}(x). \tag{7.5.38}$$

Thus, as in (7.7.12), it follows

$$(\mathrm{D} X^\theta \,|\, n_0) = \widehat{Q}(\mathrm{D} X^{\rm cat} \,|\, n_0) \quad \Rightarrow \quad \widehat{Q} = (\mathrm{D} X^\theta \,|\, n_0)(\mathrm{D} X^{\rm cat} \,|\, n_0)^{-1}\,, \tag{7.5.39}$$

and a direct computation gives the entries of $\widehat{Q}(x)$, which, indeed, coincide with (7.5.35). From the uniqueness of the Euler-Rodrigues representation it then follows that the corresponding rotation angle $\widehat{\alpha}(x)$ is constant and is given by $-\theta$ (or differs from it by an integer multiple of 2π). Indeed, with (7.3.7) we have

$$\sin(\widehat{\alpha}(x)) \overset{(7.3.7)}{=} -\frac{\mathrm{tr}\left(\mathrm{Anti}(n_0(x))\widehat{Q}(x)\right)}{2} \underset{(7.5.35)}{\overset{\widehat{Q}=Q^\theta}{=}} -\sin\theta \quad \text{and} \quad \cos(\widehat{\alpha}(x)) \overset{(7.3.7)}{=} -\frac{\mathrm{tr}\,\widehat{Q}(x) - 1}{2} \underset{(7.5.35)}{\overset{\widehat{Q}=Q^\theta}{=}} \cos\theta.$$

7.6. The small rotation case: $A \in \mathfrak{so}(3)$

For $m, y_0 \colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ we discuss

$$\mathrm{D} m(x) = Q(x)\ \mathrm{D} y_0(x)\,, \qquad Q(x) n_0(x) = n_0(x)\,, \qquad n_0 \ \text{unit normal vector field on } y_0(\omega)\,, \tag{7.6.1}$$

where $Q \in C^1(\omega, \mathrm{SO}(3))$. Consider a linear approximation of this situation for small rotation angle α, $|\alpha| \ll 1$. Then we can write

$$\mathrm{D}m(x) = \mathrm{D}y_0(x) + \mathrm{D}v(x), \qquad Q(x) = \mathbb{1} + A(x) + \text{h.o.t.}, \quad A \in C^1(\omega, \mathfrak{so}(3)). \tag{7.6.2}$$

Note that we do not assume that Dv is small. We only assume that the rotations are close to $\mathbb{1}$. Then,

$$\mathrm{D}m(x) = \mathrm{D}y_0(x) + \mathrm{D}v(x) = (\mathbb{1} + A(x) + \dots)\,\mathrm{D}y_0(x) \qquad \Rightarrow \qquad \mathrm{D}v(x) = A(x)\,\mathrm{D}y_0(x) + \dots, \tag{7.6.3}$$

hence we may consider the new problem

$$\mathrm{D}v(x) = A(x)\,\mathrm{D}y_0(x), \qquad A \in C^1(\omega, \mathfrak{so}(3)), \qquad A(x)n_0(x) = 0. \tag{7.6.4}$$

Therefore, we are led to study the problem

Lemma 7.6.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume $y_0 \in C^2(\omega, \mathbb{R}^3)$ is a regular surface and let $v \in C^2(\omega, \mathbb{R}^3)$. Moreover, assume $\alpha \in C^1(\omega, \mathbb{R})$ and consider the system*

$$\mathrm{D}v(x) = \alpha(x)\ \mathrm{Anti}(n_0(x))\ \mathrm{D}y_0(x), \qquad x \in \omega, \tag{7.6.5}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$. Then $\alpha \equiv \mathrm{const}$.

Proof. We write

$$\begin{aligned}(\partial_1 v|\partial_2 v) &= \alpha\,\mathrm{Anti}(n_0)(\partial_1 y_0|\partial_2 y_0) = \alpha\Big(\mathrm{Anti}(n_0)\partial_1 y_0\,\Big|\,\mathrm{Anti}(n_0)\partial_2 y_0\Big), \\ \iff \quad & \partial_1 v = \alpha\,[\,n_0 \times \partial_1 y_0\,], \quad \partial_2 v = \alpha\,[\,n_0 \times \partial_2 y_0\,],\end{aligned} \tag{7.6.6}$$

where we have used that $\mathrm{Anti}(n_0)\eta = n_0 \times \eta$. We proceed by taking the mixed derivatives

$$\begin{aligned}\partial_2\partial_1 v &= \partial_2\alpha\,[\,n_0 \times \partial_1 y_0\,] + \alpha\,[\,\partial_2 n_0 \times \partial_1 y_0 + n_0 \times \partial_2\partial_1 y_0\,], \\ \partial_1\partial_2 v &= \partial_1\alpha\,[\,n_0 \times \partial_2 y_0\,] + \alpha\,[\,\partial_1 n_0 \times \partial_2 y_0 + n_0 \times \partial_1\partial_2 y_0\,].\end{aligned} \tag{7.6.7}$$

Hence, by equality of the mixed derivatives in (7.6.7) for $y_0, v \in C^2(\omega, \mathbb{R}^3)$ we must have

$$\partial_2\alpha\underbrace{[\,n_0 \times \partial_1 y_0\,]}_{=:\overrightarrow{Y_0}} + \alpha\underbrace{[\,\partial_2 n_0 \times \partial_1 y_0\,]}_{=:\overrightarrow{B}} = \partial_1\alpha\underbrace{[\,n_0 \times \partial_2 y_0\,]}_{=:-\overrightarrow{X_0}} + \alpha\underbrace{[\,\partial_1 n_0 \times \partial_2 y_0\,]}_{=:\overrightarrow{A}} \in \mathbb{R}^3. \tag{7.6.8}$$

Especially we have that $\overrightarrow{A}$ and $\overrightarrow{B}$ are normal vectors whereas $\overrightarrow{X_0}$ and $\overrightarrow{Y_0}$ are linear independent tangent vectors, since $\langle\overrightarrow{X_0}\rangle n_0 = 0$, $\langle\overrightarrow{Y_0}\rangle n_0 = 0$ and

$$\begin{aligned}\overrightarrow{X_0} \times \overrightarrow{Y_0} &= -(n_0 \times \partial_2 y_0) \times (n_0 \times \partial_1 y_0) = -\langle n_0\rangle \partial_2 y_0 \times \partial_1 y_0 n_0 = \langle n_0\rangle \|\partial_1 y_0 \times \partial_2 y_0\| \cdot n_0 n_0 \\ &= \|\partial_1 y_0 \times \partial_2 y_0\| \cdot n_0 = \partial_1 y_0 \times \partial_2 y_0,\end{aligned} \tag{7.6.9}$$

where we have used that $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$. Thus, the vector fields $\overrightarrow{X_0}$, $\overrightarrow{Y_0}$ and n_0 form a 3−frame on the surface $y_0(\omega)$. However, (7.6.8) reads,

$$\partial_1\alpha \cdot \overrightarrow{X_0} + \partial_2\alpha \cdot \overrightarrow{Y_0} = \alpha \cdot (\overrightarrow{A} - \overrightarrow{B}) = \delta \cdot n_0, \tag{7.6.10}$$

with a scalar field δ, so that by the linear independence of the vector fields $\overrightarrow{X_0}$, $\overrightarrow{Y_0}$ and n_0 we must always have $\partial_1\alpha = \partial_2\alpha = \delta = 0$, which gives $\alpha \equiv \mathrm{const}$. ■

Thus, adding sufficient boundary conditions we arrive at

Proposition 7.6.2. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume $y_0 \in C^2(\omega, \mathbb{R}^3)$ is a regular surface and let $v \in C^2(\omega, \mathbb{R}^3)$. Moreover assume $\alpha \in C^1(\omega, \mathbb{R}) \cap C^0(\overline{\omega}, \mathbb{R})$ and consider the system*

$$\mathrm{D}v(x) = \alpha(x)\ \mathrm{Anti}(n_0(x))\ \mathrm{D}y_0(x), \qquad x \in \omega, \qquad \alpha|_{\gamma_d} = 0, \tag{7.6.11}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$ and γ_d is a relatively open, non-empty, subset of the boundary $\partial\omega$. Then $\alpha \equiv 0$.

Proof. By Lemma 7.6.1 it holds $\alpha \equiv \text{const}$, so that due to the vanishing boundary condition $\alpha|_{\gamma_d} = 0$ and the continuity of α we obtain $\alpha \equiv 0$. ■

Corollary 7.6.3. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\overline{\omega}, \mathbb{R}^3)$ are regular surfaces and $\alpha \in C^1(\omega, \mathbb{R}) \cap C^0(\overline{\omega}, \mathbb{R})$ is given with*

$$\mathrm{D}m(x) = (\mathbb{1} + \alpha(x)\,\mathrm{Anti}(n_0(x)))\,\mathrm{D}y_0(x)\,, \qquad x \in \overline{\omega}\,, \qquad m|_{\gamma_d} = y_0|_{\gamma_d}\,, \tag{7.6.12}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$ and γ_d is a relatively open, non-empty, subset of the boundary $\partial\omega$. Then $m \equiv y_0$.

Proof. We invoke Lemma 7.4.3 to see that $m|_{\gamma_d} = y_0|_{\gamma_d}$ implies $\alpha|_{\gamma_d} \equiv 0$. Thus, for $v = m - y_0$ we can apply Proposition 7.6.2 to conclude that $\mathrm{D}v \equiv 0$. ■

Proposition 7.6.4. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume $m, y_0 \in C^2(\omega, \mathbb{R}^3)$ are regular surfaces and $\alpha \in C^1(\omega, \mathbb{R})$ with*

$$\mathrm{D}m(x) = (\mathbb{1} + \alpha(x)\,\mathrm{Anti}(n_0(x)))\,\mathrm{D}y_0(x)\,, \qquad x \in \omega\,, \tag{7.6.13}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$. Then

$$\forall\, x \in \omega: \quad \alpha(x) = 0 \quad \text{or} \quad H(x) = 0, \tag{7.6.14}$$

where H denotes the mean curvature on the surface y_0.

Proof. Recall that it holds for the vector field

$$\overrightarrow{A} - \overrightarrow{B} = \partial_1 n_0 \times \partial_2 y_0 - \partial_2 n_0 \times \partial_1 y_0 = -2\,\mathrm{H}\,\|\partial_1 y_0 \times \partial_2 y_0\|\, n_0 = -2\mathrm{H}\,\partial_1 y_0 \times \partial_2 y_0, \tag{7.6.15}$$

cf. [42, Section 2.5, Theorem 2]. Thus, for $v = m - y_0$ the validity of (7.6.10) implies that we have pointwise either a vanishing angle α or a vanishing mean curvature H since

$$0 \underset{\alpha\equiv\text{const}}{\overset{(7.6.10)}{\equiv}} \alpha \cdot (\overrightarrow{A} - \overrightarrow{B}) \overset{(7.6.15)}{=} -2\,\alpha \cdot \mathrm{H} \cdot \partial_1 y_0 \times \partial_2 y_0\,. \qquad ■$$

Remark 7.6.5 (Symmetry of the second fundamental form)**.** It is interesting to note that conclusion (7.6.14) can also be obtained from the symmetry property of the second fundamental form on the surface $m(\omega)$. Indeed, the normal vector field on $m(\omega)$ coincides with n_0 since

$$\begin{aligned}
\partial_1 m \times \partial_2 m &\overset{(7.6.13)}{=} (\partial_1 y_0 + \alpha\ \mathrm{Anti}(n_0)\,\partial_1 y_0) \times (\partial_2 y_0 + \alpha\ \mathrm{Anti}(n_0)\,\partial_2 y_0) \\
&= (\partial_1 y_0 + \alpha\, n_0 \times \partial_1 y_0) \times (\partial_2 y_0 + \alpha\, n_0 \times \partial_2 y_0) \\
&= \partial_1 y_0 \times \partial_2 y_0 + \alpha^2 (n_0 \times \partial_1 y_0) \times (n_0 \times \partial_2 y_0) \overset{(7.6.9)}{=} (1 + \alpha^2)\,\partial_1 y_0 \times \partial_2 y_0.
\end{aligned} \tag{7.6.16}$$

Thus, for the second fundamental form on $m(\omega)$ we obtain

$$\begin{aligned}
\mathrm{Sym}(2) \ni \mathrm{II}_m &= -[\mathrm{D}m]^T\,\mathrm{D}n = -[\mathrm{D}m]^T\,\mathrm{D}n_0 \overset{(7.6.13)}{=} -[(\mathbb{1} + \alpha\,\mathrm{Anti}(n_0))\,\mathrm{D}y_0]^T\,\mathrm{D}n_0 \\
&= -[\mathrm{D}y_0]^T\,\mathrm{D}n_0 - \alpha\,[\mathrm{D}y_0]^T\,\mathrm{Anti}(n_0)^T\,\mathrm{D}n_0 = \mathrm{II}_{y_0} + \alpha\,[\mathrm{D}y_0]^T\,\mathrm{Anti}(n_0)\,\mathrm{D}n_0.
\end{aligned} \tag{7.6.17}$$

Since $\mathrm{II}_v \in \mathrm{Sym}(2)$, we are left with the single condition

$$\begin{aligned}
\alpha\,[\mathrm{D}y_0]^T\,\mathrm{Anti}(n_0)\,\mathrm{D}n_0 \in \mathrm{Sym}(2) \quad &\iff \quad \alpha \begin{pmatrix} (\partial_1 y_0)^T \\ (\partial_2 y_0)^T \end{pmatrix} (n_0 \times \partial_1 n_0 \big| n_0 \times \partial_2 n_0) \in \mathrm{Sym}(2) \\
&\iff \quad \alpha \begin{pmatrix} * & \langle \partial_1 y_0, n_0 \times \partial_2 n_0 \rangle \\ \langle \partial_2 y_0 \rangle n_0 \times \partial_1 n_0 & * \end{pmatrix} \in \mathrm{Sym}(2) \\
&\iff \quad \{\ \alpha = 0 \quad \text{or} \quad \langle \partial_1 y_0, n_0 \times \partial_2 n_0 \rangle = \langle \partial_2 y_0, n_0 \times \partial_1 n_0 \rangle \} \\
&\iff \quad \{\ \alpha = 0 \quad \text{or} \quad \langle n_0, \partial_2 n_0 \times \partial_1 y_0 \rangle = \langle n_0, \partial_1 n_0 \times \partial_2 y_0 \rangle \} \\
&\iff \quad \{\ \alpha = 0 \quad \text{or} \quad 0 = \langle n_0, \partial_1 n_0 \times \partial_2 y_0 - \partial_2 n_0 \times \partial_1 y_0 \rangle \} \\
&\overset{(7.6.15)}{\iff} \quad \{\ \alpha = 0 \quad \text{or} \quad 0 = \langle n_0, -2\mathrm{H}\,\partial_1 y_0 \times \partial_2 y_0 \rangle \} \\
&\iff \quad \{\ \alpha = 0 \quad \text{or} \quad \mathrm{H}\|\partial_1 y_0 \times \partial_2 y_0\| = 0 \} \\
&\iff \quad \{\ \alpha = 0 \quad \text{or} \quad \mathrm{H} = 0 \}.
\end{aligned} \tag{7.6.18}$$

□

Corollary 7.6.6. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume $y_0 \in C^2(\omega, \mathbb{R}^3)$ is a regular surface and let $v \in C^2(\omega, \mathbb{R}^3)$. Moreover assume $\alpha \in C^1(\omega, \mathbb{R})$ and consider the system*

$$\mathrm{D}v(x) = \alpha(x)\ \mathrm{Anti}(n_0(x))\ \mathrm{D}y_0(x)\,, \qquad x \in \omega\,, \tag{7.6.19}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$. If the mean curvature H of y_0 does not vanish at one point $y_0(x_0)$, then $\alpha \equiv 0$.

Proof. It follows from the previous Proposition 7.6.4, that if the mean curvature H does not vanish at some point $y_0(x_0)$, we must have $\alpha(x_0) = 0$ and the conclusion follows, since $\alpha \equiv$ const by Lemma 7.6.1. ■

7.7. The large rotation case: $Q \in \mathrm{SO}(3)$

Lemma 7.7.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\omega, \mathbb{R}^3)$ are regular surfaces, $Q \in C^1(\omega, \mathrm{SO}(3))$ and*

$$\mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x)\,, \qquad Q(x)n_0(x) = n_0(x)\,, \qquad x \in \omega\,, \tag{7.7.1}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$. Then $\alpha \equiv$ const, where $\alpha : \omega \to \mathbb{R}$ denotes the rotations angle in the Euler-Rodrigues representation of Q.

Proof. By the Euler-Rodrigues representation there exists a function $\alpha : \omega \to \mathbb{R}$ which fulfills

$$Q(x) = (1 - \cos\alpha(x))\, n_0(x) \otimes n_0(x) + \cos\alpha(x)\, \mathbb{1} + \sin\alpha(x)\ \mathrm{Anti}(n_0(x))\,, \tag{7.7.2}$$

so that due to the expressions from (7.3.7) for all $x \in \omega$ there exists a neighborhood U of x such that $\widetilde{\alpha} \in C^1(U \cap \omega, \mathbb{R})$.
In view of (7.7.2), the problem (7.7.1) can be recast (at least locally) as follows

$$\mathrm{D}m = \Big((1 - \cos\widetilde{\alpha})\, n_0 \otimes n_0 + \cos\widetilde{\alpha}\, \mathbb{1} + \sin\widetilde{\alpha}\ \mathrm{Anti}(n_0)\Big)\, \mathrm{D}y_0. \tag{7.7.3}$$

Since $(n_0 \otimes n_0)\, \mathrm{D}y_0 = n_0 \otimes ([\mathrm{D}y_0]^T n_0) = 0$, the latter formula simplifies to

$$\mathrm{D}m = \cos\widetilde{\alpha}\ \mathrm{D}y_0 + \sin\widetilde{\alpha}\ \mathrm{Anti}(n_0)\ \mathrm{D}y_0. \tag{7.7.4}$$

Obviously we have

$$\partial_1 m = \cos\widetilde{\alpha}\, \partial_1 y_0 + \sin\widetilde{\alpha}\, (n_0 \times \partial_1 y_0) \quad \text{and} \quad \partial_2 m = \cos\widetilde{\alpha}\, \partial_2 y_0 + \sin\widetilde{\alpha}\, (n_0 \times \partial_2 y_0)\,,$$

where for $i = 1, 2$, we used that $\mathrm{Anti}(n_0)\partial_i\, y_0 = n_0 \times \partial_i y_0$. By taking the mixed derivatives, we arrive at

$$\begin{aligned} \partial_2\partial_1 m &= \partial_2(\cos\widetilde{\alpha}\, \partial_1 y_0) + \partial_2(\sin\widetilde{\alpha}\, (n_0 \times \partial_1 y_0)) \\ &= -\sin\widetilde{\alpha}\, \partial_2\widetilde{\alpha}\, \partial_1 y_0 + \cos\widetilde{\alpha}\, \partial_2\partial_1 y_0 + \cos\widetilde{\alpha}\, \partial_2\widetilde{\alpha}\, (n_0 \times \partial_1 y_0) + \sin\widetilde{\alpha}\, \partial_2 n_0 \times \partial_1 y_0 + \sin\widetilde{\alpha}\, n_0 \times \partial_2\partial_1 y_0, \end{aligned} \tag{7.7.5}$$

as well as

$$\begin{aligned} \partial_1\partial_2 m &= \partial_1(\cos\widetilde{\alpha}\, \partial_2 y_0) + \partial_1(\sin\widetilde{\alpha}\, (n_0 \times \partial_2 y_0)) \\ &= -\sin\widetilde{\alpha}\, \partial_1\widetilde{\alpha}\, \partial_2 y_0 + \cos\widetilde{\alpha}\ \partial_1\partial_2 y_0 + \cos\widetilde{\alpha}\, \partial_1\widetilde{\alpha}\, (n_0 \times \partial_2 y_0) + \sin\widetilde{\alpha}\ \partial_1 n_0 \times \partial_2 y_0 + \sin\widetilde{\alpha}\, n_0 \times \partial_1\partial_2 y_0. \end{aligned} \tag{7.7.6}$$

By using the equality of mixed derivatives for $m, y_0 \in C^2(\omega, \mathbb{R}^3)$ we must have

$$\begin{aligned} &\partial_2\widetilde{\alpha}\underbrace{\Big(-\sin\widetilde{\alpha}\ \partial_1 y_0 + \cos\widetilde{\alpha}\, (n_0 \times \partial_1 y_0)\Big)}_{=:\overrightarrow{Y}_{\widetilde{\alpha}}} + \sin\widetilde{\alpha}\ \underbrace{\partial_2 n_0 \times \partial_1 y_0}_{=\overrightarrow{B}} \\ &\qquad = \partial_1\widetilde{\alpha}\underbrace{\Big(-\sin\widetilde{\alpha}\, \partial_2 y_0 + \cos\widetilde{\alpha}(n_0 \times \partial_2 y_0)\Big)}_{=:-\overrightarrow{X}_{\widetilde{\alpha}}} + \sin\widetilde{\alpha}\ \underbrace{\partial_1 n_0 \times \partial_2 y_0}_{\overrightarrow{A}}. \end{aligned} \tag{7.7.7}$$

As in the linear case we obtain that $\overrightarrow{A}$ and $\overrightarrow{B}$ are normal vectors whereas $\overrightarrow{X_{\widetilde{\alpha}}}$ and $\overrightarrow{Y_{\widetilde{\alpha}}}$ are linear independent tangent vectors, since $\langle \overrightarrow{X_{\widetilde{\alpha}}}, n_0 \rangle = 0, \quad \langle \overrightarrow{Y_{\widetilde{\alpha}}}, n_0 \rangle = 0$ and

$$\begin{aligned}\overrightarrow{X_{\widetilde{\alpha}}} \times \overrightarrow{Y_{\widetilde{\alpha}}} &= (\sin\widetilde{\alpha}\, \partial_2 y_0 + \cos\widetilde{\alpha}\, \overrightarrow{X_0}) \times (-\sin\widetilde{\alpha}\, \partial_1 y_0 + \cos\widetilde{\alpha}\, \overrightarrow{Y_0}) \\ &= \sin^2\widetilde{\alpha}\, \partial_1 y_0 \times \partial_2 y_0 + \cos^2\widetilde{\alpha}\, \overrightarrow{X_0} \times \overrightarrow{Y_0} + \sin\widetilde{\alpha}\cos\widetilde{\alpha}(\partial_2 y_0 \times \overrightarrow{Y_0} - \overrightarrow{X_0} \times \partial_1 y_0) \\ &\underset{*}{\overset{(7.6.9)}{=}} \sin^2\widetilde{\alpha}\, \partial_1 y_0 \times \partial_2 y_0 + \cos^2\widetilde{\alpha}\, \partial_1 y_0 \times \partial_2 y_0 = \partial_1 y_0 \times \partial_2 y_0 ,\end{aligned} \tag{7.7.8}$$

where in $*$ we also used that $\partial_2 y_0 \times \overrightarrow{Y_0} - \overrightarrow{X_0} \times \partial_1 y_0 = \partial_2 y_0 \times (n_0 \times \partial_1 y_0) + (n_0 \times \partial_2 y_0) \times \partial_1 y_0 = 0$. Thus, the vector fields $\overrightarrow{X_{\widetilde{\alpha}}}$, $\overrightarrow{Y_{\widetilde{\alpha}}}$ and n_0 form a 3−frame on the surface $y_0(\omega)$. However, (7.7.7) reads,

$$\partial_1\widetilde{\alpha} \cdot \overrightarrow{X_{\widetilde{\alpha}}} + \partial_2\widetilde{\alpha} \cdot \overrightarrow{Y_{\widetilde{\alpha}}} = \sin\widetilde{\alpha} \cdot (\overrightarrow{A} - \overrightarrow{B}) = \widetilde{\delta} \cdot n_0 , \tag{7.7.9}$$

with a scalar field $\widetilde{\delta}$, so that by the linear independence of the vector fields $\overrightarrow{X_{\widetilde{\alpha}}}$, $\overrightarrow{Y_{\widetilde{\alpha}}}$ and n_0 we must always have $\partial_1\widetilde{\alpha} = \partial_2\widetilde{\alpha} = \widetilde{\delta} = 0$, which gives $\widetilde{\alpha} \equiv \text{const}$. Consequently, we also have $\cos\alpha \equiv \text{const}$ and $\sin\alpha \equiv \text{const}$ in $U \cap \omega$ for any choice of an angle function $\alpha\colon \omega \to \mathbb{R}$ It follows form the expressions in (7.3.7) that both functions $\cos\alpha(x)$ and $\sin\alpha(x)$ are continuous, which yields the conclusion. ∎

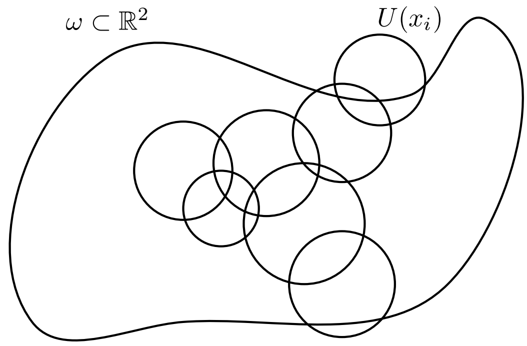


Figure 7.6: Although the rotation angle α in Q can be chosen in each point, its C^1-regularity is a priori not clear. However, it is given in a sufficient neighborhood of each point. We have seen, that the angle should be constant there, and due to the overlapping of the neighborhoods it has to be always the same constant.

Remark 7.7.2. Note that the rotation angle α has always to be constant but unconstrained, if, e.g., no further boundary conditions are appended. This is, e.g., provided by members of the same *associate family* of *minimal surfaces* (i.e., it holds $\mathrm{H} \equiv 0$). The most prominent example is the catenoid and helicoid family. More precisely, one can bend the catenoid without stretching into a portion of a helicoid in such a way that the surface normals remain unchanged. For further examples of associate families of minimal surfaces, we refer the reader to [42, chapter 3]. □

Now we are ready to turn to the large drill rotation case and prove our first main result.

Proof. (Proof of Proposition 7.0.1) Using the boundary condition $m|_{\gamma_d} = y_0|_{\gamma_d}$, Lemma 7.4.1 allows to conclude that $Q|_{\gamma_d} \equiv \mathbb{1}$ which implies $\sin\alpha|_{\gamma_d} = 0$ and $\cos\alpha|_{\gamma_d} = 1$. By Lemma 7.7.1 we have $\alpha \equiv \text{const}$, so that $\sin\alpha \equiv 0$ and $\cos\alpha \equiv 1$ on $\overline{\omega}$, and consequently by the Euler-Rodrigues representation it follows $Q \equiv \mathbb{1}$. ∎

Corollary 7.7.3. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\overline{\omega}, \mathbb{R}^3)$ are two regular surfaces and*

$$\begin{aligned}\mathrm{I}_m(x) = [\mathrm{D}m(x)]^T\, \mathrm{D}m(x) = [\mathrm{D}y_0(x)]^T\, \mathrm{D}y_0(x) = \mathrm{I}_{y_0}(x)\,, \qquad n(x) = n_0(x)\,, \qquad \forall x \in \overline{\omega}\,, \\ m|_{\gamma_d} = y_0|_{\gamma_d}\,,\end{aligned} \tag{7.7.10}$$

where $n = \frac{\partial_1 m \times \partial_2 m}{\|\partial_1 m \times \partial_2 m\|}$ and $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ are the respective normal fields and γ_d is a relatively open, non-empty subset of the boundary $\partial\omega$. Then $m \equiv y_0$.

Proof. Consider the lifted quantities $(\mathrm{D}m|n)$ versus $(\mathrm{D}y_0|n_0)$. It holds

$$(\mathrm{D}m|n)^T(\mathrm{D}m|n) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$[\mathrm{D}m]^T\,\mathrm{D}m$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad (\mathrm{D}y_0|n_0)^T(\mathrm{D}y_0|n_0) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$[\mathrm{D}y_0]^T\,\mathrm{D}y_0$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right). \tag{7.7.11}$$

By assumption $\mathrm{I}_m = \mathrm{D}m^T\,\mathrm{D}m = \mathrm{D}y_0^T\,\mathrm{D}y_0 = \mathrm{I}_{y_0}$, it follows from (7.7.11) that

$$(\mathrm{D}m|n)^T(\mathrm{D}m|n) = (\mathrm{D}y_0|n_0)^T(\mathrm{D}y_0|n_0)\,. \tag{7.7.12}$$

Then for all $x \in \omega$ there exists $Q(x) \in \mathrm{SO}(3)$ such that $(\mathrm{D}m(x)|n(x)) = Q(x)(\mathrm{D}y_0(x)|n_0(x))$. The assumption $n(x) = n_0(x)$ gives

$$(\mathrm{D}m|n_0) = Q(\mathrm{D}y_0|n_0)\,. \tag{7.7.13}$$

Multiplying both sides by e_3, we obtain $Q(x)n_0(x) = n_0(x)$ for all $x \in \omega$ and from (7.7.13) we obtain

$$Q = (\mathrm{D}m|n_0)(\mathrm{D}y_0|n_0)^{-1} = (\mathrm{D}m|n_0)\,\frac{1}{\det(\mathrm{D}y_0|n_0)}\,\mathrm{Cof}(\mathrm{D}y_0|n_0)\,. \tag{7.7.14}$$

Since by assumption $m, y_0 \in C^2(\overline{\omega}, \mathbb{R}^3)$ and y_0 is a regular surface with $\det(\mathrm{D}y_0|n_0) \geq c^+ > 0$ we observe that necessarily $Q \in C^1(\overline{\omega}, \mathrm{SO}(3))$. Thus, we are again in the situation of Proposition 7.0.1 and the proof is finished. ■

Proposition 7.7.4. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\omega, \mathbb{R}^3)$ are regular surfaces, $Q \in C^1(\omega, \mathrm{SO}(3))$ and*

$$\mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x)\,, \qquad Q(x)n_0(x) = n_0(x)\,, \qquad x \in \omega\,, \tag{7.7.15}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$. Then

$$\forall\ x \in \omega: \quad \sin\alpha(x) = 0 \quad \textit{or} \quad H(x) = 0, \tag{7.7.16}$$

where $\alpha : \omega \to \mathbb{R}$ denotes the rotations angle in the Euler-Rodrigues representation of Q.

Proof. By Lemma 7.7.1, we have $\cos\alpha \equiv$ const and $\sin\alpha \equiv$ const, so that taking and comparing the mixed derivations of m we obtain Again, in view of (7.6.15), the validity of (7.7.9) implies that we have pointwise either vanishing $\sin\alpha$ or a vanishing mean curvature H since

$$0 = \sin\alpha \cdot (\overrightarrow{A} - \overrightarrow{B}) \overset{(7.6.15)}{=} -2\,\sin\alpha \cdot \mathrm{H} \cdot \partial_1 y_0 \times \partial_2 y_0\,,$$

which implies that we have pointwise either vanishing $\sin\alpha$ or vanishing mean curvature H. ■

Remark 7.7.5 (Symmetry of the second fundamental form)**.** Conclusion (7.7.16) can also be obtained from the symmetry property of the second fundamental form on the surface $m(\omega)$. Indeed, the normal vector field on $m(\omega)$ coincides with n_0 since

$$\partial_1 m \times \partial_2 m \overset{(7.7.15)_1}{=} (Q\,\partial_1 y_0) \times (Q\,\partial_2 y_0) \overset{Q \in \mathrm{SO}(3)}{=} Q\,\partial_1 y_0 \times \partial_2 y_0 \overset{(7.7.15)_2}{=} \partial_1 y_0 \times \partial_2 y_0. \tag{7.7.17}$$

Thus, for the second fundamental form on $m(\omega)$ we obtain

$$\begin{aligned}
\mathrm{Sym}(2) \ni \mathrm{II}_m &= -[\mathrm{D}m]^T\,\mathrm{D}n = -[\mathrm{D}m]^T\,\mathrm{D}n_0 \overset{(7.7.15)_1}{=} -[\mathrm{D}y_0]^T Q^T\,\mathrm{D}n_0 \\
&\overset{(7.7.2)}{=} -[\mathrm{D}y_0]^T[(1-\cos\alpha)\,n_0 \otimes n_0 + \cos\alpha\,\mathbb{1} + \sin\alpha\ \mathrm{Anti}(n_0)]^T\,\mathrm{D}n_0 \\
&= (1-\cos\alpha)\underbrace{[\mathrm{D}y_0]^T n_0 \otimes n_0}_{=0}\mathrm{D}n_0 - \cos\alpha[\mathrm{D}y_0]^T\,\mathrm{D}n_0 - \sin\alpha[\mathrm{D}y_0]^T\,\mathrm{Anti}(n_0)^T\,\mathrm{D}n_0 \\
&= \cos\alpha\,\mathrm{II}_v + \sin\alpha\,[\mathrm{D}y_0]^T\,\mathrm{Anti}(n_0)\,\mathrm{D}n_0.
\end{aligned} \tag{7.7.18}$$

Since $\mathrm{II}_v \in \mathrm{Sym}(2)$, we are again left with the single condition

$$\sin\alpha\,[\mathrm{D}y_0]^T\,\mathrm{Anti}(n_0)\,\mathrm{D}n_0 \in \mathrm{Sym}(2) \quad \overset{(7.6.18)}{\iff} \quad \{\sin\alpha = 0 \quad \text{or} \quad \mathrm{H} = 0\}. \tag{7.7.19}$$

□

Corollary 7.7.6. *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Assume that $m, y_0 \in C^2(\omega, \mathbb{R}^3)$ are regular surfaces, $Q \in C^1(\omega, \mathrm{SO}(3))$ and*

$$\mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x)\,, \qquad Q(x)n_0(x) = n_0(x)\,, \qquad x \in \omega\,, \tag{7.7.20}$$

where $n_0 = \frac{\partial_1 y_0 \times \partial_2 y_0}{\|\partial_1 y_0 \times \partial_2 y_0\|}$ denotes the normal field on $y_0(\omega)$. If the mean curvature of y_0 does not vanish at one point $y_0(x_0)$, then $m(x) = y_0(x) + b$ or $m(x) = -y_0(x) + b$ for some constant translation $b \in \mathbb{R}^3$.

Proof. It follows from Proposition 7.7.4, that if the mean curvature H is distinct from 0 at some point $y_0(x_0)$, we must have $\sin\alpha(x_0) = 0$. Thus, $\sin\alpha \equiv 0$ and $\cos\alpha \equiv 1$ or $\sin\alpha \equiv 0$ and $\cos\alpha \equiv -1$, where the sign of $\cos\alpha$ does not change; see Lemma 7.7.1. Thus, in the first case ($\cos\alpha \equiv 1$, i.e. $\alpha \in 2\pi\mathbb{Z}$), we obtain

$$Q(x) \overset{(7.7.2)}{=} (1-\cos\alpha) n_0(x) \otimes n_0(x) + \mathbb{1}\,, \tag{7.7.21}$$

and a multiplication with $\mathrm{D}y_0$ gives

$$\mathrm{D}m(x) = Q(x)\,\mathrm{D}y_0(x) \overset{(7.7.21)}{=} (1-\cos\alpha)\underbrace{n_0(x) \otimes n_0(x)\,\mathrm{D}y_0(x)}_{=0} + \mathrm{D}y_0(x) = \mathrm{D}y_0(x).$$

In the second case ($\cos\alpha \equiv -1$, i.e. $\alpha - \pi \in 2\pi\mathbb{Z}$), we have

$$Q(x) \overset{(7.7.2)}{=} (1-\cos\alpha) n_0(x) \otimes n_0(x) - \mathbb{1} \Rightarrow \mathrm{D}m(x) = -\,\mathrm{D}y_0(x)\,. \tag{7.7.22}$$

■

Remark 7.7.7. The 'flipped' solution $m(x) = -y_0(x) + b$ appears only because no boundary conditions are present. From a mechanical point of view, this branch is irrelevant. □

Remark 7.7.8. For a C^∞-embedding y_0 a comparable result, using involved techniques from differential geometry, has been obtained in [1, 48]. □

7.8. Compatibility condition

For the description of a body, we assume that the body contained of a set of infinitesimal volumes or material points, where each volume is supposed to be connected to its neighbors, without crack or lap. Some mathematical conditions should hold which guarantees that this gap or overlap will not be developed or happened after the deformation of the body. A body that deforms without growing gap or overlap, is called a *compatible body.* In mathematical perspective, *compatibility conditions* are those conditions that characterize whether a certain deformation will drop a body in a compatible state or not. The following propositions gives a necessary and sufficient condition for compatibility of a regular surface.

Proposition 7.8.1. [***Compatbility for surfaces***] *Let $v, w \in C^1(\omega, \mathbb{R}^3)$ be given vector fields with* $\operatorname{rank}(v, w) = 2$ *everywhere and assume that ω is a bounded, simply connected domain. Then there exists a regular surface $m \in C^2(\omega, \mathbb{R}^3)$ such that*

$$\mathrm{D}m(x) = \Big(v(x)\big|w(x)\Big)\,, \qquad x \in \omega\,, \tag{7.8.1}$$

if and only if the ***compatibility condition***

$$\partial_2 v(x_1, x_2) = \partial_1 w(x_1, x_2)\,, \tag{7.8.2}$$

holds. The surface m is unique up to translation.

Proof. We only need to observe that (7.8.1) reads

$$\begin{pmatrix} m_{1,x_1} & m_{1,x_2} \\ m_{2,x_1} & m_{2,x_2} \\ m_{3,x_1} & m_{3,x_2} \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix}\,, \tag{7.8.3}$$

if and only if

$$\begin{pmatrix} m_{1,x_1} \\ m_{1,x_2} \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}\,, \qquad \begin{pmatrix} m_{2,x_1} \\ m_{2,x_2} \end{pmatrix} = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}\,, \qquad \begin{pmatrix} m_{3,x_1} \\ m_{3,x_2} \end{pmatrix} = \begin{pmatrix} v_3 \\ w_3 \end{pmatrix}\,, \tag{7.8.4}$$

and the existence of potentials $m_i \in C(\omega, \mathbb{R})$ is guaranteed if and only if

$$\partial_2 v_1 = \partial_1 w_1\,, \qquad \partial_2 v_2 = \partial_1 w_2\,, \qquad \partial_2 v_3 = \partial_1 w_3\,, \tag{7.8.5}$$

which is equivalent to

$$\partial_2 v = \partial_1 w\,. \tag{7.8.6}$$

The potentials m_i, $i = 1, 2, 3$, are unique up to constants. ■

7.9. Conclusion

We have proved some improved rigidity results for C^2-smooth regular embedded surfaces. The underlying mechanical problem, namely the possibility of a pure in-plane drill rotation field as deformation mode of a surface when boundary conditions of place are prescribed somewhere has been negatively answered, for both small and large drill rotations, assuming some natural level of smoothness for the rotation fields.

Considering classical FEM shell formulations the drilling degrees of freedom are used to obtain a precise coupling of plane shell elements in non-plane applications, in such a way, that rotations around an axis in the plane of one element are coupled to rotations around the normal of the neighboring element. Since finite elements use their shape functions as interpolations, they do not obey the kinematics everywhere. Thus, incompatibilities in the shape functions may occur, but, should not get in conflict with the convergence requirements such as the patch test and any ”torsional spring stiffness” attributed to this pseudo-deformation mode must be regarded with extreme caution (here, the Cosserat couple modulus $\mu_c \geq 0$). In fact, in many shell models such a stiffness is treated as a material parameter and many very effective finite elements use incompatible kinematic fields, mainly to overcome geometric deficiencies which may result in some kind of locking behavior. Our development suggests, however, that the fitting of the Cosserat couple modulus μ_c would depend on the imposed boundary condition, i.e., how strict drill rotations are constrained at the boundary γ_d of the shell.

Then, the mentioned stiffness is a boundary value problem dependent parameter which needs to be determined for each new problem again. Thus one should call it always a fictitious stifness and treat it accordingly, and this applies to classical FEM-shell models and Cosserat surfaces.

However, it is remarkable that in the planar Cosserat shell model [84] existence can be shown also for zero Cosserat couple modulus $\mu_c \equiv 0$ [89] (for $q > 0$ in (7.1.1) and using a generalized Korn-inequality [83]) thus disposing completely of the above described problem. In other words, the drilling degree of freedom is kept, but not connected to any fictitious torsional spring. While in a linear model this would imply that the drilling degree of freedom decouples, this is not necessarily the case in nonlinear Cosserat models based on exact rotations.

Bibliography

[1] K. Abe and J. Erbacher. "Isometric immersions with the same Gauss map". *Mathematische Annalen* 215.3 (1975). Pp. 197–201.

[2] A. Acharya. "A nonlinear generalization of the Koiter–Sanders–Budiansky bending strain measure". *International Journal of Solids and Structures* 37.39 (2000). Pp. 5517–5528.

[3] A. Acharya and J. Ginster. "Rotations with constant curl are constant". *arXiv preprint:2007.00331* (2020).

[4] I. Aganović, J. Tambača, and Z. Tutek. "Derivation and justification of the model of micropolar elastic shells from three-dimensional linearized micropolar elasticity". *Asymptotic Analysis* 51.3-4 (2007). Pp. 335–361.

[5] H. Altenbach and V. E. (editors). *Shell-like Structures: Non-classical Theories and Applications.* Vol. 15. Advanced Structured Materials. Berlin Heidelberg: Springer-Verlag, 2011.

[6] H. Altenbach and V. (Eremeyev. *Generalized Continua from the Theory to Engineering Applications.* Wien: Springer, 2013.

[7] J. Altenbach, H. Altenbach, and V. Eremeyev. "On generalized Cosserat-type theories of plates and shells: a short review and bibliography." *Arch. Appl. Mech.* 80 (2010). Pp. 73–92.

[8] S. Anicic. "Mesure des variations infinitésimales des courbures principales d'une surface". *Compte Rendus Acad. Sci. Paris, Ser. Math.* 335.3 (2002). Pp. 301–306.

[9] S. Anicic. "A shell model allowing folds". *Numerical Mathematics and Advanced Applications.* Springer, 2003, pp. 317–326.

[10] S. Anicic. "Polyconvexity and existence theorem for nonlinearly elastic shells". *J. Elasticity* 132.1 (2018). Pp. 161–173.

[11] S. Anicic. "Existence theorem for a first-order Koiter nonlinear shell model". *Discrete Contin. Dyn. Syst. Ser. S* 12.6 (2019). Pp. 1535–1545.

[12] S. Anicic and A. Léger. "Formulation bidimensionnelle exacte du modèle de coque 3D de Kirchhoff-Love". *Compte Rendus Acad. Sci. Paris, Ser. Math.* 329.8 (1999). Pp. 741–746.

[13] S. Antman. "Nonlinear Problems of Elasticity, volume 107 of Applied Mathematical Sciences, Springer". *Berlin* (1995).

[14] P. Appell. "Traité de Mécanique Rationnelle". *Gauthier-Villars, Paris* (1909).

[15] S. Bartels, M. Griehl, S. Neukamm, D. Padilla-Garza, and C. Palus. "A nonlinear bending theory for nematic LCE plates". *arXiv preprint arXiv:2203.04010* (2022).

[16] M. Bîrsan and P. Neff. "On the equations of geometrically nonlinear elastic plates with rotational degrees of freedom". *Annals of the Academy of Romanian Scientists: Series on Mathematics and its Applications* 4 (2012). Pp. 97–103.

[17] M. Bîrsan and P. Neff. "Existence theorems in the geometrically non-linear 6-parameter theory of elastic plates". *Journal of Elasticity* 112.2 (2013). Pp. 185–198.

[18] M. Bîrsan and P. Neff. "On the characterization of drilling rotation in the 6-parameter resultant shell theory". *Shell-Structures: Theory and Applications.* Ed. by J. G. W. Pietraszkiewicz. Vol. 3. CRC Press, 2013.

[19] M. Bîrsan and P. Neff. "Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations". *Mathematics and Mechanics of Solids* 19.4 (2014). Pp. 376–397.

[20] M. Bîrsan and P. Neff. "Shells without drilling rotations: A representation theorem in the framework of the geometrically nonlinear 6-parameter resultant shell theory". *International Journal of Engineering Science* 80 (2014). Pp. 32–42.

[21] M. Bîrsan and P. Neff. "On the dislocation density tensor in the Cosserat theory of elastic shells". *Advanced Methods of Continuum Mechanics for Materials and Structures.* Springer, 2016, pp. 391–413.

[22] M. Bîrsan. "Derivation of a refined 6-parameter shell model: Descent from the three-dimensional Cosserat elasticity using a method of classical shell theory". *Mathematics and Mechanics of Solids* 25(6) (2020). Pp. 1318–1339.

[23] M. Bîrsan, I.-D. Ghiba, R. J. Martin, and P. a. Neff. "Refined dimensoinal reduction for isotropic elastic Cosserat shells with initial". *Mathematics and Mechanics of Solids* 24.12 (2019). Pp. 4000–4019.

[24] F. Boyer and F. Renda. "Poincare's equations for Cosserat media: Application to shells". *Journal of Nonlinear Science* 27.1 (2017). Pp. 1–44.

[25] A. Braides. *Gamma-convergence for Beginners.* Vol. 22. Clarendon Press, 2002.

[26] J. Braun and B. Schmidt. "Existence and convergence of solutions of the boundary value problem in atomistic and continuum nonlinear elasticity theory". *Calculus of Variations and Partial Differential Equations* 55.5 (2016). Pp. 1–36.

[27] J. Chróścielewski, J. Makowski, and W. Pietraszkiewicz. *Statics and Dynamics of Multifold Shells: Nonlinear Theory and Finite Element Method (in Polish).* Warsaw: Wydawnictwo IPPT PAN, 2004.

[28] J. Chróścielewski, W. Pietraszkiewicz, and W. Witkowski. "On shear correction factors in the non-linear theory of elastic shells." *International Journal of Solids and Structures.* 47 (2010). Pp. 3537–3545.

[29] J. Chróścielewski, I. Kreja, A. Sabik, and W. Witkowski. “Modeling of composite shells in 6–parameter nonlinear theory with drilling degree of freedom”. *Mechanics of Advanced Materials and Structures* 18.6 (2011). Pp. 403–419.

[30] P. G. Ciarlet. *Mathematical Elasticity. Volume I: Three-Dimensional Elasticity.* Vol. 20. Studies in Mathematics and its Applications. Elsevier Science, 1988.

[31] P. Ciarlet. *Introduction to Linear Shell Theory.* First edition. Series in Applied Mathematics. Paris: Gauthier-Villars, 1998.

[32] P. Ciarlet. *Mathematical elasticity Vol. 1: Three dimensional elasticity.* Amsterdam, 1987.

[33] P. Ciarlet. *Three-Dimensional Elasticity.* first. Vol. 1. Studies in Mathematics and its Applications. Amsterdam: Elsevier, 1988.

[34] P. Ciarlet. *Introduction to Linear Shell Theory.* first edition. Paris: Gauthier-Villars, 1998.

[35] H. Cohen and C. DeSilva. “Nonlinear theory of elastic directed surfaces”. *Journal of Mathematical Physics* 7.6 (1966). Pp. 960–966.

[36] H. Cohen and C.-C. Wang. “A mathematical analysis of the simplest direct models for rods and shells”. *Archive for Rational Mechanics and Analysis* 108.1 (1989). Pp. 35–81.

[37] H. Cohen and C. DeSilva. “Nonlinear theory of elastic surfaces”. *Journal of Mathematical Physics* 7.2 (1966). Pp. 246–253.

[38] E. Cosserat and F. Cosserat. “Théorie des corps déformables” (1909).

[39] E. Cosserat and F. Cosserat. “Sur la théorie des corps minces”. *Comptes Rendus* 146 (1908). Pp. 169–172.

[40] E. Cosserat and F. Cosserat. *Note sur la théorie de l’action euclidienne.* Appendix in [14], pp. 557–629.

[41] G. Darboux. *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal.* Vol. 1. Paris, France, Gauthier-Villars, 1941.

[42] U. Dierkes, S. Hildebrandt, and F. Sauvigny. *Minimal surfaces. With assistance and contributions by A. Küster and R. Jakob.* 2nd revised and enlarged ed. Grundlehren der Mathematischen Wissenschaften 339. Dordrecht, Springer, 2010.

[43] H. L. Dret and A. Raoult. “The membrane shell model in nonlinear elasticity: a variational asymptotic derivation.” *Journal of Nonlinear Science* 6 (1996). Pp. 59–84.

[44] V. Eremeyev and W. Pietraszkiewicz. “The nonlinear theory of elastic shells with phase transitions.” *Journal of Elasticity* 74 (2004). Pp. 67–86.

[45] V. A. Eremeyev and W. Pietraszkiewicz. “Local symmetry group in the general theory of elastic shells”. *Journal of Elasticity* 85.2 (2006). Pp. 125–152.

[46] J. Ericksen and C. Truesdell. “Exact theory of stress and strain in rods and shells”. *Archive for Rational Mechanics and Analysis* 1.1 (1957). Pp. 295–323.

[47] A. C. Eringen. “Mechanics of continua”. *Huntington* (1980).

[48] J.-H. Eschenburg, B. Kruglikov, V. S. Matveev, and R. Tribuzy. “Compatibility of Gauss maps with metrics”. *Differential Geometry and its Applications* 28.2 (2010). Pp. 228–235.

[49] D. Fox and J. Simo. “A drill rotation formulation for geometrically exact shells”. *Computer Methods in Applied Mechanics and Engineering* 98.3 (1992). Pp. 329–343.

[50] G. Friesecke, R. D. James, M. G. Mora, and S. Müller. “Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence”. *Comptes Rendus Mathematique* 336.8 (2003). Pp. 697–702.

[51] G. Friesecke, R. D. James, and S. Müller. “A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity”. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 55.11 (2002). Pp. 1461–1506.

[52] G. Friesecke, R. D. James, and S. Müller. “The Föppl–von Kármán plate theory as a low energy Γ-limit of nonlinear elasticity”. *Comptes Rendus Mathematique* 335.2 (2002). Pp. 201–206.

[53] G. Friesecke, R. D. James, and S. Müller. “A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence”. *Archive for rational mechanics and analysis* 180.2 (2006). Pp. 183–236.

[54] A. Gastel and P. Neff. “Regularity for a geometrically nonlinear flat Cosserat micropolar membrane shell with curvature”. *in preparation* ().

[55] I. Ghiba, M. Mohammadi Saem, and P. Neff. “On the choice of third order curvature tensors in the geometrically nonlinear Cosserat micropolar model and applications”. *in preparation* ().

[56] I. D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. “The isotropic Cosserat shell model including terms up to $O(h^5)$. Part I: Derivation in matrix notation”. *Journal of Elasticity* 142 (2020). Pp. 201–262.

[57] I. D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. “The isotropic elastic Cosserat shell model including terms up to order $O(h^5)$ in the shell thickness. Part II: Existence of minimizers”. *Journal of Elasticity* 142 (2020). Pp. 263–290.

[58] I. D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. “A constrained Cosserat-shell model including terms up to $O(h^5)$”. *Journal of Elasticity* 146.1 (2021). Pp. 83–141.

[59] I. D. Ghiba, M. Bîrsan, and P. Neff. “A linear Cosserat-shell model including terms up to $O(h^5)$”. *in preparation* (2022).

[60] I. D. Ghiba, M. Bîrsan, and P. Neff. “Linear constrained Cosserat-shell models including terms up to $O(h^5)$”. *in preparation* (2022).

[61] I. D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. "A constrained Cosserat shell model up to order $O(h^5)$: Modelling, existence of minimizers, relations to classical shell models and scaling invariance of the bending tensor". *arXiv preprint,* available at arXiv:2010.14308 (2020).

[62] I. D. Ghiba and P. Neff. "On the deformation measures in shell models". *in preparation* (2022).

[63] I. D. Ghiba, O. Sander, L. J. Nebel, and P. Neff. "The classical geometrically nonlinear, physically linear Reissner-Mindlin and Kirchhoff-Love membrane-bending model is ill-posed". *in preparation* (2022).

[64] A. Green and P. Naghdi. "Shells in the light of generalized Cosserat continua." *Theory of Thin Shells.* Ed. by F. Niordson. IUTAM Symposium Copenhagen 1967. Heidelberg: Springer, 1969, pp. 39–58.

[65] A. E. Green, P. M. Naghdi, and W. Wainwright. "A general theory of a Cosserat surface". *Archive for Rational Mechanics and Analysis* 20.4 (1965). Pp. 287–308.

[66] D. A. Hoffman and R. Osserman. "The Gauss map of surfaces in $\mathbb{R}^n$ ". *Journal of Differential Geometry* 18.4 (1983). Pp. 733–754.

[67] F. Hörmander, N. Totaro, and A. V. M. Waldschmidt. *Grundlehren der mathematischen wissenschaften 332.* Vol. 5. Springer, 2006.

[68] P. Hornung, S. Neukamm, and I. Velčić. "Derivation of a homogenized nonlinear plate theory from 3d elasticity". *Calculus of Variations and Partial Differential Equations* 51.3 (2014). Pp. 677–699.

[69] T. J. Hughes and F. Brezzi. "On drilling degrees of freedom". *Computer Methods in Applied Mechanics and Engineering* 72.1 (1989). Pp. 105–121.

[70] A. Ibrahimbegović. "Stress resultant geometrically nonlinear shell theory with drilling rotations—Part I. A consistent formulation". *Computer Methods in Applied Mechanics and Engineering* 118.3-4 (1994). Pp. 265–284.

[71] H. Karcher and K. Polthier. "Construction of triply periodic minimal surfaces". *Philosophical Transactions of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* 354.1715 (1996). Pp. 2077–2104.

[72] W. Koiter. "Foundations and basic equations of shell theory. A survey of recent progress." *Theory of Thin Shells.* Ed. by F. Niordson. IUTAM Symposium Copenhagen 1967. Heidelberg: Springer, 1969, pp. 93–105.

[73] S. Kugler, P. A. Fotiu, and J. Murin. "On consistent estimations of drilling rotations in shell elements". *Shell Structures: Theory and Applications: Volume 3* 3 (2013). P. 413.

[74] J. Lankeit, P. Neff, and F. Osterbrink. "Integrability conditions between the first and second Cosserat deformation tensor in geometrically nonlinear micropolar models and existence of minimizers". *Zeitschrift für angewandte Mathematik und Physik* 68.1 (2017). P. 11.

[75] H. Le Dret and A. Raoult. "The membrane shell model in nonlinear elasticity: a variational asymptotic derivation". *Journal of Nonlinear Science* 6.1 (1996). Pp. 59–84.

[76] K. K. Liang. "Efficient conversion from rotating matrix to rotation axis and angle by extending Rodrigues' formula". *arXiv preprint,* available at arXiv:1810.02999 (2018).

[77] M. Ljulj and J. Tambača. "A Naghdi type nonlinear model for shells with little regularity". *Journal of Elasticity* (2020). Pp. 1–48.

[78] M. Mohammadi Saem, P. Lewintan, and P. Neff. "On in-plane drill rotations for Cosserat surfaces". *Proceedings of the Royal Society A* 477.2252 (2021). P. 20210158.

[79] P. M. Naghdi. *The theory of shells. In Handbuch der Physik. Machanic of Solids.* Vol. VI a/2. Springer, 1972.

[80] P. Neff. "A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus". *Continuum Mechanics and Thermodynamics* 16.6 (2004). Pp. 577–628.

[81] P. Neff. "A geometrically exact Cosserat-shell model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus." *Cont. Mech. Thermodynamics* 16.6 (DOI 10.1007/s00161-004-0182-4) (2004). Pp. 577–628.

[82] P. Neff and I. Münch. "Curl bounds Grad on SO(3)." *ESAIM: Control, Optimisation and Calculus of Variations* 14 (2008). Pp. 148–159.

[83] P. Neff. "On Korn's first inequality with non-constant coefficients". *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 132.01 (2002). Pp. 221–243.

[84] P. Neff. "A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus". *Continuum Mechanics and Thermodynamics* 16.6 (2004). Pp. 577–628.

[85] P. Neff. "Geometrically Exact Cosserat Theory for Bulk Behaviour and Thin Structures: Modelling and Mathematical Analysis". *Habilitation Thesis, TU-Darmstadt* (2004).

[86] P. Neff. "A geometrically exact viscoplastic membrane-shell with viscoelastic transverse shear resistance avoiding degeneracy in the thin-shell limit." *Zeitschrift für angewandte Mathematik und Physik ZAMP* 56.1 (2005). Pp. 148–182.

[87] P. Neff. "The Γ-limit of a finite-strain Cosserat model for asymptotically thin domains and a consequence for the Cosserat couple modulus μc". *PAMM: Proceedings in Applied Mathematics and Mechanics.* Vol. 5. 1. Wiley Online Library. 2005, pp. 629–630.

[88] P. Neff. "A geometrically exact planar Cosserat shell-model with microstructure: Existence of minimizers for zero Cosserat couple modulus". *Mathematical Models and Methods in Applied Sciences* 17.03 (2007). Pp. 363–392.

[89] P. Neff. "A geometrically exact planar Cosserat shell-model with microstructure: Existence of minimizers for zero Cosserat couple modulus". *Mathematical Models and Methods in Applied Sciences* 17.03 (2007). Pp. 363–392.

[90] P. Neff and K. Chelminski. "A geometrically exact Cosserat shell-model for defective elastic crystals. Justification via Γ-convergence". *Interfaces and Free Boundaries* 9.4 (2007). Pp. 455–492.

[91] P. Neff, K.-I. Hong, and J. Jeong. "The Reissner–Mindlin plate is the Γ-limit of Cosserat elasticity". *Mathematical Models and Methods in Applied Sciences* 20.09 (2010). Pp. 1553–1590.

[92] P. Neff, J. Lankeit, and A. Madeo. "On Grioli's minimum property and its relation to Cauchy's polar decomposition". *International Journal of Engineering Science* 80 (2014). Pp. 209–217.

[93] P. Neff, J. Lankeit, and A. Madeo. "On Grioli's minimum property and its relation to Cauchy's polar decomposition". *International Journal of Engineering Science* 80 (2014). Pp. 209–217.

[94] P. Neff and I. Münch. "Curl bounds Grad on SO(3)". *ESAIM: Control, Optimisation and Calculus of Variations* 14.1 (2008). Pp. 148–159.

[95] P. Neff and I. Münch. "Curl bounds Grad on SO (3)". *ESAIM: Control, Optimisation and Calculus of Variations* 14.1 (2008). Pp. 148–159.

[96] P. Neff, D. Pauly, and K.-J. Witsch. "Poincaré meets Korn via Maxwell: extending Korn's first inequality to incompatible tensor fields". *Journal of Differential Equations* 258.4 (2015). Pp. 1267–1302.

[97] B. Palais, R. Palais, and S. Rodi. "A disorienting look at Euler's theorem on the axis of a rotation". *The American Mathematical Monthly* 116.10 (2009). Pp. 892–909.

[98] W. Pietraszkiewicz. "Refined resultant thermomechanics of shells". *International Journal of Engineering Science* 49.10 (2011). Pp. 1112–1124.

[99] W. Pietraszkiewicz. "The resultant linear six-field theory of elastic shells: What it brings to the classical linear shell models?" *Zeitschrift für Angewandte Mathematik und Mechanik* 96.8 (2016). Pp. 899–915.

[100] W. Pietraszkiewicz and V. Konopińska. "Drilling couples and refined constitutive equations in the resultant geometrically non-linear theory of elastic shells". *International Journal of Solids and Structures* 51.11-12 (2014). Pp. 2133–2143.

[101] M. B. Rubin. *Cosserat theories: shells, rods and points.* Vol. 79. Springer Science & Business Media, 2013.

[102] M. M. Saem, I.-D. Ghiba, and P. Neff. "A geometrically nonlinear Cosserat (micropolar) curvy shell model via Gamma convergence". *arXiv preprint arXiv:2207.08541* (2022).

[103] O. Sander, P. Neff, and M. Bîrsan. "Numerical treatment of a geometrically nonlinear planar Cosserat shell model". *Computational Mechanics* 57.5 (2016). Pp. 817–841.

[104] B. Schmidt. "Linear Γ-limits of multiwell energies in nonlinear elasticity theory". *Continuum Mechanics and Thermodynamics* 20.6 (2008). Pp. 375–396.

[105] M. Šilhavỳ. "A new approach to curvature measures in linear shell theories". *Math. Mech. Solids* 26.9 (2021). Pp. 1241–1263.

[106] M. L. Szwabowicz. "Pure strain deformations of surfaces". *Journal of Elasticity* 92.3 (2008). P. 255.

[107] K. Wisniewski. "A shell theory with independent rotations for relaxed Biot stress and right stretch strain". *Computational Mechanics* 21.2 (1998). Pp. 101–122.

[108] K. Wisniewski. *Finite Rotation Shells: Basic Equations and Finite Elements for Reissner Kinematics.* Springer, 2010.

[109] K. Wisniewski and E. Turska. "Kinematics of finite rotation shells with in-plane twist parameter". *Computer Methods in Applied Mechanics and Engineering* 190.8-10 (2000). Pp. 1117–1135.

[110] K. Wisniewski and E. Turska. "Enhanced Allman quadrilateral for finite drilling rotations". *Computer Methods in Applied Mechanics and Engineering* 195.44-47 (2006). Pp. 6086–6109.

[111] J.-T. Yeh and W.-H. Chen. "Shell elements with drilling degree of freedoms based on micropolar elasticity theory". *International Journal for Numerical Methods in Engineering* 36.7 (1993). Pp. 1145–1159.

[112] P. Zhilin. *Applied Mechanics – Foundations of Shell Theory (in Russian).* Sankt Petersburg: State Polytechnical University Publisher, 2006.

A. Appendix for Part I

A.1. Calculations for the T_{Biot} stress tensor

Here we present the lengthy calculation related to the T_{Biot} stress tensor in expression (5.5.18). We have

$$\begin{aligned}
2\,\text{sym}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,c^*}-\mathbb{1}_3)n_0 &= \Big(2\,\text{sym}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})+2\,\text{sym}((0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0\\
&= \Big(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}+\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}\Big)n_0+\Big((0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1}+[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)^T\Big)n_0\\
&= \underbrace{\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}n_0}_{=0}+\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1}n_0+[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)^Tn_0\\
&= \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)e_3+[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)^Tn_0 \qquad \text{(A.1.1)}\\
&= \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(\overline{Q}_e^{\natural,T}c^*-n_0)+[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)^Tn_0,
\end{aligned}$$

and

$$\begin{aligned}
2\,\text{skew}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,c^*}-\mathbb{1}_3)n_0 &= \Big(2\,\text{skew}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})+2\,\text{skew}((0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1})\Big)n_0\\
&= \Big(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}-\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}\Big)n_0+\Big((0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1}-[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)^T\Big)n_0\\
&= -\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0+(\overline{Q}_e^{\natural,T}c^*-n_0)-[(\nabla_x\Theta)^\natural]^{-T}(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)^Tn_0. \qquad \text{(A.1.2)}
\end{aligned}$$

Calculating the trace of T_{Biot} gives

$$\begin{aligned}
\text{tr}(\text{sym}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,c^*}-\mathbb{1}_3))n_0 &= \langle\text{sym}(\overline{U}_{\varphi^\natural,\overline{Q}_e^\natural,c^*}-\mathbb{1}_3),\mathbb{1}_3\rangle n_0=\Big(\langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\mathbb{1}_3\rangle+\langle(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1},\mathbb{1}_3\rangle\Big)n_0\\
&= \langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\mathbb{1}_3\rangle n_0+(\overline{Q}_e^{\natural,T}c^*-n_0)n_0\otimes n_0, \qquad \text{(A.1.3)}
\end{aligned}$$

where we have used that

$$\langle(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1},\mathbb{1}_3\rangle_{\mathbb{R}^{3\times3}}\,n_0=\langle(\overline{Q}_e^{\natural,T}c^*-n_0),n_0\rangle_{\mathbb{R}^3}\,n_0=(\overline{Q}_e^{\natural,T}c^*-n_0)n_0\otimes n_0\,.$$

A.2. Calculations for the homogenized membrane energy

In this part we would like to do the calculations for obtaining the minimizer separately. By inserting c^* in the membrane part of the relation (5.3.8), we have

$$\begin{aligned}
\|\text{sym}(\overline{U}_h^\natural-\mathbb{1}_3)\|^2 &= \|\text{sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|c^*)[(\nabla_x\Theta)^\natural]^{-1}-\mathbb{1}_3)\|^2\\
&= \|\text{sym}\Big(\underbrace{\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural-[\nabla y_0]^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}}_{=\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}}+(0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1}\Big))\|^2\\
&= \|\text{sym}\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\|^2+\|\text{sym}((0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1})\|^2\\
&\quad+2\Big\langle\text{sym}\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\text{sym}((0|0|\overline{Q}_e^{\natural,T}c^*-n_0)[(\nabla_x\Theta)^\natural]^{-1})\Big\rangle\\
&= \|\text{sym}\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\|^2+\|\text{sym}\Big(\frac{\mu_c-\mu}{\mu_c+\mu}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0-\frac{\lambda}{2\,\mu+\lambda}\,\text{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\otimes n_0\Big)\|^2\\
&\quad+2\Big\langle\text{sym}\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\text{sym}\Big(\frac{\mu_c-\mu}{\mu_c+\mu}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0-\frac{\lambda}{2\,\mu+\lambda}\text{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\otimes n_0\Big)\Big\rangle. \qquad \text{(A.2.1)}
\end{aligned}$$

We have

$$
\begin{aligned}
&\|\mathrm{sym}\Big(\frac{\mu_c-\mu}{\mu_c+\mu}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0-\frac{\lambda}{2\,\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\otimes n_0\Big)\|^2\\
&=\frac{(\mu_c-\mu)^2}{(\mu_c+\mu)^2}\|\mathrm{sym}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0)\|^2+\frac{\lambda^2}{(2\,\mu+\lambda)^2}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})^2\|n_0\otimes n_0\|^2\\
&\quad-2\,\frac{\mu_c-\mu}{\mu_c+\mu}\,\frac{\lambda}{2\,\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})\Big\langle\mathrm{sym}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0),n_0\otimes n_0\Big\rangle\\
&=\frac{(\mu_c-\mu)^2}{(\mu_c+\mu)^2}\Big\langle\mathrm{sym}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0),\mathrm{sym}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0)\Big\rangle+\frac{\lambda^2}{(2\,\mu+\lambda)^2}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})^2\\
&\quad-\frac{\mu_c-\mu}{\mu_c+\mu}\,\frac{\lambda}{2\,\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})\Big\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0,n_0\otimes n_0\Big\rangle\\
&\quad-\frac{\mu_c-\mu}{\mu_c+\mu}\,\frac{\lambda}{2\,\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})\Big\langle n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},n_0\otimes n_0\Big\rangle \qquad \text{(A.2.2)}\\
&=\frac{(\mu_c-\mu)^2}{4(\mu_c+\mu)^2}\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0,\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0\rangle\\
&\quad+\frac{(\mu_c-\mu)^2}{4(\mu_c+\mu)^2}\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0,n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\rangle\\
&\quad+\frac{(\mu_c-\mu)^2}{4(\mu_c+\mu)^2}\langle n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0\rangle\\
&\quad+\frac{(\mu_c-\mu)^2}{4(\mu_c+\mu)^2}\langle n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\rangle+\frac{\lambda^2}{(2\,\mu+\lambda)^2}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})^2\\
&=\frac{(\mu_c-\mu)^2}{2(\mu_c+\mu)^2}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\|^2+\frac{\lambda^2}{(2\,\mu+\lambda)^2}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})^2.
\end{aligned}
$$

Since, using (5.5.23) we have

$$
\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0,n_0\otimes n_0\rangle=\langle n_0\otimes n_0,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0\rangle=0\,, \qquad \text{(A.2.3)}
$$

and since we have used the matrix expression $\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}=(*|*|0)[(\nabla_x\Theta)^\natural]^{-1}$ and $n_0\otimes n_0=(0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}$, we deduce

$$
\begin{aligned}
\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0,\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0\rangle&=\big\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}(0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1},\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}(0|0|n_0)[(\nabla_x\Theta)^\natural(0)]^{-1}\big\rangle\\
&=\big\langle(0|0|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0)^T(0|0|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0),[(\nabla_x\Theta)^\natural(0)]^{-1}[(\nabla_x\Theta)^\natural(0)]^{-T}\big\rangle\\
&=\langle(0|0|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0)^T(0|0|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0),(\widehat{\mathrm{I}}_{y_0})^{-1}\rangle\\
&=\left\langle\begin{pmatrix}0&0&0\\0&0&0\\&\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0&\end{pmatrix}(0|0|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0),\begin{pmatrix}*&*&0\\ *&*&0\\0&0&1\end{pmatrix}\right\rangle \qquad \text{(A.2.4)}\\
&=\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0,\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\rangle=\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\|^2.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&2\Big\langle\mathrm{sym}\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\mathrm{sym}(\frac{\mu_c-\mu}{\mu_c+\mu}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0-\frac{\lambda}{2\,\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\otimes n_0)\Big\rangle\\
&=\frac{1}{2}\Big\langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}+\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural},\frac{\mu_c-\mu}{\mu_c+\mu}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0+\frac{\mu_c-\mu}{\mu_c+\mu}n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}-\frac{2\lambda}{2\,\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})n_0\otimes n_0\Big\rangle\\
&=\frac{\mu_c-\mu}{2(\mu_c+\mu)}\Big\langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0\Big\rangle+\frac{\mu_c-\mu}{2(\mu_c+\mu)}\Big\langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\Big\rangle \qquad \text{(A.2.5)}\\
&\quad-\frac{\lambda}{(2\,\mu+\lambda)}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})\langle\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural},n_0\otimes n_0\rangle+\frac{\mu_c-\mu}{2(\mu_c+\mu)}\Big\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural},\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\otimes n_0\Big\rangle\\
&\quad+\frac{\mu_c-\mu}{2(\mu_c+\mu)}\Big\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural},n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\Big\rangle-\frac{\lambda}{(2\,\mu+\lambda)}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})\langle\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural},n_0\otimes n_0\rangle\\
&=\frac{\mu_c-\mu}{\mu_c+\mu}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\|^2,
\end{aligned}
$$

due to (5.5.25). Therefore, relation (A.2.1) can be reduced to

$$
\begin{aligned}
&\|\mathrm{sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|c)[(\nabla_x\Theta)^\natural]^{-1}-\mathbb{1}_3)\|^2 \qquad \text{(A.2.6)}\\
&\quad=\|\mathrm{sym}\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\|^2+\frac{(\mu_c-\mu)^2}{2(\mu_c+\mu)^2}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\|^2+\frac{\lambda^2}{(2\,\mu+\lambda)^2}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})^2+\frac{\mu_c-\mu}{(\mu_c+\mu)}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}n_0\|^2.
\end{aligned}
$$

Now we continue the calculations for the skew symmetric part,

$$
\begin{aligned}
&\|\mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|c^*)[(\nabla_x\Theta)^\natural]^{-1})\|^2\\
&\qquad= \|\mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1})\|^2 + \|\mathrm{skew}((0|0|\overline{Q}_e^{\natural,T}c^*)[(\nabla_x\Theta)^\natural]^{-1})\|^2\\
&\qquad\quad + 2\big\langle \mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}), \mathrm{skew}((0|0|\overline{Q}_e^{\natural,T}c^*)[(\nabla_x\Theta)^\natural]^{-1})\big\rangle.
\end{aligned}
\tag{A.2.7}
$$

In a similar manner, we calculate the terms separately. Since $n_0\otimes n_0$ is symmetric, we obtain

$$
\begin{aligned}
\|\mathrm{skew}((0|0|\overline{Q}_e^{\natural,T}c^*)[(\nabla_x\Theta)^\natural]^{-1})\|^2 &= \|\mathrm{skew}(n_0\otimes n_0 + \frac{\mu_c-\mu}{\mu_c+\mu}\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}\, n_0\otimes n_0 - \frac{\lambda}{2\mu+\lambda}\,\mathrm{tr}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural})\, n_0\otimes n_0)\|^2\\
&= \frac{(\mu_c-\mu)^2}{(\mu_c+\mu)^2}\|\mathrm{skew}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}\, n_0\otimes n_0)\|^2.
\end{aligned}
$$

But, we have

$$
\begin{aligned}
\|\mathrm{skew}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0)\|^2 &= \frac{1}{4}\Big\langle \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0, \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}\, n_0\otimes n_0\Big\rangle - \frac{1}{4}\Big\langle \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0, n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\Big\rangle\\
&\quad - \frac{1}{4}\Big\langle n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}, \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}\, n_0\otimes n_0\Big\rangle + \frac{1}{4}\Big\langle n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}, n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\Big\rangle\\
&= \frac{1}{2}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\|^2,
\end{aligned}
\tag{A.2.8}
$$

where we used the fact that $(n_0\otimes n_0)^2 = (n_0\otimes n_0)$. The difficulty in the skew symmetric part of (A.2.7) is solved in the following calculation

$$
\begin{aligned}
&2\big\langle \mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}), \mathrm{skew}((0|0|\overline{Q}_e^{\natural,T}c^*)[(\nabla_x\Theta)^\natural]^{-1})\big\rangle \qquad \text{(A.2.9)}\\
&\quad= 2\,\frac{(\mu_c-\mu)}{(\mu_c+\mu)}\Big\langle \mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}), \mathrm{skew}(\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0)\Big\rangle\\
&\quad= \frac{(\mu_c-\mu)}{2(\mu_c+\mu)}\langle \overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}, \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0\rangle\\
&\qquad - \frac{(\mu_c-\mu)}{2(\mu_c+\mu)}\langle \overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}, n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\rangle\\
&\qquad - \frac{(\mu_c-\mu)}{2(\mu_c+\mu)}\langle (\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1})^T, \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0\rangle\\
&\qquad + \frac{(\mu_c-\mu)}{2(\mu_c+\mu)}\langle (\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1})^T, n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}\rangle\\
&\quad= -\frac{(\mu_c-\mu)}{(\mu_c+\mu)}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\|^2.
\end{aligned}
$$

Therefore,

$$
2\big\langle\, \mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1}), \mathrm{skew}((0|0|\overline{Q}_e^{\natural,T}c^*)[(\nabla_x\Theta)^\natural]^{-1})\big\rangle = -\frac{(\mu_c-\mu)}{(\mu_c+\mu)}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\|^2\,, \tag{A.2.10}
$$

and we obtain

$$
\begin{aligned}
\|\mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|c^*)[(\nabla_x\Theta)^\natural]^{-1})\|^2 &= \|\mathrm{skew}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|0)[(\nabla_x\Theta)^\natural]^{-1})\|^2 + \frac{(\mu_c-\mu)^2}{2(\mu_c+\mu)^2}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\|^2\\
&\quad - \frac{(\mu_c-\mu)}{(\mu_c+\mu)}\|\mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\|^2.
\end{aligned}
\tag{A.2.11}
$$

The last requirement for our calculations, is

$$
\begin{aligned}
&\Big[\mathrm{tr}\Big(\mathrm{sym}(\overline{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|c^*)[(\nabla_x\Theta)^\natural]^{-1} - \mathbb{1}_3)\Big)\Big]^2\\
&= \Big(\mathrm{tr}\left(\mathrm{sym}((\overline{Q}_e^{\natural,T}\nabla_{(\eta_1,\eta_2)}\varphi^\natural - [\nabla y_0]^\natural|0)[(\nabla_x\Theta)^\natural]^{-1})\right) + \mathrm{tr}\left(\mathrm{sym}((0|0|\overline{Q}_e^{\natural,T}c^* - n_0)[(\nabla_x\Theta)^\natural]^{-1})\right)\Big)^2\\
&= \Big(\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}) + \frac{(\mu_c-\mu)}{2(\mu_c+\mu)}(\langle \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural} n_0\otimes n_0, \mathbb{1}_3\rangle + \langle n_0\otimes n_0\,\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}, \mathbb{1}_3\rangle) - \frac{\lambda}{2\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})\underbrace{\langle n_0\otimes n_0, \mathbb{1}_3\rangle}_{\langle n_0,n_0\rangle=1}\Big)^2\\
&= \Big(\frac{2\mu}{2\mu+\lambda}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}) + \frac{(\mu_c-\mu)}{2(\mu_c+\mu)}(\langle \mathcal{E}^T_{\varphi^\natural,\overline{Q}_e^\natural}, n_0\otimes n_0\rangle + \langle \mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural}, n_0\otimes n_0\rangle)\Big)^2 \qquad \text{(A.2.12)}\\
&= \frac{4\mu^2}{(2\mu+\lambda)^2}\,\mathrm{tr}(\mathcal{E}_{\varphi^\natural,\overline{Q}_e^\natural})^2.
\end{aligned}
$$

B. Appendix for Part II

B.1. Regular surfaces

By a *surface*, we mean a map $X\colon \Omega \to \mathbb{R}^3$, where Ω is a domain in $\mathbb{R}^2$. Any point of $\mathbb{R}^2$ is written like $x = (x_1, x_2)$. So, X maps any point $x \in \Omega$ onto some image point $X(x) \in \mathbb{R}^3$. The Jacobian matrix of X at point x is

$$\mathrm{D}X(x) = (\partial_{x_1} X(x), \partial_{x_2} X(x)) = \big(\frac{\partial X}{\partial x_1}(x), \frac{\partial X}{\partial x_2}(x)\big)\,. \tag{B.1.1}$$

The surface X is called *regular*, when its Jacobian matrix at each point $x \in \Omega$ has the maximal rank two.
The *tangent space* $T_x(X)$ corresponding to the parameter value x is defined by

$$T_x(X) := \mathrm{D}X(x)(\mathbb{R}^2)\,, \tag{B.1.2}$$

is the two-dimensional subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $\partial_{x_1} X(x)$ and $\partial_{x_2} X(x)$. We can say that the surface X is *regular*, if it has a well-defined tangent space at all of its points.
At a regular point x, the exterior product $\partial_{x_1} X(x) \times \partial_{x_2} X(x)$ can not vanish, i.e. $\|\partial_{x_1} X(x) \times \partial_{x_2} X(x)\| \neq 0$, where

$$\|\partial_{x_1} X(x) \times \partial_{x_2} X(x)\| = \sqrt{\|\partial_{x_1} X(x)\|^2 \|\partial_{x_2} X(x)\|^2 - \langle \partial_{x_1} X(x), \partial_{x_2} X(x)\rangle^2}\,. \tag{B.1.3}$$

The *normal vector to the surface* in a neighborhood of a regular point like x is

$$n_0 = \frac{\partial_{x_1} X \times \partial_{x_2} X}{\|\partial_{x_1} X \times \partial_{x_2} X\|}\,. \tag{B.1.4}$$

Since $\|n_0\| = 1$, we may view n_0 as a mapping of Ω into the sphere $S^2 \subset \mathbb{R}^3$

$$n_0\colon \Omega \to S^2\,. \tag{B.1.5}$$

This mapping is called *the normal map* or *Gauss map* of the surface X.

Definition B.1.1. For the surface $X\colon \Omega \to \mathbb{R}^3$, the *area* $A_\Omega(X)$ is defined by

$$A_\Omega(X) = \int_\Omega \|\partial_{x_1} X \times \partial_{x_2} X\|\, dx_1\, dx_2\,. \tag{B.1.6}$$

Definition B.1.2. Two mappings $X\colon \Omega \to \mathbb{R}^3$ and $\widetilde{X}\colon \widetilde{\Omega} \to \mathbb{R}^3$ of the class C^s, for $s \geq 1$, are named to be *equivalent*, if there is a C^s-diffeomorphism $\varphi\colon \widetilde{\Omega} \to \Omega$ such that

$$\widetilde{X} = X \circ \varphi\,. \tag{B.1.7}$$

If $J_\varphi = \det \mathrm{D}\varphi > 0$, then X and $\widetilde{X}$ are called *strictly equivalent.* If $J_\varphi < 0$, then they are called *oppositely equivalent.*

If n_0 and $\widetilde{n}_0$ denote the normal vectors of X and $\widetilde{X}$ respectively, one can see that the normal vectors of two strictly equivalent surfaces are equivalent,

$$\widetilde{n}_0 = n_0 \circ \varphi\,. \tag{B.1.8}$$

Definition B.1.3. The linear mapping $S(x)\colon T_x(X) \to T_x(X)$ of the tangent space $T_x(X)$ into itself is called *Weingarten map* such that for any tangent vector like V we have

$$S(V) = \pm \nabla_V n_0\,. \tag{B.1.9}$$

The map S is called *selfadjoint linear mapping* on the tangent space $T_x(X)$ if the following relation holds

$$\langle SV, W\rangle = \langle V, SW\rangle\,, \tag{B.1.10}$$

for arbitrary tangent vectors V, W. Therefore, we may have the following definitions

Definition B.1.4. Assume V and W are two tangent vectors and S is the Weingarten map. The bilinear forms

$$\mathrm{I}(V,W) := \langle V, W\rangle\,, \quad \mathrm{II}(V,W) := \langle SV, W\rangle\,, \quad \mathrm{III}(V,W) = \langle SV, SW\rangle\,, \tag{B.1.11}$$

are called respectively, the *first*, *second* and *third fundamental form* of the surface X at point x. In the case $V = W$, we have

$$\mathrm{I}(V) = \|V\|^2\,, \quad \mathrm{II}(V) = \langle SV, V\rangle\,, \quad \mathrm{III}(V) = \|SV\|^2\,. \tag{B.1.12}$$

Assume that $y_0 \in C^1(\omega, \mathbb{R}^3)$ is a surface. The first fundamental form of y_0, I_{y_0}, on $y_0(\omega)$ in the matrix representation is

$$\mathrm{I}_{y_0} := (\mathrm{D}y_0)^T\,\mathrm{D}y_0 = \begin{pmatrix} \|\partial_1 y_0\|^2 & \langle \partial_1 y_0, \partial_2 y_0\rangle \\ \langle \partial_1 y_0, \partial_2 y_0\rangle & \|\partial_2 y_0\|^2 \end{pmatrix} \in \mathrm{Sym}^+(2)\,. \tag{B.1.13}$$

Also, the following matrix represents the second fundamental form of y_0

$$\mathrm{II}_{y_0} := -(\mathrm{D}y_0)^T\,\mathrm{D}n_0 = -\begin{pmatrix} \langle \partial_1 y_0, \partial_1 n_0\rangle & \langle \partial_1 y_0, \partial_2 n_0\rangle \\ \langle \partial_2 y_0, \partial_1 n_0\rangle & \langle \partial_2 y_0, \partial_2 n_0\rangle \end{pmatrix}\,, \tag{B.1.14}$$

for normal vector n_0.

Remark 12. *Assume that X and $\widetilde{X}$ are two strictly equivalent surface with the corresponding diffeomorphism $\varphi\colon \Omega \to \widetilde{\Omega}$ with $J_\varphi > 0$; also assume that S and $\widetilde{S}$ are the Weingarten maps for X and $\widetilde{X}$ respectively. Then, S and $\widetilde{S}$ are also strictly equivalent and*

$$\widetilde{S} = S \circ \varphi\,. \tag{B.1.15}$$

In the case that $J_\varphi < 0$, then

$$\widetilde{S} = -S \circ \varphi\,. \tag{B.1.16}$$

B.2. Principal curvatures and fundamental forms

Assume that V is a tangent vector and X is the surface with point x on it. In [67], after some calculations it is shown that the normal curvature of the curve, κ_n, satisfies

$$\kappa_n := \frac{\mathrm{II}(V)}{\mathrm{I}(V)}\,, \tag{B.2.1}$$

where $\mathrm{I}(V), \mathrm{II}(V)$ are the values of the first and second fundamental forms of X at x. In a special case that the tangent vector V is a velocity vector of a normal section of an embedded surface X, then the principal normal and the surface normal are collinear which means

$$\kappa := \pm\frac{\mathrm{II}(V)}{\mathrm{I}(V)}\,. \tag{B.2.2}$$

where κ is the mean curvature of the surface X. This quotient measures the *curvatures* κ of all possible normal sections of the surface X at the parameter point x. The sign shows that whether the principal normal of the curve is in the same direction of the normal of the curve or in the opposite direction. We call

$$\begin{aligned}\kappa_1 &:= \min\{\frac{\mathrm{II}(V)}{\mathrm{I}(V)}\,\big|\, V \in T_x(X)\,,\, V \neq 0\} \\ &= \min\{\mathrm{II}(V)\,\big|\, V \in T_x(X)\,, \mathrm{I}(V) = 1\}\,,\end{aligned} \tag{B.2.3}$$

and

$$\begin{aligned}\kappa_2 &:= \max\{\frac{\mathrm{II}(V)}{\mathrm{I}(V)}\,\big|\, V \in T_x(X)\,,\, V \neq 0\} \\ &= \max\{\mathrm{II}(V)\,\big|\, V \in T_x(X)\,, \mathrm{I}(V) = 1\}\,,\end{aligned} \tag{B.2.4}$$

the *principal curvatures of the surface* X at x. These two principal curvatures are dependent on the point x. The *mean curvature* of the surface X at point x is

$$\mathrm{H}(x) := \frac{1}{2}(\kappa_1 + \kappa_2)\,, \tag{B.2.5}$$

and the *Gauss curvature* is

$$\mathrm{K}(x) = \kappa_1\kappa_2\,. \tag{B.2.6}$$

One can see that κ_1, κ_2 are the eigenvalues of the Weinggarten map S. That is for any tangent vector $V \in T_x(X)$ we have

$$SV = \kappa_i V\,, \quad \forall V \in T_x(X)\,. \tag{B.2.7}$$

We may choose the tangent vectors V_1, V_2 in a way that

$$SV_i = \kappa_i V_i\,, \qquad \|V_1\| = \|V_2\| = 1\,, \qquad \langle V_1, V_2\rangle = 0\,. \tag{B.2.8}$$

It means that κ_1 and κ_2 are the roots of the characteristic polynomial

$$P(\kappa) = \det(S - \kappa \mathbb{1})\,, \tag{B.2.9}$$

and therefore,

$$P(\kappa) = (\kappa - \kappa_1)(\kappa - \kappa_2) = \kappa^2 - 2\mathrm{H}\kappa + \mathrm{K}\,. \tag{B.2.10}$$

By applying the relation (0.2.25), we obtain

$$S^2 - 2\mathrm{H}S + \mathrm{K}\mathbb{1} = 0\,, \tag{B.2.11}$$

or by using the fundamental forms,

$$\mathrm{K\,I} - 2\mathrm{H\,II} + \mathrm{III} = 0\,. \tag{B.2.12}$$

Regarding to the sign of the Gauss curvature, we may have different definitions for a point in $X(x)$. If $\mathrm{K}(x) > 0$, the point $X(x)$ is *elliptic point.* If $\mathrm{K}(x) < 0$ or $\mathrm{K}(x) = 0$, then $X(x)$ is called *hyperbolic point* or *parabolic point* respectively.

Remark 13. *Assume that X and $\widetilde{X}$ are two strictly equivalent surfaces with the corresponding diffeomorphism $\varphi\colon \widetilde{\Omega} \to \Omega$ with $J_\varphi > 0$. Then, after holding the relation (B.1.8), we may have the following properties*

$$\tilde{\kappa}_1 = \kappa_1 \circ \varphi\,, \quad \tilde{\kappa}_2 = \kappa_2 \circ \varphi\,, \quad \widetilde{\mathrm{H}} = \mathrm{H} \circ \varphi\,, \quad \widetilde{\mathrm{K}} = \mathrm{K} \circ \varphi\,. \tag{B.2.13}$$

If $J_\varphi < 0$, then

$$\widetilde{N} = -N \circ \varphi\,, \qquad \widetilde{\mathrm{H}} = -\mathrm{H} \circ \varphi \qquad \text{but} \qquad \widetilde{\mathrm{K}} = \mathrm{K} \circ \varphi\,, \tag{B.2.14}$$

it means that the sign of K has an intrinsic geometrical meaning.

B.3. Minimal surfaces

The theory of *minimal surfaces* is classical and important in many branches of mathematics including the calculus of variations, partial differential equations and geometric measures theory. As mathematical examples of minimal surfaces, we may mention planes, helicoids and catenoids. In the nature, the shape of a soap film is approximately a minimal surface. Geometry of minimal surfaces is useful to solve many concepts in mathematics and in many other fields.

Definition B.3.1. [67] A regular surface $X\colon \Omega \to \mathbb{R}^3$ of class C^2 is called a *minimal surface*, if its mean curvature function H satisfies

$$\mathrm{H} = 0\,. \tag{B.3.1}$$

The mean curvature can be defined with $\mathrm{H} = \frac{k_1 + k_2}{2}$, where k_1 and k_2 are the principal curvatures. More details for mean curvature will be given in the next sections.
In particular case, we have the first variation of the area function on Ω at X like

$$\partial A_\Omega(X, Y) = -2 \int_X \langle Y, n_0\rangle \mathrm{H}\, dA\,, \quad Y \in C^\infty(\Omega, \mathbb{R}^3)\,, \tag{B.3.2}$$

The product $\langle Y, n_0\rangle$ can be seen as an arbitrary infinitely differentiable function. Therefore, we will have the following theorem

Theorem B.3.2. *The first variation $\partial A_\Omega(X, Y)$ of A_Ω at X vanishes for all vector fields $Y \in C^\infty(\Omega, \mathbb{R}^3)$* ***if and only if*** *the mean curvature* H *of X is identically zero.*

A combination between the definition of minimal surface and the relation (B.3.2), can be the following proposition

Proposition 4. *[67] If $X : \overline{\Omega} \to \mathbb{R}^3$ is a minimal surface, then the equation*

$$\partial A_\Omega(X, Y) = 0\,, \tag{B.3.3}$$

holds for all $Y \in C^1(\overline{\Omega}, \mathbb{R}^3)$ which are orthogonal to the side normal of the boundary ∂X.

Now we consider another case of surfaces. Assume that the surface X is given in *nonparametric form*, that is, as graph of a function $z = z(x, y)$. Such a surface can be described by the special parameter representation

$$X(x, y) = (x, y, z(x, y))\,, \quad (x, y) \in \Omega\,. \tag{B.3.4}$$

The relation

$$\operatorname{div}\Big(\frac{\nabla z}{\sqrt{1 + |\nabla z|^2}}\Big) = 0\,, \tag{B.3.5}$$

is called the *minimal surface equation in divergence form.*

Proposition 5. *The surface $X : \overline{\Omega} \to \mathbb{R}$ is a minimal surface if and only if $z = z(x, y)$ holds in the relation (B.3.5.)*

There are some properties related to minimal surfaces and some operators, like the following theorem

Theorem B.3.3. *Assume that $X : \Omega \to \mathbb{R}^3$ is a regular surface of class C^2 with mean curvature H and the spherical map $n_0 : \Omega \to \mathbb{R}^3$. Then*

$$\Delta_X X = 2\mathrm{H}\, n_0\,, \tag{B.3.6}$$

where $\Delta_X X$ is the Laplace-Beltrami operator on the surface X.

As a combination of this theorem and the definition of the minimal surfaces, we will have the following corollary

Corollary B.3.4. *A regular C^2-surface X is a minimal surface if and only if the following relation holds*

$$\Delta_X X = 0\,. \tag{B.3.7}$$

Assume that $N : X \subset \mathbb{R}^3 \to S^2$ is the Gauss map of the minimal surface X and Δ_X and Δ_N are the related Beltrami operators respectively. Then we have the following proposition

Proposition 6. *If N is the Gauss map of a minimal surface X, then*

$$\Delta_X = |\mathrm{K}|\Delta_N\,, \tag{B.3.8}$$

where K is the Gauss curvature and holds in relation

$$\mathrm{K}\,\mathrm{I} - 2\mathrm{H}\,\mathrm{II} + \mathrm{III} = 0\,. \tag{B.3.9}$$

Now we apply the definition of the normal vector, relation (B.1.4), in the above theorem. Hence,

Theorem B.3.5. *Let $X(x, y)$ be a regular surface of class $C^2(\Omega, \mathbb{R}^3)$ which is given by conformal parameters x and y, that is,*

$$|\partial_x X|^2 = |\partial_y X|^2\,, \quad \langle \partial_x X, \partial_y X\rangle = 0\,. \tag{B.3.10}$$

For the real valued function $\mathrm{H}(x, y)$ which represents the mean curvature of the surface X, the Rellich's equation holds if and only if

$$\Delta X = 2\mathrm{H}\,\partial_x X \times \partial_y X\,, \qquad (x, y) \in \Omega\,. \tag{B.3.11}$$

In particular, $X : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if

$$\Delta X = 0\,. \tag{B.3.12}$$

B.4. Conformal surfaces and properties

Definition B.4.1. Assume that

$$P\colon \Omega \to \mathbb{R}^{2\times 2}\,, \quad Q\colon \Omega \to \mathbb{R}^{2\times 2}\,, \tag{B.4.1}$$

are two matrix valued functions, where $\Omega \subset \mathbb{R}^2$. We say that P and Q are *conformal to each other*, if there exists a function $\mu\colon \Omega \to \mathbb{R}$ with $\mu(x) > 0$ on Ω such that

$$P(x) = \mu(x)Q(x)\,, \quad \forall x \in \Omega\,. \tag{B.4.2}$$

Regarding to this definition, we may have the definition of the *conformal surfaces* like

Remark 14. *We call two surfaces X and Y are conformal to each other if their matrix functions, respectively, are conformal to each other.*

Consequently, we can obtain that the first fundamental forms I_X and I_Y of two conformal surfaces are also conformal to each other. Therefore, for the function $\mu\colon \Omega \to \mathbb{R}$ with $\mu(x) > 0$, for every $x \in \Omega$, we have

$$\mathrm{I}_X(U) = \mu(x)\mathrm{I}_Y(V)\,, \tag{B.4.3}$$

where $U \in T_x(X)$ and $V \in T_x(Y)$.
Assume that N is the normal field of the surface $X\colon \Omega \to \mathbb{R}^3$. It is proved in [p. 36][67] that,

Remark 15. *A zero mean curvature surface X without focal points is conformal to its spherical image N.*

Now maybe we can give a more complete definition of conformal surfaces

Definition B.4.2. Two regular C^1-surfaces $X\colon \Omega \to \mathbb{R}^3$ and $Y\colon \Omega^* \to \mathbb{R}^3$ are called *conformally equivalent* if there exists a C^1-diffeomorphism $\tau\colon \Omega \to \Omega^*$ such that the surfaces $X\colon \Omega \to \mathbb{R}^3$ and $Y \circ \tau\colon \Omega \to \Omega^*$ are conformal to each other. And, the mapping τ is called the *conformal map* of X into Y.