Continuous group cohomology and Ext-groups

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Abstract

We prove that the continuous group cohomology groups of a locally profinite group G with coefficients in a smooth k-representation π of G are isomorphic to the Ext-groups $\mathrm{Ext}^i_G(\mathbb{1}, \pi)$ computed in the category of smooth k-representations of G. We apply this to show that if π is a supersingular $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the continuous group cohomology of $\mathrm{SL}_2(\mathbb{Q}_p)$ with values in π vanishes.

Furthermore, we prove that the continuous group cohomology groups of a p-adic reductive group G, with coefficients in an admissible smooth representation of G over k are finitely generated over k and similarly, the continuous cohomology groups of G with coefficients in an admissible unitary $\mathbb{Q}_p$-Banach space representation Π, are finite dimensional. We show that the continuous group cohomology of $\mathrm{SL}_2(\mathbb{Q}_p)$ with values in non-ordinary irreducible $\mathbb{Q}_p$-Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ vanishes.

We then show that the continuous cohomology groups of a p-adic reductive group with coefficients in the locally analytic vectors of an admissible $\mathbb{Q}_p$-Banach space representation are homeomorphic to those with coefficients in the Banach space representation itself. Moreover, we deduce that the canonical topologies on those continuous cohomology groups are Hausdorff and are the uniquely determined finest locally convex topologies.

Zusammenfassung

Wir beweisen, dass die stetige Gruppenkohomologiegruppen einer lokal proendlichen Gruppe G, mit Koeffizienten in einer glatten k-Darstellung π von G, isomorph zu den Ext-Gruppen $\mathrm{Ext}^i_G(\mathbb{1}, \pi)$, berechnet in der Kategorie der glatten k-Darstellungen von G, sind. Wir benutzen dies, um zu zeigen, dass für eine supersinguläre $\overline{\mathbb{F}}_p$-Darstellung π von $\mathrm{GL}_2(\mathbb{Q}_p)$ die stetigen Kohomologiegruppen von $\mathrm{SL}_2(\mathbb{Q}_p)$ mit Koeffizienten in π verschwinden.

Des Weiteren zeigen wir, dass die stetigen Kohomologiegruppen einer p-adisch reduktiven Gruppe G mit Koeffizienten in einer zulässigen glatten Darstellung von G über k endlich erzeugt über k ist und, dass die stetigen Kohomologiegruppen von G mit Koeffizienten in einer zulässigen unitären $\mathbb{Q}_p$-Banachraumdarstellung Π endlich dimensional sind. Wir zeigen, dass die stetige Gruppenkohomologie von $\mathrm{SL}_2(\mathbb{Q}_p)$ mit Werten in nicht-ordinären irreduziblen $\mathbb{Q}_p$-Banachraumdarstellungen von $\mathrm{GL}_2(\mathbb{Q}_p)$ verschwindet.

Wir zeigen außerdem, dass die stetigen Kohomologiegruppen einer p-adischen reduktiven Gruppe mit Koeffizienten in den lokalanalytischen Vektoren einer zulässigen $\mathbb{Q}_p$-Banachraumdarstellung homöomorph sind zu denen mit Koeffizienten in der Banachraumdarstellung selbst. Außerdem folgern wir daraus, dass die kanonischen Topologien auf diesen stetigen Kohomologiegruppen Hausdorffsch und somit die eindeutig bestimmten feinsten lokalkonvexen Topologien sind.

Contents

1 Introduction

Let k be a commutative ring with 1 and let G be a locally profinite group in the sense of [2], i.e. a topological group which has a fundamental system of neighborhoods of the unit element consisting of compact open subgroups.

A smooth k-representation of G is a $k[G]$-module π, such that the stabilizer $\mathrm{Stab}_G(v)$ of any element $v \in \pi$ is open in G. Denote by $\mathrm{Mod}_G^{\mathrm{sm}}(k)$ the category of all smooth k-representations of G. We prove the following:

Theorem 1.1 (Corollary 4.4)**.** *For any $\pi \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$, and any $i \geqslant 0$, we have natural isomorphisms:*

$$\mathrm{Ext}_G^i(\mathbb{1}, \pi) \cong H^i(G, \pi), \tag{1}$$

where $H^i(G, \pi)$ is the continuous group cohomology group of G with coefficients in π and the Ext-group is computed in the category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$.

The category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$ is an abelian category which moreover has enough injectives by Proposition 2.1.1 in [8]. Therefore, the Ext-groups in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$ can be defined by the right-derived functors of Hom. We prove Theorem 1.1 by showing that applying the functor of smooth vectors to the resolution of π used to compute the continuous group cohomology gives a resolution of π in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$. This question was raised by Emerton in [8, Section 2.2] and it is known in the case of compact groups ([8, Proposition 2.2.6]).

In Section 5, we apply our result to the group $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $|{-}|$ be a norm on $\mathbb{Q}_p$, normalized so that $|p| = 1/p$. Then we can define a character $\varepsilon : \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$, by $x \mapsto x|x|$. Using Theorem 1.1, we prove the following:

Theorem 1.2 (Corollary 5.2)**.** *Let k be a finite field of characteristic p and let $\pi \in \mathrm{Mod}_{\mathrm{GL}_2(\mathbb{Q}_p)}^{\mathrm{sm}}(k)$ be absolutely irreducible and not isomorphic to a twist by a character of*

$\mathbb{1}$, Sp or $\mathrm{Ind}_B^G\,\alpha$, where $\alpha : B \to k^\times$ is defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \varepsilon(ad^{-1}) \bmod p$. Then

$$H^i(\mathrm{SL}_2(\mathbb{Q}_p), \pi) = 0,$$

for $i \geqslant 0$.

This includes for example all supersingular representations $\pi \in \mathrm{Mod}^{\mathrm{sm}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(k)$. The proof makes use of calculations of Ext-groups due to Paškūnas ([14]). This result is used by Colmez, Dospinescu and Nizioł in their preprint [5].

In Section 6, we study the continuous group cohomology groups of a p-adic reductive group with coefficients in an admissible unitary L-Banach space representation, where $L/\mathbb{Q}_p$ is a finite extension. If Π is an admissible unitary L-Banach space representation of G and Π^0 is an open bounded G-invariant lattice in Π, then we show that (Proposition 6.7) for all $i \geqslant 0$ one has isomorphisms

$$H^i(G, \Pi) \cong (\varprojlim_n H^i(G, \Pi^0/\varpi^n\Pi^0))[1/\varpi],$$

for ϖ a uniformizer of L. The proof uses the Bruhat–Tits building of G to obtain a resolution of the trivial representation of G by compactly induced representations from compact-mod-center subgroups of G. Such resolutions appear in the work of Schneider–Stuhler [21] and Casselman–Wigner [4]. We use it to prove

Theorem 1.3 (Lemma 6.4)**.** *Let k be a commutative ring with 1 and let $\pi \in \mathrm{Mod}^{\mathrm{adm}}_G(k)$ be a smooth admissible representation of G over k. Then for every $i \geqslant 0$, the k-module $H^i(G, \pi)$ is finitely generated.*

We then deduce a similar statement for admissible Banach space representations:

Theorem 1.4 (Corollary 6.8)**.** *Let Π be an admissible unitary L-Banach space representation of a p-adic reductive group G, then $H^i(G, \Pi)$ is finite dimensional over L, for all $i \geqslant 0$.*

In the case where G is a compact p-adic analytic group, these isomorphisms follow directly from [6].

As in Section 5, we apply these results to the group $\mathrm{GL}_2(\mathbb{Q}_p)$ and obtain a similar statement for Banach space representations:

Theorem 1.5 (Proposition 6.13)**.** *Any absolutely irreducible admissible unitary L-Banach space representation Π of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is not isomorphic to a twist by a unitary character of $\mathbb{1}$, $\widehat{\mathrm{Sp}}$ or $\mathrm{Ind}_B^G\,\tilde{\alpha}$, has trivial continuous group cohomology groups over $\mathrm{SL}_2(\mathbb{Q}_p)$, i.e.*

$$H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi) = 0,$$

for $i \geqslant 0$.

Here, $\tilde{\alpha} : B \to L^\times$ is the representation of B, defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \varepsilon(ad^{-1})$.

Finally, in Section 7, we construct for any continuous representation of a p-adic reductive group G on an L-vector space V, a spectral sequence converging to the continuous cohomology of G with coefficients in V, whose first page terms are certain continuous cohomology groups of stabilizers of facets in the Bruhat–Tits building of G. This allows us to prove the following theorem which compares the continuous cohomology groups for admissible Banach space representations with the ones of their subrepresentation of locally analytic vectors.

Theorem 1.6 (Proposition 7.3). *Let $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$ be an admissible L-Banach space representation of G and let Π^{la} be the subspace of locally analytic vectors in Π, equipped with its natural topology. The inclusion $\Pi^{\mathrm{la}} \hookrightarrow \Pi$ induces natural isomorphisms*

$$H^i(G, \Pi^{\mathrm{la}}) \cong H^i(G, \Pi) \ \forall i \geqslant 0.$$

The result has been proved in [18, Corollary 1.6] in the case where G is a compact group. We use this as key ingredient for our proof of Theorem 7.3.

Theorems 1.6 and 1.4 imply the following corollary.

Corollary 1.7 (Corollary 7.8). *For every $\Pi \in \mathrm{Ban}_G^{\mathrm{adm}}(L)$, the continuous cohomology groups $H^i(G, \Pi^{\mathrm{la}})$ are finite dimensional over L for all $i \geqslant 0$.*

For a topological G-module V, the spaces of continuous i-cochains $C^i(G, V)$ are equipped with the compact-open topology. We thus get a quotient topology on the continuous cohomology groups $H^i(G, V)$, called the canonical topology. Moreover, if the G-module V is a locally convex L-vector space, then we show that the compact-open topology on $C^i(G, V)$, as well as the canonical topology on $H^i(G, V)$ is locally convex. The canonical topology is in general not Hausdorff. However, Theorems 1.4 and 1.6 allow us to apply a criterion for the cohomology groups being Hausdorff in [4] to deduce the following.

Corollary 1.8 (Corollary 7.9). *Suppose that G is a p-adic reductive group and Π an admissible L-Banach space representation of G. Then for all $i \geqslant 0$, the canonical topologies on the continuous cohomology groups $H^i(G, \Pi)$ and $H^i(G, \Pi^{\mathrm{la}})$ are Hausdorff.*

We use Corollary 1.8 to deduce that the isomorphisms in Theorem 1.6 are in fact homeomorphisms.

Corollary 1.9 (Corollary 7.13). *The canonical topology on the continuous cohomology groups $H^i(G, \Pi^{\mathrm{la}})$ and $H^i(G, \Pi)$ is the finest locally convex topology. In particular, for all $i \geqslant 0$, we have homeomorphisms*

$$H^i(G, \Pi^{\mathrm{la}}) \cong H^i(G, \Pi).$$

Remark 1.10. The finest locally convex topology on the finite dimensional cohomology groups $H^i(G, \Pi)$ and $H^i(G, \Pi^{\mathrm{la}})$ can be described more explicitly: Let V be a finite dimensional L-vector space. Then there is only one way to equip V with a topology making it a locally convex Hausdorff L-vector space, i. e. the finest locally convex topology. Namely, it is the topology defined by the norm $\| \sum_{i=1}^n \lambda_i e_i \| := \max_{1 \leqslant i \leqslant n} |\lambda_i|$, where $e_1, \ldots, e_n$ is an L-basis of V (Proposition 4.13 [19]).

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2 Continuous group cohomology

Following [4], we define the continuous group cohomology as follows: Let V be a topological G-module, i.e. a topological abelian group V with a G-action such that G acts on V via group automorphisms and the map $G \times V \to V$ is continuous. Then we can define the cochain complex

$$C^n(G,V) := C(G^{n+1},V) := \{f : G^{n+1} \to V \text{ continuous}\},$$

with differentials $d^n : C^n(G,V) \to C^{n+1}(G,V)$, defined by

$$d^n f(g_0,\ldots,g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0,\ldots,\widehat{g_i},\ldots,g_{n+1}).$$

We endow the spaces $C^n(G,V)$ with the compact-open topology. More explicitly, a basis of the compact-open topology is given by finite intersections of sets of the form $\Omega(K,U) = \{f \in C^n(G,V) |\ f(K) \subset U\}$, for compact subsets $K \subset G^{n+1}$ and open subsets $U \subset V$. By defining a G-action via $(gf)(g_0,\ldots,g_n) = gf(g^{-1}g_0,\ldots,g^{-1}g_n)$, for $g,g_0,\ldots,g_n \in G$ and $f \in C^n(G,V)$, these will define topological G-modules. Furthermore, there is a continuous G-equivariant injection $V \hookrightarrow C(G,V)$, defined by $v \mapsto [g \mapsto v]$.

For a closed subgroup $H \leqslant G$ and a topological G-module V, define the *induced representation* to be

$$\operatorname{Ind}_H^G V := \{f \in C(G,V) \mid f(hg) = hf(g)\ \forall h \in H,\ \forall g \in G\},$$

with the subspace topology induced from $\operatorname{Ind}_H^G V \subseteq C(G,V)$ and G-action $gf(g') := f(g'g)$, for $g,g' \in G$. Similarly, the *compact induction* is defined as

$$\text{c-}\operatorname{Ind}_H^G V := \{f \in \operatorname{Ind}_H^G V \mid \text{ the support of } f \text{ is compact modulo } H\}$$

with the same G-action.

Lemma 2.1. *For a topological G-module V, there are homeomorphisms of G-modules*

$$C^{n+1}(G,V) \cong C^0(G,C^n(G,V)) \text{ and } C^0(G,V) \cong \operatorname{Ind}_1^G V,$$

for all $n \geqslant 0$.

Proof. By [3, X.3.4 Corollaire 2], the map

$$\begin{aligned} \phi : C^{n+1}(G,V) &\to C^0(G,C^n(G,V)) \\ f &\mapsto \phi(f), \end{aligned}$$

where $\phi(f)(g_0)(g_1,\ldots,g_n) = f(g_0,\ldots,g_n)$, is a homeomorphism. It is straightforward to check that this is G-equivariant, making it an isomorphism of topological G-modules.

The assignment $f \mapsto [g \mapsto gf(g^{-1})]$ defines both maps $C^0(G,V) \to \operatorname{Ind}_1^G V$ and its inverse. One can check that both maps are continuous and G-equivariant, hence the modules are homeomorphic. □

Lemma 2.2. *For two topological G-modules V and W, one has the following isomorphism:*

$$\operatorname{Hom}_G^{\text{cts}}(V,\operatorname{Ind}_1^G W) \cong \operatorname{Hom}^{\text{cts}}(V,W),$$

where $\operatorname{Hom}^{\text{cts}}(V,W)$ are continuous group homomorphisms and on the left hand side, we take continuous G-equivariant group homomorphisms.

Proof. [4, Lemma 2]. □

Lemma 2.3. *The complex* $0 \to V \to C^\bullet(G,V)$ *is an exact complex of* G*-modules.*

Proof. Let $C^\bullet$ be the complex with $C^{-1} = V$, $C^i = 0$, for $i \leqslant -2$ and $C^i = C^i(G,V)$ for $i \geqslant 0$. To prove the exactness of this complex, we construct a cochain homotopy $s^n : C^n \to C^{n-1}$, $n \in \mathbb{Z}$ between $id_{C^\bullet}$ and the zero map on $C^\bullet$. Define $(s^n f)(g_1, \ldots, g_n) := f(1, g_1, \ldots, g_n)$ for $f \in C^n(G,V)$, $n \geqslant 0$ and $s^n = 0$ for $n \leqslant -1$. Then $s^n f : G^n \to V$ is a continuous map, since it is the composition of the continuous maps f and $\{1\} \times G^n \hookrightarrow G \times G^n$. Moreover, one can easily check that it satisfies $s^{n+1}d^n + d^{n-1}s^n = id_{C^n}$ for all n. Hence we have found the desired homotopy and the complex is exact. □

Definition 2.4. The i^{th} *continuous group cohomology group* of G with coefficients in V is defined to be $H^i(G,V) := H^i(C^\bullet(G,V)^G)$.

A homomorphism of topological G-modules $\phi : V \to W$ is said to be a *strong morphism*, if the induced morphisms $\operatorname{Ker}\phi \to V$ and $V/\operatorname{Ker}\phi \to W$ each have a continuous left inverse in the category of topological abelian groups. And if one has a short exact sequence of topological G-modules

$$0 \to V \to W \to U \to 0,$$

in which all the morphisms are strong, then the induced sequence

$$0 \to C^n(G,V) \to C^n(G,W) \to C^n(G,U) \to 0$$

is again strong and then induces a long exact sequence in cohomology.

For example, any short exact sequence

$$0 \to V \to W \to U \to 0,$$

of topological G-modules is automatically strong if U is discrete. Therefore it induces a long exact sequence in cohomology.

Definition 2.5. We say that a topological G-module V is *acyclic* (for the continuous group cohomology), if

$$H^i(G,V) = \begin{cases} V^G, & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.6.

1. The G-modules $C^n(G,V)$ are acyclic for the continuous group cohomology. Hence, the complex $0 \to V \to C^\bullet(G,V)$ is an acyclic resolution of V. (cf. [4, p. 201].)

2. A topological G-module V is said to be *continuously injective*, if for every strong G-injection $U \hookrightarrow W$ and for every morphism of topological G-modules $\phi : U \to V$, the morphism ϕ extends to a morphism from W to V. An example of continuously injective G-modules are the modules $C^n(G,V)$ (cf. [4, p. 201]).

3. If we equip a smooth representation $\pi \in \operatorname{Mod}_G^{\text{sm}}(k)$ with the discrete topology, this will give a topological G-module and we can define the continuous group cohomology groups $H^i(G,\pi)$ with coefficients in π.

Lemma 2.7. *Let* V *be a topological* G*-module which also has the structure of a topological* R*-module for some topological ring* R*, such that the* G*-action is* R*-linear. Assume that an element* $z \in Z(G)$ *in the center of* G *acts on* V *by multiplication with a scalar* $\lambda \in R$*. If* $1 - \lambda$ *is a unit in* R*, then* $H^i(G,V) = 0$ *for all* $i \geqslant 0$*.*

Proof. We consider the continuous R-linear G-module homomorphism

$$\mathrm{m}_z : V \to V, \; v \mapsto zv.$$

We can extend it to a map of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & C^0(G,V) & \longrightarrow & C^1(G,V) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \mathrm{m}_z} & & \downarrow{\scriptstyle \mathrm{m}_z} & & \downarrow{\scriptstyle \mathrm{m}_z} & & \\ 0 & \longrightarrow & V & \longrightarrow & C^0(G,V) & \longrightarrow & C^1(G,V) & \longrightarrow & \dots \end{array}$$

where m_z is the multiplication by z. Since z is central, m_z is G-equivariant.

By assumption, z acts on V by multiplication with the scalar λ, hence we can also construct the following map of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & C^0(G,V) & \longrightarrow & C^1(G,V) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & V & \longrightarrow & C^0(G,V) & \longrightarrow & C^1(G,V) & \longrightarrow & \dots \end{array}$$

where the vertical arrows are given by multipication with λ.

Therefore, we have two continuous, G-equivariant endomorphisms of the complex $0 \to V \to C^\bullet(G,V)$ extending the map m_z. But this complex is a strong resolution of V by continuously injective G-modules and by [11, Section 2], the map of m_z extends uniquely up to homotopy. Hence, the map of complexes $\mathrm{m}_z - \lambda$ is null-homotopic and induces the zero map on the cohomology groups $H^i(G,V)$ for all $i \geqslant 0$. But passing to G-invariants, the map $\mathrm{m}_z - \lambda$ is just $id_{C^\bullet(G,V)} - \lambda$. In conclusion, multiplication with $1 - \lambda$ is the zero map on $H^i(G,V)$, so that the continuous group cohomology groups $H^i(G,V)$ must be zero if $1 - \lambda$ is a unit in R. □

3 Proof for profinite groups

Proposition 3.3 below can already be found in Section 2.2 of [8]. To make the article as self-contained as possible we decided to include a proof.

First, we consider the case of a compact locally profinite group K, i.e. a profinite group, and a smooth representation $\pi \in \mathrm{Mod}_K^{\mathrm{sm}}(k)$ of K equipped with the discrete topology, so that we can define $H^i(K,\pi)$ as above. Moreover, we denote by $\mathbb{1} \in \mathrm{Mod}_K^{\mathrm{sm}}(k)$ the free k-module of rank 1 with trivial K-action and $\mathrm{Ext}_K^i(\mathbb{1},\pi)$ the Ext-group computed in $\mathrm{Mod}_K^{\mathrm{sm}}(k)$, i.e. the i^{th} right derived functor of $\mathrm{Hom}_K(\mathbb{1},-)$ applied to π.
The following lemmas will be used in several arguments throughout the paper:

Lemma 3.1 ([25, Proposition 2.3.10])**.** *Let $R : \mathcal{D} \to \mathcal{C}$ be an additive functor and suppose that it is right adjoint to an exact functor $L : \mathcal{C} \to \mathcal{D}$. Then the functor R preserves injective objects.*

Lemma 3.2. *Let $L : \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories and suppose that L admits an exact right adjoint $R : \mathcal{D} \to \mathcal{C}$. Assume that $\mathcal{D}$ has enough injectives. Then there are natural (in C and D) isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{D}}(L(C), D) \cong \mathrm{Ext}^i_{\mathcal{C}}(C, R(D)), \; \textit{for all } i \geqslant 0.$$

Proof. By assumption, $\mathcal{D}$ has enough injectives, so that we can find an injective resolution $D \hookrightarrow I^\bullet$ of D in $\mathcal{D}$. Moreover, since both functors are exact and R is right adjoint to L, applying R gives an injective resolution $R(D) \hookrightarrow R(I^\bullet)$ of $R(D)$ in $\mathcal{C}$. We get

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{D}}(L(C), D) &= H^i(\mathrm{Hom}_{\mathcal{D}}(L(C), I^\bullet)) \\ &\cong H^i(\mathrm{Hom}_{\mathcal{C}}(C, R(I^\bullet))) \\ &= \mathrm{Ext}^i_{\mathcal{C}}(C, R(D)).\end{aligned}$$

□

Proposition 3.3. *One has isomorphisms*

$$\mathrm{Ext}^i_K(\mathbb{1}, \pi) \cong H^i(K, \pi),$$

for all $i \geqslant 0$.

We split the proof into the following lemmas.

For a topological K-module V, we define the space of smooth functions $C^{\mathrm{sm}}(K, V) \subseteq C(K, V)$ to be:

$$C^{\mathrm{sm}}(K, V) := \{f : K \to V \mid \exists H \leqslant K \text{ open, s.t. } hf = f \ \forall h \in H\}.$$

Lemma 3.4. *If* V *carries the discrete topology, then we have the following equality:*

$$C(K, V) = C^{\mathrm{sm}}(K, V).$$

Proof. The inclusion $C^{\mathrm{sm}}(K, V) \subseteq C(K, V)$ is trivial. Conversely, if $f : K \to V$ is a continuous function, it is locally constant and, since K is compact, the image of f consists of finitely many elements $v_1, \ldots, v_n \in V$. Moreover, for each $h \in K$, there is an open subgroup $U_h \leqslant K$, such that $U_h h \subseteq f^{-1}(v_i)$ for some i. We get

$$K = \bigcup_{h \in K} U_h h = \bigcup_{i=1}^{m} U_{h_i} h_i.$$

Thus, $U := \bigcap_i U_{h_i} \cap \mathrm{Stab}_K(f(h_i))$ is an open subgroup of K with the property that for any $u \in U$, $uf = f$. □

Lemma 3.5. *If the topology of* V *is discrete, the compact-open topology on* $C(K, V)$ *is discrete.*

Proof. Let $f : K \to V$ be an element in $C(K, V) = C^{\mathrm{sm}}(K, V)$, i.e. there exists a compact open subgroup $H \leqslant K$, such that $f(gh) = f(g)$ for all $h \in H$, $g \in K$. Since K is compact, K is the disjoint union of finitely many compact open cosets $g_i H$, $i = 1, \ldots, n$, on which f is constant. Then $\{f\} = \bigcap_{i=1}^{n} \Omega(g_i H, \{f(g_i)\})$ is open as it is the finite intersection of the open sets

$$\Omega(g_i H, \{f(g_i)\}) = \{f' \in C(K, V) \mid \ f'(g_i H) \subseteq \{f(g_i)\}\}$$

in $C(K, V)$. □

Lemma 3.6. *If $\pi \in \mathrm{Mod}_K^{\mathrm{sm}}(k)$ is a smooth representation of K equipped with the discrete topology, then $0 \to \pi \to C^\bullet(K,\pi)$ is an acyclic resolution of π in the category of smooth representations $\mathrm{Mod}_K^{\mathrm{sm}}(k)$.*

Remark 3.7. If we say that a topological K-module is acyclic, we mean that it is acyclic for the continuous group cohomology. Whereas acyclic in $\mathrm{Mod}_K^{\mathrm{sm}}(k)$ means acyclic with respect to $\mathrm{Ext}_K^i(\mathbb{1},-)$.

Proof. We know that the complex is exact and that $C^n(K,\pi)$ is a smooth representation by the previous two lemmas. The representations $C^n(K,\pi) \cong C(K, C^{n-1}(K,\pi))$ are acyclic in the category $\mathrm{Mod}_K^{\mathrm{sm}}(k)$: Indeed, since $V := C^{n-1}(K,\pi)$ is again discrete, by Frobenius reciprocity, one has $\mathrm{Hom}_K(\mathbb{1}, C(K,V)) \cong \mathrm{Hom}_k(k,V) \cong V$. Moreover, the functor $C(K,-)$ is exact and we can apply Lemma 3.2 to get $\mathrm{Ext}_K^i(\mathbb{1}, C(K,V)) \cong \mathrm{Ext}_k^i(k,V)$, which is zero for $i > 0$ and V for $i = 0$, as expected. ☐

Proof of Proposition 3.3. By Lemma 3.6, we can compute the groups $\mathrm{Ext}_K^i(\mathbb{1},\pi)$ by using the acyclic resolution

$$0 \to \pi \to C^\bullet(K,\pi).$$

More precisely, it is the i^{th} cohomology group of the complex

$$\mathrm{Hom}_K(\mathbb{1}, C^\bullet(K,\pi)) \cong C^\bullet(K,\pi)^K$$

and this is the same as the continuous group cohomology group $H^i(K,\pi)$. This proves the proposition. ☐

4 The general case

If G is locally compact, but not compact, then the modules $C^\bullet(G,\pi)$ will not lie in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$. For example, if $G = (\mathbb{Q}_p, +)$, then the indicator function on the set $\bigcup_{n\geqslant 1}(p^{-n} + p^n\mathbb{Z}_p)$ is locally constant and hence continuous. But it is not smooth, since its stabilizer consists of $x \in \mathbb{Q}_p$, such that $x \in p^n\mathbb{Z}_p$ for all $n \geqslant 1$, and hence is zero, which is not open.

To conclude in this case, we will use the following functor from the category $\mathrm{Mod}_G(k)$ of k-representations of G to the category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$

$$(-)^{\mathrm{sm}} : \mathrm{Mod}_G(k) \to \mathrm{Mod}_G^{\mathrm{sm}}(k),$$
$$V \mapsto V^{\mathrm{sm}} := \varinjlim_K V^K,$$

where the direct limit is taken over the directed family of compact open subgroups $K \leqslant G$. Hence, V^{sm} can be seen as the subrepresentation of V consisting of the smooth vectors. Moreover, this functor is right-adjoint to the inclusion $\mathrm{Mod}_G^{\mathrm{sm}}(k) \hookrightarrow \mathrm{Mod}_G(k)$, since $\mathrm{Hom}_G(W,V) = \mathrm{Hom}_G(W,V^{\mathrm{sm}})$ for a smooth representation $W \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$. Using Lemma 3.1, this implies that the functor $(-)^{\mathrm{sm}}$ preserves injective objects.

Lemma 4.1. *For a compact open subgroup $K \leqslant G$ and a smooth representation $\pi \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$, the G-modules $C^n(G,\pi)$ are acyclic for the continuous group cohomology $H^\bullet(K,-)$.*

Moreover, the cohomology of the complex $C^\bullet(G,\pi)^K$ is $H^\bullet(K,\pi)$.

Proof. A proof is given in [4, Proposition 4 (a)]. The idea is as follows: since the quotient G/K is discrete, one has a continuous section $s : G/K \to G$ of the natural projection $G \to G/K$ with which one can define the K-invariant homeomorphism $K \times G/K \to G,\ (h, gK) \mapsto s(gK)h$. Therefore, one gets an isomorphism of K-modules

$$C(G, \pi) \cong C(K \times G/K, \pi) \cong C(K, C(G/K, \pi))$$

showing that $C(G, \pi)$ is acyclic as a K-module.

To conclude that the complex $C^\bullet(G, \pi)^K$ computes the cohomology of K, one also has to use that the resolution $\pi \hookrightarrow C^\bullet(G, \pi)$ is a strong resolution of acyclic K-modules and hence can be used to compute the cohomology of K ([4, Proposition 1]). □

Proposition 4.2. *Let $\pi \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$ be a smooth k-representation of G. Applying the functor $(-)^{\mathrm{sm}}$ to the complex $C^\bullet(G, \pi)$ gives a resolution $0 \to \pi \to C^\bullet(G, \pi)^{\mathrm{sm}}$ of π in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$.*

Proof. We want to show that the cohomology $H^i(C^\bullet(G, \pi)^{\mathrm{sm}})$ vanishes for $i > 0$. By definition, $C^n(G, \pi)^{\mathrm{sm}}$ is the direct limit over all compact open subgroups $K \leqslant G$ of K-fixed vectors $C^n(G, \pi)^K$. Note that taking direct limits in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$ is exact and hence commutes with taking cohomology of a complex. In particular, we get

$$H^i(C^\bullet(G, \pi)^{\mathrm{sm}}) \cong \varinjlim_K H^i(C^\bullet(G, \pi)^K).$$

By Lemma 4.1, this is the same as $\varinjlim_K H^i(K, \pi)$. Now, this direct limit vanishes, since any element in $\varinjlim_K H^i(K, \pi)$ comes from an element $f \in H^i(K, \pi)$ for some compact open subgroup $K \leqslant G$. By Lemma 3.4, this f is represented by a smooth, hence locally constant cocycle, hence it vanishes when restricted to some compact open subgroup $K' \leqslant K$. But since the transition maps in the direct limit are given by restrictions, this element is zero in the direct limit. Hence, $H^i(C^\bullet(G, \pi)^{\mathrm{sm}}) \cong \varinjlim_K H^i(C^\bullet(G, \pi)^K) = 0$ and $C^\bullet(G, \pi)^{\mathrm{sm}}$ gives indeed a resolution of π. □

Lemma 4.3. *For all $\pi,\ V \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$, we have isomorphisms*

$$\mathrm{Ext}_G^i(\pi, C^n(G, V)^{\mathrm{sm}}) \cong \mathrm{Ext}_k^i(\pi, C^{n-1}(G, V)),\ \textit{for all } n \geqslant 0,$$

where the Ext*-group on the right hand side is computed in the category of k-modules and $C^{-1}(G, V)$ is defined to be V.*

In particular, $\mathrm{Ext}_G^i(\mathbb{1}, C^n(G, V)^{\mathrm{sm}}) = 0$ *for all $i \geqslant 1$.*

Proof. By Lemma 2.1, we have an isomorphism $C(G, -)^{\mathrm{sm}} \cong (\mathrm{Ind}_1^G(-))^{\mathrm{sm}}$. By Frobenius reciprocity, we have

$$\mathrm{Hom}_G(\pi, C(G, V)^{\mathrm{sm}}) \cong \mathrm{Hom}_G(\pi, (\mathrm{Ind}_1^G(V))^{\mathrm{sm}}) \cong \mathrm{Hom}_k(\pi, V),$$

for any $\pi, V \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$. In particular, the functor $C(G, -)^{\mathrm{sm}}$ is right adjoint to the restriction functor which is exact. Moreover, $C(G, -)^{\mathrm{sm}}$ is left-exact, and it is even exact. Indeed, if $U \overset{\alpha}{\twoheadrightarrow} W$ is a surjection in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, then the induced map $C(G, U)^{\mathrm{sm}} \to C(G, W)^{\mathrm{sm}}$ is also surjective. To see this, note that for any smooth function $f : G \to W$, there is a compact open subgroup K of G, such that f is constant on cosets Kg. Then we can choose a set of representatives $g_i,\ i \in I$ of $K\backslash G$ and choose for each g_i an element $u_i \in U$, such that $\alpha(u_i) = f(g_i)$ and define a map in $C(G, U)^{\mathrm{sm}}$ by sending an element in the coset Kg_i to the element u_i. This will map to f.

Hence, we can apply Lemma 3.2 to get isomorphisms

$$\mathrm{Ext}^i_G(\pi, C(G,V)^{\mathrm{sm}}) \cong \mathrm{Ext}^i_k(\pi, V).$$

The claim follows from the description $C^n(G,V) \cong C(G, C^{n-1}(G,V))$ from Lemma 2.1.

Moreover, if $\pi = \mathbb{1}$, then $\mathrm{Hom}_k(\mathbb{1}, I^\bullet) \cong I^\bullet$, thus

$$\mathrm{Ext}^i_G(\mathbb{1}, C^n(G,V)^{\mathrm{sm}}) \cong \mathrm{Ext}^i_k(\mathbb{1}, C^{n-1}(G,V)) = 0$$

for all $i > 0$. □

Corollary 4.4. *Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}_G(k)$. We have isomorphisms of cohomology groups*

$$\mathrm{Ext}^i_G(\mathbb{1}, \pi) \cong H^i(G, \pi), \tag{2}$$

for all $i \geqslant 0$.

Proof. Just note that $(C^\bullet(G,\pi)^{\mathrm{sm}})^G \cong C^\bullet(G,\pi)^G$ and by Lemma 4.3, $C^\bullet(G,\pi)^{\mathrm{sm}}$ is an acyclic resolution of π in $\mathrm{Mod}^{\mathrm{sm}}_G(k)$, hence $(C^\bullet(G,\pi)^{\mathrm{sm}})^G$ computes $\mathrm{Ext}^i_G(\mathbb{1}, \pi)$, whereas the cohomology in degree i of $C^\bullet(G,\pi)^G$ is of course $H^i(G,\pi)$. □

5 Mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

From now on, let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, let k be a finite field of characteristic p. Let Z be the center of G, $\mathbb{1} \in \mathrm{Mod}^{\mathrm{sm}}_G(k)$ be the trivial representation, Sp be the Steinberg representation and let $\omega : \mathbb{Q}_p^\times \to k^\times$ be the character $\omega(x) = x|x| \pmod p$, where the absolute value $|-|$ is normalized so that $|p| = 1/p$. This character induces a representation of the subgroup $B \leqslant G$ of upper triangular matrices, $\alpha : B \to k^\times$, defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \omega(a)\omega(d)^{-1}$. And we denote by $\mathrm{Ind}_B^G \alpha$ the induced representation, given by continuous functions $f : G \to k$ with $f(bg) = \alpha(b)f(g)$ for all $b \in B$ and $g \in G$.

Moreover, we fix a continuous character $\zeta : Z \to \mathcal{O}^\times$ and we define the category $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(k)$ to be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_G(k)$ consisting of representations on which Z acts by ζ and we define the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(k)$ to be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(k)$ consisting of those representations that are locally admissible, or equivalently, that are locally of finite length (cf [14, Section 5.4]). If $\mathcal{O}$ is the ring of integers of some finite field extension $L/\mathbb{Q}_p$, such that its residue field is equal to k, then one can similarly define the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(k)$ forms a subcategory of it. Moreover, by [14, Proposition 5.34], this category decomposes into a direct product of subcategories

$$\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O}) \cong \prod_{\mathcal{B}} \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}.$$

(Note that in [14], these subcategories are denoted by $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})^{\mathcal{B}}$.) Therefore, as a subcategory, $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(k)$ will also decompose accordingly. These blocks $\mathcal{B}$ are certain equivalence classes of irreducible representations in $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(k)$. For a precise definition, see Section 5.5 in [14]. Then, for such block $\mathcal{B}$, the corresponding full subcategory $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}$ of $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ is given by those representations $\pi \in \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$, such that all its irreducible subquotients lie in the block $\mathcal{B}$.

We can use the isomorphism (2) to prove that the continuous group cohomology group $H^i(\mathrm{SL}_2(\mathbb{Q}_p), \pi)$ for some smooth irreducible representations π is trivial.

Proposition 5.1. *Let $\pi \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$ be a smooth absolutely irreducible representation of G, with central character ζ_π. Assume that for any character $\chi : \mathbb{Q}_p^\times \to k^\times$, $\pi \otimes \chi \circ \det$ is not isomorphic to $\mathbb{1}$, Sp or $\mathrm{Ind}_B^G \alpha$, then one has*

$$\mathrm{Ext}^i_{G,\zeta_\pi}(\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta_\pi, \pi) = 0,$$

for all $i \geqslant 0$, where the Ext-group is computed in the category $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta_\pi}(k)$ of smooth representations of G with central character ζ_π.

Proof. Note that if $p \neq 2$, $Z\,\mathrm{SL}_2(\mathbb{Q}_p)$ is of index 4 in G and if $p = 2$ it is of index 8, so that the representation $\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta_\pi$ is 4- or 8-dimensional. The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta_\pi$ factors through an abelian quotient, so that, after enlarging the field k, we may assume that there exists a character η_1, a 3-, respectively 7-dimensional representation $\tau \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$ and a short exact sequence of the form

$$0 \to \eta_1 \circ \det \to \mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta_\pi \to \tau \to 0.$$

This induces the long exact sequence

$$\cdots \to \mathrm{Ext}^i_{G,\zeta_\pi}(\tau, \pi) \to \mathrm{Ext}^i_{G,\zeta_\pi}(\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta_\pi, \pi) \to \mathrm{Ext}^i_{G,\zeta_\pi}(\eta_1 \circ \det, \pi) \to \dots$$

In particular, it is enough to show that $\mathrm{Ext}^i_{G,\zeta_\pi}(\eta_1 \circ \det, \pi) = 0$ and $\mathrm{Ext}^i_{G,\zeta_\pi}(\tau, \pi) = 0$. Moreover, we can repeat this procedure and find $\eta_2 \circ \det \hookrightarrow \tau$ and so on. Inductively, it will be enough to show that $\mathrm{Ext}^i_{G,\zeta_\pi}(\eta \circ \det, \pi) = 0$ for all such characters η.

The representation $\eta \circ \det$ is clearly irreducible. Moreover, π and $\eta \circ \det$ lie in the full subcategory $\mathrm{Mod}^{\mathrm{ladm}}_{G,\zeta_\pi}(k) = \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta_\pi}(k)$ of $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta_\pi}(k)$. Moreover, by Corollary 5.17 of [14], we have isomorphisms $\mathrm{Ext}^i_{G,\zeta_\pi}(\eta \circ \det, \pi) \cong \mathrm{Ext}^{\mathrm{lfin},i}_{G,\zeta_\pi}(\eta \circ \det, \pi)$, meaning that we can compute the Ext-groups in the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta_\pi}(k)$, which decomposes into a direct product of subcategories

$$\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta_\pi}(k) \cong \prod_{\mathcal{B}} \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta_\pi}(k)_{\mathcal{B}}$$

([14, Proposition 5.34]). In particular, there are no extensions between representations lying in different blocks $\mathcal{B}$. The blocks are described in Corollary 1.2 of [15] and by assumption, π does not lie in the same block of any character. Hence, $\mathrm{Ext}^{\mathrm{lfin},i}_{G,\zeta_\pi}(\eta_j \circ \det, \pi) = 0$ for all $i \geqslant 0$ and the claim follows. □

Corollary 5.2. *Under the assumptions of Proposition 5.1 we have*

$$H^i(\mathrm{SL}_2(\mathbb{Q}_p), \pi) = 0, \ \forall i \geqslant 0.$$

Proof. We may assume that $Z \cap \mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on π, since otherwise the continuous group cohomology groups $H^i(\mathrm{SL}_2(\mathbb{Q}_p), \pi)$ are zero for all $i \geqslant 0$ by Lemma 2.7. By Corollary 4.4, we know that $H^i(\mathrm{SL}_2(\mathbb{Q}_p), \pi) \cong \mathrm{Ext}^i_{\mathrm{SL}_2(\mathbb{Q}_p)}(\mathbb{1}, \pi)$. We can then consider the exact functor $\mathrm{Mod}^{\mathrm{sm}}_{\mathrm{SL}_2(\mathbb{Q}_p),\mathbb{1}}(k) \to \mathrm{Mod}^{\mathrm{sm}}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\zeta_\pi}(k)$, given by extending the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ to $Z\,\mathrm{SL}_2(\mathbb{Q}_p)$, by letting Z act via the character ζ_π. This is left adjoint to the restriction functor which is also exact. In particular, by Lemma 3.2, we obtain an isomorphism $\mathrm{Ext}^i_{\mathrm{SL}_2(\mathbb{Q}_p)}(\mathbb{1}, \pi) \cong \mathrm{Ext}^i_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\zeta_\pi}(\zeta_\pi, \pi)$ for all $i \geqslant 0$. The functor

$$\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)}(-) : \mathrm{Mod}^{\mathrm{sm}}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\zeta_\pi}(k) \to \mathrm{Mod}^{\mathrm{sm}}_{G,\zeta_\pi}(k)$$

is exact and has an exact right adjoint functor given by the restriction. Therefore, we can apply Lemma 3.2 to conclude that

$$\mathrm{Ext}^i_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\zeta_\pi}(\zeta_\pi,\pi) \cong \mathrm{Ext}^i_{G,\zeta_\pi}(\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)}\zeta_\pi,\pi),$$

which is zero by Proposition 5.1. □

Remark 5.3. For any field extension l of k, one can use Proposition 5.33 in [14], to see that the blocks are the same over l. In particular, we have the same vanishing of Ext-groups stated in Corollary 5.2.

6 Banach space representations

We want to apply Corollary 4.4 to Banach space representations of a p-adic reductive group G. More precisely, we let $F/\mathbb{Q}_p$ be a finite extension and $\mathbb{G}$ be a connected reductive group over F. Then take $G = \mathbb{G}(F)$ to be the group of its F-rational points. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k. Denote $\mathrm{Ban}^{\mathrm{adm}}_G(L)$ the category of admissible unitary L-Banach space representations of G. These are representations Π of G on L-vector spaces which are complete with respect to a G-invariant norm and which are admissible in the sense of Schneider–Teitelbaum [23]. By Proposition 9.6 in [20], this means that there is an open bounded G-invariant lattice Π^0 in Π, such that the quotient $\Pi^0/\varpi\Pi^0$ is an admissible smooth representation in $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O}/\varpi)$. Moreover, we have an isomorphism

$$\Pi \cong (\varprojlim_n \Pi^0/\varpi^n\Pi^0)[1/\varpi]. \tag{3}$$

Since the representation Π is a topological G-module, we can consider its continuous group cohomology groups and we will show that the isomorphism (3) induces an isomorphism in cohomology:

$$H^i(G,\Pi) \cong (\varprojlim_n H^i(G,\Pi^0/\varpi^n\Pi^0))[1/\varpi]. \tag{4}$$

The isomorphism (4) was proved for compact p-adic analytic groups by Emerton in [6, Proposition 1.2.19 and 1.2.20]. The main problem extending his proof is that, in general, projective limits do not commute with taking cohomology. Therefore we will first prove some finiteness properties on the cohomology groups.

6.1 Finiteness conditions

Lemma 6.1 ([8, Lemma 3.4.4])**.** *If K is a compact p-adic analytic group, π an admissible representation in $\mathrm{Mod}^{\mathrm{sm}}_K(\mathcal{O}/\varpi^n)$, $n > 0$, then the continuous group cohomology groups $H^i(K,\pi)$ are finitely generated $\mathcal{O}/\varpi^n$-modules for all $i \geqslant 0$.*

Proof. Let π be admissible representation on an $\mathcal{O}/\varpi^n$-module. Then, by Lemma 2.2.11 in [7], its Pontryagin dual $\pi^\vee = \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}}(\pi, L/\mathcal{O})$ is a finitely generated $\mathcal{O}/\varpi^n[\![K]\!]$-module (for more details see Corollary 1.8 and Proposition 1.9.(i) in [13]) and since K is compact p-adic analytic, the ring $\mathcal{O}/\varpi^n[\![K]\!]$ is Noetherian, by [7, Theorem 2.1.1]. Hence, we can find a resolution $F_\bullet \twoheadrightarrow \pi^\vee$ by free $\mathcal{O}/\varpi^n[\![K]\!]$-modules of finite rank. Then, taking the dual, we will get an injective resolution $\pi \hookrightarrow (F_\bullet)^\vee$ of π in $\mathrm{Mod}^{\mathrm{sm}}_K(\mathcal{O}/\varpi^n)$

and the K-invariants $(F_i{}^\vee)^K$ are finitely generated as $\mathcal{O}/\varpi^n$-modules. Then, since $\mathcal{O}/\varpi^n[\![K]\!]$ is Noetherian, using Proposition 3.3 we deduce that

$$H^i(K, \pi) \cong \mathrm{Ext}^i_K(\mathbb{1}, \pi) = H^i(((F_\bullet)^\vee)^K)$$

is finitely generated over $\mathcal{O}/\varpi^n$ for all i. □

Lemma 6.2. *Let H be a locally profinite group and let $\pi \in \mathrm{Mod}^{\mathrm{adm}}_H(\mathcal{O}/\varpi^n)$ be an admissible representation of H. Assume that H contains an open normal subgroup $N \leqslant H$ such that*

- *$H^i(N, \pi)$ is a finitely generated $\mathcal{O}/\varpi^n$-module for all $i \geqslant 0$;*
- *the quotient group H/N is either finite or a finitely generated abelian group.*

Then the $\mathcal{O}/\varpi^n$-modules $H^i(H, \pi)$ are finitely generated for all $i \geqslant 0$.

Proof. Again, by our main result 4.4, we can compute the continuous group cohomology as the Ext group in the category $\mathrm{Mod}^{\mathrm{sm}}_H(\mathcal{O}/\varpi^n)$ of smooth representations over $\mathcal{O}/\varpi^n$. Since the functor of taking H-invariants is the composition of the functors

$$\mathrm{Mod}^{\mathrm{sm}}_H(\mathcal{O}/\varpi^n) \xrightarrow{(-)^N} \mathrm{Mod}^{\mathrm{sm}}_{H/N}(\mathcal{O}/\varpi^n) \xrightarrow{(-)^{H/N}} \mathrm{Mod}(\mathcal{O}/\varpi^n),$$

we obtain a Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{H/N}(\mathbb{1}, \mathrm{Ext}^j_N(\mathbb{1}, \pi)) \Rightarrow \mathrm{Ext}^{i+j}_H(\mathbb{1}, \pi).$$

Hence, we obtain the same for the continuous cohomology groups:

$$E_2^{i,j} = H^i(H/N, H^j(N, \pi)) \Rightarrow H^{i+j}(H, \pi).$$

Since this is a first quadrant spectral sequence, for fixed (i, j), the limit term is $E_\infty^{i,j} = E_m^{i,j}$ for some m. Moreover, all of the modules $H^j(N, \pi)$ are finitely generated and since the quotient group H/N is discrete, $H^i(H/N, H^j(N, \pi))$ is the group cohomology of $H^j(N, \pi)$. By assumption, the quotient group H/N is either finite or finitely generated and abelian. In both cases, the group ring $\mathcal{O}/\varpi^n[H/N]$ is Noetherian and one can compute the cohomology groups $H^i(H/N, H^j(N, \pi))$ using resolutions consisting of finitely generated $\mathcal{O}/\varpi^n[H/N]$-modules. Therefore, all of the terms $E_m^{i,j}$ are finitely generated as $\mathcal{O}/\varpi^n[H/N]$-modules and, since H/N acts trivially on each $E_m^{i,j}$, they are also finitely generated as $\mathcal{O}/\varpi^n$-modules. By definition of the convergence of a spectral sequence, there is a finite filtration of $H^i(H, \pi)$ such that the graded pieces are isomorphic to one of these finitely generated modules $E_\infty^{i,j}$ and hence, $H^i(H, \pi)$ itself is also a finitely generated $\mathcal{O}/\varpi^n$-module. □

6.2 p-adic reductive groups

Let F be a finite field extension of $\mathbb{Q}_p$, let $\mathbb{G}$ be a connected reductive group over F and $G = \mathbb{G}(F)$ its group of F-rational points. Following the notations of [21], we denote by X the reduced Bruhat–Tits building associated to G and for any $q \geqslant 0$ we denote by X_q the set of all q-dimensional facets in X and by

$$X^q := \bigcup_{F \in X_q} \overline{F}$$

the q-skeleton. Then for any facet $F \in X_q$, the relative homology group $H_q(X^q, X^q \backslash F; \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 1 and we define the set of oriented q-facets to be

$$X_{(q)} := \{(F, c) | F \in X_q,\ c \in H_q(X^q, X^q \backslash F; \mathbb{Z}) \text{ a generator}\}.$$

Moreover, for a q-facet $F \in X_q$, we define $P_F^\dagger$ to be the G-stabilizer of F. This group then acts on the homology group $H_q(X^q, X^q \backslash F; \mathbb{Z}) \cong \mathbb{Z}$ and therefore, it acts via a character

$$\delta_F : P_F^\dagger \to \{\pm 1\}.$$

Following Section II.1 in [21], we obtain a G-equivariant resolution of $\mathbb{Z}$

$$\cdots \to H_1(X^1, X^0; \mathbb{Z}) \to H_0(X^0; \mathbb{Z}) = \bigoplus_{F \in X_0} \mathbb{Z} \to \mathbb{Z} \to 0,$$

since it computes the singular cohomology of X, which is trivial because X is contractible. Moreover the authors define G-equivariant isomorphisms

$$C_c^{\mathrm{or}}(X_{(q)}, \mathbb{Z}) \xrightarrow{\cong} H_q(X^q, X^{q-1}; \mathbb{Z}),$$

where

$$C_c^{\mathrm{or}}(X_{(q)}, \mathbb{Z}) = \left\{ \omega : X_{(q)} \to \mathbb{Z} \;\middle|\; \begin{array}{l} \omega \text{ has finite support} \\ \omega((F, -c)) = -\omega((F, c)) \text{ for any } (F, c) \in X_{(q)} \end{array} \right\}.$$

We fix for any $q \geqslant 0$, a set R_q containing exactly one representative from every G-orbit in X_q. Let us now fix a $q \geqslant 0$ and write $R_q = \{F_1, \ldots, F_{s_q}\}$ and fix for any F_i an orientation c_i, so that $(F_i, c_i) \in X_{(q)}$.

Lemma 6.3. *For every $q \geqslant 0$, there are G-equivariant isomorphisms*

$$C_c^{\mathrm{or}}(X_{(q)}, \mathbb{Z}) \xrightarrow{\cong} \bigoplus_{i=1}^{s_q} \text{c-Ind}_{P_{F_i}^\dagger}^G (\delta_{F_i})$$
$$\omega \mapsto (\omega_i)_i,$$

where $\omega_i : G \to \mathbb{Z}$ is given by $g \mapsto \omega((g^{-1}F_i, g^{-1}c_i))$.

Proof. Since ω has finite support, the support of each ω_i is compact modulo the stabilizer $P_{F_i}^\dagger$ and by definition,

$$\omega_i(hg) = \omega((g^{-1}h^{-1}F_i, g^{-1}h^{-1}c_i)) = \omega((g^{-1}F_i, \delta_{F_i}(h^{-1})g^{-1}c_i)) = \delta_{F_i}(h)\omega_i(g),$$

for all $h \in P_{F_i}^\dagger$, $g \in G$. In particular, ω_i defines an element of $\text{c-Ind}_{P_{F_i}^\dagger}^G (\delta_{F_i})$.

The map is therefore well-defined and it is clearly injective. Moreover, it is G-equivariant. Indeed, take any $g, h \in G$, then

$$(g\omega)_i(h) = (g\omega)((h^{-1}F_i, h^{-1}c_i)) = \omega((g^{-1}h^{-1}F_i, g^{-1}h^{-1}c_i)) = \omega_i(hg) = (g\omega_i)(h).$$

To prove surjectivity, it is enough to show that the image contains a set of generators of the G-module $\bigoplus_{i=1}^{s_q} \text{c-Ind}_{P_{F_i}^\dagger}^G (\delta_{F_i})$. We consider, for every $i \in \{1, \ldots, s_q\}$, the map $\omega^{(i)} \in C_c^{\mathrm{or}}(X_{(q)}, \mathbb{Z})$, which is supported on the set $\{(F_i, c_i), (F_i, -c_i)\}$ and maps (F_i, c_i) to 1. Then $(\omega^{(i)})_j = 0$ for $i \neq j$ and $(\omega^{(i)})_i \in \text{c-Ind}_{P_{F_i}^\dagger}^G (\delta_{F_i})$ is supported on $P_{F_i}^\dagger$ and $(\omega^{(i)})_i(1) = 1$. These elements clearly generate $\bigoplus_{i=1}^{s_q} \text{c-Ind}_{P_{F_i}^\dagger}^G (\delta_{F_i})$ as G-module, showing surjectivity.

□

We get therefore an exact resolution of $\mathbb{Z}$ in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathbb{Z})$:

$$\cdots \to \bigoplus_{F\in R_1} \text{c-Ind}_{P_F^\dagger}^G \delta_F \to \bigoplus_{F\in R_0} \text{c-Ind}_{P_F^\dagger}^G \delta_F \to \mathbb{Z} \to 0. \tag{5}$$

Note that, in general, the stabilizers $P_F^\dagger$ are not compact. However, in [21, Section 1.2], the authors construct a compact open normal subgroup R_F of $P_F^\dagger$. Moreover, they show that ZR_F is of finite index in $P_F^\dagger$, where Z denotes the F-rational points of the connected center of $\mathbb{G}$.

Lemma 6.4. *Let G be a p-adic reductive group, $\pi \in \mathrm{Mod}_G^{\mathrm{adm}}(\mathcal{O}/\varpi^n)$, then $H^i(G,\pi)$ is a finitely generated $\mathcal{O}/\varpi^n$-module.*

Proof. Since π is a smooth representation, we can apply Corollary 4.4, to get

$$H^i(G,\pi) \cong \mathrm{Ext}_G^i(\mathbb{1},\pi),$$

where the right hand side is computed in the category of smooth representations of G on $\mathcal{O}/\varpi^n$-modules. Since all terms in the resolution (5) are free abelian groups, by the Künneth formula ([25, Theorem 3.6.1]), it will remain exact after tensoring with $\mathcal{O}/\varpi^n$ and we get a resolution of the trivial representation in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O}/\varpi^n)$:

$$\cdots \to \bigoplus_{F\in R_1} \text{c-Ind}_{P_F^\dagger}^G \delta_F \to \bigoplus_{F\in R_0} \text{c-Ind}_{P_F^\dagger}^G \delta_F \to \mathbb{1} \to 0.$$

Let $\pi \hookrightarrow I^\bullet$ be an injective resolution of π in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O}/\varpi^n)$ and consider the double complex

$$C^{i,j} := \mathrm{Hom}_G(\bigoplus_{F\in R_i} \text{c-Ind}_{P_F^\dagger}^G \delta_F, I^j).$$

Since the objects I^j are injective, the complexes

$$0 \to \mathrm{Hom}_G(\mathbb{1}, I^j) \to \mathrm{Hom}_G(\bigoplus_{F\in R_\bullet} \text{c-Ind}_{P_F^\dagger}^G \delta_F, I^j)$$

are exact. Therefore, the spectral sequence associated to the horizontal filtration of $C^{\bullet,\bullet}$ is given by

$$E_{2,h}^{i,j} = \begin{cases} \mathrm{Ext}_G^j(\mathbb{1},\pi), & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

On the other hand, since the cohomology of the columns is

$$H^j(\mathrm{Hom}_G(\bigoplus_{F\in R_i} \text{c-Ind}_{P_F^\dagger}^G \delta_F, I^\bullet)) = \mathrm{Ext}_G^j(\bigoplus_{F\in R_i} \text{c-Ind}_{P_F^\dagger}^G \delta_F, \pi) \cong \bigoplus_{F\in R_i} \mathrm{Ext}_{P_F^\dagger}^j(\delta_F,\pi),$$

using that $\mathrm{Ext}_G^j(\text{c-Ind}_{P_F^\dagger}^G \delta_F,\pi) \cong \mathrm{Ext}_{P_F^\dagger}^j(\delta_F,\pi)$, by Lemma 3.2. Hence, the vertical filtration will give a spectral sequence

$$E_{2,v}^{i,j} = H^i(\bigoplus_{F\in R_\bullet} \mathrm{Ext}_{P_F^\dagger}^j(\delta_F,\pi)) \Rightarrow \mathrm{Ext}_G^{i+j}(\mathbb{1},\pi).$$

Therefore, the claim follows from the fact that $\mathrm{Ext}_{P_F^\dagger}^i(\delta_F,\pi) \cong \mathrm{Ext}_{P_F^\dagger}^i(\mathbb{1},\pi\otimes\delta_F^{-1})$ is a finitely generated $\mathcal{O}/\varpi^n$-module for every $i \geqslant 0$. Indeed, by Lemma 6.1, the groups $H^i(R_F,\pi\otimes\delta_F^{-1})$ are finitely generated over $\mathcal{O}/\varpi^n$ for all $i \geqslant 0$. Since the quotient ZR_F/R_F is a finitely generated abelian group, we can apply Lemma 6.2 to this pair of groups $R_F \leqslant ZR_F$, to get that $H^i(ZR_F,\pi\otimes\delta_F^{-1})$ is finitely generated and the Lemma 6.2 again applied to the pair $ZR_F \leqslant P_F^\dagger$ implies that $H^i(P_F^\dagger,\pi\otimes\delta_F^{-1})$ is finitely generated and the claim follows by Corollary 4.4. □

Lemma 6.5. *Let $\{\pi_i\}_{i\in\mathbb{N}}$ be a directed system of G-representations $\pi_i \in \mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$, such that the transition maps are injective. Then the natural map*

$$\varinjlim_i (\pi_i)^G \xrightarrow{\sim} (\varinjlim_i \pi_i)^G$$

is an isomorphism.

Proof. It is easy to see that the map is injective. To show surjectivity, note that two elements $v_1, v_2 \in \pi_j$ are equivalent in the limit $\varinjlim_i \pi_i$ if and only if they are equal in π_j. Indeed, by definition they are equivalent if and only if their images in some π_n are equal for some $n \geqslant j$, but all transition maps are injective and hence this is equivalent to being equal in π_j. In particular, if the image of an element $v \in \pi_j$ in the limit is G-invariant, then it is already G-invariant in π_j. □

For an $\mathcal{O}$-module M, we say that it is ϖ^∞-torsion, if every element in M is annihilated by a power of ϖ.

Lemma 6.6. *Let G be a p-adic reductive group, Π an admissible unitary Banach space representation of G over L and Π^0 an open bounded G-invariant lattice in Π. Then for all $i \geqslant 0$, we have*

$$H^i(G, \Pi/\Pi^0) \cong \varinjlim_n H^i(G, \varpi^{-n}\Pi^0/\Pi^0).$$

In particular, $H^i(G, \Pi/\Pi^0)$ is ϖ^∞-torsion, i.e. $H^i(G, \Pi/\Pi^0)[1/\varpi] = 0$.

Proof. The representation Π/Π^0 is discrete and ϖ^∞-torsion. Moreover, it lies in the category $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ and is isomorphic to the direct limit

$$\Pi/\Pi^0 \cong \varinjlim_n \varpi^{-n}\Pi^0/\Pi^0.$$

If $K \leqslant G$ is a compact open subgroup, the terms in the complex of continuous cochains $C^\bullet(K, \Pi/\Pi^0)$ commute with the direct limit, since the image of any map in $C^n(K, \Pi/\Pi^0) \cong C^n(K, \varinjlim_n \varpi^{-n}\Pi^0/\Pi^0)$ will be contained in the image of finitely many $\varpi^{-n}\Pi^0/\Pi^0$ in the limit. (Note that this argumentation does not apply to a non-compact G.) Moreover, by Lemma 6.5, we get

$$C^n(K, \Pi/\Pi^0)^K \cong \varinjlim_n C^n(K, \varpi^{-n}\Pi^0/\Pi^0)^K,$$

and since taking direct limits is exact in the category of $\mathcal{O}$-modules (cf. [25, Theorem 2.6.15]), the cohomology groups $H^i(K, -)$ commute with the direct limit, i.e.

$$H^i(K, \Pi/\Pi^0) \cong \varinjlim_n H^i(K, \varpi^{-n}\Pi^0/\Pi^0).$$

To prove the statement for a non-compact G, we proceed as in the proofs of the Lemmas 6.2 and 6.4, to reduce the proof to the compact situation. To adapt the proofs, note that if H is a discrete group, the continuous group cohomology of H is just group cohomology and if we assume that H is either finite or finitely generated and abelian, then the group ring $\mathbb{Z}[H]$ is Noetherian and therefore, $H^i(H, -) \cong \mathrm{Ext}^i_{\mathbb{Z}[H]}(\mathbb{Z}, -)$ can be computed using a projective resolution of $\mathbb{Z}$ consisting of free $\mathbb{Z}[H]$-modules of finite rank, hence $H^i(H, -)$ commutes with direct limits. Then we argue as in the proof of Lemma 6.4 to obtain a spectral sequence

$$E_2^{i,j} = H^i(\bigoplus_{F\in R_\bullet} \mathrm{Ext}^j_{P_F^\dagger}(\delta_F, \Pi/\Pi^0)) \Rightarrow \mathrm{Ext}^{i+j}_G(\mathbb{1}, \Pi/\Pi^0).$$

Moreover, by the Comparison Theorem 5.2.12 in [25], it is then enough to show that the terms in the spectral sequence commute with the direct limit, i.e. it is enough to show that

$$\mathrm{Ext}^i_{P_F^\dagger}(\delta_F, \Pi/\Pi^0) \cong \varinjlim_n \mathrm{Ext}^i_{P_F^\dagger}(\delta_F, \varpi^{-n}\Pi^0/\Pi^0),$$

for every $q \geqslant 0$ and $F \in R_q$. Or equivalently,

$$H^i(P_F^\dagger, \Pi/\Pi^0 \otimes \delta_F^{-1}) \cong \varinjlim_n H^i(P_F^\dagger, \varpi^{-n}\Pi^0/\Pi^0 \otimes \delta_F^{-1}).$$

This follows then from the fact that this is true for the compact open subgroup R_F, using as in the proof of Lemma 6.2 the Hochschild-Serre spectral sequences associated to $R_F \leqslant ZR_F$, $ZR_F \leqslant P_F^\dagger$:

$$E_2^{i,j} = H^i(ZR_F/R_F, H^j(R_F, \pi)) \Rightarrow H^{i+j}(ZR_F, \pi)$$

$$E_2^{i,j} = H^i(P_F^\dagger/ZR_F, H^j(ZR_F, \pi)) \Rightarrow H^{i+j}(P_F^\dagger, \pi)$$

The E_2-terms all commute with direct limits by the above discussion, hence so do the targets.

In particular, since each of the $H^i(G, \varpi^{-n}\Pi^0/\Pi^0)$ is ϖ^∞-torsion, so is the direct limit $\varinjlim_n H^i(G, \varpi^{-n}\Pi^0/\Pi^0) \cong H^i(G, \Pi/\Pi^0)$. □

Proposition 6.7. *Let G be a p-adic reductive group, Π an admissible unitary Banach space representation of G over L and Π^0 an open bounded G-invariant lattice in Π. Then we have*

$$H^i(G, \Pi^0) \cong \varprojlim_n H^i(G, \Pi^0/\varpi^n\Pi^0), \forall i \geqslant 0.$$

In particular,

$$H^i(G, \Pi) \cong (\varprojlim_n H^i(G, \Pi^0/\varpi^n\Pi^0))[1/\varpi].$$

Proof. We consider the tower of cochain complexes $\cdots \to C_2^\bullet \to C_1^\bullet \to C_0^\bullet$, where $C_n^i := C^i(G, \Pi^0/\varpi^n\Pi^0)^G$ is the cochain complex computing the continuous group cohomology of $\Pi^0/\varpi^n\Pi^0$. This projective system of cochain complexes satisfies the Mittag-Leffler condition, because each of the maps $C_n^i \to C_m^i$, $n \geqslant m$, is surjective. Indeed, since $\Pi^0/\varpi^m\Pi^0$ is discrete, the short exact sequence of representations

$$0 \to \varpi^m\Pi^0/\varpi^n\Pi^0 \to \Pi^0/\varpi^n\Pi^0 \to \Pi^0/\varpi^m\Pi^0 \to 0$$

induces a short exact sequence of complexes

$$0 \to C^\bullet(G, \varpi^m\Pi^0/\varpi^n\Pi^0) \to C^\bullet(G, \Pi^0/\varpi^n\Pi^0) \to C^\bullet(G, \Pi^0/\varpi^m\Pi^0) \to 0,$$

which stays exact after taking G-invariants, since the module $C^i(G, \varpi^m\Pi^0/\varpi^n\Pi^0)$ is acyclic for the continuous group cohomology for all i.

We can therefore apply Theorem 3.5.8 of [25], to get a short exact sequence

$$0 \to \varprojlim_n{}^{(1)} H^{i-1}(C_n^\bullet) \to H^i(\varprojlim_n C_n^\bullet) \to \varprojlim_n H^i(C_n^\bullet) \to 0,$$

where $\varprojlim_n^{(1)}$ denotes the right derived functor of the projective limit functor. But by Theorem 1 in [12], the functor $\varprojlim_n^{(1)}$ applied to a projective system of finitely generated modules over a complete Noetherian local ring, such as $\mathcal{O}$, vanishes. Hence, by Lemma 6.4,

$$\varprojlim_n{}^{(1)} H^{i-1}(C_n^\bullet) = \varprojlim_n{}^{(1)} H^{i-1}(G, \Pi^0/\varpi^n\Pi^0) = 0$$

and we get an isomorphism

$$H^i(\varprojlim_n C_n^\bullet) \xrightarrow{\cong} \varprojlim_n H^i(C_n^\bullet) = \varprojlim_n H^i(G, \Pi^0/\varpi^n\Pi^0).$$

Since the inverse limit $\varprojlim_n \Pi^0/\varpi^n\Pi^0$ can be seen as the inverse limit in the category of topological spaces, one has isomorphisms

$$C(G^{i+1}, \varprojlim_n \Pi^0/\varpi^n\Pi^0) \cong \varprojlim_n C(G^{i+1}, \Pi^0/\varpi^n\Pi^0),$$

inducing isomorphisms on the G-invariants

$$C^i(G, \varprojlim_n \Pi^0/\varpi^n\Pi^0)^G \cong (\varprojlim_n C^i(G, \Pi^0/\varpi^n\Pi^0))^G \cong \varprojlim_n C^i(G, \Pi^0/\varpi^n\Pi^0)^G.$$

The second isomorphism follows again from the universal property of projective limits, since $\mathrm{Hom}_G(\mathbb{1}, \varprojlim_n C^i(G, \Pi^0/\varpi^n\Pi^0)) \cong \varprojlim_n \mathrm{Hom}_G(\mathbb{1}, C^i(G, \Pi^0/\varpi^n\Pi^0))$. Therefore, the cohomology of $\varprojlim_n C_n^\bullet$ is in fact $H^i(G, \varprojlim_n \Pi^0/\varpi^n\Pi^0) = H^i(G, \Pi^0)$.

It remains to show that $H^i(G, \Pi) \cong H^i(G, \Pi^0)[1/\varpi]$. For this, consider the short exact sequence of G-modules

$$0 \to \Pi^0 \to \Pi \to \Pi/\Pi^0 \to 0.$$

The quotient Π/Π^0 is discrete, therefore we get a long exact sequence in cohomology

$$\cdots \to H^{i-1}(G, \Pi/\Pi^0) \to H^i(G, \Pi^0) \to H^i(G, \Pi) \to H^i(G, \Pi/\Pi^0) \to \ldots.$$

Since localization is exact, $H^i(G, \Pi/\Pi^0)$ is ϖ^∞-torsion by Lemma 6.6 and $H^i(G, \Pi)$ is an L-vector space, we get

$$H^i(G, \Pi^0)[1/\varpi] \cong H^i(G, \Pi)[1/\varpi] \cong H^i(G, \Pi).$$

□

Corollary 6.8. *In the notation of Proposition 6.7, the L-vector spaces $H^i(G, \Pi)$ are finite dimensional, for all $i \geqslant 0$.*

Proof. Since $H^i(G, \Pi) \cong H^i(G, \Pi^0)[1/\varpi]$, it suffices to show that $H^i(G, \Pi^0)$ is a finitely generated $\mathcal{O}$-module. We have a short exact sequence

$$0 \longrightarrow \Pi^0 \xrightarrow{\varpi} \Pi^0 \longrightarrow \Pi^0/\varpi\Pi^0 \longrightarrow 0,$$

which induces the long exact sequence of $\mathcal{O}$-modules

$$\cdots \to H^i(G, \Pi^0) \xrightarrow{\varpi} H^i(G, \Pi^0) \longrightarrow H^i(G, \Pi^0/\varpi\Pi^0) \longrightarrow \ldots.$$

In particular, the quotient $H^i(G, \Pi^0)/\varpi H^i(G, \Pi^0)$ can be embedded into the $\mathcal{O}$-module $H^i(G, \Pi^0/\varpi\Pi^0)$, which is finite by Lemma 6.4. Moreover, $H^i(G, \Pi^0)$ is profinite by Lemma 6.4 and Proposition 6.7, hence it is a compact $\mathcal{O}$-module and the claim follows from the topological Nakayama's lemma (cf. [1, §3, Corollary]). □

6.3 Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

We use Proposition 6.7 to prove an analogue of Corollary 5.2 for Banach space representations.

From now on, let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, let $\zeta : \mathbb{Q}_p^\times \to L^\times$ be a unitary character and let $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ be the full subcategory of $\mathrm{Ban}^{\mathrm{adm}}_{G}(L)$ consisting of objects which have central character ζ. This category does not have enough injectives or projectives, but we can consider the Yoneda Ext-groups in this category $\mathrm{Ext}^i_{\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)}(\Pi_1, \Pi_2)$.

We want to prove the following

Proposition 6.9. *For $\Pi_1,\ \Pi_2 \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$, with Π_1 of finite length, one has*

$$\mathrm{Ext}^i_{\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)}(\Pi_1, \Pi_2) \cong \left(\varprojlim_n \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)}(\Pi_1^0/p^n, \Pi_2^0/p^n)\right)[1/p], \forall i \geqslant 0, \qquad (6)$$

where $\bar{\zeta}$ is the composition of ζ with the projection map $\mathcal{O} \twoheadrightarrow \mathcal{O}/\varpi^n$.

By Proposition 5.34 and Proposition 5.36 of [14], we know that both categories $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ and $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})$ decompose into the direct sum, resp. product, of subcategories

$$\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L) \cong \bigoplus_{\mathcal{B}} \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}},$$
$$\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O}) \cong \prod_{\mathcal{B}} \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}.$$

Here, the blocks $\mathcal{B}$ are the same equivalence classes of irreducible representations in $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(k)$. A representation $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ lies in a subcategory $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}}$, if for an open bounded G-invariant lattice $\Pi^0 \subset \Pi$, all irreducible subquotients of the smooth k-representation $\Pi^0 \otimes_{\mathcal{O}} k$ lie in $\mathcal{B}$.

In particular, there are no extensions between representations lying in different blocks and it is enough to show (6) for each block individually. Fix such block $\mathcal{B}$ and consider the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}$ and let $\mathfrak{C}(\mathcal{O})$ be the strictly full subcategory of $\mathrm{Mod}^{\mathrm{pro\ aug}}_{G}(\mathcal{O})$ which is anti-equivalent to $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}$ via Pontryagin duality. Corollary 1.2 in [15] gives a full description of the possible blocks $\mathcal{B}$ and shows in particular, that all of the blocks contain only finitely many elements. Let $\pi_1, \ldots, \pi_s$ be representatives of the isomorphism classes of irreducible k-representations in $\mathcal{B}$. Then the representation $\oplus_{i=1}^s \pi_i^\vee$ lies in $\mathfrak{C}(\mathcal{O})$ and has a projective envelope P. As in Section 2 of [14], we define the ring $\mathrm{E} = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P})$ as the endomorphism ring of this projective envelope. This endomorphism ring is equipped with its so-called natural topology (cf. [14, p.14]), making it a pseudo-compact ring. In fact, the natural topology is the same as the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal two-sided ideal of E ([14, Corollary 3.11]). We obtain an equivalence of categories (cf. Section 4.2 in [14])

$$\mathfrak{C}(\mathcal{O}) \xrightarrow{\sim} \{\text{compact (right) } \mathrm{E}\text{-modules}\},$$
$$M \mapsto \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, M)$$

with inverse given by the completed tensor product $\mathrm{m} \mapsto \mathrm{m} \widehat{\otimes}_{\mathrm{E}} \mathrm{P}$, for a compact E-module m.

The ring E is finitely generated over its center, which is Noetherian (see Section 6.4 of [17]), and hence our setup satisfies the assumptions of Section 4.2 in [14]. As constructed in Lemma 4.9 of [14], we have a fully faithful contravariant functor

$$\mathrm{m} : \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}} \to \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{E}[1/p]},$$

where $\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{E}[1/p]}$ is the category of finitely generated $\mathrm{E}[1/p]$-modules. The functor m is defined as follows: Let $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}}$ and let Π^0 be an open bounded G-invariant lattice of Π. Then its Schikhof dual $(\Pi^0)^d := \mathrm{Hom}_{\mathcal{O}}(\Pi^0, \mathcal{O})$ is an element of $\mathfrak{C}(\mathcal{O})$, by Lemma 4.11 in [14], so by applying $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, -)$, we obtain a compact E-module. We set

$$\mathrm{m}(\Pi) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, (\Pi^0)^d) \otimes_{\mathcal{O}} L.$$

It does not depend on the choice of the lattice Π^0. This functor m induces an isomorphism

$$\mathrm{Ext}^i_{\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}}}(\Pi_1, \Pi_2) \cong \mathrm{Ext}^i_{\mathrm{E}[1/p]}(\mathrm{m}(\Pi_2), \mathrm{m}(\Pi_1)),$$

for all $i \geqslant 0$ and for $\Pi_1,\ \Pi_2 \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}}$, with Π_1 of finite length ([14, Corollary 4.48]). The Ext-group on the right hand side can then be expressed as a projective limit in the following way:

Lemma 6.10. *Let* m_1, m_2 *be finitely generated* $\mathrm{E}[1/p]$*-modules. Then there exist* E*-stable* $\mathcal{O}$*-lattices* m_1^0, m_2^0 *in* m_1 *and* m_2 *respectively, which are finitely generated as* E*-modules and we have*

$$\mathrm{Ext}^i_{\mathrm{E}[1/p]}(\mathrm{m}_2, \mathrm{m}_1) \cong \left(\varprojlim_n \mathrm{Ext}^i_{\mathrm{E}/p^n}(\mathrm{m}_2^0/p^n, \mathrm{m}_1^0/p^n)\right)[1/p],$$

where the groups $\mathrm{Ext}^i_{\mathrm{E}[1/p]}$ *and* $\mathrm{Ext}^i_{\mathrm{E}/p^n}$ *are computed in the category of finitely generated* $\mathrm{E}[1/p]$*- and* E/p^n*-modules, respectively.*

Proof. We can choose a finite set of generators of the $\mathrm{E}[1/p]$-module m_i and let m_i^0 be the E-submodule of m_i generated by those generators. Then m_i^0 is an $\mathcal{O}$-lattice in m_i. The ring E is in fact pseudo-compact (by Proposition 13 in [10, §IV.4]) and is Noetherian, as it is finitely generated over its center, which is Noetherian ([17, Corollary 6.4]). Therefore, the finitely generated E-modules m_i^0 can be presented as the cokernels of morphisms $\mathrm{E}^{\oplus n} \to \mathrm{E}^{\oplus m}$ of pseudo-compact E-modules and hence m_i^0 equipped with the quotient topology, are also pseudo-compact (using Theorem 3 in chapter IV of [10]). Moreover, by Proposition 13 in Chapter IV of [10], the quotient of E by its Jacobson radical is isomorphic to $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\oplus_{i=1}^n \pi_i^\vee) \cong \prod_{i=1}^n \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\pi_i^\vee)$, which is a finite dimensional vector space over the residue field of $\mathcal{O}$. Hence, the $\mathcal{O}$-modules m_i^0 are also pseudo-compact, thus they can be written as a projective limit $\mathrm{m}_i^0 \cong \varprojlim_n \mathrm{m}_i^0/p^n$. We obtain

$$\mathrm{m}_i \cong (\varprojlim_n \mathrm{m}_i^0/p^n) \otimes_{\mathcal{O}} L.$$

Since the ring E is Noetherian, we can compute the Ext-group $\mathrm{Ext}^i_{\mathrm{E}}(\mathrm{m}_2^0, \mathrm{m}_1^0)$ using a projective resolution

$$\cdots \to \mathrm{E}^{\oplus s_2} \to \mathrm{E}^{\oplus s_1} \to \mathrm{E}^{\oplus s_0} \to \mathrm{m}_2^0 \to 0,$$

by free E-modules of finite rank. Then

$$\mathrm{Ext}^i_{\mathrm{E}}(\mathrm{m}_2^0, \mathrm{m}_1^0) = H^i(\mathrm{Hom}_{\mathrm{E}}(\mathrm{E}^{\oplus s_\bullet}, \varprojlim_n \mathrm{m}_1^0/p^n)) \cong H^i(\varprojlim_n \mathrm{Hom}_{\mathrm{E}}(\mathrm{E}^{\oplus s_\bullet}, \mathrm{m}_1^0/p^n)).$$

But since the modules $\mathrm{Hom}_{\mathrm{E}}(\mathrm{E}^{\oplus s_\bullet}, \mathrm{m}_1^0/p^n)$ are finitely generated, the projective limit commutes with taking the cohomology and we obtain

$$\mathrm{Ext}^i_{\mathrm{E}}(\mathrm{m}_2^0, \mathrm{m}_1^0) = \varprojlim_n H^i(\mathrm{Hom}_{\mathrm{E}}(\mathrm{E}^{\oplus s_\bullet}, \mathrm{m}_1^0/p^n)) = \varprojlim_n \mathrm{Ext}^i_{\mathrm{E}}(\mathrm{m}_2^0, \mathrm{m}_1^0/p^n).$$

Moreover, since the module m_2^0 is $\mathcal{O}$-torsion free, the resolution $\mathrm{E}^{\oplus s_\bullet} \twoheadrightarrow \mathrm{m}_2^0$ stays exact after tensoring with E/p^n. In particular, the resulting sequence $(\mathrm{E}/p^n)^{\oplus s_\bullet} \twoheadrightarrow \mathrm{m}_2^0/p^n$ is a projective resolution of m_2^0/p^n as an E/p^n-module and we get

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{E}}(\mathrm{m}_2^0, \mathrm{m}_1^0/p^n) &= H^i(\mathrm{Hom}_{\mathrm{E}}(\mathrm{E}^{\oplus s_\bullet}, \mathrm{m}_1^0/p^n)) \\ &\cong H^i(\mathrm{Hom}_{\mathrm{E}/p^n}((\mathrm{E}/p^n)^{\oplus s_\bullet}, \mathrm{m}_1^0/p^n)) \\ &= \mathrm{Ext}^i_{\mathrm{E}/p^n}(\mathrm{m}_2^0/p^n, \mathrm{m}_1^0/p^n).\end{aligned}$$

□

Note that, since E is Noetherian, the groups $\mathrm{Ext}^i_{\mathrm{E}/p^n}(\mathrm{m}_2^0/p^n, \mathrm{m}_1^0/p^n)$ computed in the category of finitely generated E/p^n-modules agree with the ones computed in the category of compact E/p^n-modules, which is anti-equivalent to $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O}/\varpi^n)_{\mathcal{B}}$. Hence, we get the following lemma:

Lemma 6.11. *In the notation of the previous lemma, we have*

$$\mathrm{Ext}^i_{\mathrm{E}/p^n}(\mathrm{m}_2^0/p^n, \mathrm{m}_1^0/p^n) \cong \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)}(((\mathrm{m}_1^0/p^n)\widehat{\otimes}_{\mathrm{E}} \mathrm{P})^\vee, ((\mathrm{m}_2^0/p^n)\widehat{\otimes}_{\mathrm{E}} \mathrm{P})^\vee).$$

Proof. Via the anti-equivalence between the category of compact E/p^n-modules and the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)_{\mathcal{B}}$, the modules m_i^0/p^n correspond to $((\mathrm{m}_i^0/p^n)\widehat{\otimes}_{\mathrm{E}} \mathrm{P})^\vee$. The claim follows then from the decomposition of the category $\mathrm{Mod}^{\mathrm{lfin}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)$ into blocks and the fact that the Ext-group of locally finite representations in $\mathrm{Mod}^{\mathrm{lfin}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)$ is isomorphic to their Ext-group computed in the category of smooth representations $\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)$ (see Corollary 5.17 and 5.18 in [14]). □

Lemma 6.12. *Let Π_1 and Π_2 be admissible unitary Banach space representations in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}}$ and set $\mathrm{m}_i := \mathrm{m}(\Pi_i)$. Then there are* E*-stable $\mathcal{O}$-lattices m_1^0, m_2^0 in m_1 and m_2 respectively, such that*

$$\mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)}(\Pi_1^0/p^n, \Pi_2^0/p^n) \cong \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)}(((\mathrm{m}_1^0/p^n)\widehat{\otimes}_{\mathrm{E}} \mathrm{P})^\vee, ((\mathrm{m}_2^0/p^n)\widehat{\otimes}_{\mathrm{E}} \mathrm{P})^\vee),$$

where Π_1^0 and Π_2^0 are open bounded G-invariant lattices in Π_1 and Π_2, respectively.

Proof. Recall that for $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathcal{B}}$, $\mathrm{m}(\Pi)$ was defined as $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, (\Pi^0)^d) \otimes_{\mathcal{O}} L$. Hence, $\mathrm{m}(\Pi)^0 := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, (\Pi^0)^d)$ defines an E-stable $\mathcal{O}$-lattice in $\mathrm{m}(\Pi)$. To show that it satisfies the claimed isomorphism, we need to prove that the smooth representation $\Pi^0/p^n \in \mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}$ is isomorphic to $((\mathrm{m}(\Pi)^0/p^n)\widehat{\otimes}_{\mathrm{E}} \mathrm{P})^\vee$. But by the anti-equivalence of categories between $\mathrm{Mod}^{\mathrm{lfin}}_{G,\zeta}(\mathcal{O})_{\mathcal{B}}$ and compact E-modules, this is equivalent to having an isomorphism

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, (\Pi^0/p^n)^\vee) \cong \mathrm{m}(\Pi)^0/p^n,$$

or in other words

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, (\Pi^0/p^n)^\vee) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathrm{P}, (\Pi^0)^d)/p^n.$$

We conclude by using the isomorphism ([16, Section 2])

$$(\Pi^0)^d/p^n \cong (\Pi^0/p^n)^\vee.$$

□

These lemmas combined prove Proposition 6.9.

To state the next proposition, we need to introduce the following notation:

Let $\mathbb{1}$ be the trivial one-dimensional representation in $\mathrm{Ban}^{\mathrm{adm}}_G(L)$, $\widehat{\mathrm{Sp}}$ be the universal unitary completion of the smooth Steinberg representation of G over L. Let B be the subgroup of upper triangular matrices in G. Let $\tilde{\alpha} : B \to L^\times$ be the representation of B, defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto ad^{-1}|ad^{-1}|$. Then $|\tilde{\alpha}(b)| = 1$ for all $b \in B$ and we can define $\mathrm{Ind}_B^G \tilde{\alpha}$ to be the induced representation, given by continuous functions $f : G \to L$ with $f(bg) = \tilde{\alpha}(b)f(g)$ for all $b \in B$ and $g \in G$, endowed with the supremum norm. This defines an admissible unitary Banach space representation of G (see Section 7.2 of [14] for more details).

Proposition 6.13. *Let $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ be an absolutely irreducible admissible unitary Banach space representation of G, let Π^0 be an open bounded G-invariant lattice in Π and assume that Π is not isomorphic to a twist by a unitary character of $\mathbb{1}$, $\widehat{\mathrm{Sp}}$ or $\mathrm{Ind}_B^G \tilde{\alpha}$. Then, for all $i \geqslant 0$, we have*

$$H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi) = 0.$$

Proof. We argue as in the proof of Corollary 5.2. Namely, by Proposition 6.7 and Corollary 4.4, we have

$$\begin{aligned} H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi) &\cong (\varprojlim_n H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi^0/\varpi^n\Pi^0))[1/\varpi] \\ &\cong (\varprojlim_n \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{\mathrm{SL}_2(\mathbb{Q}_p)}(\mathcal{O}/\varpi^n)}(\mathbb{1}, \Pi^0/\varpi^n\Pi^0))[1/\varpi]. \end{aligned}$$

Let Z be the center of G. Without loss of generality, we may assume that $Z \cap \mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on Π, since otherwise, Lemma 2.7 implies that the groups $H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi)$ are zero for all $i \geqslant 0$. Then we get an exact functor

$$(-)|_{\mathrm{SL}_2(\mathbb{Q}_p)} : \mathrm{Mod}^{\mathrm{sm}}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\bar{\zeta}}(\mathcal{O}/\varpi^n) \to \mathrm{Mod}^{\mathrm{sm}}_{\mathrm{SL}_2(\mathbb{Q}_p),\mathbb{1}}(\mathcal{O}/\varpi^n),$$

where $\bar{\zeta}$ is the composition of ζ with the projection onto $\mathcal{O}/\varpi^n$. The functor $(-)|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ preserves injective objects and hence we obtain an isomorphism of Ext-groups

$$\mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\bar{\zeta}}(\mathcal{O}/\varpi^n)}(\bar{\zeta}, \Pi^0/\varpi^n\Pi^0) \cong \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{\mathrm{SL}_2(\mathbb{Q}_p),\mathbb{1}}(\mathcal{O}/\varpi^n)}(\mathbb{1}, \Pi^0/\varpi^n\Pi^0).$$

Using this, we get

$$\begin{aligned} H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi) &\cong (\varprojlim_n \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p),\bar{\zeta}}(\mathcal{O}/\varpi^n)}(\bar{\zeta}, \Pi^0/\varpi^n\Pi^0))[1/\varpi] \\ &\cong (\varprojlim_n \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)}(\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \bar{\zeta}, \Pi^0/\varpi^n\Pi^0))[1/\varpi] \\ &\cong (\varprojlim_n \mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{sm}}_{G,\bar{\zeta}}(\mathcal{O}/\varpi^n)}(\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta/\varpi^n, \Pi^0/\varpi^n\Pi^0))[1/\varpi], \end{aligned}$$

where in the bottom line, we consider the induced representation $\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta \in \mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(\mathcal{O})$ as in Proposition 5.1. Since $Z\,\mathrm{SL}_2(\mathbb{Q}_p)$ is of index 4 or 8 in G (depending on whether $p \neq 2$ or $p = 2$,) and the character ζ is unitary, $\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta$ is a 4-, or 8-dimensional unitary Banach space representation of G. Then we can apply Proposition 6.9 to get

$$H^i(\mathrm{SL}_2(\mathbb{Q}_p), \Pi) \cong \mathrm{Ext}^i_{\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)}(\mathrm{Ind}^G_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)} \zeta \otimes_{\mathcal{O}} L, \Pi).$$

Therefore, it suffices to prove that this Ext-group vanishes. We may replace L by a finite field extension so that we may assume without loss of generality, that $\operatorname{Ind}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)}^G \zeta \otimes_{\mathcal{O}} L$ is reducible. Since it is finite dimensional, we then have a short exact sequence of the form

$$0 \to \eta_1 \circ \det \to \operatorname{Ind}_{Z\,\mathrm{SL}_2(\mathbb{Q}_p)}^G \zeta \otimes_{\mathcal{O}} L \to \Pi_1 \to 0,$$

for some unitary character $\eta_1 : \mathbb{Q}_p^\times \to L^\times$ and $\Pi_1 \in \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)$ of dimension 3, respectively 7. We then repeat this as in the proof of Proposition 5.1 and see that it is enough to show that $\operatorname{Ext}^i_{\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)}(\eta \circ \det, \Pi) = 0$ for each $i \geqslant 0$ and for each unitary character $\eta : \mathbb{Q}_p^\times \to L^\times$. But by the decomposition

$$\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L) \cong \bigoplus_{\mathcal{B}} \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B}},$$

we can reduce to the case where Π is a representation in the same block $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B}}$ as $\eta \circ \det$. More precisely, this block is given by (Corollary 1.2 in [15])

$$\mathcal{B} = \{\bar{\eta} \circ \det, \mathrm{Sp} \otimes \bar{\eta} \circ \det, \operatorname{Ind}_B^G \alpha \otimes \bar{\eta} \circ \det\},$$

where $\bar{\eta} : \mathbb{Q}_p \to k$ and $\alpha : B \to k$ are the compositions of η, respectively $\tilde{\alpha}$, with the quotient map $\mathcal{O} \twoheadrightarrow k$.

And by [17, Section 6.2], the category $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B}}$ decomposes as a direct sum of subcategories

$$\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B}} \cong \bigoplus_{\mathfrak{n} \in \operatorname{MaxSpec} Z_{\mathcal{B}}[1/p]} \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B},\mathfrak{n}},$$

where $Z_{\mathcal{B}}$ is the center of the endomorphism ring E which depends on the block $\mathcal{B}$, defined in Section 6.3. The category $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B},\mathfrak{n}}$ is defined to be the full subcategory of $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B}}$ consisting of those representations which are killed by a power of the ideal $\mathfrak{n}$. Moreover, the irreducible objects of the subcategory $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)_{\mathcal{B},\mathfrak{n}}$, which contains $\eta \circ \det$, are described in Corollary 6.10 of [17] and are precisely those representations that we excluded. In particular, there are no extensions between Π and $\eta \circ \det$.

□

7 Locally analytic vectors of admissible Banach space representations

We keep the set-up of the beginning of Section 6. In particular, G is a p-adic reductive group and L a finite field extension of $\mathbb{Q}_p$. Since G is a p-adic reductive group, it is in particular a locally $\mathbb{Q}_p$-analytic group. In this section, we want to study the continuous cohomology groups of G with coefficients in the representation of locally analytic vectors in an admissible Banach space representation. For this, we use a spectral sequence, which we will first construct in a more general setting.

7.1 Construction of a spectral sequence

In Section 6.2, we used a resolution of the trivial representation $\mathbb{1} \in \mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O}/\varpi^n)$ by representations which are compactly induced from compact-mod-center stabilizer subgroups of facets in the Bruhat–Tits building of G, to construct a certain spectral sequence converging to the continuous cohomology of G with coefficients in a smooth representation. In the following, we will generalize this construction to obtain a similar spectral sequence

$$E_1^{i,j} = \bigoplus_{F \in R_i} H^j(P_F^\dagger, V \otimes_L \delta_F^{-1}) \Rightarrow H^{i+j}(G, V)., \tag{7}$$

for any G-representation V on an L-vector space such that the G-action makes it a topological G-module. In the following, we fix such a representation V. Recall that we denoted by X_q the set of q-dimensional facets in the Bruhat–Tits building X of G and R_q was a fixed set of representatives of X_q modulo the action of G. For such a facet $F \in X_q$, $P_F^\dagger$ denotes the G-stabilizer of F and $\delta_F : P_F^\dagger \to \{\pm 1\}$ is a character.

We use the resolution (5) that was constructed in Section 6.2

$$\bigoplus_{F \in R_\bullet} \text{c-Ind}_{P_F^\dagger}^G \delta_F \to \mathbb{Z} \to 0.$$

A priori, this is a resolution by smooth G-representations on free $\mathbb{Z}$-modules, but by tensoring with L, we obtain a resolution of the trivial representation by smooth G-representations on L-vector spaces.

$$\bigoplus_{F \in R_\bullet} \text{c-Ind}_{P_F^\dagger}^G \delta_F \to \mathbb{1} \to 0.$$

By abuse of notation, we now denote by δ_F the representation

$$\delta_F : P_F^\dagger \to \{\pm 1\} \subset L^\times$$

on the one-dimensional L-vector space.

Since the compactly induced representations in this resolution are smooth, we can give them the structure of topological G-modules by equipping them with the discrete topology.

Now consider the double complex

$$C^{i,j} := \text{Hom}_G(\bigoplus_{F \in R_i} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^j(G, V)), \ i, j \geqslant 0. \tag{8}$$

Note that since we equipped the compactly induced representations with the discrete topology, all the homomorphisms in $C^{i,j}$ are automatically continuous and we can also write it as $C^{i,j} = \text{Hom}_G^{\text{cts}}(\bigoplus_{F \in R_i} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^j(G, V))$.

As before, this induces two spectral sequences with the same limit term. One is coming from a horizontal filtration and the other from the vertical one. To write these down explicitly, we need to understand the cohomology of the rows and of the columns of our double complex (8). We start by studying the rows:

Fixing an $i \geqslant 0$, we can write

$$C^i(G, V) \cong C^0(G, C^{i-1}(G, V)) \cong \text{Ind}_1^G C^{i-1}(G, V) \tag{9}$$

(cf. Lemma 2.1), with the convention that $C^{-1}(G, V) = V$. In particular, we have isomorphisms $C^i(G, V)^G \cong (\text{Ind}_1^G C^{i-1}(G, V))^G \cong C^{i-1}(G, V)$. The isomorphism (9) allows us then to apply Frobenius reciprocity for topological G-modules (cf. Lemma 2.2). We obtain

$$\begin{aligned} C^{\bullet, i} &= \text{Hom}_G^{\text{cts}}(\bigoplus_{F \in R_\bullet} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^i(G, V)) \\ &\cong \text{Hom}_L^{\text{cts}}(\bigoplus_{F \in R_\bullet} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^{i-1}(G, V)) \\ &= \text{Hom}_L(\bigoplus_{F \in R_\bullet} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^{i-1}(G, V)). \end{aligned}$$

But now, the complex

$$0 \to \mathrm{Hom}_L(L, C^{i-1}(G,V)) \to \mathrm{Hom}_L(\bigoplus_{F \in R_0} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^{i-1}(G,V)) \to \dots$$

is exact, as we have taken L-linear homomorphisms of an exact sequence into the L-vector space $C^{i-1}(G,V)$. Therefore, taking cohomology of the rows $C^{\bullet,i}$ gives

$$H^j(C^{\bullet,i}) = \begin{cases} 0, \text{ if } j \neq 0, \\ \mathrm{Hom}_L(L, C^{i-1}(G,V)) \cong C^{i-1}(G,V) \cong C^i(G,V)^G, \text{ if } j = 0. \end{cases}$$

The first page terms of the spectral sequence coming from the horizontal filtration are precisely given by these cohomology groups:

$$E_{h,1}^{i,j} = H^j(C^{\bullet,i}).$$

Moreover, the first page differentials $d_{h,1}^{i,j} : E_{h,1}^{i,j} \to E_{h,1}^{i+1,j}$ are zero whenever j is non-zero and for $j = 0$, they are just the maps

$$d_{h,1}^{i,0} : C^i(G,V)^G \to C^{i+1}(G,V)^G,$$

induced by the differential maps on the complex $C^\bullet(G,V)$. We can easily compute the second page terms:

$$E_{h,2}^{i,j} = H^{i,j}(E_{h,1}^{\bullet,\bullet}) = \begin{cases} 0, \text{ if } j \neq 0 \\ H^i(C^\bullet(G,V)^G) = H^i(G,V), \text{ if } j = 0. \end{cases}$$

In conclusion, we know that the spectral sequence coming from the horizontal filtration converges to the continuous cohomology groups of G with coefficients in V. The spectral sequence induced by the vertical filtration also converges to $H^\bullet(G,V)$, since it has the same limit term.

On the other hand, we can fix an $i \geqslant 0$ and consider the columns of the double complex

$$C^{i,\bullet} = \mathrm{Hom}_G(\bigoplus_{F \in R_i} \text{c-Ind}_{P_F^\dagger}^G \delta_F, C^\bullet(G,V)).$$

By pulling out the direct sum and applying Frobenius reciprocity for the compact induction, we obtain

$$\begin{aligned} C^{i,\bullet} &\cong \bigoplus_{F \in R_i} \mathrm{Hom}_{P_F^\dagger}(\delta_F, C^\bullet(G,V)) \\ &\cong \bigoplus_{F \in R_i} \mathrm{Hom}_{P_F^\dagger}(\mathbb{1}, C^\bullet(G,V) \otimes_L \delta_F^{-1}). \end{aligned} \tag{10}$$

Lemma 7.1. *Let V be a topological G-module on an L-vector space, let $H \leqslant G$ be an open subgroup and $\delta : H \to L^\times$ a smooth character on H. We have an L-linear isomorphism of topological H-modules*

$$C^n(G,V) \otimes_L \delta \xrightarrow{\cong} C^n(G, V \otimes_L \delta), \ \forall n \geqslant 0.$$

Proof. As in Lemma 2.1, we have homeomorphisms $C^{n+1}(G,W) \to C^0(G, C^n(G,W))$ for every topological space W. These homeomorphisms are H-equivariant if W is equipped with the structure of a topological H-module. Hence, we may assume that $n = 0$.

Let $\tilde{\delta} : G \to L^\times$ be the map that sends an element $g \in G$ to $\delta(g)$ if g lies in H and to 1 otherwise. Since H is open in G, this map is locally constant (but not necessarily a character). Therefore the map

$$\alpha : \ C^0(G, V) \otimes_L \delta \to C^0(G, V \otimes_L \delta),$$

where

$$\alpha(f \otimes 1)(g) := f(g) \otimes \tilde{\delta}(g)$$

is a well-defined L-linear isomorphism. Moreover, it is H-equivariant, because for any $h \in H$ and $g \in G$ we have

$$\begin{aligned} \alpha(h(f \otimes 1))(g) &= \alpha((hf) \otimes \delta(h))(g) = f(gh) \otimes \tilde{\delta}(g)\delta(h) \\ &= f(gh) \otimes \tilde{\delta}(gh) = \alpha(f \otimes 1)(gh) = h\alpha(f \otimes 1)(g). \end{aligned}$$

Continuity can be checked on a base of the compact-open topology on the right hand side. Such a base is given by the sets $\Omega(K, U) = \{f \in C^0(G, V \otimes_L \delta) |\ f(K) \subset U\}$ for compact subsets $K \subset G$ and open subsets $U \subset V \otimes_L \delta$. Fixing such a K and U, we see that $\alpha^{-1}(\Omega(K, U))$ consists of functions f such that for any $k \in K$, $\tilde{\delta}(k)f(k)$ lies in the open subset U. But since $\tilde{\delta}$ is locally constant and K compact, we can cover K by finitely many open sets of the form $\tilde{\delta}^{-1}(\lambda_i)$ for some $\lambda_i \in L$, $i = 1, \ldots, s$. Since the intersections $K \cap \tilde{\delta}^{-1}(\lambda_i)$ are again compact, we obtain that $\alpha^{-1}(\Omega(K, U))$ is the finite intersection

$$\alpha^{-1}(\Omega(K, U)) = \bigcap_{i=1}^{s} \Omega(K_i, \lambda_i^{-1} U)$$

of open sets and therefore is open itself. Thus, α is continuous. □

We apply Lemma 7.1 to the character δ_F^{-1} in (10), and obtain

$$\begin{aligned} C^{i,\bullet} &\cong \bigoplus_{F \in R_i} \operatorname{Hom}_{P_F^\dagger}(\mathbb{1}, C^\bullet(G, V \otimes_L \delta_F^{-1})) \\ &\cong \bigoplus_{F \in R_i} C^\bullet(G, V \otimes_L \delta_F^{-1})^{P_F^\dagger}. \end{aligned}$$

As we have seen in Lemma 4.1, the complex $C^\bullet(G, V)$ is acyclic for the continuous cohomology of H for any open subgroup $H \leqslant G$ and hence, it can be used to compute the cohomology groups of H. Note that in Lemma 4.1, we assume that the subgroup is compact open, but in fact, we only use that it is open, so that the quotient G/H is discrete. Therefore, taking the cohomology of this complex gives us the following:

$$\begin{aligned} H^j(C^{i,\bullet}) &= \bigoplus_{F \in R_i} H^j(C^\bullet(G, V \otimes_L \delta_F^{-1})^{P_F^\dagger}) \\ &= \bigoplus_{F \in R_i} H^j(P_F^\dagger, V \otimes_L \delta_F^{-1}). \end{aligned}$$

We thus get a spectral sequence, converging to the continuous cohomology of G with coefficients in V, where the first page terms are given by the cohomology of the columns $C^{i,\bullet}$ of our double complex. More precisely, we obtain the spectral sequence (7).

7.2 Locally analytic vectors of admissible Banach space representations

We have already seen in Section 6, that the spectral sequence (7) can be very useful to generalize results that are known for compact groups to arbitrary p-adic reductive groups. In this section, we will show another application of this kind, by comparing the continuous cohomology with coefficients in an admissible Banach space representation to the continuous cohomology with coefficients in its subrepresentation given by locally analytic vectors. We start with a short reminder on the definition of locally analytic representations. For more details on this matter, see for example [24].

Let V be a barrelled locally convex Hausdorff L-vector space, meaning that it is a Hausdorff topological L-vector space whose topology is defined by a family of seminorms $q_i : V \to \mathbb{R}_{\geqslant 0}$, $i \in I$, in the sense that the topology on V is the coarsest topology, making all seminorms continuous maps, and in which every closed lattice is open (for more details, see Chapter 1, Section 6 in [19]). For such a space, we denote by $C^{\mathrm{la}}(G, V)$ the set of locally analytic V-valued functions on G.

Definition 7.2. A *locally analytic representation* of G over L is a representation of G on a barrelled locally convex Hausdorff L-vector space, such that G is acting on V via continuous endomorphisms and for each $v \in V$, the orbit map $\rho_v : G \to V$, $g \mapsto gv$, is a locally analytic function $\rho_v \in C^{\mathrm{la}}(G, V)$.

Let Π be an admissible L-Banach space representation of G. By definition, Π is a normed L-vector space, making it in particular a locally convex L-vector space, so that it makes sense to talk about the locally analytic vectors in Π, meaning the vectors $v \in \Pi$, whose orbit maps ρ_v are locally analytic functions. The locally analytic vectors form a locally analytic subrepresentation of Π, denoted by Π^{la}. The subrepresentation Π^{la} is equipped with the subspace topology coming from the injection

$$\begin{aligned} \Pi^{\mathrm{la}} &\to C^{\mathrm{la}}(G, \Pi), \\ v &\mapsto [g \mapsto g^{-1}v] \end{aligned}$$

making it a topological G-module (see Section 7 in [22]). The topology on Π^{la} is finer than the subspace topology coming from $\Pi^{\mathrm{la}} \subset \Pi$, so that the injection $\Pi^{\mathrm{la}} \hookrightarrow \Pi$ is a continuous map. An explanation for this is given for example in the beginning of Section 3.5 in [9]. Note that the author equips the representation Π^{la} with a finer topology but in our case of admissible Banach space representations, it coincides with the subspace topology coming from $C^{\mathrm{la}}(G, \Pi)$ (cf. [9, p. 118]).

By Corollary 1.6 in [18], for a compact p-adic Lie group K, we have isomorphisms

$$H^i(K, \Pi^{\mathrm{la}}) \cong H^i(K, \Pi).$$

Note that in [18], the condition for the existence of such isomorphism is that the representation Π has no higher locally analytic vectors, which is true for admissible Banach space representations (cf. Theorem 7.1 [22]).

We want to use the spectral sequence (7) to generalize this to the following:

Proposition 7.3. *Let Π be an admissible L-Banach space representation of G. Then for every $i \geqslant 0$, the maps*

$$H^i(G, \Pi^{\mathrm{la}}) \xrightarrow{\cong} H^i(G, \Pi),$$

induced by the inclusion $\Pi^{\mathrm{la}} \hookrightarrow \Pi$, are isomorphisms.

Before we can apply the spectral sequence argument, recall that the stabilizer groups $P_F^\dagger$ appearing in the sequence are not compact. We therefore need the following lemmas first.

Lemma 7.4. *Let* Γ *be a discrete group and let* $0 \to M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \to \dots$ *be a complex of* Γ*-modules. Then given an exact functor* $F : \mathrm{Mod}_\Gamma \to \mathrm{Mod}_\mathbb{Z}$*, we have isomorphisms*

$$H^i(F(M^\bullet)) \cong F(H^i(M^\bullet))$$

for every $i \geqslant 0$.

Proof. We fix the following notation: for any complex

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \to \dots,$$

let $Z^i(C^\bullet) := \ker(\delta^i)$ be the i-cocycles and $B^i(C^\bullet) := \mathrm{im}(\delta^{i-1}) \cong C^{i-1}/Z^{i-1}(C^\bullet)$ the i-coboundaries.

Applying the exact functor F to the exact sequence

$$0 \to Z^i(M^\bullet) \to M^i \to M^{i+1}$$

gives the isomorphism

$$Z^i(F(M^\bullet)) = F(Z^i(M^\bullet)), \ \forall i \geqslant 0. \tag{11}$$

Using this identification (11) and the exactness of F, we can also rewrite the coboundaries as

$$B^i(F(M^\bullet)) \cong \frac{F(M^{i-1})}{Z^{i-1}(F(M^\bullet))} \cong \frac{F(M^{i-1})}{F(Z^{i-1}(M^\bullet))} \cong F\left(\frac{M^{i-1}}{Z^{i-1}(M^\bullet)}\right) \cong F(B^i(M^\bullet)). \tag{12}$$

Combining these isomorphisms (11) and (12) gives us

$$H^i(F(M^\bullet)) \cong \frac{F(Z^i(M^\bullet))}{F(B^i(M^\bullet))} \cong F\left(\frac{Z^i(M^\bullet)}{B^i(M^\bullet)}\right) \cong F(H^i(M^\bullet)),$$

as claimed. □

Lemma 7.5. *Let* Γ *be a discrete group, then for every* $i \geqslant 0$*, the functor* $C^i(\Gamma, -)^\Gamma$ *is exact.*

In particular, for any complex $0 \to M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \to \dots$ *of* Γ*-modules, we have*

$$H^j(C^i(\Gamma, M^\bullet)^\Gamma) \cong C^i(\Gamma, H^j(M^\bullet))^\Gamma, \ \forall i, j \geqslant 0.$$

Proof. To show that the functor $C^i(\Gamma, -)^\Gamma$ is exact, let

$$0 \to N \to M \to Q \to 0$$

be a short exact sequence of Γ-modules. Note that, since Γ is discrete, any function on Γ (or on a product Γ^{i+1} of copies of Γ) is automatically continuous and hence, the functor $C^i(\Gamma, -)$ is exact and we obtain a short exact sequence

$$0 \to C^i(\Gamma, N) \to C^i(\Gamma, M) \to C^i(\Gamma, Q) \to 0.$$

Applying the functor $(-)^\Gamma$ to this, induces a long exact sequence in cohomology

$$0 \to C^i(\Gamma, N)^\Gamma \to C^i(\Gamma, M)^\Gamma \to C^i(\Gamma, Q)^\Gamma \to H^1(\Gamma, C^i(\Gamma, N)),$$

where the first cohomology group $H^1(\Gamma, C^i(\Gamma, N))$ vanishes, because $C^i(\Gamma, N)$ is acyclic, since by Lemma 2.1 and Shapiro's Lemma [4, Proposition 3], we have

$$H^i(\Gamma, C^i(\Gamma, N)) \cong H^i(\Gamma, \mathrm{Ind}_{\{1\}}^{\Gamma} C^{i-1}(\Gamma, N)) \cong H^i(\{1\}, C^{i-1}(\Gamma, N)) = 0$$

for $i > 0$. This proves exactness of $C^i(\Gamma, -)^\Gamma$.

We can thus apply Lemma 7.4 to the functor $C^i(\Gamma, -)^\Gamma$ and the complex $M^\bullet$, to obtain $H^j(C^i(\Gamma, M^\bullet)^\Gamma) \cong C^i(\Gamma, H^j(M^\bullet))^\Gamma$. □

Lemma 7.6. *Let V be a topological G-module and let $H \leqslant G$ be an open normal subgroup of G. Then there exists a Hochschild–Serre spectral sequence:*

$$E_2^{i,j} \cong H^i(G/H, H^j(H, V)) \Rightarrow H^{i+j}(G, V).$$

Proof. We mimic the proof of Proposition 5 in [4], where the authors prove the existence of a Hochschild–Serre spectral sequence for a closed normal subgroup H, assuming some additional conditions that are not needed for the case of an open normal subgroup. We consider the double complex

$$C^{i,j} = C^i(G/H, C^j(G, V)^H)^{G/H}, \ i, j \geqslant 0.$$

This double complex induces two spectral sequences with the same limit term. The first spectral sequence comes from the horizontal filtration of the double complex and its E_1-terms are given by:

$$E_{1,h}^{i,j} = H^j(C^{\bullet,i}) = H^j(C^\bullet(G/H, C^i(G, V)^H)^{G/H}) = H^j(G/H, C^i(G, V)^H)$$

Note that the groups $C^i(G, V)^H$ are acyclic for the cohomology of G/H. Indeed, since G/H is discrete, the continuous cohomology of G/H is just group cohomology. Now for any G/H-module W, we can give it a topological G/H-module structure by equipping it with the discrete topology and obtain

$$\begin{aligned} \mathrm{Hom}_{G/H}(W, C^i(G, V)^H) &= \mathrm{Hom}_{G/H}^{\mathrm{cts}}(W, C^i(G, V)^H) \\ &\cong \mathrm{Hom}_G^{\mathrm{cts}}(W, C^i(G, V)) \\ &\cong \mathrm{Hom}^{\mathrm{cts}}(W, C^{i-1}(G, V)) \\ &= \mathrm{Hom}(W, C^{i-1}(G, V)), \end{aligned}$$

using Lemma 2.2. Since the functor $\mathrm{Hom}(-, C^{i-1}(G, V))$ is exact, so is the functor $\mathrm{Hom}_{G/H}(-, C^i(G, V)^H)$, showing that $C^i(G, V)^H$ is acyclic for the cohomology of G/H.

Hence, we obtain

$$E_{1,h}^{i,j} = H^j(G/H, C^i(G, V)^H) = \begin{cases} 0, \text{ if } j \neq 0 \\ (C^i(G, V)^H)^{G/H} \cong C^i(G, V)^G, \text{ if } j = 0. \end{cases}$$

And the E_2-terms are given by

$$E_{2,h}^{i,j} = \begin{cases} 0, \text{ if } j \neq 0 \\ H^i(C^\bullet(G, V)^G) = H^i(G, V), \text{ if } j = 0. \end{cases}$$

Therefore, the two spectral sequences of the double complex $C^{i,j}$ converge to the continuous cohomology of G with coefficients in V.

The E_1-terms of the spectral sequence coming from the vertical filtration are given by

$$E_{1,v}^{i,j} = H^j(C^{i,\bullet}) = H^j(C^i(G/H, C^\bullet(G,V)^H)^{G/H}).$$

We can apply Lemma 7.5 to the discrete group $\Gamma = G/H$ and the complex $M^\bullet = C^\bullet(G,V)^H$ and get

$$H^j(C^i(G/H, C^\bullet(G,V)^H)^{G/H}) \cong C^i(G/H, H^j(C^\bullet(G,V)^H))^{G/H}.$$

By Lemma 4.1, the resolution $C^\bullet(G,V)$ can be used to compute the continuous cohomology of H, so that

$$E_{1,v}^{i,j} = C^i(G/H, H^j(C^\bullet(G,V)^H))^{G/H} = C^i(G/H, H^j(H,V))^{G/H}.$$

Then the E_2-terms are given by

$$E_{2,v}^{i,j} = H^{i,j}(E_{1,v}^{\bullet,\bullet}) = H^i(C^\bullet(G/H, H^j(H,V))^{G/H}) = H^i(G/H, H^j(H,V)).$$

Since we know that this spectral sequence has the same limit term as the one from the horizontal filtration, we obtain the wanted spectral sequence. □

Lemma 7.7. *Let $F \in X_q$ be a q-dimensional facet of the Bruhat–Tits building of G. Then for every $i \geqslant 0$, we have isomorphisms*

$$H^i(P_F^\dagger, \Pi^{\mathrm{la}}) \cong H^i(P_F^\dagger, \Pi).$$

Proof. Let us fix a facet $F \in X_q$. Recall that there exists a compact open subgroup $R_F \leqslant G$, which is open and normal in $P_F^\dagger$ (for a construction, see Section 1.2 in [21]).

Since the subgroup R_F is compact open in $P_F^\dagger$, we can apply Lemma 7.6, which proves the existence of a Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i(P_F^\dagger/R_F, H^j(R_F, \Pi)) \Rightarrow H^{i+j}(P_F^\dagger, \Pi).$$

By the same argument, we also obtain a spectral sequence for the locally analytic subrepresentation

$$\widetilde{E}_2^{i,j} = H^i(P_F^\dagger/R_F, H^j(R_F, \Pi^{\mathrm{la}})) \Rightarrow H^{i+j}(P_F^\dagger, \Pi^{\mathrm{la}}).$$

On the other hand, the inclusion $\Pi^{\mathrm{la}} \hookrightarrow \Pi$ induces maps between the limit terms $H^i(P_F^\dagger, \Pi^{\mathrm{la}}) \to H^i(P_F^\dagger, \Pi)$, as well as a map between all second page terms of the spectral sequence

$$\widetilde{E}_2^{i,j} = H^i(P_F^\dagger/R_F, H^j(R_F, \Pi^{\mathrm{la}})) \to H^i(P_F^\dagger/R_F, H^j(R_F, \Pi)) = E_2^{i,j}.$$

We thus obtain a map of spectral sequences $\widetilde{E}_r^{i,j} \to E_r^{i,j}$.

But since R_F is compact, Corollary 1.6 in [18] tells us that these are isomorphisms for $r = 2$ and for all $i, j \geqslant 0$. By the Comparison Theorem 5.2.12 in [25], this implies that also the map of the limit terms

$$H^i(P_F^\dagger, \Pi^{\mathrm{la}}) \to H^i(P_F^\dagger, \Pi)$$

is an isomorphism. □

We can now prove Proposition 7.3.

Proof of Proposition 7.3. The argumentation is similar to the one in the proof of Lemma 7.7. By the discussion in Section 7.1, we obtain spectral sequences like in (7) for both, Π and Π^{la}:

$$\widetilde{E}_1^{i,j} = \bigoplus_{F \in R_i} H^j(P_F^\dagger, \Pi^{\mathrm{la}} \otimes_L \delta_F^{-1}) \Rightarrow H^{i+j}(G, \Pi^{\mathrm{la}}),$$
$$E_1^{i,j} = \bigoplus_{F \in R_i} H^j(P_F^\dagger, \Pi \otimes_L \delta_F^{-1}) \Rightarrow H^{i+j}(G, \Pi).$$

But again, the inclusion $\Pi^{\mathrm{la}} \hookrightarrow \Pi$ induces maps between all the terms. And by Lemma 7.7, those maps between the E_1 terms are isomorphisms. Theorem 5.2.12 in [25] implies that the maps

$$H^i(G, \Pi^{\mathrm{la}}) \to H^i(G, \Pi)$$

are isomorphisms. □

Combining Proposition 7.3 with Corollary 6.8, the following Corollary follows immediately.

Corollary 7.8. *For any admissible L-Banach space representation Π of G, the cohomology groups $H^i(G, \Pi^{\mathrm{la}})$ with coefficients in the locally analytic vectors of Π are finite dimensional over L.*

7.3 Topology on the cohomology groups

Let V be a Hausdorff topological G-module. Note that the continuous cohomology groups $H^i(G, V)$ inherit a quotient topology from the compact-open topology on the cochains. We call this the *canonical topology*. The compact-open topology on the cochains $C^i(G, V)$ is Hausdorff, since for any pair of continuous maps $f, f' : G^{i+1} \to V$ in $C^i(G, V)$ with $f \neq f'$, we can find disjoint open neighborhoods of f, f', respectively, as follows: There is an element $g \in G^{i+1}$ such that $f(g) \neq f'(g)$ and since V is Hausdorff by assumption, we can find open disjoint neighborhoods $U \ni f(g)$, $U' \ni f'(g)$, $U \cap U' = \varnothing$. But then we have open disjoint neighborhoods $\Omega(\{g\}, U) \ni f$ and $\Omega(\{g\}, U') \ni f'$.

Since $C^i(G, V)$ is Hausdorff, so is $C^i(G, V)^G$ as a subspace of a Hausdorff space. And the space of i-cocycles $Z^i(G, V) = \ker(d^i : C^i(G, V)^G \to C^{i+1}(G, V)^G)$ is also Hausdorff and moreover, it is a closed subspace of $C^i(G, V)^G$ as it is the kernel of a continuous map between Hausdorff spaces. This also implies that the quotient topology on the i-coboundaries given by $B^i(G, V) \cong C^{i-1}(G, V)^G / Z^{i-1}(G, V)$ is Hausdorff.

The quotient topology on the cohomology groups is in general not Hausdorff. However, Proposition 6 in [4] gives a nice criterion for the cohomology groups being Hausdorff, which almost fits our setting. We need to assume though that the group G is σ-compact, which means that it can be written as a countable union of compact subsets. If F is a finite extension of $\mathbb{Q}_p$, then the group $\mathrm{GL}_n(F)$ is σ-compact. This can be shown by using the Cartan decomposition (cf. Section 3.2 in [2]).

As a consequence, we know that any p-adic reductive group $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a linear algebraic group, is also σ-compact, since it embeds into $\mathrm{GL}_n(F)$ as a closed subgroup for some n. We then obtain the following:

Corollary 7.9. *Assume that G is a p-adic reductive group and Π is an admissible L-Banach space representation of G. Then the canonical topologies on the cohomology groups $H^i(G,\Pi)$ and $H^i(G,\Pi^{\mathrm{la}})$ are Hausdorff for all $i \geqslant 0$.*

Proof. Corollary 6.8 implies that the cohomology groups are finite dimensional. Moreover, since Π is assumed to be an L-Banach space representation, it is a complete metrizable L-vector space. We can hence apply Proposition 6 in [4], which states that the cohomology groups are strongly Hausdorff, i.e. the inclusion of i-coboundaries $B^i(G,\Pi)$ into i-cocycles $Z^i(G,\Pi)$ has a continuous section, say $s : Z^i(G,\Pi) \to B^i(G,\Pi)$. We claim that the continuous L-linear bijection

$$\ker(s) \hookrightarrow Z^i(G,\Pi) \xrightarrow{\pi} H^i(G,\Pi)$$

is a homeomorphism. It suffices to show that the map is open. For this, let $U \subset \ker(s)$ be an open subset. This means that there is an open subset $V \subset Z^i(G,\Pi)$ such that $U = V \cap \ker(s)$. Since the map $\mathrm{id} - s : Z^i(G,\Pi) \to Z^i(G,\Pi)$ is continuous, the set $V' := (\mathrm{id} - s)^{-1}(V) = \{z \in Z^i(G,\Pi) | z - s(z) \in V\}$ is open in $Z^i(G,\Pi)$. Moreover, the intersection

$$\begin{aligned} V' \cap \ker(s) &= \{z \in Z^i(G,\Pi) | z - s(z) \in V \text{ and } s(z) = 0\} \\ &= \{z \in Z^i(G,\Pi) | s(z) = 0 \text{ and } z \in V\} \\ &= V \cap \ker(s) \end{aligned}$$

coincides with the intersection $V \cap \ker(s) = U$. But then we have

$$\pi(U) = \pi(V \cap \ker(s)) = \pi(V' \cap \ker(s)) = \pi(V') \text{ open},$$

since for any $z \in V'$, $\pi(z) = \pi(z) - \pi(s(z)) = \pi(z - s(z))$ with $z - s(z) \in V \cap \ker(s)$.

As the kernel of a continuous map between Hausdorff spaces, $\ker(s)$ is Hausdorff and thus so is $H^i(G,\Pi)$.

By Proposition 7.3, the continuous L-linear map $H^i(G,\Pi^{\mathrm{la}}) \to H^i(G,\Pi)$ is a bijection. This implies that the topology on $H^i(G,\Pi^{\mathrm{la}})$ is finer than (or equal to) the topology on $H^i(G,\Pi)$ which is Hausdorff, so that the groups $H^i(G,\Pi^{\mathrm{la}})$ are also Hausdorff. □

Lemma 7.10. *Let V and W be topological L-vector spaces and $\phi : V \xrightarrow{\cong} W$ be an L-linear isomorphism. Assume that $\mathcal{B}$ is a subbasis of open neighborhoods of $0 \in V$, such that the set $\phi(\mathcal{B}) = \{\phi(U) |\ U \in \mathcal{B}\}$ is a subbasis of open neighborhoods of $0 \in W$. Then the isomorphism ϕ is a homeomorphism.*

Proof. By symmetry, it is enough to show that the map ϕ is continuous. For this, let $U \subset W$ be an open set. We show that $\phi^{-1}(U)$ is open in V, by showing that each element $v \in \phi^{-1}(U)$ has an open neighborhood that is contained in $\phi^{-1}(U)$.

Since W is a topological vector space, the translated set $-\phi(v) + U$ is again open in W and contains the zero element. By assumption, we can find finitely many open neighborhoods of 0 in V, $V_1, \ldots, V_n \in \mathcal{B}$, such that the intersection $\bigcap_{i=1}^n \phi(V_i)$ is contained in $-\phi(v) + U$. This implies that the intersection of open sets $\bigcap_{i=1}^n (v + V_i)$ is contained in the inverse image $\phi^{-1}(U)$ and contains v. Therefore, the map is indeed continuous. □

Proposition 7.11. *Let V be a topological G-module on a locally convex L-vector space. Then the compact-open topology on the space of cochains $C^i(G,V)$ is locally convex, for every $i \geqslant 0$.*

Proof. By the heomeomorphism $C^i(G,V) \cong C^0(G, C^{i-1}(G,V))$ (cf. Lemma 2.1), it is enough to show that $C(G,V) = C^0(G,V)$ is locally convex.

Say the locally convex topology on V is defined by the family of seminorms $\{q_i\}_{i\in I}$. For every compact subset $K \subset G$, and $i \in I$, we define a seminorm on $C(G,V)$ by

$$q_{K,i} : C(G,V) \to \mathbb{R},$$
$$f \mapsto \sup_{k\in K} q_i(f(k))$$

Denote by $C(G,V)_{\mathrm{lc}}$ the space $C(G,V)$ equipped with the locally convex topology given by the family $\{q_{K,i}\}_{K,i}$ of seminorms and by $C(G,V)_{\mathrm{co}}$ the space $C(G,V)$ equipped with the compact-open topology. We claim that the identity map $C(G,V)_{\mathrm{co}} \to C(G,V)_{\mathrm{lc}}$ is a homeomorphism.

By Lemma 7.10, it is enough to find a set of neighborhoods of $0 \in C(G,V)$ that forms a subbasis of open neighborhoods in both topologies. By definition, a subbasis of open neighborhoods of $0 \in C(G,V)_{\mathrm{lc}}$ is given by sets of the form $q_{K,i}^{-1}(B_\epsilon)$, for $K \subset G$ compact, $i \in I$, $\epsilon > 0$. Here, B_ϵ denotes the open ball of radius ϵ around $0 \in \mathbb{R}$. We can rewrite these sets as follows:

$$\begin{aligned} q_{K,i}^{-1}(B_\epsilon) &= \{f \in C(G,V) \mid q_{K,i}(f) < \epsilon\} \\ &= \{f \in C(G,V) \mid \sup_{k\in K} q_i(f(k)) < \epsilon\} \\ &= \{f \in C(G,V) \mid q_i(f(k)) < \epsilon, \ \forall k \in K\} \\ &= \{f \in C(G,V) \mid f(k) \in q_i^{-1}(B_\epsilon) \ \forall k \in K\} \\ &= \Omega(K, q_i^{-1}(B_\epsilon)). \end{aligned}$$

These sets are open neighborhoods of 0 in $C(G,V)_{\mathrm{co}}$. We want to show that they form a subbasis of open neighborhoods of 0. Indeed, by definition of the compact-open topology, a subbasis of open neighborhoods of 0 is given by sets of the form $\Omega(K,U)$ for $K \subset G$ compact and $U \subset V$ open. But since V is locally convex, we can find finitely many open sets of the form $q_{i_j}^{-1}(B_{\epsilon_j})$, for $j = 1, \ldots, m$, with the property that

$$\bigcap_{j=1}^m q_{i_j}^{-1}(B_{\epsilon_j}) \subset U.$$

We thus obtain

$$\bigcap_{j=1}^m \Omega(K, q_{i_j}^{-1}(B_{\epsilon_j})) \subset \Omega(K,U),$$

and we see that the sets $\Omega(K, q_i^{-1}(B_\epsilon))$ form a subbasis of open neighborhoods of 0 in $C(G,V)_{\mathrm{co}}$ as stated. □

Since subspaces and quotients of locally convex vector spaces are again locally convex (see Section 5 in [19]), Proposition 7.11 implies the following corollary.

Corollary 7.12. *For every topological G-module on a locally convex L-vector space, the canonical topology on the continuous cohomology groups $H^i(G,V)$ is locally convex.*

Corollary 7.13. *For every admissible L-Banach space representation Π of G, the continuous cohomology groups $H^i(G,\Pi)$ and $H^i(G,\Pi^{\mathrm{la}})$, equipped with their canonical topologies, are finite dimensional locally convex Hausdorff L-vector spaces. In particular, their topology is uniquely determined, namely it is the finest locally convex topology.*

In particular, the isomorphisms $H^i(G,\Pi^{\mathrm{la}}) \cong H^i(G,\Pi)$ are homeomorphisms.

Proof. By Corollaries 6.8, 7.9 and 7.12, we know that the cohomology groups are finite dimensional, Hausdorff and locally convex. Therefore, we can apply Proposition 4.13 in [19], which proves that there is only one choice of locally convex Hausdorff topology on a finite dimensional L-vector spaces. The author gives an explicit description of this topology as the one defined by the norm $\| \sum_{i=1}^{n} \lambda_i e_i \| := \max_{1 \leqslant i \leqslant n} |\lambda_i|$, for any choice of L-basis $e_1, \dots, e_n$ of the vector space.

Since the canonical topologies on $H^i(G, \Pi^{\mathrm{la}})$ and $H^i(G, \Pi)$ are uniquely determined and the groups are isomorphic as L-vector spaces (Proposition 7.3), the topologies must agree and the isomorphism is a homeomorphism of topological L-vector spaces. □

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