Partial observability of stochastic semilinear systems

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Abstract

In this article, we consider an Itô stochastic semilinear differential equation with unknown initial state and a linear observation system. It is proved that under a certain condition on the observability Gramian, the initial state of the equation can be recovered. This result is demonstrated by an example.

Keywords: Stochastic system, semilinear system, partial observability, observability Gramian.

1 Introduction

In control theory, the properties of controllability and observability of systems have played an important role. These concepts were introduced by Kalman for linear dynamical systems [1–3]. It is worth noting that these concepts are dual to each other in the linear case. This subject is treated in detail in Zabczyk [4] and Curtain and Zwart [5] for deterministic and Lü and Zhang [6] for stochastic cases. Currently, there are many papers dealing with the controllability of nonlinear deterministic and stochastic systems, for example [7–17] and the references therein. Moreover, the controllability of stochastic systems in finite and infinite dimensional spaces has been studied in detail in the recent book [6]. Compared to the deterministic cases (for both finite dimensional and distributed parameter control systems), the study of stochastic controllability/observability is quite unsatisfactory, even for stochastic finite control systems (see Chapter 6 of [6] for a more detailed analysis). Moreover, one can refer to the book [32], where stochastic controllability and stochastic observability have been studied in detail. For linear stochastic control systems controlled only in the drift terms, the concept of partial approximate controllability has been studied by Dou and Lu [18].

For nonlinear systems, the duality between controllability and observability problems does not apply. Therefore, it is impossible to obtain observability conditions for nonlinear systems from controllability conditions using the duality method. The main results of [19,20] concern a partially observable linear stationary control system with an additive Gaussian white noise disturbance (the system (S)) and its deterministic part (the system (D)). In [21], it is shown that the S- controllability (the C- controllability) of a partially observable linear stationary control system with an additive Gaussian white noise disturbance on all intervals $[0,T]$ for $T>0$ is equivalent to the approximate (complete) controllability of its deterministic part on all intervals $[0,T]$ for $T>0$. As far as

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we know, there are only a few works dealing with the observability of nonlinear systems [22–26]. Basically, controllability problems for nonlinear systems are studied using fixed point theorems, or the controllability conditions are subjected to the requirements of these theorems. This method for studying controllability problems does not find a reflection in the case of observability problems. However, the recently developed method for the controllability of nonlinear systems [27], which avoids fixed point theorems, allows to make some progress in observability problems.

In this paper, we consider a stochastic semilinear system driven by a Wiener process. The following elements of the considered problem can be highlighted. First of all, we study partial observability. In this respect, we are motivated by the definition of partial controllability from [28, 29]. The issue is that some systems can be written in standard form if their dimension is increased. Examples include higher order differential equations, wave equations, delay equations, and stochastic equations driven by wide band noises [30]. Therefore, the concepts are too heavy for their controllability in the extended state space. Instead, the partial controllability concepts require the original (non-extended) state space and are achieved by projection operators (matrices). Similarly, for observability problems, we may be interested in recovering not the full initial state, but its projection onto a subspace. For the finite-dimensional linear deterministic case, this issue is known as the Kalman decomposition.

Secondly, the system is semilinear and stochastic. Semilinear systems are nonlinear systems with a specified linear part. The specification of the linear part helps clearly define sufficient conditions because controllability and observability properties heavily depend on the behavior of the linear part of semilinear systems. Most importantly, the system under consideration is stochastic equation driven by a Wiener process with the complete and continuous filtration $\{\mathcal{F}_t\}$. Briefly, under a certain sufficient condition, we give a construction of a projection of the $\mathcal{F}_0$-measurable square integrable random initial value of the system on the basis of linear observations.

The rest of the paper is organized as follows. Stochastic semilinear systems with the linear observation are described in Section 2. Main result is studied with the state and proof of Theorem 3.1 in Section 3. Next, we provide an example in Section 4. Finally, conclusions are given in Section 5.

1.1 Mathematical description

Here are some general notations used in this paper. $\mathbb{R}^n$ denotes an n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ denotes the space of $(n \times m)$-matrices. As always, $\mathbb{R} = \mathbb{R}^1$. $A^\top$ is the transpose of the matrix A. A square matrix A is symmetric if $A = A^\top$. We say that a square matrix A is nonsingular if it has a nonzero determinant, i.e., $\det A \neq 0$. In this case, A^{-1} exists. In general, the identity and null matrices are denoted by I and 0 regardless of their dimensions. $L_2(a, b; \mathbb{R}^n)$ is the space of square integrable $\mathbb{R}^n$-valued functions on $[a, b]$ with respect to the Lebesgue measure.

We always assume that the extended probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ is given. Here, $\{\mathcal{F}_t\}$ is a complete and continuous filtration generated by the underlying Wiener process. $L_2(\Omega, \mathbb{R}^n)$ is a space of square integrable random variables over the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For a sub-$\sigma$-field $\mathcal{G}$ of $\mathcal{F}$, $L_2(\Omega, \mathcal{G}, \mathbb{R}^n)$ denotes a subspace of $L_2(\Omega, \mathbb{R}^n)$ consisting of $\mathcal{G}$-measurable random variables. $L_2^{\mathcal{F}_t}(0, T; \mathbb{R}^n)$ is a subspace of $L_2([0, T] \times \Omega, \mathbb{R}^n)$ consisting of $\mathcal{F}_t$-adapted random processes. A space of random processes which are continuous from $[0, T]$ to $L_2(\Omega, \mathbb{R}^n)$ and $\mathcal{F}_t$-adapted is denoted by $C^{\mathcal{F}_t}(0, T; \mathbb{R}^n)$. We use the symbol $\mathbf{E}$ for expectation. A (finite-dimensional) Wiener process is standard if its initial value and expectation are zero and covariance matrix is identity. The dependence of the random processes on the time parameter will be shown in subscript, for example, w_t instead of $w(t)$.

2 Statement of the problem

We consider the following semilinear stochastic differential equation with the linear observation on the interval $[0, T]$:

$$\begin{cases} \mathrm{d}x_t = (Ax_t + f(t, x_t))\,\mathrm{d}t + g(t, x_t)\,\mathrm{d}w_t, \\ z_t = Cx_t, \end{cases} \tag{2.1}$$

where x and z are n- and m-dimensional state and observation processes, respectively. Throughout this paper, we assume the following conditions:

(A) $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$.

(B) f and g are nonlinear functions from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times k}$, respectively, with the properties

- f and g are measurable.
- f and g are Lipschitz continuous in x, that is, for all $0 \le t \le T$ and $x, y \in \mathbb{R}^n$,

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \le K_1\|x - y\|.$$

- f and g satisfy the linear growth condition, that is, for all $0 \le t \le T$ and $x, y \in \mathbb{R}^n$,

$$\|f(t, x)\|^2 + \|g(t, x)\|^2 \le K_2(1 + \|x\|^2).$$

(C) w is a standard k-dimensional Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $\{\mathcal{F}_t\}$ is a complete and continuous filtration generated by w.

These conditions imply the existence of a unique solution in $C^{\mathcal{F}_t}(0, T; \mathbb{R}^n)$ of the state equation in (2.1) for any initial value $x_0 \in L_2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$ (see, for example, [33,34]). In turn, this means that the stochastic integral equation

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s, x_s)\,\mathrm{d}s + \int_0^t e^{A(t-s)}g(s, x_s)\,\mathrm{d}w_s, \quad 0 \le t \le T,$$

has a unique solution in $C^{\mathcal{F}_t}(0, T; \mathbb{R}^n)$.

Definition 2.1. The equation in (2.1) is said to be observable on $[0, T]$ if the knowledge of the observation process $z(t)$, $0 \le t \le T$, uniquely determines the initial state x_0 in the sense of equality in $L_2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$.

Definition 2.2. [31] A square matrix P is called *an orthogonal projection matrix* if it is both idempotent and symmetric, that is, $P^2 = P$ and $P^\top = P$.

If a system is unobservable, then still it may be possible to extract some particular information about its initial state. The Kalman decomposition deals with this problem in the case of linear deterministic systems. To extend this issue to the semilinear and stochastic case, consider any orthogonal projection matrix P on $\mathbb{R}^n$. We can decompose $\mathbb{R}^n$ to the direct sum $\mathbb{R}^n = R \oplus R^\perp$, where R and $R^\perp$ are the range and the kernel of P, respectively. Letting the dimension of R be r, we obtain that the dimension of $R^\perp$ is $n - r$. Choose an orthonormal basis $e_1, \ldots, e_r$ in R and consider the $(n \times r)$-matrix whose columns are the vectors $e_1, \ldots, e_r$. Denote this matrix by $L^\top$, reserving the symbol L for its transposition. The matrix L is said to be an isometry, which vanishes on $R^\perp$. The relation $P = L^\top L$ holds between the matrices P and L. To stress this relation, we will denote P as P_L.

To illustrate the relation between P and L, consider the projection matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

on $\mathbb{R}^3$ with $n = 3$. Its range R and kernel $R^\perp$ are

$$R = \{[\,x \;\; y \;\; 0\,]^\top : x, y \in \mathbb{R}\} \;\text{ and }\; R^\perp = \{[\,0 \;\; 0 \;\; z\,]^\top : z \in \mathbb{R}\}.$$

So, $r = 2$ and $n - r = 1$. Consider the orthonormal basis in R consisting of the following two vectors

$$e_1 = [\,1 \;\; 0 \;\; 0\,]^\top \;\text{ and }\; e_2 = [\,0 \;\; 1 \;\; 0\,]^\top.$$

Then

$$L^\top = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \;\text{ and }\; L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

which implies

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = L^\top L.$$

Therefore, we regard this relation by letting $P = P_L$.

Definition 2.3. The system in (2.1) is said to be L-partially observable on $[0, T]$ if the knowledge of the observation process $z(t)$, $0 \le t \le T$, uniquely determines the projection $P_L x_0$ of the initial state x_0, where the uniqueness means that if the function

$$x_0 \to z_t = C\left(e^{At}x_0 + \int_0^t e^{A(t-s)} f(s, x_s)\,\mathrm{d}s + \int_0^t e^{A(t-s)} g(s, x_s)\,\mathrm{d}w_s\right),\ 0 \le t \le T, \tag{2.2}$$

sending the initial states $x_0 \in L_2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$ to the observation processes $z \in L_2(0, T; L_2(\Omega, \mathbb{R}^m))$ assigns the same observation process to two different initial values x_0 and x_0', then $P_L x_0 = P_L x_0'$ with probability 1.

To state our main result, we consider the observability Gramian

$$G(t) = \int_0^t e^{A^\top s} C^\top C e^{As}\,\mathrm{d}s$$

and the L-partial observability Gramian by $G_L(t) = LG(t)L^\top$. Let us add the followings to the conditions (A)–(C):

(D) P_L is an orthogonal projection matrix from $\mathbb{R}^n$ onto its nontrivial subspace R, $G_L(t)$ is nonsingular for all $0 < t \le T$ such that $t\|G_L(t)^{-1}\|$ is bounded on $(0, T]$.

(E) $G(t)(R^\perp) \subseteq R^\perp$ for all $0 < t \le T$.

3 Main result

The following Theorem 3.1 is the main result of this article and establishes partial observability of stochastic semilinear systems.

Theorem 3.1. *Assume that the conditions (A)–(E) hold. Then the semilinear stochastic system in* (2.1) *is L-partially observable on $[0,T]$. In particular, Lx_0 can be recovered by*

$$Lx_0 = \lim_{t\to 0^+} G_L(t)^{-1} L \int_0^t e^{A^\top s} C^\top z(s) \mathrm{d}s, \tag{3.1}$$

where the limit is in the sense of convergence in $L_2(\Omega, \mathbb{R}^n)$.

Proof. At first, note that if Lx_0 is recovered then $P_L x_0$ is also recovered since $P_L x_0 = L^\top L x_0$. Therefore, it suffices to verify the limit in (3.1). Next, since the solution $x : [0,T] \to L_2(\Omega, \mathbb{R}^n)$ of the equation in (2.1) is continuous, it is bounded. Therefore, by linear growth condition, for all $0 \le t \le T$,

$$\mathbf{E}\|f(t,x_t)\|^2 + \mathbf{E}\|g(t,x_t)\|^2 \le K_2(1 + \mathbf{E}\|x_t\|^2) \le K \tag{3.2}$$

for some $K > 0$, depending only on x_0.

Now, we fix any initial state $x_0 \in L_2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. Then

$$\begin{aligned}\int_0^t e^{A^\top s} C^\top z_s \, \mathrm{d}s = \int_0^t e^{A^\top s} C^\top C e^{As} x_0 \, \mathrm{d}s + \int_0^t \int_0^s e^{A^\top s} C^\top C e^{A(s-r)} f(r,x_r) \, \mathrm{d}r\mathrm{d}s \\ + \int_0^t \int_0^s e^{A^\top s} C^\top C e^{A(s-r)} g(r,x_r) \, \mathrm{d}w_r \mathrm{d}s.\end{aligned} \tag{3.3}$$

Here,

$$\int_0^t e^{A^\top s} C^\top C e^{As} x_0 \, \mathrm{d}s = G(t) x_0.$$

By Fubini's theorem, the second term in the right-hand side of (3.3) can be written as

$$\begin{aligned}\int_0^t \int_0^s e^{A^\top s} C^\top C e^{A(s-r)} f(r,x_r) \, \mathrm{d}r\mathrm{d}s &= \int_0^t \int_r^t e^{A^\top s} C^\top C e^{As} e^{-Ar} f(r,x_r) \, \mathrm{d}s\mathrm{d}r \\ &= \int_0^t (G(t) - G(r)) e^{-Ar} f(r,x_r) \, \mathrm{d}r.\end{aligned}$$

Similarly, by the stochastic Fubini theorem, the last term in the right-hand side of (3.3) can be written as

$$\begin{aligned}\int_0^t \int_0^s e^{A^\top s} C^\top C e^{A(s-r)} g(r,x_r) \, \mathrm{d}w_r\mathrm{d}s &= \int_0^t \int_r^t e^{A^\top s} C^\top C e^{As} e^{-Ar} g(r,x_r) \, \mathrm{d}s\mathrm{d}w_r \\ &= \int_0^t (G(t) - G(r)) e^{-Ar} g(r,x_r) \, \mathrm{d}w_r.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_0^t e^{A^\top s} C^\top z_s \, \mathrm{d}s = G(t)\left(x_0 + \int_0^t e^{-Ar} f(r,x_r) \, \mathrm{d}r + \int_0^t e^{-Ar} g(r,x_r) \, \mathrm{d}w_r \right) \\ - \int_0^t G(r) e^{-Ar} f(r,x_r) \, \mathrm{d}r - \int_0^t G(r) e^{-Ar} g(r,x_r) \, \mathrm{d}w_r.\end{aligned} \tag{3.4}$$

At this point, note that any $h \in L_2(\Omega, \mathbb{R}^n)$ can be written as $h = P_L h + (h - P_L h)$. Here, $h - P_L h$ is a random variable with values in the kernel $R^\perp$ of P_L. Then by condition (E), $G(t)(h - P_L h)$ has values in $R^\perp$ as well. Respectively, $LG(t)(h - P_L h) = 0$ because L and P_L have the same kernels. Applying this to (3.4), we obtain

$$\begin{aligned} L\int_0^t e^{A^\top s} C^\top z_s \,\mathrm{d}s = LG(t)P_L\Big(x_0 + \int_0^t e^{-Ar} f(r, x_r)\,\mathrm{d}r + \int_0^t e^{-Ar} g(r, x_r)\,\mathrm{d}w_r\Big) \\ - \int_0^t LG(r)P_L e^{-Ar} f(r, x_r)\,\mathrm{d}r - \int_0^t LG(r)P_L e^{-Ar} g(r, x_r)\,\mathrm{d}w_r. \end{aligned}$$

Here, $LG(t)P_L = LG(t)L^\top L = G_L(t)L$ for all $0 < t \le T$. Therefore,

$$\begin{aligned} G_L(t)^{-1} L\int_0^t e^{A^\top s} C^\top z_s \,\mathrm{d}s = L\Big(x_0 + \int_0^t e^{-Ar} f(r, x_r)\,\mathrm{d}r + \int_0^t e^{-Ar} g(r, x_r)\,\mathrm{d}w_r\Big) \\ - G_L(t)^{-1}\int_0^t G_L(r) L e^{-Ar} f(r, x_r)\,\mathrm{d}r \\ - G_L(t)^{-1}\int_0^t G_L(r) L e^{-Ar} g(r, x_r)\,\mathrm{d}w_r. \end{aligned}$$

We obtain that

$$\begin{aligned} \mathbf{E}\Big\| G_L(t)^{-1} L\int_0^t e^{A^\top s} C^\top z_s \,\mathrm{d}s - Lx_0\Big\|^2 \le 4\|L\|^2 \mathbf{E}\Big\|\int_0^t e^{-Ar} f(r, x_r)\,\mathrm{d}r\Big\|^2 \\ + 4\|L\|^2 \mathbf{E}\Big\|\int_0^t e^{-Ar} g(r, x_r)\,\mathrm{d}w_r\Big\|^2 \\ + 4\big\|G_L(t)^{-1}\big\|^2 \mathbf{E}\Big\|\int_0^t G_L(r) L e^{-Ar} f(r, x_r)\,\mathrm{d}r\Big\|^2 \\ + 4\big\|G_L(t)^{-1}\big\|^2 \mathbf{E}\Big\|\int_0^t G_L(r) L e^{-Ar} g(r, x_r)\,\mathrm{d}w_r\Big\|^2. \end{aligned} \tag{3.5}$$

Now, we move t to 0^+ in (3.5). The limits of the first two terms in the right-hand side of (3.5) are 0 because of the inequality (3.2) and the boundedness of e^{-Ar} on $[0, T]$. To show the same for the remaining terms, note that $G_L(0) = 0$ which implies that

$$\lim_{r\to 0^+} \frac{G_L(r)}{r} = \lim_{r\to 0^+} \frac{G_L(r) - G_L(0)}{r} = \frac{\mathrm{d}G_L(r)}{\mathrm{d}r}\Big|_{r=0} = LC^\top C L^\top.$$

Therefore, $\frac{G_L(r)}{r}$ is bounded. Let

$$\Big\|\frac{G_L(r)}{r}\Big\| \le M \text{ and } \|Le^{-Ar}\| \le M.$$

Then, by Cauchy-Schwarz's inequality, we can estimate the following as

$$\begin{aligned} \big\|G_L(t)^{-1}\big\|^2 \mathbf{E}\Big\|\int_0^t G_L(r) L e^{-Ar} f(r, x_r)\,\mathrm{d}r\Big\|^2 &\le \|G_L(t)^{-1}\|^2 t\mathbf{E}\int_0^t \|G_L(r) L e^{-Ar} f(r, x_r)\|^2\,\mathrm{d}r \\ &\le KM^4 t\|G_L(t)^{-1}\|^2 \int_0^t r^2\,\mathrm{d}r \\ &= \frac{KM^4}{3} t^4 \|G_L(t)^{-1}\|^2 = \frac{KM^4 (t\|G_L(t)^{-1}\|)^2}{3} t^2. \end{aligned}$$

Therefore, by condition (D), the third term in the right-hand side of (3.5) converges to 0 as $t \to 0^+$. Also, by Itô isometry and Cauchy-Schwartz inequality, we have the following estimate:

$$\begin{aligned}
\|G_L(t)^{-1}\|^2 \mathbf{E}\left\| \int_0^t G_L(r) L e^{-Ar} g(r, x_r)\, \mathrm{d}w_r \right\|^2 &\leq \|G_L(t)^{-1}\|^2 \int_0^t \mathbf{E}\|G_L(r) L e^{-Ar} g(r, x_r)\|^2 \,\mathrm{d}r \\
&\leq K M^4 \|G_L(t)^{-1}\|^2 \int_0^t r^2 \,\mathrm{d}r \\
&\leq \frac{KM^4}{3} t^3 \|G_L(t)^{-1}\|^2 = \frac{KM^4 (t\|G_L(t)^{-1}\|)^2}{3} t.
\end{aligned}$$

By condition (D) the forth term in the right-hand side of (3.5) converges to 0 as $t \to 0^+$. Thus, the left-hand side of (3.5) converges to 0 as $t \to 0^+$. This proves the limit in (3.1). Finally, note that the limit is a square integrable random variable because of the sense of convergence. It is also $\mathcal{F}_0$-measurable since it belongs to all $\mathcal{F}_t$ for $0 < t \leq T$, accordingly, it belongs to $\mathcal{F}_0 = \bigcap_{0<t\leq T} \mathcal{F}_t$ by continuity of the filtration. This completes the proof of Theorem 3.1. □

Remark 3.1. Note that the non-singularity of $G_L(t)$ is uniquely determined by the matrices A, C, and L which are time-independent. Therefore, the condition (D) in Theorem 3.1 about non-singularity of $G_L(t)$ for all $0 < t \leq T$ can be replaced by its non-singularity at some $0 < t \leq T$.

Remark 3.2. Theorem 3.1 extends to the time varying $A(t)$, just e^{At} should be replaced by the respective state transition matrix $S(t, s)$.

Corollary 3.1. *Assume that the conditions (A), (C), (D), and (E) hold. Additionally, let B be a bounded measurable function from $[0, T]$ to $\mathbb{R}^{n\times k}$. Then the linear stochastic system*

$$\begin{cases} \mathrm{d}x_t = Ax_t \,\mathrm{d}t + B(t)\mathrm{d}w_t, \\ z_t = Cx_t, \end{cases}$$

is L-partially observable on $[0, T]$. In particular, Lx_0 can be recovered by (3.1), in which the limit is in the sense of convergence in $L_2(\Omega, \mathbb{R}^n)$.

Proof. It is easily seen that the condition (B) with regard to g holds since B is bounded. □

Corollary 3.2. *Assume that conditions (A), (B) with regard to f, and (C)–(E) hold. Then the deterministic semilinear system*

$$\begin{cases} x_t' = Ax_t + f(t, x_t), \\ z_t = Cx_t, \end{cases}$$

is L-partially observable on $[0, T]$. In particular, Lx_0 can be recovered by (3.1), in which the limit is in the sense of convergence in $\mathbb{R}^n$.

Proof. This proof can be carried by removing g from the consideration and reducing the sense of convergence to the deterministic case. □

4 Example

Consider the system (2.1) with

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Condition (A) holds with $n = 2$ and $m = 1$. Conditions (B) and (C) are of general type. Therefore, we assume shortly that they hold. Let us verify conditions (D) and (E) for the above choice of A and C by setting an appropriate orthogonal projection matrix P_L.
To find the transition matrix e^{At}, we look to the linear part of (2.1):

$$\begin{cases} \xi'(t) = -\eta(t), \\ \eta'(t) = -\xi(t). \end{cases}$$

Letting $\xi(0) = \xi_0$ and $\eta(0) = \eta_0$, we have

$$\begin{cases} \xi(t) = \xi_0 \cosh t + \eta_0 \sinh t, \\ \eta(t) = -\xi_0 \sinh t - \eta_0 \cosh t. \end{cases}$$

Therefore,

$$e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ -\sinh t & -\cosh t \end{bmatrix},$$

implying

$$\begin{aligned} G(t) &= \int_0^t \begin{bmatrix} \cosh s & \sinh s \\ -\sinh s & -\cosh s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cosh s & -\sinh s \\ \sinh s & -\cosh s \end{bmatrix} ds, \\ &= \int_0^t (\cosh s + \sinh s)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} ds = \frac{e^{2t} - 1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned} \tag{4.1}$$

We obtain that $\det G(t) = 0$, implying the singularity of $G(t)$.
While $G(t)$ is singular, we still have chance to recover some combination of the initial data. Clearly, the kernel K and range R of $G(t)$ are defined as follows:

$$K = \{[x \quad x]^\top : x \in \mathbb{R}\} \quad \text{and} \quad R = K^\perp = \{[x \quad -x]^\top : x \in \mathbb{R}\}.$$

Theorem 3.1 does not apply to the projection matrix onto the subspace K. But, we still have a chance to recover the projection onto R. Taking $[1 \quad -1]^\top \in R$, we are seeking a formula for

$$\left\langle \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle = x_0 - y_0.$$

By doing this, consider the orthogonal projection matrix

$$P_L = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ with } L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Then

$$G_L(t) = LG(t)L^\top = \frac{e^{2t} - 1}{4} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t} - 1.$$

So, $G_L(t)^{-1} = 1/(e^{2t} - 1)$, implying

$$\lim_{t \to 0^+} t \|G_L(t)^{-1}\| = \lim_{t \to 0^+} \frac{t}{e^{2t} - 1} = \lim_{t \to 0^+} \frac{1}{2e^{2t}} = \frac{1}{2}.$$

Therefore, condition (D) holds. Condition (E) holds too since

$$G(t) \begin{bmatrix} x \\ x \end{bmatrix} = \frac{e^{2t} - 1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then by Theorem 3.1,

$$L \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{x_0 - y_0}{\sqrt{2}} = \lim_{t \to 0^+} \frac{1}{\sqrt{2}(e^{2t} - 1)} \begin{bmatrix} 1 & -1 \end{bmatrix} \int_0^t \begin{bmatrix} \cosh s & \sinh s \\ -\sinh s & -\cosh s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} z(s)\, ds.$$

Thus, $x_0 - y_0$ can be recovered by

$$x_0 - y_0 = \lim_{t \to 0^+} \frac{2}{e^{2t} - 1} \int_0^t (\cosh s + \sinh s) z(s)\, ds.$$

At the same time, Theorem 3.1 is not applicable to recover $x_0 + y_0$.

5 Conclusion

The main result of this paper is a sufficient condition of partial observability for a semilinear stochastic system in finite dimensional spaces. The contributions of our paper are as follows:

- In particular, the main result also covers the case of (non-partial) observability.
- The linear part was strengthened with the convergence rate of the partial observability Gramian and the nonsingularity of the observability Gramian.
- It is shown that the similarity between observability and controllability problems, best formulated in terms of duality in the linear case, persists also in the semilinear case. Adopting the notation of the linear system from [4], the observability condition (D) for the linear part $\Sigma(A, -, C)$ is the same as the controllability condition (E) from [27] for $\Sigma(A, C, -)$.

In summary, the technique presented in this paper is at an intersection of many important branches of research. The advantage of this work is that it can be extended to discrete and stochastic fractional order control systems.

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