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Sign Language in Light of Mathematics Education: An Exploration Within Semiotic and Embodiment Theories of Learning Mathematics

Christina M. Krause and Annika M. Wille

Research rarely focuses on how deaf and hard of hearing (DHH) students address mathematical ideas. Complexities involved in using sign language (SL) in mathematics classrooms include not just challenges, but opportunities that accompany mathematics learning in this gestural-somatic medium. The authors consider DHH students primarily as learners of mathematics, and their SL use as a special case of language in the mathematics classroom. More specifically, using SL in teaching and learning mathematics is explored within semiotic and embodiment perspectives to gain a better understanding of how using SL affects the development, conceptualization, and representation of mathematical meaning. The theoretical discussion employs examples from the authors' work and research on geometry, arithmetic, and fraction concepts with Deaf German and Austrian learners and experts. The examples inform the context of mathematics teaching and learning more generally by illuminating SL features that distinguish mathematics learning for DHH learners.

Keywords: sign language, mathematics, semiotics, embodiment, language as a resource

Ever since the paradigmatic shift from a behaviorist to a more constructivist understanding of learning, research in mathematics education has shown a strong emphasis on understanding better the processes of making meaning—on understanding better *how* students come to know what they know—and how different components shape learning processes in mathematics. In this, Deaf learners constitute a specific, albeit crucially under-researched, population.[1] Historically, the focus has been on assessing deaf students' competencies and comparing their test results to those of hearing students; the result of such assessment has been the identification of lower mathematics achievement scores and delays in mathematical achievement, starting prior to the onset of formal education (Kritzer, 2009; Traxler, 2000). These comparisons have often been described quantitatively. By contrast, explorations of the qualitative characteristics of deaf students' processes of learning mathematics have rarely been reported—for example, the ways in which they approach mathematical content and their strategies when solving mathematical problems, as well as the obstacles and pitfalls that might be related to their specific practices in the

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learning process. (For exceptions, see, e.g., Titus, 1995, on working with fractions, and Pagliaro & Ansell, 2008, and Hyde et al., 2003, on linguistic strategies for solving arithmetic word problems.) However, it is by acquiring a better understanding of these practices that we (i.e., educators and educational researchers) can become able to align teaching material and methods with the strengths and needs of Deaf learners, with our understanding always depending on the theoretical lenses we choose and the focus of our observations.

Working from theoretical perspectives of semiotics and embodiment, in the present article we focus on sign language as a specific practice significantly shaping teaching and learning for Deaf mathematics learners as a primary mode of meaning making. In this approach, the semiotic lens considers sign languages as an important semiotic resource—that is, signs in the conventional sense as representations of something—interacting with other semiotic resources such as gestures and written signs, and how this interaction contributes to the development of shared mathematical meaning. The embodiment perspective concerns how bodily and cultural experiences underlying mathematical thinking interact with signs used to refer to mathematical ideas—both in representing these experiences and shaping mathematical thought as embodied modes of learning. To capture our theoretical exploration in the present article, we adopt and adapt a conceptual framework developed in mathematics education that distinguishes different roles of language (so far only spoken) in teaching and learning mathematics. With this, we refrain from a deficit perspective, instead highlighting Deaf learners as learners of mathematics—and their employment of sign languages as a specific case of the use of languages in mathematics education.

Literature Review: The Role of Sign Language in Mathematics Thinking and Learning

Researchers increasingly emphasize the role of sign languages (SL) not only as an indirect predictor of mathematical skills (Hyde et al., 2003; Nunes, 2004) but as directly related as practice specific to the Deaf, crucially contributing to the shaping of their learning process (Kurz & Pagliaro, 2020). For example, signed algorithms considered to help carry out mental calculations have been found to be commonly employed among Deaf users of American Sign Language (ASL; Nunes & Moreno, 1998) and Finnish Sign Language (Rainò et al., 2018).[2] Similarly, Healy et al. (2016) describe a Brazilian Deaf learner's individual strategy for mental multiplication as supported by the use of LIBRAS (Brazilian Sign Language). The use of the counting string in ASL is also mentioned by Pagliaro and Ansell (2008) as an ASL-related strategy used by students to solve story problems. As reported by Kurz and Pagliaro (2020), two other successful strategies that have been observed concern the "use of the inherent cardinality of the numbers signs 1–5" and the organization of the signing space so that it can be used "like a third device (after their hands) on which to keep track of the counting strings or manipulatives" (p. 93).

Healy and colleagues investigated gestural and signed expression of Deaf learners in geometric and algebraic contexts, and in the context of engaging with educational technology, focusing on the use of sign languages in mathematics discourse (Fernandes & Healy, 2014; Healy, 2015; Healy et al., 2016; Magalhães & Healy, 2007). Reporting on a study done in a bilingual Brazilian classroom with five deaf and three hearing students, Healy

(2015) described the development and use of signs and gestures in a mixed collaboration between a hearing student and a deaf student (in which "the hearing student in this pair spoke some LIBRAS and the deaf student was partially oralised," p. 296) while the two students explored and expressed symmetry and reflection through a Logo-programmed "microworld" (expressive digital media based on principles such as invention, play, and discovery; Papert, 2002; see also Healy et al., 2010). Similarly, Fernandes and Healy (2014), in a study in Brazil with six Deaf students using a microworld "designed to encourage students to produce a variable procedure" (p. 51), observed the creation of a signed denotation of a variable *n* as a fixed unknown value by one of the students. Using LIBRAS, the student referred to *n* as the "secret number" (p. 53) and shared this interpretation of the variable, which the six students adopted for use. Fernandes and Healy point out the "process of coordinating bodily resources with visual, dynamic and linguistic signs in order to attribute meanings to mathematical objects" (p. 55) as one main aspect of the successful collaborative coordination of mathematical meaning in their teaching experiments. Inventing and negotiating ad hoc signs in order to collaborate, the students in the study embodied their experiences in signs that reflected not only the specific case, but the students' shared conceptual understanding of the mathematical idea they encountered. Healy (2015) has termed signs recalling an action through which the signed mathematical concept has been explored "imagined re-enactments" (p. 305).

In studies involving a German grade 5 geometry classroom of nine Deaf students and a Deaf teacher using German Sign Language (Deutsche Gebärdensprache, DGS), Krause (2018, 2019) found such iconic reenactments while focusing on the iconic aspects of the mathematical signs that were used. In particular, she traced how the teacher explicitly grounded his signs for "axial symmetry" and "point symmetry" in the respective actions of folding and reflecting (axial symmetry) and rotating around a point (point symmetry). Krause argued that in establishing the link between manual activity and sign, the teacher as a heritage DGS user and mathematics professional might provide a scaffold by using language as conceptual support. Although this practice is considered crucial (Kurz & Pagliaro, 2020), both its theorization and application are still in their infancy, and classroom observations of processes of learning mathematics and teaching practices are scarce.

There is a particular need for investigations of the specific features of Deaf students' processes of learning mathematics and working mathematically as connected to the different ways of thinking that are considered to be related to these learners' use of SL (Emmorey et al., 1993; Marschark, 2003; Marschark & Hauser, 2008). These different ways of thinking can be seen as linked to the different affordances of SLs in comparison to spoken languages, as summarized by Grote et al. (2018), concerning, for example, the language modality as gestural-visual versus vocal-auditive, or the degree of iconicity as strongly iconic in SL and less onomatopoeic in spoken language. The differences in regard to Deaf learners' mathematics are hence more than a matter of translation. They concern, moreover, the structuring of information caused by the modalities of signed and spoken languages differing in articulation, perception, and processing, and guided by these languages' linguistic features and rules. For example, the spatial-visual articulation of SLs enables simultaneous representation of information where spoken language expresses the same information linearly. Also, the literature suggests that early exposure to SL leads to enhanced recall of visuospatial information and that signers have generally enhanced visuospatial skills, a preference for spatial coding, and less developed sequential cuing (M. L. Hall & Bavelier, 2010).

The different ways of thinking caused by language modality not only can be expected to shape the individual learner's understanding; they also manifest in communicative situations as expression become structured on the basis of how concepts are organized cognitively. From a socio-constructivist perspective, this changes not only the quality of the learning processes but also, potentially, the quality of the mathematical knowledge as outcome, as in this perspective this knowledge becomes constructed through the student's ongoing negotiation and validation of mathematical meaning in social discourse about mathematics (Bauersfeld, 1992).

Mathematics education research increasingly acknowledges the relationship between individual and social dimensions of thinking, learning, and knowing. In the present article, we will therefore explore the potential of three lenses in mathematics education research we consider especially suitable and significant for an approach to describing and better understanding the role of SL in mathematical thinking and learning. In particular, we will consider those branches not in their entirety (as this would hardly be possible), but with respect to conceptual and theoretical perspectives influenced by our research background and current expertise in mathematics education, and by how we found them intersecting and complementing each other in compatible and harmonious ways.

We will start by building a *conceptual frame* that embeds the case of SL within the larger body of research on language in the teaching and learning of mathematics. More concretely, this framework will capture the roles of SL as a learning medium, a learning goal, a potential obstacle, a prerequisite for learning, and a resource in the mathematics classroom. We will then outline *theoretical perspectives* on aspects of semiotics and embodiment as relevant from a mathematics education perspective and discuss the several roles of SL as a language in teaching and learning mathematics in this light. The theoretical explorations will be illustrated with examples from DGS and Austrian Sign Language (Österreichische Gebärdensprache, ÖGS). We will then link the research to potential implications for practice before closing with final remarks about potential extensions and future perspectives.

Sign Language as a Case of Language in Mathematics Education: A Conceptual Framework

In recent decades, research on aspects of (so far, spoken) language in the teaching and learning of mathematics has been, and still is, increasingly gaining attention among mathematics education scholars, a development certainly not unrelated to the growing linguistic, cultural, and socioeconomic diversity in mathematics classrooms. Foci have been set on what it means to learn mathematics in a second language, how language proficiency is related to mathematics learning, bi- or multilingual settings in the mathematics classroom, and which aspects of language might support or hinder the learning of mathematics. On the basis of these developments, current research on language in mathematics education has distilled the roles of language as a *learning medium* and as a *learning goal* (Lampert & Cobb, 2003), as a *prerequisite for learning* (Prediger & Schüler-Meyer, 2017), as a potential *obstacle to learning* mathematics (Prediger et al., 2019), and as a *resource for learning* (Planas, 2018).

This research has framed mathematics learning “as a discursive practice: doing mathematics essentially entails speaking mathematically (or writing or using other communicational modes)” (Morgan et al. 2014, p. 846). Mathematical meaning can then be considered as constructed either *through* using language—understanding mathematical objects as nontangible per se and accessible only through representations, including language signs (like words in spoken languages or signs of sign languages) (Duval, 2006)—or as constructed *in* using language, understanding the mathematical discourse itself as the learning process (Sfard, 2008).

Deaf learners and their use of SLs have been neglected in the literature on language in mathematics education. However, some have implicitly suggested the potential integration of nonspoken or visual languages, for example, when “verbal and visual” languages are juxtaposed (Planas, 2018, p. 216), or if languages that are nonspoken or visual are included in a definition of language as “a system of communication used by a particular country or community” (Morgan et al., 2014, p. 844, quoting the *Oxford English Dictionary* online). We deem the current discussion in mathematics education incomplete without consideration of the distinct characteristics of learning mathematics in SLs. In the present section, we provide a background for exploring SLs as a specific case of language in the learning of mathematics and set the terminology to frame our theoretical investigation of the roles of SL in mathematical teaching and learning processes through different theoretical lenses.

The Roles of Language as a Learning Goal, a Learning Obstacle, and a Prerequisite for Learning

Communicating about mathematics and becoming proficient in mathematical discourse are considered crucial for working mathematically, as is also reflected in the integration of communication and language in mathematics as a major topic in the standards for school mathematics in the United States (National Council of Teachers of Mathematics [NCTM], 1989, 2000), and similarly in other countries, such as Germany (KMK, 2004). Language becomes a *learning goal* in the mathematics classroom, including the appropriate use of mathematical terminology and, more generally, efforts to build up *cognitive academic language proficiency* (Cummins, 2000). This can be seen as deeply linked to both *language as an obstacle* and *language as a prerequisite for learning mathematics*, as one’s lack of competence to engage in mathematical discourse—both passively and actively—can constrain learning on the individual level as well as affect the learning of the whole class on a social level. In this case, language needs to become a learning goal in order to harness the functions of language as a *learning medium*.

Language as an obstacle to learning mathematics, for example, has been widely described in the literature related to students in various settings, especially students with low levels of language proficiency (Prediger et al., 2019). Prediger et al. (2019) single out a number of potential obstacles on the word, sentence, and text levels that influence the mathematical learning process in different ways. While Deaf students have to face these obstacles with respect to written language too—maybe even more extensively, considering these learners’ reported difficulties with perceiving and processing word problems (Hyde et al., 2003)—their specificities and their relation to students' processes of learning mathematics need to be revisited for Deaf signers in the context of SL. For example, mathematical vocabulary can fall into different categories that depend on the linguistic features of a specific language, differentiated into 11 categories for English mathematical vocabulary by Riccomini et al. (2015, p. 238). Some of these categories certainly apply to mathematical signs, too. For example, the mathematical reference of some words and signs depends on the context and on having a discipline-specific technical meaning, different from the everyday meaning; or mathematical words or signs can be semantically related but show no similarity on a morphological level (see Kurz & Pagliaro, 2020). Other categories have no analogue in SL—e.g., irregularities in spelling (singular/plural)—and if mathematical meanings of math signs are more or less precise, then their everyday meaning is open to speculation. In addition, there might be features of SL that shape mathematical vocabulary as part of the *learning medium*. We will turn to signed mathematical vocabulary throughout the present article—for example, with respect to iconicity and phonological features of SL—and hence consider signed mathematical vocabulary not as an obstacle per se but as a potential resource for learning mathematics, as we describe in the following section.

Language as a Resource for Learning Mathematics

Planas (2018, 2019) describes the notion of language as a resource for learning mathematics in the context of multilingual settings, acknowledging its potential surplus in the mathematics classroom in terms of its *pedagogical, epistemic, and political value.* In terms of its pedagogical value, it is a means of orchestrating and fostering teaching and learning, including instruction and the development of learning material that takes language into account. In its epistemic value, it concerns how language contributes to "creation and exchange" (Planas 2019, p. 21), or, rather, construction and negotiation, of knowledge. The political value of language has been scarcely elaborated so far in the context of mathematics education. Because the pedagogical value concerns general pedagogical aspects—not necessarily related to disciplinary learning—our elaboration of SL as a resource for learning mathematics will focus on its epistemic value.

A Semiotic Perspective on Sign Languages in Learning Mathematics

Theories of semiotics deal with signs in a general sense[3]—distinguished, for example, by the modality in which they are produced as written signs, spoken signs, gestural signs, etc.—how they are used, and how they are endowed with meaning. These theories can provide powerful tools for better understanding students' learning of mathematics by considering the role of the signs as a constitutive part of communication, social interaction, and mathematical activity (see, e.g., Arzarello, 2006; Duval, 2006; Krause, 2016; Wille, 2020a). Understanding a communicative act in its categorical (speech or sign) and imagistic (gesture) components (Goldin-Meadow & Brentari, 2017), SL shapes learning in distinctive ways, and a semiotic lens might facilitate the understanding of how learning processes of Deaf students and hearing students relate to each other.

In the present section, we will consider the semiotic features of SLs as combining characteristics of linguistic signs, that is, certain rules of sign production and an underlying meaning structure (Ernest, 2006, pp. 69–70), with those of gesture signs as holistic, compound, and partly idiosyncratic means of expression. We will investigate the potential of SL as a semiotic resource in mathematical learning processes in the light of linguistic signs and gesture signs.[4] In particular, we will explore the idea of seeing mathematics as a sign game in which the meaning of mathematical signs arises from their use (Wittgenstein, 1956/1967). In this, the core of learning mathematics lies in becoming proficient in both engaging in mathematical activity and communicating about this activity (Wille, 2020a). With signs playing a role in being the objects of the mathematical activity as well as the means of communication, we will see how the use of SL signs influences both in specific ways.

A Peircean Understanding of Signs and the Role of Diagrams in Mathematical Activity

In the present section, we advance an understanding of signs that follows the thinking of the American mathematician, logician, semiotician, and philosopher Charles Sanders Peirce (1839–1914), as "something which stands to somebody for something in some respect or capacity" (Peirce, 1931–1958, Vol. 2, p. 228) and concerns the relationship between *that which represents* (the *representamen*, or sign-vehicle), *that which it represents* (the object), and *the respective way* in which the representamen is representing the object (the interpretant). In this triadic relation, this understanding differs from theories that assume the predetermined meaning of signs in a signifier-signified relationship (de Saussure, 1995) and instead considers the meaning of signs as depending on interpretation. In particular, the sign can have the form of an index, an icon, or a symbol in the ways the object determines the sign (Peirce; 1931–1958; for further reading, see Atkin, 2013): Indexes direct attention to something, like an arrow or a pointing finger; an icon reflects the relational structure within an object and thus creates an impression of similarity to features of the object; a symbol is interpreted on the basis of habits or conventionalized rules. These distinctions make it obvious that a sign can be a symbol for one person and an icon or an index for someone else, and can even mean different things to different people. For example, the signs of spoken and signed languages are symbols following certain linguistic conventions that form the respective languages. However, SL signs often evoke iconic relationships to actions or objects, some of these relationships transparent even to persons with no prior experience with the language (Taub, 2001, p. 19).

With respect to mathematics, Peircean semiotics take specific interest in the *diagram,* defined as "a representamen which is predominantly an icon of relations and is aided to be so by conventions.… It should be carried out upon a perfectly consistent system of representation, founded upon a simple and easily intelligible basic idea" (Peirce, 1931–1958, Vol. 4, p. 418) and defining the rules for production and manipulation of diagrams. Peirce's notion of *diagram* differs from an everyday understanding of the term, not necessarily referring to a geometric context. Examples of diagrams in mathematics are mathematical notations such as variables, algebraic terms, equations, function graphs, and geometric figures. It is through constructing diagrams, experimenting with and manipulating diagrams—through *diagrammatic activity*—that it becomes possible "to discover unnoticed and hidden relations among the parts" (Peirce, 1931–1958, Vol. 3, p. 363) and to gain new insights through observation (Hoffmann, 2007). That is, dealing with diagrams and looking at them from different perspectives can provide new ideas about the relations they can represent.

Diagrammatic Activity in SL Signs: Examples From Explanations in Austrian Sign Language (ÖGS)

Coming back to the idea of mathematics as a sign game with diagrammatic activity in the center of learning mathematics, where the mathematical meaning of signs arises in their use, diagrammatic activity is usually thought of as performed with mathematical inscriptions, for example, on paper. While diagrammatic activity can also be done purely through spoken language, we claim that this is possible to a much greater extent through SL, as we will illustrate in an example of an explanation of decimal numbers, produced as a video in the context of inclusive mathematics education for Deaf students (Wille, 2019). The baseline for the video was the following German text, adapted to ÖGS by two native ÖGS signers (English translation by A.M. Wille): "Imagine we divide each section again into 10 parts. How many parts did you divide it into?" (https://tinyurl.com/OegsVideoBrueche). Both the text and the video show a number line as a mathematical inscription; the background explanation concern the objective of getting from tenths to hundredths. In this video, it is related that a section on the number line should be divided into 10 parts. A discussion about activity on the mathematical diagram shows the ambiguity of talking *about* diagrammatic activity and the diagrammatic activity itself: In the beginning, two different ÖGS expressions are used to refer to "one tenth": One works by indicating the section on the number line—a diagram of a geometric system of representation (as shown in Figure 1a). In the other expression, the SL sign resembles spatial aspects of the diagram "1/10" in the top-bottom notation of a symbolic system of representations (Figures 1b & 1c).

The diagrammatic activity of dividing a tenth into 10 parts is first talked about with the general ÖGS sign for "dividing" (*teilen*), which resembles the action of cutting something into pieces with a knife (see Figure 2a). Then, a different signed expression for dividing the tenths part further is accomplished directly on the number line, first indicating the segment of one tenth, combined with the ÖGS sign used before together with the number line (see Figures 2b and 2c).

Finally, this ÖGS sign for dividing moves from the written number line into the signing space (see Figures 3a and 3b).

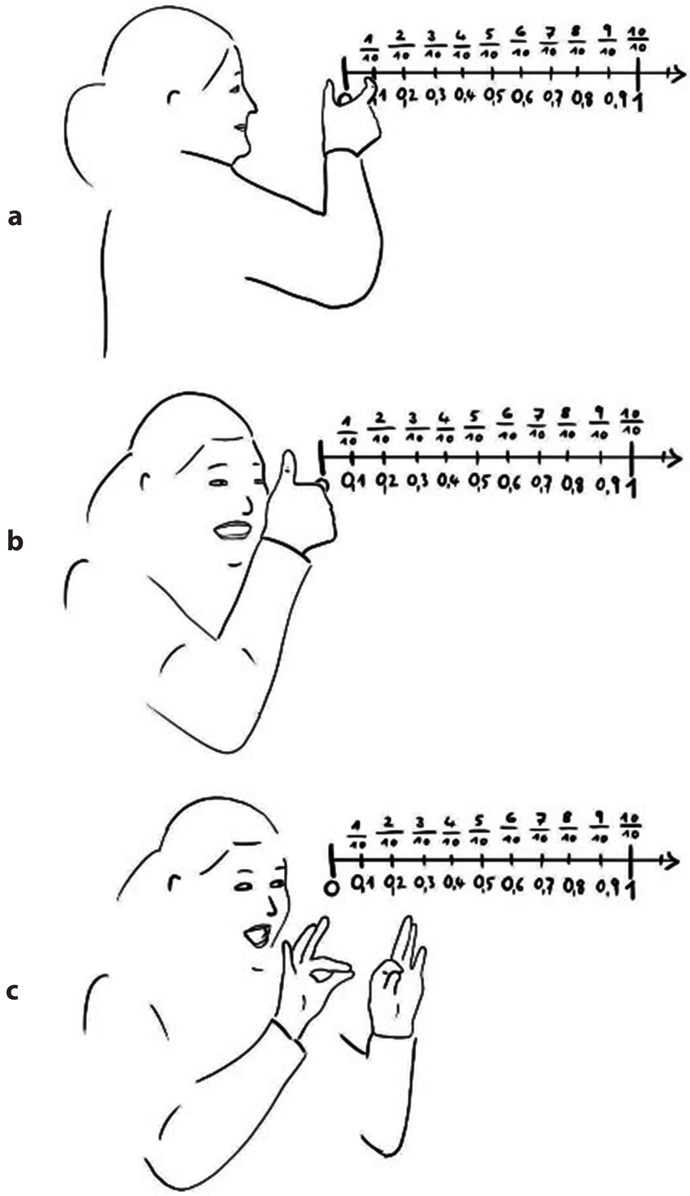
Later, a third way of referring to dividing into 10 parts in ÖGS is used, combining the ÖGS sign for 10 and a movement from left to right in the signing space (see Figures 4a, 4b, and 4c).

Hence, the signed explanation references the written diagram and its segmentation, and the ÖGS signs for "dividing" resemble the diagrammatic activity that would be carried out on paper (or on a whiteboard) on the number line.[5] This example shows that in some cases where diagrammatic activity and speaking about it can be clearly distinguished in spoken languages (leaving accompanying gestures aside for now), in SLs the line between both can be vague.

Iconicity and Indexicality of Mathematical SL Signs in the Development of Meaning

Considering the meaning of signs as emerging in and through their use in Wittgenstein's sign game as described

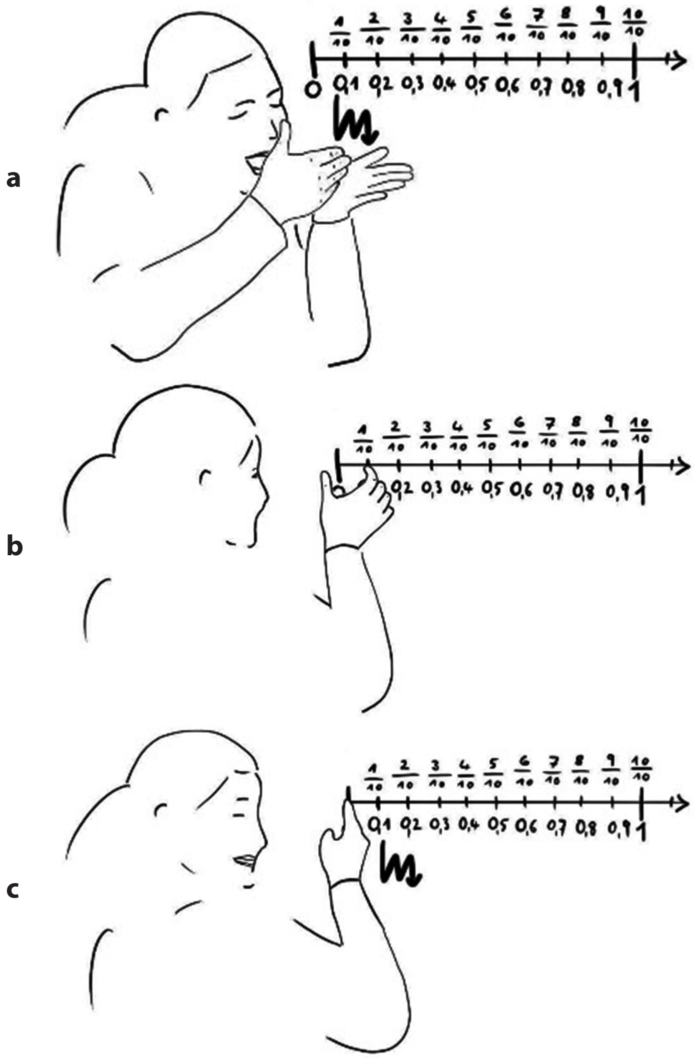
Figure 1. Two Different Expressions for "One Tenth" in ÖGS



Notes. Teacher indicating (a) the section "one tenth" on the number line, and the ÖGS signs for symbolic number (b) "one" (c) "tenth." ÖGS = Österreichische Gebärdensprache (Austrian Sign Language).

previously, the question arises of how the specific characteristics of mathematical SL signs might affect this meaning. For example, research in psycholinguistics provides evidence that "those features that are reflected in the iconic moment of sign language get a specific relevance for the whole semantic concept" (Grote, 2010, p. 316, translation by Christina M. Krause), meaning that this aspect might be associated more strongly with the concept than with those not reflected in the

Figure 2. Diagrammatic Activity of Dividing a Tenth Into 10 Parts



Notes. Teacher performing the ÖGS sign for (a) "divide" (*teilen*), (b) indicating the section "one tenth," and (c) showing the ÖGS sign for "divide" on a number line. ÖGS = Österreichische Gebärdensprache (Austrian Sign Language).

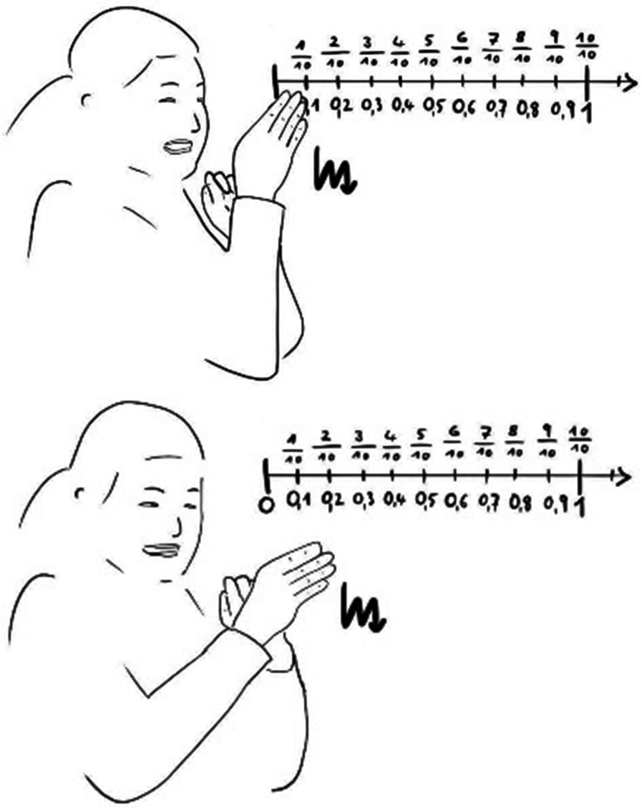
sign. The iconic aspects of SL signs used to talk about mathematical activity might hence influence the perceived meaning of these SL signs in certain ways (Krause, 2017a, 2017b; Wille, 2020b). Similarities to mathematical diagrams, for example, could highlight some properties of the diagrams or the activities done with them, and leave other properties hidden. In the examples given previously, one can see this in the different SL signs referring to the subdivision (equal segmentation; hence, *what* in terms of activity) of the number line in parts. While the SL signs in Figures 1, 2, and 3 emphasize the aspect of segmenting, the signed expression in Figure 4 integrates and highlights the value "10" (*how many*).

Wille (2020b) further explores the role of indexicality of mathematical SL signs, claiming that it can influence the meaning

Figure 3. ÖGS Sign for "Divide"

a

b

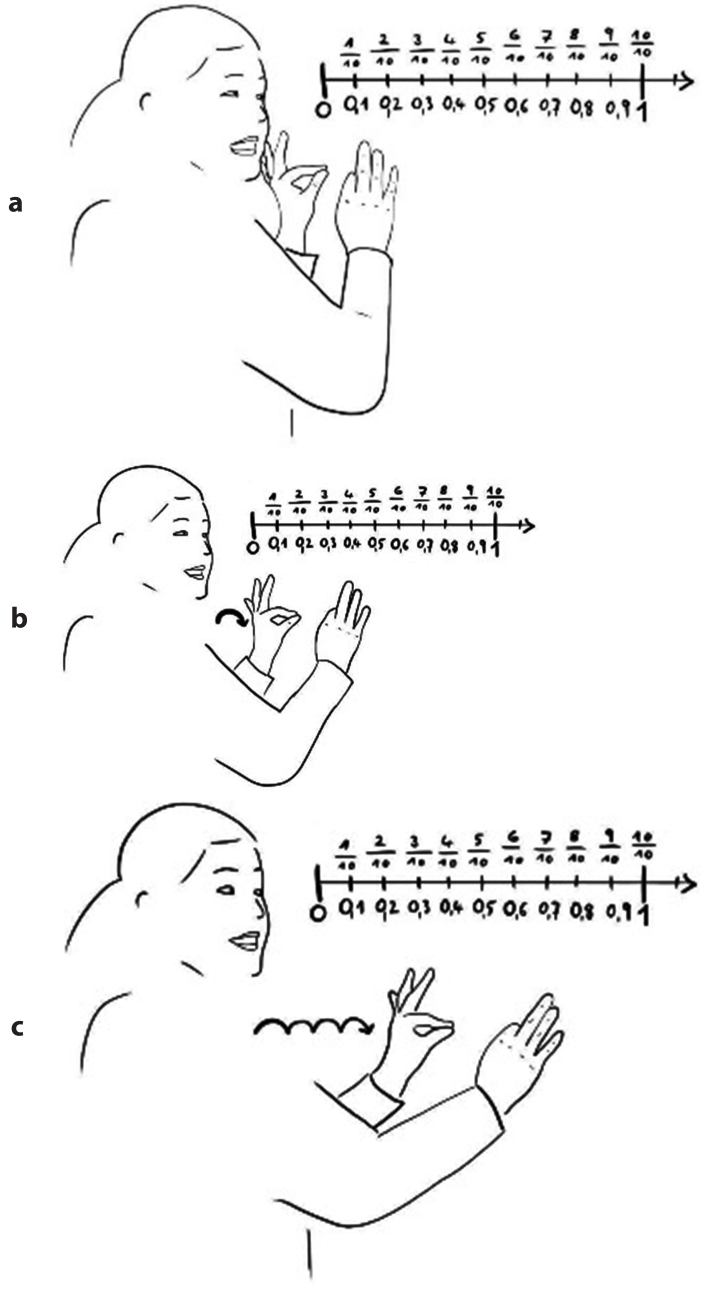


Notes. Teacher turning her body while performing the sign for "divide" in front of a number line. ÖGS = Österreichische Gebärdensprache (Austrian Sign Language).

of the signs as it develops in use in the context of talking about mathematical activity. As one form of indexicality, Wille refers to semantic fields as described in a categorization of indexical signs in DGS suggested by Kutscher (2010). Following this, SL signs can indicate semantic fields through location of performance, indicating the reference to a certain class of signed concepts, like signs associated with cognitive processes (such as thinking, forgetting, or knowing) performed on the forehead. Mathematical examples are the SL signs used for "minus," "times" (in the sense of multiplication), and "divided by" in DGS and ÖGS, indicating the symbolic notations for the operations "-,·,:" with the dominant hand in the palm of the nondominant hand.

Wille (2020b) extends this category of indexicality to resemblances in hand form, hand position, or movement, claiming that each can be interpreted as an index that directs attention to the semantic field. Krause (2017b, p. 93; 2019, p. 91) refers to this as shades of *innerlinguistic iconicity.* For example, the ÖGS signs for "formula" (*Formel*), "complicated" (*kompliziert*), and "crafting" (*basteln*) differ only in the viseme (see Figure 5), with simultaneous mouthing of the German word (not captured in Figure 5). This can be interpreted as a reference to the semantic field of "complicated things." If a learner now uses such an indexical SL sign, this should have an influence on the meaning that emerges from it. For example, the innerlinguistic

Figure 4. Signed Reference to Dividing Into 10 Parts

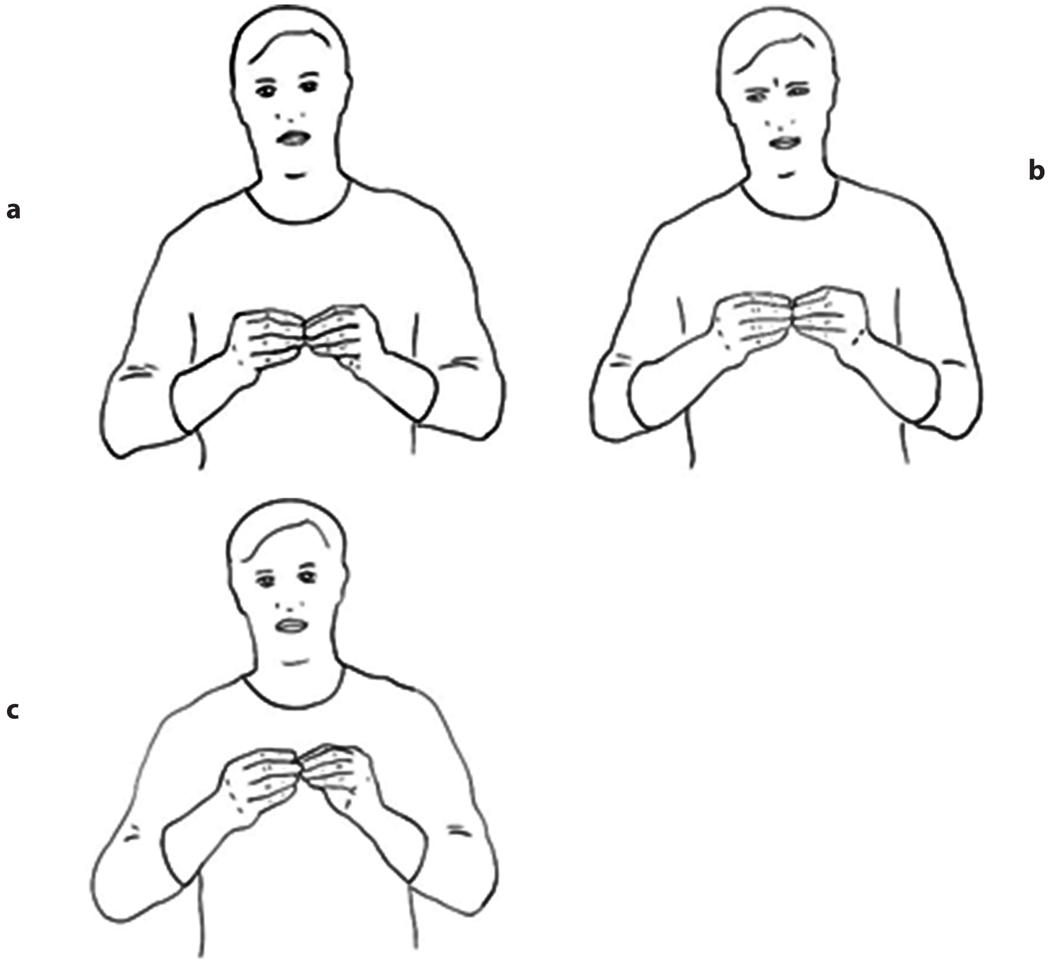


Notes. Teacher signing "each division into 10 parts" in ÖGS. ÖGS = Österreichische Gebärdensprache (Austrian Sign Language).

iconicity between the signs for "formula" and "complicated" might potentially lead the signer to perceive formulas as complicated and encourage a certain mindset toward mathematics.

Kurz and Pagliaro (2020) support this claim about indexicality (Wille, 2020b) and innerlinguistic iconicity (Krause, 2017b; 2019) and concretize it in the terminology of SL linguistics, referring to *phonological*

Figure 5. Three Signs Sharing the Same Hand Form and Location in ÖGS



Notes. ÖGS signs for (a) "formula" (*Formel*), (b) "complicated" (*kompliziert*), and (c) "crafting" (*basteln*). ÖGS = Österreichische Gebärdensprache (Austrian Sign Language).

patterns. These "often consist of one or more similar parameters (handshape, location, palm orientation, movement, and nonmanual markers) to portray a category of vocabulary or phrases that share similar characteristics, actions, or classifications" (p. 90). Kurz and Pagliaro argue that such patterns in spoken language "help the receiver to break down a word and make connections to its meaning" (p. 90). For SL users, this means that it might then become more difficult for a signer to break such patterns and link a mathematical SL sign to another representation not related to the semantic field.

The Different Roles of Sign Language in Learning Mathematics From a Semiotic Perspective

In this present section, on semiotics, we have had to distinguish SL signs from a more general notion of signs. This was not only an issue of terminology, but was essential to acquiring a better understanding of the roles of *SL as a learning medium* and *SL as a resource* *for learning mathematics* as compared to spoken language in the hearing classroom. However, from what we have discussed, we cannot identify any significant differences between SL and spoken language *as a prerequisite for learning*: In both cases, language is essential for talking about mathematical activity, and hence for learning mathematics.

One main characteristic of *SL as a learning medium* within the semiotic perspective concerns the interaction of signed expression and inscriptive (written and drawn) signs. In the Peirce-Wittgenstein approach, doing diagrammatic activities and talking about them are key components of learning mathematics, and the examples show how SL signs—much more so than spoken signs—can actively be used to experiment with and manipulate existing diagrams, and possess the potential to function as (visual) diagrams themselves. While one can argue that for spoken language this role can be fulfilled by gestures accompanying spoken expression, an important difference is the way in which conventional meaning of SL signs can be complemented with idiosyncratic integration of gesture signs. This causes a potential inseparability of doing diagrammatic activity and talking about this activity, while the same requires two rather distinct processes in the spoken classroom. In Krause and Wille (2021), we extend this semiotic perspective with a multimodal approach, describing how the diagrammatic activity of hypothetically manipulating an inscriptive diagram through gesturally simulated action becomes part of a mathematical sign eventually used in the classroom. The sign itself arises from doing diagrammatic activity and talking about it, and it becomes a diagram itself by incorporating aspects of this diagrammatic activity represented iconically.

Furthermore, two features of SL considered in psycholinguistic literature become important in the context of SL as a learning medium, also from the semiotic perspective: iconicity and indexicality of signs, the latter congruent with so-called phonological patterns. In particular, these aspects concern an important feature of mathematical vocabulary specific to SLs as they might implicitly or explicitly influence the meaning that emerges from its use. In adding a semiotic perspective, we extend the discussion by Kurz and Pagliaro (2020) of how this can be used with respect to *SL's potential as a resource for learning mathematics*.

We also provide a new perspective on past research that considers *SL as a potential learning obstacle*, supporting these observations from a theoretical perspective on learning mathematics. While these have mainly focused on iconicity (e.g., Bryant, 1995), seeing phonological patterns as indexical features of signs integrates them into the larger discourse of a semiotic understanding of learning mathematics in SL. In our example, associating functions with "complicated things" might have affective consequences for a student's approach to mathematics that we need to be sensitive to. All the more, this highlights the role of *SL as a learning goal*: Like hearing students when they use spoken language, Deaf students do not need only to be able to use SL to talk about mathematics. In order to seize the potential of SL as a resource for learning, a further goal in the mathematics classroom needs to be to foster the development and discussion of meaning in and of signed mathematical vocabulary. However, more research needs to be done to get a better understanding of how the semiotic potential of mathematical SL signs can be leveraged.

Emphasis on Sign Language as an Embodied Mode of Learning

With its meaningful integration of hand movements and bodily expression, SL also becomes relevant from the perspective of embodiment theories of learning, as we will describe and explore in more detail in the present section. The relationship between thinking and learning as being grounded in bodily experience and using sign language has been considered by researchers in psycholinguistics (e.g., Grote et al., 2018; Inoue, 2006) and in the first author's previous work in mathematics education (Krause, 2017a, 2017b, 2018, 2019), and is highlighted in another contribution in this special issue of the *American Annals of the Deaf* (Thom & Hallenbeck, 2021). In this section we aim to provide grounds for framing the embodied nature of SLs in the context of learning mathematics. In particular, we will look at how embodied experiences with the world can shape mathematical thinking and link these experiences to iconic and metaphoric features of signs and signed and gestural expression in mathematical discourse.

The Role of Metaphors

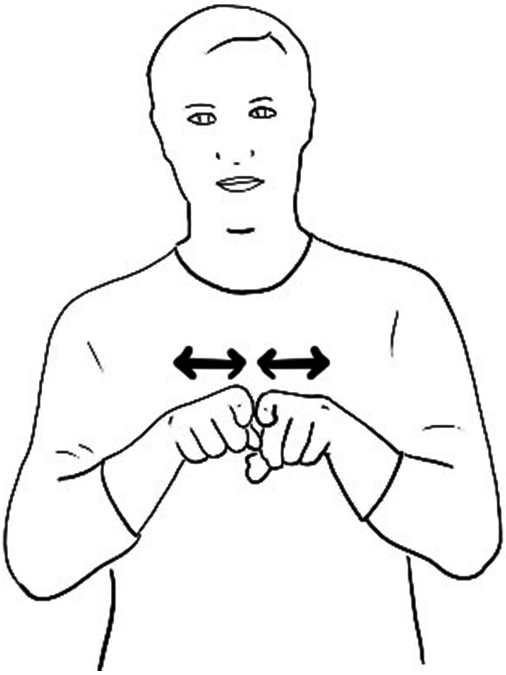
Embodiment theories root cognition in the body (Lakoff & Núñez, 2000; Nemirovsky, 2003; Shapiro, 2014; Varela et al., 1991). They build on the assumption that bodily experience in the physical and cultural world grounds cognitive processes. That is, the way human beings think and reason about mathematics emerges from the way they experience the world.

Different scholars in mathematics education consider the body in mathematics from different perspectives and with different foci: From the standpoint of metaphors, bodily experiences enable an understanding of mathematical ideas in terms of concrete physical actions (Lakoff & Núñez, 2000), *grounding* fundamental mathematical ideas in real-world experience. For example, to understand equality via the balance model—as balancing out the two sides as having equal value—and use it to reason about representing, manipulating, and solving equations (Filloy & Rojano, 1989), one needs to have experienced states of equilibrium and disequilibrium. Conversely, experiences in the real world allow one to express individual mathematical approaches and mathematical understanding in terms of metaphors, consciously or not.

While metaphors are originally a linguistic concept, they can be reflected in gestures (Edwards, 2009) as well as spoken language. In her studies of "iconicity and metaphor in American Sign Language," Taub (2001) described the relationship between iconicity and metaphor as expressed in metaphorical-iconic signs in ASL, that is, in signs that express complex ideas through visuospatial metaphor. While the sign itself reflects an idea in iconic similarity, it does not refer to this idea concretely, but uses it in a transferred, metaphoric way, employing a double mapping—first between concrete source idea and iconic referent, then between iconic referent and metaphoric goal idea (pp. 96–113). As Taub wrote, "If a metaphorical mapping exists that connects the abstract domain to a concrete domain, and if that concrete domain can be represented iconically by the language in question, the language user is in luck: He or she can construct a metaphorical–iconic linguistic item to represent the concept" (p. 110). This seems to be the case for mathematical SL: With SLs known to be rich in their iconicity, the link between a mathematical idea and the mathematical discourse about it can be much closer than is possible in spoken language. For example, the ÖGS sign for "equal to" used in mathematical contexts can be found to reflect the idea of the equilibrium in the scale in a metaphorical way (see Figure 6).

As it can also be seen as iconically representing the two bars of the equal sign in the extended index fingers, the sign for "equal to" can provide a link between the

Figure 6. Sign for "Equal" (*Gleich*) in ÖGS and DGS



Notes. Two index fingers extended and pointing forward, hands moving together and apart twice in front of the body. ÖGS = Österreichische Gebärdensprache (Austrian Sign Language). DGS = Deutsche Gebärdensprache (German Sign Language).

symbol, the concept, and the grounding metaphor of balance, as it is necessary to understand the equality of terms on two sides of the equal sign, an essential precondition for learning algebra. Integrating this into learning could possibly provide a conceptual bridge that might help educators tackle students' well-known struggle to understand the concept of equality and the multifarious meanings of the equal sign—often reduced to a signal for computation (Kieran, 2006)—and, consequently, help students learn algebra. While it remains open to speculation if and how such signs actually influence and guide students' understanding of mathematical ideas, this shows how SL sign can provide a potentially more conceptually accessible representation as compared to spoken/written language.

Understanding and Thinking as Perceptuo-Motor Activities

Nemirovsky (2003) claimed that the origin of mathematical ideas lies in bodily activities, "having the potential to refer to things and events as well as to be self-referential" (p. 106), encompassing many mathematical ideas, like, for example, the idea of measuring, originally done with body parts such as feet or forearms. This considers both the culturally and historically developed mathematics and the individual conceptualization of mathematical ideas as deeply rooted in the body, considering "understanding and thinking [as] perceptuo-motor activities" (p. 108)—for example, bodily actions, gestures, manipulation of materials—and "that of which we think emerg[ing] from and in these activities themselves" (p. 109). This resonates with an enactivist stance on embodiment focusing on the loops of perception and action that *situate* cognition as the core of thinking. Mathematical thinking and learning is then considered as shaped by the body in that it both grounds and situates mathematical thinking and the understanding of mathematical concepts by building up fundamental sensorimotor patterns and navigating them in the moment. While theories of the embodied mind certainly encompass and emphasize more aspects, we consider the described framework that embeds situated enacted mathematical cognition in grounded mathematical cognition central to the aim of understanding SL as an embodied mode of learning.

Gestures as Embodied Resources

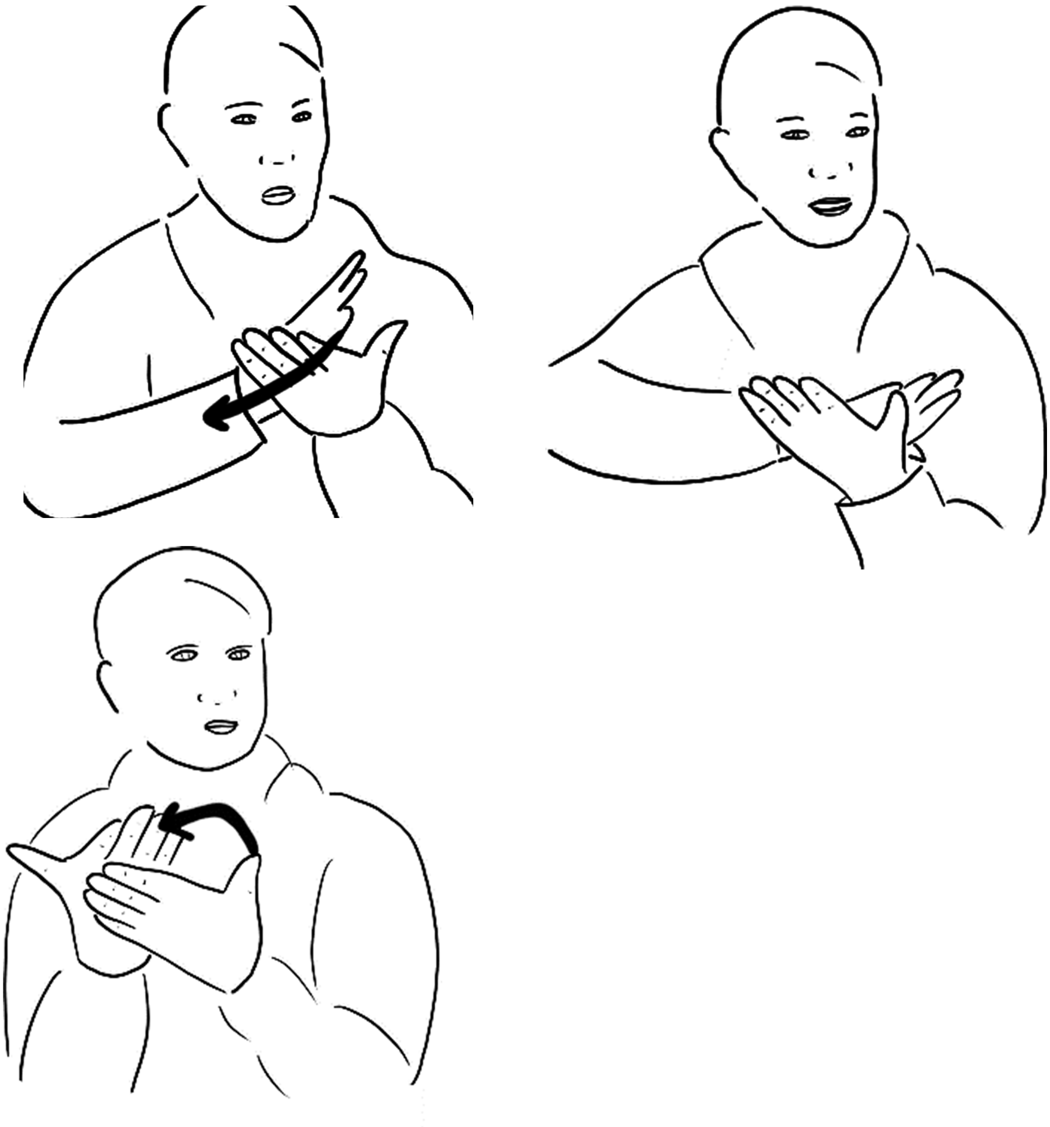
Gestures as embodied resource in mathematics have been fascinating mathematics educators both as a means of accessing mathematical thought and in consideration of gestures' roles in mathematical thinking and learning (e.g., Alibali et al., 2014; Edwards, 2009; Gerofsky, 2010; R. Hall & Nemirovsky, 2012; Krause & Salle, 2019). Signs of SLs are certainly different from gestures, but both modes of expression share the same spatial-somatic modality. In this, the more comprehensive system of language, including, as it does, the spontaneous production of idiosyncratic gestures next to signed expressions, leads to an observed hybridity of gesture and sign in mathematical discourse, potentially also grounded in action. For example, Krause (2018) described how a German Deaf mathematics teacher had his students explore the idea of axial symmetry in activities of folding and cutting paper and how the teacher moved from these activities via gestures to his mathematical sign for axial symmetry (Figure 7). In this procedure, part of the sign reflects the action of unfolding, with the hand embodying the two parts of the paper on both sides of the folding line/axis. Transitioning from action to sign, the gestures simulate the action in combination with the paper as an artifact, leading to the sign as a situationally conventionalized iconic model of the activity at hand. However, in this example, the focus is more on the teacher, less on the learner. The gesture as it unfolds is used for explanation.

The example provided by Krause (2018) can be connected to a framework that considers representational gestures—for example, the teacher's gestures as representing the action of folding—as *simulated actions* (Hostetter & Alibali, 2008, 2018). According to this framework, gestures depicting action, movement, or shape, or that indicate location or trajectory, "reflect the motor activity that occurs automatically when people think about and speak about mental simulations of motor actions and perceptual states" (Hostetter & Alibali, 2018, p. 721). Simulation is understood here as "the activation of motor and perceptual systems in the absence of external input" (p. 722), and gesture production is linked to the activation of mental images of actions and perceptual states. Representational gestures then embody actions considered to be related to the task at hand by the person producing the gesture. This can also occur in metaphorical ways, such as simulating the action of grasping and putting when elaborating the solution to a mathematical task involving substitution (Krause, 2016). Although this framework was developed in the context of co-speech gestures, its tenets also make it applicable beyond, for example, for co-thought gestures (Hostetter & Alibali, 2018) and gestures produced during signing.

Consequences for Theorizing Instructional Strategies

The framework that considers representational gestures might have interesting implications for teaching mathematics in SLs and the grounding of mathematical signs in perceptuo-motor activity in which mathematical understanding emerges. It allows for a much closer link between formal mathematical terminology, the concepts, and the activities in which the concepts are born and raised. This relates to another semiotic model that combines an enactive approach to learning with semiotic representation: Within Bruner's (1966) model of establishing a mathematical concept by moving between three representational modes—enactive, iconic, symbolic—the mathematical sign, like the teacher's sign for "axial symmetry" (Figure 7), can be considered a dynamic symbol, further bearing iconic features that can capture an aspect of the enactive representation. The symbol can hence still be enacted, and mathematical discourse about the concept can encapsulate key features of the action

Figure 7. German Teacher's DGS Sign for "Axial Symmetry" (*Achsensymmetrie*)



Notes. The teacher is performing his sign for "axial symmetry" *(Achsensymmetrie*; see Krause, 2018): The first picture represents the movement of the side of the right hand in the open left hand toward the body. The second and third pictures show the movement of the hand from the right palm directed downward on the left palm, then rotated to the palm facing up. DGS = Deutsche Gebärdensprache (German Sign Language).

as enactive representation informally in iconic gestural expression. This way, "intermodal transfer"—a transfer between the different modes that should not end once it arrives at the symbolic modality—not only becomes natural in the gestural modality of SL, but it might also provoke a closer link to the production of the mathematical sign and of representational iconic gesture as simulating the action, an outcome that argues for a potentially easier recall of perceptuo-motor activities and activation of related sensorimotor patterns through mathematical signs.

The Different Roles of Sign Language in Learning Mathematics From an Embodied Perspective

From what we have seen within the embodied perspective, SL as a *medium for learning mathematics* can be characterized by two main aspects: First, SLs are highly iconic, and with that, bear the potential for capturing metaphors through which mathematical ideas can be understood. Second, SLs are dynamic-visual and live in a modal hybridity with nonconventionalized gestural expression. These idiosyncratic gestures can be seen as simulated actions, physically enacting a motor activity or a physical state when it becomes (unconsciously) relevant to the task at hand. This modal hybridity allows for a smooth transition between an enacted approach to a mathematical idea and its conventionalized sign as bridged through the use of representational gestures (Krause & Abrahamson, 2020).

This affordance for what Krause and Abrahamson (2020) call "modal continuity" in itself reflects the great potential of SL as a *resource for learning*, guided through instruction. As described, intermodal transfer between action, iconic gestural expression, and symbolic sign becomes much more natural in SLs. However, this process needs to be initiated and supervised if its epistemic value is to be exploited, a requirement that calls for teaching methods that are both initiative and supervisory. The same must be mentioned in regard to enabling students to realize and use the representational potential of signs as conceptual bridge—it could be made, for example, for the sign for "equal," presented in Figure 6. It can hence be considered a *learning goal* to understand the metaphoric potential of mathematical SL signs in order to use them to benefit the acquisition of mathematics.

The embodiment perspective as we discussed it in this section does not allow us to make statements about *SL as a learning obstacle* and *as a prerequisite for learning*. However, the mere absence of these aspects can be seen as linked to the nature of embodiment: Within this approach, learning does not start with language but originates from the body (Nemirovsky, 2003). While language is still an important prerequisite for conceptualizing meaning, as far as it concerns embodiment, it does not seem to make a difference whether this prerequisite is in the form of spoken or signed language.

Theory to Practice: Challenges and Opportunities

In the present article, we have adapted theoretical perspectives from mathematics education to understand better how SL might influence Deaf learners' mathematical thinking and learning by focusing on aspects specific to learning the discipline. In particular, this has concerned the semiotics of mathematical learning, the embodied processes underlying the understanding and learning of mathematical concepts, and the development of mathematical meaning in and of signs.

Iconicity appeared to be a common thread, with both theoretical perspectives emphasizing its role differently. While research in Deaf education and psycholinguistics already pointed out the influence of iconicity on conceptual understanding in SL, the theoretical discussions in the present article have provided potential explanations as grounded in theories of learning specific to mathematics. With that, these theoretical discussions have also reframed learning goals, potential learning obstacles as related to SL. and, more generally, the role of SL as a medium and as a resource for learning mathematics. These discussions have furthermore provided a background for developing methods to navigate these roles in the mathematics classroom in beneficial ways.

Thoughts on the Potential of Sign Language as a Resource in the Mathematics Classroom

From what we have observed, SL offers great potential as a resource to be integrated beneficially into both the Deaf mathematics classroom and inclusive settings. Within a semiotic perspective we saw how diagrammatic activity can literally go hand in hand with talking about diagrams. It might be interesting to go further into how this might become implemented in teaching practice, as it might provide a fruitful opportunity for fostering students' diagrammatic activity which might then also open a door to diagrammatic activity with and on inscriptive diagrams. Furthermore, semantic fields related to mathematical SL signs can become an explicit focus of the interaction of talking about mathematics. As that, they can be identified and displayed on, for example, posters exhibited in the classroom. These suggestions support the idea expressed by Kurz and Pagliaro (2020) of letting the students become "language experts" and letting them seek out "patterns of meaning in specialized vocabulary and discourse" (p. 90) as a means of emphasizing the influence of the use of these patterns in the social learning process. Both from a semiotic and an embodied perspective, a discussion about "where the mathematical signs come from," how they might relate iconically or metaphorically to an underlying action, and in which respect they are "conceptually accurate" (i.e., inscription; Kurz & Pagliaro, 2020, p. 87) can be fruitful. In addition, students might think about possible alternative signs based on their understanding of the mathematical idea, and discuss these. This would not only provide a diagnostic opportunity for the teacher to access the students' understandings but would also foster the students' changing perspectives in the sense of learning from an "other knowledgeable other" (Krause, 2019, p. 95).

Making SL signs an explicit topic in the inclusive classroom can, furthermore, potentially benefit all the learners while highlighting Deaf learners' practice of signing as a strength. It can become an additional representational resource that widens access to mathematical topics. For example, Wille (2019) implemented the use of videos in which some fraction concepts become explained in ÖGS in an inclusive classroom with two Deaf signers. While the (captioned) videos were primarily used to facilitate access to the content for the Deaf students, avoiding a problematic attentional switch between the teacher's explanation and the interpreter, Wille described positive feedback not only from, the teacher but from hearing students. This also concerns the explication of the signed mathematical terms, presented by the Deaf students following the video and becoming a topic of discussion in class. In that process, the mathematical SL signs can fulfill a representational function as gestural signs (Krause, 2016) even for the hearing students, and can thereby facilitate mathematical interaction in the inclusive classroom. Signed videos such as those developed currently in ÖGS and those elaborated as multistep tools encompassing "concept–lecture (explanation)–term–definition" in ASL in the "ASL-Clear" project for several disciplines (https://aslclear.org/app/#/) might thus become a classroom tool in the sense of the basic principles of *universal design for learning* (Rose & Meyer, 2002, p. 69).

The approaches to SL for learning mathematics point to aspects important for teacher education: It is not only important to be aware of the mathematical signs used in the classroom, but also to be aware that students can endow these signs with mathematical meaning through action as activity with diagrams and as embodied experience. The semiotic and embodied lenses—and, generally, theories that focus on what characterizes the learning of mathematics and how it is related to SL—can enable a better understanding of how Deaf teachers integrate the embodied and semiotic links into their teaching potentially implicitly, opening the door for methods that can be learned and reflected upon by future teachers.

Future Directions

We have only provided a glimpse into how theories from mathematics education can enable alternative perspectives on SL in the mathematics classroom. Different theoretical approaches would shift the focus to other aspects of SL. For example, given the close link between language and culture, a sociocultural approach would focus on the relationship between SL and Deaf culture in the learning of mathematics. As Barton (2008) observed, "If mathematics is the way mathematicians talk, then the cultural influences on that talk (the language of discourse, the meanings of words and symbols at the time of talk) create different mathematics" (p. 129), and it would be worthwhile to understand better the mathematics created through signed mathematical discourse.

We also only have provided very specific perspectives within the theories we chose for our conceptual and theoretical frameworks. For example, due to space limitations, we simplified the idea of "language as a learning medium" to a commonsense understanding. Language as a learning medium also concerns the central role that language plays through its communicative function—as a tool for exchanging information through a conventualized linguistic system—and its cognitive function—as a tool for thinking mathematically—in the development of mathematical meaning (Maier & Schweiger, 1999). While this understanding of language and the roles of language provides, again, new substance for theoretical explorations and discussions from semiotic as well as embodiment sides (e.g., related to cognitive functions of gestures; Krause & Salle, 2019), we leave this to future researchers.

Space limitations have also restrained discussion of other aspects, such as further differences of mathematical explanations in SL compared to spoken language concerning structuration as related to the affordances of SL (see Wille & Schreiber, 2019) and the role of classifiers (see note 4).

Also, the Deaf mathematics classroom is inherently bilingual, as signed and written language needs to be coordinated by the Deaf learners. (In inclusive classrooms, this issue is more complex and the handling of its multilinguistic character needs further discussion.) In light of the relationship between bilingualism and logical reasoning (Secada, 1991), argumentation structures of Deaf signers might be worth exploring further. Not only is argumentation closely related to language and communication; its importance for learning mathematics is explicated as a distinct competence in, for example, the NCTM principles and standards (NCTM, 1989, 2000) and the German standards for school mathematics (e.g., KMK, 2004). Argumentation has therefore been a well-researched topic in mathematics education (Sriraman & Umland, 2014). However, the affordances of SL as a medium for learning mathematics might lead to different qualities of argumentation in the Deaf classroom, worth investigating on their own but also in relation to argumentation structures in the second (written) language in the Deaf-as-bilingual classroom.

Concluding Remarks

Our starting point in the present article was to look at SL from the perspective of mathematics education, testing how the integration of theories of mathematical thinking and learning might facilitate understanding of the role of SL in teaching and learning mathematics. The phenomena that caught our attention might not have been new, but the angle from which we considered them certainly is. New perspectives like these allow for a more comprehensive reflection about and understanding of what such phenomena might mean for Deaf students' learning of mathematics.

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Notes

1. In the present article, we use *deaf* when referring to hearing status and *Deaf* when considering identity, in particular the use of sign languages. When we are not assuming identity or SL use, we use *deaf*.
2. Interestingly, ASL and Finnish Sign Language are both SLs with one-handed signs for numbers. Whether there are similar algorithms in SLs that use two hands for number signs, such as German Sign Language, and how they might look, is an open question.
3. The "signs" in semiotics and those that are linguistic entities of SL share the same written referent. This fact complicated the writing of the present section of this article and will also complicate the reading of it. The signs of SLs being specific kinds of signs in the semiotic sense does not make this any easier. To minimize ambiguity, we will use "SL signs" when referring to the signs of SLs and "signs" in the general semiotic context throughout this section.
4. We distinguish gesture and sign language signs following the discussion in Goldin-Meadow and Brentari (2017), based on whether they are integrated into the communicative act rather than on the basis of their categorical or imagistic component, and acknowledging that at times there cannot be a clear and objective distinction between the two. We use "signed expression" to integrate expressions that include sign language as well as situated mathematical signs and gestures. However, both SL signs and gestures are clearly seen as distinguishable from manipulative actions.
5. Both handshapes and their integration into the signed mathematical explanation—and more generally into the mathematical discourse—might be further discussed against the background of the idea of classifiers (see, e.g., Emmorey, 2003). While we see great potential in investigating this connection further in the context of mathematical thinking, learning, and teaching (within this semiotic approach and beyond), we acknowledge that this could not be done sufficiently in the present article, considering both the authors' current expertise in this area and the space that would be needed to do justice to an introduction of classifier constructions and their potential integration into signed expression as either categorical (linguistic) or imagistic (gesture). To date, there is no empirical foundation to build on, and only very few links have been made in the literature between classifier handshapes and mathematical signs. (See Kurz & Pagliaro, 2020, for an exception that mentions the use of the bent-L handshape also adopted in Figures 1a and 2b as a classifier used to refer to numbers and quantities.) We therefore acknowledge that this topic would be best addressed in a future paper focusing on classifiers in mathematical signs and signed mathematics discourse.

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