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QUATERNIONIC HIDA FAMILIES AND THE TRIPLE PRODUCT p-ADIC L-FUNCTION

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Abstract. The main purpose of this thesis is to provide an algorithm for approximating the value of the balanced p-adic L-function, as constructed in [Hsi21], at the point $(2,1,1)$ which lies outside of its interpolation region. We are interested in the case where at least one of the Hida families is associated with an elliptic curve over the rationals. For ease of exposition, we consider the case where only one local sign of the functional equation is -1. The second part of this thesis, namely Section 2, deals with the above problem. The algorithmic procedure is obtained building on the work of [FM14] and considering finite length geodesics on the Bruhat–Tits tree for $GL_2(\mathbb{Q}_p)$. Section 1 is devoted to studying the behavior in families of quaternionic modular forms arising from orders defined by Pizer and Hijikata–Pizer–Shemanske. As in Section 2, we restrict our attention to a definite rational quaternion algebra ramified at a single prime ℓ. We prove a Control Theorem in the spirit of Hida, in which the novelty lies in the rank of the Hecke-eigenspaces being 2 and no more 1 as in the classical case of Eichler orders. The motivation behind Section 1 comes from the desire to consider classical weight-1 modular forms in the limit process in Section 2. Although the balanced p-adic L-function is no more available in such a situation, the analysis carried out in this thesis represents a necessary first advance in this direction.

Zusammenfassung. Das Hauptziel dieser Arbeit ist es, einen Algorithmus zur Approximation des Wertes der ausbalancierten p-adischen L-Funktion, wie in [Hsi21] konstruiert, in dem Punkt $(2,1,1)$ der außerhalb des Interpolationsbereiches liegt, zu formulieren. Wir interessieren uns für den Fall bei dem zumindest eine der Hida-Familien einer elliptischen Kurve zugeordnet ist. Zur einfacheren Darstellung betrachten wir den Fall in dem nur eines der lokalen Vorzeichen der Funktionalgleichung -1 ist. Der zweite Teil dieser Arbeit, nämlich Kapitel 2, beschäftigt sich mit dem Problem von oben. Das algorithmische Verfahren baut auf der Arbeit [FM14] auf und betrachtet geodätische Linien endlicher Länge im Bruhat–Tits Baum für $GL_2(\mathbb{Q}_p)$. Kapitel 1 ist dem Studium des Verhaltens von Familien quaternionischer Modulformen, die von Ordnungen definiert von Pizer und Hijikata–Pizer–Shemanske stammen, gewidmet. Wie in Kapitel 2 beschränken wir unsere Aufmerksamkeit auf eine definit rationale quaternionische Algebra verzweigt in einer einzelnen Primzahl ℓ. Wir beweisen einen Kontrollsatz im Sinne von Hida, bei dem die Neuheit im Rang der Hecke-Eigenräume liegt, welcher nun 2 und nicht mehr 1 wie im klassischen Fall von Eichler-Ordnungen ist. Die Motivation für Kapitel 1 entstammt dem Wunsch, Gewicht-1 Modulformen in dem Grenzwertverfahren in Kapitel 2 zu betrachten. Obwohl die ausbalancierte p-adische L-Funktion nicht mehr für solche Situationen geeignet ist, stellt die durchgeführte Analyse einen ersten nötigen Schritt in diese Richtung dar.

Contents

Introduction

This thesis is composed of two, *a priori*, distinct parts. We present here the main results and explain the relation between them.

In the first section we study the behavior in families of quaternionic modular forms arising from special orders, as defined in [Piz80b] and [HPS89b]. This situation differs markedly from the classical case of Eichler orders, where all the local non-archimedean automorphic representations are 1-dimensional.

Let ℓ be an odd prime and let $N \geq 1$ be an integer prime to ℓ. Let B be the unique (up to isomorphism) quaternion algebra over $\mathbb{Q}$ ramified exactly at ℓ and ∞. Take R to be an Eichler order of level N in B and consider, for $k \geq 2$ an integer, the space of $\mathbb{C}$-valued quaternionic newforms with level R, denoted by $\mathscr{S}_k^{new}(R, \mathbb{C})$; we refer to Section 1.1.6 for the precise definition. The Jacquet–Langlands correspondence, which ensures an injective transfer between automorphic representations for $GL_2(\mathbb{Q})$ and $B^\times$, at the level of automorphic forms takes an explicit realization, often referred to as the Eichler–Jacquet–Langlands correspondence. More precisely, there is a Hecke-equivariant isomorphism

$$\mathscr{S}_k^{new}(R, \mathbb{C}) \cong S_k^{\ell-new}(\Gamma_0(N\ell), \mathbb{C}). \tag{1}$$

On the other hand, in order to study modular forms with higher level structure at ℓ, one needs more general orders, namely the *special orders* introduced in Definition 1.1.1. In the case where R is a special order in B, corresponding to level structure $N\ell^{2r}$ with $r \geq 1$, the relation between automorphic forms changes. In order to be consistent with [HPS89a], let us fix the following notation, which will also be present throughout Section 1: for any module M we write $2M$ for the direct sum $M \oplus M$. The above Hecke-isomorphism (1), for $r \geq 2$, must be replaced by

$$2S_k^{new}(\Gamma_1(N\ell^{2r}), \mathbb{C}) \cong \mathscr{S}_k^{new}(R, \mathbb{C}) \oplus \bigoplus_{\chi} 2S_k^{new}(\Gamma_1(N\ell^r), \chi^2, \mathbb{C})^{\otimes \overline{\chi}}, \tag{2}$$

where two copies of $S_k^{new}(\Gamma_1(N\ell^{2r}), \mathbb{C})$ must be taken into account and the sum of the twisted spaces $S_k^{new}(\Gamma_1(N\ell^r), \chi^2, \mathbb{C})^{\otimes \overline{\chi}}$ (see beginning of Section 1.2.3) runs over certain primitive characters modulo ℓ^r. Equation (2) has a slightly more complicated expression for $r = 1$, but already this situation is explanatory of the phenomenon; the general statement is the content of Theorem 1.2.3. In particular, we see that any classical newform of level $N\ell^{2r}$ and twist-minimal at ℓ (as in Definition 3.2.2) lifts to two linearly independent quaternionic newforms with the same Hecke-eigenvalues away from the level. The above mentioned situation has been extensively studied in a series of works by H. Hijikata, A. Pizer, and T. Shemanske, most notably [Piz80b] and [HPS89a]. Looking at Equation (2), it is then natural to ask whether these quaternionic modular forms live in p-adic families, for p an odd prime different from ℓ and prime to N. In the first part of this thesis, we focus on this question, providing a positive answer to it. We are indebted to the work [LV12] from which we take inspiration for proving our generalization.

Let ℓ, p and N be as introduced above. For the sake of simplicity, we restrict here to the case of trivial character at ℓ and exponent $2r \geq 4$; we refer the reader to Theorem 1.3.8 for the general statement. As in Definition 2.4 of [GS93], we consider $\mathcal{R}$ to be the *universal ordinary p-adic Hecke algebra* of tame level $N\ell^{2r}$. Denoting the Iwasawa algebra by $\Lambda := \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$, $\mathcal{R}$ represents the Λ-algebra of Hecke-operators acting on Hida families of tame level $N\ell^{2r}$. For any continuous group homomorphism $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p}$, we say that κ is an arithmetic homomorphism if its restriction to $1 + p\mathbb{Z}_p \subseteq \Lambda$ defines a character of the form $z \longmapsto z^{k-2}\varepsilon(z)$, for $k \in \mathbb{Z}_{\geq 2}$ and ε a p-adic character of conductor p^n, $n \geq 0$. This homomorphisms are identified with points on the so-called weight space; we refer to Section 2.1 for further details. As usual, we associate to each arithmetic homomorphism κ the couple (k, ε). For any κ we also denote its kernel by $\mathcal{P}_\kappa$ and the corresponding localization of $\mathcal{R}$ by $\mathcal{R}_{\mathcal{P}_\kappa}$. Let f be a classical modular newform in $S_k(\Gamma_0(Np\ell^{2r}), \mathbb{C})$ and assume that f is twist-minimal at ℓ. If moreover f is p-ordinary, we can consider the unique Hida family f_∞ associated with f by the works of Hida and Wiles. For each arithmetic homomorphism κ we denote by f_κ the specialization of f_∞ at κ. We set F (resp. F_κ) to be the field extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of f (resp. f_κ) and take $\mathcal{O}$ (resp. $\mathcal{O}_\kappa$) to be its ring of integers. Fix now R to be a maximal order in the quaternion algebra B which contains the family of nested orders $\{R^n\}$,

$$\cdots \subset R^{n+1} \subset R^n \subset \cdots R^0 \subset R, \qquad R^n \text{ is a special order of level } Np^n\ell^{2r}.$$

At all places $q \neq \ell, \infty$, we fix isomorphisms $\iota_q : B \otimes_{\mathbb{Q}} \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ such that $R^n \otimes_{\mathbb{Z}} \mathbb{Z}_q$ is identified with the upper triangular matrices modulo Np^n. For each n we consider the compact open subgroup $U_n \subset \widehat{R^n} := R^n \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ defined as

$$U_n = \left\{ g = (g_q) \in \widehat{R}^{n\times} \;\middle|\; \iota_q(g_q) \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{Np^n\mathbb{Z}_q}, \text{ for } q \mid Np^n \right\}.$$

Let $V_{k-2}(\mathcal{O}_\kappa)$ be the dual of $L_{k-2}(\mathcal{O}_\kappa)$, the space of homogeneous polynomials in $\mathcal{O}_\kappa[X, Y]$ of degree $k-2$, endowed with the action $|_{u_p}$ of $GL_2(\mathbb{Z}_p)$, induced by the left multiplication $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(X, Y)^t = (aX + bY, cX + dY)^t$. Denoting $\widehat{B} = B \otimes \mathbb{A}_{\mathbb{Q},f}$ for $\mathbb{A}_{\mathbb{Q},f}$ the finite adèles of $\mathbb{Q}$, we consider, as in Definition 1.1.12, the space of quaternionic p-adic modular forms of weight $k \geq 2$, character ε (of conductor p^n) and level U_n,

$$\begin{aligned} S_k(U_n, \varepsilon, \mathcal{O}_\kappa) := \Big\{ \varphi : \widehat{B}^\times \to V_{k-2}(\mathcal{O}_\kappa) \mid \varphi(b\tilde{b}uz) = \varepsilon(z) z_p^{k-2} \varphi(\tilde{b})|_{u_p}, \\ \text{for } b \in B^\times, \tilde{b} \in \widehat{B}^\times, u \in U_n, z \in \mathbb{A}_{\mathbb{Q},f}^\times \Big\}. \end{aligned}$$

More generally, let X be the subset of *primitive vectors* in $\mathbb{Z}_p^2$, namely the subset of vectors with at least one component which is not divisible by p, and consider the space of $\mathcal{O}$-valued measures $M(\mathsf{X}, \mathcal{O})$ on X. We construct, following Definition 1.3.1, the space of measure-valued quaternionic modular forms

$$S_2(U_0, M(\mathsf{X},\mathcal{O})) := \Big\{\varphi : \widehat{B}^\times \to M(\mathsf{X},\mathcal{O}) \mid \varphi(b\tilde{b}uz) = \varphi(\tilde{b})|_{u_p} \quad \text{for } b \in B^\times,\ \tilde{b} \in \widehat{B}^\times,\ u \in U_0,\ z \in \mathbb{A}_{\mathbb{Q},f}^\times\Big\},$$

for $|_{u_p}$ the action of $GL_2(\mathbb{Z}_p)$ induced by the left multiplication on the variables. By integration, we induce, for any arithmetic homomorphism $\kappa = (k, \varepsilon)$, a specialization map

$$\nu_\kappa : S_2(U_0, M(\mathsf{X},\mathcal{O})) \longrightarrow S_k(U_n, \varepsilon, \mathcal{O})$$

such that

$$\nu_\kappa(\varphi)(\tilde{b})(P) := \int_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times} \varepsilon_{\mathbb{A}}^{-1}(y)P(x,y)d(\varphi(\tilde{b}))(x,y)$$

for ϕ and $\tilde{b}$ as above, and any $P \in L_{k-2}(\mathcal{O}_\kappa)$; we refer to Section 1.3.1 for all the details. Considering the ordinary component of $S_2(U_0, \varepsilon, M(\mathsf{X},\mathcal{O}))$, which we denote by $\mathbb{W}$, the specialization maps descend to maps between the ordinary components

$$\nu_\kappa^{ord} : \mathbb{W} \longrightarrow S_k(U_n, \varepsilon, \mathcal{O})^{ord},$$

for $S_k(U_n, \varepsilon, F_\kappa)^{ord}$ the subspace of p-ordinary quaternionic forms in $S_k(U_n, \varepsilon, F_\kappa)$. As the algebra $\mathcal{R}$ acts on the Hida family f_∞, we can consider the f_∞-isotypic component

$$\left(\mathbb{W} \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]} \mathcal{R}\right)[f_\infty],$$

that is, the component of $\mathbb{W} \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]} \mathcal{R}$ where the Hecke-operators (see Section 1.2.2 for more details) act with the same $\mathcal{R}$-eigenvalues of f_∞. Up to a mild condition on the level $N\ell^{2r}$, as explained in Remark 1.3.4, one can assume $\mathbb{W} \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]} \mathcal{R}$ to be free. We can hence state our main result under the above simplifying restrictions for the character and the level at ℓ and refer to Theorems 1.3.8 and 1.3.9 for the general statements.

Theorem A (Control theorem for special orders)**.** *With the above notation, suppose that f is twist-minimal at ℓ. For any arithmetic homomorphism $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p}$, the map ν_κ (of Proposition 1.3.5, induced by the specialization map) induces an isomorphism of 2-dimensional F_κ-vector spaces*

$$\left(\left(\mathbb{W} \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]} \mathcal{R}\right)[f_\infty]\right) \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\kappa}/\mathcal{P}_\kappa\mathcal{R}_{\mathcal{P}_\kappa} \xrightarrow{\cong} \left(S_k(U_n, \varepsilon, F_\kappa)^{ord}\right)[f_\kappa].$$

The two main ingredients needed for the proof of Theorem A are the above isomorphism (2) and the seminal paper [Hid88]; the results proved in [Hid88] for definite quaternion algebra over totally real fields different from $\mathbb{Q}$ remain true in the case of definite quaternion algebras over $\mathbb{Q}$, as already noticed in Section 3 of [LV12] and Section 4 of [Hsi21], and in the case of special orders, as remarked in Remarks 1.3.4 and 1.3.7.

Theorem A extends the foundational results of Hida theory to the case of quaternionic modular forms with special level structure, allowing to consider quaternionic p-adic families with tame level ℓ^{2r} over the quaternion algebra B, which, we remark, is ramified at ℓ. An additional motivation for obtaining such a Control Theorem originates from the second part of this thesis, namely from the desire to study points outside the interpolation region of the triple product p-adic L-function. The cases of interest emerge while considering quaternionic moudular forms arising from special orders. We will come back to this matter at the end of this introduction.

Let p be as above and suppose that $p \geq 5$. As mentioned above, Section 2 deals with the computational problem of approximating the value at $(2,1,1)$ for the balanced triple product p-adic L-function.

More precisely, consider again B to be a definite rational quaternion algebra ramified at ℓ and ∞, and take three p-adic Hida families as follows:

(a) f_∞ is the unique Hida family associated with $f \in S_2(\Gamma_0(N_1\ell p), \mathbb{Q})$, a twist-minimal primitive newform corresponding to a p-ordinary elliptic curve $E/\mathbb{Q}$. In particular, the family has tame level N_1 with trivial tame character;

(b) g_∞ and h_∞ are p-adic Hida families of tame level $N_2\ell$ with tame character ψ and ψ^{-1} respectively. We moreover suppose that ψ and ψ^{-1} are both primitive of conductor N_2.

Furthermore, we assume that N_2 is square-free, $N_2|N_1$ and that the triple $(f_\infty, g_\infty, h_\infty)$ satisfies the hypotheses in Section 2.4.1. Let $\mathcal{L}^{bal}_{F_\infty}$ be the balanced p-adic triple product L-function constructed in [Hsi21] and associated with the triple $F_\infty = (f_\infty, g_\infty, h_\infty)$. This p-adic L-function arises on the so-called balanced region, namely the region of triples of arithmetic homomorphisms $((k_1,\varepsilon_1),(k_2,\varepsilon_2),(k_3,\varepsilon_3))$ such that

$$k_1 + k_2 + k_3 \equiv 0 \pmod 2 \qquad \text{and} \qquad k_1 + k_2 + k_3 > k_i \quad \forall i = 1,2,3,$$

and it extends to the whole triple of weight spaces; we point the reader to [Hsi21] for all the precise definitions and properties. In the balanced region $\mathcal{L}^{bal}_{F_\infty}$ satisfies the interpolation problem relating its square with the complex triple product L-function. Outside this region the same relation is no more ensured; one of this points is $(2,1,1)$, where here 2 and 1 are homomorphisms associated with respectively weight-2 and weight-1 specializations. The desire to study the point $(2,1,1)$ comes from the well-known *Birch and Swinnerton–Dyer conjecture* as one expects to recover information on the L-function associated with the weight-2 specialization $f_2 = f$.

The p-adic L-function $\mathcal{L}^{bal}_{F_\infty}$ is constructed as the limit of certain theta-elements defined and studied in Section 4.6 of [Hsi21], in particular in Proposition 4.9. The aim of the second section of this thesis is to explicitly describe these theta-elements and provide an algorithm for computing its values when evaluated at a triple of arithmetic points of the form $(2,(2,\varepsilon),(2,\varepsilon))$, for ε a primitive p-adic character of conductor p^n. This allows us to approximate the value $\mathcal{L}^{bal}_{F_\infty}(2,1,1)$ as the limit over the increasing conductor p^n of such theta-elements evaluated at $(2,(2,\varepsilon),(2,\varepsilon))$.

We remark here that the restriction to a definite quaternion algebra ramified at only one rational prime is not strictly necessary and that the approach used in Section 2 can be straightforwardly generalized. On the other hand, one necessarily requires f_∞ to have trivial tame character and g_∞ and h_∞ to have opposite tame character; this last hypothesis is mandatory to isolate the component associated with f and to express the component depending on g_∞ and h_∞ as sum of functions of the finite length geodesics on the Bruhat–Tits tree. Even though Section 2 deals only with level structure associated with Eichler orders, we point out that the results in Section 2.3, and especially those contained in Section 2.3.1, are still valid in the case of special orders.

The strategy behind Section 2 relies on the fact that, considering weight-2 specializations together with the Approximation Theorem (see Lemma 2.3.1), one can interpret quaternionic p-adic modular forms (with trivial character) as functions on suitable p-adic double quotients. Section 2.2.1 is devoted to study part of these p-adic double quotients, namely $GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p)$; it is well known that it can be identified with the geodesics on the Bruhat–Tits tree of length n. Moreover, the canonical projection maps correspond to forgetting the end point of a geodesic, as observed in Lemma 2.2.1. In Section 2.3.2 and Section 2.3.3 we supply explicit matrix representatives for $GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p)$ and extend the algorithm developed in [FM14] to the case of finite length geodesics (see Algorithm 1). Consider now the three specializations $(f, g_{(2,\varepsilon)}, h_{(2,\varepsilon)})$ and the triple of eigenvalues at p, $(a_p(f), a_p(g_{(2,\varepsilon)} \otimes \varepsilon_{\mathbb{A}}^{-1}), a_p(h_{(2,\varepsilon)}))$. Let F^1 be the quaternionic modular form associated with f (which sometimes corresponds to the harmonic cocycle associated with f). Taking G^n (resp. H^n) the quaternionic modular form (for quaternionic families chosen as in [Hsi21], Theorem 4.5) associated with $g_{(2,\varepsilon)}\otimes\varepsilon_{\mathbb{A}}^{-1}$ (resp. $h_{(2,\varepsilon)}$) we can identify the product $G^n \cdot H^n$ as a function of the length-n geodesics on the Bruhat–Tits tree. We hence obtain the following theorem (see Proposition 2.4.5 and Theorem 2.4.6).

Theorem B. *Let $\underline{Q}$ be the triple of points $((2,\varepsilon), 2, (2,\varepsilon))$. The theta-element appearing in the triple product p-adic L-function $\mathcal{L}_{F_\infty}^{bal}$ evaluated at $(2,(2,\varepsilon),(2,\varepsilon))$, is equal to*

$$(3)\quad \Theta_{F_\infty^{B}{}'}(\underline{Q}) = \frac{(1-p^{-1})}{a_p(f)^n} \cdot \sum_{e\in\Gamma\backslash\mathcal{E}(\mathcal{T})} \frac{F^1(e)}{\#Stab_{(R^1[1/p])^1}(e)} \cdot$$
$$\cdot \left(\frac{\varepsilon_{\mathbb{A}}^{-1}(p^n)\cdot p^{2n}}{a_p\left(g_{(2,\varepsilon)}\otimes\varepsilon_{\mathbb{A}}^{-1}\right)^n a_p\left(h_{(2,\varepsilon)}\right)^n} \right) \sum_{g\in Geod_n(\mathcal{T})(e)} G^n(g\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right))\cdot H^n(g\begin{pmatrix}0 & p^{-n}\\ -p^n & 0\end{pmatrix}).$$

Furthermore, by construction, the value at $(2,1,1)$ of the balanced p-adic L-function (up to a unit in the fraction field of the Iwasawa algebra associated with the triple of Hida families) is

$$\mathcal{L}_{F_\infty}^{bal}(2,1,1) \doteq \lim_{\varepsilon\to 1}\Theta_{F_\infty^{B}{}'}((2,\varepsilon),2,(2,\varepsilon))$$

and the limit $\lim_{\varepsilon\to1}\Theta_{F_\infty^{B}{}'}((2,\varepsilon),2,(2,\varepsilon))$ can be algorithmically approximated with a given p-adic precision.

The outer sum in Equation (3) is independent of the limit process; it depends only on f and the representatives of the edges $\mathcal{E}(\mathcal{T})$ of the Bruhat–Tits tree, modulo the action of

Γ, the image in $PGL_2(\mathbb{Q}_p)$ of the invertible elements in a suitable Eichler $\mathbb{Z}[1/p]$-order $R^1[1/p]$. Differently, the inner component (together with $a_p(f)^n$) varies during the limit process and each summation depends on a class of an edge and only on the values of G^n and H^n on length-n geodesics.

The algorithmic procedure delineated in Section 2 is not yet supported by effective computations and examples, but we plan to address this lack in the future. Unfortunately, the procedure might require an enormous amount of computational time and resources, as the complexity grows exponentially in the length of the geodesics, hence on the precision required.

Section 3 is devoted to providing a brief account on how the hypotheses in both [GS19] and [Hsi21] do not allow to approach the point $(2, 1, 1)$ considering classical modular forms of weight 1 and level structure given by an Eichler order. This impossibility is due to the rigidity in the structure of Hida families and it can be avoided replacing Eichler orders with special ones. This limitation in the available constructions of the balanced p-adic L-function can be traced back to the crucial work of Chenevier [Che05], in which a p-adic extension of the classical Jacquet–Langlands correspondence has been developed from the point of view of automorphic forms. For this purpose, the machinery of eigenvarieties is considered and the correspondence is exploited for Eichler level structure. In this spirit, one could try to produce a suitable rigid analytic morphism between the two eigenvarieties associated respectively with the quaternion algebra B and GL_2 (with suitable level structures), and employ it to produce an analogous balanced triple product p-adic L-function, solving the interpolation problem in the case of special orders. This would allow to study the case of interest at $(2, 1, 1)$. As mentioned above, this desire represents an additional motivation for Theorem A as this theorem is the first advance towards the construction of a more general balanced p-adic L-function. We plan to further investigate this aspect in the near future.

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1. A Hida control theorem for special orders

1.1. **Quaternionic orders and modular forms.** We begin by recalling the definitions of the various quaternionic orders as well as the definition of quaternionic modular forms, both p-adic and classical, for a definite quaternion algebra over $\mathbb{Q}$. Special orders are a generalization of the classical Eichler orders, which are needed for studying both higher ramification and the presence of a character at primes where the quaternion algebra is ramified. We refer the reader to [HPS89a] and [HPS89b] for all the details.

We fix once and for all a choice of field embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$. For any prime q we denote the q-adic valuation by v_q.

1.1.1. **Special orders.** Let B the unique (up to isomorphism) quaternion algebra over $\mathbb{Q}$ with discriminant D. Fix an isomorphism $\iota_q : B_q := B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for each $q \nmid D$. We denote the reduced norm of $b \in B$ by $n(b) \in \mathbb{Q}$. Let q be an odd rational prime, and fix $u \in \mathbb{Z}$ to be a quadratic non residue modulo q. The local field $\mathbb{Q}_q$ has a unique quadratic unramified extension $\mathbb{Q}_q(\sqrt{u})$ and two quadratic ramified ones, $\mathbb{Q}_q(\sqrt{q})$ and $\mathbb{Q}_q(\sqrt{uq})$. For L_q one of these quadratic extensions, we denote by $\mathcal{O}_{L_q}$ its ring of integers. Set

$$M_2^0(N\mathbb{Z}_q) := \left\{ \gamma \in M_2(\mathbb{Z}_q) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N\mathbb{Z}_q} \right\} = M_2^0(q^{v_q(N)}\mathbb{Z}_q),$$

and, for $r \geq 1$,

$$M(L_q, r) := \mathcal{O}_{L_q} + \left\{ x \in B_q \;\middle|\; n(x) \in q\mathbb{Z}_q \right\}^{r-1} = \mathcal{O}_{L_q} + \left\{ x \in B_q \;\middle|\; n(x) \in q^{r-1}\mathbb{Z}_q \right\}.$$

We notice that for $r = 1$, $M(L_q, 1)$ is the unique maximal ideal of B_q.

Definition 1.1.1 ([HPS89b], Def. 6.1). *An order R in B is said to be a special order of level $M = N \cdot \prod_{q|D} q^{\nu(q)}$ if*

(i) $R_q := R \otimes \mathbb{Z}_q$ is conjugate to $M_2^0(N\mathbb{Z}_q)$ by an element of $B_q^\times$ (via ι_q), for each $q \nmid D\infty$;

(ii) there exists a quadratic extension L_q of $\mathbb{Q}_q$ such that R_q is conjugate to $M(L_q, \nu(q))$, for each $q|D$.

In the following, we choose for each $q \nmid D$ the isomorphism ι_q such that $\iota_l(R_q) = M_2^0(N\mathbb{Z}_q)$. If in the above definition we take the level M to be such that $D||M$, we obtain the usual definition of an Eichler order of level N (see [Piz80a], page 344).

Remark 1.1.2. *From now on, we fix a particular choice of special orders and quadratic field extensions. We follow the thorough summary given in Section 2.2 of [LRdVP18]; it is based on a careful analysis of [HPS89a] and takes into account more general orders. We take the prime decomposition of the discriminant $D = \prod_{i=1}^k \ell_i$ and let f be any newform in $S_2\left(\Gamma_1\left(N\prod_{i=1}^k \ell_i^{e_i}\right), \mathbb{C}\right)$. In order to be able to lift f to a quaternionic modular form, we fix the choice of the special order R such that the quadratic extensions L_{ℓ_i} of $\mathbb{Q}_{\ell_i}$ and the exponents m_i are as follows.*

(1) If ℓ_i is odd:

(i) e_i odd: L is the unramified extension of $\mathbb{Q}_{\ell_i}$ and $m_i = e_i$;

(ii) e_i even: L is one of the two ramified extension of $\mathbb{Q}_{\ell_i}$ and $m_i = e_i$;

(2) If $\ell_i = 2$:

(i) $e_i = 1$: L is the unramified extension of $\mathbb{Q}_2$ and $m_i = 1$;

(ii) $e_i = 2$: $L = \mathbb{Q}_2(\sqrt{3})$ or $L = \mathbb{Q}_2(\sqrt{7})$ and $m_i = 2$;

(iii) $e_i \geq 3$, odd: L is the unramified extension of $\mathbb{Q}_2$ and $m_i = e_i$;

The case $\ell_i = 2$ and e_i even presents some further difficulties and, as our main case of interest is the case of ℓ_i odd, we omit it and refer the reader to [HPS89a] and [LRdVP18].

Notation. *We set $\widehat{B} = B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},f}$ and $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ where $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q}\widehat{\mathbb{Z}}$ are the finite adèles of $\mathbb{Q}$.*

We recall some properties of special orders.

Lemma 1.1.3. *$\widehat{R}^\times$ is a compact open subgroup of $\widehat{B}^\times$. In fact, this is true for each component.*

Proof. This lemma is a classical result (we refer *e.g.* to Sections 5.1 and 5.2 of [MM06]) whenever R is an Eichler order. We consider the case of a special order. Lemma 5.1.1 of [MM06] tells us that $R_q^\times$ is compact in $B_q^\times$, independently of the order. By definition of special order (Definition 1.1.1), it is enough to consider $M(L_\ell, r)$. Since the reduced norm is continuous, $\left\{ x \in B_\ell \;\middle|\; n(x) \in \ell^{r-1}\mathbb{Z}_\ell \right\}$ is open and thus, as the sum is a continuous homomorphism, we deduce the claim. ∎

Proposition 1.1.4 ([Piz77],Proposition 2.13)**.** *All special orders have finite class number. Moreover, it depends only on the level and not on the specific choice of the special order.*

Lemma 1.1.5 ([HPS89b], Lemma 7.4)**.** *Let R be a special order of level N. Then there exists a set of ideal class representatives $\{I_1, \ldots, I_h\}$ for the left R-ideal classes, such that $I_i \otimes \mathbb{Z}_q = R_q$ for all q dividing the level.*

1.1.2. **Characters.** Let R be a special order of level $M = N \cdot \prod_{\ell | D} \ell^{\nu(\ell)}$ and let χ be a Dirichlet character with conductor C.

Assumption 1.1.6. *Assume that $v_q(N) \geq v_q(C)$ for all $q \mid M$ and that*

$$\nu(q) \geq \begin{cases} 2\, v_q(C) - 1 & \text{if } L_q \text{ is unramified over } \mathbb{Q}_q, \\ 2\, v_q(C) & \text{if } L_q \text{ is ramified over } \mathbb{Q}_q. \end{cases}$$

It is readily noticed that Assumption 1.1.6 ensures the possibility of an automorphic lifting via the Jacquet–Langlands correspondence (see also Proposition 3.2.5).

We want to extend χ to a character $\widetilde{\chi}$ of $R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and for this purpose we must deal with several sub-cases. First of all, we decompose $\chi = \prod_{q|C} \chi_q$ by the Chinese Reminder Theorem and we define each $\widetilde{\chi}_q$ as follows.

(1) If $q \mid N$ and $q \nmid C$, we set $\widetilde{\chi_q}(\alpha) = 1$ for each $\alpha \in R_q$.
(2) If $q \mid M$ and $q \mid C$, we set $\widetilde{\chi_q}(\alpha) = \chi_q(d)$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in R_q = M_2^0(N\mathbb{Z}_q)$.
(3) If q is odd, $q \mid N/M$ and $q \mid C$ we deal with three further sub-cases:
   (i) If $q \mid\mid C$ and χ_q is odd, we can always find a lift to $\mathcal{O}_{L_q}/m_{L_q} \supseteq \mathbb{Z}_q/q\mathbb{Z}_q$ and thus to $\mathcal{O}_{L_q}$, which we call χ_{L_q}; here m_{L_q} is the maximal ideal of $\mathcal{O}_{L_q}$. Because of Assumption 1.1.6, $R_q = M(L_q, \nu(q))$ is contained in $M(L_q, 2v_q(C) - 1)$ (if L_q is unramified) or in $M(L_q, 2v_q(C))$, hence we set $\widetilde{\chi_q}(\alpha + \beta) = \chi_{L_q}(\alpha)$ for each $\alpha + \beta \in R_q = M(L_q, \nu(q))$ (and $\alpha \in \mathcal{O}_{L_q}$).
   (ii) If $q^e \mid\mid C$ and χ_q is even, we can always find a character ψ such that $\psi^2 = \chi_q$ and with conductor $cond(\psi) = cond(\chi_q)$. As remarked in Section 7.2 of [HPS89b], the choice of this character is not important, but the fact that a particular choice is fixed is. We set $\widetilde{\chi_q}(\alpha) = \psi(n(\alpha))$.
   (iii) If $q^e \mid\mid C$ and χ_q is odd, we write $\chi_q = \varepsilon \cdot \phi$ for a fixed choice of characters ε odd and with $cond(\varepsilon) = q$, and ϕ even. Thus, proceeding analogously to the previous sub-cases, we set $\widetilde{\chi_q} = \widetilde{\varepsilon} \cdot \widetilde{\phi}$.
(4) If $q = 2$, $2 \mid N/M$ and $2 \mid C$, one proceeds in a similar fashion as in case 3.*i*).

Patching together the local lifts, we define

$$\widetilde{\chi}(b) := \prod_{q \mid N} \widetilde{\chi_q}(b_q)$$

for $b \in B(\mathbb{A}_\mathbb{Q})^\times$ such that $b_q \in R_q^\times \subset B_q^\times$. In particular, if I is a lattice in B such that $I_q = I \otimes \mathbb{Z}_q = R_q$ for each $q|N$, and $b \in I$, we have

$$\widetilde{\chi}(b) = \prod_{q \mid N} \widetilde{\chi_q}(b).$$

We refer to [HPS89b], Section 7.2 for all the details.

1.1.3. **Quaternionic modular forms of weight 2.** Take B as in the above Section 1.1.1 with $R \subset B$ a special order of level M and recall that we fixed isomorphisms $\iota_q : B_q := B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for each $q \nmid D$, such that $\iota_q : R_q := R \otimes \mathbb{Z}_q \cong M_2^0(M\mathbb{Z}_q)$. Set $B(\mathbb{A}_\mathbb{Q})^\times = (B \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q})^\times$ and $R(\mathbb{A}_\mathbb{Q})^\times = \{r \in B(\mathbb{A}_\mathbb{Q})^\times \mid (r_q)_{q<\infty} \in \widehat{R}^\times\}$.

Definition 1.1.7. *We define the space of weight-2 quaternionic modular forms with level structure $R(\mathbb{A}_\mathbb{Q})^\times$, character χ satisfying Assumption 1.1.6 and $\mathbb{C}$-coefficients as the $\mathbb{C}$-vector space $S_2(R, \widetilde{\chi})$ of all continuous functions*

$$\varphi : B(\mathbb{A}_\mathbb{Q})^\times \longrightarrow \mathbb{C}$$

satisfying

$$\varphi(b\tilde{b}r) = \widetilde{\chi}^{-1}(r)\varphi(\tilde{b})$$

for all $b \in B^\times$, $\tilde{b} \in B(\mathbb{A}_\mathbb{Q})^\times$ and $r \in R(\mathbb{A}_\mathbb{Q})^\times$.

As in Chapter 5 of [HPS89a], we can decompose $B(\mathbb{A}_{\mathbb{Q}})^\times$ as a finite union of distinct double cosets

$$B(\mathbb{A}_{\mathbb{Q}})^\times = \coprod_{i=1}^{h} B^\times x_i R(\mathbb{A}_{\mathbb{Q}})^\times$$

where $h = h(R)$ is the class number of R. Since B is definite, the analogous decomposition holds for $\widehat{B}^\times$, namely $\widehat{B}^\times = \coprod_{i=1}^{h} B^\times \widehat{x}_i \widehat{R}^\times$, with $\widehat{x}_i = (x_{i,l})_{l<\infty}$. By the above Lemma 1.1.5, the representatives $x_i = (x_{i,l})_l \in B(\mathbb{A}_{\mathbb{Q}})^\times$ can be taken to lie in $R(\mathbb{A}_{\mathbb{Q}})^\times$, in particular $x_{i,l} \in R_l^\times$ for each prime $l|M$. if we fix the representatives in this fashion, we have a clear expression of the quaternionic modular forms. By the definition of a quaternionic modular forms and the double coset decomposition, a quaternionic modular form φ is uniquely determined by its values on the representatives. More precisely, for $i = 1, \dots, h$, let $\widetilde{\Gamma}_{x_i} := B^\times \cap x_i^{-1} R^\times x_i$ and define

$$\mathbb{C}_{\widetilde{\chi},i} := \left\{ c \in \mathbb{C} \mid \widetilde{\chi}(\gamma) \cdot c = c, \text{ for each } \gamma \in \widetilde{\Gamma}_{x_i} \right\}.$$

As thoroughly explained in *loc.cit.*, the above observations yield the identification

$$S_2(R, \widetilde{\chi}) \cong \bigoplus_{i=1}^{h} \mathbb{C}_{\widetilde{\chi},i}$$

given by $\varphi \longmapsto (\varphi(x_1), \dots, \varphi(x_h))$. We are allowed to consider different coefficients, in fact the above identification still holds when we replace $\mathbb{C}$ by $\mathbb{Q}(\widetilde{\chi})$, the field extension of $\mathbb{Q}$ generated by the values of the character $\widetilde{\chi}$. By extension of scalars we recover $S_2(R, \widetilde{\chi}) = S_2(R, \widetilde{\chi}; \mathbb{Q}(\chi)) \otimes \mathbb{C}$ and we can consider p-adic coefficients $S_2(R, \widetilde{\chi}; \overline{\mathbb{Q}_p}) = S_2(R, \widetilde{\chi}; \mathbb{Q}(\chi)) \otimes \overline{\mathbb{Q}_p}$, for p a prime which does not divide the reduced discriminant of B.

Remark 1.1.8. *All the above constructions and definitions are, up to isomorphism, independent of the specific choice of the special order. Moreover, fixing compatible choices of the lifting characters $\widetilde{\chi}$, all the constructions are compatible with respect to the inclusion of special orders.*

We end this section with the following fact: often the groups $\widetilde{\Gamma}_{x_i}$ have cardinality 2, *i.e.* $\widetilde{\Gamma}_{x_i} = \{\pm 1\}$.

Proposition 1.1.9 ([Piz80b], Proposition 5.12)**.** *Let R be a special order of level $M\ell^2$ in the quaternion algebra over $\mathbb{Q}$ ramified exactly at ℓ and ∞. Then*

$$\#R^\times = \begin{cases} 2 & \text{if } \ell > 3, \\ \text{either } 2 \text{ or } 6 & \text{if } \ell = 3. \end{cases}$$

Moreover, if $\ell = 3$ and $2|M$ or M is divisible by a prime $q \equiv 2 \pmod 3$, then $\#R^\times = 2$.

1.1.4. **p-adic quaternionic modular forms for special orders.** Let p and ℓ be two distinct rational odd primes. From now on, we denote by B the (unique up to isomorphism) definite quaternion algebra over $\mathbb{Q}$ with discriminant ℓ and we fix R to be a maximal order in B. For N a fixed positive integer, prime to both p and ℓ, consider a family of nested special orders $\{R^n\}_{n\geq 0}$ satisfying

$$\cdots \subset R^{n+1} \subset R^n \subset \cdots R^0 \subset R, \qquad R^n \text{ is a special order of level } Np^n\ell^{2r},$$

where $r \geq 1$. Up to conjugation, we can suppose that the orders R_n are all canonical orders of level $Np^n\ell^{2r}$, that is, as in Definition 1.1.1. For any prime q different from ℓ, we can assume that the fixed isomorphism $\iota_q : B_q \cong M_2(\mathbb{Q}_q)$ satisfies $\iota_q R_q^\times \cong GL_2(\mathbb{Z}_q)$ and $\iota_q(R_q^n)^\times \cong \Gamma_0(Np^n\mathbb{Z}_q) := M_2^0(Np^n\mathbb{Z}_q) \cap GL_2(\mathbb{Z}_q)$.

Definition 1.1.10. *We define (*cf. *Lemma 1.1.3) open compact subgroups $U_n \subset \widehat{B}^\times$,*

$$U_n := U_1(R^n) := \left\{ g = (g_q) \in \widehat{R}^{n\times} \;\middle|\; \iota_q(g_q) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{Np^n\mathbb{Z}_q}, \text{ for } q \mid Np^n \right\}.$$

By construction, $U_{n+1} \subset U_n \subset \ldots \subset U_0$. Given any special order R', we denote by $U_1(R')$ the corresponding compact open defined analogously to U_n.

For any commutative ring A, we consider the left action of $M_2(A)$ on the polynomial ring $A[X,Y]$, defined as

$$\gamma \cdot P(X,Y) := P\left((X,Y)\gamma\right),$$

for $P \in A[X,Y]$ and $\gamma \in M_2(A)$. We denote by $L_m(A)$ the submodule of $A[X,Y]$ consisting of homogeneous polynomials of degree m; by definition, $L_m(A)$ is stable under the action of $M_2(\mathbb{Z})$. Its dual module $V_m(A)$ is endowed with the right action

$$\mu|_\gamma(P(X,Y)) := \mu(\gamma \cdot P((X,Y))),$$

for any $\mu \in V_m(A)$ and $P \in L_m(A)$.

We take now $\mathcal{O}$ to be a finite flat extension of $\mathbb{Z}_p$, which we assume to contain all the $\phi(Np^n\ell^{2r})$-th roots of unity, where ϕ is Euler's totient function. Given an $\mathcal{O}$-algebra A, any A-valued Dirichlet character ϵ modulo Np^n, can be lifted to $\epsilon_{\mathbb{A}} : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \longrightarrow A^\times$, its *adèlization*, that is the unique finite order Hecke character

$$\epsilon_{\mathbb{A}} : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{R}_+ (1 + Np^n\widehat{\mathbb{Z}})^\times \longrightarrow A^\times$$

such that $\epsilon_{\mathbb{A}}(\varpi_q) = \epsilon^{-1}(q)$, for every $q \nmid Np^n$ and $\varpi_q = (\varpi_{q,l} : \varpi_{q,l} = 1 \text{ if } l \neq q, \varpi_{q,q} = q)$.

We fix a Dirichlet character ψ modulo $Np^n\ell^{2r}$ with small conductor at ℓ. More precisely, as in [HPS89a], we enforce the following assumption.

Assumption 1.1.11. *The ℓ-component of ψ, ψ_ℓ, is either the trivial character modulo ℓ or an odd character of conductor exactly ℓ.*

Definition 1.1.12. *Let $k \geq 2$ be an integer and let U_n be as in Definition 1.1.10. For an $\mathcal{O}$-algebra A and an A-valued Dirichlet character $\psi = \psi_{Np^n}\psi_{\ell^{2r}}$ (for ψ_{Np^n} modulo Np^n and $\psi_{\ell^{2r}}$ modulo ℓ^{2r}) we define the space of p-adic quaternionic modular forms of weight*

k, level $Np^n\ell^{2r}$ and character ψ as

$$S_k(U_n,\psi,A):=\Big\{\varphi:\widehat{B}^\times\longrightarrow V_{k-2}(A)\mid \varphi(b\tilde{b}uz)=\psi^{-1}_{Np^n,\mathbb{A}}(z)\widetilde{\psi_{\ell^{2r}}}(z)z_p^{k-2}\widetilde{\psi_{\ell^{2r}}}(u_\ell)\varphi(\tilde{b})|_{u_p},\ \textit{for } b\in B^\times,\ \tilde{b}\in\widehat{B}^\times,\ u\in U_n,\ z\in\mathbb{A}^\times_{\mathbb{Q},f}\Big\}.$$

This space can be identified with the space of functions $\varphi: B^\times\backslash\widehat{B}^\times\longrightarrow V_{k-2}(A)$ satisfying

$$\varphi(z\tilde{b})=\psi^{-1}_{Np^n,\mathbb{A}}(z)\widetilde{\psi_{\ell^{2r}}}(z)z_p^{k-2}\varphi(\tilde{b})$$

for $\tilde{b}\in B^\times\backslash\widehat{B}^\times$ and $z\in\mathbb{A}^\times_{\mathbb{Q},f}$, and such that $(u\cdot\varphi)(\tilde{b}):=\varphi|_{u_p^{-1}}(\tilde{b}u)=\widetilde{\psi_{\ell^{2r}}}(u_\ell)\varphi(\tilde{b})$, for any $u\in U_n$ and any $\tilde{b}\in\widehat{B}^\times$.

1.1.5. **Quaternionic modular forms of higher weight.** The definition of classical quaternionic modular forms for higher weight is similar to the one for weight 2. We fix an identification $\iota_\infty: B_\infty\hookrightarrow M_2(\mathbb{C})$ in order to compare p-adic and classical quaternionic modular forms.

Definition 1.1.13. *We define the space of weight-k, $k\geq 2$, quaternionic modular forms with level structure $R^n(\mathbb{A}_\mathbb{Q})^\times$, character χ satisfying Assumption 1.1.6 and $\mathbb{C}$-coefficients as the $\mathbb{C}$-vector space $S_k(R^n,\widetilde{\chi})$ of all continuous functions $\varphi_\infty: B(\mathbb{A}_\mathbb{Q})^\times\longrightarrow V_{k-2}(\mathbb{C})$ satisfying*

$$\varphi_\infty(b\tilde{b}b_\infty r)=\widetilde{\chi}^{-1}(r)|n(b_\infty^{-1})|_{\mathbb{A}}^{(k-2)/2}b_\infty^{-1}\cdot\varphi_\infty(\tilde{b})$$

for all $b\in B^\times$, $b_\infty\in B_\infty^\times$, $\tilde{b}\in B(\mathbb{A}_\mathbb{Q})^\times$ and $r\in R^n(\mathbb{A}_\mathbb{Q})^\times$.

As explained in Chapter 2 of [Hid88], we can identify classical and p-adic modular forms. We identify $\mathbb{C}_p$ with $\mathbb{C}$ compatibly with the fixed inclusion $\overline{\mathbb{Q}}_p\hookrightarrow\mathbb{C}$ and associate (see also [Hsi21], Equation (4.4)) to $\varphi\in S_k(U_n,\psi,\mathbb{C}_p)$ the form $\Phi_\infty(\varphi)\in S_k(R^n,\widetilde{\psi})$, defined as

$$\Phi_\infty(\varphi)(\tilde{b}b_\infty)=|n(\tilde{b}b_\infty)|_{\mathbb{A}}^{(k-2)/2}b_\infty^{-1}\cdot\left(\tilde{b}_p\cdot\varphi(\tilde{b})\right),$$

for $b_\infty\in B_\infty^\times$ and $\tilde{b}\in B(\mathbb{A}_\mathbb{Q})^\times$.

1.1.6. **Quaternionic Eisenstein series and newforms.** We recall the notions of quaternionic Eisenstein series and quaternionic newforms as presented in [HPS89a], Chapters 5 and 7. For this section, we take A to be a $\mathbb{Q}_p$-module. We begin with the Eisenstein series part of $S_k(U_n,\psi,A)$, that is

$$S_k^{Eis}(U_n,\psi,A):=\left\{f\in S_k(U_n,\psi,A)\mid\exists\, g:\mathbb{A}^\times_{\mathbb{Q},f}\longrightarrow A^{k-1}\text{ s.t. } f(\tilde{b})=g(n(\tilde{b}))\right\},$$

where $n:\widehat{B}^\times\longrightarrow\mathbb{A}^\times_{\mathbb{Q},f}$ is the extension of the quaternionic norm to $\widehat{B}$. In other words, $S_k^{Eis}(U_n,\psi,A)$ is the space of quaternionic modular forms factoring through the reduced norm map. As proved by Propositions 5.2, 5.3 and the discussion after Proposition 5.4 in *loc.cit.*, this space is often trivial, in fact

$$S_k^{Eis}(U_n,\psi,A)=\begin{cases}\{0\} & \text{if } k>2 \text{ or } \psi_{Np^n} \text{ is non trivial,}\\ A^{[\mathbb{Z}_\ell^\times:n((R_\ell^n)^\times)]} & \text{if } k=2 \text{ and } \psi_{Np^n} \text{ is trivial.}\end{cases}$$

In particular, $S_k^{Eis}(U_n,\psi,A)$ has at most rank 2.

Defining the Petersson inner product as in [Shi65] or [Gro87], one can consider the orthogonal complement of $S_k^{Eis}(U_n,\psi,A)$ in $S_k(U_n,\psi,A)$, namely

$$\mathscr{S}_k(U_n,\psi,A) := \begin{cases} S_k(U_n,\psi,A)/S_k^{Eis}(U_n,\psi,A) & \text{if } k=2 \text{ and } \psi_{Np^n} \text{ is trivial,} \\ S_k(U_n,\psi,A) & \text{otherwise.} \end{cases}$$

Inside of this space, we find the so-called space of old forms, $\mathscr{S}_k^{old}(U_n,\psi,A)$, defined to be the subspace of $\mathscr{S}_k(U_n,\psi,A)$ spanned by all $\mathscr{S}_k(U_1(R'),\psi,A)$ for each special order $R'\subset R_n$ for which $S_k(U_1(R'),\psi,A)$ makes sense. One should pay attention to fix the suitable ramified extension of $\mathbb{Q}_\ell$, but we point the reader to Remarks 7.13 and 7.14 of [HPS89a] for further details. Finally, we define the space of quaternionic newforms $\mathscr{S}_k^{new}(U_n,\psi,A)$ as the orthogonal complement of $\mathscr{S}_k^{old}(U_n,\psi,A)$ inside $\mathscr{S}_k(U_n,\psi,A)$.

1.2. **Hecke algebras and lifts to quaternionic modular forms.** One of the main results of Hida's work is the duality between the Hecke algebra and the space of classical modular forms, given by the Petersson product. The analogous result can be recovered in the quaternionic setting when one considers Eichler orders or special orders with odd exponent at the primes of ramification, but in the case of special orders with even exponent, this is no more true (see Remark 1.2.4). However, even though one cannot speak about duality anymore, it is indeed possible to recover the correct dimension result for proving a rank-2 Hida theory.

1.2.1. **Hecke operators.** For any prime q, recall the element $\varpi_q\in\mathbb{A}_{\mathbb{Q},f}^\times$ such that $\varpi_{q,q}=q$ and 1 otherwise. Let A be again an $\mathcal{O}$-algebra and take $\varphi\in S_k(U_n,A)$. On this quaternionic space we have (for any $\tilde{b}\in\widehat{B}^\times$) the Hecke operators T_q

$$T_q\varphi(\tilde{b}) = \begin{cases} \varphi\left(\tilde{b}\left(\begin{smallmatrix}1&0\\0&\varpi_q\end{smallmatrix}\right)\right)+\sum_{a\in\mathbb{Z}/q\mathbb{Z}}\varphi\left(\tilde{b}\left(\begin{smallmatrix}\varpi_q&a\\0&1\end{smallmatrix}\right)\right) & \text{for each prime } q\nmid Np^n\ell^{2r}, \\ \\ \varphi|_{\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)}\left(\tilde{b}\left(\begin{smallmatrix}1&0\\0&\varpi_p\end{smallmatrix}\right)\right)+\sum_{a\in\mathbb{Z}/q\mathbb{Z}}\varphi|_{\left(\begin{smallmatrix}p&a\\0&1\end{smallmatrix}\right)}\left(\tilde{b}\left(\begin{smallmatrix}\varpi_p&a\\0&1\end{smallmatrix}\right)\right) & \text{for } q=p \text{ and } n=0, \end{cases}$$

and the Hecke operators U_l,

$$U_q\varphi(\tilde{b}) = \begin{cases} \sum_{a\in\mathbb{Z}/q\mathbb{Z}}\varphi\left(\tilde{b}\left(\begin{smallmatrix}\varpi_q&a\\0&1\end{smallmatrix}\right)\right) & \text{for } q\mid N, \\ \\ \sum_{a\in\mathbb{Z}/p\mathbb{Z}}\varphi|_{\left(\begin{smallmatrix}p&a\\0&1\end{smallmatrix}\right)}\left(\tilde{b}\left(\begin{smallmatrix}\varpi_p&a\\0&1\end{smallmatrix}\right)\right) & \text{for } q=p \text{ and } n>0. \end{cases}$$

In the end, we consider also the quaternionic operator at ℓ, $\tilde{U}_\ell$ which is defined as

$$\tilde{U}_\ell\varphi(\tilde{b})=\varphi\left(\tilde{b}\tilde{\varpi}_\ell\right)$$

for $\tilde{\varpi}_\ell$ such that $\tilde{\varpi}_{\ell,\ell}$ is a units in the maximal order at ℓ of norm $n(\tilde{\varpi}_{\ell,\ell})=\ell$, and $\tilde{\varpi}_{\ell,q}=1$ elsewhere. For each $d\in\Delta_{Np^n\ell^{2r}}:=(\mathbb{Z}/Np^n\ell^{2r}\mathbb{Z})^\times$ we also recall the diamond operator $\langle d\rangle$ with its usual definition on classical modular forms and straightforwardly extended to the

p-adic quaternionic case. On the space of classical modular forms $S_k(\Gamma_1(Np^n\ell^{2r}),\psi,A)$ we have the usual operators with a similar expression to the quaternionic ones except at ℓ, where the definition of U_ℓ is analogous to the above U_q operators.

1.2.2. **Hecke algebras.** For each $n \geq 1$, let $\mathsf{H}^1_n(A)$ be the Hecke algebra generated over A by all Hecke operators and the diamond operators away from the level, which acts on $S_k(\Gamma_1(Np^n\ell^{2r}),A)$. We denote by $\mathsf{H}_n(A)$ the direct summand of $\mathsf{H}^1_n(A)$ acting on $S_k(\Gamma_1(Np^n\ell^{2r}),\psi,A)$ and by $\mathsf{h}_n(A)$ the Hecke algebra acting on the newspace $S_k^{new}(\Gamma_1(Np^n\ell^{2r}),\psi,A)$. For each $m > n$ we have the projection maps $\mathsf{H}^1_m(A) \twoheadrightarrow \mathsf{H}^1_n(A)$ and the same holds true for the subalgebras $\mathsf{H}_n(A)$ and $\mathsf{h}_m(A)$. We construct the projective limits with respect to these maps,

$$\mathsf{H}^1_\infty(A) = \varprojlim \mathsf{H}^1_n(A), \qquad \mathsf{H}_\infty(A) = \varprojlim \mathsf{H}_n(A) \qquad \text{and} \qquad \mathsf{h}_\infty(A) = \varprojlim \mathsf{h}_n(A),$$

together with the projection maps $\mathsf{H}^1_\infty(A) \twoheadrightarrow \mathsf{H}_\infty(A) \twoheadrightarrow \mathsf{h}_\infty(A)$. For any $n \geq 1$, we define $\mathsf{H}^{1,ord}_n(A)$ to be ordinary part of $\mathsf{H}^1_n(A)$, namely the product of all the localizations of $\mathsf{H}^1_n(A)$ on which U_p is invertible, and denote by e_n the corresponding projector $\mathsf{e}_n : \mathsf{H}^1_n(A) \twoheadrightarrow \mathsf{H}^{1,ord}_n(A)$. Similarly we define $\mathsf{H}^{ord}_n(A)$ and $\mathsf{h}^{ord}_n(A)$, together with the corresponding ordinary projectors, which we denote by the same symbol e_n. Passing to the limit we obtain $\mathsf{H}^{1,ord}_\infty(A)$, $\mathsf{H}^{ord}_\infty(A)$ and $\mathsf{h}^{ord}_\infty(A)$, each of them associated with the corresponding ordinary projector $\mathsf{e}_\infty = \varprojlim \mathsf{e}_n$.

Remark 1.2.1. *It is well known (see* e.g. [Li75]*, Theorem 3) that Assumption 1.1.11 forces the Hecke operator U_ℓ to be trivial on the space of classical modular forms of level ℓ^{2r}. We can then identify each $\mathsf{h}_n(A)$ with the Hecke A-algebra*

$$\mathsf{h}^{(\ell)}_n(A) \subseteq End(S_k(\Gamma_1(Np^n\ell^{2r}),\psi,A))$$

generated by all diamond and Hecke operators, except U_ℓ.

On the quaternionic side we proceed similarly. For each $n \geq 1$, let $\mathsf{H}^B_n(A)$ be the Hecke algebra acting on $S_k(U_n,\psi,A)$, generated over A by the Hecke and diamond operators. We denote by $\mathsf{h}^B_n(A)$ the component acting on the newspace $\mathscr{S}^{new}_k(U_n,\psi,A)$. For each $n \geq 1$ we have the projection maps $\mathsf{H}^B_n(A) \twoheadrightarrow \mathsf{h}^B_n(A)$ and we construct the projective limits with respect to these maps,

$$\mathsf{H}^B_\infty(A) = \varprojlim \mathsf{H}^B_n(A) \qquad \text{and} \qquad \mathsf{h}^B_\infty(A) = \varprojlim \mathsf{h}^B_n(A),$$

together with the projection map $\mathsf{H}^B_\infty(A) \twoheadrightarrow \mathsf{h}^B_\infty(A)$. In the end, we define as above the ordinary Hecke algebras $\mathsf{H}^{B,ord}_n(A)$ and $\mathsf{h}^{B,ord}_n(A)$, and obtain $\mathsf{H}^{B,ord}_\infty(A)$ and $\mathsf{h}^{B,ord}_\infty(A)$ as inverse limit of the $\mathsf{H}^{B,ord}_n(A)$ and $\mathsf{h}^{B,ord}_n(A)$ respectively.

The Jacquet–Langlands correspondence provides a compatible morphism between the classical and the quaternionic side, that is

$$JL_\infty : \mathsf{H}_\infty(A) \longrightarrow \mathsf{H}^B_\infty(A),$$

which preserves the Hecke and diamond operators away from the discriminant of the quaternion algebra.

Let $\tilde{\Lambda} = \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ be the finite flat extension of the classical Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$. We remark that by construction, the algebra H_∞^{ord} is naturally a $\tilde{\Lambda}$-algebra and it is finitely generated. Therefore, the algebra $\mathsf{H}_\infty^B(A)$ is finitely generated over $\tilde{\Lambda}$. We define the two universal $\tilde{\Lambda}$-adic Hecke algebras

$$\begin{aligned}\mathsf{H}_{univ} :=& \tilde{\Lambda}[T_q, U_l, \langle d\rangle, \text{ for } q \nmid Np^n, l \mid Np\ell^{2r}, d \in \Delta_{Np^n\ell^{2r}}],\\ \mathsf{H}_{univ}^B :=& \tilde{\Lambda}[\tilde{U}_\ell, T_q, U_l, \langle d\rangle, \text{ for } q \nmid Np^n, l \mid Np, d \in \Delta_{Np^n\ell^{2r}}]\end{aligned}$$

and, as in [LV12], we obtain compatible morphisms

$$\mathsf{H}_{univ} \longrightarrow \mathsf{H}_\infty^{ord}(A) \qquad \text{and} \qquad \mathsf{H}_{univ}^B \longrightarrow \mathsf{H}_\infty^{B,ord}(A).$$

1.2.3. **Quaternionic lifts of modular forms and the failure of the duality.** We analyze more carefully [HPS89a], recalling the results which we need. Let $\left(\frac{-}{\ell}\right)$ be the Kronecker character at ℓ and for any space of modular forms $S_k(M, \epsilon, F)$ and each Dirichlet character modulo M, we denote by $S_k(M, \epsilon, F)^{\otimes\chi}$ the space of all the modular forms which are twists by χ of modular forms in $S_k(M, \epsilon, F)$.

Theorem 1.2.2 ([HPS89a], Theorem 7.10)**.** *Let R' be a special order of level $M\ell^{2r+1}$ (such that L_ℓ is the unramified quadratic extension of $\mathbb{Q}_\ell$). Let ϵ be a character modulo N such that ϵ_ℓ is either the trivial character modulo ℓ or an odd character modulo ℓ. Suppose moreover that ϵ is even and that $r \geq v_\ell(cond(\epsilon_\ell))$. Then there exist a Hecke–equivariant isomorphism*

$$\mathscr{S}_k^{new}(U_1(R'), \widetilde{\epsilon}, \mathbb{C}) \cong S_k^{new}(\Gamma_1(M\ell^{2r+1}), \epsilon, \mathbb{C}).$$

The above theorem proves that, as in the case of Eichler orders, there is a one-to-one correspondence for special orders with odd exponent at ℓ. The situation for even exponent is more complicated.

Theorem 1.2.3 ([HPS89a], Theorems 7.16 & 7.17)**.** *Let ℓ be an odd prime and let $r \geq 1$ and $k \geq 2$ be integers. Let ψ be a character modulo $Np^n\ell^{2r}$ such that $\psi(-1) = (-1)^k$, it satisfies Assumptions 1.1.6 and 1.1.11, and such that $cond_\ell(\psi) \leq 2r - 1$. Then the following decomposition of $\mathsf{h}_n^{(Np^n\ell^{2r})}(\mathbb{C})$-modules holds true.*

(a) If $r = 1$ and ψ_ℓ is the trivial character:

$$\begin{aligned}2S_k^{new}(\Gamma_1(Np^n\ell^2), \psi, \mathbb{C}) \cong \mathscr{S}_k^{new}(U_n, \psi, \mathbb{C}) \oplus S_k^{new}(\Gamma_1(Np^n\ell), \psi, \mathbb{C})^{\otimes\left(\frac{-}{\ell}\right)} \oplus \\ 2S_k^{new}(\Gamma_1(Np^n), \psi, \mathbb{C})^{\otimes\left(\frac{-}{\ell}\right)} \oplus \bigoplus_{\chi/\sim} 2S_k^{new}(\Gamma_1(Np^n\ell), \chi^2\psi, \mathbb{C})^{\otimes\overline{\chi}}\end{aligned}$$

where the sum $\bigoplus_{\chi/\sim}$ runs over all the $\frac{1}{2}(\ell-3)$ classes of primitive characters modulo ℓ excepting $\left(\frac{-}{\ell}\right)$, modulo the equivalence $\chi \sim \overline{\chi}$.

(b) *If $r = 1$ and ψ_ℓ is a odd character modulo ℓ:*

$$2S_k^{new}(\Gamma_1(Np^n\ell^2), \psi, \mathbb{C}) \cong \mathscr{S}_k^{new}(U_n, \widetilde{\psi}, \mathbb{C}) \oplus \bigoplus_{\chi/\sim} 2S_k^{new}(\Gamma_1(Np^n\ell), \chi^2\psi, \mathbb{C})^{\otimes\overline{\chi}}$$

where $\widetilde{\psi}$ is a lift of ψ as in Section 1.1.2 and the sum $\bigoplus_{\chi/\sim}$ runs over all the $\frac{1}{2}(\ell-3)$ classes of primitive characters modulo ℓ excepting $\overline{\psi_\ell}$, modulo the equivalence $\chi \sim \overline{\chi\psi_\ell}$.

(c) *If $r \geq 2$ and ψ_ℓ is either trivial or odd of conductor ℓ:*

$$2S_k^{new}(\Gamma_1(Np^n\ell^{2r}), \psi, \mathbb{C}) \cong \mathscr{S}_k^{new}(U_n, \widetilde{\psi}, \mathbb{C}) \oplus \bigoplus_{\chi} 2S_k^{new}(\Gamma_1(Np^n\ell^r), \chi^2\psi, \mathbb{C})^{\otimes\overline{\chi}}$$

where $\widetilde{\psi}$ is a lift of ψ as in Section 1.1.2 and the sum $\bigoplus_\chi$ runs over all the $\ell^r - 2\ell^{r-1} + \ell^{r-2}$ classes of primitive characters modulo ℓ^r, modulo the equivalence $\chi \sim \overline{\chi\psi_\ell}$.

Remark 1.2.4. (a) *In the above theorem the decomposition is given as $\mathsf{h}_n^{(Np^n\ell^{2r})}(\mathbb{C})$-modules, but the strong multiplicity one for classical modular newforms guarantees the decomposition to hold (at least) as $\mathsf{h}_n^{(\ell)}(\mathbb{C})$-modules. As already noticed in Remark 1.2.1, the Hecke algebra $\mathsf{h}_n^{(\ell)}(\mathbb{C})$ coincides with $\mathsf{h}_n(\mathbb{C})$ since the Hecke operator U_ℓ is the 0-operator on this space.*

(b) *The theorem implies that the duality between the Hecke algebra and the space of modular forms does not necessarily hold true for special orders with level ℓ^{2r}. This situation represents the main difference between this setting and the case of classical modular forms (and special orders with an odd power of ℓ). We recall that, on the contrary, the Jacquet–Langlands correspondence does hold true, as well as the multiplicity one result for automorphic representations. This phenomenon is purely local, as already remarked in Example 2.6 of [LRdVP18]. More precisely, the dimension of the local automorphic representation at ℓ is bigger than 1 in the case of level ℓ^{2r} and determined by the minimal conductor of the modular forms. We refer to Section 5 of [Car84] for all the related details.*

We recall that a modular form f is twist-minimal (see Definition 3.2.2) at the prime q if, taken $\pi_{f,q}$ its associated automorphic representation at q, the conductor of $\pi_{f,q}$ is minimal under twists, *i.e.* $cond(\pi_{f,q}) \geq cond(\pi_{f,q} \otimes \chi)$ for any q-adic character χ.

Corollary 1.2.5. *Each twist-minimal modular eigenform in $S_k^{new}(\Gamma_1(Np^n\ell^{2r}), \psi, \mathbb{C})$ lifts to (up to linear combinations) exactly two linearly independent quaternionic modular eigenforms in $\mathscr{S}_k^{new}(U_n, \widetilde{\psi}, \mathbb{C})$ with the same Hecke eigenvalues for $\mathsf{h}_n^{(\ell)}(\mathbb{C})$.*

Regardless of Remark 1.2.4.(b), one can still obtain an isomorphism between the space of quaternionic modular forms and the square of a suitable Hecke algebra, as in the following proposition.

Proposition 1.2.6. *Under the hypotheses of Theorem 1.2.3, there exists a $\mathbb{C}$-vector subspace $T_k(n,r,\psi)$ of $S_k^{new}(\Gamma_1(Np^n\ell^{2r}),\psi,\mathbb{C})$, which is a $\mathsf{h}_n^{(\ell)}(\mathbb{C})$-submodule satisfying*

$$2T_k(n,r,\psi) \cong \begin{cases} \mathscr{S}_k^{new}(U_n,\psi,\mathbb{C}) \oplus S_k^{new}(\Gamma_1(Np^n\ell),\psi,\mathbb{C})^{\otimes\left(\frac{-}{\ell}\right)} & \text{if } r=1 \text{ and } \psi_\ell \text{ is trivial,} \\ \mathscr{S}_k^{new}(U_n,\psi,\mathbb{C}) & \text{otherwise.} \end{cases}$$

Moreover, for $\mathsf{h}_n^T(\mathbb{C})$ the Hecke-subalgebra of $\mathsf{h}_n(\mathbb{C})$ acting on $T_k(n,r,\psi)$, we have an isomorphism of $\mathsf{h}_n^T(\mathbb{C}) = \mathsf{h}_n^{T,(\ell)}(\mathbb{C})$-modules,

$$\left(\mathsf{h}_n^T(\mathbb{C})\right)^2 \cong \begin{cases} \mathscr{S}_k^{new}(U_n,\psi,\mathbb{C}) \oplus S_k^{new}(\Gamma_1(Np^n\ell),\psi,\mathbb{C})^{\otimes\left(\frac{-}{\ell}\right)} & \text{if } r=1 \text{ and } \psi_\ell \text{ is trivial,} \\ \mathscr{S}_k^{new}(U_n,\psi,\mathbb{C}) & \text{otherwise.} \end{cases}$$

Proof. The first statement follows directly from Theorem 1.2.3 as noticed in Chapter 8 of [HPS89a]. The second part follows from

$$Hom\left(2T_k(n,r,\psi),\mathbb{C}\right) \cong Hom\left(T_k(n,r,\psi),\mathbb{C}\right)^2 \cong 2T_k(n,r,\psi)$$

where the first isomorphism is due to the properties of $Hom(-,\mathbb{C})$ and the second is the Hecke-duality for classical modular forms restricted to $T_k(n,r,\psi)$ (since the decomposition is Hecke-equivariant away from ℓ). ∎

As in Section 1.2.2, taken A an $\mathcal{O}$-algebra, we define $\mathsf{h}_\infty^T(A) = \varprojlim \mathsf{h}_n^T(A)$ and $\mathsf{h}_\infty^{T,ord}(A) = \varprojlim \mathsf{h}_n^{T,ord}(A)$. We obtain injective homomorphisms

$$\mathsf{h}_\infty^T(A) \hookrightarrow \mathsf{h}_\infty(A) \qquad \text{and} \qquad \mathsf{h}_\infty^{T,ord}(A) \hookrightarrow \mathsf{h}_\infty^{ord}(A).$$

Notation. *For any module M with an action of a suitable Hecke algebra, and any classical eigenform g, we denote by $A[g]$ the g-isotypic component of A,* i.e. *the biggest submodule of A on which the Hecke algebra acts with the same eigenvalues as of g.*

Proposition 1.2.7. *Let g be a newform in $S_k^{new}(\Gamma_1(Np^n\ell^{2r}),\psi,F)$ with $k \geq 2$ and $\psi(-1) = (-1)^k$. Write $\psi = \psi_{Np^n}\psi_{\ell^{2r}}$ for ψ_{Np^n} and $\psi_{\ell^{2r}}$ the component of ψ, respectively, modulo Np^n and ℓ^{2r}.*

(a) If $r=1$ and ψ_{ℓ^2} is the trivial character modulo ℓ,

$$\dim_F\left(\mathscr{S}_k^{new}(U_n,\widetilde{\psi},F)[g]\right) =$$

$$\dim_F\left(\mathscr{S}_k(U_n,\widetilde{\psi},F)[g]\right) = \begin{cases} 2 & \text{if } g \text{ is twist-minimal at } \ell, \\ 1 & \text{if } g \in S_k^{new}(\Gamma_1(Np^n\ell),\psi,F)^{\otimes\left(\frac{-}{\ell}\right)}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) If either $\psi_{\ell^{2r}}$ is a non-trivial character modulo ℓ or $r \geq 2$,

$$\dim_F\left(\mathscr{S}_k^{new}(U_n,\widetilde{\psi},F)[g]\right) = \dim_F\left(\mathscr{S}_k(U_n,\widetilde{\psi},F)[g]\right) = \begin{cases} 2 & \text{if } g \text{ is twist-minimal at } \ell. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is a straightforward consequence of Theorem 1.2.3 combined with the fact that strong multiplicity one applies to g. ∎

1.2.4. **Choice of a modular form.** Let $f \in S_2(\Gamma_1(Np\ell^{2r}), \psi, \mathbb{C})$ be a fixed p-ordinary newform, with ψ a Dirichlet character modulo $Np\ell^{2r}$ satisfying Assumptions 1.1.6 and 1.1.11. In this way, the automorphic representation associated with f admits a Jacquet–Langlands lift. Moreover, we assume that the p-adic Galois representation associated with f is residually absolutely irreducible and p-distinguished. Let $F = \mathbb{Q}_p(f)$ be the finite extension of $\mathbb{Q}_p$ defined by f and take $\mathcal{O}$ to be its ring of integers; note that $\mathcal{O}$ is a finite flat extension of $\mathbb{Z}_p$. We denote by f_∞ the unique Hida family passing through f. By duality with the Hecke algebra $\mathsf{H}_\infty^{ord}(\mathbb{C})$, we know that f_∞ defines a character, which we denote with the same symbol f_∞,

$$f_\infty : \mathsf{H}_\infty^{ord}(\mathbb{C}) \longrightarrow \mathcal{R},$$

where $\mathcal{R}$ is the *universal ordinary p-adic Hecke algebra of tame level $N\ell^{2r}$* as in Definition 2.4 of [GS93]. The Jacquet–Langlands correspondence ensures that such character factors through the morphism to $\mathsf{H}_\infty^{B}(\mathbb{C})$; we keep denoting the corresponding map by

$$f_\infty : \mathsf{H}_\infty^{B,ord}(\mathbb{C}) \longrightarrow \mathcal{R}.$$

1.3. **The control theorem.** In this section, we prove a *control theorem* for special orders of even conductor at ℓ. We begin by introducing a space suitable for the p-adic interpolation and defining some specialization maps. We consider again A to be our fixed $\mathcal{O}$-algebra.

1.3.1. **Specialization maps.** Let $\mathsf{X} = (\mathbb{Z}_p \times \mathbb{Z}_p)^{prim}$ the set of primitive row vectors, that is the vectors in $\mathbb{Z}_p \times \mathbb{Z}_p$ which have at least one component not divisible by p. Denote by $\mathscr{C}(\mathsf{X}, A)$ the space of A-valued continuous functions on X and by $M(\mathsf{X}, A)$ the space of A-valued measures on X. We have a left $M_2(\mathbb{Z}_p)$-action on $\mathscr{C}(\mathsf{X}, A)$ via

$$\gamma \cdot f(x, y) = f\left((x, y)\gamma\right),$$

for $f \in \mathscr{C}(\mathsf{X}, A)$ and $\gamma \in M_2(\mathbb{Z}_p)$, and the induced right action on $M(\mathsf{X}, A)$ as

$$\mu|_\gamma(f(x, y)) = \mu(\gamma \cdot f(x, y)),$$

for $\mu \in M(\mathsf{X}, A)$. Since we would like to consider the action of each U_n, with $n \geq 1$, on these spaces, it is enough to consider the subspace $p\mathbb{Z}_p \times \mathbb{Z}_p^\times \subset \mathsf{X}$. It is readily noticed that

$$(p\mathbb{Z}_p \times \mathbb{Z}_p^\times) \cdot (U_n)_p = (p\mathbb{Z}_p \times \mathbb{Z}_p^\times) \left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p^n\mathbb{Z}_p \end{smallmatrix}\right) = p\mathbb{Z}_p \times \mathbb{Z}_p^\times.$$

Definition 1.3.1. *Let ψ be a Dirichlet character modulo $N\ell^{2r}$ satisfying Assumptions 1.1.6 and 1.1.11. We define the measure-valued quaternionic modular forms with character ψ as the space*

$$S_2(U_0, \psi, M(\mathsf{X}, A)) := \Big\{\varphi : \widehat{B}^\times \longrightarrow M(\mathsf{X}, A) \mid \varphi(b\tilde{b}uz) = \psi_{N,\mathbb{A}}^{-1}(z)\widetilde{\psi_{\ell^{2r}}}(z)\widetilde{\psi_{\ell^{2r}}}(u_\ell)\varphi(\tilde{b})|_{u_p} \\ \text{for } b \in B^\times,\ \tilde{b} \in \widehat{B}^\times,\ u \in U_0,\ z \in \mathbb{A}_{\mathbb{Q},f}^\times\Big\}.$$

This space can be identified with the space of functions $\varphi : B^\times\backslash\widehat{B}^\times \longrightarrow M(\mathsf{X}, A)$ satisfying

$$\varphi(z\tilde{b}) = \psi_{N,\mathbb{A}}^{-1}(z)\widetilde{\psi_{\ell^{2r}}}(z)\varphi(\tilde{b}),$$

for $\tilde{b} \in B^\times\backslash\widehat{B}^\times$ and $z \in \mathbb{A}_{\mathbb{Q},f}^\times$, and such that $\varphi|_{u_p^{-1}}(\tilde{b}u) = \widetilde{\psi_{\ell^{2r}}}(u_\ell)\varphi(\tilde{b})$ for any $u \in U_0$ and any $\tilde{b} \in B^\times\backslash\widehat{B}^\times$.

Let $k \geq 2$ be any weight and let $\varepsilon : \mathbb{Z}_p^\times \longrightarrow A^\times$ be any character which factors through $(\mathbb{Z}_p/p^m\mathbb{Z}_p)^\times$; we extend ε multiplicatively to $\mathbb{Z}_p$ imposing $\varepsilon(p) = 0$. We define the specialization map

$$\nu_{k,\varepsilon} : S_2(U_0, \psi, M(\mathsf{X}, A)) \longrightarrow S_k(U_n, \psi\varepsilon, A(\varepsilon))$$

such that

$$\nu_{k,\varepsilon}(\varphi)(\tilde{b})(P) := \int_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times} \varepsilon(y)P(x,y)d(\varphi(\tilde{b}))(x,y),$$

where $n = \max\{1, m\}$, $\varphi \in S_2(U_0, \psi, M(\mathsf{X}, A))$, $\tilde{b} \in B^\times\backslash\widehat{B}^\times$ and $P \in L_{k-2}(A)$.

Proposition 1.3.2. *The specialization maps $\nu_{k,\varepsilon}$ are well-defined and Hecke-equivariant for H_{univ}^B, where the equivariance at p is meant as $\nu_{k,\varepsilon}(T_p\varphi) = U_p\nu_{k,\varepsilon}(\varphi)$.*

Proof. Let $\varphi \in S_2(U_0, \psi, M(\mathsf{X}, A))$. Then, for any $\tilde{b} \in B^\times\backslash\widehat{B}^\times$, $z \in \mathbb{A}_{\mathbb{Q},f}^\times$, $u \in U_n$ and $P \in L_{k-2}(A)$, we have

$$\begin{aligned}\nu_{k,\varepsilon}(\varphi)(\tilde{b}uz)|_{u_p^{-1}}(P) &= \int_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times} \varepsilon(y)(u_p^{-1}\cdot P(x,y))d(\varphi(\tilde{b}uz))(x,y) \\ &= \psi_{N,\mathbb{A}}^{-1}(z)\widetilde{\psi_{\ell^{2r}}}(z)\widetilde{\psi_{\ell^{2r}}}(u_\ell)\int_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times}(u_pz_p)\cdot\left(\varepsilon(y)P((x,y)u_p^{-1})\right)d(\varphi(\tilde{b}))(x,y)\end{aligned}$$

and since $(x,y)u_p^{-1} = (*, y+p*)$, ε is extended to $\mathbb{Z}_p$ and $P((x,y)z_p) = z_p^{k-2}P(x,y)$ we obtain

$$\begin{aligned}&\psi_{N,\mathbb{A}}^{-1}(z)\widetilde{\psi_{\ell^{2r}}}(z)\widetilde{\psi_{\ell^{2r}}}(u_\ell)\int_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times}\varepsilon(y)\varepsilon(z_p)\left(P((x,y)u_p^{-1}u_pz_p)\right)d(\varphi(\tilde{b}))(x,y) \\ &\quad= \psi_{N,\mathbb{A}}^{-1}(z)\varepsilon_{\mathbb{A}}(z)^{-1}\widetilde{\psi_{\ell^{2r}}}(z)\widetilde{\psi_{\ell^{2r}}}(u_\ell)z_p^{k-2}\int_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times}\varepsilon(y)\left(P((x,y))\right)d(\varphi(\tilde{b}))(x,y) \\ &\quad= \psi_{N,\mathbb{A}}^{-1}(z)\varepsilon_{\mathbb{A}}(z)^{-1}\widetilde{\psi_{\ell^{2r}}}(z)\widetilde{\psi_{\ell^{2r}}}(u_\ell)z_p^{k-2}\nu_{k,\varepsilon}(\varphi)(\tilde{b})(P).\end{aligned}$$

The equivariance with respect to the T_q operators is obvious, as well that for the U_q with $q \neq p$ (also for $\tilde{U}_\ell$). To prove the equivariance at p it is enough to note that we have

$$\begin{aligned}\nu_{k,\varepsilon}\varphi|_{\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)}\left(\tilde{b}\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\right)(P) &= \int_{\mathsf{X}}\chi_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times}(x,y)\varepsilon(y)P(x,y)\,d\left(\varphi|_{\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)}\left(\tilde{b}\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\right)\right)(x,y) \\ &= \int_{\mathsf{X}}\chi_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times}(x,py)\varepsilon(py)P(x,py)\,d\left(\varphi\left(\tilde{b}\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\right)\right)(x,y) = 0\end{aligned}$$

for $\chi_{p\mathbb{Z}_p\times\mathbb{Z}_p^\times}(x,y)$ the characteristic function of $p\mathbb{Z}_p\times\mathbb{Z}_p^\times$. ∎

We must now investigate the properties of the space $S_2(U_0,\psi,M(\mathsf{X},A))$. We begin by noticing that the action on $\mathscr{C}(\mathsf{X},A)$ is the one induced by the right action of $M_2(\mathbb{Z}_p)$ on X defined by right multiplication. We proceed similarly to Proposition 7.5 of [LRV12] or Chapter 6 of [GS93], and denoting by X_n the set of primitive vectors in $(\mathbb{Z}/p^n\mathbb{Z})^2$, we recover $\mathsf{X}=\varprojlim \mathsf{X}_n$ with respect to the canonical projection maps. We obtain then $M(\mathsf{X},A)=\varprojlim M(\mathsf{X}_n,A)$ (see *e.g.* Section 7 of [MSD74]). Since X_n is a finite set, $M(\mathsf{X}_n,A)$ is identified with the space $Hom(\mathsf{X}_n,A)$ of step functions. The action of U_0 on X_n is transitive and the stabilizer of $(0,1)$ is

$$Stab_{U_0}((0,1))=\{\gamma\in U_0 \mid (0,1)\gamma=(0,1)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=(c,d)=(0,1)\}=U_n.$$

This shows that $\mathsf{X}_n=U_0/U_n=(U_0)_p/(U_n)_p$ and then that $M(\mathsf{X}_n,A)=Hom_{U_n}(\mathcal{O}[U_0],A)$. Let $\tilde{B}^p=B^\times\backslash\widehat{B}^\times/U_0^p$ be the profinite double quotient associated with $U_0^p=U_0\cap B(\mathbb{A}_{\mathbb{Q},f}^{(p)})^\times$, for $\mathbb{A}_{\mathbb{Q},f}^{(p)}$ the finite adèles away from p. By Shapiro's Lemma we obtain

$$S_2(U_0,\mathbf{1},M(\mathsf{X}_n,A))\cong\Big(Hom_{\mathcal{O}}(\mathcal{O}[\tilde{B}^p],M(\mathsf{X}_n,A))\Big)^{(U_0)_p}\cong S_2(U_n,\mathbf{1},A), \tag{1.3.1}$$

as well as the analogous isomorphism when we consider a character ψ. Equation (1.3.1) implies that

$$S_2(U_0,\psi,M(\mathsf{X},A))=\varprojlim S_2(U_n,\psi,A), \tag{1.3.2}$$

where the identification is H^B_{univ}-equivariant. We hence deduce that $S_2(U_0,\psi,M(\mathsf{X},A))$ is a compact $\mathcal{O}$-module, since $\mathcal{O}$ is p-adically complete and each $S_2(U_n,\psi,A)$ is a finitely generated free $\mathcal{O}$-module. This allows us to define its ordinary part $S_2(U_0,\psi,M(\mathsf{X},A))^{ord}$ as usual (see Section 2.4 of [LV12] and the references therein) as its direct summand on which the Hecke operator T_p acts invertibly.

We shorten the notation and denote by $\mathbb{W}$ the space $S_2(U_0,\psi,M(\mathsf{X},\mathcal{O}))^{ord}$. In particular, the Hecke-equivariance in the inverse limit construction of $S_2(U_0,\psi,M(\mathsf{X},\mathcal{O}))$, implies that $S_2(U_0,\psi,M(\mathsf{X},\mathcal{O}))^{ord}=\varprojlim S_2(U_n,\psi,\mathcal{O})^{ord}$, where T_p is replaced by U_p on each component of the inverse limit. Proposition 1.3.2 shows that the specialization maps descend to Hecke-equivariant specialization maps between the ordinary components,

$$\nu_{k,\varepsilon}^{ord}:\mathbb{W}\longrightarrow S_k(U_n,\psi\varepsilon,\mathcal{O}(\varepsilon))^{ord},$$

with the same definition of $\nu_{k,\varepsilon}$ and where $\mathcal{O}(\varepsilon)$ is the finite extension of $\mathcal{O}$ generated by the values of ε.

Notation. *We need to introduce some more notation.*

(a) For any $m\geq 1$ and any character $\chi:\mathbb{Z}_p^\times\longrightarrow\overline{\mathbb{Q}_p}^\times$, let $\Psi_{m,\chi}:\mathsf{X}\longrightarrow\overline{\mathbb{Q}_p}^\times$ such that

$$\Psi_{m,\chi}((x,y))=\begin{cases}\chi(y) & \textit{if } x\in p^m\mathbb{Z}_p,\\ 0 & \textit{otherwise.}\end{cases}$$

In particular, $\Psi_{m,\chi}$ is homogeneous of degree χ for each m.

(b) *We also set $\mathbb{W}_\Omega := \mathbb{W} \otimes_{\widetilde{\Lambda}} \Omega$ for any $\widetilde{\Lambda}$-algebra Ω. We say that a homomorphism $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p}$ is an* arithmetic homomorphism *if its restriction to $\mathbb{Z}_p^\times$ is of the form $\kappa_{|\mathbb{Z}_p^\times}(x) = x^{k-2}\varepsilon(x)$ for $k \geq 2$ and $\varepsilon : \mathbb{Z}_p^\times \longrightarrow \overline{\mathbb{Q}_p}^\times$ a character which factors through $\mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p)$, with n minimal; we say that κ has weight k and character ε of conductor p^n (*cf. *Section 2.1).*

Lemma 1.3.3. *Let $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p}$ be an arithmetic homomorphism of weight k and character ε of conductor p^n. Let F_κ be the field extension of F containing the values of κ. The map $\nu_{k,\varepsilon}^{ord}$ induces the injective Hecke-equivariant homomorphism*

$$\nu_{k,\varepsilon}^{ord} : \mathbb{W}_{\mathcal{R}}/\mathcal{P}_\kappa \mathbb{W}_{\mathcal{R}} \hookrightarrow S_k(U_n, \psi\varepsilon, F_\kappa)^{ord},$$

where $\mathcal{P}_\kappa$ is the kernel of κ in $\mathcal{R}$.

Proof. We begin noting that $\mathcal{P}_\kappa S_2(U_0, \psi, M(\mathsf{X}, \mathcal{O})) = S_2(U_0, \psi, \mathcal{P}_\kappa M(\mathsf{X}, \mathcal{O}))$, as it can be seen by applying twice Lemma 1.2 of [AS97] to

$$S_2(U_0, \psi, M(\mathsf{X}, \mathcal{O})) = H^0(U_0^p, H^1(F[\widetilde{B}^p], M(\mathsf{X}, \mathcal{O}))).$$

We prove now that $\mathcal{P}_\kappa \mathbb{W} = ker(\nu_{k,\varepsilon}^{ord})$. Let $\varphi \in \mathcal{P}_\kappa \mathbb{W}$; therefore $\varphi(\tilde{b})$ lies in $\mathcal{P}_\kappa M(\mathsf{X}, \mathcal{O})$ for any $\tilde{b} \in B^\times \backslash \widehat{B}^\times$. Lemma 6.3 of [GS93] shows that $\varphi(\tilde{b}) \in \mathcal{P}_\kappa M(\mathsf{X}, \mathcal{O})$ if and only if $\varphi(\tilde{b})(f) = 0$ for each homogeneous function of degree κ. For each $P \in L_{k-2}(F_\kappa)$, $\varepsilon(y)P(x,y)$ is homogeneous of degree κ and hence $\nu_{k,\varepsilon}^{ord}(\varphi(\tilde{b}))(P) = 0$ for each $\tilde{b}$ and P. Take now $\varphi \in ker(\nu_{k,\varepsilon}^{ord})$ and let $m \geq 1$. Since T_p is invertible, let $\mu \in \mathbb{W}$ be such that $T_p^m \mu = \varphi$. Let $\gamma_a := \left(\begin{smallmatrix} p & a \\ 0 & 1 \end{smallmatrix}\right)$ for $a = 0, \ldots, p-1$ and $\gamma_\infty := \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. The definition of the Hecke operator T_p in Section 1.2.1 does come from the coset decomposition $U_0 \gamma_\infty U_0 = \bigsqcup_{\alpha=0,\ldots,p-1,\infty} \gamma_\alpha U_0$. Then T_p^m corresponds to a decomposition of the form $\bigsqcup_i \gamma_{m,i} U_0$, where each $\gamma_{m,i}$ is a product of m matrices γ_α, for $\alpha = 0, \ldots, p-1, \infty$. We compute

$$\int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} \Psi_{m,\kappa}(x,y) d(\varphi(\tilde{b}))(x,y) = \sum_i \int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} \Psi_{m,\kappa}\left((x,y)\gamma_{m,i}\right) d(\mu\left(\tilde{b} \cdot \gamma_{m,i}\right))(x,y).$$

For each m, $\Psi_{m,\kappa}\left((x,y)\gamma_\infty\right) = 0$ thus, $\Psi_{m,\kappa}\left((x,y)\gamma_{m,i}\right) = 0$ whenever $\gamma_{m,i}$ contains a copy of γ_∞. Therefore, only the matrices $\gamma_{m,i} = \prod_{j=0,m-1} \gamma_{\alpha_j^i} = \left(\begin{smallmatrix} p^m & \sum_j \alpha_j^i p^j \\ 0 & 1 \end{smallmatrix}\right)$ contribute to the integral and we recognize that

$$\begin{aligned} \sum_i \int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} \gamma_{m,i} \cdot \left(\varepsilon(y) y^{k-2}\right) d(\mu\left(\tilde{b} \cdot \gamma_{m,i}\right))(x,y) &= U_p^m \int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} \varepsilon(y) y^{k-2} d(\mu(\tilde{b}))(x,y) \\ &= U_p^m \nu_{k,\varepsilon}^{ord}(\mu(\tilde{b}))(y^{k-2}). \end{aligned}$$

By construction, $0 = \nu_{k,\varepsilon}^{ord}(\varphi(\tilde{b})) = \nu_{k,\varepsilon}^{ord}(T_p^m \mu(\tilde{b})) = U_p^m \nu_{k,\varepsilon}^{ord}(\mu(\tilde{b}))$ and since U_p is invertible on the space $S_k(U_n, \psi\varepsilon, F_\kappa)^{ord}$, $\nu_{k,\varepsilon}^{ord}(\mu(\tilde{b})) = 0$ and hence $\nu_{k,\varepsilon}^{ord}(\varphi(\tilde{b}))(y^{k-2}) = 0$. Lemma 6.3 of [GS93] implies that $\varphi(\tilde{b}) \in \mathcal{P}_\kappa \mathbb{W}$. ■

Remark 1.3.4. *As in the case of Eichler orders, the space of quaternionic modular forms $\mathscr{S}_k(U_n, \widetilde{\psi}, \mathcal{O})$ is often finitely generated over $\mathbb{Z}_p$ and free, as it follows from the discussion in Section 1.1.3. In particular, this holds true under the hypotheses in Proposition*

1.1.9. Lemma 1.3.3 then implies that $\mathbb{W}_{\mathcal{R}}/\mathcal{P}_\kappa\mathbb{W}_{\mathcal{R}}$ is $\mathbb{Z}_p$-finitely generated and free. The discussion in Section 1.1.3 shows also that $S_2(U_0,\psi,M(\mathsf{X},A))$ is often $\widetilde{\Lambda}$-free and finitely generated, once again, for example under the hypotheses of Proposition 1.1.9. One can argue as in the proofs of Theorem 10.1, Corollary 10.3 and Corollary 10.4 of [Hid88]*, since the results proved there for quaternionic modular forms over definite quaternion algebras hold in more generality for special orders which are split at the interpolation prime p (see also Remark 1.3.7).*

1.3.2. **The proof of the control theorem.** As in Section 1.2.4, we fix a p-ordinary newform $f \in S_2^{new}(\Gamma_1(Np\ell^{2r}),\psi,\mathbb{C})$, for ψ a Dirichlet character modulo $Np\ell^{2r}$ satisfying Assumptions 1.1.6 and 1.1.11. We also assume that its associated p-adic Galois representations is residually absolutely irreducible and p-distinguished. Let $f_\infty : \mathsf{H}_\infty^{B,ord}(F) \longrightarrow \mathcal{R}$ the homomorphism associated there with the Hida family passing through f. For any $\mathcal{P}_\kappa$ as in the above Lemma 1.3.3, we denote by f_κ^B the composition

$$f_\kappa^B : \mathsf{H}_{univ}^B \longrightarrow \mathsf{H}_\infty^{B,ord}(F) \xrightarrow{f_\infty} \mathcal{R} \longrightarrow \mathcal{R}_{\mathcal{P}_\kappa},$$

where the first map is the compatible morphism of Section 1.2.2 and the last map is the one to the localization of $\mathcal{R}$ at the prime $\mathcal{P}_\kappa$. We write

$$\widetilde{\mathbb{W}}_\kappa := \left(\mathbb{W} \otimes_{\widetilde{\Lambda}} \mathcal{R}_{\mathcal{P}_\kappa}\right)[f_\kappa^B]$$

for the isotypic component of the $\mathcal{R}_{\mathcal{P}_\kappa}$-module $\mathbb{W} \otimes_{\widetilde{\Lambda}} \mathcal{R}_{\mathcal{P}_\kappa}$, where the Hecke operators act as determined by f_κ^B.

Proposition 1.3.5. *With the notation of Lemma 1.3.3, there is an induced injective homomorphism*

$$\nu_\kappa : \widetilde{\mathbb{W}}_\kappa/\mathcal{P}_\kappa\widetilde{\mathbb{W}}_\kappa \hookrightarrow \left(S_k(U_n,\psi\varepsilon,F_\kappa)^{ord}\right)[f_\kappa],$$

for f_κ the weight-κ specialization of f_∞.

Proof. The proof is the same as of Proposition 3.5 in [LV12], since it does not depend on the choice of the quaternionic order. ■

As one can note from Theorem 1.2.3, the case of level ℓ^2 and trivial character has to be handled with more care. The theory of Hida families for classical modular forms is well known and we can restrict our attention to the Hecke-submodules $S_k^{new}(\Gamma_1(Np^n\ell),\psi,F)^{ord}$ with ψ a Dirichlet character modulo Np^n, with $n \geq 1$. We do not provide details here, but we refer to Chapter 7 of [Hid93] and Section 2 of [LV12]. We construct the space of $\widetilde{\Lambda}$-adic modular newforms, level $Np^n\ell$ and character ψ, as $\mathbb{W}^\ell := \varprojlim S_2^{new}(\Gamma_1(Np^n\ell),\psi,F)^{ord}$. Moreover, we can twist its Hecke action by the character $\left(\frac{-}{\ell}\right)$ and obtaining the corresponding space $\mathbb{W}^{\ell,\left(\frac{-}{\ell}\right)} := \varprojlim S_2^{new}(\Gamma_1(Np^n\ell),\psi,F)^{\left(\frac{-}{\ell}\right),ord}$. As in Section 1.2.2 we have an action of the universal Hecke algebra H_{univ} on $\mathbb{W}^{\ell,\left(\frac{-}{\ell}\right)}$. In particular, taking f in $S_k^{new}(\Gamma_1(Np^n\ell),\psi,F)^{\left(\frac{-}{\ell}\right),ord}$, $\left(\mathbb{W}^{\ell,\left(\frac{-}{\ell}\right)} \otimes_{\widetilde{\Lambda}} \mathcal{R}_{\mathcal{P}_{(k,\varepsilon)}}\right)[f]$ is a free rank-1 $\mathcal{R}_{\mathcal{P}_{(k,\varepsilon)}}$-module (see Proposition 2.17 and the proof of Theorem 2.18 in [LV12]).

Lemma 1.3.6. *Assume $\mathbb{W}$ to be $\widetilde{\Lambda}$-free and finitely generated (see Remark 1.3.4). Suppose that $f \in T_k(n,r,\psi)$ and set, for any arithmetic homomorphism $\kappa = (k,\varepsilon)$,*

$$\mathbb{W}_\kappa := \begin{cases} \widetilde{\mathbb{W}}_\kappa \oplus \left(\mathbb{W}^{\ell,\left(\frac{-}{\ell}\right)} \otimes_{\widetilde{\Lambda}} \mathcal{R}_{\mathcal{P}_\kappa}\right)[f_\kappa] & \text{if } r=1 \text{ and } \psi_\ell \text{ is the trivial character,} \\ \widetilde{\mathbb{W}}_\kappa & \text{otherwise,} \end{cases}$$

where we let H_{univ} act on $\mathbb{W}_\kappa$ via the homomorphism $\mathsf{H}_{univ} \to \mathsf{H}^B_{univ}$ induced by the Jacquet–Langlands correspondence. Then $\mathbb{W}_\kappa$ is a free rank-2 $\mathcal{R}_{\mathcal{P}_\kappa}$-module.

Proof. We start dealing with the case $\mathbb{W}_\kappa = \widetilde{\mathbb{W}}_\kappa$. We consider the p-divisible abelian group (*cf.* Remark 1.3.4 and Section 1.1.3)

$$\mathbb{V} := \varinjlim S_2^{ord}(U_n, \widetilde{\psi}, F/\mathcal{O}),$$

where the inductive limit is taken with respect to the restriction maps induced by the inclusions $U_{n+1} \subset U_n$. The Hecke and diamond operators (at least away from ℓ) act on $\mathbb{V}$ since, as in the case of Eichler orders, the restriction maps in [Hid88] (see Equations (2.9a), (2.9b) and (3.5)) are compatible with the Hecke action. Taking the Pontryagin dual of $\mathbb{V}$ we obtain the Hecke-equivariant isomorphism $\widehat{\mathbb{V}} \cong \mathbb{W}$ (*cf.* Equations 1.3.1 and 1.3.2), which shows it to be a free $\widetilde{\Lambda}$-module of finite rank. We denote by F_κ the field extension of F generated by the values of ε and by $\mathcal{O}_\kappa$ its ring of integers, which we can assume to be finite flat over $\mathbb{Z}_p$. Up to a scalar and up to taking the tensor product by $\mathcal{O}_\kappa$, we can suppose f_κ to have coefficients in $\mathcal{O}$. Then, we observe that $\widetilde{\mathbb{W}}_\kappa = \mathbb{W}[f_\kappa] \otimes_{\widetilde{\Lambda}} \mathcal{R}_{\mathcal{P}_\kappa}$, as the action of the Hecke algebra is on the first component and the tensor product is just an extension of scalars. We can hence apply Theorem 9.4 of [Hid88] (*cf.* Remark 1.3.7) to $\mathbb{W}[f_\kappa]$ and obtain the isomorphism of $\mathsf{h}_n^{T,ord}(\mathcal{O})$-modules,

$$\mathbb{W}[f_\kappa] \cong \widehat{\mathbb{V}[f_\kappa]} \cong S_k^{new}(U_n, \widetilde{\psi\varepsilon}, \mathcal{O})[f_\kappa]. \tag{1.3.3}$$

We remark that the last Hecke-equivariant isomorphism in the above Equation (1.3.3) (as well as in Equation (1.3.4)), comes from the restriction to $T_k(n,r,\psi\varepsilon)$ of the Pontryagin duality established in Lemma 7.1 of [Hid86a]; under the hypotheses of Lemma 1.1.9 one has the isomorphism $S_k(U_n, \widetilde{\psi\varepsilon}, F/\mathcal{O}) \cong S_k(U_n, \widetilde{\psi\varepsilon}, \mathcal{O}) \otimes F/\mathcal{O}$, as in the proof of Theorem 10.1 in [Hid88], and then Proposition 1.2.6 recovers the needed Hecke-isomorphism for quaternionic modular forms. Similarly to the above discussion for $\mathbb{W}^\ell$, we can follow Section 2 of [LV12] and construct the interpolation module $\mathbb{W}^{\ell^{2r}} = \varprojlim_n S_2(\Gamma_1(Np^n\ell^{2r}), \psi, \mathcal{O})^{ord}$, relative to the ordinary subspaces $S_k(\Gamma_1(Np^n\ell^{2r}), \psi\varepsilon, \mathcal{O})^{ord}$. We notice that under the hypothesis of Proposition 1.1.9, the space $\mathbb{W}^{\ell^{2r}}$ is free of finite rank. In particular, we can reproduce the above chain of isomorphisms and obtain $\mathsf{h}_n^{T,ord}(\mathcal{O})$-isomorphisms

$$\mathbb{W}^{\ell^{2r}}[f_\kappa] \cong S_k(\Gamma_1(Np^n\ell^{2r}), \psi\varepsilon)^{ord}[f_\kappa] \cong T_k(n,r,\psi\varepsilon)^{ord}[f_\kappa]. \tag{1.3.4}$$

Applying Propositions 1.2.6 and 1.2.7, we deduce the isomorphism of $\mathsf{h}_\infty^{T,ord}(\mathcal{O})$-modules,

$$\mathbb{W}[f_\kappa] \cong 2\mathbb{W}^{\ell^{2r}}[f_\kappa].$$

Tensoring over $\widetilde{\Lambda}$ with $\mathcal{R}_{\mathcal{P}_\kappa}$, we obtain the isomorphism of $\mathsf{h}_\infty^{T,ord}(\mathcal{O})\otimes_{\widetilde{\Lambda}}\mathcal{R}_{\mathcal{P}_\kappa}$-modules,

$$\mathbb{W}_\kappa \cong 2\left(\mathbb{W}^{\ell^{2r}}\otimes_{\widetilde{\Lambda}}\mathcal{R}_{\mathcal{P}_\kappa}\right)[f_\kappa].$$

As in the proof of Theorem 2.18 of [LV12], Proposition 2.17 of *loc.cit.* guarantees that $\left(\mathbb{W}^{\ell^{2r}}\otimes_{\widetilde{\Lambda}}\mathcal{R}_{\mathcal{P}_\kappa}\right)[f_\kappa]$ is a free $\mathcal{R}_{\mathcal{P}_\kappa}$-module of rank 1, therefore $\mathbb{W}_\kappa$ is a free $\mathcal{R}_{\mathcal{P}_\kappa}$-module of rank 2.

The case of $r=1$ and trivial character at ℓ is carried out similarly, once we define the p-divisible abelian group

$$\mathbb{V} := \varinjlim\left(S_2^{new}(U_n,\widetilde{\psi},F/\mathcal{O})^{ord}\oplus S_k^{new}(\Gamma_1(Np^n\ell),\psi,F/\mathcal{O})^{\left(\frac{\cdot}{\ell}\right),ord}\right),$$

whose Pontryagin dual is $\mathbb{W}\oplus\mathbb{W}^{\ell,\left(\frac{\cdot}{\ell}\right)}$. ■

Remark 1.3.7. *(a) The congruence subgroup we consider, away from ℓ, is the one denoted by $V(Np^n)$ in [Hid88] and one passes from this choice to the one used there by changing all the actions via $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)$.*

(b) Furthermore, we point out that Theorem 9.4 of [Hid88] is stated under more strict hypotheses but, in the case of definite quaternion algebras, such hypotheses can be relaxed; that has been already noticed in [LV12] and [Hsi21] in order to work with Eichler orders for algebras over $\mathbb{Q}$, but Theorem 9.4 of loc.cit. *holds true also for special orders. This is due to the degree of generality in which the results of Chapter 8 of [Hid88] are proved (and Lemma 1.1.3), together with the necessity of a controlled behavior only at the interpolation prime p. The case of indefinite algebras seems to require a generalization of the spectral sequences approach contained in Chapter 9 of [Hid88].*

We can finally state the control theorem in Hida theory for the case of special orders of level ℓ^{2r}.

Theorem 1.3.8 (Control theorem for special orders)**.** *With the above notation, suppose that f is twist-minimal at ℓ. For any arithmetic homomorphism $\kappa:\mathcal{R}\longrightarrow\overline{\mathbb{Q}_p}$, the map ν_κ of Proposition 1.3.5 induces an isomorphism of 2-dimensional F_κ-vector spaces*

$$\widetilde{\mathbb{W}}_\kappa/\mathcal{P}_\kappa\widetilde{\mathbb{W}}_\kappa \xrightarrow{\cong}\left(S_k(U_n,\psi\varepsilon,F_\kappa)^{ord}\right)[f_\kappa].$$

If $r=1$, f has trivial character at ℓ and lies in $S_k^{new}(\Gamma_1(Np^n\ell),\psi,F_\kappa)^{\otimes\left(\frac{\cdot}{\ell}\right)}$ (in particular, it is not twist-minimal at ℓ), then the above isomorphism still holds, but the F_κ-vector spaces are 1-dimensional.

Proof. Suppose f to be twist-minimal. Because of Propositions 1.3.5 and 1.2.7 we know that

$$\dim_{F_\kappa}\left(\widetilde{\mathbb{W}}_\kappa/\mathcal{P}_\kappa\widetilde{\mathbb{W}}_\kappa\right)\leq 2$$

and thus it is enough to prove the opposite inequality. Lemma 1.3.6 shows that $\widetilde{\mathbb{W}}_\kappa$ is a free $\mathcal{R}_\kappa$-module of rank 2. The case of $r=1$, trivial character at ℓ and $f\notin$

$S_k^{new}(\Gamma_1(Np^n\ell), \psi, F_\kappa)^{\otimes\left(\frac{\cdot}{\ell}\right)}$ follows similarly. The remaining case accounts to the fact that the Jacquet–Langlands correspondence preserves twists. ■

We can consider the finitely generated $\mathcal{R}$-module

$$\mathbb{W}_\infty := \begin{cases} \left(\left(\mathbb{W} \oplus \mathbb{W}^{\ell,\left(\frac{\cdot}{\ell}\right)}\right) \otimes_{\widetilde{\Lambda}} \mathcal{R}\right)[f_\infty] & \text{if } r = 1 \text{ and } \psi_\ell \text{ is the trivial character,} \\ \left(\mathbb{W} \otimes_{\widetilde{\Lambda}} \mathcal{R}\right)[f_\infty] & \text{otherwise.} \end{cases}$$

Proceeding similarly as in the proof of the above theorem we notice that $\mathbb{W}_\infty \otimes Frac(\widetilde{\Lambda})$ is a 2-dimensional $\mathcal{K}$-vector space, where $\mathcal{K}$ is the finite field extension of $Frac(\widetilde{\Lambda})$ called the *primitive component* associated with the Hida family f_∞ (see Section 3 in [Hid86b] in particular, Theorem 3.5 and also Theorem 2.6a of [GS93]). We can then formulate Theorem 1.3.8 highlighting this global $\mathcal{R}$-module.

Theorem 1.3.9. *With the above notation, suppose that f is twist-minimal at ℓ. For any arithmetic homomorphism $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p}$, the map ν_κ of Proposition 1.3.5 induces an isomorphism of 2-dimensional F_κ-vector spaces*

$$\mathbb{W}_\infty \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\kappa}/\mathcal{P}_\kappa \mathcal{R}_{\mathcal{P}_\kappa} \xrightarrow{\cong} \left(S_k(U_n, \psi\varepsilon, F_\kappa)^{ord}\right)[f_\kappa].$$

If $r = 1$, f has trivial character at ℓ and lies in $S_k^{new}(\Gamma_1(Np^n\ell), \psi, F_\kappa)^{\otimes\left(\frac{\cdot}{\ell}\right)}$ (in particular, it is not twist-minimal at ℓ), then holds the isomorphism of 1-dimensional F_κ-vector spaces

$$\left(\left(\mathbb{W} \otimes_{\widetilde{\Lambda}} \mathcal{R}\right)[f_\infty]\right) \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\kappa}/\mathcal{P}_\kappa \mathcal{R}_{\mathcal{P}_\kappa} \xrightarrow{\cong} \left(S_k(U_n, \psi\varepsilon, F_\kappa)^{ord}\right)[f_\kappa].$$

Corollary 1.3.10. *Let f_∞ be a primitive Hida family of tame level $N\ell^{2r}$, $r \geq 1$, tame character ψ with its ℓ-component, ψ_ℓ, as in Assumption 1.1.11. Suppose moreover f_∞ to be twist-minimal at ℓ. Then there exist two $\mathcal{R}$-linearly independent elements $\phi^1_{f_\infty}$ and $\phi^2_{f_\infty}$ in $\left(\mathbb{W} \otimes_{\widetilde{\Lambda}} \mathcal{R}\right)[f_\infty]$, which form a basis for $\left(\left(\mathbb{W} \otimes_{\widetilde{\Lambda}} \mathcal{R}\right)[f_\infty]\right) \otimes \mathcal{K}$. Moreover, for any arithmetic homomorphism κ, $\nu_\kappa(\phi^1_{f_\infty})$ and $\nu_\kappa(\phi^2_{f_\infty})$ form a F_κ-basis for $\left(S_k(U_n, \psi\varepsilon, F_\kappa)^{ord}\right)[f_\kappa]$.*

Definition 1.3.11. *We denote by $\mathcal{W}_{f_\infty}$ the $\mathcal{R}$-linear span of $\phi^1_{f_\infty}$ and $\phi^2_{f_\infty}$ and call it the* subspace of special quaternionic Hida families associated with f_∞.

1.4. **A small remark on related works and further directions.** The mathematical literature about this situation of higher ramification at the primes at which the quaternion algebra ramifies is quite little. Excluding the (singular and collective) works of A. Pizer, H. Hijikata and T. R. Shemanske, there are few other works considering special orders; they all share working with indefinite algebras. We already referred to [LRdVP18], but we wish to point the reader's attention also to the two works [Cia09] and [dVP13].

The present note leaves several unanswered questions which we wish to address carefully in the near future.

Question 1. *Do the rank-2 interpolation module split into two rank-1 eigenspaces for a suitable involution/operator?*

In his seminal work [Che05], Chenevier provides a p-adic extension of the classical Jacquet–Langlands correspondence using the machinery of eigenvarieties.

Question 2. *Is it possible to produce a suitable rigid analytic morphism between the two eigenvarieties associated with the quaternion algebra B and GL_2, and employ it to produce an analogous triple product p-adic L-function, solving the interpolation problem in the case of special orders?*

We remark that the above Question 2, which may appear quite technical, is of particular arithmetic interest; among all, we notice that the constructions in Section 2 allow further generalization considering special orders. As we remark in Section 3, one cannot hope to approach classical weight-1 specialization with the interpolation formulas provided in [Hsi21] and [GS19], as they deeply rely on [Che05]. This note lands the foundation steps for hoping in a generalization of the formulas in the above works which are more suitable to study the limit point $(2, 1, 1)$. On the other hand, in [GS19] a complete quaternionic formalism is established. Let (f, g, h) be a triple of classical Hida families associated, respectively, to an elliptic curve and two classical modular forms of weight 1. It seems then conceivable to expect certain quaternionic p-adic triple product L-functions. This observation leads to the third and last question.

Question 3. *Does this (partially conjectural) construction satisfy an analogous interpolation problem? If so, would it be possible to study the limit point $(2, 1, 1)$?*

2. Algorithmic approximation for the triple product p-adic L-function

This section is devoted to outline an algorithmic procedure for approximating the value of the balanced p-adic L-function $\mathcal{L}_{F_\infty}^{bal}$, as defined in [Hsi21], at the limit point $(2,1,1)$. We remark that this point lies outside the interpolation region. Let us fix, as before, ℓ and p two distinct primes. Here we must assume p to be odd and later on we will restrict to the case of $p \geq 5$.

2.1. Recalls on points on the weight space.

2.1. Recalls on points on the weight space. In this section we recall the main properties of the ($\mathbb{C}_p$-points of the) weight space and produce sequences of points on it. We refer the reader to Section 1.4 of [CM98] for a thorough discussion.

We are interested in the $\mathbb{C}_p$-points of the weight space, which are identified with

$$\mathcal{X}_{\mathbb{Z}_p^\times}(\mathbb{C}_p) = \mathrm{Hom}_{grp}^{cts}\left(\mathbb{Z}_p^\times, \mathbb{C}_p^\times\right).$$

The usual decomposition $\mathbb{Z}_p^\times = \mu_{p-1}(\mathbb{Q}_p) \times (1 + p\mathbb{Z}_p)$ yields the decomposition of the ($\mathbb{C}_p$-points of the) weight space into $p-1$ disjoint copies of the unit disk:

$$\mathcal{X}_{\mathbb{Z}_p^\times}(\mathbb{C}_p) \cong \coprod_{i \in \mathbb{F}_p^\times} \mathrm{Hom}_{grp}^{cts}\left(1 + p\mathbb{Z}_p, \mathbb{C}_p^\times\right).$$

We fix the dense inclusion of $\mathbb{Z}$ in $\mathcal{X}_{\mathbb{Z}_p^\times}(\mathbb{Z}_p) = \mathrm{Hom}_{grp}^{cts}\left(\mathbb{Z}_p^\times, \mathbb{C}_p^\times\right)$ via the association

$$\mathbb{Z} \ni k \longmapsto \left[x \mapsto x^{k-2}\right] \in \mathcal{X}_{\mathbb{Z}_p^\times}(\mathbb{Z}_p).$$

Due to this dense embedding, there exist points in $\mathbb{Z} \cap \mathcal{X}_{\mathbb{Z}_p^\times}$ and we refer to them as *integer weights* or *integer points* of the weight space. Let $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ be the Iwasawa algebra. We consider the following characters.

(1) The (induced) Teichmüller character $\omega : \Lambda \longrightarrow \mathbb{C}_p$:

$$\omega(1 + p\mathbb{Z}_p) = 1, \qquad \omega\left((\mathbb{Z}/p\mathbb{Z})^\times\right) = \mu_{p-1}(\mathbb{C}_p).$$

(2) The character $\eta_k : \Lambda \longrightarrow \mathbb{C}_p$ for $k \in \mathbb{Z}_p$, such that $a \longmapsto \langle\langle a \rangle\rangle^k$. Here $\langle\langle - \rangle\rangle$ is the character induced by the projection $\mathbb{Z}_p^\times \ni a \longmapsto \langle\langle a \rangle\rangle = a\omega(a)^{-1} \in 1 + p\mathbb{Z}_p$;

The fixed dense embedding can be written in terms of the above characters as $\mathbb{Z} \ni k = \omega^{k-2} \cdot \eta_{k-2}$. We define *arithmetic point* on the weight space to be a point of the form $\chi(-) \cdot k : \mathbb{Z}_p^\times \longrightarrow \mathcal{O}^\times$, where χ is a finite order character of $\mathbb{Z}_p^\times$ and k is an integer point corresponding to $k \in \mathbb{Z}_{\geq 2}$. We identify these points with the corresponding couple $(k, \chi) = \omega^{k-2} \cdot \eta_{k-2} \cdot \chi$. In the end, we say that a point is *classical*, with respect to a certain Hida family f_∞, if the specialization of f_∞ at that point is a classical modular form. We recall Hida's (and more generally Coleman's) Classicality Theorem, which ensures that arithmetic points are classical, independently of the Hida family. The inclusion is strict, *e.g.* due to classical forms of weight 1 in the ordinary case.

Aiming to consider families of classical arithmetic points converging to a triple of points with weights $(2,1,1)$, we should study limits to the weight-1 points in the weight space.

Such points correspond, under our fixed convention, to elements of the form $\varepsilon \cdot \omega^{-1} \cdot \eta_{-1}$, for ε a finite character of order a power of p.

Remark 2.1.1. *The weight-2 arithmetic points (and hence the arithmetic points in general) are dense in the weight space. In particular, there exists a sequence of weight-2 arithmetic points converging to $(1, \varepsilon)$, for ε a finite character of order a power of p.*

We point out that, taken $\{(2, \varepsilon) = \varepsilon\}$ a sequence of weight-2 points converging to a weight-1 point, the sequence of the conductors $cond(\varepsilon_n) = p^{c(\varepsilon_n)}$ has to tend to infinity, for n increasing.

2.2. **Recalls on the Bruhat–Tits tree.** Here we collect some known facts about the Bruhat–Tits tree. We focus our attention on the study of finite length geodesics and the boundary of the tree. We refer respectively to [Rho01] and the book [Ser80], and to [DT08], for all the proofs and details.

2.2.1. **Geodesics on the Bruhat–Tits tree.** Let $\mathcal{T}$ be the Bruhat–Tits tree for the group $PGL_2(\mathbb{Q}_p)$. It is the infinite $p + 1$ regular tree whose vertices are associated with the quotient space

$$GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p) = PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p).$$

We set $\mathcal{V}(\mathcal{T})$ to be set of vertices of $\mathcal{T}$ and $\mathcal{E}(\mathcal{T})$ the set of directed edges in $\mathcal{T}$. A directed edge e is represented by an ordered couple $e = (v_0, v_1)$ of two vertices, respectively the origin and the terminus of the edge e. In an analogous fashion, we define a directed n-path d, as a sequence of $n+1$ vertices $d = (v_0, \dots, v_n)$ such that for each $i = 0, \dots n$, $(v_i, v_{i+1}) \in \mathcal{E}(\mathcal{T})$; we denote the origin and terminus of d by, respectively $o(d) = v_0$ and $t(d) = v_n$. If moreover a directed n-path d satisfies the condition $v_i \neq v_{i+2}$, we say that d is a *geodesic of length* n on $\mathcal{T}$. We denote the set of all geodesics of length n by $Geod_n(\mathcal{T})$. Obviously, $\mathcal{E}(\mathcal{T}) = Geod_1(\mathcal{T})$ and $\mathcal{V}(\mathcal{T}) = Geod_0(\mathcal{T})$. Since $\mathcal{T}$ is a tree, a geodesic is a path without backtracking and it is uniquely determined by its origin and terminus, that is by an ordered couple of vertices. We have the action by left multiplication of $PGL_2(\mathbb{Q}_p)$ on the vertices of $\mathcal{T}$, which extends to an action on the set of geodesics $Geod_n(\mathcal{T})$. It is well known that the action is transitive on $\mathcal{V}(\mathcal{T})$ and $\mathcal{E}(\mathcal{T})$, but the same holds true for $Geod_n(\mathcal{T})$. Describing each geodesic as the couple consisting of origin and terminus point, it is clear that the stabilizer of a geodesic g is

$$Stab_{PGL_2(\mathbb{Q}_p)}(g) = Stab_{PGL_2(\mathbb{Q}_p)}(o(g)) \cap Stab_{PGL_2(\mathbb{Q}_p)}(t(g))$$

i.e. a matrix stabilizes g if and only if it stabilizes both its origin and terminus. We fix, for each $n \geq 0$, a privileged geodesic of length n (we call it privileged vertex and edge, respectively in the case $n = 0$ and $n = 1$)

$$g_n := \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot PGL_2(\mathbb{Z}_p), \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \cdot PGL_2(\mathbb{Z}_p), \dots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) \cdot PGL_2(\mathbb{Z}_p) \right).$$

For ease of notation we often write $g_n = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \dots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) \right)$, as well for generic geodesics, where we write a choice of representatives instead of the classes. It is readily computed that

$$Stab_{PGL_2(\mathbb{Q}_p)} \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot PGL_2(\mathbb{Z}_p) \right) = PGL_2(\mathbb{Z}_p)$$

and

$$Stab_{PGL_2(\mathbb{Q}_p)} \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) \cdot PGL_2(\mathbb{Z}_p) \right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) PGL_2(\mathbb{Q}_p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right)^{-1},$$

thus the characterization of the stabilizers of a geodesics implies that the stabilizer of the privileged geodesic g_n is

$$\begin{aligned} Stab_{PGL_2(\mathbb{Q}_p)}(g_n) &= PGL_2(\mathbb{Q}_p) \cap \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) PGL_2(\mathbb{Q}_p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right)^{-1} \\ &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PGL_2(\mathbb{Z}_p) \mid c \in p^n\mathbb{Z}_p \right\}. \end{aligned}$$

We denote the above stabilizer by $\overline{\Gamma}_0(p^n\mathbb{Z}_p)$ as it is the image in $PGL_2(\mathbb{Q}_p)$ of

$$\Gamma_0(p^n\mathbb{Z}_p) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p) \mid c \in p^n\mathbb{Z}_p \right\}.$$

Therefore, the quotient map induces the identification

$$PGL_2(\mathbb{Q}_p)/\overline{\Gamma}_0(p^n\mathbb{Z}_p) = GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \Gamma_0(p^r\mathbb{Z}_p).$$

Since the $PGL_2(\mathbb{Q}_p)$-action is transitive, we have the isomorphism

$$GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \Gamma_0(p^r\mathbb{Z}_p) \cong Geod_r(\mathcal{T})$$

given by

$$\gamma \longmapsto \gamma \cdot g_n = \left(\gamma \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \gamma \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \dots, \gamma \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) \right).$$

The inclusion of $\Gamma_0(p^n\mathbb{Z}_p)$ into $\Gamma_0(p^m\mathbb{Z}_p)$ for each $m \leq n$ provides the natural ($GL_2(\mathbb{Q}_p)$-equivariant) surjective quotient map

$$\rho_n : GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \Gamma_0(p^n\mathbb{Z}_p) \longrightarrow GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \Gamma_0(p^{n-1}\mathbb{Z}_p).$$

We can consider the surjective map

$$\alpha_n : Geod_n(\mathcal{T}) \longrightarrow Geod_{n-1}(\mathcal{T})$$

defined by

$$(v_0, \dots, v_{n-1}, v_n) \longmapsto (v_0, \dots, v_{n-1}),$$

i.e. the initial geodesic of length $n-1$; we note that the map is $GL_2(\mathbb{Q}_p)$-equivariant.

Lemma 2.2.1. *The diagrams*

$$\begin{tikzcd}
\vdots \arrow[d, dashed, two heads, "\rho_{n+1}"] & \vdots \arrow[d, dashed, two heads, "\alpha_{n+1}"] \\
GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p) \arrow[r, leftrightarrow, "\cong"] \arrow[d, two heads, "\rho_n"] & Geod_n(\mathcal{T}) \arrow[d, two heads, "\alpha_n"] \\
GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^{n-1}\mathbb{Z}_p) \arrow[r, leftrightarrow, "\cong"] \arrow[d, dashed, two heads, "\rho_{n-1}"] & Geod_{n-1}(\mathcal{T}) \arrow[d, dashed, two heads, "\alpha_{n-1}"] \\
\vdots \arrow[d, dashed, two heads, "\rho_2"] & \vdots \arrow[d, dashed, two heads, "\alpha_2"] \\
GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p\mathbb{Z}_p) \arrow[r, leftrightarrow, "\cong"] \arrow[d, two heads, "\rho_1"] & \mathcal{E}(\mathcal{T}) \arrow[d, two heads, "\alpha_1"] \\
GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times GL_2(\mathbb{Z}_p) \arrow[r, leftrightarrow, "\cong"] & \mathcal{V}(\mathcal{T})
\end{tikzcd}$$

are commutative.

Proof. Let $n > 1$ and take $\bar{A} \in GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p)$ and $\bar{B} \in GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^{n-1}\mathbb{Z}_p)$ such that $\rho_n(\bar{A}) = \bar{B}$. This means that there exists a matrix $C \in \overline{\Gamma}_0(p^{n-1}\mathbb{Z}_p) = Stab_{PGL_2(\mathbb{Q}_p)}(g_{n-1})$ such that $A = BC$ as matrices in $PGL_2(\mathbb{Q}_p)$. Thus

$$\begin{tikzcd}
\bar{A} = \bar{B}\bar{C} \arrow[r, leftrightarrow] \arrow[d, mapsto, "\rho_n"] & \left(A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \dots, A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{n-1} \end{smallmatrix}\right), A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right)\right) \arrow[d, mapsto, "\alpha_n"] \\
\bar{B} \arrow[r, leftrightarrow] & \left(A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \dots, A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{n-1} \end{smallmatrix}\right)\right)
\end{tikzcd}$$

because

$$\left(B \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), B \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \dots, B \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{n-1} \end{smallmatrix}\right)\right) = \left(A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \dots, A \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{n-1} \end{smallmatrix}\right)\right)$$

as $\alpha_n(A \cdot g_n) = A \cdot \alpha_n(g_n) = A \cdot g_{n-1} = B \cdot (C \cdot g_{n-1}) = B \cdot g_{n-1}$. The proof for the case $n = 1$ is the same, taking into account that $PGL_2(\mathbb{Z}_p)$ stabilizes $g_0 = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$. ■

Remark 2.2.2. *Since the Bruhat–Tits tree is* $p{+}1$*-regular, we have exactly* $p{+}1$ *preimages via* α_1 *and* p *via* α_n*, for each* $n \geq 2$*. Hence we readily note that, for each* $n \geq 2$*,*

$$[\Gamma_0(p^{n-1}\mathbb{Z}_p) : \Gamma_0(p^n\mathbb{Z}_p)] = [\overline{\Gamma}_0(p^{n-1}\mathbb{Z}_p) : \overline{\Gamma}_0(p^n\mathbb{Z}_p)] = p$$

and

$$[GL_2(\mathbb{Z}_p) : \Gamma_0(p\mathbb{Z}_p)] = [PGL_2(\mathbb{Z}_p) : \overline{\Gamma}_0(p\mathbb{Z}_p)] = p + 1.$$

In particular, for each fixed vertex v *in* $\mathcal{T}$*, the set* $Geod_n(v)$ *of* geodesic of length n and origin v *is the preimage of* v *via the composition* $\alpha_n \circ \cdots \circ \alpha_1$ *and thus it has cardinality* $\#Geod_n(v) = [GL_2(\mathbb{Z}_p) : \Gamma_0(p^n\mathbb{Z}_p)] = (p+1)p^{n-1}$.

2.2.2. **Ends on the Bruhat–Tits tree.** Now that we have recalled the properties of the geodesics of finite length, we can understand the behavior of the ones proceeding towards infinity. For this section, we follow thoroughly the notes [DT08].

Let $(v_0, v_1, \dots, v_n, \dots)$ be an infinite sequence of adjacent points without backtracking. We think about it as an infinite ray in the tree, heading off to the boundary of the tree. We say that two such sequences are equivalent if and only if they only differ by a finite initial sequence of vertices, i.e.

$$(v_0, v_1, \dots, v_n, \dots) \sim (v'_0, v'_1, \dots, v'_n, \dots)$$

if and only if $v_n = v'_{n+m}$ for some fixed $m \in \mathbb{Z}$, and all n great enough.

Definition 2.2.3. *An equivalence class of such sequences is called an* end *of the tree. We denote the set of ends of the Bruhat–Tits tree by $Ends(\mathcal{T})$.*

The ends represent the set of points "at infinity" for the tree $\mathcal{T}$ and must be thought of as a boundary. In fact, adding the $Ends(\mathcal{T})$ to the tree yields exactly the Borel–Serre compactification of the tree; we refer to the Appendix in [SS97], page 166, for further details.

As in Section 1.3.4 of [DT08], $Ends(\mathcal{T})$ can be endowed with a topology given by the basis of open compact subsets

$$U(e) = \Big\{ (v_0, v_1, \dots, v_n, \dots) \in Ends(\mathcal{T}) \;\Big|\; (v_0, v_1) = e \Big\}$$

for each edge e. Similarly, for a geodesic $g \in Geod_n(\mathcal{T})$, we set

$$U(g) = \Big\{ (v_0, v_1, \dots, v_n, \dots) \in Ends(\mathcal{T}) \;\Big|\; (v_0, v_1, \dots, v_n) = g \Big\} = U((v_{n-1}, v_n)).$$

Under the choice of such topology, the following lemma holds.

Lemma 2.2.4. *There exists a $GL_2(\mathbb{Q}_p)$-equivariant homeomorphism between $Ends(\mathcal{T})$ and $\mathbb{P}^1(\mathbb{Q}_p)$.*

In [DT08] the action taken on $\mathbb{P}^1(\mathbb{Q}_p)$ is a right action, while here we are considering the left one, obtained as

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [x, y] = [ax + by, cx + dy]$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Q}_p)$ and any $[x, y] \in \mathbb{P}^1(\mathbb{Q}_p)$; under this action the above Lemma remains true. It is easy to note that the stabilizer of each end is a Borel subgroup in $GL_2(\mathbb{Q}_p)$, since

$$Stab_{GL_2(\mathbb{Q}_p)}(g_\infty) = \{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in GL_2(\mathbb{Q}_p) \} =: P(\mathbb{Q}_p)$$

is the upper Borel, where we denote by g_∞ the (representative of the) *privileged end*

$$g_\infty := \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot GL_2(\mathbb{Z}_p), \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \cdot GL_2(\mathbb{Z}_p), \dots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) \cdot GL_2(\mathbb{Z}_p), \dots \right).$$

It is useful to identify $\mathbb{P}^1(\mathbb{Q}_p)$, and thus $Ends(\mathcal{T})$, with the quotient $GL_2(\mathbb{Q}_p)/P(\mathbb{Q}_p)$; the $GL_2(\mathbb{Q}_p)$-equivariant identification is given by the association

$$\mathbb{P}^1(\mathbb{Q}_p) \ni [x,y] \longmapsto \left(\begin{smallmatrix} x & * \\ y & * \end{smallmatrix}\right) P(\mathbb{Q}_p) \in GL_2(\mathbb{Q}_p)/P(\mathbb{Q}_p).$$

2.2.3. **A small aside: inverse limit of geodesics.** We can consider the inverse limit spaces of the two parallel systems of Lemma 2.2.1, obtaining the $GL_2(\mathbb{Q}_p)$-equivariant isomorphism

$$\varprojlim GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \Gamma_0(p^n\mathbb{Z}_p) \cong \varprojlim Geod_n(\mathcal{T}).$$

Each element $G \in \varprojlim Geod_n(\mathcal{T})$ is represented by an infinite sequence of the form

$$G = (g_1, g_2, \ldots, g_n, \ldots)\,,$$

where each $g_n \in Geod_n(\mathcal{T})$ such that $\alpha_{n+1}(g_{n+1}) = g_n$ for $n \geq 0$. Each of such infinite sequences can be thought of as a ray, starting from a vertex of $\mathcal{T}$ and heading towards infinity into a particular end.

Proposition 2.2.5. *We have a (canonical) $GL_2(\mathbb{Q}_p)$-isomorphism*

$$\varprojlim Geod_n(\mathcal{T}) \cong \mathcal{V}(\mathcal{T}) \times Ends(\mathcal{T}).$$

Proof. Let $G = (g_1, g_2, \ldots, g_n, \ldots)$ be an element of the inverse limit $\varprojlim Geod_n(\mathcal{T})$. We associate to G the couple $(g_1, [G] = [(g_1, t(g_2), t(g_3), \ldots)])$. The map is well-defined as $(g_1, t(g_2), t(g_3), \ldots)$ is a sequence of adjacent points without backtracking. Any couple $(v, x) \in \mathcal{V}(\mathcal{T}) \times Ends(\mathcal{T})$ can be taken such that $x = [(v, v_1, v_2, \ldots)]$ and hence the element $(v, (v, v_1), (v, v_1, v_2), \ldots)$ determines a class in the inverse limit whose image is (v, x). Moreover, the map is injective, as $\mathcal{T}$ is a tree, so there exists a unique ray starting at a vertex v and heading towards the end x. This proves that the map is bijective. The $GL_2(\mathbb{Q}_p)$-equivariance is immediate. ■

Remark 2.2.6. *We may ask whether it is possible to replace $\mathcal{V}(\mathcal{T})$ in the above proposition with $Geod_n(\mathcal{T})$, for any $n \geq 1$. One must then take into account the orientation of the finite geodesics; in each couple, the finite geodesic must have an orientation compatible with the associated end. Such restriction implies that we cannot expect a surjective map onto $Geod_n(\mathcal{T}) \times Ends(\mathcal{T})$. We deduce that $\varprojlim Geod_n(\mathcal{T}) \cong \prod_{g \in Geod_n(\mathcal{T})} U(g)$.*

2.3. **Quaternionic double quotients and geodesics on the Bruhat–Tits tree.** In this section we describe how to explicitly compute a set of representatives for the double quotient of the quaternion algebra B. We show how the procedure presented in [FM14] can be extended and applied to our case of interest.

2.3.1. **Quaternionic p-adic double coset spaces.** Let p be as above an odd prime and take B to be the (unique up to isomorphism) quaternion algebra over $\mathbb{Q}$ ramified exactly at the prime $\ell \neq p$ and ∞. Let R^n be either an Eichler order of level Np^n or a special order of level $Np^n\ell^{2r}$, with N prime to both p and ℓ, $r \geq 1$ and $n \geq 0$. Fix isomorphisms $\iota_q : B_q := B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for each prime $q \neq \ell$, such that $\iota_q : R_q :=$

$R \otimes \mathbb{Z}_q \cong M_2^0(Np^n\mathbb{Z}_q)$ and hence $\iota_p(R) = \Gamma_0(p^n\mathbb{Z}_p)$. Set $\widehat{B} = B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},f}$ and $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ where $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q}\widehat{\mathbb{Z}}$ are the finite adèles of $\mathbb{Q}$. Let $U_1(R^n)$ be as in Definition 1.1.10.

Lemma 1.1.3 allows us to consider the classical application of the *Strong Approximation Theorem* (see [Vig80], Theoreme fondamental 1.4, b)).

Lemma 2.3.1 (*p*-adic double quotient)**.** *Let $\Sigma = \prod' \Sigma_q$ be either $\widehat{R^n}^{\times}$ or $U_1(R^n)$ and let R be a maximal order containing R^n. The embedding of $GL_2(\mathbb{Q}_p) \overset{\iota_p}{\cong} B_p^\times \hookrightarrow \widehat{B}^\times$ as $b_p \longmapsto (1, \dots, 1, b_p, 1, 1 \dots)$ induces the bijection*

$$\left(R\,[1/p]^\times \cap \prod_{q \neq p} \Sigma_q \right) \backslash B_p^\times / \Sigma_p \cong B^\times \backslash \widehat{B}^\times / \Sigma.$$

Furthermore, if $\Sigma = \widehat{R^n}^{\times}$, we have

$$B^\times \backslash \widehat{B}^\times / \Sigma \cong \iota_p \left(R^n\,[1/p]^\times \right) \backslash GL_2(\mathbb{Q}_p) / \Gamma_0(p^n\mathbb{Z}_p) \cong \Gamma_n \backslash GL_2(\mathbb{Q}_p) / \mathbb{Q}_p^\times \Gamma_0(p^n\mathbb{Z}_p),$$

where Γ_n is the image of $\iota_p \left(R^n\,[1/p]^\times \right)$ in $PGL_2(\mathbb{Q}_p)$.

Proof. Since p is a split place in B, the *Strong Approximation Theorem* implies that for any Σ compact open in $\widehat{B}^\times$, $\widehat{B}^\times = B^1 B_p^1 \mathbb{A}_{\mathbb{Q},f}^\times \Sigma = B^\times B_p^\times \Sigma$. By proving the double inclusions we have $R\,[1/p]^\times \cap \prod_{q \neq p} (R_q^n)^\times = R^n\,[1/p]^\times$. In the end, $1/p$ belongs to $R^n[1/p]^\times$ and $\mathbb{Z}_p^\times \subset R_p^{n\times}$. ■

The above lemma, together with the following proposition, shows that we can focus our attention on the right quotient

$$B_p^\times / \mathbb{Q}_p^\times R_p^\times \overset{\iota_p}{\cong} GL_2(\mathbb{Q}_p) / \mathbb{Q}_p^\times \Gamma_0(p^r\mathbb{Z}_p).$$

Proposition 2.3.2. *For any $n \geq 1$, let R^n be as above. By Remark 1.1.8, we can assume that (up to conjugation) the orders R_n are encapsulated. Then $R^n\,[1/p] = R^{n+1}\,[1/p]$.*

Proof. Let $f_1, \dots, f_4$ be a $\mathbb{Z}$-basis for R^n. As thoroughly explained in [Piz80a], Section 5, any sublattice (in particular suborders) of index p in R^n, has a basis $(g_1, \dots, g_4)$ such that $(g_1, \dots, g_4) = (f_1, \dots, f_4) \cdot A$, where A varies among the matrices in Hermite normal form in $M_4(\mathbb{Z})$ and with determinant p. Such matrices are of the form

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & p \end{pmatrix},$$

for $0 \leq a, b, c < p$. These matrices belong to $M_4(\mathbb{Z}) \cap GL_4(\mathbb{Z}\,[1/p])$, thus the claim follows. ■

Notation. *We deduce that $\Gamma_n = \Gamma_{n+1}$ for each $n \geq 1$, hence we set $\Gamma := \Gamma_1$.*

2.3.2. **Classes of representatives for the finite length geodesics.** We recall that, for any prime q, we denote the q-adic valuation by v_q.

Lemma 2.3.3. *For $r \geq 1$, the quotient $GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^r\mathbb{Z}_p)$ admits a set of right-coset representatives $\{e_i\}$ consisting of matrices with coefficients in $\mathbb{Z}$ for $r = 1$, or coefficients in $\mathbb{Z}_p$ for $r \geq 2$. Moreover, there exists an effective algorithm that, given a matrix g in $GL_2(\mathbb{Q}_p)$ returns a scalar $\lambda \in \mathbb{Q}_p^\times$ and a matrix t in $\Gamma_0(p^r\mathbb{Z}_p)$, such that $\lambda \cdot g \cdot t = e_i$.*

Before giving a proof of the lemma, which provides the actual algorithm for computing λ and t, we point out that the case of $r = 1$ has already been proved in Lemma 2.2 of [FM14]. Lemma 2.3.3 indeed agrees with the result in *loc.cit.* and the computations are essentially the same performed there. Moreover, the case of $GL_2(\mathbb{Z}_p)$ is a classical computation and we refer to Section 5.3 of [MM06].

We recall the transformations:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & ax+b \\ c & cx+d \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) = \left(\begin{smallmatrix} au & bv \\ cu & dv \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ yp^r & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+byp^r & b \\ c+dyp^r & d \end{smallmatrix}\right),$$

for $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) \in \Gamma_0(p^N\mathbb{Z}_p)$ for each $N \geq 0$ and $\left(\begin{smallmatrix} 1 & 0 \\ yp^r & 1 \end{smallmatrix}\right) \in \Gamma_0(p^r\mathbb{Z}_p)$. We remark that the p-adic valuation of the determinant of these transformations is zero.

Proof of Lemma 2.3.3. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix in $GL_2(\mathbb{Q}_p)$. Up to multiply by an element in $\mathbb{Q}_p^\times$, we can suppose $M \in GL_2(\mathbb{Q}_p) \cap M_2(\mathbb{Z}_p)$. We set λ to be the power of p such that λM has coefficients in $\mathbb{Z}_p$ and at least one entry is in $\mathbb{Z}_p^\times$. Set $\lambda := p^{-\min\{v_p(a),v_p(b),v_p(c),v_p(d)\}}$. Let $N := v_p(\det(\lambda M))$ and from now on we assume that M is such that $\min\{v_p(a), v_p(b), v_p(c), v_p(d)\} = 0$. We divide the proof into three main cases. The proof is a straightforward computation once we provide a suitable matrix t, which is a product of the three transformations we introduced above.

(1) $v_p(a) \leq v_p(b)$: This first case is proved in the same fashion as in [FM14]. We have
$$t := \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{p^{v_p(a)}}{a} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{p^{N-\alpha}}{d-c\frac{a}{b}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ yp^r & 1 \end{pmatrix}$$
for $y \in \mathbb{Z}_p$ such that $\frac{p^{v_p(a)}}{a}c + yp^r$ belongs to $\mathbb{Z} \cap (0, p^{r+N-v_p(a)}]$. Let $s \in \mathbb{Z}$ such that $s \cdot \frac{a}{p^{v_p(a)}} \equiv 1 \pmod{p^{r+N-v_p(a)}}$ and $c' \in \mathbb{Z} \cap (0, p^{r+N-v_p(a)}]$ such that $c' \equiv sc \pmod{p^{r+N-v_p(a)}}$. Thus
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} p^{v_p(a)} & 0 \\ c' & p^{N-v_p(a)} \end{pmatrix} \quad (\mathrm{mod}\ \Gamma_0(p^r\mathbb{Z}_p)).$$

(2) $v_p(a) \geq v_p(b) + r$: The matrix t is of the following form
$$t := \begin{pmatrix} 1 & 0 \\ -\frac{a}{bp^r}p^r & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{p^{N-v_p(b)}}{c-\frac{a}{b}d} & 0 \\ 0 & \frac{p_p^v(b)}{b} \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
for $x \in \mathbb{Z}_p$ such that $\frac{p_p^v(b)}{b}d + xp^{N-v_p(b)}$ belongs to $\mathbb{Z} \cap (0, p^{N-v_p(b)}]$. Let $s \in \mathbb{Z}$ such that $s \cdot \frac{p_p^v(b)}{b} \equiv 1 \pmod{p^{N-v_p(b)}}$ and $d' \in \mathbb{Z} \cap (0, p^{N-v_p(b)}]$ such that $d' \equiv sd$

(mod $p^{N-v_p(b)}$). Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & p^{v_p(b)} \\ p^{N-v_p(b)} & d' \end{pmatrix} \pmod{\Gamma_0(p^r\mathbb{Z}_p)}.$$

(3) $v_p(b) < v_p(a) < v_p(b) + r$: This case appears only when $r > 1$. As in the previous case we can distinguish three cases depending on the valuations $v_p(c)$ and $v_p(d)$.

(a) $v_p(c) \le v_p(d)$: We proceed as in the case $v_p(a) \le v_p(b)$ and define a unique matrix $t \in \Gamma_0(p^r\mathbb{Z}_p)$, analogous to the one above, of the form

$$t := \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ yp^r & 1 \end{pmatrix}.$$

Let $s \in \mathbb{Z}$ be such that $s \cdot \frac{p^{v_p(c)}}{c} \equiv 1 \pmod{p^{v_p(b)+r}}$ and $a' \in \mathbb{Z} \cap (0, p^r]$ such that $a' \equiv sa/p^{v_p(b)} \pmod{p^r}$. This behavior is due to the fact that $v_p(a)$ is bigger than $v_p(b)$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a'p^{v_p(b)} & p^{v_p(b)} \\ p^{N-v_p(b)} & 0 \end{pmatrix} \pmod{\Gamma_0(p^r\mathbb{Z}_p)},$$

and, since we are supposing that one of the entries is in $\mathbb{Z}_p^\times$, we are in either one of the following two situations:

$$\begin{pmatrix} a'p^{v_p(b)} & p^{v_p(b)} \\ p^{N-v_p(b)} & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} a' & 1 \\ p^N & 0 \end{pmatrix} & \text{if } v_p(b) = 0, \\ \\ \begin{pmatrix} a'p^{v_p(b)} & p^{v_p(b)} \\ 1 & 0 \end{pmatrix} & \text{if } N = v_p(b) \text{ iff } v_p(b) \ge 1. \end{cases}$$

(b) $v_p(c) \ge v_p(d) + r$: As in the case of $v_p(a) \ge v_p(b) + r$, we have

$$t := \begin{pmatrix} 1 & 0 \\ -\frac{c}{dp^r}p^r & 1 \end{pmatrix} \cdot \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for $x \in \mathbb{Z}_p$ such that $\frac{p_p^v(d)}{d}b + xp^{N-v_p(d)}$ belongs to $\mathbb{Z} \cap (0, p^{N-v_p(d)}]$. Let $s \in \mathbb{Z}$ such that $s \cdot \frac{p_p^v(d)}{d} \equiv 1 \pmod{p^{N-v_p(d)}}$ and $b' \in \mathbb{Z} \cap (0, p^{N-v_p(d)}]$ such that $b' \equiv sb \pmod{p^{N-v_p(d)}}$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} p^{N-v_p(d)} & b' \\ 0 & p^{v_p(d)} \end{pmatrix} \pmod{\Gamma_0(p^r\mathbb{Z}_p)}.$$

(c) $v_p(d) < v_p(c) < v_p(d) + r$: This is the last case, the one in which we cannot say much about the reduction modulo $\Gamma_0(p^r\mathbb{Z}_p)$. We have

$$0 < v_p(a) - v_p(b), v_p(c) - v_p(d) < r$$

and t is the diagonal matrix with $\mathbb{Z}_p^\times$ coefficients, $t := \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$, hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} p^{v_p(a)} & 1 \\ \frac{cp^{v_p(a)}}{ap^{v_p(c)}} \cdot p^{v_p(c)} & \frac{d}{bp^{v_p(d)}} \cdot p^{v_p(d)} \end{pmatrix} & \pmod{\Gamma_0(p^r\mathbb{Z}_p)} \quad \text{if } v_p(d) \geq v_p(b), \\ \\ \begin{pmatrix} \frac{ap^{v_p(c)}}{cp^{v_p(a)}} \cdot p^{v_p(a)} & \frac{b}{dp^{v_p(b)}} \cdot p^{v_p(b)} \\ p^{v_p(c)} & 1 \end{pmatrix} & \pmod{\Gamma_0(p^r\mathbb{Z}_p)} \quad \text{if } v_p(d) < v_p(b). \end{cases}$$

■

We can explicitly write the representatives produced in the proof of Lemma 2.3.3 and restate the lemma in the following form.

Lemma 2.3.4. *There exists an effective algorithm that, given a matrix g in $GL_2(\mathbb{Q}_p)$ returns a unique scalar $\lambda \in p^{\mathbb{Z}}$ and a unique matrix t in $\Gamma_0(p^r\mathbb{Z}_p)$, such that $\lambda \cdot g \cdot t$ is, for $m, n \geq 0$, a matrix in the following set.*

$$\left\{ \begin{pmatrix} p^m & 0 \\ c & p^n \end{pmatrix} \text{ for } 0 < c \leq p^{n+r}, \begin{pmatrix} 0 & p^m \\ p^n & d \end{pmatrix} \text{ for } 0 < d \leq p^n, \begin{pmatrix} ap^m & p^m \\ 1 & 0 \end{pmatrix} \text{ for } 0 < a \leq p^r, \right.$$

$$\begin{pmatrix} a & 1 \\ p^n & 0 \end{pmatrix} \text{ for } 0 < a \leq p^r, \begin{pmatrix} p^l & 1 \\ c'p^kp^n & d'p^n \end{pmatrix} \text{ for } 0 < k, l < r,\ c', d' \in \mathbb{Z}_p^\times,$$

$$\left. \begin{pmatrix} p^m & b \\ 0 & p^n \end{pmatrix} \text{ for } 0 < b \leq p^m, \begin{pmatrix} a'p^lp^m & b'p^m \\ p^k & 1 \end{pmatrix} \text{ for } 0 < k, l < r,\ a', b' \in \mathbb{Z}_p^\times \right\}.$$

2.3.3. **The algorithm for Γ^1-classes.** Let B, R and R^n as in Section 2.3.1. We identify the p-component of R_p^n with $\Gamma_0(p^r\mathbb{Z}_p)$ via the fixed isomorphism ι_p. If there is no possibility of confusion, we will avoid to write the isomorphism ι_p. In the rest of this section we provide an immediate extension of Algorithm 1 contained in [FM14]; it allows us to compute a fundamental domain in the Bruhat–Tits tree $\mathcal{T}$ for the group $\Gamma^1 := \left(R^n\left[1/p\right]^\times\right)^1$ of elements of reduced norm 1. Moreover, Algorithm 1 computes a full set of representatives for the double quotient $\Gamma^1\backslash GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^r\mathbb{Z}_p)$. As in *loc.cit.*, Γ^1 has an increasing filtration by finite sets (because the quaternion algebra is definite), for each $m \geq 0$, given by

$$\Gamma^1(m) := \left\{ \frac{x}{p^m} \mid x \in R^n \text{ and } n(x) = p^{2m} \right\},$$

where $n(x)$ is the reduced norm map applied to x. Moreover, $\Gamma^1(m) \subset \Gamma^1(m+1)$ $(x/p^m = px/p^{m+1})$ and $\Gamma^1 = \cup_{m\geq 0}\Gamma^1(m)$.

Let u and v be two matrices in $PGL_2(\mathbb{Q}_p)$ which, by abuse of notation, we identify with two vertices on $\mathcal{T}$. Following [FM14] and [BB12], we define $Hom_{\Gamma^1}(u,v) := \left\{\gamma \in \Gamma^1 \,\middle|\, \gamma u = v\right\}$, that is the set of elements moving u to v. We note that for each vertex u, $Hom_{\Gamma^1}(u,u) = Stab_{\Gamma^1}(u)$ and u is equivalent to v modulo Γ^1 if and only if $Hom_{\Gamma^1}(u,v) \neq \emptyset$. From now on, we suppose that the matrices u and v are in one of the forms given in the above Lemma 2.3.4. Let $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ be defined as $2m = v_p\left(\det(u)\det(v)\right)$. We note that

$$\begin{aligned} Hom_{\Gamma^1}(u,v) = Hom_{GL_2(\mathbb{Q}_p)}(u,v) \cap \Gamma^1 &= \left(v \cdot Stab_{GL_2(\mathbb{Q}_p)}(g_0) \cdot u^{-1}\right) \cap \Gamma^1 \\ &= \left(v \cdot \mathbb{Q}_p^\times GL_2(\mathbb{Z}_p) \cdot u^{-1}\right) \cap \Gamma^1. \end{aligned}$$

All the lemmas and definitions needed for implementing Algorithm 1 in [FM14] are still valid in our setting without any change. Here we recall some of the results and refer the reader to *loc.cit.* for a detailed proof of them.

Lemma 2.3.5 ([FM14], Lemma 3.1)**.** *If m is not an integer, then $Hom_{\Gamma^1}(u,v) = \emptyset$. Otherwise*

$$Hom_{\Gamma^1}(u,v) = p^{-m} v M_2(\mathbb{Z}_p) u^* \cap \Gamma^1,$$

for $u^* = \frac{1}{\det u} u^{-1}$.

Definition 2.3.6 ([FM14], Definition 3.2)**.** *Let $m \geq 0$ be an integer, V and W two finite dimensional $\mathbb{Q}_p$-vector spaces, and let $\Lambda_V \subseteq V$ and $\Lambda_W \subseteq W$ be $\mathbb{Z}_p$-lattices. Let $F : V \longrightarrow W$ be a $\mathbb{Q}_p$-linear map satisfying $f(\Lambda_V) \subseteq \Lambda_W$. Then an* approximation of f to precision m *is a $\mathbb{Q}_p$-linear map $g : V \longrightarrow W$ such that $g \equiv f \pmod{p^m}$ when restricted to Λ_V.*

Lemma 2.3.7 ([FM14], Lemma 3.3)**.** *Let u and v be matrices in $GL_2(\mathbb{Q}_p) \cap M_2(\mathbb{Z}_p)$ corresponding to two vertices of $\mathcal{T}$. Let $f : M_2(\mathbb{Q}_p) \longrightarrow B_p$ be an approximation of ι_p^{-1} to p-adic precision $2m = v_p(\det(vu))$ and relative to the lattices $M_2(\mathbb{Z}_p)$ and R_p. Then $Hom_{\Gamma^1}(u,v)$ is non-empty if and only if the shortest vectors in the $\mathbb{Z}$-lattice*

$$\Lambda(u,v) := \left(f\left(vM_2(\mathbb{Z}_p)u^*\right) + p^{2m+1}R\right) \cap R^n = f\left(vM_2(\mathbb{Z}_p)u^*\right) \cap R^n + p^{2m+1}R^n$$

have reduced norm p^{2m}.

We recall furthermore that all the elements in $\Lambda(u,v)$ have reduced norm at least p^{2m}.

We are now ready to write the algorithm of [FM14], suitably adjusted for computing representatives of our double-coset space of interest.

Algorithm 1: Compute a coset decomposition for $\Gamma^1\backslash B_p^\times/\mathbb{Q}_p^\times R_p^{n\times}$ and a fundamental domain for $\Gamma^1\backslash\mathcal{T}$.

Input: The prime p, the quaternion algebra $B/\mathbb{Q}$, the order R^n and a fixed isomorphism ι_p as above.

Output: A list of coset representatives for $\iota_p(\Gamma^1)\backslash GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p)$.

```
begin
    Initialize queue W with the privileged vertex v_0;
    Initialize two empty lists E and P;
    while W ≠ ∅ do
        Pop v from W;
        for each g ∈ Geod_n(v) do
            if there is no g' ∈ Geod_n(v) which is Γ^1-equivalent to g then
                Append g to E;
                if there is a vertex v' ∈ W which is Γ^1-equivalent to t(g) then
                    Append (t(g), v', γ) to P, where γ ∈ Γ^1 is such that γt(g) = v'
                else
                    Push t(g) onto W
    return E and P.
```

Remark 2.3.8. *The complexity of the algorithm has to take into account the increased number of geodesics. We already noticed that for each vertex v, $\#Geod_n(v) = p^{n-1}(p+1)$ and thus the algorithm has to check the Γ^1-equivalence for $p^{n-1}(p+1)$-points for each v, each time for each terminus point of a chosen geodesic. Furthermore, Lemma 2.3.3 for $r \geq 2$ requires undoubtedly more time and resources compared to the case $r = 0, 1$.*

2.3.4. **The algorithm for Γ-classes.** As the above algorithm computes a set of representatives for the action of the elements of reduced norm 1 Γ^1, it remains to compute a set of representatives for the action of $(R^n[1/p])^\times$ or, more precisely, for the action of Γ. We recall that the reduced norm of any element in $B[1/p]$ is positive, as B is definite. We begin presenting an example in which Γ^1 and Γ are distinct.

Example 2.3.9. *Let B be a quaternion algebra over $\mathbb{Q}$ ramified exactly at 3 and ∞ and fix the four generators to be $\langle 1, i, j, k\rangle$ such that $i^2 = -1$, $j^2 = -3$ and $ij = k$. Example 2 in Section 9 of [Piz80a] provides an explicit basis for an Eichler order of level 15, namely*

$$R^1_{15} = \langle \tfrac{1}{2}(1+j+2k), \tfrac{1}{2}(i+5k), j+2k, 5k\rangle_{\mathbb{Z}}$$

and it is not difficult to notice that the elements $1 \pm 2i \in R^1_{15}$ have reduced norm 5. Even though they are not invertible in R^1_{15}, they can be inverted in $R^1_{15}[1/5]$ with inverse (resp.) $(1 \mp 2i)/5$. In particular, the reduced norm of $(1 \mp 2i)/5$ is $1/5$, hence $(1 \mp 2i)/5 \in (R^1_{15}[1/5])^\times - (R^1_{15}[1/5])^1$.

Lemma 2.3.10. *Let, if it exists, δ_p be an element in $(R^n[1/p])^\times$ of reduced norm p. For any $\gamma \in (R^n[1/p])^\times$ with reduced norm $n(\gamma) = p^m$, $m \in \mathbb{Z}$, set $M = m/2$ if m is even and $M = m$ otherwise. The element $\gamma \cdot \delta_p^{-M}$ belongs to $(R^n[1/p])^1$.*

Proof. The order $R^n[1/p] = R^1[1/p]$ is a $\mathbb{Z}[1/p]$-order and an element belongs to $(R^n[1/p])^\times$ if and only if its reduced norm lies in $\mathbb{Z}[1/p]^\times = \pm p^{\mathbb{Z}}$ (see [Vig80], Lemme 4.12) and moreover it is positive. If $n(\gamma) = p^{2m}$ it is obvious, otherwise, since B is definite, there are only finitely many elements of given reduced norm, hence, by finite enumeration, we can take one of these with reduced norm p (if any). In particular, this element of reduced norm p can be taken to lie in R^n. ■

Remark 2.3.11. *Let x and y be two elements in $\Gamma^1\backslash B_p^\times/\mathbb{Q}_p^\times R_p^{n\times}$ and suppose that they are expressed as in Lemma 2.3.4. If they are Γ-left-equivalent, then $n(x)/n(y) \in p^{1+2\mathbb{Z}}$.*

Proposition 2.3.12. *A set of representatives for $\Gamma\backslash Geod_n(\mathcal{T})$ can be extracted, by finite enumeration, from the a set of representatives for the quotient space $(R^n[1/p])^1\backslash Geod_n(\mathcal{T})$.*

The proof of the above proposition is the following algorithm.

Algorithm 2: Compute a coset decomposition for $\Gamma\backslash B_p^\times/\mathbb{Q}_p^\times R_p^{n\times}$.

Input: A coset decomposition provided by Algorithm 1 and the class number of R^n.

Output: A list of coset representatives for $\iota_p(\Gamma)\backslash GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p)$.

begin
- Initialize queue X with the coset representatives of $\Gamma\backslash B_p^\times/\mathbb{Q}_p^\times R_p^{n\times}$;
- Initialize empty queue Y;
- Set h as the class number of R^n;
- **while** *length*$(X) - h \neq 0$ **do**
  - **for** *each* $x \in X$ **do**
    - Compute $n(x)$;
    - **Push** x into Y;
    - **Pop** x from X;
    - **for** *each* $x' \in X$ **do**
      - Compute $n(x')$;
      - **if** $n(x)/n(x') = p^m \in p^{\mathbb{Z}}$ **then**
        - **if** $\delta_p^{-M} \cdot x'$ *is* Γ^1*-equivalent to* x **then**
          - **Pop** x' from X;
- **return** Y

We report here the following easy proposition.

Proposition 2.3.13. *The index $[\Gamma : \Gamma^1]$ is either 1 or 2. It holds $[\Gamma : \Gamma^1] = 2$ if and only if R^1 contains an element of reduced norm p.*

Proof. Recall that Γ and Γ^1 are respectively the images in $PGL_2(\mathbb{Q}_p)$ of $(R^1[1/p])^\times$ and $(R^1[1/p])^1$. In particular, we have the exact sequence

$$1 \longrightarrow (R^1[1/p])^1 \longrightarrow (R^1[1/p])^\times \xrightarrow{n} p^{\mathbb{Z}},$$

thus every element in Γ can be divided into two classes: the class of elements with reduced norm p^{2m} and the one of elements with reduced norms p^{2m+1}. Defining the character $ns : \Gamma \longrightarrow \{\pm 1\}$ as $ns([\gamma]) = (-1)^{n(\gamma)}$, we obtain the exact sequence

$$1 \longrightarrow \Gamma^1 \longrightarrow \Gamma \xrightarrow{ns} \{\pm 1\}.$$

Hence ns is surjective if and only if there exists an element of reduced norm p in $(R^1[1/p])^\times$ or, equivalently, in R^1. ∎

Remark 2.3.14 ([BD07], Section 3.1)**.** *Let K be a quadratic imaginary field and let $\mathcal{O}_K$ be its ring of integers. Assume that ℓ is inert in K and that all the primes dividing Np are split. This assumption guarantees the existence of an embedding of $\mathcal{O}_K$ into the Eichler order R^1. In particular, R^1 contains an element of norm p if $\mathcal{O}_K$ contains one of the ideals splitting (p).*

2.3.5. **A basis for the order.** In order to make the above algorithms effective we should provide a basis, or better said, provide an algorithmic procedure for computing a basis for the order R^n. Following the recipe given in Remark 1.1.2, we consider the level $N = M\ell^r$ with $(M, \ell) = 1$.

We begin with the case in which $M = 1$ and $r \in \{1, 2\}$. In this case, we have the following lemma.

Lemma 2.3.15 ([PRV05], Proposition 5 and [Piz80a], Proposition 5.2)**.** *Let B be as above and suppose that it is determined by $i^2 = a$ and $j^2 = b$; set $k = ij$. Then a maximal order R of B and a special suborder R_{ℓ^2} of level ℓ^2 are, respectively*

(1) for $a = -1$, $b = -\ell$ if $\ell \equiv 3 \pmod 4$,

$$R = \langle \frac{1}{2}(1+j), \frac{1}{2}(i+k), j, k\rangle_{\mathbb{Z}} \supset R_{\ell^2} = \langle \frac{1}{2}(1+j), \frac{1}{2}(\ell i+k), j, k\rangle_{\mathbb{Z}},$$

(2) for $a = -2$, $b = -\ell$ if $\ell \equiv 5 \pmod 8$,

$$R = \langle \frac{1}{2}(1+j+k), \frac{1}{4}(i+2j+k), j, k\rangle_{\mathbb{Z}} \supset R_{\ell^2} = \langle \frac{1}{3}(1+j+k), \frac{1}{4}(\ell i+2j+k), j, k\rangle_{\mathbb{Z}},$$

(3) for $a = -\ell$, $b = -q$ if $\ell \equiv 1 \pmod 8$ where q is a prime such that $\left(\frac{\ell}{q}\right) = -1$, $q \equiv 3 \pmod 4$ and s is an integer with $s^2 = -\ell \pmod q$ and $s \equiv -q \pmod{)}\ell$,

$$R = \langle \frac{1}{2}(1+j), \frac{1}{2}(i+k), k, \frac{1}{\ell q}(\ell j + (s+q)k)\rangle_{\mathbb{Z}}$$
$$\supset R_{\ell^2} = \langle \frac{1}{2}(1+\ell j), \frac{1}{2}(i+k), k, \frac{1}{\ell q}(\ell j + (s+q)k)\rangle_{\mathbb{Z}}.$$

We remark, following [KV14] (end of page 5) that we can always compute a $\mathbb{Z}$-basis for a given Eichler order in B. Up to conjugation, we can consider the Eichler order $R_N \subseteq R$

of level N to be locally $M_2^0(N\mathbb{Z}_q)$ as in Section 1.1.1. Factoring N into prime factors, one can compute in probabilistic polynomial time an embedding of $R \hookrightarrow M_2(\mathbb{Z}_p)$ for $p|N$, as explained in [Voi13]. Computed the embedding, one can realize the order $R_{p^{v_p(N)}}$ and produce a $\mathbb{Z}_p$-basis. Taking the intersection $R_N = \cap_{p|N} R_{p^{v_p(N)}}$ one computes (see *loc.cit.*) a $\mathbb{Z}$-basis for the Eichler order R_N. In a similar fashion it is possible to compute a $\mathbb{Z}$-basis for a special order $R_{N\ell^2}$ producing a basis of the intersection of a suitable Eichler order and an order as in the above Lemma 2.3.15.

2.4. **The triple product p-adic L-function.** The aim of this Section is to apply the algorithmic procedures of the previous Section 2.3 to the triple product balanced p-adic L-function presented in [Hsi21]; we keep the notation consistent with *loc.cit.* and denote it by $\mathcal{L}^{bal}_{F_\infty}$, for a suitable triple of Hida families F_∞. In particular, we realize the so-called *theta-element* in *loc.cit.* as a finite sum on the finite length geodesics on the Bruhat–Tits tree. This produces an effective method for the approximation of the limit of the triple product p-adic L-function at $(2, 1, 1)$.

We recall, as briefly as possible, the setting of [Hsi21]. Since we are mainly interested in considering weight-2 specializations of the formula, we recall everything that is needed for this purpose, leaving out great part of the general construction, and refer to the original paper for all the details.

2.4.1. **The setting and the hypotheses.** We take again p to be an odd prime which, for this section, is assumed to be greater equal than 5. We consider three p-adic Hida families, f_∞, g_∞ and h_∞, with tame level, respectively, N_1, N_2 and N_3, and tame character ψ_1, ψ_2 and ψ_3. We denote the Iwasawa algebra by $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, and define $\mathbf{I}_i$, for $i = 1, 2, 3$, to be the finite flat normal domain over Λ which defines the coefficients algebra of the respective family. As in Section 2.1, an arithmetic point in the weight space is induced by a morphism $Q \in \mathrm{Spec}(\mathbf{I}_i)(\overline{\mathbb{Q}_p})$ such that the restriction $Q_{|_{\mathbb{Z}_p^\times}} : \mathbb{Z}_p^\times \to \Lambda^\times \xrightarrow{Q} \overline{\mathbb{Q}_p}^\times$ is of the form $Q(x) = x^{k_Q} \cdot \varepsilon_Q(x)$ for $k_Q \in \mathbb{Z}_{\geq 2}$ and $\varepsilon_Q : \mathbb{Z}_p^\times \longrightarrow \overline{\mathbb{Q}_p}^\times$ a finite order character. We denote the exponent of its conductor by $c\,(\varepsilon_Q)$. For $N = \mathrm{lcm}\{N_1, N_2, N_3\}$ and for any $\underline{Q} = (Q_1, Q_2, Q_3)$ triple of arithmetic points, we denote

$$\Sigma^- = \left\{ q \text{ prime dividing } N \mid \varepsilon_q \left(\Pi_{\underline{Q}} \otimes \chi_{\underline{Q}}^{-1} \right) = -1 \right\}$$

where ε_q is the local epsilon factor at q. Here, $\Pi_{\underline{Q}} = \pi_{f_{Q_1}} \otimes \pi_{g_{Q_2}} \otimes \pi_{h_{Q_3}}$ is the tensor product of the three automorphic representations associated with the specializations f_{Q_1}, g_{Q_2} and h_{Q_3} and $\chi_{\underline{Q}} = \prod_{i=1}^3 \psi_i \varepsilon_{Q_i}$. We remark that Σ^- does not depend on the specific triple of arithmetic points.

We recall briefly all the hypotheses that the Hida families must satisfy:

(sf) $\gcd(N_1, N_2, N_3)$ is square-free;

(ev) $\psi_1\psi_2\psi_3 = \omega^{2a}$ for $a \in \mathbb{Z}$ and ω the Teichmüller character;

(CR,Σ^-) The residual Galois representation $\overline{\rho}_{f_\infty} : \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_p)$ (and the same holds for g_∞ and h_∞) is such that

(i) $\overline{\rho}_{f_\infty}$ is absolutely irreducible;
(ii) $\overline{\rho}_{f_\infty}$ is p-distinguished;
(iii) if $\ell \in \Sigma^-$ and $\ell \equiv 1 \pmod p$, then $\overline{\rho}_{f_\infty}$ is ramified at ℓ.

(odd) $\#\Sigma^-$ is odd.

One crucial hypothesis (*cf.* Section 3) is the hypothesis of *tame ramification* at the primes of Σ^-, that is

(TR) for $d = \prod_{q\in\Sigma^-} q$ and $N = \mathrm{lcm}(N_1, N_2, N_3)$, it must hold that $(d, N/d) = 1$.

In order to explicitly produce test vectors, in [Hsi21] it is required to introduce a permutation of the triple of families. We report the two equivalent (see Proposition 3.2.1) conditions:

(P): There exists a triple of classical points $(\kappa_1, \kappa_2, \kappa_3)$ such that for each prime $q|N$, there exists a rearrangement $\{f_1, f_2, f_3\}$ of $\{f_{\kappa_1}, g_{\kappa_2}, h_{\kappa_3}\}$ such that
(i) the conductors at q satisfy $c_q(\pi_{f_1}) \leq \min\{c_q(\pi_{f_2}), c_q(\pi_{f_3})\}$;
(ii) the local components $\pi_{f_1,q}$ and $\pi_{f_3,q}$ are minimal;
(iii) either $\pi_{f_3,q}$ is a principal series or $\pi_{f_2,q}$ and $\pi_{f_3,q}$ are both discrete series.

(P∞): For each triple of classical points $(\kappa_1, \kappa_2, \kappa_3)$ and for each prime $q|N$, there exists a rearrangement $\{f_1, f_2, f_3\}$ of $\{f_{\kappa_1}, g_{\kappa_2}, h_{\kappa_3}\}$ such that
(i) $c_q(\pi_{f_1}) \leq \min\{c_q(\pi_{f_2}), c_q(\pi_{f_3})\}$;
(ii) the local components $\pi_{f_1,q}$ and $\pi_{f_3,q}$ are minimal;
(iii) either $\pi_{f_3,q}$ is a principal series or $\pi_{f_2,q}$ and $\pi_{f_3,q}$ are both discrete series.

Remark 2.4.1. *As stated in* [Hsi21], *Remark 6.2, there always exist χ_1, χ_2 and χ_3 Dirichlet characters modulo M, with $M^2|N$ such that $\chi_1\chi_2\chi_3 = 1$ and $(\pi_{f_{\kappa_1}} \otimes \chi_1, \pi_{g_{\kappa_2}} \otimes \chi_2, \pi_{h_{\kappa_3}} \otimes \chi_3)$ satisfies (P). It is not hard to note that this condition fails if we relax condition (TR) and allow powers strictly higher than 2 at the primes of Σ^-.*

2.4.2. **Test vectors and the theta-elements.** Let B, R and R^n be as in Section 2.3.1 but we assume from now on that R^n is an Eichler order. Moreover, we fix (as in Section 1.1) a choice of field embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$ and a compatible isomorphism $i : \mathbb{C}_p \cong \mathbb{C}$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{A}^{(\ell)}_{\mathbb{Q},f})$ act on $x \in \widehat{B}^\times$ by

$$x \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := x \cdot \iota^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right),$$

for ι the induced isomorphism $GL_2(\mathbb{A}^{(\ell)}_{\mathbb{Q},f}) \cong (B \otimes \mathbb{A}^{(\ell)}_{\mathbb{Q},f})^\times$. Even though the formula presented in [Hsi21] can deal with generic arithmetic specializations, we restrict our analysis to weight-2 specializations. This situation grants easier expressions, without invalidating the study of the limit point $(2, 1, 1)$.

Assumption 2.4.2 (Choice of Hida families)**.** *We consider the triple of families $F_\infty = (g_\infty, f_\infty, h_\infty)$, where f_∞ is passing through a modular elliptic curve. We aim to let the family f_∞ to be as free as possible, thus the families g_∞ and h_∞ should be interpreted as auxiliary parameters. More precisely we take:*

(a) f_∞ *is the unique Hida family associated with* $f \in S_2^{new}(\Gamma_0(N_1\ell p), \mathbb{Q})$*, a twist-minimal primitive newform corresponding to an elliptic curve* $E/\mathbb{Q}$*, which is ordinary at* p*. In particular, the family has tame level* $N_1\ell$ *with trivial tame character;*

(b) g_∞ *and* h_∞ *are primitive* p*-adic Hida families of tame level* $N_2\ell$ *with tame character* ψ *and* ψ^{-1} *respectively. We moreover suppose that* ψ *and* ψ^{-1} *are both primitive of conductor* N_2*.*

In the end, we assume that N_2 *is square-free,* $N_2|N_1$ *and that all the three families satisfy condition* (CR,Σ^-).

Remark 2.4.3. *(a) The permutation chosen in the definition of* F_∞ *guarantees that the triple satisfies* (P)*, without any auxiliary twist.*

(b) One can also consider ψ *of smaller conductor* $N_2' \mid N_2$ *and impose the condition on* f *to have supercuspidal type at the primes dividing* N_2'*. This ensures that the set of ramified primes contains just* ℓ *(see Proposition 3.1.2).*

Let $F_\infty^B = (g_\infty^B, f_\infty^B, h_\infty^B)$ be the triple of *primitive Jacquet–Langlands lifts* as determined in Theorem 4.5 of [Hsi21].

Notation. *We fix the orders* R^n *and* R_2^n *to be nested Eichler orders of level, respectively,* $N_1\ell p^n$ *and* $N_2\ell p^n$ *such that* $R \supset R_2^n \supset R^n$ *for each* $n \geq 0$*.*

We fix a sequence of points $(2, \varepsilon)$ on the weight space, as in Section 2.1, approaching a weight-1 point with trivial character; we assume that ε is primitive of conductor p^n. Considering the notation in Section 1.1.3, at each point of the form $((2,\varepsilon), 2, (2,\varepsilon))$ we have the quaternionic test vectors

$$\left(i(g^B_{(2,\varepsilon)}) \otimes \varepsilon_{\mathbb{A}}^{-1}, i(f_2^B), i(h^B_{(2,\varepsilon)})\right) \in S_2(R^n, \widetilde{\psi\varepsilon^{-1}}) \times S_2(R_f) \times S_2(R^n, \widetilde{\psi^{-1}\varepsilon}),$$

where $\varepsilon_{\mathbb{A}}^{-1}$ is the adèlization of the p-adic character ε^{-1}, which is unramified outside p. We set ${g^B_{(2,\varepsilon)}}' = g^B_{(2,\varepsilon)} \otimes \varepsilon_{\mathbb{A}}^{-1}$ and $F_\infty^{B\prime} = ({g_\infty^B}' := g_\infty^B \otimes \varepsilon_{\mathbb{A}}^{-1}, f_\infty^B, h_\infty^B)$. Following Section 4.6 of [Hsi21], we consider the theta-element $\Theta_{F_\infty^{B\prime}}$ associated with the triple $F_\infty^{B\prime}$.

Remark 2.4.4. *(a) The specialization point* $Q = (2, \varepsilon)$ *comes from an element in* $Spec(\mathbf{I}_i)(\overline{\mathbb{Q}_p})$ *and hence,* a priori*, we should take into account the value of* $\varepsilon_{\mathbb{A}}(p) = Q(p)$*, although we can fix it to be* 1*.*

(b) The theta-element $\Theta_{F_\infty^{B\prime}}$ *is the central component of the* p*-adic triple product* L*-function* $\mathcal{L}^{bal}_{F_\infty}$ *constructed in Section 4 of* [Hsi21]*. In our situation, the difference between these two objects is give by two factors: the constant* $2^{-5/2}/\sqrt{N}$ *and an element in* $Frac\left(\widehat{\bigotimes}_{i=1,2,3}\mathbf{I}_i\right)^\times$ *(in the notation of* loc.cit. *a square root of the fudge factor). In particular, the zeros of* $\mathcal{L}^{bal}_{F_\infty}$ *are encoded by the behavior of* $\Theta_{F_\infty^{B\prime}}$*.*

By some elementary but rather tedious observations, it is not difficult to note that, up to considering the finite sum rather than the integral, Proposition 4.9 in [Hsi21] reads as

$$\Theta_{F^B_\infty{}'}(((2,\varepsilon),2,(2,\varepsilon))) = \frac{\varepsilon_{\mathbb{A}}^{-1}(p^n)\cdot p^{2n}(1-\frac{1}{p})}{a_p(f)^n a_p\left({g^B_{(2,\varepsilon)}}'\right)^n a_p\left(h^B_{(2,\varepsilon)}\right)^n}\cdot$$
$$\cdot \sum_{x\in B^\times\backslash\widehat{B}^\times/\widehat{R^n}^\times} \frac{1}{\#\Gamma_x} i\left({g^B_{(2,\varepsilon)}}'\left(x\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right)\right)\right)\cdot i\left(f^B(x)\right) i\left(h^B_{(2,\varepsilon)}\left(x\left(\begin{smallmatrix}0 & p^{-n}\\ -p^n & 0\end{smallmatrix}\right)\right)\right).$$

In the above formula we recall that $\Gamma_x = \left(B^\times \cap x\widehat{R^n}^\times x^{-1}\right)\mathbb{Q}^\times/\mathbb{Q}^\times$.

We can move the quaternionic modular forms to their exact level considering the two surjective reduction maps

$$\phi_n : B^\times\backslash\widehat{B}^\times/\widehat{R^n}^\times \longrightarrow B^\times\backslash\widehat{B}^\times/\widehat{R^n_2}^\times \qquad \text{and} \qquad \phi^1_n : B^\times\backslash\widehat{B}^\times/\widehat{R^n}^\times \longrightarrow B^\times\backslash\widehat{B}^\times/\widehat{R^1}^\times.$$

Lemma 2.3.1 allows us to identify the triple of functions

$$\left(i\left({g^B_{(2,\varepsilon)}}'(\phi_n(-))\right), i\left(f^B\left(\phi^1_n(-)\right)\right), i\left(h^B_{(2,\varepsilon)}(\phi_n(-))\right)\right)$$

with a triple of functions on $GL_2(\mathbb{Q}_p)$ which we denote by $(G^n(-), F^1(-), H^n(-))$.

Proposition 2.4.5. *With the notation of Section 2.2.1, let $\underline{Q}$ be the triple of points $((2,\varepsilon),2,(2,\varepsilon))$. It holds that*

$$\Theta_{F^B_\infty{}'}(\underline{Q}) = \frac{(1-p^{-1})}{a_p(f)^n}\cdot \sum_{e\in(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})} \frac{F^1(e)}{\#Stab_{(R^1[1/p])^\times}(e)}\cdot$$
$$\cdot\left(\frac{\varepsilon_{\mathbb{A}}^{-1}(p^n)\cdot p^{2n}}{a_p\left({g^B_{(2,\varepsilon)}}'\right)^n a_p\left(h^B_{(2,\varepsilon)}\right)^n}\right)\sum_{g\in Geod_n(\mathcal{T})(e)} G^n(g\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right))\cdot H^n(g\left(\begin{smallmatrix}0 & p^{-n}\\ -p^n & 0\end{smallmatrix}\right)).$$

Proof. We begin noting that $\Gamma_x = (B^\times \cap x\widehat{R^n}^\times x^{-1})\mathbb{Q}^\times/\mathbb{Q}^\times$ corresponds, by Lemma 2.3.1, to

$$\Gamma_{p,x_p} := \left((R^n[1/p])^\times \cap x_p\Gamma_0(p^n\mathbb{Z}_p)x_p^{-1}\right)\mathbb{Q}^\times/\mathbb{Q}^\times = \left((R^n[1/p])^\times \cap x_p\Gamma_0(p^n\mathbb{Z}_p)x_p^{-1}\right)\mathbb{Q}_p^\times/\mathbb{Q}_p^\times.$$

The theta-element becomes

$$\Theta_{F^B_\infty{}'}(\underline{Q}) = \frac{\varepsilon^{-1}(p^n)\cdot p^{2n}(1-p^{-1})}{a_p(f)^n a_p\left({g^B_{(2,\varepsilon)}}'\right)^n a_p\left(h^B_{(2,\varepsilon)}\right)^n}\cdot$$
$$\cdot\sum_{x\in\iota_p\left(R^n[1/p]^\times\right)\backslash GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times\Gamma_0(p^n\mathbb{Z}_p)} \frac{1}{\#\Gamma_{p,x_p}} F^1(x_p)\cdot G^n(x_p\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right))\cdot H^n(x_p\left(\begin{smallmatrix}0 & p^{-n}\\ -p^n & 0\end{smallmatrix}\right)),$$

and thus, under the identification with geodesics on the Bruhat–Tits tree,

$$\Theta_{F_\infty^B{}'}(\underline{Q}) = \frac{\varepsilon^{-1}(p^n)\cdot p^{2n}(1-p^{-1})}{a_p(f)^n a_p\left({g_{(2,\varepsilon)}^B}'\right)^n a_p\left(h_{(2,\varepsilon)}^B\right)^n}\cdot$$
$$\cdot \sum_{g\in\iota_p\left(R^n[1/p]^\times\right)\backslash Geod_n(\mathcal{T})} \frac{1}{\#Stab_{(R^n[1/p])^\times}(g)} F^1(g)\cdot G^n(g\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right))\cdot H^n(g\left(\begin{smallmatrix}0 & p^{-n}\\ -p^n & 0\end{smallmatrix}\right)),$$

where $Stab_{(R^n[1/p])^\times}(g)$ corresponds to Γ_{p,x_p}. In the above equation we have identified the triple $(F^1(-), G^n(-), H^n(-))$ as a triple of functions on the geodesic of the Bruhat–Tits tree as in Section 2.3.1. In order to ease the notation, we drop again the isomorphism ι_p and write $(R^n[1/p])^\times$ for the corresponding subgroup in $GL_2(\mathbb{Q}_p)$. Since the function F^1 is naturally a function on the edges of the Bruhat–Tits tree (it can indeed be identified with the harmonic cocycle associated with f) we can proceed similarly to equation (2.2) of [Rho01] and applying Lemma 2.3.2 obtain

$$\Theta_{F_\infty^B{}'}(\underline{Q}) = \frac{\varepsilon^{-1}(p^n)\cdot p^{2n}(1-p^{-1})}{a_p(f)^n a_p\left({g_{(2,\varepsilon)}^B}'\right)^n a_p\left(h_{(2,\varepsilon)}^B\right)^n}\cdot$$
$$\cdot \sum_{e\in(R^n[1/p])^\times\backslash\mathcal{E}(\mathcal{T})} \frac{F^1(e)}{\#Stab_{(R^n[1/p])^\times}(e)} \sum_{g\in Geod_n(\mathcal{T})(e)} G^n(g\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right))\cdot H^n(g\left(\begin{smallmatrix}0 & p^{-n}\\ -p^n & 0\end{smallmatrix}\right)).$$

■

Collecting all the results of this Section, we obtain the following theorem.

Theorem 2.4.6. *The value at $(2,1,1)$ of the balanced p-adic L-function associated with the triple of families F_∞ is*

$$\mathcal{L}_{F_\infty}^{bal}(2,1,1) \doteq \lim_{\varepsilon\to 1}\Theta_{F_\infty^B{}'}((2,\varepsilon),2,(2,\varepsilon))$$

up to an element in $Frac\left(\widehat{\bigotimes}_{i=1,2,3}\mathbf{I}_i\right)^\times$. In particular, the limit $\lim_{\varepsilon\to 1}\Theta_{F_\infty^B{}'}((2,\varepsilon),2,(2,\varepsilon))$ can be algorithmically approximated with a given p-adic precision, i.e. *for any ε of finite conductor we can compute $\Theta_{F_\infty^B{}'}((2,\varepsilon),2,(2,\varepsilon))$.*

Remark 2.4.7. *We can give a precise estimation of the convergence rate for the limit in Theorem 2.4.6, $\lim_{\varepsilon\to 1}\mathcal{L}_{F_\infty^B}^{bal}((2,\varepsilon),2,(2,\varepsilon))$. This comes from a straightforward application of one implication of Proposition 4.0.6 (*abstract Kummer congruences*) in* [Kat78]. *We notice that one must compute explicitly the* fudge factor *of Remark 2.4.4.*

2.4.3. **A note about effective computability.** In order to hope for effective computations of the limit value $\lim\Theta_{F_\infty^B{}'}(\underline{Q})$, we would need several steps.

Step 0: Extend the code provided by C. Franc and M. Masdeu (a task at which we plan to get back soon).

The algorithmic routine is then as follows.

Step 1: Compute some basis of the needed quaternionic orders, as in Section 2.3.5.

Step 2: Run the code in **Step 0**.

Step 3: Fix suitable Jacquet–Langlands lifts of the Hida families and compute the values of the quaternionic specializations considering the diagonalization of the relevant *Brandt matrices* (a finite amount of them are enough); we recall them briefly in Appendix B, considering special orders.

Step 4: Compute the action given by the matrices $\left(\begin{smallmatrix} 1 & p^{-n} \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & p^{-n} \\ -p^n & 0 \end{smallmatrix}\right)$ on the set of representatives.

Step 5: Match the values obtained in **Step 3** with the coset decomposition in **Step 2**, hence obtain the values corresponding with the shifts in **Step 4**.

Step 6: Compute the stabilizers, eigenvalues and then the whole theta-element.

We remark that the procedures described in Sections 2.3.3 and 2.3.4 are quite expensive, both in terms of time and resources, as the number of geodesics grows exponentially in p.

2.4.4. **Integrals on the boundary of the Bruhat–Tits tree.** As we have seen above, in the limit process we are moving through the Bruhat–Tits tree towards infinity. In particular, adjoining $Ends(\mathcal{T})$ to the tree yields exactly its Borel–Serre compactification; we refer to the Appendix in [SS97], page 166, for further details. We would like to express the limit as a limit of a sequence of locally constant functions on the boundary of the tree, integrated over it with respect to the measure associated with f^B.

2.4.4.1. **The p-adic measure associated with f^B.** Assumption 2.4.2 implies that the p-th Fourier coefficient of f is ± 1, hence we can relate the Atkin–Lehner operator W_p (given by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$) with U_p via the relation $U_p f^B = W_p f^B = a_p \cdot f^B$ (see *e.g.* Theorem 3 in [AL70]). Following Section 4 of [Gre06], we set

$$\mathrm{sign}_p(\gamma) := a_p^{v_p(\det(\gamma))}$$

for any $\gamma \in GL_2(\mathbb{Q}_p)$. By definition $\mathrm{sign}_p(\Gamma_0(p\mathbb{Z}_p)) = 1$ as well as $\mathrm{sign}_p(\mathbb{Q}_p^\times) = 1$, where the fact that a_p is ± 1 is used only for the elements of the form $p^m \in \mathbb{Q}_p^\times$. Then sign_p extends to a function of the edges of the Bruhat–Tits tree which factors through

$$\mathrm{sign}_p : \Gamma^1 \backslash \mathcal{E}(\mathcal{T}) \longrightarrow \{\pm 1\}.$$

Setting (with the notation of Section 2.2.2)

$$\mu_{f^B}(U(e)) := \mathrm{sign}_p(e) \cdot f^B(e)$$

defines a p-adic measure on $\mathbb{P}^1(\mathbb{Q}_p)$, left-invariant for Γ^1, with total measure zero.

Remark 2.4.8. *The function* $\mathrm{sign}_p(e) \cdot f^B(e)$ *of the edges defines a harmonic cocycle (up to scalar) on the Bruhat–Tits tree, the unique (up to scalar)* Γ^1*-invariant one corresponding to the newform* f*. We refer to* [DT08] *for more details.*

Assumption 2.4.9. *We assume that μ_{f^B} defines a left-Γ-invariant measure. For example, we can assume that $a_p(f)=1$ or that the index $[\Gamma:\Gamma^1]=1$.*

For any edge e and any specialization $(2,\varepsilon)$ with conductor p^n, we define the locally constant function $\eta^{\varepsilon}_{g_\infty,h_\infty}(e):U(e)\longrightarrow\mathbb{C}_p$ by setting

$$\eta^{\varepsilon}_{g_\infty,h_\infty}(e)(x)=\sum_{g\in Geod_n(\mathcal{T})(e)}\left(G^n(g\left(\begin{smallmatrix}1 & p^{-n}\\ 0 & 1\end{smallmatrix}\right))\cdot H^n(g\left(\begin{smallmatrix}0 & p^{-n}\\ -p^n & 0\end{smallmatrix}\right))\right)\cdot\mathbf{1}_{U(g)}(x),$$

where $\mathbf{1}_{U(g)}(x)$ is the characteristic function for $U(g)$. The function $\eta^{\varepsilon}_{g_\infty,h_\infty}(e)(-)$ is well-defined because $U(e)=\coprod_{g\in Geod_n(\mathcal{T})(e)}U(g)$ and its extension to 0 determines an element in the space of continuous functions $\mathscr{C}(\mathbb{P}^1(\mathbb{Q}_p),\overline{\mathbb{Q}_p})$.

In order to shorten the notation we set $\xi(\varepsilon):=\varepsilon^{-1}(p^n)\cdot p^{2n}a_p({g^B_{(2,\varepsilon)}}')^{-n}a_p(h^B_{(2,\varepsilon)})^{-n}$; we can now obtain a more aesthetically pleasing expression of the theta-element.

Corollary 2.4.10 (to Proposition 2.4.5)**.** *Let, as above, $\underline{Q}=((2,\varepsilon),2,(2,\varepsilon))$, then*

$$\Theta_{F^{B\prime}_\infty}(\underline{Q})=(1-p^{-1})\sum_{e\in(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})}\frac{\mathrm{sign}_p(e)}{\#Stab_{(R^1[1/p])^\times}(e)}\int_{\mathbb{P}^1(\mathbb{Q}_p)}\frac{\xi(\varepsilon)\eta^{\varepsilon}_{g,h}(e)(x)}{a_p(f)^n}\mu_{f^B}(x)$$
$$=(1-p^{-1})\int_{\mathbb{P}^1(\mathbb{Q}_p)}\left(\sum_{e\in(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})}\frac{\mathrm{sign}_p(e)}{\#Stab_{(R^1[1/p])^\times}(e)}\frac{\xi(\varepsilon)\eta^{\varepsilon}_{g,h}(e)(x)}{a_p(f)^n}\right)\mu_{f^B}(x).$$

2.4.4.2. **Some speculations on the limit.** One could immediately note that the limit process at $(2,1,1)$ and the integration over $\mathbb{P}^1(\mathbb{Q}_p)$ can be exchanged, but in general that does not hold for the finite sum over $(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})$. Looking at some low-level cases for the Hamilton's quaternions (see Section 6.5 and the following exercises in [Dar04]), it seems reasonable to ask the following question.

Question 4. *Let h be the class number of R^n. Does there exist a set of representatives for $(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})$, say $\{e_i\}_{i=1}^h$, such that $U(e_i)\cap U(e_j)\neq\emptyset$ if and only if $i=j$.*

If this question had a positive answer, we would obtain the point-wise limit

$$\lim_{\varepsilon\to1}\left(\sum_{e\in(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})}\frac{\mathrm{sign}_p(e)}{\#Stab_{(R^1[1/p])^\times}(e)}\frac{(1-p^{-1})}{a_p(f)^n}\xi(\varepsilon)\eta^{\varepsilon}_{g,h}(e)(x)\right)=$$
$$=\sum_{e\in(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})}\frac{\mathrm{sign}_p(e)}{\#Stab_{(R^1[1/p])^\times}(e)}\lim_{\varepsilon\to1}\frac{(1-p^{-1})}{a_p(f)^n}\xi(\varepsilon)\eta^{\varepsilon}_{g,h}(e)(x),$$

leading to the next question.

Question 5. *Does the sequence of functions $(1-p^{-1})a_p(f)^{-n}\xi(\varepsilon)\eta^{\varepsilon}_{g,h}(e)(x)$ converge (uniformly) to an element $\mathcal{N}_{g,h}(e)(-)=\lim_{\varepsilon\to1}\frac{1}{a_p(f)^n}\xi(\varepsilon)\eta^{\varepsilon}_{g,h}(e)(-)\in\mathscr{C}(\mathbb{P}^1(\mathbb{Q}_p),\mathbb{C}_p)$?*

If so, we would have the equality

$$\lim_{\varepsilon\to1}\Theta_{F^{B\prime}_\infty}(\underline{Q})=(1-p^{-1})\int_{\mathbb{P}^1(\mathbb{Q}_p)}\sum_{e\in(R^1[1/p])^\times\backslash\mathcal{E}(\mathcal{T})}\frac{\mathrm{sign}_p(e)}{\#Stab_{(R^1[1/p])^\times}(e)}\mathcal{N}_{g,h}(e)(x)\mu_{f^B}(x).$$

3. Weight-1 modular forms and the balanced p-adic triple product L-function

The aim of this conclusive section is to show the relation between the first two sections of this thesis and specifically motivate the study carried out in Section 1. We note how the balanced p-adic triple product L-function $\mathcal{L}^{bal}_{F_\infty}$, either the one constructed in [Hsi21] or in [GS19], is not suitable for the study of the limit point $(2,1,1)$ with classical weight-1 specializations. For this purpose, we analyze more carefully the local signs determining the sign of the functional equation and the automorphic type of the specializations of a Hida family.

3.1. **The local signs.** We begin by recalling that, in the balanced case, the local sign at infinity is -1 and the global sign is $+1$, forcing an odd number of local signs (at finite places) to be -1. In particular, the quaternion algebra B is ramified exactly at the places where the local sign is -1. The local signs for the triple product L-function are completely determined by the local automorphic types of the three representations. The only two situations which yield a local sign equal to -1 are collected in the following two propositions. Let ℓ be a prime.

Proposition 3.1.1 ([Pra90], Proposition 8.5)**.** *Let π_1 and π_2 be automorphic representations of $GL_2(\mathbb{Q}_\ell)$ such that $\pi_1\otimes\pi_2$ is self-dual. Then, taking π_3 a special representation of $GL_2(\mathbb{Q}_\ell)$ with trivial central character (*i.e. *the Steinberg representation),*

$$\varepsilon\left(\pi_1\otimes\pi_2\otimes\pi_3\right)=-1$$

if and only if π_1 and π_2 are supercuspidal with $\pi_1\cong\pi_2^\vee$, where $\pi_2^\vee$ denotes the contragredient of π_2.

Proposition 3.1.2 ([Pra90], Proposition 8.6)**.** *Let π_1, π_2 and π_3 be automorphic representations of $GL_2(\mathbb{Q}_\ell)$ such that the product of their central characters is trivial. If π_1 and π_2 are special representations, then*

$$\varepsilon\left(\pi_1\otimes\pi_2\otimes\pi_3\right)=-1$$

if and only if π_3 is special.

In particular, in Section 2.4 we are in the setting of Proposition 3.1.2.

3.2. **The local automorphic type.** We will observe how one can determine the local components of the automorphic representation associated with a modular form, but before that we must recall the following fundamental proposition.

Proposition 3.2.1 (Rigidity of automorphic type; [Hsi21], Remark 3.1)**.** *Let f_∞ be a primitive p-adic Hida family. For each prime $\ell\neq p$, all the arithmetic specializations of f_∞ share the same automorphic type at ℓ with the same local conductor.*

Definition 3.2.2. *Let π be a irreducible admissible representation of $GL_2(\mathbb{Q}_\ell)$. We say that π is minimal, if its conductor is minimal among all its twist,* id est, *$c(\pi) \leq c(\pi \otimes \chi)$ for each $\chi : \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times$.*

Let $f \in S_k(\Gamma_1(N\ell^r), \mathbb{C})$ be a cuspidal newform, ℓ not dividing N and $r \geq 1$. We say that f is minimal at ℓ if it is not a twist of another form of level $N' < N$ by a Dirichlet character of conductor equal to some power of ℓ.

Remark 3.2.3. *(a) Let $\pi_{f,\ell}$ be the local automorphic component associated with f at the prime ℓ. Clearly f is ℓ-minimal if and only if $\pi_{f,\ell}$ is minimal.*

(b) By definition, if the ℓ-th coefficient $a_\ell(f)$ is not zero, then f is minimal at ℓ.

In [AL78] the conditions for having a minimal modular form have been extensively studied.

Proposition 3.2.4 ([AL78], Corollary 4.4 & Theorem 4.4)**.** *Let $f \in S_k(\Gamma_1(N\ell^r), \varepsilon, \mathbb{C})$ be a cuspidal normalized newform with $r \geq 1$ and $(\ell, N) = 1$. Consider the decomposition $\varepsilon = \varepsilon_N \varepsilon_\ell$ with ε_N of modulus N and ε_ℓ of modulus ℓ^r.*

1) *If f is ℓ-primitive and $a_\ell(f) = 0$, then $cond(\varepsilon_\ell) \leq \sqrt{\ell^r}$;*
2) *If r is not even and $cond(\varepsilon_\ell) \leq \sqrt{\ell^r}$, then f is ℓ-primitive;*
3) *If r is even, $\ell = 2$ and $cond(\varepsilon_\ell) = \sqrt{\ell^r}$, then f is 2-primitive;*
4) *If $\ell = 2$, $r = 2$ and ε_2 is trivial, then f is 2-primitive and $a_2(f) = -\varepsilon_N^{-1}(2)$.*

Proposition 3.2.5 ([LW11], Proposition 2.8)**.** *Let $f \in S_k(\Gamma_1(N\ell^r), \varepsilon, \mathbb{C})$ be as in the above proposition and let $\pi_{f,\ell}$ be its automorphic representation at ℓ. Denote by $\varepsilon_{\ell,\mathbb{A}}$ the Hecke character associated with ε_ℓ (in particular $\varepsilon_{\ell,\mathbb{A}}|_{\mathbb{Z}_\ell^\times} = \varepsilon_\ell^{-1}$). Then:*

1) *If $r \geq 1$ and the conductor of $\varepsilon_{\ell,\mathbb{A}}$ is ℓ^r, then $\pi_{f,\ell} \cong \pi(\chi_1, \chi_2)$ is a principal series where χ_1 is unramified, $\chi_1(\ell) = a_\ell(f)/\ell^{\frac{(k-1)}{2}}$ and χ_2 is determined by $\chi_1\chi_2 = \varepsilon_{\ell,\mathbb{A}}$ (*i.e.* $\chi_2|_{\mathbb{Z}_\ell^\times} = \varepsilon_\ell^{-1}$ and $\chi_2(\ell) = \varepsilon_N(\ell)/\chi_1(\ell)$).*
2) *If $r = 1$ and $\varepsilon_{\ell,\mathbb{A}}$ is unramified, then $\pi_{f,\ell} \cong St \otimes \chi$ is special with χ unramified and $\chi(\ell) = a_\ell(f)/\ell^{\frac{(k-2)}{2}}$.*
3) *If neither of the above conditions hold, then $\pi_{f,\ell}$ is supercuspidal and the exponent of the conductor of $\varepsilon_{\ell,\mathbb{A}}$ is at most $\lfloor \frac{r}{2} \rfloor$.*

3.3. **Choice of families.** We fix the triple of families as in Assumption 2.4.2, that is

(1) f_∞ is the Hida family associated with the p-ordinary primitive newform $f \in S_2(\Gamma_0(N_1), \mathbb{Q})$ corresponding to an elliptic curve $E/\mathbb{Q}$;

(2) g_∞ and h_∞ are primitive Hida families with tame level N_2 and tame character ψ and ψ^{-1} respectively.

Remark 3.3.1. *(a) Since we require that there exists an odd number of primes at which the local sing is -1, it must happen that $N = gcd(N_1, N_2) \neq 1$.*

(b) By Propositions 3.2.5 and 3.2.1 we know that, for any prime q dividing N, f_∞ has automorphic type Steinberg (special with trivial central character) exactly when

$q||N_1$ and supercuspidal otherwise. For g_∞ and h_∞ we have all the possibilities, i.e. *we can have principal series type and this happens exactly when $cond_q(\psi) = q^{v_q(N_2)}$. Thus we must have an odd number of $q|N$ such that g_∞ and h_∞ have either special or supercuspidal type.*

3.4. **The hypothesis in** [Hsi21] **and** [GS19]**.** Let D be the square-free product of primes dividing $N = gcd(N_1, N_2)$ such that $q|D$ if and only if the local sign at q is -1. In [Hsi21] such D is denoted by N^- (Theorem B in *loc.cit.*) and in [GS19] by D_{JL}^- (Section 8 in *loc.cit.*). Both papers consider the following hypothesis.

Assumption 3.4.1. *For $M = \mathrm{lcm}(N_1, N_2)$, assume* $\gcd(D, M/D) = 1$.

This restriction is required in Theorem B of [Hsi21] and in Section 8 of [GS19] (D_{JL} is square-free) and has its origin in [Che05]; Chenevier establishes a 1-to-1 correspondence between p-adic modular forms and quaternionic ones extending the classical Jacquet–Langlands correspondence. The main reason is to avoid rank-2 phenomenons like the one presented in Section 1.

3.5. **Weight-1 specializations of Hida families.** The presence of classical modular forms of weight 1 (and not only p-adic) is determined, once again, by the local behavior of the automorphic representations.

Lemma 3.5.1 ([DG12], Lemma 4.4, or [Dim14], Proposition 1.8)**.** *Let f_∞ be a Hida family of tame level N and tame character ψ. If f_∞ is of special type at some ℓ dividing N (as above, if and only if $\ell||N$ and ψ is trivial at ℓ), then f_∞ has no classical weight-1 specializations.*

We deduce that our chosen g_∞ and h_∞ cannot be of special type if we want to take Hida families passing through classical weight-1 modular forms.

In [DG12] the authors give precise estimates for the number of classical weight-1 specializations in a Hida family. We refer to their results, mainly Theorem 5.1, Proposition 5.2 and Theorem 6.4, which cover the non-CM case.

3.6. **Conclusions.** The above Lemma 3.5.1 tells us that, whenever we consider the two families g_∞ and h_∞ to be associated with classical weight-1 modular forms, we must have at least a prime $q|D$ such that $q^2|M$, as the automorphic type can be either principal series or supercuspidal. In other words, the computation of the local signs must fall in the case covered by Proposition 3.1.1. On the other hand, Assumption 3.4.1 implies that the only situations in which the local sign can be -1 are those that fall under Proposition 3.1.2, *i.e.* when the exponents of the local conductors of all three representations is 1.

Remark 3.6.1. *We deduce that we cannot investigate points* $(2, 1, 1)$ *associated with classical weight-1 modular forms if Assumption 3.4.1 is in place. However, we can always study the behavior at non-classical weight-1 modular forms.*

Appendix A. Some remarks on the hypotheses of [Hsi21]

A.1. **p-distinguished modular forms.** We want to understand whether our Hida families are p-distinguished. Set $\mathbf{I} = \mathbf{I}_1$. We recall that $\overline{\rho}_{f_\infty}$ is p-distinguished if the restriction of the semi-simplification of $\overline{\rho}_{f_\infty} \pmod{m_\mathbf{I}}$ to a decomposition group at p is a sum of two characters, which are not congruent modulo the maximal ideal $m_\mathbf{I}$ of $\mathbf{I}$. However, it is once again enough to check the analogous condition on a triple (f, g, h) of classical specializations.

A.1.1. **Weight bigger equal than 2.** Let $f = \sum a_n q^n \in S_k(\Gamma_1(N), \psi, \mathbb{C})$ be a normalized cuspidal modular eigenform of weight $k \geq 2$. Take p to be a prime which does not divide the level of f and let $\overline{\rho}_{f,p}$ be the residual Galois representation associated with f at p by the works of Deligne and Serre. We denote by $\lambda(a)$ the unramified character of the absolute Galois group of $\mathbb{Q}_p$ $\mathsf{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ mapping the Frobenius element to a. Let $\varepsilon_{cyc} : \mathsf{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow \mathbb{Z}_p^\times$ be the p-adic cyclotomic character.

Definition A.1.1. *If $a_p \neq 0 \pmod{p}$ we say that f is p-distinguished if the restriction of $\overline{\rho}_{f,p}$ to a decomposition group at p is such that*

$$\overline{\rho}_{f,p}|_{\mathsf{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} \cong \begin{pmatrix} \varepsilon_{cyc}^{k-1}\lambda\left(\psi(p)/a_p\right) & * \\ 0 & \lambda(a_p) \end{pmatrix}$$

with $\varepsilon_{cyc}^{k-1}\lambda\left(\psi(p)/a_p\right) \not\equiv \lambda(a_p) \pmod{p}$.

Remark A.1.2. *Recall that the cyclotomic character ε_{cyc} is surjective. This is immediately proved since it factors, by definition, through $\mathsf{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ on which it is an isomorphism. Such a characterization allows us to provide a criterion for the p-distinguishness, since the reduction modulo p is surjective. If $k = 2$, $p > 2$ and $\psi(p)a_p(f)^2 \pmod{p}$ has order strictly less than $p - 1$, then $\psi(p)a_p(f)^2 \not\equiv \varepsilon_{cyc} \pmod{p}$.*

A.1.2. **Weight-1.** The case of weight 1 requires a bit more work. The idea is contained in the construction of the Galois representation associated with the modular form. Following the ideas in [Edi06] and [Wie06], we must embed weight-1 modular forms in weight p ones. Considering Katz modular forms, it is well-known that:

Proposition A.1.3 ([DI95], Theorem 12.3.7 & Corollary 12.3.8)**.** *Let M be a $\mathbb{Z}$-module. The natural maps*

$$M_k(\Gamma_1(N), M) \longrightarrow M_k(\Gamma_1(N), M)_{Katz} \qquad S_k(\Gamma_1(N), M) \longrightarrow S_k(\Gamma_1(N), M)_{Katz}$$

are injective, and are isomorphisms provided that M is flat over $\mathbb{Z}$ or that $k > 1$ and N is invertible in M. In particular, for $M = \mathbb{C}$ and all positive integers N and K, there exists a basis of $M_k(\Gamma_1(N), \mathbb{C})_{Katz}$ in $M_k(\Gamma_1(N), \mathbb{C})$ and the same holds true for cuspidal forms.

Consider now $f = \sum a_n q^n$ to be an element of $S_1^{new}(\Gamma_1(N), \psi, \mathbb{C})$ which is a normalized eigenform. Suppose moreover that p does not divide N. Via the above proposition we identify f with a Katz modular form. Since the prime p does not divide N and f has Fourier coefficients in $\mathbb{Q}(\psi)$, we can consider its reduction modulo p. Denote by E either a finite field of characteristic p or $\overline{\mathbb{F}_p}$. By abuse of notation we keep denoting the character with values in $E^\times$ by ψ. The E-vector space $S_1(\Gamma_1(N), \psi, E)_{Katz}$ injects in $S_p(\Gamma_1(N), \psi, E)_{Katz}$ via two operators:

$$F : S_1(\Gamma_1(N), \psi, E)_{Katz} \longrightarrow S_p(\Gamma_1(N), \psi, E)_{Katz};$$
$$A : S_1(\Gamma_1(N), \psi, E)_{Katz} \longrightarrow S_p(\Gamma_1(N), \psi, E)_{Katz}.$$

The first one is induced by the Frobenius morphism and acts on the Fourier expansion as $a_n(Fg) = a_{n/p}(g)$ with the usual convention that $a_n(Fg) = 0$ if $p \nmid n$. The second operator comes from the multiplication by the Hasse invariant and we can describe it by $a_n(Ag) = a_n(g)$. It is easily checked that these operators commute with the Hecke operators away from p. Moreover (see Equation (4.1.2) in [Edi06])

$$T_p^{(p)} A = F \qquad \text{and} \qquad AT_p^{(1)} = T_p^{(p)} A + \psi(p) F,$$

where the superscript points out the weight of the space on which the Hecke operators act. Notably, see *e.g.* Proposition 1.8.1 of [Wie06],

$$F\left(S_1(\Gamma_1(N), \psi, E)_{Katz}\right) \cap A\left(S_1(\Gamma_1(N), \psi, E)_{Katz}\right) = \{0\}.$$

Proposition A.1.4 ([Edi06], Proposition 6.2 & [Wie06], Proposition 1.8.4)**.** *With the notation as above, let $V \subseteq S_p(\Gamma_1(N), \psi, E)_{Katz}$ be a common non-zero eigenspace for all Hecke operators $T_l^{(p)}$ $l \neq p$. If the system of eigenvalues does not come from a weight-1 modular form, then* $\dim V = 1$*. Conversely, if the eigenvalues correspond to a normalized weight-1 modular form $g \in S_1(\Gamma_1(N), \psi, E)_{Katz}$, then $V = \langle Ag, Fg\rangle$ and* $\dim V = 2$*; $T_p^{(p)}$ acts on it with eigenvalues u and $\psi(p)u^{-1}$, where u satisfies the equality $u + \psi(p)u^{-1} = a_p(g)$. In particular, the weight p eigenforms coming from weight-1 are ordinary.*

We are especially interested in the proof of this proposition, because we want to understand the behavior of the eigenvalue $a_p(g)$. Let thus g be a classical normalized modular form which we think of as an element of $S_1(\Gamma_1(N), \psi, E)_{Katz}$ for a suitable E. Let f be a normalized element of V with the same eigenvalues of g away from p. We remark that this procedure returns the residual representation associated with g, since $\rho_g = \rho_f$. In particular, f has to be of the form $f = Ag + \mu F g$ for some $\mu \in E$. The coefficient in front of Ag is 1, because both f and g are normalized. Let $a_p(f)$ be the eigenvalue of f related to the operator $T_p^{(p)}$. Clearly $a_p(f) = a_p(g) + \mu$ and, as stated in the proposition it has to satisfy the relation $a_p(f)^2 - a_p(f)a_p(g) + \psi(p) = 0$. Now, at a decomposition group at p,

$$\rho_f|_{\mathsf{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} \cong \begin{pmatrix} \lambda(\psi(p)/a_p(f)) & * \\ & \lambda(a_p(f)) \end{pmatrix}$$

and recall from above that a normalized eigenform $f \in S_p(\Gamma_1(N), \psi, E)$ is p-distinguished if and only if $\psi(p) \neq a_p(f)^2$. Hence, g is not p-distinguished if and only if the two equations (for $p \neq 2$)

$$\begin{cases} a_p(f)^2 - a_p(f)a_p(g) + \psi(p) = 0, \\ a_p(f)^2 - \psi(p) = 0, \end{cases} \iff \begin{cases} 2\psi(p) - a_p(f)a_p(g) = 0, \\ a_p(f)^2 - \psi(p) = 0, \end{cases}$$

are satisfied. In particular, the system implies that $4\frac{\psi(p)}{a_p(g)^2} = 1$.

Lemma A.1.5. *If $p \neq 2$, $g \in S_1(\Gamma_1(N), \psi, E)$ is p-distinguished if $4\psi(p) \not\equiv a_p(g)^2 \pmod{p}$.*

This observation seems to be well-known to experts (*e.g.* see [CV92] and Section 2 of [Wie14], especially the discussion before Corollary 2.6).

A.2. **The hypothesis (CR,Σ^-).** Let, as above, (f, g, h) be a triple of classical specializations.

A.2.1. **Residual irreducibility.** Starting with the *absolute irreducibility* we note immediately that it is enough to check if the condition holds true for the representations associated with f, g and h. Moreover we recall also that, for $p \neq 2$, the residual Galois representation is irreducible if and only if it is absolutely irreducible. It is well know that if we deal with non-CM families, then there always exists a suitable prime p.

Theorem A.2.1 ([Rib85], Theorem 2.1)**.** *Let $f \in S_k(\Gamma_1(N), \mathbb{C})$ be a cusp form without CM and weight $k \geq 2$. For almost all primes p, the residual representation $\overline{\rho}_{f,\lambda}$ associated with f at the prime $\lambda | p$ is irreducible.*

A result of Fontaine produces a simple criterion, but it cannot be applied in the ordinary case. We report it for completeness.

Theorem A.2.2 ([Edi92], Theorem 2.6)**.** *Let f be a cusp form in $S_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ with $2 \leq k \leq p+1$, eigenform for all the Hecke operators T_l. If $a_p(f) = 0$, then the residual representation $\rho_{f,p}$ is irreducible.*

We should distinguish two cases: the elliptic curve case and the weight-1 cusp forms case.

A.2.1.1. **Reducibility of the Galois representation associated with elliptic curves.** Let E be an elliptic curve over a number field K. Suppose that E is without CM. In [Bil11] an algorithm is provided for checking whether the associated residual representation is reducible at a certain prime. These results seemed to be already known to experts in the case of $K = \mathbb{Q}$, as is pointed out in *loc.cit.*. We refer the reader to Section 1.2 of [Bil11] for the general theorems.

Theorem A.2.3 ([Bil11], Section 1.2)**.** *Let E be an elliptic curve over $\mathbb{Q}$ without CM. The number of primes p for which the residual Galois representation is reducible is finite. Furthermore, if p is such a prime, either it divides $6\Delta_E$, for Δ_E the discriminant of E, or p is of good reduction for E and divides A_l, the number of points of E on $\mathbb{F}_l$, for each prime $l \neq p$.*

The condition of being not CM implies the existence of (infinitely many) B_l which are non-zero.

A.2.1.2. **Reducibility of the Galois representation: the case of weight 1.** The weight-1 case requires more work. In particular, we need to introduce a few notions about Eisenstein series. The algorithm we are going to use has been worked out in [Ann13]. The crucial point is that a modular residual Galois representation is absolutely reducible if and only if it comes from an Eisenstein series. The difficulty in actually checking this condition is that we must rule out all the cusp forms which are congruent to Eisenstein series. We recall a couple of propositions contained in [Ste07]. First of all we need the

Definition A.2.4. *Let χ and ψ be two Dirichlet characters and k be a positive integer. Set*

$$E_{k,\chi,\psi}(q) = c_0 + \sum_{m\geq 1}\left(\sum_{n|m}\psi(n)\chi\left(\frac{m}{n}\right)n^{k-1}\right)q^m$$

for c_0 defined as in Section 5.2.2 of [Ste07].

For our purpose it is enough to know that $c_0 = 0$ if $cond(\chi) > 1$. Moreover, one has the following theorem.

Theorem A.2.5 ([Ste07],Theorem 5.8)**.** *Let t be a positive integer, χ, ψ and k as above. Suppose that k is such that $\chi(-1)\psi(-1) = (-1)^k$. Then $E_{2,1,1}(q) - tE_{2,1,1}(q^t) \in M_k(\Gamma_0(t), \mathbb{C})$ if $k = 2$, $t > 1$ and $\chi = \psi = 1$. Otherwise,*

$$E_{k,\chi,\psi}(q^t) \in M_k\left(\Gamma_0(cond(\chi)cond(\psi)t), \chi\psi, \mathbb{C}\right).$$

Theorem A.2.6 ([Ste07], Theorems 5.9 & 5.10)**.** *The Eisenstein series in the above theorem which lie in $M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ and satisfy $cond(\chi)cond(\psi)t|N$ and $\chi\psi = \varepsilon$ form a basis for the space of Eisenstein series $E_k(\Gamma_0(N), \varepsilon, \mathbb{C})$. Moreover, $E_{k,\chi,\psi}(q)$ and $E_{2,1,1}(q) - tE_{2,1,1}(q^t)$ (for $t > 1$) are normalized eigenforms.*

Assume now that p does not divide the level N. Hence, by Theorem 12.3.2 in [DI95], we know that all the modular forms modulo p are the reduction of modular forms with coefficients in a number field. Hence we can restate Theorem 7.2.3 in [Ann13] as it follows.

Theorem A.2.7. *Let N and k be positive integers and take p a prime such that p does not divide N and $2 \leq k \leq p+1$. Let $f \in S_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ and let $\overline{\rho}_{f,p} : \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ the residual Galois representation associated with f. Assume that*

- *$\overline{\rho}_{f,p}$ is reducible;*
- *f is the modular form associated with $\overline{\rho}_{f,p}$ by Serre's conjecture,* i.e. *the representation does not arise from lower level and weight.*

Assume moreover that

- *p is odd;*

- *$k \neq p$;*
- *we are neither in the case $k = 2$ and $\overline{\rho}_{f,p} \cong 1 \oplus 1$ nor in the case $k = p - 1$ and $\overline{\rho}_{f,p} \cong \chi_p^{-1} \oplus 1$.*

Then f is the reduction modulo p of an Eisenstein series as in Theorem A.2.5.

If we restrict ourselves to the case in which the starting weight-1 modular form has dihedral projective image, we have a specific result.

Lemma A.2.8 ([DG12], Lemma 4.5)**.** *Let $p > 2$ be a prime and f_∞ be a p-adic Hida family. If f_∞ is residually of dihedral or exceptional type, then the Galois representation of every classical weight-1 specialization f (if they exist) must be of the same kind (all dihedral, or all tetrahedral, or all octahedral, or all icosahedral). Moreover, in the exceptional case the projective images of ρ_f and $\overline{\rho}_{f_\infty}$ are isomorphic, while in the dihedral case, if the projective image of ρ_f is D_{2n}, then the projective image of $\overline{\rho}_{f_\infty}$ is D_{2m}, where m is the prime-to-p part of n.*

Remark A.2.9 ([DG12], Remark 4.6)**.** *Families of dihedral type with $m = 1$ have residually reducible Galois representations, whereas those with $m \geq 2$ have residually absolutely irreducible Galois representation.*

Appendix B. Brandt matrices

We consider here the notation of Section 1.1. In particular, R is a special order of level $M = N \cdot \prod_{\ell | D} \ell^{\nu(\ell)}$. Let $\{I_1 = R, \dots, I_h\}$ be a set of representatives for the classes of left R-ideals and R_i, for $i = 1, \dots, h$, the right order corresponding to I_i. Here h is (by definition) the class number of R. For $1 \leq i, j \leq h$, we set $M_{i,j} = I_j^{-1} I_i$ which is a R_j-left ideal with right order R_i. We set $n(M_{i,j})$ to be the unique positive rational number such that $n(b)/n(M_{i,j})$ are all integers with no common factor, for b varying in $M_{i,j}$. Take χ a Dirichlet character modulo N satisfying Assumption 1.1.6.

Definition B.0.1. *We define the m-*th Brandt matrix associated with the special order R and with character χ*, as the matrix*

$$B(m, R, \chi) := (b_{i,j}(m, R, \chi)),$$

where

$$b_{i,j}(m, R, \chi) := \frac{1}{2} \sum_{\substack{b \in M_{i,j}: \\ n(b) = m \cdot n(M_{i,j})}} \widetilde{\chi}(b) w_j^{-1} \quad \text{and} \quad b_{i,j}(0, R, \chi) = \begin{cases} \frac{1}{2w_j} & \text{if } \chi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

and $w_i = \#(R_i^\times / \{\pm 1\})$.

By Lemma 1.1.5, the evaluation of a Dirichlet character at $b \in M_{i,j}$ does not present any ambiguity; the lemma guarantees that, up to a suitable choice of the representatives I_i, we can define the Brandt matrices as above.

Proposition B.0.2 ([HPS89a], Theorem 4.8)**.** *With the above notation and for any couple of fixed indexes* (i,j)*, the expansion*

$$\theta_{i,j}(z) = \sum_{m=0}^{\infty} b_{i,j}(m,R,\chi)e^{2\pi imz}$$

defines a modular forms in $M_2(\Gamma_0(M\prod_{l|D} l^{e(l)}), \chi, \mathbb{C})$*, with* $e(l) = \nu(l)$ *if* L_l *is ramified over* $\mathbb{Q}_l$ *and* $e(l) = \max\{\nu(l), 2v_l(C)\}$ *otherwise. Moreover, the form is cuspidal if* χ_l *is odd for some* $l|D$ *or if* χ_l *is non trivial for some* $l|M$*.*

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