MOTIVIC EULER CHARACTERISTIC OF NEARBY CYCLES

AND

A GENERALIZED QUADRATIC CONDUCTOR FORMULA

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von
Ran Azouri, M.Sc.
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CAMPUS ESSEN

Gutachter
PROF. DR. MARC LEVINE
PROF. DR. PAUL ARNE ØSTVAER

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Dedicated to the Jews of Essen
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Abstract

We compute the motivic Euler characteristic of Ayoub's nearby cycles by strata of a semi-stable reduction, for a scheme over a discrete valuation ring of characteristic zero whose special fibre has isolated quasi-homogeneous singularities resolved by a single weighted blow-up. This allows us to compare the local picture at the singularities with the global conductor formula for hypersurfaces developed by Levine, Pepin Lehalleur and Srinivas, revealing that the formula is local in nature, thus extending it to the more general setting considered in this paper. This is a quadratic refinement for the Milnor number formula with multiple singularities of a certain type.

Zusammenfassung

Für ein Schema über einem diskreten Bewertungsring von Charakteristik Null, dessen spezielle Faser isolierte quasi-homogene Singularitäten hat, die durch eine einzige gewichtete Aufblasung aufgelöst werden können, berechnen wir die motivische Euler-Charakteristik von Ayoubs benachbarten Zykeln durch Strata einer semistabilen Reduktion. Dadurch können wir das lokale Verhalten der Singularitäten mit der globalen Führerformel von Levine, Pepin Lehalleur und Srinivas vergleichen. Insbesondere zeigen wir, dass diese Formel von lokaler Natur ist und dass sie sich für die hier betrachtete Situation verallgemeinern lässt.

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טוֹב אַחֲרִית דָּבָר מֵרֵאשִׁיתוֹ טוֹב אֶרֶךְ־רוּחַ מִגְּבַהּ־רוּחַ׃

The end of a matter is better than the beginning of it. Better a patient spirit than a haughty spirit.

Kohelet (Ecclesiates) 7, 8

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1 Introduction

1.1 The Milnor fibre, nearby cycles, and the Euler characteristic

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a non-zero polynomial function. We suppose that $X_t := f^{-1}(t)$ is smooth for $0 < |t| < 1$, and that X_0 has an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Take a small $\epsilon > 0$, and even smaller t. Take $p \in X_0$ and consider $B_{p,\epsilon}$, the open ball with radius ϵ. The Milnor fibre is defined by

$$M_{f,p} = B_{p,\epsilon} \cap X_t.$$

$M_{f,p}$ is homotopically equivalent to a bouquet of spheres, the number of spheres is defined to be the Milnor number, $\mu_{f,p}$ [Mil, Theorem 6.5]. We may also define a sheaf on X_0 by $x \mapsto H^*(M_{f,x}, \mathbb{Z})$. These concepts around the Milnor fibre can be developed in the more general setting of algebraic geometry.

Let $f : X \to B$ be a flat family of schemes. We assume that we have a distinguished closed point $\sigma \hookrightarrow B$, with complement $\eta = B \setminus \sigma \hookrightarrow B$, allowing us to talk about a generic fibre $X_\eta \to \eta$, which we assume to be smooth, and a special fibre X_σ, which may have singular points. The nearby cycles functor, defined by Deligne, see [SGA7 I, Exposé I, 2.],

$$\Psi_f : D^b_{cons}(X_\eta) \to D^b_{cons}(X_\sigma),$$

relates bounded complexes of constructible sheaves on the generic fibre X_η, and on the special fibre X_σ.

A closely related invariant in the case of proper f is the difference between the Euler characteristics of the generic fibre and that of the special fibre. Conductor formulas express this difference in terms of local invariants around the singular points of the special fibre:

$$\chi(X_\eta) - \chi(X_\sigma) = \textit{invariants related to the singular points in } X_\sigma.$$

In the setting of complex geometry, this was investigated by Milnor. Consider a complex manifold X of dimension $n+1$ and a proper holomorphic map $f : X \to D$, with D the open unit disk in $\mathbb{C}$; as above, let X_t denote the complex analytic variety $f^{-1}(t)$. Suppose that f is a submersion outside of a finite subset $\{p_1, \ldots, p_s\}$ of X_0. At each singular point p, a choice of local coordinates $s_0, ..., s_n$ for a neighbourhood of p gives us the Milnor fibre $M_{f,p}$ and the Milnor number $\mu_{f,p}$, defined as in the case of a polynomial function on $\mathbb{C}^{n+1}$. Milnor's theorem [Mil, Theorem 7.2] computes $\mu_{f,p}$ as the dimension of the Jacobian ring of f at p, that is,

$$\mu_{f,p} = \dim \mathcal{O}_{X,p}/(\partial f/\partial s_0, ..., \partial f/\partial s_n);$$

as an immediate consequence, we have the conductor formula

$$\chi^{top}(X_t) - \chi^{top}(X_0) = (-1)^n \sum_i \mu_{f,p_i}. \tag{1.1}$$

In this paper we develop a quadratic refinement for this formula in the setting of algebraic geometry.

The Deligne-Milnor conjecture [SGA7 II, Exposé XVI, Conjecture 1.9] is concerned with an algebraic version of Milnor's computation of the Milnor number, without restriction to characteristic zero. Let $X \to B$ be a separated, finite type, flat morphism of relative dimension n, where B is a henselian trait, in a setting in which we can talk about a generic fibre X_η and a special fibre X_σ. Suppose that X is regular, X_η is smooth over η, and that X_σ has a unique singular closed point p. Let l be a prime number which is invertible on $\mathcal{O}_B$. Then

$$\dim tot\Phi_p := \dim \Phi^n(\mathbb{F}_\ell)_p + \dim Sw(\Phi^n(\mathbb{F}_\ell)_p) = (-1)^n \mu_{f,p};$$

with Φ_p being the sheaf of vanishing cycles, and the *Swan conductor* $Sw(\Phi_p)$ an additional term, adjusting for the case of positive characteristic. The formula is proven in the case of equal characteristic ([SGA7 II, Exposé XVI, Proposition 2.2]), but is still a conjecture in the general case. The local formula yields as in the complex analytic case a conductor formula for a flat proper map $f : X \to B$ as above, but allowing the special fibre to have finitely many singularities.

1.2 Motivic refinements

In the context of motivic homotopy, the nearby functor cycles formalism was developed by Ayoub in [Ay07a]. Here the bounded derived category is replaced by motivic spectra over the generic and special fibres.

$$\Psi_f : \mathrm{SH}(X_\eta) \to \mathrm{SH}(X_\sigma).$$

By a motivic integration approach, Denef and Loeser [DL98], [DL00] construct a motivic Milnor fibre in a variant of the Grothendieck ring of varieties. It is expressed in terms of certain étale coverings for components of the special fibre and their intersections. Using rigid analytic motives Ayoub, Ivorra and Sebag [AIS] show that for a semi-stable scheme X, the class of the motivic nearby cycles in $K_0(\mathrm{SH}(X_\sigma))$ is equal to the one computed by those covers, the formula has the form

$$[\Psi_f] = \textit{alternating sum of étale coverings of intersections of strata of } X_\sigma.$$

Within the setting of stable $\mathbb{A}^1$-homotopy theory, we can refine the topological Euler characteristic as well. The motivic Euler characteristic of a smooth and proper scheme is defined as the categorical trace of the identity morphism of the motive of the scheme in the category of motivic spectra. A variant definable over singular schemes is the compactly supported Euler characteristic. Working in the motivic stable homotopy category $\mathrm{SH}(k)$ over a perfect field k, for every finite type k-scheme X, we get an element $\chi_c(X/k)$ in the Grothendieck-Witt group $\mathrm{GW}(k)$. That is, instead of integers we use quadratic forms, which encode more information.

Let $F(T_0, \dots, T_n) \in k[T_0, \dots, T_n]$ be a homogeneous (or weighted-homogeneous) polynomial of degree e, defining a smooth projective (or weighted projective) hypersurface. The hypersurface H^F defined by $F(T_0, \dots, T_n) - tT_{n+1}^e$ thus gives a family of hypersurfaces with a (quasi) homogeneous singularity at the special fibre $t = 0$. In this setting, Levine, Pepin Lehalleur and Srinivas [LPS, Theorem 5.2, Theorem 5.3] develop a quadratic conductor formula of the form

$$\Delta_t(F) := \mathrm{sp}_t\, \chi_c(H_t^F) - \chi_c(H_0^F) = \langle e\rangle - \langle 1\rangle + (-\langle e\rangle)^n \cdot \mu_{F,0}^q$$

in the homogeneous case. They also develop a similar formula for a weighted homogeneous F. Since $\chi_c(H_t^F) \in \mathrm{GW}(K)$, $\chi_c(H_0^F) \in \mathrm{GW}(k)$ live in different rings, one has to use the specialization map $\mathrm{sp}_t : \mathrm{GW}(K) \to \mathrm{GW}(k)$ to compare them. The invariants in the right hand side are quadratic forms defined algebraically; $\mu_{F,0}^q \in \mathrm{GW}(k)$ is a quadratic refinement of the Milnor number $\mu_{F,0} \in \mathbb{N}$. It can be defined in terms of the Jacobian ring $J(F, 0)$, by a quadratic form on this ring corresponding to a distinguished element in the ring, defined by Scheja-Storch. The main goal of this paper is to formulate and prove a generalization of this result.

1.3 Main results

We compute the quadratic Euler characteristic of the motivic nearby cycles functor in the case of a flat morphism $f : X \to S$ with a few quasi-homogeneous singularities. We compute it by pieces of the special fibre, using the étale coverings of the pieces as in the Ayoub-Ivorra-Sebag formula. Then we compare the result with the Levine-Pepin Lehalleur-Srinivas formula to obtain our generalized conductor formula.

We first give a proof of a special case of the formula by Ayoub-Ivorra-Sebag, for the case of no triple intersections, which suffices for our purpose. The coverings called here Denef-Loeser are defined in Section 3.4, but are obtained in the case considered here as strata of the special fibre of a semi-stable reduction for X.

Proposition 1.1 (Proposition 3.17)**.** *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a flat, quasi-projective morphism, with X smooth over its residue field k and with generic fibre X_η smooth over η. Suppose that the special fibre X_σ is a normal crossing divisor $X_\sigma = \sum a_i D_i$; if $\operatorname{char} k = p > 0$, we suppose in addition that $p \nmid \prod_i a_i$. Assume that for all $i \neq j$ $gcd(a_i, a_j) = 1$, and that there are no triple intersections. Denote by $\widetilde{D_i}$, $\widetilde{D_i^\circ}$, $\widetilde{D_{ij}}$ the Denef-Loeser coverings. Then*

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})) = \sum_i \chi_{\mathrm{c}}(\widetilde{D_i^\circ}) - \sum_{i<j} \chi_{\mathrm{c}}(\mathbb{G}_m \times \widetilde{D_{ij}}).$$

The type of singularities we deal with is defined as follows, for the homogeneous case:

Definition 1.2. Let $f : X \to \operatorname{Spec} \mathcal{O}$ a flat quasi-projective morphism of schemes over a discrete valuation ring $\mathcal{O}$ with quotient field K, residue field k and parameter t, with X a regular scheme and with $X_\eta \to \operatorname{Spec} K$ smooth. Let $p \in X_\sigma$ be an isolated singular point and let $F \in k(p)[T_0, \ldots, T_n]$ be a homogeneous polynomial of degree e. Let $\mathcal{O}_{X,p}$ be the stalk at p, and $m_p \subset \mathcal{O}_{X,p}$ the maximal ideal. We say that X_σ *looks like the homogeneous singularity defined by F at p* if there is a regular sequence of generators $s_0, \ldots, s_n$ for m_p such that

$$f^*(t) \equiv F(s_0, \ldots, s_n) \mod m_p^{e+1}.$$

Remark 1.3. Assuming further that the hypersurface $V_{\mathbb{P}^n}(F)$ in the projective space $\mathbb{P}^n_{k(p)}$ defined by F is smooth, the above condition is equivalent to assuming that the corresponding blow-up of (X_σ, p) resolves the singularity p of X_σ, and the intersection of the strict transform of x_σ with the exceptional divisor is smooth, isomorphic to $V(F)$. See Proposition 4.2.

Similarly we treat the more general *quasi-homogeneous* case, where the defining polynomial F at each singular point is a *weighted homogenous* polynomial with respect to a sequence of positive integer weights $a_* = (a_0, \ldots, a_n)$. The projective space $\mathbb{P}^n$ is replaced by the a_*-weighted projective space $\mathbb{P}(a_*)$ and its presentation as a finite group quotient of $\mathbb{P}^n$ is used to reduce to the homogeneous case. For the precise definition of when a point of X_σ looks like a quasi-homogeneous singularity see Definition 5.1. For the precise assumption in this case, see Assumption 5.5.

Now let $p \in X_\sigma$ be a singularity of the special fibre of $f : X \to \operatorname{Spec} \mathcal{O}$ that locally looks like a homogeneous or a a_*-weighted homogeneous singularity defined by F. We compute Ayoub's nearby cycles functor Ψ_f for the scheme X_σ locally at a point p piece by piece, using a semi-stable reduction Y for X constructed by a blow-up, base change and normalisation. Using the formalism of the nearby cycles functor and the Euler characteristics of the pieces - the strata of Y_σ - we obtain a global formula for the Euler characteristic of nearby cycles.

Theorem 1.4 (Corollary 4.4)**.** *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be as in definition 1.2, and suppose that the special fibre X_σ has finitely many singular points $p_1, \ldots, p_r$, and for each i, f looks at p_i like the singularity defined by a homogeneous polynomial $F_i \in k(p_i)[T_0, \ldots, T_n]$ of degree e_i, with $V(F_i) \subset \mathbb{P}^n_{k(p_i)}$ a smooth hypersurface, and with $\prod_i e_i$ prime to the exponential characteristic of k. Let $X_\sigma^\circ = X_\sigma \setminus \{p_1, \ldots, p_r\}$. Then*

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})) = \chi_{\mathrm{c}}(X_\sigma^\circ) + \sum_{i=1}^r \chi_{\mathrm{c}}(V(F_i - T_{n+1}^{e_i})) - \sum_{i=1}^r \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F_i)).$$

We write here the homogeneous case only not to overload notation. But we prove a similar formula in the more general case of singular points that look like quasi-homogeneous singularities, see Corollary 5.8. We then deduce the following formula.

Theorem 1.5 (Theorem 7.2). *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be as in Definition 1.2, with k of characteristic 0. Assume that an isolated singular point p of the special fibre X_σ looks like an a_*-weighted homogeneous singularity defined by a polynomial $F \in k(p)[T_0, \dots, T_n]$ of weighted degree e, and with $V(F) \subset \mathbb{P}_{k(p)}(a_*)$ smooth over $k(p)$. Then*

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})|_p) = \operatorname{Tr}_{k(p)/k}(\Delta_t(F/k(p)) + \langle 1 \rangle).$$

Note that this includes the case of a homogeneous F by taking the weights $a_* = (1, \dots, 1)$. Assume for simplicity $k = k(p)$. The formula reads

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})|_p) - \langle 1 \rangle = \Delta_t(F/k);$$

we may think of the left hand side as enumerating vanishing cycles for X around p, and the right hand side as doing the same for the hypersurface H^F. So this gives us a comparison between the two schemes, and allows us to use the main result of [LPS] for H^F, in order to get a formula for the scheme X at p. Using the sheaf properties of Ayoub's functor, we can consequently extend it to a global formula on a scheme X with several isolated homogeneous or quasi-homogeneous singularities. We state now the weighted homogeneous version.

Theorem 1.6 (Generalized quadratic conductor formula for quasi-homogeneous singularities, Corollary 7.3). *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be as in Definition 1.2, of relative dimension n with f proper. Assume X_σ satisfies Assumption 5.5 with singular points $\{p_1, ..., p_s\}$. Let e_i denote the weighted-homogeneous degree of the corresponding polynomial F_i. Then*

$$\operatorname{sp}_t \chi_c(X_\eta) - \chi_c(X_\sigma) = \sum_i Tr_{k(p_i)/k}[\langle \prod_j a_j^{(i)} \cdot e_i \rangle - \langle 1 \rangle + (-\langle e_i \rangle)^n \cdot e_{p_i}(\Omega_{X/k}, dt)]$$

Here $e_{p_i}(\Omega_{X/k}, dt)$ denotes the $\mathbb{A}^1$-local Euler class at p_i.

This settles Conjecture 5.4 in [LPS] for the case of characteristic zero and singularities resolved by a single blow-up with a smooth exceptional divisor (satisfying Assumption 4.1 or 5.5); in fact, our result handles cases not covered by Conjecture 5.4, as the type of singularities treated above are not necessarily homogeneous or weighted-homogeneous in the sense of *loc. cit.* The local Euler class for X at p_i, $e_{p_i}(\Omega_{X/k}, dt)$, is the same as the local Euler class for H^{F_i}, and that class equals to the quadratic Milnor number $\mu^q_{F_i,0}$. This is shown in Corollary 6.10.

As an application of the main formula, we deduce a quadratic formula for curves.

Corollary 1.7 (Corollary 8.3). *Let C be a reduced curve on a smooth projective surface S over a field k of characteristic zero. Suppose that $\mathcal{O}_S(C)$ admits a section s with smooth divisor C_1 that intersects C transversely. Suppose in addition that each singular point p of C is a homogeneous singularity, let e_p denote the homogeneous degree at p. Then*

$$\operatorname{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(C/k) = \sum_{p \in C_{sing}} \operatorname{Tr}_{k(p)/k}(\langle e_p \rangle - \langle 1 \rangle - \langle e_p \rangle [\mu^q_{f_p,p}]).$$

Again we write here the homogeneous version for simplicity, the more general weighted homogeneous formula is appearing in the text. This formula refines a classical formula with integers that can be deduced from the Jung-Milnor formula.

2 The motivic Euler characteristic with compact supports

A construction of central importance in this article is the *motivic Euler characteristic with compact supports*, χ_c. For a finite type separated k-scheme, $\chi_c(X/k)$ is an element in the Grothendieck-Witt ring $\mathrm{GW}(k)$ of k. Before going into a detailed discussion of $\chi_c(X/k)$ and related notions, we first give sketch of the main ideas that go into its construction.

We will use the notations and properties of the unstable and stable motivic homotopy categories to be found in [Ay07a], [CD], and [Hoy17], including the six-functor formalism for $\mathrm{SH}(-)$.

Let $p : X \to \operatorname{Spec} k$ be a proper and smooth scheme over a field k. As we shall see below, its motive with compact supports $p_!\mathbb{1}_X$ is a strongly dualisable object in the symmetric monoidal category $(\mathrm{SH}(F), \otimes)$, with dual $(p_!\mathbb{1}_X)^\vee = p_\#\mathbb{1}_X$. The Euler characteristic with compact supports of X/k is the trace of the identity endomorphism for $p_!\mathbb{1}_X \in \mathrm{SH}(k)$, this being defined as the composition

$$tr(id_{p_!\mathbb{1}_X}) : \mathbb{1}_k \xrightarrow{\delta} p_!\mathbb{1}_X \otimes (p_!\mathbb{1}_X)^\vee \xrightarrow{\tau} (p_!\mathbb{1}_X)^\vee \otimes p_!\mathbb{1}_X \xrightarrow{ev} \mathbb{1}_k$$

where δ and ev are respectively the co-evaluation and evaluation maps of the dualising data, and τ is the non-trivial permutation. Here $\mathbb{1}_X$, $\mathbb{1}_k$ denote the respective unit objects in $\mathrm{SH}(X)$ and $\mathrm{SH}(k)$. This yields an element in $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{1}_k)$, which is isomorphic as a ring to $\mathrm{GW}(k)$ via the Morel isomorphism [Mo, Lemma 6.3.8, Theorem 6.4.1]. We denote the corresponding element of $\mathrm{GW}(F)$ by $\chi_c(X/k)$; we omit k when it is obvious from the context. For more details on the motivic Euler characteristic see [Le20, §2].

Here are some useful notations and definitions.

Notation 2.1. For a field k, we usually let p denote the exponential characteristic of k, that is, p is the characteristic of k if this is positive, and is 1 if the characteristic is zero. We will always assume that the characteristic is different from 2.

For X a separated noetherian scheme, we let $\mathbf{Sch}_X$ denote the category of separated finite type schemes over X and let $\mathbf{Sm}_X$ be the full subcategory of smooth (separated and finite type) schemes over X. We will refer to an object $Y \to X$ of $\mathbf{Sch}_X$ as an *X-scheme* or *scheme over X* and similarly refer to an object $Y \to X$ of $\mathbf{Sm}_X$ as a *smooth X-scheme* or *smooth scheme over X*.

Definition 2.2. [Sch, Definition 1.9] Let k be a field. Let $M(k)$ be the monoid of equivalence classes of non-degenerate quadratic forms on k, with the operation induced by direct sum of quadratic forms, $\oplus$. Define $\mathrm{GW}(k)$ to be the Grothendieck group completion of $M(k)$. Concretely, elements of $\mathrm{GW}(k)$ are formal differences of classes of quadratic forms on k. The tensor product of quadratic forms $\otimes$ induces a well-defined multiplication on $M(k)$ that extends to $\mathrm{GW}(k)$, making $(\mathrm{GW}(k), \oplus, \otimes)$ a ring that we call *the Grothendieck-Witt ring* of k.
For $a \in k^\times$, we denote by $\langle a \rangle \in \mathrm{GW}(k)$ the class corresponding to the quadratic form $x \mapsto ax^2$.

Definition 2.3. Let $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ be a symmetric monoidal category, and take $x \in Ob(\mathcal{C})$. We say that x is strongly dualisable if there exists an object $x^\vee \in Ob(\mathcal{C})$ and morphisms $\delta_x : \mathbb{1}_{\mathcal{C}} \to x \otimes x^\vee$ and $ev_x : x^\vee \otimes x \to \mathbb{1}_{\mathcal{C}}$, called respectively co-evaluation and evaluation, such that

$$x \simeq \mathbb{1}_{\mathcal{C}} \otimes x \xrightarrow{\delta_x \otimes id} x \otimes x^\vee \otimes x \xrightarrow{id \otimes ev_x} x \otimes \mathcal{C} \simeq x$$

and

$$x^\vee \simeq x^\vee \otimes 1 \xrightarrow{id \otimes \delta_x} x^\vee \otimes x \otimes x^\vee \xrightarrow{ev_x \otimes id} 1 \otimes x^\vee \simeq x^\vee$$

are the identity morphisms. We call the object $x^\vee$ the dual of x.

For x a strongly dualisable object of $\mathcal{C}$ and $f : x \to x$ an endomorphism, the *trace* of f is the element $tr(f) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$ defined as the composition

$$\mathbb{1}_{\mathcal{C}} \xrightarrow{\delta_x} x \otimes x^\vee \xrightarrow{f \otimes id} x \otimes x^\vee \xrightarrow{\tau_{x,x^\vee}} x^\vee \otimes x \xrightarrow{ev_x} \mathbb{1}_{\mathcal{C}}.$$

In particular, taking $f = id_x$, we have the *categorical Euler characteristic* $\chi_{\mathcal{C}}(x) := tr_x(id_x)$.

Remark 2.4. It follows directly from the definitions that for x, y strongly dualisable objects of $\mathcal{C}$, we have

$$\chi_{\mathcal{C}}(x \otimes y) = \chi_{\mathcal{C}}(x) \otimes \chi_{\mathcal{C}}(y).$$

Definition 2.5 ([CD, Definition 4.2.1])**.** Define $\mathrm{SH}_c(X)$, the subcategory of constructible objects in $\mathrm{SH}(X)$, as the thick triangulated subcategory generated by the objects $\Sigma^n_{\mathbb{P}^1} f_\# \mathbb{1}_Y$, where $f : Y \to X$ is a smooth X-scheme and $n \in \mathbb{Z}$. An object in this category is called a constructible object.

Proposition 2.6. *Constructible objects are stable under f^* for any morphism f, under $f_\#$ for a smooth f, under $f^!$ for a proper f, and under $f_!$ for a separated f of finite type [CD, Proposition 4.2.4, 4.2.11, 4.2.12]*

In addition for $i : Z \hookrightarrow X$ a closed immersion and $j : U \hookrightarrow X$ its open complement, an object $\alpha \in \mathrm{SH}(X)$ is constructible if and only if $i^*\alpha$ and $j^*\alpha$ are constructible. [CD, Proposition 4.2.10]

A result of May [May, Theorem 0.1] about additivity of trace maps in triangulated categories has the follow consequences for $\mathrm{SH}(k)$.

Proposition 2.7. *Let $n > 0$ be an integer and let*

$$\alpha \to \beta \to \gamma \to \alpha[1]$$

be a distinguished triangle in $\mathrm{SH}(k)[1/n]$. Then

(1) If any two of α, β, γ, is strongly dualisable, so is the third (the subcategory of strongly dualisable objects in $\mathrm{SH}(k)[1/n]$ is thick).

(2) $\chi_{\mathrm{SH}(k)[1/n]}(\beta) = \chi_{\mathrm{SH}(k)[1/n]}(\alpha) + \chi_{\mathrm{SH}(k)[1/n]}(\gamma)$.

Proposition 2.8. *Take $\alpha \in \mathrm{SH}(k)[1/p]$, with k a perfect field of exponential characteristic p. If α is constructible then it is strongly dualisable.*

Proof. By Proposition 2.7 (1), the subcategory of strongly dualisable objects in $\mathrm{SH}(k)$ is thick. By [LYZR, Appendix B, Cor. B2], for every smooth, separated, and finite type morphism $Y \to k$ in $\mathbf{Sm}_k$, $\Sigma^n_{\mathbb{P}^1} f_\# \mathbb{1}_Y$ is strongly dualisable in $\mathrm{SH}(k)$. As elements of this type generate the thick subcategory of constructible objects we get the result. $\square$

As a consequence we can now make the following definition.

Definition 2.9. Let k be a perfect field of exponential characteristic p, $q : X \to k$ a k-scheme and $\alpha \in \mathrm{SH}(X)[1/p]$ a constructible object. Then $\chi_c(\alpha/k)$ is defined to be the categorical Euler characteristic of $q_!\alpha$ in $\mathrm{SH}(k)[1/p]$:

$$\chi_c(\alpha/k) := \chi_{\mathrm{SH}(k)[1/p]}(q_!\alpha).$$

This is well defined as $q_!\alpha \in \mathrm{SH}(k)$ is constructible by Proposition 2.6 and is strongly dualisable (in $\mathrm{SH}(k)[1/p]$) by Proposition 2.8.

In particular we define

$$\chi_c(X/k) := \chi_c(\mathbb{1}_X/k) = \chi_{\mathrm{SH}(k)[1/p]}(q_!\mathbb{1}_X) := tr(id_{q_!\mathbb{1}_X})$$

for every k-scheme $q : X \to \operatorname{Spec} k$. We write $\chi_c(\alpha)$ for $\chi_c(\alpha/k)$ when the base-field k is clear from the context.

Remark 2.10. In the case $k = \mathbb{C}$, the rank homomorphism of quadratic forms gives an isomorphism, $rk : \mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$. We recover the topological Euler characteristic under this identification

$$\chi_c(X/\mathbb{C}) = {\chi_c}^{top}(X(\mathbb{C})) = \chi^{top}(X(\mathbb{C})).$$

For the first equality see [Le20, remark 1.5]. The second equality is true for every complex algebraic variety.

A useful property of the compactly supported motivic Euler characteristic is the cut-and-paste property, which is formulated in the following proposition.

Proposition 2.11. *Let $q : X \to \operatorname{Spec} k$ be a k-scheme. Let $\alpha \in \mathrm{SH}(X)$ be a constructible object, and let*

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

be a closed embedding and its open complement. Then

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(i^*\alpha) + \chi_{\mathrm{c}}(j^*\alpha)$$

and

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(i^!\alpha) + \chi_{\mathrm{c}}(j_*j^*\alpha).$$

Proof. The distinguished triangle of endofunctors on $\mathrm{SH}(X)$

$$j_!j^! \to id_{\mathrm{SH}(X)} \to i_*i^* \to$$

gives a distinguished triangle of endofunctors on $\mathrm{SH}(k)$ after composing with $q_!$,

$$q_!j_!j^! \to q_! \to q_!i_*i^* \to$$

Applying each of these terms to α gives a constructible object in $\mathrm{SH}(k)$ by Proposition 2.6, which is therefore strongly dualisable in $\mathrm{SH}(k)[1/p]$ (Proposition 2.8) so we can apply χ_{c} and use additivity (Proposition 2.7 (2)) to get

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(j_!j^!\alpha) + \chi_{\mathrm{c}}(i_*i^*\alpha).$$

Since $i_* = i_!$, $j^* = j^!$, we have

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(j^*\alpha) + \chi_{\mathrm{c}}(i^*\alpha).$$

Doing the same with the distinguished triangle

$$i_!i^! \to id_{\mathrm{SH}(X)} \to j_*j^* \to$$

we get

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(i^!\alpha) + \chi_{\mathrm{c}}(j_*j^*\alpha).$$

□

Remark 2.12. Let k be a perfect field and let X be a k-scheme, $Y \subset X$ a closed subscheme and U the open complement $X \setminus Y$, then from Proposition 2.11 applied to $\alpha = \mathbb{1}_X$ we get

$$\chi_{\mathrm{c}}(X) = \chi_{\mathrm{c}}(Y) + \chi_{\mathrm{c}}(U).$$

From this relation it follows that the motivic Euler characteristic factorises through the Grothendieck ring of k-varieties $K_0(Var_k)$. In other words we have the following commutative diagram

$$\begin{tikzcd}
Var_k \arrow[rr, "\chi_{\mathrm{c}}(\cdot/k)"] \arrow[dr, "X\mapsto[X]"'] & & \mathrm{GW}(k) \\
& K_0(Var_k) \arrow[ur, dashed] &
\end{tikzcd}.$$

The cut and paste relation yields in the standard way a Mayer-Vietoris property with respect to a Zariski open cover for $\chi_{\mathrm{c}}(-)$.

Proposition 2.13. *Let k be a perfect field, let X be a k-scheme with a Zariski open cover $X = U_1 \cup U_2$ and let $\alpha \in \mathrm{SH}(X)$ be a constructible object. Let $U_{12} = U_1 \cap U_2$ and let $j_1 : U_1 \to X$, $j_2 : U_2 \to X$, and $j_{12} : U_{12} \to X$ be the inclusions. Then*

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(j_1^*\alpha) + \chi_{\mathrm{c}}(j_2^*\alpha) - \chi_{\mathrm{c}}(j_{12}^*\alpha).$$

Proof. Let $Z = X \setminus U_1 = U_2 \setminus U_{12}$, with reduced scheme structure, and with closed immersions $i : Z \to X$, $i_2 : Z \to U_2$. We have the canonical isomorphism $i_2^*j_2^*\alpha \cong i^*\alpha$, whence the identity

$$\chi_{\mathrm{c}}(i^*\alpha) = \chi_{\mathrm{c}}(i_2^*j_2^*\alpha).$$

By Proposition 2.11, we have the identities

$$\chi_{\mathrm{c}}(\alpha) = \chi_{\mathrm{c}}(j_1^*\alpha) + \chi_{\mathrm{c}}(i^*\alpha)$$

and

$$\chi_{\mathrm{c}}(j_2^*\alpha) = \chi_{\mathrm{c}}(j_{12}^*\alpha) + \chi_{\mathrm{c}}(i_2^*j_2^*\alpha).$$

Putting these together gives the desired result. ☐

Proposition 2.14 (Purity)**.** *Let $i : Z \to X$ be a closed immersion of smooth k-schemes, or pure codimension c, let $f : X \to \operatorname{Spec} k$, $g : Z \to \operatorname{Spec} k$ be the structure morphisms. Then for α a constructible object of $\mathrm{SH}(k)$, we have*

$$\chi_{\mathrm{c}}(i^!f^*\alpha) = \langle -1\rangle^c \cdot \chi_{\mathrm{c}}(g^*\alpha)$$

In particular,

$$\chi_{\mathrm{c}}(i^!\mathbb{1}_X) = \langle -1\rangle^c \, \chi_{\mathrm{c}}(Z/k).$$

Proof. We use the notation from [Hoy17]. Let $f : Z \to \operatorname{Spec} k$, $g : Z \to \operatorname{Spec} k$ be the structure morphisms, let Ω_f, Ω_g be the respective sheaves of relative differentials, and let $\mathcal{N}_i$ be the conormal sheaf of i. We have the purity isomorphism (see [Hoy14, Appendix A])

$$i^! \circ f^* \cong \Sigma^{-N_i} \circ g^*.$$

Using the Mayer-Vietoris property Proposition 2.13 for $\chi_{\mathrm{c}}(-)$, we reduce to the case of trivial conormal sheaf, $\mathcal{N}_i \cong \mathcal{O}_Z^c$, inducing the natural isomorphism $\Sigma^{-\mathcal{N}_i} \cong \Sigma_{\mathbb{P}^1}^{-c}$, and giving the purity isomorphism

$$i^! \circ f^* \cong \Sigma_{\mathbb{P}^1}^{-c} \circ g^*.$$

We have the projection formula [Hoy17, Theorem 6.18(7)]

$$g_!(\Sigma_{\mathbb{P}^1}^{-c}\beta) \cong \Sigma_{\mathbb{P}^1}^{-c} g_!(\beta)$$

for $\beta \in \mathrm{SH}(Z)$. Since $\Sigma_{\mathbb{P}^1}^{-c}\gamma \cong S^{-2c,-c} \wedge \gamma$ for $\gamma \in \mathrm{SH}(k)$, it follows from Remark 2.4 and [Le20, Lemma 2.2] that

$$\chi_{\mathrm{SH}(k)}(\Sigma_{\mathbb{P}^1}^{-c}\gamma) = \langle -1\rangle^{-c} \cdot \chi_{\mathrm{SH}(k)}(\gamma) = \langle -1\rangle^{c} \cdot \chi_{\mathrm{SH}(k)}(\gamma)$$

for $\gamma \in \mathrm{SH}(k)$ strongly dualisable. Thus

$$\chi_{\mathrm{c}}(i^!f^*\alpha) = \chi_{\mathrm{SH}(k)}(\Sigma_{\mathbb{P}^1}^{-c} g_!(g^*\alpha)) = \langle -1\rangle^c \cdot \chi_{\mathrm{c}}(g^*\alpha).$$

The special case $\chi_{\mathrm{c}}(i^!\mathbb{1}_X) = \langle -1\rangle^c \, \chi_{\mathrm{c}}(Z/k)$ follows by taking $\alpha = \mathbb{1}_k$. ☐

Remark 2.15 (Non-perfect fields)**.** Let F be a field of characteristic $p > 2$, and with perfect closure $F^{perf} \supset F$. The base-extension $\mathrm{GW}(F)[1/p] \to \mathrm{GW}(F^{perf})[1/p]$ is an isomorphism. For a constructible object $\beta \in \mathrm{SH}(F)$, the base-extension $\beta^{perf} \in \mathrm{SH}(F^{perf})$ is constructible. Moreover, for an F-scheme $q : X \to \operatorname{Spec} F$ and an element $\alpha \in \mathrm{SH}(X)$, we have the base-change $q^{perf} : X \times_{\operatorname{Spec} F} \operatorname{Spec} F^{perf} \to \operatorname{Spec} F^{perf}$ and $\alpha^{perf} \in \mathrm{SH}(X)$, with $q_!^{perf}(\alpha^{perf})$ canonically isomorphic to the base-change $q_!(\alpha)^{perf}$ of $q_!(\alpha)$. Thus, we may define $\chi_c(\alpha/F)$ by

$$\chi_c(\alpha/F) := \chi_c(\alpha^{perf}/F^{perf}) \in \mathrm{GW}(F^{perf})[1/p] = \mathrm{GW}(F)[1/p].$$

Having done this, it is easy to show that all the properties of $\chi_c(-/k)$ described above extend to non-perfect base-fields F, and we will use this extension to non-perfect F without further mention.

Another useful formula concerns change of base-field. For $k_1 \subset k_2$ a finite separable field extension, we have the transfer map on the Grothendieck-Witt rings

$$\mathrm{Tr}_{k_2/k_1} : \mathrm{GW}(k_2) \to \mathrm{GW}(k_1).$$

This is the so-called *Scharlau transfer*[1] with respect to the trace map $\mathrm{Tr}_{k_2/k_1} : k_2 \to k_1$ and is defined as follows. For a finite-dimensional k_2-vector space V and a non-degenerate symmetric k_2-bilinear map $b : V \times V \to k_2$, one considers V as a (finite-dimensional) k_1-vector space, giving the symmetric k_1-bilinear map $\mathrm{Tr}_{k_2/k_1} \circ b : V \times V \to k_1$; the fact that k_2 is separable over k_1 implies that Tr_{k_2/k_1} is surjective and hence $\mathrm{Tr}_{k_2/k_1} \circ b$ is non-degenerate. Sending b to $\mathrm{Tr}_{k_2/k_1} \circ b$ defines the map $\mathrm{Tr}_{k_2/k_1} : \mathrm{GW}(k_2) \to \mathrm{GW}(k_1)$.

Proposition 2.16. *Let $k_1 \subset k_2$ be a finite separable extension of fields, let $\pi : \operatorname{Spec} k_2 \to \operatorname{Spec} k_1$ be the induced morphism, and let $f : X \to \operatorname{Spec} k_2$ be a k_2-scheme, which we consider as k_1-scheme via composition with π. For a constructible object $\alpha \in \mathrm{SH}(X)$ we have*

$$\chi_c(\alpha/k_1) = \mathrm{Tr}_{k_2/k_1}(\chi_c(\alpha/k_2)) \in \mathrm{GW}(k_1).$$

Proof. This is [Hoy14, Proposition 5.2] combined with the canonical isomorphism $(\pi \circ f)_! \cong \pi_! \circ f_!$. ☐

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[1] The Scharlau transfer for the Witt groups is discussed, for example, in [Sch, Chapter 2, Section 5]; the same construction works for the Grothendieck-Witt groups.

3 Motivic nearby cycles

3.1 Ayoub's motivic nearby cycles functor

For a noetherian separated scheme X, we let $\mathbf{QProj}_X$ denote the category of quasi-projective X-schemes and $\mathbf{SmQProj}_X$ the full subcategory of smooth, quasi-projective X-schemes.

Throughout the paper we fix a discrete valuation ring $\mathcal{O}$ with residue field k, fraction field K and fixed uniformizer $t \in \mathcal{O}$; σ will denote the closed point $\operatorname{Spec} k$ and η the generic point $\operatorname{Spec} K$. We define B to be $\operatorname{Spec} \mathcal{O}$. We will assume in addition that $\mathcal{O}$ contains a subfield k_0 such that B is smooth and essentially of finite type over k_0, and the field extension $k_0 \to k$ is finite and separable.

Let $f : X \to B$ be a flat, quasi-projective B-scheme. We have the closed immersion i and the open immersion j

$$\sigma \overset{i}{\hookrightarrow} B \overset{j}{\hookleftarrow} \eta.$$

We denote the respective pullbacks by X_σ, X_η ('the special and the generic fibre') and denote the maps induced by f according to the following diagram

$$\begin{tikzcd}
X_\sigma \arrow[r, hook] \arrow[d, "f_\sigma"] & X \arrow[d, "f"] & X_\eta \arrow[l, hook'] \arrow[d, "f_\eta"] \\
\sigma \arrow[r, hook, "i"] & B & \eta \arrow[l, hook', "j"']
\end{tikzcd}$$

The motivic nearby cycles functor

$$\Psi_f : \mathrm{SH}(X_\eta) \to \mathrm{SH}(X_\sigma)$$

is constructed in [Ay07a, 3.2.1]. Fixing the parameter t defines a map $t : \operatorname{Spec} \mathcal{O} \to \operatorname{Spec} k_0[t]$. By abuse of notation we will use Ψ_f also to denote $\Psi_{t\circ f}$, with the base being $\mathbb{A}^1_{k_0}$. Among the properties satisfied by $\Psi_{(-)}$, we have the following.

Property 3.1 (see [Ay07a, Definition 3.1.1])**.** For each morphism $g : Y \to X$, of flat quasi-projective B-schemes, there are well-defined natural transformations

$$\alpha_g : g_\sigma^* \circ \Psi_f \to \Psi_{f\circ g} \circ g_\eta^*$$

and

$$\beta_g : \Psi_f \circ g_{\eta *} \to g_{\sigma *} \circ \Psi_{f\circ g}$$

such that:

(a) If g is smooth α_g is natural isomorphism.

(b) if g is projective then β_g is an natural isomorphism.

These natural transformations satisfy some compatibility conditions, for details check [Ay07a, 3.1.1, 3.1.2]

The next result describes a very useful property for computing Ψ_f.

Notation 3.2. Let X be a smooth k_0-scheme, D a reduced normal crossing divisor on X with irreducible components $D_1, \ldots, D_r$. For $I \subset \{1, \ldots, r\}$, let $D_I := \cap_{i\in I} D_i$, $D_I^\circ := \cap_{i\in I} D_i \setminus \cup_{j\notin I} D_j$, $D_{(I)} := \cup_{i\in I} D_i$, and $D_{(I)}^\circ := D_{(I)} \setminus \cup_{j\notin I} D_j$.

Proposition 3.3 ([Ay07a, Théorème 3.3.44]). *Let $f : X \to B$ be a flat quasi-projective B-scheme. Suppose that X is smooth over k_0 and that $X_\sigma := f^{-1}(0)$ is a reduced normal crossing divisor with irreducible components $D_1, \ldots, D_r$. Fix a non-empty subset $I \subset \{1, \ldots, r\}$, let $D^\circ_{(I)} \overset{v}{\hookrightarrow} D_{(I)} \overset{u}{\hookrightarrow} X_\sigma$ denote the respective open and closed immersions.*

Then composing $u^*\Psi_f f^*_\eta$ with the unit map of the adjunction $id \to v_*v^*$ induces a natural isomorphism

$$u^*\Psi_f f^*_\eta \simeq v_*v^*u^*\Psi_f f^*_\eta$$

For the rest of the section we fix I and let $D := D_{(I)}$, $D^\circ := D^\circ_{(I)}$.

Remark 3.4. We retain the notation from Proposition 3.3 Evaluating at $\mathbb{1}_\eta \in \mathrm{SH}(\eta)$ and formulating the statement slightly differently, we have

$$(\Psi_f(\mathbb{1}_{X_\eta}))|_D = v_*(\Psi_f(\mathbb{1}_{X_\eta})|_{D^\circ}).$$

Here $(\Psi_f(\mathbb{1}_{X_\eta}))|_D$ denotes the pullback $u^*\Psi_f(\mathbb{1}_{X_\eta}) \in \mathrm{SH}(D)$ via the inclusion $u : D \to X_\sigma$, and similarly $\Psi_f(\mathbb{1}_{X_\eta})|_{D^\circ} := v^*u^*\Psi_f(\mathbb{1}_{X_\eta}) \in \mathrm{SH}(D^\circ)$.

Moreover, taking $I = \{i\}$, then

$$(\Psi_f(\mathbb{1}_{X_\eta}))|_{D^\circ} = w^*\Psi_{id}(\mathbb{1}_B) = w^*(\mathbb{1}_\sigma) = \mathbb{1}_{D^\circ}$$

where $w : D^\circ \to \sigma$ is the structure morphism. This last statement follows from the compatibility of $\Psi_{(-)}$ with smooth pullback, Property 3.1, applied to the open immersion $X \setminus \cup_{j\neq i}D_j \hookrightarrow X$ and then to the smooth morphism $X \setminus \cup_{j\neq i}D_j \to B$. In addition, the identity $\Psi_{id}(\mathbb{1}_B) = \mathbb{1}_\sigma$ follows from [Ay07a, Proposition 3.4.9, Lemma 3.5.10].

Remark 3.5. The statement of the theorem appears in [Ay07a, Théorème 3.3.10, Remarque 3.3.12] for the case

$$X = Sp_1 := B[T_1, \ldots, T_k]/(T_1 \cdot \ldots \cdot T_k - t)$$

and f the obvious morphism to B. In [Ay07a, Théorème 3.3.44] the statement is essentially the same as in our Proposition 3.3, with the assumption $I = \{i\}$. This special case is in fact all we need to use later on. In the rest of this section we describe how to deduce Proposition 3.3 from [Ay07a, Théorème 3.3.10] with no claim of originality. The reader can safely skip to section 3.2.

Before we give the proof of Proposition 3.3, we prove two elementary lemmas.

Lemma 3.6 (Unit maps commute with smooth base change). *Let Z be a noetherian separated scheme and let*

$$\begin{tikzcd} W' \arrow[r, "v'"] \arrow[d, "h'"'] & W \arrow[d, "h"] \\ Z' \arrow[r, "v"] & Z \end{tikzcd}$$

be a Cartesian square in $\mathbf{Sch}_Z$ with h and v smooth morphisms. Let $\eta_v : \mathrm{id}_{\mathrm{SH}(Z)} \to v_*v^*$, $\eta_{v'} : \mathrm{id}_{\mathrm{SH}(W)} \to v'_*v'^*$ be the units of the respective adjunctions. Let $\phi : h^* \to v'_*v'^*h^*$ be the following composition

$$h^* \xrightarrow{h^*\circ\eta_v} h^*v_*v^* \xrightarrow{Ex^*_*\circ v^*} v'_*h'^*v^* \xrightarrow{\sim} v'_*v'^*h^*$$

where Ex^*_* is the smooth base-change isomorphism and the last map is given by functoriality of $(-)^*$. Then $\phi = \eta_{v'} \circ h^*$.

Proof. Since each of the morphisms involved are smooth, all the functors we are considering have left adjoints, and it suffices to prove the corresponding statement for the left adjoints. Namely, let $\theta : v_\# v^* \to \mathrm{id}_{\mathrm{SH}(Z)}$, $\theta' : v'_\# v'^* \to \mathrm{id}_{\mathrm{SH}(W)}$ be the counits of the respective adjunctions. We have the map $\psi : h_\# \circ v'_\# \circ v'^* \to h_\#$ defined as the composition

$$h_\# \circ v'_\# \circ v'^* \xrightarrow{\sim} v_\# \circ h'_\# \circ v'^* \xrightarrow{v_\# \circ Ex^*_\#} v_\# \circ v^* \circ h_\# \xrightarrow{\theta \circ h_\#} h_\#$$

defined similarly to our map ϕ. Then we need to show that $\psi = h_\# \circ \theta'$.

For this, we note that the various morphisms and adjunctions already exist at the level of categories of simplicial presheaves $\mathrm{Spc}(-)$ and the corresponding maps at the level of the motivic stable homotopy category arise from these by localization and stabilization. Thus it suffices to show that $\psi = h_\# \circ \theta'$ as natural transformations of functors

$$h_\# \circ v'_\# \circ v'^*, h_\# : \mathrm{Spc}(W) \to \mathrm{Spc}(Z)$$

Since both of these functors are left adjoints, they both commute with colimits, and both are compatible with the simplicial structure on these model categories, hence it suffices to show that they agree after evaluation on representable presheaves, in other words, on the presheaf of sets on $\mathbf{Sm}_W$ represented by an object $p : T \to W$ in $\mathbf{Sm}_W$, considered as a presheaf of constant simplicial sets.

We have

$$v'^*(p : T \to W) = p_1 : W' \times_W T \to W'$$

and thus

$$h_\# \circ v'_\# \circ v'^*(p : T \to W) = h \circ v' \circ p_1 : W' \times_W T \to Z,\ h_\#(p : T \to W) = h \circ p : T \to Z.$$

The map $(\theta \circ h_\#)(p)$ is

$$p_2 : (v \circ p_1 : Z' \times_Z T \to Z) \to (h \circ p : T \to Z)$$

and the base-change isomorphism $(v_\# \circ Ex^*_\#)$ is the morphism in $\mathbf{Sm}_Z$

$$h' \times \mathrm{id} : (v \circ h' \circ p_1 : W' \times_W T \to Z) \to (v \circ p_1 : Z' \times_Z T \to Z),$$

giving the "associativity" isomorphism

$$W' \times_W T \cong (Z' \times_Z W) \times_W T \cong Z' \times_Z T.$$

This shows that $\psi(p)$ is the composition

$$(h \circ v' \circ p_1 : W' \times_W T \to Z) = (h' \circ v \circ p_1 : W' \times_W T \to Z) \xrightarrow{h' \times \mathrm{id}} (v \circ p_1 : Z' \times_Z T \to Z) \xrightarrow{p_2} (h \circ p : T \to Z)$$

This is the same as the map

$$(h \circ v' \circ p_1 : W' \times_W T \to Z) \xrightarrow{p_2} (h \circ p : T \to Z)$$

which is $h_\# \circ \theta'(p)$.

□

Lemma 3.7 (Lifting the isomorphism over smooth maps)**.** *Let $h : Y \to X$ be a smooth morphism and let $f : X \to B$ as in Proposition 3.3. Let $E = h^{-1}(D)$, $E^\circ = h^{-1}(D^\circ)$ and let $E^\circ \xhookrightarrow{v'} E \xhookrightarrow{u'} Y_\sigma \subset Y$ denote the inclusions. Let $g = f \circ h$.*

Suppose that precomposing with the adjunction $\eta_v : id_{\mathrm{SH}(D)} \to v_* v^*$ gives an isomorphism

$$u^* \Psi_f f^*_\eta \simeq v_* v^* (u^* \Psi_f f^*_\eta).$$

Then precomposing with the adjunction $\eta_{v'} : id_{\mathrm{SH}(E)} \to v'_* v'^*$ also gives an isomorphism

$$u'^* \Psi_g g^*_\eta \simeq v'_* v'^* (u'^* \Psi_g g^*_\eta). \tag{3.1}$$

Proof. We have the following diagram with Cartesian squares:

$$\begin{tikzcd}
E^\circ \arrow[r, hook, "v'"] \arrow[d, "h_{E^\circ}"] & E \arrow[r, hook, "u'"] \arrow[d, "h_E"] & Y_\sigma \arrow[r, hook] \arrow[d, "h_\sigma"] & Y \arrow[d, "h"] \\
D^\circ \arrow[r, hook, "v"] & D \arrow[r, hook, "u"] & X_\sigma \arrow[r, hook] \arrow[d, "f_\sigma"] & X \arrow[d, "f"] \\
 & & \sigma \arrow[r, hook] & B
\end{tikzcd}$$

We have the smooth base change exchange isomorphism in the left square

$$Ex^*_* : h^*_E v_* \xrightarrow{\sim} v'_* h^*_{E^\circ}$$

Since h is smooth, Property 3.1 gives us the isomorphism

$$\alpha_h : h^*_\sigma \Psi_f \xrightarrow{\sim} \Psi_g h^*_\eta$$

This gives us the isomorphism

$$\beta : u'^* \Psi_g g^*_\eta \xrightarrow{\sim} h^*_E u^* \Psi_f f^*_\eta$$

defined as the composition

$$u'^* \Psi_g g^*_\eta \xrightarrow{(i)} u'^* \Psi_g h^*_\eta f^*_\eta \xrightarrow{(ii)} u'^* h^*_\sigma \Psi_f f^*_\eta \xrightarrow{(iii)} h^*_E u^* \Psi_f f^*_\eta.$$

Here (i) and (iii) are induced by the canonical isomorphisms $g^*_\eta \cong h^*_\eta f^*_\eta$, $u'^* h^*_\sigma \cong h^*_E u^*$ following from the functoriality of *, and (ii) is induced by the inverse of α_h. Composing with the adjunction unit $\eta_{v'}$ gives the commutative diagram

$$\begin{tikzcd}
u'^* \Psi_g g^*_\eta \arrow[r, "\beta"] \arrow[d, "\eta_{v'} \circ id"] & h^*_E u^* \Psi_f f^*_\eta \arrow[d, "\eta_{v'} \circ id"] \\
v'_* v'^* u'^* \Psi_g g^*_\eta \arrow[r, "id \circ \beta"] & v'_* v'^* h^*_E u^* \Psi_f f^*_\eta
\end{tikzcd}$$

Let $\gamma : v'_* v'^* h^*_E u^* \Psi_f f^*_\eta \to h^*_E v_* v^* u^* \Psi_f f^*_\eta$ be the isomorphism defined as the composition

$$v'_* v'^* h^*_E u^* \Psi_f f^*_\eta \xrightarrow{(iv)} v'_* h^*_{E^\circ} v^* u^* \Psi_f f^*_\eta \xrightarrow{(v)} h^*_E v_* v^* u^* \Psi_f f^*_\eta$$

where (iv) is induced by the functoriality isomorphism $v'^* h^*_E \cong h^*_{E^\circ} v^*$ and (v) is induced by the inverse of the smooth base-change isomorphism $Ex^*_* : h^*_E v_* \to v'_* h^*_{E^\circ}$. It follows from Lemma 3.6 that the diagram

$$\begin{tikzcd}
h^*_E u^* \Psi_f f^*_\eta \arrow[d, "\eta_{v'} \circ id"] \arrow[dr, "h^*_E \circ \eta_v \circ id"] & \\
v'_* v'^* h^*_E u^* \Psi_f f^*_\eta \arrow[r, "\gamma"] & h^*_E v_* v^* u^* \Psi_f f^*_\eta
\end{tikzcd}$$

commutes. Putting these two diagrams together gives the commutative diagram

$$\begin{tikzcd}
u'^* \Psi_g g^*_\eta \arrow[r, "\beta"] \arrow[d, "\eta_{v'} \circ id"] & h^*_E u^* \Psi_f f^*_\eta \arrow[d, "h^*_E \circ \eta_v \circ id"] \\
v'_* v'^* u'^* \Psi_g g^*_\eta \arrow[r, "\gamma \circ (id \circ \beta)"] & h^*_E v_* v^* u^* \Psi_f f^*_\eta
\end{tikzcd} \tag{3.2}$$

in which the horizontal maps are isomorphisms. By assumption, the map $h^*_E \circ \eta_v \circ id$ is an isomorphism, hence $\eta_{v'} \circ id$ is an isomorphism as well, which is what we wanted to show. $\square$

Proof of Proposition 3.3. Let $p \in X_\sigma$, and order the components of X_σ such that p is contained in the first m, $D_1, \dots, D_m$, and not contained in the rest $D_{m+1}, \dots, D_r$. There is an affine neighbourhood $U \subset X$ of p, and étale local coordinates $t_1, \dots, t_n \in \mathcal{O}_X(U)$ $(n = dim X)$, $u \in \mathcal{O}_X(U)^\times$, such that $\{t_i = 0\}$ defines $D_i \cap U$ on U, $f^*(t) = ut_1 \cdot \dots \cdot t_m$, and $D \cap U$ is defined on U by $\bigcup_{i \in I} V(t_i)$ for some $I \subset \{1, \dots, n\}$. We may assume $u = 1$ by replacing t_1 by ut_1 and shrinking U if necessary.

Recall that X is a scheme over $k[t]$. We have the $k[T]$-algebra homomorphism

$$k[t][T_1, \dots, T_n]/(T_1 \dots, T_m - t) \to \mathcal{O}_X(U)$$

sending T_i to t_i; since the t_i are étale local coordinates, this gives us the smooth morphism

$$\phi : U \to \operatorname{Spec} k[t][T_1, \dots, T_n]/(T_1 \dots, T_m - t).$$

Let now $q : \operatorname{Spec} k[t, T_1, \dots, T_n]/(T_1 \cdot \dots \cdot T_m - t) \to \operatorname{Spec} k[t, T_1, \dots, T_m]/(T_1 \cdot \dots \cdot T_m - t)$ be the smooth projection induced by the inclusion of polynomial rings. In the diagram below the vertical arrows in the upper part are restrictions of ϕ, and the vertical arrows in the lower part are restrictions of q.

$$\begin{tikzcd}
D \cap U \arrow[r, hook] \arrow[d] & X_\sigma \cap U \arrow[r, hook] \arrow[d] & U \arrow[d, "\phi"] \\
\bigcup_{i \in I} V(T_i) \arrow[r, hook, "u"] \arrow[d] & \operatorname{Spec} k[T_1, \dots, T_n]/(T_1 \cdot \dots \cdot T_m) \arrow[r, hook] \arrow[d] & \operatorname{Spec} k[t, T_1, \dots, T_n]/(T_1 \cdot \dots \cdot T_m - t) \arrow[d, "q"] \\
\bigcup_{i \in I} V(T_i) \arrow[r, hook] & \operatorname{Spec} k[T_1, \dots, T_n]/(T_1 \cdot \dots \cdot T_m) \arrow[r, hook] & \operatorname{Spec} k[t, T_1, \dots, T_m]/(T_1 \cdot \dots \cdot T_m - t)
\end{tikzcd}$$

The statement proven in [Ay07a, Thm 3.3.10] gives the desired identity for

$$X = \operatorname{Spec} k[t, T_1, \dots, T_m]/(T_1 \cdot \dots \cdot T_m - t),$$

with $X_\sigma = k[T_1, \dots, T_m]/(T_1 \cdot \dots \cdot T_d)$ and $D = \bigcup_{i \in I} V(t_i)$, just as appears in the bottom row of the diagram.

From Lemma 3.7 we can 'lift' the statement to hold for the upper row of the diagram. Thus, the proposition holds for the restriction $f_U : U \to B$ of f.

Let $j : U \to X$ be the inclusion. We denote by D_U, D_U°, v_U, and so on, the restrictions to U of the various objects involved in our discussion. For instance, let $j_{DU} : D_U \to D$ be the open immersion induced by j_U and let $u_U : D_U \to U_\sigma$ be the closed immersion induced by $u : D \to X_\sigma$. We apply the commutative diagram (3.2) of Lemma 3.7 with $h = j_U$. This gives us the commutative diagram

$$\begin{tikzcd}
u_U^* \Psi_{f_U} f_{U\eta}^* \arrow[r] \arrow[d, "\eta_{v_U} \circ id"] & j_{DU}^* u^* \Psi_f f_\eta^* \arrow[d, "j_{DU}^* \circ \eta_v \circ id"] \\
v_{U*} v_U^* u_U^* \Psi_{f_U} f_{U\eta}^* \arrow[r] & j_{DU}^* v_* v^* u^* \Psi_f f_\eta^*
\end{tikzcd}$$

with horizontal arrows being isomorphisms. We have shown that $\eta_{v_U} \circ id$ is an isomorphism, hence the restriction $j_{DU}^* \circ \eta_v \circ id$ of $\eta_v \circ id : u^* \Psi_f f_\eta^* \to v_* v^* u^* \Psi_f f_\eta^*$ is an isomorphism. Thus the natural transformation $u^* \Psi_f f_\eta^* \to \eta_v \circ id v_* v^* u^* \Psi_f f_\eta^*$ is an isomorphism after restriction to a finite open covering $\mathcal{U}$ of X. If we apply $\eta_v \circ id$ to an object $\alpha \in \mathrm{SH}(B)$, the Mayer-Vietoris spectral sequence for the cover $\mathcal{U}$ shows that $(\eta_v \circ id)(\alpha)$ is an isomorphism, hence $\eta_v \circ id$ is a natural isomorphism.

□

3.2 The Euler characteristic of nearby cycles

We take $\mathcal{O}$ and $B = \operatorname{Spec}\mathcal{O}$ as in Section 3.1.

Let $f : X \to B$ be a flat quasi-projective morphism with X smooth over k_0 and X_η smooth over η. We retain the notation from Section 3.1, and make some first computations of $\chi_c(\Psi_f(\mathbb{1}_{X_\sigma}))$.

Proposition 3.8. *(1) $f_{\sigma !}\Psi_f(\mathbb{1}_{X_\eta})$ is a strongly dualisable object in $SH(k)$.*

(2) $\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) \in \mathrm{GW}(k)$ is well-defined.

Proof. For the first assertion, Ψ_f sends constructible objects to constructible objects [Ay07a, Théorème 3.5.14] and constructibles are stable under $f_!$ [Ay07a, Corollaire 2.2.20], hence $f_{\sigma !}\Psi_f(\mathbb{1}_{X_\eta})$ is constructible and therefore strongly dualisable (Proposition 2.8).

(2) follows from (1) and Definition 2.9. □

For $i : Z \to Y$ the inclusion of a locally closed subscheme, and $\alpha \in \mathrm{SH}(Y)$, we sometimes write $\alpha|_Z$ for $i^*(\alpha) \in \mathrm{SH}(Z)$.

By the following formal consequence of the properties of Ψ_f one can compute $\chi_c(\Psi_f)$ by just investigating Ψ_f around isolated singularities.

Proposition 3.9. *[LPS, Proposition 8.3] Assume $P = \{p_1, \ldots, p_s\}$ is the (finite) set of singular points in X_σ. Then*

$$\chi_c(\Psi_f(\mathbb{1}_{X\eta})) = \sum_i \chi_c(\Psi_f(\mathbb{1}_{X\eta})|_{p_i}) + \chi_c(X_\sigma \setminus P)$$

Proof. Denote by $j : X \setminus P \hookrightarrow X$, then by Property 3.1,

$$\Psi_f(\mathbb{1}_{X\eta})|_{X_\sigma \setminus P} \simeq \Psi_{f\circ j}(j_\eta^* \mathbb{1}_X) = \Psi_{f\circ j}((\mathbb{1}_{X\setminus P})_\eta) = \mathbb{1}_{X_\sigma \setminus P}$$

the last equality being since $X \setminus P$ is smooth (e.g. by 3.3).

Then by cut-and paste (2.11)

$$\chi_c(\Psi_f(\mathbb{1}_{X\eta})) = \sum_i \chi_c(\Psi_f(\mathbb{1}_{X\eta})|_{p_i}) + \chi_c(\Psi_f(\mathbb{1}_{X\eta})|_{X_\sigma \setminus P})$$

and we get the result. □

The following example illustrates how we can use Proposition 3.3 to compute $\chi_c(\Psi_f)$ on a reduced normal crossing divisor stratum by stratum.

Example 3.10. Suppose X_σ is a reduced normal crossing divisor on X that can be written as $X_\sigma = D_1 + D_2$ with D_1 and D_2 smooth over σ and with transverse intersection $D_{12} := D_1 \cap D_2$. Let $D_i^\circ := D_i \setminus D_{12}$, $i = 1, 2$.

We have the close-open complements

$$D_1 \overset{u_1}{\hookrightarrow} X_\sigma \overset{j}{\hookleftarrow} D_2^\circ.$$

Then by Proposition 2.11

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(\Psi_f(\mathbb{1}_{X_\eta})|_{D_1}) + \chi_c(\Psi_f(\mathbb{1}_{X_\eta})|_{D_2^\circ}).$$

Using Proposition 3.3

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(v_{1*}\mathbb{1}_{D_1^\circ}) + \chi_c(\mathbb{1}_{D_2^\circ}),$$

applying both equations of Proposition 2.11 to $\mathbb{1}_{D_1}$ and the close-open complements

$$D_{12} \overset{i}{\hookrightarrow} D_1 \overset{v_1}{\hookleftarrow} D_1^0$$

gives

$$\chi_{\mathrm{c}}(\mathbb{1}_{D_1}) = \chi_{\mathrm{c}}(i^*\mathbb{1}_{D_1}) + \chi_{\mathrm{c}}(v_1^*\mathbb{1}_{D_1}) = \chi_{\mathrm{c}}(\mathbb{1}_{D_{12}}) + \chi_{\mathrm{c}}(\mathbb{1}_{D_1^\circ})$$

and

$$\chi_{\mathrm{c}}(v_{1*}\mathbb{1}_{D_1^\circ}) = \chi_{\mathrm{c}}(v_{1*}v_1^*\mathbb{1}_{D_1}) = \chi_{\mathrm{c}}(\mathbb{1}_{D_1}) - \chi_{\mathrm{c}}(i^!\mathbb{1}_{D_1}).$$

Applying Proposition 2.14, we have

$$\chi_{\mathrm{c}}(v_{1*}\mathbb{1}_{D_1^\circ}) = \chi_{\mathrm{c}}(\mathbb{1}_{D_1}) - \langle -1\rangle\, \chi_{\mathrm{c}}(\mathbb{1}_{D_{12}}).$$

Combining the equations we get

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_{\mathrm{c}}(\mathbb{1}_{D_{12}}) + \chi_{\mathrm{c}}(\mathbb{1}_{D_1^\circ}) - \langle -1\rangle\, \chi_{\mathrm{c}}(\mathbb{1}_{D_{12}}) + \chi_{\mathrm{c}}(\mathbb{1}_{D_2^\circ}).$$

We obtain the nice formula

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_{\mathrm{c}}(D_1^\circ) + \chi_{\mathrm{c}}(D_2^\circ) - (\langle -1\rangle - \langle 1\rangle) \cdot \chi_{\mathrm{c}}(D_{12}).$$

This exhibits how Proposition 3.3 enables us to compute the Euler characteristic of the nearby cycles functor of the unit when the special fibre is a reduced normal crossing divisor. We would like to be able reduce to that case also when the fibre is not reduced.

3.3 Semi-stable reduction

Let $\mathcal{O}$ be our discrete valuation ring, and $B = \operatorname{Spec}\mathcal{O}$ as in Section 3.1. Let $f : X \to B$ be as in Section 3.2 a flat quasi-projective morphism with X smooth over k_0 and X_η smooth over η. Let $\mathcal{O}_e := \mathcal{O}[s]/(s^e - t)$, $B_e := \operatorname{Spec}\mathcal{O}_e$ and $b_e : B_e \to B$ the projection. Let $X_e := X \times_B B_e$. Note that $\sigma_e = \sigma$ as the residue field does not change by adding a root, but $\eta_e \to \eta$ may not be trivial.

Definition 3.11. A semi-stable reduction datum for f consists of a natural number e and a projective birational map $p_e : Y \to X_e$, such that Y is smooth over k_0, Y_σ is a **reduced** normal crossings divisor and $p_{e\eta} : Y_\eta \to X_{\eta_e}$ is an isomorphism. In addition, we will require that the cover $B_e \to B$ is *tame*, that is, that e is prime to the exponential characteristic of k.

$$\begin{tikzcd}
Y_\sigma \arrow[rr, hook] \arrow[d] & & Y \arrow[d, "p_e"] & & Y_\eta \arrow[ll, hook'] \arrow[d, "p_{e\sigma}"] & \\
X_{\sigma e} \arrow[rr, hook] \arrow[dr] \arrow[dd] & & X_e \arrow[dr, "\pi"] \arrow[dd, "f_e"] & & X_{\eta_e} \arrow[ll, hook'] \arrow[dr] \arrow[dd] & \\
& X_\sigma \arrow[rr, hook] \arrow[dd] & & X \arrow[dd, "f"] & & X_\eta \arrow[ll, hook'] \arrow[dd] \\
\sigma_e \arrow[rr, hook] \arrow[dr] & & B_e \arrow[dr] & & \eta_e \arrow[ll, hook'] \arrow[dr] & \\
& \sigma \arrow[rr, hook] & & B & & \eta \arrow[ll, hook']
\end{tikzcd}$$

A theorem by Kempf, Knudsen, Mumford, and Saint-Donat [KKMSD] asserts that over a field of characteristic 0, and base B a smooth curve, every variety X admits a semi-stable reduction.

Proposition 3.12. *Assume* $f : X \to B$ *admits a semi-stable reduction* $Y \xrightarrow{p_e} X_e \xrightarrow{f_e} B_e$ *for some* e*. Let* $\pi : X_e \to X$ *be the projection, and let* $f_Y = f_e \circ p_e$*. Then*

$$\Psi_f(\mathbb{1}_{X_\eta}) \simeq (\pi \circ p_e)_{\sigma *} \circ \Psi_{f_Y}(\mathbb{1}_{Y_\eta})$$

Proof. By [Ay07a, Proposition 3.5.9], we have the natural isomorphism $\Psi_f \simeq p_{\sigma *} \circ \Psi_{f_e} \circ p_\eta^*$. Since $p_{e\eta}$ is an isomorphism, the natural map $\mathrm{id}_{\mathrm{SH}(X_{e\eta})} \to p_{e\eta *} \circ p_{e\eta}^*$ is an isomorphism. This together with the pushforward property of Ψ for projective maps, Property 3.1(b), gives the sequence of isomorphisms

$$\begin{aligned}\Psi_f(\mathbb{1}_{X_\eta}) \simeq p_{\sigma *} \circ \Psi_{f_e}(\mathbb{1}_{X_{e\eta}}) \simeq p_{\sigma *} \circ \Psi_{f_e} \circ p_{e\eta *} \circ p_{e\eta}^*(\mathbb{1}_{X_{e\eta}}) \\ \simeq p_{\sigma *} \circ p_{e *} \circ \Psi_{f_Y}(\mathbb{1}_{Y_\eta}) \simeq (\pi \circ p_e)_{\sigma *} \circ \Psi_{f_Y}(\mathbb{1}_{Y_\eta}).\end{aligned}$$

□

As a consequence we can compute $\chi_c(\Psi_f)$ on a semi-stable reduction.

Corollary 3.13. $\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(\Psi_{f_Y}(\mathbb{1}_{f_Y}))$.

Proof. $(\pi \circ p_e)_\sigma$ is proper, so $(\pi \circ p_e)_{\sigma *} = (\pi \circ p_e)_{\sigma !}$. Since $\sigma_e = \sigma$, we thus have

$$\begin{aligned}\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_{\mathrm{SH}(k)}(f_{\sigma !} \circ \Psi_f(\mathbb{1}_{X_\eta})) = \chi_{\mathrm{SH}(k)}(f_{\sigma !} \circ (p \circ p_e)_{\sigma !} \circ \Psi_{f_Y}(\mathbb{1}_{Y_\eta})) \\ = \chi_{\mathrm{SH}(k)}(f_{Y\sigma !} \circ \Psi_{f_Y}(\mathbb{1}_{Y_\eta})) = \chi_c(\Psi_{f_Y}(\mathbb{1}_{f_Y})).\end{aligned}$$

□

Example 3.14. Assume X has a normal crossing special fibre with two components

$$X_\sigma = aD_1 + bD_2$$

which are not necessarily reduced, and X admits a semi-stable reduction Y, with a (reduced) special fibre $Y_\sigma = \widetilde{D_1} + \widetilde{D_2}$; let $\widetilde{D_{12}}$ denote the intersection $\widetilde{D_1} \cap \widetilde{D_2}$. Then by Example 3.10 and Corollary 3.13 we get

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(\Psi_{f_Y}(\mathbb{1}_{f_Y})) = \chi_c(\widetilde{D_1^\circ}) + \chi_c(\widetilde{D_2^\circ}) - (\langle -1 \rangle - \langle 1 \rangle) \cdot \chi_c(\widetilde{D_{12}})$$

Since $\chi_c(\mathbb{G}_m) = \chi_c(\mathbb{A}^1) - \chi_c(pt) = \langle -1 \rangle - \langle 1 \rangle$, the formula can be rewritten as

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(\widetilde{D_1^\circ}) + \chi_c(\widetilde{D_2^\circ}) - \chi_c(\mathbb{G}_m \times \widetilde{D_{12}})$$

3.4 Denef-Loeser coverings

In the course of their work on motivic integration and motivic Zeta functions, Denef and Loeser define a motivic Milnor fibre of morphism $f : X \to \mathbb{A}^1$ [DL00, 3.3], [DL98, 4] as an element in the Grothendieck ring of varieties, defined by certain coverings of the strata of the special fibre of a resolution of f. Ayoub, Ivorra and Sebag prove that the class of Ayoub's functor at the identity in this ring can be computed by an alternating sum involving these coverings [AIS, Thm. 8.6]; their proof relies on the use of motivic stable homotopy categories for rigid analytic sheaves. We treat here some simple cases in which semi-stable reduction can be achieved by a simple construction, and then the formula can be proven by purely geometric means, relying on the properties of the nearby cycles functor developed in the previous section. We take $\mathcal{O}$ and $B = \operatorname{Spec} \mathcal{O}$ as in Section 3.1. Let $\sigma \hookrightarrow B \hookleftarrow \eta$ be the closed point and generic point of B.

We recall the construction of the Denef-Loeser covers following the description in [IS, 3.1]:

Let $f : X \to B$ be a flat quasi-projective morphism with X smooth over k_0 and X_η smooth over η, and suppose X_σ is a simple normal crossing divisor. We write $X_\sigma = a_1 D_1 + \ldots + a_r D_r$ with $D_1, \ldots, D_r$ the reduced irreducible components and assume that if $char k = p > 1$, then $p \nmid a_i$ for each i.

Let I be a non-empty subset of $\{1, \ldots, r\}$, giving us the closed stratum $u_I : D_I \to X_\sigma$ and open substratum $v_I : D_I^\circ \to D_I$. f may be described on some affine open neighbourhood U of some point of D_I as

$$f = u \cdot \prod_{i \in I} t_i^{a_i}$$

with $t_i \in \mathcal{O}_X(U)$, $u \in \mathcal{O}_X(U)^\times$, and D_i being $V(t_i)$ in U.

Let $N_I = gcd_{i \in I}(a_i)$. We have the finite étale cover

$$\widetilde{D_{I,U}} := \mathrm{Spec}(\mathcal{O}_{D_I^\circ \cap U}[T]/(T^{N_I} - u)) \to D_I^\circ \cap U.$$

The finite morphism $\widetilde{D_I} \to D_I$ is defined as the normalisation of D_I in $\widetilde{D_{I,U}}$ and $\widetilde{D_I^\circ} \subset D_I$ is defined to be the open subscheme $\widetilde{D_I} \times_{D_i} D_I^\circ$ of $\widetilde{D_I}$. One shows that this construction is independent of the choice of U and hence $\widetilde{D_I^\circ} \to D_I^\circ$ is étale.

We call the coverings $\widetilde{D_I} \to D_I$, $\widetilde{D_I^\circ} \to D_I^\circ$ the Denef-Loeser coverings of D_I, D_I°, respectively. These coverings are well-defined up to isomorphism and do not depend on the choice of open neighbourhood and local coordinates.

In some cases semi-stable reduction can be achieved by taking $p : Y \to X_e$ to be the normalisation of a base change X_e of X, and the components of the special fibre $Y_\sigma = \widetilde{D_1} + \ldots + \widetilde{D_r}$ which lie above $D_1, \ldots, D_r$ give indeed the Denef-Loeser coverings described here. We address such a situation in the following proposition.

Proposition 3.15. *Let $f : X \to B = \mathrm{Spec}\,\mathcal{O}$ be a flat morphism, essentially of finite type. We assume that X is smooth over k_0, with the generic fibre X_η smooth over η. Suppose X_σ is a normal crossing divisor, $X_\sigma = aD_1 + bD_2$, with each D_i smooth over σ. Suppose in addition that $gcd(a, b) = 1$, and if $char k = p > 0$ then $p \nmid a, b$. Let $e = ab$.*

Form the base-change X_e as defined above and let $Y \to \mathrm{Spec}\,\mathcal{O}_e$ be the normalisation of X_e, with the induced morphism $h : Y \to X$. Let $E_i = h^{-1}(D_i)_{red}$, $i = 1, 2$. Then

(1) Y is a smooth k_0-scheme.

(2) E_1 and E_2 are smooth divisors on Y, intersecting transversally. In particular, $Y_\sigma = E_1 + E_2$ is a reduced normal crossing divisor and Y is a semi-stable reduction of X.

(3) The maps $E_I \to D_I$, $\emptyset \neq I \subset \{1, 2\}$, are isomorphic to the Denef-Loeser covers $\tilde{D}_I \to D_I$.

Proof. Let m, n be integers such that $1 = ma + nb$.

For the first assertion, take $q \in Y$; we will show that Y is smooth over k_0 at q. For $q \in Y_{\eta_e} \simeq X_{\eta_e}$, then as B is smooth over k_0 and $B_e \to B$ is tame, B_e is also smooth over k_0. Since X_{η_e} is smooth over η_e, we see that Y is smooth over k_0 at q.

If q is a point of Y_σ, let $p = h(q)$. We deal separately with the cases $p \in D_{12}$, $p \in D_1^\circ$, $p \in D_2^\circ$.

For $p \in D_{12}$, f may be locally described on some affine open $U \ni p$ by $t = ux^a y^b$, $x, y \in \mathcal{O}_X(U)$ local coordinates on U with $V(x) = D_1 \cap U$, $V(y) = D_2 \cap U$ and $u \in \mathcal{O}_X(U)^\times$.

We may assume $u = 1$ as $ux^a y^b = u^{ma+nb} x^a y^b = (u^m x)^a (u^n y)^b$ and so by replacing x and y by unit multiples we can get rid of u.

In the e-base change scheme X_e, $s^e = t$, the defining equation on U_e becomes $s^e = x^a y^b$.

Normalisation can be achieved by adjoining roots $z^b = x$, $w^a = y$ as follows. Set $z = \frac{s^{am} x^n}{y^m}$, $w = \frac{s^{bn} y^m}{x^n}$ and let $V = h^{-1}(U)$. Then z and w are in $Frac(\mathcal{O}_{X_e}(U_e))$ and satisfy the integral equations above, so z and w are in the normalisation $\mathcal{O}_Y(V)$, and satisfy the equation $z \cdot w = s$.

Now consider the ring $\mathcal{O}_X(U)[s,z,w] \subset \mathcal{O}_Y(V)$. We claim that in fact $\mathcal{O}_X(U)[s,z,w] = \mathcal{O}_Y(V)$ and that V is smooth over k. Indeed, since x, y are local coordinates they define an étale map $\operatorname{Spec} \mathcal{O}_X(U) \to \mathbb{A}^2_k$. This gives the étale ring extension $k[X,Y] \to \mathcal{O}_X(U)$. The algebraic picture after adjoining s, z, w to the ring $\mathcal{O}_X(U)$ is described by the following commutative diagram:

$$\begin{tikzcd}
k_0[X,Y] \arrow[r] \arrow[d] & \mathcal{O}_X(U) \arrow[d] \\
k_0[X,Y,S,Z,W]/(S-ZW, S^e - X^aY^b, Z^b - X, W^b - Y) \arrow[r] & \mathcal{O}_X(U)[s,z,w]
\end{tikzcd}$$

which induces a surjective homomorphism

$$\phi : \mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[X,Y,S,Z,W]/(S-ZW, S^e - X^aY^b, Z^b - X, W^a - Y) \to \mathcal{O}_X(U)[s,z,w].$$

We claim that ϕ is an isomorphism. To see this, denote the quotient ring in the left lower corner by C. Of the equations defining C, the second is redundant as it follows from the other three, the first one makes the variable S redundant, and the last two makes X and Y redundant, so we can write

$$C \simeq k_0[Z,W].$$

Since $k_0[X,Y] \to \mathcal{O}_X(U)$ is smooth, the homomorphism $k_0[Z,W] \to \mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$ is smooth as well, hence $\mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$ is smooth over k_0, of Krull dimension equal to the Krull dimension of $\mathcal{O}_X(U)$. From the equations defining C we can deduce the further relations

$$ZY^m = X^n S^{am}, WX^n = S^{bn}Y^m. \tag{3.3}$$

From the relations $S^e = X^aY^b$, $S = ZW$, and $t = x^ay^b$, we see that canonical map $\mathcal{O}_X(U) \to \mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$ extends to $\mathcal{O}_X(U)[s]/(s^e - t) \to \mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$ by sending s to $1 \otimes ZW$. After inverting x and y, the relations (3.3) and the universal property of the localization yield an extension of this homomorphism to a homomorphism

$$\psi : \mathcal{O}_X(U)[x^{-1}, y^{-1}][s,z,w] \to \mathcal{O}_X(U)[x^{-1}, y^{-1}] \otimes_{k_0[X,Y]} k_0[Z,W]$$

sending z to $1 \otimes Z$, w to $1 \otimes W$, and it is easy to see that ψ defines a inverse to ϕ, after inverting x and y. Furthermore, the extension $k_0[X,Y] \to k_0[Z,W]$ is flat, so $\mathcal{O}_X(U) \to \mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$ is flat as well, and thus x and y are non-zero divisors on $\mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$. As $\mathcal{O}_X(U) \otimes_{k_0[X,Y]} k_0[Z,W]$ and $\mathcal{O}_X(U)[s,z,w]$ have the same Krull dimension and both rings are finite type k_0-algebras, the surjective, birational k_0-algebra homomorphism ϕ has zero kernel (by Krull's principal ideal theorem), hence is an isomorphism, as claimed.

In addition, this shows that $\mathcal{O}_X(U)[s,z,w]$ is a smooth k_0-algebra. Since $\mathcal{O}_X(U)[s,z,w]$ contains $\mathcal{O}_{X_e}(U_e)$ and is contained in the normalisation $\mathcal{O}_Y(V)$ we have the desired equality $\mathcal{O}_X(U)[s,z,w] = \mathcal{O}_Y(V)$ and hence $V \subset Y$ is smooth. This also verifies that $Y_\sigma \cap U$, defined by $s = 0 = z \cdot w$, is a reduced divisor, $Y_\sigma \cap U \simeq \operatorname{Spec} \mathcal{O}_X(U)[z,w]/(zw)$, with $V \cap \widetilde{D_1} = V(z)$ and $V \cap \widetilde{D_2} = V(w)$.

By definition of the Denef-Loeser covers, since $gcd(a,b) = 1$, $\widetilde{D_{12}} \simeq D_{12} \xrightarrow{id} D_{12}$. But also

$$\begin{aligned} E_{12} \cap V \simeq \operatorname{Spec} \mathcal{O}_X(U)[z,w]/(z,w) &\simeq \operatorname{Spec}(\mathcal{O}_X(U) \otimes_{k[X,Y]} k[Z,W])/(Z,W) \\ &\simeq \operatorname{Spec} \mathcal{O}_X(U)/(x,y) \simeq D_{12} \cap U \end{aligned}$$

Thus E_{12} coincides with the Denef-Loeser cover $\widetilde{D_{12}} \simeq D_{12} \simeq E_{12}$.

We now consider the case $p \in D_1^\circ$; the case $p \in D_2^\circ$ is handled the same way. There is a neighbourhood $U \ni p$ on which f is described as $f^*(t) = t = u \cdot x^a$ with $u \in \Gamma(U, \mathcal{O}_X)^\times$ and

$U \cap \widetilde{D_1} = V(x)$. After e-base change we have the equation $s^e = u \cdot x^a$. Set $v = \frac{s^b}{x}$, then $v^a = u$, so v is in $\mathcal{O}_Y(V)$.

In a similar manner to the previous case we wish to describe the ring $\mathcal{O}_Y(V)$, to ascertain that $V \subset Y$ is smooth. We have to show that the inclusion $\mathcal{O}_X(U)[s,v] \subset \mathcal{O}_Y(V)$ is an equality. For this, we define a commutative square

$$\begin{tikzcd}
k_0[W, W^{-1}, X] \arrow[r] \arrow[d] & \mathcal{O}_X(U) \arrow[d] \\
k_0[W, W^{-1}, X, V, S]/(V^a - W, S^b - VX) \arrow[r] & \mathcal{O}_X(U)[v,s]
\end{tikzcd}$$

where the upper horizontal morphism is defined by $W \mapsto u$, $X \mapsto x$, and the lower one by $V \mapsto v$, $S \mapsto s$. We have the isomorphism

$$k_0[W, W^{-1}, X, V, S]/(V^a - W, S^b - VX) \simeq k_0[V, V^{-1}, X, S]/(S^b V^{-1} - X) \simeq k_0[V, V^{-1}, S].$$

As in the previous case, one shows that the square induces an isomorphism

$$\mathcal{O}_X(U) \otimes_{k_0[W,W^{-1},X]} k_0[V, V^{-1}, S] \xrightarrow{\sim} \mathcal{O}_X(U)[v,s],$$

so $\mathcal{O}_X(U)[v,s]$ is a smooth k_0-algebra and is therefore equal to the normalisation $\mathcal{O}_Y(V)$. Thus $V \subset Y$ is smooth and $Y_\sigma \cap V$, being defined by $s = 0$, is a smooth divisor on V.

We can now show that $\widetilde{D_1} \simeq E_1$ over D_1. Let $\pi : \widetilde{D_1} \to D_1$ be the Denef-Loser covering, U being the same neighbourhood of $p \in D_1^\circ$ as above. Then by definition $\pi^{-1}(D_1 \cap U) = \operatorname{Spec}(\mathcal{O}_X(U)[T]/(T^a - u))/(x) \simeq \mathcal{O}_X(U)(v)/(x)$. On the other hand

$$E_1 \cap V = \operatorname{Spec} \mathcal{O}_X(U)[v,s]/(s) \simeq \operatorname{Spec} \mathcal{O}_X(U)[v]/(x)$$

. We get $E_1 \cap V \simeq \pi^{-1}(D_1 \cap U)$. Since E_1 is normal and $\widetilde{D_1}$ is the normalisation of D_1 in $\pi^{-1}(D_1 \cap U)$, we get $\widetilde{D_1} \simeq E_1$.

The case $p \in D_2^\circ$ is the same as the case $p \in D_1^\circ$, and so $\widetilde{D_2} \simeq E_2$. This completes the proof of (1), (2) and (3). ☐

Remark 3.16. With $f : X \to B = \operatorname{Spec} \mathcal{O}$ and a, b and $e = ab$ as in Proposition 3.15, suppose that X is irreducible and that $a = 1$. We retain the notation of Proposition 3.15. We claim that the base-change X_e is integral. To see this, let x be a generic point of D_1. Since X is smooth, D_1 is a Cartier divisor on X and thus the local ring $\mathcal{O}_{X,x}$ is a dvr. Moreover, since $a = 1$, t is a parameter for $\mathcal{O}_{X,x}$. Let $y \in X_e$ be the unique point lying over x. Then

$$\mathcal{O}_{X_e,y} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}} \mathcal{O}[s]/s^e - t = \mathcal{O}_{X,x}[s]/s^e - t.$$

Since e is prime to the characteristic, $\mathcal{O}_{X_e,y}$ is smooth over k, so $\mathcal{O}_{X_e,y}$ is a normal local ring, hence integral. Since $X_e \to X$ is finite and flat, each irreducible component of X_e dominates X, and thus X_e is irreducible and is also reduced in a neighbourhood of y. Since X_e is a hypersurface in the smooth k-scheme $X \times_k \operatorname{Spec} k[s]$, X is Cohen-Macaulay, and the fact that X_e is irreducible and generically reduced then implies that X_e is integral.

Proposition 3.17. *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a flat, quasi-projective morphism, with X smooth over k_0 and with generic fibre X_η smooth over η. Suppose that the special fibre X_σ is a normal crossing divisor $X_\sigma = \sum a_i D_i$; if char$k = p > 0$, we suppose in addition that $p \nmid \prod_i a_i$. Assume that for all $i \neq j$ $gcd(a_i, a_j) = 1$, and that there are no triple intersections, i.e. for each triple of distinct indices i, j, k, $D_i \cap D_j \cap D_k = \emptyset$.*

Denote by $\widetilde{D_i}$, $\widetilde{D_i^\circ}$, $\widetilde{D_{ij}}$ the Denef-Loeser coverings.

Then

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \sum_i \chi_c(\widetilde{D_i^\circ}) - \sum_{i<j} \chi_c(\mathbb{G}_m \times \widetilde{D_{ij}}).$$

Remark 3.18. This is a special case of the formula by Ayoub-Ivorra-Sebag [AIS, Thm. 8.6] which is proven there in a more general setting, relying on the theory of rigid analytic motives. The case considered here suffices for our use in this paper and follows from the same general construction as in our main result so we include it here.

Proof. To analyse each intersection separately consider

$$X_{ij} := X \setminus \bigcup_{k \neq i,j} D_k,$$

and set $D'_\ell := D_\ell \setminus \bigcup_{k\neq i,j} D_k$. Then ${X_{ij}}_\sigma = a_i D'_i + a_j D'_j$.Define $f_{ij} = f|_{X_{ij}} : X_{ij} \to B$.

By Proposition 3.15, X_{ij} admits a semi-stable reduction Y_{ij} with components of the special fibre giving the Denef-Loeser coverings $\widetilde{D'_i} \to D'_i$, $\widetilde{D'_j} \to D'_j$ and $\widetilde{D_{ij}} = D_{ij}$. Note that $(D'_i)^\circ = D^\circ_i$ and $(D'_j)^\circ = D^\circ_j$, so $\widetilde{D'^\circ_i} = \widetilde{D^\circ_i}$ and $\widetilde{D'^\circ_j} = \widetilde{D^\circ_j}$.

We can use Example 3.14 to get

$$\chi_{\mathrm{c}}(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})) = \chi_{\mathrm{c}}(\widetilde{D^\circ_i}) + \chi_{\mathrm{c}}(\widetilde{D^\circ_j}) - \chi_{\mathrm{c}}(\mathbb{G}_m \times \widetilde{D_{ij}}).$$

By the same argument applied to $X_{ij} \setminus D_{ij}$, we find

$$\chi_{\mathrm{c}}(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{X_{ij}\setminus D_{ij}}) = \chi_{\mathrm{c}}(\widetilde{D^\circ_i}) + \chi_{\mathrm{c}}(\widetilde{D^\circ_j}),$$

and by cut and paste, we have

$$\chi_{\mathrm{c}}(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{D_{ij}}) = \chi(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})) - \chi_{\mathrm{c}}(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{X_{ij}\setminus D_{ij}}),$$

so

$$\chi_{\mathrm{c}}(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{D_{ij}}) = -\chi_{\mathrm{c}}(\mathbb{G}_m \times \widetilde{D_{12}}).$$

Similarly,

$$\chi_{\mathrm{c}}(\Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{D^\circ_i}) = \chi_{\mathrm{c}}(\widetilde{D^\circ_i}).$$

Since X_{ij} is an open neighbourhood of D_{ij} in X, the compatibility of $\Psi_{(-)}$ with respect to the smooth morphism $X_{ij} \hookrightarrow X$ (Property 3.1) implies

$$\Psi_f(\mathbb{1}_{X_\eta})|_{D_{ij}} = \Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{D_{ij}}.$$

Similarly,

$$\Psi_f(\mathbb{1}_{X_\eta})|_{D^\circ_i} = \Psi_{f_{ij}}(\mathbb{1}_{{X_{ij}}_\eta})|_{D^\circ_i}.$$

By the cut and paste along $X_\sigma = \coprod_i D^\circ_i \coprod \coprod_{i<j} D_{ij}$ we have

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})) = \sum_i \chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})|_{D^\circ_i}) + \sum_{ij} \chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})|_{D_{ij}}) = \sum \chi_{\mathrm{c}}(\widetilde{D^\circ_i}) - \sum \chi_{\mathrm{c}}(\mathbb{G}_m \times \widetilde{D_{ij}})$$

□

Remark 3.19. Suppose that we drop the hypothesis that the components D_i are smooth, but assume that the same construction as in Proposition 3.15 applied to the schemes X_{ij} yield a semi-stable reduction Y_{ij} with smooth components for $(Y_{ij})_\sigma$, $\widetilde{D_i}$, $\widetilde{D_j}$, $\widetilde{D_{ij}}$ (so Y_{ij} is smooth, $(Y_{ij})_\sigma$ is a smooth normal crossing divisor and $(Y_{ij})_\eta \simeq (X_{ij})_\eta)$. In that case all the arguments in the proof of Proposition 3.17 still hold, and so the concluded formula does as well; however the terms $\widetilde{D_i}$ are no longer Denef-Loeser covers.

3.5 Nearby cycles at the base

We continue to use our discrete valuation ring $\mathcal{O}$, with subfield k_0, residue field k, fraction field K and parameter t, and let $B = \operatorname{Spec}\mathcal{O}$, as in Section 3.1; in this section, however, we assume in addition that k_0 has characteristic zero.

We have a ring homomorphism sp_t, (see for example [LPS, Remark 5.1]) from the Grothendieck-Witt ring of the fraction field K to that of the residue field k, characterised as the unique map

$$\mathrm{sp}_t : \mathrm{GW}(K) \to \mathrm{GW}(k)$$

satisfying:

(1) $\mathrm{sp}_t(t) = \langle 1 \rangle$ for the uniformizer t.

(2) $\mathrm{sp}_t(u) = \langle \bar{u} \rangle$ for all invertible elements $u \in O^\times$ where $\bar{u}$ denotes the image of u under the quotient map $\mathcal{O} \to k$.

Given a strongly dualisable object $\alpha \in \mathrm{SH}(K)$, the motivic Euler characteristic $\chi(\alpha)$ is an endomorphism of $\mathrm{SH}(K)$, and so the functor $\Psi_{id} : \mathrm{SH}(K) \to \mathrm{SH}(k)$ can be applied to it and produce an endomorphism of the unit in $\mathrm{SH}(k)$. Via the Morel isomorphism we get an object in $\mathrm{GW}(k)$. We state results from [LPS], which follows from the fact that Ψ_{id} is a monoidal functor in characteristic 0 [Ay07a, Corollaire 3.5.19].

Proposition 3.20 ([LPS, Lemma 8.1]). *For $\alpha \in \mathrm{SH}(K)$, we have $\Psi_{id*}(\chi(\alpha)) = \chi(\Psi_{id}(\alpha))$.*

In fact, sp_t computes Ayoub's functor Ψ_{id}.

Proposition 3.21 ([LPS, Proposition 8.2]). *The following diagram commutes.*

$$\begin{tikzcd}
\mathrm{End}_{\mathrm{SH}(K)}(\mathbb{1}_K) \arrow[r, "\Psi_{id*}"] \arrow[d, "\sim"] & \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{1}_k) \arrow[d, "\sim"] \\
\mathrm{GW}(K) \arrow[r, "sp_t"] & \mathrm{GW}(k)
\end{tikzcd}$$

Here the vertical arrows are Morel's isomorphisms.

4 The case of a homogeneous singularity

We continue to use our discrete valuation ring $\mathcal{O}$ and base-scheme $B := \operatorname{Spec}\mathcal{O}$, and retain the notations and assumptions from Section 3.1.

Let $f : X \to B$ be a flat quasi-projective morphism with X smooth over k_0 and with X_η is smooth over η. We make the following assumption on the special fibre.

Assumption 4.1. The reduced special fibre X_σ has only isolated singularities $p_1, \dots, p_r$. Moreover, if $\hat{X} = Bl_P(X)$ is the blow up of X at $P := \{p_1, \dots, p_r\}$, $E = E_{p_1} \amalg \dots \amalg E_r$ the exceptional divisor and $\pi^{-1}[X_\sigma] := \overline{\pi^{-1}(X_\sigma \setminus \{p_1, \dots, p_r\})}$ the proper transform, then $\pi^{-1}[X_\sigma]$ is smooth over k and intersects each E_i transversally.

We show that Assumption 4.1 is equivalent to having an 'analytic expansion' of f at each each singular point p of the form

$$f^*(t) = F(s_0, \dots, s_n) + h$$

with $s_0, \dots, s_n$ local coordinates at p, F a homogeneous polynomial of degree e defining a smooth projective hypersurface over $k(p)$, and $h \in m_p^{e+1}$, where m_p is the maximal ideal in $\mathcal{O}_{X,p}$.

We say then that locally at p, f looks like the homogeneous singularity defined by F (see Definition 1.2).

Proposition 4.2. *Assumption 4.1 above is equivalent to the following:*

(1) The special fibre X_σ has only isolated singularities.

(2) At each singular point p, let $\mathcal{O}_{X,p}$ denote the local ring at p, with maximal ideal m_p, let e_p be the maximal integer with $f^*(t) \in m_p^{e_p}$, and let $\overline{f^*(t)}_p$ be the image of $f^*(t)$ in $m_p^{e_p}/m_p^{e_p+1}$. Then $\overline{f^*(t)}_p$ defines a smooth hypersurface in $\operatorname{Proj}\operatorname{Sym}^*(m_p/m_p^2) \simeq \mathbb{P}^n_{k(p)}$.

Moreover, if Assumption 4.1 is satisfied then for each singular point p there is a neighbourhood U such that, letting $\hat{U} \to U$ denote the blow-up of U at p, the special fibre $\hat{U}_\sigma$ decomposes as $\hat{U}_\sigma = e_p D_1 + D_2$ with $D_1 \simeq \mathbb{P}^n_{k(p)}$ the reduced exceptional divisor and $D_2 = \pi^{-1}[U_\sigma]$ the strict transform of U_σ. Both D_1 and D_2 are smooth and intersect transversely, with $D_1 \cap D_2 \subset D_1$ the hypersurface defined by $\overline{f^*(t)}_p$.

Proof. Let p be a singularity and let $(s_0, \dots, s_n) = m_p$ be a regular sequence of parameters on the maximal ideal m_p of $\mathcal{O}_{X,p}$. We write

$$f^*(t) = F(s_0, \dots, s_n) + h$$

with F a homogeneous polynomial of degree e with coefficients in $\mathcal{O}_{X,p}$, and $h \in m_p^{e+1}$.

$\overline{f^*(t)} = F(\bar{s_0}, \dots, \bar{s}_n)$ is a homogeneous equation defining an hypersurface in $\mathbb{P}^n_{k(p)}$, $k(p)$ the residue field of $\mathcal{O}_{X,p}$. We show that this hypersurface is isomorphic to the intersection D_{12}.

Define

$$\hat{X} = Bl_p(\operatorname{Spec}\mathcal{O}_{X,p}) = \operatorname{Proj}\mathcal{O}_{X,p}[T_0, \dots, T_n]/(s_iT_j - s_jT_i)_{i<j}$$

Let $\hat{X} = \bigcup U_i$ be the standard covering of the blow up, where U_i is defined by $T_i \neq 0$.

For simplicity of notation we describe U_0 but the argument is similar for each of the U_i. Use s_0, $t_1 = T_1/T_0, \dots, t_n = T_n/T_0$ as coordinates on U_0.

$$U_0 = \operatorname{Spec}\mathcal{O}_{X.p}[T_1/T_0, .., T_n/T_0]/(s_iT_j - s_jT_i)_{i,j} = \operatorname{Spec}\mathcal{O}_{X,p}[t_1, \dots, t_n]/(s_0t_1 - s_1, \dots, s_0t_n - s_n)$$

We may write now

$$f^*(t) = s_0^e \cdot (F(1, t_1, \dots, t_n) + s_0\tilde{h}) =: s_0^e \cdot g_0$$

with $\tilde{h} \in m_p$. Then $D_1 \cap U_0 = V_{U_0}(s_0)$, $D_2 \cap U_0 = V_{U_0}(g_0)$ and $D_{12} = V_{U_0}(s_0, g_0)$; We have $(U_0)_\sigma = e \cdot (D_1 \cap U_0) + D_2 \cap U_0$ and similarly for all i, so $\hat{X}_\sigma = e \cdot D_1 + D_2$.

So $D_1 \cap U_0 \simeq \operatorname{Spec} \mathcal{O}_{X,p}[t_1,\dots,t_n]/(s_0,s_1,..,s_n) \simeq \operatorname{Spec} k(p)[t_1,\dots,t_n]$. We have a similar computation for each i. This shows that the $D_1 \cap U_i$ form the standard affine chart for the projective space $\mathbb{P}^n_{k(p)}$, giving the isomorphism $D_1 \simeq \mathbb{P}^n_{k(p)} = \operatorname{Proj} k(p)[T_0,\dots,T_n]$, with $D_1 \cap U_i$ defined as usual as the open subscheme $T_i \neq 0$.

$D_{12} \cap U_0$ is defined then by $F(1,t_1,\dots,t_n) = 0$ inside $D_1 \cap U_0$; making the same construction for general i shows that $D_{12} \cap U_i$ is defined by $F(t_1,\dots,t_{i-1},1,t_i,\dots,t_n) = 0$ inside $D_1 \cap U_i = \operatorname{Spec} k(p)[t_1,\dots,t_n]$, with $t_j = T_{j-1}/T_i$ for $j = 1,\dots,i$ and $t_j = T_j/T_i$ for $j = i+1,\dots,n$. This shows that D_{12} is globally defined in $D_1 \simeq \mathbb{P}^n_{k(p)}$ by F, as claimed. Thus the condition in the statement of the proposition is equivalent to the smoothness of D_{12}

Now, since the blow-up of X is smooth, $\operatorname{codim}(D_1) = \operatorname{codim}(D_2) = 1$ in the blow-up, and $\operatorname{codim}(D_{12}) = 2$ being a hypersurface in D_1, the condition of the proposition is equivalent to Assumption 4.1.

□

Theorem 4.3. *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a flat quasi-projective morphism with X smooth of dimension $n+1$ over k_0. Suppose that X_σ has a single singular point p and that locally at p, f looks like the homogeneous singularity defined by $F \in k(p)[T_0,\dots,T_n]$ of degree e, and that $V(F) \subset \mathbb{P}^n_{k(p)}$ is a smooth hypersurface. We suppose in addition that e is prime to the exponential characteristic of k_0.*

Let $q : \hat{X} \to X$ be the blow-up of X and let $\mathcal{O}_e = \mathcal{O}[s]/(s^e - t)$. Let $D_1 \subset \hat{X}$ be the exceptional divisor and let $D_2 \subset \hat{X}$ be the proper transform of X_σ. Then there exists a quasi-projective morphism $Y \to \operatorname{Spec} \mathcal{O}_e$ and a morphism $\pi : Y \to \hat{X}$ over $\operatorname{Spec} \mathcal{O}_e \to \operatorname{Spec} \mathcal{O}$ such that

(1) π defines a semi-stable reduction of X.

(2) The special fibre Y_σ is of the form $\widetilde{D_1} + \widetilde{D_2}$ with $\widetilde{D_1}$ and $\widetilde{D_2}$ smooth, with intersection $\widetilde{D_{12}}$, and with π mapping $\widetilde{D_1}$ to D_1, and $\widetilde{D_2}$ to D_2.

(3)

$$\widetilde{D_1} \simeq V(F - T^e_{n+1}) \subset \mathbb{P}^{n+1}_{k(p)}$$

The maps

$$\pi : \widetilde{D_2} \to D_2,\ \pi : \widetilde{D_{12}} \to D_{12} := D_1 \cap D_2$$

are isomorphisms, and

$$\widetilde{D_{12}} \simeq V(F) \subset \mathbb{P}^n_{k(p)}.$$

The morphism $\widetilde{D_1} \to D_1 = \mathbb{P}^n_{k(p)}$ is the evident cyclic cover, induced by the projection $\mathbb{P}^{n+1}_{k(p)} \setminus \{(0,\dots,0,1)\} \to \mathbb{P}^n_{k(p)}$ from $(0,\dots,0,1)$.

(4) $\widetilde{D_1} \to D_1$, $\widetilde{D_2} \to D_2$ and $\widetilde{D_{12}} \to D_{12}$ are the Denef-Loeser coverings.

Proof. By Proposition 4.3, $\hat{X}_\sigma = eD_1 + D_2$ with $D_1 \simeq \mathbb{P}^n$ and $D_2 \to X_\sigma$ a resolution of singularities of X_σ, and so $f \circ q : \hat{X} \to \operatorname{Spec} \mathcal{O}$ satisfies the requirements of Proposition 3.15 (with $a = e$, $b = 1$). Then we have the scheme Y constructed by first forming the base-change by $\mathcal{O} \to \mathcal{O}_e$, and then taking the normalisation. By Proposition 3.15, Y is a semi-stable reduction for $\hat{X}$. That is, Y is smooth over k_0 and $Y_\sigma = \widetilde{D_1} + \widetilde{D_2}$ is a reduced simple normal crossing divisor. Also if we denote by h the composition

$$h : Y \to \hat{X}_e \to \hat{X} \to X,$$

then $\widetilde{D_I} = h^{-1}(D_I) \to D_I$ are the Denef-Loeser coverings for all $\emptyset \neq I \subset \{1,2\}$. The only thing we have left to do is to give the explicit description of those coverings.

By definition of Denef-Loeser covers and since $b = 1$, $\widetilde{D_{12}} \simeq D_{12}$ and $\widetilde{D_2} \simeq D_2$. By Proposition 4.2 then, $\widetilde{D_{12}} \simeq V(F) \subset \mathbb{P}^n_{k(p)}$. In the remaining part of the proof we shall describe $\widetilde{D_1}$.

We only need to check the explicit description of the covering $\widetilde{D_1} \to D_1$ after restriction over some neighbourhood of p in X. Thus, we may replace X with the local scheme $\operatorname{Spec} \mathcal{O}_{X,p}$; we change notation and assume that $X = \operatorname{Spec} \mathcal{O}_{X,p}$ is local.

Take the standard covering of the blow-up $\hat{X} = \bigcup U_i$, where U_i is defined by $T_i \neq 0$. Write again

$$f^*(t) = F(s_0, \ldots, s_n) + h$$

with F a homogeneous polynomial of degree e and $h \in m_p^{e+1}$. Take s_0, $t_1 = T_1/T_0, \ldots, t_n = T_n/T_0$ as coordinates on U_0. Then

$$U_0 \simeq \operatorname{Spec} \mathcal{O}_{X,p}[t_1, \ldots, t_n]/(s_i - s_0 t_i).$$

On U_0, $f^*(t) = s_0^e \cdot (F(1, t_1, \ldots, t_n) + s_0 \tilde{h}) =: s_0^e \cdot g_0$ with $\tilde{h} \in m_p$ and $g_0 = F(1, t_1, \ldots, t_n) + s_0 \tilde{h}$. After the base change, on $U_{0,e} = U_0 \times_{\mathcal{O}} \operatorname{Spec} \mathcal{O}[t']/(t'^e - t)$ we have

$$U_{0,e} \simeq \mathcal{O}_{X,p}[t_1, \ldots, t_n, t']/(s_i - s_0 t_i, s_0^e \cdot g_0 - (t')^e).$$

Normalising amounts to adjoining $t_{n+1} = t'/s_0$, which is an integral element as $t_{n+1}^e = g_0$ [see the proof of Proposition 3.15]. So on V_0, the inverse image of U_0 in Y, we have

$$V_0 = \operatorname{Spec}(\mathcal{O}_{X,p}[t_1, \ldots, t_n, t_{n+1}]/(\{s_i - s_0 t_i\}_{1 \leq i \leq n}, g_0 - (t_{n+1})^e)).$$

The special fibre Y_σ then is covered by the $V_i = h^{-1}(U_i)$.

The exceptional divisor $\widetilde{D_1}$ is the fibre along $\operatorname{Spec} k(p) \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,p}$, defined by $s_0 = 0$ on V_0, and so

$$\widetilde{D_1} \cap V_0 = \operatorname{Spec} k(p)[t_1, \ldots, t_{n+1}]/(\bar{g}_0 - t_{n+1}^e)$$

where $\bar{g}_0 = F(1, t_1, .., t_n)$. Set $\mathbb{P}^{n+1}_{k(p)} = \operatorname{Proj} k(p)[T_0, \ldots, T_{n+1}] = \bigcup_{i=0}^{n+1} W_i$ to be the standard affine covering, with W_i corresponding to $T_i \neq 0$, and identify $\widetilde{D_1} \cap V_0$ as embedded in the affine space $W_0 = \operatorname{Spec} k(p)[t_1, \ldots, t_{n+1}]$ with $t_j = T_j/T_0$.

In order to describe the cover $\widetilde{D_1} \cap V_0 \to D_1 \cap U_0$, we also use the identification $D_1 = \mathbb{P}^n_{k(p)} = \operatorname{Proj} k(p)[T_0, \ldots, T_n]$ as in Proposition 4.2, with $D_1 \cap U_0$ being $\operatorname{Spec} k(p)[t_1, \ldots, t_n]$, still with $t_j = T_j/T_0$. We then get the restriction of the cover $\widetilde{D_1} \to D_1$ to V_0 to be

$$\begin{tikzcd}
\operatorname{Spec} k(p)[t_1, \ldots, t_n, t_{n+1}]/(F(1, t_1, \ldots, t_n) - t_{n+1}^e) \arrow[r] \arrow[d, hook] & \operatorname{Spec} k(p)[t_1, \ldots, t_n] \arrow[d, hook] \\
V(F - T_{n+1}^e) \subset \mathbb{P}^{n+1}_{k(p)} \arrow[r] & \mathbb{P}^n_{k(p)}.
\end{tikzcd}$$

This is the restriction of the cover $V_{\mathbb{P}^{n+1}_{k(p)}}(F - T_{n+1}^e) \to \mathbb{P}^n_{k(p)}$ over W_0.

Similarly, for each $i = 0, .., n$, the cover $\widetilde{D_1} \cap V_i \to D_1 \cap U_i$ is

$$\operatorname{Spec} k(p)[t_1, \ldots, t_n, t_{n+1}]/(F(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) - t_{n+1}^e) \to \operatorname{Spec} k(p)[t_1, \ldots, t_n]$$

with $t_j := T_j/T_i$ as in the proof of Proposition 4.2. Here we are considering $V_i \cap \widetilde{D_1}$ as a closed subscheme of W_i. We get $\widetilde{D_1} \cap V_i = V(F - T_{n+1}^e) \cap W_i$ in $\mathbb{P}^{n+1}_{k(p)}$. These restrictions of $\widetilde{D_1} \to D_1$ to V_i, patch together then to give exactly the desired cover

$$V_{\mathbb{P}^{n+1}_{k(p)}}(F(T_0, \ldots, T_n) - T_{n+1}^e) \to \mathbb{P}^n_{k(p)}.$$

To be precise, the open subschemes we described here are $V(F - T_{n+1}^e) \cap W_i$ for $i = 0, \dots, n$, and in principle we should also consider the remaining open $V(F - T_{n+1}^e) \cap W_{n+1}$. This open is defined by $F(y_0, \dots, y_n) - 1 = 0$ on $W_{n+1} = \operatorname{Spec} k(p)[y_0, \dots, y_n]$ with $y_i = T_i/T_{n+1}$, $i = 0, \dots, n$. But since F is homogeneous, $y_0, \dots, y_n$ satisfying this equation cannot be all 0, so at least one $T_i \neq 0$, $i < n+1$, and the point falls in some W_i, $i < n+1$. So this remaining open is contained in the union of the others, and is therefore redundant for our covering describing $V(F - T_{n+1}^e)$.

$\widetilde{D_{12}}$ is given locally on V_i by both $s_i = 0$ and $t_{n+1} = 0$, and so by the description of $\widetilde{D_1} \to D_1$ above it is contained in the $\mathbb{P}^n_{k(p)} \subset \mathbb{P}^{n+1}_{k(p)}$ given by $T_{n+1} = 0$. We have

$$\widetilde{D_{12}} \simeq D_{12} \simeq V(F) \subset \mathbb{P}^n_{k(p)}$$

as we saw in Proposition 4.2.

□

Corollary 4.4. *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a flat quasi-projective morphism with X smooth over k_0 and with X_η smooth over η. Suppose that the special fibre X_σ has finitely many singular points $p_1, \dots, p_r$, and for each i, f looks at p_i like the homogeneous singularity defined by a homogeneous polynomial $F_i \in k(p_i)[T_0, \dots, T_n]$ of degree e_i, with $V(F_i) \subset \mathbb{P}^n_{k(p_i)}$ a smooth hypersurface, and with $\prod_i e_i$ prime to the exponential characteristic of k_0. Let $X_\sigma^\circ = X_\sigma \setminus \{p_1, \dots, p_r\}$. Then*

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(X_\sigma^\circ) + \sum_{i=1}^r \chi_c(V(F_i - T_{n+1}^{e_i})) - \sum_{i=1}^r \chi_c(\mathbb{A}^1 \times V(F_i)).$$

Proof. For the case $r = 1$, with notation as in the previous theorem,

$$\widetilde{D_1^\circ} = \widetilde{D_1} \setminus D_{12} \simeq V(F - T_{n+1}^e) \setminus V(F)$$

. Then Theorem 4.3 and Proposition 3.17 tell us that

$$\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(X_\sigma^\circ) + \chi_c(V(F - T_{n+1}^e) \setminus V(F)) - \chi_c(\mathbb{G}_m \times V(F)).$$

with $F = F_1$. But $\mathbb{G}_m \times V(F) = \mathbb{A}^1 \times V(F) \setminus 0 \times V(F)$, so by cut and paste, we have

$$\begin{aligned}\chi_c(V(F - T_{n+1}^e) \setminus V(F)) - \chi_c(\mathbb{G}_m \times V(F)) &= \chi_c(V(F - T_{n+1}^e)) - \chi_c(V(F)) - \chi_c(\mathbb{A}^1 \times V(F)) + \chi_c(V(F)) \\ &= \chi_c(V(F - T_{n+1}^e)) - \chi_c(\mathbb{A}^1 \times V(F)),\end{aligned}$$

which is what we want.

In general, we proceed by induction on r. Let $U_1 = X \setminus \{p_1\}$, $U_2 = X \setminus \{p_2, \dots, p_r\}$ and let $U_{12} = U_1 \cap U_2$, with open immersions $j_1 : U_1 \to X$, $j_2 : U_2 \to X$ and $j_{12} : U_{12} \to X$. By our induction hypothesis together with Property 3.1 applied to the smooth morphisms j_1, j_2, j_{12}, we have

$$\chi_c(j_1^* \Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(X_\sigma^\circ) + \sum_{i=2}^r \chi_c(V(F_i - T_{n+1}^{e_i})) - \sum_{i=1}^r \chi_c(\mathbb{G}_m \times V(F_i)),$$

$$\chi_c(j_2^* \Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(X_\sigma^\circ) + \chi_c(V(F_1 - T_{n+1}^{e_1})) - \chi_c(\mathbb{G}_m \times V(F_1)),$$

and

$$\chi_c(j_{12}^* \Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(X_\sigma^\circ).$$

The Mayer-Vietoris property (Proposition 2.13) then yields the result. □

5 The quasi-homogeneous case

We can extend the results of the previous section to the case in which the defining polynomial at the singularity is weighted homogeneous. The usual blow-up should be replaced by a weighted blow-up, but treating it is not as straightforward as in the homogeneous case. For example, the exceptional divisor is a weighted projective space, which is not smooth in the case of non-trivial weights. Presenting the scheme in the weighted case as a quotient of a scheme with a homogeneous singularity modulo a finite group allows us to use the results of the previous section, once we show that the quotient defines a semi-stable reduction of our original degeneration. The end result is completely parallel to the homogeneous case.

We retain our assumptions on the discrete valuation ring $\mathcal{O}$ with residue field k and parameter t from Section 3.1.; as before, we let $\sigma \hookrightarrow B := \operatorname{Spec} \mathcal{O} \hookleftarrow \eta$ denote the closed and generic points of $B := \operatorname{Spec} \mathcal{O}$, respectively, and we have the subfield k_0 of $\mathcal{O}$, with B smooth and essentially of finite type over k_0, and with $k_0 \to k$ finite and separable.

Let X be a separated k_0-scheme, essentially of finite type, and $p \in X$ a smooth closed point with maximal ideal $m_p \subset \mathcal{O}_{X,p}$. Let $(s_0, \dots, s_n)$ be a regular sequence generating m_p and let $(a_0, \dots, a_n)$ be a system of positive integral weights with $\gcd(a_i, a_j) = 1$ for every i, j. Define the ideal $m^{(\ell)}_{p,s_*,a_*} \subset \mathcal{O}_{X,p}$ to be the ideal generated by monomials of weighted homogeneous degree ℓ, that is, by monomials $s_0^{i_0} \cdot \ldots \cdot s_n^{i_n}$ with $\ell = \sum_j a_j i_j$.

Definition 5.1. Let $f : X \to \operatorname{Spec} \mathcal{O}$ a flat proper morphism of schemes with X smooth over k_0 and X_η smooth over η. Let $p \in X_\sigma$ be an isolated singular point and let $F \in k(p)[T_0, \dots, T_n]$ be a homogeneous polynomial of weighted degree e for some weights $a_* = (a_0, \dots, a_n)$ as above. We say that (X_σ, p) *locally looks like the weighted homogeneous singularity defined by* F if there is a regular sequence of generators for m_p such that

$$f^*(t) \equiv F(s_0, \dots, s_n) \mod m_p \cdot m^{(e)}_{p,s_*,a_*}.$$

Here we have implicitly chosen a splitting of $\mathcal{O}_{X,p}/m_p \cdot m^{(e)}_{p,s_*,a_*} \to k(p)$.

5.1 Weighted projective space

First let us review the notion of weighted projective space as in [LPS]. Let R be a ring and $a = (a_0, .., a_n)$ a sequence of positive integers, which we call *weights*. Let $R[X_0, .., X_n]$ be the graded ring with X_i having degree a_i. Define

$$\mathbb{P}_R(a) = \operatorname{Proj} R[X_0, \dots, X_n].$$

An alternate description of $\mathbb{P}_R(a)$ is as a quotient of $\mathbb{P}^n$ by the group scheme $\mu_a = \mu_{a_0} \times \ldots \times \mu_{a_n}$.

Let $\iota_a : R[X_0, .., X_n] \to R[Y_0, \dots, Y_n]$ be the graded ring homomorphism mapping X_i to $Y_i^{a_i}$, where the ring $R[X_0, .., X_n]$ is with the a-grading, and $R[Y_0, \dots, Y_n]$ is with the usual grading on a polynomial ring. Let μ_a act on $R[Y_0, \dots, Y_n]$ by $Y_i \mapsto \zeta_{a_i} Y_i$, for $\zeta_{a_i} \in \mu_{a_i}$. Then the image of ι_a can be identified with the fixed ring $R[Y_0, \dots, Y_n]^{\mu_a}$, hence defining

$$\pi : \mathbb{P}^n \to \mathbb{P}(a)$$

as a quotient $\mathbb{P}(a) \simeq \mathbb{P}^n / \mu_a$.

We may view the projective space $\mathbb{P}^n$ at the source of π as achieved from $\mathbb{P}(a)$ by adjoining for each i the a_i-th root of X_i. We now describe a similar construction of a local version of a "weighted blow-up" of our scheme X in Definition 5.1, retaining the notations from that definition.

As our construction will be local around the given point $p \in X_\sigma$, we replace X with an affine open neighbourhood U of p in X, such that the local parameters $s_0, \dots, s_n$ of Definition 5.1 extend to étale coordinates on U, that is, the morphism $(s_0, \dots s_n) : U \to \mathbb{A}^{n+1}_k$ is étale. We change notation and suppose $X = U$, and let A denote the ring of functions on the affine scheme $X = \operatorname{Spec} A$. We let $\mathfrak{m}_p \subset A$ denote the maximal ideal of p and following Definition 5.1, we define $\mathfrak{m}^{(e)}_{p,s_*,a_*} \subset \mathfrak{m}_p$ as the ideal defined by monomials of weighted degree e in the s_i.

Construction 5.2. With $p \in X = \operatorname{Spec} A$, $a_* = (a_0, \dots, a_n)$ and $s_0, \dots, s_n \in \mathfrak{m}_p$ étale coordinates on X as above, define $A[s^{1/a}] := A[\sigma_0, \dots, \sigma_n]/(\sigma_0^{a_0} - s_0, \dots, \sigma_n^{a_n} - s_n)$ and let $Z = \operatorname{Spec} A[s^{1/a}]$. Let $\mu_a = \mu_{a_0} \times \dots \times \mu_{a_n}$. We have the μ_a-action on $A[s^{1/a}]$, where $\zeta \in \mu_{a_i}$ acts by

$$\zeta \cdot \sigma_j := \begin{cases} \zeta\sigma_i & \text{for } j = i \\ \sigma_j & \text{for } j \neq i. \end{cases}$$

Then A is equal to the subring of μ_a-invariants in $A[s^{1/a}]$, $A = A[s^{1/a}]^{\mu_a}$, and so the map

$$\pi : Z \to X$$

realises X as the quotient of Z by the action of the group scheme μ_a. Also, there is a unique point $q \in Z$ lying over Z, and we have $k(q) = k(p)$. We let $\mathfrak{m}_q \subset A[s^{1/a}]$ denote the maximal ideal of $q \in Z$.

An argument similar to that given in Remark 3.16 shows that Z is smooth over k and if X is integral, then so is Z.

From Definition 5.1, we have

$$f^*(t) = F(s_0, .., s_n) + h.$$

After shrinking X if necessary, and changing notation, we may assume that h is in $\mathfrak{m}_p \cdot \mathfrak{m}^{(e)}_{p,s_*,a_*} \subset A$. Letting $g := \pi \circ f : Z \to \operatorname{Spec} \mathcal{O}$, we have

$$g^*(t) = F(\sigma_0^{a_0}, \dots, \sigma_n^{a_n}) + h'$$

with $h' \in \mathfrak{m}_q^{e+1} \subset B$. Let $G(Z_0, \dots, Z_n) \in k(p)[Z_0, \dots, Z_n]$ be the degree e polynomial with $G(\sigma_0, .., \sigma_n) = F(\sigma_0^{a_0}, \dots, \sigma_n^{a_n})$.

Definition 5.3 ([LPS, Def. 4.2]). Let F, G be defined as in the above Construction 5.2. We say that $V(F) \subset \mathbb{P}_{k(p)}(a)$ is a *smooth quotient hypersurface* if the polynomial G defines a smooth hypersurface $V(G) \subset \mathbb{P}^n_{k(p)}$ and in addition $V(F) \subset \mathbb{P}_{k(p)}(a)$ is smooth. Furthermore, letting $v_i \in \mathbb{P}_{k(p)}(a)$ be the point with ith homogeneous coordinate 1 and all other coordinates 0, we require that $F(v_i) \neq 0$ if $a_i > 1$. Finally, we require that the weights a_i are pairwise relatively prime, each a_i divides e, and e is prime to the exponential characteristic of k.

Remark 5.4. The condition that each a_i divides e implies that $V(F)$ is a Cartier divisor on $\mathbb{P}_{k(p)}(a)$. This being the case, the assumption that $V(F) \subset \mathbb{P}_{k(p)}(a)$ is smooth implies that $V(F)$ does not contain any singular point of $\mathbb{P}_{k(p)}(a)$. If $n \geq 2$, and if the a_i are pairwise relatively prime, then v_i is a singular point of $\mathbb{P}_{k(p)}(a)$ if $a_i > 1$, so in case $n \geq 2$, the last condition in the definition above is superfluous.

5.2 The nearby cycles of a quasi-homogeneous singularity

As before, we take $\mathcal{O}$ and $B = \operatorname{Spec} \mathcal{O}$ as in Section 3.1 and we fix a flat quasi-projective morphism $f : X \to B$ with X smooth over k_0 and X_η smooth over η.

We formulate out conditions for the singularities in the quasi-homogeneous case, similar to Assumption 4.1.

Assumption 5.5. (1) The special fibre X_σ has only isolated singularities $p_1, \ldots, p_r$.

(2) For each $p \in \{p_1, \ldots, p_r\}$ there is a polynomial $F \in k(p)[T_0, \ldots, T_n]$ of weighted degree e_p with respect to some weights a_*, with $gcd(a_*) = 1$ and $lcm(a_*)$ dividing e_p, such that F defines a smooth quotient hypersurface in $\mathbb{P}_{k(p)}(a_*)$ (Definition 5.3 above), and (X_σ, p) locally looks like the weighted homogeneous singularity defined by F (see Definition 5.1).

For later use we need the following fact:

Lemma 5.6. *Let k be a field and let Y be a k-scheme, separated and essentially of finite type over k. Let D be an effective Cartier divisor on Y. Suppose that both D and $Y \setminus D$ are smooth over k. Then Y is smooth over k.*

Proof. Since smoothness is invariant under field extensions we may assume k is algebraically closed. Let y be a point in Y. Since D is a closed subscheme of Y, if $y \notin D$ then it has a smooth neighbourhood. We have to show that also $y \in D$ is a smooth point in Y. Since D is an effective Cartier divisor, there is a neighbourhood U of y in Y, and a non-zero divisor f on U such that $D \cap U$ is defined by the vanishing of f. The exact sequence of sheaves

$$0 \to \mathcal{O}_U(-D \cap U) \xrightarrow{\cdot f} \mathcal{O}_U \to \mathcal{O}_{D \cap U} \to 0$$

gives on stalks at y

$$0 \to \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{D,y} \to 0$$

Let $d = dimY$ so $dimD = d-1$. D is smooth so $\mathcal{O}_{D,y}$ is a regular local ring of dimension $d-1$, so we can write the maximal ideal $m_{D,y}$ as generated by a regular sequence, $m_{D,y} = (\bar{f}_1, \ldots, \bar{f}_{d-1})$. The $\bar{f}_i$ lift to $f_1, \ldots, f_{d-1}$ in $m_{Y,y}$. Now since $ker(\mathcal{O}_{Y,y} \to \mathcal{O}_{D,y}) = (f)\mathcal{O}_{Y,y}$ we get from the exact sequence that $m_{Y,y} = (f, f_1, \ldots, f_{d-1})$, with $(f, f_1, \ldots, f_{d-1})$ a regular sequence. Then $\mathcal{O}_{Y,y}$ is a regular local ring and hence y is a smooth point of Y. ☐

Assuming that our only singularity is $p = p_1$, the main result of the section is an analogue of Theorem 4.3.

Theorem 5.7. *Let $f : X \to \mathcal{O}$ be a flat quasi-projective morphism such that the generic fibre X_η is smooth over η and with X smooth over k_0. Suppose in addition that $p \in X_\sigma$ is the only singular point of X_σ and that f satisfies Assumption 5.5. Let $e = e_p$, let F be as in Assumption 5.5 for p, with respect to weights a_*, and let $\mathcal{O}_e = \mathcal{O}[t']/(t'^e - t)$. Finally, we assume that $X = \operatorname{Spec} A$ is affine with a system of étale coordinates $s_0, \ldots, s_n \in \mathfrak{m}_p$, and that the all the steps in Construction 5.2 can be carried out for (X, p, F, s_*, a_*) without having to shrink X to a smaller affine neighbourhood of p.*

Let $\pi : Z \to X \simeq Z/\mu_a$ be the μ_a-quotient map given by Construction 5.2 and let $q \in Z$ be the unique point lying over p; note that $k(p) = k(q)$. Let $\hat{Z} = Bl_q(Z)$ and let $Y_Z \to \hat{Z}$ be the normalisation of the base-change $\hat{Z}_e := \hat{Z} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathcal{O}_e$. Then the μ_a-action on Z extends to a μ_a-action on Y_Z. Moreover, letting $Y := Y_Z/\mu_a$ and letting $Y \to X$ be the resulting map on the quotients, we have

(1) Y is smooth over k and $Y \to \operatorname{Spec} \mathcal{O}_e$ is a semi-stable reduction of $X \to \operatorname{Spec} \mathcal{O}$.

(2) Let $F \in k(p)[T_0, \ldots, T_n]$ be the weighted weighted-homogeneous polynomial of weighted degree e as given by Assumption 5.5 for (X_σ, p). Then the special fibre $Y_\sigma \subset Y$ is a reduced normal crossing divisor, $Y_\sigma = \widetilde{D_1} + \widetilde{D_2}$ with $\widetilde{D_1}, \widetilde{D_1}$ smooth. Letting $\widetilde{D_{12}} := \widetilde{D_1} \cap \widetilde{D_2}$, we have

$$\widetilde{D_1} \simeq V(F - T_{n+1}^e) \subset \mathbb{P}_{k(p)}(a, 1)$$

$$\widetilde{D_{12}} \simeq V(F) \subset \mathbb{P}_{k(p)}(a).$$

Moreover, the projection $q : \widetilde{D_2} \to X_\sigma$ *is an isomorphism over* $X_\sigma \setminus \{p\}$ *and defines a resolution of singularities of* X_σ*, with* $q^{-1}(p) = \widetilde{D_{12}}$*.*

(3) We have the identity

$$\chi_c(\Psi_f(\mathbb{1}_{X\eta})) = \chi_c(\widetilde{D_1^\circ}) + \chi_c(\widetilde{D_2^\circ}) - (\langle -1\rangle - \langle 1\rangle) \cdot \chi(\widetilde{D_{12}}).$$

Proof. We may assume that X is integral and we retain the notation from Construction 5.2.

Let

$$A[s^{1/a}] = A[\sigma_0, \dots, \sigma_n]/(\sigma_0^{a_0} - s_0, \dots, \sigma_n^{a_n} - s_n).$$

We have $Z = \operatorname{Spec} A[s^{1/a}]$, Z is integral and is smooth over k, and we have a μ_a-action on Z with quotient X. Let $\pi : Z \to X = Z/\mu_a$ be the quotient map, induced by the inclusion $A \hookrightarrow B$. Let $q \in Z$ be the unique point lying over $p \in X$. (Z_σ, q) satisfies Assumption 4.1 for locally looking like a homogeneous singularity defined by $G(\sigma_0, .., \sigma_n) := F(\sigma_0^{a_0}, \dots, \sigma_n^{a_n})$ (see Construction 5.2). G has degree e and $V(G)$ is smooth by our assumption on F. We apply Theorem 4.3 and construct the semi-stable reduction $Y_Z \to \operatorname{Spec} \mathcal{O}_e$ of $Z \to \operatorname{Spec} \mathcal{O}$ by forming the blow-up $\hat{Z} := Bl_q Z$, and letting Y_Z be the normalisation of the base-change $\hat{Z}_e := \hat{Z} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathcal{O}_e$.

Since the μ_a-action on Z fixes q, this action lifts canonically to an action on $\hat{Z}$, which gives a μ_a-action on $\hat{Z}_e$ over $\operatorname{Spec} \mathcal{O}_e$ and finally induces a μ_a-action on the normalisation Y_Z. Let $Y := Y_Z/\mu_a$ and let $\pi : Y_Z \to Y$ denote the quotient map. Since $Y_Z \to Z_e$ is proper, it follows that the induced map on the quotients $Y \to X_e$ is also proper.

Let $E_1 \subset \hat{Z}$ be the exceptional divisor, let $E_2 \subset \hat{Z}$ be the strict transform of Z_σ and let $E_{12} = E_1 \cap E_2$. Denote by $\widetilde{E_1}$, $\widetilde{E_2}$, $\widetilde{E_{12}}$ their respective coverings in $(Y_Z)_\sigma$, as in the proof of Theorem 4.3. Let $\widetilde{D_i} := \pi(\widetilde{E_i}) = \widetilde{E_i}/\mu_a \subset Y_\sigma$.

Since Z is integral, it follows from Remark 3.16 that Y_Z is integral and thus the quotient scheme $Y = Y_Z/\mu_a$ is integral as well.

We use the standard presentation of the blow-up $\hat{Z}$ as

$$\hat{Z} = \operatorname{Proj} A[s^{1/a}][Z_0, \dots, Z_n]/(\{\sigma_i Z_j - \sigma_j Z_i\}_{0 \le i,j \le n})$$

giving the standard open cover of $\hat{Z}$ by the affine open subsets $Z_i \neq 0$. This induces the affine open cover $\{V_0, \dots, V_n\}$ of Y_Z. As in the proof of Theorem 4.3, we have the explicit description of the V_i, for instance,

$$V_0 = \operatorname{Spec}(B_e[z_1, \dots, z_n, z_{n+1}]/(\{\sigma_i - \sigma_0 z_i\}_{1 \le i \le n+1}, g_0 - z_{n+1}^e))$$

with $B_e := B \otimes_{\mathcal{O}} \mathcal{O}_e$, $z_i = Z_i/Z_0$ for $i = 1, \dots, n$, $z_{n+1} = t'/\sigma_0$ and $g_0 = G(1, z_1, \dots, z_n) + \sigma_0 h'$ for suitable h'. Letting $A_e := A \otimes_{\mathcal{O}} \mathcal{O}_e$, we can rewrite this as

$$V_0 = \operatorname{Spec}(A_e[\sigma_0, z_1, \dots, z_n, z_{n+1}]/(\{s_i - \sigma_0^{a_i} z_i^{a_i}\}_{1 \le i \le n+1}, g_0 - z_{n+1}^e, s_0 - \sigma_0^{a_0})).$$

Again referring to Theorem 4.3 and its proof, we have the global description of $\widetilde{E_1}$ as the closed subscheme $V(G(Z_0, \dots, Z_n) - Z_{n+1}^e)$ of $\mathbb{P}_{k(q)}^{n+1} := \operatorname{Proj} k(q)[Z_0, \dots, Z_{n+1}]$, with $\widetilde{E_{12}} \subset \widetilde{E_1}$ defined by $Z_{n+1} = 0$. Finally, the projection $Y_Z \to Z$ restricts to a morphism $\pi_2 : \widetilde{E_2} \to Z_\sigma$, π_2 is an isomorphism over $Z_\sigma \setminus \{q\}$ and the reduced inverse image $\pi_2^{-1}(q)$ is $\widetilde{E_{12}}$.

Taking the μ_a-quotients $U_i := V_i/\mu_a$ gives the affine open cover $\{U_0, \dots, U_n\}$ of Y.

We have the commutative diagram

$$\begin{tikzcd}
V_i \cap \widetilde{E_1} \arrow[rr] \arrow[dr, hook] \arrow[dd, hook] & & U_i \cap \widetilde{D_1} \arrow[dr, hook] \arrow[dd, hook] & & \\
& \widetilde{E_1} \arrow[rr] \arrow[dd, hook] & & \widetilde{D_1} \arrow[dd, hook] & \\
V_i \arrow[rr] \arrow[dr, hook] & & U_i \arrow[dr, hook] & & \\
& Y_Z \arrow[rr] \arrow[d] & & Y \arrow[dd] \arrow[ddr] & \\
& \hat{Z}_e \arrow[d] \arrow[ddl] & & & \\
& Z_e \arrow[rr] \arrow[ddl] & & X_e \arrow[r] \arrow[ddl] & \operatorname{Spec} \mathcal{O}_e \arrow[ddl] \\
\hat{Z} \arrow[d] & & & & \\
Z \arrow[rr, "\pi"] & & X \arrow[r] & \operatorname{Spec} \mathcal{O} &
\end{tikzcd}$$

Let us now describe the μ-action on $\hat{Z}_e$ and on V_0. For $\zeta \in \mu_{a_i}$, and $j = 0, \ldots, n$, we have

$$\zeta \cdot Z_j = \begin{cases} \zeta Z_i & \text{for } j = i \\ Z_j & \text{for } j \neq i. \end{cases}$$

and

$$\zeta \cdot \sigma_j = \begin{cases} \zeta \sigma_i & \text{for } j = i \\ \sigma_j & \text{for } j \neq i. \end{cases}$$

On the affine piece V_0, and for $\zeta \in \mu_{a_i}$, $i = 1, \ldots, n$ and for $j = 1, \ldots, n+1$, we thus have

$$\zeta \cdot z_j = \begin{cases} \zeta z_i & \text{for } j = i \\ z_j & \text{for } j \neq i, \end{cases}$$

and $\zeta \cdot \sigma_0 = \sigma_0$. For $\zeta \in \mu_{a_0}$, we have $\zeta \cdot \sigma_0 = \zeta \sigma_0$ and

$$\zeta \cdot z_j = \zeta^{-1} z_j$$

for all $j = 1, \ldots, n+1$. The μ_a-action on the other open subschemes V_i is defined similarly. We also have a global description of the μ_a-action on $\widetilde{E_1} \subset \mathbb{P}^{n+1}_{k(q)} = \operatorname{Proj} k(q)[Z_0, \ldots, Z_{n+1}]$ by having μ_a act trivially on Z_{n+1}; one can easily check that this restricts to the action on each $V_i \cap \widetilde{E_1}$ defined above.

We will describe the quotient by μ_a in two steps - first taking the quotient by the subgroup $\mu_{a_{>0}} := \mu_{a_1} \times \ldots \times \mu_{a_n}$ and then by the remaining factor μ_{a_0}.

Proof of (1). The assertion (1) is local on Y, so it suffices to prove (1) after restricting to $U_i \subset Y$; we give the proof for U_0. We assume at first that $a_0 > 1$; the case $a_0 = 1$ is easier and will be dealt with at the end of the argument.

Let

$$C_0 = A_e[\sigma_0, z_1, \ldots, z_n, z_{n+1}]/((s_i - \sigma_0^{a_i} z_i^{a_i})_{1 \leq i \leq n+1}, g_0 - (z_{n+1})^e, s_0 - \sigma_0^{a_0})$$

and let $C \subset C_1 \subset C_0$ be the rings of invariants

$$C_1 = C_0^{\mu_{a_{>0}}}, C = C_0^{\mu_a} = C_1^{\mu_{a_0}},$$

so $V_0 = \operatorname{Spec} C_0$ and $U_0 = \operatorname{Spec} C \subset Y$. Since V_0 is smooth over k and is integral, the invariant subrings C, C_1 are both integral and normal.

We have $s_0 \in C$ and $\sigma_0 \in C_1$. We first show that $C[s_0^{-1}]$ is a smooth $\mathcal{O}_e$-algebra. To see this, note that the special fibre X_σ has only p as singular point, so $A[s_0^{-1}]$ is a smooth $\mathcal{O}$-algebra. Thus the base extension $A_e[s_0^{-1}] = A[s_0^{-1}] \otimes_{\mathcal{O}} \mathcal{O}_e$ is a smooth $\mathcal{O}_e$-algebra. Moreover, since localization commutes with taking invariants, $A_e[s_0^{-1}]$ is the μ_a-invariants in $B_e[\sigma_0^{-1}]$, and since σ_0 defines $\widetilde{E_1} \cap V_0$ in V_0, $V_0 \to \operatorname{Spec} B_e$ is an isomorphism over $\operatorname{Spec} B_e[\sigma_0^{-1}]$. This shows that $C[s_0^{-1}] = A_e[s_0^{-1}]$ and hence $C[s_0^{-1}]$ is a smooth $\mathcal{O}_e$-algebra.

The $\mu_{a>0}$-invariant subring of $A_e[\sigma_0, z_1, \dots, z_n, z_{n+1}]/(s_0 - \sigma_0^{a_0})$ is

$$[A_e[\sigma_0, z_1, \dots, z_n, z_{n+1}]/s_0 - \sigma_0^{a_0})]^{\mu_{a>0}} = A_e[\sigma_0, t_1, \dots, t_n, z_{n+1}]/(s_0 - \sigma_0^{a_0}),$$

with $t_i = z_i^{a_i}$. From this it follows that

$$C_1 := C_0^{\mu_{a>0}} = A_e[\sigma_0, t_1, \dots, t_n, z_{n+1}]/(\{s_i - \sigma_0^{a_i} t_i\}_{i=1,\dots,n}, f_0 - z_{n+1}^e, s_0 - \sigma_0^{a_0}),$$

where $f_0 = F(1, t_1, \dots, t_n) + \sigma_0 \cdot h$ for suitable h. Note that μ_{a_0} now acts by $\zeta \cdot t_i = \zeta^{-a_i} t_i$.

Our assumption that F defines a smooth quotient hypersurface in $\mathbb{P}(a)$ and our assumption $a_0 > 1$ implies that $F(1, 0, \dots, 0) \neq 0$, that is

$$\emptyset = V(\sigma_0, t_1, \dots, t_n, z_{n+1}) \cap V(f_0 - z_{n+1}^e) \subset \operatorname{Spec} A_e[\sigma_0, t_1, \dots, t_n, z_{n+1}]/(s_0 - \sigma_0^{a_0}, \{s_i - \sigma_0^{a_i} t_i\}_{i=1,\dots,n}).$$

The μ_{a_0}-action on $\operatorname{Spec} A_e[\sigma_0, t_1, \dots, t_n, z_{n+1}]/(s_0 - \sigma_0^{a_0}, \{s_i - \sigma_0^{a_i} t_i\}_{i=1,\dots,n})$ is free outside the origin $V(\sigma_0, t_1, \dots, t_n, z_{n+1})$. Thus the μ_{a_0}-action on $\operatorname{Spec} C_1$ is free and hence the ring extension $C \hookrightarrow C_1$ is étale. In particular, $C_1[\sigma_0^{-1}] = C_1[s_0^{-1}]$ is étale over the smooth k-algebra $C[s_0^{-1}]$ and hence $C_1[\sigma_0^{-1}]$ is a smooth k-algebra.

Since σ_0 is $\mu_{a>0}$-invariant, it follows that $(\sigma_0)C_1$ is the $\mu_{a>0}$-invariants in $(\sigma_0)C_0$, in other words

$$(\sigma_0)C_1 = C_1 \cap (\sigma_0)C_0.$$

This implies that the evident ring homomorphism $C_1/(\sigma_0) \to C_0/(\sigma_0)$ is injective and since e is prime to the characteristic of k, taking $\mu_{a>0}$ invariants is an exact functor, and thus

$$C_1/(\sigma_0) = [C_0/(\sigma_0)]^{\mu_{a>0}}.$$

Explicitly,

$$C_0/(\sigma_0) = k(p)[z_1, \dots, z_n, z_{n+1}]/(G(1, z_1, \dots, z_n) - z_{n+1}^e)$$

Since $G(1, z_1, \dots, z_n) = F(1, z_1^{a_1}, \dots, z_n^{a_n})$, $G(1, z_1, \dots, z_n) - z_{n+1}^e$ is $\mu_{a>0}$ invariant, so as above, we have

$$\begin{aligned} C_1/(\sigma_0) = [k(p)[z_1, \dots, z_n, z_{n+1}]/(G(1, z_1, \dots, z_n) - z_{n+1}^e)]^{\mu_{a>0}} \\ = k(p)[t_1, \dots, t_n, z_{n+1}]/(F(1, t_1, \dots, t_n) - z_{n+1}^e). \end{aligned}$$

Using again our smoothness assumption on F, we see that $C_1/(\sigma_0)$ is a smooth k-algebra. By Lemma 5.6, C_1 itself is a smooth k-algebra and since $C \hookrightarrow C_1$ is étale, C is also a smooth k-algebra.

Similarly, to see that $V_0 \to \operatorname{Spec} \mathcal{O}_e$ is a semi-stable reduction, it suffices to see that the special fibre $\operatorname{Spec} C_1/(t')C_1$ is a reduced normal crossing divisor on $\operatorname{Spec} C_1$. For this, we have $t' = \sigma_0 z_{n+1}$. We have already seen that $C_1/(\sigma_0)$ is a smooth k-algebra, in other words, the Cartier divisor $V(\sigma_0)$ on $\operatorname{Spec} C_1$ is smooth. We have

$$C_1/(z_{n+1}, \sigma_0) = k(p)[t_1, \dots, t_n]/(F(1, t_1, \dots, t_n))$$

which again by our assumption on F is a smooth k-algebra. This implies that the Cartier divisors $V(\sigma_0), V(z_{n+1}) \subset \operatorname{Spec} C_1$ intersect transversely on $\operatorname{Spec} C_1$, which implies that $V(z_{n+1})$ is smooth in a neighbourhood of $V(\sigma_0)$ in $\operatorname{Spec} C_1$; this also implies that $(t') = (\sigma_0) \cap (z_{n+1})$. We have also shown that $C[s_0^{-1}]$ is smooth over $\mathcal{O}_e$, which implies that $C_1[\sigma_0^{-1}]$ is also smooth over $\mathcal{O}_e$, so $V(z_{n+1}) \setminus V(\sigma_0)$ is smooth. Thus the Cartier divisor $V(t')$ on $\operatorname{Spec} C_1$ is $V(\sigma_0) + V(z_{n+})$, which we have just shown is a reduced normal crossing divisor. This completes the proof of (1), and also shows that Y_σ is a union of two smooth components, intersecting transversely, proving the first part of (2).

In case $a_0 = 1$, we have $C = C_1$ and a much simpler version of the arguments given above takes care of this case.

Proof of (2). We have just shown that Y_σ is the Cartier divisor $\widetilde{D_1} + \widetilde{D_2}$, with $\widetilde{D_1}, \widetilde{D_2}$ both smooth and with transverse intersection $\widetilde{D_{12}}$. We have the global description of $\widetilde{E_1}$ given by Theorem 4.3, namely $\widetilde{E_1}$ is the closed subscheme $V(G(Z_0, \ldots, Z_n) - Z_{n+1}^e)$ of $\mathbb{P}^{n+1}_{k(q)}$. We have

$$\widetilde{D_1} = \widetilde{E_1}/\mu_a.$$

The μ_a-action on $\widetilde{E_1}$ extends to an action on $\mathbb{P}^{n+1}_{k(q)} = \operatorname{Proj} k(q)[Z_0, \ldots, Z_n, Z_{n+1}]$ as described in the proof of (1) by having μ_a act trivially on Z_{n+1}. Then

$$\mathbb{P}^{n+1}_{k(q)}/\mu_a = \mathbb{P}(a_0, \ldots, a_n, 1).$$

Let $a_{n+1} = 1$, let $T_i = Z_i^{a_i}$, $i = 0, \ldots, n+1$, and let $D_1 \subset \mathbb{P}(a_0, \ldots, a_n, 1)$ be the hypersurface $V(F - T_{n+1}^e)$. Let $W_i \subset \mathbb{P}(a_0, \ldots, a_n, 1)$ be the open subscheme $T_i \neq 0$. Giving T_j weight a_j, we have

$$W_i = \operatorname{Spec} k(p)[T_0, \ldots, T_{n+1}][T_i^{-1}]_0$$

We concentrate on the case $i = 0$ to simplify the notation. Let $R_0 := k(p)[T_0, \ldots, T_{n_1}][T_0^{-1}]_0$ and let $R_0' := [k(p)[t_1, \ldots, t_n, z_{n+1}]]^{\mu_{a_0}}$, with the μ_{a_0} action as defined in the proof of (1). A direct computation shows that $R_0 \cong R_0'$. Indeed, a monomial $\prod_j t_j^{b_j} \cdot z_{n+1}^{b_{n+1}}$ is μ_{a_0}-invariant if and only if $\sum_{j=1}^{n+1} a_j b_j$ is divisible by a_0. Similarly, a monomial $\prod_{j=1}^{n+1} T_j^{b_j} \cdot T_0^{-b_0}$ has weighted degree zero if and only if $\sum_{j \geq 1} a_j b_j = a_0 b_0$. So, sending $\prod_{j=1}^{n+1} T_j^{b_j} \cdot T_0^{-b_0}$ to $\prod_j t_j^{b_j} \cdot z_{n+1}^{b_{n+1}}$ gives an isomorphism of R_0 with R_0'.

Similarly, recalling that a_0 divides e, the weighted homogeneous polynomial $F(T_0, \ldots, T_n) - T_{n+1}^e$ gives the element $F(T_0, \ldots, T_n)/T_0^{e/a_0} - T_{n+1}^e/T_0^{e/a_0}$ in R_0, which corresponds to the element $F(1, t_1, \ldots, t_n) - z_{n+1}^e$ of $[k(p)[t_1, \ldots, t_n, z_{n+1}]]^{\mu_{a_0}}$.

Let $W_0' := \operatorname{Spec} k(p)[t_1, \ldots, t_n, z_{n+1}]$. The finite extension

$$[k(p)[t_1, \ldots, t_n, z_{n+1}]]^{\mu_{a_0}} \to k(p)[t_1, \ldots, t_n, z_{n+1}]$$

defines a finite morphism $p : W_0' \to W_0$. By our computations in the proof of (1) and that given in the previous paragraph, we see that

$$p^{-1}(D_1 \cap W_0) = V(F(1, t_1, \ldots, t_n) - z_{n+1}^e) = \operatorname{Spec} C_1/(\sigma_0) = (\widetilde{E_1} \cap V_0)/\mu_{a>0},$$

and thus

$$D_1 \cap W_0 = (\widetilde{E_1} \cap V_0)/\mu_a = \widetilde{D_1} \cap W_0$$

An analogous computation shows that $D_1 \cap W_i = \widetilde{D_1} \cap W_i$ for $i = 1, \ldots, n+1$, so $\widetilde{D_1} = V(F - T_{n+1}^e)$, as desired.

A similar argument shows that $\widetilde{D_{12}} = V(F - T_{n+1}^e) \cap V(T_{n+1})$, in other words, $\widetilde{D_{12}} = V(F) \subset \mathbb{P}(a)$.

In the proof of (1), we showed that the projection $U_0 \setminus V(s_0) \to X_e \setminus V(s_0)$ is an isomorphism; a similar argument shows that $U_i \setminus V(s_i) \to X_e \setminus V(s_i)$ is an isomorphism for all i. This shows that $Y \setminus \widetilde{D_1} \to X_e \setminus \{p\}$ is an isomorphism. Passing to the fibre over the closed point of $\operatorname{Spec} \mathcal{O}_e$, it follows that $\widetilde{D_2} \setminus \widetilde{D_{12}} \to X_\sigma \setminus \{p\}$ is an isomorphism. Since $\widetilde{D_2}$ is smooth, $\widetilde{D_2} \setminus \widetilde{D_{12}}$ is dense in $\widetilde{D_2}$ and $\widetilde{D_2} \to X_\sigma$ is proper, we see that $q : \widetilde{D_2} \to X_\sigma$ is a resolution of singularities of X_σ, with $q^{-1}(p) = \widetilde{D_{12}}$, proving (2).

Proof of (3). The formula for $\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta}))$ is a consequence of (1), (2) and Proposition 3.17.

□

Corollary 5.8. *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a quasi-projective flat morphism with X smooth over k_0. Suppose that X_η is smooth over η and X_σ has finitely many singular points $p_1, \ldots, p_r$. Suppose in addition that for each i, (X_σ, p_i) locally looks like the weighted homogeneous singularity defined by a weighted homogeneous polynomial $F_i \in k(p_i)[T_0, \ldots, T_n]$ of weighted degree e_i for weights $a_*^{(i)}$, such that F_i defines a smooth quotient hypersurface in $\mathbb{P}_{k(p)}(a_*^{(i)})$. Let $X_\sigma^\circ = X_\sigma \setminus \{p_1, \ldots, p_r\}$. Then*

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})) = \chi_{\mathrm{c}}(X_\sigma^\circ) + \sum_{i=1}^{r} \chi_{\mathrm{c}}(V(F_i - T_{n+1}^{e_i})) - \sum_{i=1}^{r} \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F_i)).$$

Proof. We proceed by induction on r. The induction step is exactly as the proof of Corollary 4.4.

Suppose X_σ has the single singular point $p = p_1$, let $e = e_1$, $F = F_1$. By Theorem 5.7, there is an affine open neighbourhood U of p in X such that the restriction $f_U : U \to \operatorname{Spec} \mathcal{O}$ admits a semi-stable reduction $Y \to \operatorname{Spec} \mathcal{O}_e$, with special fibre $Y_\sigma = \widetilde{D_1} + \widetilde{D_2}$ and where $\tilde{D} \to U_\sigma$ is an isomorphism over $U_\sigma^\circ := U_\sigma \setminus \{p\}$, with $\widetilde{D_1} \cong V(F \setminus T_{n+1}^e)$, and with $\widetilde{D_{12}} = V(F)$. By Example 3.10 and Corollary 3.13, we have

$$\chi_{\mathrm{c}}(\Psi_{f_U}(\mathbb{1}_U)) = \chi_{\mathrm{c}}(U_\sigma^\circ) + \chi_{\mathrm{c}}(V(F - T_{n+1}^e) \setminus V(F)) - \chi_{\mathrm{c}}(\mathbb{G}_m \times V(F)).$$

Using cut and paste, as in the proof of Corollary 4.4, we have

$$\chi_{\mathrm{c}}(\Psi_{f_U}(\mathbb{1}_U)) = \chi_{\mathrm{c}}(U_\sigma^\circ) + \chi_{\mathrm{c}}(V(F - T_{n+1}^e)) - \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F)).$$

Let $V = X \setminus \{p\}$ with morphism $f_V : V \to \operatorname{Spec} \mathcal{O}$. Then $V_\sigma = X_\sigma^\circ$ and since V is smooth over $\mathcal{O}$, we have

$$\chi_{\mathrm{c}}(\Psi_{f_V}(\mathbb{1}_V)) = \chi_{\mathrm{c}}(V_\sigma) = \chi_{\mathrm{c}}(X_\sigma^\circ).$$

For $U \cap V = U_\sigma^\circ$, we similarly have

$$\chi_{\mathrm{c}}(\Psi_{f_{U \cap V}}(\mathbb{1}_{U \cap V})) = \chi_{\mathrm{c}}(U_\sigma^\circ).$$

Using Mayer-Vietoris for the cover $X = U \cup V$, as in the proof of Corollary 4.4, we thus have

$$\begin{aligned}\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_X) = \chi_{\mathrm{c}}(U_\sigma^\circ) + \chi_{\mathrm{c}}(V(F - T_{n+1}^e)) - \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F)) + \chi_{\mathrm{c}}(X_\sigma^\circ) - \chi_{\mathrm{c}}(U_\sigma^\circ) \\ = \chi_{\mathrm{c}}(X_\sigma^\circ) + \chi_{\mathrm{c}}(V(F - T_{n+1}^e)) - \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F)).\end{aligned}$$

□

6 Comparison of local Euler classes

In this section we introduce the local invariant, the local Euler class, that will give an effective tool for computing the quadratic Euler characteristic of the nearby cycles. We will show that, for the type of morphism $f : X \to \operatorname{Spec} \mathcal{O}$ that we have been considering, and if f looks like a (weighted) homogeneous singularity defined by a (weighted) homogeneous polynomial $F(T_0, \dots, T_n)$, locally near a point $p \in X_\sigma$, then the local Euler class at p for df is the same as the local Euler class for the map $F : \mathbb{A}^{n+1} \to \mathbb{A}^1$ at the origin $0 \in \mathbb{A}^{n+1}$ (see Definition 6.7 and Corollary 6.10 for a precise statement).

6.1 The local Euler class

We recall here some preliminary definitions and define $\mathbb{A}^1$-local Euler class with respect to a section of a vector bundle following [BW, 5.1].

Definition 6.1. For a vector bundle $p : V \to X$ with zero section $s_0 : X \to V$, and dual bundle V^*, define the functor $\Sigma^{V^*} : \mathrm{SH}(X) \to \mathrm{SH}(X)$ by $\Sigma^{V^*} := p_\# s_{0*}$.

We have the identity $\Sigma^{V^*} \mathbb{1}_X = V/(V \setminus X) \in SH(X)$, see [Hoy17, 5.2].

Definition 6.2. Let S be a scheme, let $E \in \mathrm{SH}(S)$, $f : X \to S$ an S-scheme, $i : Z \hookrightarrow X$ a closed subscheme, and $p : V \to X$ a vector bundle. We define the V-twisted E-cohomology of X with support on Z, which we denote $E^V_Z(X)$, to be

$$E^V_Z(X) = [\mathbb{1}_S, f_* i_! \Sigma^{i^* V} i^! f^* E]_{\mathrm{SH}(S)} \simeq [X/(X \setminus Z), \Sigma^V f^* E]_{\mathrm{SH}(X)}.$$

See [BW, 4.2.1].

When $Z = X$, we drop Z from the notation. We also denote $E^n_Z(X) = E_Z^{\mathcal{O}_X^{\otimes n}}(X)$.

For $\mathcal{L}$ a line bundle over X, we put $E^n_Z(X, \mathcal{L}) = E^{n-1+\mathcal{L}}_Z(X)$.

Definition 6.3. Let $E \in \mathrm{SH}(S)$ be a motivic ring spectrum. We denote by (V, ρ) pairs consisting of a vector bundle $p : V \to X$ and an isomorphism $\rho : detV \xrightarrow{\sim} \mathcal{O}_X$.

An SL-orientation on E is an assignment of an element $th(V, \rho) \in E_0^{p^* V^*}(V)$ for each such pair (V, ρ), satisfying the axioms of [LR, Definition 3.4].

An SL-oriented ring spectrum E is a motivic ring spectrum $E \in \mathrm{SH}(S)$ together with a fixed SL-orientation $th(-, -)$.

If E is an SL-oriented motivic spectrum, and $p : V \to X$ is a vector bundle of rank n, we have $E^V_Z(X) = E^n(X, detV)$.

The motivic ring spectrum we use in this paper is $E = H\mathcal{K}^{MW}$ representing the Milnor-Witt homotopy module $\mathcal{K}^{MW}_*$, and admitting a canonical SL-orientation. For details on the construction of this motivic spectrum and its SL-orientation see [Le20, Section 2].

Let X be a smooth scheme over a perfect field k and $p \in X$ a closed point. Then we have an isomorphism $(H\mathcal{K}^{MW})^n(X, \omega_{X/k}) \simeq \mathrm{GW}(k(p))$ ([Le20, Cor. 3.3]), so by using classes in cohomology groups defined by this motivic ring spectrum we can express invariants in quadratic forms. We also use the notation $H^n_Z(X, \mathcal{K}^{MW}(\mathcal{L}))$ for the group $(H\mathcal{K}^{MW})^n_Z(X, \mathcal{L})$.

Definition 6.4. Let $V \to X$ be a vector bundle of rank n, $s : X \to V$ a section and $i : Z = Z(s) \hookrightarrow X$ the zero locus of s. The local Euler class of (V, s), also called the refined Euler class, is the element $e(V, s) \in E^{V^*}_Z(X)$ defined by the composition

$$X/X \setminus Z \xrightarrow{s} V/V \setminus 0 \simeq \Sigma^{V^*} \mathbb{1}_k \to \Sigma^{V^*} E|_X \in \mathrm{SH}(X).$$

Remark 6.5. In the case of an SL-oriented theory E, and a rank n bundle V, we have $E_Z^{V^*}(X) = E_Z^n(X, det^{-1}V)$, giving the local Euler class $e(V,s) \in E_Z^{V^*}(X) = E_Z^n(X, det^{-1}V)$.

We also have the *Thom class* $th(V) \in E_{0_V}^{p*V^*}(V)$, defined as the local Euler class $e(t, p^*V)$, where $t : V \to p^*V$ is the tautological section (with zero-locus the zero-section in V). In general,

$$e_Z(V,s) = s^*th(V) \in E_Z^V(X),$$

see [BW, Def. 5.12].

Example 6.6. In the case the section s has zero locus Z a single point p, then for $E = H\mathcal{K}^{MW}$, $V = \Omega_{X/k}$, we have $e_p(\Omega_{X/k}, s) \in H\mathcal{K}_p^{MW}(X, \omega_{X/k})$. By the purity isomorphism for $H\mathcal{K}^{MW}$, this latter group is canonically isomorphic to $\mathrm{GW}(k(p))$ and this element can be computed as the Scheja-Storch quadratic form on the Jacobian ring at the point, see [Le20, Cor. 3.3] and below 8.2.

6.2 Comparing Euler classes

Definition 6.7. Let κ be a field, let $a_* = (a_0, \ldots, a_n)$ be a sequence of weights and let $F(T_0, \ldots, T_n) \in \kappa[T_0, \ldots, T_n]$ be an a_*-weighted homogeneous polynomial of weighted degree e. Let $\mathcal{O}_\kappa = \kappa[t]_{(t)}$; we denote the closed point of $\operatorname{Spec} \mathcal{O}_\kappa$ by σ_κ and the generic point by η_κ.

We assume that the a_i are pairwise relatively prime, that a_i divides e for all i and that $V(F) \subset \mathbb{P}_\kappa(a_*)$ is a smooth quotient hypersurface; in particular e is prime to the exponential characteristic of κ.

Define $H^F \subset \mathbb{P}_{\mathcal{O}_\kappa}(a_*, 1)$ to be the hypersurface $V(F - tT_{n+1}^e)$, and let $f_F : H^F \to \operatorname{Spec} \mathcal{O}_\kappa$ denote the projection.

One can see that H^F is smooth over κ, the generic fibre $H^F_{\eta_\kappa}$ is smooth over $\eta = \operatorname{Spec} \kappa(t)$ and the special fibre $H^F_{\sigma_\kappa}$ has a single isolated singular point $0 := (0 : \ldots : 0 : 1)$.

We return to our main object of study, a quasi-projective flat map $f : X \to \operatorname{Spec} \mathcal{O}$ with an isolated critical point $p \in X$. Our goal is to show that, under the assumption that f locally looks near p like a quasi-homogeneous singularity defined by a polynomial $F \in k(p)[T_0, \ldots, T_n]$, the local Euler class $e_p(\Omega_{X/k_0}, df)$ at the critical point $p \in X$ is equal to the local Euler class $e_0(\Omega_{H_F/k(p)}, df_F)$. By df, we really mean the section $d(f^*(t))$ of Ω_{X/k_0}, and define df_F similarly. We first make some elementary simplifications.

First of all, due to the Nisnevich descent properties enjoyed by all motivic cohomology theories, the local Euler class $e_p(\Omega_{X/k_0}, df) \in \mathrm{GW}(k(p))$ is unchanged if we replace (X, p) by a Nisnevich neighbourhood $(X', p) \to (X, p)$, and also depends only on df restricted to $\operatorname{Spec} \mathcal{O}_{X,p}$. Thus, we may replace X with $\operatorname{Spec} \mathcal{O}_{X,p}$, and, changing notation, assume that $X = \operatorname{Spec} \mathcal{O}_{X,p}$ is local. Similarly, we may assume that the local ring $\mathcal{O}_{X,p}$ contains its residue field $k(p)$; changing notation, we may assume that $k(p) = k$. The special fibre X_σ is just the subscheme of X defined by $f \in \mathcal{O}_{X,p}$, so we may replace $\mathcal{O}$ with $k[t]$, with morphism $f : X \to \operatorname{Spec} k[t]$ given by the k-algebra homomorphism $t \mapsto f^*(t)$. Choosing a system of parameters $s_0, \ldots, s_n$ so that $f^*(t) = F(s_0, \ldots, s_n) + h$ as in Assumption 5.5, we have the morphism over $\operatorname{Spec} k[\lambda]$,

$$f_\lambda : X \times \operatorname{Spec} k[\lambda] \to \operatorname{Spec} k[t, \lambda],$$

defined by $f_\lambda^*(t) := F(s_0, \ldots, s_n) + \lambda \cdot h$.

Proposition 6.8. *Let $X = \operatorname{Spec} \mathcal{O}_{X,p}$ be an $\mathbb{A}^1$-family $f : X \to \operatorname{Spec} k[t]$, flat and essentially of finite type, satisfying Assumption 4.1 (homogeneous case) or 5.5 (quasi-homogeneous case). Define $\mathcal{X} = \operatorname{Spec} \mathcal{O}_{X,p}[\lambda] = X \times \mathbb{A}^1$. Let $f_\lambda : X \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ be defined as above, and let $\mathcal{X}_\sigma = f_\lambda^{-1}(0 \times \mathbb{A}^1)$ with induced morphism $(f_\lambda)_\sigma : \mathcal{X}_\sigma \to \mathbb{A}^1$. Then there exists an open neighbourhood $U \supset p \times \mathbb{A}^1$ in $\mathcal{X}_\sigma$, such that $U \setminus (p \times \mathbb{A}^1)$ is smooth over $\mathbb{A}^1$.*

Proof. We start with the homogeneous case. Let $\rho : Bl_{p\times\mathbb{A}^1}\mathcal{X} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $p\times\mathbb{A}^1$. Let $q : \bigcup U_i = Bl_{p\times\mathbb{A}^1}\mathcal{X} \to X$ be the standard covering and denote by $\mathcal{D}_{12}$ the intersection of the strict transform of $\mathcal{X}_\sigma$ and the exceptional divisor $\mathcal{D}_1 \subset Bl_{p\times\mathbb{A}^1}\mathcal{X}$.

We can describe the morphism $\mathcal{D}_{12} \to \mathbb{A}^1$ similarly to our description of D_{12} in Proposition 4.3, just adding the variable λ. The blow-up $Bl_{p\times X}(X\times\mathbb{A}^1)$ is $(Bl_pX)\times\mathbb{A}^1$ and is covered by the open subsets $U_i\times\mathbb{A}^1$, with the U_i as in 4.3.

Over $U_0\times\mathbb{A}^1$ we have

$$f_\lambda = s_0^e(F(1,t_1,\ldots,t_n) + s_0\lambda h')$$

and since the exceptional divisor is defined by s_0 on $U_0\times\mathbb{A}^1$, we see that $(\mathcal{D}_{12}\cap U_0)\times\mathbb{A}^1 = (V(F)\cap U_0)\times\mathbb{A}^1 \subset (Bl_pX)\times\mathbb{A}^1$. Thus $\mathcal{D}_{12} = V(F)\times\mathbb{A}^1$, and that scheme is smooth by our assumption on F.

In the quasi-homogenous case we go through the same construction as in the last section. First let

$$\mathcal{O}_{X,p}[s^{1/a}] := \mathcal{O}_{X,p}[\sigma_0,\ldots,\sigma_n]/(\{\sigma_i^{a_i} - s_i\}_i).$$

and let $g^*(t) \in \mathcal{O}_{X,p}[s^{1/a}]$ be the image of $f^*(t) = F(s_0,\ldots,s_n) + h$ under the inclusion $\mathcal{O}_{X,p} \subset \mathcal{O}_{X,p}[s^{1/a}]$. Letting $Z = \operatorname{Spec}\mathcal{O}_{X,p}[a^{1/a}]$, we have the usual μ_a-action on Z with $X = Z/\mu_a$. The element $g^*(t) \in \mathcal{O}_{X,p}[s^{1/a}]$ defines the morphism

$$Z = \operatorname{Spec}\mathcal{O}_{X,p}[a^{1/a}] \xrightarrow{g} \operatorname{Spec} k[t],$$

making the the diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr, "g"] & X \arrow[d, "f"] \\
 & \operatorname{Spec} k[t]
\end{tikzcd}$$

commute. Moreover, $g^*(t) = F(\sigma_0^{a_0},\ldots,\sigma_n^{a_n}) + h'$ with $G(\sigma_0,\ldots,\sigma_n) = F(\sigma_0^{a_0},..,\sigma_n^{a_n})$ homogeneous of degree e, and with $h' \in m_q^{e+1}$. Define the morphism

$$g_\lambda : \mathcal{Z} := Z\times\mathbb{A}^1 \to \operatorname{Spec} k[t,\lambda]$$

by $g_\lambda^*(t) = G + \lambda\cdot h'$. Then $\mathcal{X} = \mathcal{Z}/\mu_a$ and we have a commutative diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[dr, "g_\lambda"] & \mathcal{X} \arrow[d, "f_\lambda"] \\
 & \operatorname{Spec} k[t,\lambda]
\end{tikzcd}$$

Next, blow up $\mathcal{Z} = Z\times\mathbb{A}^1$ at $p\times\mathbb{A}^1$ to get $\hat{\mathcal{Z}}$ and denote by $\hat{\mathcal{X}}$ the quotient by the action of μ_a. Let $q : \hat{\mathcal{X}} \to \mathcal{X}$ be the natural map. Denote the intersection of the strict transform of $\mathcal{Z}_\sigma$ and the exceptional divisor in $\hat{\mathcal{Z}}$ by $\mathcal{E}_{12} = V_{\mathbb{P}^n}(F)\times\mathbb{A}^1$ (see the paragraph above) and its image under the μ_a-quotient map by $\mathcal{D}_{12}$. Then we get $\mathcal{D}_{12} = V_{\mathbb{P}(a)}(F)\times\mathbb{A}^1$ which is smooth by our Assumption 5.5.

Let $\mathcal{X}_\sigma := f_\lambda^{-1}(\sigma\times\mathbb{A}^1) \subset \mathcal{X}$. We have in both cases the proper map $q : \hat{\mathcal{X}} \to \mathcal{X}$, which is an isomorphism over $\mathcal{X}\setminus p\times\mathbb{A}^1$. Let $q^{-1}[\mathcal{X}_\sigma]$ be the closure of $q^{-1}(\mathcal{X}_\sigma\setminus p\times\mathbb{A}^1)$ in $\hat{\mathcal{X}}$.

In both cases, the Cartier divisor $\mathcal{D}_{12}$ on the reduced scheme $q^{-1}[\mathcal{X}_\sigma]$ is smooth over $\mathbb{A}^1$. Let $r : q^{-1}[\mathcal{X}_\sigma] \to \mathbb{A}^1$ be the morphism induced by f_λ. Then r is flat and the set W of points $x \in q^{-1}[\mathcal{X}_\sigma]$ such that x is a smooth point of the fibre $r^{-1}(r(x))$ is an open subset of $q^{-1}[\mathcal{X}_\sigma]$, and is equal to the set of points of $q^{-1}[\mathcal{X}_\sigma]$ at which r is a smooth morphism. By Lemma 5.6, W is an open neighbourhood of $\mathcal{D}_{12}$ in $q^{-1}[\mathcal{X}_\sigma]$. Letting F be the closed complement of W in $q^{-1}[\mathcal{X}_\sigma]$, and noting the q is proper, $q(F)$ is a closed subset of $\mathcal{X}_\sigma$, disjoint from $p\times\mathbb{A}^1$. Set $U := \mathcal{X}_\sigma\setminus q(F)$. Then U is open and $U\setminus(p\times\mathbb{A}^1) \simeq W\setminus D_{12}$ is smooth over $\mathbb{A}^1$.

□

Proposition 6.9. *Let X be a smooth quasi-projective scheme over a field k, with $Z \subset X$ closed, let $p : V \to X$ be a vector bundle, and let $s_1, s_2 : X \to V$ be two sections.*

Consider $\tilde{p} : \pi^*V \to X \times \mathbb{A}^1$ with π the projection $\pi : X \times \mathbb{A}^1 \to X$. Define a section $s : X \times \mathbb{A}^1 \to \pi^*V$ by $s = \lambda s_1 + (1-\lambda)s_2$ and assume that we have an open neighbourhood U of $Z \times \mathbb{A}^1$ in $X \times \mathbb{A}^1$ such that $Z(s) \cap U = Z \times \mathbb{A}^1$. Then

$$e_Z(X, s_1) = e_Z(X, s_2).$$

Proof. Let $E = H\mathcal{K}^{MW}$. Let $s_0 : X \to V$ be the zero section. We have the Thom class

$$th(V) = s_{0*}\mathbb{1}_X \in E_{0_V}^{V^*}(V)$$

We have the two embeddings $i_1 : X \hookrightarrow X \times 0 \subset X \times \mathbb{A}^1$ and $i_2 : X \hookrightarrow X \times 1 \subset X \times \mathbb{A}^1$. By homotopy invariance the two maps

$$i_1^*, i_2^* : E_{Z\times\mathbb{A}^1}^{\pi^*V^\vee}(X \times \mathbb{A}^1) \to E_Z^{V^\vee}(X)$$

are equal.

Using excision property in cohomology we can remove the piece $(X \times \mathbb{A}^1) \setminus U$ to get

$$\alpha : E_{Z\times\mathbb{A}^1}^{V_U^*}(U) \simeq E_{Z\times\mathbb{A}^1}^{\pi^*V^*}(X \times \mathbb{A}^1).$$

Here V_U is the pullback of V over $U \hookrightarrow X \times \mathbb{A}^1$.

We denote $s' = s|_U : U \to V_U$ and $\tilde{p}' = \tilde{p}|_{V_U} : V_U \to U$. Since $Z(s') = Z(s) \cap U = Z \times \mathbb{A}^1$, we have a map

$$s'^* : E_0^{p'^*V_U^*}(V_U) \to E_{Z\times\mathbb{A}^1}^{V_U^*}(U).$$

Denote by $\tilde{\pi}$ the pullback map $V_U \to V$ of the vector bundle $V \to X$ along $U \hookrightarrow X \times \mathbb{A}^1 \to X$ and consider the following commutative diagram -

$$\begin{tikzcd}
E_0^{p'^*V_U}(V_U) \arrow[d, "s'^*"] & E_{0_V}^{p^*V^*}(V) \arrow[l, "\tilde{\pi}^*"'] \arrow[dd, shift right, "s_1^*"'] \arrow[dd, shift left, "s_2^*"] \\
E_{Z\times\mathbb{A}^1}^{V_U^*}(U) \arrow[d, "\alpha", "\simeq"'] & \\
E_{Z\times\mathbb{A}^1}^{\pi^*V^*}(X \times \mathbb{A}^1) \arrow[r, shift left, "i_1^*"] \arrow[r, shift right, "i_2^*"'] & E_Z^{V^*}(X)
\end{tikzcd}$$

We have

$$s_1^* th(V) = i_1^* \circ \alpha \circ s'^* \circ \tilde{\pi}^* th(V) = i_2^* \circ \alpha \circ s'^* \circ \tilde{\pi}^* th(V) = s_2^* th(V)$$

which gives the desired equality of local Euler classes.

$\square$

Corollary 6.10. *Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a flat quasi-projective morphism with X smooth over k_0 and with an isolated critical point $p \in X_\sigma$. Suppose that f locally looks like $F = F(T_0, \ldots, T_n)$ at p (see 1.2). Then*

$$e_p(\Omega_{X/k(p)}, df) = e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}/k(p)}, d(F(t_0, \ldots, t_n)) = e_0(\Omega_{H^F/k(p)}, df_F)$$

in $\mathrm{GW}(k(p))$*.*

Proof. Proposition 6.8 proves that the assumptions in Proposition 6.9 are satisfied for $E = H\mathcal{K}^{MW}$, $Z = \{p\}$, $V = \Omega_{X/k} \to X$, $s_1 = df$, $s_2 = dF$. This gives the identity in $\mathrm{GW}(k(p))$,

$$e_p(\Omega_{X/k(p)}, df) = e_p(\Omega_{X/k(p)}, d(F(s_0, \ldots, s_n))).$$

The parameters $s_0, \ldots, s_n \in \mathcal{O}_{X,p}$ define an étale map $\alpha : \operatorname{Spec} \mathcal{O}_{X,p} \to \mathbb{A}^{n+1}_{k(p)} := \operatorname{Spec} k(p)[t_0, \ldots, t_n]$ which maps p to 0 and with $\alpha^* F(t_0, \ldots, t_n) = F(s_0, \ldots, s_n)$.

Thus $(s_0, \ldots, s_n)$ expresses (X, p) as a Nisnevich neighbourhood of $(\mathbb{A}^{n+1}_{k(p)}, 0)$. Since

$$(s_0, \ldots, s_n)^*(F(t_0, \ldots, t_n)) = F(s_0, \ldots, s_n),$$

we have

$$e_p(\Omega_{X/k(p)}, d(F(s_0, \ldots, s_n))) = \overline{(s_0, \ldots, s_n)}^*(e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}/k(p)}, d(F(t_0, \ldots, t_n))$$

where $\overline{(s_0, \ldots, s_n)}^* : \mathrm{GW}(k(p)(0)) \to \mathrm{GW}(k(p))$ is the isomorphism induced by $(s_0, \ldots, s_n) : p \to 0$; this is just the identity map on $\mathrm{GW}(k(p))$, so we can write this as an identity

$$e_p(\Omega_{X/k(p)}, d(F(s_0, \ldots, s_n))) = e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}/k(p)}, d(F(t_0, \ldots, t_n)).$$

The singular point $0 = (0 : \ldots : 0 : 1)$ of $H^F_{\sigma_{k(p)}}$ is in the affine open subscheme $U_{n+1} \subset \mathbb{P}_{\mathcal{O}_{k(-)}}(a_*, 1)$, so to compute $e_0(\Omega_{H^F/k(p)}, df_F)$, we can restrict to U_{n+1}. We have

$$U_{n+1} = \operatorname{Spec} \mathcal{O}_{k(p)}[T_0, \ldots, T_n, T_{n+1}][T_{n+1}^{-1}]_0$$

and $\mathcal{O}_{k(p)}[T_0, \ldots, T_n, T_{n+1}][T_{n+1}^{-1}]_0$ is the polynomial ring $\mathcal{O}_{k(p)}[t_0, \ldots, t_n]$, with $t_i = T_i/T_{n+1}^{a_i}$. On U_{n+1}, H^F has defining equation

$$(F(T_0, \ldots, T_n) - tT_{n+1})/T_{n+1}^e = F(t_0, \ldots, t_n) - t.$$

Thus, $H^F \cap U_{n+1}$ is just the graph of the morphism

$$F(t_0, \ldots, t_n) : \mathbb{A}^{n+1}_{k(p)} = \operatorname{Spec} k(p)[t_0, \ldots, t_n] \to \operatorname{Spec} k[t]_{(t)}$$

If we replace the graph $H^F \cap U_{n+1}$ with the isomorphic scheme $\operatorname{Spec} k(p)[t_0, \ldots, t_n]$ via the isomorphism given by the first projection, then f_F transforms to the map $F(t_0, \ldots, t_n)$ and 0 goes to the origin $(0, \ldots, 0) \in \mathbb{A}^{n+1}_{k(p)}$. In other words,

$$e_0(\Omega_{H^F/k(p)}, df_F) = e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}/k(p)}, d(F(t_0, \ldots, t_n)).$$

□

7 The generalized conductor formula

In this section, we retain our notations and assumptions for $\mathcal{O}$ and $B = \operatorname{Spec} \mathcal{O}$ as in Section 3.1, assume in addition that the subfield $k_0 \subset \mathcal{O}$ has characteristic zero. We have the characteristic zero residue field k and fraction field K of $\mathcal{O}$.

Let $f : X \to B$ be a flat, quasi-projective morphism such that X is smooth over k_0, X_η is smooth over η and such that X_σ has finitely many singular points.

Fix a sequence of pairwise relative prime weights $a := (a_0, \ldots, a_n)$ and a field κ, and let $F \in \kappa[T_0, \ldots, T_n]$ be a degree e a-weighted homogeneous polynomial such that $V(F) \subset \mathbb{P}_\kappa(a)$ is a smooth quotient hypersurface, in the sense of Definition 5.3. We have the discrete valuation ring $\mathcal{O}_\kappa := \kappa[t]_{(t)}$, the hypersurface $H^F := V(F - tT_{n+1}^e) \subset \mathbb{P}_{\mathcal{O}_\kappa}(a, 1)$ with projection $f_F : H^F \to \operatorname{Spec} \mathcal{O}_\kappa$. H^F is smooth over κ, $H^F_{\eta_k}$ is smooth over η_κ, and $H^F_{\sigma_\kappa}$ has a single singularity at $p := (0 : \ldots : 0 : 1)$. In fact, $H^F_{\sigma_\kappa}$ is the cone over $V(F, T_{n+1}) \subset V(T_{n+1}) = \mathbb{P}(a)_\kappa$ with vertex p.

In [LPS], Levine, Pepin Lehalleur and Srinivas consider the invariant

$$\Delta_t(F) := \mathrm{sp}_t(\chi_{\mathrm{c}}(H^F_{\eta_\kappa}/\kappa(t))) - \chi_{\mathrm{c}}(H^F_{\sigma_\kappa}/\kappa) \in \mathrm{GW}(\kappa)$$

and derive an expression, which they call a *conductor formula*, for $\Delta_t(F)$ in terms of the local Euler class $e_p(\Omega_{H^F/\kappa}, dt) \in \mathrm{GW}(\kappa)$.

Note that $f_F : H^F \to \operatorname{Spec} \mathcal{O}_\kappa$ looks locally at $p = (0 : \ldots : 0 : 1)$ like the weighted homogeneous singularity defined by F. A generalization of the conductor formulas for $\Delta_t(F)$ for degenerations with finitely many singularities of a certain type is conjectured in *loc. cit.*.

In this section we use the results of the previous sections computing $\chi(\Psi_f(\mathbb{1}_{X\eta})|_p)$ at a singular point p and reinterpret them in terms of the difference $\Delta_t(F)$ considered in [LPS]. Using the sheaf properties of the functor Ψ, this allows to generalize the formula proven in [LPS] to the case of $f : X \to \operatorname{Spec} \mathcal{O}$ with finitely many isolated critical points, all satisfying Assumptions 4.1 or 5.5. In particular, this verifies the conjecture formulated in [LPS] in characteristic zero, for a somewhat wider class of singularities than what was considered there.

We record the formal definition of the invariant $\Delta_t(F)$.

Definition 7.1. Let κ be a field, let $\mathcal{O}_\kappa = \kappa[t]_{(t)}$, let $a = (a_0, \ldots, a_n)$ be a sequence of positive integral weights, and let $F(T_0, \ldots, T_n) \in \kappa[T_0, \ldots, T_n]$ be an a-weighted homogeneous polynomial of weighted degree e. We assume that $V(F) \subset \mathbb{P}_\kappa(a_*)$ is a smooth quotient hypersurface; in particular, the a_i are pairwise relative prime and e is prime to the exponential characteristic of κ.

Define $\Delta_t(F/\kappa) \in \mathrm{GW}(\kappa)$ by

$$\Delta_t(F/\kappa) := \mathrm{sp}_t(\chi_{\mathrm{c}}(H^F_{\eta_\kappa})) - \chi_{\mathrm{c}}(H^F_{\sigma_\kappa}).$$

This includes the homogeneous case by taking $a = (1, \ldots, 1)$.

Levine, Pepin Lehalleur and Srinivas prove the formula

$$\Delta_t(F/\kappa) = \langle e \rangle - \langle 1 \rangle + (-\langle e \rangle)^n \cdot e_0(\Omega_{H^F/\kappa}, dt) \in \mathrm{GW}(\kappa)$$

for a homogeneous F ([LPS, Theorem 5.2 (2)]). Here $e_0(\Omega_{H^F/k(p)}, dt)$ is the local Euler class at $0 := (0 : \ldots : 0 : 1)$ [LPS, 5].

For an a-weighted homogeneous F they prove a similar formula, taking the weights into account [LPS, Theorem 5.3].

$$\Delta_t(F/\kappa) = \langle \prod_j a_j \cdot e \rangle - \langle 1 \rangle + (-\langle e \rangle)^n \cdot e_0(\Omega_{H^F/\kappa}, dt) \in \mathrm{GW}(\kappa)$$

We wish to extend this to a formula to the case of a morphism with isolated critical points that locally look like a homogeneous or quasi-homogeneous singularity. Our main theorem, comparing the $\chi_{\mathrm{c}}(\Psi_f|_p)$ of the scheme and the χ_{c} of the hypersurface defined by F can now be proven.

Recall from Section 2 that for a finite separable field extension $k_1 \subset k_2$, we have the transfer map $\mathrm{Tr}_{k_2/k_1} : \mathrm{GW}(k_2) \to \mathrm{GW}(k_1)$.

Theorem 7.2. *Let $\mathcal{O}$ and $B := \operatorname{Spec} \mathcal{O}$ be as in Section 3.1, and assume in addition that the subfield $k_0 \subset \mathcal{O}$ has characteristic zero.*

Let $f : X \to B$ be a flat quasi-projective morphism with X smooth over k_0 and with X_η smooth over η, and let $p \in X_\sigma$ be an isolated critical point of f, satisfying 4.1 or 5.5. Let $F \in k(p)[T_0, \ldots, T_n]$ be corresponding (weighted) homogeneous polynomial. Then

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})|_p) = \mathrm{Tr}_{k(p)/k}(\Delta_t(F/k(p)) + \langle 1 \rangle) \in \mathrm{GW}(k).$$

If $(a_0, \ldots, a_n)$ are the weights and e is the weighted degree for F, then

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})|_p) = \mathrm{Tr}_{k(p)/k}(\langle \prod_j a_j \cdot e \rangle + (-\langle e \rangle)^n \cdot e_0(\Omega_{\mathbb{A}^{n+1}/k(p)}, dF)) \in \mathrm{GW}(k).$$

Proof. The homogeneous case is a special case of the weighted homogeneous case, with all weights equal to 1, so we need only handle the weighted homogeneous case.

The second formula follows from the first, together with the formula of [LPS, Theorem 5.3] mentioned above, and the identity

$$e_0(\Omega_{\mathbb{A}^{n+1}/k(p)}, dF) = e_0(\Omega_{H^F/\kappa}, dt),$$

from Corollary 6.10.

For the proof of the first formula, note that we have families $f : X \to \operatorname{Spec} \mathcal{O}$ and $f_F : H_F \to \operatorname{Spec} k(p)[t]_{(t)}$ over different bases, so we will need to keep track of the base-fields for the Euler characteristics and the base-schemes for the nearby cycles functors.

Since $\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X_\eta})|_p)$ is determined by a neighbourhood of p we can assume p is the only critical point of f.

First we show that the terms in the difference $\Delta_t(F/k(p))$ are closely related to the Denef-Loeser covers we computed in Theorem 4.3 and Theorem 5.7.

By Property 3.1, Proposition 3.20 and Proposition 3.21, we have

$$\begin{aligned}
\mathrm{sp}_t(\chi_{\mathrm{c}}(H^F_{\eta_{k(p)}}/k(p)(t))) &= \Psi_{\mathrm{id}_{k(p)[t]_{(t)}}*}(\chi_{\mathrm{c}}(H^F_{\eta_{k(p)}}/k(p)(t))) \\
&= \Psi_{\mathrm{id}_{k(p)[t]_{(t)}}*}(\chi(f_{F\eta_{k(p)}*}(\mathbb{1}_{H^F_\eta}))) \\
&= \chi(\Psi_{\mathrm{id}_{k(p)[t]_{(t)}}}(f_{F\eta_{k(p)}*}(\mathbb{1}_{H^F_{\eta_{k(p)}}}))) \\
&= \chi(f_{F\sigma_{k(p)}*}(\Psi_{f_F}(\mathbb{1}_{H^F_{\eta_{k(p)}}}))) \\
&= \chi_{\mathrm{c}}(\Psi_{f_F}(\mathbb{1}_{H^F_{\eta_{k(p)}}})/k(p)).
\end{aligned}$$

On the other hand, we can apply Corollary 5.8 to give

$$\chi_{\mathrm{c}}(\Psi_{f_F}(\mathbb{1}_{H^F_{\eta_{k(p)}}}))/k(p)) = \chi_{\mathrm{c}}(V(F - T^e_{n+1})/k(p)) + \chi_{\mathrm{c}}(H^{F\circ}_{\sigma_{k(p)}}/k(p)) - \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F)/k(p)).$$

However, $H^{F\circ}_{\sigma_{k(p)}}$ is an $\mathbb{A}^1$-bundle over $V_{\mathbb{P}_{k(p)}(a)}(F) \cong V_{\mathbb{P}_{k(p)}(a,1)}(F, T_{n+1}) \subset \mathbb{P}_{k(p)}(a, 1)$, so we have

$$\chi_{\mathrm{c}}(H^{F\circ}_{\sigma_{k(p)}/k(p)}) = \chi_{\mathrm{c}}(V(F)/k(p)) \cdot \chi_{\mathrm{c}}(\mathbb{A}^1/k(p)) = \chi_{\mathrm{c}}(\mathbb{A}^1 \times V(F)/k(p)),$$

which yields

$$\chi_{\mathrm{c}}(\Psi_{f_F}(\mathbb{1}_{H^F_{\eta_{k(p)}}})/k(p)) = \chi_{\mathrm{c}}(V(F - T^e_{n+1})/k(p)).$$

Thus

$$\mathrm{sp}_t\, \chi_{\mathrm{c}}(H^F_{\eta_{k(p)}}/k(p)(t)) = \chi_{\mathrm{c}}(V(F - T^e_{n+1})/k(p)) = \chi_{\mathrm{c}}(\widetilde{D_1}/k(p)).$$

Now $H^F_{\sigma_{k(p)}} = H^{F\circ}_{\sigma_{k(p)}} \amalg (0 : \ldots : 0 : 1)_{k(p)}$, and $V_{\mathbb{P}_{k(p)}(a,1)}(F, T_{n+1}) \simeq V_{\mathbb{P}(a)_{k(p)}}(F) \simeq \widetilde{D_{12}}$, so

$$\chi_{\mathrm{c}}(H^F_{\sigma_{k(p)}}/k(p)) = \chi_{\mathrm{c}}(\widetilde{D_{12}}/k(p)) \cdot \langle -1 \rangle + \langle 1 \rangle \in \mathrm{GW}(k(p)).$$

Adding this up (or rather subtracting) we have the formula

$$\Delta_t(F/k(p)) = \chi_{\mathrm{c}}(\widetilde{D_1}/k(p)) - \chi_{\mathrm{c}}(\widetilde{D_{12}}/k(p)) \cdot \langle -1 \rangle - \langle 1 \rangle \in \mathrm{GW}(k(p)).$$

Applying Proposition 2.16, this gives

$$\mathrm{Tr}_{k(p)/k}(\Delta_t(F/k(p))) = \chi_{\mathrm{c}}(\widetilde{D_1}/k) - \chi_{\mathrm{c}}(\widetilde{D_{12}}/k) \cdot \langle -1 \rangle - \mathrm{Tr}_{k(p)/k}(\langle 1 \rangle) \in \mathrm{GW}(k).$$

On the other hand, by Proposition 3.9 and Theorem 4.3 (5.7), we have

$$\begin{aligned}\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})|_p/k) &= \chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})/k) - \chi_{\mathrm{c}}(X_\sigma \setminus \{p\}/k) \\ &= \chi(\widetilde{D_1}/k) + \chi(\widetilde{D_2^\circ}/k) - \chi(\widetilde{D_{12}}/k) \cdot \langle -1 \rangle - \chi(\widetilde{D_2^\circ}/k) = \chi(\widetilde{D_1}/k) - \chi(\widetilde{D_{12}}/k) \cdot \langle -1 \rangle\end{aligned}$$

So comparing both terms, we have

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})|_p) := \chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})|_p/k) = \mathrm{Tr}_{k(p)/k}(\Delta_t(F/k(p)) + \langle 1 \rangle),$$

concluding the proof. □

We would like now to obtain a global formula in the general case, when X has multiple singular points at the special fibre $p_1, \ldots, p_r$ satisfying Assumption 4.1 (or 5.5).

We state our main result in weighted homogeneous setting as this also includes the homogeneous case.

Corollary 7.3 (Generalized quadratic conductor formula)**.** *Let* $X \to \operatorname{Spec} \mathcal{O}$ *be a flat projective morphism of relative dimension* n*, with* X *smooth over* k_0 *and* X_η *smooth over* η*. Suppose that the special fibre* X_σ *has isolated singularities* $p_1, .., p_r$ *satisfying Assumption 5.5 with* $F_i \in k(p_i)[T_0, \ldots, T_n]$ *an* $a^{(i)}_*$*-weighted homogeneous polynomial of degree* e_i*. Then*

$$\mathrm{sp}_t(\chi_{\mathrm{c}}(X_\eta/k(\eta))) - \chi_{\mathrm{c}}(X_\sigma/k) = \sum_i Tr_{k(p_i)/k}[\langle \prod_j a^{(i)}_j \cdot e_i \rangle - \langle 1 \rangle + (-\langle e_i \rangle)^n \cdot e_{p_i}(\Omega_{X/k(p_i)}, dt)]$$

Proof. By applying Proposition 2.16, Proposition 3.9 and Theorem 7.2 we obtain the formula

$$\begin{aligned}\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})) &= \sum_i \chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})|_{p_i}) + \chi_{\mathrm{c}}(X_\sigma \setminus \{p_1, \ldots, p_k\}/k) \\ &= \sum_i Tr_{k(p_i)/k}(\Delta_t(F_i/k(p_i)) + \langle 1 \rangle) + \chi_{\mathrm{c}}(X_\sigma) - \sum_i Tr_{k(p_i)/k}(\langle 1 \rangle)\end{aligned}$$

This gives the global formula

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})) - \chi_{\mathrm{c}}(X_\sigma) = \sum_i Tr_{k(p_i)/k}(\Delta_t(F_i/k(p_i)))$$

Substituting Levine-Pepin Lehalleur-Srinivas's conductor formula [LPS, Theorem 5.3] gives

$$\chi_{\mathrm{c}}(\Psi_f(\mathbb{1}_{X\eta})) - \chi_{\mathrm{c}}(X_\sigma) = \sum_i Tr_{k(p_i)/k}[\langle \prod_j a^{(i)}_j \cdot e_i \rangle - \langle 1 \rangle + (-\langle e_i \rangle)^n \cdot e_0(\Omega_{H^{F_i}/k(p_i)}, dt)]$$

But as we proved in Section 6, Corollary 6.10 we can replace $e_0(\Omega_{H^{F_i}/k(p_i)}, dt)$ with $e_{p_i}(\Omega_{X/k(p_i)}, dt)$. Then by [LPS, Proposition 8.3] which states that $\chi_c(\Psi_f(\mathbb{1}_{X\eta})) = \mathrm{sp}_t\, \chi_c(X_\eta/k(\eta))$, we get the desired result

$$\mathrm{sp}_t\, \chi_c(X_\eta/k(\eta)) - \chi_c(X_\sigma/k) = \sum_i Tr_{k(p_i)/k}[\langle \prod_j a_j^{(i)} \cdot e_i\rangle - \langle 1\rangle + (-\langle e_i\rangle)^n \cdot e_{p_i}(\Omega_{X/k(p_i)}, dt)]$$

□

8 Interpretations, applications and examples

8.1 The Jacobian ring, Milnor number and quadratic refinements

As mentioned in the introduction, the conductor formula is expressed in terms of quadratic forms related to algebraic invariants of the singularities. We recall here a construction of a distinguished quadratic form related to the Scheja-Storch element, which gives the local Euler class $e_p(\Omega_{X/k}, s)$ of Definition 6.4.

Definition 8.1. Let k be a field and X be a smooth finite type scheme over k. Let $p \in X$ be a closed point, take $f \in \mathcal{O}_{X,p}$, and let $s_0, \dots, s_n \in m_p$ be a regular system of parameters at p. Suppose that $\sqrt{(\partial f/\partial s_0 \dots \partial f/\partial s_n)} = m_p$, so df has an isolated zero at p; note that the ideal $(\partial f/\partial s_0 \dots \partial f/\partial s_n)$ does not depend on the choice of the s_i. Let $k(p)$ be the residue field of $\mathcal{O}_{X,p}$.

The *Jacobian ring of f at p*, $J(f,p)$, is defined as

$$J(f,p) := \mathcal{O}_{X,p}/(\partial f/\partial s_0 \dots \partial f/\partial s_n).$$

For k algebraically closed, the dimension of $J(f,p)$ over k is the *Milnor number* $\mu_{f,p}$.

Since $\partial f/\partial s_i$ is in $m_p = (s_0, \dots, s_n)$, we can write for each i,

$$\partial f/\partial s_i = \sum_j a_{ij} s_j$$

with $a_{ij} \in \mathcal{O}_{X,p}$. The *Scheja-Storch element* $e_{f,p} \in J(f,p)$ is defined as the image of the determinant $det(a_{ij})$ in $J(f,p)$; $e_{f,p}$ is independent of the choices made. Since $J(f,p)$ is an Artinian local k-algebra, $J(f,p)$ contains the residue field $k(p)$.

Let $Tr : J(f,p) \to k(p)$ be a $k(p)$-linear map sending $e_{f,p}$ to 1. Define

$$B_{f,p} : J(f,p) \times_{k(p)} J(f,p) \to k(p)$$

by $B_{f,p}(x,y) = Tr(xy)$. The class $[B_{f,p}] \in \mathrm{GW}(k(p))$ does not depend on the choices of generators $(s_0, ..., s_n)$ or the map Tr, see [Le20, Theorem 3.1]. If $char k \neq 2$ we denote the quadratic form corresponding to the bilinear form $B_{f,p}$ by $\mu^q_{f,p}$. By taking the rank of the quadratic form, $\mathrm{rk}\, \mu^q_{f,p} = dim J(f,p) = \mu_{f,p}$, so the class $[\mu^q_{f,p}] \in \mathrm{GW}(k(p))$ can be viewed as a quadratic refinement of the Euler number $\mu_{f,p} \in \mathbb{Z}$.

Theorem 8.2 ([BW, Proposition 2.32 and Theorem 7.6] and [Le20, Corollary 3.3])**.** *Take $X \in \mathbf{Sm}_k$ and let $\Omega_{X/k}$ be the sheaf of Kähler differentials. Let $f : X \to \operatorname{Spec} \mathcal{O}$ be a flat morphism with an isolated critical point $p \in X_\sigma$, so the section $df \in H^0(X, \Omega_{X/k})$ has zero locus $Z(s) = \{p\}$ in a neighbourhood of p. Let $e_p(\Omega_{X/k}, df) \in \mathrm{GW}(k(p))$ be the local Euler class as in Example 6.6, and let $[\mu^q_{f,p}] \in \mathrm{GW}(k(p))$ be as defined above. Then*

$$e_p(\Omega_{X/k}, df) = [\mu^q_{f,p}].$$

Rewriting our main result Corollary 7.3 then, we get

$$\mathrm{sp}_t(\chi_c(X_\eta/k(\eta))) - \chi_c(X_\sigma/k) = \sum_i Tr_{k(p_i)/k}\left(\langle \prod_j a_j^{(i)} \cdot e_i \rangle - \langle 1 \rangle + (-\langle e_i \rangle)^n \cdot \mu^q_{F_i,p_i}\right);$$

the left hand side is defined purely algebraically by invariants of the polynomials $F_i \in k(p_i)[T_0, \dots T_n]$.

Notice that this formula refines in quadratic forms the formula by Milnor (1.1) mentioned in the introduction. Assume $k = \mathbb{C}$, and let $f : X \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$ be a flat family of varieties, X being an $n+1$-dimensional smooth $\mathbb{C}$-scheme, and let $X_t = f^{-1}(\mathbb{G}_m)$, $X_0 = f^{-1}(0)$. Suppose that $f|_{X_t} : X_t \to \mathbb{G}_m$ is smooth, and $f|_{X_0} : X_0 \to \mathbb{C}$ has isolated F_i-weighted-homogeneous singular points p_i. We can specialize to $X \to \operatorname{Spec} k[t]_{(t)}$ and use our formula above. Then since $\operatorname{rk} \mu^q_{F_i,p_i} = dim J(F_i, p_i) = \mu_{F_i,p_i}$, and from Remark 2.10, taking ranks on both sides of the equation in the formula above gives

$$\chi^{top}(X_t) - \chi^{top}(X_0) = (-1)^n \sum_i \mu_{F_i,p_i}.$$

which is Milnor's formula mentioned in the introduction (1.1). Note that at each point, the difference $\langle \prod_j a_j^{(i)} \cdot e_i \rangle - \langle 1 \rangle$ vanishes under the rank map, as a difference of two rank 1 quadratic forms; similarly, the term $(-\langle e_i \rangle)^n$ maps to $(-1)^n$. This simplification also occurs for $k = \mathbb{R}$, as $\prod_j a_j^{(i)} \cdot e_i$ and e_i are squares in $\mathbb{R}$. Thus, these terms are only apparent in the refined formulas; see also [LPS, Section 1 and Remark 5.5].

8.2 The case of curves on a surface

As an application of our main theorem, we develop here a formula for the difference between the quadratic Euler characteristic of curves on a surface, refining a formula for complex varieties deduced from the formula of Jung-Milnor.

Let C be a reduced curve on a smooth projective surface S over an algebraically closed field k of characteristic zero. Let $\pi : \tilde{C} \to C$ be the normalisation. Let p be a singular point of C.
Let r_p be the the number of points in $\pi^{-1}(p)$.
Let δ_p be the length of the (finite length) $\mathcal{O}_{C,p}$-module $\pi_*(\mathcal{O}_{\tilde{C},\pi^{-1}(p)})/\mathcal{O}_{C,p}$.
Let μ_p be the Milnor number defined above for the local defining equation for C, $f_p \in \mathcal{O}_{S,p}$, at p.

The Jung-Milnor formula [Mil, Chapter 10] states that

$$2\delta_p = \mu_p + r_p - 1.$$

If C is irreducible, we have $h^0(C, \mathcal{O}_C) = 1 = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}})$ and the short exact sequence

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{\tilde{C}} \to \pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C \to 0$$

gives

$$h^1(C, \mathcal{O}_C) = h^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) + \sum_{x \in C_{sing}} \delta_p.$$

Let f_0 be the canonical section of the invertible sheaf $\mathcal{O}_S(C)$ and assume that $\mathcal{O}_S(C)$ has a section f_1 whose divisor is a smooth curve C_1, such that each point of $C \cap C_1$ is a smooth point of C, and that the intersection is transverse. In case $S = \mathbb{P}^2$, and C is a curve of degree e, then $\mathcal{O}_S(C) \cong \mathcal{O}_{\mathbb{P}^2}(e)$, the canonical section is just the section given by the defining equation f_0 of C, and a general homogeneous polynomial f_1 of degree e will have the desired properties.

C_1 is a smooth deformation of C, and so we have $g(C_1) = h^1(C, \mathcal{O}_C)$; $g(\tilde{C}) = h^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$. The classical formula obtained, relating the genus of $\tilde{C}$ and of C_1 in case C is irreducible, is then

$$g(\tilde{C}) - g(C_1) = \sum_{p \in C_{sing}} (1/2)(1 - \mu_p - r_p)$$

or in terms of the topological Euler characteristic ($= 2 - 2g(-)$) of C_1 and $\tilde{C}$

$$\chi^{top}(C_1) - \chi^{top}(\tilde{C}) = \sum_{p \in C_{sing}} 1 - \mu_p - r_p. \tag{8.1}$$

which holds even if C is not irreducible.

We can also compare with $\chi^{top}(C)$. Since for a curve we have

$$C \setminus C_{sing} \cong \tilde{C} \setminus \pi^{-1}(C_{sing}),$$

we deduce

$$\chi^{top}(\tilde{C}) - \sum_{p \in C_{sing}} r_p = \chi^{top}(C) - \sum_{p \in C_{sing}} 1.$$

Putting this into the genus formula above, we see that this formula is equivalent to

$$\chi^{top}(C_1) - \chi^{top}(C) = \sum_{p \in C_{sing}} (-\mu_p) = -\sum_{p \in C_{sing}} dim J(f_p, p), \tag{8.2}$$

where we use some local defining equation $f_p \in \mathcal{O}_{S,p}$ for C to define the Jacobian ring.

Using our main result we can deduce a refinement of formulas 8.1, 8.2 with quadratic forms.

Corollary 8.3. *Let C be a reduced curve on a smooth projective surface S over a field k of characteristic zero. Suppose that $\mathcal{O}_S(C)$ admits a section s with smooth divisor C_1 that intersects C transversely. Suppose in addition that each singular point p of C is a quasi-homogeneous singularity; let a_0^p, a_1^p denote the homogeneous weights (with a_0^p, a_1^p relatively prime), let e_p denote the homogeneous degree at p. Let $\pi : \tilde{C} \to C$ be the normalisation of C. Then*

$$\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(C/k) = \sum_{p \in C_{sing}} \mathrm{Tr}_{k(p)/k}(\langle a_0^p a_1^p e_p\rangle - \langle 1\rangle - \langle e_p\rangle[\mu^q_{f_p,p}]),$$

refining (8.2) *by taking the rank; and*

$$\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(\tilde{C}/k) = \sum_{p \in C_{sing}} \mathrm{Tr}_{k(p)/k}\left(\langle a_0^p a_1^p e_p\rangle - \langle e_p\rangle[\mu^q_{f_p,p}] - (\sum_{q \in \pi^{-1}(p)} \mathrm{Tr}_{k(q)/k(p)}\langle 1\rangle)\right),$$

refining (8.1) *by taking the rank.*

Proof. Let f_0 be the canonical section of $\mathcal{O}_S(C)$ and s as in the statement. Let $B := \mathrm{Spec}\, k[t]_{(t)}$, let $H = ts + (1-t)f_0$, form the surface $X := V(H) \subset S \times B$, and let $f : X \to B$ be the projection. $H_t = s - f_0$, the assumption on $C \cap C_1$ implies that X is smooth over k with generic fibre X_η a smooth curve over $\eta = \mathrm{Spec}\, k(t)$, and with special fibre C. Since each singular point p looks like a weighted homogeneous singularity of degree e_p with weights a_0^p, a_1^p, the formula of Corollary 7.3 for $f : X \to B$ becomes

$$\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(C/k) = \sum_{p \in C_{sing}} \mathrm{Tr}_{k(p)/k}(\langle a_0^p a_1^p e_p\rangle - \langle 1\rangle - \langle e_p\rangle e_p(\Omega_{X/k}, dt)).$$

Note that $e_p(\Omega_{X/k}, dt) = e_p(\Omega_{S/k}, df_p)$, where $f_p \in \mathcal{O}_{S,p}$ is any local expression for f_0 (this is independent of choice of local expression, since $\Omega_{S,p}$ has rank 2). Using Theorem 8.2 this is the first formula

$$\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(C/k) = \sum_{p \in C_{sing}} \mathrm{Tr}_{k(p)/k}(\langle a_0^p a_1^p e_p\rangle - \langle 1\rangle - \langle e_p\rangle[\mu^q_{f_p,p}]).$$

For the second formula , we just have to recall that since the normalisation of a curve, $\tilde{C} \to C$, satisfies $\tilde{C} \setminus \pi^{-1}(C_{sing}) \simeq C \setminus C_{sing}$, and using cut and paste property, we have

$$\chi_c(\tilde{C}/k) - \chi_c(C/k) = \chi_c(\pi^{-1}(C_{sing})/k) - \chi_c(C_{sing}/k) = \sum_{p \in C_{sing}} \mathrm{Tr}_{k(p)/k}(\sum_{q \in \pi^{-1}(p)} \mathrm{Tr}_{k(q)/k(p)}\langle 1\rangle - \langle 1\rangle);$$

this gives the last formula for the difference

$$\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(\tilde{C}/k) = (\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(C/k)) - (\chi_c(\tilde{C}/k) - \chi_c(C/k)),$$

$$\mathrm{sp}_t(\chi_c(C_\eta/\eta)) - \chi_c(\tilde{C}/k) = \sum_{p\in C_{sing}} \mathrm{Tr}_{k(p)/k}\left(\langle a_0^p a_1^p e_p\rangle - \langle e_p\rangle[\mu^q_{f_p,p}] - (\sum_{q\in\pi^{-1}(p)} \mathrm{Tr}_{k(q)/k(p)}\langle 1\rangle)\right).$$

To see that those formulas refine the classical formulas over $\mathbb{C}$ by taking ranks, use remark 2.10, note that C_η is a smooth deformation of C_1, so C_η and C_1 have the same topological Euler characteristic after choosing an embedding of $k(p)$ into $\mathbb{C}$, and that $\mathrm{rk}\, q_{f_p,p} = dim J(f,p) = \mu_{F_p,p}$. □

We conclude with the following identity in the Witt ring $W(k)$.

Corollary 8.4. *Let C be a reduced curve on a smooth projective surface S over a field k of characteristic zero. Suppose that $\mathcal{O}_S(C)$ admits a section s with smooth divisor C_1 that intersects C transversely. Suppose in addition that each singular point p of C is a quasi-homogeneous singularity; let a_0^p, a_1^p denote the homogeneous weights (with a_0^p, a_1^p relatively prime), let e_p denote the homogeneous degree at p. Let $\pi : \tilde{C} \to C$ be the normalisation of C. Then*

$$\sum_{p\in C_{sing}} \mathrm{Tr}_{k(p)/k}\left(\langle a_0^p a_1^p e_p\rangle - \langle e_p\rangle[\mu^q_{f_p,p}] + \sum_{q\in\pi^{-1}(p)} \mathrm{Tr}_{k(q)/k(p)}\langle 1\rangle\right) = 0$$

in $W(k)$.

Proof. For Y smooth and projective of odd dimension over k, $\chi_c(Y/k) = 0$ in $W(k)$ (see [Le20, Example 1.7, 2.]).

□

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Universitat Duisburg-Essen, Fakultät Mathematik
Campus Essen, 45117 Essen, Germany
ran.azouri@uni-due.de