Essays in Applied Microeconomics

Von der Mercator School of Management, Fakultät für Betriebswirtschaftslehre, der

Universität Duisburg-Essen

zur Erlangung des akademischen Grades

eines Doktors der Wirtschaftswissenschaft (Dr. rer. oec.)

genehmigte Dissertation

von

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Tag der Mündlichen Prüfung: 25.03.2021

Acknowledgements

First, I want to thank my advisor Eugen Kovac, to whom I am very grateful for his support and encouragement. In numerous discussions, he helped me develop and shape my ideas, which ultimately emerged in this thesis. He encouraged me from the beginning to present my research at conferences to get valuable feedback and training and get in touch with the scientific community. He patiently listened to my sometimes admittedly crazy ideas on revolutionizing economics (for instance, my idea to ban the concept of utility from economics and consider gravity between products and individuals) to carefully convince me to rethink my ideas. Luckily, from an ex-post point of view, he was successful. Furthermore, he greatly enhanced my writing, presentation, and reasoning skills by steadily giving me feedback concerning my presentations and research drafts. I also want to thank Tobias Seidel for agreeing to be my second supervisor and for his numerous comments during my brown-bag presentations, which reminded me not to get lost in the math (which happens very quickly if you are a theorist) and to always think about the economic intuition behind my ideas.

Second, I want to especially thank two colleagues, namely Dongyu Guo and Kangkan Choudhury. Dongyu Guo helped me, especially at the beginning of my Ph.D., with numerous discussions and suggestions, and I enjoyed the time that we spent together in Duisburg. Kangkan Choudhury, my co-author for the third chapter of this thesis and with whom I shared an office in Duisburg, helped me a lot through long discussions and hours of scribbling derivations on the white-board in our office. I enjoyed working together with him, and I appreciate the effort that he put into our paper.

Third, I also want to thank my other colleagues from the Mercator School of Management, who are too numerous to be listed up here (besides, since the risk of forgetting somebody is high, I choose the safe option to simply omit everybody), and all participants who attended my presentations during brown bags, seminars and conferences, for their helpful comments and suggestions.

Fourth, I want to thank two teachers/professors of mine. I want to thank Helmut Cremer, who was my teacher in social sciences and is the reason why I study economics. I will never forget his encouragement, which unfortunately was not always appreciated by me back then (I do it now, Herr Cremer). I also want to thank Prof. Dr. Lorens Imhof, whose courses I attended during my studies at the University of Bonn, and who greatly enhanced my quantitative skills through his way of teaching.

Most importantly, I want to thank my mother Sima, my father Nassi, and my brother Shahab for their constant support and love, which encouraged me and without whom this thesis would not have been possible.

I dedicate this thesis to my beloved friend Arman Galali and my beloved aunt Massoumeh Joshaghani, who unfortunately passed away. I will never forget the joy you brought me during your lifetimes, and you will always have a special place in my heart. This thesis is for you.

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Introduction

This dissertation contains three chapters that are concerned with the fields of behavioral economics and industrial organization.

The first chapter of this thesis represents my job market paper. The chapter starts by observing that the choice sets with which individuals are confronted in various situations have grown significantly during the last decades. Large choice sets might benefit individuals due to the increased probability of picking options that better suit their preferences. However, empirical literature in the field of behavioral economics has identified circumstances under which individuals dislike making a choice or refuse to choose at all when confronted with large choice sets. These two somehow paradoxical findings gave rise to the terms "Choice Overload" and "Status Quo Bias", respectively. A possible explanation for those findings can be found in the well-established field of Regret Theory.

Regret Theory assumes that individuals do not solely care about the final outcomes of their decisions, but they suffer from regret if they find out or believe that they could have made a better decision in the past. The connection between Regret Theory and the phenomena of Choice Overload and Status Quo Bias becomes apparent when considering that the benefit that a large choice set exerts on an individual can be harmful to an individual if he suffers from regret. The key lies within the nature of the decision-making process itself. To utilize the benefit of a large choice set, an individual must first gain information about the choice set's options. However, there might be boundaries that limit the extent to which an individual can gain information. These might be reflected in the cognitive costs of the information gaining process or opportunity costs of time. Given these boundaries, there is a limited benefit of a large choice set for an individual. The individual only gains information about a small subset of the whole choice set. A problem arises now if the individual suffers from regret. The large subset of the choice set that the individual does not analyze might contain several elements, yielding potential improvement possibilities. This can harm an agent who suffers from regret in two ways and lead to Choice Overload and Status Quo Bias. First, he might simply be unsatisfied with his choice because he suffers from the belief that the remaining big subset contains improvement possibilities. Second, this belief can tempt the agent to engage in exhausting search, ultimately leading to dissatisfaction. If the agent is aware that the second effect will strongly harm him, he might simply refuse to choose at all. However, there might be a desire for the agent to limit the choice set's size in all cases.

In the first chapter, I formalize the previously mentioned mechanisms by studying a sequential search model, where the agent suffers from regret. I identify conditions, such that a Choice Overload effect and a Status Quo Bias occur within my framework. This chapter's main contribution is given by an extension of regret preferences from a static to a dynamic framework, which involves the non-trivial incorporation of search costs into the agent's regret preferences. The agent's resulting equilibrium behavior yields predictions that significantly depart from an agent's behavior with Expected Utility preferences.

In the second chapter, I am concerned with a prediction that emerges from the traditional literature on vertical differentiation. Classical models of vertical differentiation predict that on oligopolistic markets, where firms not only engage in price or quantity competition but also quality competition, firms try to differentiate their products' quality in the maximum possible way from each other. This paradigm of maximum differentiation relaxes price competition, which in turn increases profits. While some markets might indeed confirm this prediction (for instance, the market of grocery retailers in Germany), it does not apply to several industries, where all firms are steadily increasing the quality level of their products to become the innovation leader (for instance, the market of semiconductors).

I develop a model that can predict both market situations. I extend a classical vertical differentiation model by assuming that competing firms are uncertain about their rival's fixed costs of quality improvements. This single model yields two equilibria, which predict the previously mentioned market situations. In one equilibrium, firms choose similar quality levels whenever they are similarly efficient. Consequently, if both firms are highly efficient, a market situation arises where both choose high-quality levels for their products and compete to become the innovation leader in the industry. The second equilibrium is reminiscent of the maximum differentiation paradigm in the classical literature of vertical differentiation. In this equilibrium, firms maximally differentiate their products. As a novel, I derive a sufficient and necessary condition for this equilibrium to exists, which depends on the primitives of the classical models of vertical differentiation and the primitives shaping the cost uncertainty in my model. This chapter's final contribution shows that within the concepts of stochastic orders, the monotone likelihood ratio order is a very useful tool to study comparative statics in models of vertical differentiation when considering fairly general assumptions concerning the primitives. Among many results, I showed that an increase in the probability that consumers have a high willingness to pay for quality could increase the consumer rent for all types of consumers in the population.

The third and final chapter of this thesis is joint work with Kangkan Choudhury. This chapter is connected to the second chapter in that we embedded my second chapter's model in a dynamic framework. We are concerned with how firms choose to innovate over time when they have uncertainty concerning their rival's fixed costs of quality improvements in general and about their own fixed costs of quality improvement in the future. We find that a Markov Perfect equilibrium exists where, surprisingly, firms will never choose intermediate quality levels but will either stay at the lowest possible quality level or choose the highest possible quality level.

Chapter 1

Why Less can be More: Choice Overload in a Sequential Searching Model

Abstract

We use an extension of Regret Theory, particularly Forward-Looking Regret, to study under which conditions Choice Overload occurs in a sequential discrete-time searching model with perfect recall. We find that whenever our agent suffers disproportionately from regret, more choice elements drive him into an excessive, costly search to minimize regret, which leads to ex-post dissatisfaction and a desire to limit the number of available options. Furthermore, we identify conditions under which the agent refuses to choose at all and sticks to his outside option, giving rise to a Status Quo Bias, even though the choice set solely contains desirable options that are superior to the outside option. We find two behavioral deviations from an agent with Expected Utility preferences concerning our agent's optimal stopping behavior. First, we find that the optimal stopping strategy consists of non-stationary behavior since the reward the agent demands to stop the decision problem is strictly increasing in the amount of search he conducted. Second, we find that our agent continues to search in states of the world where an agent with Expected Utility preferences would stop.

1.1 Introduction

We face choice problems in our everyday lives. Many of the choices are sequential, for instance, when searching for a job or a product. Consider an agent who wants to buy a new smartphone. So he visits an electronic store and starts trying out different models. When he finds a model he likes, the agent has in mind that there are many models that he has not tried out yet. He might be afraid of missing out on a model, which suits him more, even if it requires further search. Therefore, he keeps on searching until he finally buys a smartphone. Even if he found a very nice smartphone from an ex-ante point of view, ex-post, he still might be unhappy for two reasons. First, he conducted an exhausting search until he made the final purchase decision, which lowers his final evaluation of the smartphone. Second, he might regret that he did not continue the search since he still could have found a smartphone, which he would have preferred even when considering the additional amount of search costs. This way of thinking and the resulting behavior might lead to dissatisfaction with the choice (Choice Overload). In an extreme case, the agent might avoid searching for a new smartphone and continue to use his old smartphone (Status Quo Bias).

Consider that people have to make several important decisions during their lifetime. While those decisions might include less important ones on a daily frequency, such as consumption, they also include serious decisions that have a significant long-term impact on an individual's life journey, as an occupation or field of study. Due to globalization, trade liberalization, and the rapid development of the internet in recent decades, individuals' choice possibilities have grown significantly. An individual who desired to find a job 40 years ago was usually restricted to vacancies in magazines and newspapers and information he could get verbally. Concerning consumption, one was usually restricted to the local stores offer. Today, the same individual can visit several websites with thousands of vacancies or websites containing a much higher product variety. Due to an ongoing digitalization, individuals in the future might even be confronted with much larger choice sets in many different situations, and hence Choice Overload might become an even more severe problem over time.

In this work, we study that Choice Overload and Status Quo Bias occurs in a sequential discrete-time searching model with perfect recall in the following sense: when adding additional "desired" elements to the choice set, which in a sequential discrete-time screening model is equivalent to adding additional decision stages, the ex-ante expected utility of the decision problem can decrease.[1] This is counterintuitive at first glance since adding additional stages to a sequential screening model should weakly increase the ex-ante expected utility of the decision problem. Due to the decision problem's sequential nature, the agent is not obliged to search for these additional elements. He will do so only if the expected value of searching in any stage is positive. We will show that Choice Overload/Status Quo Bias cannot occur within our framework when considering an agent with Expected Utility preferences.

We consider an agent with regret preferences in the sense of Bell (1982) and Loomes and Sugden (1982): when making a decision, the agent does not solely gain utility from his ultimate choice, but besides, he will suffer from regret if his choice turns out to be inferior ex-post.

[1]By desired elements, we mean that the (expected) utility of each element accounting for search costs is positive.

However, due to our decision problem's sequential structure, we will extend the classic idea of regret from a static into a dynamic framework. In particular, our agent regrets stopping the decision process too early in the following sense: in any stage of the decision process, he compares his current best option with the expected value of the maximum of all remaining options, accounting for additional search costs. He engages in some counterfactual thinking: what could have happened if he would have continued searching? Whenever the probability of finding a better option is high, this will drive him to continue searching to minimize the regret of not having found the best option and not only to maximize the intrinsic value he derives from his final choice. This behavior might lead to dissatisfaction because search is costly, and the agent searched too long from an ex-post point of view. If he anticipates this behavior ex-ante, he might not even start the searching process and decide to choose his outside option, even though all other choice set elements are more desirable, which we will make precise in the Model section. An increase in the number of options increases the regret an agent feels in case of stopping in any stage of the decision problem, which in turn leads to the excessive search behavior just described. Additionally, we find that our agent's optimal stopping strategy is not stationary. The reward he demands for stopping the decision problem is strictly increasing the more options he has already searched. We offer a novel explanation for this kind of behavior.

The paper is organized as follows. First, we will discuss the relevant literature. In section 1.3, we will introduce our model and then describe the main mechanisms in the paper using a simplified model version. We then continue to analyze a benchmark framework in a more general setup, where we consider an Expected Utility maximizing agent who feels no regret. Next, we will allow for regret and show how the results differ from the previous framework. In section 1.4, we will discuss some extensions of our model and their implications on our results. Finally, in section 1.5, we will conclude.

1.2 Literature

Some of the first studies finding evidence for Choice Overload effects include Samuelson and Zeckhauser (1988) and Redelmeier and Shafir (1995). They find support for a Status Quo Bias in case of an increasing amount of options. Other studies find that sales increase when customers are confronted with a smaller assortment size (Iyengar and Lepper (2000), Boutwright and Nunes (2001)), individuals report less satisfaction (Iyengar (2000), Botti and Iyengar (2004)) and are less confident of having chosen the best option available leading to increased ex-post regret (Haynes (2009), Inbar et al. (2011)) when confronted with larger choice sets. In a meta-analysis by Chernev et al. (2015), the authors find that - among other things - more difficult tasks related to the choice procedure, more complex choice set elements, and uncertainty about own preferences foster the appearance of Choice Overload.

The theoretical literature on Choice Overload so far has focused on axiomatic and static approaches. Sarver (2008) shows that Choice Overload occurs in a two-stage model when considering an agent with preferences over menus. In the first stage, the agent commits to a menu, while he picks a single alternative of this menu in the second stage. Since the agent suffers from regret, if he finds out that there have been superior alternatives in his menu ex post his ultimate decision in stage 2, he will sometimes commit to smaller menus. However, one problem with his approach is that the agent does not regret the choice of his menu. Restricting the menu size always comes with an increased risk of ending up with all menu options being unsatisfying. In such a case, the agent might regret picking a relatively small menu size. Note that this should counteract the agent's desire to limit his choice set and hence mitigate Choice Overload's occurrence within his framework. Butarak and Evren (2017) propose a static model with an agent who suffers from "asymmetric regret". Their agent does not feel regret whenever he picks his outside option. They show that an increase in the number of options increases the agent's incentive to pick his outside option. Gerasimou (2017) shows that the agent in his model behaves as if he had an individual complexity function, where its value depends on the menu he chooses and a complexity threshold. As long as the complexity function is below the complexity threshold, he will maximize his utility by choosing an element of a menu. Otherwise, he will defer his choice. When assuming that the complexity threshold is a fixed number of options, and the complexity function is strictly increasing in the number of options, Choice Overload occurs in his model.

Our work differs from the previously mentioned papers for the following reasons: First, in contrast to Sarver (2008), our agent regrets the menu size. Whenever he decides to stop the decision process, he feels regret if his choice was inferior relative to the choice set's remaining options. Besides, we explicitly model the agent's choice within a menu by considering a sequential search model and derive results concerning his choice behavior. Second, in contrast to Butarak and Evren (2016), Choice Overload occurs in our model even in the absence of asymmetric regret. Even though a default bias cannot occur in our framework when the agent feels regrets when choosing his outside option, he still might be unsatisfied with his choice due to an excessive search caused by a large choice set. The difference to the framework of Gerasimou (2017) lies in the underlying assumption of the agent's behavior. We assume that our agent suffers from regret, while his agent has complexity constraints, which is more of a bounded rationality approach. A difference to all the previously mentioned papers is given by the dynamic framework in this work. In our framework, regret becomes more endogenous in the sense that the agent is able to minimize the ultimate regret he feels by continuing to search.

Let us come to the theoretical foundation of our model, which lies within regret theory, first introduced by Loomes and Sugden (1982), Bell (1982) and Fishburn (1982) in a static context.[2] When making a decision, a Regret Agent does not solely gain utility from his ultimate choice, but besides, he will suffer from regret if his choice turned out to be inferior ex-post. When considering regret in a dynamic context, particularly in a search model, some novel issues arise. The agent could suffer from two types of regret. First, he might suffer from Backward-Looking-Regret if he searched too long. Suppose that the agent got a decent option in the early stages and still decided to continue the searching process without finding a better option. If he would stop the decision process, he might regret the additional search costs he suffered without finding a better option. Therefore, he fails to ignore sunk costs. Second, he might suffer from Forward-Looking-Regret in case he stopped the decision process too early. Consider that the agent stops the decision process at some point and receives information ex-post, that there have been better options he did not search for, even when accounting for the additional search costs he would have

[2]For a survey of regret theory in static frameworks, see Bleichrodt and Wakker (2015).

had to reveal them. As in static Regret Theory, one implicitly assumes that ex-post his decision, the agent receives information about the values of all options he did not screen for and regrets if his choice turns out to be inferior. However, even if the agent does not receive information on whether his choice was inferior, one could assume that solely the existence of unscreened options, consequently derived expectations and the possibility of still improving, make the agent feel regret. Furthermore, note that as long as the agent knows the distribution of the options' quality levels, he is able to form correct expectations about the actual realizations.

From a conceptual perspective, we are closest to Strack and Viefers (2017), who analyze an optimal stopping problem considering an agent with dynamic regret preferences. They show that whenever the agent refuses an offer and decides to continue, he will not stop before finding an offer, which is at least as high as the offer he denied before. Our paper differs from theirs in three ways. First and most importantly, the underlying research questions differ. While we are interested under which circumstances Choice Overload results in a dynamic framework, they are interested in the agent's actual stopping behavior. Second, contrary to their framework, we consider search with perfect recall. Whenever our agent decides to continue searching, he can still choose any option he searched before, which increases the search's value in contrast to their paper. Third, our focus lies on Forward-Looking-Regret, while they focus on Backward-Looking-Regret. Therefore, while our results regarding the agent's stopping behavior are similar, the explanations differ, and the impact of additional options on the stopping behavior is different.

Next, we will discuss the relevant empirical literature on regret preferences in dynamic frameworks.[3] Strack and Viefers (2017) find empirical evidence for the existence of Backward-Looking-Regret. In their lab experiment, they find that whenever participants who face an optimal stopping problem without perfect recall refuse an option in the past, which leads to a particular outcome, will not stop the searching process until they found an option that leads to a similar outcome as the past peak. Fioretti et al. (2017), conducting a similar lab experiment, additionally provide evidence for the existence of Forward-Looking-Regret. In one of the setups, they inform participants beforehand that the outcome of unscreened options will be revealed whenever they decide to stop the searching process. This information leads to a prolonged search of the participants to avoid the possibility of feeling regret due to unscreened, superior options. Filiz-Ozbay and Ozbay (2007, 2010) find overbidding in first-price and third price auctions when manipulating feedback information. They conclude that this behavior might be driven from anticipated losers regret, which to some degree is equivalent to our definition of Forward-Looking-Regret.[4]

1.3 Model

In this section, we will present the model, present the model's main mechanisms within a simple example, solve benchmark cases, and finally solve for the main model. When not stated differently, the proofs of all results can be found in the appendix.

[3]For evidence on regret in general, see Bleichrodt and Wakker (2015).

[4]Anticipated-losers regret refers to the regret that an agent feels if he lost an auction while the winning bid was below his valuation, also known as the "Losers Curse".

An agent desires to pick a single option from a set consisting of $N \in \mathbb{N}_+$ options. Each option $t \in \{1, ..., N\}$ is attributed with a quality level X_t, which is unknown to the agent before undertaking some searching process. We assume that the quality levels X_t are independently and identically distributed continuous random variables according to some cumulative distribution function (henceforth denoted by CDF) $F(\cdot)$ with strictly positive and log-concave density $f(\cdot)$ and compact support given by $\chi = [0, 1]$.[5] The agent reveals the exact quality level x_t of each option by searching for it, each time suffering constant search costs $s > 0$. Search costs s can be interpreted as opportunity costs of time or as costs of investigation. Finally, we assume that there exists a deterministic outside option with quality level x_0. We interpret picking the outside option as not searching or refusing to make a choice. To make the decision problem less trivial, we will assume that $x_0 = 0$, such that the quality level of any option in the choice set is almost surely higher than the quality level of the outside option. In case that x_0 would be too high, the decision to not search might be simply driven by the fact that the outside option is too attractive.

The agent is confronted with the following discrete-time optimal stopping problem, where the total amount of decision stages corresponds to the number of options N:

At decision stage 0, the agent decides whether to pick the outside option or search for the first option at cost s. When deciding to pick the outside option, the decision problem ends, else he suffers search costs s and moves on to the next stage, where he investigates the first option and learns the quality level. At stage 1, the agent decides whether to pick the first option which he already searched for or to search for an additional option, again suffering costs s. The remaining parts of the decision problem proceed in the same matter. We will assume perfect recall such that whenever the agent stops searching at some stage, he can pick any of the previously screened options.[6] The decision problems ends either when the agent decides to stop further searching or when he searched for all available options.

We will now describe the preferences of the agent, which will be represented by a utility function which consists of two parts. First, there will be an intrinsic utility function. Second, there will be a regret term which captures the regret that the agent suffers from, when making ex-post wrong decisions. The expected utility of stopping the decision problem at a particular stage $t < N$ with information set $I_t = \{x_1, ..., x_t\}$ and choosing the option with the highest quality level in the screened bundle, $x_t^* := \max\{x_1, ..., x_t\}$, is given by:

$$\mathbb{E}_t[U_t(x_t^*)] = x_t^* - t \cdot s + \gamma \cdot \mathbb{E}_t[x_t^* - \max\{x_t^*, X_{t+1} - s, ..., X_N - (N - t) \cdot s\}]. \tag{1.1}$$

When screening all available options such that the agent reaches the last stage of the decision problem, his utility is given by:

$$U_N(x_N^*) = x_N^* - N \cdot s. \tag{1.2}$$

The first term in (1.1) and (1.2) represents the intrinsic utility that the agent derives whenever he stops the decision problem at stage t. It is given by the difference between the quality level

[5]Restricting the support on the unit interval is without loss of generality. The paper's results will hold for any compact support in $\mathbb{R}_+$.

[6]Implicitly, this will make the agent choose the option with the highest quality in his screened bundle. Furthermore, he will never pick the outside option once he started searching since the quality levels of all options in the choice set are by assumption almost surely higher than $x_0 = 0$.

of the option with the highest quality level among all the options that the agent searched so far and the total search costs that the agent suffered until t.[7] The last term in (1.1) represents the Forward-Looking-Regret component of the utility function. We model Forward-Looking-Regret by assuming that the preferences of our agent are represented by a reference dependent utility function (for instance, see Köszegi and Rabin (2006)), where the reference points captures Forward-Looking-Regret in the following way: whenever the agent stops the decision problem in stage t, he compares his current highest quality level x_t^* with the expected maximum quality level of all remaining options. However, he accounts for the additional search costs he would have had, in case he would have searched for them:

$$Z_{t,N} := \max\{X_{t+1} - s, ..., X_N - (N - t) \cdot s\}. \tag{1.3}$$

For all states of the world, where it holds that $\max\{x_{t+1} - s, ..., x_N - (N - t) \cdot s\} > x_t^*$, the agent will suffer from Forward-Looking-Regret, proportional to

$$x_t^* - \max\{x_{t+1} - s, ..., x_N - (N - t) \cdot s\}, \tag{1.4}$$

while for $\max\{x_{t+1} - s, ..., x_N - (N - t) \cdot s\} \leq x_t^*$ the agent will feel no regret.[8] The parameter $\gamma \geq 0$ measures how much the agent suffers from Forward-Looking-Regret, relative to the intrinsic utility of his current best option. If $\gamma = 0$, our agent behaves like an agent with Expected Utility preferences (hereafter referred to as EU Agent) who only cares about the intrinsic utility of outcomes. The higher γ, the more he suffers from Forward-Looking-Regret. Consider that when the agent searched for all options, he does not suffer from Forward-Looking-Regret, since there are simply no options left which could yield an improvement possibility.

Note that $Z_{t,N}$ is a continuous random variable with support $[-s, 1 - s]$ for all t and N, where the CDF is given by:[9]

$$P(Z_{t,N} < z) = P(\max\{X_{t+1} - s, ..., X_N - (N - t) \cdot s\} < z) := H_{t,N}(z). \tag{1.5}$$

[7] The qualitative results in this paper will hold when extending the intrinsic utility to $v(x_t^*) - t \cdot s$, where $v(\cdot)$ is a continuously differentiable and strictly increasing function, which maps from [0,1] to $\mathbb{R}_+$.

[8] Note that by including x_t^* into the maximum function in (1) we exclude the possibility that the agent feels joy for all states of the world where $x_t^* > \max\{x_{t+1} - s, ..., x_N - (N - t) \cdot s\}$. In section 1.3.5 we will discuss possible effects that will occur when including joy into our model.

[9] Both the highest possible and the low possible value of $Z_{t,N}$ are determined by the first element $X_{t+1} - s$. The lowest value that $Z_{t,N}$ can attain is given by the lowest value that $X_{t+1} - s$ can attain, which is given by $-s$. On the other hand, the highest value that $Z_{t,N}$ can attain is given by the highest value that $X_{t+1} - s$ can attain, which is given by $1 - s$.

$H_{t,N}(\cdot)$ is a piecewise defined (but continuous) CDF, where

$$H_{t,N}(z) = \begin{cases} 1 & , if \;\; z > 1-s \\ F(z+s) & , if \;\; 1-s > z > 1-2\cdot s \\ F(z+s)\cdot F(z+2s) & , if \;\; 1-2\cdot s > z > 1-3\cdot s \\ \vdots & \vdots \\ \prod_{i=1}^{N-t} F(z+i\cdot s) & , if \;\; 1-(N-t)\cdot s > z > -s \\ 0 & , if \;\; -s > z \end{cases}$$

Therefore, the probability of missing out better outcomes when stopping the decision process at a particular stage t with quality level x_t^*, is given by $1 - H_{t,N}(x_t^*)$.[10] The expected utility of stopping the decision problem in any stage $t < N$ with quality level x_t^* can now be written in a more compact way:

$$\mathbb{E}_t[U_t(x_t^*)] = x_t^* - t\cdot s + \int_{-s}^{1-s} \gamma\cdot(x_t^* - \max\{x_t^*, z\})\, dH_{t,N}(z). \tag{1.6}$$

When deriving the optimal stopping strategy of the agent, the expected utility of a single search in any stage $t < N$, given quality level x_t^* will be an important factor and is given by the following for all stages $t < N-1$:

$$\mathbb{E}_t[\Delta U_t(x_t^*)] = \int_0^1 \max\{x_t^*, x_{t+1}\} - s - x_t^* \, dF(x_{t+1}) \tag{1.7}$$

$$+ \int_0^1 \int_{-s}^{1-s} \gamma\cdot(\max\{x_t^*, x_{t+1}\} - \max\{x_t^*, x_{t+1}, z\})\, dH_{t+1,N}(z)\, dF(x_{t+1}) \tag{1.8}$$

$$- \int_{-s}^{1-s} \gamma\cdot(x_t^* - \max\{x_t^*, z\})\, dH_{t,N}(z). \tag{1.9}$$

The expected utility of a single search in stage $N-1$ is given by:

$$\mathbb{E}_{N-1}[\Delta U_{N-1}(x_{N-1}^*)] = \int_0^1 \max\{x_{N-1}^*, x_N\} - s - x_{N-1}^* \tag{1.10}$$

$$-\gamma\cdot(x_{N-1}^* - \max\{x_{N-1}^*, x_N - s\})\, dF(x_N). \tag{1.11}$$

The expected utility of a single search for the Regret Agent can be decomposed in the following way. First, he faces the same tradeoff of searching as the EU Agent concerning the intrinsic utility of outcomes. Both agent types benefit (suffer) from searching if the next option's quality turns out to be higher (lower) than the maximum quality of previously searched options, accounting for one time searching costs s. This tradeoff is captured by (1.7) and (1.10). Second, the Regret Agent wants to minimize the exposure to Forward-Looking-Regret, captured by (1.8)

[10] Due to Assumption 2, all events which are contained in the definition of $H_{t,N}(z)$ are possible.

and (1.9), and by (1.11). In particular, (1.8) and (1.9) represent the tradeoff that the agent faces with respect to the exposure to Forward-Looking-Regret, when deciding whether to stop the decision problem or to continue searching. In case of stopping in stage t, the agent can suffer from Forward-Looking-Regret due to $N-t$ options which he did not search for, whereas if he continues the search, he reveals X_{t+1} and hence there are only $N-t-1$ options left. We will show, that the second tradeoff induces an optimal stopping behavior for the Regret Agent, which is different from the EU Agent's behavior, which in turn can lead to Choice Overload and Status Quo Bias.

Another important expression is given by the continuation value of the decision problem at any stage t of the decision problem, which takes into account that for any stage $t < N-1$, the agent has the possibility to engage in more than a single search. In particular, the agents decision on whether to search or to stop will depend on the sign of this expression. We iteratively define the continuation value of the decision problem at any stage t, given the best option with quality level x_t^*, as follows:

$$C_t(x_t^*) = \begin{cases} \mathbb{E}_t[\Delta U_t(x_t^*) + \max\{0, C_{t+1}(x_t^*)\}] & , if \ \ t \in \{0, 1, ..., N-2\} \\ \mathbb{E}_{N-1}[\Delta U_{N-1}(x_{N-1}^*)] & , if \ \ t = N-1 \end{cases} \tag{1.12}$$

We will refer to $C_t(x_t^*)$ as the expected value of searching in stage t.

As standard in the search literature, we characterize the agents search behavior by a stopping strategy:

Definition 1 (*Stopping Strategy*)**.** **i)** A *stopping strategy* is a sequence of cut off values $\{\hat{x}_t\}_{t=1}^{N-1}$, such that the agent will stop searching in stage t if and only if $x_t^* > \hat{x}_t$.
ii) A *stopping strategy* is *stationary* if the cut off values are equal in all stages, i.e., if $\hat{x}_t = \hat{x} \, \forall t$.
iii) A *stopping strategy* $\{\hat{x}_t\}_{t=1}^{N-1}$ is *optimal* if and only if $C_t(\hat{x}_t) = 0$ for all $t \in \{1, ...N\}$.

Note that our definition of an optimal stopping strategy makes sure that the agent behaves dynamically consistent, therefore he has no incentive to deviate from his optimal stopping strategy in any stage of the game, for any possible history. Our agent will search, if the expected value of searching is strictly positive, else he will stop searching.

Given an optimal stopping strategy, we denote the ex-ante expected value of the decision problem with N options (beside the outside option) from stage 0 perspective by $\Gamma_N(\gamma, s)$. We define Choice Overload as the following:

Definition 2 (*Choice Overload*)**.** *Choice Overload* occurs if and only if there exists $\tilde{N} \in \mathbb{N}_+$ such that $\Gamma_{N+1}(\gamma, s) - \Gamma_N(\gamma, s) < 0$ for all $N > \tilde{N}$.

We define Choice Overload by a situation, where the ex-ante expected value of the decision problem is strictly decreasing in the amount of options N after some threshold is surpassed. Therefore, if Choice Overload occurs, the agent would like to limit the choice set's number of elements.

We define the Status Quo Bias as follows:

Definition 3 (*Status Quo Bias*)**.** *Status Quo Bias* occurs if and only if there exists $\tilde{N} \in \mathbb{N}_+$ such that $\Gamma_N(\gamma, s) < \mathbb{E}_0[U_0(x_0)] = 0$ for all $N > \tilde{N}$.

Whenever the ex-ante expected value of the decision problem is lower than the utility of choosing the outside option, the agent will not start the searching process and choose his outside option. Alternatively, one can interpret picking the outside option as refusing to make an active choice, which would involve searching. Note that $\mathbb{E}_0[U_0(x_0)]$, which represents the expected utility of picking the outside option, is equal to 0, since we assumed that $x_0 = 0$ and Assumption 3 implies that the agent does not suffer from Forward-Looking-Regret in case he picks his outside option. Furthermore, note that the Status Quo Bias can never occur when there is one option besides the outside option, due to Assumption 1 and since the agent does not suffer from Forward-Looking-Regret when he searched for the only option in the choice set.
Throughout the paper, we will consider the following three assumption:

Assumption 1. $\mathbb{E}(X_t) > s$ *for all* $t \in 1, ...N$.

This assumption guarantees the existence of an optimal stopping strategy by ensuring that searching is attractive for the agent in some states of the world. It implies that the agent will always search if his current best option is of the lowest possible quality level 0.

Assumption 2. $1 - (N - t + 1) \cdot s > 0$ *for all* $t \in \{1, ..., N\}$.

This assumption makes sure that independent of the size of the choice set, there exists a non-zero probability that searching for all available options is better than stopping the decision problem after the first search for very low realizations of X_1. Note that this assumption implies that when stopping the decision problem in any stage, the agent can potentially suffer from Forward-Looking-Regret with respect to all remaining options in the choice set. While this assumption obviously imposes a negative impact on the Regret Agent, it ensures at the same time that search costs are sufficiently small for large choice sets. As a consequence, additional search in order to minimize Forward-Looking-Regret is not too costly for the Regret Agent. Therefore, the assumption is not too restrictive. Ignoring this assumption would significantly complicate the analysis of the model, since one would have to consider many different special cases.

Assumption 3. *The agent does not suffer from Forward-Looking-Regret when choosing his outside option in decision stage 0.*

As in Butarak and Evren (2017) we assume that our agent does not suffer from regret, in our case Forward-Looking-Regret, when he decides to pick his outside option. This assumption can be justified by considering that in our model, starting the decision process by searching for the first option is an active choice, while picking the outside option can be interpreted as picking a default option, which is more of a passive choice. Note that all our results concerning the Choice Overload effect do not rely on this assumption, whereas it is necessary for the Status Quo Bias to arise. However, we think that the results concerning Status Quo Bias are still interesting, since Assumption 1 guarantees that the agent will always prefer to have one option in the choice set, instead of having only the outside option. In addition, every option in the choice set is strictly preferred to the outside option, even after accounting for the costs of a single search.

1.3.1 Illustrative example

Before analyzing the general model, we will describe the mechanisms which lead to the Choice Overload effect in a simplified model. We consider an outside option with a known quality level

$x_0 = 0$ and two options with unknown quality levels X_1 and X_2, which are independently and uniformly distributed on the unit interval. Search costs are given by s, where $0 < s < 0.5$. Note that this restriction on s will make sure that both Assumptions 1 and 2 are fulfilled.
First, assume that there exists one additional option with unknown quality level X_1, beside the outside option, therefore $N = 1$. The agent will search for option 1 if and only if:

$$\mathbb{E}_0[\Delta U_0(x_0)] > 0, \tag{1.13}$$

which is equivalent to

$$\int_0^1 x_1 - s - \gamma \cdot (0 - \max\{0, x_1 - s\})\, dx_1 > 0. \tag{1.14}$$

Due to Assumption 1, this inequality is always fulfilled for all non negative values of γ. As a consequence, both the Regret Agent and the EU Agent will always screen for the first option. For both agents, the ex-ante expected value of the decision problem with one option beside the outside option is given by:

$$\Gamma_1(\gamma, s) = \Gamma_1(0, s) = \mathbb{E}_0[X_1 - s] = 0.5 - s > 0. \tag{1.15}$$

Next, we add the second option with unknown quality level X_2 into the choice set. We will solve this decision problem by backward induction, which gives us the cut off value $\hat{x}_1$ for stage 1.
Consider stage 1, where the agent already searched and revealed the true quality level x_1 of the first option. He will search for the second (and last) option if and only if:

$$\mathbb{E}_1[\Delta U_1(x_1)] > 0, \tag{1.16}$$

which is equivalent to

$$\int_0^1 (\max\{x_1, x_2\} - x_1 - s)\, dx_2 - \int_0^1 \gamma \cdot (x_1 - \max\{x_1, x_2 - s\})\, dx_2 > 0. \tag{1.17}$$

Note that the agent will never search for the second option if $x_1 > 1 - s$, since there are no improvement possibilities left. Solving this inequality for x_1 yields the cut off value of stage 1 which is given by:

$$\hat{x}_1 := 1 - \frac{\gamma}{1+\gamma} \cdot s - \frac{\sqrt{s \cdot (2 \cdot (1+\gamma) - \gamma \cdot s)}}{1+\gamma} \in \left[1 - \sqrt{2 \cdot s}, 1 - s\right). \tag{1.18}$$

Therefore, at stage 1 the agent will search for the second option if and only if $x_1 > \hat{x}_1$. First, note that $\hat{x}_1$ is strictly increasing in γ. The more the agent suffers from forward looking regret, the stronger is the negative impact of improvements possibilities on his final utility. Therefore, with increasing γ, the quality level that the agent demands to stop the decision problem is also increasing. Second, note that whenever $\gamma = 0$, the agent feels no regret and hence behaves like an EU Agent, where the cut off value is given by $\hat{x}_1 = \hat{x}_{EU} := 1 - \sqrt{2 \cdot s}$. Third, if $\gamma \to \infty$, the

agent will search the second option for any value of x_1 given there are improvement possibilities left, such that $\lim_{\gamma\to\infty} \hat{x}_1 = 1-s$.
Given the optimal stopping strategy of the agent, we now derive the ex-ante expected value of the decision problem with two options beside the outside option from stage 0 perspective, which is given by:

$$\Gamma_2(\gamma, s) = P(X_1 > \hat{x}_1) \cdot \mathbb{E}_0[X_1 - s + \gamma \cdot (X_1 - \max\{X_1, X_2 - s\} | X_1 > \hat{x}_1] \tag{1.19}$$

$$+ P(X_1 < \hat{x}_1) \cdot \mathbb{E}_0[\max\{X_1, X_2\} - 2 \cdot s | X_1 < \hat{x}_1]. \tag{1.20}$$

In order to describe the mechanisms which lead to the choice overload effect, we decompose $\Gamma_2(\gamma, s)$ in the following way:

$$\begin{aligned}\Gamma_2(\gamma, s) =&\ \Gamma_2(0, s) + \int_{\hat{x}_{EU}}^{\hat{x}_1}\int_0^1 \max\{x_1, x_2\} - x_1 - s\, dx_2 dx_1 + \int_{\hat{x}_1}^{1-s}\int_{x_1+s}^{1} \gamma \cdot (x_1 + s - x_2)\, dx_2 dx_1 \\ =&\ \Gamma_1(\gamma, s) + \Gamma_2(0, s) - \Gamma_1(0, s) \\ &+ \int_{\hat{x}_{EU}}^{\hat{x}_1}\int_0^1 \max\{x_1, x_2\} - x_1 - s\, dx_2 dx_1 + \int_{\hat{x}_1}^{1-s}\int_{x_1+s}^{1} \gamma \cdot (x_1 + s - x_2)\, dx_2 dx_1.\end{aligned}$$

The first term on the right hand side describes the expected utility of the decision problem for a Regret Agent and an EU Agent with one option beside the outside option.

The difference between the second and the third term captures the positive impact that the second option has on both the Regret Agent and the EU Agent, i.e., the possibility of searching for one more option whenever the quality level of the first option was relatively low. It is given by:

$$\Gamma_2(0, s) - \Gamma_1(0, s) = \frac{1}{6} + \frac{(2s)^{\frac{3}{2}}}{3} - s > 0 \qquad \forall s \in (0, 0.5), \tag{1.21}$$

and is strictly positive, since the EU Agent only cares about the intrinsic value of the options.

The fourth and the fifth terms capture the negative impact that the second option exerts on the Regret Agent. First, note that the fourth term describes the intrinsic expected utility of searching the second option whenever $\hat{x}_{EU} < x_1 < \hat{x}_1$. For those values of x_1, the Regret Agent will search for the second option, while the EU Agent will not search. Since the EU Agent only cares about the intrinsic utility, this term will always be strictly negative.[11] In particular, the fourth term captures the indirect negative impact of Forward-Looking-Regret on the final utility of the agent, caused by a prolonged search compared to the EU Agent, i.e., searching whenever the intrinsic expected utility of searching is negative.

The fifth term captures the direct negative impact that Forward-Looking-Regret exerts on the agent's final utility. Whenever the Regret Agent stops the decision problem in the first stage, when there are improvements possibilities left, i.e., when $\hat{x}_1 < x_1 < 1-s$, he suffers from Forward-Looking-Regret if the unscreened option turns out to be superior, accounting for the

[11] To see this, note that $\hat{x}_{EU} = 1 - \sqrt{2 \cdot s}$ is the root of $\int_0^1 (\max\{x_1, x_2\} - x_1 - s)\, dx_2$ and note that this expression is strictly decreasing in x_1.

cost of one additional search. Note that this term is also strictly negative.
For Choice Overload to occur in this simple model, the negative impact of adding the second option must outweigh the positive impact. In particular it must hold that:

$$\begin{aligned}\Gamma_2(\gamma, s) - \Gamma_1(\gamma, s) = \Gamma_2(0, s) - \Gamma_1(0, s) \\ + \int_{\hat{x}_{EU}}^{\hat{x}_1} \int_0^1 (\max\{x_1, x_2\} - x_1 - s)\, dx_2 dx_1 \\ + \int_{\hat{x}_1}^{1-s} \int_{x_1+s}^{1} \gamma \cdot (x_1 + s - x_2)\, dx_2 dx_1 \\ = \frac{1}{6} - s + \frac{[s \cdot (2 \cdot (1+\gamma) - \gamma \cdot s)]^{\frac{3}{2}}}{3 \cdot (1+\gamma)^2} + \frac{\gamma}{1+\gamma} \cdot s^2 - \frac{\gamma \cdot (\gamma - 1)}{6 \cdot (1+\gamma)^2} \cdot s^3 < 0\end{aligned}$$

The inequality will be fulfilled for large values of s and γ, since in those cases the Regret Agent will search for the second option (due to a high γ) even when the expected value of searching for it is negative (due to high search costs s). In those cases, the Regret Agent would like to eliminate the second option and thus restrict the size of the choice set such that it only contains one additional option besides the outside option. Figure 1.1 shows the difference between the ex-ante expected value of the decision problem with two and a single option as a function of γ for fixed search costs s. As one can see, the difference is strictly larger than 0 for low values of s and γ and will become strictly negative when increasing both s and γ.

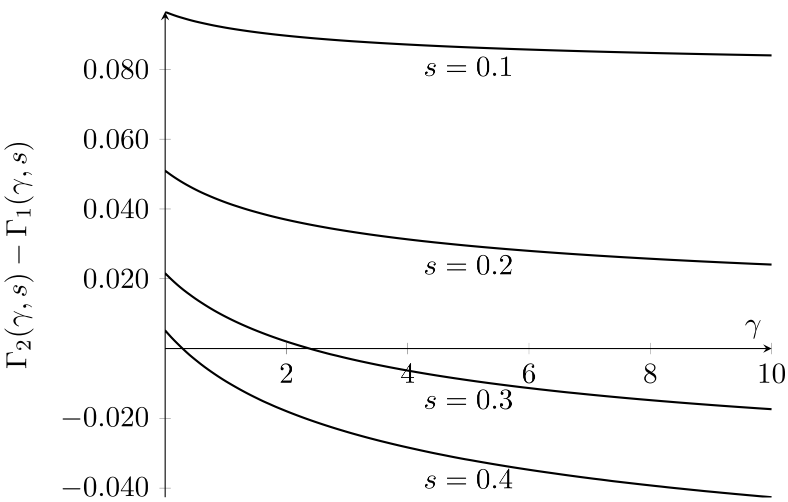

Figure 1.1: Difference between the ex-ante expected value of the decision problem with one and two options, for different values of search costs.

In the following sections, we will examine three different frameworks to study whether Choice Overload/Status Quo Bias can occur. First, we consider two limit cases. By setting $\gamma = 0$, we consider an EU Agent, who only cares about the intrinsic value of his choice. On the other hand, when $\gamma \to \infty$, we consider an Extreme Regret Agent (hereafter referred to as ER Agent) who only cares about minimizing Forward-Looking-Regret. Finally, in our main framework, when $0 < \gamma < \infty$, we consider a Regret Agent who anticipates that he will feel regret if his choice turns out to be inferior ex-post and who wants to maximize the intrinsic outcome of his choice and to minimize the exposure to Forward-Looking-Regret.

1.3.2 Benchmark 1: Expected Utility Agent

Let us consider the case where the agent only cares about the intrinsic value of the options. Therefore we set $\gamma = 0$. As we will see, the decision problem reduces to a simple stationary optimal stopping problem, which can be easily solved. We will use the outcomes of this framework for comparison to the main model in the next subsection. In the following, we will solve the decision problem with N options besides the outside option.
First, note that the expected utility of a single search in any stage t is time invariant and given by:

$$\mathbb{E}_t[\Delta U_t(x_t^*)] = \int_0^1 (\max\{x_t^*, x_{t+1}\} - s - x_t^*)\, dF(x_{t+1}). \tag{1.22}$$

Setting $\Delta \mathbb{E}_t(x_t^*) = 0$ and solving for x_t^* leads to the following result which characterizes the optimal stopping strategy of the EU Agent:

Lemma 1. *The unique optimal stopping strategy of the EU Agent exists, is stationary and is given by searching until he hits a cut off quality level $\hat{x}_{EU} \in [0,1]$, which is implicitly given by*

$$\int_{\hat{x}_{EU}}^1 (x - \hat{x}_{EU})\, dF(x) = s. \tag{1.23}$$

Hence, the optimal stopping strategy of the agent is given by searching until he found an option with a quality level higher than $\hat{x}$. The cut-off value is time invariant and describes a quality level where the expected gain of searching in terms of a higher quality level equals the search costs. The stationarity property of the optimal stopping strategy is due to the perfect recall assumption. Note that the cut off value is strictly decreasing in search costs and is increasing with more probability mass being shifted to the right tail of the distribution.

Given the optimal stopping strategy, the ex-ante expected value of the decision problem for the EU Agent with N options beside the outside option is given by:

$$\begin{aligned}\Gamma_N(0,s) =\ & \mathbb{P}(X_t < \hat{x}_{EU}\ \forall t < N)\cdot \mathbb{E}_{I_0}[U_N(X_N^*)|X_t < \hat{x}_{EU}\ \forall t < N] \\ & + \sum_{t=1}^{N-1} \mathbb{P}(X_j < \hat{x}_{EU}\ < X_t\ \forall j < t)\cdot \mathbb{E}_{I_0}[U_t(X_{t+1})|X_j < \hat{x}_{EU}\ < X_t\ \forall j < t] \\ & = \int_0^{\hat{x}} \cdots \int_0^{\hat{x}} \int_0^1 (\max\{x_1, ..., x_N\} - N\cdot s)\, dF(x_N)\cdots dF(x_1) \\ & + \sum_{t=1}^{N-1} F(\hat{x})^{t-1} \int_{\hat{x}}^1 (x - t\cdot s)\, dF(x).\end{aligned} \tag{1.24}$$

The first expression describes the conditional expected utility of searching all the options, while the second expression describes the sum of the conditional expected utilities of stopping at any stage before the last stage. The next result shows, that increasing the amount of options can never make the EU Agent worse off:

Proposition 1. *Given the optimal stopping strategy, the ex-ante expected value of the decision problem for the EU Agent with N options beside the outside option is given by*

$$\Gamma_N(0,s) = \hat{x}_{EU} - \int_0^{\hat{x}_{EU}} F(x)^N\, dx \tag{1.25}$$

The EU Agent does not suffer from Choice Overload nor from Status Quo Bias.

The intuition for this result was already given in the introduction: additional options affect an EU Agent only in a positive way, since he will use the additional search possibilities whenever the expected value of searching for them is positive. To see this, note that the ex-ante expected value of the decision problem with $N+1$ options can be written in the following way:

$$\Gamma_{N+1}(0,s) = \Gamma_N(0,s) + \underbrace{\int_0^{\hat{x}_{EU}} F(x)^N(1-F(x))\, dx}_{>0} \tag{1.26}$$

The last term, which is strictly positive, describes the benefit of being able to search one more time in contrary to the decision problem with N options. In particular, it measures how likely it is that the agent benefits from the last option in the decision problem with $N+1$ options, given he reaches the last stage.

Therefore, we find that the EU Agent's optimal stopping strategy is stationary and that he does not suffer from Choice Overload or Status Quo Bias.

1.3.3 Benchmark: Extreme Regret Agent

As a second benchmark, we consider the limit case where we first let $\gamma \to \infty$ and then solve the sequential search model. In this case, the ER Agent wants to eliminate the possibility of regret exposure by using the following optimal stopping strategy:

Lemma 2. *A unique optimal stopping strategy of the ER Agent exists, is stationary, and is given by searching until he hits a cut off quality level given by* $\hat{x}_{ER} = 1 - s$.

The intuition behind this result is given by the fact that the Extreme Regret Agent will have an expected utility of $-\infty$ whenever he stops the decision problem in any stage t when there are improvement possibilities left, i.e., whenever $P(Z_{t,N} > x_t^*) > 0$. Therefore, his optimal stopping strategy consists of stopping the decision problem in any stage before the last stage only if $x_t^* \geq 1 - s$, such that there indeed no improvement possibilities left, or by simply searching for all available options.

Given the optimal stopping behavior, we find the following result for the ER Agent concerning Choice Overload:

Proposition 2. *Given the optimal stopping strategy, the ex-ante expected value of the decision problem for the ER Agent with* N *options besides the outside option is given by*

$$\Gamma_N(\infty, s) = \frac{1 - F(1-s)^N}{1 - F(1-s)} \cdot \left(\int_{1-s}^{1} x \, dF(x) - s \right) + \int_{0}^{1-s} u \, dF(u)^N. \tag{1.27}$$

The ER Agent suffers from Choice Overload.

Due to the desire to eliminate the exposure to Forward-Looking-Regret, the ER Agent will engage in an excessive search, which ultimately might lead to an option with a high quality level. However, at the same time, he will also suffer from high searching cots. In particular, the ER Agent will search for an additional option, even if the expected intrinsic value of searching for it is strongly negative. Therefore, there is a limited benefit of adding an option to an ER Agent's choice set.

The following shows that under some conditions, the ER Agent will also suffer from Status Quo Bias:

Proposition 3. *If*

$$\int_{1-s}^{1} x \, dF(x) - s < 0, \tag{1.28}$$

the ER Agent suffers from the Status Quo Bias.

The condition in (1.28) is sufficient for the Status Quo Bias to be fulfilled whenever the amount of options is going to infinity. It compares the conditional expected utility of stopping the ER Agent's decision problem with the search costs. Whenever the search costs dominate the first term, the expected utility of the decision problem from stage zero perspective will become negative, and the ER Agent will pick his outside option instead of starting the searching process.

1.3.4 Regret Agent

In this section, we will consider our main framework, where the agent not only cares about the intrinsic value of the options but where he suffers from Forward-Looking-Regret. Therefore we assume that $0 < \gamma < \infty$. As we will see, this will cause behavioral deviations from the EU Agent framework, which ultimately can lead to Choice Overload and Status Quo Bias. Besides, there will be a behavioral deviation from the optimal stopping behavior of the ER Agent due to the non-stationarity of the searching model in this framework. We start with a result concerning the expected utility of a single search:

Lemma 3. *For any $t \in \{1, ..., N-2\}$ and any quality level x in $[0,1]$ it holds that:*

$$\mathbb{E}_{t+1}[\Delta U_{t+1}(x)] \geq \mathbb{E}_t[\Delta U_t(x)]. \tag{1.29}$$

With strict inequality if $x < 1 - (N-t-1) \cdot s$.

This result shows that the expected utility of a single search is weakly increasing with each stage. In particular, it shows a weakly higher incentive to search in later stages of the decision problem compared to early stages. The reason for this effect is given by the consideration of search costs in the regret term of the agents' utility function. On the contrary, for the EU Agent, a single search's expected utility is constant for all stages. Our next result describes the optimal stopping behavior of the Regret Agent:

Proposition 4. *A unique optimal stopping strategy of the Regret Agent exists. It is represented by a strictly increasing sequence $\{\hat{x}_t\}_{t=1}^{N-1}$, which is bounded below by $\hat{x}_{EU}$ and above by $1-s$.*

This result has two implications. First, whenever a Regret Agent decides to start the decision process, he will search at least as long as the EU Agent[12], simply because he regrets missing out on better options, which will drive him to search more often. Second, the quality level the Regret Agent demands to stop the decision process is increasing with higher stages. The reason for this is given by the fact that in the early stages of the decision problem, the risk of missing out unscreened options of higher quality when stopping in a particular stage t, is almost as high as the risk of missing them out in stage $t+1$. When getting closer to the end stage, the risk difference is more and more increasing, which explains why the agent demands a higher quality level to stop the decision problem in later stages of the game. This offers an alternative explanation of the time-varying cut off behavior in Strack and Viefers (2017). In their model, the agent tries to minimize backward looking regret, while here, the agent wants to minimize forward looking regret.

Since we know that the cut off values are strictly increasing and bounded, the expected utility of the decision problem from stage 0 perspective is given by:

$$\begin{aligned}\Gamma_N(\gamma, s) = \; & P(X_t < \hat{x}_t \,\forall t < N)\mathbb{E}_0[X_N^* - N \cdot s | X_t < \hat{x}_t \,\forall t < N] \\ & + \sum_{t=1}^{N-1} P(X_j < \hat{x}_j \forall j < t, X_t > \hat{x}_t)\mathbb{E}_0[X_t^* - t \cdot s + \\ & \gamma(X_t^* - \max\{X_t^*, X_{t+1} - s, ..., X_N - (N-t) \cdot s\}) | X_j < \hat{x}_j \forall j < t, X_t > \hat{x}_t],\end{aligned}$$

which can be written in a compact way:

Lemma 4. *The ex-ante expected value of the decision problem with N options for a Regret*

[12] For any possible realization of $\{X_1, ..., X_N\}$.

Agent is given by:

$$\Gamma_N(\gamma, s) = 1 - s - \int_{\hat{x}_1}^{1} F(x)\,dx - \int_{0}^{\hat{x}_1} F(x)^N\,dx - \int_{\hat{x}_1}^{1-s} \gamma \cdot (1 - H_{1,N}(x)) \cdot F(x)\,dx \tag{1.30}$$

Reconsider that $1 - H_{1,N}(x)$ represents the probability of improvement possibilities among $N - 1$ options which the agent did not search for, accounting for the additional search costs. This result shows, that we can rewrite $\Gamma_N(\gamma, s)$ in a simple way, where it does only depend on the cut-off value of the first stage, instead of all cut-off values of the decision problem. The reason for this is given by the fact that the implicit equation which defines $\hat{x}_1$ already contains all the information about the continuation value from stage 1 perspective and hence all the information about the remaining cut-off values. This representation will allow us more easily to see how a change in the amount of options will impact the ex-ante expected value of the decision problem. In addition, we can more easily make a comparison to EU Agent framework. To see this, we rewrite the ex-ante expected value of the decision problem from stage 0 perspective for the Regret Agent in the following way:

$$\Gamma_N(\gamma, s) = \Gamma_N(0, s) - \int_{\hat{x}_{EU}}^{\hat{x}_1} F(x) \cdot [1 - F(x)^{N-1}]\,dx - \gamma \cdot \int_{\hat{x}_1}^{1-s} F(x) \cdot [1 - H_{1,N}(x)]\,dx. \tag{1.31}$$

Therefore, the ex-ante expected value of the decision problem can be decomposed in the following way: the first term simply described the ex-ante expected value of the decision problem for the EU Agent. The second term captures the expected negative effects of over-searching, conducted by the Regret Agent. Since the Regret Agent searches in situations where the EU Agent would not search, in particular for $x \in (\hat{x}_{EU}, \hat{x}_1)$, he searches whenever the intrinsic expected utility of searching is strictly negative, which lowers his expected final outcome. The third term captures the direct negative effect of all remaining options in the choice set, when the agent stops the decision problem after searching for one option. The following describes the main result of our paper:

Proposition 5. *For sufficiently high values of the regret parameter γ, the Regret Agent suffers from Choice Overload.*

When adding more elements to the choice set, two countervailing effects occur in our framework, possibly leading to an inverted U shape of $\Gamma_N(\gamma, s)$, the ex-ante expected value of the decision problem, concerning N, the number of elements in the choice set.

First, additional elements in a dynamic search model increase the option value of searching at any stage of the game. Suppose that there is only one element in the choice set beside the outside option, such that the agent can only search once. In such a case, the risk of receiving an option of low quality is notably high. Having more elements in the choice set diminishes the risk that the decision problem ends with an option of relatively low quality. This positive effect of increasing the choice set is strong when the choice set is small and negligible when it is large.

On the other hand, when the agent suffers from Forward-Looking-Regret, adding more elements to the choice set has a direct and an indirect negative effect on the ex-ante expected value of the decision problem.

The direct effect is given by the increased probability of missing out on better options, in case of stopping the decision problem in any stage, which directly decreases the utility the agent derives from his decision.

The indirect effect is given by the agent trying to minimize Forward-Looking-Regret, which in turn leads to a prolonged searching process, which turns out to be suboptimal ex-post. Even if the agent finally finds an option of high quality, he might be unsatisfied due to the high amount of search he conducted. Whenever search costs s are high (low), the magnitude of these negative effects is strong (weak) when adding elements to a small choice set, increases with an increasing amount of elements, and finally diminishes.[13]

The negative effects overweight the positive effect, particularly when the agent cares a lot about minimizing regret and when search costs are high. For such agents, an increase of the choice set might have substantial negative effects on welfare. It is important to note that the possibility of suffering from Forward-Looking-Regret is not sufficient for the occurrence of Choice Overload. Since the maximum amount of regret an agent can feel is bounded above, the previously mentioned positive effect might predominate the negative ones, in particular when both γ and s are low. For very high values of γ, however, the behavior of the Regret Agent will be equivalent to the behavior of the Extreme Regret Agent and consequently he will suffer from Choice Overload, when confronted with very large choice sets.

Let us come to another interesting prediction of our model. Within our framework, there is also the possibility of a Status Quo Bias:

Proposition 6. *For sufficiently high values of the regret parameter γ and if*

$$\int_{1-s}^{1} x \, dF(x) - s < 0, \tag{1.32}$$

the Regret Agent suffers from the Status Quo Bias.

If the agent anticipates that he will conduct an exhausting search, which ultimately leads to dissatisfaction, he will avoid this by not even starting to search. This circumstance is paradoxical since, within our framework, all elements in the choice set are better on expectation than the outside option. If there would be only one option to search, besides the outside option, the agent would search for it due to Assumption 1. Therefore, the rejection of searching is purely implied by the choice set's size and not by the elements themselves. However, this result crucially depends on Assumption 3, which stated that the Regret Agent does not suffer Forward-Looking-Regret in case he picks his outside option.

1.3.5 Extensions

In this section, we will extend our models (simultaneously) in two ways: First, we will allow for the possibility that the agent suffers from Backward-Looking-Regret, i.e., suffering from search without getting any better options, accounting for additional search costs. Even in a search

[13] The effect diminishes since for fixed γ, the regret an agent can feel is bounded above when N is increasing.

model with perfect recall, the agent might suffer from Backward-Looking-Regret due to ex-post unnecessary search as the following example shows: when the agent considers stopping the decision process after searching for ten times, while he received the best option already after his first search, he might regret ex-post that he unnecessarily searched for additional nine times. The agent might engage in additional search to minimize this kind of regret, which would reinforce the excessive search behavior he already has due to Forward-Looking-Regret. However, the opposite is possible too: if the agent suffers disproportionally from Backward-Looking-Regret, relative to Forward-Looking-Regret, he might not search to avoid the possibility of additional exposure to Backward-Looking-Regret.

Second, note that the imposed structure on the regret term of the utility function implies that the agent only feels regret but no Joy. One could argue that the agent feels Joy if he finds an option of high quality in the early stages and the probability to find a higher option, after accounting for search costs, is relatively low. Therefore the agent feels Joy that he avoided additional searching. In what follows, we will allow for Joy in addition to regret. Now, the agent has a high probability of feeling Joy whenever the quality level of his current best option x_t^* is high. In such a case, Joy has an opposed effect on searching than Forward-Looking-Regret and mitigates excessive searching, which might nullify the choice overload effect. Also, the direct negative effect that each additional option imposed on the Regret Agent in the previous section might now become positive. On the other hand, Joy, concerning Backward-Looking-Regret could reinforce excessive search behavior, leading to the Choice Overload effect.

We will assume that the agent suffers from Loss Aversion, such that for both types of regret, he will suffer more from regret than he benefits from Joy.

In what follows, we will extend our main framework by allowing for Backward-Looking-Regret and the possibility to feel Joy. The expected utility of the agent when stopping in a particular stage t with highest quality level x_t^* is given by:

$$\begin{aligned}\mathbb{E}_t[U_t(x_t^*)] = x_t^* - t \cdot s + \mathbb{E}_t[\gamma \cdot (x_t^* - \max\{X_{t+1} - s, ..., X_N - (N-t) \cdot s)] \\ + \lambda \cdot (x_t^* - \max\{x_0 + t \cdot s, ..., X_{t-1} + s\}),\end{aligned}$$

where now the parameter λ represents the weight that the agent attributes to Backward-Looking-Regret. Due to the possibility of feeling Joy, both regret terms are piecewise defined, depending on whether the expression in the brackets is positive or negative:

$$\gamma(x) = \begin{cases} \gamma \cdot x & , x \geq 0 \\ \gamma \cdot \eta \cdot x & , x < 0 \end{cases} \qquad \lambda(x) = \begin{cases} \lambda \cdot x & , x \geq 0 \\ \lambda \cdot \eta \cdot x & , x < 0 \end{cases}$$

The parameter η, which is strictly larger than 1, measures the magnitude of Loss Aversion. We will restrict the analysis to compare a framework with one and two options, besides the outside option. We will assume that the quality levels are uniformly distributed on the unit interval. Furthermore, we assume that the outside option's quality level is $x_0 = 0$ and that Assumption 1 still holds. As we will see, the possibility of suffering from Backward-Looking-Regret will complicate the derivation of an optimal stopping strategy for decision problems with $N \geq 2$

options, since the cut-off values will become path dependent.[14]

Consider a framework with one option beside the outside option. From stage 0 perspective, the expected utility of picking the outside option is given by:

$$\mathbb{E}_0[U_0(x_0)] = \int_0^1 \gamma \cdot (s - x_1)\, dx_1, \tag{1.33}$$

while the expected utility of searching and picking the first option is given by:

$$\mathbb{E}_0[U_1(x_1)] = \int_0^1 (x_1 - s) + \lambda \cdot (x_1 - s)\, dx_1. \tag{1.34}$$

The Regret Agent will search for the first option if and only if:

$$\mathbb{E}_0[U_1(x_1) - U_0(x_0)] \geq 0 \Leftrightarrow (1 + \gamma \cdot \eta + \lambda) \cdot \left(\frac{1}{2} - s\right) + \frac{s^2 \cdot (\gamma - \lambda) \cdot (\eta - 1)}{2} \geq 0. \tag{1.35}$$

From this expression, one can see that an increase in γ will increase the agents incentive to engage in search, as in the case without Joy and Backward-Looking-Regret. However, an increase in λ will increase the incentive to search for low values of η and s, while it will decrease the incentives for high values of η and s. Whenever s is low, there is a high probability that the agent will feel Joy instead of Backward-Looking-Regret, whenever he is searching for the first option. If in addition η is low, the agent will suffer less from ex-post unnecessary search. In those cases, an increase in λ makes searching more attractive since the probability to feel Joy is high, while the harm from Backward-Looking-Regret is low. The opposite will hold for high values of s and η.

If

$$s \leq \underbrace{\frac{1 + \gamma \cdot \eta + \lambda - \sqrt{(1 + \gamma + \lambda \cdot \eta) \cdot (1 + \lambda + \gamma \cdot \eta)}}{(\gamma - \lambda) \cdot (\eta - 1)}}_{=\hat{s}}, \tag{1.36}$$

the Regret Agent will search for the first option such that the expected utility of the decision problem with one option beside the outside option is given by:

$$\Gamma_1(\gamma, \lambda, \eta, s) = \begin{cases} -\gamma \cdot \eta \cdot \left(\frac{1}{2} - s\right) - \frac{\gamma \cdot (\eta - 1) \cdot s^2}{2} & , s > \hat{s} \\ (1 + \lambda) \cdot \left(\frac{1}{2} - s\right) - \frac{\lambda \cdot (\eta - 1) \cdot s^2}{2} & , s \leq \hat{s} \end{cases}$$

Next, consider the framework with two options beside the outside option. Given x_1, the agent will search for the second option if and only if:

$$\mathbb{E}_1[U_2(x_2^*) - U_1(x_1)] \geq 0, \tag{1.37}$$

[14] In particular, the cut-off value of a stage will depend on $\max\{x_0 + t \cdot s, ..., x_{t-1} + s\}$.

which is equivalent to

$$\int_0^1 \Bigg(\max\{x_1, x_2\} - s - x_1 + \lambda \cdot (\max\{x_1, x_2\} - \max\{2 \cdot s, x_1 + s\}) \tag{1.38}$$

$$-\lambda \cdot (x_1 - s) - \gamma \cdot (x_1 + s - \max\{x_1, x_2\}) \Bigg) \, dx_2 > 0. \tag{1.39}$$

Now, the cut-off value $\hat{x}_1$ will depend on whether the agent suffers from Backward-Looking-Regret or Joy in the first stage, i.e., whether x_1 is larger or smaller than s. We denote the cut-off values for the former and latter case by $\hat{x}_1^I$ and $\hat{x}_1^{II}$, respectively. Existence and uniqueness in each case is guaranteed since:

$$\frac{\partial \mathbb{E}_1[U_2(x_2^*) - U_1(x_1)]}{\partial x_1} < 0$$

$$\mathbb{E}_1[U_2(x_2^*) - U_1(1)] = -\lambda - s \cdot (1 + \lambda \cdot (\eta - 1) + \gamma) < 0$$

$$\mathbb{E}_1[U_2(x_2^*) - U_1(0)] = \left(\frac{1}{2} - s\right) (\lambda \cdot (1 + 2 \cdot s \cdot (\eta - 1)) + 1 + \gamma \cdot \eta)) + \frac{\gamma \cdot s^2 \cdot (\eta - 1)}{2} > 0$$

where the third inequality holds due to Assumption 1.

Depending on the particular values of the exogenous parameters in the model, the cut-off value, which will be used to derive the ex-ante expected value of the decision problem, will differ. In particular, whenever it holds that $\min\{\hat{x}_1^I, \hat{x}_1^{II}\} \geq s$, the cut-off value is given by $\hat{x}_1^I$, while for $\max\{\hat{x}_1^I, \hat{x}_1^{II}\} < s$,the cut-off value is given by $\hat{x}_1^{II}$.[15] Therefore, the ex-ante expected value of the decision problem with two options besides the outside option can be written in the following way:

$$\Gamma_2(\gamma, \lambda, \eta, s) = \begin{cases} \Gamma_2^I(\gamma, \lambda, \eta, s) & , \min\{\hat{x}_1^I, \hat{x}_1^{II}\} \geq s \\ \Gamma_2^{II}(\gamma, \lambda, \eta, s) & , \max\{\hat{x}_1^I, \hat{x}_1^{II}\} < s \end{cases}$$

In this model, Choice Overload occurs whenever it holds that $\Gamma_2(\gamma, \lambda, \eta, s) < \Gamma_1(\gamma, \lambda, \eta, s)$. We find that Choice Overload can still occur in this model for a wide range of parameters. Generally, we find that an increase in η leads to the Choice Overload effect, given that λ is not substantially higher than γ. This occurs since an increase in η, while λ is sufficiently close to γ, increases the agents' incentive to search for the second option in the two option framework to an extent, where he searches whenever the intrinsic expected utility of searching is negative. This corresponds to the indirect negative impact of an additional option. Simultaneously, whenever the agent searches for the second option, there is still a probability that he will suffer from Backward-Looking-Regret if his decision to search turned out to be inferior ex-post. An increase in η increases the negative impact of Backward-Looking-Regret relative to Joy and hence the expected utility of the decision problem.

Figure 1.2 shows a comparison between the expected utility of the decision problem with

[15]Note that the other two cases are not possible, since they either violate uniqueness or existence of an optimal stopping strategy. For example, $\hat{x}_1^I > s$ and $\hat{x}_1^{II} < s$ would violate uniqueness, while $\hat{x}_1^I < s$ and $\hat{x}_1^{II} > s$ would violate existence. Therefore, for any values of the exogenous parameters, exactly one of the two cases which were mentioned in the text must hold. Hence, we can restrict our analysis to those two cases.

one and two options, respectively when holding γ, λ and s fixed, while varying η. One can see that an increase in η decreases the expected utility of the decision problem in both frameworks. However, the negative impact of an increase in η is stronger in the two option framework compared to the one option framework. This occurs because, in the two option framework, there are more possibilities to suffer from regret than in the one option framework, and a high value of η increases the harm that regret causes compared to the benefit through Joy.

Consider that the agent can get potentially harmed only from one type of regret in the one option framework, depending on whether he searches for the option or sticks to the outside option. On the contrary, in the two option framework, the agent can suffer from Forward-Looking-Regret concerning two unscreened options if he picks his outside option. He can suffer from Backward-Looking-Regret and Forward-Looking-Regret if he searches and picks the first option. Finally, he can only suffer from Backward Looking Regret with respect to the outside option and the first option if he searches for the second option. Therefore, whenever the agent suffers sufficiently from Loss Aversion, adding additional options simply increases the possibility to suffer from regret, which corresponds to the direct negative impact of adding options to the choice set.

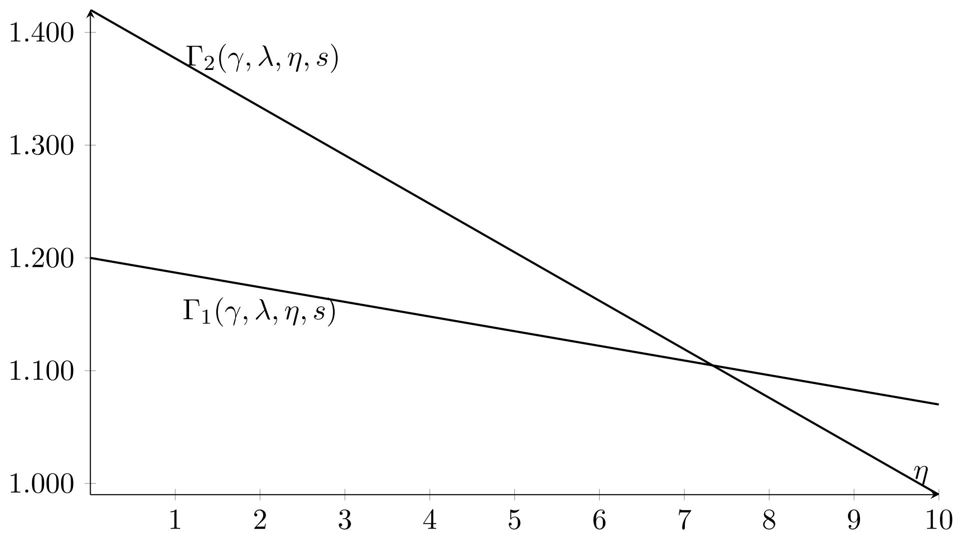

Figure 1.2: The expected value of the decision problem for one and two options for $(\gamma, \lambda, s) = (1, 2, 0.1)$ while varying η.

1.4 Conclusion

This paper's primary purpose was to show that Choice Overload can occur in a discrete time-sequential searching model when an agent suffers from Forward-Looking-Regret. We extended Regret Theory to a dynamic framework and showed that it implies behavioral predictions, which differ from the behavior of an Expected Utility Maximizer. First, we showed that an increasing amount of elements in the choice set could lead to dissatisfaction of the Regret Agent, caused by an indirect negative effect of excessive search and a direct negative effect of improvement possibilities due to regret exposure. This should increase the agent's desire to limit the choice set whenever it is possible. Second, a Status Quo Bias might occur when the Regret Agent does not suffer from Forward-Looking-Regret if he chooses his outside option. Even though the choice set consists of more desirable elements than the outside option, he refuses to make a choice. Third, we showed that the Regret Agent's optimal stopping strategy consists of searching until he hits a time-varying cut off value. Finally, in a simple model, where we allowed that the agent can feel Joy when his decisions turned out to be superior ex-post and that the agent can suffer from Backward-Looking-Regret, we showed that Choice Overload can still occur.

The existence of choice overloaded agents in a population introduces a new trade-off dimension for firms or regulators confronted with those types of agents. Typically, firms that sell a particular product do not sell single but multiple versions of that product. The reason for that is given by customers typically being heterogeneous concerning their preferences and firms trying to match those preferences by providing a large variety of product versions. When exposed to the risk of losing choice overloaded customers, the amount of product versions becomes an interesting strategic variable, besides prices and product differentiation. To the best of our knowledge, the only papers incorporating choice overloaded agents into a competition model are given by Kamenica (2008) and Gerasimou and Papi (2018).

1.5 Appendix: Proofs

Lemma 1. First, we will show that there exists a unique cutoff $\hat{x}_{EU} \in (0, 1)$ in stage $N-1$ of the decision problem such that the agent is indifferent between searching and stopping. Next, we will show that this cutoff is stage independent such that the optimal stopping strategy is stationary.

i) Existence and uniqueness

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_x^1 \Big(x_N - x\Big)\, dF(x_N) - s \tag{1.40}$$

where $g(x)$ is continuous in x. The function $g(x)$ describes the expected utility of searching in the last stage or the expected utility of a single search in any stage $t < N-1$, ignoring the possibility of additional search in the following stages, given the current best option with quality level x. Next, note that

$$g(1) = \int_1^1 \Big(x_N - 1\Big)\, dF(x_N) - s = -s < 0 \tag{1.41}$$

$$g(0) = \int_0^1 x_N\, dF(x_N) - s = \mathbb{E}[X_N] - s > 0 \tag{1.42}$$

where the inequality in the second row follows by A1.
Furthermore, it can be shown that x is strictly decreasing in x within $[0, 1)$ since

$$g'(x) = \frac{\partial}{\partial x} \int_x^1 \Big(x_N - x\Big)\, dF(x_N) = -[1 - F(x)] < 0 \tag{1.43}$$

where the second equality follows by Leibniz's integral rule. Since $g(x)$ is continuous, strictly decreasing and has a sign change within $[0,1]$ it follows that there exists a unique $\hat{x}_{EU}$ such that $g(\hat{x}_{EU}) = 0$, where $\hat{x}_{EU}$ denotes the quality level which makes the agent indifferent between searching and stopping in stage $N-1$ of the decision problem. It is implicitly given by

$$\int_{\hat{x}_{EU}}^{1} \left(x - \hat{x}_{EU}\right) dF(x) = s. \tag{1.44}$$

ii) Stationarity

Consider a candidate optimal strategy given by a sequence of cut of values $\{\hat{x}_t\}_{t=1}^{N-1}$, such that the EU Agent stops searching at stage t if $x_t^* > \hat{x}_t$. Consider any stage t and assume $x_t^* > \hat{x}_{t+1}$, therefore the current best option is higher than the cut off value of the next stage. Due to the perfect recall assumption, the agent will not screen again once he reached the next stage, independent of the realization of X_{t+1}. Hence, the option value of searching again in the next stages is zero. The expected value of searching in stage t, breaks down to the comparison of the current outcome and the outcome of searching one more time, which was described by the function $g(x)$. Since we already showed that the zero of this function is given by $\hat{x}_{EU}$, it follows that the EU Agent's optimal stopping strategy is stationary. □

Proposition 1. We want to show that $\Gamma_{N+1}(0,s) - \Gamma_N(0,s) > 0 \quad \forall\ N \geq 1$.
Using assumption 1, it follows that

$$\Gamma_1(0,s) - \Gamma_0(0,s) = \mathbb{E}(X) - s > 0. \tag{1.45}$$

For the cases where $N \geq 2$, we first rewrite the ex-ante expected value of the decision problem for an N in the following way:

$$\int_0^{\hat{x}_{EU}} \cdots \int_0^{\hat{x}_{EU}} \int_0^1 (max\{x_1, ..., x_N\} - N \cdot s)\, dF(x_N) \cdots dF(x_1) + \sum_{t=1}^{N-1} F(\hat{x}_{EU})^{t-1} \cdot \int_{\hat{x}_{EU}}^1 (x - t \cdot s)\, dF(x)$$

$$= \int_0^{\hat{x}_{EU}} \cdots \int_0^{\hat{x}_{EU}} (max\{x_1, ..., x_N\} - N \cdot s)\, dF(x_N) \cdots dF(x_1) + \sum_{t=1}^{N} F(\hat{x}_{EU})^{t-1} \cdot \int_{\hat{x}_{EU}}^1 (x - t \cdot s)\, dF(x)$$

$$= \int_0^{\hat{x}_{EU}} (u - N \cdot s)\, dF(u)^N + \frac{1 - F(\hat{x}_{EU})^N}{1 - F(\hat{x}_{EU})} \cdot \int_{\hat{x}_{EU}}^1 x\, dF(x) - s\frac{1 + N \cdot F(\hat{x}_{EU})^{N+1} - (N+1) \cdot F(\hat{x}_{EU})^N}{1 - F(\hat{x}_{EU})}$$

$$= \int_0^{\hat{x}_{EU}} u\, dF(u)^N + \frac{1 - F(\hat{x}_{EU})^N}{1 - F(\hat{x}_{EU})} \cdot \left(\int_{\hat{x}_{EU}}^1 x\, dF(x) - s \right)$$

Adding and subtracting $\hat{x}_{EU}$ in the second integral yields:

$$\int_0^{\hat{x}_{EU}} u\, dF(u)^N + \frac{1 - F(\hat{x}_{EU})^N}{1 - F(\hat{x}_{EU})} \cdot \Bigg(\underbrace{\int_{\hat{x}_{EU}}^1 (x - \hat{x}_{EU})\, dF(x) - s}_{=0} + (1 - F(\hat{x}_{EU})) \cdot \hat{x}_{EU} \Bigg), \tag{1.46}$$

where the term is 0 by the definition of $\hat{x}$. We finally have:

$$\Gamma_N(0,s) = \int_0^{\hat{x}_{EU}} u\, dF(u)^N + (1 - F(\hat{x}_{EU})^N) \cdot \hat{x}_{EU} = \hat{x}_{EU} - \int_0^{\hat{x}_{EU}} F(x)^N\, dx, \tag{1.47}$$

where the last equality follows by using integration by parts.
Next, consider that:

$$\Gamma_{N+1}(0,s) - \Gamma_N(0,s) = \int_0^{\hat{x}_{EU}} F(x)^N \cdot (1 - F(x))\, dx > 0. \tag{1.48}$$

Therefore, the expected utility of the decision problem is strictly increasing in N, implying that Choice Overload and Status Quo Bias do not occur in the EU Agent framework. □

Lemma 2. Note that for the analysis of the ER Agent, we first consider$\gamma \to \infty$ and then solve for his optimal stopping strategy. First, consider that any stopping strategy which tells the ER agent to stop lower than $1-s$ in a particular stage gives him a utility of $-\infty$. This occurs since whenever $1-s$ and the agent stops searching, the random variable $Z_{t,N}$, consisting of the unsearched bundle, accounting for additional search costs, can take values between x_t^* and $1-s$ with strictly positive probability. But this cannot be optimal, since the agent could deviate to a strategy which consists of screening all options in the choice set, leading to an expected utility of

$$\int_0^1 \cdots \int_0^1 \max\{x_1, ..., x_N\} - N \cdot s \, dF(x_1)...dF(x_N) > -\infty. \tag{1.49}$$

On the other hand, any strategy that tells the agent to continue screening in a particular stage whenever $x_t^* > 1-s$ cannot be optimal either. In such a case, the possibility of suffering from Forward-Looking Regret is eliminated because the agent cannot find an option that yields a higher utility when he accounts for the additional search costs. Therefore, any additional search would strictly lower his final utility, such that he could profitably deviate by stopping the searching process. □

Proposition 2. The Proof consists of three steps. First, we will simplify the ex-ante expected value of the decision problem for the ER Agent with N options. Next, we will show that there exists $\tilde{N} \in \mathbb{N}_+$ such that $\Gamma_{\tilde{N}+1}(\infty, s) - \Gamma_{\tilde{N}}(\infty, s) < 0$. Note that $\Gamma_N(\infty, s)$ means that we first consider $\gamma \to \infty$ and then consider the ex-ante expected utility of the devision problem with N options. Finally, we will show that whenever such $\tilde{N}$ exists, $\Gamma_{N+1}(\infty, s) - \Gamma_N(\infty, s) < 0$ will hold for all $N > \tilde{N}$.

The ex-ante expected value of the decision problem with N options beside the outside option is given by:

$$\begin{aligned}
\Gamma_N(\infty, s) = &\sum_{t=1}^{N-1} P(X_{N-1}^* < 1-s, X_t > 1-s) \cdot \mathbb{E}[X_t - t \cdot s | X_{t-1}^* < 1-s, X_t > 1-s] \\
&+ P(X_{N-1}^* < 1-s) \cdot \mathbb{E}[X_N^* - N \cdot s | X_{N-1}^* < 1-s] \\
= &\sum_{t=1}^{N} P(X_{N-1}^* < 1-s, X_t > 1-s) \cdot \mathbb{E}[X_t - t \cdot s | X_{t-1}^* < 1-s, X_t > 1-s] \\
&+ P(X_N^* < 1-s) \cdot \mathbb{E}[X_N^* - N \cdot s | X_N^* < 1-s] \\
= &\frac{1 - F(1-s)^N}{1 - F(1-s)} \left(\int_{1-s}^{1} x \, dF(x) - s \right) + \int_0^{1-s} u \, dF(u)^N
\end{aligned} \tag{1.50}$$

Using integration by parts on both integrals in (1.50) and simplifying finally yields:

$$\Gamma_N(\infty, s) = 1 - s - \frac{1 - F(1-s)^N}{1 - F(1-s)} \int_{1-s}^{1} F(x) \, dx - \int_0^{1-s} F(u)^N \, du$$

Next, consider the difference between the decision problem with $N+1$ and N options:

$$\Gamma_{N+1}(\infty, s) - \Gamma_N(\infty, s) = \int\limits_0^{1-s} F(u)^N (1 - F(u))\, du - F(1-s)^N \int\limits_{1-s}^{1} F(u)\, du$$

where this difference is smaller than 0 whenever it holds that:

$$\int\limits_0^{1-s} \left(\underbrace{\frac{F(u)}{F(1-s)}}_{<1\, a.e} \right)^N (1 - F(u))\, du < \int\limits_{1-s}^{1} F(u)\, du \tag{1.51}$$

Note that the term $\frac{F(u)}{F(1-s)}$ is almost everywhere strictly smaller than 0. Therefore, whenever $N \to \infty$ the left hand side will converge to 0 such that the inequality is fulfilled. Therefore, there exists $\tilde{N} \in \mathbb{N}_+$ such that $\Gamma_{\tilde{N}+1}(\infty, s) - \Gamma_{\tilde{N}}(\infty, s) < 0$.

For the last step of the proof, note that the left hand side of (1.51) is simply decreasing in N, such that whenever $\tilde{N}$ which fulfills (1.51) exists, it follows that:

$$\int\limits_0^{1-s} \left(\frac{F(u)}{F(1-s)} \right)^{\tilde{N}+1} (1 - F(u))\, du < \int\limits_0^{1-s} \left(\frac{F(u)}{F(1-s)} \right)^{\tilde{N}} (1 - F(u))\, du < \int\limits_{1-s}^{1} F(u)\, du \tag{1.52}$$

Therefore, it follows that whenever there exists $\tilde{N}$ such that $\Gamma_{\tilde{N}+1}(\infty, s) - \Gamma_{\tilde{N}}(\infty, s) < 0$ it will hold for any $N > \tilde{N}$. □

Proposition 3. Consider the ex-ante expected value of the decision problem and let $N \to \infty$ such that the ex-ante expected value of the decision problem will converge to:

$$\frac{\int\limits_{1-s}^{1} x\, dF(x) - s}{1 - F(1-s)} \tag{1.53}$$

Note that, in case that the ER Agent picks his outside option without suffering from regret, by construction he will have a utility of 0. Therefore, whenever it holds that

$$\int\limits_{1-s}^{1} x\, dF(x) < s, \tag{1.54}$$

the ER Agent will pick his outside option for $N \to \infty$. Next, note that by construction the ex-ante expected value of the decision problem is strictly positive if there is one option beside the outside option. Therefore, there indeed exists $\tilde{N}$ such that $\Gamma_N(\infty, s) < 0$ for all $N \geq \tilde{N}$ whenever the condition in (1.54) is fulfilled. □

Lemma 3. First, we will rewrite the expected utility of a single search, given quality level $x \in [0,1]$, for any stage $t < N-1$ in the following way:

$$\begin{aligned}
\mathbb{E}_t[\Delta U_t(x)] &= \int_0^1 (\max\{x, x_{t+1}\} - s - x)\, dF(x_{t+1}) - \int_{-s}^{1-s} \gamma \cdot (x - \max\{x, z\})\, dH_{t,N}(z) \\
&\quad + \int_0^1 \int_{-s}^{1-s} \gamma(\max\{x, x_{t+1}\} - \max\{x, x_{t+1}, z\})\, dH_{t+1,N}(z) dF(x_{t+1}) \\
&= (1+\gamma) \cdot \int_0^1 (\max\{x, x_{t+1}\} - s - x)\, dF(x_{t+1}) + \int_{-s}^{1-s} \gamma \cdot (\max\{x, z\})\, dH_{t,N}(z) \\
&\quad - \int_0^1 \int_{-s}^{1-s} \gamma(\max\{x - s, x_{t+1} - s, z - s\})\, dH_{t+1,N}(z) dF(x_{t+1}) \\
&= (1+\gamma) \cdot \int_0^1 \max\{x, x_{t+1}\} - s - x\, dF(x_{t+1}) \\
&\quad + \int_{-s}^{1-s} \gamma \cdot (\max\{x, z\} - \max\{x - s, z\})\, dH_{t,N}(z)
\end{aligned}$$

In the first step we simply added and subtracted s in the second line and rearranged terms. In the second step we simply used the fact that:

$$\int_0^1 \int_{-s}^{1-s} \gamma \cdot (\max\{x - s, x_{t+1} - s, z - s\})\, dH_{t+1,N}(z) dF(x_{t+1}) = \int_{-s}^{1-s} \gamma \cdot (\max\{x - s, z\})\, dH_{t,N}(z)$$

Further simplification finally yields:

$$\begin{aligned}
\mathbb{E}_t[\Delta U_t(x)] = (1+\gamma) \cdot &\int_0^1 (\max\{x, x_{t+1}\} - s - x)\, dF(x_{t+1}) \\
&+ \int_{x-s}^{x} \gamma \cdot (x - z)\, dH_{t,N}(z) + \gamma \cdot s \cdot H_{t,N}(x - s) \\
= (1+\gamma) &\int_0^1 (\max\{x, x_{t+1}\} - s - x)\, dF(x_{t+1}) + \gamma \cdot \int_{x-s}^{x} H_{t,N}(z)\, dz
\end{aligned}$$

Note that the first term does not depend on the particular stage, it cancels out when we consider the difference between the expected utility of a single search for two successive stages, for a fixed quality level x:

$$\mathbb{E}_{t+1}[\Delta U_{t+1}(x)] - \mathbb{E}_t[\Delta U_t(x)] = \gamma \int_{x-s}^{x} H_{t+1,N}(z) - H_{t,N}(z)\, dz$$

Now, whenever $x - s < 1 - (N-t) \cdot s$ it follows that $H_{t,N}(x - s) < H_{t+1,N}(x - s)$ such that $\mathbb{E}_{t+1}[\Delta U_{t+1}(x)] - \mathbb{E}_t[\Delta U_t(x)] > 0$. For the case where $x - s \geq 1 - (N-t) \cdot s$ it follows that

$H_{t,N}(x-s) = H_{t+1,N}(x-s)$ and hence $\mathbb{E}_{t+1}[\Delta U_{t+1}(x)] - \mathbb{E}_t[\Delta U_t(x)] = 0$, which completes the proof. $\square$

Proposition 4. We want to show that there exists a strictly increasing sequence of cut-off values $\{\hat{x}_t\}_{t=1}^{N-1}$, where each value fulfills $C_t(\hat{x}_t) = 0$. The proof consists of four steps. First, we will show that the expected utility of searching in any stage t of the decision problem, $C_t(x_t^*)$ is strictly decreasing in the current best option x_t^*. Second, we will show that a unique cut-off value $\hat{x}_t$, such that $C_t(\hat{x}_t) = 0$, exists in each stage. Third, we will show that the sequence of cut-off values $\{\hat{x}_t\}_{t=1}^{N-1}$ is strictly increasing. Finally, we show that the sequence of cut-off values is bounded from below by $\hat{x}_{EU}$ and from above by $1-s$.

i) Strictly decreasing expected utility of searching

First, consider the expected utility of a single search in any sage $t \in \{1, ..., N-1\}$, given the current best option x_t^*, which is given by:

$$\mathbb{E}_t[\Delta U_t(x_t^*)] = (1+\gamma) \int_0^1 \max\{x_t^*, x_{t+1}\} - s - x_t^* \, dF(x_{t+1}) + \gamma \int_{x_t^*-s}^{x_t^*} H_{t,N}(z) \, dz$$

Differentiating with respect to x_t^* yields:

$$\frac{\partial \mathbb{E}_t[\Delta U_t(x_t^*)]}{\partial x_t^*} = -(1+\gamma)F(x_t^*) + \gamma(H_{t,N}(x_t^*) - H_{t,N}(x_t^* - s)) \tag{1.55}$$

$$< -(1+\gamma)F(x_t^*) + \gamma(F(x_t^*) - H_{t,N}(x_t^* - s)) < 0 \tag{1.56}$$

The inequality follows since $H_{t,N}(x_t^*) < F(x_t^*)$ by the definition of $H_{t,N}$. Therefore, we find that the expected utility of a single search is strictly decreasing in the current best option x_t^*. Note, that this also means that the expected utility of a single search in stage t from stage $t' < t$ perspective is also strictly decreasing in $x_{t'}^*$, namely that $\frac{\partial \mathbb{E}_{t'}[\Delta U_t(X_t^*)]}{\partial x_{t'}^*} < 0$.

Next, consider the expected utility of searching in any stage $t \in \{1, ..., N-1\}$ which was iteratively defined by:

$$C_t(x_t^*) = \begin{cases} \mathbb{E}_t[\Delta U_t(x_t^*) + \max\{0, C_{t+1}(x_t^*)\}] & , t \in \{1, ..., N-2\} \\ \mathbb{E}_{N-1}[\Delta U_{N-1}(x_{N-1}^*)] & , t = N-1 \end{cases} \tag{1.57}$$

Since there is no option value of further search in stage $N-1$, the expected utility of searching in stage $N-1$ is simply given by the expected utility of a single search for which we have already shown that it is strictly decreasing in x_{N-1}^* . Now, consider any stage $t < N-1$ and note that we can rewrite $C_t(x_t^*)$ in the following way:

$$C_t(x_t^*) = \mathbb{E}_t[\Delta U_t(x_t^*) + \max\{0, \mathbb{E}_{t+1}[\Delta U_{t+1}(X_{t+1}^*) + \max\{0, C_{t+2}(x_{t+1}^*)\}]\}] \tag{1.58}$$

This shows, that the expected utility of searching in any stage t can be written as the weighted sum of the expected utilities of single searches in the current and forthcoming stages. Differentiation with respect to x_t^* yields:

$$\frac{\partial C_t(x_t^*)}{\partial x_t^*} = \underbrace{\frac{\partial \mathbb{E}_t[\Delta U_t(x_t^*)]}{\partial x_t^*}}_{<0} + \underbrace{\frac{\partial \mathbb{E}_t[\max\{0, \mathbb{E}_{t+1}[\Delta U_{t+1}(X_{t+1}^*) + \max\{0, C_{t+2}(X_{t+1}^*)\}]\}]}{\partial x_t^*}}_{\leq 0} < 0$$

The second expression is weakly negative since for the cases where we have $C_{t+1}(x_t^*) > 0$, the expected utility of a single search in stage $t+1$ from stage t perspective is strictly decreasing in x_t^*. For cases where $C_{t+1}(x_t^*) \leq 0$, the impact of an increase in x_t^* on $C_t(x_t^*)$ is given through its impact on $\mathbb{E}_t[\Delta U_t(x_t^*)]$, which is strictly negative. Therefore, we find that the expected utility of searching in any stage t of the decision problem is strictly decreasing in the current best option x_t^*.

ii) Existence and Uniqueness

Before, we showed that $C_t(x_t^*)$ is strictly decreasing in x_t^*. Furthermore, note that $C_t(x_t^*)$ is continuous in x_t^*. Therefore, in order to show uniqueness and existence of a cut off value $\hat{x}_t$ such that $C_t(\hat{x}_t) = 0$, it suffices to show that $C_t(0) > 0$ and that $C_t(1) < 0$.

Consider that

$$C_t(0) \geq \mathbb{E}_t[\Delta U_t(0)] = (1+\gamma) \underbrace{\int_0^1 x_{t+1} - s \, dF(x_{t+1})}_{>0} + \gamma \underbrace{\int_{-s}^0 H_{t,N}(z) \, dz}_{>0} > 0 \tag{1.59}$$

where the first expression is strictly larger than 0 due to Assumption 1 and the second expression is strictly larger than 0 since $H_{t,N}(z) \in (0,1)$ for $0 > z > -s$. Next, note that $\mathbb{E}_{t'}[\Delta U_t(1)] = -s$ for any $t' \leq t$, since there are simply no improvement possibilities left. Hence, the agent would only suffer from search costs in case of searching. Since $C_t(x_t^*)$ is the weighted sum of expected utilities of single searches, it follows that $C_t(1) < 0$. Therefore, since $C_t(x_t^*)$ is strictly decreasing and continuous in x_t^*, $C_t(0) > 0$ and $C_t(0) < 0$ it follows that for each t there exists a unique $\hat{x}_t$ such that $C_t(\hat{x}_t) = 0$.

iii) Strictly increasing sequence of cut-off values

Consider stage $N-1$ of the decision problem. Given x_{N-1}^*, the agent will search if and only if:

$$C_{N-1}(x_{N-1}^*) = (1+\gamma) \int_0^1 \max\{x_{N-1}^*, x_N\} - s - x_{N-1}^* \, dF(x_N) + \gamma \int_{x_{N-1}^* - s}^{x_{N-1}^*} H_{N-1,N}(z) \, dz \geq 0$$

which leads to a cut-off value $\hat{x}_{N-1}$ implicitly defined by

$$(1+\gamma) \int_0^1 \max\{\hat{x}_{N-1}, x_N\} - s - \hat{x}_{N-1} \, dF(x_N) + \gamma \int_{\hat{x}_{N-1} - s}^{\hat{x}_{N-1}} H_{N-1,N}(z) \, dz = 0 \tag{1.60}$$

Note that it must hold that $\hat{x}_{N-1} < 1-s$, since any quality level over $1-s$ leaves no room for improvement possibilities. Therefore, the cut-off value in stage $N-1$ must be strictly lower than $1-s$.

Next, consider stage $N-2$ of the decision problem. Given x^*_{N-2}, the agent will search if and only if:

$$C_{N-2}(x^*_{N-2}) = \mathbb{E}_{N-2}[\Delta U_{N-2}(x^*_{N-2}) + \max\{0, \mathbb{E}_{N-1}[\Delta U_{N-1}(X^*_{N-1})]\}] \geq 0 \tag{1.61}$$

First, assume that $x^*_{N-2} \geq \hat{x}_{N-1}$. In this case, independent of the realization of X_{N-1} if the agent searches once, he will stop in stage $N-1$ since x^*_{N-2} is weakly higher than the cut-off value of this stage. In particular, $x^*_{N-2} \geq \hat{x}_{N-1}$ implies $\mathbb{E}_{N-2}[\Delta U_{N-1}(X^*_{N-1})] \leq 0$ such that there is no option value of being able to search an additional time anymore. Therefore, the decision of searching in stage $N-2$ will only depend on the expected value of a single search $\mathbb{E}_{N-2}[\Delta U_{N-2}(x^*_{N-2})]$. In particular, for $x^*_{N-2} = \hat{x}_{N-1}$ the agent will search if and only if $\mathbb{E}_{N-2}[\Delta U_{N-2}(\hat{x}_{N-1})] \geq 0$, which is fulfilled whenever:

$$(1+\gamma)\int_0^1 \max\{\hat{x}_{N-1}, x_N\} - s - \hat{x}_{N-1}\, dF(x_N) + \gamma \int_{\hat{x}_{N-1}-s}^{\hat{x}_{N-1}} H_{N-2,N}(z)\, dz \geq 0 \tag{1.62}$$

Adding and subtracting $\gamma \int_{\hat{x}_{N-1}-s}^{\hat{x}_{N-1}} H_{N-1,N}(z)\, dz$ to the left hand side of (1.62) yields:

$$\begin{aligned}
&(1+\gamma)\int_0^1 \max\{\hat{x}_{N-1}, x_N\} - s - \hat{x}_{N-1}\, dF(x_N) + \gamma \int_{\hat{x}_{N-1}-s}^{\hat{x}_{N-1}} H_{N-1,N}(z)\, dz \\
&+\gamma \int_{\hat{x}_{N-1}-s}^{\hat{x}_{N-1}} H_{N-2,N}(z) - H_{N-1,N}(z)\, dz \geq 0 \\
&\Leftrightarrow \gamma \int_{\hat{x}_{N-1}-s}^{\hat{x}_{N-1}} H_{N-2,N}(z) - H_{N-1,N}(z)\, dz \geq 0
\end{aligned}$$

But consider that since

$$H_{N-2,N}(z) - H_{N-1,N}(z) = \begin{cases} 0 & , z > 1-2\cdot s \vee -s > z \\ -F(z+s)\cdot F(z+2\cdot s) & , 1-2\cdot s > z > -s \end{cases}$$

and $1-2\cdot s > \hat{x}_{N-1} - s > -s$, it follows that there exists some $z \in (\hat{x}_{N-1}-s, \hat{x}_{N-1})$ such that $H_{N-2,N}(z) - H_{N-1,N}(z) < 0$ and we find that:

$$\gamma \int_{\hat{x}_{N-1}-s}^{\hat{x}_{N-1}} H_{N-2,N}(z) - H_{N-1,N}(z)\, dz < 0$$

Therefore, we find that the agent will not search in stage $N-2$ of the decision problem whenever $x^*_{N-2} = \hat{x}_{N-1}$ and since the expected utility of a single screening is strictly decreasing in x^*_{N-2}, he will obviously not search for any value $x^*_{N-2} > \hat{x}_{N-2}$. But since we know that a cut-off value $\hat{x}_{N-2}$ exists, it must follow that $\hat{x}_{N-2} < \hat{x}_{N-1}$ and that $\hat{x}_{N-2}$ is implicitly defined by:

$$\int_0^{\hat{x}_{N-1}} \int_0^1 \max\{\hat{x}_{N-2}, x_{N-1}, x_N\} - N \cdot s \, dF(x_N) dF(x_{N-1})$$
$$+ \int_{\hat{x}_{N-1}}^1 \int_0^1 x_{N-1} - (N-1) \cdot s + \gamma(x_{N-1} - \max\{x_{N-1}, x_N - s\}) \, dF(x_N) dF(x_{N-1})$$
$$= \hat{x}_{N-1} - (N-2) \cdot s + \int_0^1 \int_0^1 \gamma(\hat{x}_{N-2} - \max\{\hat{x}_{N-2}, x_{N-1} - s, x_N - 2 \cdot s\}$$

which can be written as:

$$(1+\gamma) \int_0^1 \max\{\hat{x}_{N-2}, x_{N-1}\} - s - \hat{x}_{N-2} \, dF(x_{N-1}) + \gamma \int_{\hat{x}_{N-2}-s}^{\hat{x}_{N-2}} H_{N-2,N}(z) \, dz$$
$$+(1+\gamma) \int_0^{\hat{x}_{N-1}} \int_0^1 \max\{\hat{x}_{N-2}, x_{N-1}, x_N\} - s - \max\{\hat{x}_{N-2}, x_{N-1}\} \, dF(x_N) dF(x_{N-1}) \tag{1.63}$$
$$+\gamma \int_0^{\hat{x}_{N-1}} \int_{\max\{\hat{x}_{N-2}, x_{N-1}\}-s}^{\max\{\hat{x}_{N-2}, x_{N-1}\}} H_{N-1,N}(z) \, dz \, dF(X_{N-1}) = 0$$

Now, consider stage $N-3$ of the decision problem, and suppose that $x^*_{N-3} = \hat{x}_{N-2}$, such that the agent will search if and only if:

$$(1+\gamma) \int_0^1 \max\{\hat{x}_{N-2}, x_{N-2}\} - s - \hat{x}_{N-2} \, dF(x_{N-2}) + \gamma \int_{\hat{x}_{N-2}-s}^{\hat{x}_{N-2}} H_{N-3,N}(z) \, dz \geq 0 \tag{1.64}$$

Adding and subtracting the second expression in the first line and the whole second and third line on the left hand side of (1.63) to the left hand side of (1.64) and rearranging we find:

$$
\begin{aligned}
&(1{+}\gamma)\int_0^1 \max\{\hat{x}_{N-2}, x_{N-2}\} - s - \hat{x}_{N-2}\, dF(x_{N-2}) + \gamma \int_{\hat{x}_{N-2}-s}^{\hat{x}_{N-2}} H_{N-2,N}(z)\, dz \\
&+(1{+}\gamma)\int_0^{\hat{x}_{N-1}}\int_0^1 \max\{\hat{x}_{N-2}, x_{N-1}, x_N\} - s - \max\{\hat{x}_{N-2}, x_{N-1}\}\, dF(x_N)dF(x_{N-1}) \\
&+\gamma \int_0^{\hat{x}_{N-1}} \int_{\max\{\hat{x}_{N-2},x_{N-1}\}-s}^{\max\{\hat{x}_{N-2},x_{N-1}\}} H_{N-1,N}(z)\, dz dF(X_{N-1}) \\
&+\gamma \int_{\hat{x}_{N-2}-s}^{\hat{x}_{N-2}} H_{N-3,N}(z) - H_{N-2,N}(z)\, dz \\
&-(1{+}\gamma)\int_0^{\hat{x}_{N-1}}\int_0^1 \max\{\hat{x}_{N-2}, x_{N-1}, x_N\} - s - \max\{\hat{x}_{N-2}, x_{N-1}\}\, dF(x_N)dF(x_{N-1}) \\
&-\gamma \int_0^{\hat{x}_{N-1}} \int_{\max\{\hat{x}_{N-2},x_{N-1}\}-s}^{\max\{\hat{x}_{N-2},x_{N-1}\}} H_{N-1,N}(z)\, dz dF(X_{N-1})
\end{aligned}
$$

Note that the sum of the first three lines is equal to 0 due to the definition of $\hat{x}_{N-2}$ in (1.63) such that we finally find:

$$
\begin{aligned}
&\gamma \int_{\hat{x}_{N-2}-s}^{\hat{x}_{N-2}} H_{N-3,N}(z) - H_{N-2,N}(z)\, dz \\
&-(1{+}\gamma)\int_0^{\hat{x}_{N-1}}\int_0^1 \max\{\hat{x}_{N-2}, x_{N-1}, x_N\} - s - \max\{\hat{x}_{N-2}, x_{N-1}\}\, dF(x_N)dF(x_{N-1}) \\
&-\gamma \int_0^{\hat{x}_{N-1}} \int_{\max\{\hat{x}_{N-2},x_{N-1}\}-s}^{\max\{\hat{x}_{N-2},x_{N-1}\}} H_{N-1,N}(z)\, dz dF(x_{N-1}) < 0
\end{aligned}
\tag{1.65}
$$

The first line in (1.65) is weakly negative since:

$$
H_{N-2,N}(z) - H_{N-1,N}(z) = \begin{cases} 0 & ,if\;\; z > 1 - 3 \cdot s \vee -s > z \\ -F(z+s) \cdot F(z + 2 \cdot s) \cdot F(z + 3 \cdot s) & ,if\;\; 1 - 3 \cdot s > z > -s \end{cases}
$$

Note that it is only weakly negative, since in cases where $\hat{x}_{N-2} > 1 - 2 \cdot s$ or when $s > \frac{1}{2}$ the difference will always be 0. The second and third line in (1.65) represents the conditional expected utility of a single search in stage $N-1$ (with a negative sign), given a quality level of $\max\{\hat{x}_{N-2}, x_{N-1}\}$, where x_{N-1} is bounded above by $\hat{x}_{N-1}$. Since we know that $\hat{x}_{N-2} < \hat{x}_{N-1}$ and that $\hat{x}_{N-1}$ is the zero of the expected utility of a single search in stage in $N-1$, it must follow that the expected utility of a single search in stage $N-1$ given $\max\{\hat{x}_{N-2}, x_{N-1}\}$ must be strictly positive and hence the second and third line in (1.65) are strictly negative (due to the negative sign). As a consequence, it must follow that $\hat{x}_{N-3} < \hat{x}_{N-2}$. Using the same reasoning for any of the remaining stages, yields the desired result, that the sequence of cut-off values is strictly increasing.

iv) Boundaries

Since we know that $\{\hat{x}_t\}_{t=1}^{N-1}$ is a strictly increasing sequence, it suffices to show that $\hat{x}_1 > \hat{x}_{EU}$ and that $\hat{x}_{N-1} < 1 - s$. For the lower boundary, consider that $\hat{x}_1$ is the zero of $C_1(x_1)$, $C_1(x_1)$ is strictly decreasing and continuous in x_1 and that the following holds when $x_1 = \hat{x}_{EU}$:

$$C_1(\hat{x}_{EU}) = (1+\gamma)\underbrace{\int_0^1 \max\{\hat{x}_{EU}, x_2\} - s - \hat{x}_{EU}\, dF(x_2)}_{=0}$$
$$+\underbrace{\gamma\int_{\hat{x}_{EU}-s}^{\hat{x}_{EU}} H_{1,N}(z)\, dz}_{>0} + \underbrace{\mathbb{E}_1[\max\{0, C_2(X_2^*)\}]}_{\geq 0} > 0$$

As a consequence, it must hold that $\hat{x}_1 > \hat{x}_{EU}$. To derive the upper boundary, simply note that if it would hold that $\hat{x}_{N-1} \geq 1 - s$, the agent would search in stage $N-1$ when there is no improvement possibility left with respect to the final outcome. Therefore, the agent could deviate by simply stopping the decision problem which guarantees him a higher final outcome. □

Lemma 4. Given the strictly increasing sequence of cut-off values, the ex-ante expected value of the decision problem from stage 0 perspective is given by:

$$\begin{aligned}
\Gamma_N(\gamma, s) = &\int_{\hat{x}_1}^1 x_1 - s + \int_{-s}^{1-s} \gamma(x_1 - \max\{x_1, z\})\, dH_{1,N}(z) dF(x_1)\\
&+ F(\hat{x}_1)\left(\int_{\hat{x}_2}^1 x_2 - 2\cdot s + \int_{-s}^{1-s} \gamma(x_2 - \max\{x_2, z\})\, dH_{2,N}(z) dF(x_2)\right)\\
&+ \cdots\\
&+ F(\hat{x}_1)\cdots F(\hat{x}_{N-2})\Bigg(\int_{\hat{x}_{N-1}}^1 x_{N-1} - (N-1)\cdot s\\
&+ \int_{-s}^{1-s} \gamma(x_{N-1} - \max\{x_{N-1}, z\})\, dH_{N-1,N}(z) dF(x_{N-1})\Bigg)\\
&+ \int_0^{\hat{x}_1} \cdots \int_0^{\hat{x}_{N-1}} \int_0^1 \max\{x_1, ..., x_N\} - N\cdot s\, dF(x_N)\cdots dF(x_1)
\end{aligned} \tag{1.66}$$

Next, consider that $\hat{x}_1$ is implicitly defined by:

$$
\begin{aligned}
&\hat{x}_1 - s + \int_{-s}^{1-s} \gamma(\hat{x}_1 - \max\{x_1, z\})\, dH_{1,N}(z) \\
&= \int_{\hat{x}_2}^{1} x_2 - 2 \cdot s + \int_{-s}^{1-s} \gamma(x_2 - \max\{x_2, z\})\, dH_{2,N}(z) dF(x_2) \\
&+ \cdots \\
&+ F(\hat{x}_2) \cdots F(\hat{x}_{N-2}) \Bigg(\int_{\hat{x}_{N-1}}^{1} x_{N-1} - (N-1) \cdot s \\
&+ \int_{-s}^{1-s} \gamma(x_{N-1} - \max\{x_{N-1}, z\})\, dH_{N-1,N}(z) dF(x_{N-1}) \Bigg) \\
&+ \int_{0}^{\hat{x}_2} \cdots \int_{0}^{\hat{x}_{N-1}} \int_{0}^{1} \max\{\hat{x}_1, ..., x_N\} - N \cdot s\, dF(x_N) \cdots dF(x_1)
\end{aligned}
$$

Multiplying both sides with $F(\hat{x}_1)$ and rearranging terms yields:

$$
\begin{aligned}
&F(\hat{x}_1) \left(\hat{x}_1 - s + \int_{-s}^{1-s} \gamma(\hat{x}_1 - \max\{x_1, z\})\, dH_{1,N}(z) \right) \\
&- \int_{0}^{\hat{x}_1} \cdots \int_{0}^{\hat{x}_{N-1}} \int_{0}^{1} \max\{\hat{x}_1, ..., x_N\} - N \cdot s\, dF(x_N) \cdots dF(x_1) \\
&= F(\hat{x}_1) \left(\int_{\hat{x}_2}^{1} x_2 - 2 \cdot s + \int_{-s}^{1-s} \gamma(x_2 - \max\{x_2, z\})\, dH_{2,N}(z) dF(x_2) \right) \\
&+ \cdots \\
&+ F(\hat{x}_1) \cdots F(\hat{x}_{N-2}) \Bigg(\int_{\hat{x}_{N-1}}^{1} x_{N-1} - (N-1) \cdot s \\
&+ \int_{-s}^{1-s} \gamma(x_{N-1} - \max\{x_{N-1}, z\})\, dH_{N-1,N}(z) dF(x_{N-1}) \Bigg)
\end{aligned}
$$

Note that the RHS of this expression is equal to the second till second last row in $\Gamma_N(\gamma, s)$ which can be found in (1.66). Plugging in yields:

$$\Gamma_N(\gamma, s) = \int_{\hat{x}_1}^{1} x_1 - s + \int_{-s}^{1-s} \gamma(x_1 - \max\{x_1, z\})\, dH_{1,N}(z) dF(x_1)$$
$$+ \int_0^{\hat{x}_1} \cdots \int_0^{\hat{x}_{N-1}} \int_0^1 \max\{x_1, ..., x_N\} - N \cdot s\, dF(x_N) \cdots dF(x_1)$$
$$+ F(\hat{x}_1) \left(\hat{x}_1 - s + \int_{-s}^{1-s} \gamma(\hat{x}_1 - \max\{\hat{x}_1, z\})\, dH_{1,N}(z) \right)$$
$$- \int_0^{\hat{x}_1} \cdots \int_0^{\hat{x}_{N-1}} \int_0^1 \max\{\hat{x}_1, ..., x_N\} - N \cdot s\, dF(x_N) \cdots dF(x_1)$$

Simplifying and rearranging terms we have:

$$\Gamma_N(\gamma, s) = \int_{\hat{x}_1}^{1} x_1 - s\, dF(x_1) + F(\hat{x}_1)(\hat{x}_1 - s)$$
$$+ \int_0^{\hat{x}_1} \cdots \int_0^{\hat{x}_{N-1}} \int_0^1 \max\{x_1, ..., x_N\} - \max\{\hat{x}_1, ..., x_N\}\, dF(x_N) \cdots dF(x_1)$$
$$+ \int_{\hat{x}_1}^{1-s} \int_{x_1}^{1-s} \gamma(x_1 - z)\, dH_{1,N}(z) dF(x_1) + F(\hat{x}_1) \int_{\hat{x}_1}^{1-s} \gamma(\hat{x}_1 - z)\, dH_{1,N}(z)$$

Using integration by parts with respect to x_1 on the first and fourth term, noting that the second row is 0 whenever $\max\{x_2, ..., x_N\} > \hat{x}_1$ and simplifying yields:

$$\Gamma_N(\gamma, s) = 1 - s - \int_{\hat{x}_1}^{1} F(x_1)\, dx_1 + \int_0^{\hat{x}_1} u - \hat{x}_1\, dF(u)^N - \int_{\hat{x}_1}^{1-s} \gamma(1 - H_{1,N}(x_1))F(x_1)\, dx_1 \quad (1.67)$$

Finally, using integration by parts on the second integral with respect to u we have:

$$\Gamma_N(\gamma, s) = 1 - s - \int_{\hat{x}_1}^{1} F(x_1)\, dx_1 - \int_0^{\hat{x}_1} F(u)^N\, du - \int_{\hat{x}_1}^{1-s} \gamma(1 - H_{1,N}(x_1))F(x_1)\, dx_1 \quad (1.68)$$

which completes the proof. □

Proposition 5. We want to show that whenever γ is sufficiently large, there exists $\tilde{N}$ such that $\Gamma_{N+1}(\gamma, s) - \Gamma_N(\gamma, s) < 0$ for all $N \geq \tilde{N}$. In order to distinguish the decision problem with $N+1$ and N options, we will denote the cut-off values for each decision problem by $\hat{x}_{t,N+1}$ and $\hat{x}_{t,N}$, respectively. When adding an additional option to a choice set which consisted of N options, the difference in the ex-ante expected value of the decision problems is given by:

$$\begin{aligned}\Gamma_{N+1}(\gamma, s) - \Gamma_N(\gamma, s) = &\int_{\hat{x}_{1,N}}^{1} F(x_1)\, dx_1 + \int_{0}^{\hat{x}_{1,N}} F(u)^N\, du + \int_{\hat{x}_{1,N}}^{1-s} \gamma(1 - H_{1,N}(x_1))F(x_1) \\ &- \int_{\hat{x}_{1,N+1}}^{1} F(x_1)\, dx_1 - \int_{0}^{\hat{x}_{1,N+1}} F(u)^{N+1}\, du - \int_{\hat{x}_{1,N+1}}^{1-s} \gamma(1 - H_{1,N+1}(x_1))F(x_1)\end{aligned}$$

Note that $\hat{x}_{1,N} = \hat{x}_{2,N+1}$ since both cut-off values are making the agent indifferent between stopping and the possibility to search for $N-1$ times. Furthermore, due to Proposition 4, we know that $\hat{x}_{2,N+1} > \hat{x}_{1,N+1}$. The difference can now be written in the following way:

$$\begin{aligned}\Gamma_{N+1}(\gamma, s) - \Gamma_N(\gamma, s) = &\int_{\hat{x}_{1,N+1}}^{\hat{x}_{2,N+1}} F(x_1)[F(x)^{N-2} + \gamma H_{1,N+1}(x_1) - (1+\gamma)]\, dx_1 \\ + &\int_{0}^{\hat{x}_{1,N+1}} F(u)^{N-1}(1 - F(u))\, du + \int_{\hat{x}_{2,N+1}}^{1-s} \gamma F(x_1)[H_{1,N+1}(x_1) - H_{1,N}(x_1)]\, dx_1\end{aligned}$$

The first and the third term are strictly negative, while the second term is strictly positive. First, note that when increasing $N \to \infty$, the difference will ultimately go to 0. The first terms tends to 0 due to the fact that $\{\hat{x}_t\}_{t=1}^{N}$ is bounded below and strictly increasing, such that the difference $\hat{x}_{1,N+1} - \hat{x}_{2,N+1}$ will become closer to each other for an increasing amount of options. The second term will converge to 0 due to the monotone convergence theorem. The last term will also converge to 0 due to the fact that the difference $H_{1,N+1}(x_1) - H_{1,N}(x_1)$ gets arbitrarily close to each other. Second, due to Assumption 1, adding one option to the outside option always makes the agent strictly better off. Third, note that for very high values of γ, the negative parts in the difference are disproportionally high compared to the positive parts. When increasing N for high values of γ, the positive parts in the difference will shrink faster to 0 than the negative parts, which will lead to a decrease in the expected utility of the decision problem. However, since the amount of regret that an agent can feel is bounded above, the difference will ultimately converge to 0, as stated in the first point. However, for very large values of γ, the behavior of the Regret Agent will be equivalent to the behavior of the ER Agent and hence he will suffer from Choice Overload. □

Proposition 6. This Proposition simply follows from the fact that the sufficient conditions for the Extreme Regret Agent to suffer from Status Quo Bias are also sufficient for the Regret Agent to suffer from Status Quo Bias. When increasing γ, the behavior of the Regret Agent and the Extreme Regret Agent will be equivalent. □

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Chapter 2

Cost Uncertainty and Stochastic Orders in a Static Model of Vertical Differentiation

Abstract

We analyze a canonical model of vertical differentiation, where two firms compete in prices and qualities. In contrast to classical vertical differentiation models, we assume that a firm is uncertain about its rival's fixed costs of quality improvement, captured by an efficiency parameter. We find two types of equilibria, which substantially differ in the features of the products which are sold. In one equilibrium, the products are maximally differentiated in the quality dimension, reminiscent of the equilibrium with no cost uncertainty. Besides, we identify a second equilibrium in which firms are more competitive in the quality dimension in that similar efficiency parameters lead to similar products concerning the quality dimension. This, in turn, leads to lower prices compared to the first equilibrium, to the benefit of consumers. Finally, we conduct comparative statics using stochastic orders. Under weak assumptions, we find that shifting probability mass in the distribution of consumers' willingness to pay for quality from the lower to the upper boundary increases consumer rent for all consumers, independent of their willingness to pay for quality.

2.1 Introduction

Many markets are dominated by two firms, each offering a product not perceived as superior to the other product. This occurs since both firms are steadily engaging in innovation. Examples include Boeing and Airbus in the market of airplane manufacturers or Intel and AMD in the market of CPUs. While engaging in price or quantity competition, there is another dimension in which competition occurs, which substantially impacts customers' purchasing decision: the quality of a product. For an airline company that needs to purchase airplanes, an airplane's quality might reflect itself in a measure of efficiency, e.g., the fuel economy, which an airplane manufacturer can influence through R&D expenses. Therefore, when a firm is developing a new product and deciding about its R&D expenses, it should consider the R&D expenditures of its rival firm since the market share might crucially depend on the final product's quality level, besides the market price.[1]

However, R&D is costly, and there always exists the risk of a product getting crowded out (e.g., Blu-ray, which crowded out HD DVD) or even failing (e.g., Boeing 737 MAX). The question arises, why we often do not see a partition of the market in duopoly markets, as predicted in classical models of vertical differentiation, introduced by Gabszewicz and Thisse (1979), Shaked and Sutton (1982), or Tirole (1988). One firm could invest heavily in R&D to innovate its product and sell it with a premium to customers with a high willingness to pay for quality. In contrast, the other firm sells some basic version to customers with a lower willingness to pay for quality. Independent of whether there are other product differentiation sources, besides the quality, this differentiation in the products' quality dimension should relax prices compared to a situation where both firms are selling innovative but similar products. The innovative firm benefits by having access to customers with a high willingness to pay, while the other firm benefits by not suffering from high R&D expenses. An example for oligopolistic markets, where firms are typically differentiated in the quality dimension, is given by the airline industry. Given that quality in the airline industry is reflected for instance by service and comfort, most customers would rank Lufthansa over Ryanair.

Our paper provides an explanation for the observed pattern of firms steadily engaging in innovation: First, in case of market segmentation, the market for the high-tech product might be much more profitable compared to the market of the low-tech product, which increases the incentive that a firm wants to become rather a high- than a low-tech firm. Second, a firm might be uncertain about the innovation efficiency of a rival, e.g., about the size of the rival's R&D expenditure to reach a particular quality level for a product. Even if R&D expenses for a rival's future product would be perfectly observable, there might still be some uncertainty involved with the product's final quality level. Generally, a product's final quality level is a function of labor and capital, e.g., the size of the research staff, the amount and quality of machines/equipment/resources which are used to conduct experiments and to build and test prototypes, combined with some random component. While a firm might have information about the rival's staff size and the machines used, it might still have no information about the actual productivity concerning innovation, e.g., the research staff's quality and within-compatibility.

[1] For instance, in 2019, Boeing spent 3.2 billion dollars on R&D, accounting for approx. 4% of the revenue in 2019, while Airbus SE spent 4.3 billion dollars, accounting for approx. 5.4% of the revenue.

This might translate into a firm being uncertain about its rivals' efficiency and, depending on its own efficiency, to more aggressively investing into R&D to become a high-tech firm, especially if the market for a high-tech product is more profitable than the market of a low-tech product.

In this paper, we revisit the canonical model of vertical differentiation introduced by Tirole (1988). We consider two firms, which compete in prices and qualities and a mass of consumers, where each consumer exclusively buys from one of the two firms in each period. Consumers are heterogeneous concerning their marginal utility of quality, which implies that their willingness to buy a particular product with a given price and quality level differs. We assume that firms suffer from strictly convex fixed costs of quality improvements.

Our model's key assumption is given by uncertainty about the fixed costs: a firm does not know its rival's fixed costs of quality improvements. Due to consumer heterogeneity, firms want to sell their products to consumers with the highest marginal utility for quality since their willingness to pay is the highest. Hence, the firms prefer to be the high quality- rather than the low-quality firm for given fixed costs. Uncertainty about the rivals' fixed costs translates into the uncertainty of who becomes the high-quality firm. This mechanism gives rise to a competitive equilibrium, where both firms choose similar quality levels when they have similar efficiency parameters and where both firms choose strictly positive quality levels for a wide range of efficiency parameters. Since we assume that quality is the only source of product heterogeneity in the model, this equilibrium can lead to less product differentiation and lower prices, while the quality level of both products is high, ultimately benefiting the consumers.

This equilibrium is in stark contrast to a second equilibrium that we find. In the second equilibrium, which is similar to the equilibrium of the vertical differentiation model without uncertainty about the rival's fixed costs, product differentiation is maximized by one firm choosing the highest possible- and the other firm choosing the lowest possible quality level. In this equilibrium, the low-quality firm acts as a free rider: whenever a firm chooses the lowest possible quality level, it benefits from an increasing quality level of its rival since the products become more heterogeneous, which relaxes price competition. At the same time, it has low fixed costs of product innovation. However, in contrast to the first equilibrium, this equilibrium only exists under certain conditions.

In our model, the primitives are given by two distributions: the distribution of consumers' marginal utility of quality and the distribution of efficiency parameters, which determines the degree of convexity for a firm's fixed cost function of quality improvements. Besides log concavity of the densities and a compact support, we do not impose any further restrictions. Nevertheless, we can conduct comparative statics in a quite general sense for the equilibrium outcomes by using stochastic orders. In particular, we use a stochastic order in the likelihood ratio sense, which, roughly speaking, shifts probability mass from the lower to the upper bound of a random variables support when keeping the support fixed. We then compare the impact on the different equilibria and the implications for consumer rent and firms' profits.

The paper is organized as follows. In the next section we give an overview of the literature. In section 2.3 we introduce and solve the model. In section 2.4 we will conduct comparative statics. Finally, in section 2.5, we will conclude.

2.2 Literature

The literature on vertical differentiation starts with the seminal papers by Gabszewicz and Thisse (1979) and Shaked and Sutton (1982, 1983). They analyze a price competition model where firms can endogenously choose the quality level of their product. In their models, consumers are heterogeneous concerning their income, which determines their willingness to pay for quality.[2] While those papers were the first papers on the topic of vertical differentiation, most literature, including this paper, is based on the reduced form model of vertical differentiation, which can be found in Tirole (1988). In Tirole (1988), consumers are heterogeneous concerning their willingness to pay for quality, represented by some parameter. In an alternative interpretation, this parameter can be seen as the marginal rate of substitution between income and quality, which is positively correlated with income. A common result in all the previously mentioned models is given by the existence of a unique equilibrium, where the firms' products are maximally differentiated. One firm chooses the highest possible quality level, while the other firm chooses the lowest possible quality level, relaxing price competition and leading to strictly positive profits for both firms. However, real-life examples have shown that, especially in duopoly markets, firms engage in innovation and sell products that are quite similar in their quality level, which is at odds with the early models' theoretical predictions.

Since then, the models have been extended in various directions, and in some cases, the results partially deviated from the Maximum Differentiation Equilibrium. One strand of the literature considers non-uniform distributions of income or the possibility of the market to be uncovered (Wauthy (1996), Choi and Shin (1992), Benassi et al. (2006), Bonnisseau and Lahmandi-Ayed (2007), Liao (2008), Furth (2011), Benassi et al. (2019)). While sole consideration of more general distributions does not necessarily lead to a deviation from the maximum differentiation result, additionally allowing for the market to be uncovered partially mitigates it.[3] However, in those models, the firms' quality levels can still not be close to each other. Besides, allowing for the possibility of the market to be uncovered while considering general distributions makes the model highly intractable, and it becomes difficult to conduct comparative statics.

A second strand of the literature focuses on introducing costs of quality improvements. Lehmann-Grube (1997) shows that the high-quality firm's advantage persists when introducing fixed costs, Wang (2003) shows that the advantage can disappear when introducing variable costs of quality improvements. Brecard (2010) allows for both types of costs and finds that product differentiation decreases when variable costs are low. In all papers, however, the products cannot be similar in the quality dimension, and the main analysis is restricted to consumers' willingness to pay for quality being uniformly distributed. All three papers build upon Tirole (1988), such that the models implicitly assume income to be uniformly distributed, which is not in line with empirical income distributions.

Other papers argued that products typically have multidimensional attributes, which gives

[2] The income level of an individual is of private information. The firms only know that income is uniformly distributed among the population.

[3] The reason for this is given by the fact that allowing for the market to be uncovered can be seen as introducing a fringe competitor, who exogenously chooses the lowest possible quality level and sells at a price of 0. This increases the low-quality firm's competitive pressure, increasing its quality level, while the high-quality firm does not change its behavior.

firms the possibility to differentiate their products among several dimensions. In a multidimensional vertical differentiation model, one could think of a similar result as in Irmen and Thisse (1998). They find that in a multidimensional horizontal differentiation model, there is minimum product differentiation among all but one dimension, where they find maximal vertical differentiation. Dos Santos Ferreira and Thisse (1996) and Gabszewicz and Wauthy (2012) analyze a model where products are differentiated across a vertical and a horizontal dimension. While in Gabszewicz and Wauthy (2012), the quality choice is exogenous, and the focus lies on price effects, Dos Santos Ferreira and Thisse (1996) show that an equilibrium exists, where products are minimally differentiated across the vertical dimension, given that horizontal differentiation is large enough. However, this model's location choices are exogenous since price equilibria failed to exist for some location pairs. Therefore, it is unclear whether the consideration of endogenous location choice would yield maximum horizontal differentiation, which in turn would imply minimal vertical differentiation. Also, due to tractability, the parameter measuring consumer heterogeneity is uniformly distributed. Garella and Lambertini (2014) consider a model with two dimensions of vertical differentiation, showing that equilibria with quality leadership in both dimensions and cross-leadership equilibria exist. Barigozzi and Ma (2018) consider a n-dimensional attribute model of vertical differentiation, assuming a general distribution for the parameters measuring consumers' willingness to pay for improvements among each quality dimension and convex variable costs of production.

To establish a result, which matches the empirical observations that firms sell products of similar quality levels, our paper takes a different approach than previous papers. By assuming uncertainty concerning a rival's fixed costs of quality improvements, we establish an equilibrium in pure strategies, predicting that firms' quality levels are similar whenever they are similarly efficient. To the best of our knowledge, the only paper which finds similar results is given by Wang and Yang (2001). They consider a standard textbook model of vertical differentiation as in Tirole (1988) and show that there exists an infinite amount of mixed equilibria, where in one equilibrium, there is a non-zero probability that firm's quality levels will be equal to each other in equilibrium when both firms are choosing strictly positive quality levels. However, in contrast to their paper, we establish this result and even more results within a pure strategy equilibrium and for fairly general distributions for the model's primitives, without relying on the uniform distribution. More importantly, our paper explains why the firms' qualities are close or far away from each other: the more the firms differ in their efficiency, the more differentiated their products are. Furthermore, our paper allows the firms' products not only to be maximally or minimally differentiated in equilibrium, but there can also be intermediate product differentiation. We establish all those results within one pure strategy equilibrium, and we derive conditions such that it is the unique pure strategy equilibrium. In contrast, in Wang and Yang (2001), no single mixed equilibrium can yield all our predictions. Furthermore, there is no justification which mixed equilibrium within the infinite set of mixed equilibria will be played.

From a methodological perspective, our paper is closest to Moldovanu and Sela (2001), who analyze a multi-prize all-pay auction. Our model's quality stage, where the firms simultaneously choose their quality levels, can be seen as some adjusted version of an all-pay contest, which is analyzed in Siegel (2009). The quality levels chosen by the firms in our model can be seen as a "score" in Siegel (2009) or a "bid" in Moldovanu and Sela (2001). The revenues of the high and low-quality firm in our model can be seen as "prizes" in their models. However, in contrast to their models, the "prizes" in our paper are endogenous, i.e., they depend on the competing firms' actions. In our model, increasing the quality level of a product not only harms the outcome through higher costs, but it also has a negative effect when a firm loses the innovation contest due to decreased product differentiation, which lowers the prizes of both firms. Furthermore, when a firm wins the innovation contest in our paper, there is close to no benefit if the chosen quality level of the winner is larger but arbitrarily close to the quality level chosen by the firm, which loses the contest. In this case, the products of the firms will be homogenous. Hence, the prizes of both firms will be 0. In Moldovanu and Sela (2001) and Siegel (2009), however, the profit of the auction/contest winner is highest when its bid/score is as close as possible to the bid/score of the loser. Therefore, firms' incentives to increase their quality level in our model are different from the incentives in their papers. For instance, in both their models, the equilibrium bid/score is strictly increasing in the auction/contest loser's prize. In contrast, our model's equilibrium quality function is strictly decreasing in the low-quality firm's prize.

Other papers, which introduce some additional uncertainty into a model of vertical differentiation, are given by Bester (1998), Cheng (2014), and Candio and Gall (2018). However, the uncertainty in those papers is related to the demand side, and the research questions differ from ours.

Our paper makes three contributions to the vertical differentiation literature. First, by assuming that firms are uncertain about each other's innovation costs, we show the existence of a pure strategy equilibrium that allows the firms' quality levels to be close to each other, which matches the empirical observation on some markets. The resulting model is highly tractable, and we show this result, although we consider relatively weak assumptions concerning the model's primitives. Second, we prove the existence of a second pure strategy equilibrium in which the firms' products are maximally differentiated, reminiscent of equilibrium in the classical vertical differentiation literature without cost uncertainty. However, we derive a necessary and sufficient condition for this quality equilibrium to exist, using our model's primitives, which, to the best of our knowledge, no other paper has done in the literature. Third, we show that the monotone likelihood order within the concept of stochastic orders (see chapter 1.C in Shaked and Shanthikumar (2007)) is a useful tool to conduct comparative statics in models of vertical differentiation when considering weak assumptions concerning the distribution of the primitives. To the best of our knowledge, no other paper in the literature has utilized this particular stochastic order, despite its usefulness.

2.3 Model

In this section, we will describe and solve the model of the paper. The proofs of all the results in this and the next section can be found in the Appendix.

There are two single good firms, denoted by $i \in \{A, B\}$, which compete in prices and qualities in a vertically differentiated market. Firms choose price p_i and quality level q_i of their good to maximize expected profits. Firms face strictly increasing and strictly convex fixed costs of quality improvement in each period, which are given by

$$C(q_i) = \frac{1}{2} \cdot c_i \cdot q_i^2. \tag{2.1}$$

The cost-efficiency parameter $c_{t,i}$, which determines the slope and the curvature of the fixed cost function of a firm, is of private information and is drawn independently and identically across firms from an atomless distribution according to a CDF $G(\cdot)$ with strictly positive density $g(\cdot)$ on a compact interval $\mathcal{C} = [\underline{c}, \overline{c}] \subset \mathbb{R}_{++}$. Hence, a firm is uncertain about its rival's cost efficiency.[4] A firm's good is characterized by its price p_i and its quality level $q_{t,i}$. There is no other source of heterogeneity. However, a firm cannot decrease its quality level. Whenever the quality level of a firm's good is higher than its rivals, all consumers strictly prefer its good when prices are equal.

The demand side of the market is given by a unit mass of consumers who are characterized by a parameter θ which measures their marginal utility of quality and is of private information for each consumer. However firms know that θ is drawn independently and identically across consumers from an atomless distribution according to a CDF $F(\cdot)$ with strictly positive density $f(\cdot)$ on a compact interval $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}_{++}$. A (representative) consumer's utility of buying firm i's product with parameter θ is given by:

$$U_i(\theta) = \theta \cdot q_i - p_i. \tag{2.2}$$

We assume that the market is covered, such that each consumer buys from either of the firms. A consumer will buy firm i's good if and only if his indirect utility of buying from firm i is higher than buying from its rival. The game proceeds in the following way:

Stage 0 Nature draws c_A and c_B according to $G(\cdot)$. Each Firm i privately observes its cost parameter.

Stage 1 Firms simultaneously choose (q_A, q_B).

Stage 2 Firms observe (q_A, q_B) and simultaneously choose (p_A, p_B).

Stage 3 Consumers observe (q_A, q_B) and (p_A, p_B) and decide from which firm to buy.

We assume that firms choose qualities before prices since the price of a good, in general, can be adjusted more readily than its quality.

In what follows, we will solve the game by backward induction. We start with stage 3, where consumers exclusively decide from which firm to buy.

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[4]The model can be extended to allow for variable costs, which are independent of the quality level. All qualitative results in the paper will hold when making a slight change in Assumption 2, which guarantees the existence of a price equilibrium where both firms are active on the market. The resulting prices in equilibrium will simply include the marginal variable costs as an additive term, which does not interact with any of the other primitives. Consequently, it will simply drop out from the firms' profit functions and have no impact on the analysis of the quality stage in the model. Furthermore, considering a more general fixed cost function $C(q_i) = c_i \cdot \gamma(q_i)$, where $\gamma(0) = 0$, $\gamma'(\cdot) > 0$ and $\gamma''(\cdot) > 0$ will not change any of the qualitative results, since the objective function of a firm in the quality stage will still be quasi concave in its quality level for a fixed cost parameter.

2.3.1 Stage 3: Consumer Choice

Given (p_A, q_A, p_B, q_B) a consumer with parameter θ will buy from firm A iff

$$U_A(\theta) > U_B(\theta) \Leftrightarrow v + \theta \cdot q_A - p_A > v + \theta \cdot q_B - p_B. \tag{2.3}$$

Assume that $q_A > q_B$, which in equilibrium implies that $p_A > p_B$[5]. The consumer, who is indifferent between buying from firm A or B, is characterized by the parameter $\hat{\theta}$ which is given by:

$$\hat{\theta} = \frac{p_A - p_B}{q_A - q_B}. \tag{2.4}$$

All consumers with $\theta > \hat{\theta}$ will buy from firm A, while all consumers with $\theta < \hat{\theta}$ will buy from firm B, which leads to the following demand functions for the firms

$$D_A = P(\theta > \hat{\theta}) = 1 - F(\hat{\theta}), \tag{2.5}$$

$$D_B = P(\theta < \hat{\theta}) = F(\hat{\theta}). \tag{2.6}$$

Therefore, the firms' revenue functions are given by[6]:

$$R_A = (1 - F(\hat{\theta})) \cdot p_A, \tag{2.7}$$

$$R_B = F(\hat{\theta}) \cdot p_B. \tag{2.8}$$

Whenever it is the case that $q_A = q_B$, the products are homogenous, and the firms engage in Bertrand competition. Therefore, all consumers will buy from the firm at the lower price. For the cases where we have $q_A = q_B$ and $p_A = p_B$, all consumers will randomly buy from any of the firms.

2.3.2 Stage 2: Price Equilibrium

In this stage, firms simultaneously choose their prices to maximize their revenues, given qualities (q_A, q_B). Consider any subgame where $q_A = q_B$. Since quality is the only source of differentiation for the goods, an equal quality level leads to homogenous goods. Hence, firms end up in the Bertrand paradox where prices equal marginal costs. The analysis of the cases where it holds that $q_A > q_B$ or $q_A < q_B$ are equivalent, the only relevant aspect is given by the fact that there is one high-quality firm and one low-quality firm. Therefore, in what follows, we will denote by (p_H, q_H) and (p_L, q_L) the quality and price of the high-quality firm and low-quality firm, respectively. Throughout the paper, we will make two assumptions, which are sufficient for the existence of a unique price equilibrium, where both firms are active on the market.

Assumption 1. $f(\cdot)$ *is log concave.*

[5]In any subgame perfect equilibrium, it cannot hold that $q_A > q_B$ but $p_A \leq p_B$, since in such a case, all consumers would buy from firm A and there is no demand for firm Bs good. Both firm A and firm B would have an incentive to deviate.

[6]At this point, we talk about revenue functions to distinguish them from the profit functions in the first stage since firms suffer from fixed costs of quality improvements.

This assumption will ensure that the profit function of each firm is quasi concave in its own price, see Caplin and Nalebuff (1991). [7]

Assumption 2. $f(\underline{\theta}) \cdot \underline{\theta} < 1$.

This assumption, together with Assumption 1, ensures that both firms are active on the market. In particular it ensures that there is sufficient heterogeneity of consumers on the market, such that both firms can coexist in equilibrium.[8] If the willingness to pay for quality for the consumer with the lowest willingness to pay for quality is high (i.e., if $\underline{\theta}$ is high) or the probability mass on consumers with a low willingness to pay for quality is high (i.e., if $f(\underline{\theta})$ is high) the high quality firm has an incentive to decrease its prices to serve those consumers, which drives the low quality firm out of the market.

We can now state the first result:

Proposition 1. *There exists a unique price equilibrium, where both firms have strictly positive demand, which is given by:*

$$p_H^*(q_H, q_L) = \frac{1 - F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_H - q_L), \tag{2.9}$$

$$p_L^*(q_H, q_L) = \frac{F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_H - q_L). \tag{2.10}$$

where $\hat{\theta}^* \in (\underline{\theta}, \overline{\theta})$, which describes the parameter of the indifferent consumer in equilibrium, is the unique solution to

$$\hat{\theta}^* = \frac{1 - 2 \cdot F(\hat{\theta}^*)}{f(\hat{\theta}^*)}. \tag{2.11}$$

One can make four observations. First, the high quality firm sets a higher price than the low quality firm.[9] Second, prices are set in a way that for each firm there is unit price elasticity of demand, independent of the chosen qualities. Third, both firms benefit from an increasing difference in quality levels. This occurs due to the fact that quality is the only source of product heterogeneity and an increase in the level of quality difference relaxes price competition. Fourth, the indifference consumers parameter $\hat{\theta}$ is independent of the chosen quality levels in equilibrium. The equilibrium revenues are given by

$$R_H(q_H, q_L) = \frac{(1 - F(\hat{\theta}^*))^2}{f(\hat{\theta}^*)} \cdot (q_H - q_L) := \pi_H \cdot (q_H - q_L), \tag{2.12}$$

$$R_B(q_H, q_L) = \frac{F(\hat{\theta}^*)^2}{f(\hat{\theta}^*)} \cdot (q_H - q_L) := \pi_L \cdot (q_H - q_L). \tag{2.13}$$

[7] Many common distributions and their truncated versions belong to the family of distributions with log concave densities, see Bagnoli and Bergstrom (2005).

[8] When considering a uniform distribution on $[\underline{\theta}, \overline{\theta}]$, Assumption 2 is given by $2 \cdot \underline{\theta} < \overline{\theta}$. For a triangular distribution, where the modus is not equal to $\underline{\theta}$, the condition is always fulfilled, while for the other case it is fulfilled only if $\overline{\theta} - \underline{\theta} > 1$.

[9] This holds since $p_H^*(q_H, q_L) - p_L^*(q_H, q_L) = \frac{1-2F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_H - q_L) = \hat{\theta}^* \cdot (q_H - q_L) > \underline{\theta} \cdot (q_H - q_L) > 0$

where π_H and π_L can be interpreted as the marginal revenue of a quality difference increase for the high-quality firm and the low-quality firm, respectively. The higher π_H, relative to π_L, the more profitable it is to become the high-quality firm relative to becoming the low-quality firm. One can easily see that depending on whether a firm becomes a high or a low-quality firm, it either increases or decreases product differentiation when it increases its own quality. This tradeoff will become important when considering uncertainty about the rival's cost parameter since uncertainty about the rival's cost parameter translates into the uncertainty of who becomes the quality leader. Note that the revenue functions are linear in the qualities since both π_H and π_L do not depend on the qualities. This will drastically simplify the analysis of the quality stage.

2.3.3 Stage 1: Quality Equilibrium

In this stage, firms simultaneously choose their quality levels to maximize profits, determining who will be the high-, and who will be the low-quality firm in the next stage. First, we will derive a symmetric quality equilibrium, where a firm's equilibrium quality level is weakly decreasing in its cost parameter c_i. A strategy of firm i is given by a mapping $q_i : \mathcal{C} \to \mathbb{R}_+$. Second, we will show that, depending on the distribution of the primitives, there also exists a Maximum Differentiation Equilibrium, where one firm chooses a quality level of 0, whereas the other firm chooses quality level such that the marginal revenue of an increase in quality equals marginal costs. Due to the uncertainty about the rival's cost parameter, firms will maximize their expected profit. By R_H and R_L, we will denote the high-quality and low-quality firm's revenues, respectively.

If a firm increases its quality level, there are four channels through which its profit will be affected: First, it increases the probability of becoming a high-quality firm. Second, conditional on becoming a high-quality firm, its revenue will increase. On the other hand, conditional on becoming a low-quality firm, its revenue will decrease. Fourth, its costs will increase.

This stage of the game can be seen as a particular type of a Tullock Contest, where the agents in the contest choose efforts to win either a high or a low prize while suffering from quadratic effort costs, independent of which prize they win. However, the particular shape of the cost function is private information. Besides, the value of both prizes and the probability of getting any of the prizes is endogenously determined by the chosen efforts.

Note that, since we will first derive a symmetric equilibrium, we will focus on the perspective of firm A. From the perspective of firm A, the quality function of firm B will depend on firm B's cost parameter and hence it will be stochastic. Therefore, given firm A's own cost parameter c_A and the stochastic quality function of firm B, denoted by $q_B(c_B)$, the optimization problem of firm A is given by:

$$\max_{q_A \geq 0} \Pi_A = P(q_B(c_B) = 0) \cdot \mathbb{E}[R_H | q_B(c_B) = 0] + P(0 < q_B(c_B) \leq q_A) \cdot \mathbb{E}[R_H | 0 \leq q_B(c_B) \leq q_A] \\ + P(q_B(c_B) > q_A) \cdot \mathbb{E}[R_L | q_B(c_B) > q_A] - \frac{1}{2} \cdot c_A \cdot q_A^2.$$

The first term describes the conditional expected revenue of firm A, when firm B sets a quality level of zero. The second term describes the conditional expected revenue of firm A, when firm B sets a lower, but strictly positive quality level. The third term is the conditional expected

revenue of firm A, when it sets a lower quality than firm B. As we will see, in equilibrium, there will be a cost threshold $\tilde{c}$, above which a firm will choose a quality of of zero. Therefore, $q_B^{-1}(0)$ will not be a singleton and hence $P(q_B(c_2) = 0)$ is nonzero, even-though the distribution of cost parameters is continuous.
We can write the optimization problem in the following way:

$$\begin{aligned}\max_{q_A \geq 0} \Pi_A = \pi_H \cdot \int_{\tilde{c}}^{\bar{c}} q_A \, dG(c_B) + \pi_H \cdot \int_{q_B^{-1}(q_A)}^{\tilde{c}} \Big(q_A - q_B(c_B) \Big) \, dG(c_B) \\ + \pi_L \cdot \int_{\underline{c}}^{q_B^{-1}(q_A)} \Big(q_B(c_B) - q_A \Big) \, dG(c_B) - \frac{1}{2} \cdot c_A \cdot q_A^2,\end{aligned} \tag{2.14}$$

and state our second result:

Proposition 2. *There exists a symmetric quality equilibrium where each firms sets it quality according to:*

$$q^*(c_i) = \max\left\{ \frac{\pi_H \cdot (1 - G(c_i)) - \pi_L \cdot G(c_i)}{c_i}, 0 \right\}, \tag{2.15}$$

where $q^*(c_i) = 0$ *if*

$$c_i \geq \tilde{c} = G^{-1}\left(\frac{\pi_H}{\pi_H + \pi_L} \right) \in (\underline{c}, \bar{c}). \tag{2.16}$$

The optimal quality level reflects a quality level where the expected marginal revenue of increasing quality equals the marginal costs. As stated before, a firm only benefits from increasing its quality level, whenever it is the high quality firm. In case of becoming the low quality firm, its revenue will decrease in addition to higher fixed costs when it increases its quality. One can see that the optimal quality level is strictly decreasing in the firms cost parameter until it hits the threshold value $\tilde{c}$, after which it will set a quality level of 0. An interesting property of this equilibrium is given by the possibility that the firms' products are similar, whenever the cost efficiency parameters are similar, or whenever both firms' cost parameters are larger than $\tilde{c}$. This equilibrium yields a possible explanation for the observed pattern on some duopoly markets, where both firms are steadily engaging in innovation.

The threshold cost parameter has the following intuition: it reflects the cost parameter in equilibrium, which sets the expected marginal revenue of increasing the quality to 0. Hence, at cost parameters weakly above this threshold, the probability of becoming the low-quality firm in a symmetric equilibrium is so high that a firm simply sets its quality level to 0 as if it was certain that it would become the low-quality firm.
The threshold value depends on the two primitives of the model.

First, it depends on the distribution of the cost parameter. Whenever the density of the distribution is left-skewed, therefore there is more probability mass on higher cost parameters, the threshold is higher compared to a distribution with right-skewed density. This follows since, in equilibrium, for a particular cost parameter of a firm, which in turn determines a fixed quality level, the probability of becoming the high-quality firm increases, the more probability mass is shifted to high cost parameters. Hence, a firm is willing to set a strictly positive quality level for higher values of the cost parameter for densities that are more left-skewed than right-skewed ones.

Second, it depends on the distribution of consumers' willingness to pay for quality, which determines the marginal revenue of a quality difference increase for the high- and the low-quality firm, namely π_H and π_L, respectively. The ratio $\frac{\pi_H}{\pi_H+\pi_L}$ measures the relative benefit a firm has in equilibrium when it becomes a high-quality firm for a given cost parameter. The higher this relative benefit, the higher are the incentives of a firm to increase the probability of becoming a high-quality firm. Hence, for a given distribution of the cost parameter, the threshold increases whenever the relative benefit is increasing. When the ratio approaches 1, the threshold value will get arbitrary close to the support's upper boundary, and hence firms will almost always choose a strictly positive quality level. To see when the relative benefit is high, we rewrite the ratio in the following way:

$$\frac{\pi_H}{\pi_H + \pi_L} = \frac{1}{1 + \frac{\pi_L}{\pi_H}} = \frac{1}{1 + \left(\frac{\int_{\underline{\theta}}^{\hat{\theta}^*} f(\theta)\, d\theta}{\int_{\hat{\theta}^*}^{\bar{\theta}} f(\theta)\, d\theta} \right)^2}. \tag{2.17}$$

The ratio in the denominator describes the squared odds of a consumer buying the low-quality firm's good as opposed to buying the high-quality firm's good. In particular, it is the squared ratio of the firms' demand functions in equilibrium. This ratio will be low, and hence the benefit of becoming a high-quality firm high, whenever the number of consumers with a high willingness to pay for quality is high. This is the case whenever the density of θ is left-skewed or whenever we increase the upper boundary of the support of θ while holding the lower boundary fixed.

Given the equilibrium qualities, we can derive the expected equilibrium profit of a firm which is given by

$$\Pi_i^* = \pi_L \cdot \int_{\underline{c}}^{\tilde{c}} \frac{\pi_H \cdot (1 - G(c_{-i})) - \pi_L \cdot G(c_{-i})}{c_{-i}}\, dG(c_{-i}), \tag{2.18}$$

if it sets a quality level of 0, i.e., if $c_i \geq \tilde{c}$. In this case it acts as a free rider, who benefits from higher quality chosen by the high quality firm. This occurs due to the fact that any increase in product differentiation, which relaxes price competition and hence increases profits for both firms, comes from the chosen quality level of the high quality firm. The higher it is, the more the low quality firm benefits, without suffering from any costs.

Whenever the firm chooses a strictly positive quality level, i.e., $c_i < \tilde{c}$, its expected equilibrium profit is given by

$$\begin{aligned}\Pi^*(c_i) = & \frac{(\pi_H \cdot (1 - G(c_i)) - \pi_L \cdot G(c_i))^2}{2 \cdot c_i} \\ & - \pi_H \cdot \int_{c_i}^{\tilde{c}} \frac{\pi_H \cdot (1 - G(c_{-i})) - \pi_L \cdot G(c_{-i})}{c_{-i}} \, dG(c_{-i}) \\ & + \pi_L \cdot \int_{\underline{c}}^{c_i} \frac{\pi_H \cdot (1 - G(c_{-i})) - \pi_L \cdot G(c_{-i})}{c_{-i}} \, dG(c_{-i}).\end{aligned} \tag{2.19}$$

The first term in (2.19) represents the tradeoff due to a strictly positive quality level, ignoring the direct effects of a strictly positive quality level chosen by the rival. In particular, it reflects the benefit of a strictly positive quality level in case of becoming the high quality firm, the loss in case of becoming the low quality and the loss caused by the fixed costs. The second term represents the loss of a firm, caused by a strictly positive quality level of the rival, in case the firm becomes the high quality firm, since it decreases product differentiation. On the other hand, the third term represents the gain of a firm, caused by a strictly positive quality level of the rival, in case the firm becomes the low quality firm, since it increases product differentiation.

Given this equilibrium profits, the question arises whether firms expected profits are positive before learning about their costs. If this is not the case, adding an entry decision stage before stage 0 would lead only one firm to enter if such an equilibrium is played. Our next result shows that we do not need to worry about this:

Corollary 1. *The ex-ante expected profit of a firm, when the equilibrium as described in Proposition 2 is played, is strictly positive.*

Intuitively, a firm could always guarantee itself a non-negative expected profit by just entering and setting a quality level of 0, independent of what the other firm is doing. In the equilibrium of Proposition 2, the firms will only set a positive quality if the expected profit of doing so is strictly positive. Besides, its expected profit when choosing a quality level of 0 is also strictly positive. Hence, its total expected profit in the equilibrium of Proposition 2 is strictly positive.

Next we will show that, depending on the models primitives, namely the distribution of consumers willingness to pay for quality and the distribution of firms cost parameter, there can exist two asymmetric Maximum Differentiation Equilibria, where one firm sets a quality level of 0, while the other firm sets a quality level such that the marginal revenue of increasing its quality conditional on the other firm setting a quality level of 0 equals the marginal costs, independent of the both firms cost parameters:

Proposition 3. *There exist two asymmetric equilibria where the equilibrium qualities are given by*

$$q_i^* = \frac{\pi_H}{c_i} \qquad q_{-i}^* = 0, \tag{2.20}$$

if and only if

$$\frac{\pi_H}{\pi_H + \pi_L} \leq 2 \cdot \underline{c} \cdot \mathbb{E}\left[\frac{1}{c}\right]. \tag{2.21}$$

The expected equilibrium profits are given by

$$\Pi_i^* = \frac{1}{2} \cdot \frac{\pi_H^2}{c_i} \qquad \Pi_{-i}^* = \pi_L \cdot \pi_H \cdot \mathbb{E}\left[\frac{1}{c}\right]. \tag{2.22}$$

The equilibria in Proposition 3 correspond to the equilibria in classical vertical differentiation models with no private information about a firm's cost parameter. In the private information framework, however, these equilibria's existence crucially depends on the distributions of cost parameters and consumers' willingness to pay for quality. In particular, for the equilibria to exist, firms must have no incentive to deviate for any possible cost parameter. Obviously, the firm which sets a strictly positive quality level has no incentive to deviate. In contrast, the other firm could have an incentive to deviate if it has a cost parameter that is close or equal to or $\underline{c}$. When deviating from a quality level of 0 to a high-quality level, a firm has a high probability of becoming a high-quality firm. However, it suffers from higher costs, and the products could possibly become less differentiated. The sufficient and necessary condition for these equilibria to exist reflects this tradeoff for a firm with the highest possible efficiency, i.e., if it has a cost parameter of $\underline{c}$.

Whether the condition holds depends on three factors: First, it will hold whenever the relative benefit of becoming the high-quality firm, $\frac{\pi_H}{\pi_H+\pi_L}$, is low. Second, it holds whenever the lowest possible cost parameter $\underline{c}$ is high, since in this case, the high fixed costs in case of a deviation, even when having the lowest possible cost parameter, are too high. Finally, it depends on the skewness of the cost parameter distribution. Whenever the distribution is right-skewed, and hence the probability of a firm having small cost parameters is high, the expected quality level of the firm which sets a strictly positive quality in the equilibrium of Proposition 3 is relatively high. Therefore, when the other firm deviates to a high-quality level, the expected quality level difference between the firms' products will be low. Hence, its expected profits of a deviation will be low.

2.4 Comparative Statics

The equilibria in Propositions 2 and 3 have different properties concerning changes in the model's primitives. In what follows, we will consider changes in the distribution of the firms' cost parameter and consumers' marginal utility of quality concerning stochastic orders and the implications on equilibrium outcomes.

Denote by F_1 and F_2 two different distributions for θ, by $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ the indifferent consumers and by $\pi_{H,1}$, $\pi_{L,1}$, $\pi_{H,2}$ and $\pi_{L,2}$ the marginal revenues of an increase in the quality difference for the high and the low quality firms in the F_1 and F_2 environment, respectively. Denote by G_1 and G_2 two different distributions for firms cost parameter c. Note that F_1 and F_2 are representing the distribution of consumer preferences, while G_1 and G_2 are representing the distribution of firms' technology and that θ and c are independent of each other. To distinguish the different equilibria in Propositions 2 and 3, we will refer to the former as the Competitive Equilibrium and to the latter as the Maximum Differentiation Equilibrium. We will focus on stochastic orders in the likelihood ratio sense:

Definition 1. *Let X_1 and X_2 be two continuous random variables with CDFs W_1, W_2, densities w_1, w_2 and supports χ_1,χ_2, respectively. X_1 stochastically dominates X_2 in the likelihood ratio order, denoted by $W_1 \geq_{lr} W_2$, if for all $x, x' \in \chi_1 \cup \chi_2$ s.t. $x < x'$:*

$$\frac{w_1(x)}{w_2(x)} < \frac{w_1(x')}{w_2(x')}. \tag{2.23}$$

The likelihood ratio order between two random variables has, among others[10], three interesting properties: First, when both variables have the same support, the density of X_2 starts strictly above X_1. Second, the density of X_1 will be strictly above X_2 at the upper boundary of the support. Third, the densities will cross exactly once. The first two properties imply that X_1 is more likely to take on large values than X_2, while the reverse holds for small values. The third property will give us sufficient structure to make comparative statics for changes in densities, which are not restricted to a particular type of distribution.[11]

Let us first consider a change in the distribution of θ. By $q_1(c)$ and $q_2(c)$ we denote the equilibrium quality functions in the Competitive equilibrium for the F_1 and F_2 environment, respectively. By $\tilde{c}_1$ and $\tilde{c}_2$ we denote the threshold cost parameters above which the firms will choose a quality level of 0, in the F_1 and F_2 environment, respectively. By $q_1^M(c)$ and $q_2^M(c)$, we denote the quality functions of the high quality firm in the Maximum Differentiation Equilibrium for the F_1 and F_2 environment, respectively.

We find the following:

Proposition 4. *If $F_1 \geq_{lr} F_2$, while holding the support constant, the following holds:*

i) $\hat{\theta}_1^* > \hat{\theta}_2^*$.

ii) *If $\pi_{H,1} > \pi_{H,2}$ and $F_1(\hat{\theta}_1^*) \leq F_2(\hat{\theta}_2^*) \Rightarrow q_1(c) \geq q_2(c)$ for all c and $\tilde{c}_1 \geq \tilde{c}_2$.*

iii) *If $\pi_{H,1} > \pi_{H,2}$ and $F_1(\hat{\theta}_1^*) > F_2(\hat{\theta}_2^*) \Rightarrow \tilde{c}_1 \leq \tilde{c}_2$. There exists c' such that for all $c \geq c'$ it holds that $q_1(c) \leq q_2(c)$ and for all $c < c'$ it holds that $q_1(c) > q_2(c)$.*

iv) *If $\pi_{H,1} \leq \pi_{H,2} \Rightarrow \tilde{c}_1 \geq \tilde{c}_2 \Rightarrow$ There exists c'' such that for all $c \geq c''$ it holds that $q_1(c) \geq q_2(c)$ and for all $c < c'$ it holds that $q_1(c) < q_2(c)$.*

v) *If $\pi_{H,1} > \pi_{H,2}$ and $F_1(\hat{\theta}_1^*) \leq F_2(\hat{\theta}_2^*) \Rightarrow$ Condition (2.21) in Proposition 3 becomes more restrictive and $q_1^M(c) > q_2^M(c)$ for all c. Both firms weakly increase their prices. For $F_1(\hat{\theta}_1^*) > F_2(\hat{\theta}_2^*)$, condition (2.21) in Proposition 3 becomes less restrictive.*

vi) *If $\pi_{H,1} \leq \pi_{H,2} \Rightarrow$ Condition (2.21) in Proposition 3 becomes more restrictive and $q_1^M(c) \leq q_2^M(c)$ for all c. Both firms will weakly decrease their prices.*

[10]Note that $X_1 \geq_{lr} X_2$ implies that X_1 stochastically dominates X_2 also in the first-, the hazard rate- and the reversed hazard rate order (see Theorem 1.C.1 in Shaked and Shanthikumar (2007)).

[11]There are several distributions, which can be ranked in the likelihood ratio order. For instance, consider two normally distributed random variables with the same variance but different means. The random variable with the higher mean stochastically dominates the other variable in the likelihood ratio order. Another example is given by random variables that have a Weibull distribution. Holding the shape parameter constant, the random variable with the lower scale parameter stochastically dominates the other variable in the likelihood ratio order. This also holds for truncated versions of those distributions. For other examples, see Shaked and Shanthikumar (2007).

When shifting probability mass from the lower- to the upper boundary of the distribution, the probability that a consumer has a high willingness to pay for quality increases. This will reduce (increase) the price elasticity of demand for the high (low) - quality firm. Consequently, in the pricing stage of the game, for any fixed level of qualities, the high (low) - quality firm increases (decreases) its price. This will shift the $\hat{\theta}^*$ parameter, characterizing an indifferent consumer, closer to the upper boundary since he now needs a higher willingness to pay for quality to be indifferent between the two goods. When this happens, it will have a negative (positive) impact on the high (low) - quality firm's equilibrium price.[12]

In addition, when changing from the F_1 to the F_2 environment, there will be a negative and a positive impact on both firms equilibrium demands, using similar reasoning as in the case of equilibrium prices. Therefore, the total effect on the firms' equilibrium revenue functions in the pricing stage is unclear. But in order to be able to make clear statements about how a change in the distribution affects the equilibrium quality levels, we need to figure out the impact on the firms' equilibrium revenue functions in the pricing stage, which are anticipated in the quality stage. Depending on how the equilibrium revenue of a high quality firm changes relative to the equilibrium revenue of a low quality firm will determine whether equilibrium quality levels will increase. In any case, however, the revenue of the high quality firm increases relative to the revenue of the low quality firm in absolute terms, for fixed quality levels.[13]

Statement **ii)** in Proposition 4 describes a situation where the equilibrium revenue and the demand of the high-quality firm increases when changing from the F_2 to the F_1 environment.[14] In this case, the benefit of becoming a high-quality firm increases both in absolute and relative terms compared to the low-quality firm. Consequently, a firm will set a weakly higher quality level for all cost parameters and choose strictly positive quality levels for a broader range of cost parameters. Note that since this will impact both firms in the same way, it will, on expectation, yield a higher average quality level, while the expected difference in qualities does not increase. As a consequence, qualities increase, while prices do not increase on expectation. This will benefit all consumers, independent of whether they buy from the high-quality firm or the low-quality firm, and therefore consumer rent will increase.

In statement **iii)** we consider a situation where the benefit of becoming the high-quality firm only increases in absolute terms relative to the low-quality firm while decreasing relatively.[15] This will imply that the prize of becoming the low-quality firm increases in the F_1, relative to the F_2 environment, and as a consequence, the threshold parameter under F_1 will be lower. Therefore, for high cost parameters, the quality level chosen by a firm will be lower under the F_1 environment.

[12]To see this, note that the equilibrium prices of a high-quality and a low-quality firm are given by $p_H = \frac{1-F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_H - q_L)$ and $p_L = \frac{F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_H - q_L)$, respectively. When changing from the F_1 to the F_2 environment, for any θ it holds that $\frac{1-F_1(\theta)}{f_1(\theta)} > \frac{1-F_2(\theta)}{f_2(\theta)}$ and $\frac{F_1(\theta)}{f_1(\theta)} < \frac{F_1(\theta)}{f_1(\theta)}$. However, in both environments, it holds that $\frac{1-F_i(\theta)}{f_i(\theta)}$ is strictly decreasing in θ, while $\frac{F_i(\theta)}{f_i(\theta)}$ is strictly increasing. Therefore, when $\hat{\theta}_1^* > \hat{\theta}_2^*$ there will be a negative (positive) effect on the high (low) quality firm's equilibrium price.

[13]In the proof of statement **iii)** in Proposition 1 we show that $\hat{\theta}_1^* > \hat{\theta}_2^*$ is equivalent to $\pi_{H,1} - \pi_{L,1} > \pi_{H,2} - \pi_{L,2}$.

[14]We found that the conditions in statement **ii)** are fulfilled for several distributions. For instance, when considering a truncated normal distribution or a Weibull distribution with appropriate shape parameters, such that Assumptions 1 and 2 are not violated.

[15]The only distribution we found, which satisfies assumption 1 and 2, that fulfill these conditions, was given by truncated Weibull distributions with high values for the lower boundary of the truncated support.

In statement **iv)** we consider a situation where the equilibrium revenue of the high-quality firm decreases in the F_1 environment.[16]. However, since we know that the high-quality firm's revenue, relative to the low-quality firm, increases in absolute terms when changing from the F_2 to the F_1 environment, it must hold that the revenue of the low-quality firm decreases in absolute terms. Together, those effects will lead to a higher relative benefit of becoming the high quality firm, relative to the low quality firm in the F_1 environment, i.e $\frac{\pi_{H,1}}{\pi_{L,1}} > \frac{\pi_{H,2}}{\pi_{L,2}}$. A consequence of this will be that the threshold value will be higher under F_1, such that for high cost parameters, a firm will choose strictly higher quality levels under F_1, while the opposite holds for low cost parameters.

The last two statements, **v)** and **vi)** are related to the Maximum Differentiation Equilibria. Since in **v)** the revenue of a high-quality firm is higher under F_1, it will strictly increase its quality level. Due to increased product differentiation, the prices of both firms will increase. In this case, consumer rent will strictly decrease for all consumers buying from the low-quality firm. If the equilibrium demand of the high-quality firm is higher, as well, the benefit of being the high-quality firm relative to the low-quality firm is higher in absolute and relative terms. This makes the condition for the existence of this equilibrium more restrictive. The firm that sets a quality level of 0 has a stronger incentive to deviate from this equilibrium under F_1, whenever it has a very low cost parameter. The opposite holds when its equilibrium demand is lower.

Finally, case **vi)** considers a situation where the revenue of a high-quality firm decreases in the F_1 environment. In this case, the high-quality firm will weakly decrease its quality level for any cost parameter, and as a consequence, the products become more homogenous, and hence prices decrease. The consumer rent of all individuals who buy from the low-quality firm will weakly increase. As already shown in **iv)**, also, in this case, the relative benefit of becoming the high-quality firm relative to the low-quality firm is higher in the F_1 environment. Therefore, the condition for the existence of this equilibrium is more restrictive under F_1.

Next, we will consider changes in the distribution of c. By $q_1(c)$ and $q_2(c)$ we denote the equilibrium quality functions in the Competitive equilibrium for the G_1 and G_2 environment, respectively. By $\tilde{c}_1$ and $\tilde{c}_2$ we denote the threshold cost parameters above which the firms will choose a quality level of 0, in the G_1 and G_2 environment, respectively.

Proposition 5. *If $G_1 \geq_{lr} G_2$ while holding the support constant, the following holds:*

i') *$q_1(c) \geq q_2(c)$ for all c.*

ii') *$\tilde{c}_1 \geq \tilde{c}_2$.*

iii') *There exist $\{c^*, c'\} \in [\underline{c}, \overline{c}]$ such that whenever $\max\{c_A, c_B\} < c^*$, the Competitive Equilibrium prices of both firms under G_1 are strictly lower, while the opposite holds when both cost parameters are above c'.*

vi') *Condition (2.21) in Proposition 3 becomes weaker.*

Since for a given cost parameter, the probability of the other firm having higher costs, and hence choosing a lower quality increment level in equilibrium, is higher under G_1 than under

[16] While we could not find any distribution which fulfills this condition, we provide it for a complete analysis.

G_2, both firms will increase their quality levels for given cost parameters in equilibrium and also choose a strictly positive quality level for a larger interval of cost parameters.

Statement **iii')** tells us that whenever both firms' cost parameters are below some threshold, the firm with the higher cost parameter will increase its quality more than the other firm when switching from the G_2 to the G_1 environment. This leads to a reduction in product differentiation, which ultimately reduces the firms' prices in equilibrium. Since both quality levels are increasing at the same time, the consumer rent increases for all consumers, independent of their marginal utility of quality. For the last result, consider that in the G_1 environment, there is more probability mass on higher values of the cost parameter. Hence whenever the firm which sets a quality level of 0 has a cost parameter of $\underline{c}$ and decides to deviate to $\frac{\pi_H}{\underline{c}}$, it is less concerned with a reduction of product differentiation due to high-quality levels chosen by its rival and hence the condition for the existence of the Maximum Differentiation Equilibria becomes weaker and is more likely to be fulfilled.

2.5 Conclusion

This paper revisited a classical vertical differentiation model and assumed that firms are uncertain about their rival's fixed costs of quality improvements. With this assumption, the quality competition stage becomes an all-pay contest that can be easily analyzed and is highly tractable. We established the existence of an equilibrium, which deviates from typical theoretical predictions of maximum product differentiation. In this equilibrium, firms set similar quality levels whenever they have similar cost-efficiency parameters, resulting in lower profits due to lack of product differentiation. We derived a threshold, depending on the model's primitives, which determines whether a firm will innovate at all. Furthermore, we identified conditions for the existence of a second type of equilibrium, where firms maximally differentiate their products. This equilibrium is in line with the results of classical models of vertical differentiation. Finally, we conducted comparative statics in a quite general way, using the concept of stochastic orders, and showed that the two identified types of equilibria react differently due to changes in the model's underlying primitives.

An interesting extension of this model could be given by given by assuming that firms' cost parameters are correlated. This would adjust a firm's belief about the probability of the other firm having a higher (lower) cost parameter whenever it has a low (high) cost parameter, potentially leading to lower quality levels provided by the firms. Furthermore, since the model is tractable, one could consider a dynamic version, where the four stages of the game are repeated in each period. The state variable in such a model could be given by the firms' aggregated quality difference in the previous period, which determines which firm has an advantage in the form of a head-start at the beginning of each period. Identifying Markov Equilibria of this model could lead to interesting insights into how R&D competition evolves.

2.6 Appendix: Proofs

Proposition 1. Taking the derivates of R_A and R_B with respect to p_A and p_B and rearranging the first order conditions yields the following system of implicit equations:

$$p_A^* = \frac{1 - F\left(\frac{p_A^* - p_B^*}{q_A - q_B}\right)}{f\left(\frac{p_A^* - p_B^*}{q_A - q_B}\right)} \cdot (q_A - q_B) \tag{2.24}$$

$$p_B^* = \frac{F\left(\frac{p_A^* - p_B^*}{q_A - q_B}\right)}{f\left(\frac{p_A^* - p_B^*}{q_A - q_B}\right)} \cdot (q_A - q_B) \tag{2.25}$$

Due to the quasi concavity of the revenue functions, the first-order conditions become sufficient and describe a maximum (see Caplin and Nalebuff (1991)). Next, note that the indifferent consumer was characterized by

$$\hat{\theta} = \frac{p_A - p_B}{q_A - q_B} \tag{2.26}$$

Plugging in p_A^* and p_B^* we receive the following implicit equation:

$$\hat{\theta}^* = \frac{1 - 2 \cdot F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \tag{2.27}$$

which describes the indifferent consumer in equilibrium. Note that $\hat{\theta}^*$ is fully determined by the distribution of consumers willingness to pay for quality and does not depend on the quality levels, chosen by the firms. Plugging in $\hat{\theta}^*$ for $\frac{p_A^* - p_B^*}{q_A - q_B}$ into the equilibrium prices finally yields

$$p_A^*(q_A, q_B) = \frac{1 - F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_A - q_B) \tag{2.28}$$

$$p_B^*(q_A, q_B) = \frac{F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_A - q_B) \tag{2.29}$$

Next, we need to verify that both firms are active on the market, i.e., we need to show that $\hat{\theta}^* \in (\underline{\theta}, \overline{\theta})$. Consider the function

$$m(\theta) = \theta - \frac{1 - 2 \cdot F(\theta)}{f(\theta)} \tag{2.30}$$

and note that we can rewrite the function to get

$$m(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)} + \frac{1}{\frac{\partial}{\partial \theta} \ln\left(F(\theta)\right)} \tag{2.31}$$

Note that due to log concavity of $f(\theta)$, $m(\theta)$ is strictly increasing. This follows due to the fact that log concavity of $f(\theta)$ implies a monotone hazard rate which in turn implies regularity and hence $\theta - \frac{1-F(\theta)}{f(\theta)}$ is weakly increasing. In addition, log concavity of $f(\theta)$ implies log concavity of $F(\theta)$ such that $\frac{1}{\frac{\partial}{\partial\theta}\ln(F(\theta))}$ is strictly increasing in θ.[17] Next, consider that

$$m(\overline{\theta}) = \overline{\theta} + \frac{1}{f(\overline{\theta})} > 0 \tag{2.32}$$

$$m(\underline{\theta}) = \underline{\theta} - \frac{1}{f(\underline{\theta})} < 0 \tag{2.33}$$

where the second inequality follows due to to Assumption 2. Since $m(\theta)$ is strictly increasing, continuous, and has a sign change within $\underline{\theta}$ and $\overline{\theta}$, we know that a zero of the function exists and is given by $\hat{\theta}^* \in (\underline{\theta}, \overline{\theta})$. This shows that both firms are active on the market, since both firms have strictly positive demand. □

Proposition 2. First, we will derive a symmetric candidate equilibrium. Next, we will show that no firm has an incentive to deviate.

Differentiating Π_A with respect to q_A and considering that due to the Leibniz rule some of the terms drop out yields:

$$\frac{\partial \Pi_A}{\partial q_A} = \pi_H \cdot (G(\bar{c}) - G(\tilde{c})) + \pi_H \cdot (G(\tilde{c}) - G(q_B^{-1}(q_A)) - \pi_L \cdot (G(q_B^{-1}(q_A)) - G(\underline{c})) - c_A \cdot q_A \overset{!}{=} 0$$

Due to symmetry, it follows that in equilibrium $q_B^*(c) = q_A^*(c) = q^*(c)$ and hence $q_B^{-1}(q_A(c_A)) = c$, such that we have:

$$q^*(c) = \frac{\pi_H \cdot (1 - G(c)) - \pi_L \cdot G(c)}{c} \tag{2.34}$$

Note that $q^*(c)$ is decreasing in c since the numerator is decreasing whereas the denominator is increasing in c, and will become 0 if

$$\pi_H \cdot (1 - G(c)) - \pi_L \cdot G(c) \leq 0 \Leftrightarrow c \geq G^{-1}\left(\frac{\pi_H}{\pi_H + \pi_L}\right) := \tilde{c} \tag{2.35}$$

Next, consider the second-order condition which is given by

$$-(\pi_H + \pi_L) \cdot g(q_B^{-1}(q_A)) \cdot \frac{\partial q_B^{-1}(q_A)}{\partial q_A} - c_A < 0 \tag{2.36}$$

Due to symmetry it follows that

$$\frac{\partial q_B^{-1}(q_A)}{\partial q_A} = \frac{1}{q_A^{*'}(c_A)} = -\frac{c_A^2}{(\pi_H + \pi_L) \cdot g(c_A) \cdot c_A + \pi_H - (\pi_H + \pi_L) \cdot G(c_A)} \tag{2.37}$$

Plugin (2.37) into (2.36) and noting that in in equilibrium it holds that $g(q_B^{-1}(q_A)) = g(c_A)$ yields:

$$\frac{(\pi_H + \pi_L) \cdot g(c_A) \cdot c_A^2}{(\pi_H + \pi_L) \cdot g(c_A) \cdot c_A + \pi_H - (\pi_H + \pi_L) \cdot G(c_A)} - c_A < 0 \tag{2.38}$$

[17] It is strictly increasing, since we assume the density to be strictly positive

which is equivalent to

$$c_A \leq G^{-1}\left(\frac{\pi_H}{\pi_H + \pi_L}\right) \tag{2.39}$$

Therefore, for a firm with $c_A < \tilde{c}$ it follows that $q_A^*(c_A)$ is a local maximizer of the profit function. Now we need to show that this candidate equilibrium is indeed an equilibrium. Therefore, we need to show that for any cost parameter a firm has, it has no incentive to deviate from the candidate equilibrium, given its rival sticks to it. We will distinguish the two cases where a firm sets a quality level of 0 and the case where it sets a strictly positive quality level.
For the case where firm i sets a quality level of 0, i.e., if $c_i \geq \tilde{c}$, it can deviate in two ways. First, it could set a strictly positive quality level such that there is still a possibility that it becomes the low quality firm, i.e., $q_i \in (0, \frac{\pi_H}{\underline{c}})$. Denote the deviation of i by $\hat{q}_i$ and define $\hat{c}$ by the following equation:

$$\hat{q}_i = \frac{\pi_H - (\pi_H + \pi_L) \cdot G(\hat{c})}{\hat{c}} \tag{2.40}$$

Therefore, for any $\hat{q}_i \in (0, \frac{\pi_H}{\underline{c}})$, $\hat{c}$ describes the cost value at which the rival firm, which sticks to the equilibrium, sets a quality level equal to $\hat{q}_i$.[18] Note that by construction it follows that $\hat{c} < \tilde{c}$. The expected profit of firm i when having costs $c_i \geq \tilde{c}$ and setting $\hat{q}_i \in (0, \frac{\pi_H}{\underline{c}})$ is given by:

$$\begin{aligned}\hat{\Pi}_i(c_i) = {} & \pi_H \cdot \int_{\tilde{c}}^{\bar{c}} \hat{q}_i \, dG(c_{-i}) + \pi_H \cdot \int_{\hat{c}}^{\tilde{c}} \hat{q}_i - \frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}} \, dG(c_{-i}) \\ & + \pi_L \cdot \int_{\underline{c}}^{\hat{c}} \frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}} - \hat{q}_i \, dG(c_{-i}) - \frac{1}{2} \cdot c_i \cdot \hat{q}_i^2\end{aligned} \tag{2.41}$$

Note that the marginal expected profit with respect to $\hat{q}_i$ is strictly negative since:

$$\frac{\partial \hat{\Pi}_i(c_i)}{\partial \hat{q}_i} = \pi_H - (\pi_H + \pi_L) \cdot G(\hat{c}) - c_i \hat{q}_i < \pi_H - (\pi_H + \pi_L) \cdot G(\hat{c}) - \hat{c} \cdot \hat{q}_i = 0 \tag{2.42}$$

for any $c_i \geq \tilde{c}$ and $\hat{q}_i > 0$. Therefore, any firm with $c_i \geq \tilde{c}$ has no incentive to deviate to a quality level $\hat{q}_i \in (0, \frac{\pi_H}{\underline{c}})$.
As the second deviation possibility we consider a corner solution, which guarantees that the firm will become the high quality firm, i.e., the case where it sets $\hat{q}_i = \frac{\pi_H}{\underline{c}}$. In this case, its profit is given by

$$\hat{\Pi}_i\left(\frac{\pi_H}{\underline{c}}, c_i\right) = \frac{\pi_H^2}{\underline{c}} - \pi_H \cdot \int_{\underline{c}}^{\tilde{c}} \frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}} \, dG(c_{-i}) - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 \tag{2.43}$$

and we need to show that

$$\hat{\Pi}_i\left(\frac{\pi_H^2}{\underline{c}}, c_i\right) \leq \pi_L \cdot \int_{\underline{c}}^{\tilde{c}} \frac{\pi_H - \pi_H + \pi_L \cdot G(c_{-i})}{c_{-i}} \, dG(c_{-i}) \tag{2.44}$$

[18] Existence and uniqueness of $\hat{c}$ is guaranteed since we considered $\hat{q}_i \in (0, \frac{\pi_H}{\underline{c}})$ and since $\frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}}$ is strictly decreasing in c_{-i}.

Plugging in the right hand side of (2.43) for $\hat{\Pi}_i\left(\frac{\pi_H}{\underline{c}}, c_i\right)$ in (2.44) and rearranging yields:

$$\frac{\pi_H^2}{\underline{c}} - (\pi_H + \pi_L) \cdot \int_{\underline{c}}^{\tilde{c}} \frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}} \, dG(c_{-i}) - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 \leq 0 \tag{2.45}$$

Using integration by parts on the middle term in (2.45), we have

$$\frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - \underbrace{(\pi_H + \pi_L) \cdot \Big|_{\underline{c}}^{\tilde{c}} \frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}} G(c_{-i}) \, dc_{-i}}_{=0}$$

$$+ (\pi_H + \pi_L) \cdot \int_{\underline{c}}^{\tilde{c}} \frac{\partial \frac{\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}}}{\partial c_{-i}} G(c_{-i}) \, dc_{-i}$$

$$= \frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - (\pi_H + \pi_L) \cdot \int_{\underline{c}}^{\tilde{c}} \frac{(\pi_H + \pi_L) \cdot g(c_{-i}) c_{-i} + \pi_H - (\pi_H + \pi_L) \cdot G(c_{-i})}{c_{-i}^2} G(c_{-i}) \, dc_{-i}$$

$$< \frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - (\pi_H + \pi_L) \cdot \int_{\underline{c}}^{\tilde{c}} \frac{(\pi_H + \pi_L) \cdot g(c_{-i}) c_{-i} G(c_{-i})}{c_{-i}^2} \, dc_{-i}$$

$$= \frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - (\pi_H + \pi_L)^2 \cdot \int_{\underline{c}}^{\tilde{c}} \frac{g(c_{-i}) G(c_{-i})}{c_{-i}} \, dc_{-i} \tag{2.46}$$

Where the inequality holds since $\pi_H - (\pi_H + \pi_L) \cdot G(c_{-i}) \geq 0$ for $c_{-i} \leq \tilde{c}$. Noting that $g(c_{-i}) \cdot G(c_{-i}) = \frac{\partial \frac{G(c_{-i})^2}{2}}{\partial c_{-i}}$ and using integration by parts on the last term in (2.46) we have

$$\frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - (\pi_H + \pi_L)^2 \cdot \left(\Big|_{\underline{c}}^{\tilde{c}} \frac{G(c_{-i})^2}{2 c_{-i}} + \int_{\underline{c}}^{\tilde{c}} \frac{G(c_i)^2}{2 \cdot c_{-i}^2} \, dc_{-i} \right)$$

$$= \frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot c_i \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - (\pi_H + \pi_L)^2 \cdot \frac{G(\tilde{c})^2}{2\tilde{c}} - (\pi_H + \pi_L)^2 \cdot \int_{\underline{c}}^{\tilde{c}} \frac{G(c_i)^2}{2 \cdot c_{-i}^2} \, dc_{-i}$$

$$\leq \frac{\pi_H^2}{\underline{c}} - \frac{1}{2} \cdot \tilde{c} \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 - (\pi_H + \pi_L)^2 \cdot \frac{G(\tilde{c})^2}{2\tilde{c}} - (\pi_H + \pi_L)^2 \cdot \int_{\underline{c}}^{\tilde{c}} \frac{G(c_i)^2}{2 \cdot c_{-i}^2} \, dc_{-i} \tag{2.47}$$

where the last inequality follows due to the fact that we considered the deviation of a firm with $c_i \geq \tilde{c}$. Noting that $G(\tilde{c})^2 = \left(\frac{\pi_H}{\pi_H + \pi_L}\right)^2$ and simplifying the first three terms in (2.47) finally yields

$$\frac{\pi_H^2 \cdot 2 \cdot \tilde{c} \cdot \underline{c} - \tilde{c}^2 \cdot \pi_H^2 - \underline{c}^2 \cdot \pi_H^2}{2 \cdot \underline{c}^2 \cdot \tilde{c}} - (\pi_H + \pi_L)^2 \cdot \int_{\underline{c}}^{\tilde{c}} \frac{G(c_i)^2}{2 \cdot c_{-i}^2} \, dc_{-i} \tag{2.48}$$

$$= -\frac{\pi_H^2}{2 \cdot \underline{c}^2 \cdot \tilde{c}} \cdot (\tilde{c} - \underline{c})^2 - (\pi_H + \pi_L)^2 \cdot \int_{\underline{c}}^{\tilde{c}} \frac{G(c_i)^2}{2 \cdot c_{-i}^2} \, dc_{-i} < 0 \tag{2.49}$$

Therefore, inequality (2.44) holds and no firm with $c_i \geq \tilde{c}$ has an incentive to deviate from the

optimal quality level as described in proposition 2.
Next, let us consider the case where a firm has a cost parameter $c_i < \tilde{c}$, i.e., when it sets a strictly positive quality level. We showed before, that the second order condition is fulfilled for $q^*(c_i)$ whenever $c_i \in [\underline{c}, \tilde{c})$. Hence, a firm with cost parameter $c_i < \tilde{c}$ has no incentive to deviate. □

Proposition 3. Obviously, the firm that sets the strictly positive quality level has no incentive to deviate whenever the other firm sets a quality level of 0. To see this, note that the profit function of firm i, in case that $q^*_{-i} = 0$, is given by

$$\pi_H \cdot q_i - \frac{1}{2} \cdot c_i \cdot q_i^2 \tag{2.50}$$

which is maximized for $q_i^* = \frac{\pi_H}{c_i}$.

For firm $-i$, there might be a profitable deviation whenever it has very low cost parameters, in particular when its cost parameter is close or equal to $\underline{c}$. The optimization problem for finding the best profitable deviation is given by

$$\max_{\hat{q}_{-i}} \ \hat{\Pi}_{-i}\left(\hat{q}_{-i}, c_{-i}\right) = \pi_H \cdot \int_{\frac{\pi_H}{\hat{q}_{-i}}}^{\bar{c}} \left(\hat{q}_{-i} - \frac{\pi_H}{c_i}\right) dG(c_i) + \pi_L \cdot \int_{\underline{c}}^{\frac{\pi_H}{\hat{q}_{-i}}} \left(\frac{\pi_H}{c_i} - \hat{q}_{-i}\right) dG(c_i) - \frac{1}{2} \cdot c_{-i} \cdot \hat{q}_{-i}^2$$

Taking the derivative of $\hat{\Pi}_{-i}$ with respect to $\hat{q}_{-i}$ and deriving the first-order condition yields:

$$\pi_H - c_{-i} \cdot \hat{q}_{-i}(c_{-i}) - (\pi_H + \pi_L) \cdot G\left(\frac{\pi_H}{\hat{q}_{-i}(c_{-i})}\right) \stackrel{!}{=} 0 \tag{2.51}$$

which is equivalent to

$$\hat{q}_{-i}(c_{-i}) \stackrel{!}{=} \frac{\pi_H - (\pi_H + \pi_L) \cdot G\left(\frac{\pi_H}{\hat{q}_{-i}(c_{-i})}\right)}{c_{-i}} \tag{2.52}$$

In what follows, we will first derive a condition such that a firm with the lowest possible cost parameter $\underline{c}$ has an incentive to deviate. We will then proceed to show that if a firm with the lowest possible cost parameter has an incentive to deviate, any firm with a cost parameter arbitrarily close to $\underline{c}$ has an incentive to deviate as well. As a consequence, from the perspective of the other firm, deviation happens with a strictly positive probability. Plugging in $\underline{c}$ for c_{-i} in (2.52) yields:

$$\hat{q}_{-i}(\underline{c}) \stackrel{!}{=} \frac{\pi_H - (\pi_H + \pi_L) \cdot G\left(\frac{\pi_H}{\hat{q}_{-i}(\underline{c})}\right)}{\underline{c}} \tag{2.53}$$

which is fulfilled only if $\hat{q}_{-i}(\underline{c}) = \frac{\pi_H}{\underline{c}}$. Setting a higher quality level higher than $\frac{\pi_H}{\underline{c}}$ yields a situation where the marginal costs of increasing the quality are higher than the marginal revenue of a firm while having no impact on the probability of becoming the high-quality firm, since firm $-i$ will become the high-quality firm almost surely. The first-order condition rules out any lower quality level in case that the deviation is profitable.

The expected profit of the deviation of firm $-i$ with costs $c_{-i} = \underline{c}$ when setting a quality level of $\frac{\pi_H}{\underline{c}}$ is given by:

$$\hat{\Pi}_{-i}\left(\frac{\pi_H}{\underline{c}}, \underline{c}\right) = \pi_H \cdot \int_{\underline{c}}^{\bar{c}} \left(\frac{\pi_H}{\underline{c}} - \frac{\pi_H}{c}\right) dG(c) - \frac{1}{2} \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 \cdot \underline{c} \tag{2.54}$$

The expected profit of choosing quality level 0 is given by:

$$\pi_L \cdot \int_{\underline{c}}^{\bar{c}} \frac{\pi_H}{c} \, dG(c) \tag{2.55}$$

Therefore, the deviation is profitable if and only if:

$$\pi_H \cdot \int_{\underline{c}}^{\bar{c}} \left(\frac{\pi_H}{\underline{c}} - \frac{\pi_H}{c}\right) dG(c) - \frac{1}{2} \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 \cdot \underline{c} > \pi_L \cdot \int_{\underline{c}}^{\bar{c}} \frac{\pi_H}{c} \, dG(c) \tag{2.56}$$

which is equivalent to

$$\frac{\pi_H}{\pi_H + \pi_L} > 2 \cdot \underline{c} \cdot \int_{\underline{c}}^{\bar{c}} \frac{1}{c} \, dG(c) = 2 \cdot \underline{c} \cdot \mathbb{E}\left[\frac{1}{c}\right] \tag{2.57}$$

Therefore, Firm $-i$ with cost parameter $\underline{c}$ has an incentive to deviate from $q_{-i} = 0$ to $\hat{q}_{-i}(\underline{c}) = \frac{\pi_H}{\underline{c}}$ if and only if condition (2.57) is fulfilled. Else it will choose a quality level of 0.

Finally, note that if a firm with cost parameter $\underline{c}$ has a profitable deviation, due to continuity of $\hat{\Pi}_{-i}$ in c_{-i} there exists some $\epsilon > 0$ such that a firm with a cost parameter of $c_{-i} \in (\underline{c}, \underline{c} + \epsilon)$ will also have a profitable deviation. To see this, consider a firm with cost parameter c_{-i} which mimics the optimal deviation of a firm with cost parameter $\underline{c}$ by choosing $q^*_{-i} = \frac{\pi_H(\theta^*)}{\underline{c}}$. In such a case, its expected value of deviating is given by:

$$\hat{\Pi}_{-i}\left(\frac{\pi_H}{\underline{c}}, c_{-i}\right) = \pi_H \cdot \int_{\underline{c}}^{\bar{c}} \left(\frac{\pi_H}{\underline{c}} - \frac{\pi_H}{c}\right) dG(c) - \frac{1}{2} \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 \cdot c_{-i} \tag{2.58}$$

The deviation is profitable if only if

$$\pi_H \cdot \int_{\underline{c}}^{\bar{c}} \left(\frac{\pi_H}{\underline{c}} - \frac{\pi_H}{c}\right) dG(c) - \frac{1}{2} \cdot \left(\frac{\pi_H}{\underline{c}}\right)^2 \cdot c_{-i} > \pi_L \cdot \int_{\underline{c}}^{\bar{c}} \frac{\pi_H}{c} \, dG(c) \tag{2.59}$$

But note that since the left hand side in (2.59) is continuous in c_{-i} we conclude that if (2.56) is fulfilled, we can find c_{-i} sufficiently close to $\underline{c}$ such that (2.59) is fulfilled.

As a consequence, when condition (2.57) is violated, a firm with the lowest possible cost parameter does not have an incentive to deviate from the Maximum Differentiation Equilibrium, which implies that for any cost parameter a firm does not have an incentive to deviate. □

Proposition 4. We will prove each of the statements separately:

i) For each distribution, $\hat{\theta}_e^*$ is the zero of:

$$m_e(\theta) = \theta - \frac{1 - F_e(\theta)}{f_e(\theta)} + \frac{F_e(\theta)}{f_e(\theta)}, \tag{2.60}$$

where $e \in \{1, 2\}$. Existence of the zero is guaranteed by Assumptions 1 and 2. Since F_1 stochastically dominates F_2 in the likelihood ratio order, F_1 also dominates F_2 in the hazard rate and the reverse hazard rate order (see Theorem 1.C.1 in Shaked and Shanthikumar (2007)) which imply that:

$$\frac{1 - F_1(\theta)}{f_1(\theta)} > \frac{1 - F_2(\theta)}{f_2(\theta)} \quad \text{and} \quad \frac{F_1(\theta)}{f_1(\theta)} < \frac{F_2(\theta)}{f_2(\theta)} \text{ for all } \theta \in [\underline{\theta}, \overline{\theta}] \tag{2.61}$$

Therefore, it holds that $m_1(\theta) < m_2(\theta)$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Since both functions are strictly increasing due to log-concavity of the underlying distributions, the zero of $m_1(\theta)$ must be closer to the upper boundary of the support compared to the zero of $m_2(\theta)$. Therefore, we find that $\hat{\theta}_1^* > \hat{\theta}_2^*$.

ii) We start by showing that the threshold cost parameter is higher under F_1. We will use the following definitions throughout this proof, for each of the two environments $e \in \{1, 2\}$:

$$\pi_{H,e} := \frac{(1 - F_e(\hat{\theta}_e^*))^2}{f_e(\hat{\theta}_e^*)} \qquad \pi_{L,e} := \frac{F_e(\hat{\theta}_e^*)^2}{f_e(\hat{\theta}_e^*)} \qquad q_e(c) = \frac{\pi_{H,e} - G(c) \cdot (\pi_{H,e} + \pi_{L,e})}{c}$$

The threshold cost parameter for each of the environments will be defined by

$$\tilde{c}_e := G^{-1}\left(\frac{\pi_{H,e}}{\pi_{H,e} + \pi_{L,e}}\right) \tag{2.62}$$

We need to show that $\tilde{c}_1 > \tilde{c}_2$ which is equivalent to showing that

$$G^{-1}\left(\frac{\pi_{H,1}}{\pi_{H,1} + \pi_{L,1}}\right) > G^{-1}\left(\frac{\pi_{H,2}}{\pi_{H,2} + \pi_{L,2}}\right) \Leftrightarrow \pi_{H,1} \cdot \pi_{L,2} > \pi_{H,2} \cdot \pi_{L,1} \Leftrightarrow F_1(\hat{\theta}_1^*) < F_2(\hat{\theta}_2^*)$$

therefore, we find that $\tilde{c}_1 > \tilde{c}_2$ iff $F_1(\hat{\theta}_1^*) < F_2(\hat{\theta}_2^*)$. As a consequence, under environment 1, a firm will set a strictly positive quality level for a wider range of cost parameters whenever $F_1(\hat{\theta}_1^*) < F_2(\hat{\theta}_2^*)$, i.e., it holds that $q_1(\tilde{c}_2) > q_2(\tilde{c}_2) = 0$.

We now show that the quality level under environment 1 is weakly higher for all cost parameters. Obviously, whenever the cost parameter of a firm is weakly higher than $\tilde{c}_1$, it will set a quality level of 0 in both environments. For a cost parameter between $\tilde{c}_2$ and $\tilde{c}_1$, it will choose a strictly positive quality level in environment 1, while choosing a quality level of 0 in environment 2. For a cost parameter equal to $\tilde{c}_2$, we showed that $q_1(\tilde{c}_2) > q_2(\tilde{c}_2)$. Finally, a firm will choose a strictly higher quality levels for cost parameter c between $\underline{c}$ and $\tilde{c}_2$ iff:

$$q_1(c) > q_2(c) \tag{2.63}$$

which is equivalent to

$$(1 - G(c)) \cdot (\pi_{H,1} - \pi_{H,2}) > G(c) \cdot (\pi_{L,1} - \pi_{L,2}) \tag{2.64}$$

We know that the LHS of (2.64) is strictly positive since $\pi_{H,1} > \pi_{H,2}$. Therefore, if the RHS of (2.64) is weakly negative, the inequality would be fulfilled. Suppose the RHS is strictly positive. We already know that the inequality holds for a cost parameter equal to $\tilde{c}_2$, i.e.,

$$(1 - G(\tilde{c}_2)) \cdot (\pi_{H,1} - \pi_{H,2}) > G(\tilde{c}_2) \cdot (\pi_{L,1} - \pi_{L,2}) \tag{2.65}$$

Such that for any $c < \tilde{c}_2$ the following holds:

$$(1 - G(c)) \cdot (\pi_{H,1} - \pi_{H,2}) > (1 - G(\tilde{c}_2)) \cdot (\pi_{H,1} - \pi_{H,2}) \tag{2.66}$$

$$> G(\tilde{c}_2) \cdot (\pi_{L,1} - \pi_{L,2}) > G(c) \cdot (\pi_{L,1} - \pi_{L,2}) \tag{2.67}$$

which completes the proof.

iii) Since $F_1(\hat{\theta}_1^*) \geq F_2(\hat{\theta}_2^*)$, it follows that $\tilde{c}_1 \leq \tilde{c}_2$. Therefore, for cost parameters between $\tilde{c}_1$ and $\tilde{c}_2$, a firm will set a strictly positive quality level in environment 2, while choosing a quality level of 0 in environment 1. However, consider that $\hat{\theta}_1^* > \hat{\theta}_2^*$ and that we can rewrite $\hat{\theta}_e^*$ for both environments $e \in \{1, 2\}$ in the following way:

$$\hat{\theta}_e^* = \frac{1 - 2 \cdot F_e(\hat{\theta}_e^*)}{f_e(\hat{\theta}_e^*)} = \frac{(1 - F_e(\hat{\theta}_e^*))^2}{f_i(\hat{\theta}_e^*)} - \frac{F_e(\hat{\theta}_e^*)^2}{f_e(\hat{\theta}_e^*)} = \pi_{H,e} - \pi_{L,e}$$

Therefore, it follows that

$$\pi_{H,1} - \pi_{L,1} > \pi_{H,2} - \pi_{L,2} \tag{2.68}$$

Next, define the median of the cost distribution by c^{med}, such that $G(c^{med}) = 1 - G(c^{med}) = \frac{1}{2}$. Dividing both sides by c^{med} and multiplying by $\frac{1}{2}$ yields:

$$\frac{(1 - G(c^{med})) \cdot \pi_{H,1} - G(c^{med}) \cdot \pi_{L,1}}{c^{med}} > \frac{(1 - G(c^{med})) \cdot \pi_{H,2} - G(c^{med}) \cdot \pi_{L,2}}{c^{med}} \tag{2.69}$$

which implies that given a cost parameter equal to the median, a firm will always choose a strictly higher quality level in environment 1. Since in both environments the equilibrium quality levels are continuous and decreasing in a firms cost parameter, and since $q_2(\tilde{c}_1) > q_1(\tilde{c}_1) = 0$, there will exist some $c' \in (c^{med}, \tilde{c}_1)$ such that $q_1(c) > q_2(c)\ \forall c \in [c^{med}, c')$.[19] Since we know that such c' exists and since $\pi_{H,1} > \pi_{H,2}$, using the same reasoning as in **ii)**, we conclude that $q_1(c) > q_2(c)\ \forall c \in [\underline{c}, c')$.

iv) Note that since $\pi_{H,2} > \pi_{H,1}$ and since we know from the proofs of **i)** and **iii)** that $\pi_{H,1} - \pi_{L,1} > \pi_{H,2} - \pi_{L,2}$, it must follow that $\pi_{L,2} > \pi_{L,1}$. Therefore, it holds that: $\pi_{L,2} > \pi_{L,1} + \pi_{H,2} - \pi_{H,1}$. As a consequence, it must hold that $\pi_{H,1} \cdot \pi_{L,2} > \pi_{H,2} \cdot \pi_{L,1}$, which is, as shown in the proof of **ii)** equivalent to $\tilde{c}_1 > \tilde{c}_2$. Next, consider that $\pi_{H,2} > \pi_{H,1}$ implies that the quality level of a firm

[19]Note that $c^{med} < \tilde{c}_1$ since $G(\tilde{c}_1) = \frac{\pi_{H,1}}{\pi_{H,1} + \pi_{L,1}} > \frac{1}{2}$ due to $\pi_{H,1} > \pi_{L,1}$.

for very low cost parameters is higher in environment 2. However, since $\tilde{c}_1 > \tilde{c}_2$, and since the equilibrium quality level is strictly decreasing and continuous in the cost parameter, it follows that $q_1(c)$ and $q_2(c)$ intersect at some cost parameter c'' which is explicitly defined by:

$$q_1(c'') = q_2(c'') \Leftrightarrow \frac{\pi_{H,1} - G(c'') \cdot (\pi_{H,1} + \pi_{L,1})}{c''} = \frac{\pi_{H,2} - G(c'') \cdot (\pi_{H,2} + \pi_{L,2})}{c''}$$
$$\Leftrightarrow c'' = G^{-1}\left(\frac{\pi_{H,2} - \pi_{H,1}}{(\pi_{H,2} + \pi_{L,2}) - (\pi_{H,1} + \pi_{L,1})}\right)$$

where $\frac{\pi_{H,2}-\pi_{H,1}}{(\pi_{H,2}+\pi_{L,2})-(\pi_{H,1}+\pi_{L,1})} < 1$ since $\pi_{L,2} > \pi_{L,1}$. Therefore, for any cost parameter below c'', it follows that $q_2(c) > q_1(c)$, while the opposite holds above c''.

v) As shown before, $F_1(\hat{\theta}_1^*) < F_2(\hat{\theta}_2^*)$ implies $\frac{\pi_{H,1}}{\pi_{L,1}} \geq \frac{\pi_{H,2}}{\pi_{L,2}}$ such that the LHS of the condition for existence of the Maximum Differentiation Equilibrium in (2.21) will be strictly higher under F_1, such that the condition becomes more restrictive. Obviously, the opposite holds when $F_1(\hat{\theta}_1^*) \geq F_2(\hat{\theta}_2^*)$. The quality level of the high-quality firm strictly increases simply due to the fact that $\pi_{H,1} > \pi_{H,2}$. Therefore, both firms increase their prices due to an increase in product differentiation.

vi) As shown in **iv)**, $\pi_{H,1} \leq \pi_{H,2}$, while $\pi_{H,1} - \pi_{L,1} > \pi_{H,2} - \pi_{L,2}$ implies that $\frac{\pi_{H,1}}{\pi_{L,1}} \geq \frac{\pi_{H,2}}{\pi_{L,2}}$ which, as stated in **v)**, increases the LHS of the condition for existence of the Maximum Differentiation Equilibrium in (2.21). The quality of the high-quality firm weakly decreases since $\pi_{H,1} \leq \pi_{H,2}$ and hence both firms decrease their prices due to a reduction in product differentiation. □

Proposition 5. We will proof each of the statements separately:

i') Since $G_1 \geq_{lr} G_2$ it follows that G_1 first order stochastically dominates G_2, in particular $G_1(c) \leq G_2(c)$ for all $c \in [\underline{c}, \overline{c}]$ and hence for any cost parameter it holds that

$$q^1(c) = \frac{\pi_H - (\pi_H + \pi_L) \cdot G_1(c)}{c} \geq \frac{\pi_H - (\pi_H + \pi_L) \cdot G_2(c)}{c} = q^2(c) \tag{2.70}$$

ii') This follows simply from the fact that whenever $G_1 \geq_{lr} G_2$ it follows that $G_1^{-1}(x) \geq G_2^{-1}(x)$ for all $x \in (0,1)$ (see Theorem 1.C.1 and statement 1.A.12 in Shaked and Shanthikumar (2007)). Therefore

$$\tilde{c}_1 = G_1^{-1}\left(\frac{\pi_H}{\pi_H + \pi_L}\right) \geq G_2^{-1}\left(\frac{\pi_H}{\pi_H + \pi_L}\right) = \tilde{c}_2 \tag{2.71}$$

iii') In **i')** we showed that the qualities of both firms are increasing. The prices of both firms will decrease whenever the difference between the quality levels decreases. Hence, when fixing two cost parameters $c_A < c_B$ such that firm A will be the high quality firm in equilibrium, it must hold that when switching from the G_2 to the G_1 environment, firm B must increase its quality more relative to firm A. To show this, consider the difference of chosen quality levels in the G_1 compared to the G_2 environment of a firm with cost parameter c:

$$\Delta(c) = q^1(c) - q^2(c) = \frac{\pi_H - (\pi_H + \pi_L) \cdot G_1(c)}{c} - \frac{\pi_H - (\pi_H + \pi_L) \cdot G_2(c)}{c} \tag{2.72}$$

$$= \frac{(\pi_H + \pi_L) \cdot (G_2(c) - G_1(c))}{c} \tag{2.73}$$

We know that the difference is 0 whenever $c = \underline{c}$ and whenever $c \geq \tilde{c}_1$ [20] We now need to show that for some interval in $[\underline{c}, \overline{c}]$ the difference is increasing in c, since this would mean that whenever the two firms cost parameters are in this interval, the firm with the higher cost parameter increases its quality more relative to the firm with the lower cost parameter, when changing from the G_2 to the G_1 environment. Taking the derivative of the difference yields:

$$\frac{\partial \Delta(c)}{\partial c} = \frac{(\pi_H + \pi_L) \cdot (g_2(c) - g_1(c)) \cdot c - (\pi_H + \pi_L) \cdot (G_2(c) - G_1(c))}{c^2} \tag{2.74}$$

Consider that:

$$\left.\frac{\partial \Delta(c)}{\partial c}\right|_{c=\underline{c}} = \frac{(\pi_H + \pi_L) \cdot (g_2(\underline{c}) - g_1(\underline{c}))}{\underline{c}} > 0 \tag{2.75}$$

where the inequality follows since $G_1 \geq_{lr} G_2$ and hence $g_2(\underline{c}) > g_1(\underline{c})$. Hence, starting at the lower boundary of the support, the difference is increasing. However, due to $G_1 \geq_{lr} G_2$ we know that $g_2(c) - g_1(c)$ is decreasing and will become 0 at some $\hat{c} \in [\underline{c}, \overline{c}]$ and stay negative afterwards. Therefore, $\Delta(c)$ will reach its maximum c^* in the interval of $[\underline{c}, \hat{c}]$ which shows that $\Delta(c)$ is strictly increasing in c for all $c \in [\underline{c}, c^*)$ and strictly decreasing for all $c \in [c^*, \tilde{c}_1)$.

iv') Considering the RHS of condition (2.21) and taking the difference between the RHS in the G_1 and the G_2 environment yields:

$$2 \cdot \underline{c} \cdot \int_{\underline{c}}^{\overline{c}} \frac{1}{c} \cdot (g_1(c) - g_2(c))dc = 2 \cdot \underline{c} \cdot \int_{\underline{c}}^{\overline{c}} \frac{1}{c^2} \cdot (G_1(c) - G_2(c))dc < 0$$

The equality follows from using integration by parts. The differences between the CDFs is negative since G_1 first order stochastically dominates G_2 (see Theorem 1.C.1 in Shaked and Shanthikumar (2007). Therefore, the RHS in the G_1 environment is strictly lower. □

[20]Since $\tilde{c}_1 \geq \tilde{c}_2$, the difference will be positive as long as $c \in [\tilde{c}_2, \tilde{c}_1)$. As soon as $c \geq \tilde{c}_1$ a firm will choose a quality level of 0 in both environments.

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Chapter 3

A Dynamic Model of Vertical Differentiation with Cost Uncertainty

This chapter is joint work with Kangkan Choudhury.

Abstract

This paper extends a classical vertical differentiation model to a dynamic framework while assuming that firms are uncertain about the rival's and their own future fixed cost of quality improvement. The dynamic dimension allows firms to stepwise increase their quality, while uncertainty about costs translates into uncertainty about how profitable it is to increase quality. As a consequence of cost uncertainty, firms condition their strategy on their own costs, which gives rise to an Markov Perfect equilibrium, where both the principle of maximum- and minimum quality differentiation hold. We identify conditions such that the game ends almost surely in either maximum- or minimum differentiation. The latter case only occurs, if both firms decide to choose the highest possible quality level for their product. This is surprising since in this case the firms make zero profits forever within our model, even though the time dimension allows the firms to stepwise increase their quality and end up in states of intermediate product differentiation. A direct consequence of this result is that leap-frogging does not occur in the Markov Perfect equilibrium of our model.

3.1 Introduction

Many Oligopolistic industries are replete with instances where the participating firms are engaged in a constant battle for technological superiority. Space X and Blue Origin in the commercial space flight industry, Tesla and Nio in the Electric car industry, Intel and AMD in the Microprocessor industry are examples of firms that engaged in intense competition in the race for technological superiority. Such industries are characterized by large expenditure in Research and Development (henceforth denoted by R&D) to achieve an edge in technical capabilities. [1] The edge in technological capabilities enables firms to develop a superior product in quality and deliver more value to the consumers. As a result, competition in such emerging, high-technology industries is hardly ever limited to price or quantity choices. Product quality emerges as an important strategic choice for the firm. This paper analyses the dynamics of quality choice in a duopolistic setting, where firm and consumer behavior is based on the well established vertical differentiation paradigm (see chapter 7 in Tirole (1988)). We assume that firms are uncertain about their rival's fixed costs of quality improvement and derive a Markov Perfect equilibrium where both firms are investing into R&D to become the innovation leader in the industry.

The static setting has been extensively studied in the vertical differentiation literature. Two important results are emerging from this line of research. The first key result is given by firms choosing to produce different qualities when consumers are heterogeneous concerning their willingness to pay for quality. The quality-leader charges a higher price for its product and earns more profit. Furthermore, the analysis in the static setting has shown that quality choice is not a race. The profit of an individual firm does not necessarily increase with an increase in the own product quality. Although the trailing firm has a lower profit, it's profit increases as the difference in qualities increases. Thus in equilibrium, one firm serves the high end of the market by producing a more expensive high quality good. In contrast, the other firm serves the low end of the market by producing a lower quality but cheaper version of the good. This result is often referred to as the principle of maximum differentiation.

The static models of vertical differentiation have greatly added to our understanding of quality choice. Maximum differentiation is a reasonable approximation of quality-choice behavior in the context of retail and restaurant markets. However, this seems to be a bad approximation for intense and ongoing competition patterns that one observes in the high-tech industries. Specific attributes of the static vertical differentiation framework limit its applicability in studying oligopolistic competition in the context described above. The first major limitation is that firms are able to generally adjust their qualities over time. A pizza restaurant may contemplate switching from a brick to a more expensive wood-fired oven. However, the nature of the choice problem is very different in many emerging industries. For example, a microprocessor manufacturer like Intel or AMD may have to weigh between hundreds of potential improvements to incorporate in their products. Continuous quality adjustments in response to competitor behavior is a crucial characteristic of many oligopolistic high-tech industries. The second major limitation comes from the informational structure of the underlying model. Private information is an essential facet of many of the industries that we are interested in. Although firms may be

[1] For example, in 2019, Intel and AMD spend 17 and 20 percent of their revenue on R&D, respectively.

competing to achieve progress along the same general direction, there may be many different methods to achieve the same technological advance[2] and the firm's relative expertise in pursuing the different options is often private information. Private-information creates an incentive for the firms to "diversify" their quality choice from among the available alternatives. In this paper, we attempt to understand the implications of these two attributes within the context of a vertical differentiation framework.

We consider a Duopolistic setting in a covered market where interactions occur in discrete time. Consumers are heterogeneous concerning their marginal utility of quality, which implies that their willingness to buy a particular product with a given price and quality level differs. Firms choose their product qualities from a finite set of possible qualities in each period. Due to consumers heterogeneity, firms want to sell their products to consumers with the highest marginal utility for quality since their willingness to pay is the highest. Hence, the firms would prefer to be the higher quality firm and have maximum differences in product quality. Due to the dynamic framework in the model, a firm will not only benefit from being the high quality in a particular time period, but it also benefits in later periods. Uncertainty about the rival's fixed costs of quality-improvement implies that the quality choice of the rival in every possible quality-state is uncertain. This, along with the cost structure's linear nature, gives rise to a Markov Perfect equilibrium, where firms split their choices between the highest and lowest available quality levels. Thus both maximum and minimum differentiation can be an equilibrium phenomenon in our model, whereas intermediate product differentiation cannot occur in any Markov Perfect equilibrium. As a consequence, leap-frogging cannot occur in our model.

The paper is organized as follows. In the next section, we review the relevant literature on the static and dynamic models of vertical differentiation and locate our contribution to it. The following section describes the model and some key results that act as our stepping stone for the analysis in the subsequent section, which presents the main results and their interpretations. In the final section, we conclude by highlighting the key takeaways and possible extensions.

3.2 Literature

Starting with the seminal papers by Gabszewicz and Thisse (1979) and Shaked and Sutton (1982, 1983), models of vertical differentiation have been used and extended in various ways to analyze markets, where competition among firms does not solely take place on the price or quantity dimension, but also concerning innovation. While the early literature was concerned with static models, it has been recognized that some features of real innovation processes in markets can only be captured by considering dynamic models. For instance, static models cannot capture leap-frogging behavior, where an innovation leader gets surpassed by some rival, nor are they able to predict how innovation changes over time if, for instance, characteristics of the demand side are changing over time. Besides the necessity of models to better match real innovation processes, the development of new equilibrium concepts, such as Markov Perfect equilibria (Masking and Tirole (2001)), provided researchers with the analytical tools to study innovation behavior in dynamic models. In what follows, we will first briefly review static models

[2]For example the storage capacity of an Electric vehicle battery can be affected by choice of the electrolyte or the material composition of the capacitor plates.

of vertical differentiation and then proceed to the literature on dynamic models of innovation, to which this paper contributes.

Concerning static models, extensions include non-uniform distribution of consumers marginal utility of quality (Benassi et al. (2006) and (2019)), multiple qualities (Barigozzi and Ma (2018)), uncovered markets (Liao (2008), Wauthy (1996)), variations of innovation costs (Brecard (2010), Lehmann-Grube (1997), Wang (2003)), uncertainty about innovation costs (Jouchaghani (2021)), nesting of horizontal and vertical differentiation (Dos Santos Ferreira and Thisse (1996), Gabszewicz and Wauthy (2012)) and consideration of mixed strategies (Wang and Yang (2001)). Our paper is related to Jouchaghani (2021), since our model is a dynamic version of the static vertical differentiation model with cost uncertainty in his paper. He shows the existence of an equilibrium, in which firms behave more competitive with respect to innovation, which is in stark contrast to the maximum differentiation paradigm as predicted by classical static models of vertical differentiation. Our paper goes one step further, in that we analyze the implications of cost uncertainty within a framework, where firms can steadily increase their quality level over time. For instance, we are interested whether leap-frogging can occur within our model and are able to provide a negativity result. This and all questions related to innovation patterns over time go beyond the scope of the static model analyzed in Jouchaghani (2021).

Concerning dynamic models of innovation, there are several strands of literature within the field of industrial organization, shedding light on different aspects of innovation. Models of entry are concerned with the interplay of market entry and innovation (e.g., Bergemann and VÏimk̈i (2002), Fudenberg and Tirole (1984), Goettler and Gordon (2014)). For instance, Fudenberg and Tirole (1984) consider a duopoly model with an incumbent firm and a potential entrant firm and find that, under some stability assumptions on the reaction curves, the incumbent has an incentive to underinvest in the first period to look "lean and hungry" in order to more aggressively engage in innovation once the rival firm enters. Another strand of literature endogenizes the time at which a firm can engage in innovation (e.g., Hoppe and Lehmann-Grube (2005), Milliou and Petrakis (2011), Dutta et al. (2013)). Instead of assuming that all firms in a model can only innovate at the same fixed set of points in time, these so-called timing models allow firms to innovate at different time points within the time horizon. Hoppe and Lehmann-Grube (2005) analyze the tradeoff between being an early adopter and capturing high initial profits or being a late adopter who has access to better technology, which yields high profits, as well. They find that, in general, there is a second-mover advantage, which is increasing in the costs of innovation. A third strand of the literature is concerned with the relationship between competition, imitation, and innovation (e.g., Aghion et al. (2001), Aghion et al. (2005)). Aghion et al. (2001) show that, as long as the imitation possibility is limited to some extent, more competition leads to more innovation and more growth in an economy since firms innovate to avoid fierce competition.

Our paper differs from the previously mentioned papers in that we are interested in how incomplete information and the possibility that firms can steadily engage in innovation over time influence firms' innovation decisions in a fixed market structure. Therefore, our paper is related to the literature which analyzes the innovation behavior of duopolies over time (e.g., Hörner (2004), Anderson and Cabral (2007), Goettler and Gordon (2011), Baron (2020)). Our paper is closest to Baron (2020), who analyzes a dynamic duopoly model of vertical differentiation where innovation success is stochastic. In his model, he identifies three different equilibria types: The first types are Running-Coasting equilibria, where the innovation leader engages in innovation, whole the trailer acts as a free-rider who is not engaging in innovation. The second types are Leap-Frogging equilibria, where the trailer engages innovation, while the leader does not, such that the trailer surpasses the leader. Finally, there are Racing Equilibria, where both firms engage in innovation.

Our paper differs from his paper in at least two ways. First, we assume that the firms are ex-ante symmetric when the game starts, while in Baron (2020), one firm initially has a head-start. As we will see, even though in our model, the game could enter states where one firm has a head-start, our analysis shows that those states are not feasible in equilibrium when starting from a state where no firm has a head-start. As we will see in our model, this result is driven by the fact that the fixed costs of innovation are linear. Since in Baron (2020) costs are linear as well, the same result could hold, such that the game he analyzes is not feasible if firms would be symmetric. Second, we assume that firms are uncertain about their rival's and their own future costs of quality improvement, while the uncertainty in Baron (2020) stems from uncertainty about the success of innovation. Uncertainty concerning costs allows a Racing Equilibrium to occur in our model, even though when both firms engage in innovation, they might end up in the worst possible state where they make 0 profits forever. In Baron (2020), firms would suffer due to less product differentiation, however, both firms would still make strictly positive profits.

Our paper makes three contributions to the literature. First, it provides a tractable dynamic model of innovation with a micro-foundation of the "prizes" that firms receive when becoming the innovation leader or follower. Second, we establish a negativity result for dynamic models of vertical differentiation that assume linear fixed costs of innovation. For this case, we show that firms' behavior in our dynamic model does not differ from the behavior in a static model. There will be either maximum- or minimum product differentiation. The firms will not engage in stepwise innovation, leap-frogging cannot occur in equilibrium. We believe that this result will hold for any bounded quality space with an arbitrary partition, as long as fixed costs of innovation are linear. Third, we show that when assuming cost uncertainty, firms' quality levels can be close to each other whenever they have similar levels of technology (measured by a cost parameter), and in the worst-case, firms end up making zero profits forever.

3.3 Model

We consider a dynamic vertical differentiation model, where the baseline model is based on Tirole (1988). The market consists of two single good firms, denoted by $i \in \{A, B\}$, which compete in prices and qualities, and a unit mass of consumers, where each consumer decides exclusively from which firm to buy. Time is discrete, and the time horizon is infinite. A time period is denoted by $t \in \mathcal{T} = \{1, 2, ...\}$, where each time period consists of 3 stages.

In each period, firms are characterized by a cost parameter $c_{i,t}$, which is drawn from a two point distribution with support $\mathcal{C} = \{c_L, c_H\}$ and describes a firms marginal fixed costs of innovation. With a probability of p, a firm will have a cost parameter of c_H, while it will have a cost parameter of c_L with a probability of $1 - p$. We assume that $c_H > c_L > 0$, such that a firm with a cost parameter of c_L has lower marginal fixed costs of innovation. We assume that there are no production costs and that there are no variable innovation costs.[3] Furthermore, we assume that the cost parameters are iid across time periods and firms. As a consequence, a firm is not only uncertain about its rival's cost of innovation in the current and future periods but also about its own innovation costs in the future. A firm's product is characterized by its quality level $q_{i,t}$ and price $p_{i,t}$. The quality space is given by $\mathcal{Q} = \{q_L, q_M, q_H\}$, where $0 < q_L < q_M < q_H$. We assume that both firms start with a quality level of q_L, i.e., $q_{i,0} = q_L$. In each period, a firm first chooses a quality increment $\Delta q_{i,t} = q_{i,t} - q_{i,t-1}$ and then chooses a price $p_{i,t}$. The quality level of a firm's product in any period t can be express as the sum of all quality increments that a firm has chosen up to and including t, i.e., $\sum_{k=1}^{t} \Delta q_{i,k} = q_{i,t}$. When choosing a quality increment of $\Delta q_{i,t}$, a firm incurs fixed costs which are given by

$$C(\Delta q_{i,t}) = c_{i,t} \cdot \Delta q_{i,t}. \tag{3.1}$$

The demand side of the market is given by a mass of consumers in each period, where each consumer is characterized by his marginal utility of quality, measured by a parameter θ. For each consumer, this parameter is of private information. Firms only know that θ is drawn independently and identically according to a CDF $F(\cdot)$ with strictly positive density $f(\cdot)$ on a compact interval $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}_{++}$. A (representative) consumer's indirect utility of consuming firm i's product when having parameter θ is given by:

$$U_i(\theta) = \theta \cdot q_{i,t} - p_{i,t}. \tag{3.2}$$

We assume that the market is covered and that consumers are myopic in the sense that they do not time their purchase decision. Therefore, in any period t, every consumer buys exactly one of the products and leaves the market.[4]

The timing in any period is as follows:

Stage 1 Nature draws $c_{i,t}$, firms privately observe $c_{i,t}$ and simultaneously choose $\Delta q_{i,t}$.

Stage 2 Firms observe $q_{i,t}$ and simultaneously choose $p_{i,t}$.

Stage 3 Consumers observe $q_{i,t}$ and $p_{i,t}$ and decide from which firm to buy.

Fixing the firms' quality levels, the second and third stage will be time-invariant, such that we do not need to solve those stages within a dynamic framework. In particular, the following two assumptions will guarantee the existence of a price equilibrium, for any pair of qualities $(q_A, q_B) \in \mathcal{Q} \times \mathcal{Q}$ where $q_A \geq q_B$:

Assumption 1. *$f(\cdot)$ is log-concave.*

[3] All results in this paper will still hold when introducing variable production costs which are independent of the quality level. The marginal variable production costs will simply appear as a mark-up in the prices that consumers have to pay for the products in equilibrium. We only need to consider a slight change with respect to Assumption 2, such that both firms will have strictly positive market shares in equilibrium.

[4] Dropping the assumption of market coverage, while allowing for a general distribution of consumers marginal utility of quality, already complicates the analysis already in static models. Therefore we assume market coverage such that our model is tractable.

Assumption 2. $f(\underline{\theta}) \cdot \underline{\theta} < 1$.

Assumption 1 will ensure that the profit function of each firm is quasi-concave in its own price, such that a price equilibrium exists (see Caplin and Nalebuff (1991)). Assumption 2 will ensure that in equilibrium both firms are active on the market. In particular, it ensures that there is sufficient heterogeneity of consumers on the market, such that both firms can coexist.[5]

Stages 2 and 3 are fully characterized by the following lemma:

Lemma 1. *Let $q_A \geq q_B$. There exists a unique price equilibrium, in which both firms have strictly positive demand and which is given by:*

$$p_A^*(q_A, q_B) = \frac{1 - F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_A - q_B), \tag{3.3}$$

$$p_B^*(q_A, q_B) = \frac{F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_A - q_B), \tag{3.4}$$

where $\hat{\theta}^* \in (\underline{\theta}, \overline{\theta})$, which describes the parameter of the indifferent consumer in equilibrium, is the unique solution to

$$\hat{\theta}^* = \frac{1 - 2F(\hat{\theta}^*)}{f(\hat{\theta}^*)}. \tag{3.5}$$

Proof: See Theorem 2 in Caplin & Nalebuff (1991).[6]

Given this price equilibrium, the equilibrium revenues are given by:

$$R_A(q_A, q_B) = \frac{(1 - F(\hat{\theta}^*))^2}{f(\hat{\theta}^*)} \cdot (q_A - q_B) := \pi_H \cdot (q_A - q_B), \tag{3.6}$$

$$R_B(q_A, q_B) = \frac{F(\hat{\theta}^*)^2}{f(\hat{\theta}^*)} \cdot (q_A - q_B) := \pi_L \cdot (q_A - q_B). \tag{3.7}$$

For our purposes, there are three important things to consider: First, for any fixed quality difference between the firms' products, the high-quality firm has a higher revenue than the low-quality firm, i.e., $\pi_H > \pi_L$.[7] Therefore, firms have an incentive to rather be the high- than the low-quality firm. Second, both firms' revenues increase with increasing quality difference and are equal to 0 when the firms' products have the same quality level. As a consequence, firms prefer to be the low-quality firm, rather than choosing the same strictly positive quality level as their rival. Third, revenues are linear in the quality difference, which will drastically simplify the analysis of the quality stage within a dynamic framework.

We will now come to the analysis of the first stage for an infinite time horizon. First, we will define the underlying game. In what follows, we will refer to quality increments $\Delta q_{i,t}$ as actions. In any period t, the history $h_t = \{(q_{i,k}, q_{-i,k})\}_{k=0}^{t-1}$ is a sequence of the firms' quality

[5] Many common distributions and their truncated versions, as for instance the normal distribution, belong to the family of distributions with log concave densities, see Bagnoli and Bergstrom (2005). When considering a uniform distribution on $[\underline{\theta}, \overline{\theta}]$, Assumption 2 is given by $2 \cdot \underline{\theta} < \overline{\theta}$. For a triangular distribution, where the modus is not equal to $\underline{\theta}$, the condition is always fulfilled, while for the other case it is fulfilled only if $\overline{\theta} - \underline{\theta} > 1$.

[6] If not stated otherwise, all proofs can be found in the Appendix.

[7] This holds since $p_A^*(q_A, q_B) - p_B^*(q_A, q_B) = \frac{1 - 2 \cdot F(\hat{\theta}^*)}{f(\hat{\theta}^*)} \cdot (q_A - q_B) = \hat{\theta}^* \cdot (q_A - q_B) > \underline{\theta} \cdot (q_A - q_B) > 0$

levels from the previous periods. By $\mathcal{H}$, we denote the set of all possible histories for all periods. Furthermore, we denote the state of a period t by s_t, where $s_t = (q_{i,t-1}, q_{-i,t-1})$. The state of a period t is therefore given by the quality level of the products from the previous period, which is the only part of the history h_t which directly impacts the firms' profit in t. The set of states is denoted by $\mathcal{S} = \mathcal{Q} \times \mathcal{Q}$, such that there is a total of 9 states. We assume that firms can either increase- or stay at their current quality level.[8] As a consequence, the action set $\mathcal{A}_{i,s_t}$ of a firm will depend on the state in the following way:

$$\mathcal{A}_{i,s_t} = \begin{cases} \{0, q_M - q_L, q_H - q_L\} & ,if \;\; q_{i,t-1} = q_L \\ \{0, q_H - q_M\} & ,if \;\; q_{i,t-1} = q_M \\ \{0\} & ,if \;\; q_{i,t-1} = q_H \end{cases} \tag{3.8}$$

In addition, the set of possible states $\mathcal{S}_t$ in a particular period t will also depend on the state s_{t-1} in the following way:

$$\mathcal{S}_t = \begin{cases} \mathcal{S} & ,if \;\; s_{t-1} = (q_L, q_L) \\ \mathcal{S} \setminus \{(q_H, q_L), (q_M, q_L), (q_L,_L)\} & ,if \;\; s_{t-1} = (q_L, q_M) \\ \mathcal{S} \setminus \{(q_L, q_H), (q_L, q_M), (q_L,_L)\} & ,if \;\; s_{t-1} = (q_M, q_L) \\ \{(q_H, q_M), (q_M, q_H), (q_H, q_H)\} & ,if \;\; s_{t-1} = (q_M, q_M) \\ \{(q_L, q_H), (q_M, q_H), (q_H, q_H)\} & ,if \;\; s_{t-1} = (q_L, q_H) \\ \{(q_H, q_L), (q_H, q_M), (q_H, q_H)\} & ,if \;\; s_{t-1} = (q_H, q_L) \\ \{(q_M, q_H), (q_H, q_H)\} & ,if \;\; s_{t-1} = (q_M, q_H) \\ \{(q_H, q_M), (q_H, q_H)\} & ,if \;\; s_{t-1} = (q_H, q_M) \\ \{(q_H, q_H)\} & ,if \;\; s_{t-1} = (q_H, q_H) \end{cases} \tag{3.9}$$

Conditional on the state s_t and a firm's cost parameter $c_{i,t}$, the firm's profit in the current period for an action profile $(\Delta q_{i,t}, \Delta q_{-i,t})$ is given by:

$$\pi_i(\Delta q_{i,t}, \Delta q_{-i,t} | s_t, c_{i,t}) = \begin{cases} \pi_H \cdot (q_{i,t} - q_{-i,t}) - c_{i,t} \cdot \Delta q_{i,t} & ,if \;\; q_{i,t} > q_{-i,t} \\ -c_{i,t} \cdot \Delta q_{i,t} & ,if \;\; q_{i,t} = q_{-i,t} \\ \pi_L \cdot (q_{-i,t} - q_{i,t}) - c_{i,t} \cdot \Delta q_{i,t} & ,if \;\; q_{i,t} < q_{-i,t} \end{cases} \tag{3.10}$$

As stated before, since $\pi_H > \pi_L$, a firm prefers to be the high quality- rather than the low-quality firm. Nevertheless, a firm would rather prefer to be a low-quality firm when there is product differentiation than having homogenous products. Furthermore, the firm's choice of quality increment has consequences not only for its current period profit but also for its future profits. The optimal choice of quality increment has to take the downstream effects into account.

[8] Allowing for firms to decrease their quality would make the analysis of our model very difficult. First, it would eliminate all absorbing states. Second, it would yield that any state can be reached from another state. As a consequence, we would not be able to derive closed form expressions for our equilibrium and we would need to rely on numerical methods. From an empirical perspective, it is also rarely the case that firms decrease the quality level of their products.

Thus the quality choice problem of the two firms can be considered as a Dynamic Stochastic game. As is standard in the literature, we restrict our attention to looking for Markovian or Stationary strategies in which the firms condition their strategy on only the pay-off relevant variables of the current period. We will focus on Markov Perfect equilibria, in which firms only use Markovian strategies (see Maskin and Tirole (1988) and (2001)). As a consequence, firms are not allowed to use strategies which condition on pay-off irrelevant aspects in the history, as for instance grim-trigger or tit-for-tat strategies.

Note that when the firms choose their quality increments, they have observed their own cost realization for the given period, but not the cost realization of their opponent. The only variables that affect its current period pay-off are its current costs and the two firms' current quality. In any given time period, the firm conditions their choice of quality increment on both the firms' existing product quality and their own cost realization. Since the action space and the state space are both discrete, we restrict our attention to mixed strategies to ensure equilibrium existence.[9] Specifically, we are looking for $\sigma_i(c_i, s) : \mathcal{C} \times \mathcal{Q}^2 \mapsto \Delta\mathcal{A}_s$ where Δ indicates the probability simplex over $\mathcal{A}_{i,s}$, the set of feasible actions given state $s \in \mathcal{S}$. Let $\hat{\Sigma}(\mathcal{A}_{i,s_t})$ denote the set of all probability simplexes over the set $\mathcal{A}_{i,s_t}$. Let $\Pi_{i,t}(\sigma_i, \sigma_{-i}, s_t, c_t)$ denote the expected profit of firm i in period t when it is playing a stationary strategy σ_i, its opponent is playing σ_{-i}, the quality state inherited is s_t and its current realized cost is $c_{i,t}$, i.e., :

$$\Pi_{i,t}(\sigma_i, \sigma_{-i}, s_t, c_{i,t}) =$$

$$\sum_{c \in \mathcal{C}} \sum_{\Delta q_i \in \mathcal{A}_{i,s_t}} \sum_{\Delta q_{-i} \in \mathcal{A}_{-i,s_t}} P(c_{-i} = c) \cdot \sigma_i(\Delta q_i | s_t, c_{i,t}) \cdot \sigma_{-i}(\Delta q_{-i} | s_t, c) \cdot \pi_i(\Delta q_i, \Delta q_{-i} | s_t, c_{i,t}). \tag{3.11}$$

The expectation in $\Pi_{i,t}(\sigma_i, \sigma_{-i}, s_t, c_{i,t})$ is taken with respect to the rival firm's cost parameter, his strategy σ_{-i} and firm $i's$ own strategy σ_i. The first three factors in $\Pi_{i,t}(\sigma_i, \sigma_{-i}, s_t, c_{i,t})$ represent the probability that a particular action profile $(\Delta q_i, \Delta q_{-i})$ will be played. The last factor represents the payoff of firm i if the action profile $(\Delta q_i, \Delta q_{-i})$ is realized, given that the state is s_t and the firm has cost parameter $c_{i,t}$.

Let $V_i(s_t, c_{i,t}, \sigma_{-i})$ denote the maximum of the present value of stream of profits that the firm i can achieve given that it is entering the current period with inherited qualities s_t, has observed a cost realization of $c_{i,t}$ and its opponent is playing a Markovian strategy σ_{-i}. An equilibrium in this dynamic game would correspond to a pair of strategies $\sigma_i^* : \mathcal{Q}^2 \times \mathcal{C} \mapsto \mathcal{A}_s$ and pair of value functions $V_i : \mathcal{S} \times \mathcal{C} \times \hat{\Sigma} \mapsto \mathbb{R}$ such that for each $i = A, B$ the following hold

$$\sigma_i^* \in \operatorname*{argmax}_{\sigma \in \hat{\Sigma}} \left(\Pi_{i,t}(\sigma_i, \sigma_{-i}^*, s_t, c_t) + \beta \cdot \mathbb{E}(V_i(s_{t+1}, c_{i,t+1}, \sigma_{-i}^*) | s_t, \sigma_i, \sigma_{-i}^*) \right). \tag{3.12}$$

$$V_i(s_t, c_{i,t}, \sigma_{-i}^*) = \max_{\sigma_i \in \hat{\Sigma}} \left(\Pi_{i,t}(\sigma_i, \sigma_{-i}^*, s_t, c_t) + \beta \cdot \mathbb{E}(V_i(s_{t+1}, c_{i,t+1}, \sigma_{-i}^*) | s_t, \sigma_i, \sigma_{-i}^*) \right). \tag{3.13}$$

In what follows, we focus on finding an equilibrium of the form described above for the quality choice game. We will consider two additional assumptions for this stage:

[9] See Theorem 2 in Fink (1964).

Assumption 3. $\frac{\pi_H}{1-\beta} > c_L$.

This assumption is necessary for innovation to be profitable. Absence of this assumption would lead to a trivial unique equilibrium where firms play the pure strategy 0 in every period, such that the game remains in state (q_L, q_L) forever. To see this, consider that a firm knows with certainty, that the rival firm does not increase it's quality level. Therefore, it knows for sure that it will become the high-quality firm when it increases the quality level. Now if Assumption 3 is violated, increasing the quality level is strictly dominated by playing the pure action 0. When jumping to quality level $q > q_L$, both the revenue of a firm and the costs of a firm will be proportional to $(q - q_L)$, such that a violation of Assumption 3 yields strictly negative profits.

Assumption 4. $q_M - q_L \geq q_H - q_M$.

Four our purposes, this assumption is purely technical, since it drastically simplifies the derivation of some of our results. In particular it will simplify the proofs of Lemmas 5 and 6. Simulations suggest that this assumption is not necessary for the results, i.e., we believe that all our results hold even if Assumption 4 is violated. However, it turned out to be difficult to prove Lemmas 5 and 6 in an explicit way without it. Note that this assumption includes the case where the distance between the quality levels is equal, i.e., $q_M - q_L = q_H - q_M$, which is a common assumption in the literature. Relative to the literature (see for instance Hörner (2004) or Aghion et al. (2005)), our assumption can therefore be even seen as a generalization, rather than a restriction.

We begin our analysis by partitioning our state space into a set of absorbing and non-absorbing states. Absorbing states correspond to all quality-states, such that, once the game reaches one of these absorbing states, it remains there forever. As a consequence, the analysis of strategic behavior in the absorbing states is straight forward. Then we prove a few small results that help us narrow down the set of equilibrium strategies for the non-absorbing states. Next, we state some obvious claims about the equilibrium strategy and value functions for the absorbing states. Thereafter, we begin our analysis of the non-absorbing states by looking at the state where both firms have an intermediate quality level. Thereafter, we work recursively through the other quality-states, considering the state where both firms have the lowest quality state at the very end.

3.4 Analysis

We will solve the dynamic quality choice game by solving the sub-games related to each of the states. In what follows, the state vector $s_t = (q_{A,t-1}, q_{B,t-1})$ will refer to the current quality levels of firm A and firm B, respectively, before choosing their quality increments in period t. We start by deriving some results, which apply to several states:

Lemma 2. *In any state where at least one firm has a quality level of q_H, the unique equilibrium for the state is given by both firms choosing a quality increment of* 0.

This follows since if one firm has chosen the highest possible quality level, the other firm cannot surpass it anymore. Therefore, any increase in its quality level would decrease product differentiation in addition to an increase in costs and hence decrease profit. A direct consequence of this result and our assumption that firms cannot decrease their quality level is that all states where at least one of the firms has a quality level of q_H are absorbing states. Once the game enters one of those states, it stays there forever. As a result, the state space can be partitioned into the set of 5 absorbing states $\{(q_L, q_H), (q_H, q_L), (q_M, q_H), (q_H, q_M), (q_H, q_H)\}$ and the set of 4 non-absorbing states $\{(q_L, q_L), (q_L, q_M), (q_M, q_L), (q_M, q_M)\}$.[10] In what follows, we refer to the two non-absorbing states where firms have different quality, i.e., $\{(q_L, q_M), (q_M, q_L)\}$, as the asymmetric non-absorbing states. The next two results simplify the analysis of non-absorbing states:

Lemma 3. *There exists no equilibrium in the non-absorbing states, where firms mix between all possible actions for both cost parameters.*

This result applies even when a firm would mix with different probabilities, depending on its cost parameter. The reason for this is given by the fact that the present value of expected profits that firm i gets when choosing a quality increment of 0, is independent of the cost parameter.[11] As a consequence, the necessary conditions for an equilibrium where firms are mixing for both cost parameters yield that the present value of expected profits that firm i gets when choosing a strictly positive quality increment is independent of the cost parameter as well. This, however, cannot hold since the profit of the current period will be different for different cost parameters when choosing a strictly positive quality increment. The following result is related to Lemma 3:

Lemma 4. *In any equilibrium, where both firms mix between the available actions if they have a low cost parameter, they will not increase their quality level when having a high cost parameter.*

Together with Lemma 3, this result will restrict the set of candidate equilibria that we need to consider in the non-absorbing states. The possible equilibria are either of the form where the firms play pure strategies or where they play a pure strategy when having a high cost parameter and a mixed strategy when having a low cost parameter. Besides, Lemma 4 tells us that if we find a candidate equilibrium where a firm does not have an incentive to deviate from its mixed strategy when having a low cost parameter, it will not have an incentive to deviate from its pure strategy in case of having a high cost parameter. Having decreased the set of possible candidate equilibria, we will first start deriving the value functions of the absorbing states. We then continue to derive the equilibria of the non-absorbing states.

3.4.1 Absorbing states

As shown in Lemma 2, both firms choose a quality increment of 0 in equilibrium in any absorbing state, such that they do not incur costs and there is no transition to another state. Therefore,

[10] Note that the existence of absorbing states is not due to our restriction to three different quality levels. As long as firms cannot decrease their quality level, absorbing states will exists as long as the set of qualities is finite

[11] This follows since we assume that the cost parameters are iid over time and that in future periods firms condition their behavior only on the future state.

the value functions are simply given by:

$$\mathcal{V}_i(q_i, q_{-i}) = \begin{cases} \sum_{t=0}^{\infty} \beta^t \cdot \pi_H \cdot (q_i - q_{-i}) = \frac{\pi_H \cdot (q_i - q_{-i})}{1-\beta} & , if \;\; q_i = q_H. \\ \sum_{t=0}^{\infty} \beta^t \cdot \pi_L \cdot (q_{-i} - q_i) = \frac{\pi_L \cdot (q_{-i} - q_i)}{1-\beta} & , if \;\; q_{-i} = q_H. \end{cases} \tag{3.14}$$

As a result, both the best and the worst possible states are among the absorbing states for a firm. When a firm has chosen a quality level of q_H while its rival has chosen a quality level of q_L, it will receive the highest possible profit in each period due to maximum product differentiation. However, when both firms have chosen a quality level of q_H, the firms end up in an absorbing state where the products are homogenous forever, and hence they will receive 0 profits in each period. This creates a tradeoff that a firm faces when choosing a quality increment, which leads to a quality level of q_H. In what follows, the value functions for absorbing states will impact the firms' decision-making in all non-absorbing states since there is simply no way back once the game enters one of the stages. Note that for each non-absorbing state, we can find an absorbing state which yields strictly higher profits for at least one of the firms. A consequence of this will be that the game will almost surely end in one of those states.

3.4.2 State (q_M, q_M)

We start analyzing the non-absorbing states with the state where both firms have a quality level of q_M. This state represents a market situation, where after the initial introduction of the products, both firms decided to upgrade their product on an intermediate level. The action set of the firms is given by $\{0, q_H - q_L\}$, the set of feasible states is given by $\{(q_M, q_M), (q_M, q_H), (q_H, q_M), (q_H, q_H)\}$. As long as the firms remain in this state, they will make 0 profits. This occurs since the products are homogenous, and hence price competition drives the profits down to 0. Any of the other states but (q_H, q_H) is a Pareto improvement, since both firms will make strictly positive profits for ever.[12] However, reaching state (q_H, q_H) leaves the firms with 0 profits forever, while both suffer from one time fixed costs of quality improvement. Therefore, a firm only has an incentive to increase its quality level whenever the other firm does not. The following result describes an equilibrium for this state:

Proposition 1. *There exists a symmetric equilibrium in which for the state (q_M, q_M), firms choose quality increment 0, when having cost parameter c_H and mix between 0 and $q_H - q_M$ when having cost parameter c_L according to:*

$$\sigma^*(0|c_L, q_M, q_M) = \max\left\{0, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta \cdot (1-p)} - \frac{p}{1-p}\right\}, \tag{3.15}$$

where

$$\lambda = \pi_H + \pi_L + \beta \cdot (1-\beta) \cdot c_L. \tag{3.16}$$

$$\gamma = 2 \cdot (\pi_L + (1-\beta) \cdot c_L). \tag{3.17}$$

$$\eta = 2 \cdot \beta \cdot \pi_H. \tag{3.18}$$

[12] Since $\frac{\pi_H}{1-\beta} > c_L$, the quality leader will also improve, even though he suffers from one time fixed costs.

The unconditional probability that a firm chooses a quality increment of 0 *is given by:*

$$\tilde{\sigma}^*(0|q_M, q_M) := p + (1-p) \cdot \sigma^*(0|c_L, q_M, q_M) = \max\left\{p, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right\}. \quad (3.19)$$

Note that for increasing p, the probability that a firm will play 0 when having a cost parameter of c_L is decreasing and will become 0 eventually.[13] This occurs due to the fact that for high values of p, is is very likely that the rival firm has a high cost parameter and will choose a quality increment of 0. Therefore, when a firm draws a low cost parameter, knowing that p is high, the probability that the game will end in the worst possible state (q_H, q_H) is low. As a consequence, it has a high incentive of increasing its quality level to end up in the best feasible state (q_H, q_M). Furthermore, note that the unconditional probability that a firm will play 0 is bounded below by p, since it will choose a quality increment of 0 with certainty, in case it has a high cost parameter. The next results describe some comparative statics with respect to the unconditional probability to play 0 in the equilibrium of state (q_M, q_M):

Corollary 1. *With respect to the equilibrium described in Proposition 1, the following holds:*

- $\tilde{\sigma}^*(0|q_M, q_M)$ *is weakly decreasing in* π_H.
- $\tilde{\sigma}^*(0|q_M, q_M)$ *is weakly increasing in* π_L *and* c_L.

Consider that π_H and π_L denote the marginal revenue of the high- and low-quality firm, respectively, whenever the quality differential is increasing. Therefore, when π_H is increasing, it becomes relatively more attractive to become a high-quality firm, while the opposite holds whenever π_L is increasing. Consequently, the probability that a firm will play 0 weakly decreases when π_H is increasing, while it is weakly increasing as π_L is increasing. Furthermore, the incentive to not innovate increases as the marginal cost of a strictly positive quality increment increases. An increase in the discount factor β has two conflicting effects on the equilibrium probability of playing 0. First, it increases the net present value of a firm being the low quality in the absorbing states (q_H, q_M) and (q_M, q_H), since $\frac{\pi_L \cdot (q_H - q_M)}{1-\beta}$ is increasing. Hence, this effect should increase $\tilde{\sigma}^*(0|q_M, q_M)$. Second, it increases the net present value of a firm being the high quality firm in the absorbing states (q_H, q_M) and (q_M, q_H), since $\frac{\pi_H \cdot (q_H - q_M)}{1-\beta}$ is increasing. Therefore, this effect should decrease $\tilde{\sigma}^*(0|q_M, q_M)$. The sign of the total effect will crucially depend on the difference between π_H and π_L. For large differences between π_H and π_L the second effect will dominate such that an increase in β will have a negative total effect on $\tilde{\sigma}^*(0|q_M, q_M)$, while the opposite holds whenever π_H and π_L are close to each other. Therefore, when firms become more patient, there is no clear effect on the probability that innovation will happen. Our micro-foundation of π_H and π_L allows us to link the difference between the expressions to the primitives of our model:

$$\pi_H - \pi_L = \frac{(1 - F(\hat{\theta}^*))^2}{f(\hat{\theta}^*)} - \frac{F(\hat{\theta}^*)^2}{f(\hat{\theta}^*)} = \frac{1 - 2 \cdot F(\hat{\theta}^*)}{f(\hat{\theta}^*)} = \hat{\theta}^*. \quad (3.20)$$

[13]This statement always holds, independent of the other primitives but p, since in the proof of Proposition 1 we show that $\frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta} < 1$.

Therefore, the difference between π_H and π_L is equal to the indifferent consumer's marginal utility of quality in equilibrium. Considering the implicit equation which defines $\hat{\theta}^*$ in Lemma 1, one can see that $\hat{\theta}^*$ and therefore $\pi_H - \pi_L$ is entirely determined by the distribution of the consumers' willingness to pay for quality, i.e., by $F(\cdot)$. Note that $\hat{\theta}^*$ increases, the more probability mass is shifted to the upper boundary of the distribution (see Proposition 4 in Jouchaghani (2021)). Consequently, when firms become more patient, the probability that innovation occurs increases whenever the population of consumers consists of individuals with a high willingness to pay for quality. On the other hand, when the population consists of individuals with a low willingness to pay for quality, the probability that innovation occurs decreases when firms become more patient.

Let $\mathcal{V}(c, q_i, q_{-i})$ denote the value function (in the Bellman sense) of a firm, when it has quality q_i, its opponent has quality q_{-i} and it has received a cost realization of c. Denote the value function of the firm in state (q_i, q_{-i}) prior to observing its own cost realization by $\mathcal{V}(q_i, q_{-i})$ (henceforth, we will refer to this expression as the expected value function of state (q_i, q_{-i})). Using the unconditional probability of playing 0 in state (q_M, q_M) in equilibrium, we can derive the expected value function for state (q_M, q_M):

$$\begin{aligned}
&\mathcal{V}(q_M, q_M) := p \cdot \mathcal{V}(c_H, q_M, q_M) + (1-p) \cdot \mathcal{V}(c_L, q_M, q_M) \\
= &\sum_{t=0}^{\infty} \tilde{\sigma}^*(0|q_M, q_M)^{2t} \cdot \beta^t \cdot (1 - \tilde{\sigma}^*(0|q_M, q_M)) \cdot \left(\tilde{\sigma}^*(0|q_M, q_M) \cdot \frac{(\pi_H + \pi_L)}{1-\beta} - c_L \right) \cdot (q_H - q_M) \\
= &\frac{1 - \tilde{\sigma}^*(0|q_M, q_M)}{1 - \beta \cdot \tilde{\sigma}^*(0|q_M, q_M)^2} \cdot \left(\frac{\tilde{\sigma}^*(0|q_M, q_M) \cdot (\pi_H + \pi_L)}{1-\beta} - c_L \right) \cdot (q_H - q_M). \qquad (3.21)
\end{aligned}$$

The first factor in the sum simply describes the probability that both firms have chosen a quality increment of 0 until time period t, while the second factor is simply the discount factor for period t. The remaining factors describe the two possible net present value profits that a firm can achieve when one firm is increasing its quality level, while the other firm maintains the current quality level.[14] The first event is given by firm i increasing its quality level in t, while the other firm maintains the current quality level. In this case, the net present value profit is given by

$$(1 - \tilde{\sigma}^*(0|q_M, q_M)) \cdot \left(\tilde{\sigma}^*(0|q_M, q_M) \cdot \frac{\pi_H}{1-\beta} - c_L \right) \cdot (q_H - q_M). \qquad (3.22)$$

The second event is given by firm i maintaining its current quality level, while the other firm increases its quality level. Now the net present value of profits is given by:

$$(1 - \tilde{\sigma}^*(0|q_M, q_M)) \cdot \tilde{\sigma}^*(0|q_M, q_M) \cdot \frac{\pi_L}{1-\beta} \cdot (q_H - q_M). \qquad (3.23)$$

We will continue with the analysis of the asymmetric non-absorbing states.

[14]Note that there is a third possibility occurring with probability $(1 - \tilde{\sigma}^*(0|q_M, q_M))^2$, where both firms are choosing a quality increment of $q_H - q_M$. Since the profits, in this case, are equal to zero, it does not appear in the expected value function.

3.4.3 Asymmetric states (q_M, q_L) and (q_L, q_M)

States (q_M, q_L) and (q_L, q_M) represent market situations where starting from the initial state (q_L, q_L), one firm decided to proceed to sell the basic version of the product. In contrast, the other firm decided to upgrade its product on an intermediate level. Since both firms are ex-ante symmetric, the analysis of the asymmetric states (q_M, q_L) and (q_L, q_M) will be equivalent. We, therefore, restrict ourselves to the analysis of state (q_M, q_L). The action set of firm A is given by $\{0, q_H - q_M\}$, while the action set of firm B is given by $\{0, q_M - q_L, q_H - q_L\}$. The set of feasible states is given by $\{(q_M, q_L), (q_H, q_L), (q_M, q_M), (q_M, q_H)\}$. In this state, firm A has a head-start. However, both firms are making strictly positive profits since the products are heterogeneous. As long as the firms remain in this state, the profits of firm A and B are given by $\pi_H \cdot (q_M - q_L)$ and $\pi_L \cdot (q_M - q_L)$ in each period, respectively. First, we will show that we can restrict the set of actions that can be played in any equilibrium:

Lemma 5. *In equilibrium, firm B will not put positive probability on incrementing by $q_M - q_L$ in state (q_M, q_L).*

This holds, since for firm B, the pure strategy $q_M - q_L$ is a never best response for any (mixed strategy) of firm A in state. Given any mixed strategy of firm A in state q_M, q_L, it cannot simultaneously hold that the pure strategy $q_M - q_L$ yields a weakly higher expected payoff than playing either off the pure strategies 0 and $q_H - q_L$. The intuition is given by the following: whenever firm A puts a lot of probability mass on 0, firm B would rather play 0 as well or choose a quality increment of $q_H - q_L$. Choosing a quality increment of $q_M - q_L$ in this case would lead to a high probability that the products become homogenous and hence profits will be 0. On the other hand, if firm A puts a lot of probability mass on the pure strategy $q_H - q_M$, firm B would rather play 0 than choosing a quality increment of $q_M - q_L$. In both cases firm B would remain the low-quality quality firm. In the first case, however, there would be higher product differentiation at no cost. Besides, since the costs are linear, there is no benefit in step-wise increasing the quality level to reach the highest quality level with lower costs, as it would be possible when considering strictly convex quality increment costs. A direct consequence of this result is that in this state, there does not exist any equilibrium where firm B is playing $q_M - q_L$, since it would always have an incentive to deviate to another strategy. Therefore, when deriving an equilibrium, we can restrict the choice set of firm B to $\{0, q_H - q_L\}$, which simplifies the analysis.

Note that in this state, there is the possibility that for firm B the pure strategy $q_H - q_L$ is strictly dominated by 0 which yields the following result:

Proposition 2. *For*

$$\frac{\pi_L \cdot (q_M - q_L)}{1-\beta} + c_L \cdot (q_H - q_L) \geq \frac{\pi_H \cdot (q_H - q_M)}{1-\beta},$$

there exists a unique equilibrium, in which for state (q_M, q_L) firm B plays the pure strategy 0 with certainty for both cost parameters. If $\frac{\pi_H}{1-\beta} \geq c_H$, firm A plays the pure strategy $q_H - q_M$ with certainty for both cost parameters. If $\frac{\pi_H}{1-\beta} < c_H$, firm A will play the pure strategy $q_H - q_M$ only when having cost parameter c_L, while playing 0 when having cost parameter c_H.

The sufficient condition for existence and uniqueness of this equilibrium states that for firm B, the opportunity costs of becoming the high-quality firm when in state (q_M, q_L) are so high that even in the case that firm A is playing 0 with certainty, firm B would still prefer to play 0 over $q_H - q_L$. This holds in particular when $\frac{\pi_H}{1-\beta}$ is relatively close to $\frac{\pi_L}{1-\beta} + c_L$, or when q_M is relatively close to q_H. In the former case, the market of a high-quality product is not substantially more profitable than the market of the low-quality product when accounting for the additional costs. In the latter case, there is almost no product differentiation in state (q_M, q_H), which is the only state where firm B can be the high-quality firm. This implies that the game will almost surely proceed to state (q_H, q_L), whenever it reaches state (q_M, q_L), which eliminates the possibility of leap-frogging. However, if the condition below holds, there exists another equilibrium which substantially differs from the equilibrium as described in Proposition 2:

$$\frac{\pi_L \cdot (q_M - q_L)}{1-\beta} + c_L \cdot (q_H - q_L) < \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} \tag{3.24}$$

Proposition 3. *If the condition in (3.24) holds, there exists an asymmetric equilibrium where firms choose quality increment* 0*, when having cost parameter* c_H*. On the other hand, if firms have a cost parameter* c_L*, the firms maintain their quality level with the following probabilities:*

$$\sigma_A^*(0|c_L, q_M, q_L) = \min\left\{1, \max\left\{0, \frac{\lambda + \frac{q_M-q_L}{q_H-q_M}\cdot\mu - \sqrt{[\lambda + \frac{q_M-q_L}{q_H-q_M}\cdot\mu]^2 - \eta\cdot\gamma\cdot\frac{q_H-q_L}{q_H-q_M}}}{\eta\cdot(1-p)} - \frac{p}{1-p}\right\}\right\}$$

$$\sigma_B^*(0|c_L, q_M, q_L) = \max\left\{0, \frac{\lambda - \frac{q_M-q_L}{q_H-q_L}\cdot\alpha - \sqrt{[\lambda - \frac{q_M-q_L}{q_H-q_L}\cdot\alpha]^2 - \eta\cdot\gamma\cdot\frac{q_H-q_M}{q_H-q_L}}}{\eta\cdot(1-p)} - \frac{p}{1-p}\right\}$$

where λ*,*γ *and* η *are defined as in (3.16) - (3.18) and*

$$\mu = \beta\cdot(\pi_L + (1-\beta)\cdot c_L).$$
$$\alpha = (1-\beta)\cdot\pi_H + \pi_L + \beta\cdot(1-\beta)\cdot c_L.$$

The unconditional probability that a firm chooses a quality increment of 0 is given by:

$$\tilde{\sigma}_A^*(0|q_M, q_L) = \min\left\{1, \max\left\{p, \frac{\lambda + \frac{q_M-q_L}{q_H-q_M}\cdot\mu - \sqrt{[\lambda + \frac{q_M-q_L}{q_H-q_M}\cdot\mu]^2 - \eta\cdot\gamma\cdot\frac{q_H-q_L}{q_H-q_M}}}{\eta}\right\}\right\}.$$

$$\tilde{\sigma}_B^*(0|q_M, q_L) = \max\left\{p, \frac{\lambda - \frac{q_M-q_L}{q_H-q_L}\cdot\alpha - \sqrt{[\lambda - \frac{q_M-q_L}{q_H-q_L}\cdot\alpha]^2 - \eta\cdot\gamma\cdot\frac{q_H-q_M}{q_H-q_L}}}{\eta}\right\}.$$

Note that contrary to the equilibrium of state (q_M, q_M), the difference between the quality levels impacts the equilibrium probabilities. This occurs since the level of product differentiation will be different within the set of feasible absorbing states, where both firms are making strictly positive profits. As long as the game remains in state (q_M, q_L), firm A's profit is higher than the profit of firm B. The game can also enter state (q_H, q_L), which is the most profitable state for firm A, while state (q_L, q_H) is infeasible. Consequently, the incentives for innovation differ, and hence the firms are mixing with different probabilities in equilibrium. Due to the same logic described in state (q_M, q_M), firm A and firm B will play $q_H - q_M$ and $q_H - q_L$, respectively, with certainty when having cost parameter c_L, whenever p is large enough. The next result shows, maybe surprisingly, that when considering the equilibrium in Proposition 3, the head-start of firm A leads to a more competitive outcome than the equilibrium in Proposition 2:

Corollary 2. *In the equilibrium of Proposition 3 the following holds:*

$$\tilde{\sigma}_A^*(0|q_M, q_L) \geq \tilde{\sigma}^*(0|q_M, q_M) \geq \tilde{\sigma}_B^*(0|q_M, q_L).$$

This result implies that whenever the game arrives in state (q_M, q_L) and both firms are mixing in equilibrium, state (q_M, q_H) will be reached with a higher probability than (q_H, q_L). While the equilibrium in Proposition 2 suggested that leap-frogging cannot occur, but there will almost surely be a further increase in product differentiation, this equilibrium states that leap-frogging will occur with a higher probability than further product differentiation. To see this, note that Corollary 2 implies that $1 - \tilde{\sigma}_B^*(0|q_M, q_L) > 1 - \tilde{\sigma}_A^*(0|q_M, q_L)$ and hence:

$$(1 - \tilde{\sigma}_B^*(0|q_M, q_L)) \cdot \tilde{\sigma}_A^*(0|q_M, q_L) \geq (1 - \tilde{\sigma}_A^*(0|q_M, q_L)) \cdot \tilde{\sigma}_B^*(0|q_M, q_L).$$

This result is a consequence of firms being indifferent between the available pure strategies in a mixed equilibrium. The transition from state (q_M, q_L) to (q_H, q_L) yields a higher benefit for firm A, then a transition from state (q_M, q_L) to (q_M, q_H) yields for firm B. Therefore, in order for firm A to be indifferent between the pure strategies 0 and $q_H - q_M$, firm B has to put more probability mass on its pure strategy $q_H - q_L$, compared to the probability mass that firm A attaches to its pure strategy $q_H - q_M$. Considering the condition for the existence of the equilibrium in Proposition 3, we expect leap-frogging to occur, especially when the benefit of becoming the high-quality firm relative to the low-quality firm is very high. In those cases, the low-quality firm is willing to take the risk to become the high-quality firm, even though the game could end up in the worst possible state (q_H, q_H). From the consumers' perspective, this equilibrium is preferred to the equilibrium in Proposition 2. On expectation, the products will have higher quality levels, and there will be less product differentiation, such that the prices are lower.

We find the following results with respect to comparative statics:

Corollary 3. *In the equilibrium discussed in Proposition 3, the following holds:*

- $\tilde{\sigma}_A^*(0|q_L, q_M)$ *and* $\tilde{\sigma}_B^*(0|q_L, q_M)$ *are weakly decreasing in* π_H*.*
- $\tilde{\sigma}_A^*(0|q_M, q_M)$ *and* $\tilde{\sigma}_B^*(0|q_M, q_M)$ *are weakly increasing in* π_L *and* c_L*.*

The qualitative results for the equilibrium of Proposition 3 are equivalent to the equilibrium of state (q_M, q_M). Besides, the reasoning concerning changes in β applies here as well: depending on the difference between π_H and π_L, an increase in β can have positive or negative effects on the firms' probability to play 0 in equilibrium. With slight abuse of notation, where $\tilde{\sigma}_A^*(0|q_M, q_L) := \tilde{\sigma}_A$ and $\tilde{\sigma}_B^*(0|q_M, q_L) := \tilde{\sigma}_B$, the expected value functions of the firms for state (q_M, q_L) are given by:

$$\mathcal{V}_A(q_M, q_L) = \frac{\tilde{\sigma}_A}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}\cdot\Big(\tilde{\sigma}_B\cdot\pi_H\cdot(q_M-q_L)+(1-\tilde{\sigma}_B)\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta}\Big)$$
$$+\frac{1-\tilde{\sigma}_A}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}\cdot\Big(\tilde{\sigma}_B\cdot\frac{\pi_H\cdot(q_H-q_L)}{1-\beta}-c_L\cdot(q_H-q_M)\Big).$$

$$\mathcal{V}_B(q_M, q_L) = \frac{\tilde{\sigma}_B}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}\cdot\Big(\tilde{\sigma}_A\cdot\pi_L\cdot(q_M-q_L)+(1-\tilde{\sigma}_A)\cdot\frac{\pi_L\cdot(q_H-q_L)}{1-\beta}\Big)$$
$$+\frac{1-\tilde{\sigma}_B}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}\cdot\Big(\tilde{\sigma}_A\cdot\frac{\pi_H\cdot(q_H-q_M)}{1-\beta}-c_L\cdot(q_H-q_L)\Big).$$

The first term in each expression represents the expected lifetime profit, conditional on playing the pure strategy 0. The second term in $\mathcal{V}_A(q_M, q_L)$ represents the expected lifetime profit of firm A when playing pure strategy $q_H - q_M$, whereas the second term in $\mathcal{V}_B(q_M, q_L)$ represents the expected lifetime profit of firm $B$when playing pure strategy $q_H - q_L$.

3.4.4 State (q_L, q_L)

State (q_L, q_L) is the initial state of the game. It can be interpreted as an early market phase for a new product category. Both firms have already introduced their basic version products and decide on whether they should upgrade their products. The firms' action set is given by $\{0, q_M - q_L, q_H - q_L\}$, and all states are feasible. Therefore, our model allows firms to either continue selling a basic version of their product to upgrade it to an intermediate or high level. As long as the game remains in this state, both firms will make 0 profits. We will start the analysis with a result that restricts the set of actions that can be played in equilibrium, which drastically simplifies the analysis and has some interesting implications:

Lemma 6. *In equilibrium, no firm will put positive probability on incrementing by $q_M - q_L$ in state (q_L, q_L).*

This result manifests the principles of maximum and minimum product differentiation within a dynamic framework. Since the game starts in state (q_L, q_L), this result implies that the game will never enter any state where the product of one of the firms has quality level q_M. The game will almost surely either end in a state where the firms' products are maximally differentiated, namely (q_H, q_L) and (q_L, q_H) or in a state where the products are minimally differentiated, (q_H, q_H), without ever entering a state of intermediate product differentiation. Consequently, leap-frogging cannot occur in our model: the probability that one firm is surpassing another firm is equal to 0. The intuition behind this result is similar to the intuition behind Lemma 5. Suppose the rival firm has assigned a lot of probability mass on 0. In that case, a firm has an incentive to jump to the highest possible quality level rather than to choose an intermediate quality level. In both cases, it will become the high-quality firm with a high probability. However, in the former case, there is higher product differentiation, and the game enters the best possible state, which is the absorbing state (q_H, q_L). On the contrary, when the rival firm has assigned a lot of probability mass on $q_H - q_L$, a firm has an incentive to play 0 rather than $q_M - q_L$. In both situations, it will become the low-quality firm with a high probability. However, as in the case before, there is higher product differentiation in the former situation, and the game enters a preferable absorbing state. Besides, since the costs of a quality increment are linear rather than strictly convex, there is no benefit of step-wise increasing the quality level to reduce the total amount of costs to reach the highest quality level.

Nevertheless, it is surprising that this result occurs for state (q_L, q_L). Note that when firms are mixing only between 0 and $q_H - q_L$, there is a non-zero probability that the game ends in the worst possible state (q_H, q_H), where the game remains forever. Both firms suffer from one-time costs $c_L \cdot (q_H - q_L)$ and will make 0 profits forever. Any state where at least one of the firms has a product level of q_M is Pareto dominant to the state q_H, q_H. The absorbing states (q_M, q_H) and (q_H, q_M) yield strictly positive profits for both firms, the states (q_M, q_L) and (q_L, q_M) yield strictly positive profits and the possibility to reach states (q_M, q_H) and (q_H, q_M). While the firms make 0 profits as long as they remain in state (q_M, q_M), they can still reach states (q_M, q_H) and (q_H, q_M). An intermediate increase of a product's quality level can be seen as insurance against landing in state (q_H, q_H). However, Lemma 6 suggests that the firms are willing to take the risk rather than step-wise increase their quality levels. We believe that this result holds for any bounded quality space with an arbitrary partition, as long as the fixed costs of quality increment are linear since the same logic with three possible qualities should apply. Lemma 6 simplifies the derivation of the equilibrium for state (q_L, q_L) which is described in our next result:

Proposition 4. *There exists a symmetric equilibrium in state q_L, q_L, where firms choose quality increment 0, when having cost parameter c_H and mix between 0 and $q_H - q_L$ when having cost parameter c_L according to:*

$$\sigma^*(0|c_L, q_L, q_L) = \max\left\{0, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta \cdot (1-p)} - \frac{p}{1-p}\right\},$$

where λ, γ and η are defined as in (3.16) - (3.18).
The unconditional probability that a firm chooses a quality increment of 0 is given by:

$$\tilde{\sigma}^*(0|q_L, q_L) := p + (1-p) \cdot \sigma^*(0|q_M, q_M, c_L) = \max\left\{p, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right\}.$$

Note that this equilibrium is equivalent to the equilibrium of state (q_M, q_M). The reason for this is given by the fact that due to Lemma 6, the firms will either make 0 profits or make profits which are proportional to $q_H - q_L$. As a consequence, the profits are homogenous of degree 1 with respect to $q_H - q_L$, such that the equilibrium probabilities will not be affected by changes in $q_H - q_L$. Additionally, we have already noted that linearity of the costs of drives probability mass away from middle quality choices. These two results suggest that an equilibrium where both maximum and minimum differentiation can occur should be still feasible in a setting where the quality space consists of many equidistant levels.

The expected value function for state (q_L, q_L) is given by:

$$\mathcal{V}(q_L, q_L) = \frac{1 - \tilde{\sigma}^*(0|q_M, q_M)}{1 - \beta \cdot \tilde{\sigma}^*(0|q_M, q_M)^2} \cdot \left(\frac{\tilde{\sigma}^*(0|q_M, q_M) \cdot (\pi_H + \pi_L)}{1 - \beta} - c_L \right) \cdot (q_H - q_L). \tag{3.25}$$

Surprisingly, the analysis of state (q_L, q_L) turned out to be as simple as the analysis of state (q_M, q_M). In addition, all results turned out to be almost equivalent. This is remarkable, since one could have expected that the analysis of state (q_L, q_L) will be difficult since all states of the game are, at least in principle, feasible from from this state. However, Lemma 6 facilitated the analysis and provided us with an interesting insight into what could potentially expect when increasing the size of the quality spectrum.

3.5 Conclusion

In this paper, we study the dynamics of quality choice in a Duopolistic setting. The classic model of vertical differentiation serves as our starting point. We incorporate into the classic model two features that we consider to be important characteristics of competition in the technologically-intensive industries: quality adjustments over time and private information concerning R&D costs.

Maximum differentiation emerges as an equilibrium outcome in our framework. This parallels what one observes in the static framework. However, our framework also allows minimum differentiation to emerge in equilibrium, where both firms choose to sell products having the highest possible quality level. The emergence of these two qualitatively different outcomes can occur even though firms use symmetric equilibrium strategies and are ex-ante symmetric concerning all model primitives.

Uncertainty about the rival's costs incentives firms to spread their choices between the highest possible and the lowest possible quality levels. Thus, although intermediate quality choice is possible, ex-ante symmetric firms never choose to increment their quality only to an intermediate level. When the firms in our model decide to improve product quality, they tend to jump to the highest possible quality level. One implication of this behavior is that the quality choice trajectory can never enter into a state where leap-frogging is possible. However, on the other hand, if one firm starts exogenously with a quality advantage, leap-frogging becomes likely. The trailing firm implements product-quality increment with a higher probability than the leading firm.

Our model framework uses several simplifying assumptions to enable analysis in a dynamic setting. Our set of possible qualities has only three elements. We believe this assumption is not crucial. Our analysis suggests that using a quality ladder framework with equidistant quality levels would not alter our results' key qualitative features. Albeit, this claim remains a conjecture at this point. Secondly, we work with a simple linear cost structure. We believe this is a less innocuous modeling assumption. We expect that introducing convexities in the costs of quality improvement will allow the game to enter into states where one firm has an intermediate product quality. We plan to address these limitations in future research.

3.6 Appendix: Proofs

Lemma 2. Consider any period t' where $q_{A,t'} = q_H$. Since firm A cannot decrease it's quality level, it follows that $q_{A,t} = q_H$ for all $t > t'$. For any state where firm B has a quality of q_H, the best response is trivially given by 0 since firm B cannot decrease its quality level and has the highest possible quality level. For any state where firm B has a quality level of q_L or q_M, increasing the quality level will only decrease product differentiation and hence profits. At the same time, firm B will also suffer from costs. Therefore, whenever the rival has a quality level of q_H, the best response is given by playing the pure strategy 0. $\square$

Lemma 3. For this proof, we will consider the non-absorbing state (q_M, q_M), such that the set of possible actions is given by $\{0, q_H - q_M\}$. The reasoning for this state holds for any of the other non-absorbing states. Denote by $\mathbb{E}[\Delta q_{i,t}|c_{i,t}, q_M, q_M, \sigma_{-i,t}]$ the expected present value profit of playing the action $\Delta q_{i,t}$, given cost parameter $c_{i,t}$ and firm $-i$'s mixed strategy $\sigma_{-i,t}$ in period t. In particular:

$$\mathbb{E}[\Delta q_{i,t}|c_{i,t}, (q_M, q_M), \sigma_{-i,t}] = \Pi_{i,t}(\Delta q_{i,t}, \sigma_{-i,t}, q_M, q_M, c_{i,t}) \tag{3.26}$$

$$+\beta \cdot \mathbb{E}(V_i(s_{t+1}, c_{i,t+1}, \sigma_{-i,t+1})|q_M, q_M, q_{i,t}, \sigma_{i,t}) \tag{3.27}$$

We will consider a proof by contradiction. Therefore, let us assume that there exists a mixed equilibrium, where firms are mixing for both cost parameters. A necessary condition for a mixed equilibrium is given by a firm being indifferent between all possible pure actions, given the other firm is playing according to the mixed equilibrium. Therefore, the following two conditions must hold:

$$\begin{aligned} \mathbb{E}[0|c_H, q_M, q_M, \sigma_{-i,t}] &\overset{!}{=} \mathbb{E}[q_H - q_M|c_H, q_M, q_M, \sigma_{-i,t}] \\ \mathbb{E}[0|c_L, q_M, q_M, \sigma_{-i,t}] &\overset{!}{=} \mathbb{E}[q_H - q_M|c_L, q_M, q_M, \sigma_{-i,t}] \end{aligned} \tag{3.28}$$

Note that the current cost parameter of firm i has no impact on future periods since in future periods, both firms condition their behavior on the corresponding state and their cost parameter in those periods. This holds due to the restriction on Markovian strategies and due to the assumption that a firm's cost parameter is iid over time. Therefore, independent of whether firm A has a cost parameter of c_L or c_H, the future expected profits would be the same. A consequence of this is that when firm i is playing the action 0, therefore incurring no costs in the current period, the expected present value profit of playing 0 does not depend on the cost parameter, such that

$$\mathbb{E}[0|c_H, q_M, q_M, \sigma_{-i,t}] = \mathbb{E}[0|c_L, q_M, q_M, \sigma_{-i,t}] \tag{3.29}$$

Since the left hand sides in (3.28) are equivalent, the right hand sides must be equivalent, too. Therefore, we have:

$$\mathbb{E}[q_H - q_M|c_H, q_M, q_M, \sigma_{-i,t}] - \mathbb{E}[q_H - q_M|c_H, q_M, q_M, \sigma_{-i,t}] \overset{!}{=} 0 \tag{3.30}$$

Considering again that future payoffs do not depend on the current cost parameter of firm i, (3.30) reduces to:

$$\sum_{c\in\mathcal{C}} \sum_{\Delta q_{-i}\in\mathcal{A}_{-i,(q_M,q_M)}} P(c_{-i}=c)\cdot\sigma_{-i}(\Delta q_{-i}|c,q_M,q_M)$$
$$\cdot\Big(\pi_i(q_H-q_M,\Delta q_{-i,t}|c_L,q_M,q_M)-\pi_i(q_H-q_M,\Delta q_{-i,t}|c_H,q_M,q_M)\Big)\overset{!}{=}0 \tag{3.31}$$

whish is equivalent to

$$(q_H-q_M)\cdot(c_H-c_L)\cdot\underbrace{\sum_{c\in\mathcal{C}} \sum_{\Delta q_{-i}\in\mathcal{A}_{-i,(q_M,q_M)}} P(c_{-i}=c)\cdot\sigma_{-i}(\Delta q_{-i}|c,q_M,q_M)}_{=1}\overset{!}{=}0 \tag{3.32}$$

This is a contradiction, since $c_L<c_H$ and $q_H>q_M$. □

Lemma 4. We want to show that:

$$\mathbb{E}[0|c_H,q_M,q_M,\sigma_{-i,t}]>\mathbb{E}[q_H-q_M|c_H,q_M,q_M,\sigma_{-i,t}] \tag{3.33}$$

such that given a mixed strategy of its rival, the expected present value profit of firm i is strictly higher when choosing a quality increment of 0, while having a cost parameter of c_H. This simply follows from the fact that when firms are mixing whenever they have a cost parameter c_L, for firm i it must hold that:

$$\mathbb{E}[0|c_L,q_M,q_M,\sigma_{-i,t}]\overset{!}{=}\mathbb{E}[q_H-q_M|c_L,q_M,q_M,\sigma_{-i,t}] \tag{3.34}$$

We know from the proof of lemma 2 that:

$$\mathbb{E}[0|c_L,q_M,q_M,\sigma_{-i,t}]=\mathbb{E}[0|c_H,q_M,q_M,\sigma_{-i,t}] \tag{3.35}$$

$$\mathbb{E}[q_H-q_M|c_L,q_L,q_M,\sigma_{-i,t}]>\mathbb{E}[q_H-q_M|c_H,q_M,q_M,\sigma_{-i,t}] \tag{3.36}$$

Combinging (3.34), (3.35) and (3.36) yields:

$$\begin{aligned}\mathbb{E}[0|c_H,q_L,q_M,\sigma_{-i,t}]=\mathbb{E}[0|c_L,q_M,q_M,\sigma_{-i,t}]&=\mathbb{E}[q_H-q_M|c_L,q_M,q_M,\sigma_{-i,t}]\\&>\mathbb{E}[q_H-q_M|c_H,q_M,q_M,\sigma_{-i,t}]\end{aligned}$$

which yields the desired result. □

Proposition 1. Too shorten the notation, we omit the state in the mixing probabilities of the firms. Therefore, the probability for a firm $i\in\{A,B\}$ to play the pure strategy 0 or q_H-q_M for cost parameter $c\in\{c_L,c_H\}$ is given by $\sigma_i(0|c)$ and $\sigma_i(q_H-q_M|c)$, respectively. Given

$c \in \{c_L, c_H\}$, the Bellman equations for firm A in state 4 are given by:

$$\mathcal{V}_A(c, q_M, q_M) =$$

$$p \cdot \Big\{\sigma_A(0|c) \cdot \Big[\sigma_B(0|c_H) \cdot \Big(0 + \beta \cdot \Big[p \cdot \mathcal{V}_A(c_H, q_M, q_M) + (1-p) \cdot \mathcal{V}_A(c_L, q_M, q_M)\Big]\Big)$$

$$+\sigma_B(q_H - q_M|c_H) \cdot \Big(\pi_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_M, q_H)\Big)\Big]$$

$$+ \sigma_A(q_H - q_M|c) \cdot \Big[\sigma_B(0|c_H) \cdot \Big((\pi_H - c) \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_H, q_M)\Big)$$

$$+\sigma_B(q_H - q_M|c_H) \cdot \Big(0 - c \cdot (q_H - q_M) + \beta \cdot \mathcal{V}(q_H, q_H)\Big)\Big]\Big\}$$

$$+(1-p) \cdot \Big\{\sigma_A(0|c) \cdot \Big[\sigma_B(0|c_L) \cdot \Big(0 + \beta \cdot \Big[p \cdot \mathcal{V}_A(c_H, q_M, q_M) + (1-p) \cdot \mathcal{V}_A(c_L, q_M, q_M)\Big]\Big)$$

$$+\sigma_B(q_H - q_M|c_L) \cdot \Big(\pi_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_M, q_H)\Big)\Big]$$

$$+ \sigma_A(q_H - q_M|c) \cdot \Big[\sigma_B(0|c_L) \cdot \Big((\pi_H - c) \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_H, q_M)\Big)$$

$$+\sigma_B(q_H - q_M|c_L) \cdot \Big(0 - c \cdot (q_H - q_M) + \beta \cdot \mathcal{V}(q_H, q_H)\Big)\Big]\Big\}$$

Solving the system of bellman equations for $\mathcal{V}_A(c_H, q_M, q_M)$ and $\mathcal{V}_A(c_L, q_M, q_M)$ yields:

$$\begin{aligned}
\mathcal{V}_A(c_H, q_M, q_M) = (q_H - q_M) \cdot \Big(&\frac{\pi_H \cdot \tilde{\sigma}_B}{1-\beta} \cdot \Big[1 - \frac{1 - \beta \cdot \tilde{\sigma}_B}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A} \cdot \sigma_A(0|c_H)\Big] \\
&+ \frac{\pi_L \cdot (1 - \tilde{\sigma}_B) \cdot \sigma_A(0|c_H)}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A} \\
&- \frac{c_H \cdot (1 - \sigma_A(0|c_H)) \cdot (1 - \sigma_A(0|c_L) \cdot \beta \cdot (1-p) \cdot \tilde{\sigma}_B)}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A} \\
&- \frac{c_L \cdot \sigma_A(0|c_H) \cdot (1 - \sigma_A(0|c_L)) \cdot \beta \cdot (1-p) \cdot \tilde{\sigma}_B}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A}\Big) \\
\mathcal{V}_A(c_L, q_M, q_M) = (q_H - q_M) \cdot \Big(&\frac{\pi_H \cdot \tilde{\sigma}_B}{1-\beta} \cdot \Big[1 - \frac{1 - \beta \cdot \tilde{\sigma}_B}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A} \cdot \sigma_A(0|c_L)\Big] \\
&+ \frac{\pi_L \cdot (1 - \tilde{\sigma}_B) \cdot \sigma_A(0|c_L)}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A} \\
&- \frac{c_H \cdot \sigma_A(0|c_L) \cdot (1 - \sigma_A(0|c_H)) \cdot \beta \cdot p \cdot \tilde{\sigma}_B}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A} \\
&- \frac{c_L \cdot (1 - \sigma_A(0|c_L)) \cdot (1 - \sigma_A(0|c_H) \cdot \beta \cdot p \cdot \tilde{\sigma}_B)}{1 - \beta \cdot \tilde{\sigma}_B \cdot \tilde{\sigma}_A}\Big)
\end{aligned}$$

where $\tilde{\sigma}_i := (1-p)\cdot\sigma_i(0|c_L)+p\cdot\sigma_i(0|c_H),\ \ i\in\{A,B\}$. Note that $\tilde{\sigma}_i$ describes the unconditional (unconditional with respect to costs) probability that a firm maintains it's quality.
Next, we want to derive a symmetric equilibrium, where firms play 0 whenever they have a high cost parameter, while mixing whenever they have a low cost parameter. Let us derive the expected profit of firm A, when playing the pure strategies, given firm B is mixing with

$$\sigma_B = \{\sigma_B(0|c_H), \sigma_B(q_H-q_M|c_H), \sigma_B(0|c_L), \sigma_B(q_H-q_M|c_L)\} = \{1, 0, \sigma_B(0|c_L), \sigma_B(q_H-q_M|c_L)\}$$

Whenever A plays the pure strategy 0, the expected profit is given by:

$$\mathbb{E}[0|c_L, q_M, q_M, \sigma_B] = \Big(p+(1-p)\cdot\sigma_B(0|c_L)\Big)\cdot\Big(0+\beta\cdot[p\cdot\mathcal{V}_A(c_H,q_M,q_M)+(1-p)\cdot\mathcal{V}_A(c_L,q_M,q_M)]\Big)$$
$$+(1-p)\cdot\sigma_B(q_H-q_M|c_L)\cdot\Big(\pi_L\cdot(q_H-q_M)+\beta\cdot\mathcal{V}_A(q_M,q_H)\Big)$$

Whenever A plays the pure strategy $q_H - q_M$, the expected profit is given by:

$$\mathbb{E}[q_H-q_M|c_L,q_M,q_M,\sigma_B] = [p+(1-p)\cdot\sigma_B(0|c_L)]\cdot[(\pi_H-c_L)\cdot(q_H-q_M)+\beta\cdot\mathcal{V}_A(q_H,q_M)]$$
$$+(1-p)\cdot\sigma_B(q_H-q_M|c_L)\cdot[0-c_L\cdot(q_H-q_M)+\beta\cdot\mathcal{V}(q_H,q_H)]$$

Setting $\mathbb{E}[0|c_L,q_M,q_M,\sigma_B] = \mathbb{E}[q_H-q_M|c_L,q_M,q_M,\sigma_B]$, plugging in for the value functions, considering symmetry and that $\sigma_B(q_H-q_M|c_L) = 1-\sigma_B(0|c_L)$ and solving for $\sigma_B(0|c_L)$ yields:

$$\sigma_B(0|c_L) = \sigma(0|c_L) = \frac{\pi_H+\pi_L+\beta\cdot(1-\beta)\cdot c_L}{2\cdot\beta\cdot(1-p)\cdot\pi_H} - \frac{p}{1-p} \tag{3.37}$$
$$-\frac{\sqrt{(\pi_H+\pi_L+\beta\cdot(1-\beta)\cdot c_L)^2+4\cdot\beta\cdot\pi_H\cdot(\pi_L+(1-\beta)\cdot c_L)}}{2\cdot\beta\cdot(1-p)\cdot\pi_H} \tag{3.38}$$

Using the following substitutions:

$$\lambda = \pi_H+\pi_L+\beta\cdot(1-\beta)\cdot c_L \tag{3.39}$$
$$\gamma = 2\cdot(\pi_L+(1-\beta)\cdot c_L) \tag{3.40}$$
$$\eta = 2\cdot\beta\cdot\pi_H \tag{3.41}$$

we can simplify the expression to get:

$$\sigma(0|c_L) = \frac{\lambda-\sqrt{\lambda^2-\eta\cdot\gamma}}{\eta\cdot(1-p)} - \frac{p}{1-p} \tag{3.42}$$

Note that when p is getting closer to 1, the expression becomes negative, such that the firms do not mix anymore, but will play the strategy $q_H - q_L$ with certainty whenever they have a low cost parameter. On the other hand, the expression will never be strictly larger than 1 since:

$$\frac{\lambda-\sqrt{\lambda^2-\eta\cdot\gamma}}{\eta\cdot(1-p)} - \frac{p}{1-p} < 1 \Leftrightarrow \frac{\pi_H}{1-\beta} > c_L \tag{3.43}$$

Therefore, the equilibrium probability to play 0 in state (q_M, q_M) when having cost parameter c_L is equal to:

$$\sigma^*(0|c_L, q_M, q_M) = \max\left\{0, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta \cdot (1-p)} - \frac{p}{1-p}\right\} \tag{3.44}$$

In order to get the unconditional probability to play 0 in equilibrium, we multiply $\sigma^*(0|q_M, q_M, c_L)$ with $(1-p)$ and add p such that we finally have:

$$\tilde{\sigma}^*(0|q_M, q_M) := p + (1-p) \cdot \sigma^*(0|q_M, q_M, c_L) = \max\left\{p, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right\} \tag{3.45}$$

□

Corollary 1. Note that, whenever $p \geq \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}$, the unconditional probability is independent of π_H, π_L and c_L. Whenever $p < \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}$, we find the following:

$$\frac{\partial\left(\frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right)}{\partial \pi_H} < 0 \Leftrightarrow 4 \cdot \pi_H^2 \cdot \pi_L \cdot \beta \cdot (1-\beta) \cdot (\pi_L + (1-\beta) \cdot c_L) > 0 \tag{3.46}$$

$$\frac{\partial\left(\frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right)}{\partial \pi_L} > 0 \Leftrightarrow 4 \cdot \pi_H \cdot \beta \cdot (1-\beta) \cdot (\pi_H - c_L \cdot (1-\beta)) > 0 \tag{3.47}$$

$$\frac{\partial\left(\frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right)}{\partial c_L} > 0 \Leftrightarrow 4 \cdot \pi_H \cdot \pi_L \cdot (1-\beta) > 0 \tag{3.48}$$

Therefore, the unconditional probability of playing 0 is weakly decreasing in π_H, while it is weakly increasing in π_L and c_L.

□

Lemma 5. Suppose that for firm B, $q_M - q_L$ is a best response for some Markovian strategy of firm A, where firm A is mixing between 0 and $q_H - q_M$ with unconditional probabilities ϕ and $1-\phi$, respectively, whenever the state is (q_M, q_L). In order for $q_M - q_L$ to be a best response against ϕ in state (q_M, q_L), it must hold that the expected net present value profit of playing the pure strategy $q_M - q_L$ is weakly higher than playing either 0 or $q_H - q_L$. Let us first consider the expected net present value profits for the different pure strategies, given that firm A mixes as described before. Playing the pure strategy 0 yields:

$$\phi \cdot \left[\pi_L \cdot (q_M - q_L) + \beta \cdot \mathbb{E}_B[0|q_L, q_M, \phi]\right] + (1-\phi) \cdot \frac{\pi_L \cdot (q_H - q_L)}{1-\beta} \tag{3.49}$$

where $\mathbb{E}_B[0|q_L, q_M, \phi]$ denotes the expected net present value profit for firm B when playing 0 every time the state remains in (q_L, q_M), while firm A is mixing with ϕ and $1-\phi$ and is given by:

$$\mathbb{E}_B[0|q_L, q_M, \phi] = \frac{\phi}{1-\beta\cdot\phi}\cdot\pi_L\cdot(q_M - q_L) + \frac{1-\phi}{1-\beta\cdot\phi}\cdot\frac{\pi_L\cdot(q_H-q_L)}{1-\beta} \tag{3.50}$$

Plugging this expression into (3.50) into (3.49) and simplifying yields:

$$F_L(\phi) := \frac{\pi_L\cdot(q_M-q_L)}{1-\beta} + \frac{1-\phi}{1-\beta\cdot\phi}\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta} \tag{3.51}$$

Playing the pure strategy $q_H - q_L$ yields:

$$F_H(\phi) := \phi\cdot\frac{\pi_H\cdot(q_H-q_M)}{1-\beta} - c_L\cdot(q_H-q_L) \tag{3.52}$$

Finally, playing the pure strategy $q_M - q_L$ yields:

$$F_M(\phi) := \phi\cdot\beta\cdot\mathcal{V}(q_M,q_M) + (1-\phi)\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta} - c_L\cdot(q_M-q_L) \tag{3.53}$$

Where $\mathcal{V}(q_M, q_M)$ denotes the expected value function for state (q_M, q_M). Plugging in the expression for $\mathcal{V}(q_M, q_M)$(the explicit expression can be found in (3.21)) in (3.53), while considering that $\tilde{\sigma}$ denotes the unconditional probability to play 0 in the equilibrium of state (q_M, q_M) yields:

$$F_M(\phi) = \frac{\phi\cdot\beta\cdot(1-\tilde{\sigma})}{1-\beta\cdot\tilde{\sigma}^2}\cdot\left(\tilde{\sigma}\cdot\frac{\pi_H+\pi_L}{1-\beta} - c_L\right)\cdot(q_H-q_M) + (1-\phi)\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta}$$
$$-c_L\cdot(q_M-q_L)$$

In order for $q_M - q_L$ to be a best response against some ϕ, it must simultaneously hold that $F_M(\phi)$ is weakly larger than $F_L(\phi)$ and $F_H(\phi)$. Therefore, it must hold that:

$$F_M(\phi) \geq \max\{F_L(\phi), F_H(\phi)\} \tag{3.54}$$

In what follows, we want to show that for any $\phi \in [0,1]$ the inequality described in (3.54) cannot hold. First, note that $F_H(\phi)$ is strictly increasing and linear in ϕ, $F_L(\phi)$ is strictly decreasing and strictly concave in ϕ and $F_M(\phi)$ is linear and either strictly increasing, strictly decreasing or constant in ϕ. For the proof, we will distinguish two cases. First, we consider the case where some $\bar{\phi}$ exists such that $F_L(\bar{\phi}) = F_H(\bar{\phi})$. Second, we consider the case where such $\bar{\phi}$ does not exist.

Case 1: $\bar{\phi}$ exists

$\bar{\phi}$ exists if

$$\frac{\pi_H\cdot(q_H-q_M)}{1-\beta} - c_L\cdot(q_H-q_L) \geq \frac{\pi_L\cdot(q_M-q_L)}{1-\beta} \tag{3.55}$$

since this implies that $F_H(1) \geq F_L(1)$. In addition, it generally holds that $F_H(0) < F_L(0)$ and as stated before, $F_H(\phi)$ is strictly increasing, while $F_L(\phi)$ is strictly decreasing in ϕ. Therefore,

$\bar{\phi}$ exists and is defined by the following implicit equation:

$$\frac{\pi_L \cdot (q_M - q_L)}{1-\beta} + \frac{1-\bar{\phi}}{1-\beta \cdot \bar{\phi}} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} = \bar{\phi} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L) \tag{3.56}$$

For the case where $\bar{\phi}$ exists, the function $\max\{F_L(\phi), F_H(\phi)\}$ in (3.54) is strictly decreasing and strictly concave in ϕ for $0 \leq \phi < \bar{\phi}$, attains its minimum at $\phi = \bar{\phi}$ and is strictly increasing and linear for $\bar{\phi} < \phi \leq 1$. In addition, note that $F_L(\phi) > F_H(\phi)$ for $\phi \in [0, \bar{\phi})$ and $F_L(\phi) < F_H(\phi)$ for $\phi \in (\bar{\phi}, 1]$. To show that the inequality (3.54) is violated, it suffices to show that the following three conditions are true:

1. $F_L(0) > F_M(0)$

2. $F_H(1) > F_M(1)$

3. $F_L(\bar{\phi}) = F_H(\bar{\phi}) > F_M(\bar{\phi})$

Condition 1 and 3 together imply that for any $\phi \in [0, \bar{\phi}]$ it holds that $F_L(\phi) > F_M(\phi)$. To see this, suppose first that $F_M(\phi)$ is weakly increasing. In that case, condition 3 together with the fact that $F_L(\phi)$ is strictly decreasing yields

$$F_L(\phi) > F_L(\bar{\phi}) > F_M(\bar{\phi}) \geq F_M(\phi) \tag{3.57}$$

for any $\phi \in [0, \bar{\phi})$. For the case where $F_M(\phi)$ is strictly decreasing, condition 1 and 3, together with the facts that $F_L(\phi)$ is strictly concave, while $F_M(\phi)$ is linear yield:

$$\begin{aligned} F_L(\phi) = F_L\Big((1 - \frac{\phi}{\bar{\phi}}) \cdot 0 + \frac{\phi}{\bar{\phi}} \cdot \bar{\phi}\Big) &> (1 - \frac{\phi}{\bar{\phi}}) \cdot F_L(0) + \frac{\phi}{\bar{\phi}} \cdot F_L(\bar{\phi}) \\ &> (1 - \frac{\phi}{\bar{\phi}}) \cdot F_M(0) + \frac{\phi}{\bar{\phi}} \cdot F_M(\bar{\phi}) = F_M(\phi) \end{aligned}$$

for any $\phi \in [0, \bar{\phi})$.
Conditions 2 and 3, the fact that both $F_M(\phi)$ and $F_H(\phi)$ are linear, yield, independent of whether $F_M(\phi)$ is strictly increasing, decreasing or constant:

$$F_H(\phi) = \frac{\phi - \bar{\phi}}{1-\bar{\phi}} \cdot F_H(1) + \frac{1-\phi}{1-\bar{\phi}} \cdot F_H(\bar{\phi}) > \frac{\phi - \bar{\phi}}{1-\bar{\phi}} \cdot F_M(1) + \frac{1-\phi}{1-\bar{\phi}} \cdot F_M(\bar{\phi}) = F_M(\phi)$$

for any $\phi \in (\bar{\phi}, 1]$.
As a consequence, inequality (3.54) can not hold for the case where $\bar{\phi}$ exists and when conditions 1 - 3 are true. Note that condition 1 always holds, since:

$$F_L(0) = \frac{\pi_L \cdot (q_H - q_L)}{1-\beta} > \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_M - q_L) = F_M(0) \tag{3.58}$$

Next, we want to show that condition 2 holds, i.e., we want to show that:

$$\begin{aligned}
&\frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L) \\
&> \frac{\beta \cdot (1-\tilde{\sigma})}{1-\beta \cdot \tilde{\sigma}^2} \cdot \left(\tilde{\sigma} \cdot \frac{\pi_H + \pi_L}{1-\beta} - c_L \right) \cdot (q_H - q_M) - c_L \cdot (q_M - q_L) \\
&\Leftrightarrow (1-\beta \cdot \tilde{\sigma}) \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} \\
&> \beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} + (1 - \beta \cdot \tilde{\sigma}^2 - \beta \cdot (1-\tilde{\sigma})) \cdot c_L \cdot (q_H - q_M)
\end{aligned} \tag{3.59}$$

Note that for this Case 1 we assumed that $\frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L) \geq \frac{\pi_L \cdot (q_M - q_L)}{1-\beta}$ such that (3.59) holds whenever:

$$\begin{aligned}
&(1-\beta \cdot \tilde{\sigma}) \cdot \left(\frac{\pi_L \cdot (q_M - q_L)}{1-\beta} + c_L \cdot (q_H - q_L) \right) \\
&> \beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} + (1 - \beta \cdot \tilde{\sigma}^2 - \beta \cdot (1-\tilde{\sigma})) \cdot c_L \cdot (q_H - q_M)
\end{aligned} \tag{3.60}$$

The inequality in (3.60) is fulfilled since:

$$(1-\beta \cdot \tilde{\sigma}) \cdot \frac{\pi_L}{1-\beta} \cdot (q_M - q_L) > \underbrace{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}_{<1-\beta \cdot \tilde{\sigma}} \cdot \frac{\pi_L}{1-\beta} \cdot \underbrace{(q_H - q_M)}_{\leq q_M - q_L}$$

$$(1-\beta \cdot \tilde{\sigma}) \cdot c_L \cdot (q_H - q_L) > \underbrace{(1 - \beta \cdot \tilde{\sigma}^2 - \beta \cdot (1-\tilde{\sigma}))}_{<2 \cdot (1-\beta \cdot \tilde{\sigma})} \cdot c_L \cdot \underbrace{(q_H - q_M)}_{\leq \frac{1}{2} \cdot (q_H - q_L)}$$

Therefore, inequality (3.59) holds and hence condition 2 is fulfilled.

Condition 3 holds if:

$$\begin{aligned}
&\bar{\phi} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L) \\
&> \frac{\bar{\phi} \cdot \beta \cdot (1-\tilde{\sigma})}{1-\beta \cdot \tilde{\sigma}^2} \cdot \left(\tilde{\sigma} \cdot \frac{\pi_H + \pi_L}{1-\beta} - c_L \right) \cdot (q_H - q_M) + (1-\bar{\phi}) \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_M - q_L) \\
&\Leftrightarrow (1-\beta \cdot \tilde{\sigma}) \cdot \bar{\phi} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} \\
&> (1 - \bar{\phi} + \beta \cdot \tilde{\sigma} \cdot (\bar{\phi} - \tilde{\sigma})) \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} + (1 - \beta \cdot \tilde{\sigma}^2 - \beta \cdot \bar{\phi} \cdot (1-\tilde{\sigma})) \cdot c_L \cdot (q_H - q_M)
\end{aligned} \tag{3.61}$$

By the definition of $\bar{\phi}$ it holds that

$$\bar{\phi} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} = \frac{\pi_L \cdot (q_M - q_L)}{1-\beta} + \frac{1-\bar{\phi}}{1-\beta \cdot \bar{\phi}} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} + c_L \cdot (q_H - q_L)$$

Plugging in for $\bar{\phi} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta}$ in (3.61) yields:

$$
\begin{aligned}
&(1-\beta\cdot\tilde{\sigma})\cdot\left(\frac{\pi_L\cdot(q_M-q_L)}{1-\beta}+\frac{1-\bar{\phi}}{1-\beta\cdot\bar{\phi}}\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta}+c_L\cdot(q_H-q_L)\right)\\
>\ &(1-\bar{\phi}+\beta\cdot\tilde{\sigma}\cdot(\bar{\phi}-\tilde{\sigma}))\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta}+(1-\beta\cdot\tilde{\sigma}^2-\beta\cdot\bar{\phi}\cdot(1-\tilde{\sigma}))\cdot c_L\cdot(q_H-q_M)\\
\Leftrightarrow\ &(1-\beta\cdot\tilde{\sigma})\cdot\left(\frac{\pi_L\cdot(q_M-q_L)}{1-\beta}+c_L\cdot(q_H-q_L)\right)\\
>\ &\frac{\beta\cdot(\tilde{\sigma}-\bar{\phi})\cdot(1-\tilde{\sigma}-\bar{\phi}+\beta\cdot\bar{\phi}\cdot\tilde{\sigma})}{1-\beta\cdot\bar{\phi}}\cdot\frac{\pi_L\cdot(q_H-q_M)}{1-\beta}\\
+\ &(1-\beta\cdot\tilde{\sigma}^2-\beta\cdot\bar{\phi}\cdot(1-\tilde{\sigma}))\cdot c_L\cdot(q_H-q_M)
\end{aligned}
\tag{3.62}
$$

Note that (3.62) holds since:

$$
(1-\beta\cdot\tilde{\sigma})\cdot\frac{\pi_L\cdot(q_M-q_L)}{1-\beta}>\underbrace{\frac{\beta\cdot(\tilde{\sigma}-\bar{\phi})}{1-\beta\cdot\bar{\phi}}}_{<1}\cdot\underbrace{(1-\tilde{\sigma}-\bar{\phi}+\beta\cdot\bar{\phi}\cdot\tilde{\sigma})}_{<1-\beta\cdot\tilde{\sigma}}\cdot\frac{\pi_L}{1-\beta}\cdot\underbrace{(q_H-q_M)}_{\leq q_m-q_L}
$$

$$
(1-\beta\cdot\tilde{\sigma})\cdot c_L\cdot(q_H-q_L)>\underbrace{(1-\beta\cdot\tilde{\sigma}^2-\beta\cdot\bar{\phi}\cdot(1-\tilde{\sigma}))}_{<2\cdot(1-\beta\cdot\tilde{\sigma})}\cdot c_L\cdot\underbrace{(q_H-q_M)}_{\leq\frac{1}{2}\cdot(q_H-q_L)}
$$

Therefore, condition 3 is fulfilled as well. As a consequence, for the case where $\bar{\phi}$ exists, for any mixed strategy of firm A in state (q_M, q_L) where A is unconditionally mixing between 0 and $q_H - q_L$ with probabilities ϕ and $1-\phi$, respectively, the pure strategy $q_M - q_L$ is never a best response for firm B.

Case 2: $\bar{\phi}$ does not exist

$\bar{\phi}$ does not exist, if:

$$
\frac{\pi_H\cdot(q_H-q_M)}{1-\beta}-c_L\cdot(q_H-q_L)<\frac{\pi_L\cdot(q_M-q_L)}{1-\beta} \tag{3.63}
$$

A consequence of this is that the pure strategy $q_H - q_L$ for firm B is strictly dominated by the pure strategy 0. Therefore, inequality (3.54) reduces to $F_M(\phi) \geq F_L(\phi)$ and the following two conditions are sufficient for this inequality to be violated:

1. $F_L(0) > F_M(0)$
2. $F_L(1) > F_M(1)$

These conditions, together with the fact that $F_L(\phi)$ is strictly concave, while $F_M(\phi)$ is linear, yield:

$$
F_L(\phi)=F_L((1-\phi)\cdot 0+\phi\cdot 1)>(1-\phi)\cdot F_L(0)+\phi\cdot F_L(1)>\underbrace{(1-\phi)\cdot F_M(0)+\phi\cdot F_M(1)}_{=F_M(\phi)}
$$

for any $\phi \in [0,1]$. We already showed that $F_L(0) > F_M(0)$, therefore condition 1 if fulfilled. For condition 2, i.e. for $F_L(1) > F_L(1)$ we need to show that:

$$
\begin{aligned}
&\frac{\pi_L \cdot (q_M - q_L)}{1-\beta} > \frac{\beta \cdot (1-\tilde{\sigma})}{1-\beta \cdot \tilde{\sigma}^2} \cdot \left(\tilde{\sigma} \cdot \frac{\pi_H + \pi_L}{1-\beta} - c_L\right) \cdot (q_H - q_M) - c_L \cdot (q_M - q_L) \\
\Leftrightarrow\ &\frac{\pi_L \cdot (q_M - q_L)}{1-\beta} > \frac{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2} \cdot \left(\frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L)\right) \\
+\ &\frac{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2} \cdot \frac{\pi_L}{1-\beta} \cdot (q_H - q_M) - c_L \cdot (q_M - q_L) - \frac{\beta \cdot (1-\tilde{\sigma})}{1-\beta \cdot \tilde{\sigma}^2} \cdot c_L \cdot (q_H - q_M) \\
+\ &\frac{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2} \cdot c_L \cdot (q_H - q_L)
\end{aligned} \tag{3.64}
$$

Note that the highest value that $\left(\frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L)\right)$ can attain in case 2 is given by $\frac{\pi_L \cdot (q_M - q_L)}{1-\beta}$. Plugging in $\frac{\pi_L \cdot (q_M - q_L)}{1-\beta}$ for $\left(\frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L)\right)$ in (3.64) and simplifying yields:

$$
\begin{aligned}
&\frac{1-\beta \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2} \cdot \frac{\pi_L \cdot (q_M - q_L)}{1-\beta} > \frac{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2} \cdot \frac{\pi_L}{1-\beta} \cdot (q_H - q_M) \\
&-c_L \cdot (q_M - q_L) - \frac{\beta \cdot (1-\tilde{\sigma})}{1-\beta \cdot \tilde{\sigma}^2} \cdot c_L \cdot (q_H - q_M) + \frac{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2} \cdot c_L \cdot (q_H - q_L)
\end{aligned} \tag{3.65}
$$

which is fulfilled since:

$$
(1-\beta \cdot \tilde{\sigma}) \cdot \frac{\pi_L \cdot (q_M - q_L)}{1-\beta} > \underbrace{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}_{<1-\beta \cdot \tilde{\sigma}} \cdot \frac{\pi_L}{1-\beta} \cdot \underbrace{(q_H - q_M)}_{\leq (q_M - q_L)}
$$

$$
c_L \cdot (q_M - q_L) > \underbrace{\frac{\beta \cdot (1-\tilde{\sigma}) \cdot \tilde{\sigma}}{1-\beta \cdot \tilde{\sigma}^2}}_{<\frac{1}{2}} \cdot c_L \cdot \underbrace{(q_H - q_L)}_{<2 \cdot (q_M - q_L)}
$$

Therefore, inequality (3.64) holds and hence condition 2 for case 2 is fulfilled. As a consequence, for the case where $\bar{\phi}$ does not exist, for any mixed strategy of firm A in state (q_M, q_L) where A is unconditionally mixing between 0 and $q_H - q_L$ with probabilities ϕ and $1-\phi$, respectively, the pure strategy $q_M - q_L$ is strictly dominated by the pure strategy 0.
Putting together the results for case 1 and case 2 finishes the proof. We can conclude that in state (q_M, q_L), the pure strategy $q_M - q_L$ is a never best response for any mixed strategy of firm A and hence firm B will not put positive probability mass on $q_M - q_L$ in any equilibrium. □

Proposition 2. From case 2 in the proof of Lemma 5, we know that for $\frac{\pi_H \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_H - q_L) < \frac{\pi_L \cdot (q_M - q_L)}{1-\beta}$ the pure strategy $q_H - q_L$ is strictly dominated by 0 for firm B. The reason for this is given by the fact that even if firm A plays the pure strategy 0 with certainty, the expected payoff of playing 0 is strictly higher than playing $q_H - q_L$. As a consequence, elimination of strictly dominated strategies yields that firm B will play the pure strategy 0 for both cost parameters. Given this behavior, the pure strategy 0 is strictly dominated by $q_H - q_M$ for firm A when having a low cost parameter, since:

$$\frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_M) > \frac{\pi_H \cdot (q_M - q_L)}{1-\beta} \Leftrightarrow \frac{\pi_H}{1-\beta} > c_L \tag{3.66}$$

In addition, 0 will be strictly dominated by $q_H - q_M$ for firm A when having a high cost parameter if:

$$\frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_H \cdot (q_H - q_M) > \frac{\pi_H \cdot (q_M - q_L)}{1-\beta} \Leftrightarrow \frac{\pi_H}{1-\beta} > c_H \tag{3.67}$$

which finishes the proof. □

Proposition 3. Note that Lemma 5 implies that firm B is not putting any probability mass on the pure strategy $q_M - q_L$ in equilibrium. Therefore, for the derivation of the equilibrium we can restrict the action set of firm B to $\{0, q_H - q_L\}$. As in the derivation of the equilibrium of state (q_M, q_M), we omit the state in the mixing probabilities of the firms to shorten notation. The probability for firm A to play the pure strategy 0 or $q_H - q_M$ for cost parameter $c \in \{c_L, c_H\}$ is given by $\sigma_A(0|c)$ and $\sigma_A(q_H - q_M|c)$, respectively. The probability for firm B to play the pure strategy 0 or $q_H - q_L$ for cost parameter $c \in \{c_L, c_H\}$ is given by $\sigma_B(0|c)$ and $\sigma_A(q_H - q_L|c)$, respectively. Given $c \in \{c_L, c_H\}$, the Bellman equations for firm A in state (q_M, q_L) are given by:

$$\mathcal{V}_A(c, q_M, q_L) =$$

$$p \cdot \Big\{\sigma_A(0|c) \cdot \Big[\sigma_B(0|c_H) \cdot \Big(\pi_H \cdot (q_M - q_L) + \beta \cdot \Big[p \cdot \mathcal{V}_A(c_H, q_M, q_L) + (1-p) \cdot \mathcal{V}_A(c_L, q_M, q_L)\Big]\Big)$$

$$+\sigma_B(q_H - q_L|c_H) \cdot \Big(\pi_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_M, q_H)\Big)\Big]$$

$$+ \sigma_A(q_H - q_M|c) \cdot \Big[\sigma_B(0|c_H) \cdot \Big(\pi_H \cdot (q_H - q_L) - c \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_H, q_L)\Big)$$

$$+\sigma_B(q_H - q_L|c_H) \cdot \Big(- c \cdot (q_H - q_M) + \beta \cdot \mathcal{V}(q_H, q_H)\Big)\Big]\Big\}$$

$$+(1-p) \cdot \Big\{\sigma_A(0|c) \cdot \Big[\sigma_B(0|c_L) \cdot \Big(\pi_H \cdot (q_M - q_L) + \beta \cdot \Big[p \cdot \mathcal{V}_A(c_H, q_M, q_L) + (1-p) \cdot \mathcal{V}_A(c_L, q_M, q_L)\Big]\Big)$$

$$+\sigma_B(q_H - q_L|c_L) \cdot \Big(\pi_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_M, q_H)\Big)\Big]$$

$$+ \sigma_A(q_H - q_M|c) \cdot \Big[\sigma_B(0|c_L) \cdot \Big(\pi_H \cdot (q_H - q_L) - c \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_H, q_L)\Big)$$

$$+\sigma_B(q_H - q_L|c_L) \cdot \Big(- c \cdot (q_H - q_M) + \beta \cdot \mathcal{V}(q_H, q_H)\Big)\Big]\Big\}$$

Given $c \in \{c_L, c_H\}$, the Bellman equations for firm B in state (q_M, q_L) are given by:

$$\mathcal{V}_B(c, q_M, q_L) =$$

$$p \cdot \Big\{\sigma_B(0|c) \cdot \Big[\sigma_A(0|c_H) \cdot \Big(\pi_L \cdot (q_M - q_L) + \beta \cdot \Big[p \cdot \mathcal{V}_B(c_H, q_M, q_L) + (1-p) \cdot \mathcal{V}_B(c_L, q_M, q_L)\Big]\Big)$$

$$+\sigma_A(q_H - q_M|c_H) \cdot \Big(\pi_L \cdot (q_H - q_L) + \beta \cdot \mathcal{V}_B(q_H, q_L)\Big)\Big]$$

$$+ \sigma_B(q_H - q_L|c) \cdot \Big[\sigma_A(0|c_H) \cdot \Big(\pi_H \cdot (q_H - q_M) - c \cdot (q_H - q_L) + \beta \cdot \mathcal{V}_B(q_M, q_H)\Big)$$

$$+\sigma_A(q_H - q_M|c_H) \cdot \Big(- c \cdot (q_H - q_L) + \beta \cdot \mathcal{V}(q_H, q_H)\Big)\Big]\Big\}$$

$$+(1-p) \cdot \Big\{\sigma_B(0|c) \cdot \Big[\sigma_A(0|c_L) \cdot \Big(\pi_L \cdot (q_M - q_L) + \beta \cdot \Big[p \cdot \mathcal{V}_B(c_H, q_M, q_L) + (1-p) \cdot \mathcal{V}_B(c_L, q_M, q_L)\Big]\Big)$$

$$+\sigma_A(q_H - q_M|c_L) \cdot \Big(\pi_L \cdot (q_H - q_L) + \beta \cdot \mathcal{V}_B(q_H, q_L)\Big)\Big]$$

$$+ \sigma_B(q_H - q_L|c) \cdot \Big[\sigma_A(0|c_L) \cdot \Big(\pi_H \cdot (q_H - q_M) - c \cdot (q_H - q_L) + \beta \cdot \mathcal{V}_B(q_M, q_H)\Big)$$

$$+\sigma_A(q_H - q_M|c_L) \cdot \Big(- c \cdot (q_H - q_L) + \beta \cdot \mathcal{V}(q_H, q_H)\Big)\Big]\Big\}$$

Solving the system of bellman equations for $\mathcal{V}_i(c, q_M, q_L)$, $c \in \{c_L, c_H\}$ and $i \in \{A, B\}$ yields, where $\tilde{\sigma}_i := (1-p)\cdot\sigma_i(0|c_L) + p\cdot\sigma_i(0|c_H)$:

$$\mathcal{V}_A(c_H, q_M, q_L) =$$

$$\frac{\pi_H\tilde{\sigma}_B\Big((q_H-q_L)[\beta(1-p)\tilde{\sigma}_B[\sigma_A(0|c_H)-\sigma_A(0|c_L)]+\sigma_A(q_H-q_M|c_H)]+(q_M-q_L)(1-\beta)\sigma_A(0|c_H)\Big)}{(1-\beta)\cdot(1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B)}$$

$$+\frac{\pi_L\cdot(q_H-q_M)\cdot\sigma_A(0|c_H)\cdot(1-\tilde{\sigma}_B)}{(1-\beta)\cdot(1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B)}$$

$$-\frac{c_H\cdot(q_H-q_M)\cdot\sigma_A(q_H-q_M|c_H)\cdot(1-(1-p)\cdot\beta\cdot\sigma_A(0|c_L)\cdot\tilde{\sigma}_B)}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}$$

$$-\frac{c_L\cdot(q_H-q_M)\cdot(1-p)\cdot\beta\cdot\sigma_A(0|c_H)\cdot\sigma_A(q_H-q_M|c_L)\cdot\tilde{\sigma}_B}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}$$

$$\mathcal{V}_A(c_L, q_M, q_L) =$$

$$\frac{\pi_H\tilde{\sigma}_B\Big((q_H-q_L)[\beta(1-p)\cdot\tilde{\sigma}_B[\sigma_A(0|c_L)-\sigma_A(0|c_H)]+\sigma_A(q_H-q_M|c_L)]+(q_M-q_L)(1-\beta)\sigma_A(0|c_L)\Big)}{(1-\beta)\cdot(1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B)}$$

$$+\frac{\pi_L\cdot(q_H-q_M)\cdot\sigma_A(0|c_L)\cdot(1-\tilde{\sigma}_B)}{(1-\beta)\cdot(1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B)}$$

$$-\frac{c_H\cdot(q_H-q_M)\cdot(1-p)\cdot\beta\cdot\sigma_A(0|c_L)\cdot\sigma_A(q_H-q_M|c_H)\cdot\tilde{\sigma}_B}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}$$

$$-\frac{c_L\cdot(q_H-q_M)\cdot\sigma_A(q_H-q_M|c_L)\cdot(1-(1-p)\cdot\beta\cdot\sigma_A(0|c_H)\cdot\tilde{\sigma}_B)}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}$$

$$\mathcal{V}_B(c_H, q_M, q_L) = \frac{\pi_H\cdot(q_H-q_M)\cdot\tilde{\sigma}_A\cdot\Big(\beta\cdot(1-p)\cdot[\sigma_B(0|c_H)-\sigma_B(0|c_L)]\cdot\tilde{\sigma}_A+\sigma_B(q_H-q_L|c_H)\Big)}{(1-\beta)\cdot(1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B)}$$

$$+\frac{\pi_L\cdot\sigma_B(0|c_H)\cdot[(q_H-q_L)\cdot(1-\tilde{\sigma}_A)+(q_M-q_L)\cdot(1-\beta)\cdot\tilde{\sigma}_A]}{(1-\beta)\cdot(1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B)}$$

$$-\frac{c_H\cdot(q_H-q_L)\cdot\sigma(q_H-q_L|c_H)\cdot(1-\beta\cdot(1-p)\cdot\sigma_B(0|c_L)\cdot\tilde{\sigma}_A}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}$$

$$-\frac{c_L\cdot(q_H-q_L)\cdot(1-p)\cdot\sigma_B(0|c_H)\cdot\sigma_B(q_H-q_L|c_L)\cdot\tilde{\sigma}_A}{1-\beta\cdot\tilde{\sigma}_A\cdot\tilde{\sigma}_B}$$

$$\mathcal{V}_B(c_L, q_M, q_L) = \frac{\pi_H \cdot (q_H - q_M) \cdot \tilde{\sigma}_A \cdot \Big(\beta \cdot (1-p) \cdot [\sigma_B(0|c_L) - \sigma_B(0|c_H)] \cdot \tilde{\sigma}_A + \sigma_B(q_H - q_L|c_L)\Big)}{(1-\beta) \cdot (1 - \beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B)}$$

$$+ \frac{\pi_L \cdot \sigma_B(0|c_L) \cdot [(q_H - q_L) \cdot (1 - \tilde{\sigma}_A) + (q_M - q_L) \cdot (1-\beta) \cdot \tilde{\sigma}_A)]}{(1-\beta) \cdot (1 - \beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B)}$$

$$- \frac{c_H \cdot (q_H - q_L) \cdot (1-p) \cdot \sigma_B(0|c_L) \cdot \sigma_B(q_H - q_L|c_H) \cdot \tilde{\sigma}_A}{1 - \beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B}$$

$$- \frac{c_L \cdot (q_H - q_L) \cdot \sigma(q_H - q_L|c_L) \cdot (1 - \beta \cdot (1-p) \cdot \sigma_B(0|c_H) \cdot \tilde{\sigma}_A)}{1 - \beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B}$$

Next, we want to derive an equilibrium, where firms play 0 whenever they have a high cost parameter, while mixing whenever they have a low cost parameter. Therefore, let the mixed strategies of the firms be given by:

$$\sigma_A = \{\sigma_A(0|c_H), \sigma_A(q_H - q_M|c_H), \sigma_A(0|c_L), \sigma_A(q_H - q_M|c_L)\} = \{1, 0, \sigma_A(0|c_L), \sigma_A(q_H - q_M|c_L)\}$$

$$\sigma_B = \{\sigma_B(0|c_H), \sigma_B(q_H - q_L|c_H), \sigma_B(0|c_L), \sigma_B(q_H - q_L|c_L)\} = \{1, 0, \sigma_B(0|c_L), \sigma_B(q_H - q_L|c_L)\}$$

The expected profit of firm A, when playing pure strategy 0 is given by:

$$\mathbb{E}_A[0|c_L, \sigma_B] = [p + (1-p) \cdot \sigma_B(0|c_L)] \cdot [\pi_H \cdot (q_M - q_L) + \beta \cdot (p \cdot \mathcal{V}_A(c_H, q_M, q_L) + (1-p) \cdot \mathcal{V}_A(c_L, q_M, q_L))]$$

$$+ (1-p) \cdot \sigma_B(q_H - q_L|c_L) \cdot [\pi_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_M, q_H)]$$

The expected profit of firm A, when playing pure strategy $q_H - q_M$ is given by:

$$\mathbb{E}_A[q_H - q_M|c_L, \sigma_B] = [p + (1-p) \cdot \sigma_B(0|c_L)] \cdot [\pi_H \cdot (q_H - q_L) - c_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}_A(q_H, q_L)]$$

$$+ (1-p) \cdot \sigma_B(q_H - q_L|c_L) \cdot [-c_L \cdot (q_H - q_M) + \beta \cdot \mathcal{V}(q_H, q_H)]$$

The expected profit of firm B, when playing pure strategy 0 is given by:

$$\mathbb{E}_B[0|c_L, \sigma_A] = [p + (1-p) \cdot \sigma_A(0|c_L)] \cdot [\pi_L \cdot (q_M - q_L) + \beta \cdot (p \cdot \mathcal{V}_B(c_H, q_M, q_L) + (1-p) \cdot \mathcal{V}_B(c_L, q_M, q_L))]$$

$$+ (1-p) \cdot \sigma_A(q_H - q_M|c_L) \cdot [\pi_L \cdot (q_H - q_L) + \beta \cdot \mathcal{V}_B(q_H, q_L)]$$

The expected profit of firm B, when playing pure strategy $q_H - q_L$ is given by:

$$\mathbb{E}_B[q_H - q_L|c_L, \sigma_A] = [p + (1-p) \cdot \sigma_A(0|c_L)] \cdot [\pi_H \cdot (q_H - q_M) - c_L \cdot (q_H - q_L) + \beta \cdot \mathcal{V}_B(q_M, q_H)]$$

$$+ (1-p) \cdot \sigma_A(q_H - q_M|c_L) \cdot [-c_L \cdot (q_H - q_L) + \beta \cdot \mathcal{V}(q_H, q_H)]$$

Setting $\mathbb{E}_A[0|c_L, \sigma_B] = \mathbb{E}_A[q_H - q_M|c_L, \sigma_B]$, $\mathbb{E}_B[0|c_L, \sigma_A] = \mathbb{E}_B[q_H - q_L|c_L, \sigma_A]$,plugging in for the value functions, considering that $\sigma_A(q_H - q_M|c_L) = 1 - \sigma_A(0|c_L)$,$\sigma_B(q_H - q_L|c_L) = 1 - \sigma_B(0|c_L)$

and solving for $\sigma_A(0|c_L)$ and $\sigma_B(0|c_L)$ yields:

$$\sigma_A(0|c_L) = \frac{(q_H - q_M)\cdot(\pi_H + \pi_L) + \beta\cdot(1-\beta)\cdot(q_H - q_L)\cdot c_L + (q_M - q_L)\cdot\beta\cdot\pi_L}{2\cdot\beta\cdot(1-p)\cdot(q_H - q_M)\cdot\pi_H} - \frac{p}{1-p}$$

$$-\frac{\sqrt{\begin{array}{l}\pi_L^2(q_H - q_M + \beta(q_M - q_L))^2 + \pi_H(\pi_H + 2\pi_L)(q_H - q_M)^2 - 2\beta\pi_H\pi_L(q_H - q_M)(2q_H - q_M - q_L)\\ -c_L(q_H - q_L)\beta(1-\beta)[2(\pi_H - \pi_L)(q_H - q_M) - 2\pi_L\beta(q_M - q_L) - c_L(q_H - q_L)\beta(1-\beta)]\end{array}}}{2\cdot\beta\cdot(1-p)\cdot\pi_H\cdot(q_H - q_M)}$$

$$\sigma_B(0|c_L) = \frac{(q_H - q_M)\cdot[\pi_H + \pi_L + \beta\cdot(1-\beta)\cdot c_L] + (q_M - q_L)\cdot\beta\cdot\pi_H}{2\cdot\beta\cdot(1-p)\cdot(q_H - q_L)\cdot\pi_H} - \frac{p}{1-p}$$

$$-\frac{\sqrt{\begin{array}{l}\pi_H^2(q_H - q_M + \beta(q_M - q_L))^2 + \pi_L(2\pi_H + \pi_L)(q_H - q_M)^2 - 2\beta\pi_H\pi_L(q_H - q_M)(2q_H - q_M - q_L)\\ -c_L(q_H - q_M)\beta(1-\beta)[(2(\pi_H - \pi_L) - c_L\beta(1-\beta))(q_H - q_M) + 2\pi_H(q_M - q_L)(2-\beta)]\end{array}}}{2\cdot\beta\cdot(1-p)\cdot\pi_H\cdot(q_H - q_L)}$$

Using the following substitutions:

$$\lambda = \pi_H + \pi_L + \beta\cdot(1-\beta)\cdot c_L$$

$$\gamma = 4\cdot\pi_H\cdot\pi_L\cdot(1-\beta)$$

$$\eta = 2\cdot\beta\cdot(1-p)\cdot\pi_H$$

$$\mu = \beta\cdot(\pi_L + (1-\beta)\cdot c_L)$$

$$\alpha = (1-\beta)\cdot\pi_H + \pi_L + \beta\cdot(1-\beta)\cdot c_L$$

We can simplify the expressions to get:

$$\sigma_A(0|c_L) = \frac{\lambda + \frac{q_M - q_L}{q_H - q_M}\cdot\mu - \sqrt{[\lambda + \frac{q_M - q_L}{q_H - q_M}\cdot\mu]^2 - \eta\cdot\gamma\cdot\frac{q_H - q_L}{q_H - q_M}}}{\eta\cdot(1-p)} - \frac{p}{1-p}$$

$$\sigma_B(0|c_L) = \frac{\lambda - \frac{q_M - q_L}{q_H - q_L}\cdot\alpha - \sqrt{[\lambda - \frac{q_M - q_L}{q_H - q_L}\cdot\alpha]^2 - \eta\cdot\gamma\cdot\frac{q_H - q_M}{q_H - q_L}}}{\eta\cdot(1-p)} - \frac{p}{1-p}$$

Note that both expressions can be strictly smaller than 0, for instance for large values of p, while $\sigma_A(0|c_L)$ can also be strictly larger than 1, for instance when q_M is close to q_H. Therefore, the probabilities to play 0 in the equilibrium of state (q_M, q_L) are given by:

$$\sigma_A^*(0|c_L, q_M, q_L) = \min\left\{1, \max\left\{0, \frac{\lambda + \frac{q_M - q_L}{q_H - q_M}\cdot\mu - \sqrt{[\lambda + \frac{q_M - q_L}{q_H - q_M}\cdot\mu]^2 - \eta\cdot\gamma\cdot\frac{q_H - q_L}{q_H - q_M}}}{\eta\cdot(1-p)} - \frac{p}{1-p}\right\}\right\}$$

$$\sigma_B^*(0|c_L, q_M, q_L) = \max\{0, \frac{\lambda - \frac{q_M - q_L}{q_H - q_L}\cdot\alpha - \sqrt{[\lambda - \frac{q_M - q_L}{q_H - q_L}\cdot\alpha]^2 - \eta\cdot\gamma\cdot\frac{q_H - q_M}{q_H - q_L}}}{\eta\cdot(1-p)} - \frac{p}{1-p}\Big\}$$

Finally, the unconditional probabilities to play 0 in the equilibrium of state (q_M, q_L) are given by:

$$\tilde{\sigma}_A^*(0|q_M, q_L) = \min\left\{1, \max\left\{p, \frac{\lambda + \frac{q_M - q_L}{q_H - q_M} \cdot \mu - \sqrt{[\lambda + \frac{q_M - q_L}{q_H - q_M} \cdot \mu]^2 - \eta \cdot \gamma \cdot \frac{q_H - q_L}{q_H - q_M}}}{\eta}\right\}\right\}$$

$$\tilde{\sigma}_B^*(0|q_M, q_L) = \max\left\{p, \frac{\lambda - \frac{q_M - q_L}{q_H - q_L} \cdot \alpha - \sqrt{[\lambda - \frac{q_M - q_L}{q_H - q_L} \cdot \alpha]^2 - \eta \cdot \gamma \cdot \frac{q_H - q_M}{q_H - q_L}}}{\eta}\right\}$$

□

Corollary 2. The steps are similar to the derivation of the results in Corollary 1 and will be omitted. □

Corollary 3. This follows simply by comparison of the unconditional equilibrium probabilities to play 0 in each of the propositions. □

Lemma 6. Suppose that for firm A, $q_M - q_L$ is a best response for some Markovian strategy of firm B, where firm B is mixing between 0 and $q_H - q_L$ with unconditional probabilities ϕ and $1 - \phi$, respectively, whenever the state is (q_L, q_L). In order for $q_M - q_L$ to be a best response against ϕ in state (q_L, q_L), it must hold that the expected net present value profit of playing the pure strategy $q_M - q_L$ is weakly higher than playing either 0 or $q_H - q_L$. Let us first consider the expected net present value profits for the different pure strategies, given that firm B mixes as described before. Playing the pure strategy 0 yields:

$$\phi \cdot \beta \cdot \mathbb{E}[0|q_L, q_L, \phi] + (1 - \phi) \cdot \frac{\pi_L \cdot (q_H - q_L)}{1 - \beta} \tag{3.68}$$

where $\mathbb{E}[0|q_L, q_L, \phi]$ denotes the expected net present value profit of firm A when playing 0 every-time the state remains in (q_L, q_M), while firm B is mixing with ϕ and $1 - \phi$, and is given by:

$$\mathbb{E}[0|q_L, q_L, \phi] = \frac{1 - \phi}{1 - \beta \cdot \phi} \cdot \frac{\pi_L \cdot (q_H - q_L)}{1 - \beta} \tag{3.69}$$

Plugging (3.69) into (3.68) and simplifying yields:

$$G_L(\phi) := \frac{1 - \phi}{1 - \beta \cdot \phi} \cdot \frac{\pi_L \cdot (q_H - q_L)}{1 - \beta} \tag{3.70}$$

Playing the pure strategy $q_H - q_L$ yields:

$$G_H(\phi) := \phi \cdot \frac{\pi_H \cdot (q_H - q_L)}{1 - \beta} - c_L \cdot (q_H - q_L) \tag{3.71}$$

Finally, playing the pure strategy $q_M - q_L$ yields:

$$G_M(\phi) := \phi \cdot (\pi_H \cdot (q_M - q_L) + \beta \cdot \mathcal{V}_A(q_M, q_L)) + (1-\phi) \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_M - q_L) \tag{3.72}$$

Where $\mathcal{V}_A(q_M, q_L)$ denotes the expected value function of firm A for state (q_M, q_L). In order for $q_M - q_L$ to be a best response against some ϕ, it must simultaneously hold that $G_M(\phi)$ is weakly larger than $G_L(\phi)$ and $G_H(\phi)$. Therefore, it must hold that:

$$G_M(\phi) \geq \max\{G_L(\phi), G_H(\phi)\} \tag{3.73}$$

In what follows, we want to show that for any $\phi \in [0,1]$ the inequality described in (3.73) cannot hold. First, note that $G_H(\phi)$ is strictly increasing and linear in ϕ, $G_L(\phi)$ is strictly decreasing and strictly concave in ϕ and $G_M(\phi)$ is linear and either strictly increasing or strictly decreasing in ϕ. Consider that for this state, $\hat{\phi}$, such that $G_L(\hat{\phi}) = G_H(\hat{\phi})$, exists if $\frac{\pi_H}{1-\beta} > c_L$, which holds by assumption. Too see this note that:

$$G_L(0) = \frac{\pi_L \cdot (q_H - q_L)}{1-\beta} > -c_L \cdot (q_H - q_L) = G_H(0) \tag{3.74}$$

$$G_L(1) = 0 < \frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_L) = G_H(1) \tag{3.75}$$

Due to the monotonicity of both functions they cross exactly once at $\hat{\phi}$, which is implicitly defined by:

$$\hat{\phi} \cdot \frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_L) = \frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L \cdot (q_H - q_L)}{1-\beta} \tag{3.76}$$

Note that, as in the proof of Lemma 5, $\hat{\phi}$ in this case describes the minimum of the function $\max\{G_L(\phi), G_H(\phi)\}$. First we derive a set of sufficient conditions for the inequality in (3.73) to be violated:

1. $G_L(0) > G_M(0)$
2. $G_H(1) > G_M(1)$
3. $G_L(\hat{\phi}) = G_H(\hat{\phi}) > G_M(\hat{\phi})$

The steps to show that those conditions are sufficient for (3.73) to be violated are equivalent to the steps in case 1 within the proof of Lemma 5 and will be omitted here. Condition 1 holds generally, since:

$$G_L(0) = \frac{\pi_L \cdot (q_H - q_L)}{1-\beta} > \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_M - q_L) = G_M(0) \tag{3.77}$$

whereas condition 2 holds if:

$$G_H(1) = \frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_L) > \underbrace{\pi_H \cdot (q_M - q_L) + \beta \cdot \mathcal{V}_A(q_M, q_L) - c_L \cdot (q_M - q_L)}_{= G_M(1)}$$

Note that an upper boundary for $\mathcal{V}(q_M, q_L)$ is given by $\frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_M)$, such that condition 2 holds if:

$$\frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_L) > \pi_H \cdot (q_M - q_L) + \beta \cdot \left(\frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_M) \right) - c_L \cdot (q_M - q_L)$$

which is equivalent to

$$\frac{\pi_H \cdot (q_M - q_L)}{1-\beta} > c_L \cdot (q_H - q_M) \tag{3.78}$$

which is fulfilled since we assume that $q_M - q_L \geq q_H - q_M$ and $\frac{\pi_H}{1-\beta} > c_L$.

It remains to show that $G_L(\hat{\phi}) = G_H(\hat{\phi}) > G_M(\hat{\phi})$. In particular, we want to show that:

$$\begin{aligned} &\hat{\phi} \cdot \frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - c_L \cdot (q_H - q_L) > \hat{\phi} \cdot (\pi_H \cdot (q_M - q_L) + \beta \cdot \mathcal{V}_A(q_M, q_M)) \\ &+(1-\hat{\phi}) \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} - c_L \cdot (q_M - q_L) \\ \Leftrightarrow &\hat{\phi} \cdot \frac{\pi_H \cdot (q_H - q_L)}{1-\beta} > \hat{\phi} \cdot \Big(\pi_H \cdot (q_M - q_L) + \beta \cdot \mathcal{V}_A(q_M, q_M) \Big) \\ &+\beta \cdot \hat{\phi} \cdot c_L \cdot (q_H - q_M) + (1 - \beta \cdot \hat{\phi}) \cdot \left(\frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L}{1-\beta} + c_L \right) \cdot (q_H - q_M) \end{aligned} \tag{3.79}$$

By the definition of $\hat{\phi}$, it holds that

$$\frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L}{1-\beta} + c_L = \hat{\phi} \cdot \frac{\pi_H}{1-\beta} \tag{3.80}$$

Plugging in for $\frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L}{1-\beta} + c_L$ in (3.79) and simplifying yields:

$$\frac{\pi_H \cdot (q_M - q_L)}{1-\beta} + \left(\hat{\phi} \cdot \frac{\pi_H}{1-\beta} - c_L \right) \cdot (q_H - q_M) > \mathcal{V}_A(q_M, q_L) \tag{3.81}$$

$$\Leftrightarrow \frac{\pi_H \cdot (q_M - q_L)}{1-\beta} + \left(\frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L}{1-\beta} \right) \cdot (q_H - q_M) > \mathcal{V}_A(q_M, q_L) \tag{3.82}$$

where from (3.81) to (3.82) we again used the definition of $\hat{\phi}$. In what follows, we will bring $\mathcal{V}_A(q_M, q_L)$ in a suitable form to show that inequality (3.82) holds. Let $\tilde{\sigma}_A$ and $\tilde{\sigma}_B$ denote the unconditional probabilities of the firms to play 0 in the equilibrium of state (q_M, q_L), the expected value function $\mathcal{V}_A(q_M, q_L)$ is then given by:

$$
\begin{aligned}
\mathcal{V}_A(q_M, q_L) =& \frac{\tilde{\sigma}_A \cdot \tilde{\sigma}_B}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \pi_H \cdot (q_M - q_L) + \frac{\tilde{\sigma}_A \cdot (1-\tilde{\sigma}_B)}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} \\
&+ \frac{(1-\tilde{\sigma}_A) \cdot \tilde{\sigma}_B}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_H \cdot (q_H - q_L)}{1-\beta} - \frac{1-\tilde{\sigma}_A}{1-\beta \cdot \tilde{\sigma}_A} \cdot c_L \cdot (q_H - q_M) \\
=& \frac{(1-\beta \cdot \tilde{\sigma}_A) \cdot \tilde{\sigma}_B}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_H \cdot (q_M - q_L)}{1-\beta} + \frac{\tilde{\sigma}_A \cdot (1-\tilde{\sigma}_B)}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} \\
&+ \frac{1-\tilde{\sigma}_A}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \left((\tilde{\sigma}_B - \hat{\phi}) \cdot \frac{\pi_H}{1-\beta} + \hat{\phi} \cdot \frac{\pi_H}{1-\beta} - c_L \right) \cdot (q_H - q_M)
\end{aligned}
\tag{3.83}
$$

Using the fact that $\hat{\phi} \cdot \frac{\pi_H}{1-\beta} - c_L = \frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L}{1-\beta}$ and plugging in for $\hat{\phi} \cdot \frac{\pi_H}{1-\beta} - c_L$ in the last column of (3.83) we finally have:

$$
\begin{aligned}
\mathcal{V}_A(q_M, q_L) =& \frac{(1-\beta \cdot \tilde{\sigma}_A) \cdot \tilde{\sigma}_B}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_H \cdot (q_M - q_L)}{1-\beta} + \frac{\tilde{\sigma}_A \cdot (1-\tilde{\sigma}_B)}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta} \\
&+ \frac{(1-\tilde{\sigma}_A) \cdot (\tilde{\sigma}_B - \hat{\phi})}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{\pi_H \cdot (q_H - q_M)}{1-\beta} + \frac{1-\tilde{\sigma}_A}{1-\beta \cdot \tilde{\sigma}_A \cdot \tilde{\sigma}_B} \cdot \frac{1-\hat{\phi}}{1-\beta \cdot \hat{\phi}} \cdot \frac{\pi_L \cdot (q_H - q_M)}{1-\beta}
\end{aligned}
$$

Plugging this expression into (3.82) and simplifying yields:

$$
(1-\tilde{\sigma}_B) \cdot \frac{\pi_H \cdot (q_M - q_L)}{1-\beta} > (\tilde{\sigma}_B - \hat{\phi}) \cdot \left((1-\tilde{\sigma}_A) \cdot \frac{\pi_H}{1-\beta} - \tilde{\sigma}_A \cdot \frac{\pi_L}{1-\beta \cdot \hat{\phi}} \right) \cdot (q_H - q_M) \tag{3.84}
$$

Consider first that $\hat{\phi} \leq \tilde{\sigma}_B$. In this case (3.84) holds since:

$$
(1-\tilde{\sigma}_B) \cdot \frac{\pi_H}{1-\beta} \cdot (q_M - q_L) > \underbrace{(\tilde{\sigma}_B - \hat{\phi})}_{<1} \cdot \underbrace{(1-\tilde{\sigma}_A)}_{\leq 1-\tilde{\sigma}_B} \cdot \frac{\pi_H}{1-\beta} \cdot \underbrace{(q_H - q_M)}_{\leq q_M - q_L}
$$

If $\hat{\phi} > \tilde{\sigma}_B$, (3.84) holds as well since:

$$
(1-\tilde{\sigma}_B) \cdot \frac{\pi_H}{1-\beta} \cdot (q_M - q_L) > \underbrace{(\hat{\phi} - \tilde{\sigma}_B)}_{<1-\tilde{\sigma}_B} \cdot \underbrace{\tilde{\sigma}_A}_{\leq 1} \cdot \underbrace{\frac{\pi_L}{1-\beta \cdot \hat{\phi}}}_{<\frac{\pi_H}{1-\beta}} \cdot \underbrace{(q_H - q_M)}_{\leq q_M - q_L}
$$

As a consequence (3.84) holds such that condition 3 is fulfilled and therefore, inequality (3.73) does not hold. Therefore, for any mixed strategy of firm B in state (q_L, q_L) where B is unconditionally mixing between 0 and $q_H - q_L$ with probabilities ϕ and $1-\phi$, respectively, the pure strategy $q_M - q_L$ is never a best response for firm A in state (q_L, q_L). As a consequence firm A will not put positive probability mass on $q_M - q_L$ in any equilibrium. Note that Lemma 6 also implies that $q_M - q_L$ is a never best response for firm A if firm B is mixing between all three pure strategies. Using the same logic for firm B, we can conclude that no equilibrium exists, where the firms put positive probability mass on the pure strategy $q_M - q_L$. □

Proposition 4. Note that since no firm will put probability mass on the pure strategy $q_M - q_L$

in equilibrium, the equilibrium probabilities for state (q_L, q_L) are equivalent to the equilibrium in state (q_M, q_M).
The reason is as follows: in state (q_M, q_M), firms make either zero profits, or (possibly negative) profits which are proportional to $q_H - q_M$. The same holds for state q_L, q_L when firms are mixing only between $q_H - q_L$ and 0, however, the profits are proportional to $q_H - q_L$. As a consequence, the following relationship holds with respect to the value functions of those states for cost parameter $c \in \{c_L, c_H\}$:

$$\mathcal{V}(c, q_L, q_L) = \frac{q_H - q_L}{q_H - q_M} \cdot \mathcal{V}(c, q_M, q_M) \tag{3.85}$$

Hence, the following relationship will hold with respect to the expected profits of firm A when playing the pure strategies 0 and $q_H - q_L$ in state (q_L, q_L), given that firm B is playing the pure strategy 0 when having cost parameter c_H, while mixing between 0 and $q_H - q_L$ ($q_H - q_M$ for state (q_M, q_M)) with σ_B and $1 - \sigma_B$, respectively, when having cost parameter c_L:

$$\mathbb{E}[0|c_L, q_L, q_L, \sigma_B] = \frac{q_H - q_L}{q_H - q_M} \cdot \mathbb{E}[0|c_L, q_M, q_M, \sigma_B]$$

$$\mathbb{E}[q_H - q_L|c_L, q_L, q_L, \sigma_B] = \frac{q_H - q_L}{q_H - q_M} \cdot \mathbb{E}[q_H - q_M|c_L, q_M, q_M, \sigma_B]$$

Therefore, the symmetric equilibrium in both states is equivalent and hence for state (q_L, q_L) we have the following equilibrium probability to play 0 when having a low cost parameter:

$$\sigma^*(0|c_L, q_L, q_L) = \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta \cdot (1-p)} - \frac{p}{1-p} \tag{3.86}$$

where

$$\lambda = \pi_H + \pi_L + \beta \cdot (1-\beta) \cdot c_L \tag{3.87}$$

$$\gamma = 2 \cdot (\pi_L + (1-\beta) \cdot c_L) \tag{3.88}$$

$$\eta = 2 \cdot \beta \cdot \pi_H \tag{3.89}$$

while the unconditional equilibrium probability to play 0 is given by

$$\tilde{\sigma}^*(0|q_L, q_L) = \max\left\{p, \frac{\lambda - \sqrt{\lambda^2 - \eta \cdot \gamma}}{\eta}\right\}. \tag{3.90}$$

□

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Conclusion

In this thesis, I was concerned with the fields of behavioral economics and industrial organization.

In the first chapter, I studied a sequential search model and identified conditions, such that a choice overload effect and a status quo bias could occur within my framework. I showed that both effects could not occur within my framework given that the agent behaves like an Expected Utility maximizer. I then proceeded and assumed that the agent suffers from Forward-Looking-Regret, which I defined as an aversion to missing out on better options in case of stopping the searching process too early. Surprisingly, I found that the agent's optimal stopping behavior is non-stationary: the reward he demands to stop the searching process increases, the more search he has already conducted. I showed that whenever the agent suffered disproportionally from regret, increasing the choice set's size would lead to an excessive search that would end with the agent being dissatisfied with his choice. If possible, the agent would ex-ante commit to decreasing the size of the choice set. Furthermore, I identified conditions where the agent would not even start the searching process when increasing the choice set's size. He would stick to his outside option, giving rise to a status quo bias, even though the choice set would consist only of desirable options, all of which are superior to his outside option. Finally, I considered an extension by introducing a second type of regret while simultaneously allowing the agent to feel joy and showed that the choice overload can still occur.

In the second chapter, I revisited a classical vertical differentiation model and assumed that the competing firms are uncertain about their rival's fixed costs of quality improvements. I showed how this assumption would lead to an equilibrium, where firms would choose similar quality levels whenever they would be similarly efficient. Consequently, the firms could end up with close to homogenous products, yielding close to zero revenues for both firms, while both suffered from fixed costs of quality improvements. This equilibrium was in stark contrast to a second equilibrium, which I identified, which is reminiscent of the maximum differentiation paradigm in the classical literature of vertical differentiation. In this equilibrium, firms maximally differentiate their products. One of the firms acts as a free rider, benefiting from a high degree of product differentiation while not suffering from any costs of quality improvements. However, in contrast to the classical literature, I derive a sufficient and necessary condition for the existence of such an equilibrium. I then continued to analyze the qualitative features of both equilibria and showed that, within the concepts of stochastic orders, the monotone likelihood ratio order is a very useful tool to study comparative statics in models of vertical differentiation. Among many results, I showed that an increase in the probability that consumers have a high willingness to pay for quality could increase the consumer rent for all types of consumers in the population.

In the third and final chapter, which is joint work with Kangkan Choudhury, we embedded my second chapter's model in a dynamic framework. By doing so, we could go beyond the second chapter's scope by analyzing which type of innovation patterns emerge over time. We identified a Markov Perfect equilibrium, which, to our surprise, yielded a negative result concerning some innovation patterns. In particular, we found that even-though firms could choose intermediate quality levels in our model, one property of the equilibrium was given by firms either staying at the lowest possible quality level or firms choosing to the highest possible quality level. This is surprising since in the case where firms would choose the highest possible quality level, they end up in an absorbing state where they make zero profits forever. On the contrary, by choosing intermediate quality levels, the game could end up in states where both firms would make strictly positive profits forever. A direct consequence of the negativity result was that leap-frogging was not possible, given the game starts in a state where both firms are ex-ante symmetric and start from the lowest possible quality level.

DOI: 10.17185/duepublico/74365
URN: urn:nbn:de:hbz:464-20210518-110107-3