Equimomental Polygonal Systems

Stefan Gössner

Mechanical Engineering Dept., University of Applied Sciences, Dortmund, Germany.

Feb. 2021

Keywords: equimomental systems; dynamic equivalence; point-mass; mass moment of inertia; centroid; polygon; symplectic geometry;

Abstract

Two systems of rigid bodies are called *equimomental* if their mass, the position of their center of mass and their moment of inertia are identical with respect to any straight line in space.

During a synthesis of planar mechanisms, ternary, quaternary, quinary (...) members are initially abstracted as polygons. Now, in order to be able to make well-approximated statements about the dynamic behavior at a rather early stage of link design, the placement of suitable point-masses in joint centers and link centroids is a simple and effective measure.

Against such an application-specific background, a dimensioning procedure for those point masses is now discussed in this paper. Their respective masses can be derived pragmatically from the area and shape of the polygonal links used. The presented method is robust, efficient and can also be applied to non-simple polygons.

1. Introduction

> *Two mass-systems are said to be* equimomental *if they have equal second moments (moments of inertia) about any line in space* [1].
> — A. Talbot (1952)

Talbot also shows more concretely: "*Two equimomental systems will also have the same total mass, and the same centroid then*". What Talbot said is valid even today: "Standard textbooks on Statics or Mechanics say very little about *equimomental systems*". The curious reader is recommended to read the paper of L.P. Laus and J.M. Selig [3], where they are giving a good summary of aspects and history of the study of these systems.

More recent literature uses equimomental systems for dynamic balancing spatial mechanisms [4] or takes mechanism topology of planar structures into account [5]. In this paper polygonal link geometry is discussed exclusively.

It was already shown in 1897 by Routh [2], that a mimimum of four point-masses is required to be equimomental to a spatial rigid body. For laminae, in the planar case, a minimum of two point-masses is needed. There is no upper limit for the number of point-masses.

An arbitrary planar body – a lamina with constant density and thickness – has given mass M and mass moment of inertia J about its centroid (Fig. 1a).

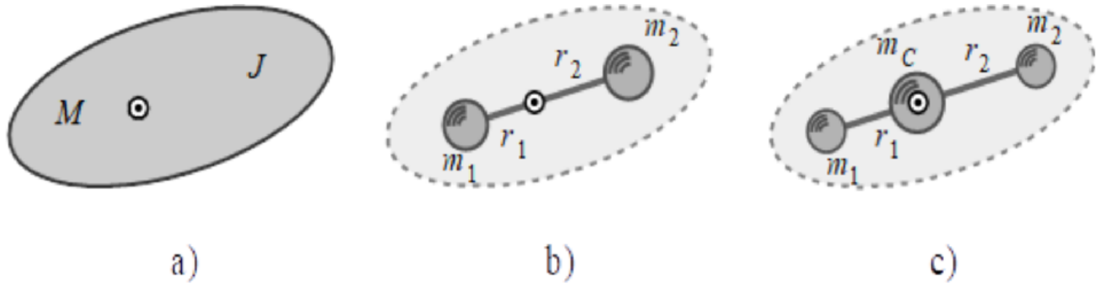

Fig. 1: Equimomental systems – Planar rigid body vs. Point-masses

Now in an equimomental system with two point-masses, m_1 and m_2 must be located on a line through the centroid, with distances r_1 and r_2 to the center of mass – on different sides if we want positive masses (Fig. 1b).

$$\begin{aligned} m_1 + m_2 &= M \\ m_1 r_1 + m_2 r_2 &= 0 \\ m_1 r_1^2 + m_2 r_2^2 &= J \end{aligned} \tag{1}$$

Those three expressions (1) contain four unknowns – the masses and their locations. Thus substituting $m_1 = m$ and setting $m_2 = \lambda m$ to a given multiple λ of it, we get

$$m = \frac{M}{1+\lambda}, \quad r_1 = \sqrt{\frac{\lambda J}{M}}, \quad r_2 = -\sqrt{\frac{J}{\lambda M}}.$$

However, if we now want to specify the positions of both point masses, then the system of equations (1) is overdetermined. To clear up this condition, we place another point mass m_C in the center of gravity (Fig. 1c). Now the first of the three equations (1) reads $m_1 + m_2 + m_C = M$, which then leads to the three point masses

$$m_1 = \frac{J}{r_1^2 - r_1 r_2}, \quad m_2 = \frac{J}{r_2^2 - r_1 r_2}, \quad m_C = M - m_1 - m_2 = M + \frac{J}{r_1 r_2}.$$

The general conclusion of this discussion is:

> *If point mass positions in an equimomental system are explicitly specified, a certain centroidal mass is needed. The occurrence of negative masses cannot be excluded.*

2. Two Equimomental Polygonal Systems

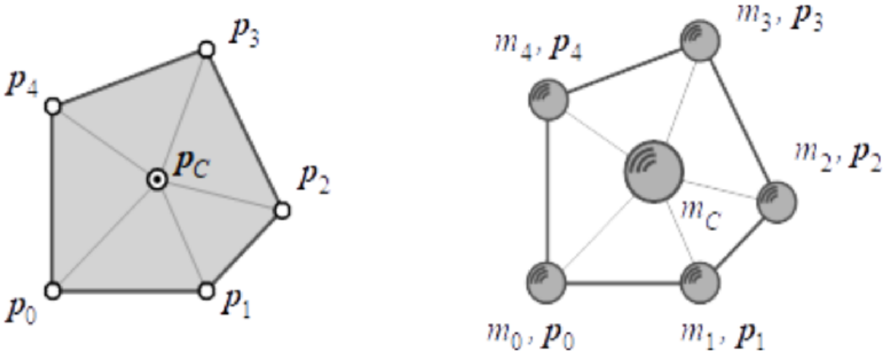

Fig. 2: Two planar equimomental polygonal systems

We start with a polygon, supposed to have constant density ρ and constant thickness t, perform its radial triangulation with respect to its centroid (Fig 2) and then proceed to an equimomental point-mass system, where point-masses are required to be placed at the polygon verticies.

2.1 Triangular Area Model

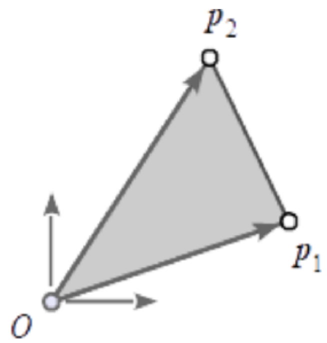

Fig. 3: Triangle area model

The triangle according to Fig.3 is defined by two side vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ from its root point $\mathbf{O}$ located in origin. Its area is

$$\mathrm{A} = \frac{1}{2}(\tilde{\mathbf{p}}_1\mathbf{p}_2)\,. \tag{2}$$

Please note, that A due to the symplectic product $\tilde{\mathbf{p}}_1\mathbf{p}_2$ in expression (2) is a *signed area*, which is positiv, if orientation from $\mathbf{p}_1$ to $\mathbf{p}_2$ is mathematically positive (counterclockwise), otherwise negative [6]. It is well known, that the triangle centroid $\mathbf{p}_C$ is located at two third of the length of the median from O to the midpoint between $\mathbf{p}_1$ and $\mathbf{p}_2$ [7]

$$\mathbf{p}_C = \frac{1}{3}(\mathbf{p}_1 + \mathbf{p}_2)\,. \tag{3}$$

Adopting the results of treating simple polygons by Green's theorem in these well written article [8], we have the triangular polar (second) moment of inertia (MoI) with respect to origin O

$$\mathrm{I} = \frac{\mathrm{A}}{6}(\mathbf{p}_1^2 + \mathbf{p}_1\mathbf{p}_2 + \mathbf{p}_2^2)\,.$$

Assuming constant density ρ and thickness t of the triangular object we deduce its mass M from its area A and its moment of inertia J by its polar 2$^{\text{nd}}$ moment I.

$$\mathrm{M} = \mathrm{A}\,\rho\,t \quad \text{and} \quad \mathrm{J} = \frac{\mathrm{M}}{6}(\mathbf{p}_1^2 + \mathbf{p}_1\mathbf{p}_2 + \mathbf{p}_2^2) \tag{4}$$

Please note, that masses can get negative values due to the signed area A in expression (2).

2.2 Triangular Particle Model

> *"It is well known that, as regards moment of inertia about any line in its plane, a uniform triangular lamina may be replaced by a set of three particles, each with mass equal to one-third of the mass of the triangle, and placed at the midpoints of the sides."* [1].
> — A. Talbot (1952)

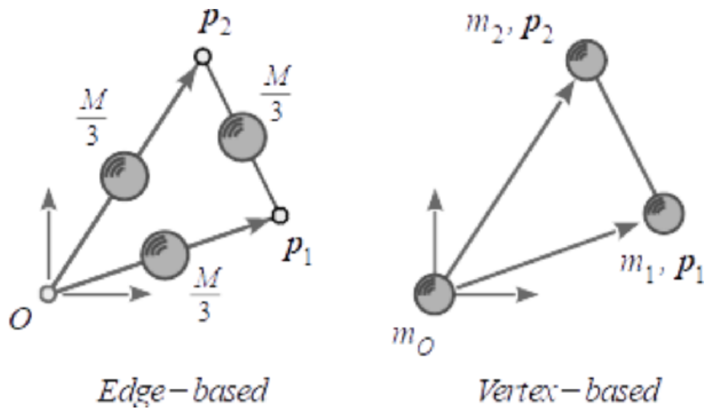

Fig. 4: Triangle Particle Model

In order to proof Talbot's quote above with edge-based triangle model in Fig.4 (left), we formulate the sum of the particle mass moments of inertia about origin

$$\mathrm{J} = \frac{\mathrm{M}}{3}\,\frac{(\mathbf{p}_1+\mathbf{p}_2)^2+\mathbf{p}_1^2+\mathbf{p}_2^2}{4}\,,$$

which can then easily be simplified into equation (4).

However this is not our preferred mass distribution, since we want to compose the polygonal particle system in Fig.2 to vertex-based triangular particles. So we intend to distribute the total triangle mass into verticies (Fig.4 right), while meeting the requirement

$$\mathrm{m}_0+\mathrm{m}_1+\mathrm{m}_2 = \mathrm{M}\,. \tag{5}$$

The mass moment of inertia with respect to origin must be

$$\mathrm{m}_1\mathbf{p}_1^2+\mathrm{m}_2\mathbf{p}_2^2 = \mathrm{J}\,. \tag{6}$$

Now we have two equations (5) and (6) for three unknowns $\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2$. Since lacking another equation we want to introduce two alternative pragmatic assumptions regarding the relationship between $\mathrm{m}_1, \mathrm{m}_2$.

Table 1: Alternative Mass Relationship

Assumption	Result	Equation
Equal vertex masses (1) $\mathrm{m}_1 = \mathrm{m}_2 = \mathrm{m}$	$\mathrm{m} = \frac{\mathrm{J}}{\mathrm{p}_1^2+\mathrm{p}_2^2}$	(7)
Equal vertex mass MoI's(2) $\mathrm{m}_1\,\mathrm{p}_1^2 = \mathrm{m}_2\,\mathrm{p}_2^2$	$\mathrm{m}_1 = \frac{\mathrm{J}}{2\,\mathrm{p}_1^2}\,,\ \mathrm{m}_2 = \frac{\mathrm{J}}{2\,\mathrm{p}_2^2}$	(8)

2.3 Polygon

We are considering closed polygons according to Fig.5 exclusively. Their n verticies are numbered from 0 to $n-1$. Index n is accessing vertex 0 then again.

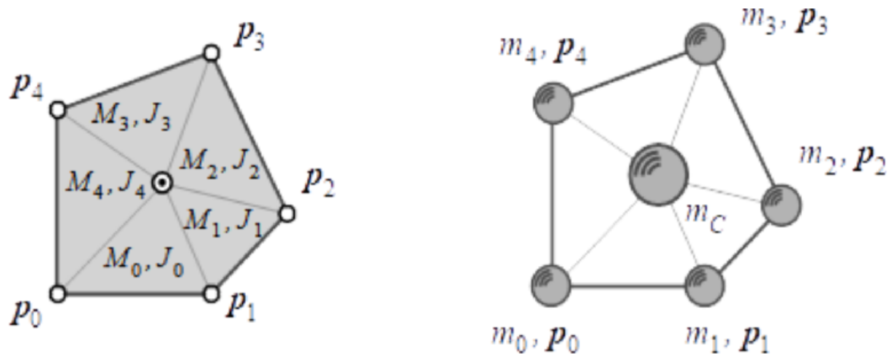

Fig. 5: Deriving particle polygon from areal polygon

The polygon area results from the sum of all triangle areas (2)

$$A = \frac{1}{2}\sum_{i=0}^{n-1} \tilde{\mathbf{p}}_i \mathbf{p}_{i+1} \,. \tag{9}$$

Area A is positive, if verticies are ordered counterclockwise, otherwise negative of the same amount. Knowing this we are allowed to correct that sign when necessary, without manually reordering the verticies. By using relation (3) its centroid comes out to be [5]

$$\mathbf{p}_C = \frac{1}{3A}\sum_{i=0}^{n-1} A_i(\mathbf{p}_i + \mathbf{p}_{i+1}) \,. \tag{10}$$

Now – for convenience only – we want to move the coordinate system's origin into that centroid. All following values depend on that origin location. So each centroidal triangular area now has a mass

$$M_i = \frac{\rho\, t}{2} \tilde{\mathbf{p}}_i \mathbf{p}_{i+1} \tag{11}$$

as well as a mass moment of inertia

$$J_i = \frac{M_i}{6}(\mathbf{p}_i^2 + \mathbf{p}_i \mathbf{p}_{i+1} + \mathbf{p}_{i+1}^2) \,. \tag{12}$$

Then both adjacent triangle masses contribute to their enclosed vertex mass m_i via

$$m_i = \frac{1}{2\mathbf{p}_i^2}(J_{i-1} + J_i) \tag{13}$$

while using mass relationship (8) from table 1 above. Yet it's easy to show, that areal and particle model do have identical mass moment of inertia

$$J = \sum_{i=0}^{n-1} J_i = \sum_{i=0}^{n-1} m_i \mathbf{p}_i^2 \,,$$

by inserting mass expression (13) to the right side of that equation. Both models already share the same centroid location. So now we merely need to ensure same total masses via

$$M = \sum_{i=0}^{n-1} M_i = m_C + \sum_{i=0}^{n-1} m_i \,,$$

which finally gives us the necessary centroidal point-mass

$$m_C = M - \sum_{i=0}^{n-1} m_i \,. \tag{14}$$

Herewith the vertex based point-mass distribution process is successfully completed.

5. Conclusion

Discussion of dynamically equivalent alias equimomental systems has a long tradition back into 19th century. Despite that, this subject has only a low value in today's engineering education.

This paper is addressed to the consideration of polygons and equimomental systems of point-masses, without taking into account the dynamic conditions in mechanisms. It is of high advantage to place the point-masses in the vertices of the polygon. However, restricting the mass locations comes at the expense of an additional point mass at the polygon centroid.

This approach of consideration of equimomental polygonal systems is new to the best of the author's knowledge and belief. It should also work for non-simple polygons, but that has little practical value in mechanism engineering. Future work can focus on negative masses occurring here and their avoidance.

References

[1] A. Talbot, *Equimomental systems*, Math. Gaz. 36(316), 95–110 (1952).
[2] E.J. Routh, *Dynamics of a System of Bodies*, MacMillan and Co., London (1897).
[3] L.P. Laus, J.M. Selig, *Rigid body dynamics using equimomental systems of point-masses*, DOI: https://doi.org/10.1007/s00707-019-02543-3 (2019).
[4] K. Chaudhary, H. Chaudhary, *Concept of Equimomental System for Dynamic Balancing of Mechanisms*, ICAMS2013.
[5] J.J. de Jong, J. van Dijk, J.L. Herder, *On the dynamic equivalence of planar mechanisms, an inertia decomposition method*, University of Twente.
[6] S. Gössner, Symplectic Geometry for Engineers - Fundamentals, DOI: 10.13140/RG.2.2.25568.43526, [https://tinyurl.com/y78mgv22]
[7] S. Gössner, *Symplectic Geometry for Engineers – Triangle*, DOI: 10.13140/RG.2.2.24288.02569, [https://tinyurl.com/y9jug9ar]
[8] Green's Theorem and Section Properties
[https://leancrew.com/all-this/2018/01/greens-theorem-and-section-properties/]

DOI: 10.17185/duepublico/74044
URN: urn:nbn:de:hbz:464-20210216-190052-2