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Well-posedness theory for electromagnetic obstacle problems

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Abstract

This paper develops a well-posedness theory for hyperbolic Maxwell obstacle problems generalizing the result by Duvaut and Lions (1976) [5]. Building on the recently developed result by Yousept (2020) [30], we prove an existence result and study the uniqueness through a local H(curl)-regularity analysis with respect to the constraint set. More precisely, every solution is shown to locally satisfy the Maxwell-Ampère equation (resp. Faraday equation) in the region where no obstacle is applied to the electric field (resp. magnetic field). By this property, along with a structural assumption on the feasible set, we are able to localize the obstacle problem to the underlying constraint regions. In particular, the resulting localized problem does not employ the electric test function (resp. magnetic test function) in the area where the L^2-regularity of the rotation of the electric field (resp. magnetic field) is not a priori guaranteed. This localization strategy is the main ingredient for our uniqueness proof. After establishing the well-posedness, we consider the case where the electric permittivity is negligibly small in the electric constraint region and investigate the corresponding eddy current obstacle problem. Invoking the localization strategy, we derive an existence result under an L^2-boundedness assumption for the electric constraint region along with a compatibility assumption on the initial data. The developed theoretical results find applications in electromagnetic shielding.
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1. Introduction

More than four decades ago, Duvaut and Lions [5, Chapter 7, Section 8] proposed and analyzed a (hyperbolic) Maxwell obstacle problem describing the propagation of electromagnetic waves in a polarizable medium with an obstacle constraint on the electric field of the form

$$|\boldsymbol{E}(x,t)| \leq d(x) \quad \text{a.e. in } \Omega \times (0,T) \tag{1.1}$$

for some $d : \Omega \to [0,\infty]$. Based on the method of vanishing (curl-curl) viscosity and constraint penalization, they proved a global well-posedness result for the proposed obstacle problem [5, Theorem 8.1]. Some years later, Milani [15,16] extended their theory to the case of a time-dependent upper bound $d = d(x,t)$. See also Miranda and Santos [19] for the non-Hilbertian extension of the electromagnetic antenna problem in [5, Chapter 7]. Maxwell (quasi)-variational inequalities play as well a profound role in the mathematical modeling of type-II superconductivity. See Bossavit [2], Prigozhin [22], Barrett and Prigozhin [1], Elliott and Kashima [6], Jochmann [11], Rodrigues and Santos [24,25], Pan [20,21], Yin et al. [29], and our previous works [27,32] for results in this direction. From among many other contributions towards obstacle problems, we refer to the monographs [9,12,23] and the pioneering works by Fichera [7,8], Brézis and Stampacchia [3], and Lions and Stampacchia [13,14].

Quite recently, the author [30] examined the mathematical analysis for Maxwell variational inequalities of the second kind. Based on a local boundedness assumption for the governing subdifferential, [30, Theorem 3.3] proved a global well-posedness result. However, as shown in [30, Example 3.6], the local boundedness assumption fails to hold for indicator functionals. As a remedy, through the use of the minimal section operator and the Nemytskii operator of the governing subdifferential, a more refined existence result was derived in [30, Theorem 3.11]. Though this result applies to a wider class of nonlinearities, including indicator functionals, it merely affirms the existence of a rather weak solution, which does not necessarily belong to the effective domain of the nonlinearity. More crucially, the uniqueness of the solution is not guaranteed.

This paper discusses and explores obstacle problems for Maxwell's equations with a general feasible set structure. Our study is mainly motivated by electromagnetic shielding applications to block or redirect undesired electromagnetic fields in a certain domain of interest by means of barriers (obstacles) made of conductive or magnetic materials. Typical materials used for the barriers in electric shielding include highly conductive sheet metals and metallic alloys, while materials with high magnetic permeability such as ferromagnetic materials are typically used for magnetic obstacles (see [4]). From the mathematical point of view, electromagnetic shielding phenomena fall into the class of obstacle problems: In the free region, the electromagnetic fields satisfy the fundamental Maxwell equations, whereas in the shielded area obstacle constraints are applied to the electric field (electric shielding) or the magnetic field (magnetic shielding).

Let $\Omega \subset \mathbb{R}^3$ be an open set (not necessarily bounded, Lipschitz, or connected) representing an anisotropic medium where the electromagnetic fields are acting. Inside Ω, we consider two open subsets $\Omega_E^c, \Omega_H^c \subset \Omega$. The set Ω_E^c (resp. Ω_H^c) represents the region where an obstacle constraint is imposed on the electric field $\boldsymbol{E}$ (resp. magnetic field $\boldsymbol{H}$). The free regions for the electric and magnetic fields are denoted, respectively, by

$$\Omega_E := \Omega \setminus \overline{\Omega_E^c} \quad \text{and} \quad \Omega_H := \Omega \setminus \overline{\Omega_H^c}.$$

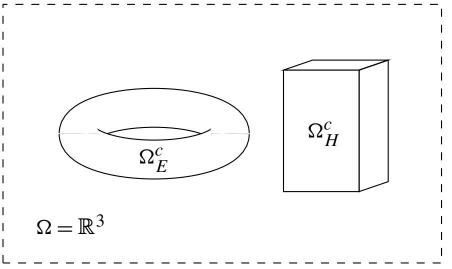

Fig. 1. The material Ω_H^c with a magnetic shielding property blocks the penetration ($\boldsymbol{H} = 0$ in Ω_H^c) of the magnetic field $\boldsymbol{H}$ generated by the electromagnetic coil Ω_E^c surrounded by electric insulation material.

The exact mathematical formulation of the Maxwell obstacle problem under consideration is presented in (P).

The novelties of this paper include existence and uniqueness results for (P) (Theorems 1 and 2) generalizing [5, Theorem 8.1] since [5] solely considers the case (1.1) and $\Omega_H^c = \emptyset$, i.e., in the absence of an obstacle for the magnetic field. Furthermore, our results hold true without the higher regularity assumption on the initial data [5, Eq. (89), p. 370] and without the piecewise constant assumption on the material parameters [5, Eq. (3.19), p. 338]. Differently from [5], our analysis is not based on the method of vanishing viscosity. Here, the proof of the existence result is built on the recently developed result [30, Theorem 3.11]. Along with the existence result, we show that every solution locally satisfies

(i) the Maxwell-Ampère equation in the electric free region Ω_E,
(ii) the Faraday equation in the magnetic free region Ω_H.

On this basis, the uniqueness analysis is studied. First, in the case where the obstacle constraint is applied only either to the magnetic field $\boldsymbol{H}$ or to the electric field $\boldsymbol{E}$, i.e., $\Omega_E^c = \emptyset$ or $\Omega_H^c = \emptyset$, a uniqueness result is obtained from (i)-(ii) (see Theorem 2). The uniqueness question becomes more challenging if both Ω_E^c and Ω_H^c have positive measure. To tackle this case, we propose structural assumptions on the constraint set ($\Omega_E^c \cap \Omega_H^c = \emptyset$) and the tangential components across the interfaces between the free and obstacle regions (Assumption 1.1). From the physical point of view, the proposed separation assumption $\Omega_E^c \cap \Omega_H^c = \emptyset$ is reasonable since the electric and magnetic fields are coupled to each other by Maxwell's equations. See Fig. 1 and Example 1.1 for an exemplary physical model satisfying Assumption 1.1 related to electromagnetic shielding. Making use of Assumption 1.1, we are able to localize (P) into the constraint regions Ω_E^c and Ω_H^c. In particular, the resulting localized problem does not employ the magnetic test function (resp. electric test function) in Ω_E^c (resp. Ω_H^c), i.e., in the region where the L^2-regularity of the rotation of the corresponding field is not a priori guaranteed. This localization strategy is the central ingredient of our uniqueness proof under Assumption 1.1.

The final part of this paper considers the case where the electric permittivity ϵ is negligibly small in the electric constraint region Ω_E^c. We investigate the resulting eddy current obstacle problem (1.19) by neglecting ϵ in Ω_E^c and derive an existence result for (1.19) (Theorem 3). The proof is based on the above-mentioned localization strategy together with an L^2-boundedness assumption for the constraint set on Ω_E^c and a compatibility assumption on the initial data (Assumption 1.2). For earlier contributions towards eddy current (semistatic) approximations for nonlinear Maxwell's equations, we refer to Milani and Picard [17], Jochmann [10], and Yin [28].

The remainder of this paper is organized as follows. In the upcoming subsection, we introduce our notation and formulate the Maxwell obstacle problem (P) under investigation. The first two main results are Theorems 1 and 2 regarding existence and uniqueness results for the Maxwell obstacle problem (P). The final main result is Theorem 3 concerning an existence result for the eddy current obstacle problem (1.19). The proofs for these three theorems are presented, respectively, in Sections 2, 3, and 4.

1.1. Problem formulation and main results

For a given Hilbert space V, we use the notation $\|\cdot\|_V$ and $(\cdot,\cdot)_V$ for a standard norm and a standard scalar product in V. A bold typeface is used to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions.

The electric permittivity and the magnetic permeability in the medium Ω are matrix-valued functions: $\epsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$. Moreover, they are assumed to be of class $L^\infty(\Omega)^{3\times 3}$, symmetric, and uniformly positive definite in the sense that there exist positive constants $\underline{\epsilon}, \underline{\mu} > 0$ such that

$$\xi^T \epsilon(x)\xi \geq \underline{\epsilon}|\xi|^2 \quad \text{and} \quad \xi^T \mu(x)\xi \geq \underline{\mu}|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3. \tag{1.2}$$

Given a symmetric and uniformly positive definite matrix-valued function $\alpha \in L^\infty(\Omega)^{3\times 3}$, let $\boldsymbol{L}^2_\alpha(\Omega)$ denote the weighted $\boldsymbol{L}^2(\Omega)$-space endowed with the weighted scalar product $(\alpha\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$. The pivot Hilbert space for our analysis is

$$\boldsymbol{X} := \boldsymbol{L}^2_\epsilon(\Omega) \times \boldsymbol{L}^2_\mu(\Omega),$$

equipped with the scalar product

$$((\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{X}} = (\epsilon\boldsymbol{e},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + (\mu\boldsymbol{h},\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \quad \forall(\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{X}. \tag{1.3}$$

Next, let $\mathcal{O} \subset \mathbb{R}^3$ be an open set. If $\mathcal{O}$ is compactly contained in Ω, then we write $\mathcal{O} \subset\subset \Omega$. Furthermore, we introduce the Hilbert space

$$\boldsymbol{H}(\mathbf{curl},\mathcal{O}) := \left\{\boldsymbol{q} \in \boldsymbol{L}^2(\mathcal{O}) \;\middle|\; \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{L}^2(\mathcal{O})\right\},$$

where the operator **curl** is understood in the sense of distributions. As usual, $\boldsymbol{C}_0^\infty(\mathcal{O})$ stands for the space of all infinitely differentiable three-dimensional vector functions with compact support contained in $\mathcal{O}$. We denote the closure of $\boldsymbol{C}_0^\infty(\mathcal{O})$ with respect to the $\boldsymbol{H}(\mathbf{curl},\mathcal{O})$-topology by $\boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) := \overline{\boldsymbol{C}_0^\infty(\mathcal{O})}^{\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\mathcal{O})}}$. It is well known that the Hilbert space $\boldsymbol{H}_0(\mathbf{curl},\mathcal{O})$ satisfies

$$\boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) = \left\{\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl},\mathcal{O}) \;\middle|\; \int_{\mathcal{O}} \mathbf{curl}\,\boldsymbol{q}\cdot\boldsymbol{v}\,dx = \int_{\mathcal{O}} \boldsymbol{q}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx \;\forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\mathcal{O})\right\}. \tag{1.4}$$

Note that this characterization does not require any regularity assumption on the open set $\mathcal{O}$ (see e.g. [30, Appendix A] for a proof of (1.4)). If $\mathcal{O} = \Omega$, then we simply write $\boldsymbol{H}(\mathbf{curl}) = \boldsymbol{H}(\mathbf{curl},\Omega)$ and $\boldsymbol{H}_0(\mathbf{curl}) = \boldsymbol{H}_0(\mathbf{curl},\Omega)$.

Let us now introduce the (unbounded) Maxwell operator

$$\mathcal{A} : D(\mathcal{A}) \subset \boldsymbol{X} \to \boldsymbol{X}, \quad \mathcal{A}(\boldsymbol{v}, \boldsymbol{w}) := (\epsilon^{-1} \mathbf{curl}\, \boldsymbol{w}, -\mu^{-1} \mathbf{curl}\, \boldsymbol{v}) \tag{1.5}$$

with the effective domain

$$D(\mathcal{A}) := \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}).$$

The choice of the domain $D(\mathcal{A})$ is motivated by the perfectly conducting electric boundary condition, which specifies that the tangential component of the electric field vanishes on the boundary. It is well known that the Maxwell operator $\mathcal{A} : D(\mathcal{A}) \subset \boldsymbol{X} \to \boldsymbol{X}$ is skew-adjoint, i.e., $\mathcal{A}^* = -\mathcal{A}$ and $D(\mathcal{A}^*) = D(\mathcal{A}) = \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})$.

In the following, let $\boldsymbol{K} \subset \boldsymbol{X}$ be a closed and convex set containing the origin $(0, 0) \in \boldsymbol{K}$. This set denotes the feasible set for the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$. As pointed out in the introduction, the open subsets $\Omega_E, \Omega_H \subset \Omega$ represent the free region for $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively. In other words, $\boldsymbol{K}$ satisfies

$$(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K} \Rightarrow (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}) \in \boldsymbol{K} \quad \forall \tilde{\boldsymbol{v}} = \begin{cases} \boldsymbol{v}_E & \text{in } \Omega_E \\ \boldsymbol{v} & \text{elsewhere} \end{cases} \qquad \forall \tilde{\boldsymbol{w}} = \begin{cases} \boldsymbol{w}_H & \text{in } \Omega_H \\ \boldsymbol{w} & \text{elsewhere} \end{cases} \tag{1.6}$$

for any $(\boldsymbol{v}_E, \boldsymbol{w}_H) \in \boldsymbol{L}^2(\Omega_E) \times \boldsymbol{L}^2(\Omega_H)$. Having introduced all the required function spaces, let us now formulate the Maxwell obstacle problem under investigation: Let $T \in (0, \infty)$. Given initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A}) \cap \boldsymbol{K}$ and $(\mathbf{f}, \mathbf{g}) \in W^{1,\infty}((0, T), \boldsymbol{X})$, find $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0, T), \boldsymbol{X})$ such that

$$\begin{cases} \int_0^T \int_\Omega \epsilon \partial_t \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E}) + \mu \partial_t \boldsymbol{H} \cdot (\boldsymbol{w} - \boldsymbol{H}) - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w} \, dx\, dt \\ \geq \int_0^T \int_\Omega \mathbf{f} \cdot (\boldsymbol{v} - \boldsymbol{E}) + \mathbf{g} \cdot (\boldsymbol{w} - \boldsymbol{H}) \, dx\, dt \\ \text{for all } (\boldsymbol{v}, \boldsymbol{w}) \in L^2((0, T), D(\mathcal{A})) \text{ with } (\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K} \text{ for a.e. } t \in (0, T), \\ (\boldsymbol{E}, \boldsymbol{H})(t) \in \boldsymbol{K} \text{ for all } t \in [0, T], \\ (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases} \tag{P}$$

By (1.5) and (1.3), the variational inequality in (P) can be compactly written as

$$\begin{aligned} &\int_0^T (\partial_t (\boldsymbol{E}, \boldsymbol{H})(t), (\boldsymbol{v}, \boldsymbol{w})(t) - (\boldsymbol{E}, \boldsymbol{H})(t))_{\boldsymbol{X}} + ((\boldsymbol{E}, \boldsymbol{H})(t), \mathcal{A}(\boldsymbol{v}, \boldsymbol{w})(t))_{\boldsymbol{X}} \, dt \\ &\geq \int_0^T ((\epsilon^{-1} \mathbf{f}, \mu^{-1} \mathbf{g})(t), (\boldsymbol{v}, \boldsymbol{w})(t) - (\boldsymbol{E}, \boldsymbol{H})(t))_{\boldsymbol{X}} \, dt. \end{aligned} \tag{1.7}$$

Theorem 1 *(Existence). Let $\Omega \subset \mathbb{R}^3$ be an open set and $\boldsymbol{K} \subset \boldsymbol{X}$ be a closed and convex subset satisfying $(0,0) \in \boldsymbol{K}$. Then, for all $(\mathbf{f}, \mathbf{g}) \in W^{1,\infty}((0,T), \boldsymbol{X})$ and $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A}) \cap \boldsymbol{K}$, the Maxwell obstacle problem (P) admits a solution $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{X})$. If $\boldsymbol{K} \subset \boldsymbol{X}$ additionally satisfies (1.6) for some (possibly empty) open subsets $\Omega_E, \Omega_H \subset \Omega$, then every solution of (P) satisfies*

$$\boldsymbol{E}_{|\Omega_H} \in L^\infty((0,T), \boldsymbol{H}(\mathbf{curl}, \Omega_H)) \quad \textit{and} \quad \boldsymbol{H}_{|\Omega_E} \in L^\infty((0,T), \boldsymbol{H}(\mathbf{curl}, \Omega_E)) \tag{1.8}$$

and fulfills the Maxwell-Ampère equation in Ω_E and the Faraday equation in Ω_H, i.e.,

$$\begin{cases} \epsilon \partial_t \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{H} = \mathbf{f} & \textit{a.e. in } \Omega_E \times (0,T), \\ \mu \partial_t \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = \mathbf{g} & \textit{a.e. in } \Omega_H \times (0,T). \end{cases} \tag{1.9}$$

If additionally $\Omega_H = \Omega$, then $\boldsymbol{E} \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}))$.

Let us point out that Theorem 1 yields a rather weak existence result for (P), which does not necessarily satisfy the prescribed electric boundary condition and the global $D(\mathcal{A})$-regularity, i.e., $(\boldsymbol{E}, \boldsymbol{H}) \in L^2((0,T), D(\mathcal{A}))$ is not guaranteed by Theorem 1. This is the reason why it is difficult to derive a uniqueness result in Theorem 1 since classical energy arguments cannot be directly applied here. Note that Theorem 1 affirms that the electric boundary condition $\boldsymbol{E} \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}))$ holds in the case of $\Omega_H = \Omega$, i.e., if there is no obstacle constraint imposed on the magnetic field. Of course, if $\boldsymbol{K}$ is assumed to additionally satisfy $\boldsymbol{K} \subset D(\mathcal{A})$, then the prescribed electric boundary condition and the global $D(\mathcal{A})$-regularity are pointwisely satisfied. However, the additional assumption $\boldsymbol{K} \subset D(\mathcal{A})$ is rather restrictive since our analysis requires that $\boldsymbol{K}$ is closed in $\boldsymbol{X}$. Note that a closed set in $D(\mathcal{A})$ is not necessarily closed in $\boldsymbol{X}$. For this reason, we do not focus on the additional restrictive assumption $\boldsymbol{K} \subset D(\mathcal{A})$ in our analysis.

As pointed out earlier, if the obstacle constraint is applied only either to the magnetic field or to the electric field, i.e., if $\Omega_E = \Omega$ or $\Omega_H = \Omega$, then (1.9) leads to a uniqueness result for (P) (see Theorem 2). For a more general case, we propose the following structural assumption on the feasible set:

Assumption 1.1. Suppose that $\Omega_H^c \subset \Omega$ and $\Omega_E^c \subset\subset \mathcal{U} \subset \Omega$ are open sets such that $\mathcal{U}_E := \mathcal{U} \setminus \overline{\Omega_E^c}$ is a bounded Lipschitz domain and

$$\Omega_E^c \cap \Omega_H^c = \emptyset, \quad |\partial \Omega_E^c| = |\partial \Omega_H^c| = 0. \tag{1.10}$$

Furthermore, we introduce $\boldsymbol{K}^c := \{(\boldsymbol{v}_{|\Omega_E^c}, \boldsymbol{w}_{|\Omega_H^c}) \mid (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K}\} \subset \boldsymbol{L}^2(\Omega_E^c) \times \boldsymbol{L}^2(\Omega_H^c)$ and assume that

$$\boldsymbol{K}^c \cap \left\{ \boldsymbol{H}(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}(\mathbf{curl}, \Omega_H^c) \right\} \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c). \tag{1.11}$$

There is no particular physical reason why the magnetic constraint region $\Omega_H^c \subset \Omega$ is chosen just to be open (not necessarily Lipschitz). This choice is considered solely to make our result mathematically more general. On the other hand, the Lipschitz regularity for the domain $\mathcal{U}_E = \mathcal{U} \setminus \overline{\Omega_E^c} \subset \Omega_E$ is required for the application of the extension theorem [10, Appendix] in (3.6). In the real applications, both electric and magnetic constraint regions Ω_E^c, Ω_H^c are typically given by bounded Lipschitz polyhedral domains.

We note that the assumption (1.11) is also related to obstacle problems with curl constraints (see Miranda et al. [18] for recent mathematical results on parabolic nonlinear obstacle problems with curl constraints). An example for such a set is given in Example 1.2. As a consequence of Assumption 1.1, if $\partial\Omega \subset \partial\Omega_H^c$, then the solution to (P) satisfies the magnetic boundary condition, i.e., $\boldsymbol{H} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c)$.

Example 1.1 *(cf. Fig. 1).* Let $\Omega = \mathbb{R}^3$, $\Omega_H^c = (4,5) \times (-1,1) \times (-0.5, 0.5)$ and $\Omega_E^c = \big\{x \in \mathbb{R}^3 \mid \big(\sqrt{x_1^2 + x_2^2} - 2\big)^2 + x_3^2 < 1\big\}$. Furthermore, let $d : \Omega_E^c \to [0, \infty]$ with $d = 0$ in $\big\{x \in \mathbb{R}^3 \mid 1 - \delta < \big(\sqrt{x_1^2 + x_2^2} - 2\big)^2 + x_3^2 < 1\big\}$ for some $\delta \in (0,1)$. Then,

$$\boldsymbol{K} = \{(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X} \mid |\boldsymbol{v}(x)| \leq d(x) \text{ for a.e. } x \in \Omega_E^c \text{ and } \boldsymbol{w}(x) = 0 \text{ for a.e. } x \in \Omega_H^c\}$$

is closed, convex and satisfies $(0,0) \in \boldsymbol{K}$, (1.6), and Assumption 1.1. In this case, Ω_H^c models a medium with a magnetic shielding property, and Ω_E^c describes an electromagnetic coil with electric insulation on $\partial\Omega_E^c$ (for $\delta \approx 0$).

Example 1.2. Let $\Omega_H^c, \Omega_E^c \subset \mathbb{R}^3 = \Omega$ be as in Assumption 1.1, and let $d_E \in L^2(\Omega_E^c)$ and $d_H \in L^2(\Omega_H^c)$ be nonnegative functions. We consider

$$\begin{aligned}\boldsymbol{K} = \{&(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X} \mid (\boldsymbol{v}_{|\Omega_E^c}, \boldsymbol{w}_{|\Omega_H^c}) \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c),\\ &|\mathbf{curl}\, \boldsymbol{v}(x)| \leq d_E(x) \text{ for a.e. } x \in \Omega_E^c \text{ and } |\mathbf{curl}\, \boldsymbol{w}(x)| \leq d_H(x) \text{ for a.e. } x \in \Omega_H^c\}.\end{aligned}$$

By the above construction, we see that (1.11) is readily satisfied, and $\boldsymbol{K}^c$ is a closed and convex subset of $\boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c)$. Moreover, it is also obvious that $\boldsymbol{K} \subset \boldsymbol{X}$ is convex and contains $(0,0)$. Let us now show that $\boldsymbol{K} \subset \boldsymbol{X}$ is closed. To this aim, let $\{\boldsymbol{v}_n, \boldsymbol{w}_n\}_{n=1}^\infty \subset \boldsymbol{K}$ be a strongly converging sequence in $\boldsymbol{X}$, i.e.,

$$(\boldsymbol{v}_n, \boldsymbol{w}_n) \to (\boldsymbol{v}, \boldsymbol{w}) \quad \text{in } \boldsymbol{X} \tag{1.12}$$

for some $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X}$. We have to show that $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K}$. By the definition of $\boldsymbol{K}$ and in view of (1.12), the sequence $\{(\boldsymbol{v}_{n|\Omega_E^c}, \boldsymbol{w}_{n|\Omega_H^c})\}_{n=1}^\infty \subset \boldsymbol{K}^c$ is bounded in $\boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c)$, and so there exists a subsequence of $\{(\boldsymbol{v}_{n|\Omega_E^c}, \boldsymbol{w}_{n|\Omega_H^c})\}_{n=1}^\infty$ such that

$$(\boldsymbol{v}_{n_j|\Omega_E^c}, \boldsymbol{w}_{n_j|\Omega_H^c}) \rightharpoonup (\boldsymbol{y}, \boldsymbol{z}) \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c) \tag{1.13}$$

for some $(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{K}^c$ since $\boldsymbol{K}^c \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \times \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c)$ is closed and convex. Altogether, (1.12)-(1.13) yield

$$(\boldsymbol{v}_{|\Omega_E^c}, \boldsymbol{w}_{|\Omega_H^c}) = (\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{K}^c \quad \Rightarrow \quad (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K}.$$

In conclusion, $\boldsymbol{K} \subset \boldsymbol{X}$ is closed, convex and satisfies $(0,0) \in \boldsymbol{K}$, (1.6), and Assumption 1.1.

Theorem 2 *(Uniqueness). Let $\Omega \subset \mathbb{R}^3$ be an open set and $\boldsymbol{K} \subset \boldsymbol{X}$ be a closed and convex subset satisfying $(0,0) \in \boldsymbol{K}$ and (1.6) for some (possibly empty) open subsets $\Omega_E, \Omega_H \subset \Omega$. Suppose that one of the following conditions holds true:*

(i) Assumption 1.1 with $\Omega_E = \Omega \setminus \overline{\Omega_E^c}$ and $\Omega_H = \Omega \setminus \overline{\Omega_H^c}$,
(ii) $\Omega_H = \Omega$ or $\Omega_E = \Omega$.

Then, for all $(\mathbf{f}, \mathbf{g}) \in W^{1,\infty}((0,T), \boldsymbol{X})$ and $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A}) \cap \boldsymbol{K}$, (P) admits a unique solution $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{X})$.

Our final theoretical finding is an existence result for the eddy current approximation to (P) in the case where the electric permittivity is negligibly small in the electric constraint region Ω_E^c. As pointed out in the introduction, our result relies on the following additional assumption:

Assumption 1.2. There exists a constant $C_K > 0$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega_E^c)} \leq C_K \quad \forall(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K} \quad \text{and} \quad \mathbf{f}(x,t) = 0 \quad \forall(x,t) \in \Omega_E^c \times [0,T] \tag{1.14}$$

with an open set $\Omega_E^c \subset \Omega$ satisfying $|\partial\Omega_E^c| = 0$ and $\Omega_E = \Omega \setminus \overline{\Omega_E^c}$. If Assumption 1.1 holds, then the initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A}) \cap \boldsymbol{K}$ is assumed to satisfy the following compatibility condition:

$$\int_{\Omega_H^c} \boldsymbol{E}_0 \cdot \mathbf{curl}\,(\boldsymbol{w} - \boldsymbol{H}_0)\,dx - \int_{\Omega_E^c} \boldsymbol{H}_0 \cdot \mathbf{curl}\,(\boldsymbol{v} - \boldsymbol{E}_0)\,dx \geq \int_{\Omega_H^c} \mathbf{g}(0) \cdot (\boldsymbol{w} - \boldsymbol{H}_0)\,dx \qquad \forall(\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}) \cap \boldsymbol{K}, \tag{1.15}$$

$$-\mathbf{curl}\,\boldsymbol{H}_0 = \mathbf{f}(0) \quad \text{a.e. in } \Omega_E, \qquad \mathbf{curl}\,\boldsymbol{E}_0 = \mathbf{g}(0) \quad \text{a.e. in } \Omega_H. \tag{1.16}$$

If $\Omega_H = \Omega$, then the initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A}) \cap \boldsymbol{K}$ is assumed to satisfy the following compatibility condition:

$$-\int_{\Omega} \boldsymbol{H}_0 \cdot \mathbf{curl}\,(\boldsymbol{v} - \boldsymbol{E}_0)\,dx \geq \int_{\Omega_E} \mathbf{f}(0) \cdot (\boldsymbol{v} - \boldsymbol{E}_0)\,dx \quad \forall(\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}) \cap \boldsymbol{K}, \tag{1.17}$$

$$\mathbf{curl}\,\boldsymbol{E}_0 = \mathbf{g}(0). \tag{1.18}$$

Let us remark that the L^2-boundedness assumption (1.14) is reasonable since Ω_E^c is exactly the region where the obstacle constraint is applied to the electric field.

Theorem 3. *Suppose that all the assumptions of Theorem 2 and Assumption 1.2 are satisfied with $(\mathbf{f}, \mathbf{g}) \in W^{1,\infty}((0,T), \boldsymbol{X})$ and $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A}) \cap \boldsymbol{K}$. Then, the eddy current obstacle problem*

$$
\begin{cases}
\int_0^T \int_{\Omega_E} \epsilon \partial_t \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx + \int_{\Omega} \mu \partial_t \boldsymbol{H} \cdot (\boldsymbol{w} - \boldsymbol{H}) - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx\, dt \\
\geq \int_0^T \int_{\Omega} \mathbf{f} \cdot (\boldsymbol{v} - \boldsymbol{E}) + \mathbf{g} \cdot (\boldsymbol{w} - \boldsymbol{H})\, dx\, dt \\
\textit{for all } (\boldsymbol{v}, \boldsymbol{w}) \in L^2((0,T), D(\mathcal{A})) \textit{ with } (\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K} \textit{ for a.e. } t \in (0,T), \\
(\boldsymbol{E}, \boldsymbol{H})(t) \in \boldsymbol{K} \textit{ for a.e. } t \in (0,T), \\
\boldsymbol{E}(0) = \boldsymbol{E}_0 \textit{ in } \Omega_E, \quad \boldsymbol{H}(0) = \boldsymbol{H}_0 \textit{ in } \Omega
\end{cases}
\tag{1.19}
$$

admits a solution $(\boldsymbol{E}, \boldsymbol{H}) \in L^\infty((0,T), \boldsymbol{X}) \cap W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega_E) \times \boldsymbol{L}^2(\Omega))$ *satisfying* (1.8) *and* (1.9).

1.2. Preliminaries

Let us first make a preparation by recalling some well-known results. For a nonempty, convex, and closed subset $\boldsymbol{K} \subset \boldsymbol{X}$, let $\mathcal{I}_{\boldsymbol{K}} : \boldsymbol{X} \to \overline{\mathbb{R}} := (-\infty, \infty]$ denote the indicator functional of $\boldsymbol{K}$:

$$
\mathcal{I}_{\boldsymbol{K}}(\boldsymbol{p}, \boldsymbol{q}) := \begin{cases} 0 & \text{if } (\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{K} \\ \infty & \text{if } (\boldsymbol{p}, \boldsymbol{q}) \notin \boldsymbol{K}. \end{cases} \tag{1.20}
$$

By definition, for every $(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{X}$, the subdifferential $\partial \mathcal{I}_{\boldsymbol{K}}(\boldsymbol{p}, \boldsymbol{q})$ is given by

$$
\begin{aligned}
\partial \mathcal{I}_{\boldsymbol{K}}(\boldsymbol{p}, \boldsymbol{q}) = \{(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{X} \mid ((\boldsymbol{y}, \boldsymbol{z}), (\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{p}, \boldsymbol{q}))_{\boldsymbol{X}} + \mathcal{I}_{\boldsymbol{K}}(\boldsymbol{p}, \boldsymbol{q}) \leq \mathcal{I}_{\boldsymbol{K}}(\boldsymbol{v}, \boldsymbol{w}) \\
\forall (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X}\}.
\end{aligned} \tag{1.21}
$$

Furthermore, for every $\lambda > 0$, let $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ and $\Phi_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ denote, respectively, the resolvent and the Yosida approximation of the subdifferential $\partial \mathcal{I}_{\boldsymbol{K}}$, i.e.,

$$
J_\lambda := (I_d + \lambda \partial \mathcal{I}_{\boldsymbol{K}})^{-1} \quad \text{and} \quad \Phi_\lambda := \lambda^{-1}\big(I_d - J_\lambda\big), \tag{1.22}
$$

where $I_d : \boldsymbol{X} \to \boldsymbol{X}$ denotes the identity operator. Since $\boldsymbol{K} \subset \boldsymbol{X}$ is nonempty, convex and closed, the indicator functional $\mathcal{I}_{\boldsymbol{K}} : \boldsymbol{X} \to \overline{\mathbb{R}}$ is proper, convex, and lower semicontinuous. As a consequence (see [26, Proposition 1.5, p. 157]), the subdifferential $\partial \mathcal{I}_{\boldsymbol{K}} : \boldsymbol{X} \to 2^{\boldsymbol{X}}$ is m-accretive, which implies that $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is non-expansive, and $\Phi_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is m-accretive and Lipschitz-continuous with the Lipschitz constant $L_\lambda = \lambda^{-1}$ (see [26, Theorem 1.1, p. 161]). We also make use of the Hilbert projection operator on $\boldsymbol{K}$ denoted by $\mathcal{P}_{\boldsymbol{K}} : \boldsymbol{X} \to \boldsymbol{K}$, i.e., for every $(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{X}$, $\mathcal{P}_{\boldsymbol{K}}(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{K}$ is given by the unique minimizer of

$$
\min_{(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K}} F(\boldsymbol{v}, \boldsymbol{w}) := \|(\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{p}, \boldsymbol{q})\|_{\boldsymbol{X}}^2. \tag{1.23}
$$

Lemma 4. *Let* $\boldsymbol{K} \subset \boldsymbol{X}$ *be nonempty, convex and closed. Then, it holds that* $J_\lambda = \mathcal{P}_{\boldsymbol{K}}$ *for all* $\lambda > 0$.

Proof. Let $\lambda > 0$ and $(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{X}$. According to (1.22),

$$\lambda^{-1}((\boldsymbol{p}, \boldsymbol{q}) - J_\lambda(\boldsymbol{p}, \boldsymbol{q})) \in \partial \mathcal{I}_{\boldsymbol{K}} (J_\lambda(\boldsymbol{p}, \boldsymbol{q})),$$

which implies by (1.21) that $J_\lambda(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{K}$ and

$$\lambda^{-1}((\boldsymbol{p}, \boldsymbol{q}) - J_\lambda(\boldsymbol{p}, \boldsymbol{q}), (\boldsymbol{v}, \boldsymbol{w}) - J_\lambda(\boldsymbol{p}, \boldsymbol{q}))_{\boldsymbol{X}} \leq 0 \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K},$$

or equivalently

$$F'(J_\lambda(\boldsymbol{p}, \boldsymbol{q}))((\boldsymbol{v}, \boldsymbol{w}) - J_\lambda(\boldsymbol{p}, \boldsymbol{q})) \geq 0 \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K}.$$

In conclusion, $J_\lambda(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{K}$ satisfies the necessary and sufficient optimality condition for the quadratic minimization problem (1.23), and so $J_\lambda(\boldsymbol{p}, \boldsymbol{q}) = \mathcal{P}_{\boldsymbol{K}}(\boldsymbol{p}, \boldsymbol{q})$. □

2. Proof of Theorem 1

We split the proof into two parts.

1. Step: Existence for (P). Let $\{\lambda_n\}_{n=1}^\infty \subset (0, \infty)$ be a null sequence. For every $n \in \mathbb{N}$, let $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}([0, T], D(\mathcal{A})) \cap \mathcal{C}^1([0, T], \boldsymbol{X})$ denote the unique solution to

$$\begin{cases} \partial_t(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) - \mathcal{A}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) = (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) - \Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) \quad \forall t \in [0, T], \\ (\boldsymbol{E}_n, \boldsymbol{H}_n)(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases} \tag{2.1}$$

As shown in [30, Theorem 3.11 & Remark 3.14] for $\varphi = \mathcal{I}_{\boldsymbol{K}}$, the sequence $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ is bounded in $\mathcal{C}^1([0, T], \boldsymbol{X})$, and there exists a subsequence of $\{\lambda_n\}_{n=1}^\infty$, denoted again by $\{\lambda_n\}_{n=1}^\infty$, such that

$$(\boldsymbol{E}_n, \boldsymbol{H}_n) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H}) \qquad \text{weakly star in } L^\infty((0, T), \boldsymbol{X}) \text{ as } n \to \infty, \tag{2.2}$$

$$(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H})(t) \quad \text{weakly in } \boldsymbol{X} \text{ as } n \to \infty \text{ for all } t \in [0, T] \tag{2.3}$$

with $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0, T), \boldsymbol{X})$ satisfying (1.7) for all $(\boldsymbol{v}, \boldsymbol{w}) \in L^2((0, T), D(\mathcal{A}))$ with $(\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K}$ a.e. in $(0, T)$. After a modification on a subset of $[0, T]$ with measure zero, we have that $(\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{C}([0, T], \boldsymbol{X})$. It remains to show that

$$(\boldsymbol{E}, \boldsymbol{H})(t) \in \boldsymbol{K} \quad \forall t \in [0, T]. \tag{2.4}$$

To this aim, we make use the fact that $\mathcal{A}$ is skew adjoint and (2.1) to deduce that

$$\big((I_d - J_{\lambda_n})(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w})\big)_{\boldsymbol{X}} \underbrace{=}_{(1.22)} \lambda_n \big(\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w})\big)_{\boldsymbol{X}}$$

$$= \lambda_n \Big[((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) - \partial_t(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} - ((\boldsymbol{E}_n, \boldsymbol{H}_n)(t), \mathcal{A}(\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \Big] \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}) \quad \forall t \in [0, T].$$

Thus, since $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^{\infty} \subset \mathcal{C}^1([0,T], \boldsymbol{X})$ is bounded, it follows that

$$\lim_{n\to\infty} \big((I_d - J_{\lambda_n})(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w})\big)_{\boldsymbol{X}} = 0 \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}) \quad \forall t \in [0,T]. \tag{2.5}$$

By the monotonicity of Φ_{λ_n}, it holds for all $t \in [0,T]$, $n \in \mathbb{N}$ and $(\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A})$ that

$$\begin{aligned} &((I_d - J_{\lambda_n})(\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}} \\ &= \lambda_n (\Phi_{\lambda_n}(\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}} \geq \\ &\lambda_n (\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}} \\ &= ((I_d - J_{\lambda_n})(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} - \lambda_n (\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}}. \end{aligned} \tag{2.6}$$

Using again the fact that $\mathcal{A}$ is skew-adjoint, we obtain from (2.1) that

$$(\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}} = ((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) - \partial_t(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}}$$

for all $n \in \mathbb{N}$ and all $t \in [0,T]$. Therefore, in view of the boundedness of $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^{\infty} \subset \mathcal{C}^1([0,T], \boldsymbol{X})$, it follows that

$$\lambda_n (\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{E}_n, \boldsymbol{H}_n)(t))_{\boldsymbol{X}} \to 0 \quad \text{as n } \to \infty \text{ for all } t \in [0,T]. \tag{2.7}$$

On the other hand, from Lemma 4, we know that

$$J_{\lambda_n} = \mathcal{P}_{\boldsymbol{K}} \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Altogether, due to (2.3), (2.5), (2.7) and (2.8), passing to the limit $n \to \infty$ in (2.6) leads to

$$((I_d - \mathcal{P}_{\boldsymbol{K}})(\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{E}, \boldsymbol{H})(t))_{\boldsymbol{X}} \geq 0 \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}) \quad \forall t \in [0,T].$$

Consequently, as $D(\mathcal{A}) \subset \boldsymbol{X}$ is dense and $(I_d - \mathcal{P}_{\boldsymbol{K}}) : \boldsymbol{X} \to \boldsymbol{X}$ is continuous, we obtain that

$$((I_d - \mathcal{P}_{\boldsymbol{K}})(\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{E}, \boldsymbol{H})(t))_{\boldsymbol{X}} \geq 0 \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X} \quad \forall t \in [0,T].$$

Inserting $(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{E}, \boldsymbol{H})(t) + \tau(\boldsymbol{p}, \boldsymbol{q})$ with $\tau > 0$ and $(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{X}$, we have

$$((I_d - \mathcal{P}_{\boldsymbol{K}})((\boldsymbol{E}, \boldsymbol{H})(t) - \tau(\boldsymbol{p}, \boldsymbol{q})), (\boldsymbol{p}, \boldsymbol{q}))_{\boldsymbol{X}} \geq 0.$$

Then, letting $\tau \to 0$, we deduce from the continuity of $(I_d - \mathcal{P}_{\boldsymbol{K}}) : \boldsymbol{X} \to \boldsymbol{X}$ that

$$((I_d - \mathcal{P}_{\boldsymbol{K}})((\boldsymbol{E}, \boldsymbol{H})(t)), (\boldsymbol{p}, \boldsymbol{q}))_{\boldsymbol{X}} \geq 0 \quad \forall (\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{X} \quad \forall t \in [0,T],$$

from which it follows that

$$(I_d - \mathcal{P}_{\boldsymbol{K}})((\boldsymbol{E}, \boldsymbol{H})(t)) = 0 \quad \forall t \in [0,T] \quad \Leftrightarrow \quad (\boldsymbol{E}, \boldsymbol{H})(t) \in \boldsymbol{K} \quad \forall t \in [0,T].$$

2. Step: We prove (1.9). Let $\Pi_E : \boldsymbol{L}^2(\Omega_E) \to \boldsymbol{L}^2(\Omega)$ and $\Pi_H : \boldsymbol{L}^2(\Omega_H) \to \boldsymbol{L}^2(\Omega)$ denote the zero extension operators, i.e.,

$$\Pi_E(\boldsymbol{v}_E) = \begin{cases} \boldsymbol{v}_E & \text{in } \Omega_E \\ 0 & \text{elsewhere,} \end{cases} \qquad \Pi_H(\boldsymbol{w}_H) = \begin{cases} \boldsymbol{w}_H & \text{in } \Omega_H \\ 0 & \text{elsewhere} \end{cases} \tag{2.9}$$

for all $(\boldsymbol{v}_E, \boldsymbol{w}_H) \in \boldsymbol{L}^2(\Omega_E) \times \boldsymbol{L}^2(\Omega_H)$. Since $(0, 0) \in \boldsymbol{K}$, (1.6) and (2.9) imply that

$$(0, \Pi_H(\boldsymbol{w}_H)) \in \{\boldsymbol{C}_0^\infty(\Omega) \times \boldsymbol{C}_0^\infty(\Omega)\} \cap \boldsymbol{K} \subset D(\mathcal{A}) \cap \boldsymbol{K} \quad \forall \boldsymbol{w}_H \in \boldsymbol{C}_0^\infty(\Omega_H).$$

Let $\boldsymbol{w}_H \in \boldsymbol{C}_0^\infty(\Omega_H)$, $\tau \in [0, T)$ and $h \in (0, T - \tau)$ be arbitrarily fixed. Then,

$$(\boldsymbol{v}, \boldsymbol{w}) := (0, \chi_{[\tau,\tau+h]}\Pi_H(\boldsymbol{w}_H)) \in L^2((0, T), D(\mathcal{A})) \quad \text{and} \quad (\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K} \quad \forall t \in [0, T],$$

where $\chi_{[\tau,\tau+h]} : [0, T] \to \{0, 1\}$ denotes the characteristic function of the time interval $[\tau, \tau + h]$. Now, let $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0, T), \boldsymbol{X})$ denote a solution to (P). Then, applying the above test function to (P) results in

$$\int_\tau^{\tau+h}\int_{\Omega_H} \mu\partial_t \boldsymbol{H} \cdot \boldsymbol{w}_H + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}_H - \mathbf{g} \cdot \boldsymbol{w}_H \, dxdt \geq \int_0^T\int_\Omega \epsilon\partial_t \boldsymbol{E} \cdot \boldsymbol{E} + \mu\partial_t \boldsymbol{H} \cdot \boldsymbol{H} - \mathbf{f} \cdot \boldsymbol{E} - \mathbf{g} \cdot \boldsymbol{H} \, dxdt.$$

As $\boldsymbol{w}_H \in \boldsymbol{C}_0^\infty(\Omega_H)$ was chosen arbitrarily, it follows that

$$\int_\tau^{\tau+h}\int_{\Omega_H} \mu\partial_t \boldsymbol{H} \cdot \boldsymbol{w}_H + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}_H - \mathbf{g} \cdot \boldsymbol{w}_H \, dxdt = 0 \quad \forall \boldsymbol{w}_H \in \boldsymbol{C}_0^\infty(\Omega_H). \tag{2.10}$$

Multiplying (2.10) with $\frac{1}{h}$ and letting $h \to 0$, we deduce from the Lebesgue differentiation theorem that

$$\int_{\Omega_H} \mu\partial_t \boldsymbol{H}(\tau) \cdot \boldsymbol{w}_H + \boldsymbol{E}(\tau) \cdot \mathbf{curl}\, \boldsymbol{w}_H - \mathbf{g}(\tau) \cdot \boldsymbol{w}_H \, dx = 0 \quad \forall \boldsymbol{w}_H \in \boldsymbol{C}_0^\infty(\Omega_H)$$

for a.e. $\tau \in (0, T)$. Thus, the distributional definition of the **curl**-operator implies

$$\boldsymbol{E}(\tau) \in \boldsymbol{H}(\mathbf{curl}, \Omega_H) \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{E}(\tau) = \mathbf{g}(\tau) - \mu\partial_t \boldsymbol{H}(\tau) \text{ in } \Omega_H \tag{2.11}$$

for a.e. $\tau \in (0, T)$. Analogously, making use of the test function

$$(\boldsymbol{v}, \boldsymbol{w}) := (\chi_{[\tau,\tau+h]}\Pi_E(\boldsymbol{v}_E), 0) \quad \forall \boldsymbol{v}_E \in \boldsymbol{C}_0^\infty(\Omega_E),$$

we deduce that

$$\boldsymbol{H}(\tau) \in \boldsymbol{H}(\mathbf{curl}, \Omega_E) \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{H}(\tau) = \epsilon\partial_t \boldsymbol{E}(\tau) - \mathbf{f}(\tau) \text{ in } \Omega_E \quad \text{for a.e. } \tau \in (0, T).$$

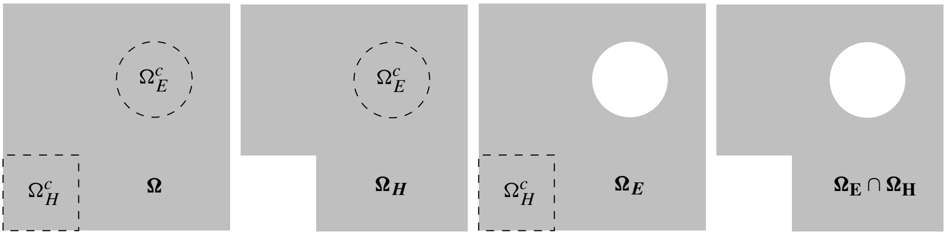

Fig. 2. Example for Ω and its subsets Ω_E^c, Ω_H^c, Ω_E, Ω_H and $\Omega_E \cap \Omega_H$.

Suppose now that $\Omega_H = \Omega$. In this case, (1.6) implies

$$(0, \boldsymbol{w}) \in D(\mathcal{A}) \cap \boldsymbol{K} \quad \forall \boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}).$$

For this reason, as before, applying the test function $(0, \chi_{[\tau,\tau+h]}\boldsymbol{w})$ with $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$ to (P) and using the Lebesgue differentiation theorem, we obtain

$$\int_\Omega \mu \partial_t \boldsymbol{H}(\tau) \cdot \boldsymbol{w} + \boldsymbol{E}(\tau) \cdot \mathbf{curl}\, \boldsymbol{w} - \mathbf{g}(\tau) \cdot \boldsymbol{w}\, dx = 0 \tag{2.12}$$

for a.e. $\tau \in (0, T)$ and all $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$. Combining (2.11)–(2.12) together yields

$$\int_\Omega \boldsymbol{E}(\tau) \cdot \mathbf{curl}\, \boldsymbol{w}\, dx = \int_\Omega (\mathbf{g}(\tau) - \mu \partial_t \boldsymbol{H}(\tau)) \cdot \boldsymbol{w}\, dx = \int_\Omega \mathbf{curl}\, \boldsymbol{E}(\tau) \cdot \boldsymbol{w}\, dx$$

for a.e. $\tau \in (0, T)$ and all $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$. Thus, in view of (1.4), $\boldsymbol{E}(\tau) \in \boldsymbol{H}_0(\mathbf{curl})$ holds true for a.e. $\tau \in (0, T)$. In conclusion, the assertion is valid. □

3. Proof of Theorem 2

We split the proof into three steps.

Step 1: Localization of (P) *under Assumption 1.1*. Let $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0, T), \boldsymbol{X})$ denote a solution to (P). According to Theorem 1, it holds that

$$\boldsymbol{E}(t)_{|\Omega_H} \in \boldsymbol{H}(\mathbf{curl}, \Omega_H) \quad \text{and} \quad \boldsymbol{H}(t)_{|\Omega_E} \in \boldsymbol{H}(\mathbf{curl}, \Omega_E) \quad \text{for a.e. } t \in (0, T). \tag{3.1}$$

Furthermore, since $\Omega_E = \Omega \setminus \overline{\Omega_E^c}$ and $\Omega_H = \Omega \setminus \overline{\Omega_H^c}$, (1.10) implies (cf. Fig. 2) that

$$\Omega_E^c \subset \Omega_H, \quad \Omega_H^c \subset \Omega_E. \tag{3.2}$$

Thus, as $(\boldsymbol{E}, \boldsymbol{H})(t) \in \boldsymbol{K}$ holds true for all $t \in [0, T]$, it follows from (1.11) and (3.1)–(3.2) that

$$\boldsymbol{E}(t)_{|\Omega_E^c} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega_E^c) \quad \text{and} \quad \boldsymbol{H}(t)_{|\Omega_H^c} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega_H^c) \quad \text{for a.e. } t \in (0, T). \tag{3.3}$$

Let now $(\boldsymbol{v}, \boldsymbol{w}) \in L^2((0, T), D(\mathcal{A}))$ with $(\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K}$ a.e. in $(0, T)$. In view of (1.10) (cf. Fig. 2), we may write

$$\Omega = \Omega_E^c \cup \Omega_H^c \cup (\Omega_E \cap \Omega_H) \cup \partial\Omega_E^c \cup (\partial\Omega_H^c \cap \Omega). \tag{3.4}$$

Therefore, as $|\partial\Omega_E^c| = |\partial\Omega_H^c| = 0$, (P) can be written as

$$\int_0^T \Bigg[\int_{\Omega_E^c} \epsilon\partial_t \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E}) + (\mu\partial_t \boldsymbol{H} - \mathbf{g}) \cdot (\boldsymbol{w} - \boldsymbol{H}) - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx +$$
$$\int_{\Omega_H^c} (\epsilon\partial_t \boldsymbol{E} - \mathbf{f}) \cdot (\boldsymbol{v} - \boldsymbol{E}) + \mu\partial_t \boldsymbol{H} \cdot (\boldsymbol{w} - \boldsymbol{H}) - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx +$$
$$\int_{\Omega_E \cap \Omega_H} (\epsilon\partial_t \boldsymbol{E} - \mathbf{f}) \cdot (\boldsymbol{v} - \boldsymbol{E}) + (\mu\partial_t \boldsymbol{H} - \mathbf{g}) \cdot (\boldsymbol{w} - \boldsymbol{H}) - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v}$$
$$+ \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx \Bigg] dt \geq \int_0^T \int_{\Omega_E^c} \mathbf{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx + \int_{\Omega_H^c} \mathbf{g} \cdot (\boldsymbol{w} - \boldsymbol{H})\, dx dt.$$

Then, thanks to (3.2), we may apply (1.9) to obtain

$$\int_0^T \Bigg[\int_{\Omega_E^c} \epsilon\partial_t \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E}) - \boldsymbol{H} \cdot \mathbf{curl}\, (\boldsymbol{v} - \boldsymbol{E}) - \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{w} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx +$$
$$\int_{\Omega_H^c} \mu\partial_t \boldsymbol{H} \cdot (\boldsymbol{w} - \boldsymbol{H}) + \boldsymbol{E} \cdot \mathbf{curl}\, (\boldsymbol{w} - \boldsymbol{H}) + \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx +$$
$$\int_{\Omega_E \cap \Omega_H} \mathbf{curl}\, \boldsymbol{H} \cdot (\boldsymbol{v} - \boldsymbol{E}) - \mathbf{curl}\, \boldsymbol{E} \cdot (\boldsymbol{w} - \boldsymbol{H}) - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx \Bigg] dt$$
$$\geq \int_0^T \int_{\Omega_E^c} \mathbf{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx + \int_{\Omega_H^c} \mathbf{g} \cdot (\boldsymbol{w} - \boldsymbol{H})\, dx dt.$$

Consequently, by the local regularity property (3.3), (1.4) implies

$$\int_0^T \Bigg[\int_{\Omega_E^c} \epsilon\partial_t \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E}) - \boldsymbol{H} \cdot \mathbf{curl}\, (\boldsymbol{v} - \boldsymbol{E}) + \int_{\Omega_H^c} \mu\partial_t \boldsymbol{H} \cdot (\boldsymbol{w} - \boldsymbol{H}) + \tag{3.5}$$
$$\boldsymbol{E} \cdot \mathbf{curl}\, (\boldsymbol{w} - \boldsymbol{H})\, dx + \int_{\Omega_E \cap \Omega_H} \mathbf{curl}\, \boldsymbol{H} \cdot (\boldsymbol{v} - \boldsymbol{E}) - \mathbf{curl}\, \boldsymbol{E} \cdot (\boldsymbol{w} - \boldsymbol{H}) -$$

$$\boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx \Bigg] dt \geq \int_0^T \int_{\Omega_E^c} \mathbf{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx + \int_{\Omega_H^c} \mathbf{g} \cdot (\boldsymbol{w} - \boldsymbol{H})\, dx dt.$$

Let us underline that in (3.5) the test function $\boldsymbol{w}$ (resp. $\boldsymbol{v}$) does not appear in the region Ω_E^c (resp. Ω_H^c), i.e., in the region where the L^2-regularity of $\mathbf{curl}\, \boldsymbol{H}$ (resp. $\mathbf{curl}\, \boldsymbol{E}$) is not guaranteed. This particular structure allows us to deduce that the third integral in (3.5) over $\Omega_E \cap \Omega_H$ vanishes. To this aim, let $\boldsymbol{\eta} \in \boldsymbol{H}(\mathbf{curl}, \Omega_E)$ be arbitrarily fixed satisfying $\boldsymbol{\eta}_{|\Omega_H^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c)$ (recall that $\Omega_H^c \subset \Omega_E$). According to Assumption 1.1, $\Omega_E^c \subset\subset \mathcal{U} \subset \Omega$, and $\mathcal{U}_E = \mathcal{U} \setminus \overline{\Omega_E^c}$ is a bounded Lipschitz domain. Thus, since $\Omega_E = \Omega \setminus \overline{\Omega_E^c}$, the vector field $\boldsymbol{\eta} \in \boldsymbol{H}(\mathbf{curl}, \Omega_E)$ can be extended to a $\boldsymbol{H}(\mathbf{curl})$-function in Ω (see [10, Appendix]), i.e.,

$$\exists\, \hat{\boldsymbol{\eta}} \in \boldsymbol{H}(\mathbf{curl}, \Omega): \quad \hat{\boldsymbol{\eta}} = \boldsymbol{\eta} \text{ in } \Omega_E. \tag{3.6}$$

We modify the extended vector field into

$$\tilde{\boldsymbol{\eta}} := \begin{cases} \hat{\boldsymbol{\eta}} & \text{in } \Omega_H \\ 0 & \text{in } \Omega_H^c. \end{cases} \tag{3.7}$$

The modified vector field $\tilde{\boldsymbol{\eta}}$ belongs as well to $\boldsymbol{H}(\mathbf{curl})$. Indeed, as $\hat{\boldsymbol{\eta}} \in \boldsymbol{H}(\mathbf{curl})$, the distributional definition of the $\mathbf{curl}$-operator yields

$$\begin{aligned} &\int_{\Omega_H} \mathbf{curl}\, \hat{\boldsymbol{\eta}} \cdot \boldsymbol{\varphi}\, dx + \int_{\Omega_H^c} \mathbf{curl}\, \boldsymbol{\eta} \cdot \boldsymbol{\varphi}\, dx \underbrace{=}_{\Omega_H^c \subset \Omega_E \& (3.6)} \int_{\Omega} \mathbf{curl}\, \hat{\boldsymbol{\eta}} \cdot \boldsymbol{\varphi}\, dx \\ &= \int_{\Omega} \hat{\boldsymbol{\eta}} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx \underbrace{=}_{\Omega_H^c \subset \Omega_E \& (3.6)} \int_{\Omega_H} \hat{\boldsymbol{\eta}} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx + \int_{\Omega_H^c} \boldsymbol{\eta} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx \\ &\underbrace{=}_{\boldsymbol{\eta}_{|\Omega_H^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c) \& (1.4)} \int_{\Omega_H} \hat{\boldsymbol{\eta}} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx + \int_{\Omega_H^c} \mathbf{curl}\, \boldsymbol{\eta} \cdot \boldsymbol{\varphi}\, dx \quad \forall \boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\Omega). \end{aligned}$$

Thus,

$$\int_{\Omega_H} \mathbf{curl}\, \hat{\boldsymbol{\eta}} \cdot \boldsymbol{\varphi}\, dx = \int_{\Omega_H} \hat{\boldsymbol{\eta}} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx \underbrace{=}_{(3.7)} \int_{\Omega} \tilde{\boldsymbol{\eta}} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx \quad \forall \boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\Omega),$$

from which it follows that $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}(\mathbf{curl})$ and $\mathbf{curl}\, \tilde{\boldsymbol{\eta}} = \Pi_H(\mathbf{curl}\, \hat{\boldsymbol{\eta}}_{|\Omega_H})$ where $\Pi_H : \boldsymbol{L}^2(\Omega_H) \to \boldsymbol{L}^2(\Omega)$ denotes the zero extension operator (2.9). As a consequence, since $(0, 0) \in \boldsymbol{K}$, (1.6) ensures that $(0, \tilde{\boldsymbol{\eta}}) \in D(\mathcal{A}) \cap \boldsymbol{K}$. Therefore, for arbitrarily fixed $\tau \in [0, T)$ and $h \in (0, T - \tau)$, we may insert

$$(\boldsymbol{v}, \boldsymbol{w}) = (0, \chi_{[\tau, \tau+h]} \tilde{\boldsymbol{\eta}}) \in L^2((0, T), D(\mathcal{A}))$$

in (3.5) to obtain

$$\int_{\tau}^{\tau+h} \int_{\Omega_E \cap \Omega_H} \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{\eta} - \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{\eta}\, dx dt \geq C$$

with $C = \int_0^T \int_{\Omega_E \cap \Omega_H} \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{H} - \int_{\Omega_E^c} \mathbf{f} \cdot \boldsymbol{E}\, dx - \int_{\Omega_H^c} \mathbf{g} \cdot \boldsymbol{H}\, dx + \int_{\Omega_E^c} \epsilon \partial_t \boldsymbol{E} \cdot \boldsymbol{E} - \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{E} + \int_{\Omega_H^c} \mu \partial_t \boldsymbol{H} \cdot \boldsymbol{H} + \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{H}\, dx dt$. Since $\boldsymbol{\eta}$ was chosen arbitrarily, similarly to the proof of Theorem 1 (2. Step), it follows that

$$\int_{\Omega_E \cap \Omega_H} \boldsymbol{E}(\tau) \cdot \mathbf{curl}\, \boldsymbol{\eta}\, dx = \int_{\Omega_E \cap \Omega_H} \mathbf{curl}\, \boldsymbol{E}(\tau) \cdot \boldsymbol{\eta}\, dx \tag{3.8}$$
$$\text{for a.e. } \tau \in (0,T) \text{ and all } \boldsymbol{\eta} \in \boldsymbol{H}(\mathbf{curl}, \Omega_E) \text{ with } \boldsymbol{\eta}_{|\Omega_H^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c).$$

In particular, thanks to (3.1) and (3.3), we may insert $\boldsymbol{\eta} = \boldsymbol{H}(\tau)_{|\Omega_E}$ in (3.8) to obtain

$$\int_{\Omega_E \cap \Omega_H} \boldsymbol{E}(\tau) \cdot \mathbf{curl}\, \boldsymbol{H}(\tau)\, dx = \int_{\Omega_E \cap \Omega_H} \mathbf{curl}\, \boldsymbol{E}(\tau) \cdot \boldsymbol{H}(\tau)\, dx \quad \text{for a.e. } \tau \in (0,T). \tag{3.9}$$

Next, let $\boldsymbol{\xi} \in \boldsymbol{H}_0(\mathbf{curl})$ with $\boldsymbol{\xi}_{|\Omega_E^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c)$. Similarly to the case of $\tilde{\boldsymbol{\eta}}$, the modified function

$$\tilde{\boldsymbol{\xi}} = \begin{cases} \boldsymbol{\xi} & \text{in } \Omega_E \\ 0 & \text{in } \Omega_E^c \end{cases}$$

belongs to $\boldsymbol{H}_0(\mathbf{curl})$, and so thanks to $(0,0) \in \boldsymbol{K}$, (1.6) ensures that $(\tilde{\boldsymbol{\xi}}, 0) \in D(\mathcal{A}) \cap \boldsymbol{K}$. Therefore, as before, applying of the test function $(\boldsymbol{v}, \boldsymbol{w}) = (\chi_{[\tau,\tau+h]} \tilde{\boldsymbol{\xi}}, 0) \in L^2((0,T), D(\mathcal{A}))$ to (3.5), we obtain that

$$\int_{\Omega_E \cap \Omega_H} \boldsymbol{H}(\tau) \cdot \mathbf{curl}\, \boldsymbol{\xi}\, dx = \int_{\Omega_E \cap \Omega_H} \mathbf{curl}\, \boldsymbol{H}(\tau) \cdot \boldsymbol{\xi}\, dx \tag{3.10}$$
$$\text{for a.e. } \tau \in (0,T) \text{ and all } \boldsymbol{\xi} \in \boldsymbol{H}_0(\mathbf{curl}) \text{ with } \boldsymbol{\xi}_{|\Omega_E^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c).$$

On the other hand, due to the assumption (1.11), every $(\boldsymbol{v}, \boldsymbol{w}) \in L^2((0,T), D(\mathcal{A}))$ with $(\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K}$ a.e. in $(0,T)$ satisfies

$$\boldsymbol{v}(t)_{|\Omega_E^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) \quad \text{and} \quad \boldsymbol{w}(t)_{|\Omega_H^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c) \quad \text{for a.e. } t \in (0,T).$$

Thus, applying (3.8)-(3.10) to (3.5), we conclude that $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{X})$ satisfies

$$\int_0^T \Big[\int_{\Omega_E^c} \epsilon \partial_t \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E}) - \boldsymbol{H} \cdot \mathbf{curl}\, (\boldsymbol{v} - \boldsymbol{E})\, dx + \int_{\Omega_H^c} \mu \partial_t \boldsymbol{H} \cdot (\boldsymbol{w} - \boldsymbol{H}) \tag{3.11}$$

$$+\boldsymbol{E}\cdot\mathbf{curl}\,(\boldsymbol{w}-\boldsymbol{H})\,dx\Big]dt \geq \int_0^T \Big[\int_{\Omega_E^c} \mathbf{f}\cdot(\boldsymbol{v}-\boldsymbol{E})\,dx + \int_{\Omega_H^c} \mathbf{g}\cdot(\boldsymbol{w}-\boldsymbol{H})\,dx\Big]dt$$

for all $(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),D(\mathcal{A}))$ with $(\boldsymbol{v},\boldsymbol{w})(t) \in \boldsymbol{K}$ a.e. in $(0,T)$.

Let us now prove that (3.11) holds true without the time integration. To this aim, we first note that, in view of (3.2) and (1.9),

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{E}_{|\Omega_E^c} &= (\mathbf{g}-\mu\partial_t\boldsymbol{H})_{|\Omega_E^c} \in L^\infty((0,T),\boldsymbol{L}^2(\Omega_E^c))\\ \mathbf{curl}\,\boldsymbol{H}_{|\Omega_H^c} &= (\epsilon\partial_t\boldsymbol{E}-\mathbf{f})_{|\Omega_H^c} \in L^\infty((0,T),\boldsymbol{L}^2(\Omega_H^c)),\end{aligned}$$

and so

$$\boldsymbol{E}_{|\Omega_E^c} \in L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl},\Omega_E^c)) \quad \text{and} \quad \boldsymbol{H}_{|\Omega_H^c} \in L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl},\Omega_H^c)). \tag{3.12}$$

Introducing the zero extension operators $\Pi_E^c : \boldsymbol{L}^2(\Omega_E^c) \to \boldsymbol{L}^2(\Omega)$ and $\Pi_H^c : \boldsymbol{L}^2(\Omega_H^c) \to \boldsymbol{L}^2(\Omega)$,

$$\Pi_E^c(\boldsymbol{\eta}) = \begin{cases} \boldsymbol{\eta} & \text{in } \Omega_E^c \\ 0 & \text{in } \Omega_E \end{cases} \qquad \Pi_H^c(\boldsymbol{\xi}) = \begin{cases} \boldsymbol{\xi} & \text{in } \Omega_H^c \\ 0 & \text{in } \Omega_H, \end{cases} \tag{3.13}$$

it follows by (3.12), (1.4) and (1.6) that

$$\begin{cases} (\Pi_E^c(\boldsymbol{E}_{|\Omega_E^c}), \Pi_H^c(\boldsymbol{H}_{|\Omega_H^c})) \in L^2((0,T),D(\mathcal{A})), \\ (\Pi_E^c(\boldsymbol{E}_{|\Omega_E^c}), \Pi_H^c(\boldsymbol{H}_{|\Omega_H^c}))(t) \in \boldsymbol{K} \quad \forall t \in [0,T]. \end{cases} \tag{3.14}$$

Let now $\tau \in (0,T)$, $h \in (0,T-\tau)$ and $(\boldsymbol{y},\boldsymbol{z}) \in D(\mathcal{A}) \cap \boldsymbol{K}$. By virtue of (3.14), it holds that

$$(\boldsymbol{v},\boldsymbol{w}) := \chi_{[\tau,\tau+h]}(\boldsymbol{y},\boldsymbol{z}) + \chi_{[0,T]\setminus[\tau,\tau+h]}(\Pi_E^c(\boldsymbol{E}_{|\Omega_E^c}), \Pi_H^c(\boldsymbol{H}_{|\Omega_H^c})) \in L^2((0,T),D(\mathcal{A}))$$

and $(\boldsymbol{v},\boldsymbol{w})(t) \in \boldsymbol{K}$ for all $t \in [0,T]$. Therefore, considering this specific test function in (3.11) leads to

$$\begin{aligned}&\frac{1}{h}\int_\tau^{\tau+h}\Big[\int_{\Omega_E^c} \epsilon\partial_t\boldsymbol{E}(t)\cdot(\boldsymbol{y}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}\,(\boldsymbol{y}-\boldsymbol{E}(t))\,dx\\ &+\int_{\Omega_H^c} \mu\partial_t\boldsymbol{H}(t)\cdot(\boldsymbol{z}-\boldsymbol{H}(t)) + \boldsymbol{E}(t)\cdot\mathbf{curl}\,(\boldsymbol{z}-\boldsymbol{H}(t))\,dx\Big]dt\\ &\geq \frac{1}{h}\int_\tau^{\tau+h}\Big[\int_{\Omega_E^c} \mathbf{f}(t)\cdot(\boldsymbol{y}-\boldsymbol{E}(t))\,dx + \int_{\Omega_H^c} \mathbf{g}(t)\cdot(\boldsymbol{z}-\boldsymbol{H}(t))\,dx\Big]dt,\end{aligned}$$

from which it follows, after passing to the limit $h \downarrow 0$, that

$$\int_{\Omega_E^c} \epsilon\partial_t \boldsymbol{E}(t)\cdot(\boldsymbol{y}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot \mathbf{curl}\,(\boldsymbol{y}-\boldsymbol{E}(t))\,dx + \int_{\Omega_H^c} \mu\partial_t \boldsymbol{H}(t)\cdot(\boldsymbol{z}-\boldsymbol{H}(t))$$
$$+\boldsymbol{E}(t)\cdot \mathbf{curl}\,(\boldsymbol{z}-\boldsymbol{H}(t))\,dx \geq \int_{\Omega_E^c} \mathbf{f}(t)\cdot(\boldsymbol{y}-\boldsymbol{E}(t))\,dx + \int_{\Omega_H^c} \mathbf{g}(t)\cdot(\boldsymbol{z}-\boldsymbol{H}(t))\,dx$$
$$\text{for a.e. } t\in(0,T) \text{ and all } (\boldsymbol{y},\boldsymbol{z})\in D(\mathcal{A})\cap \boldsymbol{K}. \quad (3.15)$$

Step 2.: Uniqueness for (P) *under Assumption 1.1.* Suppose that $(\boldsymbol{E}^j, \boldsymbol{H}^j) \in W^{1,\infty}((0,T), \boldsymbol{X})$, $j = 1, 2$, are solutions to (P). We set $(\boldsymbol{e}, \boldsymbol{h}) := (\boldsymbol{E}^1 - \boldsymbol{E}^2, \boldsymbol{H}^1 - \boldsymbol{H}^2)$. According to (1.9), we have

$$\begin{cases} \epsilon\partial_t \boldsymbol{e} - \mathbf{curl}\,\boldsymbol{h} = 0 & \text{a.e. in } \Omega_E\times(0,T), \\ \mu\partial_t \boldsymbol{h} + \mathbf{curl}\,\boldsymbol{e} = 0 & \text{a.e. in } \Omega_H\times(0,T). \end{cases} \quad (3.16)$$

Furthermore, thanks to (3.14), we may insert $(\boldsymbol{y},\boldsymbol{z}) = (\Pi_E^c(\boldsymbol{E}^2_{|\Omega_E^c}), \Pi_H^c(\boldsymbol{H}^2_{|\Omega_H^c}))(t)$ (resp. $(\boldsymbol{y},\boldsymbol{z}) = (\Pi_E^c(\boldsymbol{E}^1_{|\Omega_E^c}), \Pi_H^c(\boldsymbol{H}^1_{|\Omega_H^c}))(t)$) in the variational inequality (3.15) for $(\boldsymbol{E},\boldsymbol{H}) = (\boldsymbol{E}^1,\boldsymbol{H}^1)$ (resp. $(\boldsymbol{E},\boldsymbol{H}) = (\boldsymbol{E}^2,\boldsymbol{H}^2)$) to obtain after adding the resulting inequalities that

$$0 \geq \frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E^c)} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c)} - \int_{\Omega_E^c} \boldsymbol{h}(t)\cdot \mathbf{curl}\,\boldsymbol{e}(t)\,dx$$
$$+\int_{\Omega_H^c} \mathbf{curl}\,\boldsymbol{h}(t)\cdot\boldsymbol{e}(t)\,dx \underbrace{=}_{(3.16)\ \&\ (3.2)} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E^c)} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c)}$$
$$+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_E^c)} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)} \quad \text{for a.e. } t\in(0,T),$$

from which it follows, due to $(\boldsymbol{e},\boldsymbol{h})(0) = (0,0)$, that $(\boldsymbol{e},\boldsymbol{h})(t) = 0$ in $\Omega_E^c\cup\Omega_H^c$ for all $t\in[0,T]$. To show the uniqueness in the subset $\Omega_E\cap\Omega_H$, we first notice that (3.8) holds true for $\boldsymbol{E} = \boldsymbol{E}^j$, $j = 1, 2$. Also, (3.1) and (3.3) are satisfied for $\boldsymbol{H} = \boldsymbol{H}^j$, $j = 1, 2$. Inserting $\boldsymbol{\eta} = \boldsymbol{h}$ in (3.8) for $\boldsymbol{E} = \boldsymbol{E}^j$ for $j = 1, 2$ and then subtracting the resulting equalities, we obtain

$$\int_{\Omega_E\cap\Omega_H} \boldsymbol{e}(t)\cdot \mathbf{curl}\,\boldsymbol{h}(t)\,dx = \int_{\Omega_E\cap\Omega_H} \mathbf{curl}\,\boldsymbol{e}(t)\cdot\boldsymbol{h}(t)\,dx \quad \text{for a.e. } t\in(0,T).$$

Multiplying the second equation in (3.16) with $\boldsymbol{h}$ and then integrating the resulting equality over $\Omega_E\cap\Omega_H$, we obtain from the above identity that

$$0 = \int_{\Omega_E\cap\Omega_H} \mu\partial_t\boldsymbol{h}(t)\cdot\boldsymbol{h}(t) + \mathbf{curl}\,\boldsymbol{e}(t)\cdot\boldsymbol{h}(t)\,dx = \int_{\Omega_E\cap\Omega_H} \mu\partial_t\boldsymbol{h}(t)\cdot\boldsymbol{h}(t) + \boldsymbol{e}(t)\cdot\mathbf{curl}\,\boldsymbol{h}(t)\,dx$$
$$\underbrace{=}_{(3.16)} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_E\cap\Omega_H)} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E\cap\Omega_H)} \quad \text{for a.e. } t\in(0,T).$$

Again, due to $(\boldsymbol{e}, \boldsymbol{h})(0) = (0, 0)$, it follows that $(\boldsymbol{e}, \boldsymbol{h})(t) = 0$ in $\Omega_E \cap \Omega_H$ for all $t \in [0, T]$. In conclusion, $(\boldsymbol{e}, \boldsymbol{h})(t) = 0$ holds true for all $t \in [0, T]$.

Step 3: Uniqueness for (P) *in the case of* $\Omega_H = \Omega$ *or* $\Omega_E = \Omega$. We only consider the case $\Omega_H = \Omega$. The proof for the case $\Omega_E = \Omega$ is completely analogous. In view of $\Omega_H = \Omega$, we have that $\boldsymbol{K} = \boldsymbol{K}_E \times \boldsymbol{L}^2_\mu(\Omega)$ for some closed and convex subset $\boldsymbol{K}_E \subset \boldsymbol{L}^2_\epsilon(\Omega)$ containing the zero element. Let $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0, T), \boldsymbol{X})$ denote a solution to (P). According to Theorem 1, since $\Omega_H = \Omega$, we have that

$$\boldsymbol{E} \in L^\infty((0, T), \boldsymbol{H}_0(\mathbf{curl})),$$

and

$$\mu\partial_t \boldsymbol{H}(t) \cdot (\boldsymbol{w} - \boldsymbol{H}(t)) + \mathbf{curl}\, \boldsymbol{E}(t) \cdot (\boldsymbol{w} - \boldsymbol{H}(t)) = \mathbf{g}(t) \cdot (\boldsymbol{w} - \boldsymbol{H}(t)) \quad \forall \boldsymbol{w} \in \boldsymbol{L}^2_\mu(\Omega) \tag{3.17}$$

holds for a.e. $t \in (0, T)$. Then, applying (3.17) to (P) leads to the following variational inequality for the electric field $\boldsymbol{E}$:

$$\int_0^T \int_\Omega \epsilon\partial_t \boldsymbol{E}(t) \cdot (\boldsymbol{v}(t) - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}\, (\boldsymbol{v}(t) - \boldsymbol{E}(t))\, dxdt$$
$$\geq \int_0^T \int_\Omega \mathbf{f}(t) \cdot (\boldsymbol{v}(t) - \boldsymbol{E}(t))\ dxdt$$

for all $\boldsymbol{v} \in L^2((0, T), \boldsymbol{H}_0(\mathbf{curl}))$ with $\boldsymbol{v}(t) \in \boldsymbol{K}_E$ for a.e. $t \in (0, T)$. Then, as in Step 1, employing the test function

$$\boldsymbol{v} := \chi_{[\tau,\tau+h]}\boldsymbol{z} + \chi_{[0,T]\setminus[\tau,\tau+h]}\boldsymbol{E} \in L^2((0, T), \boldsymbol{H}_0(\mathbf{curl}))$$

with $\tau \in (0, T)$, $h \in (0, T - \tau)$ and $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{K}_E$, we get the following variational inequality for $\boldsymbol{E}$ without the time integration:

$$\int_\Omega \epsilon\partial_t \boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}\, (\boldsymbol{z} - \boldsymbol{E}(t))\, dx \geq \int_\Omega \mathbf{f}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t))\ dx$$
$$\text{for a.e. } t \in (0, T) \text{ and all } \boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{K}_E. \tag{3.18}$$

Suppose that $(\boldsymbol{E}^j, \boldsymbol{H}^j) \in W^{1,\infty}((0, T), \boldsymbol{X})$, $j = 1, 2$, are solutions to (P). We set $(\boldsymbol{e}, \boldsymbol{h}) := (\boldsymbol{E}^1 - \boldsymbol{E}^2, \boldsymbol{H}^1 - \boldsymbol{H}^2)$. Setting $\boldsymbol{z} = \boldsymbol{E}^2(t)$ (resp. $\boldsymbol{z} = \boldsymbol{E}^1(t)$) in the variational inequality (3.18) for $\boldsymbol{E} = \boldsymbol{E}^1$ (resp. $\boldsymbol{E} = \boldsymbol{E}^2$) and adding the resulting inequalities, we obtain

$$\int_\Omega \mu\partial_t \boldsymbol{e}(t) \cdot \boldsymbol{e}(t) - \boldsymbol{h}(t) \cdot \mathbf{curl}\, \boldsymbol{e}(t)\, dx \leq 0 \quad \text{for a.e. } t \in (0, T).$$

On the other hand, inserting $\boldsymbol{w} = \boldsymbol{H}^2(t)$ (resp. $\boldsymbol{w} = \boldsymbol{H}^1(t)$) in the variational equality (3.17) for $\boldsymbol{H} = \boldsymbol{H}^1$ (resp. $\boldsymbol{H} = \boldsymbol{H}^2$) and adding the resulting equalities, we obtain

$$\mu \partial_t \boldsymbol{h}(t) \cdot \boldsymbol{h}(t) = -\mathbf{curl}\, \boldsymbol{e}(t) \cdot \boldsymbol{h}(t) \quad \text{for a.e. } t \in (0,T).$$

In conclusion, it holds that $\frac{d}{dt}\|\boldsymbol{e}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{d}{dt}\|\boldsymbol{h}(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \leq 0$ for a.e. $t \in (0,T)$, which yields that (P) admits at most only one solution. This completes the proof. □

4. Proof of Theorem 3

Let $\{\epsilon_n\}_{n=1}^\infty \subset L^\infty(\Omega)^{3\times 3}$ be defined by

$$\epsilon_n(x) := n^{-1} \quad \forall x \in \Omega_E^c \quad \text{and} \quad \epsilon_n(x) := \epsilon(x) \quad \text{a.e. in } \Omega_E. \tag{4.1}$$

In the following let (P_n) denote (P) with ϵ replaced by ϵ_n.

Step 1: Suppose that Assumption 1.1 is satisfied. According to Theorems 1 and 2, (3.3) and (3.15) along with (1.14), for every $n \in \mathbb{N}$, (P_n) admits a unique solution $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in W^{1,\infty}((0,T), \boldsymbol{X})$ satisfying

$$\int_{\Omega_E^c} \frac{1}{n}\partial_t \boldsymbol{E}_n(t)\cdot(\boldsymbol{y} - \boldsymbol{E}_n(t)) - \boldsymbol{H}_n(t)\cdot\mathbf{curl}\,(\boldsymbol{y} - \boldsymbol{E}_n(t))\, dx + \int_{\Omega_H^c} \mu\partial_t \boldsymbol{H}_n(t)\cdot(\boldsymbol{z} - \boldsymbol{H}_n(t))$$
$$+\boldsymbol{E}_n(t)\cdot\mathbf{curl}\,(\boldsymbol{z} - \boldsymbol{H}_n(t))\, dx \geq \int_{\Omega_H^c} \mathbf{g}(t)\cdot(\boldsymbol{z} - \boldsymbol{H}_n(t))\, dx$$
$$\text{for a.e. } t \in (0,T) \text{ and all } (\boldsymbol{y},\boldsymbol{z}) \in D(\mathcal{A}) \cap \boldsymbol{K}, \tag{4.2}$$

$$\begin{cases} \boldsymbol{E}_n(t)_{|\Omega_E^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_E^c) & \text{for a.e. } t \in (0,T), \\ \boldsymbol{H}_n(t)_{|\Omega_H^c} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_H^c) & \text{for a.e. } t \in (0,T), \end{cases} \tag{4.3}$$

$$\begin{cases} \epsilon\partial_t \boldsymbol{E}_n - \mathbf{curl}\, \boldsymbol{H}_n = \mathbf{f} & \text{a.e. in } \Omega_E \times (0,T), \\ \mu\partial_t \boldsymbol{H}_n + \mathbf{curl}\, \boldsymbol{E}_n = \mathbf{g} & \text{a.e. in } \Omega_H \times (0,T). \end{cases} \tag{4.4}$$

Inserting $(\boldsymbol{y},\boldsymbol{z}) = (0,0)$ in (4.2) and applying (4.4) along with $\Omega_H^c \subset \Omega_E$ and $\Omega_E^c \subset \Omega_H$, we obtain after integrating over the time interval $[0,\tau]$ with $\tau \in [0,T]$ that

$$\frac{1}{2}\Bigg[\frac{1}{n}\|\boldsymbol{E}_n(\tau)\|^2_{\boldsymbol{L}^2(\Omega_E^c)} + \|\boldsymbol{E}_n(\tau)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)} + \|\boldsymbol{H}_n(\tau)\|^2_{\boldsymbol{L}^2_\mu(\Omega_E^c\cup\Omega_H^c)} - \frac{1}{n}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega_E^c)} - \|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}$$
$$-\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega_E^c\cup\Omega_H^c)}\Bigg] \leq \int_0^\tau \int_{\Omega_H^c} \mathbf{f}(t)\cdot \boldsymbol{E}_n(t)\, dx + \int_{\Omega_H^c\cup\Omega_E^c} \mathbf{g}(t)\cdot\boldsymbol{H}_n(t)\, dx\, dt$$

$$\leq \int_0^\tau \frac{1}{2\underline{\epsilon}}\|\mathbf{f}(t)\|^2_{\boldsymbol{L}^2(\Omega_H^c)} + \frac{1}{2\underline{\mu}}\|\mathbf{g}(t)\|^2_{\boldsymbol{L}^2(\Omega_H^c\cup\Omega_E^c)} + \frac{1}{2}\|\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)} + \frac{1}{2}\|\boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\, dt.$$

Thus, applying the Gronwall lemma leads to

$$\|\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)} + \|\boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_E^c\cup\Omega_H^c)} \leq c_1 \quad \forall n \in \mathbb{N} \quad \forall t \in [0,T] \tag{4.5}$$

with a constant $c_1 > 0$, depending only on $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\mathbf{f}$, $\mathbf{g}$, $\underline{\epsilon}$, $\underline{\mu}$ and T. On the other hand, according to (3.9), we know that

$$\int_{\Omega_E\cap\Omega_H} \boldsymbol{E}_n(t)\cdot\mathbf{curl}\,\boldsymbol{H}_n(t)\,dx = \int_{\Omega_E\cap\Omega_H} \mathbf{curl}\,\boldsymbol{E}_n(t)\cdot\boldsymbol{H}_n(t)\,dx \tag{4.6}$$

holds for a.e. $t\in(0,T)$ and all $n\in\mathbb{N}$. Then, applying (4.6) to (4.4), we obtain that

$$\|\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H\cap\Omega_E)} + \|\boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_H\cap\Omega_E)} \le c_2 \quad \forall n\in\mathbb{N} \quad \forall t\in[0,T] \tag{4.7}$$

with a constant $c_2 > 0$, depending only on $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\mathbf{f}$, $\mathbf{g}$, $\underline{\epsilon}$, $\underline{\mu}$ and T. Combining (1.14), (4.5) and (4.7), we conclude that there exists a constant $c_3 > 0$ such that

$$\|(\boldsymbol{E}_n,\boldsymbol{H}_n)\|_{C([0,T],\boldsymbol{X})} \le c_3 \quad \forall n\in\mathbb{N}. \tag{4.8}$$

Next, thanks to (4.3) and (1.6), we have that

$$(\Pi^c_E(\boldsymbol{E}_{n|\Omega^c_E}),\Pi^c_H(\boldsymbol{H}_{n|\Omega^c_H}))(t)\in D(\mathcal{A})\cap\boldsymbol{K} \quad \text{for a.e. } t\in(0,T)$$

such that we may apply the above test function to (1.15) to obtain

$$\begin{aligned} -\int_{\Omega^c_H} \boldsymbol{E}_0\cdot\mathbf{curl}\,(\boldsymbol{H}_n(t)-\boldsymbol{H}_0) + \int_{\Omega^c_E} \boldsymbol{H}_0\cdot\mathbf{curl}\,(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)\,dx \le \\ -\int_{\Omega^c_H} \mathbf{g}(0)\cdot(\boldsymbol{H}_n(t)-\boldsymbol{H}_0)\,dx \quad \text{for a.e. } t\in(0,T). \end{aligned} \tag{4.9}$$

Hereafter, setting $(\boldsymbol{y},\boldsymbol{z}) = (\boldsymbol{E}_0,\boldsymbol{H}_0)\in D(\mathcal{A})\cap\boldsymbol{K}$ in (4.2) and adding the resulting inequality with (4.9), we obtain for a.e. $t\in(0,T)$ that

$$\begin{aligned} &\int_{\Omega^c_E} \frac{1}{n}\partial_t(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)\cdot(\boldsymbol{E}_n(t)-\boldsymbol{E}_0) + (\boldsymbol{H}_0-\boldsymbol{H}_n(t))\cdot\mathbf{curl}\,(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)\,dx + \\ &\int_{\Omega^c_H} \mu\partial_t(\boldsymbol{H}_n(t)-\boldsymbol{H}_0)\cdot(\boldsymbol{H}_n(t)-\boldsymbol{H}_0) + (\boldsymbol{E}_n(t)-\boldsymbol{E}_0)\cdot\mathbf{curl}\,(\boldsymbol{H}_n(t)-\boldsymbol{H}_0)\,dx \\ &\le \int_{\Omega^c_H} (\mathbf{g}(t)-\mathbf{g}(0))\cdot(\boldsymbol{H}_n(t)-\boldsymbol{H}_0)\,dx. \end{aligned} \tag{4.10}$$

On the other hand, combining (1.16) and (4.4) together yields for a.e. $t\in(0,T)$ that

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{H}_n(t)-\boldsymbol{H}_0) = \epsilon\partial_t(\boldsymbol{E}_n(t)-\boldsymbol{E}_0) - \mathbf{f}(t)+\mathbf{f}(0) & \text{a.e. in } \Omega_E, \\ \mathbf{curl}\,(\boldsymbol{E}_n(t)-\boldsymbol{E}_0) = -\mu\partial_t(\boldsymbol{H}_n(t)-\boldsymbol{H}_0) + \mathbf{g}(t)-\mathbf{g}(0) & \text{a.e. in } \Omega_H. \end{cases} \tag{4.11}$$

Since $\Omega_E^c \subset \Omega_H$ and $\Omega_H^c \subset \Omega_E$ hold true (see (3.2)), applying (4.11) to (4.10), we obtain that

$$\begin{aligned}\frac{1}{2n}\frac{d}{dt}\|\boldsymbol{E}_n(t)-\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{E}_n(t)-\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{H}_n(t)-\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\\ \leq \|\mathbf{f}(t)-\mathbf{f}(0)\|_{\boldsymbol{L}^2(\Omega_H^c)}\|\boldsymbol{E}_n(t)-\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega_H^c)}+\\ \|\mathbf{g}(t)-\mathbf{g}(0)\|_{\boldsymbol{L}^2(\Omega_E^c\cup\Omega_H^c)}\|\boldsymbol{H}_n(t)-\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega_E^c\cup\Omega_H^c)}\\ \leq L_f t\|\boldsymbol{E}_n(t)-\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega_H^c)}+L_g t\|\boldsymbol{H}_n(t)-\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega_E^c\cup\Omega_H^c)}\quad \text{for a.e. } t\in(0,T),\end{aligned}$$

where $L_f > 0$ and $L_g > 0$ denote the Lipschitz constants for $\mathbf{f}$ and $\mathbf{g}$. Integrating the above inequality over the time interval $[0, h]$ with $h \in [0, T]$ leads to

$$\begin{aligned}&\frac{1}{2n}\|\boldsymbol{E}_n(h)-\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\frac{1}{2}\|\boldsymbol{E}_n(h)-\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}+\frac{1}{2}\|\boldsymbol{H}_n(h)-\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\\ &\leq h\max\{L_f,L_g\}\int_0^h \|\boldsymbol{E}_n(t)-\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega_H^c)}+\|\boldsymbol{H}_n(t)-\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega_E^c\cup\Omega_H^c)}\,dt.\end{aligned}$$

Then, by the Gronwall lemma, it follows that

$$\begin{aligned}\frac{1}{n}\left\|\frac{\boldsymbol{E}_n(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\left\|\frac{\boldsymbol{E}_n(h)-\boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}+\left\|\frac{\boldsymbol{H}_n(h)-\boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\\ \leq c_4\quad \forall h\in(0,T]\quad \forall n\in\mathbb{N}\end{aligned}\tag{4.12}$$

with a constant $c_4 > 0$, depending only on $L_f, L_g, \underline{\epsilon}, \underline{\mu}$ and T.

Inserting $(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{E}, \boldsymbol{H})(t + h)$ in (4.2) and $(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{E}, \boldsymbol{H})(t)$ in (4.2) with t replaced by $t + h$, adding the resulting inequalities, and then applying (4.4) along with $\Omega_H^c \subset \Omega_E$ and $\Omega_E^c \subset \Omega_H$, we obtain similarly as above that

$$\begin{aligned}\frac{1}{2n}\|\boldsymbol{E}_n(t+h)-\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\frac{1}{2}\|\boldsymbol{E}_n(t+h)-\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}\\ +\frac{1}{2}\|\boldsymbol{H}_n(t+h)-\boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\\ \leq\frac{1}{2n}\|\boldsymbol{E}_n(h)-\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\frac{1}{2}\|\boldsymbol{E}_n(h)-\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}+\frac{1}{2}\|\boldsymbol{H}_n(h)-\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\\ +h\max\{L_f,L_g\}\int_0^t\|\boldsymbol{E}_n(s+h)-\boldsymbol{E}_n(s)\|_{\boldsymbol{L}^2(\Omega_H^c)}+\|\boldsymbol{H}_n(s+h)-\boldsymbol{H}_n(s)\|_{\boldsymbol{L}^2(\Omega_E^c\cup\Omega_H^c)}\,ds\\ \forall t\in(0,T),h\in(0,T-t).\end{aligned}$$

Then, thanks to (4.12), the Gronwall lemma implies that

$$\frac{1}{n}\left\|\frac{\boldsymbol{E}_n(t+h)-\boldsymbol{E}_n(t)}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\left\|\frac{\boldsymbol{E}_n(t+h)-\boldsymbol{E}_n(t)}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}$$
$$+\left\|\frac{\boldsymbol{H}_n(t+h)-\boldsymbol{H}_n(t)}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\le c_5 \quad \forall t\in(0,T), h\in(0,T-t), n\in\mathbb{N}$$

with a constant $c_5>0$, depending only on $L_f, L_g, \underline{\epsilon}, \underline{\mu}$ and T. Passing to the limit $h\to 0$, we deduce that

$$\frac{1}{n}\|\partial_t\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2(\Omega_E^c)}+\|\partial_t\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_H^c)}+\|\partial_t\boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_H^c\cup\Omega_E^c)}\le c_5 \quad \text{for a.e. } t\in(0,T) \text{ and all } n\in\mathbb{N}. \tag{4.13}$$

By analogous arguments using (4.4), (4.6), Assumption 1.1 and (1.16), we find a constant $c_6>0$ such that

$$\|\partial_t\boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2(\Omega_E\cap\Omega_H)}+\|\partial_t\boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega_E\cap\Omega_H)}\le c_6 \quad \text{for a.e. } t\in(0,T) \text{ and all } n\in\mathbb{N}. \tag{4.14}$$

Concluding from (4.8), (4.13), (4.14) and (4.4), standard arguments imply the existence of a subsequence of $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty$ denoted again by $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty$ such that

$$\begin{aligned}
(\boldsymbol{E}_n,\boldsymbol{H}_n)&\rightharpoonup(\boldsymbol{E},\boldsymbol{H}) && \text{weakly star in } L^\infty((0,T),\boldsymbol{X})\\
\mathbf{curl}\,\boldsymbol{H}_n&\rightharpoonup\mathbf{curl}\,\boldsymbol{H} && \text{weakly star in } L^\infty((0,T),\boldsymbol{L}^2(\Omega_E))\\
\mathbf{curl}\,\boldsymbol{E}_n&\rightharpoonup\mathbf{curl}\,\boldsymbol{E} && \text{weakly star in } L^\infty((0,T),\boldsymbol{L}^2(\Omega_H))\\
\partial_t\boldsymbol{H}_n&\rightharpoonup\partial_t\boldsymbol{H} && \text{weakly star in } L^\infty((0,T),\boldsymbol{L}^2(\Omega))\\
\partial_t\boldsymbol{E}_n&\rightharpoonup\partial_t\boldsymbol{E} && \text{weakly star in } L^\infty((0,T),\boldsymbol{L}^2(\Omega_E))\\
\frac{1}{\sqrt{n}}\partial_t\boldsymbol{E}_n&\rightharpoonup\boldsymbol{d} && \text{weakly star in } L^\infty((0,T),\boldsymbol{L}^2(\Omega_E^c))
\end{aligned} \tag{4.15}$$

for some $\boldsymbol{d}\in L^\infty((0,T),\boldsymbol{L}^\infty(\Omega_E^c))$ and $(\boldsymbol{E},\boldsymbol{H})\in L^\infty((0,T),\boldsymbol{X})\cap W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega_E)\times\boldsymbol{L}^2(\Omega))$ satisfying

$$\boldsymbol{E}_{|\Omega_H}\in L^\infty((0,T),\boldsymbol{H}(\mathbf{curl},\Omega_H)),\quad \boldsymbol{H}_{|\Omega_E}\in L^\infty((0,T),\boldsymbol{H}(\mathbf{curl},\Omega_E)) \tag{4.16}$$

and

$$\begin{cases}\epsilon\partial_t\boldsymbol{E}-\mathbf{curl}\,\boldsymbol{H}=\mathbf{f} & \text{a.e. in } \Omega_E\times(0,T),\\ \mu\partial_t\boldsymbol{H}+\mathbf{curl}\,\boldsymbol{E}=\mathbf{g} & \text{a.e. in } \Omega_H\times(0,T).\end{cases} \tag{4.17}$$

Possibly after a modification on a subset of $[0,T]$ with measure zero, the weak limit also satisfies $(\boldsymbol{E},\boldsymbol{H})\in\mathcal{C}([0,T],\boldsymbol{L}^2(\Omega_E)\times\boldsymbol{L}^2(\Omega))$. Hereafter, in view of (4.15), well-known arguments imply that

$$\boldsymbol{E}_n(t) \rightharpoonup \boldsymbol{E}(t) \quad \text{weakly in } \boldsymbol{L}^2(\Omega_E), \qquad \boldsymbol{H}_n(t) \rightharpoonup \boldsymbol{H}(t) \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \tag{4.18}$$

for all $t \in [0, T]$. In particular, setting $t = 0$ in (4.18) yields that

$$\boldsymbol{E}(0) = \boldsymbol{E}_0 \text{ in } \Omega_E \quad \text{and} \quad \boldsymbol{H}(0) = \boldsymbol{H}_0 \text{ in } \Omega. \tag{4.19}$$

On the other hand, applying (4.18) with $t = T$, we infer that

$$\begin{aligned}\limsup_{n\to\infty} -\int_0^T\int_{\Omega_E} \epsilon\partial_t \boldsymbol{E}_n \cdot \boldsymbol{E}_n \,dxdt &= -\liminf_{n\to\infty} \frac{1}{2}\|\boldsymbol{E}_n(T)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} + \frac{1}{2}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} \\ &\le -\frac{1}{2}\|\boldsymbol{E}(T)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} + \frac{1}{2}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} = -\int_0^T\int_{\Omega_E} \epsilon\partial_t \boldsymbol{E}\cdot\boldsymbol{E}\,dxdt\end{aligned} \tag{4.20}$$

and analogously

$$\limsup_{n\to\infty} -\int_0^T\int_{\Omega} \mu\partial_t \boldsymbol{H}_n \cdot \boldsymbol{H}_n \,dxdt \le -\int_0^T\int_{\Omega} \mu\partial_t \boldsymbol{H}\cdot\boldsymbol{H}\,dxdt. \tag{4.21}$$

From (4.15), (4.20) and (4.21), we obtain after passing to the limit sup $n \to \infty$ in (P_n) that the weak limit $(\boldsymbol{E}, \boldsymbol{H})$ satisfies

$$\begin{aligned}&\int_0^T \int_{\Omega_E} \epsilon\partial_t \boldsymbol{E}\cdot(\boldsymbol{v}-\boldsymbol{E})\,dx + \int_{\Omega} \mu\partial_t\boldsymbol{H}\cdot(\boldsymbol{w}-\boldsymbol{H}) - \boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{v} + \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{w}\,dx\,dt \\ &\qquad \ge \int_0^T\int_{\Omega} \mathbf{f}\cdot(\boldsymbol{v}-\boldsymbol{E}) + \mathbf{g}\cdot(\boldsymbol{w}-\boldsymbol{H})\,dx\,dt\end{aligned} \tag{4.22}$$

for all $(\boldsymbol{v}, \boldsymbol{w}) \in L^2((0, T), D(\mathcal{A}))$ with $(\boldsymbol{v}, \boldsymbol{w})(t) \in \boldsymbol{K}$ for a.e. $t \in (0, T)$.

It remains to prove that the weak limit satisfies

$$(\boldsymbol{E}, \boldsymbol{H})(t) \in \boldsymbol{K} \quad \text{for a.e. } t \in (0, T). \tag{4.23}$$

To this aim, we introduce $\widehat{\boldsymbol{K}} := \{\boldsymbol{U} \in L^2((0, T), \boldsymbol{X}) \mid \boldsymbol{U}(t) \in \boldsymbol{K} \text{ for a.e. } t \in (0, T)\}$. Since $\boldsymbol{K} \subset \boldsymbol{X}$ is convex, it follows that

$$\begin{aligned}\boldsymbol{U}_1, \boldsymbol{U}_2 \in \widehat{\boldsymbol{K}} \quad &\Rightarrow \quad \lambda\boldsymbol{U}_1(t) + (1-\lambda)\boldsymbol{U}_2(t) \in \boldsymbol{K} \quad \text{for all } \lambda \in [0, 1] \text{ and a.e. } t \in (0, T) \\ &\Rightarrow \quad \lambda\boldsymbol{U}_1 + (1-\lambda)\boldsymbol{U}_2 \in \widehat{\boldsymbol{K}} \quad \text{for all } \lambda \in [0, 1].\end{aligned}$$

Also, the set $\widehat{\boldsymbol{K}} \subset L^2((0, T), \boldsymbol{X})$ is closed. Indeed, let $\{\boldsymbol{U}_n\}_{n=1}^\infty \subset \widehat{\boldsymbol{K}}$ be a sequence converging strongly in $L^2((0, T), \boldsymbol{X})$ towards an element $\boldsymbol{U} \in L^2((0, T), \boldsymbol{X})$. Then, we can extract a subsequence $\{\boldsymbol{U}_{n_j}\}_{j=1}^\infty \subset \{\boldsymbol{U}_n\}_{n=1}^\infty$ such that

$$\boldsymbol{U}_{n_j}(t) \to \boldsymbol{U}(t) \text{ in } \boldsymbol{X} \text{ for a.e. } t \in (0,T) \Rightarrow \boldsymbol{U}(t) \in \boldsymbol{K} \text{ for a.e. } t \in (0,T) \Rightarrow \boldsymbol{U} \in \widehat{\boldsymbol{K}},$$

where we have used our assumption that $\boldsymbol{K} \subset \boldsymbol{X}$ is closed. Altogether, $\widehat{\boldsymbol{K}} \subset L^2((0,T),\boldsymbol{X})$ is closed and convex, and consequently, since $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \widehat{\boldsymbol{K}}$ for all $n \in \mathbb{N}$, (4.15) implies that $(\boldsymbol{E}, \boldsymbol{H}) \in \widehat{\boldsymbol{K}}$, i.e., (4.23) is valid.

Step 2: For the case of $\Omega_E = \Omega$, the claim is nothing but Theorem 1. Suppose now that $\Omega_H = \Omega$. In this case, we have that $\boldsymbol{K} = \boldsymbol{K}_E \times \boldsymbol{L}^2_\mu(\Omega)$ for some closed and convex subset $\boldsymbol{K}_E \subset \boldsymbol{L}^2_\epsilon(\Omega)$ containing the zero element. According to Theorems 1 and 2 as well as (3.18) and (1.14), for every $n \in \mathbb{N}$, (P$_n$) (cf. (4.1)) admits a unique solution $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in W^{1,\infty}((0,T),\boldsymbol{X})$ satisfying

$$\boldsymbol{E}_n \in L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})), \quad \boldsymbol{H}_{n|\Omega_E} \in L^\infty((0,T),\boldsymbol{H}(\mathbf{curl},\Omega_E)), \tag{4.24}$$

$$\int_{\Omega_E^c} \frac{1}{n}\partial_t \boldsymbol{E}_n(t)\cdot(z-\boldsymbol{E}_n(t))\,dx + \int_{\Omega_E} \epsilon\partial_t \boldsymbol{E}_n(t)\cdot(z-\boldsymbol{E}_n(t))\,dx - \int_\Omega \boldsymbol{H}_n(t)\cdot\mathbf{curl}\,(z-\boldsymbol{E}_n(t))dx$$
$$\geq \int_{\Omega_E} \mathbf{f}(t)\cdot(z-\boldsymbol{E}_n(t))\,dx \quad \text{for a.e. } t\in(0,T) \text{ and all } z \in \boldsymbol{H}_0(\mathbf{curl})\cap\boldsymbol{K}_E, \tag{4.25}$$

$$\begin{cases} \epsilon\partial_t \boldsymbol{E}_n - \mathbf{curl}\,\boldsymbol{H}_n = \mathbf{f} & \text{a.e. in } \Omega_E\times(0,T),\\ \mu\partial_t \boldsymbol{H}_n + \mathbf{curl}\,\boldsymbol{E}_n = \mathbf{g} & \text{a.e. in } \Omega\times(0,T). \end{cases} \tag{4.26}$$

Setting $z = 0$ in (4.25) and using the second equality in (4.26), we obtain that

$$\int_{\Omega_E^c} \frac{1}{n}\partial_t \boldsymbol{E}_n(t)\cdot\boldsymbol{E}_n(t)dx + \int_{\Omega_E} \epsilon\partial_t \boldsymbol{E}_n(t)\cdot\boldsymbol{E}_n(t)dx + \int_\Omega \mu\partial_t \boldsymbol{H}_n(t)\cdot\boldsymbol{H}_n(t)dx$$
$$\leq \int_{\Omega_E} \mathbf{f}(t)\cdot\boldsymbol{E}_n(t)\,dx + \int_\Omega \mathbf{g}(t)\cdot\boldsymbol{H}_n(t)\,dx \quad \text{for all } n\in\mathbb{N} \text{ and a.e. } t\in(0,T).$$

Then, applying the Gronwall lemma and (1.14) (similarly to Step 1), we find a constant $c_7 > 0$ such that

$$\|(\boldsymbol{E}_n, \boldsymbol{H}_n)\|_{C([0,T],\boldsymbol{X})} \leq c_7 \quad \forall n \in \mathbb{N}. \tag{4.27}$$

Setting now $z = \boldsymbol{E}_0$ in (4.25) and adding the resulting inequality with the inequality (1.17) for $\boldsymbol{v} = \boldsymbol{E}_n(t)$, we obtain that

$$\int_{\Omega_E^c} \frac{1}{n}\partial_t(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)\cdot(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)dx + \int_{\Omega_E} \epsilon\partial_t(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)\cdot(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)dx$$
$$-\int_\Omega (\boldsymbol{H}_n(t)-\boldsymbol{H}_0)\cdot\mathbf{curl}\,(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)dx \leq \int_{\Omega_E} (\mathbf{f}(t)-\mathbf{f}(0))\cdot(\boldsymbol{E}_n(t)-\boldsymbol{E}_0)dx$$

for all $n \in \mathbb{N}$ and a.e. $t \in (0,T)$. On the other hand, (1.18) combined with (4.26), it holds that

$$\mathbf{curl}\,(\boldsymbol{E}_n(t) - \boldsymbol{E}_0) = \mathbf{g}(t) - \mathbf{g}(0) - \mu\partial_t(\boldsymbol{H}_n(t) - \boldsymbol{H}_0) \quad \text{for all } n \in \mathbb{N} \text{ and a.e. } t \in (0, T).$$

Applying this equality to the previous inequality leads to

$$\begin{aligned}&\frac{1}{2n}\frac{d}{dt}\|\boldsymbol{E}_n(t) - \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega_E^c)} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{E}_n(t) - \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{H}_n(t) - \boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\ &\le \|\mathbf{f}(t) - \mathbf{f}(0)\|_{\boldsymbol{L}^2(\Omega_E)}\|\boldsymbol{E}_n(t) - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega_E)} + \|\mathbf{g}(t) - \mathbf{g}(0)\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{H}_n(t) - \boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}\\ &\qquad \le L_f t\|\boldsymbol{E}_n(t) - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega_E)} + L_g t\|\boldsymbol{H}_n(t) - \boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)} \quad \text{for a.e. } t \in (0, T).\end{aligned}$$

Then, invoking the Gronwall lemma (similarly to Step 1), we find a constant $c_8 > 0$ such that

$$\begin{aligned}&\frac{1}{n}\left\|\frac{\boldsymbol{E}_n(h) - \boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2(\Omega_E^c)} + \left\|\frac{\boldsymbol{E}_n(h) - \boldsymbol{E}_0}{h}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} + \left\|\frac{\boldsymbol{H}_n(h) - \boldsymbol{H}_0}{h}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\ &\qquad \le c_8 \quad \forall h \in (0, T] \quad \forall n \in \mathbb{N}.\end{aligned} \tag{4.28}$$

Thanks to (4.28), we may proceed as in Step 1 to deduce the existence of a constant $c_9 > 0$ such that

$$\begin{aligned}&\frac{1}{n}\|\partial_t \boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2(\Omega_E^c)} + \|\partial_t \boldsymbol{E}_n(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega_E)} + \|\partial_t \boldsymbol{H}_n(t)\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le c_9\\ &\qquad \text{for a.e. } t \in (0, T) \text{ and all } n \in \mathbb{N}.\end{aligned} \tag{4.29}$$

Finally, by analogous arguments as in Step 1, the stability estimates (4.27) and (4.29) along with (4.26) allow us to extract a weakly star converging subsequence of $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$, and after passing to the limit sup $n \to \infty$ in (P_n) we obtain that the associated weak limit $(\boldsymbol{E}, \boldsymbol{H}) \in L^\infty((0, T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \cap W^{1,\infty}((0, T), \boldsymbol{L}^2(\Omega_E) \times \boldsymbol{L}^2(\Omega))$ is a solution to the eddy current problem (1.19) satisfying (1.9).

5. Conclusion

In this paper, we developed an existence and uniqueness theory for the electromagnetic obstacle problem (P). While the existence is guaranteed for a general closed and convex set $\boldsymbol{K} \subset \boldsymbol{X}$ containing the origin, the uniqueness is shown under two different assumptions. The general one is based on a localization strategy, leading to a localized variational inequality (3.15) on the electric and magnetic constraint regions Ω_E^c and Ω_H^c. The established well-posedness result finds applications in the mathematical modeling of electromagnetic shielding. Therefore, it serves as a fundament for the numerical simulation and shape design of electromagnetic shielding materials that requires a substantial extension of the developed techniques [31,33]. In particular, the numerical analysis of (P) requires a Sobolev regularity property for the electric field of the type $\boldsymbol{E} \in L^1((0, T), \boldsymbol{H}^s(\Omega))$ for some $s > 0$. We aim at investigating this Sobolev regularity issue in our future research related to the numerical analysis of (P).

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