Hodge-Newton filtration for p-divisible groups with quadratic ramified endomorphism structure

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Abstract

Let $\mathcal{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field. We prove the existence of the Hodge-Newton filtration for p-divisible groups over $\mathcal{O}_K$ with additional endomorphism structure for the ring of integers of a finite field extension of $\mathbb{Q}_p$ with ramification index at most 2. The argument is based on the Harder-Narasimhan theory for finite flat group schemes over $\mathcal{O}_K$. In particular, we describe a sufficient condition for the existence of a filtration of p-divisible groups over $\mathcal{O}_K$ associated to a break point of the Harder-Narasimhan polygon.

Zusammenfassung

Sei $\mathcal{O}_K$ ein vollständiger diskreter Bewertungsring mit gemischter Charakteristik $(0,p)$ und perfektem Restklassenkörper. Wir beweisen die Existenz der Hodge-Newton-Filtrierung für p-dividierbare Gruppen über $\mathcal{O}_K$ mit zusätzlicher Endomorphismusstruktur für den Ganzheitsring einer endlichen Körpererweiterung von $\mathbb{Q}_p$ mit Verzweigungsindex höchstens 2. Das Argument stützt sich auf die Harder-Narasimhan-Theorie für endliche flache Gruppenschemata über $\mathcal{O}_K$. Insbesondere beschreiben wir eine hinreichende Bedingung für die Existenz einer Filtrierung von p-dividierbaren Gruppen über $\mathcal{O}_K$ im Zusammenhang mit einem Bruchpunkt des Harder-Narasimhan-Polygons.

Contents

Introduction

Let p be a prime number. This work concerns p-divisible groups, alias Barsotti-Tate groups, over a complete discrete valuation ring $\mathcal{O}_K$ in mixed characteristic, with perfect residue field k of characteristic p. More generally, we consider p-divisible groups H endowed with additional endomorphism structure $\iota\colon \mathcal{O}_F \to \mathrm{End}(H)$, where $\mathcal{O}_F$ is the ring of integers of a finite field extension F of $\mathbb{Q}_p$. To any such pair (H, ι) we associate two invariants up to ($\mathcal{O}_F$-equivariant) isogeny, the *Hodge polygon* $\mathrm{Hdg}(H, \iota)$ and the *Newton polygon* $\mathrm{Newt}(H, \iota)$; each of them consists in a collection of "slopes" (with multiplicities) and can be visualised as a concave polygonal curve, starting from the origin, in the Euclidean plane. The main property relating these two polygons is that they lie one above the other and share the end point. When they share a further point z and this is a break point of the Newton polygon, we say that (H, ι) is *Hodge-Newton reducible* (*at* z). In this case, assuming that the ramification index of F over $\mathbb{Q}_p$ is at most 2, we prove that there exists an ι-stable sub-p-divisible group H_1 of H corresponding to the division of the polygons in their parts before and after z, that is, a *Hodge-Newton filtration* of (H, ι). More precisely:

Theorem 3.8. *Assume that the ramification index of F over $\mathbb{Q}_p$ is at most 2. Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and suppose that (H, ι) is Hodge-Newton reducible at $z = (x, y)$. Then, there exists a unique ι-stable sub-p-divisible group H_1 of H such that, if ι_1 denotes the restriction of ι to H_1, then* $\mathrm{Newt}(H_1, \iota_1)$, $\mathrm{Hdg}(H_1, \iota_1)$ *and* $\mathrm{HN}(H_1, \iota_1)$ *equal respectively the restriction of* $\mathrm{Newt}(H, \iota)$, $\mathrm{Hdg}(H, \iota)$ *and* $\mathrm{HN}(H, \iota)$ *to $[0, x]$. Furthermore, if H_2 denotes the quotient of H by H_1, with induced $\mathcal{O}_F$-action ι_2, then* $\mathrm{Newt}(H_2, \iota_2)$, $\mathrm{Hdg}(H_2, \iota_2)$ *and* $\mathrm{HN}(H_2, \iota_2)$ *equal respectively the rest of* $\mathrm{Newt}(H, \iota)$, $\mathrm{Hdg}(H, \iota)$ *and* $\mathrm{HN}(H, \iota)$ *after z.*

The third polygon featuring in the statement, the *Harder-Narasimhan polygon* $\mathrm{HN}(H, \iota)$, is related to the strategy behind the proof. This, indeed, is based on the *Harder-Narasimhan theory* for finite flat group schemes developed by Fargues, but let us postpone this discussion to a second moment.

In presence of an $\mathcal{O}_F$-equivariant polarisation $\lambda\colon H \xrightarrow{\sim} H^\vee$ (where the dual p-divisible group $H^\vee$ is endowed with the dual $\mathcal{O}_F$-action, possibly twisted by a field involution of F), the three polygons gain some symmetry. This leads to an enhanced version of the theorem, where a division of the polygons into *three* parts corresponds to the existence of an ι-stable filtration $H_1 \subseteq H_1' \subseteq H$, with λ inducing crossed isomorphisms between the graded pieces and their duals (see Corollary 3.11).

In order to put the above results into context, let us give some historical coordinates. The notion of Hodge-Newton reducibility first appeared in the work of Katz, [15, §1.6] (1979), in the context of F-crystals over a perfect field k of characteristic p. To these objects are associated a Hodge polygon and a Newton polygon, satisfying the same main property as described above (in this context, this is known as *Mazur's inequality*). Under the same Hodge-Newton reducibility hypothesis on the polygons, Katz proves the existence of a *Hodge-Newton decomposition* of the relative F-crystal in two components, whose polygons correspond respectively to the part before and after the internal contact point given by the assumption. In case such an F-crystal is the Dieudonné module of a p-divisible group over k, this result recovers the multiplicative-bilocal-étale decomposition.

This finding was later generalised by Kottwitz, in [17] (2003), to F-crystals with additional endomorphism structure for the ring of integers of a finite unramified extension F of $\mathbb{Q}_p$ and possibly endowed with a polarisation. By this time, however, new points of view on the subject had developed. Kottwitz's result is formulated in terms of *affine Deligne-Lusztig sets*, objects constructed by means of a reductive group over a local field and determined by a defining datum. The additional structure is encoded in the reductive group, in this case the restriction of scalars from F to $\mathbb{Q}_p$ of a general linear group GL_n, or a general symplectic group GSp_{2n} in presence of a polarisation. The definitions of the Hodge and the Newton polygons, as well as the notion of Hodge-Newton reducibility, are also translated in the group-theoretic language, thus taking into account the additional structure; for the groups in question, all this can still be visualised in terms of polygons. An affine Deligne-Lusztig set can then be seen as a set of F-crystals with additional structure, whose Hodge and Newton polygons are fixed by the defining datum. From this point of view, the Hodge-Newton decomposition is expressed as a bijection between the affine Deligne-Lusztig set associated to a Hodge-Newton reducible datum and one relative to a Levi subgroup, which collects F-crystals admitting a decomposition.

Mantovan and Viehmann, in [20] (2010), further generalised Kottwitz's result to endomorphism structures for more general unramified $\mathbb{Z}_p$-algebras. In addition, they proved that their Hodge-Newton decomposition can be lifted to a filtration of p-divisible groups over a complete Noetherian local $W(k)$-algebra (here, $W(k)$ denotes the ring of Witt vectors with coefficients in k). Their argument is based on an explicit description of the universal deformation of a p-divisible group over k with unramified endomorphism structure.

This result is used in Mantovan's work, [19] (2008), to study the generic fibre of certain *Rapoport-Zink spaces*. First conceived in the book [25] (1996), to whose authors they owe their name, these spaces are formal schemes over (a finite extension of) $\breve{\mathbb{Z}}_p$, parametrising p-divisible groups with additional structure and rigidified by a quasi-isogeny to a fixed "frame" object modulo p (here, $\breve{\mathbb{Z}}_p$ denotes the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$). A Rapoport-Zink space is determined by a *local PEL datum*, which prescribes the kind of additional structure in form of polarisation, endomorphism structure and level structure (whence the acronym "PEL") and fixes a number of combinatorial invariants, including the Hodge and the Newton polygons.

In fact, the connection with Rapoport-Zink spaces extends to a broader level. If, on the one hand, the Hodge-Newton filtration over a good class of $W(k)$-algebras leads to properties of their generic fibre (for Hodge-Newton reducible PEL data), on the other hand, the Hodge-Newton decomposition over k corresponds to properties of their special fibre (here, k is the algebraic closure of the finite field $\mathbb{F}_p$, so $W(k) = \breve{\mathbb{Z}}_p$). Indeed, the k-valued points of a Rapoport-Zink space are in bijection with a corresponding affine Deligne-Lusztig set, at least in the cases concerned up to this point; we will come back to this matter in a more general framework.

Further progress in the same direction as Mantovan and Viehmann was made by Shen, in [28] (2013). Here, besides an application to the generic fibre of Rapoport-Zink spaces, we find a new proof of the existence of the Hodge-Newton filtration, for p-divisible groups over a complete valuation ring (of rank 1) $\mathcal{O}_K$ in mixed characteristic, with perfect residue field k of characteristic p. The setup is still that of unramified endomorphism structure and the Hodge and the Newton polygons are defined through the reduction of the p-divisible group to k. However, instead of obtaining the Hodge-Newton filtration over $\mathcal{O}_K$ by lifting the Hodge-Newton decomposition over k, Shen proves its existence directly, by means of the Harder-Narasimhan theory for finite flat group schemes, developed by Fargues in [7] (2010).

In the more recent articles [13] (2018) and [14] (2019), Hong further extended the conclusions of Mantovan, Viehmann and Shen to a wider class of additional structures and the relative Rapoport-Zink spaces (namely of unramified Hodge type). For this purpose, the same author developed a tool, called "EL realisation", to reduce the problem to the previously known cases.

All the results mentioned so far deal with additional endomorphism structures of unramified type. The notion of Hodge-Newton reducibility, however, was meanwhile formalised in a more general context by Rapoport and Viehmann, in [24, Definition 4.28] (2014), including possible ramification. The definition is formulated in a group-theoretic way, building upon Kottwitz's language, as a property of *local Shimura data*, a more general version of local PEL data. A local Shimura datum determines a *local Shimura variety*, a concept developed in the same paper, which, in the PEL case, should be realised as the generic fibre of a Rapoport-Zink space. In the work of Rapoport and Viehmann, it is also explained how the Hodge-Newton reducibility should affect the geometry of local Shimura varieties, in the very general context of the Harris-Viehmann conjecture. In this sense, the results of Mantovan, Shen and Hong represent advances in this direction, in the unramified case.

The notion of Hodge-Newton reducibility in a possibly ramified setup features in the work of Görtz, He and Nie, [12] (2019). Here, the Hodge-Newton decomposition for a very large class of affine Deligne-Lusztig *varieties* is proved; in fact, these objects acquired in the meantime the geometric structure of perfect schemes over k, inside the Witt vector affine flag variety (cf [2], note that we are considering here the mixed characteristic version of affine Deligne-Lusztig varieties). In the article it is conjectured, in full generality, that the bijection between the k-valued points of Rapoport-Zink spaces and the corresponding affine Deligne-Lusztig sets upgrades to an isomorphism of perfect schemes, between the perfection of the special fibre on one side and an affine Deligne-Lusztig variety on the other. This brings us to the motivation of our own work.

The Hodge-Newton decomposition in a possibly ramified setup proved in [12], along with the conjectural consequences on the geometry of the special fibre of Hodge-Newton reducible Rapoport-Zink spaces, suggest that, in a similar fashion as in the unramified case, a corresponding statement could hold at the level of the generic fibre. As we saw in the previous overview, this investigation starts from the existence of the Hodge-Newton filtration for p-divisible groups over mixed characteristic base rings. In this sense, despite being limited to quadratic ramification, the present work could lead to interesting consequences on the geometry of the generic fibre of Hodge-Newton reducible

Rapoport-Zink spaces in a ramified setting and, possibly, further progress towards the Harris-Viehmann conjecture. For instance, the polarised setup, taking F a quadratic ramified extension of $\mathbb{Q}_p$ with the involution given by the unique Galois transformation over $\mathbb{Q}_p$, applies to $\mathcal{O}_K$-valued points of Rapoport-Zink spaces associated to PEL data of ramified unitary type (for $\mathcal{O}_K$ a finite extension of $\breve{\mathbb{Z}}_p$). For the developments in this direction, moreover, one may now rely on the modern methods elaborated by Chen, Fargues and Shen in [3], by means of vector bundles over the Fargues-Fontaine curve.

Let us mention, in addition, that the conclusions of this work can be interpreted in terms of p-adic Galois representations. Indeed, the category of p-divisible groups over $\mathcal{O}_K$ is equivalent to the category of Galois stable $\mathbb{Z}_p$-lattices in certain crystalline representations of the absolute Galois group of K (cf [27, Corollary 6.2.3]).

The first issue that we address in our work are the definitions of the Newton polygon and the Hodge polygon for p-divisible groups over $\mathcal{O}_K$ with additional endomorphism structure. In order to do this, we make use of the equivalence of categories between p-divisible groups over $\mathcal{O}_K$ up to isogeny and a certain abelian subcategory of weakly admissible filtered isocrystals over K, the fraction field of $\mathcal{O}_K$ (cf Remark 2.1). The definitions, as well as the main property relating the two polygons, can be dealt with at the level of (weakly admissible) filtered isocrystals, for which reason the first section is devoted to these objects.

As for the Newton polygon, it arises from a functorial formalism (namely the *slope decomposition* for isocrystals) and is therefore not affected by the additional endomorphism structure, except for a simple rescaling process. The situation is different for the Hodge polygon, which is based on the group-theoretic definition of $\bar{\mu}$ from [24, 2.4]; here, μ is a dominant geometric cocharacter of the reductive group over $\mathbb{Q}_p$ encoding the additional structure (cf Remark 1.11 for the translation to the group-theoretic setting). Passing from μ to $\bar{\mu}$ amounts to an averaging process over the Galois conjugates of μ; in this case, the additional endomorphism structure plays a more decisive role. This discrepancy between the definitions of the two polygons is also behind the fact that a p-divisible group might be Hodge-Newton reducible only if considered with some additional endomorphism structure, losing this property when neglecting the same additional structure. In the group-theoretic language of [24], the Newton polygon corresponds to the *Newton point* from loc. cit. 2.1. In this sense, our definitions of the Newton polygon and the Hodge polygon agree with those of the corresponding invariants of local Shimura data.

Let us remark that the Newton polygon is actually an invariant of the reduction of the p-divisible group to k (or, in terms of filtered isocrystals, an invariant of the underlying isocrystal). For unramified endomorphism structures, even the Hodge polygon can be defined at the level of p-divisible groups over k (or, more generally, of F-crystals over k), although it is in general not an invariant up to isogeny at this level. This is in fact the approach that we find in the previous literature (in particular [20] and [28]), based on the group-theoretic notion of *Hodge point* from [23, 4.1]. Our definition of the Hodge polygon recovers this approach in the unramified case, but it does not give an invariant of the reduction in general, as illustrated in Example 2.4.

Concerning the proof of Theorem 3.8, we followed essentially the same approach as in Shen's work (cf [28]), adding some details about the main input from Harder-Narasimhan theory, even for the unramified case (see Corollary 3.4 below). Given its rather formal nature, in fact, the argument in loc. cit. adapts well to the more general situation considered here. Let us summarise the underlying strategy.

The Hodge-Newton reducibility assumption for p-divisible groups over $\mathcal{O}_K$ with endomorphism structure concerns whole equivariant isogeny classes of objects. One can then use the fact that the category of p-divisible groups over $\mathcal{O}_K$ up to isogeny admits a *Harder-Narasimhan formalism* (cf [8]); in other words, to each object is associated a third polygon, the Harder-Narasimhan polygon, whose break points correspond to unique sub-p-divisible groups up to isogeny.

The first step consists in proving that the point where the Hodge-Newton reducibility assumption is realised is also a break point of the Harder-Narasimhan polygon (or rather of a rescaled version of it, as determined by the endomorphism structure). This yields a sub-p-divisible group up to isogeny, which, due to the functorial nature of the formalism, is stable under any additional endomorphism structure. This step is taken care of in §3.2, at the level of weakly admissible filtered isocrystals; in fact, both the Harder-Narasimhan formalism and the notion of Hodge-Newton reducibility can be set up in this context.

In order to upgrade this subobject up to isogeny to an actual sub-p-divisible group, we make use of the fact that finite flat group schemes of p-power order over $\mathcal{O}_K$ also admit a Harder-Narasimhan formalism (cf [7]). Moreover, the polygons associated to the p-power torsion parts of a p-divisible group converge from above to the Harder-Narasimhan polygon of the p-divisible group itself. This family of polygons provides now a finer invariant, bounded from below by the Harder-Narasimhan polygon of the p-divisible group (hence uniformly over an isogeny class).

The next step consists in finding a (similarly uniform) upper bound for the family of polygons under consideration. This upper bound is indeed represented by the Hodge polygon, as proved in §2.4. This section is the technical heart of the whole argument, as well as its bottleneck, where the assumption on the ramification index comes into play. In fact, this is essentially the only point where the endomorphism structure really plays a role, as functoriality takes care of it in the other steps. Anyway, we believe that the crucial Proposition 2.13 could hold beyond the quadratic ramified case; this would automatically unlock a similarly more general version of the theorem.

The double bound obtained on the Harder-Narasimhan polygons of the p-power torsion parts forces these polygons to pass through the critical point as well. In order to conclude the existence of the required sub-p-divisible group, we finally prove the following statement, which is of independent interest for the study of p-divisible groups over $\mathcal{O}_K$ via Harder-Narasimhan theory.

Corollary 3.4. *Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. Suppose that $z = (x, y)$ is a break point of $\mathrm{HN}(H, \iota)$ which also lies on $\mathrm{HN}(H[p], \iota)$. Then, there exists a unique ι-stable sub-p-divisible group H_1 of H such that, if ι_1 denotes the restriction of ι to H_1, then $\mathrm{HN}(H_1, \iota_1)$ equals the restriction of $\mathrm{HN}(H, \iota)$ to $[0, x]$. Furthermore, if H_2 denotes the quotient of H by H_1, with induced $\mathcal{O}_F$-action ι_2, then $\mathrm{HN}(H_2, \iota_2)$ equals the rest of $\mathrm{HN}(H, \iota)$ after z.*

Here, $\mathrm{HN}(H[p], \iota)$ denotes the Harder-Narasimhan polygon of the p-torsion part of (H, ι). This result is already contained implicitly in the proof of [28, Theorem 5.4], under the additional assumption that z is a break point of the (renormalised) Harder-Narasimhan polygon of $(H[p^i], \iota)$, for some $i \geq 1$. To obtain the more general statement, we combined the argument in loc. cit. with some methods from the algorithm in [8, §3] (cf Remark 3.3).

Prerequisites. We assume that the reader is familiar with the basics of finite (flat) group schemes and p-divisible groups (possibly viewed as formal groups), including the notions of *augmentation ideal*, *Cartier duality* and *(quasi-)isogeny.* The notion of exact sequence for these objects comes from viewing them in the category of fppf abelian sheaves. Let us recall a few facts from this perspective:

- a map of finite flat group schemes is a monomorphism (respectively an epimorphism) if and only if it is a closed embedding (respectively faithfully flat);
- the kernel of a map of finite flat group schemes is always representable by a finite group scheme; the kernel of an epimorphism is flat;
- the cokernel of a monomorphism $H' \hookrightarrow H$ of finite flat group schemes (respectively p-divisible groups) is always representable by a finite flat group scheme (respectively a p-divisible group), denoted by H/H' (the *quotient* of H by H');
- a sequence $0 \to H' \xrightarrow{u} H \xrightarrow{v} H'' \to 0$ of finite flat group schemes is exact if and only if v is faithfully flat with kernel u, if and only if u is a closed embedding and v induces an isomorphism $H/H' \xrightarrow{\sim} H''$;
- a sequence of p-divisible groups is exact if and only if it induces exact sequences of finite flat group schemes by restriction to the p^i-torsion parts for every $i \geq 1$;
- finite flat group schemes over a field form themselves an abelian category.

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Setup and notation

The following setup will be in force throughout the document:

- p is a fixed prime number;
- k is a perfect field of characteristic p;
- $W(k)$ is the ring of Witt vectors with coefficients in k, with fraction field K_0;
- K is a totally ramified extension of K_0 of degree e, with ring of integers $\mathcal{O}_K$;
- $\sigma\colon W(k) \to W(k)$ denotes the lift of the Frobenius map $(x \mapsto x^p)\colon k \to k$, the same notation is used for the extension of σ to K_0;
- F is a finite extension of $\mathbb{Q}_p$ of degree d, with ring of integers $\mathcal{O}_F$.

The Newton set. For $n \in \mathbb{N}$, we denote by $\mathbb{Q}^n_+ := \{ (a_i)_{i=1}^n \in \mathbb{Q}^n \mid a_1 \geq \cdots \geq a_n \}$ the set of decreasing n-tuples of rational numbers. An element $(a_i)_{i=1}^n \in \mathbb{Q}^n_+$ can be interpreted as the concave polygon $[0, n] \to \mathbb{R}$ starting at $(0, 0)$ and proceeding with slope a_i on $[i - 1, i]$. Here, by a *concave polygon* we mean a piecewise affine linear, continuous, concave function $[0, N] \to \mathbb{R}$ (for some $N \in \mathbb{N}$), such that $0 \mapsto 0$; we will often make no distinction between the function and its graph. There is an obvious notion of *break point*, from which we exclude the extremal points. The concave polygons corresponding to elements of $\mathbb{Q}^n_+$ are those defined on $[0, n]$ and whose break points lie in $\mathbb{Z} \times \mathbb{Q}$.

The set $\mathbb{Q}^n_+$ is partially ordered by the following rule:

$$(a_i)_{i=1}^n \leq (b_i)_{i=1}^n \qquad \text{if} \qquad \sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} b_i \quad \text{for all } 1 \leq j \leq n \text{ and } \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

In terms of the corresponding polygons, the relation means “lying below” and sharing the end point. With this meaning, we extend the partial order to the set of concave polygons defined on $[0, n]$.

Note on the normalisation. Our choice of working with *concave* polygons forced us to invert the sign of the slopes defining the Newton polygon (compared to the classical definition) and the type of a filtration (compared to the definition in [5, §I.1]). The alternative would be working with *convex* polygons (as in [10, §4.3]), in which case no sign change is necessary. A second choice was to have the polygons associated to a p-divisible group having slopes within the interval $[0, 1]$. This, in turn, forced us to normalise the functor to weakly admissible filtered isocrystals in such a way that it produces objects with jumps in $\{ -1, 0 \}$ and Newton slopes in $[-1, 0]$. This normalisation agrees with the one adopted in [28].

1 Filtered isocrystals

1.1 Isocrystals

Recall that an *isocrystal* over k is a finite dimensional K_0-vector-space N, together with a σ-linear bijective endomorphism $\varphi\colon N \to N$. The *height* of (N,φ) and its *dimension* are respectively:

$$\operatorname{ht}(N,\varphi) := \dim_{K_0} N \in \mathbb{N} \qquad \text{and} \qquad \dim(N,\varphi) := v_p(\det\varphi) \in \mathbb{Z};$$

here, choosing a K_0-basis of N, one can write the matrix of φ in the usual way and the p-adic valuation of its determinant will be independent of the chosen basis. The isocrystals over k form a $\mathbb{Q}_p$-linear abelian category $\operatorname{Isoc}(k)$, with morphisms given by K_0-linear maps compatible with φ. The dimension function is additive on short exact sequences.

An isocrystal (N,φ) is *isotypical* (or *isoclinic*) if there exist integers r,s with $s>0$ and a $W(k)$-lattice $M \subseteq N$ such that $\varphi^s M = p^r M$. In this case, we have $r/s = \dim(N,\varphi)/\operatorname{ht}(N,\varphi) \in \mathbb{Q}$ and call this number the *slope* of (N,φ).

The Newton polygon. Because k is perfect, every isocrystal (N,φ) over k has a unique *slope decomposition*:

$$(N,\varphi) = \bigoplus_{\lambda\in\mathbb{Q}} (N_\lambda, \varphi_\lambda) \tag{1.1}$$

into isotypical sub-isocrystals $(N_\lambda,\varphi_\lambda)$ of slope λ (cf [31, 6.22]); this is functorial in the sense that there are no nonzero morphisms between isotypical isocrystals of different slope (cf loc. cit. 6.20). Say that $\lambda_1 < \cdots < \lambda_m$ are the slopes appearing nontrivially in the decomposition (called the *Newton slopes* of (N,φ)) and let $h_i := \operatorname{ht}(N_{\lambda_i}, \varphi_{\lambda_i})$, $i = 1,\ldots,m$, and $h := \operatorname{ht}(N,\varphi) = h_1 + \cdots + h_m$. Then, we define the *Newton polygon* of (N,φ) to be:

$$\operatorname{Newt}(N,\varphi) := (-\lambda_1^{(h_1)}, \ldots, -\lambda_m^{(h_m)}) \in \mathbb{Q}_+^h,$$

the superscript denoting the number of repetitions. This polygon is the concave envelope of the points $(\operatorname{ht}(N',\varphi'), -\dim(N',\varphi'))$ over all sub-isocrystals (N',φ') of (N,φ). Note that the break points of $\operatorname{Newt}(N,\varphi)$ lie in $\mathbb{Z}\times\mathbb{Z}$ and we have $\operatorname{Newt}(N,\varphi)(h) = -\dim(N,\varphi)$.

1.2 Isocrystals with coefficients

Definition 1.1. An *isocrystal* over k *with coefficients* in F is a triple (N,φ,ι) consisting of an isocrystal (N,φ) over k and a map of $\mathbb{Q}_p$-algebras $\iota\colon F \to \operatorname{End}(N,\varphi)$.

The isocrystals over k with coefficients in F form an F-linear abelian category $\operatorname{Isoc}(k)_F$, with morphisms given by maps of isocrystals compatible with ι. For objects of this category we still have the notions of height and dimension, which simply refer to those of the underlying isocrystal.

It is sometimes useful (particularly in view of Remark 1.4 below) to consider isocrystals with coefficients from another point of view, which we borrow from [5, §VIII.5]. First of all, note that the underlying vector space of an isocrystal over k with coefficients in F has a module structure over $K_0 \otimes_{\mathbb{Q}_p} F$, thanks to the $\mathbb{Q}_p$-linear F-action. Let us study this ring in more detail.

Let $f_F := f(F|\mathbb{Q}_p)$ be the inertia degree of F over $\mathbb{Q}_p$ and write $K_0^{\sigma^{f_F}}$ for the σ^{f_F}-fixed subfield of K_0; this is a finite unramified extension of $\mathbb{Q}_p$, say of degree f, with f dividing f_F. In fact, we have $K_0^{\sigma^{f_F}} = K_0^{\sigma^f}$, the σ^f-fixed subfield of K_0. Note that f can be strictly smaller than f_F, without necessarily $K_0 = K_0^{\sigma^f}$ being the case (e.g. if $k = \mathbb{F}_p(T^{1/p^\infty})$, the field obtained from the field of rational functions $\mathbb{F}_p(T)$ over $\mathbb{F}_p$ adjoining the p^j-th root of T for all $j \geq 1$).

Fix now an embedding $\tau_0 \colon K_0^{\sigma^f} \to F$. We obtain all the embeddings of $K_0^{\sigma^f}$ in F as $\tau_i := \tau_0 \circ \sigma^{-i} \colon K_0^{\sigma^f} \to F$, for $i \in \mathbb{Z}/f\mathbb{Z}$. Set then $K_F^{(i)} := K_0 \otimes_{K_0^{\sigma^f}, \tau_i} F$, for $i \in \mathbb{Z}/f\mathbb{Z}$, and $K_F := K_F^{(0)}$; all these are unramified field extensions of F. Indeed, $\tau_i(K_0^{\sigma^f})$ is contained in the maximal unramified subextension F^{nr} of $F|\mathbb{Q}_p$ and the minimal polynomial of $F^{\mathrm{nr}}|K_0^{\sigma^f}$ is irreducible over K_0, because the coefficients of any nontrivial monic factor would be fixed by σ^{f_F} and hence would lie in $K_0^{\sigma^{f_F}} = K_0^{\sigma^f}$. Thus, $K_{F^{\mathrm{nr}}}^{(i)} := K_0 \otimes_{K_0^{\sigma^f}, \tau_i} F^{\mathrm{nr}}$ is a field and, in particular, an unramified extension of F^{nr}. Then, $K_F^{(i)} = K_{F^{\mathrm{nr}}}^{(i)} \otimes_{F^{\mathrm{nr}}} F$ is an unramified field extension of F. We have an isomorphism:

$$\begin{aligned} K_0^{\sigma^f} \otimes_{\mathbb{Q}_p} F &\cong \prod_{i=0}^{f-1} F \\ a \otimes b &\mapsto (\tau_i(a)b)_i, \end{aligned}$$

which extends to a decomposition:

$$K_0 \otimes_{\mathbb{Q}_p} F \cong \prod_{i=0}^{f-1} K_F^{(i)}.$$

The automorphism $\sigma \otimes \mathrm{id}$ on the left-hand side corresponds to the product of the isomorphisms:

$$\sigma \otimes \mathrm{id} \colon K_F^{(i)} = K_0 \otimes_{K_0^{\sigma^f}, \tau_i} F \longrightarrow K_0 \otimes_{K_0^{\sigma^f}, \tau_{i+1}} F = K_F^{(i+1)}$$

on the right-hand side. In particular, $\sigma^f \otimes \mathrm{id}$ induces an automorphism $\sigma_F \colon K_F \to K_F$, the Frobenius of K_F over F.

Definition 1.2. A *σ_F-K_F-space* is a pair (N_F, φ_F) consisting of a finite dimensional K_F-vector-space N_F and a σ_F-linear bijective endomorphism $\varphi_F \colon N_F \to N_F$.

The σ_F-K_F-spaces form an F-linear abelian category σ_F-K_F-Sp, with morphisms given by K_F-linear maps compatible with φ_F. This category is in fact equivalent to $\mathrm{Isoc}(k)_F$, as we will see shortly.

Consider the following maps of rings:

$$\varepsilon_0\colon K_0 \otimes_{\mathbb{Q}_p} F \cong \prod_{i=0}^{f-1} K_F^{(i)} \xrightarrow{\mathrm{pr}_0} K_F, \qquad \rho\colon K_F \longrightarrow \prod_{i=0}^{f-1} K_F^{(i)} \cong K_0 \otimes_{\mathbb{Q}_p} F$$
$$x \mapsto ((\sigma^i \otimes \mathrm{id})(x))_i.$$

We use them to define functors:

$$\varepsilon_{0,*}\colon \mathrm{Isoc}(k)_F \longrightarrow \sigma_F\text{-}K_F\text{-}\mathrm{Sp}$$
$$(N, \varphi, \iota) \mapsto (N \otimes_{K_0 \otimes_{\mathbb{Q}_p} F, \varepsilon_0} K_F, \varphi^f \otimes \sigma_F)$$

and:

$$\rho_*\colon \sigma_F\text{-}K_F\text{-}\mathrm{Sp} \longrightarrow \mathrm{Isoc}(k)_F$$
$$(N_F, \varphi_F) \mapsto (N_F \otimes_{K_F,\rho} (K_0 \otimes_{\mathbb{Q}_p} F), \varphi, \iota).$$

Here, $N_F \otimes_{K_F,\rho} (K_0 \otimes_{\mathbb{Q}_p} F)$ has an obvious structure of K_0-vector-space and φ is given with respect to the decomposition:

$$N_F \otimes_{K_F,\rho} (K_0 \otimes_{\mathbb{Q}_p} F) \cong \prod_{i=0}^{f-1} N_F \otimes_{K_F, \sigma^i \otimes \mathrm{id}} K_F^{(i)}$$

as the product of the maps:

$$\mathrm{id} \otimes (\sigma \otimes \mathrm{id})\colon N_F \otimes_{K_F} K_F^{(i)} \longrightarrow N_F \otimes_{K_F} K_F^{(i+1)} \quad \text{for } i = 0, \dots, f-2,$$
$$\varphi_F \otimes (\sigma \otimes \mathrm{id})\colon N_F \otimes_{K_F} K_F^{(f-1)} \longrightarrow N_F \otimes_{K_F} K_F = N_F.$$

Finally, ι is given by the natural F-multiplication on $K_0 \otimes_{\mathbb{Q}_p} F$.

Lemma 1.3 ([5, 8.5.4])**.** *The functors $\varepsilon_{0,*}$ and ρ_* are quasi-inverse F-linear equivalences of categories between* $\mathrm{Isoc}(k)_F$ *and* σ_F*-*K_F*-*Sp*.*

Remark 1.4. As an important consequence of this lemma, given any $(N, \varphi, \iota) \in \mathrm{Isoc}(k)_F$, we may always write $N \cong N_F \otimes_{K_F,\rho} (K_0 \otimes_{\mathbb{Q}_p} F)$ for some K_F-vector-space N_F. In particular, N is free as a module over $K_0 \otimes_{\mathbb{Q}_p} F$, so its K_0-dimension is a multiple of $\dim_{K_0}(K_0 \otimes_{\mathbb{Q}_p} F) = [F : \mathbb{Q}_p] = d$. In other words, $\mathrm{ht}(N, \varphi) \in d\mathbb{N}$.

Note that if F is a totally ramified extension of $\mathbb{Q}_p$, this is more easily granted by the fact that $K_0 \otimes_{\mathbb{Q}_p} F$ is itself a field. At the other extreme, assume that F is unramified over $\mathbb{Q}_p$ and admits an embedding into K_0 (e.g. if k is algebraically closed). Then, fixing $F \subseteq K_0$, we have a decomposition into sub-K_0-vector-spaces:

$$N = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} N_i, \quad \text{with } N_i = \left\{ v \in N \;\middle|\; \forall a \in F\colon \iota(a)(v) = \sigma^i(a)v \right\}.$$

Moreover, φ induces σ-linear bijections $N_i \to N_{i+1}$, so that $\dim_{K_0} N_i$ is constant for $i \in \mathbb{Z}/d\mathbb{Z}$ and again $\dim_{K_0} N = d \cdot \dim_{K_0} N_0 \in d\mathbb{N}$. The formalism above combines these last two arguments in the general case.

The Newton polygon. Let (N, φ, ι) be an isocrystal over k with coefficients in F, of height $h = dn$ (note that $h \in d\mathbb{N}$ by the previous remark). By functoriality of the slope decomposition (1.1) of (N, φ), the action of F via ι restricts to each isotypical component $(N_\lambda, \varphi_\lambda)$. These components are hence again isocrystals with coefficients in F, whose height is then a multiple of d. Thus, in the Newton polygon of (N, φ), each entry is repeated a multiple of d times. In light of this, if $\mathrm{Newt}(N, \varphi) = (-\lambda_1^{(h_1)}, \ldots, -\lambda_m^{(h_m)}) \in \mathbb{Q}_+^h$, we define the *Newton polygon* of (N, φ, ι) to be:

$$\mathrm{Newt}(N, \varphi, \iota) := (-\lambda_1^{(h_1/d)}, \ldots, -\lambda_m^{(h_m/d)}) \in \mathbb{Q}_+^n.$$

Equivalently:

$$\mathrm{Newt}(N, \varphi, \iota)\colon x \longmapsto \frac{1}{d}\mathrm{Newt}(N, \varphi)(dx).$$

Note that the break points of $\mathrm{Newt}(N, \varphi, \iota)$ lie in $\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$. Furthermore, we have that $\mathrm{Newt}(N, \varphi, \iota)(n) = -\dim(N, \varphi)/d$.

1.3 Filtered vector spaces

Let here $K_2|K_1$ be any field extension and let us recall the category $\mathrm{FilVect}_{K_2|K_1}$ of *K_2-filtered K_1-vector-spaces*. Objects are K_1-vector-spaces V equipped with a $\mathbb{Z}$-filtration $\mathrm{Fil}^\bullet V_{K_2} = (\mathrm{Fil}^i V_{K_2})_{i\in\mathbb{Z}}$ of $V_{K_2} := V \otimes_{K_1} K_2$ by sub-K_2-vector-spaces, which is decreasing (i.e. $\mathrm{Fil}^i V_{K_2} \supseteq \mathrm{Fil}^{i+1} V_{K_2}$ for every $i \in \mathbb{Z}$), exhaustive and separated (i.e. respectively $\mathrm{Fil}^i V_{K_2} = V_{K_2}$ and $\mathrm{Fil}^j V_{K_2} = 0$ for some integers $i \leq j$). A morphism between two objects $(V', \mathrm{Fil}^\bullet V'_{K_2}), (V, \mathrm{Fil}^\bullet V_{K_2})$ is given by a K_1-linear map $f\colon V' \to V$ whose base change $f_{K_2}\colon V'_{K_2} \to V_{K_2}$ is compatible with the filtrations, meaning that $f_{K_2}(\mathrm{Fil}^i V'_{K_2}) \subseteq \mathrm{Fil}^i V_{K_2}$ for all $i \in \mathbb{Z}$; if $f_{K_2}(\mathrm{Fil}^i V'_{K_2}) = f_{K_2}(V'_{K_2}) \cap \mathrm{Fil}^i V_{K_2}$ for all $i \in \mathbb{Z}$, then we say that f is a *strict* morphism.

The category $\mathrm{FilVect}_{K_2|K_1}$ is K_1-linear and quasi-abelian, with short exact sequences given by short sequences of strict morphisms which are exact on the underlying vector spaces. In this case, the first term and the last term of the sequence are called respectively a *subobject* and a *quotient object* of the middle term.

For $(V, \mathrm{Fil}^\bullet V_{K_2}) \in \mathrm{FilVect}_{K_2|K_1}$ and $i \in \mathbb{Z}$, we set:

$$\mathrm{gr}^i V_{K_2} := \mathrm{Fil}^i V_{K_2}/\mathrm{Fil}^{i+1} V_{K_2} \qquad \text{and} \qquad \deg(V, \mathrm{Fil}^\bullet V_{K_2}) := \sum_{i\in\mathbb{Z}} i \cdot \dim_{K_2} \mathrm{gr}^i V_{K_2} \in \mathbb{Z},$$

called respectively the *i-th graded piece* and the *degree* of $(V, \mathrm{Fil}^\bullet V_{K_2})$. Note that a short sequence is exact if and only if it induces exact sequences on all the i-th graded pieces. In particular, the degree function is additive.

The indices $i \in \mathbb{Z}$ such that $\mathrm{gr}^i V_{K_2} \neq 0$ are called the *jumps* of the filtration. If these are, say, $i_1 < \cdots < i_m$ and we let $n_j := \dim_{K_2} \mathrm{gr}^{i_j} V_{K_2}$, $j = 1, \ldots, m$, and $n := \dim_{K_1} V = n_1 + \cdots + n_m$, then the *type* of $(V, \mathrm{Fil}^\bullet V_{K_2})$ is defined to be:

$$f(V, \mathrm{Fil}^\bullet V_{K_2}) := (-i_1^{(n_1)}, \ldots, -i_m^{(n_m)}) \in \mathbb{Q}_+^n.$$

The break points of $f(V, \mathrm{Fil}^\bullet V_{K_2})$ lie in $\mathbb{Z} \times \mathbb{Z}$ and $f(V, \mathrm{Fil}^\bullet V_{K_2})(n) = -\deg(V, \mathrm{Fil}^\bullet V_{K_2})$. Moreover, if:

$$0 \longrightarrow (V', \mathrm{Fil}^\bullet V'_{K_2}) \longrightarrow (V, \mathrm{Fil}^\bullet V_{K_2}) \longrightarrow (V'', \mathrm{Fil}^\bullet V''_{K_2}) \longrightarrow 0$$

is a short exact sequence of K_2-filtered K_1-vector-spaces, then:

$$f(V', \mathrm{Fil}^\bullet V'_{K_2}) = (-i_1^{(n'_1)}, \dots, -i_m^{(n'_m)}) \qquad \text{and} \qquad f(V'', \mathrm{Fil}^\bullet V''_{K_2}) = (-i_1^{(n''_1)}, \dots, -i_m^{(n''_m)})$$

with $n'_j + n''_j = n_j$ for all $1 \le j \le m$.

We remark that if K'_2 is a field extension of K_2 and K'_1 is a field extension of K_1 contained in K'_2, then the obvious base change functor $\mathrm{FilVect}_{K_2|K_1} \to \mathrm{FilVect}_{K'_2|K'_1}$ is exact and preserves the type.

Lemma 1.5. *Let $(V, \mathrm{Fil}^\bullet V_{K_2})$ be a K_2-filtered K_1-vector-space and $(V', \mathrm{Fil}^\bullet V'_{K_2})$ a subobject. Set $n' := \dim_{K_1} V'$. Then:*

$$-\deg(V', \mathrm{Fil}^\bullet V'_{K_2}) \le f(V, \mathrm{Fil}^\bullet V_{K_2})(n'),$$

with equality if and only if the type of $(V', \mathrm{Fil}^\bullet V'_{K_2})$ equals the restriction of $f(V, \mathrm{Fil}^\bullet V_{K_2})$ to $[0, n']$.

Proof. It is enough to observe that the type of $(V', \mathrm{Fil}^\bullet V'_{K_2})$ is a polygon with the same slopes as $f(V, \mathrm{Fil}^\bullet V_{K_2})$, but with lower (possibly zero) multiplicity.

In numbers, let $i_1 < \cdots < i_m$ be the jumps of $(V, \mathrm{Fil}^\bullet V_{K_2})$ and $n_j := \dim_{K_2} \mathrm{gr}^{i_j} V_{K_2}$, for $j = 1, \dots, m$. The jumps of $(V', \mathrm{Fil}^\bullet V'_{K_2})$ are among those of $(V, \mathrm{Fil}^\bullet V_{K_2})$, with $n'_j := \dim_{K_2} \mathrm{gr}^{i_j} V'_{K_2} \le n_j$, $j = 1, \dots, m$, and $n' = n'_1 + \cdots + n'_m$. Let $l \in \{ 1, \dots, m \}$ be such that $n_1 + \cdots + n_{l-1} < n' \le n_1 + \cdots + n_l$ (the case $n' = 0$ being trivial). Then, the claimed inequality reads:

$$-\sum_{j=1}^{m} i_j n'_j \le -\sum_{j=1}^{l-1} i_j n_j - i_l(n' - (n_1 + \cdots + n_{l-1})).$$

Now, for $j \le l-1$ we have $i_j < i_l$, whereas for $j \ge l+1$ we have $i_j > i_l$. Thus:

$$\sum_{j=1}^{l-1} i_j(n_j - n'_j) \le i_l \sum_{j=1}^{l-1}(n_j - n'_j) \qquad \text{and} \qquad \sum_{j=l+1}^{m} i_j n'_j \ge i_l \sum_{j=l+1}^{m} n'_j. \tag{1.2}$$

Altogether:

$$\begin{aligned} &\sum_{j=1}^{l-1} i_j n_j + i_l(n' - (n_1 + \cdots + n_{l-1})) - \sum_{j=1}^{m} i_j n'_j = \\ &\qquad = \sum_{j=1}^{l-1} i_j(n_j - n'_j) + i_l(n' - (n_1 + \cdots + n_{l-1}) - n'_l) - \sum_{j=l+1}^{m} i_j n'_j \le \\ &\qquad\qquad \le i_l(n' - (n'_1 + \cdots + n'_l)) = 0, \end{aligned} \tag{1.3}$$

which is the desired inequality. If $f(V', \mathrm{Fil}^\bullet V'_{K_2})$ equals the restriction of $f(V, \mathrm{Fil}^\bullet V_{K_2})$ to $[0, n']$, then:

$$-\deg(V', \mathrm{Fil}^\bullet V'_{K_2}) = f(V', \mathrm{Fil}^\bullet V'_{K_2})(n') = f(V, \mathrm{Fil}^\bullet V_{K_2})(n').$$

Conversely, if $-\deg(V', \mathrm{Fil}^\bullet V'_{K_2}) = f(V, \mathrm{Fil}^\bullet V_{K_2})(n')$, then we have equality in (1.3) and hence in (1.2). In particular, $n'_j = n_j$ for $j \leq l-1$ and $n'_j = 0$ for $j \geq l+1$, which implies that $f(V', \mathrm{Fil}^\bullet V'_{K_2})$ equals the restriction of $f(V, \mathrm{Fil}^\bullet V_{K_2})$ to $[0, n']$. □

Remark 1.6. As explained in [5, §I.3], the category of K_2-filtered K_1-vector-spaces admits a Harder-Narasimhan formalism for the slope function $\mu = \deg / \dim$, which associates to each nonzero object $(V, \mathrm{Fil}^\bullet V_{K_2}) \in \mathrm{FilVect}_{K_2|K_1}$ an element of the Newton set $\mathbb{Q}^n_+$, where $n = \dim_{K_1} V$, called the *HN-vector* of $(V, \mathrm{Fil}^\bullet V_{K_2})$. This is not to be confused with the type of $(V, \mathrm{Fil}^\bullet V_{K_2})$; in fact, the previous lemma implies that, setting up the formalism with respect to $-\mu$ instead (in order to comply with our normalisation), then the HN-vector of $(V, \mathrm{Fil}^\bullet V_{K_2})$ is less than or equal to its type, with respect to the partial order of $\mathbb{Q}^n_+$. We will not make use of this notion in the sequel, but we will later see other examples of Harder-Narasimhan formalisms in more detail.

1.4 Filtered isocrystals

A *filtered isocrystal* over K is an isocrystal (N, φ) over k, together with a decreasing, exhaustive and separated $\mathbb{Z}$-filtration $\mathrm{Fil}^\bullet N_K$ of $N_K = N \otimes_{K_0} K$ by sub-K-vector-spaces. The filtered isocrystals over K form a $\mathbb{Q}_p$-linear category $\mathrm{FilIsoc}_K$, with morphisms given by maps of isocrystals whose base change to K is compatible with the filtrations; a morphism is *strict* if so is the resulting map in $\mathrm{FilVect}_{K|K_0}$. In fact, $\mathrm{FilIsoc}_K$ is a quasi-abelian category too, with short exact sequences given by short sequences of strict morphisms which are exact on the underlying isocrystals; we deduce the meaning of *subobject* and *quotient object*.

The forgetful functors:

$$\begin{array}{rcl} \mathrm{FilIsoc}_K & \longrightarrow & \mathrm{Isoc}(k) \\ (N, \varphi, \mathrm{Fil}^\bullet N_K) & \longmapsto & (N, \varphi) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{FilIsoc}_K & \longrightarrow & \mathrm{FilVect}_{K|K_0} \\ (N, \varphi, \mathrm{Fil}^\bullet N_K) & \longmapsto & (N, \mathrm{Fil}^\bullet N_K) \end{array}$$

are exact. Given a filtered isocrystal $(N, \varphi, \mathrm{Fil}^\bullet N_K)$ over K, say with $\mathrm{ht}(N, \varphi) = h$, we define its *Newton polygon* to be:

$$\mathrm{Newt}(N, \varphi, \mathrm{Fil}^\bullet N_K) := \mathrm{Newt}(N, \varphi) \in \mathbb{Q}^h_+$$

and its *Hodge polygon* to be:

$$\mathrm{Hdg}(N, \varphi, \mathrm{Fil}^\bullet N_K) := f(N, \mathrm{Fil}^\bullet N_K) \in \mathbb{Q}^h_+.$$

Furthermore, we define its *Newton number* to be:

$$t_N(N, \varphi, \mathrm{Fil}^\bullet N_K) := \dim(N, \varphi) \in \mathbb{Z}$$

and its *Hodge number* to be:

$$t_H(N,\varphi,\mathrm{Fil}^\bullet N_K) := \deg(N,\mathrm{Fil}^\bullet N_K) \in \mathbb{Z}.$$

In this way, the Hodge polygon and number only depend on the underlying filtered vector space, whereas the Newton polygon and number are invariants of the underlying isocrystal. By the properties already known, we see that the break points of both polygons lie in $\mathbb{Z}\times\mathbb{Z}$ and that the Newton number and the Hodge number are additive on short exact sequences. Moreover:

$$\mathrm{Newt}(N,\varphi,\mathrm{Fil}^\bullet N_K)(h) = -t_N(N,\varphi,\mathrm{Fil}^\bullet N_K)$$

and:

$$\mathrm{Hdg}(N,\varphi,\mathrm{Fil}^\bullet N_K)(h) = -t_H(N,\varphi,\mathrm{Fil}^\bullet N_K).$$

1.5 Weakly admissible filtered isocrystals

A *filtered isocrystal* $(N,\varphi,\mathrm{Fil}^\bullet N_K)$ over K is called *weakly admissible* if for all subobjects $(N',\varphi',\mathrm{Fil}^\bullet N'_K)$ we have:

$$t_H(N',\varphi',\mathrm{Fil}^\bullet N'_K) \le t_N(N',\varphi',\mathrm{Fil}^\bullet N'_K),$$

with equality holding for $(N,\varphi,\mathrm{Fil}^\bullet N_K)$ itself. Let $\mathrm{FilIsoc}_K^{\text{w-a}}$ denote the full subcategory of $\mathrm{FilIsoc}_K$ consisting of the weakly admissible objects. This is an abelian category, with kernels and cokernels coinciding with those in $\mathrm{FilIsoc}_K$; moreover, if two out of three objects of a short exact sequence in $\mathrm{FilIsoc}_K$ are weakly admissible, then the third object is weakly admissible too (cf [10, §4.2]).

A fundamental implication of weak admissibility is expressed by the following inequality of polygons, which is part of the characterisation given in [10, §4.3] (taking into account a different normalisation). It can also be seen as an easy consequence of Lemma 1.5 (see the proof of Proposition 1.10 for the argument in a more general setting).

Proposition 1.7. *Let $(N,\varphi,\mathrm{Fil}^\bullet N_K)$ be a weakly admissible filtered isocrystal over K. Then:*

$$\mathrm{Newt}(N,\varphi,\mathrm{Fil}^\bullet N_K) \le \mathrm{Hdg}(N,\varphi,\mathrm{Fil}^\bullet N_K).$$

The Harder-Narasimhan polygon. As explained in [8, §5.2.3], the category of weakly admissible filtered isocrystals over K admits a Harder-Narasimhan formalism for the slope function $\mu = -t_N/\,\mathrm{ht} = -t_H/\,\mathrm{ht}$. More precisely, to each nonzero object $\mathcal{N} = (N,\varphi,\mathrm{Fil}^\bullet N_K) \in \mathrm{FilIsoc}_K^{\text{w-a}}$ we associate its *slope*:

$$\mu(\mathcal{N}) := \frac{-t_N(N,\varphi,\mathrm{Fil}^\bullet N_K)}{\mathrm{ht}(N,\varphi)} = \frac{-t_H(N,\varphi,\mathrm{Fil}^\bullet N_K)}{\mathrm{ht}(N,\varphi)} \in \mathbb{Q}$$

and say that $\mathcal{N}$ is *semi-stable* of slope $\mu(\mathcal{N})$ if for every subobject $\mathcal{N}' \subseteq \mathcal{N}$ we have $\mu(\mathcal{N}') \le \mu(\mathcal{N})$. In general, there exists a unique *Harder-Narasimhan filtration*:

$$0 = \mathcal{N}_0 \subsetneq \mathcal{N}_1 \subsetneq \cdots \subsetneq \mathcal{N}_m = \mathcal{N} \tag{1.4}$$

in $\mathrm{FilIsoc}_K^{\text{w-a}}$, such that each $\mathcal{N}_i/\mathcal{N}_{i-1}$ is semi-stable, say of slope μ_i, with $\mu_1 > \cdots > \mu_m$. Some functoriality of this filtration follows from the fact that if $\mathcal{N}'$ and $\mathcal{N}''$ are two semi-stable objects with $\mu(\mathcal{N}') > \mu(\mathcal{N}'')$, then there are no nontrivial morphisms $\mathcal{N}' \to \mathcal{N}''$. Now, for $\mathcal{N}$ as above, with Harder-Narasimhan filtration as in (1.4), let $h := \mathrm{ht}(N, \varphi)$ and let h_i be the height of the underlying isocrystal of $\mathcal{N}_i/\mathcal{N}_{i-1}$, for $i = 1, \ldots, m$. Then, we define the *Harder-Narasimhan polygon* of $\mathcal{N}$ to be:

$$\mathrm{HN}(\mathcal{N}) := (\mu_1^{(h_1)}, \ldots, \mu_m^{(h_m)}) \in \mathbb{Q}_+^h.$$

This polygon is the concave envelope of the points $(\mathrm{ht}(N', \varphi'), -t_N(\mathcal{N}'))$ over all subobjects $\mathcal{N}' = (N', \varphi', \mathrm{Fil}^\bullet N'_K)$ of $\mathcal{N}$ in $\mathrm{FilIsoc}_K^{\text{w-a}}$. Its break points lie in $\mathbb{Z} \times \mathbb{Z}$ and we have $\mathrm{HN}(\mathcal{N})(h) = -t_N(\mathcal{N}) = -t_H(\mathcal{N})$.

Given $\mathcal{N} = (N, \varphi, \mathrm{Fil}^\bullet N_K) \in \mathrm{FilIsoc}_K^{\text{w-a}}$, every subobject $\mathcal{N}' = (N', \varphi', \mathrm{Fil}^\bullet N'_K)$ gives rise to a sub-isocrystal (N', φ') of (N, φ), hence the point $(\mathrm{ht}(N', \varphi'), -t_N(\mathcal{N}')) = (\mathrm{ht}(N', \varphi'), -\dim(N', \varphi')$ lies below $\mathrm{Newt}(N, \varphi) = \mathrm{Newt}(\mathcal{N})$. We easily deduce the following inequality of polygons.

Proposition 1.8. *Let* $(N, \varphi, \mathrm{Fil}^\bullet N_K)$ *be a weakly admissible filtered isocrystal over* K*. Then:*

$$\mathrm{HN}(N, \varphi, \mathrm{Fil}^\bullet N_K) \leq \mathrm{Newt}(N, \varphi, \mathrm{Fil}^\bullet N_K).$$

1.6 Filtered isocrystals with coefficients

Definition 1.9. A *filtered isocrystal* over K *with coefficients* in F is a pair $(\mathcal{N}, \iota)$ consisting of a filtered isocrystal $\mathcal{N} = (N, \varphi, \mathrm{Fil}^\bullet N_K)$ over K and a map of $\mathbb{Q}_p$-algebras $\iota\colon F \to \mathrm{End}(N, \varphi, \mathrm{Fil}^\bullet N_K)$.

The filtered isocrystals over K with coefficients in F form an F-linear quasi-abelian category $\mathrm{FilIsoc}_{K,F}$, with morphisms given by maps of filtered isocrystals compatible with ι; the notions of exactness, subobject and quotient object come from those in $\mathrm{FilIsoc}_K$ by neglecting ι.

We have an exact forgetful functor:

$$\begin{aligned} \mathrm{FilIsoc}_{K,F} &\longrightarrow \mathrm{Isoc}(k)_F \\ (N, \varphi, \mathrm{Fil}^\bullet N_K, \iota) &\longmapsto (N, \varphi, \iota), \end{aligned}$$

where ι also denotes the induced F-action on the underlying isocrystal (N, φ). Given $(\mathcal{N}, \iota) = (N, \varphi, \mathrm{Fil}^\bullet N_K, \iota) \in \mathrm{FilIsoc}_{K,F}$, say with $\mathrm{ht}(N, \varphi) = dn$, we define its *Newton polygon* to be:

$$\mathrm{Newt}(\mathcal{N}, \iota) := \mathrm{Newt}(N, \varphi, \iota) \in \mathbb{Q}_+^n.$$

Equivalently:

$$\mathrm{Newt}(\mathcal{N}, \iota)\colon x \longmapsto \frac{1}{d}\mathrm{Newt}(\mathcal{N})(dx).$$

Thus, the Newton polygon only depends on the underlying isocrystal. Moreover, as we already know for isocrystals with coefficients in F, the break points of $\mathrm{Newt}(\mathcal{N}, \iota)$ lie in $\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$ and we have $\mathrm{Newt}(\mathcal{N}, \iota)(n) = -\dim(N, \varphi)/d = -t_N(\mathcal{N})/d$.

The Hodge polygon. Let $(\mathcal{N}, \iota) = (N, \varphi, \mathrm{Fil}^\bullet N_K, \iota)$ be a filtered isocrystal over K with coefficients in F and pick a field extension K' of K containing all embeddings τ of F in an algebraic closure of K. Then, ι extends linearly to an action on the K'-vector-space $N_{K'} = N \otimes_{K_0} K'$, which respects the induced filtration $\mathrm{Fil}^\bullet N_{K'}$. We obtain decompositions of K'-vector-spaces:

$$\begin{aligned} N_{K'} &= \bigoplus_{\tau\colon F \to K'} N_\tau, \quad \text{with } N_\tau = \{\, v \in N_{K'} \mid \forall a \in F\colon \iota(a)(v) = \tau(a)v \,\}\,; \\ \mathrm{Fil}^\bullet N_{K'} &= \bigoplus_{\tau\colon F \to K'} \mathrm{Fil}^\bullet N_\tau, \quad \text{with } \mathrm{Fil}^\bullet N_\tau = N_\tau \cap \mathrm{Fil}^\bullet N_{K'}. \end{aligned} \tag{1.5}$$

Now, as seen in Remark 1.4, N is free as a module over $K_0 \otimes_{\mathbb{Q}_p} F$, so that $N_{K'}$ is free over $K' \otimes_{\mathbb{Q}_p} F \cong \prod_\tau K'$ $(b \otimes a \mapsto (b\tau(a))_\tau)$; in particular, $\dim_{K'} N_\tau$ is constant over all embeddings τ, say equal to n (thus, $\mathrm{ht}(N, \varphi) = dn$). Let $f_\tau \in \mathbb{Q}^n_+$ be the type of $(N_\tau, \mathrm{Fil}^\bullet N_\tau) \in \mathrm{FilVect}_{K'|K'}$. Then, we define the *Hodge polygon* of $(\mathcal{N}, \iota)$ to be:

$$\mathrm{Hdg}(\mathcal{N}, \iota) := \frac{1}{d} \sum_\tau f_\tau \in \mathbb{Q}^n_+.$$

This polygon only depends on the underlying filtered vector space of $\mathcal{N}$ and the compatible F-action on it; however, we used that the same action respects the operator φ on N, in order to ensure the equidimensionality of N_τ for varying τ. Finally, note that the break points of $\mathrm{Hdg}(\mathcal{N}, \iota)$ lie in $\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$ and we have $\mathrm{Hdg}(\mathcal{N}, \iota)(n) = -t_H(\mathcal{N})/d$.

1.7 Weakly admissible filtered isocrystals with coefficients

A filtered isocrystal over K with coefficients in F is called *weakly admissible* if the underlying filtered isocrystal is weakly admissible. We denote by $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$ the full subcategory of $\mathrm{FilIsoc}_{K,F}$ consisting of the weakly admissible objects. This is an abelian category, with kernels and cokernels coinciding with those in $\mathrm{FilIsoc}_{K,F}$.

Using Lemma 1.5, we can easily generalise the fundamental inequality between the Newton polygon and the Hodge polygon to this setting.

Proposition 1.10. *Let $(\mathcal{N}, \iota) = (N, \varphi, \mathrm{Fil}^\bullet N_K, \iota)$ be a weakly admissible filtered isocrystal over K with coefficients in F. Then:*

$$\mathrm{Newt}(\mathcal{N}, \iota) \leq \mathrm{Hdg}(\mathcal{N}, \iota).$$

Proof. Let $n := \mathrm{ht}(N, \varphi)/d$ and note first that:

$$\mathrm{Newt}(\mathcal{N}, \iota)(n) = -t_N(\mathcal{N})/d = -t_H(\mathcal{N})/d = \mathrm{Hdg}(\mathcal{N}, \iota)(n),$$

the central equality due to weak admissibility; thus, the two polygons share their end point. By concavity, it is then enough to show that each break point of $\mathrm{Newt}(\mathcal{N}, \iota)$ lies below $\mathrm{Hdg}(\mathcal{N}, \iota)$.

Let (x, y) be a break point of $\mathrm{Newt}(\mathcal{N}, \iota)$; by definition, we find an F-stable sub-isocrystal (N', φ') of (N, φ) such that $x = \mathrm{ht}(N', \varphi')/d$ and $y = -\dim(N', \varphi')/d$. Let

$\mathrm{Fil}^\bullet N'_K := N'_K \cap \mathrm{Fil}^\bullet N_K$ be the induced filtration, so that $\mathcal{N}' := (N', \varphi', \mathrm{Fil}^\bullet N'_K)$ is a sub-filtered-isocrystal of $\mathcal{N}$; moreover, ι restricts to an F-action ι' on $\mathcal{N}'$.

Fix now a field extension K' of K containing all embeddings τ of F in an algebraic closure of K and let:

$$N_{K'} = \bigoplus_{\tau\colon F\to K'} N_\tau, \qquad \mathrm{Fil}^\bullet N_{K'} = \bigoplus_{\tau\colon F\to K'} \mathrm{Fil}^\bullet N_\tau,$$
$$N'_{K'} = \bigoplus_{\tau\colon F\to K'} N'_\tau, \qquad \mathrm{Fil}^\bullet N'_{K'} = \bigoplus_{\tau\colon F\to K'} \mathrm{Fil}^\bullet N'_\tau$$

be the decompositions as in (1.5). Note that $N'_\tau = N'_{K'} \cap N_\tau$ and $\mathrm{Fil}^\bullet N'_\tau = N'_\tau \cap \mathrm{Fil}^\bullet N_\tau$, so $(N'_\tau, \mathrm{Fil}^\bullet N'_\tau)$ is a subobject of $(N_\tau, \mathrm{Fil}^\bullet N_\tau)$ in $\mathrm{FilVect}_{K'|K'}$, with $\dim_{K'} N'_\tau = x$. By Lemma 1.5, then:

$$-\deg(N'_\tau, \mathrm{Fil}^\bullet N'_\tau) \le f_\tau(x)$$

for every τ, where f_τ is the type of $(N_\tau, \mathrm{Fil}^\bullet N_\tau)$. On the other hand, weak admissibility gives:

$$dy = -\dim(N', \varphi') = -t_N(\mathcal{N}') \le -t_H(\mathcal{N}') = -\deg(N', \mathrm{Fil}^\bullet N'_K).$$

Finally, since $\deg(N', \mathrm{Fil}^\bullet N'_K) = \sum_\tau \deg(N'_\tau, \mathrm{Fil}^\bullet N'_\tau)$, we get:

$$y \le \frac{1}{d}\sum_\tau f_\tau(x) = \mathrm{Hdg}(\mathcal{N}, \iota)(x),$$

which means exactly that the point (x, y) lies below the polygon $\mathrm{Hdg}(\mathcal{N}, \iota)$. □

Remark 1.11. If $K_0 = \breve{\mathbb{Q}}_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$, then the inequality proved above is an instance of the more general fact that the p-adic period domain associated to a pair $(b, \{\mu\})$ for a reductive group G over $\mathbb{Q}_p$ is nonempty if and only if the σ-conjugacy class $[b]$ is "acceptable" with respect to $\{\mu\}$ (cf [24, 3.1]). Here, the group G is given by the restriction of scalars $\mathrm{Res}_{F|\mathbb{Q}_p} \mathrm{GL}(V)$, where V is any F-vector-space such that $V \otimes_{\mathbb{Q}_p} K_0$ identifies with the $K_0 \otimes_{\mathbb{Q}_p} F$-module N. Then, φ corresponds to an element $b \in G(K_0)$ and $\{\mu\}$ is the conjugacy class of any geometric cocharacter of G whose induced grading on $N_{K'}$ splits the filtration $\mathrm{Fil}^\bullet N_{K'}$, for a suitable field extension K' of K. In this setup, the filtration $\mathrm{Fil}^\bullet N_K$ determines a K-valued point of the p-adic period domain and $[b]$ being acceptable with respect to $\{\mu\}$ translates into the inequality stated in the proposition.

The Harder-Narasimhan polygon. Let $(\mathcal{N}, \iota) = (N, \varphi, \mathrm{Fil}^\bullet N_K, \iota)$ be a weakly admissible filtered isocrystal over K with coefficients in F, say with $\mathrm{ht}(N, \varphi) = h = dn$. By functoriality of the Harder-Narasimhan filtration (1.4) of $\mathcal{N}$, the action of F via ι restricts to each piece of this filtration, forcing the height of the respective underlying isocrystal to be a multiple of d. Thus, in the Harder-Narasimhan polygon of $\mathcal{N}$, each entry is repeated a multiple of d times. In light of this, if $\mathrm{HN}(\mathcal{N}) = (\mu_1^{(h_1)}, \dots, \mu_m^{(h_m)}) \in \mathbb{Q}_+^h$, we define the *Harder-Narasimhan polygon* of $(\mathcal{N}, \iota)$ to be:

$$\mathrm{HN}(\mathcal{N}, \iota) := (\mu_1^{(h_1/d)}, \dots, \mu_m^{(h_m/d)}) \in \mathbb{Q}_+^n.$$

Equivalently:

$$\mathrm{HN}(\mathcal{N},\iota)\colon x \longmapsto \frac{1}{d}\,\mathrm{HN}(\mathcal{N})(dx).$$

Note that this polygon is the concave envelope of the points $(\mathrm{ht}(N',\varphi')/d, -t_N(\mathcal{N}')/d)$ over all subobjects $(\mathcal{N}',\iota') = (N',\varphi',\mathrm{Fil}^\bullet N'_K,\iota')$ of $(\mathcal{N},\iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$. Moreover, its break points lie in $\mathbb{Z}\times\frac{1}{d}\mathbb{Z}$ and we have $\mathrm{HN}(\mathcal{N},\iota)(n) = -t_N(\mathcal{N})/d = -t_H(\mathcal{N})/d$.

Because both the Newton polygon and the Harder-Narasimhan polygon of a weakly admissible filtered isocrystal with coefficients in F are a rescaled version (by the same factor $1/d$) of their counterparts obtained neglecting the action of F, Proposition 1.8 implies directly the same inequality in this setting.

Proposition 1.12. *Let $(\mathcal{N},\iota) = (N,\varphi,\mathrm{Fil}^\bullet N_K,\iota)$ be a weakly admissible filtered isocrystal over K with coefficients in F. Then:*

$$\mathrm{HN}(\mathcal{N},\iota) \le \mathrm{Newt}(\mathcal{N},\iota).$$

2 p-divisible groups

Let R be a complete Noetherian commutative local ring, with residue field of characteristic p. We denote by $p\text{-div}_R$ the category of p-divisible groups, alias Barsotti-Tate groups, over $\operatorname{Spec} R$ (or "over R" for short); we write $\operatorname{ht} H$ for the height and $H^\vee$ for the Cartier dual of any object $H \in p\text{-div}_R$. This is a $\mathbb{Z}_p$-linear category, with the hom-sets being torsion free $\mathbb{Z}_p$-modules. Let then $p\text{-div}_R \otimes \mathbb{Q}_p$ denote the $\mathbb{Q}_p$-linear category of p-divisible groups over R "up to isogeny": objects are again p-divisible groups over R, but for H_1, H_2 two of them we set $\operatorname{Hom}_{p\text{-div}_R \otimes \mathbb{Q}_p}(H_1, H_2) := \operatorname{Hom}_{p\text{-div}_R}(H_1, H_2) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \operatorname{Hom}_{p\text{-div}_R}(H_1, H_2)[\frac{1}{p}]$ and extend the composition maps by linearity. We have an obvious functor $p\text{-div}_R \to p\text{-div}_R \otimes \mathbb{Q}_p$.

Recall that every p-divisible group $H = (H[p^i])_{i \geq 1}$ over R, say with $H[p^i] = \operatorname{Spec} A_i$, can be viewed as a formal group $\operatorname{Spf} A$ over R, where $A = \varprojlim_{i \geq 1} A_i$. If H is connected, then $\operatorname{Spf} A$ is a formal Lie group, i.e. A is isomorphic to the profinite R-algebra of formal power series $R[[X_1, \ldots, X_r]]$, for some $r \geq 1$ called the *dimension* of $\operatorname{Spf} A$ (cf [29, §2.2]). In general, there exists a unique exact sequence of p-divisible groups over R:

$$0 \longrightarrow H^\circ \longrightarrow H \longrightarrow H^{\text{ét}} \longrightarrow 0$$

where H° is connected and $H^{\text{ét}}$ is étale (cf loc. cit.). The *dimension* of H, denoted by $\dim H$, is by definition the dimension of the formal Lie group corresponding to H°; we have $\dim H + \dim H^\vee = \operatorname{ht} H$ (loc. cit. Proposition 3). Next, let $I \subseteq A$ be the augmentation ideal and set $\omega_H := I/I^2$. Then, $\omega_H = \omega_{H^\circ}$; in particular, this is a free R-module of rank $\dim H$. On the other hand, set $\operatorname{Lie}(H) := \operatorname{Ker}((\operatorname{Spf} A)(R[\varepsilon]/(\varepsilon^2)) \to (\operatorname{Spf} A)(R))$, the map being induced by $\varepsilon \mapsto 0$; this is again a free R-module of rank $\dim H$, which in fact identifies naturally with the dual of ω_H. The constructions of ω_H and $\operatorname{Lie}(H)$ define additive functors to the category of finitely generated R-modules and are compatible with base change. Finally, note that if p is nilpotent in R, then H° is the same as the formal completion of H along the identity section, as considered in [22, §II] (this follows from loc. cit. 4.4, 4.7, 4.11).

Let us next analyse some specific cases of our setup, namely when $R = k$ is a perfect field of characteristic p or $R = \mathcal{O}_K$ is the ring of integers of a finite totally ramified extension K of $K_0 = W(k)[\frac{1}{p}]$.

p-divisible groups over k. Recall that to each p-divisible group H over k Dieudonné theory associates a finite free $W(k)$-module $\mathbb{D}(H)$ of rank $\operatorname{ht} H$, together with an injective σ-linear endomorphism $\varphi_H \colon \mathbb{D}(H) \to \mathbb{D}(H)$ such that $\varphi_H \mathbb{D}(H) \supseteq p\mathbb{D}(H)$. This kind of objects are called *Dieudonné modules* over k; they form a $\mathbb{Z}_p$-linear category, with morphisms given by $W(k)$-linear maps compatible with the σ-linear endomorphism. The association above is in fact functorial and induces a $\mathbb{Z}_p$-linear equivalence of categories between $p\text{-div}_k$ and Dieudonné modules over k; moreover, we have a natural identification of k-vector-spaces:

$$\mathbb{D}(H)/\varphi_H \mathbb{D}(H) \cong \omega_{H^\vee} \tag{2.1}$$

(cf [9, §III]; we are considering here the covariant version of the equivalence, which is obtained by composing the contravariant functor with duality). Now, every Dieudonné module $(\mathbb{D},\varphi)$ gives rise to an isocrystal $(N,\varphi) := (\mathbb{D}\otimes_{W(k)} K_0, \varphi\otimes \mathrm{id})$, whose Newton slopes lie in $[0,1]$. Indeed, each isotypical component $(N_\lambda,\varphi_\lambda)$ of (N,φ) contains a $W(k)$-lattice $M_\lambda := \mathbb{D}\cap N_\lambda$ with the property that $pM_\lambda \subseteq \varphi_\lambda M_\lambda \subseteq M_\lambda$. Here, the second inclusion implies that the slope λ of $(N_\lambda,\varphi_\lambda)$ is nonnegative, while the first inclusion (which is equivalent to $p\varphi_\lambda^{-1}M_\lambda \subseteq M_\lambda$) implies that $\lambda \leq 1$. In fact, the converse statement also holds, namely, if $(N_\lambda,\varphi_\lambda)$ is an isotypical isocrystal of slope $\lambda\in[0,1]$, then it contains a $W(k)$-lattice $M_\lambda\subseteq N_\lambda$ such that $pM_\lambda\subseteq\varphi_\lambda M_\lambda\subseteq M_\lambda$. Indeed, by definition there exist integers r,s and a $W(k)$-lattice $M\subseteq N_\lambda$ such that $\varphi_\lambda^s M = p^r M$, with $s>0$ and $r/s=\lambda\in[0,1]$, i.e. $0\leq r\leq s$. Then, the $W(k)$-lattice $M_\lambda := \sum_{i=0}^{s-r}\varphi_\lambda^i M + \sum_{i=1}^{r-1} p^{-i}\varphi_\lambda^{s-r+i}M\subseteq N_\lambda$ satisfies the requirements. Thus, shifting the slopes by -1 (for normalisation reasons), Dieudonné theory induces a $\mathbb{Q}_p$-linear equivalence of categories:

$$\begin{aligned} p\text{−div}_k\otimes\mathbb{Q}_p &\xrightarrow{\sim} \mathrm{Isoc}(k)^{[-1,0]} \\ H &\longmapsto (\mathbb{D}(H)\otimes_{W(k)} K_0, p^{-1}(\varphi_H\otimes\mathrm{id})), \end{aligned} \tag{2.2}$$

where $\mathrm{Isoc}(k)^{[-1,0]}$ denotes the full subcategory of $\mathrm{Isoc}(k)$ consisting of the isocrystals whose Newton slopes lie in $[-1,0]$, an abelian subcategory of $\mathrm{Isoc}(k)$. Note that, if (N,φ) is the isocrystal associated to a p-divisible group H via the functor above, then, taking k-dimensions in (2.1), we get that $\dim(N,\varphi) = -\operatorname{ht} H + \dim H^\vee = -\dim H$.

p-divisible groups over $\mathcal{O}_K$. Every p-divisible group H over $\mathcal{O}_K$ gives rise to a natural exact sequence of finite free $\mathcal{O}_K$-modules:

$$0\longrightarrow \omega_{H^\vee}\longrightarrow M(H)\longrightarrow \mathrm{Lie}(H)\longrightarrow 0, \tag{2.3}$$

where $M(H)$ is the Lie algebra of the universal extension of H (cf [22, §IV.1]; we obtain the sequence above by taking the projective limit of all the similar sequences over $\mathcal{O}_K/p^r\mathcal{O}_K$, for $r\geq 1$). In addition, there is a natural isomorphism of $\mathcal{O}_K$-modules $M(H)\cong \mathbb{D}(H_k)\otimes_{W(k)}\mathcal{O}_K$, whose reduction to k identifies the image of $\omega_{H_k^\vee}$ in $M(H)\otimes_{\mathcal{O}_K} k$ with $V\mathbb{D}(H_k)/p\mathbb{D}(H_k)\subseteq \mathbb{D}(H_k)/p\mathbb{D}(H_k)$; here, H_k denotes the reduction of H to k and $V = p\varphi_{H_k}^{-1}$ the "Verschiebung" map of $\mathbb{D}(H_k)$ (cf [21, II.15.3], [1, 4.3.10]). Thus, if (N,φ) is the isocrystal over k associated to H_k as in (2.2), we get a filtered isocrystal $\mathcal{N} := (N,\varphi,\mathrm{Fil}^\bullet N_K)$ over K by setting $\mathrm{Fil}^0 N_K$ to be the image of $\omega_{H^\vee}\otimes_{\mathcal{O}_K} K$ in $N_K\cong M(H)\otimes_{\mathcal{O}_K} K$, with $\mathrm{Fil}^{-i}N_K = N_K$ and $\mathrm{Fil}^i N_K = 0$ for all $i\geq 1$; let us check that this is weakly admissible. First of all:

$$\begin{aligned} t_H(\mathcal{N}) &= -(\dim_K N_K - \dim_K(\omega_{H^\vee}\otimes_{\mathcal{O}_K} K)) \\ &= -\operatorname{ht} H + \dim H^\vee = -\dim H = \dim(N,\varphi) = t_N(\mathcal{N}). \end{aligned}$$

Next, a subobject $\mathcal{N}'\subseteq\mathcal{N}$ is given by a sub-isocrystal $(N',\varphi')\subseteq(N,\varphi)$, together with the induced filtration $\mathrm{Fil}^\bullet N'_K = N'_K\cap \mathrm{Fil}^\bullet N_K$. Set $D' := N'\cap\mathbb{D}(H_k)$, a direct summand of the free $W(k)$-module $\mathbb{D}(H_k)$, and $E' := N'_K \cap \omega_{H^\vee}$, a direct summand of the free $\mathcal{O}_K$-module $\omega_{H^\vee}$. Let then $\bar{D}' := D'/pD'$ and $\bar{E}' := E' \otimes_{\mathcal{O}_K} k$. We have a commutative diagram of inclusions:

$$\begin{array}{ccccc} \omega_{H_k^\vee} & \xrightarrow{\sim} & V\mathbb{D}(H_k)/p\mathbb{D}(H_k) & \longrightarrow & \mathbb{D}(H_k)/p\mathbb{D}(H_k) \\ \uparrow & & & & \uparrow \\ \bar{E}' & & \longrightarrow & & \bar{D}', \end{array}$$

which shows that $\bar{E}' \subseteq VD'/pD'$ (as $VD' = D' \cap V\mathbb{D}(H_k)$). In addition, V^{-1} induces an isomorphism $VD'/pD' \cong D'/\varphi_{H_k}D'$ of k-vector-spaces, so:

$$\begin{aligned} \dim_k(\bar{E}') \leq \dim_k(D'/\varphi_{H_k}D') = v_p(\det(\varphi_{H_k}|D')) = v_p(\det p\varphi') = \\ = \mathrm{ht}(N', \varphi') + \dim(N', \varphi'). \end{aligned}$$

Then:

$$\begin{aligned} t_H(\mathcal{N}') = -\dim_K N'_K + \dim_K(N'_K \cap \omega_{H^\vee} \otimes_{\mathcal{O}_K} K) = -\,\mathrm{ht}(N', \varphi') + \dim_k(\bar{E}') \leq \\ \leq \dim(N', \varphi') = t_N(\mathcal{N}'). \end{aligned}$$

We obtain a $\mathbb{Q}_p$-linear functor:

$$\begin{aligned} p\text{−div}_{\mathcal{O}_K} \otimes \mathbb{Q}_p &\longrightarrow \mathrm{FilIsoc}_K^{\text{w-a},[-1,0]} \\ H &\longmapsto (N, \varphi, \mathrm{Fil}^\bullet N_K), \end{aligned} \tag{2.4}$$

where $\mathrm{FilIsoc}_K^{\text{w-a},[-1,0]}$ denotes the full subcategory of $\mathrm{FilIsoc}_K^{\text{w-a}}$ consisting of those weakly admissible filtered isocrystals whose underlying filtration has jumps in $\{-1, 0\}$, an abelian subcategory of $\mathrm{FilIsoc}_K^{\text{w-a}}$; note that this restriction on the jumps implies that the Newton slopes of the underlying isocrystal lie in $[-1, 0]$, by Proposition 1.7 (the converse does not hold: for example, assume that k is algebraically closed and consider the simple isotypical isocrystal of height 2 and dimension −1 as in [31, 6.27], together with any filtration with jumps −2 and 1). The functor just defined is compatible with (2.2), if we reduce p-divisible groups along $\mathcal{O}_K \to k$ on one side and forget the filtration on the other. Moreover, if $H \in p\text{−div}_{\mathcal{O}_K} \otimes \mathbb{Q}_p$ maps to the filtered isocrystal $\mathcal{N}$, we have $t_H(\mathcal{N}) = t_N(\mathcal{N}) = -\dim H$.

Remark 2.1. The functor (2.4) is in fact an equivalence of categories. This was first conjectured by Fontaine in [10, §5.2], who already observed fully faithfulness. Later, in [11], he introduced the category of "crystalline" p-adic representations $\mathrm{Rep}_{\mathrm{cris}}(G_K)$ of the absolute Galois group G_K of K, together with a functor $D_{\mathrm{cris}}\colon \mathrm{Rep}_{\mathrm{cris}}(G_K) \to \mathrm{FilIsoc}_K^{\text{w-a}}$ such that, for H a p-divisible group over $\mathcal{O}_K$ and $V_p(H)$ its rational Tate module, $D_{\mathrm{cris}}(V_p(H))$ is the filtered isocrystal associated to $H^\vee$ as in (2.4) (after shifting the filtration and the Newton slopes of the latter by +1). Now, a theorem of Colmez and Fontaine (cf [4] and consider there the case $N = 0$) ensures that every weakly admissible filtered isocrystal over K is “admissible”, meaning that it belongs to the essential image of D_{cris}. In turn, a result of Kisin says that every crystalline representation V of G_K with Hodge-Tate weights in $\{0,1\}$ (which is equivalent to the underlying filtration of $D_{\mathrm{cris}}(V)$ having jumps in $\{0,1\}$) is isomorphic to the rational Tate module of a p-divisible group over $\mathcal{O}_K$ (cf [16, 2.2.6]). It follows that (2.4) is also essentially surjective.

2.1 p-divisible groups with endomorphism structure

Definition 2.2. Let R be a complete Noetherian commutative local ring, with residue field of characteristic p. A *p-divisible group* over R *with endomorphism structure* for $\mathcal{O}_F$ is a pair (H,ι) consisting of a p-divisible group H over R and a map of $\mathbb{Z}_p$-algebras $\iota\colon \mathcal{O}_F \to \mathrm{End}(H)$.

Note that, because $\mathrm{End}(H)$ is a torsion free $\mathbb{Z}_p$-module, the map ι is automatically injective; in other words, the $\mathcal{O}_F$-action on H corresponding to ι is faithful. We denote by $p\text{−}\mathrm{div}_{R,\mathcal{O}_F}$ the $\mathcal{O}_F$-linear category formed by the objects just defined, with morphisms given by maps of p-divisible groups over R compatible with ι (or $\mathcal{O}_F$-*equivariant*). Let then $p\text{−}\mathrm{div}_{R,\mathcal{O}_F}\otimes F$ be the F-linear category of p-divisible groups over R with endomorphism structure for $\mathcal{O}_F$ “up to equivariant isogeny”: this is constructed, as before, from $p\text{−}\mathrm{div}_{R,\mathcal{O}_F}$ by inverting p in the homomorphisms.

The functors (2.2) and (2.4) introduced above upgrade to F-linear equivalences of categories:

$$p\text{−}\mathrm{div}_{k,\mathcal{O}_F}\otimes F \xrightarrow{\sim} \mathrm{Isoc}(k)_F^{[-1,0]}, \tag{2.5}$$

where $\mathrm{Isoc}(k)_F^{[-1,0]}$ denotes the full subcategory of $\mathrm{Isoc}(k)_F$ consisting of objects whose underlying isocrystal has Newton slopes in $[-1,0]$, an abelian subcategory of $\mathrm{Isoc}(k)_F$, and:

$$p\text{−}\mathrm{div}_{\mathcal{O}_K,\mathcal{O}_F}\otimes F \xrightarrow{\sim} \mathrm{FilIsoc}_{K,F}^{\text{w-a},[-1,0]}, \tag{2.6}$$

where $\mathrm{FilIsoc}_{K,F}^{\text{w-a},[-1,0]}$ denotes the full subcategory of $\mathrm{FilIsoc}_{K,F}^{\text{w-a}}$ consisting of objects whose underlying filtration has jumps in $\{-1,0\}$, an abelian subcategory of $\mathrm{FilIsoc}_{K,F}^{\text{w-a}}$. The fact that these functors are equivalences of categories is a formal consequence of (2.2) and (2.4) being themselves equivalences (see Remark 2.1 for the latter), apart from a small consideration about essential surjectivity. Namely, an object of the right-hand side category corresponds, by what we formally know, to some p-divisible group H together with a map $\iota\colon F \to \mathrm{End}(H)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ of $\mathbb{Q}_p$-algebras. Now, since $\mathrm{End}(H)$ is a finitely generated $\mathbb{Z}_p$-module, e.g. because we know that the hom-sets of (filtered) isocrystals are finite-dimensional $\mathbb{Q}_p$-vector-spaces, we have that $p^r\iota(\mathcal{O}_F)\subseteq \mathrm{End}(H)$ for $r\geq 0$ sufficiently large. Then, $(H,p^r\iota)$ belongs to the left-hand side category and maps to some object which is isomorphic (via p^{-r}) to our target.

The two functors (2.5) and (2.6) are compatible with respect to the reduction of p-divisible groups along $\mathcal{O}_K\to k$ on one side and forgetting the filtration on the other.

The polygons. For (H, ι) a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, we define its *Newton polygon*, its *Hodge polygon* and its *Harder-Narasimhan polygon* through the functor (2.6), to be those of the corresponding weakly admissible filtered isocrystal $(\mathcal{N}, \iota)$ over K with coefficients in F; they live in $\mathbb{Q}^n_+$, where $dn = \operatorname{ht} H$ is also the height of the underlying isocrystal of $\mathcal{N}$, a multiple of d by Remark 1.4. By definition, all these are invariants up to $\mathcal{O}_F$-equivariant isogeny. Moreover, we have $\operatorname{Newt}(H, \iota)(n) = \operatorname{Hdg}(H, \iota)(n) = \operatorname{HN}(H, \iota)(n) = \dim H/d$. Finally, note that $\operatorname{Newt}(H, \iota)$ only depends on the reduction (H_k, ι) of (H, ι) to k (denoting again by ι the induced $\mathcal{O}_F$-action on H_k).

Remark 2.3. If F is unramified over $\mathbb{Q}_p$, the Hodge polygon is classically defined at the level of p-divisible groups over k as follows. Assume for simplicity that k is algebraically closed and fix an embedding $F \subseteq K_0$; note that we have an isomorphism of $W(k)$-algebras:

$$W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \prod_{i \in \mathbb{Z}/d\mathbb{Z}} W(k), \quad b \otimes a \mapsto (b\sigma^i(a))_i.$$

Now, for $(H, \iota) \in p\text{--}\operatorname{div}_{k, \mathcal{O}_F}$, say with $\operatorname{ht} H = dn$, the Dieudonné module $(\mathbb{D}, \varphi)$ of H inherits a $\mathbb{Z}_p$-linear $\mathcal{O}_F$-action from ι, which makes $\mathbb{D}$ a module over the above ring. We obtain a decomposition of $W(k)$-modules:

$$\mathbb{D} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathbb{D}_i, \quad \text{with } \mathbb{D}_i = \left\{ v \in \mathbb{D} \;\middle|\; \forall a \in \mathcal{O}_F\colon \iota(a)(v) = \sigma^i(a)v \right\}.$$

Moreover, φ restricts to σ-linear injective maps $\varphi\colon \mathbb{D}_i \to \mathbb{D}_{i+1}$, so that $\operatorname{rk}_{W(k)} \mathbb{D}_i = n$ for all $i \in \mathbb{Z}/d\mathbb{Z}$. Let then $r_i := \dim_k \mathbb{D}_i/\varphi\mathbb{D}_{i-1}$ and $f_i := (1^{(n-r_i)}, 0^{(r_i)}) \in \mathbb{Q}^n_+$ and set:

$$\operatorname{Hdg}(H, \iota) := \frac{1}{d} \sum_{i \in \mathbb{Z}/d\mathbb{Z}} f_i \in \mathbb{Q}^n_+.$$

This definition agrees with the one for p-divisible groups over $\mathcal{O}_K$, in the sense that for $(H, \iota) \in p\text{--}\operatorname{div}_{\mathcal{O}_K, \mathcal{O}_F}$ we have $\operatorname{Hdg}(H, \iota) = \operatorname{Hdg}(H_{\bar{k}}, \iota)$, where $(H_{\bar{k}}, \iota)$ is the reduction of (H, ι) to an algebraic closure $\bar{k}$ of k. Indeed, let K' be a finite field extension of K containing F and fix $F \subseteq K'$. The $\mathcal{O}_{K'}$-module $\omega_{H^\vee, \mathcal{O}_{K'}} := \omega_{H^\vee} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ has a $\mathbb{Z}_p$-linear $\mathcal{O}_F$-action induced by ι and, since $\mathcal{O}_{K'} \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \prod_{i \in \mathbb{Z}/d\mathbb{Z}} \mathcal{O}_{K'}$ (similarly as before), we obtain a decomposition of $\mathcal{O}_{K'}$-modules:

$$\omega_{H^\vee, \mathcal{O}_{K'}} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \omega_{H^\vee, i}, \quad \text{with } \omega_{H^\vee, i} = \left\{ v \in \omega_{H^\vee, \mathcal{O}_{K'}} \;\middle|\; \forall a \in \mathcal{O}_F\colon \iota(a)(v) = \sigma^i(a)v \right\},$$

where σ denotes, without ambiguity, the Frobenius automorphism of F over $\mathbb{Q}_p$. Now, on the one hand $\omega_{H^\vee, i} \otimes_{\mathcal{O}_{K'}} \bar{k} \cong \mathbb{D}(H_{\bar{k}})_i/\varphi_{H_{\bar{k}}}\mathbb{D}(H_{\bar{k}})_{i-1}$ via $\omega_{H^\vee, \mathcal{O}_{K'}} \otimes_{\mathcal{O}_{K'}} \bar{k} = \omega_{H^\vee_{\bar{k}}}$ and the natural identification (2.1). On the other hand, if $(N, \varphi, \operatorname{Fil}^\bullet N_K, \iota)$ is the filtered isocrystal with coefficients in F associated to (H, ι), we see that $\omega_{H^\vee, i} \otimes_{\mathcal{O}_{K'}} K' \cong \operatorname{Fil}^0 N_{\sigma^i}$, with notation as in (1.5). Hence, for all $i \in \mathbb{Z}/d\mathbb{Z}$, we have that $\dim_{\bar{k}} \mathbb{D}(H_{\bar{k}})_i/\varphi_{H_{\bar{k}}}\mathbb{D}(H_{\bar{k}})_{i-1} = \dim_{K'} \operatorname{Fil}^0 N_{\sigma^i}$, which yields the desired equality of polygons.

In particular, this shows that if F is unramified over $\mathbb{Q}_p$, then the Hodge polygon of a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ only depends on its reduction (H_k, ι) to k. This is not true for a general extension $F|\mathbb{Q}_p$, as the following example illustrates.

Example 2.4. Let $k = \bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, so that $K_0 = \breve{\mathbb{Q}}_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. Choose $\sqrt{p}$ a square root of p and let $K = K_0(\sqrt{p})$. Let then $F = \mathbb{Q}_p(\pi)$, also with $\pi^2 = p$. We have two embeddings $\tau_0 \colon \pi \mapsto \sqrt{p}$ and $\tau_1 \colon \pi \mapsto -\sqrt{p}$ of F in K.

Consider the Dieudonné module $(\mathbb{D}, \varphi)$ over k given by $\mathbb{D} = W(k)^2$ and the σ-linear endomorphism $\varphi \colon e_1 \mapsto e_2, e_2 \mapsto pe_1$, where e_1, e_2 is the standard basis. We let π act on $(\mathbb{D}, \varphi)$ by $\pi \colon e_1 \mapsto e_2, e_2 \mapsto pe_1$; since π^2 acts as p, this defines a map of $\mathbb{Z}_p$-algebras $\iota \colon \mathcal{O}_F \to \mathrm{End}(\mathbb{D}, \varphi)$. Set now $v_0 := \sqrt{p}e_1 + e_2$ and $v_1 := -\sqrt{p}e_1 + e_2$, elements of $\mathbb{D}_{\mathcal{O}_K} := \mathbb{D} \otimes_{W(k)} \mathcal{O}_K = \mathcal{O}_K^2$; note that these are eigenvectors for the operator π, with eigenvalue respectively $\sqrt{p}$ and $-\sqrt{p}$. In particular, the sub-$\mathcal{O}_K$-modules $L_0 := \mathcal{O}_K \cdot v_0$ and $L_1 := \mathcal{O}_K \cdot v_1$ of $\mathbb{D}_{\mathcal{O}_K}$ are π-stable. Observe, in addition, that these submodules are direct summands of $\mathbb{D}_{\mathcal{O}_K}$.

Define now a new Dieudonné module $(\mathbb{D}', \varphi') := (\mathbb{D}, \varphi) \oplus (\mathbb{D}, \varphi)$ over k and let H' be the corresponding p-divisible group over k. We endow $(\mathbb{D}', \varphi')$ with the diagonal $\mathcal{O}_F$-action induced by ι; this reflects into a $\mathbb{Z}_p$-linear $\mathcal{O}_F$-action on H' (which we denote again by ι), making (H', ι) a p-divisible group with endomorphism structure for $\mathcal{O}_F$. Consider, for $i = 0, 1$, the filtration $L_{0,i} \subseteq \mathbb{D}'_{\mathcal{O}_K} := \mathbb{D}' \otimes_{W(k)} \mathcal{O}_K = \mathbb{D}_{\mathcal{O}_K} \oplus \mathbb{D}_{\mathcal{O}_K}$ given by L_0 in the first factor and L_i in the second factor; we have:

$$L_{0,i} \otimes_{\mathcal{O}_K} k = k \cdot e_2 \oplus k \cdot e_2 = V\mathbb{D}/p\mathbb{D} \oplus V\mathbb{D}/p\mathbb{D} = V'\mathbb{D}'/p\mathbb{D}',$$

where $V = p\varphi^{-1}$, $V' = p\varphi'^{-1}$. Then, by Grothendieck-Messing theory, there exist p-divisible groups H_0, H_1 over $\mathcal{O}_K$ which reduce to H' via $\mathcal{O}_K \to k$ and whose corresponding exact sequence as in (2.3) is given by the filtration $L_{0,i} \subseteq \mathbb{D}'_{\mathcal{O}_K}$, for $i = 0, 1$ respectively (cf [22, V.1.6]; we apply the theory over all the rings $\mathcal{O}_K/p^r\mathcal{O}_K$, $r \geq 1$; the resulting compatible families can be assembled to p-divisible groups over $\mathcal{O}_K$ by [22, II.4.16]). Since the $\mathcal{O}_F$-action on $\mathbb{D}'_{\mathcal{O}_K}$ respects these filtrations, H_0 and H_1 will carry a $\mathbb{Z}_p$-linear $\mathcal{O}_F$-action lifting ι (and hence deserving the same name again), making (H_0, ι) and (H_1, ι) two p-divisible groups over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. Finally, although these two objects have the same reduction to k, we see that:

$$\mathrm{Hdg}(H_0, \iota) = (1/2, 1/2) \in \mathbb{Q}_+^2, \quad \text{while } \mathrm{Hdg}(H_1, \iota) = (1, 0) \in \mathbb{Q}_+^2.$$

Indeed, setting $N := \mathbb{D} \otimes_{W(k)} K_0$, and taking $K' = K$ in (1.5), we have that $N_{\tau_0} = L_0 \otimes_{\mathcal{O}_K} K$ and $N_{\tau_1} = L_1 \otimes_{\mathcal{O}_K} K$.

Back to the main discussion, let us specialise Propositions 1.10 and 1.12 to the context of p-divisible groups and write the following cumulative statement.

Proposition 2.5. *Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. Then:*

$$\mathrm{HN}(H, \iota) \leq \mathrm{Newt}(H, \iota) \leq \mathrm{Hdg}(H, \iota).$$

Remark 2.6. If F is unramified over $\mathbb{Q}_p$, then the inequality $\mathrm{Newt}(H,\iota) \leq \mathrm{Hdg}(H,\iota)$ is classically seen as a consequence of the generalised Mazur inequality (cf [23, 4.2]). Indeed, as we saw in Remark 2.3, the Hodge polygon of (H,ι) is determined in this case by the "relative position" of the pair of $W(\bar{k})$-lattices $(\mathbb{D}(H_{\bar{k}}), \varphi_{H_{\bar{k}}}\mathbb{D}(H_{\bar{k}}))$ in $\mathbb{D}(H_{\bar{k}})[\frac{1}{p}]$. The two statements, however, are conceptually of a different nature. Indeed, on the one hand the inequality stated here relates to the nonemptiness of p-adic period domains (cf Remark 1.11), which are geometric objects living over (a finite extension of) K_0. On the other hand, Mazur's inequality relates to the nonemptiness of affine Deligne-Lusztig varieties (cf [18]), which live over k.

2.2 p-groups

Let R be a complete Noetherian commutative local ring, with residue field of characteristic p. We denote by $p\mathrm{-gr}_R$ the category of finite flat group schemes of p-power order over $\mathrm{Spec}\, R$ and call *p-groups* over R its objects. For $X \in p\mathrm{-gr}_R$, we write $X^\vee$ for its Cartier dual and $\mathrm{ht}\, X$ for the *height* of X, i.e. the logarithm to base p of its order; the height function is additive on short exact sequences. Moreover, if X has height h, then $X = X[p^h] := \mathrm{Ker}(p^h \colon X \to X)$, that is, X is p^h-torsion (cf [30, §1]).

Given a p-group $X = \mathrm{Spec}\, A$ over R, with augmentation ideal $I \subseteq A$, set $\omega_X := I/I^2$; this defines an additive functor to the category of finitely generated R-modules, which is compatible with base change. If X is the kernel of an isogeny $H \to H'$ of p-divisible groups over R, then we have an exact sequence of R-modules:

$$\omega_{H'} \longrightarrow \omega_H \longrightarrow \omega_X \longrightarrow 0. \tag{2.7}$$

This follows essentially from the fact that, on the level of formal groups, the isogeny $H \to H'$ is a "topologically flat" map (cf [6, $\mathrm{VII_B}$, 1.3.1] and loc. cit. 2.4).

Assume now that R is a discrete valuation ring. For $X \in p\mathrm{-gr}_R$, denote by X_η its generic fibre, that is, its base change to the field of fractions of R. Every closed sub-group-scheme Y of X_η yields a closed sub-p-group $\bar{Y}$ of X, through the operation of schematic closure in X (cf [26, §2]). This induces a bijection between the closed sub-group-schemes of X_η and the closed sub-p-groups of X, preserving closed embeddings; the inverse is given by taking the generic fibre. Finally, if X is a closed sub-p-group of another object $X' \in p\mathrm{-gr}_R$, then the schematic closure in X coincides with that in X'.

The degree. Let X be a p-group over $\mathcal{O}_K$. Then, because X is p-power torsion and the functor ω is additive, ω_X is a finitely generated torsion $\mathcal{O}_K$-module. Thus, we may define the *degree* of X to be:

$$\deg X := \frac{1}{e} \lg \omega_X \in \frac{1}{e}\mathbb{Z},$$

where $\lg \omega_X$ means the length of ω_X as $\mathcal{O}_K$-module (and remember that $e = [K : K_0]$). Explicitely,

$$\text{if:} \quad \omega_X \cong \bigoplus_{i=1}^{r} \mathcal{O}_K/b_i\mathcal{O}_K, \qquad \text{then:} \quad \deg X = \sum_{i=1}^{r} v(b_i),$$

where the b_i's are elements of $\mathcal{O}_K$ and v is the valuation of K, normalised at $v(p) = 1$. The degree function is additive on short exact sequences and satisfies $\deg X + \deg X^\vee = \operatorname{ht} X$ (cf [7, Lemme 4]). Moreover, if $X' \xrightarrow{u} X \xrightarrow{v} X''$ is a sequence of p-groups over $\mathcal{O}_K$ such that u is a closed embedding, $v \circ u = 0$ and v induces an isomorphism $X_\eta / X'_\eta \xrightarrow{\sim} X''_\eta$ on the generic fibre, then $\deg X \leq \deg X' + \deg X''$, with equality if and only if the sequence:

$$0 \longrightarrow X' \xrightarrow{u} X \xrightarrow{v} X'' \to 0$$

is exact (cf [7, Corollaire 3]).

The Harder-Narasimhan polygon. As explained in [7, §4], the category of p-groups over $\mathcal{O}_K$ admits a Harder-Narasimhan formalism for the slope function $\mu = \deg / \operatorname{ht}$. More precisely, to each nontrivial object $X \in p{-}\mathrm{gr}_{\mathcal{O}_K}$ we associate its *slope*:

$$\mu(X) := \frac{\deg X}{\operatorname{ht} X} \in \mathbb{Q}$$

and say that X is *semi-stable* of slope $\mu(X)$ if for every closed sub-p-group $X' \subseteq X$ we have $\mu(X') \leq \mu(X)$. In general, there exists a unique *Harder-Narasimhan filtration*:

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_m = X \tag{2.8}$$

by closed sub-p-groups, such that each X_i/X_{i-1} is semi-stable, say of slope μ_i, with $\mu_1 > \cdots > \mu_m$. Some functoriality of this filtration follows from the fact that if X' and X'' are two semi-stable objects with $\mu(X') > \mu(X'')$, then there are no nontrivial morphisms $X' \to X''$. Now, for X as above, with Harder-Narasimhan filtration as in (2.8), let $h := \operatorname{ht} X$ and let $h_i := \operatorname{ht} X_i/X_{i-1}$ for $i = 1, \ldots, m$. Then, we define the *Harder-Narasimhan polygon* of X to be:

$$\mathrm{HN}(X) := (\mu_1^{(h_1)}, \ldots, \mu_m^{(h_m)}) \in \mathbb{Q}_+^h.$$

This polygon is the concave envelope of the points $(\operatorname{ht} X', \deg X')$ over all closed sub-p-groups X' of X. Its break points lie in $\mathbb{Z} \times \frac{1}{e}\mathbb{Z}$ and we have $\mathrm{HN}(X)(h) = \deg X$. Moreover, the following compatibility with respect to duality holds (cf [7, Corollaire 8]):

$$\begin{aligned} &\mathrm{HN}(X^\vee) = ((1-\mu_m)^{(h_m)}, \ldots, (1-\mu_1)^{(h_1)}), \qquad \text{or:} \\ &\mathrm{HN}(X^\vee)\colon x \longmapsto x + \mathrm{HN}(X)(h-x) - \deg X. \end{aligned}$$

2.3 p-groups with endomorphism structure

Definition 2.7. Let R be a complete Noetherian commutative local ring, with residue field of characteristic p. A *p-group* over R *with endomorphism structure* for $\mathcal{O}_F$ is a pair (X, ι) consisting of a p-group X over R and a map of rings $\iota\colon \mathcal{O}_F \to \mathrm{End}(H)$.

We denote by $p{-}\mathrm{gr}_{R,\mathcal{O}_F}$ the category formed by the objects just defined, with morphisms given by maps of p-groups over R compatible with ι (or $\mathcal{O}_F$-*equivariant*).

The Harder-Narasimhan polygon. Let (X, ι) be a p-group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, say with $\operatorname{ht} X = h$. We define the *Harder-Narasimhan polygon* of (X, ι) to be:

$$\begin{aligned} \operatorname{HN}(X, \iota)\colon [0, h/d] &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{d} \operatorname{HN}(X)(dx). \end{aligned}$$

This is a concave polygon, whose break points lie in $\frac{1}{d}\mathbb{Z} \times \frac{1}{de}\mathbb{Z}$; in particular, we are not talking about an element of the Newton set, in general. Anyway, the $\mathcal{O}_F$-action on X given by ι restricts to each piece of its Harder-Narasimhan filtration (2.8), by functoriality of the latter (see also [7, §5.3]); this means that $\operatorname{HN}(X, \iota)$ is the concave envelope of the points $(\operatorname{ht} X'/d, \deg X'/d)$ over the closed sub-p-groups X' of X that are stable under ι. Finally, we have $\operatorname{HN}(X, \iota)(h/d) = \deg X/d$ and the following compatibility with respect to duality:

$$\operatorname{HN}(X^\vee, \iota^\vee)\colon x \longmapsto x + \operatorname{HN}(X, \iota)(h/d - x) - \deg X/d,$$

where $\iota^\vee$ denotes the dual action induced by ι.

Truncated p-divisible groups. Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, say with $\operatorname{ht} H = dn$. For $i \geq 1$, the p-groups $H[p^i]$ over $\mathcal{O}_K$ inherit a linear $\mathcal{O}_F$-action from ι; denoting this action again by ι, we obtain a family $(H[p^i], \iota)_{i \geq 1}$ of p-groups over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. We have $\operatorname{ht} H[p^i] = idn$. On the other hand, looking at the exact sequence (2.7) associated to the isogeny $p^i\colon H \to H$, we see that $\omega_{H[p^i]} = \omega_H / p^i \omega_H$; since ω_H a free $\mathcal{O}_K$-module of rank $\dim H$, we get $\deg H[p^i] = i \dim H$. Now, for $i \geq 1$, consider the *renormalised Harder-Narasimhan polygons*:

$$\begin{aligned} \operatorname{HN}^{\mathrm{r}}(H[p^i], \iota)\colon [0, n] &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{i} \operatorname{HN}(H[p^i], \iota)(ix). \end{aligned}$$

These are concave polygons, whose break points lie in $\frac{1}{id}\mathbb{Z} \times \frac{1}{ide}\mathbb{Z}$. Moreover, we have $\operatorname{HN}^{\mathrm{r}}(H[p^i], \iota)(n) = \dim H/d$ for every $i \geq 1$. The following result from [8] relates these polygons to the Harder-Narasimhan polygon of (H, ι).

Proposition 2.8. *Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and set $n := \operatorname{ht} H/d$. Then, for $i \to \infty$, the sequence of functions:*

$$\operatorname{HN}^{\mathrm{r}}(H[p^i], \iota)\colon [0, n] \to \mathbb{R}$$

converges uniformly to the function:

$$\operatorname{HN}(H, \iota)\colon [0, n] \to \mathbb{R},$$

which is equal to their infimum. In fact, for $i \geq 1$ and $k \geq 1$, we have:

$$\operatorname{HN}^{\mathrm{r}}(H[p^{ki}], \iota) \leq \operatorname{HN}^{\mathrm{r}}(H[p^i], \iota).$$

Proof. Note first that the polygons under consideration are just a rescaled version (by the same factor $1/d$) of their counterparts obtained neglecting the action of $\mathcal{O}_F$; since the rescaling process does not affect the properties stated here, we may indeed neglect the mentioned action.

The convergence statement is [8, Théorème 1]. Then, it follows from loc. cit. Théorème 5 that the limit coincides with the Harder-Narasimhan polygon of H as it is defined here; indeed, the functor (2.4) is an equivalence of categories (cf Remark 2.1), which makes the function $\dim / \operatorname{ht}$ on $p\text{−div}_{\mathcal{O}_K} \otimes \mathbb{Q}_p$ correspond to the slope function μ on $\mathrm{FilIsoc}_K^{\text{w-a}}$. For the final statement, see the proof of loc. cit. Proposition 2. □

Remark 2.9. From the convergence statement and the compatibility of the Harder-Narasimhan polygon of p-groups with respect to duality, we deduce that similarly, for $(H, \iota) \in p\text{−div}_{\mathcal{O}_K,\mathcal{O}_F}$, we have:

$$\mathrm{HN}(H^\vee, \iota^\vee)\colon x \longmapsto x + \mathrm{HN}(H, \iota)(\operatorname{ht} H/d - x) - \dim H/d,$$

where $\iota^\vee$ denotes the dual action induced by ι.

The family of polygons $(\mathrm{HN}^{\mathrm{r}}(H[p^i], \iota))_{i\geq 1}$ concerns (H, ι) as an object of $p\text{−div}_{\mathcal{O}_K,\mathcal{O}_F}$ and not just up to isogeny; in this sense, it provides more expendable information about the structure of our p-divisible group. As we just saw, these polygons are bounded from below by $\mathrm{HN}(H, \iota)$, which is instead determined up to isogeny. We now come to finding an upper bound, similarly uniform over the whole isogeny class, under the assumption that the ramification index of F over $\mathbb{Q}_p$ is at most 2 (cf Proposition 2.13).

2.4 Comparison between Harder-Narasimhan and Hodge polygon

Given a p-divisible group (H, ι) over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, we would like to compare the two polygons $\mathrm{HN}(H[p], \iota)$ and $\mathrm{Hdg}(H, \iota)$ and ultimately prove Proposition 2.13. The way the endomorphism structure affects the definitions of these two polygons is radically different, namely a rescaling process for $\mathrm{HN}(H[p], \iota)$ and an averaging process for $\mathrm{Hdg}(H, \iota)$. In order to address this discrepancy, we make use of different factors depending on whether the endomorphism structure is of an unramified or a ramified nature.

Before proving the proposition, let us recall from [28] the main argument for the unramified case. Afterwards, we will review the algebra required to isolate the ramified component in the general case and gather some tools that will help us deal with it.

Let $H \to H'$ be an isogeny of p-divisible groups over k with kernel $X \in p\text{−gr}_k$ and consider the corresponding map $\mathbb{D}(H'^\vee) \to \mathbb{D}(H^\vee)$ of contravariant Dieudonné modules. This induces an isomorphism on the isocrystals; in particular, it is injective and its cokernel C is a $W(k)$-module of finite length. More precisely, we have $\lg_{W(k)} C = \operatorname{ht} X$ (in fact, C is the contravariant Dieudonné module of the p-group X, cf [9, §III.1]). We obtain an exact sequence of $W(k)$-modules:

$$0 \longrightarrow \mathbb{D}(H'^\vee) \longrightarrow \mathbb{D}(H^\vee) \longrightarrow C \longrightarrow 0. \tag{2.9}$$

Assume now that H and H' are endowed with endomorphism structure for $\mathcal{O}_F$ and that $H \to H'$ is $\mathcal{O}_F$-equivariant. Then, the sequence above has an induced $\mathbb{Z}_p$-linear $\mathcal{O}_F$-action, which makes it a sequence of $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$-modules.

If we assume further that F is unramified over $\mathbb{Q}_p$ and that K_0 contains F, then, fixing $F \subseteq K_0$, we have an isomorphism of $W(k)$-algebras:

$$W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \prod_{i \in \mathbb{Z}/d\mathbb{Z}} W(k), \qquad b \otimes a \mapsto (b\sigma^i(a))_i. \tag{2.10}$$

Thus, the sequence (2.9) splits as a direct sum of exact sequences:

$$0 \longrightarrow \mathbb{D}(H'^{\vee})_i \longrightarrow \mathbb{D}(H^{\vee})_i \longrightarrow C_i \longrightarrow 0, \qquad i \in \mathbb{Z}/d\mathbb{Z}, \tag{2.11}$$

with $\mathcal{O}_F$ acting through $\sigma^i \colon \mathcal{O}_F \to W(k)$ on the i-th component.

The following lemma applies when these sequences arise from a situation over $\mathcal{O}_K$. It is the core of the proof of [28, 3.10]; for convenience, we report it here as an isolated statement, together with its argument.

Lemma 2.10. *Assume that F is unramified over $\mathbb{Q}_p$ and that K_0 contains F; fix $F \subseteq K_0$. Let (H, ι) and (H', ι') be p-divisible groups over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and $H \to H'$ an $\mathcal{O}_F$-equivariant isogeny with kernel X. Let then $H_k \to H'_k$ be its reduction to k and consider the exact sequences of $W(k)$-modules:*

$$0 \longrightarrow \mathbb{D}(H_k'^{\vee})_i \longrightarrow \mathbb{D}(H_k^{\vee})_i \longrightarrow C_i \longrightarrow 0, \qquad i \in \mathbb{Z}/d\mathbb{Z},$$

as in (2.11)*. Then, $\lg_{W(k)} C_i = \frac{1}{d} \operatorname{ht} X$ for every $i \in \mathbb{Z}/d\mathbb{Z}$.*

Proof. For $i \in \mathbb{Z}/d\mathbb{Z}$, the σ-linear endomorphism $\varphi := \varphi_{H_k^{\vee}}$ of $\mathbb{D}(H_k^{\vee})$ restricts to injective maps $\varphi \colon \mathbb{D}(H_k^{\vee})_{i-1} \to \mathbb{D}(H_k^{\vee})_i$. Reasoning similarly for $\mathbb{D}(H_k'^{\vee})$ and using that the map $\mathbb{D}(H_k'^{\vee}) \to \mathbb{D}(H_k^{\vee})$ is compatible with the σ-linear endomorphisms, we obtain commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{D}(H_k'^{\vee})_{i-1} & \longrightarrow & \mathbb{D}(H_k^{\vee})_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & \mathbb{D}(H_k'^{\vee})_{i} & \longrightarrow & \mathbb{D}(H_k^{\vee})_{i} & \longrightarrow & C_{i} & \longrightarrow & 0
\end{array}$$

with exact rows and the first two columns being injective. Now, the argument of Remark 2.3 shows that the numbers $\dim_k \mathbb{D}(H_k^{\vee})_i/\varphi\mathbb{D}(H_k^{\vee})_{i-1}$ are determined by the filtered isocrystal with coefficients in F associated to $(H^{\vee}, \iota^{\vee})$, where $\iota^{\vee}$ denotes the dual action induced by ι. However, since $H^{\vee}$ and $H'^{\vee}$ are $\mathcal{O}_F$-equivariantly isogenous, the respective associated filtered isocrystals with coefficients in F are isomorphic. In particular:

$$\dim_k \mathbb{D}(H_k^{\vee})_i/\varphi\mathbb{D}(H_k^{\vee})_{i-1} = \dim_k \mathbb{D}(H_k'^{\vee})_i/\varphi\mathbb{D}(H_k'^{\vee})_{i-1}$$

for every $i \in \mathbb{Z}/d\mathbb{Z}$. By the commutative diagram above, we conclude that $\lg C_i = \lg C_{i-1}$, i.e. $\lg C_i$ is constant over $i \in \mathbb{Z}/d\mathbb{Z}$. Finally, since these lengths sum to $\operatorname{ht} X$, we must have $\lg C_i = \frac{1}{d} \operatorname{ht} X$ for every $i \in \mathbb{Z}/d\mathbb{Z}$. □

Let now F be any finite extension of $\mathbb{Q}_p$. Denote by F^{nr} the maximal unramified subextension (or *inertia subfield*) of $F|\mathbb{Q}_p$ and by $\mathcal{O}_{F^{\mathrm{nr}}}$ its ring of integers, so that $F^{\mathrm{nr}}|\mathbb{Q}_p$ is an unramified extension of degree $f(F|\mathbb{Q}_p)$, the inertia index of $F|\mathbb{Q}_p$, and $F|F^{\mathrm{nr}}$ is a totally ramified extension of degree $e(F|\mathbb{Q}_p)$, the ramification degree of $F|\mathbb{Q}_p$. Write $\mathcal{O}_F = \mathcal{O}_{F^{\mathrm{nr}}}[\pi]$, with π the root of an Eisenstein polynomial of degree $e(F|\mathbb{Q}_p)$ with coefficients in $\mathcal{O}_{F^{\mathrm{nr}}}$.

Assume that K contains all embeddings τ of F in an algebraic closure. In particular, K_0 contains F^{nr}; note that, given one embedding $\upsilon_0 \colon F^{\mathrm{nr}} \to K_0$, we obtain all the embeddings of F^{nr} in K_0 as $\sigma^i \circ \upsilon_0$ for $1 \leq i \leq f(F|\mathbb{Q}_p)$. Then, applying (2.10) to F^{nr} and base changing along $\mathcal{O}_{F^{\mathrm{nr}}} \to \mathcal{O}_F$, we get the following isomorphism of $W(k)$-algebras:

$$W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \prod_{\upsilon \colon F^{\mathrm{nr}} \to K_0} W(k) \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F, \tag{2.12}$$

where υ runs through all the embeddings of F^{nr} in K_0. Thus, every $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$-module P decomposes as:

$$P = \bigoplus_{\upsilon} P_\upsilon, \quad \text{with } P_\upsilon = \{\, w \in P \mid \forall a \in \mathcal{O}_{F^{\mathrm{nr}}} \colon (1 \otimes a)w = (\upsilon(a) \otimes 1)w \,\}, \tag{2.13}$$

where each P_υ is a module over the ring $W(k) \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$. Note that this is a discrete valuation ring; in fact, it is the ring of integers of a totally ramified extension of K_0 of degree $e(F|\mathbb{Q}_p)$. After base change along $W(k) \to \mathcal{O}_K$, the isomorphism (2.12) extends to:

$$\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \prod_{\upsilon} \mathcal{O}_K \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F, \tag{2.14}$$

inducing a similar decomposition of every $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_F$-module into a direct sum of modules over the rings $\mathcal{O}_K \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$.

Fix now $\upsilon \colon F^{\mathrm{nr}} \to K_0$ and let Q be an $\mathcal{O}_K \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module; such an object can be studied as an $\mathcal{O}_K$-module together with an $\mathcal{O}_K$-linear operator Π, corresponding to the multiplication by the element $1 \otimes \pi \in \mathcal{O}_K \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$. We have the following decomposition of $Q_K := Q \otimes_{\mathcal{O}_K} K$ into sub-K-vector-spaces:

$$Q_K = \bigoplus_{\tau|\upsilon} Q_{K,\tau}, \quad \text{with } Q_{K,\tau} = \{\, w \in Q_K \mid \forall a \in F \colon (1 \otimes a)w = (\tau(a) \otimes 1)w \,\}, \tag{2.15}$$

where $\tau|\upsilon$ stands for the embeddings of F in K which restrict to υ on F^{nr} (note that Q_K is automatically a module over $K \otimes_{\upsilon, F^{\mathrm{nr}}} F \cong \prod_{\tau|\upsilon} K$, $b \otimes a \mapsto (b\tau(a))_\tau$). In fact, this is the eigenspace decomposition of Q_K for the K-linear operator induced by Π, each $Q_{K,\tau}$ being associated to the eigenvalue $\tau(\pi)$. Assume now that Q is free as an $\mathcal{O}_K$-module and fix $\xi := \tau(\pi) \in \mathcal{O}_K$ for some $\tau|\upsilon$. The following lemma relates the action induced by Π on $Q/\xi Q$ to the eigenspace decomposition of Q_K considered above.

Lemma 2.11. *Assume that K contains all embeddings τ of F in an algebraic closure and fix $\upsilon\colon F^{\mathrm{nr}} \to K_0$. Let Q be a finitely generated $\mathcal{O}_K \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module which is free as an $\mathcal{O}_K$-module and set $h_\tau := \dim_K Q_{K,\tau}$ for $\tau|\upsilon$, with notation as in* (2.15)*. Let then $\xi := \tau(\pi) \in \mathcal{O}_K$ for some $\tau|\upsilon$ and consider the induced action of Π, i.e. multiplication by the element $1 \otimes \pi \in \mathcal{O}_K \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$, on $Q/\xi Q$. Then, for $0 \le i \le d' := e(F|\mathbb{Q}_p)$ and any selection of pairwise distinct elements $\tau_1, \dots, \tau_{d'-i} \in \{\, \tau\colon F \to K \mid \tau|\upsilon \,\}$, we have:*

$$\lg_{\mathcal{O}_K} \Pi^i(Q/\xi Q) \le \frac{e}{d'} \sum_{j=1}^{d'-i} h_{\tau_j}.$$

Proof. Let us begin with a general observation. For $f\colon Q \to Q$ any $\mathcal{O}_K$-linear homomorphism, consider on the one hand its reduction $\bar{f}\colon Q/\xi Q \to Q/\xi Q$ modulo ξ and, on the other hand, the induced map $f_K := f \otimes \mathrm{id}_K\colon Q_K \to Q_K$. Note that $\operatorname{Ker} f = Q \cap \operatorname{Ker} f_K$ is a free $\mathcal{O}_K$-submodule of Q of rank equal to the dimension of the K-vector-space $\operatorname{Ker} f_K$. Then, the reduction of $\operatorname{Ker} f$ modulo ξ is a free $\mathcal{O}_K/\xi\mathcal{O}_K$-submodule of $Q/\xi Q$, which is clearly contained in the kernel of $\bar{f}$. Computing the lengths, we obtain:

$$\lg_{\mathcal{O}_K} \operatorname{Ker} \bar{f} \ge \lg_{\mathcal{O}_K} \operatorname{Ker} f/\xi \operatorname{Ker} f = (\lg_{\mathcal{O}_K} \mathcal{O}_K/\xi\mathcal{O}_K)(\dim_K \operatorname{Ker} f_K).$$

Set now $f := (\Pi - \tau_{d'-i+1}(\pi)) \circ \dots \circ (\Pi - \tau_{d'}(\pi))$, where we numbered $\tau_{d'-i+1}, \dots, \tau_{d'}$ the elements of $\{\, \tau\colon F \to K \mid \tau|\upsilon \,\}$ other than $\tau_1, \dots, \tau_{d'-i}$. On the one hand, we have that $\bar{f} = \Pi^i\colon Q/\xi Q \to Q/\xi Q$. Indeed, for varying $\tau|\upsilon$, the elements $\tau(\pi) \in \mathcal{O}_K$ differ from each other by a unit of $\mathcal{O}_K$, as they all have the same valuation; in particular, they are all divisible by ξ. On the other hand, the kernel of f_K contains all the eigenspaces Q_{K,τ_j} for $j = d' - i + 1, \dots, d'$, which gives:

$$\dim_K \operatorname{Ker} f_K \ge \sum_{j=d'-i+1}^{d'} h_{\tau_j}.$$

Moreover, since $\xi^{d'} = \tau(\pi^{d'}) \in \mathcal{O}_K$ has the same valuation as p and $\lg_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K = [K : K_0] = e$, we have that $\lg_{\mathcal{O}_K} \mathcal{O}_K/\xi\mathcal{O}_K = e/d'$. Plugging all this information in the first inequality, we get:

$$\lg_{\mathcal{O}_K} \operatorname{Ker}(\Pi^i\colon Q/\xi Q \to Q/\xi Q) \ge \frac{e}{d'} \sum_{j=d'-i+1}^{d'} h_{\tau_j}.$$

Observe finally that $Q/\xi Q$ is a free $\mathcal{O}_K/\xi\mathcal{O}_K$-module of rank $\sum_{\tau|\upsilon} h_\tau$, so that:

$$\lg_{\mathcal{O}_K} \Pi^i(Q/\xi Q) = \lg_{\mathcal{O}_K} Q/\xi Q - \lg_{\mathcal{O}_K} \operatorname{Ker}(\Pi^i\colon Q/\xi Q \to Q/\xi Q) \le \frac{e}{d'} \sum_{j=1}^{d'-i} h_{\tau_j},$$

as claimed in the statement. $\square$

A special case is represented by free $\mathcal{O}_K \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-modules. For instance, let P be a finitely generated $W(k) \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module which is free as a $W(k)$-module. In particular, P is torsion free and hence free over $W(k) \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$, as this is a discrete valuation ring. Its base change $P \otimes_{W(k)} \mathcal{O}_K$ will then be a free $\mathcal{O}_K \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module.

Lemma 2.12. *Assume that K contains all embeddings τ of F in an algebraic closure and fix $v\colon F^{\mathrm{nr}} \to K_0$. Let Q be a free $\mathcal{O}_K \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module. Let then $\xi := \tau(\pi) \in \mathcal{O}_K$ for some $\tau|v$ and consider the induced action of Π, i.e. multiplication by the element $1 \otimes \pi \in \mathcal{O}_K \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$, on $Q/\xi Q$. Then, for $0 \le i \le d' := e(F|\mathbb{Q}_p)$, we have:*

$$\Pi^i(Q/\xi Q) = (Q/\xi Q)[\Pi^{d'-i}] := \operatorname{Ker}\left(\Pi^{d'-i}\colon Q/\xi Q \to Q/\xi Q\right).$$

Proof. It is enough to consider the case $Q = \mathcal{O}_K \otimes_{v,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$. We view this as a free $\mathcal{O}_K$-module of rank d', with basis $(1 \otimes \pi^i)_{0 \le i \le d'-1}$. Then, the action of Π is represented by a matrix of the form:

$$\begin{pmatrix} 0 & \cdots & 0 & * \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & * \end{pmatrix},$$

the entries of the last column being divisible by p. On the free $\mathcal{O}_K/\xi\mathcal{O}_K$-module $Q/\xi Q$, the action of Π is represented (with respect to the induced basis) by a similar matrix, but with the last column being zero, as $\xi = \tau(\pi)$ divides p. Given this, the result follows from a simple computation. □

We can now prove the following crucial proposition.

Proposition 2.13. *Assume that the ramification index of F over $\mathbb{Q}_p$ is at most 2 and let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. Then:*

$$\mathrm{HN}(H[p], \iota) \le \mathrm{Hdg}(H, \iota).$$

Proof. Set $h := \operatorname{ht} H$ and let $(\mathcal{N}, \iota)$ be the filtered isocrystal over K with coefficients in F associated to (H, ι) via (2.6), so that $\mathrm{Hdg}(H, \iota) = \mathrm{Hdg}(\mathcal{N}, \iota)$. Up to replacing K by a finite extension K', we may assume that K contains all embeddings τ of F in an algebraic closure. Indeed, on the one hand, this step is anyway embedded in the definition of $\mathrm{Hdg}(\mathcal{N}, \iota)$; moreover, the underlying filtration of $\mathcal{N}$ is determined by H through the embedding $\omega_{H^\vee} \hookrightarrow M(H)$ as in (2.3), which is compatible with respect to base change along $\mathcal{O}_K \to \mathcal{O}_{K'}$. On the other hand, $\mathrm{HN}(H[p], \iota)$ can only increase after the same base change (but in fact it does not, cf [7, Proposition 6]).

We need to prove that for every ι-stable closed sub-p-group X' of $H[p]$ we have:

$$\frac{\deg X'}{d} \le \mathrm{Hdg}(\mathcal{N}, \iota)\left(\frac{\operatorname{ht} X'}{d}\right). \tag{2.16}$$

Set $X := H[p]/X'$ and consider the dual p-group $X^\vee$ and the dual p-divisible group $H^\vee$, with the dual action $\iota^\vee$ induced by ι. Then, $X^\vee$ is an $\iota^\vee$-stable closed sub-p-group of $H[p]^\vee$, inducing an $\mathcal{O}_F$-equivariant isogeny $H^\vee \to H^\vee/X^\vee$ with kernel $X^\vee$. After reducing to k, we obtain an exact sequence of $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}_F$-modules:

$$0 \longrightarrow \mathbb{D}(H'_k) \longrightarrow \mathbb{D}(H_k) \longrightarrow C \longrightarrow 0 \tag{2.17}$$

as in (2.9), where $H' := (H^\vee/X^\vee)^\vee$. Moreover, because $X^\vee \subseteq H[p]^\vee$ and the functor $\mathbb{D}$ is $\mathbb{Z}_p$-linear, we have that C is a p-torsion module.

Let now F^{nr} denote the inertia subfield of $F|\mathbb{Q}_p$, with ring of integers $\mathcal{O}_{F^{\mathrm{nr}}}$. By the decomposition in (2.13), the exact sequence above splits as a direct sum of exact sequences:

$$0 \longrightarrow \mathbb{D}(H'_k)_\upsilon \longrightarrow \mathbb{D}(H_k)_\upsilon \longrightarrow C_\upsilon \longrightarrow 0 \tag{2.18}$$

of $W(k)\otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}}\mathcal{O}_F$-modules, where υ runs through all the embeddings of F^{nr} in K_0. This decomposition coincides with that in (2.11), if we restrict ι to $\mathcal{O}_{F^{\mathrm{nr}}}$. In particular, by Lemma 2.10, we have:

$$\lg_{W(k)} C_\upsilon = \frac{1}{[F^{\mathrm{nr}}:\mathbb{Q}_p]}\,\mathrm{ht}\, X^\vee = \frac{1}{[F^{\mathrm{nr}}:\mathbb{Q}_p]}\,\mathrm{ht}\, X \tag{2.19}$$

for all υ's.

The $\mathcal{O}_F$-equivariant isogeny $H^\vee \to H^\vee/X^\vee$ also induces an exact sequence of modules over $\mathcal{O}_K\otimes_{\mathbb{Z}_p}\mathcal{O}_F$:

$$0 \longrightarrow \omega_{H^\vee/X^\vee} \longrightarrow \omega_{H^\vee} \longrightarrow \omega_{X^\vee} \longrightarrow 0$$

as in (2.7), the first map being injective because $\omega_{H^\vee/X^\vee}$ and $\omega_{H^\vee}$ are free $\mathcal{O}_K$-modules of the same rank and $\omega_{X^\vee}$ is torsion. Taking into consideration the natural exact sequence (2.3), we obtain a commutative diagram of $\mathcal{O}_K\otimes_{\mathbb{Z}_p}\mathcal{O}_F$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{H^\vee/X^\vee} & \longrightarrow & \omega_{H^\vee} & \longrightarrow & \omega_{X^\vee} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(H') & \longrightarrow & M(H) & \longrightarrow & C_{\mathcal{O}_K} & \longrightarrow & 0
\end{array}$$

with exact rows, where the first two columns are direct summands (as free $\mathcal{O}_K$-modules) and hence the last column is injective as well; the notation $C_{\mathcal{O}_K}$ for the quotient of $M(H)$ by $M(H')$ is justified by the fact that the lower row identifies with the base change of (2.17) along $W(k)\to\mathcal{O}_K$. Due to the isomorphism (2.14), this diagram splits as a direct sum of commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{H^\vee/X^\vee,\upsilon} & \longrightarrow & \omega_{H^\vee,\upsilon} & \longrightarrow & \omega_{X^\vee,\upsilon} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(H')_\upsilon & \longrightarrow & M(H)_\upsilon & \longrightarrow & C_{\mathcal{O}_K,\upsilon} & \longrightarrow & 0
\end{array} \tag{2.20}$$

of modules over $\mathcal{O}_K\otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}}\mathcal{O}_F$, for υ varying as before. The properties of the previous diagram are preserved and the lower row identifies now with the base change of (2.18) along $W(k)\to\mathcal{O}_K$.

Recall that the underlying filtered vector space of $(\mathcal{N}, \iota)$ is given by the K_0-vector-space $N := \mathbb{D}(H_k)[\frac{1}{p}]$ (whose base change $N_K := N \otimes_{K_0} K$ identifies with $M(H)_K := M(H) \otimes_{\mathcal{O}_K} K$), with $\mathrm{Fil}^0 N_K = \omega_{H^\vee,K} := \omega_{H^\vee} \otimes_{\mathcal{O}_K} K$, $\mathrm{Fil}^{-i} N_K = M(H)_K$ and $\mathrm{Fil}^i N_K = 0$ for all $i \geq 1$. Write:

$$M(H)_K = \bigoplus_{\tau\colon F \to K} M(H)_{K,\tau}, \qquad \omega_{H^\vee,K} = \bigoplus_{\tau\colon F \to K} \omega_{H^\vee,K,\tau}$$

as in (1.5), set $h_\tau := \dim_K \omega_{H^\vee,K,\tau}$ and remember that $\dim_K M(H)_{K,\tau} = h/d$ for all τ's. Then:

$$\mathrm{Hdg}(\mathcal{N}, \iota)\left(\frac{\mathrm{ht}\, X'}{d}\right) = \mathrm{Hdg}(\mathcal{N}, \iota)\left(\frac{h - \mathrm{ht}\, X}{d}\right) = \frac{1}{d}\sum_\tau \min\left\{\frac{h}{d} - h_\tau, \frac{h - \mathrm{ht}\, X}{d}\right\}.$$

On the other side:

$$\frac{\deg X'}{d} = \frac{1}{d}(\dim H - \deg X) = \frac{1}{d}(h - \dim H^\vee - \mathrm{ht}\, X + \deg X^\vee).$$

However, $\dim H^\vee = \mathrm{rk}_{\mathcal{O}_K} \omega_{H^\vee} = \dim_K \omega_{H^\vee,K} = \sum_\tau h_\tau$, so:

$$h - \dim H^\vee - \mathrm{ht}\, X = \sum_\tau \left(\frac{h - \mathrm{ht}\, X}{d} - h_\tau\right).$$

Thus, (2.16) may be rewritten as:

$$\deg X^\vee \leq \sum_\tau \min\left\{\frac{\mathrm{ht}\, X}{d}, h_\tau\right\}. \tag{2.21}$$

Observe now that:

$$\deg X^\vee = \frac{1}{e}\lg \omega_{X^\vee} = \frac{1}{e}\sum_\upsilon \lg \omega_{X^\vee,\upsilon}$$

and, using (2.19):

$$\frac{\mathrm{ht}\, X}{d} = \frac{\lg C_\upsilon}{d'} = \frac{\lg C_{\mathcal{O}_K,\upsilon}}{ed'},$$

where $d' := [F : F^{\mathrm{nr}}] \leq 2$ is the ramification index of F over $\mathbb{Q}_p$ and the lengths are meant as $W(k)$-modules or, when applicable, as $\mathcal{O}_K$-modules (in fact, since K is totally ramified over K_0, this does not really matter, as the only simple module is k in both cases). As a consequence, grouping the right-hand side of (2.21) to a sum over υ, it suffices to show that for every embedding υ of F^{nr} in K_0 we have:

$$\lg \omega_{X^\vee,\upsilon} \leq \sum_{\tau|\upsilon} \min\left\{\frac{\lg C_{\mathcal{O}_K,\upsilon}}{d'}, eh_\tau\right\}, \tag{2.22}$$

where $\tau|\upsilon$ are the embeddings of F in K which restrict to υ on F^{nr}.

Fix $\upsilon\colon F^{\mathrm{nr}} \to K_0$ and, for the sake of brevity, set:

$$\omega := \omega_{H^\vee,\upsilon}, \qquad \bar\omega := \omega_{X^\vee,\upsilon}, \qquad M := M(H)_\upsilon, \qquad \bar M := C_{\mathcal{O}_K,\upsilon}$$

for the $\mathcal{O}_K \otimes_{\upsilon, \mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-modules in the right square of (2.20). Note that:

$$M_K := M \otimes_{\mathcal{O}_K} K = \bigoplus_{\tau|\upsilon} M(H)_{K,\tau}, \qquad \omega_K := \omega \otimes_{\mathcal{O}_K} K = \bigoplus_{\tau|\upsilon} \omega_{H^\vee,K,\tau}$$

and this coincides with the decomposition described in (2.15). Now, on the one hand the embedding $\bar{\omega} \hookrightarrow \bar{M}$ ensures that $\lg \bar{\omega} \leq \lg \bar{M}$. On the other hand, the surjection $\omega \twoheadrightarrow \bar{\omega}$ factors as $\omega/p\omega \twoheadrightarrow \bar{\omega}$, as $\bar{\omega}$ is p-torsion, so that $\lg \bar{\omega} \leq e \operatorname{rk}_{\mathcal{O}_K} \omega = e \sum_{\tau|\upsilon} h_\tau$. If $d' = 1$ (that is, if F is unramified over $\mathbb{Q}_p$), then the right-hand side of (2.22) reduces to a single summand and this argument shows that the inequality is verified. If $d' = 2$, the same argument covers the cases when both summands in the right-hand side of (2.22) are equal to $\lg C_{\mathcal{O}_K,\upsilon}/d' = \lg \bar{M}/2$ or they are both equal to eh_τ (for the respective τ). Therefore, we are only left with showing that for every choice of $\tau_1|\upsilon$ we have:

$$\lg \bar{\omega} \leq \frac{\lg \bar{M}}{2} + eh_{\tau_1}.$$

Fix $\tau_1|\upsilon$. Write $\mathcal{O}_F = \mathcal{O}_{F^{\mathrm{nr}}}[\pi]$, with π the root of an Eisenstein polynomial of degree 2 with coefficients in $\mathcal{O}_{F^{\mathrm{nr}}}$, and set $\xi := \tau(\pi) \in \mathcal{O}_K$ for some $\tau|\upsilon$. Let then $\bar{\omega}[\xi]$ denote the kernel of the multiplication by ξ (i.e. the element $\xi \otimes 1 \in \mathcal{O}_K \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$) on $\bar{\omega}$, so that $\lg \bar{\omega} = \lg \bar{\omega}[\xi] + \lg \xi\bar{\omega}$. Note that $\xi^2 = \tau(\pi^2)$ is a multiple of p, hence, since $\bar{\omega}$ is p-torsion, both modules $\bar{\omega}[\xi]$ and $\xi\bar{\omega}$ are ξ-torsion. We are going to analyse separately their contribution to the total length.

Starting with $\bar{\omega}[\xi]$, let $\bar{\omega}[\xi][\Pi]$ denote the kernel of the map $\Pi\colon \bar{\omega}[\xi] \to \bar{\omega}[\xi]$ given by the multiplication by the element $1 \otimes \pi \in \mathcal{O}_K \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$, so that $\lg \bar{\omega}[\xi] = \lg \bar{\omega}[\xi][\Pi] + \lg \Pi(\bar{\omega}[\xi])$. On the one hand, the embedding $\bar{\omega} \hookrightarrow \bar{M}$ restricts to $\bar{\omega}[\xi][\Pi] \hookrightarrow \bar{M}[\xi][\Pi]$ (adopting for $\bar{M}$ the same notations as for $\bar{\omega}$), ensuring that $\lg \bar{\omega}[\xi][\Pi] \leq \lg \bar{M}[\xi][\Pi]$. On the other hand, let $\omega' \subseteq \omega$ be the preimage of $\bar{\omega}[\xi]$ in ω, that is, the kernel of $\xi\colon \omega \to \bar{\omega}$. The surjection $\omega' \twoheadrightarrow \bar{\omega}[\xi]$ factors through $\omega'/\xi\omega'$ and induces $\Pi(\omega'/\xi\omega') \twoheadrightarrow \Pi(\bar{\omega}[\xi])$. Thus, we have that $\lg \Pi(\bar{\omega}[\xi]) \leq \lg \Pi(\omega'/\xi\omega')$. In turn, by Lemma 2.11, we know that $\lg \Pi(\omega'/\xi\omega') \leq (e/2)h_{\tau_1}$ (note that $\omega' \otimes_{\mathcal{O}_K} K = \omega_K$, so the numbers h_τ do not change by passing from ω to ω'). In summary:

$$\lg \bar{\omega}[\xi] \leq \lg \bar{M}[\xi][\Pi] + \frac{e}{2} h_{\tau_1}.$$

As for $\xi\bar{\omega}$, we have a surjective map $\xi\colon \omega \twoheadrightarrow \xi\bar{\omega}$, which factors through $\omega/\xi\omega$. Let $(\omega/\xi\omega)[\Pi]$ denote the kernel of $\Pi\colon \omega/\xi\omega \to \omega/\xi\omega$ (i.e. the multiplication by $1 \otimes \pi \in \mathcal{O}_K \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$ as before). Let then T be the image of $(\omega/\xi\omega)[\Pi]$ in $\xi\bar{\omega}$ and $S := \xi\bar{\omega}/T$ the quotient, so that $\lg \xi\bar{\omega} = \lg T + \lg S$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega/\xi\omega[\Pi] & \longrightarrow & \omega/\xi\omega & \xrightarrow{\Pi} & \Pi(\omega/\xi\omega) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \xi & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & \xi\bar{\omega} & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

On the one hand, the induced surjection $\Pi(\omega/\xi\omega) \twoheadrightarrow S$ shows, using Lemma 2.11 again, that $\lg S \leq (e/2)h_{\tau_1}$. On the other hand, the surjective map $\xi\colon M \to \xi\bar{M}$ also factors through $M/\xi M$ (as $\bar{M}$ is p-torsion) and fits in the commutative diagram:

$$\begin{array}{ccc} \omega/\xi\omega & \xrightarrow{\ \xi\ } & \xi\bar{\omega} \\ \downarrow & & \downarrow \\ M/\xi M & \xrightarrow{\ \xi\ } & \xi\bar{M}. \end{array}$$

Recall now that $M \cong \mathbb{D}(H_k)_\upsilon \otimes_{W(k)} \mathcal{O}_K$ and that $\mathbb{D}(H_k)_\upsilon$ is a finitely generated $W(k) \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module which is free as a $W(k)$-module. In particular, $\mathbb{D}(H_k)_\upsilon$ is torsion free and hence free over $W(k) \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$, as this is a discrete valuation ring. Therefore, M is a free $\mathcal{O}_K \otimes_{\upsilon,\mathcal{O}_{F^{\mathrm{nr}}}} \mathcal{O}_F$-module. By Lemma 2.12, then, we have that $(M/\xi M)[\Pi] = \Pi(M/\xi M)$ (adopting for $M/\xi M$ the same notations as for $\omega/\xi\omega$) and so its image in $\xi\bar{M}$ is $\Pi(\xi\bar{M})$. Thus, the embedding $\xi\bar{\omega} \hookrightarrow \xi\bar{M}$ restricts to $T \hookrightarrow \Pi(\xi\bar{M})$, ensuring that $\lg T \leq \lg \Pi(\xi\bar{M})$. In summary:

$$\lg \xi\bar{\omega} \leq \lg \Pi(\xi\bar{M}) + \frac{e}{2}h_{\tau_1}.$$

Recall, in addition, that $\bar{M} \cong C_\upsilon \otimes_{W(k)} \mathcal{O}_K$, with C_υ a p-torsion $W(k)$-module, i.e. a k-vector-space. Hence, $\bar{M}$ is a free $\mathcal{O}_K/p\mathcal{O}_K$-module. In particular, we have that $\xi\bar{M} = \bar{M}[\xi]$ and $\lg \bar{M}[\xi] = \lg \bar{M}/2$. Thus:

$$\lg \bar{M}[\xi][\Pi] + \lg \Pi(\xi\bar{M}) = \lg \bar{M}[\xi][\Pi] + \lg \Pi(\bar{M}[\xi]) = \lg \bar{M}[\xi] = \frac{\lg \bar{M}}{2}.$$

Altogether:

$$\lg \bar{\omega} = \lg \bar{\omega}[\xi] + \lg \xi\bar{\omega} \leq \lg \bar{M}[\xi][\Pi] + \frac{e}{2}h_{\tau_1} + \lg \Pi(\xi\bar{M}) + \frac{e}{2}h_{\tau_1} = \frac{\lg \bar{M}}{2} + eh_{\tau_1}.$$

This concludes the proof of the proposition. □

Remark 2.14. We conjecture that the previous proposition holds even without the limitation on the ramification index of F over $\mathbb{Q}_p$. At present time, however, proving such a more general statement seems to the author strictly more demanding than the case covered here.

3 Hodge-Newton filtration

3.1 Filtrations of p-divisible groups via Harder-Narasimhan theory

Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and let $(\mathcal{N}, \iota)$ be its associated filtered isocrystal over K with coefficients in F, via (2.6). Suppose that $z = (x, y)$ is a break point of the Harder-Narasimhan polygon $\mathrm{HN}(H, \iota) = \mathrm{HN}(\mathcal{N}, \iota)$. By definition (and compatibility of the Harder-Narasimhan filtration with the F-action), this corresponds to a subobject $(\mathcal{N}_1, \iota_1) \subseteq (\mathcal{N}, \iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$, such that $\mathrm{HN}(\mathcal{N}_1, \iota_1)$ is the restriction of $\mathrm{HN}(\mathcal{N}, \iota)$ to $[0, x]$ and, if $(\mathcal{N}_2, \iota_2)$ denotes the quotient of $(\mathcal{N}, \iota)$ by $(\mathcal{N}_1, \iota_1)$, then $\mathrm{HN}(\mathcal{N}_2, \iota_2)$ is the rest of $\mathrm{HN}(\mathcal{N}, \iota)$ after z, i.e.:

$$\mathrm{HN}(\mathcal{N}_2, \iota_2)\colon t \mapsto \mathrm{HN}(\mathcal{N}, \iota)(t + x) - y, \qquad t \in [0, \mathrm{ht}\, H/d - x].$$

A crucial question for our purpose is the following: when is $(\mathcal{N}_1, \iota_1)$ the filtered isocrystal with coefficients in F associated to an ι-stable sub-p-divisible group H_1 of H?

For instance, this is the case when H is "of HN type", that is, when $\mathrm{HN}(H) = \mathrm{HN}(H[p^i])$ for all $i \geq 1$ (cf [8, §2.3]). In this situation, the Harder-Narasimhan filtrations of the p-groups $H[p^i]$ build up to a filtration of H by sub-p-divisible groups, which are ι-stable by functoriality of the former filtrations. Taking the associated filtered isocrystals, we obtain the Harder-Narasimhan filtration of $\mathcal{N}$. More generally, there may exist a sub-p-divisible group H_1 of H satisfying our requirements, even without the p-groups $H_1[p^i]$ being necessarily part of the Harder-Narasimhan filtration of $H[p^i]$. The next proposition provides a sufficient condition in this sense, but let us first make some further observations.

Remark 3.1. First of all, the property required in the question characterises H_1 uniquely among the ι-stable sub-p-divisible groups of H. Indeed, if H_1' is another candidate, then the identity of $(\mathcal{N}_1, \iota_1)$ corresponds, by fully faithfulness of (2.6), to morphisms between H_1 and H_1' that are compatible with the inclusion in H, so we must have $H_1' = H_1$. Furthermore, the composition of functors $p{-}\mathrm{div}_{\mathcal{O}_K} \to p{-}\mathrm{div}_{\mathcal{O}_K} \otimes \mathbb{Q}_p \to \mathrm{FilIsoc}^{\text{w-a},[-1,0]}_K$ sends exact sequences of p-divisible groups to exact sequences of filtered isocrystals; this can be checked formally, considering that the target is an abelian category and that the induced maps on the hom-sets are just given by inverting p in finite free $\mathbb{Z}_p$-modules (recall that the second functor is an equivalence of categories, cf Remark 2.1). In particular, if H_2 denotes the quotient of H by H_1, with induced $\mathcal{O}_F$-action ι_2, then $(\mathcal{N}_2, \iota_2)$ is the filtered isocrystal with coefficients in F associated to (H_2, ι_2).

Suppose now that H_1 is an ι-stable sub-p-divisible group of H with the property that $(\mathrm{ht}\, H_1/d, \dim H_1/d) = z$ and denote by ι_1 the restriction of ι to H_1. Let $(\mathcal{N}', \iota')$ be the filtered isocrystal with coefficients in F associated to (H_1, ι_1); this is a subobject of $(\mathcal{N}, \iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$. Then, the Harder-Narasimhan polygon of $(\mathcal{N}', \iota')$ lies below $\mathrm{HN}(\mathcal{N}, \iota)$; moreover, its end point is z, so that $\mathrm{HN}(\mathcal{N}', \iota') \leq \mathrm{HN}(\mathcal{N}_1, \iota_1)$. This implies that the minimal slope of $\mathrm{HN}(\mathcal{N}', \iota')$ is at least that of $\mathrm{HN}(\mathcal{N}_1, \iota_1)$, which in turn is strictly greater than the maximal slope of $\mathrm{HN}(\mathcal{N}_2, \iota_2)$. By functoriality of the Harder-Narasimhan filtration, it follows that $(\mathcal{N}', \iota') \subseteq (\mathcal{N}_1, \iota_1)$. However, since the underlying isocrystals have the same height dx, this is in fact an equality. Thus, the condition that

$(\mathrm{ht}\, H_1/d, \dim H_1/d) = z$ is enough to ensure that $(\mathcal{N}_1, \iota_1)$ is the filtered isocrystal with coefficients in F associated to (H_1, ι_1).

Proposition 3.2. *Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and let $(\mathcal{N}, \iota)$ be its associated filtered isocrystal over K with coefficients in F. Suppose that $z = (x, y)$ is a break point of $\mathrm{HN}(H, \iota) = \mathrm{HN}(\mathcal{N}, \iota)$ and let $(\mathcal{N}_1, \iota_1) \subseteq (\mathcal{N}, \iota)$ be the corresponding subobject in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$. If z lies on $\mathrm{HN}(H[p], \iota)$, then there exists a unique ι-stable sub-p-divisible group H_1 of H whose associated filtered isocrystal with coefficients in F is $(\mathcal{N}_1, \iota_1)$.*

Proof. We first reduce to the following situation: either z is a break point of $\mathrm{HN}(H[p], \iota)$ or the slope of $\mathrm{HN}(H[p], \iota)$ at z is strictly greater than the first slope of $\mathrm{HN}(H, \iota)$ after z.

If none of these is the case, then, since $\mathrm{HN}(H, \iota) \leq \mathrm{HN}(H[p], \iota)$ by Proposition 2.8 and z is a break point of $\mathrm{HN}(H, \iota)$, the slope of $\mathrm{HN}(H[p], \iota)$ at z must be strictly less than the last slope of $\mathrm{HN}(H, \iota)$ before z. Consider then the dual p-divisible group $H^\vee$, with the dual action $\iota^\vee$ induced by ι. Recalling the compatibility of the Harder-Narasimhan polygon with respect to duality (cf Remark 2.9), we see that $(H^\vee, \iota^\vee)$ satisfies the assumptions of the proposition at the point $z^\vee = (x^\vee, y^\vee)$ given by:

$$\begin{aligned} x^\vee &= \mathrm{ht}\, H/d - x, \\ y^\vee &= \mathrm{ht}\, H/d - x + \mathrm{HN}(H, \iota)(x) - \dim H/d = \dim H^\vee/d - x + y. \end{aligned}$$

Moreover, the slope of $\mathrm{HN}(H[p]^\vee, \iota^\vee)$ at $z^\vee$ is strictly greater than the first slope of $\mathrm{HN}(H^\vee, \iota^\vee)$ after $z^\vee$, so we are in the situation described above.

Suppose now that we find an $\iota^\vee$-stable sub-p-divisible group H_1' of $H^\vee$ as claimed in the statement relative to $(H^\vee, \iota^\vee)$ and $z^\vee$. Then, $H_1 := (H^\vee/H_1')^\vee$ is an ι-stable sub-p-divisible group of H, with:

$$\mathrm{ht}\, H_1 = \mathrm{ht}\, H - \mathrm{ht}\, H_1' = \mathrm{ht}\, H - dx^\vee = dx$$

and:

$$\begin{aligned} \dim H_1 &= \mathrm{ht}\, H_1 - \dim H_1^\vee \\ &= dx - \dim H^\vee + \dim H_1' = dx - \dim H^\vee + dy^\vee = dy. \end{aligned}$$

By the previous remark, this is enough to prove the proposition for (H, ι). Thus, by possibly passing to the dual p-divisible group, we may assume that we are in the situation described above.

Let us now define a sequence of ι-stable closed sub-p-groups G_i of $H[p^i]$, for $i \geq 1$; these will form a first approximation of H_1.

Since $\mathrm{HN}(H, \iota)$ and $\mathrm{HN}(H[p], \iota)$ share the point z and they meet at their end point, there must be a break point of $\mathrm{HN}(H[p], \iota)$ at z or after (by the previous reduction step); let $z_1 = (x_1, y_1)$ be the first such point (possibly $z_1 = z$). We denote by μ the last slope of $\mathrm{HN}(H[p],\iota)$ before z_1, so that μ is strictly greater than the first slope of $\mathrm{HN}(H,\iota)$ after z. Recall that, by Proposition 2.8, we have:

$$\mathrm{HN}(H,\iota) \leq \mathrm{HN}^{\mathrm{r}}(H[p^i],\iota) \leq \mathrm{HN}(H[p],\iota)$$

for every $i \geq 1$. In particular, z lies on every polygon $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)$, which then has both a slope valued at least μ (before z) and a slope with value strictly less than μ (as it meets the other polygons at the end point). Let $z_i = (x_i, y_i)$ be the break point of $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)$ such that all its slopes before z_i are greater than or equal to μ and all its slopes after z_i are strictly less than μ; then, $x \leq x_i \leq x_1$ and z_i lies on $\mathrm{HN}(H[p],\iota)$, i.e. $\mathrm{HN}(H[p],\iota)(x_i) = y_i$, for all $i \geq 1$. Note that, because the polygons $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)$ converge uniformly to $\mathrm{HN}(H,\iota)$ as $i \to \infty$ by Proposition 2.8 again, we also have that $z = \lim_{i\to\infty} z_i$.

For $i \geq 1$, let $G_i \subseteq H[p^i]$ be the ι-stable closed sub-p-group corresponding to the break point z_i of $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)$; by definition, $\mathrm{ht}\, G_i = idx_i$ and $\deg G_i = idy_i$. For every other index $j > i$, the choice of z_i implies that the minimal slope of G_i is at least μ, which in turn, by the choice of z_j, is strictly greater than the maximal slope of $H[p^j]/G_j$. By functoriality of the Harder-Narasimhan filtration (see also [7, Proposition 8]), the closed embedding $H[p^i] \to H[p^j]$ restricts to $G_i \to G_j$; similarly, the map $p^i\colon H[p^j] \to H[p^{j-i}]$ restricts to $p^i\colon G_j \to G_{j-i}$.

We claim that, for $j > i \geq 1$, we have:

$$G_i = G_j[p^i] := \mathrm{Ker}\left(p^i\colon G_j \to G_{j-i}\right).$$

Consider the restriction $p^i_\eta\colon G_{j,\eta} \to G_{j-i,\eta}$ of the map $p^i\colon G_j \to G_{j-i}$ to the generic fibre (that is, its base change to K). Let $\mathcal{C}$ be the schematic closure in G_j of the kernel $\mathrm{Ker}(p^i_\eta)$ and let $\mathcal{D}$ be the schematic closure in G_{j-i} of the image $\mathrm{Im}(p^i_\eta)$. Note that $G_{i,\eta} \subseteq \mathrm{Ker}(p^i_\eta) \subseteq H[p^i]_\eta$, which gives the sequence of closed embeddings $G_i \subseteq \mathcal{C} \subseteq H[p^i]$ of p-groups over $\mathcal{O}_K$. Moreover, the map $p^i\colon G_j \to G_{j-i}$ factors through the closed sub-p-group $\mathcal{D} \subseteq G_{j-i}$ and we have a sequence:

$$0 \longrightarrow \mathcal{C} \xrightarrow{u} G_j \xrightarrow{p^i} \mathcal{D} \longrightarrow 0 \tag{3.1}$$

of p-groups over $\mathcal{O}_K$, with u a closed embedding, $p^i \circ u = 0$ and such that p^i induces an isomorphism $G_{j,\eta}/\mathcal{C}_\eta \xrightarrow{\sim} \mathcal{D}_\eta$ on the generic fibre, as $\mathcal{C}_\eta = \mathrm{Ker}(p^i_\eta)$ and $\mathcal{D}_\eta = \mathrm{Im}(p^i_\eta)$. Thus:

$$\mathrm{ht}\, G_j = \mathrm{ht}\, \mathcal{C} + \mathrm{ht}\, \mathcal{D} \tag{3.2}$$

and:

$$\deg G_j \leq \deg \mathcal{C} + \deg \mathcal{D}, \tag{3.3}$$

with equality if and only if the sequence (3.1) is exact (cf [7, Corollaire 3]). Now, because $\mathcal{C}$ and $\mathcal{D}$ are closed sub-p-groups of $H[p^i]$ and $H[p^{j-i}]$ respectively, we have:

$$\begin{aligned} \deg \mathcal{C} &\leq \mathrm{HN}(H[p^i])(\mathrm{ht}\, \mathcal{C}) = id\, \mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)(\mathrm{ht}\, \mathcal{C}/id) \\ &\leq id\, \mathrm{HN}(H[p],\iota)(\mathrm{ht}\, \mathcal{C}/id) \end{aligned} \tag{3.4}$$

and:

$$\begin{aligned}\deg\mathcal{D} &\leq \mathrm{HN}(H[p^{j-i}])(\operatorname{ht}\mathcal{D}) = (j-i)d\,\mathrm{HN}^{\mathrm{r}}(H[p^{j-i}],\iota)(\operatorname{ht}\mathcal{D}/(j-i)d)\\ &\leq (j-i)d\,\mathrm{HN}(H[p],\iota)(\operatorname{ht}\mathcal{D}/(j-i)d).\end{aligned}\tag{3.5}$$

Altogether, using the concavity of $\mathrm{HN}(H[p],\iota)$ and (3.2):

$$\begin{aligned}\deg\mathcal{C}+\deg\mathcal{D} &\leq jd\left(\frac{i}{j}\,\mathrm{HN}(H[p],\iota)(\operatorname{ht}\mathcal{C}/id)+\frac{j-i}{j}\,\mathrm{HN}(H[p],\iota)(\operatorname{ht}\mathcal{D}/(j-i)d)\right)\\ &\leq jd\,\mathrm{HN}(H[p],\iota)(\operatorname{ht}\mathcal{C}/jd+\operatorname{ht}\mathcal{D}/jd)\\ &= jd\,\mathrm{HN}(H[p],\iota)(\operatorname{ht}G_j/jd)\\ &= jd\,\mathrm{HN}(H[p],\iota)(x_j)\\ &= jdy_j = \deg G_j.\end{aligned}\tag{3.6}$$

Hence, we have equality in (3.3) and the sequence (3.1) is exact. In particular, $\mathcal{C} = G_j[p^i]$; we remark that this already implies the flatness of $G_j[p^i]$. Since G_i is a closed sub-p-group of $\mathcal{C}$, it suffices to show that $\operatorname{ht}\mathcal{C} \leq \operatorname{ht}G_i$ in order to conclude that $G_i = \mathcal{C} = G_j[p^i]$.

Note that, as a consequence of the previous argument, we have equality all over in (3.6) and hence in (3.4) and (3.5) as well. In particular, the polygon $\mathrm{HN}(H[p],\iota)$ is a straight line between $\operatorname{ht}\mathcal{D}/(j-i)d$ and $\operatorname{ht}\mathcal{C}/id$, with x_j lying in the interior of this segment, unless the three points coincide.

We first show that $\operatorname{ht}\mathcal{C}/id \leq x_1$. Assume by contradiction that $\operatorname{ht}\mathcal{C}/id > x_1$. Since $\mathcal{D}$ is a closed sub-p-group of G_{j-i}, we have:

$$\operatorname{ht}\mathcal{D}/(j-i)d \leq \operatorname{ht}G_{j-i}/(j-i)d = x_{j-i} \leq x_1 < \operatorname{ht}\mathcal{C}/id,$$

with $z_1 = (x_1, y_1)$ a break point of $\mathrm{HN}(H[p],\iota)$. But this polygon is a straight line on $[\operatorname{ht}\mathcal{D}/(j-i)d, \operatorname{ht}\mathcal{C}/id]$, so we must have $\operatorname{ht}\mathcal{D}/(j-i)d = x_1$. However, this would contradict x_j being in the interior of the mentioned segment, as $x_j \leq x_1$.

We can now show that $\operatorname{ht}\mathcal{C} \leq \operatorname{ht}G_i$ or, equivalently, that $\operatorname{ht}\mathcal{C}/id \leq x_i$. Assume by contradiction that $\operatorname{ht}\mathcal{C}/id > x_i$. The polygons $\mathrm{HN}(H[p],\iota)$ and $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)$ share the point $z_i = (x_i, y_i)$; however, the slope of $\mathrm{HN}(H[p],\iota)$ on $[x_i, \operatorname{ht}\mathcal{C}/id]$ is at least μ (as $\operatorname{ht}\mathcal{C}/id \leq x_1$), whereas the slope of $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)$ on the same segment is strictly less than μ (by definition of z_i). Therefore, $\mathrm{HN}^{\mathrm{r}}(H[p^i],\iota)(\operatorname{ht}\mathcal{C}/id) < \mathrm{HN}(H[p],\iota)(\operatorname{ht}\mathcal{C}/id)$, contradicting the fact that we have an equality in (3.4). This proves that $\operatorname{ht}\mathcal{C} \leq \operatorname{ht}G_i$ and hence that $G_i = \mathcal{C} = G_j[p^i]$ as claimed.

We now proceed to refining the sequence of p-groups G_i, $i \geq 1$, to an ι-stable sub-p-divisible group $H_1 = (K_i)_{i\geq 1}$ of H, with $(\operatorname{ht}H_1/d, \dim H_1/d) = z$. Given the observations of Remark 3.1, this is sufficient in order to conclude the proof of the proposition.

By the previous step, multiplication by p induces closed embeddings:

$$p\colon G_{i+1}/G_i \longrightarrow G_i/G_{i-1}$$

for $i > 1$. Thus, the sequence of numbers $a_i := \operatorname{ht} G_{i+1}/G_i = \operatorname{ht} G_{i+1} - \operatorname{ht} G_i$, $i \geq 1$, is nonincreasing; since we are talking about natural numbers, there exists an index $i_0 \geq 1$ such that $a_i = a_{i_0}$ for every $i \geq i_0$. We obtain the following formula:

$$\operatorname{ht} G_i = \operatorname{ht} G_{i_0} + (i - i_0)a_{i_0} = ia_{i_0} + \operatorname{ht} G_{i_0} - i_0 a_{i_0}$$

for every $i \geq i_0$. Then:

$$x_i = \frac{\operatorname{ht} G_i}{id} = \frac{a_{i_0}}{d} + \frac{\operatorname{ht} G_{i_0} - i_0 a_{i_0}}{id} \xrightarrow[i\to\infty]{} \frac{a_{i_0}}{d}.$$

At the same time, we already observed that the sequence of points $z_i = (x_i, y_i)$, $i \geq 1$, converges to $z = (x, y)$ as $i \to \infty$; hence, $a_{i_0}/d = x$.

For $i \geq 1$, we define the following p-group over $\mathcal{O}_K$:

$$K_i := G_{i+i_0}/G_{i_0}.$$

Note that:

$$\operatorname{ht} K_i = \operatorname{ht} G_{i+i_0} - \operatorname{ht} G_{i_0} = ia_{i_0} = idx.$$

Moreover, because $x_{i_0}, x_{i+i_0} \in [x, x_1]$ and, unless $x = x_1$, the polygon $\mathrm{HN}(H[p], \iota)$ is a straight line (of slope μ) on this segment, we also have:

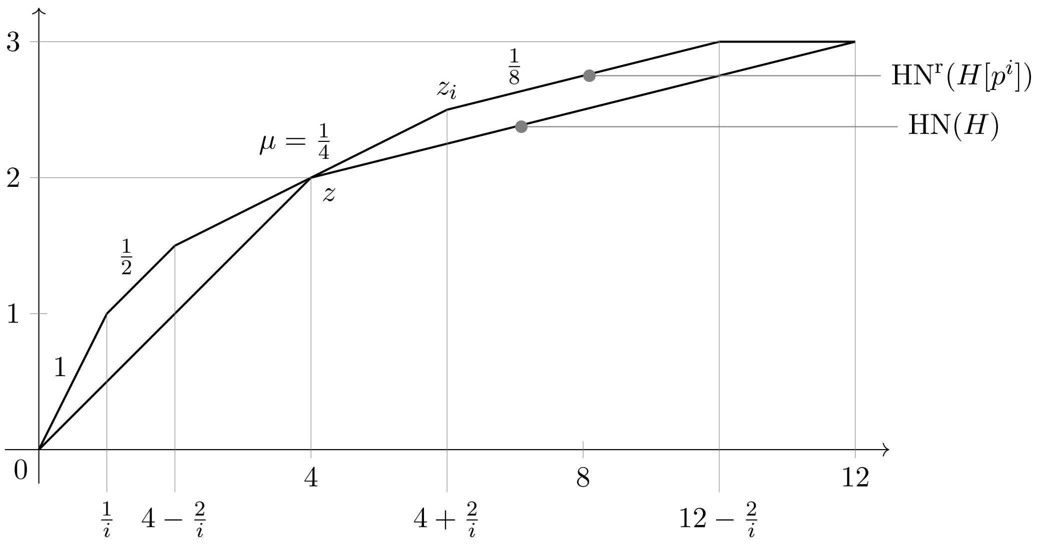
$$\begin{aligned}
\deg K_i &= \deg G_{i+i_0} - \deg G_{i_0} = (i + i_0)dy_{i+i_0} - i_0 dy_{i_0} \\
&= id\left(\frac{i+i_0}{i}\mathrm{HN}(H[p],\iota)(x_{i+i_0}) - \frac{i_0}{i}\mathrm{HN}(H[p],\iota)(x_{i_0})\right) \\
&= id\,\mathrm{HN}(H[p],\iota)\left(\frac{i+i_0}{i}x_{i+i_0} - \frac{i_0}{i}x_{i_0}\right) \\
&= id\,\mathrm{HN}(H[p],\iota)(x) = idy.
\end{aligned}$$

For $j > i \geq 1$, we have obvious closed embeddings $K_i \to K_j$ which induce the identification $K_i = K_j[p^i]$. Thus, the family $(K_i)_{i\geq 1}$ of p-groups over $\mathcal{O}_K$ is a p-divisible group H_1 of height dx and dimension dy, with $H_1[p^i] = K_i$. Finally, for $i \geq 1$, the map:

$$p^{i_0}\colon K_i = G_{i+i_0}/G_{i_0} \longrightarrow G_i \subseteq H[p^i]$$

is a closed embedding, which defines an ι-stable closed sub-p-group of $H[p^i]$. Therefore, H_1 is an ι-stable sub-p-divisible group of H, with $(\operatorname{ht} H_1/d, \dim H_1/d) = (x, y) = z$. This concludes the proof of the proposition. □

Remark 3.3. The proof of the previous proposition follows a similar procedure to the first step of the algorithm in [8, §3], with the difference that here we "jump" to a given break point z of the Harder-Narasimhan polygon. The crucial point is to obtain a family of closed sub-p-groups G_i of $H[p^i]$, with the property that $G_i = G_j[p^i]$ for $j > i \geq 1$. In order to show that this property holds for the constructed family, we used a similar argument to that in the proof of [28, 5.4]. Here, we also deal with the case that the polygon $\mathrm{HN}^{\mathrm{r}}(H[p^i])$ does not have a break point at z for any index $i \geq 1$, as for instance in the following configuration (for e even).



The previous proposition has its own relevance for the study of p-divisible groups over $\mathcal{O}_K$ via Harder-Narasimhan theory. Let us make this more explicit by reformulating it in a more self-contained fashion.

Corollary 3.4. *Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. Suppose that $z = (x, y)$ is a break point of $\mathrm{HN}(H, \iota)$ which also lies on $\mathrm{HN}(H[p], \iota)$. Then, there exists a unique ι-stable sub-p-divisible group H_1 of H such that, if ι_1 denotes the restriction of ι to H_1, then $\mathrm{HN}(H_1, \iota_1)$ equals the restriction of $\mathrm{HN}(H, \iota)$ to $[0, x]$. Furthermore, if H_2 denotes the quotient of H by H_1, with induced $\mathcal{O}_F$-action ι_2, then $\mathrm{HN}(H_2, \iota_2)$ equals the rest of $\mathrm{HN}(H, \iota)$ after z, i.e.:*

$$\mathrm{HN}(H_2, \iota_2)\colon t \mapsto \mathrm{HN}(H, \iota)(t + x) - y, \qquad t \in [0, \mathrm{ht}\, H/d - x].$$

3.2 Hodge-Newton reducible filtered isocrystals

Let us now introduce the Hodge-Newton reducibility hypothesis in the discussion. This hypothesis concerns the Hodge and the Newton polygon and can therefore by formulated at the level of filtered isocrystals. We first analyse its effects in this setting, following the same argument as in [28, §5.1].

Definition 3.5. A weakly admissible filtered isocrystal $(\mathcal{N}, \iota)$ over K with coefficients in F is *Hodge-Newton reducible* if there exists a break point of $\mathrm{Newt}(\mathcal{N}, \iota)$ which also lies on $\mathrm{Hdg}(\mathcal{N}, \iota)$. If z is such a point, we say that $(\mathcal{N}, \iota)$ is Hodge-Newton reducible *at z*.

Proposition 3.6. *Let $(\mathcal{N}, \iota)$ be a weakly admissible filtered isocrystal over K with coefficients in F and suppose that $(\mathcal{N}, \iota)$ is Hodge-Newton reducible at $z = (x, y)$. Then, z is also a break point of the Harder-Narasimhan polygon* $\mathrm{HN}(\mathcal{N}, \iota)$. *Furthermore, if* $(\mathcal{N}_1, \iota_1)$ *denotes the corresponding subobject in* $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$, *then* $\mathrm{Newt}(\mathcal{N}_1, \iota_1)$ *and* $\mathrm{Hdg}(\mathcal{N}_1, \iota_1)$ *equal respectively the restriction of* $\mathrm{Newt}(\mathcal{N}, \iota)$ *and* $\mathrm{Hdg}(\mathcal{N}, \iota)$ *to* $[0, x]$. *If* $(\mathcal{N}_2, \iota_2)$ *denotes the quotient of* $(\mathcal{N}, \iota)$ *by* $(\mathcal{N}_1, \iota_1)$, *then* $\mathrm{Newt}(\mathcal{N}_2, \iota_2)$ *and* $\mathrm{Hdg}(\mathcal{N}_2, \iota_2)$ *equal respectively the rest of* $\mathrm{Newt}(\mathcal{N}, \iota)$ *and* $\mathrm{Hdg}(\mathcal{N}, \iota)$ *after* z.

Proof. Write $\mathcal{N} = (N, \varphi, \mathrm{Fil}^\bullet N_K)$. Since z is a break point of $\mathrm{Newt}(\mathcal{N}, \iota) = \mathrm{Newt}(N, \varphi, \iota)$, we have a decomposition:

$$(N, \varphi, \iota) = (N_1, \varphi_1, \iota_1) \oplus (N_2, \varphi_2, \iota_2)$$

in $\mathrm{Isoc}(k)_F$, with $\mathrm{Newt}(N_1, \varphi_1, \iota_1)$ equal to the restriction of $\mathrm{Newt}(\mathcal{N}, \iota)$ to $[0, x]$ and $\mathrm{Newt}(N_2, \varphi_2, \iota_2)$ equal to the rest of $\mathrm{Newt}(\mathcal{N}, \iota)$ after z; in particular, $\dim_{K_0} N_1 = \mathrm{ht}(N_1, \varphi_1) = dx$.

We endow $N_{1,K}$ with the induced filtration $\mathrm{Fil}^\bullet N_{1,K} := N_{1,K} \cap \mathrm{Fil}^\bullet N_K$, so that $\mathcal{N}_1 := (N_1, \varphi_1, \mathrm{Fil}^\bullet N_{1,K})$ is a sub-filtered-isocrystal of $\mathcal{N}$. Since ι_1 respects this filtration, we obtain a subobject $(\mathcal{N}_1, \iota_1)$ of $(\mathcal{N}, \iota)$ in $\mathrm{FilIsoc}_{K,F}$. Let then $(\mathcal{N}_2, \iota_2) \in \mathrm{FilIsoc}_{K,F}$ be the quotient of $(\mathcal{N}, \iota)$ by $(\mathcal{N}_1, \iota_1)$ and note that the underlying isocrystal with coefficients in F of $(\mathcal{N}_2, \iota_2)$ identifies with $(N_2, \varphi_2, \iota_2)$. Thus, $\mathrm{Newt}(\mathcal{N}_1, \iota_1) = \mathrm{Newt}(N_1, \varphi_1, \iota_1)$ equals the restriction of $\mathrm{Newt}(\mathcal{N}, \iota)$ to $[0, x]$ and $\mathrm{Newt}(\mathcal{N}_2, \iota_2) = \mathrm{Newt}(N_2, \varphi_2, \iota_2)$ equals the rest of $\mathrm{Newt}(\mathcal{N}, \iota)$ after z.

We claim that $(\mathcal{N}_1, \iota_1)$ is weakly admissible; note that this implies that $(\mathcal{N}_2, \iota_2)$ is weakly admissible too. In addition, the following argument allows us to find the Hodge polygon of these two objects.

Since $\mathcal{N}_1$ is a sub-filtered-isocrystal of $\mathcal{N}$, which is weakly admissible, we have that $t_H(\mathcal{N}_1) \leq t_N(\mathcal{N}_1)$ and only need to check the opposite inequality. Pick a field extension K' of K containing all embeddings τ of F in an algebraic closure of K and let:

$$N_{K'} = \bigoplus_{\tau\colon F \to K'} N_\tau, \qquad \mathrm{Fil}^\bullet N_{K'} = \bigoplus_{\tau\colon F \to K'} \mathrm{Fil}^\bullet N_\tau,$$
$$N_{1,K'} = \bigoplus_{\tau\colon F \to K'} N_{1,\tau}, \qquad \mathrm{Fil}^\bullet N_{1,K'} = \bigoplus_{\tau\colon F \to K'} \mathrm{Fil}^\bullet N_{1,\tau}$$

be the decompositions as in (1.5). Note that $N_{1,\tau} = N_{1,K'} \cap N_\tau$ and $\mathrm{Fil}^\bullet N_{1,\tau} = N_{1,\tau} \cap \mathrm{Fil}^\bullet N_\tau$, so $(N_{1,\tau}, \mathrm{Fil}^\bullet N_{1,\tau})$ is a subobject of $(N_\tau, \mathrm{Fil}^\bullet N_\tau)$ in $\mathrm{FilVect}_{K'|K'}$, with $\dim_{K'} N_{1,\tau} = \dim_{K_0} N_1/d = x$. By Lemma 1.5, then:

$$-\deg(N_{1,\tau}, \mathrm{Fil}^\bullet N_{1,\tau}) \leq f_\tau(x) \tag{3.7}$$

for every τ, where f_τ is the type of $(N_\tau, \mathrm{Fil}^\bullet N_\tau)$. Thus, using that $z = (x, y)$ also lies on $\mathrm{Hdg}(\mathcal{N},\iota)$:

$$\begin{aligned} t_N(\mathcal{N}_1) &= -d\,\mathrm{Newt}(\mathcal{N}_1,\iota_1)(x) = -dy \\ &= -d\,\mathrm{Hdg}(\mathcal{N},\iota)(x) = -\sum_\tau f_\tau(x) \\ &\le \sum_\tau \deg(N_{1,\tau},\mathrm{Fil}^\bullet N_{1,\tau}) \\ &= \deg(N_1,\mathrm{Fil}^\bullet N_{1,K}) = t_H(\mathcal{N}_1). \end{aligned} \tag{3.8}$$

This proves the claim.

As a consequence, we have an equality in (3.8) and hence in (3.7) for every τ. By Lemma 1.5 again, the type of $(N_{1,\tau},\mathrm{Fil}^\bullet N_{1,\tau})$ equals then the restriction of f_τ to $[0,x]$. Summing over τ and dividing by d, we obtain that $\mathrm{Hdg}(\mathcal{N}_1,\iota_1)$ equals the restriction of $\mathrm{Hdg}(\mathcal{N},\iota)$ to $[0,x]$.

Denote now by $\mathrm{Fil}^\bullet N_{2,K}$ the filtration of $\mathcal{N}_2$ and let:

$$N_{2,K'} = \bigoplus_{\tau\colon F\to K'} N_{2,\tau}, \qquad \mathrm{Fil}^\bullet N_{2,K'} = \bigoplus_{\tau\colon F\to K'} \mathrm{Fil}^\bullet N_{2,\tau}$$

be the decomposition as in (1.5), with respect to the F-action given by ι_2. Note that $(N_{2,\tau},\mathrm{Fil}^\bullet N_{2,\tau})$ is the quotient of $(N_\tau,\mathrm{Fil}^\bullet N_\tau)$ by $(N_{1,\tau},\mathrm{Fil}^\bullet N_{1,\tau})$ in $\mathrm{FilVect}_{K'|K'}$ for every τ. Since we already know that the type of $(N_{1,\tau},\mathrm{Fil}^\bullet N_{1,\tau})$ equals the restriction of $f_\tau = f(N_\tau,\mathrm{Fil}^\bullet N_\tau)$ to $[0,x]$, it follows from the behaviour of the type in short exact sequences that $f(N_{2,\tau},\mathrm{Fil}^\bullet N_{2,\tau})$ equals the rest of f_τ after $(x,f_\tau(x))$. Summing over τ and dividing by d, we obtain that $\mathrm{Hdg}(\mathcal{N}_2,\iota_2)$ equals the rest of $\mathrm{Hdg}(\mathcal{N},\iota)$ after z.

We can finally show that z is a break point of $\mathrm{HN}(\mathcal{N},\iota)$ and that $(\mathcal{N}_1,\iota_1)$ is the corresponding subobject of $(\mathcal{N},\iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$. This is enough to finish the proof of the proposition, as we already know that the Newton polygon and the Hodge polygon of $(\mathcal{N}_1,\iota_1)$ and $(\mathcal{N}_2,\iota_2)=(\mathcal{N},\iota)/(\mathcal{N}_1,\iota_1)$ are as claimed in the statement.

Because $(\mathcal{N}_1,\iota_1)$ is a subobject of $(\mathcal{N},\iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$, we have:

$$-t_N(\mathcal{N}_1)/d \le \mathrm{HN}(\mathcal{N},\iota)(\mathrm{ht}(N_1,\varphi_1)/d) = \mathrm{HN}(\mathcal{N},\iota)(x).$$

Now, on the one hand:

$$-t_N(\mathcal{N}_1)/d = \mathrm{Newt}(\mathcal{N}_1,\iota_1)(x) = \mathrm{Newt}(\mathcal{N},\iota)(x) = y;$$

on the other hand, by Proposition 1.12:

$$\mathrm{HN}(\mathcal{N},\iota)(x) \le \mathrm{Newt}(\mathcal{N},\iota)(x) = y.$$

Thus, equality holds and, in particular, $z=(x,y)$ lies on $\mathrm{HN}(\mathcal{N},\iota)$. But z is a break point of $\mathrm{Newt}(\mathcal{N},\iota)$ and, by Proposition 1.12 again, $\mathrm{HN}(\mathcal{N},\iota)\le \mathrm{Newt}(\mathcal{N},\iota)$. Hence, z is break point of $\mathrm{HN}(\mathcal{N},\iota)$ as well.

Lastly, note that $(\mathcal{N}_1,\iota_1)$ is a subobject of $(\mathcal{N},\iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$ with the property that $(\mathrm{ht}(N_1,\varphi_1)/d, -t_N(\mathcal{N}_1)/d) = z$. By functoriality of the Harder-Narasimhan filtration (see also the argument in Remark 3.1), it follows that $(\mathcal{N}_1,\iota_1)$ is the subobject of $(\mathcal{N},\iota)$ corresponding to z. This concludes the proof of the proposition. □

3.3 Hodge-Newton reducible p-divisible groups

Definition 3.7. A p-divisible group (H, ι) over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ is *Hodge-Newton reducible* if its associated filtered isocrystal over K with coefficients in F is Hodge-Newton reducible, i.e. there exists a break point of $\mathrm{Newt}(H, \iota)$ which also lies on $\mathrm{Hdg}(H, \iota)$. If z is such a point, we say that (H, ι) is Hodge-Newton reducible *at* z.

The Hodge-Newton reducibility is an assumption that regards the whole equivariant isogeny class of a p-divisible group (H, ι) over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$. Using Propositions 2.8 and 2.13, we deduce a constraint on $\mathrm{HN}(H[p], \iota)$, hence concerning (H, ι) itself. We can then apply Proposition 3.2 and obtain the following theorem.

Theorem 3.8. *Assume that the ramification index of F over $\mathbb{Q}_p$ is at most 2. Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and suppose that (H, ι) is Hodge-Newton reducible at $z = (x, y)$. Then, there exists a unique ι-stable sub-p-divisible group H_1 of H such that, if ι_1 denotes the restriction of ι to H_1, then $\mathrm{Newt}(H_1, \iota_1)$, $\mathrm{Hdg}(H_1, \iota_1)$ and $\mathrm{HN}(H_1, \iota_1)$ equal respectively the restriction of $\mathrm{Newt}(H, \iota)$, $\mathrm{Hdg}(H, \iota)$ and $\mathrm{HN}(H, \iota)$ to $[0, x]$. Furthermore, if H_2 denotes the quotient of H by H_1, with induced $\mathcal{O}_F$-action ι_2, then $\mathrm{Newt}(H_2, \iota_2)$, $\mathrm{Hdg}(H_2, \iota_2)$ and $\mathrm{HN}(H_2, \iota_2)$ equal respectively the rest of $\mathrm{Newt}(H, \iota)$, $\mathrm{Hdg}(H, \iota)$ and $\mathrm{HN}(H, \iota)$ after z.*

Proof. Let $(\mathcal{N}, \iota)$ be the filtered isocrystal with coefficients in F associated to (H, ι). By Proposition 3.6, z is a break point of $\mathrm{HN}(\mathcal{N}, \iota) = \mathrm{HN}(H, \iota)$. Let then $(\mathcal{N}_1, \iota_1)$ be the corresponding subobject of $(\mathcal{N}, \iota)$ in $\mathrm{FilIsoc}^{\text{w-a}}_{K,F}$ and let $(\mathcal{N}_2, \iota_2)$ be the quotient of $(\mathcal{N}, \iota)$ by $(\mathcal{N}_1, \iota_1)$.

By Propositions 2.8 and 2.13, we have:

$$\mathrm{HN}(H, \iota) \leq \mathrm{HN}(H[p], \iota) \leq \mathrm{Hdg}(H, \iota).$$

Since z lies on both $\mathrm{HN}(H, \iota)$ and $\mathrm{Hdg}(H, \iota)$, this implies that z also lies on $\mathrm{HN}(H[p], \iota)$. Then, by Proposition 3.2, there exists a unique ι-stable sub-p-divisible group H_1 of H whose associated filtered isocrystal with coefficients in F is $(\mathcal{N}_1, \iota_1)$. Furthermore, if H_2 denotes the quotient of H by H_1, with induced $\mathcal{O}_F$-action ι_2, then $(\mathcal{N}_2, \iota_2)$ is the filtered isocrystal with coefficients in F associated to (H_2, ι_2). Denoting by ι_1 the restriction of ι to H_1, it follows then from Proposition 3.6 that the polygons of (H_1, ι_1) and (H_2, ι_2) are as claimed.

As for uniqueness, the prescription on the Harder-Narasimhan polygon of (H_1, ι_1) ensures that its associated filtered isocrystal with coefficients in F is the $(\mathcal{N}_1, \iota_1)$ considered above. The uniqueness of H_1 follows then from Proposition 3.2. □

Remark 3.9. The reduction $H_{1,k} \subseteq H_k$ to k of the filtration obtained above is part of the slope filtration of H_k from [32, Corollary 13]. This is immediate from the configuration of the Newton polygons stated in the theorem; in fact, these polygons are really an invariant of the reduction of the respective p-divisible group and they are only affected by ι in terms of a rescaling factor. If F is unramified over $\mathbb{Q}_p$, then $H_{1,k}$ is one piece of the Hodge-Newton decomposition of H_k from [20, Corollary 7] (using Remark 2.3 to compare our definition of the Hodge polygon with that in loc. cit.); in particular, the filtration $H_{1,k} \subseteq H_k$ is split. We ignore, at present time, whether this splitting holds more generally in our setup.

3.4 The polarised case

Assume here that F carries a field involution $(\ )^*\colon F \to F$, possibly equal to the identity of F. For $n \in \mathbb{N}$ and an element $f = (a_i)_{i=1}^n \in \mathbb{Q}_+^n$ of the Newton set, we define its *dual* element to be $f^\vee := (1 - a_{n+1-i})_{i=1}^n \in \mathbb{Q}_+^n$ or, as a polygon:

$$f^\vee\colon x \mapsto x + f(n-x) - f(n).$$

We clearly have $f^{\vee\vee} = f$. If $f \in \mathbb{Q}_+^n$ satisfies $f = f^\vee$, we say that f is *symmetric*.

For R a complete Noetherian commutative local ring with residue field of characteristic p and (H, ι) a p-divisible group over R with endomorphism structure for $\mathcal{O}_F$, we have an induced $\mathcal{O}_F$-action $\iota^\vee\colon \mathcal{O}_F \to \mathrm{End}(H^\vee)$ on the dual p-divisible group $H^\vee$. Denote by $\iota^{\vee,*}$ the composition $\iota^\vee \circ (\ )^*$. Then, $(H^\vee, \iota^{\vee,*})$ is a new p-divisible group with endomorphism structure for $\mathcal{O}_F$.

When $R = \mathcal{O}_K$, Remark 2.9 shows that the Harder-Narasimhan polygon of $(H^\vee, \iota^{\vee,*})$ is the dual of $\mathrm{HN}(H, \iota) \in \mathbb{Q}_+^n$, where $n = \mathrm{ht}\, H/d$ (and recall that $\mathrm{HN}(H,\iota)(n) = \dim H/d$). In fact, the twist introduced by $(\ )^*$ in the endomorphism structure does not really affect the Harder-Narasimhan polygon, which depends on the $\mathcal{O}_F$-action only by means of the rescaling factor $1/d = 1/[F : \mathbb{Q}_p]$. We will now see that the other polygons behave in the same way with respect to duality.

Isocrystals and duality. Let (N, φ) be an isocrystal over k. The *dual isocrystal* is given by the dual K_0-vector-space $N^\vee := \mathrm{Hom}_{K_0}(N, K_0)$, together with the σ-linear endomorphism $\varphi^\vee\colon f \mapsto \sigma \circ f \circ V$, where $V = p\varphi^{-1}$ is the Verschiebung map of (N, φ). If (N, φ) is isotypical of slope $\lambda = r/s$, then $(N^\vee, \varphi^\vee)$ is isotypical of slope $1 - \lambda = (s-r)/r$. Indeed, if $M \subseteq N$ is a $W(k)$-lattice with $\varphi^s M = p^r M$, then the $W(k)$ lattice $M^\vee := \{\, f \in N^\vee \mid f(M) \subseteq W(k) \,\} \subseteq N^\vee$ satisfies $\varphi^{s-r} M^\vee = p^r M^\vee$. In general, if the Newton slopes of (N, φ) are $\lambda_1 < \cdots < \lambda_m$, then those of $(N^\vee, \varphi^\vee)$ will be $1 - \lambda_m < \cdots < 1 - \lambda_1$, with the height of the isotypical component associated to $1 - \lambda_i$ equal to that of the isotypical component of (N, φ) associated to λ_i, for $1 \le i \le m$.

The Newton polygon and duality. Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$ and recall that $\mathrm{Newt}(H, \iota)$ is defined to be the Newton polygon of the isocrystal $(\mathbb{D}(H_k) \otimes_{W(k)} K_0, p^{-1}(\varphi_{H_k} \otimes \mathrm{id}))$ over k, with the induced F-action, where $(\mathbb{D}(H_k), \varphi_{H_k})$ is the Dieudonné module of the reduction H_k of H to k. Now, if $(\mathbb{D}(H_k^\vee), \varphi_{H_k^\vee})$ is the Dieudonné module of the dual p-divisible group $H_k^\vee$ over k, then the isocrystal $((\mathbb{D}(H_k^\vee) \otimes_{W(k)} K_0, (\varphi_{H_k^\vee} \otimes \mathrm{id}))$ identifies naturally with the dual of $((\mathbb{D}(H_k) \otimes_{W(k)} K_0, (\varphi_{H_k} \otimes \mathrm{id}))$, as defined in the previous paragraph (cf [9, §III, Proposition 6.4]). Thus, shifting the slopes by -1 and computing the Newton polygons (including the rescaling due to the endomorphism structure), we obtain that $\mathrm{Newt}(H^\vee, \iota^{\vee,*}) = \mathrm{Newt}(H, \iota)^\vee \in \mathbb{Q}_+^n$, where $n = \mathrm{ht}\, H/d$ (and recall that $\mathrm{Newt}(H, \iota)(n) = \dim H/d$).

The Hodge polygon and duality. Let (H,ι) be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, pick a field extension K' of K containing all embeddings τ of F in an algebraic closure of K and consider the exact sequence of K'-vector-spaces:

$$0 \longrightarrow \omega_{H^\vee,K'} \longrightarrow M(H)_{K'} \longrightarrow \mathrm{Lie}(H)_{K'} \longrightarrow 0,$$

obtained as the base change of (2.3) along $\mathcal{O}_K \to K'$. This sequence carries a $\mathbb{Q}_p$-linear F-action induced by ι and splits as a direct sum of exact sequences:

$$0 \longrightarrow \omega_{H^\vee,K',\tau} \longrightarrow M(H)_{K',\tau} \longrightarrow \mathrm{Lie}(H)_{K',\tau} \longrightarrow 0 \tag{3.9}$$

indexed by the embeddings τ of F in K', with F acting through $\tau\colon F \to K'$ on the respective component. Recall that $\mathrm{Hdg}(H,\iota)$ is defined to be the average, over all τ's, of the types of the filtered vector spaces $(M(H)_{K',\tau}, \mathrm{Fil}^\bullet M(H)_{K',\tau})$ given by $\mathrm{Fil}^0 M(H)_{K',\tau} = \omega_{H^\vee,K',\tau}$, with $\mathrm{Fil}^{-i} M(H)_{K',\tau} = M(H)_{K',\tau}$ and $\mathrm{Fil}^i M(H)_{K',\tau} = 0$ for $i \geq 1$. Remember now that we have a natural identification $\omega_H \cong \mathrm{Hom}_{\mathcal{O}_K}(\mathrm{Lie}(H), \mathcal{O}_K)$ of finite free $\mathcal{O}_K$-modules, which, after base change along $\mathcal{O}_K \to K'$, gives $\omega_{H,K'} \cong \mathrm{Hom}_{K'}(\mathrm{Lie}(H)_{K'}, K')$. By compatibility with respect to the F-action induced by ι on both sides, this splits as a direct sum of isomorphisms:

$$\omega_{H,K',\tau} \cong \mathrm{Hom}_{K'}(\mathrm{Lie}(H)_{K'}, K')_\tau = \mathrm{Hom}_{K'}(\mathrm{Lie}(H)_{K',\tau}, K')$$

indexed by τ as above and with F acting through $\tau\colon F \to K'$ on the respective component. Twisting the F-action on $\omega_{H,K'}$ by $(\ )^*$ and using the exactness of (3.9), we get that:

$$\begin{aligned}\dim_{K'} \omega_{H,K',\tau\circ(\ )^*} &= \dim_{K'} \mathrm{Hom}_{K'}(\mathrm{Lie}(H)_{K',\tau}, K') = \dim_{K'} \mathrm{Lie}(H)_{K',\tau} \\ &= \dim_{K'} M(H)_{K',\tau} - \dim_{K'} \omega_{H^\vee,K',\tau} = \mathrm{ht}\, H/d - \dim_{K'} \omega_{H^\vee,K',\tau}\end{aligned}$$

for all τ's. Then, computing the relevant types and averaging over τ (clearing thus the influence of the twist), we obtain that $\mathrm{Hdg}(H^\vee, \iota^{\vee,*}) = \mathrm{Hdg}(H,\iota)^\vee \in \mathbb{Q}_+^n$, where $n = \mathrm{ht}\, H/d$ (and recall that $\mathrm{Hdg}(H,\iota)(n) = \dim H/d$).

Definition 3.10. Let R be a complete Noetherian commutative local ring with residue field of characteristic p, let (H,ι) be a p-divisible group over R with endomorphism structure for $\mathcal{O}_F$ and consider the $\mathcal{O}_F$-action given by $\iota^{\vee,*} = \iota^\vee \circ (\ )^*$ on the dual p-divisible group $H^\vee$. A *polarisation* on (H,ι) is an $\mathcal{O}_F$-equivariant isomorphism $\lambda\colon H \to H^\vee$ such that, under the natural identification $H^{\vee\vee} \cong H$, we have $\lambda^\vee = -\lambda$ (i.e. λ is *antisymmetric*). We call (H,ι,λ) a *polarised* p-divisible group over R with endomorphism structure for $\mathcal{O}_F$.

If (H,ι,λ) is a polarised p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, then the isomorphism $(H,\iota) \cong (H^\vee, \iota^{\vee,*})$ given by the polarisation λ implies that the Newton polygon, the Hodge polygon and the Harder-Narasimhan polygon of (H,ι) coincide with the respective polygons of $(H^\vee, \iota^{\vee,*})$. According to the compatibility relations found above, this means that these polygons are symmetric elements of $\mathbb{Q}_+^n$, where $n = \mathrm{ht}\, H/d$. In particular, if (H,ι) is Hodge-Newton reducible at $z = (x,y)$, then it is also Hodge-Newton reducible at the symmetric point $z^\vee = (x^\vee, y^\vee)$ given by:

$$x^\vee = \mathrm{ht}\, H/d - x, \qquad y^\vee = \dim H/d - x + y. \tag{3.10}$$

Thus, applying Theorem 3.8 for both points and using the uniqueness property, we find the following corollary.

Corollary 3.11. *Assume that the ramification index of F over $\mathbb{Q}_p$ is at most 2. Let (H, ι, λ) be a polarised p-divisible group over $\mathcal{O}_K$ with endomorphism structure for $\mathcal{O}_F$, suppose that (H, ι) is Hodge-Newton reducible at $z = (x, y)$ and let $z^\vee = (x^\vee, y^\vee)$ be the symmetric point as in (3.10). Assume without loss of generality that $x \leq x^\vee$. Then, there exists a unique filtration of (H, ι) by sub-p-divisible groups with endomorphism structure for $\mathcal{O}_F$:*

$$(H_1, \iota_1) \subseteq (H'_1, \iota'_1) \subseteq (H, \iota)$$

satisfying the following property: if H_2 denotes the quotient of H'_1 by H_1, with induced $\mathcal{O}_F$-action ι_2, and H_3 denotes the quotient of H by H'_1, with induced $\mathcal{O}_F$-action ι_3, then $\mathrm{Newt}(H_i, \iota_i)$, $\mathrm{Hdg}(H_i, \iota_i)$ and $\mathrm{HN}(H_i, \iota_i)$ equal respectively the parts of $\mathrm{Newt}(H, \iota)$, $\mathrm{Hdg}(H, \iota)$ and $\mathrm{HN}(H, \iota)$ between the origin and z if $i = 1$, between z and $z^\vee$ if $i = 2$ and between $z^\vee$ and $(\mathrm{ht}\, H/d, \dim H/d)$ if $i = 3$. Furthermore, if H'_2 denotes the quotient of H by H_1, endowed with the induced $\mathcal{O}_F$-action, then λ induces $\mathcal{O}_F$-equivariant isomorphisms:

$$H_3 \cong H_1^\vee \qquad \text{and} \qquad H'_2 \cong H_1'^\vee,$$

where $H_1^\vee$ and $H_1'^\vee$ carry the $\mathcal{O}_F$-action given respectively by $\iota_1^{\vee,*}$ and $\iota_1'^{\vee,*}$. In addition, λ induces a polarisation on (H_2, ι_2).

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