Johannes Oertel

Pair production in spacetime-dependent fields using WKB-like techniques

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Pair production in spacetime-dependent fields using WKB-like techniques

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Dissertation

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von
Johannes Oertel
aus Freiberg

Gutachter:
Prof. Dr. Ralf Schützhold,
Prof. Dr. Dr. Carsten Müller,
Prof. Dr. Dr. Klaus Hornberger

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Abstract

The WKB method, well-known from quantum mechanics, has been sucessfully applied to the problem of non-perturbative pair production in time-dependent background fields, such as in cosmological particle creation or in the Sauter-Schwinger effect, i.e. particle creation due to a strong electromagnetic field. It can be used to calculate the pair production probability for a wide range of field profiles and to obtain the momentum spectrum of the produced pairs as well.

We present a perturbative approach to the dynamically-assisted Sauter-Schwinger effect, i.e. exponential enhancement of the Sauter-Schwinger effect by an additional weak and fast electric field. In this scenario, we treat only the strong field non-perturbatively using the WKB method while the weak field enters via a perturbation series. This approach can be used to explain qualitative differences in the enhancement mechanism when comparing different field profiles for the weak field. We find that these differences can be attributed to different forms of the weak field's Fourier transform.

Furthermore, we propose another setup which gives exponential enhancement that consists of a strong, slowly varying electric field, a weak, fast varying electric field and a high-energy photon. We again employ the WKB method; this time treating strong and weak field non-perturbatively. The exponential enhancement always turns out to be larger than when any of the three ingredients is missing. In this case, the considered electromagnetic field is spacetime-dependent but this dependency enters the calculation only perturbatively (via the photon).

Truly spacetime-dependent background fields, however, can not be considered using the WKB method in its original form. As the main result of this thesis, we propose a new method based on the relativistic eikonal (or Hamilton-Jacobi) equation that reduces to the WKB method in time-dependent backgrounds but can also be employed for spacetime-dependent fields. We calculate corrections to the locally homogeneous field approximation for a weakly space-dependent mass and find that the spatial inhomogeneity decreases the pair production density exponent for the chosen field profile.

Zusammenfassung

Die aus der Quantenmechanik bekannte WKB-Methode wurde schon erfolgreich auf Probleme nichtperturbativer Paarerzeugung in zeitabhängigen externen Feldern, wie zum Beispiel im Falle kosmologischer Teilchenerzeugung oder des Sauter-Schwinger-Effekts (Teilchenerzeugung durch starke elektromagnetische Felder), angewandt. Mit Hilfe der Methode kann die Paarerzeugungswahrscheinlichkeit für eine breite Auswahl an Feldprofilen berechnet werden. Zudem erhält man ebenfalls das Spektrum der erzeugten Paare.

Es wird ein perturbativer Ansatz zur Berechnung des dynamisch assistierten Sauter-Schwinger-Effektes (exponentielle Verstärkung des Effektes durch ein zusätzliches schwaches, aber schnell veränderliches elektrisches Feld) vorgestellt. In diesem Fall wird nur das starke Feld nichtperturbativ mit Hilfe der WKB-Methode berücksichtigt während das schwache Feld über eine Störungsreihe eingeht. Der Ansatz kann qualitative Unterschiede des Verstärkungsmechanismus erklären, die bei Vergleich unterschiedlicher Feldprofile offenkundig werden. Es stellt sich heraus, dass die Fourier-Transformation eine bedeutende Rolle spielt bei der Erklärung dieser Unterschiede.

Darüber hinaus wird ein anderer Aufbau, der zu einer exponentiellen Verstärkung führt, vorgeschlagen, der aus einem starken, langsam veränderlichen elektrischen Feld, einem schwachen, schnell veränderlichen elektrischen Feld und einem hochenergetischen Photon besteht. Wieder wird die WKB-Method angewendet; diesmal um das starke und das schwache Feld nichtperturbativ zu berücksichtigen. Im Ergebnis zeigt sich, dass die Verstärkung in diesem Fall immer größer ist als in den Fällen, bei denen einer der drei Bestandteile fehlt. Das hier betrachtete elektromagnetische Feld ist durchaus raumzeitabhängig, wenn auch diese Abhängigkeit nur perturbativ (über das Photon) berücksichtigt wird.

Die WKB-Methode in ihrer ursprünglichen Form kann jedoch nicht auf Probleme mit echt raumzeitabhängigen externen Feldern angewandt werden. In dieser Arbeit wird eine neue Methode entwickelt, die auf der relativistischen Eikonalgleichung (oder Hamilton-Jacobi-Gleichung) be-

ruht und die im Grenzfall rein zeitabhängiger Felder in die WKB-Methode übergeht, aber auch für echt raumzeitabhängige Felder eingesetzt werden kann. Für den Fall einer raumzeitabhängigen Masse werden Korrekturen zu einer Näherung berechnet, die das externe Feld als lokal homogen annimmt. Es ergibt sich, dass für das betrachtete Feldprofil die räumliche Abhängigkeit den Exponenten der Paarerzeugungsdichte verringert.

List of publications

This thesis contains results that were published in the following articles:

[OS19] J. Oertel and R. Schützhold. 'WKB approach to pair creation in spacetime-dependent fields: The case of a spacetime-dependent mass'. *Physical Review D* **99**, 125014 (2019). DOI: 10.1103/physrevd.99.125014.
This contains the main result of my thesis. I did all of the calculations and wrote the bulk of the article.

[Tor+17] G. Torgrimsson, C. Schneider, J. Oertel and R. Schützhold. 'Dynamically assisted Sauter-Schwinger effect — non-perturbative versus perturbative aspects'. *Journal of High Energy Physics* **2017**, 043 (2017). DOI: 10.1007/jhep06(2017)043.
I verified and cross-checked the WKB calculation of the first-order results, contributed in discussions and proof-read the final version of the article.

[TOS16] G. Torgrimsson, J. Oertel and R. Schützhold. 'Doubly assisted Sauter-Schwinger effect'. *Physical Review D* **94**, 065035 (2016). DOI: 10.1103/PhysRevD.94.065035.
I did the initial WKB derivation for the expression of the relevant amplitude [eq. (7)]. I contributed in discussions and proof-read the final version of the article.

The following article was prepared during my doctoral studies as well but is unrelated to this thesis as its content was already part of my Master's thesis:

[OS15] J. Oertel and R. Schützhold. 'Inverse approach to solutions of the Dirac equation for space-time dependent fields'. *Physical Review D* **92**, 025055 (2015). DOI: 10.1103/physrevd.92.025055.

Danksagung

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Desweiteren gilt mein Dank den anderen Mitgliedern der Arbeitgruppe Schützhold in Duisburg, die für eine angenehme Arbeitsatmosphäre gesorgt haben und mir immer hilfsbereit zur Seite gestanden haben; insbesondere: Kay Eibl für ihre überaus freundliche Hilfe bei allen Problemen organisatorischer Art und nette Gespräche über England und den Rest der Welt; Stefan Thomae für angeregte Diskussionen über Wissenschaft, Gesellschaft, Kultur, Religion, Politik und Sprache; Nikodem Szpak für den Austausch über Physik, Computer, Technik und Finanzen; Andreas Osterloh für Einblicke in die italienische Sprache; Nicolai ten Brinke, Malte Linder, Christian Schneider und Sascha Lang für unvergessliche DPG-Tagungen und das nötige Quäntchen Humor. Besonders Christian Schneider möchte ich für den heiteren Austausch über den Sauter-Schwinger-Effekt, OpenGL, Rust, Haskell, Kaffee, Tee, Flaschen, Laptops, Server und vieles mehr in den Jahren in MG386 danken.

Für ihre fortwährende Unterstützung in den letzten Jahren danke ich meiner Familie und meinen Freunden und unter den letzteren besonders: Manuel, Thomas, Tim und Lizzie. Zu guter Letzt gilt mein allergrößter Dank meiner wunderbaren Frau: Ohne dich, Melissa, hätte ich das niemals geschafft!

Contents

Chapter 1

Introduction

1.1 Pair production

Symmetry is and always has been a cornerstone of physics. A system exhibits a symmetry if it is invariant under certain transformations such as translations, rotations or time reversal. By Noether's theorem, every continuous symmetry of a system has an associated conserved quantity. For example, momentum is the conserved quantity associated with translation invariance. Most predictions made by physical theories rest on such conserved quantities being available and, if they are not, analytic results are elusive as in the famous three-body problem of classical mechanics [GBL10]. One of the fundamental conservation laws is the conservation of energy. A particularly interesting consequence of this is that, given the equivalence of energy and mass manifest in Einstein's famous formula

$$E = mc^2 \tag{1.1}$$

from special relativity, it is possible, in principle, to convert energy in any form to matter and vice versa. For example, one could imagine that electromagnetic energy in the form of a photon with energy ω could be converted into a massive particle like the electron with the same energy:

$$\omega \longrightarrow e^- \tag{1.2}$$

The astute reader might feel a bit uneasy looking at this equation. Although we assumed the energies on both sides to be equal there are other quantities that are not conserved by this reaction, rendering it unphysical. First, from classical electrodynamics it is known that total charge is a conserved quantity[1]. Because photons do not have any charge the electron's charge has to be compensated by another particle on the right-hand side. This other particle – the electron's anti-particle – is the positron:

$$\omega \longrightarrow e^+ e^- \tag{1.3}$$

[1] Conservation of charge is related to invariance under global $U(1)$ transformations.

Although energy and charge are now conserved by this reaction it still violates another fundamental conservation law namely the conservation of momentum. To see this we change to the center-of-mass frame of the electron and the positron. There, by the very definition of this reference frame, the total momentum vanishes. Thus, the total momentum on the reaction's left-hand side has to vanish as well. However, a single photon always has a finite momentum $p = E/c$. Therefore, at least two photons are needed on the left-hand side:

$$\omega_1 \omega_2 \longrightarrow e^+ e^- \tag{1.4}$$

This reaction is known as the *Breit-Wheeler process*, named after the two physicists who in 1934 first calculated the probability for this effect to happen [BW34]. It may also be described as *perturbative pair production* as the probability may be obtained using a perturbation series expansion for small electric charges or field strengths. The reaction has been observed experimentally in 1997 at SLAC [Bur+97]. Mostly due to experimental constraints, Burke et al. observed the multi-photon Breit-Wheeler process

$$\omega_1 + n\omega_2 \longrightarrow e^+ e^-, \tag{1.5}$$

i.e. the reaction where a high energy photon of frequency ω_1 collides with n photons from a high-intensity laser beam with frequency ω_2. Again, the total energy of the photons has to be in the order of two times the electron's rest mass but each photon may need a lower energy than in the basic Breit-Wheeler reaction (1.4).

1.2 Dirac sea picture

So far we have only argued using a formula from special relativity. However, special relativity is a classical theory that does not contain explanations for certain phenomena that occur with particles like the electron. For example, the double-slit experiment showed that electrons may exhibit wave-like or particle-like behaviour depending on the specific experimental setup. This quantum nature of the electron in some regimes may be described using quantum mechanics based on the Schrödinger equation. However, the Schrödinger equation is a non-relativistic wave equation that cannot account for relativistic effects. Therefore, several physicists tried to find a relativistic formulation of quantum mechanics in the 1920s. At the forefront of these attempts was Paul Dirac who in 1928 derived a relativistic wave equation which came to bear his

name [Dir28a; Dir28b]. The Dirac equation successfully reproduces the hydrogen spectrum as predicted by the Schrödinger equation but also describes other phenomena not governed by the Schrödinger equation like the electron's spin and the fine structure[2].

All relativistic wave equations including the Dirac equation have solutions with negative energy. This is due to the relativistic energy-momentum relation

$$\epsilon^2 = \left(mc^2\right)^2 + (\mathbf{p}c)^2 \tag{1.6}$$

where ϵ is the energy of particle with mass m, $\mathbf{p}$ is its momentum and c is the speed of light. Solving for the energy one finds

$$\epsilon = \pm\sqrt{\left(mc^2\right)^2 + (\mathbf{p}c)^2} \tag{1.7}$$

which obviously can be both positive and negative. The solutions with the highest negative energy and the lowest positive energy are separated by a gap of $2mc^2$. In the classical theory of special relativity the negative-energy case is usually neglected based on the assumption that real particles have positive energy. Once in the positive energy regime, no physical process can change the energy continuously over to the negative regime because of the gap. When researchers in the 1920s tried to come up with relativistic quantum theories they were struggling with the negative-energy solutions. In quantum mechanics, one cannot simply exclude negative-energy solutions as in the classical theory because perturbations may cause transitions even between states separated by a gap.

After Dirac presented his work on the Dirac equation he also came up with a (at the time) satisfying interpretation for the states with negative energy [Dir30a; Dir30b]. Assume that in the vacuum, where there are no particles present, all states with negative energy are filled and all states with positive energy are empty (see fig. 1.1). This works because of the Pauli exclusion principle that does not permit more than one particle in any state[3]. In Dirac's interpretation, the vacuum can be thought of as an infinite sea of negative-energy particles – the *Dirac sea* – and only local deviations in the charge can be measured. Now imagine a transition from a negative-energy state to a positive-energy state (see fig. 1.2). We now have a filled positive-energy state and an empty negative-energy one. The empty state with negative energy (also called hole) has positive energy (with respect to the vacuum) but opposite charge than the positive-energy state. Because of that, the hole may be interpreted as a real particle with charge opposite to the electron that

[2] It should also be noted that relativistic quantum mechanics based on the Dirac equation fails to predict other quantities like the Lamb shift in the hydrogen spectrum or the exact value of the electron's g-factor. These can be calculated using quantum electrodynamics (QED) which itself uses the Dirac equation as well.

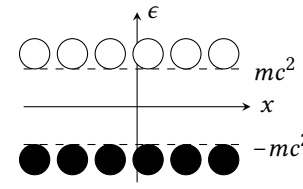

FIGURE 1.1 – In the vacuum, all negative-energy states are filled and all positive-energy states are empty.

[3] This is also the reason why Dirac's interpretation does not work as well for particles arising from a relativistic quantum theory that do not obey the Pauli exclusion principles, i.e. bosons instead of fermions.

is represented by a filled state with positive energy. Dirac first suggested that this particle might be a proton but later on – after Oppenheimer had pointed out that matter would be very instable that way [Opp30] – he stated that it must be some 'anti-electron'. Finally, in 1932 Anderson could verify the existence of that particle experimentally which from then on was named 'positron' [And33].

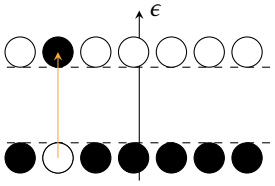

FIGURE 1.2 – A transition from the negative continuum to the positive continuum corresponds to the creation of an electron and a hole that is interpreted as a positron.

Thus, in fig. 1.2 the filled positive-energy state is interpreted as an electron while the hole in the Dirac sea is seen as a positron. To enable this transition the energy difference between the transition states has to be provided for example in the form of radiation. In the case depicted in fig. 1.2 this is just the gap energy of $2mc^2$. Hence, the process depicted in fig. 1.2 could describe the creation of an electron-positron pair by radiation (i.e. photons). This is actually the Breit-Wheeler reaction we discussed in the last section. Indeed, Breit and Wheeler used Dirac's theory to calculate the probability for the process.

1.3 Feynman diagrams

Dirac's work greatly furthered the development of quantum electrodynamics (QED) which describes interactions between electromagnetic fields on one hand and electrons and positrons on the other hand. In more modern terms, Breit and Wheeler's calculation would be carried out using Feynman diagrams which are named after its biggest proponent, Richard Feynman (who together with Julian Schwinger and Sin-Itiro Tomonaga won the Nobel prize in 1965 for their contributions to quantum electrodynamics [Nob18]). Feynman diagrams are a clever way to express the integrals that occur when calculating probabilities for processes such as the Breit-Wheeler processes using simple pictures[4]. For our purposes here it is enough to know that a wavy line depicts photon propagation while straight lines stand for propagating fermions like the electron and the positron. An arrow on the line signifies the direction of (electron) charge propagation. Then, a fermion line with a forwards-pointing arrow (in time) is interpreted as an electron whereas a fermion line with a backwards-pointing arrow is a positron. Fermions may interact with photons. Such an interaction is depicted by a vertex where a photon line, an incoming fermion line and an outgoing fermion line meet (see fig. 1.3). We employ the convention that time increases from the bottom to the top of the diagram.

[4] See standard text books on quantum field theory, e.g. [Zee10], for a more thorough treatment on Feynman diagrams.

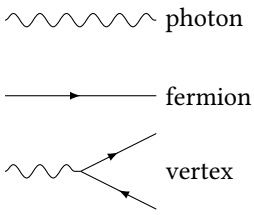

FIGURE 1.3 – Basic elements of Feynman diagrams for quantum electrodynamics

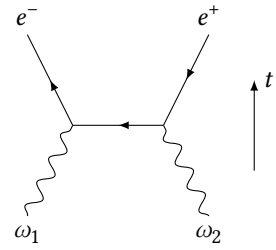

FIGURE 1.4 – Feynman diagram for the Breit-Wheeler process given in (1.4)

Figure 1.4 shows a Feynman diagram for the Breit-Wheeler process given in (1.4). At the beginning, there are two photons with frequen-

cies ω_1 and ω_2 respectively. At some point in space and time they each interact with two fermions. Finally, we end up with an electron and a positron. The Feynman diagram gives a very clear picture of the reaction we are considering. To calculate the probability for the process there are rules which map each graphical element (lines and vertices) to mathematical expressions that are combined to calculate the amplitude associated with the process. The probability then is the absolute square of the amplitude. For example, every vertex contributes a factor of q where q is the electron's charge. Therefore, the amplitude for fig. 1.4 is proportional to q^2. But there are other diagrams that also describe Breit-Wheeler pair production. For example, consider the two diagrams in fig. 1.5. The upper one is essentially the same as fig. 1.4 but with swapped vertices. The lower Feynman diagram in fig. 1.5 is completely different and has four interaction vertices instead of just two. To calculate the probability for the Breit-Wheeler process it is necessary to sum up the amplitudes for all the Feynman diagrams where initially there are two photons and finally there is an electron and a positron. The probability is then the absolute square of the total amplitude.

The amplitude for the four-vertex diagram in fig. 1.5 is proportional to q^4. So if we consider the electron charge q to be a small quantity, to lowest order we may neglect the contribution of the four-vertex diagram and all other diagrams that have more than two vertices. That is what we mean when we speak about the Breit-Wheeler process as a perturbative process: We may expand the total amplitude for the process in powers of q (or equivalently in powers of the fine structure constant $\alpha_{\text{QED}} = q^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$) and only keep terms up to a certain order. We can make our calculation more precise by including the contributions of higher-order terms.

Feynman diagrams for the multi-photon process given in (1.5) that involves $n + 1$ photons will have at least $n + 1$ vertices and thus be of order q^{n+1}. All else being equal, the probability for a process involving more photons will thus be less than for one with fewer photons.

Quantum electrodynamics in the perturbative regime (where probabilities can be expanded for small q or α_{QED}) is rather well understood. However, there are effects which cannot be described using such perturbation series. Consequently, the framework of Feynman diagrams and the associated methods cannot be applied for these problems. Moreover, our knowledge about effects in this non-perturbative regime is, both experimentally and theoretically, far more limited than in the perturbative one (see e.g. [Dun09]). The rest of this thesis will be con-

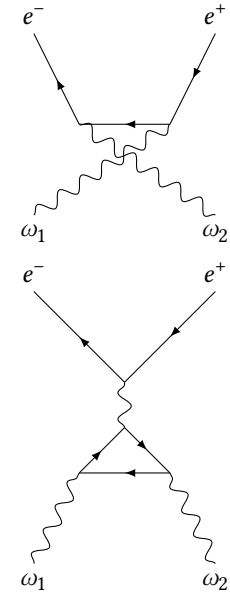

FIGURE 1.5 – Other Feynman diagrams for the Breit-Wheeler process given in (1.4)

cerned with such an effect; namely non-perturbative pair production.

1.4 Sauter-Schwinger effect

Besides the (perturbative) Breit-Wheeler process there is another, non-perturbative mechanism for electron-positron pair production in quantum electrodynamics: If one applies a strong electric field to the vacuum, electrons and positrons will be created in pairs. Although Fritz Sauter first calculated solutions of the Dirac equation in the presence of such a strong, constant electric field [Sau31; Sau32] in 1931 (and even less known Szczeniowski; see [Szc32; Sau32]), it was Julian Schwinger who treated this case correctly within the realms of quantum electrodynamics and found an expression for the pair production probability using the proper-time method [Sch51]. From his results one can deduce that the probability behaves like[5]

$$P_{e^+e^-} \propto \exp\left(-\pi \frac{m^2}{qE}\frac{c^3}{\hbar}\right) = \exp\left(-\pi \frac{E_S}{E}\right) \tag{1.8}$$

where

$$E_S = \frac{m^2}{q}\frac{c^3}{\hbar} \approx 1.3 \times 10^{18}\,\frac{\mathrm{V}}{\mathrm{m}} \tag{1.9}$$

is known as the *critical field strength* or *Schwinger field strength*.

[5] Heisenberg and Euler gave this expression in their 1936 article as well ('Dies ist die Größenordnung der Glieder, die von der Paarerzeugung im elektrischen Felde Rechenschaft geben'). Their calculation built on Sauter's result but they also included an additional constant magnetic field [HE36] (see also [Dun12] for a review).

The possibility of electron-positron pairs popping out of the vacuum has far-reaching consequences. It actually means that the vacuum is a polarisable medium which allows for effects like vacuum birefringence; see e.g. [KH16]. The vacuum may be seen as a sea of virtual electrons and positrons that can turn into real particles under the right circumstances. Therefore, the vacuum Maxwell equations have to be modified to account for the vacuum's polarisability (see [EK35; HE36] and [Dun12] for a review).

From Schwinger's expression (1.8) it is evident that, contrary to the Breit-Wheeler process, a series expansion for small charge or field strength is not possible, thus making Sauter-Schwinger pair production a non-perturbative effect. Intuitively, a static electric field could be understood as consisting of photons with vanishing frequency. Infinitely many of those photons would be needed to obtain twice the electron's rest energy. Hence, the associated Feynman diagram would show an infinite number of vertices thus making the probability inaccessible for a perturbative calculation.

There is an intuitive explanation for the Sauter-Schwinger effect in

the Dirac sea picture. The four-potential

$$A_0 = -Ex, \quad \mathbf{A} = 0 \tag{1.10}$$

generates a constant electric field $\mathbf{E} = E\mathbf{e}_x$. The energy-momentum relation (1.7) then reads

$$\epsilon = qEx \pm \sqrt{\left(mc^2\right)^2 + (\mathbf{p}c)^2}. \tag{1.11}$$

Thus, the levels in the Dirac sea picture will be tilted. This enables tunnelling from one continuum of states to the other through the forbidden region. After such a tunneling process an electron-positron pair has been produced (see fig. 1.6). To estimate the probability for this tunnelling effect, remember that the transmission probability for tunnelling through a barrier of length L and height H in non-relativistic quantum mechanics scales like

$$|T|^2 \propto \exp\left(-2\frac{L}{\hbar}\sqrt{2mH}\right). \tag{1.12}$$

In our case, the length of the barrier is $L = 2mc^2/(qE)$ and the height H is of order mc^2. We thus get

$$|T|^2 \propto \exp\left(-4\sqrt{2}\frac{m^2}{qE}\frac{c^3}{\hbar}\right). \tag{1.13}$$

which is astonishingly close to the Schwinger result (1.8).

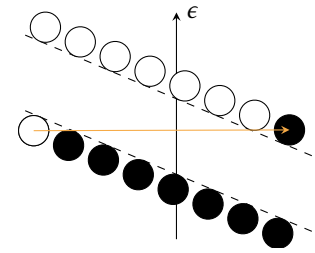

FIGURE 1.6 – A constant electric field (here with $qE < 0$) tilts the energy levels thereby allowing for tunnelling from the negative to the positive continuum.

The Sauter-Schwinger effect is much more difficult to verify experimentally than Breit-Wheeler or perturbative pair production due to the enormous critical field strength (1.9). Translated into an intensity, the critical intensity is in the order of $I_S = \varepsilon_0 c E_S^2 \approx 4.65 \times 10^{29}\,\mathrm{W/cm^2}$. It has been estimated that the number of produced particles is of order one at an even lower intensity of $I = 5 \times 10^{27}\,\mathrm{W/cm^2}$ due to the prefactor that we omitted in (1.8) [Nar+04]; see also [Rin01]. However, this is still beyond the capabilities of currently available technology. The European Extreme Light Infrastructure project (ELI) aims to achieve laser intensities in the order of $10^{25}\,\mathrm{W/cm^2}$ to $10^{26}\,\mathrm{W/cm^2}$. At the latter, the pair production probability is in the order of 10^{-61} which is too small to be observable in practice [Kor+11]. This motivates the search for mechanisms that improve the pair production probability. One mechanism that has been proposed is the one described by the Feynman diagram in fig. 1.7. There, the double fermion lines denote electrons and positrons

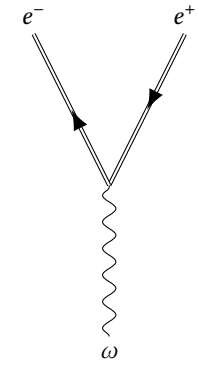

FIGURE 1.7 – Photon-assisted Sauter-Schwinger effect. The double fermion lines denote propagation under the influence of the strong external field.

propagating under the influence of a strong external electromagnetic field. Again, there are several other diagrams that have exactly the same outcome as in fig. 1.7 but they will be of higher order (i.e. have more vertices). Without the external field this process would be forbidden – as we have discussed before – due to momentum conservation. In the presence of a strong electric field however, it yields a dramatically higher number of pairs than the basic Sauter-Schwinger effect [DGS09] (see also [MV10]).

So far we have not considered energy aspects of the Sauter-Schwinger effect. Assuming that the electric field E performs work over a distance L then pairs can only be created if the work done can at least provide the rest mass of the electron and the positron, i.e. $qEL \geq 2mc^2$. For example, at the critical field strength $E = E_S$, the field has to do work over a distance in the order of the Compton wave length $\lambda_c = h/(mc)$. For a constant electric field we actually expect an infinite number of pairs to be produced as it extends over an infinite spacetime region. We have actually hidden this infinite spacetime volume factor in the prefactor of the pair production probability in (1.8). By dividing the pair production probability by this spacetime volume factor one obtains the pair production rate or density. So to calculate the pair production probability for a constant electric field confined in a finite spacetime volume we would have to integrate the pair production density over this spacetime volume. However, such an electric field would be discontinuous at the borders of the volume. Clearly, this is not a very realistic experimental situation. In a real experiment, a strong electric field would probably be created by focusing or colliding strong lasers such as planned at ELI[6]. However, the electromagnetic field of such an experimental setup is far more complex than just a constant electric field.

[6] The field strength of an atomic nucleus's Coulomb field may actually exceed the critical field strength as $E_{\text{nucleus}}/E_S = Z\alpha_{\text{QED}}\lambda_c^2/(4\pi^2 r^2)$ where Z is the number of protons and r is the distance from the nucleus. However, Sauter-Schwinger pair production is not possible in this field as super-critical field strengths are only realised on length scales below the Compton wave length λ_c.

This motivates the investigation of pair production using spacetime-dependent electromagnetic fields. First, one could consider time-dependent electric fields (i.e. still space-independent or homogeneous fields). We still expect an infinite number of pairs in that case as the spatial extent is still infinite. Nevertheless, the influence of the time dependence on the pair production density can be investigated. It can be calculated analytically for the often-used Sauter pulse $E(t) = E_0 \operatorname{sech}^2(\omega t)$ [Sau32; Heb11]. One may also look at the superposition of different pulses and determine their combined impact. An especially striking result is obtained in the case of the *dynamically assisted Sauter-Schwinger effect*: If one uses a strong, slowly varying electric field and superimposes a weak, rapidly varying electric field, an exponential enhance-

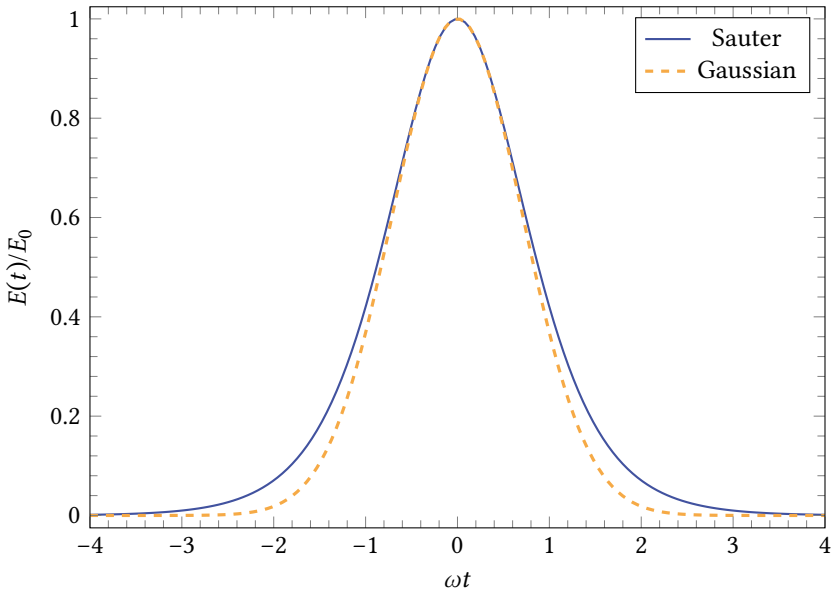

FIGURE 1.8 – Pulse shape comparison between a Sauter pulse $E(t) = E_0 \operatorname{sech}^2(\omega t)$ and a Gaussian $E(t) = E_0 \exp(-\omega^2 t^2)$.

ment of the pair production density is found [SGD08]. Subsequent studies showed that there are tremendous qualitative differences between different pulse shapes even for the Sauter pulse and the Gaussian $E(t) = E_0 \exp(-\omega^2 t^2)$ which, as can be seen in fig. 1.8, are visually very similar [Lin+15; Tor+17]. Furthermore, when adding a high-energy photon to this setup, an even stronger improvement of the pair production density can be observed [TOS16]. We will discuss these enhancement mechanisms in chapter 3.

The aforementioned qualitative differences for different pulse shapes suggest that there may be even more surprises for electromagnetic fields that depend both on time and space. At present, our knowledge about the influence of truly spacetime-dependent fields on vacuum pair production is very limited, partly because of limitations in the methods presently available. There are two different approaches to quantum electrodynamics or quantum field theories in general. One is the path integral formulation and the other one is canonical quantisation. A method to calculate the pair production probability based on the path integral approach is the worldline formalism, often in conjunction with instanton methods [Vaĭ+82; AAM82; DS05; Dun+06; Dun08]. It works for a wide range of field configurations [DW06; DD11a; SX14; Ild14; ITW15] and lends itself well to numerical calculations [GK05; STS18]; there are even some analytic results for special spacetime-dependent field configurations using the worldline instanton method [SS16; Dum16; TSS18]. Although there are ideas on how to obtain the momentum distribution of the produced pairs using this

method [DD11a], in its most basic form it only yields the total pair production probability. Other methods based on canonical quantisation like the quantum kinetic approach [Heb+09; HAG10] or the WKB method [Pop72; MP77; DD11b; Lin+15] are more commonly used to obtain the momentum spectrum[7]. However, these are based on analytical solutions of the Dirac equation and those can practically only be found for field profiles that only depend on a single coordinate like the time t [Di 04; KRX08; DD11b; KS11; SX14; Lin+15], a spatial coordinate x [Sau32; Sau32; Nik70; Ken02; KP02; KRX08] or a light-cone coordinate $x_\pm = t \pm x$ [TTW01; AFG03; HIM11]. This is due to the concept of symmetry we already mentioned at the very beginning of this chapter: If the electromagnetic field exhibits a certain symmetry the associated conserved quantities can be used to simplify the problem. For example, for a time-dependent electric field, the (canonical) momentum is conserved. Then, the Dirac equation may be reduced to an ordinary differential equation by a Fourier transformation. This is no longer the case if the electric field may also depend on a spatial coordinate as then the momentum is not a conserved quantity anymore. This reliance on symmetries is present in most methods and therefore about the same class of fields can be treated with all of these.

[7] It can be shown that the quantum kinetic approach and the WKB method are actually equivalent [Dum09].

So most of the results we have today for the pair production probability in the presence of truly spacetime-dependent fields were obtained numerically (see for example [Ruf+09; HAG11; Jia+11; Jia+12; WBK15; APS16; APS17]). For analytical results, often the locally constant field approximation has been used [Nar+04; Bul+06]. The approximation rests on the assumption that the pair production process happens on length scales[8] much smaller than the length scale on which the electric field varies. Then, the pair production probability is calculated by replacing the field strength E in the result for the pair production density for a constant field with the spacetime-dependent electric field $E(t, x)$ and integrating this density over the whole spacetime. It has been found that for purely space-dependent fields the locally constant field approximation overestimates the pair production probability while it underestimates it for time-dependent fields. However, for fields depending on a single light-cone coordinate the approximation is exact [GK05; Ild14].

[8] The tunnelling length $L = \lambda_c E_S/(\pi E)$ introduced above may be used as a first estimate for the length scale on wich the pair production process happens.

In this thesis, we develop a method which can be used to go beyond the locally constant field approximation. The method is based on canonical quantisation. It resembles the WKB method and reproduces its results (see chapter 4). We will mostly work in 1+1 spacetime dimensions. This simplifies the calculation as two-dimensional spinors are

sufficient. However, in 1+1 dimensions there is no magnetic field and no spin. The electric field has only a single component and the only vacuum solutions are constant electric fields (waves are not vacuum solutions). This is qualitatively different from what we know about our universe and therefore the 1+1-dimensional spacetime should be seen as a toy model. Still, we expect the results to give valuable insights about the influence of spacetime-dependent fields on pair production. Furthermore, it should also be possible to generalise the results to the case of an electric field with a single spacetime-dependent component in 3+1 dimensions. Even simpler could be a generalisation to 2+1 dimensions because two-dimensional spinors can still be used in that case.

All the interactions so far have been between electrons and positrons on one hand and electromagnetic fields in the form of photons or classical backgrounds on the other hand. But the notion of the vacuum as a sea of virtual electrons and positrons suggests that any interaction or external influence that can provide a pair's rest energy may turn such an electron-positron pair into a real one. Expansion and contraction of the universe is such an external influence that may produce pairs, an effect known as *cosmological particle creation.* In 1939, Erwin Schrödinger was the first to note that this might occur [Sch39] (although he seems to have been quite concerned about this 'alarming' phenomenon). About thirty years later, Parker used modern quantum field theory in curved spacetime to calculate the effect and estimate the rate at which pairs are produced with the current expansion rate of the universe [Par68]. Interestingly, as we will see in chapter 5, in 1+1 dimensions an expanding universe is actually equivalent to a static universe in which the electron mass grows over time (see also [KNA16]). Therefore, we may use the formalism we develop in chapter 4 to investigate cosmological particle creation by using a spacetime-dependent mass in a flat spacetime. This makes this case easier to handle than the Sauter-Schwinger effect because we only have to deal with a scalar potential $m(t, x)$ instead of a vector potential $A_\mu(t, x)$.

1.5 Outline of thesis

The thesis is structured as follows: We start by reviewing quantum electrodynamics with an external background field using canonical quantisation in chapter 2. We will see how the solutions of the (covariant) Dirac equation can be used to calculate the pair production probability. In chapter 3 we assume that the external field only depends on the time

t. We review the WKB method for time-dependent fields and show how to obtain the Schwinger exponent (1.8) using this method. We then look at setups with enhancement mechanisms like the dynamically-assisted Sauter-Schwinger effect and the doubly-assisted Sauter-Schwinger effect (as published in [TOS16]); we also discuss a perturbative approach to dynamical assistance which can explain qualitative differences in the results for seemingly similar pulse shapes (as published in [Tor+17]). Finally, we consider the case of a time-dependent mass. Chapter 4 contains the main work of the thesis where we develop a method closely related to the WKB method from chapter 3 but for spacetime-dependent fields (as published in [OS19]). The method uses a similar decomposition into exponent and prefactor as the WKB method and we show that our method actually reduces to the WKB method for time-dependent fields. In chapter 5 we apply this method to the case of a spacetime-dependent mass. First, we show the equivalence of a non-Minkowskian metric and a spacetime-dependent mass in a flat spacetime in 1+1 dimensions. We discuss problems that occur when solving the eikonal (or Hamilton-Jacobi) equation in spacetime-dependent fields and finally consider the case of a weakly space-dependent mass. We outline the strategy for applying the method from chapter 4 to spacetime-dependent electric fields in chapter 6.

1.6 Notation and conventions

Throughout this thesis unless otherwise noted we will use units where $\hbar = c = 1$. We use q to denote the electric charge of a particle. When coupling the electromagnetic field to the spinor field, q denotes the charge of the electron, i.e. $q = e_- < 0$.

We will refer to x^μ either as the Lorentz vector (i.e. with all of its components) or as a specific component with index μ; what is meant should be clear from the context. We use the Einstein summmation convention, i.e. a summation is implied for indices that occur once as an upper and once as a lower index in a term. Greek indices of Lorentz vectors or tensors run over all spacetime coordinates while Roman indices only run over the spatial coordinates.

The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ in 3+1 dimensions or analogously $\eta_{\mu\nu} = \mathrm{diag}(1, -1)$ in 1+1 dimensions, while general metric tensors are denoted by $g_{\mu\nu}$. Additionally, η and g designate the respective determinants.

We will make extensive use of the Levi-Civita tensor in 1+1 dimen-

sions $\varepsilon^{\mu\nu}$, which is defined by $\varepsilon^{01} = -\varepsilon^{10} = 1$ and $\varepsilon^{00} = \varepsilon^{11} = 0$. As for other tensors, its indices can be lowered using the metric, such that, for example, $\varepsilon_0{}^1 = \eta_{00}\varepsilon^{01} = 1$, $\varepsilon_1{}^0 = \eta_{11}\varepsilon^{10} = 1$ and $\varepsilon_0{}^0 = \varepsilon_1{}^1 = 0$.

We will use ∂_μ as a shorthand for $\frac{\partial}{\partial x^\mu}$. Additionally, a dot above a quantity denotes the derivative with respect to time, i.e. $\dot{A} = \partial_t A$. Derivatives operate on everything to their right in a term, except if the derivative is enclosed in brackets. For example, the product rule could be written as $\partial_\mu uv = (\partial_\mu u)v + u\partial_\mu v$. When going over to curved coordinates, the partial derivative ∂_μ has to be replaced with the usual covariant derivative known from general relativity. For example, the covariant derivative of a vector field w_ν is given by $\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\lambda_{\mu\nu} w_\lambda$ where $\Gamma^\lambda_{\mu\nu}$ denote the Christoffel symbols of the second kind that are defined as $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$.

Quantum mechanical operators are indicated by a hat, i.e. $\hat{a}$ and $\hat{\Psi}$. The commutator of two operators is written using square brackets

$$\left[\hat{a}, \hat{b}\right] = \hat{a}\hat{b} - \hat{b}\hat{a} \tag{1.14}$$

while we use curly brackets for the anticommutator

$$\left\{\hat{a}, \hat{b}\right\} = \hat{a}\hat{b} + \hat{b}\hat{a}. \tag{1.15}$$

We will use $\psi^\dagger = \psi^{*\mathsf{T}}$ for the conjugate transpose of ψ.

Chapter 2

Quantum electrodynamics with external background

In this chapter, we review the theory of quantum electrodynamics in the presence of an external electromagnetic field (see [FGS91] and [GMR85] for extensive treatments on this topic). Additionally, we will consider the electron mass to be spacetime-dependent. We will work in 3+1 dimensions first as an adaptation to lower-dimensional spacetime can be carried out in a straightforward manner. As already hinted at in the introduction we will employ the canonical quantisation formalism.

2.1 Lagrangian density

As usual, to formulate the quantum field theory of electrons, positrons and photons that is quantum electrodynamics we start with a classical field theory. For quantum electrodynamics we have two fields: The spinor field ψ and the electromagnetic or Maxwell field A_μ. If we consider quantum electrodynamics with an external background, we have an additional field A_μ^{ext} which is not subject to any dynamics. Basically, the idea is that this additional external field is not affected by the presence of electrons and positrons that are contained in the spinor field (see [GMR85, pp. 197 sqq.] for a detailed discussion of this approach). This is a reasonable assumption in the case of strong (high-intensity) electromagnetic fields in the order of the Schwinger field strength. The Lagrangian density for such a theory

$$\mathscr{L} = \mathscr{L}_e + \mathscr{L}_\gamma + \mathscr{L}_{\text{int}} \tag{2.1}$$

consists of three terms: The Lagrangian density for the spinor field in the presence of the external field

$$\mathscr{L}_e = \overline{\psi}[i\gamma^\mu(\partial_\mu + iqA_\mu^{\text{ext}}) - m]\psi, \tag{2.2}$$

the Lagrangian density for the Maxwell field

$$\mathscr{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.3}$$

and the Lagrangian density for the interaction between the spinor field and the Maxwell field

$$\mathscr{L}_{\text{int}} = -q\overline{\psi}\gamma^\mu A_\mu \psi. \tag{2.4}$$

The γ^μ are the usual gamma matrices satisfying the Clifford algebra's anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{2.5}$$

and the adjoint Dirac spinor is defined as $\overline{\psi} = \psi^\dagger\gamma^0$. Using the Euler-Lagrange equations of field theory

$$\partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathscr{L}}{\partial\phi} = 0, \tag{2.6}$$

where ϕ stands for a generic field in question, we can calculate the equations of motion. Because the spinor field ψ is complex, we describe it using the two fields ψ and $\overline{\psi}$. The Euler-Lagrange equation for $\overline{\psi}$ is

$$[i\gamma^\mu(\partial_\mu + iqA_\mu^{\text{ext}} + iqA_\mu) - m]\psi = 0 \tag{2.7}$$

which is just the covariant Dirac equation. From the Euler-Lagrange equation for ψ we find the conjugate Dirac equation

$$i(\partial_\mu - iqA_\mu^{\text{ext}} - iqA_\mu)\overline{\psi}\gamma^\mu + m\overline{\psi} = 0. \tag{2.8}$$

Finally, the Euler-Lagrange equations for the Maxwell field A_μ leads to

$$\partial_\mu F^{\mu\nu} = q\overline{\psi}\gamma^\nu\psi = j^\nu \tag{2.9}$$

which are just the inhomogeneous Maxwell equations. The theory so far is a gauge theory, meaning that it is invariant under local $U(1)$ transformations of the form

$$qA_\mu \longrightarrow qA_\mu - \partial_\mu\vartheta, \quad \psi \longrightarrow \psi e^{i\vartheta} \tag{2.10}$$

where $\vartheta(x^\mu)$ is an arbitrary function of space and time. For later quantisation of the theory it is beneficial to fix a specific gauge. Instead of

adding constraints for the gauge field A_μ like when using the Lorenz gauge or Coulomb gauge, one can also add a term to the Lagrangian density that breaks the gauge symmetry. A popular choice is the *Feynman gauge* where one adds the term $-\frac{1}{2}(\partial_\mu A^\mu)^2$ to the Lagrangian density. Thus, instead of (2.3) we use

$$\mathscr{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{2.11}$$

as the Lagrangian for the Maxwell field. Then, the equations of motion are

$$\partial_\mu \partial^\mu A^\nu = \partial_t \dot{A}^\nu + \partial_k \partial^k A^\nu = j^\nu. \tag{2.12}$$

However, these equations of motion may also be obtained from the Lagrangian density

$$\mathscr{L}_\gamma = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \tag{2.13}$$

which only differs from the one in (2.11) by a four-divergence term. As the Lagrangian density in (2.13) will be easier to deal with later we will use it instead of (2.11).

Even with the gauge fixed, the Lagrangian density still exhibits a global $U(1)$ symmetry, i.e. it is invariant under transformations of the form (2.10) with ϑ constant. According to the Noether theorem, for every continuous symmetry there is a conserved current J^μ for which $\partial_\mu J^\mu = 0$. The conserved current of the global $U(1)$ symmetry is the current j^μ as defined in (2.9). Integrating its zeroth component over the whole space gives rise to a conserved quantity

$$Q = \int \mathrm{d}^3 r\, j^0 = q \int \mathrm{d}^3 r\, \overline{\psi}\gamma^0\psi = q \int \mathrm{d}^3 r\, \psi^\dagger \psi \tag{2.14}$$

which is interpreted as the total charge.

2.2 Hamiltonian formulation

We now want to transform from the Lagrangian density to the Hamiltonian density. First, we need to calculate the conjugate momentum for every field in the theory. For the spinor field we find

$$\Pi_\psi = \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = i\psi^\dagger, \quad \Pi_{\overline{\psi}} = \frac{\partial \mathscr{L}}{\partial \dot{\overline{\psi}}} = 0. \tag{2.15}$$

Because the momenta do not depend on $\dot{\psi}$ or $\dot{\overline{\psi}}$ it is impossible to express these time derivatives in terms of the momenta Π_ψ and $\Pi_{\overline{\psi}}$. Hence, the equations (2.15) are constraints on the spinor field. Nevertheless, the Hamiltonian density can be calculated as usual because – as we will see shortly – the terms with $\dot{\psi}$ and $\dot{\overline{\psi}}$ drop out anyway.

From the Lagrangian density (2.13), we get the conjugate momentum for the Maxwell field

$$\Pi_A^\mu = \frac{\partial \mathscr{L}}{\partial \dot{A}_\mu} = -\dot{A}^\mu. \tag{2.16}$$

Using these momenta we find the Hamiltonian density

$$\mathscr{H} = \Pi_\psi \dot{\psi} + \Pi_{\overline{\psi}} \dot{\overline{\psi}} + \Pi_A^\mu \dot{A}_\mu - \mathscr{L} = \mathscr{H}_e + \mathscr{H}_\gamma + \mathscr{H}_{\text{int}} \tag{2.17}$$

which consists of the Hamiltonian density for the spinor field in the presence of the external field

$$\mathscr{H}_e = \Pi_\psi \dot{\psi} + \Pi_{\overline{\psi}} \dot{\overline{\psi}} - \mathscr{L}_e = \overline{\psi}[-i\gamma^k \partial_k + \gamma^\mu q A_\mu^{\text{ext}} + m]\psi = \psi^\dagger H_D \psi, \tag{2.18}$$

the Hamiltonian density for the Maxwell field

$$\mathscr{H}_\gamma = \Pi_A^\mu \dot{A}_\mu - \mathscr{L}_\gamma = -\frac{1}{2}\eta_{\mu\nu}\Pi_A^\mu \Pi_A^\nu + \frac{1}{2}(\partial_k A_\nu)(\partial^k A^\nu) \tag{2.19}$$

and the Hamilton density for the interaction between spinor field and Maxwell field

$$\mathscr{H}_{\text{int}} = -\mathscr{L}_{\text{int}} = q\overline{\psi}\gamma^\mu A_\mu \psi. \tag{2.20}$$

Before proceeding with the quantisation of the theory, we first want to take a look at the Dirac equation and some of its solutions. We start with some remarks about the gamma matrices in the following section.

2.3 Gamma matrices

From the anticommutation relation (2.5) it can be seen that different gamma matrices anticommute, i.e. $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$ for $\mu \neq \nu$. Additionally, the square of γ^0 is the identity matrix, i.e. $\gamma^0 = (\gamma^0)^{-1}$ is self-inverse, whereas the square of the other gamma matrices γ^k, $k \neq 0$, is the negative identity matrix, i.e. $(\gamma^k)^{-1} = -\gamma^k$. Furthermore, there are exactly as many gamma matrices as there are spacetime dimensions. The condition for the square of any gamma matrix γ^μ can be expressed

conveniently as

$$\frac{1}{\eta^{\mu\mu}}\gamma^\mu\gamma^\mu = \mathbb{1} \tag{2.21}$$

where no summation over μ is implied.

This condition immediately leads to statements about the eigenvalues of the gamma matrices: Assume that $u^{(\mu)}$ is an eigenvector of γ^μ for the eigenvalue $\lambda^{(\mu)}$, i.e. $\gamma^\mu u^{(\mu)} = \lambda^{(\mu)} u^{(\mu)}$. Then

$$\begin{aligned} u^{(0)} &= \gamma^0\gamma^0 u^{(0)} = \left(\lambda^{(0)}\right)^2 u^{(0)}, \\ -u^{(k)} &= \gamma^k\gamma^k u^{(k)} = \left(\lambda^{(k)}\right)^2 u^{(k)}, \quad k \neq 0 \end{aligned} \tag{2.22}$$

shows that the eigenvalues of γ^0 are $\lambda^{(0)} = \pm 1$ and the eigenvalues of γ^k, $k \neq 0$, are $\lambda^{(k)} = \pm i$.

On the other hand, eq. (2.21) may be used to prove that the gamma matrices are traceless:

$$\begin{aligned} \operatorname{tr}(\gamma^\mu) &= \frac{1}{\eta^{\nu\nu}}\operatorname{tr}(\gamma^\nu\gamma^\nu\gamma^\mu) = \frac{1}{\eta^{\nu\nu}}\operatorname{tr}(\gamma^\nu\gamma^\mu\gamma^\nu) \\ &= -\frac{1}{\eta^{\nu\nu}}\operatorname{tr}(\gamma^\nu\gamma^\nu\gamma^\mu) = -\operatorname{tr}(\gamma^\mu) = 0 \end{aligned} \tag{2.23}$$

where $\nu \neq \mu$ so that γ^μ and γ^ν anticommute.

The conjugate transpose of any gamma matrix is given by the relation

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{2.24}$$

which follows from the hermiticity of the Dirac Hamiltonian (see section 2.4).

One may ask which dimension the matrices should at least have depending on the dimension of the spacetime[9]. Because the gamma matrices are traceless and their possible eigenvalues are either ± 1 or $\pm i$ the gamma matrices have to be of even dimension: There have to be as many eigenvalues with a plus sign as eigenvalues with a minus sign for the trace to vanish. Thus, we need at least 2×2 matrices. There is a set of three anticommuting 2×2 matrices that is often used in quantum mechanics – the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.27}$$

Each Pauli matrix is self-inverse and thus may be used as γ^0. By multiplying a Pauli matrix by the imaginary unit i the square of the resulting

[9] Clearly, one can always use matrices of higher dimensions than needed. For example, if γ^μ are a set of gamma matrices then the matrices

$$\tilde{\gamma}^\mu = \begin{bmatrix} \gamma^\mu & \mathrm{O} \\ \mathrm{O} & \gamma^\mu \end{bmatrix}, \tag{2.26}$$

where O is the zero matrix of appropriate dimension, will satisfy the anticommutation relation. Thus, spinors will have more components than in the original representation. However, there will be no further physical information in these additional components.

matrix will be the negative identity matrix. Hence, $i\sigma_k$ may be used as one of the γ^k, $k \neq 0$. For example, in 1+1 dimensions we may simply choose two Pauli matrices, use one of them as γ^0 and the other one multipled by i as γ^1, e.g.

$$\gamma^0 = \sigma_z, \quad \gamma^1 = i\sigma_x. \tag{2.28}$$

The choice which Pauli matrices to use is completely arbitrary and just corresponds to a different choice of basis for the spinors. This procedure also works in 2+1 dimensions where we need three gamma matrices: We just use the remaining Pauli matrix multiplied by i as γ^2, e.g.

$$\gamma^0 = \sigma_z, \quad \gamma^1 = i\sigma_x, \quad \gamma^2 = i\sigma_y. \tag{2.29}$$

In 3+1 dimensions, however, 2×2 matrices are not enough: There are only three anticommuting and self-inverse 2×2 matrices[10]. Thus, gamma matrices in 3+1 dimensions have to be at least 4×4. Often, the Dirac representation

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{bmatrix}, \quad \gamma^k = \begin{bmatrix} \mathbb{O} & \sigma_k \\ -\sigma_k & \mathbb{O} \end{bmatrix}, \quad k \neq 0, \tag{2.30}$$

is used, where $\mathbb{O}$ is the 2×2 zero matrix. Generally speaking, it can be shown that in $d+1$ dimensions the gamma matrices are square matrices with a minimum dimension of $2^{\lfloor (d+1)/2 \rfloor}$ (see [Cor89, Appendix L]).

[10] There are at most four linear independent 2×2 matrices in general; the requirement of them being self-inverse adds another constraint that reduces this set to three matrices.

2.4 Inner product

We define the following inner product for solutions of the (covariant) Dirac equation (2.7) with an external field A_μ^{ext} only (i.e. $A_\mu = 0$)

$$(\phi, \psi) = \int \mathrm{d}^3 r \, \phi^\dagger(t, \mathbf{r}) \, \psi(t, \mathbf{r}) \tag{2.31}$$

where $\phi^\dagger$ is the conjugate transpose of ϕ. We want to prove that this inner product does not depend on the time t at which we evaluate its operands. To do this, we need the time derivative of a solution to the Dirac equation. Solving the covariant Dirac equation (2.7) with just the external field for $i\partial_t \psi$ we find

$$i\partial_t \psi = \left[-i\gamma^0 \gamma^k \left(\partial_k + iqA_k^{\text{ext}} \right) + \gamma^0 m + qA_0^{\text{ext}} \right] \psi, = H_D \psi \tag{2.32}$$

where H_D is the Hamilton operator for the Dirac equation (in analogy to the Hamilton operator from the time-dependent Schrödinger equation in quantum mechanics). We require that H_D is self-adjoint in the sense that $H_D^\dagger = H_D$. Therefore, both γ^0 and $\gamma^0\gamma^k$ have to be self-adjoint which is equivalent to the hermiticity condition for the gamma matrices (2.24).

We now calculate the time derivative of (ϕ, ψ),

$$\begin{aligned}\partial_t(\phi,\psi) &= \int \mathrm{d}^3 r \left[\left(\partial_t\phi^\dagger\right)\psi + \phi^\dagger\partial_t\psi\right] \\ &= \int \mathrm{d}^3 r \left[\left(-iH_D\phi\right)^\dagger\psi - i\phi^\dagger H_D\psi\right] \\ &= i\int \mathrm{d}^3 r \left[\phi^\dagger H_D^\dagger\psi - \phi^\dagger H_D\psi\right] \\ &= 0,\end{aligned} \tag{2.33}$$

where we have used the self-adjointness of H_D in the last step.

2.5 Plane-wave solutions

We now want to look at solutions of the free Dirac equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0 \tag{2.34}$$

as this is the relevant equation in the case that any external field has been switched off. We make a plane-wave ansatz, i.e. $\psi \propto e^{-ip_\mu x^\mu}$, and find that $p_\mu = (p_0, -\mathbf{p})$ has to satisfy the relativistic energy-momentum relation

$$p_\mu p^\mu = m^2 \iff p_0 = \pm\epsilon_\mathbf{p} = \pm\sqrt{m^2 + \mathbf{p}^2}. \tag{2.35}$$

Thus there are two independent solutions ψ^+ and ψ^- with either positive or negative energy respectively[11]:

$$\psi^+ = ue^{-ip_\mu x^\mu} = ue^{-i\epsilon_\mathbf{p} t + i\mathbf{p}\cdot\mathbf{r}}, \quad \psi^- = ve^{ip_\mu x^\mu} = ve^{i\epsilon_\mathbf{p} t - i\mathbf{p}\cdot\mathbf{r}}. \tag{2.36}$$

[11] Note that the sign of the momentum term $i\mathbf{p}\cdot\mathbf{r}$ is chosen by convention and one could also use the same sign for the momentum term for both of the two solutions.

Inserting $\psi^\pm$ into the free Dirac equation (2.34) we get

$$\gamma^\mu p_\mu u = mu, \quad -\gamma^\mu p_\mu v = mv. \tag{2.37}$$

These equations are equivalent to saying that u is an eigenvector of $\gamma^\mu p_\mu$ with eigenvalue m and v is an eigenvector of $-\gamma^\mu p_\mu$ with eigenvalue m (or an eigenvector of $\gamma^\mu p_\mu$ with eigenvalue $-m$). Depending on the dimension of the gamma matrices (see section 2.3) there may be several

eigenvectors. For example, in 3+1 dimensions we have two independent eigenvectors for both u and v, corresponding to spin up or down respectively. In addition, the solutions will in general depend on the momentum $\mathbf{p}$. We will often refer to spin and momentum collectively using a mode index $j = (s_j, \mathbf{p}_j)$. Therefore, we write the solutions of the free Dirac equation as[12]

[12] The energy ϵ_j is a shorthand for $\epsilon_{\mathbf{p}_j} = \sqrt{m^2 + \mathbf{p}_j^2}$.

$$\psi^+(j;t,\mathbf{r}) = u(j)e^{-i\epsilon_j t + i\mathbf{p}_j\cdot\mathbf{r}}, \quad \psi^-(j;t,\mathbf{r}) = v(j)e^{i\epsilon_j t - i\mathbf{p}_j\cdot\mathbf{r}}. \tag{2.38}$$

We orthonormalise the spinors using

$$\begin{aligned} \bar{u}(s,\mathbf{p})u(s,\mathbf{p}) &= 1, \quad & \bar{v}(s,\mathbf{p})v(s,\mathbf{p}) &= -1, \\ \bar{u}(s,\mathbf{p})v(s,\mathbf{p}) &= 0, \quad & \bar{v}(s,\mathbf{p})u(s,\mathbf{p}) &= 0. \end{aligned} \tag{2.39}$$

[13] See for example [Zee10, pp. 108 sq.] for an intuitively accessible derivation of these expressions or [Pal15, p. 15] for one without using a specific representation.

In addition, one can show that[13]

$$\begin{aligned} \sum_s u(s,\mathbf{p})\bar{u}(s,\mathbf{p}) &= \frac{\gamma^\mu p_\mu + m}{2m}, \\ \sum_s v(s,\mathbf{p})\bar{v}(s,\mathbf{p}) &= \frac{\gamma^\mu p_\mu - m}{2m} \end{aligned} \tag{2.40}$$

holds. The relations (2.39) and (2.40) imply the relations

$$\begin{aligned} u^\dagger(s,\mathbf{p})u(s',\mathbf{p}) = v^\dagger(s,\mathbf{p})v(s',\mathbf{p}) &= \frac{\epsilon_\mathbf{p}}{m}\delta_{ss'}, \\ u^\dagger(s,\mathbf{p})v(s',-\mathbf{p}) = v^\dagger(s,-\mathbf{p})u(s',\mathbf{p}) &= 0, \\ \sum_s \left[u(s,\mathbf{p})u^\dagger(s,\mathbf{p}) + v(s,-\mathbf{p})v^\dagger(s,-\mathbf{p})\right] &= \frac{\epsilon_\mathbf{p}}{m}, \end{aligned} \tag{2.41}$$

which in turn can be used to show that the solutions $\{\psi^\kappa(j;t,\mathbf{r})\}$ form an orthogonal and complete set, i.e. they fulfil the orthogonality condition

$$\left(\psi^\kappa(j), \psi^\lambda(l)\right) = (2\pi)^3 \frac{\epsilon_j}{m}\delta_{\kappa\lambda}\delta_{jl}, \tag{2.42}$$

and the completeness relation

$$\sum_{j,\kappa} \frac{m}{(2\pi)^3 \epsilon_j} \psi^\kappa(j;t,\mathbf{r}) \left(\psi^\kappa\right)^\dagger(j;t,\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{2.43}$$

where the delta function for mode indices is $\delta_{jl} = \delta_{s_j s_l}\delta(\mathbf{p}_j - \mathbf{p}_l)$ and the sum over mode indices is $\sum_j = \sum_{s_j}\int \mathrm{d}^3 p_j$. The greek indices κ and λ used here do not run over all spacetime coordinates but instead are denoting the energy of a solution (i.e. they are either + or −).

At the beginning of this section we stated that the free Dirac equation is the relevant equation in the case that the external field is switched off. However, a constant electromagnetic potential A_μ does not generate an electromagnetic field, too. Thus, we should have looked at solutions to the covariant Dirac equation

$$\left[i\gamma^\mu\left(\partial_\mu + qA_\mu\right) - m\right]\psi = 0 \tag{2.44}$$

with A_μ constant. Yet, a gauge transformation

$$\begin{aligned} qA_\mu &\mapsto qA'_\mu = 0, \\ \psi &\mapsto \psi' = \psi\exp\left(iqA_\mu x^\mu\right) \end{aligned} \tag{2.45}$$

turns this equation into the free Dirac equation (2.34). Put another way, solutions to (2.44) are given by $\psi^\pm\exp\left(-iqA_\mu x^\mu\right)$ where $\psi^\pm$ are the solutions to the free Dirac equation (2.36). Inserting these into (2.44) we recover the equations for the spinors u and v from (2.37). Thus, the same spinors as in the plane-wave case may be used and all the relations presented above still hold with a constant electromagnetic potential. When using the other sign convention for the momentum term where

$$\psi^+ = ue^{-i\epsilon_\mathbf{p}t+i\mathbf{p}\cdot\mathbf{r}}, \quad \psi^- = ve^{i\epsilon_\mathbf{p}t+i\mathbf{p}\cdot\mathbf{r}}, \tag{2.46}$$

one might also introduce a new momentum variable $\tilde{\mathbf{p}} = \mathbf{p} - q\mathbf{A}$. Then, solutions to (2.44) are of the form

$$\begin{aligned} \psi_A^+ &= u(s,\mathbf{p})e^{-i(\epsilon_\mathbf{p}+qA_0)t+i(\mathbf{p}-q\mathbf{A})\cdot\mathbf{r}} &&= \tilde{u}(s,\tilde{\mathbf{p}})e^{-i(\tilde{\epsilon}_{\tilde{\mathbf{p}}}+qA_0)t+i\tilde{\mathbf{p}}\cdot\mathbf{r}}, \\ \psi_A^- &= v(s,\mathbf{p})e^{i(\epsilon_\mathbf{p}-qA_0)t+i(\mathbf{p}-q\mathbf{A})\cdot\mathbf{r}} &&= \tilde{v}(s,\tilde{\mathbf{p}})e^{i(\tilde{\epsilon}_{\tilde{\mathbf{p}}}-qA_0)t+i\tilde{\mathbf{p}}\cdot\mathbf{r}}, \end{aligned} \tag{2.47}$$

where $\tilde{\epsilon}_{\tilde{\mathbf{p}}} = \sqrt{m^2 + \left(\tilde{\mathbf{p}} + q\mathbf{A}\right)^2}$, $\tilde{u}(s,\tilde{\mathbf{p}}) = u(s,\tilde{\mathbf{p}} + q\mathbf{A})$ and $\tilde{v}(s,\tilde{\mathbf{p}}) = v(s,\tilde{\mathbf{p}} + q\mathbf{A})$. These new spinors $\tilde{u}$ and $\tilde{v}$ satisfy all of the relations (2.41) just with $\epsilon_\mathbf{p}$ replaced by $\tilde{\epsilon}_\mathbf{p}$. Thus, the solutions $\psi_A^\pm$ also satisfy the orthogonality condition (2.42) and the completeness relation (2.43) if $\tilde{\epsilon}_\mathbf{p}$ is used instead of $\epsilon_\mathbf{p}$. We will often use this form of the solutions (see chapter 3).

2.6 Canonical quantisation

So far, we have only been dealing with a classical field theory (albeit one with a field that is not known in classical physics, i.e. the spinor field). To obtain a quantum field theory we use the procedure of canonical

quantisation: We promote the fields and momenta to field operators denoted by a hat,

$$\begin{aligned} \psi &\to \hat{\Psi}, \qquad \overline{\psi} \to \hat{\overline{\Psi}}, \qquad A_\mu \to \hat{A}_\mu, \\ \Pi_\psi &\to \hat{\Pi}_\Psi, \quad \Pi_{\overline{\psi}} \to \hat{\Pi}_{\overline{\Psi}}, \quad \Pi_A^\mu \to \hat{\Pi}_A^\mu, \end{aligned} \tag{2.48}$$

and we impose commutation or anticommutation relations on the fields and their respective momenta. For the (fermionic) spinor field we use the equal-time anticommutation relation

$$\left\{\hat{\Psi}_a(t,\mathbf{r}), (\hat{\Pi}_\Psi)_b(t,\mathbf{r}')\right\} = i\delta_{ab}\delta(\mathbf{r}-\mathbf{r}') \tag{2.49}$$

which is equivalent to

$$\left\{\hat{\Psi}_a(t,\mathbf{r}), \hat{\Psi}_b^\dagger(t,\mathbf{r}')\right\} = \delta_{ab}\delta(\mathbf{r}-\mathbf{r}'). \tag{2.50}$$

On the other hand, for the Maxwell field we use the bosonic commutation relation

$$\left[\hat{A}_\mu(t,\mathbf{r}), (\hat{\Pi}_A^\nu)(t,\mathbf{r}')\right] = i\delta_\mu^\nu\delta(\mathbf{r}-\mathbf{r}'). \tag{2.51}$$

Because the Maxwell field and the spinor field are independent, their field operators commute with each other.

We will use the interaction picture in the following. To do this we divide the Hamiltonian

$$\hat{H} = \int \mathrm{d}^3 r\, (\hat{\mathscr{H}}_e + \hat{\mathscr{H}}_\gamma + \hat{\mathscr{H}}_{\mathrm{int}}) \tag{2.52}$$

into two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}} \tag{2.53}$$

where

$$\hat{H}_0 = \int \mathrm{d}^3 r\, (\hat{\mathscr{H}}_e + \hat{\mathscr{H}}_\gamma), \quad \hat{H}_{\mathrm{int}} = \int \mathrm{d}^3 r\, \hat{\mathscr{H}}_{\mathrm{int}}. \tag{2.54}$$

In the interaction picture the dynamics due to $\hat{H}_{\mathrm{int}}$ is encoded in the state vectors whereas the $\hat{H}_0$ dynamics is carried by the operators. This divison is of course completely arbitrary but we have chosen it that way based on the assumption that we are able to calculate exact solutions (i.e. state vectors) for the Hamiltonian $\hat{H}_0$. Then, the interaction between the fields is taken into account approximately using perturbation theory.

To see how this works, consider the amplitude of going from a state $|i\rangle$ to a state $\langle f|$

$$\mathscr{M}_{i\to f} = \langle f|\, \hat{S}\, |i\rangle \tag{2.55}$$

The states $|i\rangle$ and $\langle f|$ are eigenstates of $\hat{H}_0$ and the time evolution in between is governed by the scattering operator $\hat{S}$

$$\hat{S} = \mathscr{T} \exp\left(-i \int \mathrm{d}t\, \hat{H}_{\mathrm{int}}\right) \tag{2.56}$$

where $\mathscr{T}$ denotes that the following product of operators should be time-ordered, i.e. operators at earlier times should appear to the right of operators taken at later times. Using the series expansion of the exponential function and the definition of the interaction hamiltonian according to (2.54) and (2.20) we get[14]

$$\hat{S} = 1 - iq \int \mathrm{d}t \int \mathrm{d}^3 r\, \hat{\bar{\Psi}} \gamma^\mu \hat{A}_\mu \hat{\Psi} + \mathcal{O}(q^2). \tag{2.57}$$

Hence, we may find an expression for the amplitude $\mathscr{M}_{i\to f}$ as a power series expansion in the charge q (or equivalently in the fine structure constant $\alpha_{\mathrm{QED}} = q^2/(4\pi\varepsilon_0)$).

Because we have included the interaction of the spinor field with the external electromagnetic field in the Hamiltonian $\hat{H}_0$ the influence of the external classical field is taken into account exactly. In that sense, Sauter-Schwinger pair production due to an external field is a non-perturbative effect. On the other hand, interactions of electrons and positrons with single photons are governed by the Hamiltonian $\hat{H}_{\mathrm{int}}$ and hence are only correct to a specific order in perturbation theory. Therefore, effects like the Breit-Wheeler effect or enhancement of Sauter-Schwinger pair production using a high-energy photon are perturbative effects.

[14] The product of the field operators here is supposed to be in (generalized) normal ordered form, i.e. annihilation operators are to the right of creation operators. By Wick's theorem, the time-ordered product of operators is equal to the normal-ordered product plus normal-ordered products after all possible contractions [PS95, pp. 88 sqq.]. For a product of field operators all taken at the same point there are no contractions and thus the time-ordered product is equal to the normal-ordered product.

2.7 Mode expansion

As stated before, the time dependence of the operators in the interaction picture is governed by $\hat{H}_0$. Consequently, the time evolution of the field operator $\hat{\Psi}$ is given by the Heisenberg equation of motion

$$i\partial_t \hat{\Psi} = -\left[\hat{H}_0, \hat{\Psi}\right] = -\left[\hat{H}_e, \hat{\Psi}\right] \tag{2.58}$$

where in the last step we used that $\hat{H}_\gamma$ only contains the Maxwell field operator $\hat{A}_\mu$ which commutes with $\hat{\Psi}$. Using the anticommutation relation (2.50) one can show that the remaining commutator reduces to

$$i\partial_t \hat{\Psi} = H_D \hat{\Psi} \tag{2.59}$$

which is the Dirac equation with an external field as in (2.32). The only difference is that this time the solution of the Dirac equation is a field operator instead of a single-particle wave function.

However, this means that we can expand the field operators in terms of (single-particle) solutions of the covariant Dirac equation. We assume that the electromagnetic field is switched off for $t \longrightarrow -\infty$ and $t \longrightarrow \infty$ and the mass is constant at these times. Let there be a set of solutions $\{\psi^{\pm}_{\text{in}}(j;t,\mathbf{r})\}$ with the initial condition that $\psi^{+}_{\text{in}}(j;t,\mathbf{r})$ and $\psi^{-}_{\text{in}}(j;t,\mathbf{r})$ initally (i.e. for $t \longrightarrow -\infty$) behave like positive- and negative-energy plane-wave solutions, respectively, where j denotes the inital mode. The set should be orthogonal and complete as defined in (2.42) and (2.43). Analogously, assume that there is another complete and orthogonal set of solutions $\{\psi^{\pm}_{\text{out}}(j;t,\mathbf{r})\}$ that contains solutions that behave like plane waves for $t \longrightarrow \infty$. These two sets can be used as bases for the space of solutions of the Dirac equation, i.e. the field operator $\hat{\Psi}$ can be expanded in terms of functions from either set, i.e.

$$\begin{aligned}\hat{\Psi}(t,\mathbf{r}) &= \sum_j \sqrt{\frac{m_{\text{in}}}{(2\pi)^3 \epsilon^{\text{in}}_j}}\left[\hat{a}_{\text{in}}(j)\psi^{+}_{\text{in}}(j;t,\mathbf{r}) + \hat{b}^{\dagger}_{\text{in}}(j)\psi^{-}_{\text{in}}(j;t,\mathbf{r})\right] \\ &= \sum_j \sqrt{\frac{m_{\text{out}}}{(2\pi)^3 \epsilon^{\text{out}}_j}}\left[\hat{a}_{\text{out}}(j)\psi^{+}_{\text{out}}(j;t,\mathbf{r}) + \hat{b}^{\dagger}_{\text{out}}(j)\psi^{-}_{\text{out}}(j;t,\mathbf{r})\right];\end{aligned} \tag{2.60}$$

where $\hat{a}_{\text{in}}(j)$, $\hat{b}_{\text{in}}(j)$ and $\hat{a}_{\text{out}}(j)$, $\hat{b}_{\text{out}}(j)$ are the initial and final electron and positron annihilation operators, respectively[15]. Moreover, m_{in} and m_{out} is the value of the mass for $t \longrightarrow -\infty$ and for $t \longrightarrow \infty$, respectively. Similarly, ϵ^{in}_j and ϵ^{out}_j is the initial and final energy, respectively. Depending on the exact form of the solutions they are either

[15] When there is no electric field and the mass is constant, the two sets of solutions actually coincide and so do the in- and out-operators. In this case, we recover ordinary QED without any external field.

$$\epsilon^{\text{in}}_j = \sqrt{m^2_{\text{in}} + \mathbf{p}^2_j}, \quad \epsilon^{\text{out}}_j = \sqrt{m^2_{\text{out}} + \mathbf{p}^2_j} \tag{2.61}$$

or

$$\epsilon^{\text{in}}_j = \sqrt{m^2_{\text{in}} + \left(\mathbf{p}_j + q\mathbf{A}_{\text{in}}\right)^2}, \quad \epsilon^{\text{out}}_j = \sqrt{m^2_{\text{out}} + \left(\mathbf{p}_j + q\mathbf{A}_{\text{out}}\right)^2}, \tag{2.62}$$

see section 2.5.

We require the positron and electron operators to satisfy the canonical anticommutation relations

$$\left\{\hat{a}(j), \hat{a}^{\dagger}(l)\right\} = \delta_{jl}, \quad \left\{\hat{b}(j), \hat{b}^{\dagger}(l)\right\} = \delta_{jl}, \tag{2.63}$$

with all other anticommutators vanishing. Then, the expansions (2.60)

fulfil the anticommutation relation (2.50) for the field operator. This can be checked easily by inserting the expansions into the anticommutation relation and using the completeness relation of the mode functions.

We define two vacuum states which are annihilated by either the initial annihilation operators or the final ones:

$$\hat{a}_{\text{in}}(j)\,|0_{\text{in}}\rangle = \hat{b}_{\text{in}}(j)\,|0_{\text{in}}\rangle = 0, \quad \hat{a}_{\text{out}}(j)\,|0_{\text{out}}\rangle = \hat{b}_{\text{out}}(j)\,|0_{\text{out}}\rangle = 0. \tag{2.64}$$

To see that these operators actually are related to electrons and positrons, we calculate the Hamilton operator $\hat{H}_e$ and the charge operator $\hat{Q}$ using the mode expansion (2.60). First observe that by making use of the equation of motion for the field operator, we can write the Hamiltonian $\hat{H}_e$ as

$$\hat{H}_e = \int \mathrm{d}^3 r\, \hat{\Psi}^\dagger H_D \hat{\Psi} = i \int \mathrm{d}^3 r\, \hat{\Psi}^\dagger \partial_t \hat{\Psi}. \tag{2.65}$$

We insert the mode expansion (2.60) in terms of the in-solutions into this. Because $\psi^{\pm}_{\text{in}}$ is a plane wave solution for $t \longrightarrow -\infty$ we have

$$\lim_{t\longrightarrow-\infty} \partial_t \psi^{\pm}_{\text{in}}(j;t,\mathbf{r}) = \mp i\epsilon_j \lim_{t\longrightarrow-\infty} \psi^{\pm}_{\text{in}}(j;t,\mathbf{r}). \tag{2.66}$$

We employ the orthonormality of the $\psi^{\pm}_{\text{in}}$ and obtain

$$\begin{aligned} \lim_{t\longrightarrow-\infty} \hat{H}_e &= \sum_j \epsilon_j \big[\hat{a}^\dagger_{\text{in}}(j)\hat{a}_{\text{in}}(j) - \hat{b}_{\text{in}}(j)\hat{b}^\dagger_{\text{in}}(j)\big] \\ &= \sum_j \epsilon_j \big[\hat{a}^\dagger_{\text{in}}(j)\hat{a}_{\text{in}}(j) + \hat{b}^\dagger_{\text{in}}(j)\hat{b}_{\text{in}}(j)\big] - \delta(0). \end{aligned} \tag{2.67}$$

Because $\hat{a}^\dagger_{\text{in}}(j)\hat{a}_{\text{in}}(j)$ and $\hat{b}^\dagger_{\text{in}}(j)\hat{b}_{\text{in}}(j)$ are the number operators for electrons and positrons, respectively, we see that each electron and each positron contributes an energy of ϵ_j to the total energy. The divergent term $-\delta(0)$ corresponds to the zero point energy of the vacuum. We could get rid of it by using normal ordering in the Hamiltonian. In a similar manner, we can show that

$$\lim_{t\longrightarrow\infty} \hat{H}_e = \sum_j \epsilon_j \big[\hat{a}^\dagger_{\text{out}}(j)\hat{a}_{\text{out}}(j) + \hat{b}^\dagger_{\text{out}}(j)\hat{b}_{\text{out}}(j)\big] - \delta(0). \tag{2.68}$$

Thus, the Hamilton operator is diagonalised by the in-solutions at $t \longrightarrow -\infty$ and by the out-solutions at $t \longrightarrow \infty$. Now consider the charge operator

$$\hat{Q} = q \int \mathrm{d}^3 r\, \hat{\Psi}^\dagger \Psi \tag{2.69}$$

which we get from the classical theory (2.14) by replacing ψ with $\hat{\Psi}$. In complete analogy to the calculation of the Hamiltonian above, we then find

$$\begin{aligned}\hat{Q} &= q\sum_j \left[\hat{a}^\dagger_{\text{in}}(j)\hat{a}_{\text{in}}(j) - \hat{b}^\dagger_{\text{in}}(j)\hat{b}_{\text{in}}(j)\right] + \delta(0)\\ &= q\sum_j \left[\hat{a}^\dagger_{\text{out}}(j)\hat{a}_{\text{out}}(j) - \hat{b}^\dagger_{\text{out}}(j)\hat{b}_{\text{out}}(j)\right] + \delta(0).\end{aligned} \tag{2.70}$$

As expected, an electron (created by $\hat{a}^\dagger_{\text{in}}(j)$ or $\hat{a}^\dagger_{\text{out}}(j)$) contributes a charge of q to the total charge while a positron (created by $\hat{b}^\dagger_{\text{in}}(j)$ or $\hat{b}^\dagger_{\text{out}}(j)$) contributes $-q$.

Instead of expressing the field operator in terms of the electron and positron operators, using the orthogonality property of the mode functions we can also write the electron and positron operators in terms of the mode functions and the field operator:

$$\begin{aligned}\hat{a}_{\text{in}}(j) &= \sqrt{\frac{m_{\text{in}}}{(2\pi)^3\epsilon^{\text{in}}_j}}\left(\psi^+_{\text{in}}(j),\hat{\Psi}\right), \qquad \hat{b}^\dagger_{\text{in}}(j) = \sqrt{\frac{m_{\text{in}}}{(2\pi)^3\epsilon^{\text{in}}_j}}\left(\psi^-_{\text{in}}(j),\hat{\Psi}\right),\\ \hat{a}_{\text{out}}(j) &= \sqrt{\frac{m_{\text{out}}}{(2\pi)^3\epsilon^{\text{out}}_j}}\left(\psi^+_{\text{out}}(j),\hat{\Psi}\right), \quad \hat{b}^\dagger_{\text{out}}(j) = \sqrt{\frac{m_{\text{out}}}{(2\pi)^3\epsilon^{\text{out}}_j}}\left(\psi^-_{\text{out}}(j),\hat{\Psi}\right).\end{aligned} \tag{2.71}$$

If we use the respective other expansion of the field operator in every expression, we get a relationship between in- and out-operators

$$\begin{aligned}\hat{a}_{\text{in}}(j) &= \sum_l \left[B^{++}_{jl}\,\hat{a}_{\text{out}}(l) + B^{+-}_{jl}\,\hat{b}^\dagger_{\text{out}}(l)\right],\\ \hat{b}^\dagger_{\text{in}}(j) &= \sum_l \left[B^{-+}_{jl}\,\hat{a}_{\text{out}}(l) + B^{--}_{jl}\,\hat{b}^\dagger_{\text{out}}(l)\right],\\ \hat{a}_{\text{out}}(j) &= \sum_l \left[(B^{++}_{lj})^*\,\hat{a}_{\text{in}}(l) + (B^{-+}_{lj})^*\,\hat{b}^\dagger_{\text{in}}(l)\right],\\ \hat{b}^\dagger_{\text{out}}(j) &= \sum_l \left[(B^{+-}_{lj})^*\,\hat{a}_{\text{in}}(l) + (B^{--}_{lj})^*\,\hat{b}^\dagger_{\text{in}}(l)\right],\end{aligned} \tag{2.72}$$

where the Bogoliubov coefficients are defined as

$$B^{\kappa\lambda}_{jl} = \frac{1}{(2\pi)^3}\sqrt{\frac{m_{\text{in}}m_{\text{out}}}{\epsilon^{\text{in}}_j\epsilon^{\text{out}}_l}}\left(\psi^\kappa_{\text{in}}(j),\,\psi^\lambda_{\text{out}}(l)\right). \tag{2.73}$$

From their definition it is clear that the Bogoliubov coefficients not only constitute the transformation between in- and out-operators but also

between in- and out-solutions, i.e.

$$\begin{aligned}\psi^{\lambda}_{\text{out}}(l) &= \sum_{j,\kappa}\sqrt{\frac{\epsilon^{\text{out}}_{l}\, m_{\text{in}}}{\epsilon^{\text{in}}_{j}\, m_{\text{out}}}}\, B^{\kappa\lambda}_{jl}\,\psi^{\kappa}_{\text{in}}(j),\\ \psi^{\kappa}_{\text{in}}(j) &= \sum_{l,\lambda}\sqrt{\frac{\epsilon^{\text{in}}_{j}\, m_{\text{out}}}{\epsilon^{\text{out}}_{l}\, m_{\text{in}}}}\,(B^{\kappa\lambda}_{jl})^{*}\,\psi^{\lambda}_{\text{out}}(l)\end{aligned} \tag{2.74}$$

By making use of the orthogonality condition (2.42) and the completeness relation (2.43), we can prove the following identity for the Bogoliubov coefficients:

$$\sum_{n,\xi} B^{\kappa\xi}_{jn}\left(B^{\lambda\xi}_{ln}\right)^{*} = \sum_{n,\xi} B^{\xi\kappa}_{nj}\left(B^{\xi\lambda}_{nl}\right)^{*} = \delta_{\kappa\lambda}\delta_{jl} \tag{2.75}$$

This unitarity condition ensures that the relation between in- and out-operators given in (2.72) is one-to-one. If we define matrices $B^{\kappa\lambda}$ with components $B^{\kappa\lambda}_{jl}$ the condition (2.75) may also be written using matrix products:

$$\sum_{\xi} B^{\kappa\xi}\left(B^{\lambda\xi}\right)^{\dagger} = \sum_{\xi} B^{\xi\kappa}\left(B^{\xi\lambda}\right)^{\dagger} = \delta_{\kappa\lambda}. \tag{2.76}$$

We write these out for all combinations of $\kappa = +, -$ and $\lambda = +, -$,

$$\begin{aligned}B^{++}\left(B^{++}\right)^{\dagger} + B^{+-}\left(B^{+-}\right)^{\dagger} &= B^{++}\left(B^{++}\right)^{\dagger} + B^{-+}\left(B^{-+}\right)^{\dagger} = 1,\\ B^{-+}\left(B^{-+}\right)^{\dagger} + B^{--}\left(B^{--}\right)^{\dagger} &= B^{+-}\left(B^{+-}\right)^{\dagger} + B^{--}\left(B^{--}\right)^{\dagger} = 1,\\ B^{++}\left(B^{-+}\right)^{\dagger} + B^{+-}\left(B^{--}\right)^{\dagger} &= B^{++}\left(B^{+-}\right)^{\dagger} + B^{-+}\left(B^{--}\right)^{\dagger} = 0,\end{aligned} \tag{2.77}$$

which immediately lead to

$$B^{++}\left(B^{++}\right)^{\dagger} = B^{--}\left(B^{--}\right)^{\dagger}, \quad B^{+-}\left(B^{+-}\right)^{\dagger} = B^{-+}\left(B^{-+}\right)^{\dagger}. \tag{2.78}$$

With these matrices we can also write the relationship between in- and out-operators (2.72) using matrix-vector products

$$\begin{aligned}\hat{a}_{\text{in}} &= B^{++}\hat{a}_{\text{out}} + B^{+-}\hat{b}^{\dagger}_{\text{out}},\\ \hat{b}^{\dagger}_{\text{in}} &= B^{-+}\hat{a}_{\text{out}} + B^{--}\hat{b}^{\dagger}_{\text{out}},\\ \hat{a}_{\text{out}} &= \left(B^{++}\right)^{\dagger}\hat{a}_{\text{in}} + \left(B^{-+}\right)^{\dagger}\hat{b}^{\dagger}_{\text{in}},\\ \hat{b}^{\dagger}_{\text{out}} &= \left(B^{+-}\right)^{\dagger}\hat{a}_{\text{in}} + \left(B^{--}\right)^{\dagger}\hat{b}^{\dagger}_{\text{in}},\end{aligned} \tag{2.79}$$

where, for example, $\hat{a}_{\text{in}}$ is a tuple that contains the operators $\hat{a}_{\text{in}}(j)$ for all modes j.

For calculating amplitudes of processes we want to write these relations in a form where the annihilation out-operators and the creation in-operators only contain operators that either annihilate the out-vacuum to their left or the in-vacuum to their right, i.e.

$$\begin{aligned}
\hat{a}^\dagger_{\text{in}} &= \hat{a}^\dagger_{\text{out}}(B^{++})^{-1} - \hat{b}_{\text{in}}B^{-+}(B^{++})^{-1},\\
\hat{b}^\dagger_{\text{in}} &= -[(B^{--})^\dagger]^{-1}(B^{+-})^\dagger\hat{a}_{\text{in}} + [(B^{--})^\dagger]^{-1}\hat{b}^\dagger_{\text{out}},\\
\hat{a}_{\text{out}} &= (B^{++})^{-1}\hat{a}_{\text{in}} - (B^{++})^{-1}B^{+-}\hat{b}^\dagger_{\text{out}},\\
\hat{b}_{\text{out}} &= -\hat{a}^\dagger_{\text{out}}(B^{-+})^\dagger[(B^{--})^\dagger]^{-1} + \hat{b}_{\text{in}}[(B^{--})^\dagger]^{-1}.
\end{aligned} \tag{2.80}$$

These can then be used to write down another mode expansion for the spinor field operator and its adjoint: To calculate amplitudes as described in section 2.6 we need to write the field operators in (generalized) normal form, i.e. they should only contain operators that annihilate the in-vacuum to their right or the out-vacuum to their left (see [FGS91, sections 2.2 and 2.4]). Starting from the original mode expansions (2.60) we may obtain such a (generalized) normal form by employing the relations (2.80) and (2.75). After some manipulations, we get

$$\begin{aligned}
\hat{\Psi}(t,\mathbf{r}) = \sum_{j,l}\Bigg\{&\sqrt{\frac{m_{\text{out}}}{(2\pi)^3\epsilon^{\text{out}}_j}}\Big[(B^{++})^{-1}\Big]_{jl}\hat{a}_{\text{in}}(l)\psi^+_{\text{out}}(j;t,\mathbf{r})\\
&+\sqrt{\frac{m_{\text{in}}}{(2\pi)^3\epsilon^{\text{in}}_j}}\Big[(B^{--})^{-1}\Big]^*_{lj}\hat{b}^\dagger_{\text{out}}(l)\psi^-_{\text{in}}(j;t,\mathbf{r})\Bigg\}
\end{aligned} \tag{2.81}$$

and

$$\begin{aligned}
\hat{\overline{\Psi}}(t,\mathbf{r}) = \sum_{j,l}\Bigg\{&\sqrt{\frac{m_{\text{out}}}{(2\pi)^3\epsilon^{\text{out}}_j}}\Big[(B^{--})^{-1}\Big]^*_{jl}\hat{b}_{\text{in}}(l)\overline{\psi}^-_{\text{out}}(j;t,\mathbf{r})\\
&+\sqrt{\frac{m_{\text{in}}}{(2\pi)^3\epsilon^{\text{in}}_j}}\Big[(B^{++})^{-1}\Big]_{lj}\hat{a}^\dagger_{\text{out}}(l)\overline{\psi}^+_{\text{in}}(j;t,\mathbf{r})\Bigg\}.
\end{aligned} \tag{2.82}$$

When calculating amplitudes that contain products of these operators, we use the (generalized) normal form and the contractions to calculate the time-ordered product of the operators. To obtain the (generalized) normal form of any functional of the spinor field operators, simply use the mode expansions (2.81) and (2.82) and order all terms such that out-creation operators are to the left of in-annihilation operators.

The dynamics of the Maxwell field operator is also determined by the

Heisenberg equation

$$\partial_t \hat{A}^\mu = i\left[\hat{H}_0, \hat{A}^\mu\right] = i\left[\hat{H}_\gamma, \hat{A}^\mu\right]. \tag{2.83}$$

Using the commutation relation (2.51) we get

$$\partial_t \hat{A}^\mu = -\hat{\Pi}^\mu_A \tag{2.84}$$

which is exactly as in the classical theory (compare the definition of the canonical momentum (2.16)). Similarly, we obtain the equation of motion for $\hat{\Pi}^\mu_A$ itself,

$$\partial_t \hat{\Pi}^\mu_A = \partial_j \partial^j \hat{A}^\mu \tag{2.85}$$

which is identical to the classical equation of motion with no external current (compare (2.12)). Thus, $\hat{A}^\mu$ can be expanded into classical solutions of the free Maxwell equations. One finds the expansion to be

$$\hat{A}_\mu = \int \mathrm{d}^3k \sum_\lambda \left[\hat{c}(\lambda, \mathbf{k}) f_\mu(\lambda, \mathbf{k}; t, \mathbf{r}) + \hat{c}^\dagger(\lambda, \mathbf{k}) f^*_\mu(\lambda, \mathbf{k}; t, \mathbf{r})\right] \tag{2.86}$$

where $f_\mu(\lambda, \mathbf{k}; t, \mathbf{r})$ is a plane wave with momentum (or wave vector) $\mathbf{k}$ and polarisation λ, i.e.

$$f_\mu(\lambda, \mathbf{k}; t, \mathbf{r}) = \frac{1}{\sqrt{(2\pi)^3 2\,|\mathbf{k}|}} e^\mu(\lambda, \mathbf{k}) e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{r}} \tag{2.87}$$

where $e^\mu(\lambda, \mathbf{k})$ is the polarisation vector to the polarisation λ. These functions are orthonormal in a specific sense which can be used to show that the operators $\hat{c}(\lambda, \mathbf{k})$ satisfy the commutation relation

$$\left[\hat{c}(\lambda, \mathbf{k}), \hat{c}^\dagger(\lambda', \mathbf{k}')\right] = -\eta_{\lambda\lambda'}\delta(\mathbf{k} - \mathbf{k}'). \tag{2.88}$$

The operators $\hat{c}(\lambda, \mathbf{k})$ and $\hat{c}^\dagger(\lambda, \mathbf{k})$ are the annihilation and creation operators for the photon respectively. Thus,

$$\hat{c}(\lambda, \mathbf{k})\,|0_{\text{in}}\rangle = 0, \quad \hat{c}(\lambda, \mathbf{k})\,|0_{\text{out}}\rangle = 0. \tag{2.89}$$

2.8 Amplitudes

In the preceding sections we have described a framework that enables us to calculate probabilities for many processes. For completeness, we first

briefly consider the Breit-Wheeler process. There, no external field is present. Therefore, in- and out-solutions are identical and consequently the vacuum states and the annihilation and creation operators are the same initially and finally, i.e.

$$|0_{\text{in}}\rangle = |0_{\text{out}}\rangle = |0\rangle\,, \quad \hat{a}_{\text{in}} = \hat{a}_{\text{out}} = \hat{a}, \quad \hat{b}_{\text{in}} = \hat{b}_{\text{out}} = \hat{b}. \tag{2.90}$$

In the initial state $|i\rangle$ we have two photons,

$$|i\rangle = \hat{c}^\dagger(\lambda, \mathbf{k})\hat{c}^\dagger(\lambda', \mathbf{k}')\,|0\rangle\,, \tag{2.91}$$

and in the final state $|f\rangle$ an electron-positron pair has been produced,

$$|f\rangle = \hat{a}^\dagger(l)\hat{b}^\dagger(l')\,|0\rangle\,. \tag{2.92}$$

Thus, the amplitude for a system going from state $|i\rangle$ to state $|f\rangle$ is

$$\mathscr{M}_{\text{BW}} = \langle 0|\,\hat{b}(l')\hat{a}(l)\,\hat{S}\,\hat{c}^\dagger(\lambda, \mathbf{k})\hat{c}^\dagger(\lambda', \mathbf{k}')\,|0\rangle \tag{2.93}$$

with the scattering operator $\hat{S}$ as given in (2.56). Employing the series expansion of $\hat{S}$ in q (2.57) we see that the lowest-order term[16] is linear in q:

$$\mathscr{M}_{\text{BW}} = -iq \int \mathrm{d}t \int \mathrm{d}^3 r\, \langle 0|\,\hat{b}(l')\hat{a}(l)\hat{\overline{\Psi}}\gamma^\mu \hat{A}_\mu \hat{\Psi}\hat{c}^\dagger(\lambda, \mathbf{k})\hat{c}^\dagger(\lambda', \mathbf{k}')\,|0\rangle + \mathscr{O}(q^2) \tag{2.94}$$

where the field operators have to be evaluated at t and $\mathbf{r}$. Using the mode expansions of the field operators this can be further reduced to an expression involving only annihilation and creation operators. The strategy then is to appropriately commute and anticommute the operators such that the annihilation operators are brought into a position where they annihilate the vacuum. The remaining terms produced by this process are the desired result.

This straightforward procedure is not possible anymore when an external field is switched on. In general, due to Sauter-Schwinger pair production because of the external field the initial and the final vacuum state will not be identical anymore. This is conveniently expressed using the *vacuum persistence amplitude*

$$\mathscr{M}_V = \langle 0_{\text{out}}|0_{\text{in}}\rangle\,. \tag{2.95}$$

which measures how much of the original vacuum state is still part of the final vacuum state. Sometimes $1 - |\mathscr{M}_V|^2$ is used to quantify the pair

[16] The q^0 term vanishes because the photon operator $\hat{c}^\dagger$ commutes with both fermion operators $\hat{a}$ and $\hat{b}$.

production probability, for example in the worldline instanton formalism. However, this does not yield any information about how many pairs were produced with which momentum.

Instead, we could calculate the amplitude for going from a state with no electrons or positrons to a state with a single electron-positron pair

$$\mathscr{M}_{e^+e^-} = \langle 0_{\text{out}}|\, \hat{a}_{\text{out}}(j)\hat{b}_{\text{out}}(l)\hat{S}\,|0_{\text{in}}\rangle\,. \tag{2.96}$$

This time, the leading term is of order q^0 because the annihilation out-operators do not annihilate the in-vacuum state. We write $\hat{a}_{\text{out}}(j)$ using the appropriate relation from (2.80) and $\hat{b}_{\text{out}}(l)$ using the appropriate relation from (2.79) and get

$$\begin{aligned}
&\langle 0_{\text{out}}|\, \hat{a}_{\text{out}}(j)\, \hat{b}_{\text{out}}(l)\, |0_{\text{in}}\rangle \\
&= \sum_{h,n} \left[(B^{++})^{-1}\right]_{jh} B^{+-}_{nl}\, \langle 0_{\text{out}}|\, \hat{a}_{\text{in}}(h)\hat{a}^{\dagger}_{\text{in}}(n)\, |0_{\text{in}}\rangle \\
&= \sum_{n} \left[(B^{++})^{-1}\right]_{jn} B^{+-}_{nl}\, \mathscr{M}_V \\
&= \left[(B^{++})^{-1} B^{+-}\right]_{jl} \mathscr{M}_V .
\end{aligned} \tag{2.97}$$

and thus

$$\mathscr{M}_{e^+e^-} = \left[(B^{++})^{-1} B^{+-}\right]_{jl} \mathscr{M}_V + \mathcal{O}(q). \tag{2.98}$$

The probability that a single electron-positron pair is created in the presence of the external field is $P_{e^+e^-} = |\mathscr{M}_{e^+e^-}|^2$. Note that because we are neglecting terms with higher powers of q this result does not include any effects due to the interaction between the created particles mediated by the quantised Maxwell field (i.e. photons).

The probabilities for processes that produce multiple pairs can be obtained analogously. To calculate the number of produced particles we have to compute the expectation value of the final number operators in the initial state (at $t \longrightarrow \infty$),

$$N_{e^+}(j) = \langle 0_{\text{in}}|\, \hat{S}^{\dagger}\hat{b}^{\dagger}_{\text{out}}(j)\hat{b}_{\text{out}}(j)\hat{S}\,|0_{\text{in}}\rangle\,. \tag{2.99}$$

Again, neglecting interactions with the quantised Maxwell field, we have $\hat{S} = 1$. Then, by using the Bogoliubov transformation (2.79), we find

$$N_{e^+}(j) = \sum_{l,n} \left(B^{+-}_{lj}\right)^{*} B^{+-}_{nj}\, \langle 0_{\text{in}}|\, \hat{a}_{\text{in}}(l)\hat{a}^{\dagger}_{\text{in}}(n)\, |0_{\text{in}}\rangle = \sum_{l} \left|B^{+-}_{lj}\right|^2 . \tag{2.100}$$

Analogously, the number of electrons is

$$N_{e^-}(j) = \langle 0_{\text{in}}| \hat{a}^\dagger_{\text{out}}(j)\hat{a}_{\text{out}}(j) |0_{\text{in}}\rangle = \sum_l \left|B^{-+}_{lj}\right|^2 . \tag{2.101}$$

Hence, the total number of created electrons and positrons can be computed by summing the previous two expressions over all modes, i.e.

$$N_{e^+} = \sum_j N_{e^+}(j) = \sum_{j,l} \left|B^{+-}_{lj}\right|^2 , \quad N_{e^-} = \sum_j N_{e^-}(j) = \sum_{j,l} \left|B^{-+}_{lj}\right|^2 . \tag{2.102}$$

Using the relationship for the Bogoliubov coefficients (2.75) we can show that the number of created electrons and positrons is actually equal

$$\begin{aligned} N_{e^+} &= \sum_{j,l} \left|B^{+-}_{lj}\right|^2 \\ &= \sum_j \Big[\sum_{l,\xi} B^{\xi-}_{lj} \left(B^{\xi-}_{lj}\right)^* - \sum_l B^{--}_{lj} \left(B^{--}_{lj}\right)^* \Big] \\ &= \sum_{j,l,\xi} B^{-\xi}_{jl} \left(B^{-\xi}_{jl}\right)^* - \sum_{j,l} B^{--}_{lj} \left(B^{--}_{lj}\right)^* \\ &= \sum_{j,l} B^{-+}_{jl} \left(B^{-+}_{jl}\right)^* \\ &= N_{e^-} \end{aligned} \tag{2.103}$$

and thus charge conservation holds.

Chapter 3

Time-dependent WKB method

In this chapter, we investigate pair production due to time-dependent external fields. In the preceding chapter, we showed that the pair production probability can be obtained from single-particle solutions of the Dirac equation. To calculate these, we use what is often called the WKB method (see [Pop72; DD11b; Lin+15]). The WKB method is most often presented in the context of calculating (semi-classical) approximate solutions of the Schrödinger equation in non-constant potentials. This is actually how the method was used by Wentzel, Kramers and Brillouin[17] whose surnames the name of the method refers to [Wen26; Kra26; Bri26]. However, the method is far more general than that. It may be used for linear differential equations where the term with the highest-order derivative is small in some sense. Then, an ansatz for the solution of the form $\exp(iA)$ is inserted into the equation where A is expanded into a power series for the small parameter. By balancing terms, an approximate solution can then be found [BO78, pp. 484 sqq.].

[17] There are even more scholars who deserve the method to bear their name, for example Liouville, Green or Jeffreys; see [Din73, pp. 317 sqq.] for an historical overview.

The method we use in this chapter for approximating solutions of the Dirac equation proceeds similarly. First, the space dependence of the Dirac spinor is separated. We then insert an ansatz of the form $\alpha \exp(iS) + \beta \exp(-iS)$ into the remaining time-dependent ordinary differential equation, leading to equations for the coefficients α and β. Then, suitable approximations to solve these are used. The difference to the general WKB ansatz described above is that here the phase S is only the leading-order term of the phase A in the general WKB theory. The remainder of the power series for A is contained in the prefactors α and β. This procedure essentially separates different scales of oscillations in the wave function: The fastest oscillations are contained in the exponential $\exp(\pm iS)$ whereas the prefactors α or β are (in comparison) only slowly varying. The function S is closely related to the classical action and Hamilton-Jacobi theory; we will return to this once we develop the spacetime-dependent theory (see chapter 4).

3.1 Riccati equation

We want to calculate solutions of the Dirac equation in the presence of a time-dependent electric field in 1+1 dimension. We work in the gauge

$$A_0 = 0, \quad A_1 = A(t) \quad \Longrightarrow E(t) = \dot{A}(t). \tag{3.1}$$

Additionally, we assume the electron mass $m(t)$ to be time-dependent as well as we want to consider the case of a spacetime-dependent mass later (see chapter 5 which also contains a motivation for the physical relevance of a non-constant electron mass). Thus, the coefficients in the covariant Dirac equation

$$\Big\{ i\gamma^\mu \big[\partial_\mu + iqA_\mu(t)\big] - m(t) \Big\} \psi = 0 \tag{3.2}$$

are only time-dependent. We separate the spatial dependence of the wave function, i.e.

$$\psi(t,x) = \psi_p(t) e^{ipx}, \tag{3.3}$$

and obtain

$$\Big\{ i\gamma^0 \partial_t - \gamma^1 \big[p + qA(t)\big] - m(t) \Big\} \psi_p = 0 \tag{3.4}$$

as an ordinary differential equation for the Fourier coefficients $\psi_p(t)$. Note that the different modes are decoupled, i.e. there is no scattering to other modes. Multiplying by γ^0 and rearranging terms yields the Hamiltonian form

$$i\partial_t \psi_p = \Big\{ \gamma^0\gamma^1 \big[p + qA(t)\big] + \gamma^0 m(t) \Big\} \psi_p = H_p(t)\psi_p. \tag{3.5}$$

Because $H_p^2(t) = m^2(t) + \big[p + qA(t)\big]^2 = \Omega_p^2(t)$, the eigenvalues of $H_p(t)$ are $\pm\Omega_p(t)$ with

$$\Omega_p(t) = \sqrt{m^2(t) + \big[p + qA(t)\big]^2}. \tag{3.6}$$

The eigenvectors of $H_p(t)$ are defined by

$$H_p(t) u_{\pm,p}(t) = \pm\Omega_p(t) u_{\pm,p}(t). \tag{3.7}$$

Since $H_p^\dagger(t) = H_p(t)$, the Hamilton operator $H_p(t)$ is self-adjoint with respect to the inner product $\langle u, v \rangle = u^\dagger v$. Thus, by the spectral theorem the eigenvectors $u_{+,p}$ and $u_{-,p}$ are orthogonal

$$u_{\pm,p}^\dagger u_{\mp,p} = 0 \tag{3.8}$$

and can be normalised according to

$$u_{\pm,p}^{\dagger} u_{\pm,p} = 1. \tag{3.9}$$

We expand the spinor ψ_p in terms of $u_{+,p}$ and $u_{-,p}$,

$$\psi_p(t) = \alpha_p(t) u_{+,p}(t) e^{-i\varphi_p(t)} + \beta_p(t) u_{-,p}(t) e^{i\varphi_p(t)}, \tag{3.10}$$

where we have separated the phase[18]

$$\varphi_p(t) = \int_{t_0}^{t} \mathrm{d}t'\, \Omega_p(t'). \tag{3.11}$$

[18] The lower limit of integration has no influence on pair production. Shifting it corresponds to adding a constant phase to α and β. As we will see in the next section only the moduli of α and β are relevant for the pair production probability (see also the remarks after (3.46)).

Upon inserting the expansion for ψ_p into the Hamiltonian form of the Dirac equation (3.5) and multiplying it once by $u_{+,p}^{\dagger}$ and once by $u_{-,p}^{\dagger}$ from the left we get the two coupled equations

$$\begin{aligned} \dot{\alpha_p} &= -\alpha_p\, u_{+,p}^{\dagger} \dot{u}_{+,p} - \beta_p e^{2i\varphi_p}\, u_{+,p}^{\dagger} \dot{u}_{-,p}, \\ \dot{\beta_p} &= -\beta_p\, u_{-,p}^{\dagger} \dot{u}_{-,p} - \alpha_p e^{-2i\varphi_p}\, u_{-,p}^{\dagger} \dot{u}_{+,p}. \end{aligned} \tag{3.12}$$

To obtain the coefficients in these equations we need to calculate the inner products of $u_{\pm}$ with their time derivatives. Differentiating (3.8) and (3.9) with respect to time gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} u_{\pm,p}^{\dagger} u_{\mp,p} = \dot{u}_{\pm,p}^{\dagger} u_{\mp,p} + u_{\pm,p}^{\dagger} \dot{u}_{\mp,p} \quad \Longrightarrow \quad \dot{u}_{\pm,p}^{\dagger} u_{\mp,p} = -u_{\pm,p}^{\dagger} \dot{u}_{\mp,p}, \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t} u_{\pm,p}^{\dagger} u_{\pm,p} = \dot{u}_{\pm,p}^{\dagger} u_{\pm,p} + \underbrace{u_{\pm,p}^{\dagger} \dot{u}_{\pm,p}}_{=(\dot{u}_{\pm,p}^{\dagger} u_{\pm,p})^*} \quad \Longrightarrow \quad \Re\left(\dot{u}_{\pm,p}^{\dagger} u_{\pm,p}\right) = 0. \end{aligned} \tag{3.13}$$

Additionally, we can calculate

$$\begin{aligned} u_{+,p}^{\dagger} \dot{u}_{-,p} &= \frac{1}{2}\Big[u_{+,p}^{\dagger} \dot{u}_{-,p} - u_{+,p}^{\dagger} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Omega_p} H_p u_{-,p}\Big] \\ &= \frac{1}{2}\Big[u_{+,p}^{\dagger} \dot{u}_{-,p} - \frac{1}{\Omega_p} u_{+,p}^{\dagger} \dot{H}_p u_{-,p} - \frac{1}{\Omega_p} u_{+,p}^{\dagger} H_p \dot{u}_{-,p}\Big] \\ &= -\frac{1}{2\Omega_p} u_{+,p}^{\dagger} \dot{H}_p u_{-,p} \\ &= \left(\dot{u}_{-,p}^{\dagger} u_{+,p}\right)^* \end{aligned} \tag{3.14}$$

Using these identities, the equations (3.12) for α_p and β_p can be written as

$$\begin{aligned}\dot{\alpha_p} &= -i\alpha_p\,\Im\Big(u_{+,p}^\dagger \dot{u}_{+,p}\Big) + \frac{1}{2\Omega_p}\beta_p e^{2i\varphi_p}\, u_{+,p}^\dagger \dot{H}_p u_{-,p},\\ \dot{\beta_p} &= -i\beta_p\,\Im\Big(u_{-,p}^\dagger \dot{u}_{-,p}\Big) - \frac{1}{2\Omega_p}\alpha_p e^{-2i\varphi_p}\left(u_{+,p}^\dagger \dot{H}_p u_{-,p}\right)^*.\end{aligned} \tag{3.15}$$

Note that $\Im\Big(\dot{u}_\pm^\dagger u_\pm\Big)$ is nonzero only when $u_\pm$ contains a time-dependent phase. A phase may always be shifted to α or β by redefining them; such a phase change is irrelevant for pair production as we will see later. Consequently, the terms that contain $\Im\Big(\dot{u}_\pm^\dagger u_\pm\Big)$ in (3.15) can be removed by redefining α and β. Therefore, in the following we will just neglect the terms with $\Im\Big(\dot{u}_\pm^\dagger u_\pm\Big)$.

Using the definition $\mathscr{R}_p = \beta_p/\alpha_p$ we can derive a single differential equation from the equations for α_p and β_p,

$$\begin{aligned}\dot{\mathscr{R}}_p = \frac{\dot{\beta_p}}{\alpha_p} - \mathscr{R}_p^2\frac{\dot{\alpha_p}}{\beta_p} = &-\frac{\Re\Big(u_{+,p}^\dagger \dot{H}_p u_{-,p}\Big)}{2\Omega_p}\Big[e^{-2i\varphi_p} + \mathscr{R}_p^2 e^{2i\varphi_p}\Big]\\ &+\frac{i\Im\Big(u_+^\dagger \dot{H}_p u_-\Big)}{2\Omega_p}\quad\Big[e^{-2i\varphi_p} - \mathscr{R}_p^2 e^{2i\varphi_p}\Big].\end{aligned} \tag{3.16}$$

Depending on the phase difference between $u_{+,p}$ and $u_{-,p}$ this may be reduced to a form where we only have the term with the plus sign in the square brackets or the one with the minus sign[19]. For example when we use $\gamma^0 = \sigma_z$ and $\gamma^1 = i\sigma_x$ and assume vanishing pase difference between the spinors, the orthonormalised eigenvectors $u_{\pm,p}$ may be written as

$$u_{+,p} = \frac{1}{\sqrt{1+\lambda_p^2}}\begin{pmatrix}1\\ -i\lambda_p\end{pmatrix}, \qquad u_{-,p} = \frac{1}{\sqrt{1+\lambda_p^2}}\begin{pmatrix}-i\lambda_p\\ 1\end{pmatrix}, \tag{3.17}$$

where

$$\lambda_p(t) = \frac{p + qA(t)}{m(t) + \Omega_p(t)}. \tag{3.18}$$

Then (3.16) simplifies to

$$\dot{\mathscr{R}}_p = i\frac{mqE - \big(p+qA\big)\dot{m}}{2\Omega_p^2}\Big[e^{-2i\varphi_p} - \mathscr{R}_p^2 e^{2i\varphi_p}\Big]. \tag{3.19}$$

At this point, it should be stressed that up to here this is still exact. The Riccati equation (3.19) is completely equivalent to the covariant Dirac equation we started with as long as the separation into exponential and prefactor does not break down (compare section 5.1).

[19] This reveals that the sign within the brackets has no physical meaning and is not related to the fermionic or bosonic nature of the particles in question (compare [DD11b; Lin+15]).

However, under the assumption that $|\mathscr{R}_p(t)|^2$ is small, a linearised version of the Riccati equation (3.19), where the term proportional to $\mathscr{R}_p^2$ is neglected, is often used. $\mathscr{R}_p(t)$ is then given by

$$\mathscr{R}_p(t) \approx i \int \mathrm{d}t \, \frac{mqE - \big(p + qA\big)\dot{m}}{2\Omega_p^2} e^{-2i\varphi_p}. \tag{3.20}$$

The Bogoliubov coefficients α and β are not completely independent. In fact, they are related via the conservation law that stems from the global $U(1)$ symmetry. For a purely time-dependent problem, the conserved current j^μ from (2.9) does not depend on x. Thus, the conservation law simplifies to $\partial_t j^0 = 0$ which is equivalent to $j^0 = q\psi^\dagger\psi = \text{const}$. We assume that initially we only have a single positive-energy plane wave with momentum p, i.e.

$$\alpha_p(t \longrightarrow -\infty) = 1, \quad \beta_p(t \longrightarrow -\infty) = 0. \tag{3.21}$$

Then, we get

$$j^0 = q\big[|\alpha_p(t)|^2 + |\beta_p(t)|^2\big] = q \tag{3.22}$$

or equivalently

$$|\alpha_p(t)|^2 + |\beta_p(t)|^2 = 1. \tag{3.23}$$

Then, the following relations between $\mathscr{R}_p$ and α_p or β_p hold:

$$|\alpha_p(t)|^2 = \frac{1}{1 + |\mathscr{R}_p(t)|^2}, \qquad |\beta_p(t)|^2 = \frac{|\mathscr{R}_p(t)|^2}{1 + |\mathscr{R}_p(t)|^2}. \tag{3.24}$$

On the other hand, if, for example, $|\beta_p(t)|^2$ is known one can calculate $|\mathscr{R}_p(t)|^2$ from

$$|\mathscr{R}_p(t)|^2 = \frac{|\beta_p(t)|^2}{1 - |\beta_p(t)|^2}. \tag{3.25}$$

Additionally, when $|\beta_p^{\text{out}}|^2 \ll 1$ then

$$\left|\mathscr{R}_p(t \longrightarrow \infty)\right| = \left|\mathscr{R}_p^{\text{out}}\right|^2 \approx \left|\beta_p^{\text{out}}\right|^2. \tag{3.26}$$

3.2 Pair production probability

To calculate the pair production probability we need to compute the Bogoliubov coefficients $B_{jl}^{\kappa\lambda}$ as given in (2.73). In 1+1 dimensions there

is no spin degree of freedom and the momentum has only a single component. So instead of the combined mode index introduced in section 2.5 we can label the modes using their momentum p only. The sum over mode indices reduces to an integral over the momentum $\sum_j = \int \mathrm{d}p_j = \int \mathrm{d}p$. Factors of $(2\pi)^3$ are changed to just 2π because of the reduced dimensionality of space. Taking all this into account, we have

$$B^{+-}_{p'p} = \frac{1}{2\pi}\sqrt{\frac{m_{\text{in}}\, m_{\text{out}}}{\epsilon^{\text{in}}_{p'}\epsilon^{\text{out}}_{p}}}\left(\psi^{+}_{\text{in}}(p'), \psi^{-}_{\text{out}}(p)\right). \tag{3.27}$$

We can use the single-particle solutions that we derived in the preceding section as the solutions with positive energy initially, i.e.[20]

$$\psi^{+}_{\text{in}}(p;t,x) = \alpha_p(t)\sqrt{\frac{\epsilon^{\text{in}}_p}{m_{\text{in}}}}\, u_{+,p}(t) e^{-i\varphi_p(t)+ipx} + \beta_p(t)\sqrt{\frac{\epsilon^{\text{in}}_p}{m_{\text{in}}}}\, u_{-,p}(t) e^{i\varphi_p(t)+ipx}, \tag{3.28}$$

where $\alpha_p(t)$ and $\beta_p(t)$ are subject to the initial condition (3.21).

Additionally, we know that the solutions corresponding to plane waves finally have the following asymptotic behaviour,

$$\psi^{\pm}_{\text{out}}(p;t,x) \overset{t\to\infty}{\longrightarrow} \sqrt{\frac{\epsilon^{\text{out}}_p}{m_{\text{out}}}}\, u^{\text{out}}_{\pm,p}\, e^{ipx} \tag{3.29}$$

where $u^{\text{out}}_{\pm,p} = \lim_{t\to\infty} u_{\pm,p}(t)$. Because the inner product in (3.27) is time-independent, we may evaluate it at any instant of time. Using $t \longrightarrow \infty$ we find

$$\begin{aligned}\left(\psi^{+}_{\text{in}}(p'), \psi^{-}_{\text{out}}(p)\right) = \int \mathrm{d}x \sqrt{\frac{\epsilon^{\text{in}}_{p'}\epsilon^{\text{out}}_{p}}{m_{\text{in}}\, m_{\text{out}}}}\Big[&\left(\alpha^{\text{out}}_{p'}\right)^{*}\left(u^{\text{out}}_{+,p'}\right)^{\dagger} u^{\text{out}}_{-,p} e^{i\varphi^{\text{out}}_{p'}} \\ &+ \left(\beta^{\text{out}}_{p'}\right)^{*}\left(u^{\text{out}}_{-,p'}\right)^{\dagger} u^{\text{out}}_{-,p} e^{-i\varphi^{\text{out}}_{p'}}\Big] e^{-i(p'-p)x}\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}\alpha^{\text{out}}_{p} &= \lim_{t\to\infty} \alpha_p(t), \\ \beta^{\text{out}}_{p} &= \lim_{t\to\infty} \beta_p(t), \\ \varphi^{\text{out}}_{p} &= \lim_{t\to\infty} \varphi_p(t).\end{aligned} \tag{3.31}$$

Performing the x-integration gives $2\pi\delta(p' - p)$ and using the orthonormality of the $u_{\pm,p}$ we get

$$\left(\psi^{+}_{\text{in}}(p'), \psi^{-}_{\text{out}}(p)\right) = 2\pi\sqrt{\frac{\epsilon^{\text{in}}_{p}\epsilon^{\text{out}}_{p}}{m_{\text{in}}\, m_{\text{out}}}}\left(\beta^{\text{out}}_{p}\right)^{*} e^{-i\varphi^{\text{out}}_{p}}\delta(p' - p). \tag{3.32}$$

[20] The additional factor of $\sqrt{\epsilon_p/m_{\text{in}}}$ is to fulfill the orthogonality condition (2.42) and the completeness relation (2.43). It is explicitly needed here because of the chosen normalisation for the spinors $u_{\pm,p}$ and the relationship between $\alpha_p(t)$ and $\beta_p(t)$ given in (3.23). The factor could also be moved into the definition of either the spinors or the functions $\alpha_p(t)$ and $\beta_p(t)$.

and thus

$$B^{+-}_{p'p} = \left(\beta^{\text{out}}_p\right)^* e^{-i\varphi^{\text{out}}_p}\,\delta(p'-p). \tag{3.33}$$

In complete analogy, we can also calculate $B^{++}_{p'p}$. This time, we need to compute the inner product

$$\begin{aligned}\left(\psi^+_{\text{in}}(p'),\psi^+_{\text{out}}(p)\right) = \int \mathrm{d}x \sqrt{\frac{\epsilon^{\text{in}}_{p'}\epsilon^{\text{out}}_p}{m_{\text{in}}m_{\text{out}}}}\Big[&\left(\alpha^{\text{out}}_{p'}\right)^*\left(u^{\text{out}}_{+,p'}\right)^\dagger u^{\text{out}}_{+,p}e^{i\varphi^{\text{out}}_{p'}}\\ &+\left(\beta^{\text{out}}_{p'}\right)^*\left(u^{\text{out}}_{-,p'}\right)^\dagger u^{\text{out}}_{+,p}e^{-i\varphi^{\text{out}}_{p'}}\Big]e^{-i(p'-p)x}\end{aligned} \tag{3.34}$$

The spatial integration again gives a delta distribution and therefore we get

$$B^{++}_{p'p} = \frac{1}{2\pi}\sqrt{\frac{m_{\text{in}}m_{\text{out}}}{\epsilon^{\text{in}}_{p'}\epsilon^{\text{out}}_p}}\left(\psi^+_{\text{in}}(p'),\psi^+_{\text{out}}(p)\right) = \left(\alpha^{\text{out}}_p\right)^* e^{i\varphi^{\text{out}}_p}\,\delta(p'-p). \tag{3.35}$$

We see that B^{++} and B^{+-} are diagonal (due to the factor of $\delta(p'-p)$). This can be traced back to the problem being only time-dependent which implies that the canonical momentum p is conserved (i.e. no scattering to other modes occurs). Thus, the matrices B^{--} and B^{-+} are diagonal as well. Using the unitarity condition (2.75) we can show that[21]

$$\begin{aligned}B^{--}_{p'p} &= \alpha^{\text{out}}_p e^{-i\varphi^{\text{out}}_p}\delta(p'-p) = \left(B^{++}_{p'p}\right)^*,\\ B^{-+}_{p'p} &= -\beta^{\text{out}}_p e^{i\varphi^{\text{out}}_p}\delta(p'-p) = -\left(B^{+-}_{p'p}\right)^*.\end{aligned} \tag{3.36}$$

[21] There is some freedom in choosing the phase of these as the unitarity condition (2.75) only fixes the phase difference.

Finally, using (2.100) we can calculate the number of produced positrons in the mode labelled p

$$N_{e^+}(p) = \int \mathrm{d}p'\,\left|B^{+-}_{p'p}\right|^2 = |\beta^{\text{out}}_p|^2\,\delta(0). \tag{3.37}$$

This clearly diverges due to the factor of $\delta(0)$. Physically, this is because the field is non-zero in an infinitely large spacetime region, thereby producing infinitely many particles. Hence, the factor $|\beta^{\text{out}}_p|^2$ can be interpreted as the density of created positrons and should be used to compare the effect of different field configurations. Furthermore, when only few pairs are created, i.e. when $|\beta^{\text{out}}_p|^2 \ll 1$, then using (3.26) we find

$$N_{e^+}(p) \approx |\mathscr{R}^{\text{out}}_p|^2\,\delta(0) \tag{3.38}$$

which justifies the linearisation of the Riccati equation (3.20) in that case. We may already infer some qualitative aspects about the depend-

ence of the produced-pair density on the momentum p from the linearised Riccati equation (3.20). For instance, when p goes to infinity the denominator of the prefactor, Ω_p^2 goes to infinity such that $|\mathscr{R}_p^{\text{out}}|^2$ vanishes. Physically this can be understood because much more energy is needed to produce a high-momentum electron-positron pair thus making this process less likely to happen. Furthermore, when there is no electric field, i.e. $A(t) = 0$, then $\mathscr{R}_{p=0}(t) = 0$ for all times. Thus, a time-dependent mass does not create particles with vanishing momentum. In contrast, for a symmetric electric field ($A(-t) = -A(t)$) with constant mass ($\dot{m} = 0$) we expect the maximum density of produced pairs where the denominator is smallest, i.e. for $p = 0$.

3.3 Basic Sauter-Schwinger effect

Using the formalism derived so far it is – in principle – possible to calculate the density of produced pairs for different electric field configurations $E(t)$ like a constant electric field, a Sauter pulse $E(t) = E_0 \operatorname{sech}^2(\omega t)$ or a Gaussian pulse $E(t) = E_0 \exp\left[-(\omega t)^2\right]$. We are only interested in the exponent of the produced-pair density in the subcritical regime, where few pairs are being created. Hence, we may use the linearised Riccati equation (3.20). With the aforementioned initial conditions

$$\alpha_p(t \longrightarrow -\infty) = 1, \beta_p(t \longrightarrow -\infty) = 0 \quad \Longrightarrow \quad \mathscr{R}_p(t \longrightarrow -\infty) = 0 \tag{3.39}$$

we have to calculate

$$\mathscr{R}_p^{\text{out}} \approx i \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{mqE}{2\Omega_p^2} e^{-2i\varphi_p} \tag{3.40}$$

to obtain the (approximate) density of produced pairs $|\mathscr{R}_p^{\text{out}}|^2$. The standard procedure to compute this integral uses analytic continuation and the fact that the integrand is exponentially suppressed in the lower half-plane. We then deform the integration contour into the lower half-plane until we encounter a singularity at t_* like a pole of the prefactor where $\Omega_p(t_*) = 0$, a branch point or any other point where the integrand is not analytic (see fig. 3.1). Then, the value of the integral is dominated by the value of the exponential at that singularity. Therefore, we have

$$\mathscr{R}_p^{\text{out}} \sim e^{-2i\varphi_p(t_*)} \tag{3.41}$$

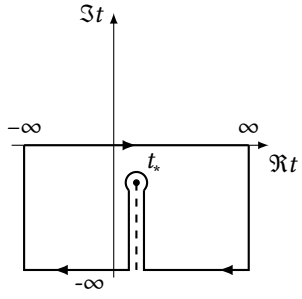

FIGURE 3.1 – Integration contour used to calculate the integral in (3.40). A branch cut goes from t_* to infinity because of the square root in the exponent of the integrand.

and we find the density of produced pairs to be

$$|\beta_p^{\text{out}}|^2 \approx |\mathscr{R}_p^{\text{out}}|^2 \sim e^{4\Im[\varphi_p(t_*)]}. \tag{3.42}$$

Note that this procedure does not produce the prefactor but only the exponent. Additionally, this approximation holds only if there is a single dominant pole, otherwise interference effects have to be taken into account. However, the field configurations mentioned above each consist of only a single pulse for which no interference effects occur [DD11b] (see also [Heb+09; Dum10; DD10] for fields with strong interference effects).

For example, for a constant electric field $A(t) = Et$ with E constant we have

$$\Omega_p^2(t_*) = m^2 + (p + qEt_*)^2 = 0 \iff t_* = \frac{-im - p}{qE}. \tag{3.43}$$

To estimate the density of produced pairs we need to compute the phase

$$\varphi_p(t) = \int_{t_0}^{t} \mathrm{d}t'\, \Omega_p(t') = \int_{t_0}^{t} \mathrm{d}t \sqrt{m^2 + (p + qEt')^2}. \tag{3.44}$$

We substitute $u = (p + qEt')/m$ and get

$$\varphi_p(t) = \frac{m^2}{qE} \int_{u(t_0)}^{u(t)} \mathrm{d}u \sqrt{1 + u^2} = \frac{1}{2}\frac{m^2}{qE}\left[\phi\left(\frac{p + qEt}{m}\right) - \phi\left(\frac{p + qEt_0}{m}\right)\right] \tag{3.45}$$

where

$$\phi(u) = u\sqrt{1 + u^2} + \operatorname{arsinh} u. \tag{3.46}$$

If t lies on the real axis, $\phi((p + qEt)/m)$ is also real. Thus, if we are only interested in the imaginary part of $\varphi_p(t)$ the lower limit of integration t_0 is completely irrelevant, as was mentioned before in section 3.1. We find

$$\Im\varphi_p(t_*) = \frac{1}{2}\frac{m^2}{qE}\Im\phi\left(\frac{p + qEt_*}{m}\right) = -\frac{1}{2}\frac{m^2}{qE}\arcsin 1 = -\frac{1}{4}\pi\frac{m^2}{qE}. \tag{3.47}$$

Thus we find for the density of produced pairs

$$\left|\beta_p^{\text{out}}\right|^2 \sim e^{4\Im[\varphi_p(t_*)]} = e^{-\pi\frac{m^2}{qE}} \tag{3.48}$$

which indeed is the well-known Schwinger result (compare (1.8)). The density is exponentially suppressed and the critical field strength $E_S =$

$m^2/q \approx 1.3 \times 10^{18}$ V/m at which the exponent is of order 1 is beyond the reach of currently available technology. Therefore, several enhancement mechanisms have been proposed some of which we will review in the following sections.

3.4 Dynamically-assisted Sauter-Schwinger effect

Schützhold, Gies and Dunne first proposed using a combination of a strong, slowly varying field and a weak, rapidly varying field to enhance the pair production probability [SGD08]. The potential can be written as

$$A(t) = \frac{E_{\text{slow}}}{\omega_{\text{slow}}} F(\omega_{\text{slow}} t) + \frac{E_{\text{fast}}}{\omega_{\text{fast}}} G(\omega_{\text{fast}} t) \tag{3.49}$$

where $0 < E_{\text{fast}} \ll E_{\text{slow}} \ll E_S$ and $0 < \omega_{\text{slow}} \ll \omega_{\text{fast}} \ll m$ so that the electric field is a combination of a slow and a fast pulse:

$$E(t) = \dot{A}(t) = E_{\text{slow}} f(\omega_{\text{slow}} t) + E_{\text{fast}} g(\omega_{\text{fast}} t) \tag{3.50}$$

where $f(\tau) = F'(\tau)$ and $g(\tau) = G'(\tau)$. Different choices for the shape of the pulses $f(\tau)$ and $g(\tau)$ have been investigated. For example, in the original paper [SGD08] the authors have used Sauter pulses for both the strong and the weak field, i.e. $f(\tau) = g(\tau) = \operatorname{sech}^2(\tau)$. In [Lin+15] different choices for $g(\tau)$ were compared while using a constant strong field $f(\tau) = 1$. As an example, we will also consider a constant strong field[22] and a Sauter pulse as the weak field, i.e.

$$f(\tau) = 1, \quad g(\tau) = \operatorname{sech}^2 \tau \tag{3.51}$$

and consequently

$$F(\tau) = \tau, \quad G(\tau) = \tanh \tau. \tag{3.52}$$

We proceed as before and approximate $\mathscr{R}_p^{\text{out}}$ from (3.40) by identifying the dominant pole. In this case, the position of the poles where

$$0 = \Omega_p^2(t_*) = m^2 + \left[p + qE_{\text{slow}} t_* + \frac{qE_{\text{fast}}}{\omega_{\text{fast}}} \tanh(\omega_{\text{fast}} t_*) \right]^2 \tag{3.53}$$

[22] In fact, as an approximation the strong field can always be taken to be constant because on the time scale of the weak field the strong field does not change much.

are given by

$$\omega_{\text{fast}} t_* + \frac{E_{\text{fast}}}{E_{\text{slow}}} \tanh(\omega_{\text{fast}} t_*) = \omega_{\text{fast}} \frac{\pm im - p}{qE_{\text{slow}}}. \tag{3.54}$$

Because $E_{\text{fast}}/E_{\text{slow}} \ll 1$ we find

$$t_* = \frac{-im - p}{qE_{\text{slow}}} + \mathcal{O}\left(\frac{E_{\text{fast}}}{E_{\text{slow}}}\right) \tag{3.55}$$

which to lowest order is exactly the same singularity as in the constant field case (3.43). However, in addition to this singularity we also find that the electric field diverges at

$$t_*^{\cosh} = -i\frac{\pi}{2\omega_{\text{fast}}} \tag{3.56}$$

where $\cosh(\omega_{\text{fast}} t)$ vanishes[23]. The existence of this additional pole means that – depending on the values of the frequencies and field strengths – the dominant singularity may be at $t_*^{\cosh}$ instead of t_*. The dependence on the parameters can be expressed best using a single dimensionless quantity, namely the combined Keldysh parameter

$$\gamma = \frac{m\omega_{\text{fast}}}{qE_{\text{slow}}}. \tag{3.57}$$

With this definition, the positions of the singularities can be expressed as

$$\omega_{\text{fast}} t_* \approx -i\gamma - \gamma\frac{p}{m}, \quad \omega_{\text{fast}} t_*^{\cosh} = -i\frac{\pi}{2}. \tag{3.58}$$

Thus, there exists a threshold $\gamma_{\text{crit}} = \pi/2$ at which the two singularities are equally close to the real axis. Near the threshold, contributions from both singularities to the value of $\mathcal{R}_p^{\text{out}}$ have to be taken into account, i.e. it will be given by

$$\mathcal{R}_p^{\text{out}} \approx c_* e^{-2i\varphi_p(t_*)} + c_*^{\cosh} e^{-2i\varphi_p(t_*^{\cosh})} \tag{3.59}$$

and interference effects occur [OHA11; FS12]. Below the threshold $\gamma < \gamma_{\text{crit}}$ the strong-field singularity t_* dominates and above the threshold $\gamma > \gamma_{\text{crit}}$ the weak-field singularity dominates. To estimate the density of produced pairs we need to calculate the phase $\varphi_p(t)$. Again using $E_{\text{fast}}/E_{\text{slow}} \ll 1$ we may – to lowest order – neglect the weak field term

[23] Actually, the roots of $\cosh(\omega_{\text{fast}} t)$ are at $t = i\frac{\pi}{2\omega_{\text{fast}}}(2n+1)$, $n \in \mathbb{Z}$, but $t_*^{\cosh}$ is the one with negative imaginary part that is closest to the real axis.

in $\Omega_p(t)$ and approximate

$$\begin{aligned}\varphi_p(t) &\approx \int_{t_0}^{t} \mathrm{d}t' \sqrt{m^2 + \left(p + qE_{\text{slow}}t'\right)^2} \\ &= \frac{1}{2}\frac{m^2}{qE_{\text{slow}}}\left[\phi\left(\frac{p + qE_{\text{slow}}t}{m}\right) - \phi\left(\frac{p + qE_{\text{slow}}t_0}{m}\right)\right]\end{aligned} \tag{3.60}$$

with the function $\phi(u)$ from (3.46).

Therefore, sufficiently below the threshold $\gamma < \gamma_{\text{crit}}$ where the dominant singularity is at t_* and where interference effects can be ignored, the density of produced pairs

$$\left|\beta_p^{\text{out}}\right|^2 \sim \exp\{4\Im[\varphi_p(t_*)]\} = \exp\left\{2\frac{m^2}{qE_{\text{slow}}}\Im[\phi(-i)]\right\} = \exp\left(-i\pi\frac{m^2}{qE_{\text{slow}}}\right) \tag{3.61}$$

is exactly the same as in the constant field case (3.48). However, sufficiently above the threshold $\gamma > \gamma_{\text{crit}}$ the singularity at $t_*^{\cosh}$ dominates and we find the density of produced pairs to be

$$\left|\beta_p^{\text{out}}\right|^2 \sim \exp\{4\Im[\varphi_p(t_*^{\cosh})]\} = \exp\left\{2\frac{m^2}{qE_{\text{slow}}}\Im\left[\phi\left(\frac{p}{m} - i\frac{\pi}{2\gamma}\right)\right]\right\}. \tag{3.62}$$

For $p = 0$, we get

$$\begin{aligned}\left|\beta_{p=0}^{\text{out}}\right|^2 &\sim \exp\left\{-2\frac{m^2}{qE_{\text{slow}}}\left[\frac{\pi}{2\gamma}\sqrt{1 - \left(\frac{\pi}{2\gamma}\right)^2} + \arcsin\left(\frac{\pi}{2\gamma}\right)\right]\right\} \\ &= \exp\left\{-\frac{m^2}{qE_{\text{slow}}}\chi(\gamma)\right\}.\end{aligned} \tag{3.63}$$

For $\gamma \longrightarrow \gamma_{\text{crit}} = \pi/2$ we recover the constant field result (3.48). On the other hand, for $\gamma \longrightarrow \infty$ the leading-order exponent is

$$\left|\beta_{p=0}^{\text{out}}\right|^2 \sim \exp\left(-2\frac{\pi}{\gamma}\frac{m^2}{qE_{\text{slow}}}\right) = \exp\left(-\frac{2\pi m}{\omega_{\text{fast}}}\right) \tag{3.64}$$

which agrees with the result given in equation (30) in [Pop72]. In this case, the exponent falls off as $1/\gamma$ which is expected (compare with equation (49) in [BI70]) as we are in the (perturbative) multi-photon regime here where the weak field provides enough energy to bridge the gap in the Dirac sea directly. As can be seen in fig. 3.2, the additional weak field leads to an exponential enhancement in the regime $\gamma > \gamma_{\text{crit}}$. - This can be understood intuitively using the simple tunnelling picture already employed in the introduction: The weak, fast field provides en-

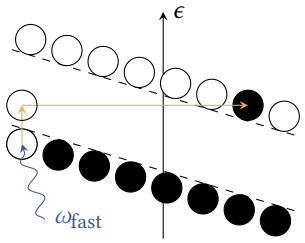

FIGURE 3.3 – The weak field provides energy to lift states into the forbidden region which reduces the way to tunnel. Thus, the tunnelling probability is enhanced exponentially.

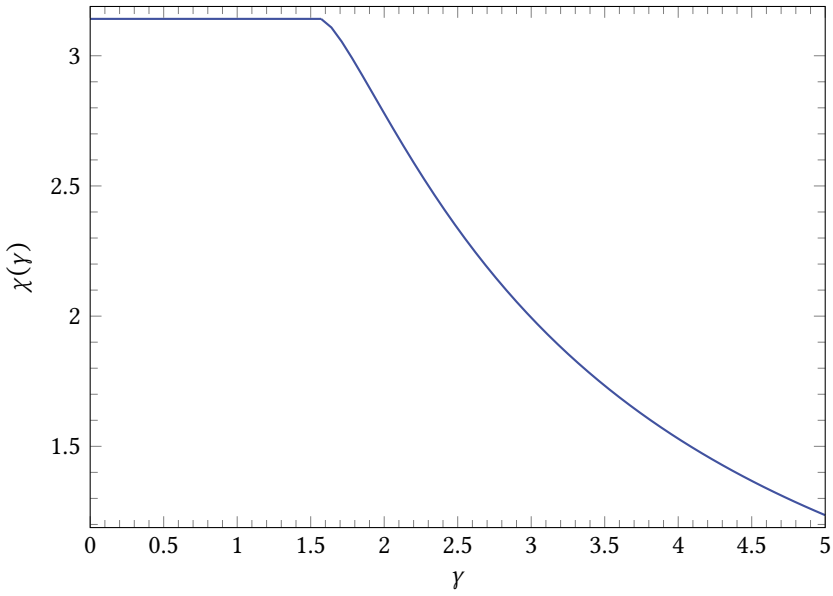

FIGURE 3.2 – Dependence of the produced-pairs density's exponent on the combined Keldysh parameter γ as given in (3.63). A smaller value for χ corresponds to a higher pair production probability (see (3.63)). Below the threshold $\gamma < \gamma_{\text{crit}}$ we find the same behaviour as in the constant field case ($\chi = \pi$) whereas above the threshold an exponential enhancement due to the weak field can be observed.

ergy to lift states from the negative continuum into the forbidden region and thereby reduces the way to tunnel (see fig. 3.3). Thus, the probability for this process to happen is increased exponentially. However, this picture fails to explain the threshold behaviour: Any additional photon even with vanishingly low frequency ω_{fast} should be able to provide a lift into the forbidden region and thus reduce the way to tunnel. Due to the threshold behaviour that is central to the dynamically-assisted Sauter-Schwinger effect though, the effect is only operative if $\omega_{\text{fast}} > \frac{qE_{\text{slow}}}{m}\gamma_{\text{crit}}$. Furthermore, the threshold behaviour is qualitatively different for seemingly similar shapes of the weak field. We will further discuss and explain these differences in section 3.6.

3.5 Doubly-assisted Sauter-Schwinger effect

Another enhancement mechanism that has been proposed recently is the doubly-assisted Sauter-Schwinger effect [TOS16]. This setup is based on two other ideas: Photon-assisted Sauter-Schwinger pair production and the dynamically-assisted Sauter-Schwinger effect we discussed in the preceding section. The former has been discussed by Dunne, Gies and Schützhold and its setup consists of a strong electric field and an additional high-energy photon with an energy in the order of the electron mass [DGS09]. Both mechanism facilitate an exponential enhancement of the pair production probability. However, combining both gives an even greater enhancement as we will see in this section.

From the original paper [DGS09] it is known that the maximum enhancement for photon-assisted Sauter-Schwinger pair creation is attained in the case where the photon travels perpendicular to the electric field. Thus, we now want to work in a 3+1-dimensional spacetime. The background field is essentially the same as in the last section and directed in z-direction, i.e.

$$A_0 = 0, \quad \mathbf{A} = \mathbf{e}_z A(t) \tag{3.65}$$

with $A(t)$ from (3.49). We want to calculate the amplitude for the process where we start with a single photon with momentum $\mathbf{K}$ and end up with an electron with momentum $\mathbf{p}$ and a positron with momentum $\mathbf{p}'$, i.e.

$$\mathscr{M}_{\text{doubly}} = \langle 0_{\text{out}}| \, \hat{a}_{\text{out}}(s, \mathbf{p}) \hat{b}_{\text{out}}(s', \mathbf{p}') \, \hat{S} \, \hat{c}^\dagger(\lambda, \mathbf{K}) \, |0_{\text{in}}\rangle \, . \tag{3.66}$$

To lowest order in q, we find (see (2.57) for the series expansion of $\hat{S}$)

$$\mathscr{M}_{\text{doubly}} \approx -iq \int \mathrm{d}t \int \mathrm{d}^3 r \, \langle 0_{\text{out}}| \, \hat{a}_{\text{out}}(s, \mathbf{p}) \hat{b}_{\text{out}}(s', \mathbf{p}') \hat{\bar{\Psi}} \gamma^\mu \hat{A}_\mu \hat{\Psi} \hat{c}^\dagger(\lambda, \mathbf{K}) \, |0_{\text{in}}\rangle \, . \tag{3.67}$$

We are only interested in the exponent of the amplitude here so we will just sketch the next few steps of the derivation. We insert the mode expansion in (generalized) normal form for the spinor field operator (2.81) and its adjoint (2.82), and the mode expansion for the Maxwell field operator (2.86). To this order in q the product of the field operators gives no contractions and we just need to look at the normal form of the product (i.e. reorder every term such that out-creation operators are to the left of in-annihilation operators). Because there are no fermions in the initial state, only a single term from the product remains. By using the commutation and anticommutation relations for the particle creation and annihilation operators we get several delta distributions that make the momentum integrals that come from the mode expansions simple to do. We arrive at

$$\begin{aligned} \mathscr{M}_{\text{doubly}} \approx iq \int \mathrm{d}t \int \frac{\mathrm{d}^3 r}{(2\pi)^3} \sum_{\lambda'} & \tilde{M}(s, s', \mathbf{p}, \mathbf{p}') \eta_{\lambda\lambda'} f_\mu(\lambda', \mathbf{K}; t, \mathbf{r}) \\ & \times \overline{\psi}^+_{\text{in}}(s, \mathbf{p}; t, \mathbf{r}) \gamma^\mu \psi^-_{\text{in}}(s', \mathbf{p}'; t, \mathbf{r}) \end{aligned} \tag{3.68}$$

where $\tilde{M}$ contains several Bogoliubov coefficients. From section 3.2 we know that we can write the solutions with positive energy initially in the form (3.28), at least in 1+1 dimensions. In 3+1 dimensions a similar

form can be used:

$$\begin{aligned}\psi^{+}_{\text{in}}(s,\mathbf{p};t,x) = \alpha_{s,\mathbf{p}}(t)\sqrt{\frac{\epsilon^{\text{in}}_{\mathbf{p}}}{m}}u_{+,s,\mathbf{p}}(t)e^{-i\varphi_{\mathbf{p}}(t)+i\mathbf{p}\cdot\mathbf{r}} \\ + \beta_{s,\mathbf{p}}(t)\sqrt{\frac{\epsilon^{\text{in}}_{\mathbf{p}}}{m}}u_{-,s,\mathbf{p}}(t)e^{i\varphi_{\mathbf{p}}(t)+i\mathbf{p}\cdot\mathbf{r}}.\end{aligned} \tag{3.69}$$

This is essentially the same expression as (3.28) just with an additional spin index s and a three-dimensional momentum vector $\mathbf{p}$. We can do a similar expansion for ψ^{-}_{in}:

$$\begin{aligned}\psi^{-}_{\text{in}}(s,\mathbf{p};t,x) = \tilde{\alpha}_{s,-\mathbf{p}}(t)\sqrt{\frac{\epsilon^{\text{in}}_{-\mathbf{p}}}{m}}u_{+,s,-\mathbf{p}}(t)e^{-i\varphi_{-\mathbf{p}}(t)-i\mathbf{p}\cdot\mathbf{r}} \\ + \tilde{\beta}_{s,-\mathbf{p}}(t)\sqrt{\frac{\epsilon^{\text{in}}_{-\mathbf{p}}}{m}}u_{-,s,-\mathbf{p}}(t)e^{i\varphi_{-\mathbf{p}}(t)-i\mathbf{p}\cdot\mathbf{r}}.\end{aligned} \tag{3.70}$$

with the only major difference being that $\tilde{\alpha}_{s,\mathbf{p}}$ and $\tilde{\beta}_{s,\mathbf{p}}$ have to satisfy different boundary conditions, i.e. $\alpha_{s,\mathbf{p}}(t \longrightarrow -\infty) = 0$, $\beta_{s,\mathbf{p}}(t \longrightarrow -\infty) = 1$. As an additional change, we have modified the sign of the momentum term in the exponent so that the positron's momentum is actually $\mathbf{p}'$ and not $-\mathbf{p}'$. After inserting these expressions for $\psi^{\pm}_{\text{in}}$ and the definition of the mode function f_{μ} for the Maxwell field operator (2.87) into (3.68) we can carry out the spatial integral which gives a delta distribution $\delta(\mathbf{K} - \mathbf{p} - \mathbf{p}')$. This is just an expression of momentum conservation: the initial momentum (i.e. just the photon's momentum) and the final momentum (i.e. the momentum of the electron-positron pair) have to be equal. After eliminating $\mathbf{p}'$ using momentum conservation, we find the dominant contribution to the amplitude to be

$$\mathscr{M}_{\text{doubly}} \approx \int \mathrm{d}t\, M(t) \exp\left[-i\,|\mathbf{K}|\, t + i\varphi_{\mathbf{p}}(t) + i\varphi_{\mathbf{p}-\mathbf{K}}(t)\right] \tag{3.71}$$

where $M(t)$ contains $\tilde{M}$ and several of the spinors u_{+} and u_{-}. These are only slowly varying with time in contrast to the exponential. In fact, there are other in terms in the amplitude with different prefactors and exponents as the dominant one. The differences in the exponent amount to different signs in front of $\varphi_{\mathbf{p}}$ and $\varphi_{\mathbf{p}-\mathbf{K}}$. Only looking at the term above is equivalent to using the lowest-order WKB approximation for $\psi^{\pm}_{\text{in}}$,

$$\psi^{\pm}_{\text{in}}(s,\mathbf{p};t,x) \approx \sqrt{\frac{\epsilon^{\text{in}}_{\mathbf{p}}}{m}}u_{\pm,s,\mathbf{p}}(t)e^{\mp i\varphi_{\mathbf{p}}(t)\pm i\mathbf{p}\cdot\mathbf{r}}, \tag{3.72}$$

i.e. using $\alpha_{s,\mathbf{p}}(t) \approx 1$, $\beta_{s,\mathbf{p}}(t) \approx 0$, $\tilde{\alpha}_{s,\mathbf{p}}(t) \approx 0$ and $\tilde{\beta}_{s,\mathbf{p}}(t) \approx 1$.

To get the total probability for a pair to be produced from a photon with momentum $\mathbf{K}$ we need to integrate the probability over all electron momenta

$$\begin{aligned} P^{\text{tot}}_{\text{doubly}} &= \int \mathrm{d}^3p \left|\mathscr{M}_{\text{doubly}}\right|^2 \\ &= \int \mathrm{d}^3p \left|\int \mathrm{d}t\, M(t) \exp\left[-i\,|\mathbf{K}|\, t + i\varphi_{\mathbf{p}}(t) + i\varphi_{\mathbf{p}-\mathbf{K}}(t)\right]\right|^2 . \end{aligned} \tag{3.73}$$

We evaluate the integrals using the saddle point method[24] and keep only the exponential contribution. For the time integral the saddle point t_* is a solution of

$$0 = -|\mathbf{K}| + \dot{\varphi}_{\mathbf{p}}(t_*) + \dot{\varphi}_{\mathbf{p}-\mathbf{K}}(t_*). \tag{3.76}$$

Using $\dot{\varphi}_{\mathbf{p}}(t) = \sqrt{m^2 + \left[\mathbf{p} + q\mathbf{A}(t)\right]^2}$ we get

$$|\mathbf{K}| - \sqrt{m^2 + \left[\mathbf{p} + q\mathbf{A}(t_*)\right]^2} = \sqrt{m^2 + \left[\mathbf{p} + q\mathbf{A}(t_*) - \mathbf{K}\right]^2}. \tag{3.77}$$

After some manipulations which involve squaring the equation we arrive at

$$m^2 + \left[\mathbf{p} + q\mathbf{A}(t_*)\right]^2 = \left\{\frac{\mathbf{K}}{|\mathbf{K}|} \cdot \left[\mathbf{p} + q\mathbf{A}(t_*)\right]\right\}^2 . \tag{3.78}$$

Thus, the position of the saddle point does not depend on the photon's energy $|\mathbf{K}|$ but only on its propagation direction $\mathbf{K}/|\mathbf{K}|$. Because deriving (3.78) involved squaring the original equation (3.77) solutions of the former are not necessarily solutions of the latter. Inserting (3.78) into (3.77) we get

$$|\mathbf{K}| - \sqrt{\left\{\frac{\mathbf{K}}{|\mathbf{K}|} \cdot \left[\mathbf{p} + q\mathbf{A}(t_*)\right]\right\}^2} = \sqrt{\left\{|\mathbf{K}| - \frac{\mathbf{K}}{|\mathbf{K}|} \cdot \left[\mathbf{p} + q\mathbf{A}(t_*)\right]\right\}^2} . \tag{3.79}$$

For any complex number z we have

$$\sqrt{z^2} = \sqrt{\left|z^2\right| e^{2i\arg z + 2\pi i n}} = \sqrt{|z|^2\, e^{2i\arg z + 2\pi i n}} = |z|\, e^{i\arg z + \pi i n} \tag{3.80}$$

where $n = 0, 1$ numbers the different branches of the square root. With the standard choice for the branch cut of the square root along the negative real axis (i.e. $-\pi \le \arg z < \pi$) we then find for the principal branch ($n = 0$)

$$\Re\sqrt{z^2} = |z| \cos(\arg z) = |\Re z| . \tag{3.81}$$

[24] The saddle point method (or method of steepest descent, also stationary phase approximation) is a way to approximate integrals related to Laplace's method. It states that integrals with a large exponent can be estimated as

$$\int_C \mathrm{d}z\, g(z) e^{-\lambda f(z)} \approx \sqrt{\frac{2\pi}{\lambda f''(z_0)}}\, g(z_0) e^{-\lambda f(z_0)} \tag{3.74}$$

where $f'(z_0) = 0$ and $\lambda \gg 1$. This assumes a single saddle point z_0. Otherwise, the expression on the right side has to be summed over all saddle points. In higher dimensions, $f''(z_0)$ needs to be replaced with the determinant of the Hessian matrix and instead of the factor $(2\pi)^{1/2}$ we have a factor of $(2\pi)^{1/2}$ for every dimension [Won01].

Thus, taking the real part of (3.79) leads to

$$|\mathbf{K}| - \left|\Re\left\{\frac{\mathbf{K}}{|\mathbf{K}|}\cdot\left[\mathbf{p}+q\mathbf{A}(t_*)\right]\right\}\right| = \left||\mathbf{K}| - \Re\left\{\frac{\mathbf{K}}{|\mathbf{K}|}\cdot\left[\mathbf{p}+q\mathbf{A}(t_*)\right]\right\}\right| - \tag{3.82}$$

This only has a solution if

$$0 \leq \Re\left\{\frac{\mathbf{K}}{|\mathbf{K}|}\cdot\left[\mathbf{p}+q\mathbf{A}(t_*)\right]\right\} \leq |\mathbf{K}| \tag{3.83}$$

which therefore is a condition for solutions of (3.78) to be solutions of (3.77). Taking the imaginary part of (3.79) imposes no further restrictions. Considering that the vector potential in our case only has a z-component we may write the photon's wave vector as[25]

$$\mathbf{K} = |\mathbf{K}|\left(\mathbf{e}_x \sin\theta + \mathbf{e}_z \cos\theta\right) \tag{3.84}$$

where θ is the angle between the photon's direction of propagation and the electric field ($0 \leq \theta \leq \pi$). The conditions for the saddle point of the time integral (3.78) and (3.83) then may be expressed as

$$\begin{aligned}\left\{p_1\cos\theta - \left[p_3 + qA(t_*)\right]\sin\theta\right\}^2 &= -\left(m^2 + p_2^2\right),\\ 0 \leq p_1\sin\theta + \left[p_3 + q\Re A(t_*)\right]\cos\theta &\leq |\mathbf{K}|\,.\end{aligned} \tag{3.85}$$

Finally, the total probability given in (3.73) can be approximated to be

$$P^{\text{tot}}_{\text{doubly}} \approx \int \mathrm{d}^3p\,\Xi\exp\left[2\,|\mathbf{K}|\,\Im t_* - 2\Im\varphi_{\mathbf{p}}(t_*) - 2\Im\varphi_{\mathbf{p}-\mathbf{K}}(t_*)\right] \tag{3.86}$$

where

$$\Xi = |M(t_*)|^2\,\frac{2\pi}{\left|\ddot{\varphi}_{\mathbf{p}}(t_*) + \ddot{\varphi}_{\mathbf{p}-\mathbf{K}}(t_*)\right|}. \tag{3.87}$$

The saddle point $\mathbf{p}_*$ of the remaining momentum integral has to satisfy

$$\Im\int_0^{t_*}\mathrm{d}t\left\{\frac{\mathbf{p}+q\mathbf{A}(t)}{\sqrt{m^2+\left[\mathbf{p}+q\mathbf{A}(t)\right]^2}} + \frac{\mathbf{p}-\mathbf{K}+q\mathbf{A}(t)}{\sqrt{m^2+\left[\mathbf{p}-\mathbf{K}+q\mathbf{A}(t)\right]^2}}\right\} = 0 \tag{3.88}$$

where we have already used the saddle point condition for t_* from (3.76). This condition is not as straightforward to handle as the one for t_* because of the remaining time integral. Nevertheless, if we consider a symmetric electric field $A(-t) = -A(t)$ we have a saddle point at $\mathbf{p}_* = \mathbf{K}/2$. To see this, first consider the conditions for the temporal saddle point (3.85)

[25] The y-component of the wave vector may be set to zero without loss of generality by rotating the coordinate system appropriately.

with $\mathbf{p} = \mathbf{p}_* = \mathbf{K}/2$,

$$qA(t_*) = \pm i\frac{m}{\sin\theta}, \quad 0 \leq \frac{1}{2}|\mathbf{K}| + q\underbrace{\Re A(t_*)}_{=0}\cos\theta \leq |\mathbf{K}|. \tag{3.89}$$

The inequality is satisfied because $A(t_*)$ is purely imaginary. Because of the antisymmetry of $A(t)$ we can conclude that t_* has to be purely imaginary as well[26]. Thus, we may use the parametrisation $t = iu$, $0 \leq u \leq \Im t_*$, to calculate the integral in (3.88). After inserting $\mathbf{p} = \mathbf{p}_*$ in the integrand we find

$$\begin{aligned}
&\Im\int_0^{t_*} \mathrm{d}t \left\{ \frac{\mathbf{p} + q\mathbf{A}(t)}{\sqrt{m^2 + \left[\mathbf{p} + q\mathbf{A}(t)\right]^2}} + \frac{\mathbf{p} - \mathbf{K} + q\mathbf{A}(t)}{\sqrt{m^2 + \left[\mathbf{p} - \mathbf{K} + q\mathbf{A}(t)\right]^2}} \right\} \\
&= \Re\int_0^{\Im t_*} \mathrm{d}u \left\{ \frac{\frac{1}{2}\mathbf{K} + q\mathbf{A}(iu)}{\sqrt{m^2 + \left[\frac{1}{2}\mathbf{K} + q\mathbf{A}(iu)\right]^2}} - \frac{\frac{1}{2}\mathbf{K} + q\mathbf{A}(-iu)}{\sqrt{m^2 + \left[\frac{1}{2}\mathbf{K} + q\mathbf{A}(-iu)\right]^2}} \right\}
\end{aligned} \tag{3.90}$$

which is of the form $\Re(z - z^*)$ and thus vanishes. Hence, for symmetric electric fields with potentials of the form (3.65) $\mathbf{p}_* = \mathbf{K}/2$ is indeed a saddle point of the momentum integral in (3.86). Thus, we may approximate the total probability as

$$P^{\text{tot}}_{\text{doubly}} \sim \exp\left[2|\mathbf{K}|\,\Im t_* - 2\Im\varphi_{\mathbf{K}/2}(t_*) - 2\Im\varphi_{-\mathbf{K}/2}(t_*)\right]. \tag{3.91}$$

We now proceed as with the dynamically-assisted Sauter-Schwinger effect. We use a constant field as the strong background field ($f(\tau) = 1$) and a Sauter pulse as the weak field ($g(\tau) = \operatorname{sech}^2\tau$). The equation for the saddle point then reads

$$\omega_{\text{fast}} t_* + \frac{E_{\text{fast}}}{E_{\text{slow}}}\tanh(\omega_{\text{fast}} t_*) = \pm\frac{i}{\sin\theta}\frac{m\omega_{\text{fast}}}{qE_{\text{slow}}}. \tag{3.92}$$

Again, for $E_{\text{fast}} \ll E_{\text{slow}}$ we can approximate the saddle point as

$$t_* \approx \pm\frac{i}{\sin\theta}\frac{m}{qE_{\text{slow}}}. \tag{3.93}$$

Furthermore, we still have the additional singularities at zeros of the hyperbolic cosine (see (3.56)). In terms of the combined Keldysh parameter $\gamma = m\omega_{\text{fast}}/(qE_{\text{slow}})$ the relevant singularities are at[27]

$$\omega_{\text{fast}} t_* = i\frac{\gamma}{\sin\theta}, \quad \omega_{\text{fast}} t_*^{\cosh} = i\frac{\pi}{2}. \tag{3.94}$$

[26] To prove this use $A^*(t_*) = A(t_*^*)$ ($A(t) = A^*(t^*)$ for $t \in \mathbb{R}$, thus this has to hold for all $t \in \mathbb{C}$). Then $A(-t_*^*) = -A(t_*^*) = -A^*(t_*) = A(t_*)$ and therefore $t_* = -t_*^*$.

[27] Here, the integral is exponentially suppressed in the upper half-plane, thus $\Im t_* > 0$ and $\Im t_*^{\cosh} > 0$.

Thus, the threshold value for the Keldysh parameter where the dominant singularity changes from t_* to $t_*^{\cosh}$ is $\gamma_{\text{crit}} = \frac{\pi}{2}\sin\theta$. To evaluate the exponent as given in (3.91) we need to calculate the phase

$$\varphi_{\mathbf{p}}(t) = \int \mathrm{d}t \sqrt{m^2 + \left[\mathbf{p} + q\mathbf{A}(t)\right]^2}. \tag{3.95}$$

As before, we neglect the term with the weak field and find

$$\begin{aligned}
\varphi_{\mathbf{p}}(t) &\approx \int \mathrm{d}t \sqrt{m^2 + p_1^2 + p_2^2 + \left[p_3 + qE_{\text{slow}}t\right]^2} \\
&= \int \mathrm{d}t \sqrt{m_\perp^2(\mathbf{p}) + \left[p_3 + qE_{\text{slow}}t\right]^2} \\
&= \frac{1}{2}\frac{m_\perp^2(\mathbf{p})}{qE_{\text{slow}}}\phi\left(\frac{p_3 + qE_{\text{slow}}t}{m_\perp(\mathbf{p})}\right) \\
&= \frac{1}{2}\frac{m_\perp^2(\mathbf{p})}{qE_{\text{slow}}}\phi\left(\frac{p_3}{m_\perp(\mathbf{p})} + \frac{m}{m_\perp(\mathbf{p})}\frac{\omega_{\text{fast}}t}{\gamma}\right)
\end{aligned} \tag{3.96}$$

where we have used the function $\phi(u)$ from (3.46) and have defined

$$m_\perp(\mathbf{p}) = \sqrt{m^2 + p_1^2 + p_2^2}. \tag{3.97}$$

Putting everything together, we get for the total probability (3.91) above the threshold ($\gamma > \gamma_{\text{crit}}$)

$$\begin{aligned}
P_{\text{doubly}}^{\text{tot}} \sim \exp\Bigg\{\pi\frac{|\mathbf{K}|}{\omega_{\text{fast}}} - \frac{m_\perp^2(\mathbf{K}/2)}{qE_{\text{slow}}}\Im\Bigg[\phi\left(\frac{|\mathbf{K}|\cos\theta}{2m_\perp(\mathbf{K}/2)} + i\frac{\pi}{2\gamma}\frac{m}{m_\perp(\mathbf{K}/2)}\right) \\
+\phi\left(-\frac{|\mathbf{K}|\cos\theta}{2m_\perp(\mathbf{K}/2)} + i\frac{\pi}{2\gamma}\frac{m}{m_\perp(\mathbf{K}/2)}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{3.98}$$

To compare with known limiting cases we write the exponent as

$$P_{\text{doubly}}^{\text{tot}} \sim \exp\left\{-\frac{m^2}{qE_{\text{slow}}}\chi(\gamma, |\mathbf{K}|, \theta)\right\} \tag{3.99}$$

where

$$\begin{aligned}
\chi(\gamma, |\mathbf{K}|, \theta) = &-\frac{\pi}{\gamma}\frac{|\mathbf{K}|}{m} + \frac{\pi}{\gamma}C_+ + \frac{|\mathbf{K}\cos\theta|}{m}C_- \\
&+ \left[\frac{m_\perp(\mathbf{K}/2)}{m}\right]^2(\arctan D_+ + \arctan D_-)
\end{aligned} \tag{3.100}$$

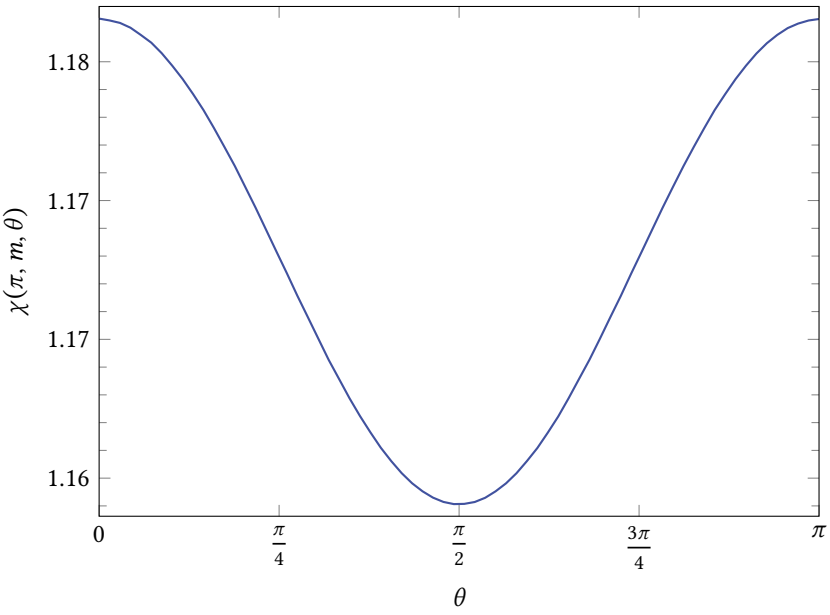

FIGURE 3.4 – This plot shows the dependence of the pair production probability exponent on the angle between the photon and the electric field θ as given in (3.100) with $|\mathbf{K}| = m$ and $\gamma = \pi$. A smaller value for χ corresponds to a higher pair production probability (see (3.99)). The pair production probability is maximal at $\theta = \pi/2$ but overall the dependence on θ is rather weak and becomes even less pronounced for higher values of γ; see fig. 3.5.

and

$$
\begin{aligned}
m_* &= m_\perp(\mathbf{K}/2)|_{\theta=\pi/2} = \sqrt{m^2 + (\mathbf{K}/2)^2} \\
C_\pm &= \sqrt{\sqrt{\left[\left(\frac{m_*}{m}\right)^2 - \left(\frac{\pi}{2\gamma}\right)^2\right]^2 + \left(\frac{|\mathbf{K}\cos\theta|}{m}\frac{\pi}{2\gamma}\right)^2} \pm \left[\left(\frac{m_*}{m}\right)^2 - \left(\frac{\pi}{2\gamma}\right)^2\right]} \\
D_\pm &= \left(\frac{\frac{\pi}{2\gamma} \pm C_- \operatorname{sgn}\cos\theta}{\pm\frac{|\mathbf{K}|\cos\theta}{2m} + C_+}\right).
\end{aligned}
\tag{3.101}
$$

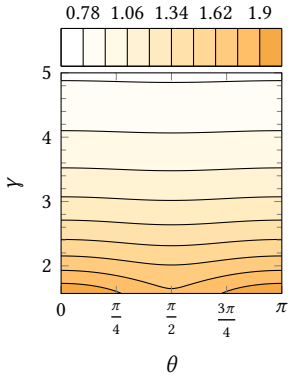

FIGURE 3.5 – Plot of $\chi(\gamma, m, \theta)$ as given in (3.100). The minimum is at $\theta = \pi/2$ but this is less pronounced for larger values of γ.

The minimum of χ (i.e. the maximum of the pair production probability) is at $\theta = \pi/2$ (i.e. when $\mathbf{K} \perp \mathbf{E}$) as can be seen in fig. 3.4. The dependence on the angle θ is actually rather weak and becomes even less pronounced for higher values of γ; see fig. 3.5. For $\theta = \pi/2$, the expression for the pair production probability (3.98) reduces to

$$
\begin{aligned}
P^{\text{tot,max}}_{\text{doubly}} &\sim \exp\left[-\pi\frac{|\mathbf{K}|}{\omega_{\text{fast}}} - 2\frac{m_*^2}{qE_{\text{slow}}}\Im\phi\left(i\frac{\pi}{2\gamma}\frac{m}{m_*}\right)\right] \\
&= \exp\left\{-2\frac{m_*^2}{qE_{\text{slow}}}\left[\frac{\pi}{2\gamma_*}\sqrt{1 - \left(\frac{\pi}{2\gamma_*}\right)^2} + \arcsin\left(\frac{\pi}{2\gamma_*}\right) - \frac{\pi}{2\gamma_*}\frac{|\mathbf{K}|}{m_*}\right]\right\}
\end{aligned}
\tag{3.102}
$$

where we have used

$$
\gamma_* = \gamma\frac{m_*}{m}. \tag{3.103}
$$

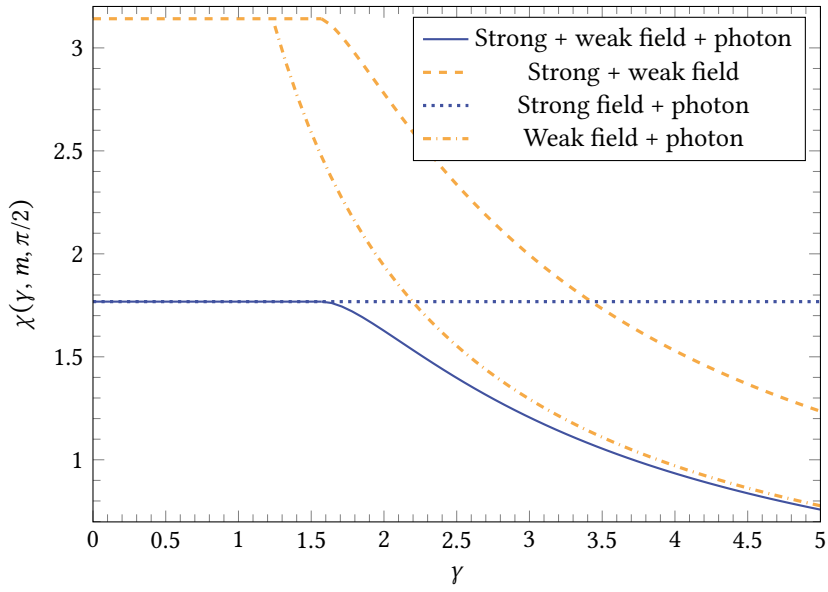

FIGURE 3.6 – This plot shows the dependence of the pair production probability exponent on γ as given in (3.102) and (3.104) for different setups with $|\mathbf{K}| = m$. A smaller value for χ corresponds to a higher pair production probability (see (3.102)). The basic Sauter-Schwinger effect has $\chi = \pi$. We see that the pair production probability for the doubly-assisted effect is always larger than when any of the three ingredients is missing.

Equivalently, this can be expressed using

$$\chi(\gamma, |\mathbf{K}|, \pi/2) = \frac{\pi}{\gamma}\sqrt{\left(\frac{m_*}{m}\right)^2 - \left(\frac{\pi}{2\gamma}\right)^2} + 2\left(\frac{m_*}{m}\right)^2 \arcsin\left(\frac{\pi}{2\gamma}\frac{m}{m_*}\right) - \frac{\pi}{\gamma}\frac{|\mathbf{K}|}{m} \tag{3.104}$$

The expression (3.102) for the total probability is very similar to the expression for the density of produced pairs in the dynamically-assisted Sauter-Schwinger effect (compare (3.63)) just with γ replaced with γ_*, m replaced with m_* and the additional term due to the high-energy photon. In fact, if the photon's energy $|\mathbf{K}|$ vanishes we recover the exponent for dynamical assistance only (3.63). On the other hand, when $\gamma \longrightarrow \gamma_{\text{crit}} = \pi/2$ (or equivalently $\gamma_* \longrightarrow \pi m_*/(2m)$), the influence of the weak field becomes irrelevant and after some manipulations we get

$$P^{\text{tot,max}}_{\text{doubly}} \sim \exp\left\{-2\frac{m^2}{qE_{\text{slow}}}\left[\left(1 + \frac{|\mathbf{K}|^2}{4m^2}\right)\arctan\left(\frac{2m}{|\mathbf{K}|}\right) - \frac{|\mathbf{K}|}{2m}\right]\right\}. \tag{3.105}$$

This the same as equation (5) in the original paper on photon-assisted Sauter-Schwinger pair production [DGS09]. Therefore, the result for the doubly-assisted Sauter-Schwinger effect contains both cases where the strong background field is only assisted by a single ingredient (a weak field or a photon). However, when both assistance mechanisms are operative the probability for pair production is enhanced even further. To compare the different setups we plotted the function $\chi(\gamma, |\mathbf{K}|, \pi/2)$ for a fixed value of $|\mathbf{K}|$; see fig. 3.6.

In the limit of very high photon energy ($|\mathbf{K}| \longrightarrow \infty$) the influence

of the weak field should be negligible as the photon frequency (and also the frequency of both the electron's and the positron's wavefunction) is too large to resolve the temporal variations in the weak field. From the literature it is known that the pair production probability in a constant field with a high-energy photon in that limit scales like $\exp[-8m^3/(3qE_{\text{slow}}|\mathbf{K}|)]$; see equation (36') in [NR64]. This corresponds to the locally constant field approximation as this result is obtained in the presence of only a constant background field. However, from our result (3.102) we find the leading-order result to be

$$\lim_{|\mathbf{K}|\to\infty} P_{\text{doubly}}^{\text{tot,max}} \sim \exp\left\{-\frac{8m^3}{3qE_{\text{slow}}|\mathbf{K}|}\frac{1}{2}\left(\frac{\pi}{2\gamma}\right)^3\left[3\left(\frac{2\gamma}{\pi}\right)^2 - 1\right]\right\}. \tag{3.106}$$

This is always larger than the result from [NR64] (remember that we are above the threshold $\gamma > \gamma_{\text{crit}} = \pi/2$). Thus, in this case the locally constant field approximation underestimates the pair production probability. This can be understood in the following way: Although the oscillations of the particle wavefunctions perpendicular to the electric field are too high to resolve the temporal variation in $E(t)$, the oscillations parallel to the electric field are much slower and thus 'feel' the weak field's influence.

Finally, when the frequency of the weak field is very high $\omega_{\text{fast}} \longrightarrow \infty$ (or equivalently $\gamma \longrightarrow \infty$) we get

$$\lim_{\gamma\to\infty} P_{\text{doubly}}^{\text{tot,max}} \sim \exp\left\{-\frac{m^2}{qE_{\text{slow}}}\frac{\pi}{\gamma}\frac{2m_* - |\mathbf{K}|}{m}\right\} = \exp\left\{-\frac{\pi}{\omega_{\text{fast}}}(2m_* - |\mathbf{K}|)\right\}. \tag{3.107}$$

This result is independent of the slow field's field strength and thus corresponds to the case where the strong field can be neglected (i.e. the perturbative regime). Consequently, the result in (3.107) coincides with (3.64) when $|\mathbf{K}| \longrightarrow 0$.

Comparing these different limits with the full result from (3.102) (see fig. 3.6) we find that the doubly-assisted Sauter-Schwinger effect always yields a larger pair production probability than any of the cases where one of the three ingredients (strong field, weak field, photon) is missing. The results we derived here using the WKB method may also be obtained using the worldline instanton method (see the original article [TOS16]).

3.6 Perturbative approach to dynamical assistance

We now want to return to the dynamically-assisted Sauter-Schwinger effect. As was mentioned before, different choices for the weak field have been considered which lead to results that are quite different qualitatively (see [Lin+15] for a detailed study). For example, for the Sauter pulse we considered in section 3.4 the threshold for dynamical assistance is a constant whereas it depends on the relative strength of the weak field $\varepsilon = E_{\text{fast}}/E_{\text{slow}}$ for a Gaussian pulse ($\gamma_{\text{crit}} \sim \sqrt{|\ln \varepsilon|}$). This fundamental qualitative difference is surprising as these two field shapes are visually almost indistinguishable (see fig. 1.8). In [Tor+17] we pursued an perturbative approach to explain these qualitative differences which we want to review in the following.

We will work in 1+1 dimensions and use the same form of the electromagnetic potential (3.49) as in the previous two sections. We employ the interaction picture but this time the split into $\hat{H}_0$ problem and $\hat{H}_1$ problem will be different[28]: The strong field gives the $\hat{H}_0$-dynamics while the weak field will be included in $\hat{H}_1$. Thus, the interaction Hamiltonian is given by

$$\hat{H}_{\text{int}} = q \int \mathrm{d}x \, \hat{\overline{\Psi}} \gamma^\mu A_\mu^{\text{fast}} \hat{\Psi} \tag{3.108}$$

[28] Note that we do not consider interactions with the quantized Maxwell field in this section.

where

$$A_\mu^{\text{fast}} = \left(0, \frac{\varepsilon E_{\text{slow}}}{\omega_{\text{fast}}} G(\omega_{\text{fast}} t)\right) \tag{3.109}$$

is not a field operator but just a function of time. Then, the scattering operator up to linear order in the weak field is

$$\hat{S} = 1 - iq \int \mathrm{d}t \int \mathrm{d}x \, \hat{\overline{\Psi}} \gamma^\mu A_\mu^{\text{fast}} \hat{\Psi} + \mathcal{O}(\varepsilon^2). \tag{3.110}$$

Here, the spinor field operator $\hat{\Psi}$ has to satisfy the equations of motion for the $\hat{H}_0$ problem, that is the covariant Dirac equation with the strong background field. We calculate the amplitude for the creation of a single pair as a power series in the relative field strength ε,

$$\begin{aligned} \mathcal{M}_{e^+e^-} &= \langle 0_{\text{out}} | \, \hat{a}_{\text{out}}(p) \hat{b}_{\text{out}}(p') \hat{S} \, | 0_{\text{in}} \rangle \\ &= \mathcal{M}^{(0)}_{e^+e^-} + \varepsilon \mathcal{M}^{(1)}_{e^+e^-} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.111}$$

To obtain the total probability, one would have to integrate the absolute square of the amplitude $\mathcal{M}_{e^+e^-}$ over all momenta p and p'. We already

know the zeroth-order amplitude $\mathscr{M}^{(0)}_{e^+e^-}$ from eq. (2.98),

$$\mathscr{M}^{(0)}_{e^+e^-} = \left[(B^{++})^{-1}B^{+-}\right]_{pp'}\mathscr{M}_V, \tag{3.112}$$

and using the results for B^{++} and B^{+-} from section 3.2 we get

$$\mathscr{M}^{(0)}_{e^+e^-} = \delta(p-p')\mathscr{M}_V\frac{\left(\beta_p^{\text{out}}\right)^*}{\left(\alpha_p^{\text{out}}\right)^*} = \delta(p-p')\mathscr{M}_V\left(\mathscr{R}_p^{\text{out}}\right)^*. \tag{3.113}$$

For the first-order amplitude we need to calculate

$$\varepsilon\mathscr{M}^{(1)}_{e^+e^-} = -iq\int \mathrm{d}t \int \mathrm{d}x \,\langle 0_{\text{out}}|\,\hat{a}_{\text{out}}(p)\hat{b}_{\text{out}}(p')\hat{\overline{\Psi}}\gamma^\mu A_\mu^{\text{fast}}\hat{\Psi}\,|0_{\text{in}}\rangle\,. \tag{3.114}$$

We insert the mode expansion for the spinor field operator and its adjoint in (generalized) normal form (see (2.81) and (2.82)) into this and write the product of field operators in (generalized) normal form. We thus get

$$\begin{aligned}\varepsilon\mathscr{M}^{(1)}_{e^+e^-} = &\int\mathrm{d}t\int\frac{\mathrm{d}x}{2\pi}\int\mathrm{d}q\int\mathrm{d}q'\int\mathrm{d}k\int\mathrm{d}k'\,\frac{-iqm}{\sqrt{\epsilon_q^{\text{in}}\epsilon_k^{\text{in}}}}\left[(B^{++})^{-1}\right]_{q'q}\left[(B^{--})^{-1}\right]^*_{k'k}\\ &\times\,\overline{\psi}^+_{\text{in}}(q)\gamma^\mu A_\mu^{\text{fast}}\psi^-_{\text{in}}(k)\,\langle 0_{\text{out}}|\,\hat{a}_{\text{out}}(p)\hat{b}_{\text{out}}(p')\hat{a}^\dagger_{\text{out}}(q')\hat{b}^\dagger_{\text{out}}(k')\,|0_{\text{in}}\rangle\end{aligned} \tag{3.115}$$

where we have not explicitly written out the dependence of $\overline{\psi}^+_{\text{in}}(q)$, $\psi^-_{\text{in}}(k)$ and A_μ^{fast} on t or x. Using the anticommutation relations (2.63) for the out-operators gives two delta distributions which makes the q' and k' integrals easy to carry out. As the spatial dependence of $\psi^\pm_{\text{in}}(p;t,x)$ just consists of the phase factor e^{ipx} we can do the spatial integral as well and get another delta distribution that can be used to get rid of another momentum integral:

$$\begin{aligned}\varepsilon\mathscr{M}^{(1)}_{e^+e^-} = -iq\int\mathrm{d}t\int\mathrm{d}k\,\frac{m}{\sqrt{\epsilon_k^{\text{in}}\epsilon_k^{\text{in}}}}\mathscr{M}_V\left[(B^{++})^{-1}\right]_{pk}\left[(B^{--})^{-1}\right]^*_{p'k}\\ \times\,\overline{\psi}^+_{p,\text{in}}(k;t)\gamma^\mu A_\mu^{\text{fast}}(t)\psi^-_{p,\text{in}}(k;t)\end{aligned} \tag{3.116}$$

where $\psi^\pm_{p,\text{in}}$ denotes the Fourier transform of $\psi^\pm_{p,\text{in}}$ [see (3.3)]. Finally, we use the expressions for B^{--} and B^{++} from section 3.2, perform the last momentum integration and get

$$\varepsilon\mathscr{M}^{(1)}_{e^+e^-} = -iq\frac{m}{\epsilon_p^{\text{in}}}\frac{\mathscr{M}_V\delta(p-p')}{\left(\alpha_p^{\text{out}}\right)^*\left(\alpha_{p'}^{\text{out}}\right)^*}\int\mathrm{d}t\,\overline{\psi}^+_{p,\text{in}}(p;t)\gamma^\mu A_\mu^{\text{fast}}(t)\psi^-_{p,\text{in}}(p;t). \tag{3.117}$$

We now insert the Fourier representation of the weak field

$$\begin{aligned} A_1^{\text{fast}}(t) &= \int \mathrm{d}t\, E^{\text{fast}}(t) \\ &= \int \mathrm{d}t \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t} \tilde{E}^{\text{fast}}(\omega) \\ &= \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega} \tilde{E}^{\text{fast}}(\omega) \\ &= \frac{\varepsilon E_{\text{slow}}}{\omega_{\text{fast}}} \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega} \tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \end{aligned} \tag{3.118}$$

where $\tilde{g}$ is the Fourier transform of g. Additionally, we approximate the mode functions as

$$\psi^{\pm}_{p,\text{in}}(p;t) \approx \sqrt{\frac{\epsilon_p^{\text{in}}}{m}} u_{\pm,p}(t) e^{\mp i\varphi_p(t)}. \tag{3.119}$$

Comparing with (3.28) we see that this approximation is equivalent to assuming $\alpha_p(t) \approx 1$ and $\beta_p(t) \approx 0$ which should hold in the subcritical regime where $E_{\text{slow}} \ll E_S$; see also (3.72). Consequently, we also assume that $\alpha_p^{\text{out}} \approx 1$ in (3.117). Putting everything together, we find

$$\mathscr{M}^{(1)}_{e^+e^-} \approx i\mathscr{M}_V \delta(p-p') \frac{qE_{\text{slow}}}{\omega_{\text{fast}}} \int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{\omega} \tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \int \mathrm{d}t\, \frac{m}{\Omega_p(t)} e^{-i\omega t + 2i\varphi_p(t)} \tag{3.120}$$

where we also used $\overline{u}_{+,p}(t)\gamma^1 u_{-,p}(t) = im/\Omega_p(t)$ (see (3.17)). We approximate the time integral using the saddle point method. The saddle point t_* is given by the equation

$$2\dot{\varphi}_p(t_*) = 2\Omega_p(t_*) = \omega. \tag{3.121}$$

Differentiating with respect to ω we get

$$2\dot{\Omega}_p(t_*) t_*'(\omega) = 1. \tag{3.122}$$

Therefore,

$$\mathscr{M}^{(1)}_{e^+e^-} \approx i\mathscr{M}_V \delta(p-p') \frac{qE_{\text{slow}}}{\omega_{\text{fast}}} \int \frac{\mathrm{d}\omega}{2\pi} \frac{2m}{\omega^2} \tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \sqrt{\frac{i\pi}{\dot{\Omega}_p(t_*)}}\, e^{-i\omega t_* + 2i\varphi_p(t_*)}. \tag{3.123}$$

To calculate the frequency integral we have to choose a shape for the weak pulse. As before, we use a Sauter pulse $g(\tau) = \operatorname{sech}^2 \tau$. The Fourier transform is

$$\tilde{g}(\omega) = \int \mathrm{d}\tau\, e^{i\omega\tau} g(\tau) = \frac{\pi\omega}{\sin(\pi\omega/2)}. \tag{3.124}$$

For $\omega \gg \omega_{\text{fast}}$, we get

$$\tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \approx \frac{2\pi\omega}{\omega_{\text{fast}}} \exp\left(-\frac{\pi\omega}{2\omega_{\text{fast}}}\right). \tag{3.125}$$

Assuming that we are in this regime we may estimate the frequency integral

$$\begin{aligned} &\int \frac{\mathrm{d}\omega}{2\pi} \frac{2m}{\omega^2} \tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \sqrt{\frac{i\pi}{\dot{\Omega}_p(t_*)}} e^{-i\omega t_* + 2i\varphi_p(t_*)} \\ &\approx \frac{2m}{\omega_{\text{fast}}} \int \mathrm{d}\omega \frac{1}{\omega} \sqrt{\frac{i\pi}{\dot{\Omega}_p(t_*)}} \exp\left(-\frac{\pi\omega}{2\omega_{\text{fast}}} - i\omega t_* + 2i\varphi_p(t_*)\right) \end{aligned} \tag{3.126}$$

using the saddle point method as well. The saddle point ω_* is a solution to

$$-\frac{\pi}{2\omega_{\text{fast}}} - it_*(\omega_*) - i\omega_* t_*'(\omega_*) + \underbrace{2i\dot{\varphi}_p(t_*(\omega_*))}_{=i\omega_*} t_*'(\omega_*) = 0 \tag{3.127}$$

which reduces to

$$t_*(\omega_*) = i\frac{\pi}{2\omega_{\text{fast}}}. \tag{3.128}$$

This is (up to a sign) the same expression as for the singularity $t_*^{\cosh}$ we already encountered in section 3.4 and section 3.5. Upon inserting this into the exponent, we find that the first two terms in the exponent cancel. Thus, the frequency integral in (3.126) approximately reduces to

$$\begin{aligned} &\int \frac{\mathrm{d}\omega}{2\pi} \frac{2m}{\omega^2} \tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \sqrt{\frac{i\pi}{\dot{\Omega}_p(t_*)}} e^{-i\omega t_* + 2i\varphi_p(t_*)} \\ &\approx \frac{2m}{\omega_{\text{fast}}} \frac{1}{\omega_*} \sqrt{\frac{i\pi}{\dot{\Omega}_p(t_*(\omega_*))}} \sqrt{\frac{2\pi}{it_*'(\omega_*)}} \exp\{2i\varphi_p[t_*(\omega_*)]\} \\ &= \frac{2m}{\omega_{\text{fast}}} \frac{2\pi}{\omega_*} \exp\{2i\varphi_p[t_*(\omega_*)]\} \end{aligned} \tag{3.129}$$

where we have used (3.122) in going from the second to the third line. Therefore, we get for the first-order amplitude

$$\begin{aligned} \mathscr{M}_{e^+e^-}^{(1)} &\approx i\mathscr{M}_V \delta(p-p') \frac{qE_{\text{slow}}}{\omega_{\text{fast}}} \frac{2m}{\omega_{\text{fast}}} \frac{2\pi}{\omega_*} \exp\{2i\varphi_p[t_*(\omega_*)]\} \\ &= i\mathscr{M}_V \delta(p-p') \frac{2\pi}{\gamma} \frac{2m}{\omega_{\text{fast}}} \frac{m}{\omega_*} \exp\{2i\varphi_p[t_*(\omega_*)]\} \end{aligned} \tag{3.130}$$

where we used the definition of the combined Keldysh parameter (3.57)

in the last step. If we assume a constant field for the strong field we have [see (3.45)]

$$\Omega_p(t) = m\sqrt{1+\left(\frac{p+qE_{\text{slow}}t}{m}\right)^2}, \quad \varphi_p(t) = \frac{1}{2}\frac{m^2}{qE_{\text{slow}}}\phi\left(\frac{p+qE_{\text{slow}}t}{m}\right) \tag{3.131}$$

and therefore

$$\Omega_p[t_*(\omega_*)] = m\sqrt{1+\left(\frac{p}{m}+i\frac{\pi}{2\gamma}\right)^2}, \quad \varphi_p[t_*(\omega_*)] = \frac{1}{2}\frac{m^2}{qE_{\text{slow}}}\phi\left(\frac{p}{m}+i\frac{\pi}{2\gamma}\right) \tag{3.132}$$

In the region where the assistance due to the weak field is strong we expect that $\varepsilon\mathscr{M}^{(1)}_{e^+e^-} \gg \mathscr{M}^{(0)}_{e^+e^-}$. Thus, we may approximate the total probability

$$P_{e^+e^-} = \int \mathrm{d}p \int \mathrm{d}p' \, |\mathscr{M}_{e^+e^-}|^2 \approx \int \mathrm{d}p \int \mathrm{d}p' \, \left|\varepsilon\mathscr{M}^{(1)}_{e^+e^-}\right|^2. \tag{3.133}$$

Inserting (3.130) into this and carrying out one momentum integration, we obtain the total probability density relative to the vacuum persistence amplitude

$$\frac{P_{e^+e^-}}{\delta(0)\,|\mathscr{M}_V|^2} \approx \varepsilon^2 \int \mathrm{d}p \left(\frac{2\pi}{\gamma}\right)^2 \left(\frac{2m}{\omega_{\text{fast}}}\right)^2 \frac{m^2}{\omega_*^2} \exp\{-4\Im\varphi_p[t_*(\omega_*)]\}. \tag{3.134}$$

The integrand has a saddle point at $p = 0$ and therefore[29]

$$\begin{aligned}\frac{P_{e^+e^-}}{\delta(0)\,|\mathscr{M}_V|^2} \approx{}& 8\varepsilon^2 \frac{m}{\gamma^2}\sqrt{2\pi}\left(\frac{m^2}{qE_{\text{slow}}}\right)^{3/2}\left[\left(\frac{2\gamma}{\pi}\right)^2 - 1\right]^{-3/4} \\ &\times \exp\left\{-\frac{2m^2}{qE_{\text{slow}}}\left[\frac{\pi}{2\gamma}\sqrt{1-\left(\frac{\pi}{2\gamma}\right)^2} + \arcsin\left(\frac{\pi}{2\gamma}\right)\right]\right\}\end{aligned} \tag{3.135}$$

which has exactly the same exponent as the expression we found in section 3.4 for the pair production density [compare (3.63)]. The result here rests on the underlying assumption that the Fourier transform of the weak pulse has the form (3.125) which is only valid when $\omega \gg \omega_{\text{fast}}$ (and consequently also $\omega_* \gg \omega_{\text{fast}}$). To see how this reflects in our result (3.135), consider

$$\frac{\omega_*}{\omega_{\text{fast}}} = \pi\frac{m^2}{\omega_{\text{fast}}^2}\frac{qE_{\text{slow}}}{m^2}\sqrt{\left(\frac{2\gamma}{\pi}\right)^2 - 1}. \tag{3.136}$$

[29] Comparing with equation (2.13) in [Tor+17] we find the same exponent but a different prefactor. This is due to the different dimensionality of space: Here we worked in 1+1 dimensions whereas we used a 3+1-dimensional spacetime in [Tor+17].

From the initial assumptions for the setup of the dynamically-assisted Sauter-Schwinger effect we have $\omega_{\text{fast}} \ll m$ and $qE_{\text{slow}} \ll m^2$. Thus, in the regime where $\omega_*/\omega_{\text{fast}} \gg 1$ we demand that

$$\sqrt{\left(\frac{2\gamma}{\pi}\right)^2 - 1} \gg \frac{m^2}{qE_{\text{slow}}} \gg 1 \tag{3.137}$$

which is equivalent to saying that $\gamma \gg \gamma_{\text{crit}}$. This is consistent with the result in (3.135) as the prefactor diverges when $\gamma \longrightarrow \gamma_{\text{crit}} = \pi/2$. Thus, the expression for the pair production rate (3.135) is only valid when γ lies sufficiently above γ_{crit}. Closer to the threshold we expect that the strong field starts to dominate and thus we would have to include the contribution from the zeroth-order amplitude $\mathscr{M}^{(0)}_{e^+e^-}$.

We see that using the perturbative approach just described we may recover results that have been obtained in a non-perturbative way before. But what is even more interesting is the fact that the result in (3.135) rests on the high-frequency behaviour of the weak pulse's Fourier transform. This suggests that a different field shape than the Sauter pulse that also has a exponentially-decaying Fourier transform will give a very similar result[30]. Consider the function

$$g(\tau) = -\frac{\sinh\tau}{\cosh^2\tau}. \tag{3.138}$$

which has the Fourier transform

$$\tilde{g}(\omega) = -i\frac{\pi\omega}{\cosh(\pi\omega/2)}. \tag{3.139}$$

For $\omega \gg \omega_{\text{fast}}$, we then find

$$\tilde{g}\left(\frac{\omega}{\omega_{\text{fast}}}\right) \approx -i\frac{2\pi\omega}{\omega_{\text{fast}}}\exp\left(-\frac{\pi\omega}{2\omega_{\text{fast}}}\right). \tag{3.140}$$

which is up to the factor $-i$ the same as the Fourier transform for the Sauter pulse (3.125). Consequently, the first-order amplitude just gets the same factor of $-i$. Thus, the pair production probability in the regime where the assistance is strong (3.135) will be exactly the same for this pulse although it is visually very different from the Sauter pulse (see fig. 3.7).

This shows that the frequency spectrum of a pulse seems to be more important to its effect on dynamical assistance than its actual shape in

[30] We expect differences between pulses of this type to be restricted to the prefactor and the value of the threshold γ_{crit} [note that γ always occurs in the form $\gamma/\gamma_{\text{crit}}$ in (3.135)].

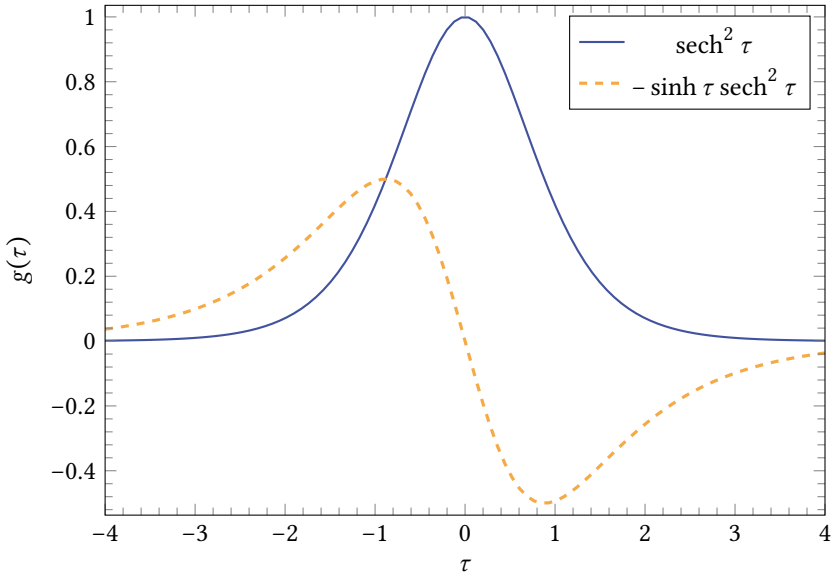

FIGURE 3.7 – Comparison of Sauter pulse $g(\tau) = \operatorname{sech}^2 \tau$ and the pulse from (3.138).

the time domain. For example, a Gaussian pulse

$$g(\tau) = \exp\left(-\tau^2\right) \tag{3.141}$$

looks very similar to a Sauter pulse, as was already mentioned in the introduction (see section 1.4). However, its Fourier transform, which is also a Gaussian

$$\tilde{g}(\omega) = \sqrt{\pi}\exp\left(-\omega^2/4\right), \tag{3.142}$$

behaves quite differently for large ω. The saddle point of the frequency integral in (3.123) for this pulse is

$$\omega_* = \frac{2m}{1 + \left(qE_{\text{slow}}/\omega_{\text{fast}}^2\right)^2}\left[i\frac{p}{m}\frac{qE_{\text{slow}}}{\omega_{\text{fast}}^2} \pm \sqrt{1 + \left(\frac{p}{m}\right)^2 + \left(\frac{qE_{\text{slow}}}{\omega_{\text{fast}}^2}\right)^2}\right] \tag{3.143}$$

and the saddle point of the time integral is

$$t_*(\omega_*) = i\frac{\omega_*}{2\omega_{\text{fast}}^2}. \tag{3.144}$$

Using the exact same steps as before, we approximate the pair production density using the saddle point method for the momentum integral

with the saddle point $p = 0$ and obtain

$$\frac{P_{e^+e^-}}{\delta(0)\left|\mathscr{M}_V\right|^2} \approx \frac{1}{2}\varepsilon^2 m\left(\frac{\pi}{2}\right)^{3/2} \frac{qE_{\text{slow}}}{m^2}\frac{qE_{\text{slow}}}{m\omega_{\text{fast}}}\left[1+\left(\frac{qE_{\text{slow}}}{\omega_{\text{fast}}^2}\right)^2\right]^{3/2} \times \exp\left[-\frac{2m^2}{qE_{\text{slow}}}\arctan\left(\frac{qE_{\text{slow}}}{\omega_{\text{fast}}^2}\right)\right]. \tag{3.145}$$

Here, the prefactor is well defined for all values of $\omega_{\text{fast}} > 0$. To see where the regime of strong dynamical assistance begins, we expand the exponent in a Taylor series for small $\omega_{\text{fast}}^2/qE_{\text{slow}}$:

$$\exp\left[-\frac{2m^2}{qE_{\text{slow}}}\arctan\left(\frac{qE_{\text{slow}}}{\omega_{\text{fast}}^2}\right)\right] \approx \exp\left(-\frac{\pi m^2}{qE_{\text{slow}}} + 2\gamma^2\right). \tag{3.146}$$

For $\gamma \longrightarrow 0$ we recover Schwinger's result (1.8). At the threshold γ_{crit} we expect the contributions of the zeroth-order term and of the first-order term to be of roughly the same magnitude. Thus,

$$\varepsilon^2 \exp\left(2\gamma_{\text{crit}}^2\right) \sim 1 \tag{3.147}$$

which is equivalent to

$$\gamma_{\text{crit}} \sim \sqrt{|\ln \varepsilon|} \tag{3.148}$$

as expected; compare with equation (30) in [Lin+15].

So far, we only looked at the regime where dynamical assistance is strong. However, when calculating the probability density including both the zeroth-order amplitude and the first-order amplitude there is interference between these contributions. This analytic approximation almost perfectly resembles the numerical data for the Sauter pulse but does so only for certain choices of parameters for the Gaussian pulse (see figures 1–3 in [Tor+17]). As it turns out, the numeric results for the Gaussian pulse can be reproduced if one considers higher order-terms in the perturbation expansion (3.110). The order whose contribution is dominant can be determined analytically and is found to become quite large for certain parameters (see [Tor+17, section 3.2]). Once again, this qualitatively different situation in comparison with the Sauter pulse, where already the first-order amplitude is an excellent approximation, can be traced back to the qualitatively different Fourier transforms.

3.7 Time-dependent mass

We now want to spend some time on the case of a purely timedependent mass with no electric field ($A(t) = 0$) as we did in [OS19]. This is useful, as our first example for a truly spacetime-dependent field will be that of a spacetime-dependent mass (see chapter 5). The Riccati equation (3.19) for the time-dependent mass has the form

$$\dot{\mathscr{R}}_p^{\text{out}} = -i\frac{p\dot{m}}{2\Omega_p^2}\left[e^{-2i\varphi_p} - \mathscr{R}_p^2 e^{2i\varphi_p}\right] \tag{3.149}$$

where

$$\varphi_p = \int_{-\infty}^{t} \mathrm{d}t'\, \Omega_p(\tau') = \int_{-\infty}^{t} \mathrm{d}t'\, \sqrt{m^2(t') + p^2}. \tag{3.150}$$

We assume that the mass $m(t)$ is of the form

$$m(t) = m_0\sqrt{1 + \left[\frac{f(\omega t)}{\gamma}\right]^2}. \tag{3.151}$$

Here, m_0 is a constant mass, e.g. the electron mass m_e, ω sets the time scale of the pulse and γ is a dimensionless parameter that determines the amplitude of the pulse. If γ is large, the relative change of $m(t)$ is small. Thus, this case can be treated within perturbation theory. However, if γ is small the change of $m(t)$ will be large and we need to use non-perturbative methods like the WKB approach. More generally speaking, the WKB method can be applied successfully if the mass (or the potential) changes only adiabatically, i.e. if $\dot{m} \ll m^2$. Thus, for the mass given in (3.151), we require that $\gamma\omega/m_0 \ll 1$. Hence, if $\omega \ll m_0$ we can indeed use the WKB approach to treat the case of small γ.

Furthermore, the mass (3.151) closely resembles the form of an electron's effective mass in a laser pulse

$$m_{\text{eff}} = m\sqrt{1 - \frac{\langle qA_\mu qA^\mu\rangle}{m^2}}, \tag{3.152}$$

see e.g. [Kib66; DF08; KGA14]. Comparing (3.151) and (3.152) we see that the parameter γ plays a role very similar to the the Keldysh parameter

$$\gamma = \frac{m\omega}{qE} \tag{3.153}$$

which, as we saw in the preceding sections, plays a crucial role in pair production processes using strong electromagnetic fields. However, the mathematical similarity between the case with a time-dependent mass

and a strong electromagnetic field is not exact and the pair production probability for the latter should be computed using the electromagnetic potential A_μ directly.

For a mass of the form (3.151) we have

$$\dot{m} = 2\frac{m_0\omega}{\gamma^2}\frac{f(\omega t)f'(\omega t)}{\sqrt{1+\left[\frac{f(\omega t)}{\gamma}\right]^2}} \tag{3.154}$$

and

$$\Omega_p^2 = m^2 + p^2 = (m_0^2 + p^2)\left\{1+\left[\frac{f(\omega t)}{\tilde{\gamma}}\right]^2\right\} \tag{3.155}$$

where $\tilde{\gamma} = \gamma\sqrt{1+\left(\frac{p}{m_0}\right)^2}$. We will use the dimensionless time $\tau = \omega t$ in the following. Then the Riccati equation (3.149) can be written as

$$\frac{\mathrm{d}\mathscr{R}_p}{\mathrm{d}\tau} = -\frac{i}{\tilde{\gamma}^2}\frac{p}{m_0}\frac{f(\tau)f'(\tau)}{\sqrt{1+\left[\frac{f(\tau)}{\gamma}\right]^2}\left\{1+\left[\frac{f(\tau)}{\tilde{\gamma}}\right]^2\right\}}\left[e^{-2i\varphi_p} - \mathscr{R}_p^2 e^{2i\varphi_p}\right] \tag{3.156}$$

where

$$\varphi_p = \frac{m_0}{\omega}\sqrt{1+\left(\frac{p}{m_0}\right)^2}\int_{-\infty}^{\tau}\mathrm{d}\tau'\sqrt{1+\left[\frac{f(\tau')}{\tilde{\gamma}}\right]^2}. \tag{3.157}$$

Again, we will use a linearised approximation to the Riccati equation to calculate $\mathscr{R}_p^{\mathrm{out}}$:

$$\mathscr{R}_p^{\mathrm{out}} \approx -\frac{i}{\tilde{\gamma}^2}\frac{p}{m_0}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\frac{f(\tau)f'(\tau)}{\sqrt{1+\left[\frac{f(\tau)}{\gamma}\right]^2}\left\{1+\left[\frac{f(\tau)}{\tilde{\gamma}}\right]^2\right\}}e^{-2i\varphi_p} \tag{3.158}$$

The integrand has the following singularities[31]:

- At $\tau = \tau_1$ where $f(\tau_1) = \pm i\gamma$,
- at $\tau = \tau_2$ where $f(\tau_1) = \pm i\tilde{\gamma}$,
- (possibly) at $\tau = \tau_3$ where $f(\tau_3)$ is singular.

As in the previous cases, we expect that the value of the integral (3.158) is dominated by the value of the exponential at one of these singularities; if multiple of them give equally large contributions, interference effects will occur.

[31] The point $\tau = \tau_2$ is also a saddle point (or point with stationary phase) of the integrand because $\Omega_p(\tau_2/\omega) = \dot{\varphi}_p(\tau_2/\omega) = 0$.

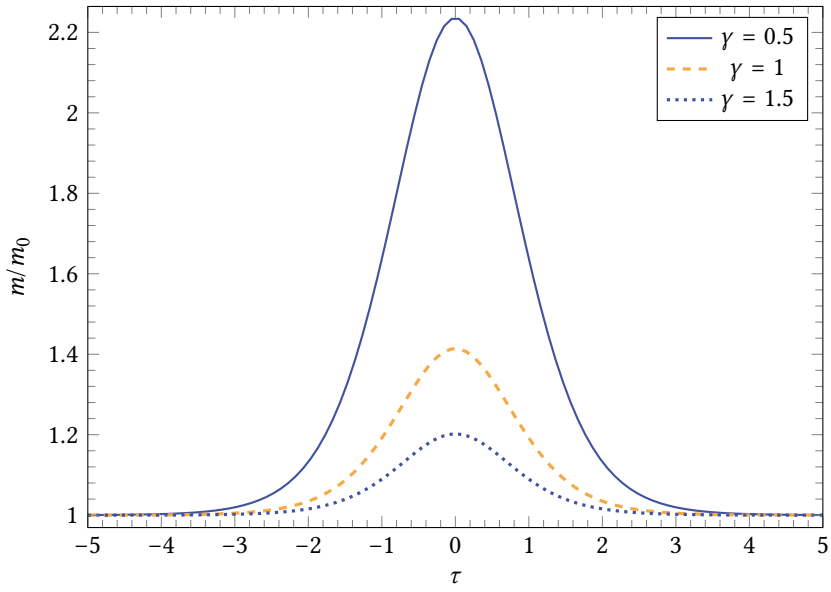

FIGURE 3.8 – A mass of the form given in (3.151) using $f(\tau) = \operatorname{sech}\tau$ for different values of γ.

We consider a pulse with

$$f(\tau) = \operatorname{sech}\tau, \tag{3.159}$$

as an example; see fig. 3.8. We then find for the positions of the singularities listed above

$$\begin{aligned}
\tau_1 &= \operatorname{arcosh}\left(\pm\frac{i}{\gamma}\right) = \ln\left[\frac{1}{\gamma} + \sqrt{1+\left(\frac{1}{\gamma^2}\right)}\right] - i\frac{\pi}{2} + i\pi n \\
\tau_2 &= \operatorname{arcosh}\left(\pm\frac{i}{\tilde{\gamma}}\right) = \ln\left[\frac{1}{\tilde{\gamma}} + \sqrt{1+\left(\frac{1}{\tilde{\gamma}^2}\right)}\right] - i\frac{\pi}{2} + i\pi n \\
\tau_3 &= \operatorname{arcosh}(0) = -i\frac{\pi}{2} + i\pi n
\end{aligned} \tag{3.160}$$

where $n \in \mathbb{Z}$. It should be noted that for fixed n, the singularities have the same imaginary part and are actually identical when looking at special limits, e.g. $\tau_1 = \tau_2$ for $\tilde{\gamma} \longrightarrow \gamma$ (i.e. $p \longrightarrow 0$) and $\tau_1 = \tau_2 = \tau_3$ for $\gamma \longrightarrow \infty$.

The phase integral (3.157) can be solved analytically, giving

$$\varphi_p = \frac{m_0}{\omega}\sqrt{1+\left(\frac{p}{m_0}\right)^2}\left[\phi(\tau) - \phi(-\infty)\right] \tag{3.161}$$

where

$$\phi(\tau) = \frac{1}{\tilde{\gamma}}\arctan\left[\frac{\sinh\tau}{\sqrt{1+\tilde{\gamma}^2\cosh^2\tau}}\right] + \operatorname{artanh}\left[\frac{\tilde{\gamma}\sinh\tau}{\sqrt{1+\tilde{\gamma}^2\cosh^2\tau}}\right]. \tag{3.162}$$

This has branch cuts parallel to the real axis at multiples of $\pi/2$. We write the prefactor of the integrand in (3.158) as

$$\frac{f(\tau)f'(\tau)}{\sqrt{1+\left[\frac{f(\tau)}{\gamma}\right]^2}\left\{1+\left[\frac{f(\tau)}{\tilde{\gamma}}\right]^2\right\}} = \frac{-\sinh\tau}{\sqrt{\cosh^2\tau+1/\gamma^2}\left(\cosh^2\tau+1/\tilde{\gamma}^2\right)} \tag{3.163}$$

Thus, we see that the prefactor is actually well-defined at $\tau = \tau_3$. Additionally, at $\tau = -i\pi/2 + 2i\pi n$, $n \in \mathbb{Z}$, the imaginary part of $\phi(\tau)$ goes to $-\infty$ while at $\tau = i\pi/2 + 2i\pi n$, $n \in \mathbb{Z}$, it goes to $+\infty$. Thus, the integrand is exponentially suppressed at the former and only the latter are singularities that have to be avoided.

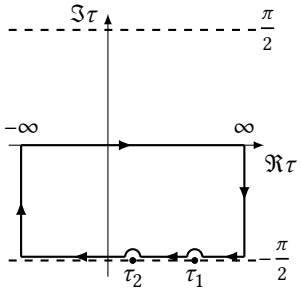

FIGURE 3.9 – Integration contour used to calculate the integral in (3.158).

We thus choose a rectangular contour for the integral (3.158) in the lower half-plane as pictured in fig. 3.9. The lines parallel to the imaginary axis do not contribute anything because the prefactor vanishes for $\Re\tau \longrightarrow \pm\infty$. Because the contour does not enclose any poles, the integral over the whole contour vanishes. Thus, the original integral is equivalent to the integral along the branch cut at $\tau = -i\pi/2$. However, the semicircle around the square-root singularity at $\tau = \tau_1$ does not contribute anything, too. We thus have identified the dominant singularity to be at τ_2 and consequently get

$$\mathscr{R}_p^{\text{out}} \propto e^{-2i\varphi_p(\tau_2)}. \tag{3.164}$$

Because $\phi(\tau_2) = -\frac{\pi}{2\tilde{\gamma}} - i\frac{\pi}{2}$, the density of produced pairs is approximately

$$\left|\mathscr{R}_p^{\text{out}}\right|^2 \propto \exp\left[4\frac{m_0}{\omega}\sqrt{1+\left(\frac{p}{m_0}\right)^2}\,\Im\phi(\tau_2)\right] = \exp\left[-2\pi\frac{m_0}{\omega}\sqrt{1+\left(\frac{p}{m_0}\right)^2}\right]. \tag{3.165}$$

This result might seem a bit odd at first as the exponent does not depend on γ – when γ goes to infinity, the pulse's amplitude goes to zero and therefore no pairs should be produced in that case. The desired behaviour, however, is not achieved by the exponent but by the prefactor. Looking at (3.158) again, we see a factor of γ^{-2} in front of the integral and thus for large $\gamma \gg 1$ we expect the density of produced pairs to go like γ^{-4}. This is consistent with our earlier observation that the WKB approach breaks down for γ large and perturbation theory has to be used.

We confirmed our results by numerically computing $\mathscr{R}_p^{\text{out}}$ from the full Riccati equation (3.156). We used `TIDES` [Aba+12] for the actual integration. This supports arbitrary-precision computations using `MPFR`

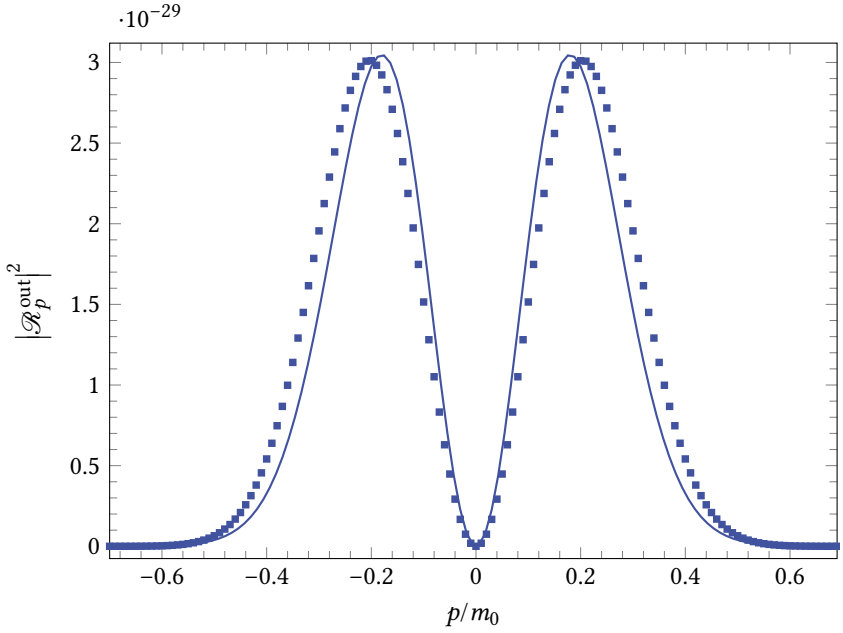

FIGURE 3.10 – Plot of the density of produced pairs $\left|\mathscr{R}_p^{\text{out}}\right|$ for the time-dependent mass (3.151) with $f(\tau) = \operatorname{sech}\tau$, $\omega = 0.1m_0$ and $\gamma = 0.1$. The squares are the numerically calculated result while the solid line represents the analytical result (3.165) with $5p^2/m_0^2$ as the prefactor.

[Fou+07] which is beneficial for the highly oscillatory integral we are facing here. To parallelise the calculation, we employed `GNU Parallel` [Tan11].

A typical result for $\mathscr{R}_p^{\text{out}}$ is shown in fig. 3.10 together with the analytical estimate from (3.165) for $\omega = 0.1m_0$ and $\gamma = 0.1$. Our analytical approximation did not yield the prefactor. However, from (3.158) we see that the prefactor has to be proportional to p^2/m_0^2. To produce fig. 3.10 we used $5p^2/m_0^2$ for that specific choice of parameters to fit the height of the peaks in the numerical result. The agreement between the numerical and the analytical result is very good. Additionally, the analytical approximation is within an order of magnitude of the numerical result even without the heuristically determined prefactor. Indeed, when plotting the dependence of the maximum of $\left|\mathscr{R}_p^{\text{out}}\right|^2$ on ω for a fixed value of $\gamma = 0.1$, the numerically-calculated points fall nicely on the curve predicted by the analytical approximation from (3.165); see fig. 3.11.

Furthermore, we investigated the dependence on γ for fixed $\omega = 0.1m_0$. In fig. 3.12 we plotted the maximum of $\left|\mathscr{R}_p^{\text{out}}\right|^2$ for a wide range of values for γ in a log-log plot. We fitted a line with slope −3.91 to the data points for $\gamma \gg 1$. Thus, in agreement with the analytic considerations, we see that $\left|\mathscr{R}_p^{\text{out}}\right|^2$ approximately behaves like γ^{-4} for large values of γ. For small $\gamma \ll 1$, the numerically calculated $\left|\mathscr{R}_p^{\text{out}}\right|^2$ seems to be constant which is consistent with the prediction from (3.165). In the region where $\gamma \approx 1$ fluctuations can be observed. However, these do

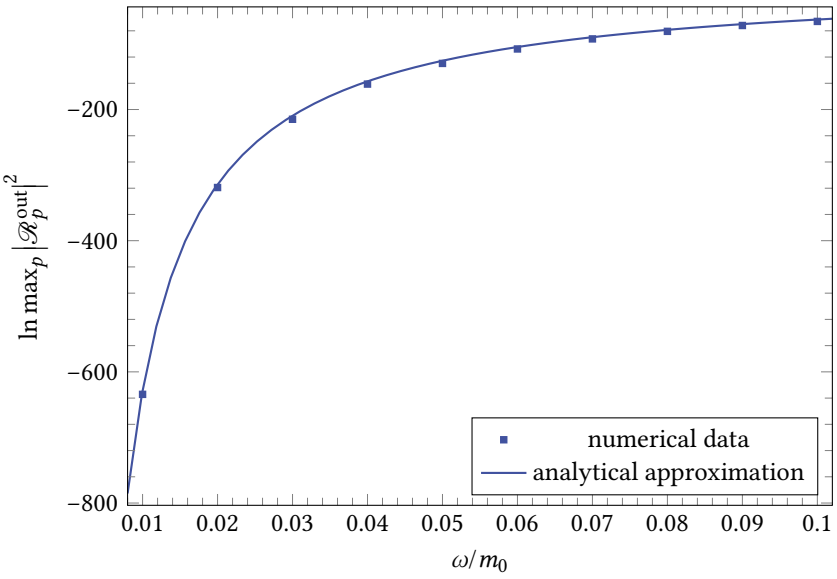

FIGURE 3.11 – Plot of the logarithmum of the maximum of $\left|\mathscr{R}_p^{\text{out}}\right|^2$ for different values of ω and $\gamma = 0.1$. The plot shows both the numerical results and the analytical approximation from (3.165).

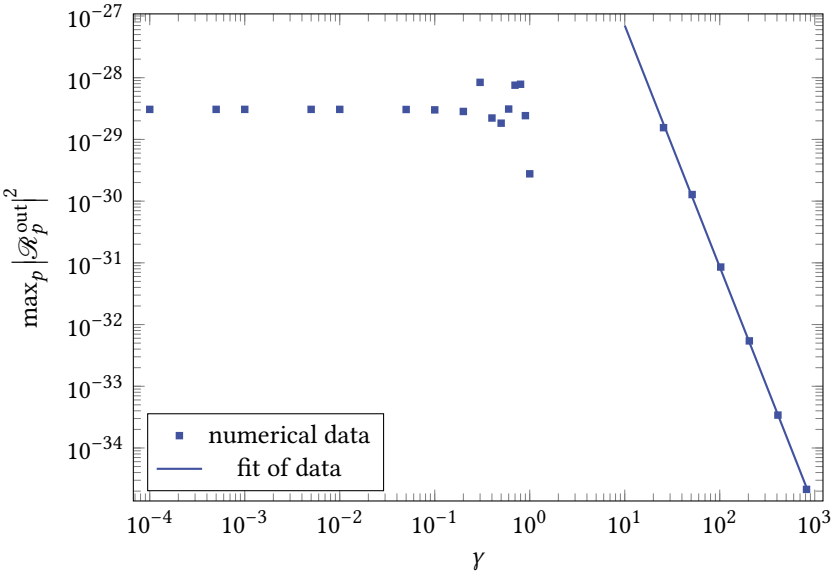

FIGURE 3.12 – Log-log plot of the (numerically-calculated) maximum of $\left|\mathscr{R}_p^{\text{out}}\right|^2$ for different values of γ with $\omega = 0.1m_0$. The slope of the fitted line is −3.91. There is a crossover from the non-perturbative regime where (3.165) holds to the perturbative regime where the prefactor $\sim \gamma^{-4}$ dominates.

not change the order of magnitude and thus can be concluded to come from the prefactor. These fluctuations mark the crossover between the non-perturbative regime ($\gamma \ll 1$) where (3.165) holds to the perturbative regime ($\gamma \gg 1$) where the prefactor, which is proportional to γ^{-4}, dominates.

Chapter 4

Spacetime-dependent WKB method

In the last chapter we looked at pair production processes in time-dependent external fields. In this chapter, we want to investigate pair production with spacetime-dependent backgrounds. As before, we need to find (approximate) solutions to the Dirac equation. We would like to derive a method that is similar to the WKB approach we used before. However, as the external field does not depend on only a single coordinate we cannot reduce the Dirac equation to an ordinary differential equation. Nevertheless, we make an ansatz with an exponential $\exp(iS)$ as before and require the phase S to be a solution to the eikonal or Hamilton-Jacobi equation for a relativistic particle in the prescribed external field. As in the solely time-dependent case, we then obtain equations for coefficients α and β but this time these are still partial differential equations. As we will show at the end of this chapter, these equations reproduce the correct equations in the purely time-dependent and the purely space-dependent case. Furthermore, they provide a better starting point for calculating approximations to the pair production probability than the Dirac equation, which actually is also a system of partial differential equations.

This method could als be motivated by inserting an ansatz $\psi \sim \exp(iS/\hbar)$ into the Dirac equation and making an expansion in terms of small $\hbar$. Then, the leading-order terms produce the eikonal or Hamilton-Jacobi equation while higher-order terms produce the equations for the Bogoliubov coefficients α and β (see also [Di 14a]).

Finding solutions to the eikonal equation is crucial for this method to work. If the problem effectively depends only on a single spacetime coordinate (e.g. time or a spatial coordinate) such solutions are straightforward to find. In fact, we already encountered solutions to the eikonal equation in the time-dependent case in the form of $px \pm \varphi_p(t)$ which is the phase of a term with momentum p in the spinor before Fourier expansion (compare (3.3) and (3.10)).

The use of the eikonal equation is a concept originating from optics. There, the wave equation governs the dynamics of an optical system in the same way as quantum mechanical wave equations like the Schrödinger equation or the Dirac equation do for quantum mechanical systems. Making an ansatz proportional to $\exp(iS)$ for the wave field leads to geometrical (or ray) optics which can be used to describe many optical phenomena but not diffraction or interference. To include these, too, the dynamics of the wave field's prefactor has to be considered as well. Similarly, if we insert an ansatz proportional to $\exp(iS)$ into the Dirac equation we recover the Hamiltonian-Jacobi equation for a classical relativistic electron but to observe quantum mechanical behaviour we need to include the prefactor of the spinor, too.

The method that we present in this chapter and the results that are contained in the following chapters were published in [OS19].

4.1 Eikonal equation

We start with the covariant Dirac equation

$$\Big[i\gamma^\mu\big(\partial_\mu + iqA_\mu\big) - m\Big]\psi = 0. \tag{4.1}$$

where now both the components of the electromagnetic potential A_μ and the electron mass m may be spacetime-dependent. We define two operators

$$M_\pm = -\gamma^\mu\big(\partial_\mu S_\pm + qA_\mu\big), \tag{4.2}$$

with $S_\pm$ being two independent solutions of the eikonal equation (or Hamilton-Jacobi equation)

$$\eta^{\mu\nu}\big(\partial_\mu S_\pm + qA_\mu\big)\big(\partial_\nu S_\pm + qA_\nu\big) = m^2. \tag{4.3}$$

We will often use $\Pi^\pm_\mu = \partial_\mu S_\pm + qA_\mu$ as an abbreviation[32] so that $M_\pm = -\gamma^\mu\Pi^\pm_\mu$.

As we will see later on, the choice of which solution gets labelled S_+ or S_- depends on the physical problem at hand and influences how to interpret certain quantities (see section 4.4 for a particular example). However, usually – especially when dealing with electromagnetic fields depending on time – we will employ the convention that $\Pi^+_t = \partial_t S_+ + qA_0$ is strictly negative while $\Pi^-_t = \partial_t S_- + qA_0$ is strictly positive, i.e.

$$\Pi^\pm_t = \partial_t S_\pm + qA_0 = \mp\sqrt{m^2 + \big(\nabla S_\pm + q\mathbf{A}\big)^2}. \tag{4.4}$$

[32] Note that $\Pi^\pm_\mu$ is manifestly gauge-invariant: A local gauge transformation $\psi \mapsto e^{i\vartheta}\psi$ can also be understood as adding ϑ to $S_\pm$. At the same time, the gauge field qA_μ transforms according to $qA_\mu \mapsto qA_\mu - \partial_\mu\vartheta$, thus leaving the sum $\Pi^\pm_\mu = \partial_\mu S_\pm + qA_\mu$ invariant.

We introduce this convention so that if A_μ vanishes, S_+ and S_- correspond to plane-wave solutions with positive and negative energy as introduced in section 2.5, respectively, and the operators $M_\pm$ reduce to

$$M_\pm \overset{A_\mu \to 0}{\longrightarrow} \mp \gamma^\mu p_\mu. \tag{4.5}$$

In general, squaring the operators $M_\pm$ gives

$$M_\pm^2 = \gamma^\mu \gamma^\nu \, \Pi_\mu^\pm \Pi_\nu^\pm = \eta^{\mu\nu} \, \Pi_\mu^\pm \Pi_\nu^\pm = m^2. \tag{4.6}$$

Thus, the operators $M_\pm$ both have the two distinct eigenvalues $\pm m$. Let $u_\pm$ and $v_\pm$ be their respective eigenvectors defined as follows

$$M_+ u_\pm = \pm m u_\pm, \quad M_- v_\pm = \pm m v_\pm. \tag{4.7}$$

There may actually be several linear independent eigenvectors to each eigenvalue because of the spin degree of freedom. Thus, we might denote eigenvectors corresponding to spin s by $u_\pm(s)$ and $v_\pm(s)$ when needed. We will leave out the spin argument when all the eigenvectors in an expression have the same spin.

We now want to show some properties of the operators and their eigenvectors that will prove to be useful afterwards. We introduce the usual inner product for spinors[33]

$$\langle \psi, \phi \rangle = \overline{\psi} \phi = \psi^\dagger \gamma^0 \phi. \tag{4.8}$$

[33] Although $\psi^\dagger \phi$ is also a perfectly valid choice for an inner product mathematically, we choose $\overline{\psi}\phi$ here because it is invariant under Lorentz transformations.

Next observe that the gamma matrices γ^μ are self-adjoint with respect to this inner product, as can be seen from the following:

$$\overline{\psi} \gamma^\mu \phi = \psi^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 \phi = \psi^\dagger \left(\gamma^\mu\right)^\dagger \gamma^0 \phi = \overline{\left(\gamma^\mu \psi\right)} \phi. \tag{4.9}$$

Therefore, as the operators $M_\pm$ are sums of self-adjoint operators, they are itself self-adjoint. Moreover, by the spectral theorem their respective eigenvectors u_+ and u_- as well as v_+ and v_- are orthogonal with respect to this inner product, i.e.

$$\overline{u}_+ u_- = \overline{u}_- u_+ = \overline{v}_+ v_- = \overline{v}_- v_+ = 0. \tag{4.10}$$

Additionally, we assume that the eigenvectors are normalised according to

$$\overline{u}_+ u_+ = -\overline{u}_- u_- = -\overline{v}_+ v_+ = \overline{v}_- v_- = 1. \tag{4.11}$$

With this normalisation, if A_μ vanishes, the eigenvectors u_+ and v_+ are

exactly the same as in the plane-wave case, i.e. $u_+ = u$ and $v_+ = v$. In fact, even in the general case we may derive completeness relations for the eigenvectors similar to the ones we found for plane waves [see (2.40)]: Consider the operator

$$\mathbb{A}_{u_+} = \sum_s u_+(s)\overline{u}_+(s). \tag{4.12}$$

Acting with it on the eigenvector $u_+(s')$ gives

$$\mathbb{A}_{u_+} u_+(s') = \sum_s u_+(s)\overline{u}_+(s)u_+(s') = u_+(s) \tag{4.13}$$

where we have used the normalisation from (4.11). Similarly, acting on $u_-(s')$ gives

$$\mathbb{A}_{u_+} u_-(s') = 0 \tag{4.14}$$

because of the orthogonality of u_+ and u_-. We observe that $\mathbb{A}_{u_+}$ acts on u_+ and u_- exactly like $(M_+ + m)/2m$ does. Because u_+ and u_- can be used as a basis for spinors – for they are eigenvectors of a self-adjoint operator – we conclude that $\mathbb{A}_{u_+}$ will act like $(M_+ + m)/2m$ on *any* spinor. Therefore,

$$\mathbb{A}_{u_+} = \sum_s u_+(s)\overline{u}_+(s) = \frac{M_+ + m}{2m}. \tag{4.15}$$

In complete analogy, we can show that

$$\sum_s v_+(s)\overline{v}_+(s) = -\frac{M_- + m}{2m}. \tag{4.16}$$

We now want to calculate several inner products that will be of use in the next section. We start by calculating the anticommutator of a gamma matrix and $M_\pm$ using the Clifford algebra's anticommutation relation:

$$\{\gamma^\mu, M_\pm\} = -\{\gamma^\mu, \gamma^\nu\}\Pi^\pm_\nu = -2\eta^{\mu\nu}\Pi^\pm_\nu. \tag{4.17}$$

Using (4.17), the eigenvalue equation from (4.7) and the normalisation of the eigenvectors (4.11), the Lorentz vector $\overline{u}_+\gamma^\mu u_+$ can be calculated

as follows:

$$\begin{aligned}\overline{u}_+\gamma^\mu u_+ &= \frac{1}{2}\left(\overline{u}_+\gamma^\mu u_+ + \overline{u}_+\gamma^\mu u_+\right)\\ &= \frac{1}{2m}\left(\overline{u}_+\gamma^\mu M_+ u_+ + \overline{u}_+ M_+\gamma^\mu u_+\right)\\ &= \frac{1}{2m}\overline{u}_+\left\{\gamma^\mu, M_+\right\}u_+\\ &= -\frac{1}{m}\eta^{\mu\nu}\Pi^+_\nu.\end{aligned} \tag{4.18}$$

In complete analogy, we find

$$\overline{v}_+\gamma^\mu v_+ = \frac{1}{m}\eta^{\mu\nu}\Pi^-_\nu. \tag{4.19}$$

Similarly, another identity, which will be used in the next section, can be derived[34]:

$$\begin{aligned}\overline{u}_+\gamma^\mu\partial_\mu u_+ &= \frac{1}{2}\left[\overline{u}_+\gamma^\mu\partial_\mu\frac{1}{m}M_+u_+ + \frac{1}{m}\overline{u}_+M_+\gamma^\mu\partial_\mu u_+\right]\\ &= \frac{1}{2}\left[-\frac{1}{m^2}(\partial_\mu m)\overline{u}_+\gamma^\mu M_+u_+ + \frac{1}{m}\overline{u}_+\gamma^\mu\left(\partial_\mu M_+\right)u_+ + \frac{1}{m}\overline{u}_+\left\{\gamma^\mu, M_+\right\}\partial_\mu u_+\right]\\ &= \frac{1}{2m}\left[\frac{1}{m}\eta^{\mu\nu}\Pi^+_\nu\partial_\mu m - \left(\partial_\mu\Pi^+_\nu\right)\overline{u}_+\gamma^\mu\gamma^\nu u_+ - 2\eta^{\mu\nu}\Pi^+_\nu\,\overline{u}_+\partial_\mu u_+\right].\end{aligned} \tag{4.20}$$

[34] Remember that we are keeping the derivative of m here and in the next section because we want to investigate the case of a spacetime-dependent mass later on (see chapter 5).

By using the derivative of the eikonal equation,

$$m\partial_\mu m = \eta^{\nu\lambda}\Pi^\pm_\nu\partial_\mu\Pi^\pm_\lambda, \tag{4.21}$$

we can replace $\partial_\mu m$ in (4.20) and get

$$\begin{aligned}&\overline{u}_+\gamma^\mu\partial_\mu u_+\\ &= \frac{1}{2m}\left[\frac{1}{m^2}\left(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\rho\lambda}\overline{u}_+\gamma^\mu\gamma^\nu u_+\right)\Pi^+_\rho\Pi^+_\lambda\partial_\mu\Pi^+_\nu - 2\eta^{\mu\nu}\Pi^+_\nu\,\overline{u}_+\partial_\mu u_+\right].\end{aligned} \tag{4.22}$$

Again, we can find a similar expression for v_+, namely

$$\begin{aligned}&\overline{v}_+\gamma^\mu\partial_\mu v_+\\ &= -\frac{1}{2m}\left[\frac{1}{m^2}\left(\eta^{\mu\rho}\eta^{\nu\lambda} + \eta^{\rho\lambda}\overline{v}_+\gamma^\mu\gamma^\nu v_+\right)\Pi^-_\rho\Pi^-_\lambda\partial_\mu\Pi^-_\nu + 2\eta^{\mu\nu}\Pi^-_\nu\,\overline{v}_+\partial_\mu v_+\right].\end{aligned} \tag{4.23}$$

Depending on the choice of coordinates and gamma matrices, these identities will simplify a lot as we will see in the next section.

Finally, we expand the spinor ψ in terms of the solutions of the eikonal equation[35],

[35] Depending on the dimension of the spacetime there actually may be multiple eigenvectors to the same eigenvalue, so the expansion should in fact contain a term $\alpha_s u^{(s)}_+ e^{iS_+}$ and $\beta_s v^{(s)}_+ e^{iS_-}$ for every eigenvector $u^{(s)}_+$ and $v^{(s)}_+$ respectively.

$$\psi = \alpha u_+ e^{iS_+} + \beta v_+ e^{iS_-}, \tag{4.24}$$

so that the Dirac equation reduces to

$$\gamma^\mu \partial_\mu (\alpha u_+) e^{iS_+} + \gamma^\mu \partial_\mu (\beta v_+) e^{iS_-} = 0. \tag{4.25}$$

In terms of the small-$\hbar$ expansion mentioned in the introduction of this chapter, this equation contains the higher-order terms (compare eq. (2) in [Di 14b]).

By multiplying (4.25) by suitable spinors partial differential equations for α and β can be derived. We will do this in 1+1 dimensions in the following section.

4.2 1+1 dimensions

In 1+1 dimensions it suffices to use 2×2 matrices for the gamma matrices; see section 2.3. Therefore, the operators $M_\pm$ each only have two eigenvectors – one with eigenvalue $+m$ and one with $-m$. If we multiply (4.25) once by $\overline{u}_+$ and once by $\overline{v}_+$, we get two coupled partial differential equations for the functions α and β, namely

$$\begin{aligned} (\partial_\mu \alpha)\, \overline{u}_+ \gamma^\mu u_+ + \alpha\, \overline{u}_+ \gamma^\mu \partial_\mu u_+ &= -\overline{u}_+ \gamma^\mu \partial_\mu (\beta v_+)\; e^{-i(S_+ - S_-)}, \\ (\partial_\mu \beta)\, \overline{v}_+ \gamma^\mu v_+ + \beta\, \overline{v}_+ \gamma^\mu \partial_\mu v_+ &= -\overline{v}_+ \gamma^\mu \partial_\mu (\alpha u_+)\; e^{i(S_+ - S_-)}. \end{aligned} \tag{4.26}$$

We have already calculated the coefficients on the left-hand side in a representation-independent manner in the last section. To simplify the expressions for these and compute the remaining coefficients on the right-hand side we need to choose a concrete representation for the gamma matrices,

$$\gamma^0 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.27}$$

Using these, the operators $M_\pm$ have the following form:

$$M_\pm = \begin{pmatrix} -\Pi_t^\pm & -i\Pi_x^\pm \\ -i\Pi_x^\pm & +\Pi_t^\pm \end{pmatrix}. \tag{4.28}$$

The eigenvectors $u_\pm$ and $v_\pm$ can then be written as[36]

[36] The relationship between u_+ and u_- (and analogously for v_+ and v_-) can also be expressed using the charge conjugation operator C, $u_- = \gamma^0 C u_+^*$ (see also [Pal11]). In this representation, $C = \gamma^2$ and additionally $u_+^* = \gamma^0 u_+$.

$$
\begin{aligned}
u_+ &= N_+ \begin{pmatrix} m - \Pi_t^+ \\ -i\Pi_x^+ \end{pmatrix}, \quad u_- = N_+ \begin{pmatrix} i\Pi_x^+ \\ m - \Pi_t^+ \end{pmatrix} = i\gamma^0\gamma^1 u_+, \\
v_+ &= N_- \begin{pmatrix} -i\Pi_x^- \\ m + \Pi_t^- \end{pmatrix}, \quad v_- = N_- \begin{pmatrix} m + \Pi_t^- \\ i\Pi_x^- \end{pmatrix} = i\gamma^0\gamma^1 v_+,
\end{aligned}
\tag{4.29}
$$

with the normalisation constants[37]

$$
N_\pm = \frac{1}{\sqrt{2m(m \mp \Pi_t^\pm)}}. \tag{4.30}
$$

[37] Note that with the sign convention from (4.4), $m \mp \Pi_t^\pm$ is always greater than or equal to $2m$. Therefore the normalisation constants $N_\pm$ are real.

In this representation, derivatives of u_+ and v_+ are given by

$$
\begin{aligned}
\partial_\mu u_+ &= \frac{i}{2m^2}(\Pi_t^+ \partial_\mu \Pi_x^+ - \Pi_x^+ \partial_\mu \Pi_t^+) u_- = \frac{i}{2m^2} \varepsilon^{\nu\lambda} \Pi_\nu^+ (\partial_\mu \Pi_\lambda^+) u_-, \\
\partial_\mu v_+ &= -\frac{i}{2m^2}(\Pi_t^- \partial_\mu \Pi_x^- - \Pi_x^- \partial_\mu \Pi_t^-) v_- = -\frac{i}{2m^2} \varepsilon^{\nu\lambda} \Pi_\nu^- (\partial_\mu \Pi_\lambda^-) v_-
\end{aligned}
\tag{4.31}
$$

where we have used the derivative of the eikonal equation (4.21). Thus, we have

$$
\overline{u}_+ \partial_\mu u_+ = \overline{v}_+ \partial_\mu v_+ = 0. \tag{4.32}
$$

Furthermore, we can check that

$$
\overline{u}_+ \gamma^\mu \gamma^\nu u_+ = -\overline{v}_+ \gamma^\mu \gamma^\nu v_+ = \eta^{\mu\nu} \tag{4.33}
$$

in this representation. Therefore, the expressions from (4.22) and (4.23) simplify to

$$
\begin{aligned}
\overline{u}_+ \gamma^\mu \partial_\mu u_+ &= \frac{1}{2m^3}\left(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\rho\lambda}\right) \Pi_\rho^+ \Pi_\lambda^+ \partial_\mu \Pi_\nu^+, \\
\overline{v}_+ \gamma^\mu \partial_\mu v_+ &= -\frac{1}{2m^3}\left(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\rho\lambda}\right) \Pi_\rho^- \Pi_\lambda^- \partial_\mu \Pi_\nu^-.
\end{aligned}
\tag{4.34}
$$

We now turn to the coefficients on the right-hand side of (4.26). First, by explicitly using the expressions for the eigenvectors we may show that[38]

$$
\overline{u}_+ \gamma^\mu v_+ = i\kappa^\mu, \quad \overline{v}_+ \gamma^\mu u_+ = -i\kappa^\mu, \tag{4.35}
$$

[38] As $\overline{u}_+ \gamma^\mu v_+ = (\overline{v}_+ \gamma^\mu u_+)^*$, it is sufficient to check one of these.

where we defined κ^μ as an abbreviation according to

$$
\kappa^\mu = N_+ N_- \begin{pmatrix} \Pi_x^+(m + \Pi_t^-) - \Pi_x^-(m - \Pi_t^+) \\ (m - \Pi_t^+)(m + \Pi_t^-) - \Pi_x^+ \Pi_x^- \end{pmatrix}. \tag{4.36}
$$

We observe that

$$
\gamma^\mu \gamma^0 \gamma^1 = \varepsilon_\nu{}^\mu \gamma^\nu \tag{4.37}
$$

and thus find

$$\begin{aligned}\overline{u}_+\gamma^\mu v_- &= i\overline{u}_+\gamma^\mu\gamma^0\gamma^1 v_+ = i\varepsilon_\rho{}^\mu\,\overline{u}_+\gamma^\rho v_+ = -\varepsilon_\rho{}^\mu\kappa^\rho,\\ \overline{v}_+\gamma^\mu u_- &= i\overline{v}_+\gamma^\mu\gamma^0\gamma^1 u_+ = i\varepsilon_\rho{}^\mu\,\overline{v}_+\gamma^\rho u_+ = \varepsilon_\rho{}^\mu\kappa^\rho.\end{aligned} \tag{4.38}$$

Using these expressions and using the derivatives from (4.31) we get

$$\begin{aligned}\overline{u}_+\gamma^\mu\partial_\mu v_+ &= \frac{i}{2m^2}\varepsilon^{\lambda\nu}\varepsilon_\rho{}^\mu\kappa^\rho\Pi^-_\lambda\partial_\mu\Pi^-_\nu,\\ \overline{v}_+\gamma^\mu\partial_\mu u_+ &= \frac{i}{2m^2}\varepsilon^{\lambda\nu}\varepsilon_\rho{}^\mu\kappa^\rho\Pi^+_\lambda\partial_\mu\Pi^+_\nu.\end{aligned} \tag{4.39}$$

Inserting the explicit expressions for the inner products from (4.18), (4.19), (4.34) and (4.39) appearing in (4.26) that we just derived, we get the following two coupled partial differential equations for α and β

$$\begin{aligned}&\eta^{\mu\nu}\Pi^+_\mu\partial_\nu\alpha - \frac{1}{2m^2}\left(\eta^{\mu\rho}\eta^{\nu\lambda}-\eta^{\mu\nu}\eta^{\rho\lambda}\right)\Pi^+_\rho\Pi^+_\lambda(\partial_\mu\Pi^+_\nu)\alpha\\ &= ime^{-i(S_+-S_-)}\left[\kappa^\mu\partial_\mu\beta + \frac{1}{2m^2}\varepsilon^{\lambda\nu}\varepsilon_\rho{}^\mu\kappa^\rho\Pi^-_\lambda(\partial_\mu\Pi^-_\nu)\beta\right],\\ &\eta^{\mu\nu}\Pi^-_\mu\partial_\nu\beta - \frac{1}{2m^2}\left(\eta^{\mu\rho}\eta^{\nu\lambda}-\eta^{\mu\nu}\eta^{\rho\lambda}\right)\Pi^-_\rho\Pi^-_\lambda(\partial_\mu\Pi^-_\nu)\beta\\ &= ime^{i(S_+-S_-)}\left[\kappa^\mu\partial_\mu\alpha - \frac{1}{2m^2}\varepsilon^{\lambda\nu}\varepsilon_\rho{}^\mu\kappa^\rho\Pi^+_\lambda(\partial_\mu\Pi^+_\nu)\alpha\right].\end{aligned} \tag{4.40}$$

These equations are fully equivalent to the covariant Dirac equation we started with. However, they might offer computational advantages for numerical simulations or analytical approximations. For example, in the time-dependent case it is known that the quantum kinetic approach or methods based on the Riccati equation lend themselves better to numerical calculations than the Dirac equation itself. Especially when the functions $S_\pm$ are known analytically (or can be implemented efficiently in a numerical computation) similar advantages may be available here.

Still, the above equations are arduous if not impossible to solve exactly for fields other than the most simple ones. Thus, we want to consider approximations to the equations (4.40) that work well for fields where the pair production probability is small. One approximation we might use is to assume that for the pair creation ratio being small, the left-hand side of the first equation in (4.40) is much larger than the right-hand side (because β is small) so that we can neglect the right-hand side completely. We thus determine an approximation for α by solving

$$\eta^{\mu\nu}\Pi^+_\mu\partial_\nu\alpha - \frac{1}{2m^2}\left(\eta^{\mu\rho}\eta^{\nu\lambda}-\eta^{\mu\nu}\eta^{\rho\lambda}\right)\Pi^+_\rho\Pi^+_\lambda(\partial_\mu\Pi^+_\nu)\alpha \approx 0. \tag{4.41}$$

The solution can then be inserted into the second equation in (4.40) to get an approximation for β. As we will show in the following section, in a purely time-dependent problem this approximation is equivalent to the linearisation of the Riccati equation.

In the next two sections we want to show that the equations (4.40) reproduce the correct ordinary differential equations for purely time-dependent or purely space-dependent problems. This could be extended even further: For problems that depend only on a single coordinate (time, spatial coordinate, light-cone coordinate, et cetera) it should be possible to reduce the partial differential equations (4.40) to a set of ordinary differential equations.

4.3 Purely time-dependent potentials

Assume that we have a time-dependent mass $m = m(t)$ and a time-dependent electric field in temporal gauge

$$A_0 = 0, \quad A_1 = A(t), \quad E = \dot{A}(t). \tag{4.42}$$

Then, two independent solutions to the eikonal equation (4.3) are given by

$$S_\pm = \mp\varphi_p(t) + px, \tag{4.43}$$

where

$$\varphi_p(t) = \int_{-\infty}^{t} \Omega_p(t')\mathrm{d}t', \quad \Omega_p(t) = \sqrt{m^2(t) + \big[p + qA(t)\big]^2}. \tag{4.44}$$

Thus, we have

$$\Pi_t^\pm = \mp\Omega_p(t), \quad \Pi_x^\pm = p + qA(t). \tag{4.45}$$

Using the expressions given in (4.30) and (4.36), we can calculate

$$N_\pm = \frac{1}{\sqrt{2m(t)[m(t) + \Omega_p(t)]}}, \quad \kappa^\mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.46}$$

Therefore, in this case the equations (4.40) for α and β read

$$\begin{aligned}
\Omega_p\partial_t\alpha + \big(p+qA\big)\partial_x\alpha + \frac{1}{2}\dot{\Omega}_p\alpha &= -ime^{2i\varphi_p}\left[\partial_x\beta + \frac{1}{2}\left(\frac{qE}{\Omega_p} - \frac{p+qA}{\Omega_p}\frac{\dot{m}}{m}\right)\beta\right], \\
\Omega_p\partial_t\beta - \big(p+qA\big)\partial_x\beta + \frac{1}{2}\dot{\Omega}_p\beta &= ime^{-2i\varphi_p}\left[\partial_x\alpha + \frac{1}{2}\left(\frac{qE}{\Omega_p} - \frac{p+qA}{\Omega_p}\frac{\dot{m}}{m}\right)\alpha\right].
\end{aligned} \tag{4.47}$$

We are looking for a solution with only positive energy initially, so we choose $\alpha(t \to -\infty) = \text{const.}$ and $\beta(t \to -\infty) = 0$ as our boundary condition. Because these and all coefficients in (4.47) are independent of x, $\partial_x\alpha$ and $\partial_x\beta$ vanish for all times. We introduce the ratio $\mathscr{R}(t) = \beta(t)/\alpha(t)$ and calculate its time derivative

$$\partial_t\mathscr{R} = i\frac{mqE - \big(p + qA\big)\dot{m}}{2\Omega_p^2}\Big[e^{-2i\varphi_p} + \mathscr{R}^2 e^{2i\varphi_p}\Big], \tag{4.48}$$

which is a Riccati equation equivalent to the one in ordinary time-dependent WKB [compare (3.19)][39]. From here, we could proceed exactly the same as in chapter 3 and would obtain the same results as there. We now want to show that we directly obtain the linearised Riccati equation if we use the approximation described at the end of section 4.2; see (4.41). In this case here, the approximated equation for α (4.41) reads

$$\Omega_p\partial_t\alpha + \big[p + qA\big]\partial_x\alpha \approx -\frac{1}{2}\dot{\Omega}_p\alpha. \tag{4.49}$$

As before, we impose the boundary condition that α is a constant for $t \to -\infty$. Hence, α is independent of x for all times, i.e. $\partial_x\alpha = 0$, and (4.49) reduces to a first-order ordinary differential equation whose general solution is given by

$$\alpha(t) \approx \frac{\tilde{\alpha}}{\sqrt{\Omega_p(t)}}, \quad \tilde{\alpha} = \text{const.} \tag{4.50}$$

Finally, upon rescaling β according to $\beta(t) = \tilde{\beta}(t)/\sqrt{\Omega_p(t)}$, the approximated equation for β in this case simplifies to

$$\partial_t\sqrt{\Omega_p}\,\beta = \partial_t\tilde{\beta} \approx i\tilde{\alpha}\frac{mqE - \big(p + qA\big)\dot{m}}{2\Omega_p^2}e^{-2i\varphi_p}. \tag{4.51}$$

Because $\tilde{\alpha}$ is a constant, the time derivative of the ratio $\mathscr{R} = \beta/\alpha \approx \tilde{\beta}/\tilde{\alpha}$ is

$$\partial_t\mathscr{R} \approx \frac{\partial_t\tilde{\beta}}{\tilde{\alpha}} \approx i\frac{mqE - \big(p + qA\big)\dot{m}}{2\Omega_p^2}e^{-2i\varphi_p}. \tag{4.52}$$

Thus, we get exactly the same result as if we had linearised the Riccati equation (4.48).

[39] The different sign within the brackets in comparison with (3.19) is because here the equations (4.40) for α and β were derived by multiplying with the Dirac adjoints $\overline{u}_+$ and $\overline{v}_+$ while in section 3.1 we used the conjugate transpose $u_\pm^\dagger$.

4.4 Purely space-dependent potentials

In complete analogy to the last section, we can derive an expression for the ratio $\mathscr{R} = \beta/\alpha$ in the presence of an electric field that only depends on x (compare [Nik70; KP02; KRX08]). We work in the gauge

$$A_0 = \phi(x), \quad A_1 = 0, \quad E = -\phi'(x). \tag{4.53}$$

The physical situation is now quite different from the one in the time-dependent case. This is essentially just scattering at a non-constant barrier as in ordinary quantum mechanics but with taking relativistic effects into account[40]. In the last section, the problem was independent of x and thus its conjugate momentum p was conserved. Here, the problem is independent of t and hence its conjugate momentum (the null-component of the momentum vector, i.e. the energy or frequency) ω is conserved. This is reflected in the form of the solutions to the eikonal equation

$$S_\pm = -\omega t \pm \varphi_\omega(x), \tag{4.54}$$

where

$$\varphi_\omega(x) = \int_{-\infty}^{x} P_\omega(x')\mathrm{d}x', \quad P_\omega(x) = \sqrt{\left[\omega - q\phi(x)\right]^2 - m^2}. \tag{4.55}$$

Thus we have

$$\Pi_t^\pm = -\omega + q\phi(x), \quad \Pi_x^\pm = \pm P_\omega(x). \tag{4.56}$$

We again calculate $N_\pm$ and κ^μ using (4.30) and (4.36)[41]:

$$N_\pm = \frac{1}{\sqrt{2m\{m \pm [\omega - q\phi(x)]\}}}, \quad \kappa^\mu = \begin{pmatrix} -i \\ 0 \end{pmatrix}. \tag{4.57}$$

Here, the equations for α and β simplify to

$$\begin{aligned} \big(q\phi - \omega\big)\partial_t\alpha - P_\omega\partial_x\alpha - \frac{1}{2}P'_\omega\alpha &= me^{-2i\varphi_\omega}\left(\partial_t\beta - \frac{1}{2}\frac{qE}{P_\omega}\beta\right), \\ \big(q\phi - \omega\big)\partial_t\beta + P_\omega\partial_x\beta + \frac{1}{2}P'_\omega\beta &= me^{2i\varphi_\omega}\left(\partial_t\alpha - \frac{1}{2}\frac{qE}{P_\omega}\alpha\right). \end{aligned} \tag{4.58}$$

We are looking for a solution that corresponds to a wave incoming from the left, i.e. we want $\alpha(x \longrightarrow -\infty) = \text{const.}$ and $\beta(x \longrightarrow -\infty) = 0$. Because all coefficients in (4.58) are independent of t, $\partial_t\alpha = \partial_t\beta = 0$ for all x.

[40] This is actually the kind of problem Sauter (and previously Klein) was considering in his original articles that started the investigation of pair creation due to strong electromagnetic fields [Sau31; Sau32; Kle29].

[41] In this case we do not have $m \mp \Pi_t^\pm > 0$, depending on the value of ω and $\phi(x)$. This is the reason why κ^μ is purely imaginary here.

Thus, when calculating the derivative of $\mathscr{R} = \beta/\alpha$ we get

$$\partial_x \mathscr{R} = \frac{\partial_x \beta}{\alpha} - \mathscr{R}^2 \frac{\partial_x \alpha}{\beta} = -\frac{mqE}{2P_\omega^2}\left[e^{2i\varphi_\omega} + \mathscr{R}^2 e^{-2i\varphi_\omega}\right], \tag{4.59}$$

Again, if we use the approximated equation (4.41) for α we recover the linearised Riccati equation: With the appropriate boundary conditions, the equation for α (4.41) in this case reduces to an ordinary differential equation in x,

$$\partial_x \alpha \approx -\frac{1}{2}\alpha \partial_x \ln P_\omega, \tag{4.60}$$

which has the solution

$$\alpha(x) \approx \frac{\tilde{\alpha}}{\sqrt{P_\omega(x)}}, \quad \tilde{\alpha} = \text{const.} \tag{4.61}$$

The equation for $\tilde{\beta}(x) = \beta(x)/\sqrt{P_\omega(x)}$ then reads

$$\partial_x \sqrt{P_\omega}\, \beta = \partial_x \tilde{\beta} \approx \tilde{\alpha} \frac{mqE}{2P_\omega^2} e^{2i\varphi_\omega}. \tag{4.62}$$

Thus,

$$\partial_x \mathscr{R} \approx \frac{\partial_x \tilde{\beta}}{\tilde{\alpha}} \approx \frac{mqE}{2P_\omega^2} e^{2i\varphi_\omega}, \tag{4.63}$$

which is the linearisation of the Riccati equation given in (4.59).

In complete analogy with the time-dependent case, we expect $\mathscr{R}_{\text{out}} = R(x \longrightarrow \infty)$ to be dominated by the value of the exponential at the classical turning points x_* where $P_\omega^2(x_*) = 0$. It should be noted however that these turning points can be real for sub-barrier tunnelling whereas they are imaginary in the time-dependent case. Nevertheless, for over-the-barrier scattering, x_* is also complex.

4.5 Pair production probability

Assuming that we can calculate α and β from (4.40) we now want to determine the number of created electron-positron pairs. As in section 3.2, we need to calculate the Bogoliubov coefficients $B^{\kappa\lambda}_{pp'}$, especially $B^{+-}_{pp'}$ for the number of produced positrons

$$N_{e^+}(p) = \int \mathrm{d}p' \left|B^{+-}_{p'p}\right|^2 = \int \mathrm{d}p' \frac{1}{(2\pi)^2} \frac{m_{\text{in}} m_{\text{out}}}{\epsilon^{\text{in}}_{p'} \epsilon^{\text{out}}_{p}} \left|\left(\psi^+_{\text{in}}(p'), \psi^-_{\text{out}}(p)\right)\right|^2. \tag{4.64}$$

First, we see that upon choosing

$$\alpha(t \longrightarrow -\infty, x) = 1, \quad \beta(t \longrightarrow -\infty, x) = 0 \tag{4.65}$$

as the initial conditions for (4.40) and $S_\pm(t \longrightarrow -\infty, x) = px$ as the initial condition for the eikonal equation, the spinor $\psi = \alpha u_+ e^{iS_+} + \beta v_+ e^{iS_-}$ is a solution that has positive energy initially, i.e.

$$\psi^+_{\text{in}}(p;t,x) = \alpha(p;t,x)u_+(p;t,x)e^{iS_+(p;t,x)} + \beta(p;t,x)v_+(p;t,x)e^{iS_-(p;t,x)}. \tag{4.66}$$

The main difference to (3.28) is that the functions α and β as well as the spinors u_+ and v_+ now depend on x, too. This already suggests that the spatial integration in the inner product will not be as straightforward to carry out as in the time-dependent case. On the other hand, the solutions which are negative-energy plane waves finally have the asymptotic form

$$\psi^-_{\text{out}}(p;t,x) \overset{t\longrightarrow\infty}{\longrightarrow} \tilde{v}^{\text{out}}_+(p)e^{ipx} \tag{4.67}$$

where $\tilde{v}^{\text{out}}_+$ is a spinor v^{out}_+ where solutions $\tilde{S}_\pm$ with outgoing boundary conditions $S_\pm(t \longrightarrow \infty, x) = px$ were used.

As we are free to evaluate the inner product in (4.64) at any instant of time (see section 2.4), we will evaluate it at $t \longrightarrow \infty$ where we assume any external field to be switched off. At $t \longrightarrow \infty$ we find

$$\begin{aligned}\left(\psi^+_{\text{in}}(p'), \psi^-_{\text{out}}(p)\right) = \int \mathrm{d}x \Big\{ &\alpha^*_{\text{out}}(p';x)\left(u^{\text{out}}_+\right)^\dagger(p';x)\tilde{v}^{\text{out}}_+(p)e^{-i[S^{\text{out}}_+(p';x)-px]} \\ &+ \beta^*_{\text{out}}(p';x)\left(v^{\text{out}}_+\right)^\dagger(p';x)\tilde{v}^{\text{out}}_+(p)e^{-i[S^{\text{out}}_-(p';x)-px]}\Big\}.\end{aligned} \tag{4.68}$$

For time-dependent problems, the situation is exactly the same as in section 3.2: The canonical momentum p is conserved and therefore $S^{\text{out}}_\pm(p;x) = px \mp \vartheta(p)$ where $\vartheta(p)$ is an only momentum-dependent phase. The spatial integral then just gives a delta distribution and we get

$$N_{e^+}(p) = \frac{m_{\text{in}}}{m_{\text{out}}}\frac{\epsilon^{\text{out}}_p}{\epsilon^{\text{in}}_p}\left|\beta_{\text{out}}(p)\right|^2 \delta(0). \tag{4.69}$$

However, when the problem is truly spacetime-dependent, we have to take the spatial dependence of α_{out} and β_{out} into account. It should be stressed that up to this point, all expressions are still exact. That means if one would be able solve the equations (4.40) for α and β and calculate the inner product, one would obtain the exact number of produced pairs.

However, this is rather complicated in general. Motivated by the results in the time-dependent case, one could assume that for only weakly space-dependent background fields the number of produced pairs can be approximated as

$$N_{e^+}(p) \approx \frac{m_{\text{in}}}{m_{\text{out}}} \frac{\epsilon_p^{\text{out}}}{\epsilon_p^{\text{in}}} \int \mathrm{d}x \; |\beta_{\text{out}}(p;x)|^2 \,. \tag{4.70}$$

This estimate may be obtained from the expression in the time-dependent case (4.69) by replacing $\delta(0)$ with a spatial integral. It should be a good approximation if scattering to other modes than the original one is low, for example when the problem is only weakly space-dependent (see section 5.3).

Chapter 5

Spacetime-dependent mass

In this chapter we want to apply the method developed in the preceding chapter to problems with truly spacetime-dependent fields. We will investigate cases with a spacetime-dependent mass only instead of ones with spacetime-dependent electromagnetic potentials. There are several reasons for proceeding like this: First, it simplifies the matter because we need to be only concerned with a scalar field instead of a vector field. Second, in 1+1 dimensions a spacetime-dependent mass is equivalent to a conformal scale factor in the spacetime's metric. Thus, pair production due to a spacetime-dependent mass in 1+1 dimensions is actually cosmological particle creation because of a curved spacetime metric (see appendix A and B). Last but not least, the motion of a charged particle in a laser field may be described using an effective spacetime-dependent mass (see e.g. [Kib66; DF08; KGA14]).

We start by discussing difficulties in solving the eikonal equation (4.3) that occur when truly spacetime-dependent fields are present because of the nonlinear character of the eikonal equation. We then apply the spacetime WKB method to the case of a spacetime-dependent mass and introduce new coordinates that follow directly from solutions of the eikonal equation. Finally, we consider a spacetime-dependent mass where the spatial dependence is much weaker than the temporal dependence.

5.1 Caustics

Finding solutions to the eikonal equation (4.3) is central for the spacetime WKB method. However, for nonlinear partial differential equations like the eikonal equation it may not be possible to find global solutions in a classical sense. Classical solutions in this context are those for which all derivatives that occur in the partial differential equation are defined at every point. For first-order partial differential equations like the eikonal equation non-classical solutions can be understood and

identified intuitively using the *method of characteristics*. This is a well-known method for solving first-order partial differential equations by which the partial differential equation is transformed into an equivalent system of ordinary differential equations. The basic idea is to find certain *projected characteristic curves* in the t-x plane (in our case) such that the solution of the partial differential equation can be integrated easily along those curves. Combining these curves into a surface, where the height is given by the sought-after function, gives the solution of the original partial differential equation.

Closely following [Eva98, pp. 97 sq.], the method of characteristics for the general nonlinear case works as follows: For a first-order partial differential equation[42]

$$F(Du, u, x) = 0, \quad x \in U,\ u : U \longrightarrow \mathbb{R},\ U \subset \mathbb{R}^n \tag{5.1}$$

[42] Du denotes the gradient, i.e. $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)^\mathsf{T}$.

subject to the boundary condition[43]

$$u(x_1, \dots, x_{n-1}, x_n^{(0)}) = g(x_1, \dots, x_{n-1}), \tag{5.2}$$

[43] Without loss of generality, we impose the boundary value on a hypersurface at $x_n = x_n^{(0)} = \text{const.}$. Equivalently, we could have fixed any other coordinate.

we assume that we have a parametrised curve $x(\tau) = (x_1(\tau), \dots, x_n(\tau))$ and define

$$z(\tau) \coloneqq u(x(\tau)), \quad p(\tau) \coloneqq Du(x(\tau)). \tag{5.3}$$

Then, the system of *characteristic* ordinary differential equations equivalent to (5.1) is given by

$$\begin{aligned} \dot{x}_j &= \frac{\mathrm{d}\, x_j}{\mathrm{d}\, \tau} = \partial_{p_j} F, \\ \dot{p}_j &= \frac{\mathrm{d}\, p_j}{\mathrm{d}\, \tau} = -\partial_{x_j} F - (\partial_z F) p_j, \\ \dot{z} &= \frac{\mathrm{d}\, z}{\mathrm{d}\, \tau} = p_j \partial_{p_j} F, \end{aligned} \tag{5.4}$$

where $j = 1, \dots, n$ and summation over j is implied in the last equation. The boundary condition (5.2) translates into the initial conditions

$$\begin{aligned} x(0) &= (x_1^{(0)}, \dots, x_{n-1}^{(0)}, x_n^{(0)}), \\ p_j(0) &= \partial_{x_j} g(x_1^{(0)}, \dots, x_{n-1}^{(0)}),\ j = 1, \dots, n-1 \\ z(0) &= g(x_1^{(0)}, \dots, x_{n-1}^{(0)}), \\ F(p(0), z(0), x(0)) &= 0, \end{aligned} \tag{5.5}$$

where the $x_j^{(0)}$, $j = 1, \dots, n-1$, are constants that fix the starting point

of each characteristic curve. Note that the derivatives of the function at the boundary determine all but the last of the $p_j(0)$. The remaining $p_n(0)$ is fixed by the last equation in (5.5) which requires the initial values to satisfy the original partial differential equation. By varying the constants $x_j^{(0)}$, $j = 1, \dots, n-1$ we obtain a set of characteristic curves that can be used to construct the solution surface $(x, u(x))$. This essentially amounts to a change of coordinates in the original partial differential equation: Instead of $(x_1, \dots, x_n)$ we now use $(\tau, x_1^{(0)}, \dots x_{n-1}^{(0)})$. However, this does not work for arbitrary boundary conditions. The characteristic ordinary differential equations (5.4) together with the initial conditions (5.5) can only be used to construct a solution to the original partial differential equation if the *noncharacteristic boundary condition*

$$\partial_{p_n} F(p(0), z(0), x(0)) \neq 0 \tag{5.6}$$

holds. This conditions is not satisfied if the boundary lies on a projected characteristic curve. If that would be the case, we could not construct the solution surface $(x, u(x))$ by combining different characteristic curves as all curves that started from the boundary would stay on the boundary. Thus, no additional information about the solution elsewhere could be gained by integrating the solution along the characteristic curves (see fig. 5.1).

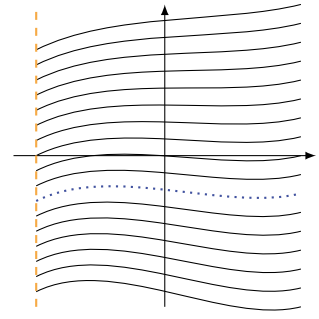

Figure 5.1 – Let the solid black curves be the projected characteristic curves for some first-order partial differential equation. The dashed orange curve is a boundary that satisfies the noncharacteristic boundary condition (5.6) whereas the dotted blue curve does not. This is the case because the latter lies on a projected characteristic curve: If we start on a point on the boundary and follow the characteristic emanating from there we will always stay on the boundary and thus gain no additional information about the solution elsewhere. On the other hand, if we start from a point on the orange dashed curve we obtain many different characteristic curves and can construct the solution surface by combining them.

We now apply the method of characteristics to the eikonal equation (4.3) with a spacetime-dependent mass and no electromagnetic field ($A_\mu = 0$) in 1+1 dimensions,

$$\eta^{\mu\nu}\big(\partial_\mu S_\pm\big)\big(\partial_\nu S_\pm\big) = m^2(t, x). \tag{5.7}$$

We write this in the form given in (5.1)

$$F(\partial_t S, \partial_x S, S, t, x) = \frac{1}{m_0}\big[(\partial_t S)^2 - (\partial_x S)^2 - m^2(t, x)\big]. \tag{5.8}$$

We have included the constant factor $1/m_0$ so that the dimension of t and the new coordinate τ is the same. The constant m_0 is the value of the mass initially, i.e.

$$\lim_{t \longrightarrow -\infty} m(t, x) = m_0 = \text{const.} \tag{5.9}$$

If the mass is constant, the solutions of the eikonal equation correspond to plane waves. Therefore, as a boundary condition we require the solu-

tion S to satisfy

$$\lim_{t\longrightarrow-\infty} S(t,x) = px = g(x). \tag{5.10}$$

Furthermore, we use

$$p_0(\tau) = \partial_t S, \quad p_1(\tau) = \partial_x S \tag{5.11}$$

to denote the values of the partial derivatives and $z(\tau) = S(t(\tau), x(\tau))$ as the value of the eikonal on a projected characteristic curve $(t(\tau), x(\tau))$.. Thus, the characteristic ordinary differential equations (5.4) in this case are given by

$$\begin{aligned}
\dot{t} &= \partial_{p_0} F &&= \frac{2}{m_0} p_0,\\
\dot{x} &= \partial_{p_1} F &&= -\frac{2}{m_0} p_1,\\
\dot{p}_0 &= -\partial_t F - (\partial_z F) p_0 &&= \frac{1}{m_0} \partial_t m^2,\\
\dot{p}_1 &= -\partial_x F - (\partial_z F) p_1 &&= \frac{1}{m_0} \partial_x m^2,\\
\dot{z} &= p_0 \partial_{p_0} F + p_1 \partial_{p_1} F &&= \frac{2}{m_0}\left(p_0^2 - p_1^2\right).
\end{aligned} \tag{5.12}$$

Using (5.5), we find initial conditions

$$\begin{aligned}
t(\tau_0) &= -\infty,\\
x(\tau_0) &= x_0,\\
p_0(\tau_0) &= \pm\sqrt{m_0^2 + p^2} = \pm\epsilon_p,\\
p_1(\tau_0) &= \partial_x g(x_0) = p,\\
z(\tau_0) &= p x_0.
\end{aligned} \tag{5.13}$$

Here, $p_0(\tau_0)$ was obtained from the last equation in (5.5). We see that two qualitatively different solutions are possible: a solution may have either $\partial_t S > 0$ or $\partial_t S < 0$ initially. These are the two independent solutions that we were referring to in section 4.1; S_+ corresponds to the former while S_- corresponds to the latter.

We can verify that the noncharacteristic boundary condition (5.6) holds:

$$\partial_{p_0} F(p_0(\tau_0), p_1(\tau_0), z(\tau_0), t(\tau_0), x(\tau_0)) = \frac{2}{m_0} p_0(\tau_0) = \pm\frac{2\epsilon_p}{m_0} \neq 0. \tag{5.14}$$

The functions $p_0(\tau)$ and $p_1(\tau)$ can actually be eliminated from (5.12)

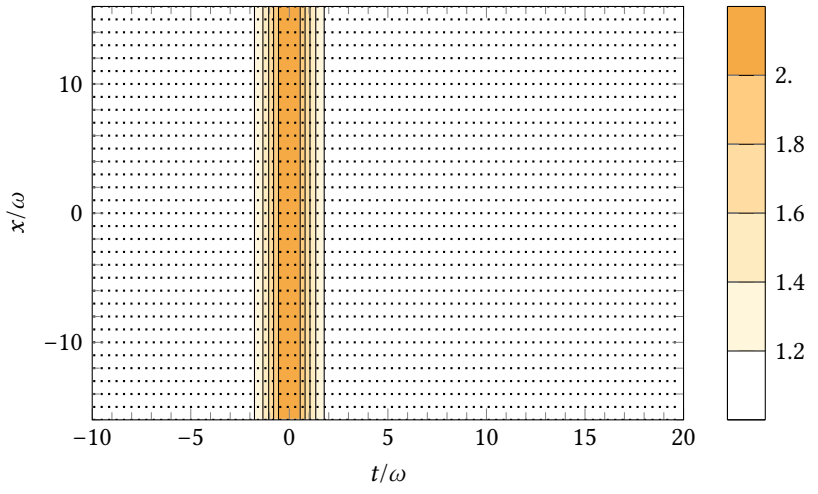

FIGURE 5.2 – Projected characteristic curves (dotted) for the $m(t)$ from (3.151) together with $m(t)$ itself (contour lines) using $f(\tau) = \operatorname{sech}\tau$, $m_0 = 1$, $p = 0$, $\omega = 0.8$, and $\gamma = 0.5$.

by differentiating the first two equations in (5.12). We then obtain the following system of second-order ordinary differential equations that is equivalent to the eikonal equation with a spacetime-dependent mass

$$\begin{aligned}
\ddot{t} &= \frac{2}{m_0^2}\partial_t m^2, & t(\tau_0) &= -\infty,\ \dot{t}(\tau_0) = \pm 2\frac{\epsilon_p}{m_0},\\
\ddot{x} &= -\frac{2}{m_0^2}\partial_x m^2, & x(\tau_0) &= x_0,\ \dot{x}(\tau_0) = -2\frac{p}{m_0},\\
\dot{z} &= \frac{2}{m_0}m^2 = \frac{m_0}{2}\left(\dot{t}^2 - \dot{x}^2\right), & z(\tau_0) &= \pm\omega t_0 + p x_0.
\end{aligned} \tag{5.15}$$

We see that $x(\tau_0) = x_0$ primarily labels different characteristic curves. In principle, we first want to solve the two first equations to obtain a parameterisation $(t(\tau, x_0), x(\tau, x_0))$ for the *projected characteristic curves* along which we then can integrate the last equation to get the solution $S(t(\tau, x_0), x(\tau, x_0)) = z(\tau, x_0)$. As was already mentioned for the general case, this essentially amounts to performing a coordinate transformation from Cartesian coordinates to τ-x_0 coordinates such that integrating the eikonal equation is possible.

In practice however, it is still infeasible to solve (5.15) analytically for arbitrary spacetime-dependent fields. Instead, we integrate these equations numerically using well-established methods to gain some insight about how exact solutions of the eikonal equation look like. For example, for the time-dependent mass that we considered in section 3.7 the projected characteristic curves for vanishing momentum are shown in fig. 5.2. In that case, the projected characteristic curves are simply lines parallel to the time axis. This can directly be seen from the characteristic differential equations (5.15): For $p = 0$, $x = x_0$ and the equation

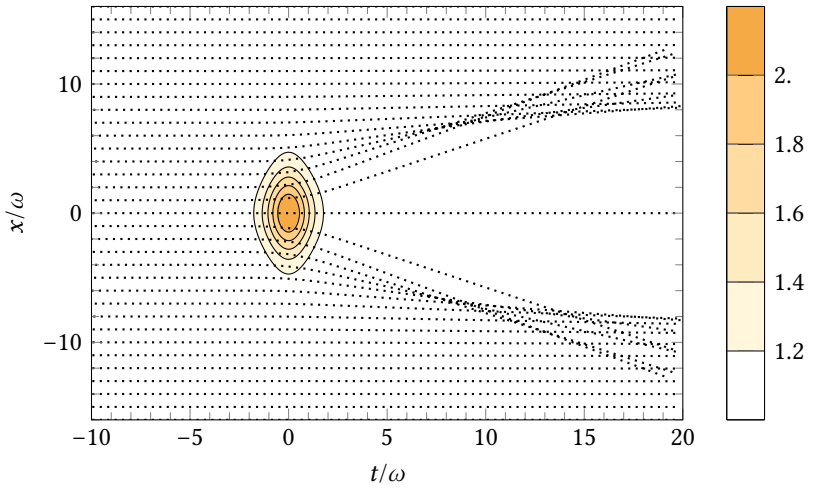

FIGURE 5.3 – Projected characteristic curves (dotted) for the $m(t, x)$ from (5.90) together with $m(t, x)$ itself (contour lines) using $f(\tau) = \operatorname{sech} \tau$, $g(\chi) = \operatorname{sech} \chi$, $m_0 = 1$, $p = 0$, $\omega = 0.8$, $\varepsilon = 0.375$ and $\gamma = 0.5$.

for $t(\tau)$ is independent of x. Thus, τ is essentially just a rescaled time variable. This holds for any only time-dependent mass as the differential equations for $t(\tau)$ and $x(\tau)$ always decouple in that case. For a truly spacetime-dependent mass this is no longer true: The projected characteristic curves are no longer parallel everywhere but might cross each other as can be seen in fig. 5.3 for the spacetime-dependent mass from (5.90). The parameters are deliberately chosen like this to demonstrate the crossing of characteristic curves, i.e. the formation of caustics. The spacetime-dependent mass has a focusing/defocusing effect on the characteristic curves just like an optical lens focuses light rays. At such a focus point, the value of S is not defined uniquely anymore because it may be different on each of the crossing characteristic curves. However, this does not (necessarily) mean that the corresponding solution to the Dirac equation is ill-defined. It is just an indication that the separation of the wave function into exponential and prefactor as done in the WKB method breaks down. Similar situations are well-known for caustics in geometric optics: The density of light rays is singular at any point of the caustic surface while the solution to the full wave equation is perfectly regular.

For the mass given in (5.90), these caustics tend to form earlier in time, all else being equal, for larger values of ε or, more generally speaking, if the x-dependence of the mass $m(t, x)$ is stronger. Thus, we are not able to calculate a solution $S(t, x)$ that is smooth everywhere in the case considered here.

However, we can estimate the onset of the caustics. If they form far away from the spacetime-region where the mass is not constant – where we expect the pair production density to be localised – we might still

use the solution to the eikonal equation in that region to estimate the pair production probability. To estimate the location of the caustics we try to approximate the form of the projected characteristic curves after the region of non-constant mass. For a solution that has positive energy initially, the initial projected characteristic curves are parallel and can be parameterised by

$$t = t_0 - 2\frac{\epsilon_p}{m_0}\tau, \quad x = x_0 - 2\frac{p}{m_0}\tau. \tag{5.16}$$

Due to the inhomogeneity in the mass, the rays are deflected, i.e. after passing the region of non-constant mass their slope $\mathrm{d}x/\mathrm{d}t$ has changed. From the ordinary differential equations (5.15) we find that the change of the slope with the parameter τ is giveny by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\dot{x}}{\dot{t}} = -\frac{2}{\dot{t}}\frac{\partial_x m^2}{m_0^2} - \frac{2\dot{x}}{\dot{t}^2}\frac{\partial_t m^2}{m_0^2}. \tag{5.17}$$

We now want to approximate the change of the slope due to the inhomogeneity. To do so, we use the initial form of the projected characteristic curves (5.16) in (5.17) and integrate over all τ. This should be a good approximation if the mass is only weakly space-dependent as then the change of slope should be small. For $p = 0$ we thus find

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{1}{m_0^2}\partial_x m^2(t_0 - 2\tau, x_0). \tag{5.18}$$

Integrating this over all τ gives

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t\to\infty} \approx -\frac{1}{2m_0^2}\int_{-\infty}^{\infty}\mathrm{d}t\ \ \partial_x m^2(t, x_0) \tag{5.19}$$

and therefore we get the following rough approximation for the form of the projected characteristics after the region of non-constant mass:

$$x_{\text{after}}(x_0; t) \approx x_0 - \frac{t}{2m_0^2}\int_{-\infty}^{\infty}\mathrm{d}t\ \partial_x m^2(t, x_0). \tag{5.20}$$

To determine the position of the caustics we calculate the time t_* at which two neighbouring characteristics cross. For two characteristic curves that start with a spatial separation of δ, we need to solve

$$x_{\text{after}}(x_0 + \delta; t_*) = x_{\text{after}}(x_0; t_*) \tag{5.21}$$

for t_*. Using (5.20), we find

$$t_* = \frac{2m_0^2\delta}{\int_{-\infty}^{\infty} \mathrm{d}t \left[\partial_x m^2(t, x_0 + \delta) - \partial_x m^2(t, x_0)\right]} \tag{5.22}$$

which in the limit $\delta \longrightarrow 0$ turns into

$$t_* = \frac{2m_0^2}{\int_{-\infty}^{\infty} \mathrm{d}t\, \partial_x^2 m^2(t, x_0)}. \tag{5.23}$$

To estimate the earliest time at which a crossing of characteristics occurs we have to find the minimum of t_* with respect to x_0. For example, for a mass of the form given in (5.90), we find that

$$\begin{aligned} &\frac{1}{2m_0^2}\int_{-\infty}^{\infty} \mathrm{d}t\, \partial_x^2 m^2(t, x_0) \\ &= \frac{\varepsilon^2\omega^2}{\gamma^2\omega}\left\{g(\varepsilon\omega x_0)g''(\varepsilon\omega x_0) + \left[g'(\varepsilon\omega x_0)\right]^2\right\}\int_{-\infty}^{\infty} \mathrm{d}\tau \left[f(\tau)\right]^2 \end{aligned} \tag{5.24}$$

Thus, the minimum of t_* scales according to

$$t_f = \min_{x_0} t_* \sim \frac{\gamma^2\omega}{\varepsilon^2\omega^2} \tag{5.25}$$

This confirms our earlier analysis: If the space-dependence of the mass grows (ε gets bigger) the caustics occur earlier, all else being equal. The expression for t_f in (5.25) is strikingly similar to the focal length of a thin, biconvex spherical lens[44]

$$f \propto \frac{L^2}{D(n_2 - n_1)}, \tag{5.26}$$

where L is the length of the lens, D is its thickness, and n_1 and n_2 is the refractive index of the lens and the surrounding medium respectively. Comparing to our expression for t_f we see that $L \sim 1/(\varepsilon\omega)$, $D \sim 1/\omega$ and $n_2 - n_1 \sim 1/\gamma^2$.

For the functions we will use in section 5.3, $f(\tau) = \operatorname{sech}\tau$ and $g(\chi) = \operatorname{sech}\chi$ we actually find that the minimum is at

$$t_f = \frac{3}{2}\frac{\gamma^2\omega}{\varepsilon^2\omega^2}. \tag{5.27}$$

By choosing ε to be small enough, we can move the focal point far away from the region where pairs will most likely be created. As a more formal resolution of the problem, one could consider adding another

[44] This follows from the *lensmaker's formula* [HZ79, p. 107] using $R_1 = -R_2 \propto \frac{L^2}{D}$ for the radii of curvature.

spacetime-dependence to the mass behind the original pulse that does not create pairs but reverses the focusing/defocusing effect of the original pulse such that no caustics occur.

5.2 Choice of coordinates

We now apply the formalism that we derived in chapter 4 to the case of a spacetime-dependent mass $m = m(t, x)$ with no electromagnetic field ($A_\mu = 0$). In general, we may also write the two independent solutions $S_\pm$ to the eikonal equation (5.7) using two different functions $R(t, x)$ and $S(t, x)$ defined according to

$$S_\pm = R \pm S \tag{5.28}$$

thereby splitting $S_\pm$ into a symmetric and an antisymmetric part. The inverse transformation is given by

$$R = \frac{1}{2}(S_+ + S_-), \quad S = \frac{1}{2}(S_+ - S_-). \tag{5.29}$$

Inserting this definition into the eikonal equation (5.7) yields the two equations

$$m^2 = \eta^{\mu\nu}(\partial_\mu R)(\partial_\nu R) + \eta^{\mu\nu}(\partial_\mu S)(\partial_\nu S) \pm 2\eta^{\mu\nu}(\partial_\mu R)(\partial_\nu S). \tag{5.30}$$

We calculate the sum and difference of these two equations and get

$$\begin{aligned} m^2 &= \eta^{\mu\nu}(\partial_\mu R)(\partial_\nu R) + \eta^{\mu\nu}(\partial_\mu S)(\partial_\nu S), \\ 0 &= \eta^{\mu\nu}(\partial_\mu R)(\partial_\nu S). \end{aligned} \tag{5.31}$$

Using Cartesian coordinates this can be written as

$$m^2 = (\partial_t R)^2 - (\partial_x R)^2 + (\partial_t S)^2 - (\partial_x S)^2, \tag{5.32a}$$

$$0 = (\partial_t R)(\partial_t S) - (\partial_x R)(\partial_x S). \tag{5.32b}$$

We solve the second equation[45] for $\partial_t R$,

$$\partial_t R = (\partial_x R)\frac{\partial_x S}{\partial_t S}. \tag{5.33}$$

[45] We assume $\partial_t S \neq 0$ here. This is motivated by the purely time-dependent case where $\partial_t S = -\sqrt{m^2 + p^2} \neq 0$.

We now introduce the quantity

$$\lambda = \frac{\partial_x R}{\partial_t S} \tag{5.34}$$

which relates the derivatives of R and S as follows,

$$\partial_t R = \lambda \partial_x S, \quad \partial_x R = \lambda \partial_t S \tag{5.35}$$

where the first equation is just (5.33) and the second equation is the definition of λ in (5.34). We insert the expressions for $\partial_t R$ and $\partial_x R$ into the first equation (5.32a) and get

$$\lambda^2 = 1 - \frac{m^2}{(\partial_t S)^2 - (\partial_x S)^2}. \tag{5.36}$$

Using the form of $S_\pm$ introduced above we have

$$\Pi^\pm_\mu = \partial_\mu R \pm \partial_\mu S. \tag{5.37}$$

and therefore

$$N_\pm = \frac{1}{\sqrt{2m(m \mp \partial_t R - \partial_t S)}}. \tag{5.38}$$

Employing the relationship between derivatives of R and derivatives of S given in (5.35) we then find after some manipulations

$$\kappa^0 = \sqrt{1-\lambda^2}\frac{\partial_x S}{m}, \quad \kappa^1 = -\sqrt{1-\lambda^2}\frac{\partial_t S}{m} \tag{5.39}$$

or written in a more concise way

$$\kappa^\mu = \frac{\sqrt{1-\lambda^2}}{m}\varepsilon^{\mu\nu}\partial_\nu S. \tag{5.40}$$

We could now write down the equations for α and β from (4.40). However, we first want to introduce new coordinates which simplify the equations considerably. The equations for α and β contain the rapidly oscillating phase $\exp[\pm i(S_+ - S_-)] = \exp(\pm 2iS)$. Thus, we choose one of our new coordinates $s = S(t,x)/m_0$ to be in that direction. This coordinate can be thought of as the new time coordinate. To simplify any inner products the other coordinate (the new spatial coordinate) should be locally orthogonal to s. As we will see shortly, this is the case if we use $r = R(t,x)/m_0$ as the other coordinate[46]. In these coordinates

$$\partial_s S = m_0, \quad \partial_r S = 0, \quad \partial_s R = 0, \quad \partial_r R = m_0 \tag{5.41}$$

and thus the components of $\Pi^\pm_\mu$ are

$$\Pi^\pm_s = \pm m_0, \quad \Pi^\pm_r = m_0. \tag{5.42}$$

[46] We introduce the constant mass m_0 to ensure that s and r each have the dimension of length. For example, assuming that $m(t,x)$ is a constant (i.e. independent of x) initially, we could use $m_0 = \lim_{t\to-\infty} m$.

We now need to express all terms in the equations (4.40) for α and β in these new coordinates. We start with the metric tensor itself which by now will be denoted by $g_{\mu\nu}$. If $\Lambda^{\mu'}{}_{\mu} = \partial_\mu x^{\mu'}$ is the Jacobian matrix of the coordinate transformation then the components of the inverse metric tensor transform as

$$g^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu}\eta^{\mu\nu}. \tag{5.43}$$

Thus, the inverse metric tensor's components in the new coordinates are

$$\begin{aligned} g^{ss} &= (\partial_t s)^2 - (\partial_x s)^2 = \frac{1}{1-\lambda^2}\left(\frac{m}{m_0}\right)^2, \\ g^{rr} &= (\partial_t r)^2 - (\partial_x r)^2 = -\frac{\lambda^2}{1-\lambda^2}\left(\frac{m}{m_0}\right)^2, \\ g^{sr} &= g^{rs} = (\partial_t s)(\partial_t r) - (\partial_x s)(\partial_x r) = 0 \end{aligned} \tag{5.44}$$

where we have made use of (5.35); the last equation is a direct consequence of (5.32b). Hence the metric tensor is again diagonal in these coordinates. This is equivalent to saying that the curves of constant s and constant r are locally orthogonal, just as intended. However, it should be noted that the metric tensor is singular for $\lambda = 0$. We will return to this point later.

Similarly we find the components of the Levi-Civita tensor in s-r coordinates to be

$$\varepsilon^{ss} = \varepsilon^{rr} = 0, \qquad \varepsilon^{sr} = -\varepsilon^{rs} = \frac{\lambda}{1-\lambda^2}\left(\frac{m}{m_0}\right)^2. \tag{5.45}$$

The partial derivative ∂_μ acting on the vectors $\Pi^{\pm}_{\nu}$ in (4.40) has to be replaced with the usual covariant derivative ∇_μ which involves the Christoffel symbols of the second kind

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{5.46}$$

Using the components of the inverse metric tensor given in (5.44) we

get

$$\begin{aligned}
\Gamma^s_{ss} &= -\frac{1}{2}\partial_s \ln g^{ss} &&= -\partial_s \ln m - \frac{\lambda^2}{1-\lambda^2}\partial_s \ln \lambda,\\
\Gamma^s_{sr} = \Gamma^s_{rs} &= -\frac{1}{2}\partial_r \ln g^{ss} &&= -\partial_r \ln m - \frac{\lambda^2}{1-\lambda^2}\partial_r \ln \lambda,\\
\Gamma^s_{rr} &= -\frac{1}{2\lambda^2}\partial_s \ln g^{rr} &&= -\frac{1}{\lambda^2}\partial_s \ln m - \frac{1}{\lambda^2(1-\lambda^2)}\partial_s \ln \lambda,\\
\Gamma^r_{rr} &= -\frac{1}{2}\partial_r \ln g^{rr} &&= -\partial_r \ln m - \frac{1}{1-\lambda^2}\partial_r \ln \lambda,\\
\Gamma^r_{sr} = \Gamma^r_{rs} &= -\frac{1}{2}\partial_s \ln g^{rr} &&= -\partial_s \ln m - \frac{1}{1-\lambda^2}\partial_s \ln \lambda,\\
\Gamma^r_{ss} &= -\frac{1}{2}\lambda^2\partial_r \ln g^{ss} &&= -\lambda^2\partial_r \ln m - \frac{\lambda^4}{1-\lambda^2}\partial_r \ln \lambda.
\end{aligned} \tag{5.47}$$

Together with the expressions for $\Pi^\pm_\mu$ given in (5.42) we thus find

$$\begin{aligned}
\nabla_\mu \Pi^\pm_s &= \underbrace{\partial_\mu \Pi^\pm_s}_{=\pm\partial_\mu m_0 = 0} - \Gamma^\nu_{\mu s}\Pi^\pm_\nu = -m_0\big(\pm\Gamma^s_{\mu s} + \Gamma^r_{\mu s}\big),\\
\nabla_\mu \Pi^\pm_r &= \underbrace{\partial_\mu \Pi^\pm_r}_{=\partial_\mu m_0 = 0} - \Gamma^\nu_{\mu r}\Pi^\pm_\nu = -m_0\big(\pm\Gamma^s_{\mu r} + \Gamma^r_{\mu r}\big).
\end{aligned} \tag{5.48}$$

Therefore, after some manipulations we get

$$\begin{aligned}
&\frac{1}{2m^2}\big(g^{\mu\rho}g^{\nu\lambda} - g^{\mu\nu}g^{\rho\lambda}\big)\Pi^\pm_\rho \Pi^\pm_\lambda \nabla_\mu \Pi^\pm_\nu\\
&= -\frac{1}{2}\frac{m^2}{m_0}\frac{1}{1-\lambda^2}\big[\mp\partial_s \ln(\lambda m) + \lambda^2\partial_r \ln(\lambda m) - 2\lambda^2\partial_r \ln\lambda\big]
\end{aligned} \tag{5.49}$$

and

$$\begin{aligned}
&\frac{1}{2m^2}\varepsilon^{\lambda\nu}{}_\rho{}^\mu \kappa^\rho \Pi^\pm_\lambda \nabla_\mu \Pi^\pm_\nu\\
&= \frac{1}{2}\frac{m}{m_0}\frac{1}{1-\lambda^2}\left(\mp\frac{\lambda}{\sqrt{1-\lambda^2}}\partial_s \ln\lambda - \lambda\sqrt{1-\lambda^2}\,\partial_r \ln m - \frac{\lambda^3}{\sqrt{1-\lambda^2}}\partial_r \ln\lambda\right).
\end{aligned} \tag{5.50}$$

Putting it all together, we can now write the equations (4.40) for α and

β using r-s coordinates:

$$\begin{aligned}
&\partial_s\alpha - \lambda^2\partial_r\alpha - \frac{1}{2}\alpha\big[\partial_s \ln(\lambda m) - \lambda^2\partial_r \ln(\lambda m) + 2\lambda^2\partial_r \ln\lambda\big] \\
&= -ie^{-2iS}\lambda\sqrt{1-\lambda^2}\bigg[\partial_r\beta - \frac{1}{2}\beta\bigg(\frac{1}{1-\lambda^2}\partial_s \ln\lambda - \partial_r \ln m - \frac{\lambda^2}{1-\lambda^2}\partial_r \ln\lambda\bigg)\bigg], \\
&\partial_s\beta + \lambda^2\partial_r\beta - \frac{1}{2}\beta\big[\partial_s \ln(\lambda m) + \lambda^2\partial_r \ln(\lambda m) - 2\lambda^2\partial_r \ln\lambda\big] \\
&= ie^{2iS}\lambda\sqrt{1-\lambda^2}\bigg[\partial_r\alpha - \frac{1}{2}\alpha\bigg(\frac{1}{1-\lambda^2}\partial_s \ln\lambda + \partial_r \ln m + \frac{\lambda^2}{1-\lambda^2}\partial_r \ln\lambda\bigg)\bigg].
\end{aligned} \tag{5.51}$$

The left-hand side can be written even more concise by rescaling α and β according to

$$\alpha = \tilde{\alpha}\sqrt{\lambda m}, \quad \beta = \tilde{\beta}\sqrt{\lambda m}. \tag{5.52}$$

We then have

$$\begin{aligned}
\partial_\mu\alpha &= \sqrt{\lambda m}\partial_\mu\tilde{\alpha} + \frac{1}{2}\alpha\partial_\mu \ln(\lambda m), \\
\partial_\mu\beta &= \sqrt{\lambda m}\partial_\mu\tilde{\beta} + \frac{1}{2}\beta\partial_\mu \ln(\lambda m).
\end{aligned} \tag{5.53}$$

Thus, rewriting (5.51) in terms of $\tilde{\alpha}$ and $\tilde{\beta}$ yields

$$\begin{aligned}
\partial_s\tilde{\alpha} - \lambda^2\partial_r\tilde{\alpha} - \lambda^2\tilde{\alpha}\partial_r \ln\lambda &= -ie^{-2iS}\lambda\sqrt{1-\lambda^2}\bigg(\partial_r\tilde{\beta} - \frac{1}{2}\tilde{\beta}\chi_\beta\bigg), \\
\partial_s\tilde{\beta} + \lambda^2\partial_r\tilde{\beta} + \lambda^2\tilde{\beta}\partial_r \ln\lambda &= ie^{2iS}\lambda\sqrt{1-\lambda^2}\bigg(\partial_r\tilde{\alpha} - \frac{1}{2}\tilde{\alpha}\chi_\alpha\bigg)
\end{aligned} \tag{5.54}$$

where

$$\begin{aligned}
\chi_\beta &= \frac{1}{1-\lambda^2}\partial_s \ln\lambda - 2\partial_r \ln m - \frac{1}{1-\lambda^2}\partial_r \ln\lambda, \\
\chi_\alpha &= \frac{1}{1-\lambda^2}\partial_s \ln\lambda - \frac{1-2\lambda^2}{1-\lambda^2}\partial_r \ln\lambda.
\end{aligned} \tag{5.55}$$

We already mentioned that the case where $\lambda = 0$ is a singular one. This is also apparent in the definition of $\tilde{\alpha}$ and $\tilde{\beta}$ (5.52): If λ vanishes, then $\alpha = \beta = 0$ regardless of the value of $\tilde{\alpha}$ and $\tilde{\beta}$. To understand the physical relevance of this singularity we consider the usual initial conditions for solutions of the eikonal equation

$$\lim_{t\to-\infty} S_\pm = px, \tag{5.56}$$

i.e. the solutions correspond to plane waves initially. Equivalently, we may also write

$$\lim_{t\to-\infty} R = px, \qquad \lim_{t\to-\infty} S = 0. \tag{5.57}$$

Hence, initially we have $\partial_x R = p$ and $\partial_x S = 0$. Furthermore, $\lambda = \partial_x R/\partial_t S$

is proportional to p initially. Thus, when $p = 0$, λ will vanish initially. From (5.35) we find that R will not change (i.e. $\partial_t R = 0$) in that case. Thus, R (and λ) will vanish for all times. This is problematic as we used $R(t, x)$ to define a coordinate transformation which is not well-defined when $R = 0$ for all t and x. Thus, we avoid the case of vanishing momentum $p = 0$ (and equivalently $\lambda = 0$) in the following.

It should be stressed that the new equations (5.54) for α and β are still exact for $p \neq 0$. In comparison to the covariant Dirac equation (2.7) we started with, however, they have several advantages. First of all, the rapidly oscillating phase $\exp(\pm iS)$ only depends on the new time coordinate s as in the time-dependent case [compare (4.47)]. Second, for small values of λ, the equations (5.54) can be written as

$$\begin{aligned} \partial_s \tilde{\alpha} &= -ie^{-2iS}\lambda\left(\partial_r \tilde{\beta} - \frac{1}{2}\tilde{\beta}\chi_\beta\right) + \mathcal{O}(\lambda^2), \\ \partial_s \tilde{\beta} &= ie^{2iS}\lambda\left(\partial_r \tilde{\alpha} - \frac{1}{2}\tilde{\alpha}\chi_\alpha\right) + \mathcal{O}(\lambda^2). \end{aligned} \tag{5.58}$$

Furthermore, if we assume $\tilde{\alpha} \gg \tilde{\beta}$, which should hold in the subcritical regime where few pairs are created, we may approximate $\tilde{\alpha} \approx \tilde{\alpha}(r)$ using the first equation in (5.58). This corresponds to a wave packet $\tilde{\alpha}(r)e^{im_0 r + im_0 s}$ moving along lines of constant r (i.e. in s-direction) where the shape of the wave packet is given by $\tilde{\alpha}(r)$. If we go back to Cartesian coordinates t and x, we find this to be equivalent to a wave packet travelling at varying speed where the form of the wave packet changes over time (i.e. becomes narrower or wider). Using that approximation for $\tilde{\alpha}$ we may obtain $\tilde{\beta}(r)$ by integrating the second equation in (5.58) over all s for a fixed value of r. Thus, we expect the value of the exponent to be determined by $S(t, x)$ whereas the path of integration in the spacetime depends on $R(t, x)$.

In principle, this works exactly the same as in the time-dependent case. We perform the integral using analytic continuation in s and expect the pair production density exponent to be dominated by the value of S at the singularity closest to the real axis. However, this requires all quantities to be re-written as functions of s and r instead of functions of t and x and thus analytical solutions to the eikonal equation (5.7) with a spacetime-dependent mass are needed. As this is only possible to do fully analytically for some special cases, we employ an approximation for fields that are only weakly space-dependent in the following section. Another analytical approach that could be used is an inverse one (see also [OS15]): Given functions $R(t, x)$ and $S(t, x)$ that are locally ortho-

gonal one can calculate the associated mass from (5.32a). The functions R and S could, for example, be obtained by choosing S such that R can be calculated easily from the orthogonality condition (5.32b). Most likely, this will only be possible for a single mode p for each associated mass m, though.

5.3 Weakly space-dependent mass

We now want to turn to the case of a mass whose dependence on x is much weaker than its dependence on t, i.e. $m = m(t, \varepsilon x)$ with $\varepsilon \ll 1$. We introduce the length scale $\xi = \varepsilon x$ and thus have $\partial_x = \varepsilon \partial_\xi$. Writing the sum and difference of the eikonal equations (5.32) using t and ξ we have

$$m^2(t,\xi) = (\partial_t R)^2 + (\partial_t S)^2 - \varepsilon^2 (\partial_\xi R)^2 - \varepsilon^2 (\partial_\xi S)^2, \tag{5.59a}$$

$$0 = (\partial_t R)(\partial_t S) - \varepsilon^2 (\partial_\xi R)(\partial_\xi S). \tag{5.59b}$$

The initial condition (5.56) then reads

$$\lim_{t \to -\infty} S_\pm = px = \frac{1}{\varepsilon} p\xi. \tag{5.60}$$

We now want to make a formal expansion of both $S_\pm$ and the equations for α and β (5.54) for small ε. To satisfy the initial condition (5.60) the lowest-order term in R has to be of order ε^{-1} whereas the expansion of S starts with a term of order ε^0:

$$\begin{aligned} R(t,\xi) &= \frac{1}{\varepsilon} R_0(t,\xi) + R_1(t,\xi) + \varepsilon R_2(t,\xi) + \varepsilon^2 R_3(t,\xi) + \dots, \\ S(t,\xi) &= \qquad\qquad S_0(t,\xi) + \varepsilon S_1(t,\xi) + \varepsilon^2 S_2(t,\xi) + \dots. \end{aligned} \tag{5.61}$$

We insert this expansion into (5.59) and balance terms of the same order. The lowest order that occurs is ε^{-2}. From the first equation (5.59a) we get

$$0 = (\partial_t R_0)^2. \tag{5.62}$$

Thus the lowest-order term only depends on ξ, i.e. $R_0 = R_0(\xi)$. Comparing with the initial condition (5.60) we can conclude that

$$R_0 = p\xi. \tag{5.63}$$

The second equation (5.59b) does not have any term proportional to ε^{-2}. The terms of order ε^{-1} are

$$\begin{aligned} 0 &= 2\,\underbrace{(\partial_t R_0)}_{=0}(\partial_t R_1), \\ 0 &= \underbrace{(\partial_t R_0)}_{=0}(\partial_t S_0) \end{aligned} \tag{5.64}$$

which are already satisfied by the result from order ε^{-2}. We now look at the non-vanishing terms of order ε^0

$$\begin{aligned} m^2 &= (\partial_t R_1)^2 + (\partial_t S_0)^2 - (\partial_\xi R_0)^2, \\ 0 &= (\partial_t R_1)(\partial_t S_0). \end{aligned} \tag{5.65}$$

Assuming that S_0 depends on the time t – as otherwise it would just be constant due to the initial condition (5.60) – we find from the second equation that $R_1 = R_1(\xi)$ has to be time-independent. However, there is no symmetric term of order ε^0 in the initial condition (5.60) and thus R_1 has to vanish identically. Therefore, we get from the first equation

$$S_0 = -\int_{-\infty}^{t} \mathrm{d}t' \sqrt{m^2(t', \xi) + p^2} \tag{5.66}$$

where the constant of integration was chosen such that the initial condition (5.60) is satisfied. Thus, to lowest order in ε, R and S have exactly the same form as in the time-dependent case; compare (4.43). This is expected as for $\varepsilon \longrightarrow 0$ we just have a time-dependent mass. Next, we consider the non-vanishing terms of order ε^1

$$\begin{aligned} 0 &= 2(\partial_t S_0)(\partial_t S_1), \\ 0 &= (\partial_t R_2)(\partial_t S_0) - (\partial_\xi R_0)(\partial_\xi S_0). \end{aligned} \tag{5.67}$$

The first equation can be treated analogously to the equation for R_1 above: Although the equation implies that S_1 only depends on ξ there is no term of order ε^1 in the initial condition (5.60). Thus, S_1 has to vanish identically as well. From the second equation we find

$$\partial_t R_2 = \frac{(\partial_\xi R_0)(\partial_\xi S_0)}{\partial_t S_0} = \frac{p}{\sqrt{m^2 + p^2}} \int_{-\infty}^{t} \mathrm{d}t' \, \frac{m \partial_\xi m}{\sqrt{m^2 + p^2}} \tag{5.68}$$

Going to order ε^2, we have

$$\begin{aligned} 0 &= (\partial_t R_2)^2 + 2(\partial_t S_0)(\partial_t S_2) - 2(\partial_t R_0)(\partial_t R_2) - (\partial_\xi S_0)^2, \\ 0 &= (\partial_t R_3)(\partial_t S_0). \end{aligned} \tag{5.69}$$

Again, from the second equation we can conclude that $R_3 = R_3(\xi) = 0$ because of the initial condition (5.60) while the first equation leads to

$$\begin{aligned} \partial_t S_2 &= \frac{(\partial_\xi S_0)^2 + 2(\partial_\xi R_0)(\partial_\xi R_2) - (\partial_t R_2)^2}{2\partial_t S_0} \\ &= \frac{1}{2}\frac{(\partial_\xi S_0)^2}{\partial_t S_0} + \left(\frac{\partial_\xi R_0}{\partial_t S_0}\right)\left[\partial_\xi R_2 - \frac{1}{2}\frac{\partial_\xi R_0}{(\partial_t S_0)^2}\right]. \end{aligned} \tag{5.70}$$

Finally, we look at the terms of order ε^3

$$\begin{aligned} 0 &= 2(\partial_t S_0)(\partial_t S_3), \\ 0 &= (\partial_t R_2)(\partial_t S_2) + (\partial_t R_4)(\partial_t S_0) - (\partial_\xi R_0)(\partial_\xi S_2) - (\partial_\xi R_2)(\partial_\xi S_0). \end{aligned} \tag{5.71}$$

From the first equation we just get $S_3 = S_3(\xi) = 0$. The second equation could be used to calculate R_4 but an expansion up to terms of order ε^2 is sufficient for our purposes. Overall, we have

$$R(t,\xi) = \frac{1}{\varepsilon}R_0(\xi) + \varepsilon R_2(t,\xi) + \mathcal{O}(\varepsilon^3), \tag{5.72a}$$

$$S(t,\xi) = S_0(t,\xi) + \varepsilon^2 S_2(t,\xi) + \mathcal{O}(\varepsilon^4) \tag{5.72b}$$

where R_0, R_2, S_0 and S_2 are defined by (5.63), (5.68), (5.66) and (5.70) respectively. We see that the expansions contain either only odd or only even powers of ε. This is because the original eikonal equation (5.7) only contains squares of the derivatives of $S_\pm$.

Using these expressions we may expand λ into powers of ε:

$$\begin{aligned} \lambda &= \frac{\partial_x R}{\partial_t S} = \varepsilon\frac{\partial_\xi R}{\partial_t S} = \frac{\partial_\xi R_0}{\partial_t S_0} + \varepsilon^2\left[\frac{\partial_\xi R_2}{\partial_t S_0} - \frac{(\partial_\xi R_0)(\partial_t S_2)}{(\partial_t S_0)^2}\right] + \mathcal{O}(\varepsilon^4) \\ &= \lambda_0 + \varepsilon^2\lambda_2 + \mathcal{O}(\varepsilon^4) \end{aligned} \tag{5.73}$$

Furthermore, it is instructive to expand ∂_s and ∂_r as well. To do so, we first use the chain rule to get

$$\partial_s = (\partial_s t)\partial_t + (\partial_s x)\partial_x, \quad \partial_r = (\partial_r t)\partial_t + (\partial_r x)\partial_x \tag{5.74}$$

and then employ the inverse function theorem to determine the derivatives $\partial_s t$, $\partial_s x$, $\partial_r t$ and $\partial_r x$. Consider the coordinate transform $\zeta(t,x) =$

$(s(t,x), r(t,x))^\mathsf{T}$. Its Jacobian matrix is

$$J_\zeta(t,x) = \begin{pmatrix} \partial_t s & \partial_x s \\ \partial_t r & \partial_x r \end{pmatrix} \tag{5.75}$$

and its determinant is

$$\det J_\zeta(t,x) = \left(\frac{m}{m_0}\right)^2 \frac{\lambda}{1-\lambda^2}. \tag{5.76}$$

By the inverse function theorem, if $\det J_\zeta$ is non-zero then the Jacobian matrix of the inverse coordinate transform, $\zeta^{-1}(s,r) = (t(s,r), x(s,r))^\mathsf{T}$, is given by[47]

$$J_{\zeta^{-1}}(s,r) = \left[J_\zeta(t(s,r), x(s,r))\right]^{-1} = \left(\frac{m_0}{m}\right)^2 \frac{1-\lambda^2}{\lambda} \begin{pmatrix} \partial_x r & -\partial_x s \\ -\partial_t r & \partial_t s \end{pmatrix}. \tag{5.77}$$

[47] Actually, for $\lambda = 0$, the determinant of $J_\zeta(t,x)$ vanishes and thus the assumption of the inverse function theorem does not hold. However, this is the just the singular case of $R = 0$ (or equivalently $p = 0$) that we have discussed before (see section 5.2).

On the other hand,

$$J_{\zeta^{-1}}(s,r) = \begin{pmatrix} \partial_s t & \partial_r t \\ \partial_s x & \partial_r x \end{pmatrix} \tag{5.78}$$

and therefore

$$\begin{aligned} \partial_s &= \left(\frac{m_0}{m}\right)^2 \frac{1-\lambda^2}{\lambda}\big[(\partial_x r)\partial_t - (\partial_t r)\partial_x\big] \\ &= \frac{m_0}{m^2}(1-\lambda^2)\big[(\partial_t S)\partial_t - (\partial_x S)\partial_x\big] \\ &= \frac{m_0}{m^2}(1-\lambda^2)\big[(\partial_t S)\partial_t - \varepsilon^2(\partial_\xi S)\partial_\xi\big] \end{aligned} \tag{5.79}$$

where we have used (5.35) to arrive at the last expression. Similarly we obtain

$$\begin{aligned} \partial_r &= \left(\frac{m_0}{m}\right)^2 \frac{1-\lambda^2}{\lambda}\big[-(\partial_x s)\partial_t + (\partial_t s)\partial_x\big] \\ &= \frac{m_0}{m^2}\frac{1-\lambda^2}{\lambda}\varepsilon\big[-(\partial_\xi S)\partial_t + (\partial_t S)\partial_\xi\big]. \end{aligned} \tag{5.80}$$

Using the expansion for S from (5.72b), we finally find the expansions of ∂_s and ∂_r to be

$$\begin{aligned} \partial_s =& \frac{m_0}{m^2}(1-\lambda_0^2)\Bigg\{(\partial_t S_0)\partial_t \\ &+ \varepsilon^2\left[\left(-\lambda_0\partial_\xi R_2 + \frac{1}{2}(1+\lambda_0^2)\frac{(\partial_\xi S_0)^2}{\partial_t S_0}\right)\partial_t - (\partial_\xi S_0)\partial_\xi\right] + \mathcal{O}(\varepsilon^4)\Bigg\} \end{aligned} \tag{5.81}$$

and

$$\partial_r = \frac{m_0}{m^2}(1-\lambda_0^2)\left\{\varepsilon\left[-\frac{(\partial_\xi S_0)}{\lambda_0}\partial_t + \frac{(\partial_t S_0)}{\lambda_0}\partial_\xi\right] + \mathcal{O}(\varepsilon^3)\right\}. \tag{5.82}$$

We see that to order ε^0, ∂_s is proportional to ∂_t and thus our new time coordinate s is indeed just a rescaled time variable to this order of approximation.

We now want expand the equations for $\tilde{\alpha}$ and $\tilde{\beta}$ (5.54) in powers of ε as well. We first restrict our expansion to terms of order ε^0. As the lowest-order term in ∂_r is already of order ε, we drop all the terms with ∂_r from (5.54). Furthermore, to this order of approximation we have $\lambda = \lambda_0 = -p/\sqrt{m^2+p^2}$. Then the equations for $\tilde{\alpha}$ and $\tilde{\beta}$ read

$$\begin{aligned}\partial_s\tilde{\alpha} &= \frac{i}{2}\tilde{\beta}e^{-2iS_0}\frac{\partial_s\lambda_0}{\sqrt{1-\lambda_0^2}} = \frac{i}{2}\tilde{\beta}e^{-2iS_0}\frac{p\partial_s m}{m^2+p^2},\\ \partial_s\tilde{\beta} &= -\frac{i}{2}\tilde{\alpha}e^{2iS_0}\frac{\partial_s\lambda_0}{\sqrt{1-\lambda_0^2}} = -\frac{i}{2}\tilde{\alpha}e^{2iS_0}\frac{p\partial_s m}{m^2+p^2}.\end{aligned} \tag{5.83}$$

These are exactly the same equations as in the purely time-dependent case [compare (3.15)] with s as the variable instead of t. There are no derivatives with respect to the spatial variable x so these equations are just a system of ordinary differential equations with t as the variable and x as a parameter. As before, by defining $\mathscr{R} = \beta/\alpha = \tilde{\beta}/\tilde{\alpha}$ we can cast these in a single Riccati equation and upon linearising it, we get

$$\mathscr{R}_{\text{out}} \approx -\frac{i}{2}\int \mathrm{d}s\,\frac{p\partial_s m}{m^2+p^2}e^{2iS_0} = -\frac{i}{2}\int \mathrm{d}t\,\frac{p\partial_t m}{m^2+p^2}e^{2iS_0}. \tag{5.84}$$

The integral has to be evaluated with the other coordinate r (or equivalently ξ) held constant. Then, the value of the integral is dominated[48] by the pole where $m^2(t_*(\xi),\xi)+p^2=0$. Hence,

$$|\mathscr{R}_{\text{out}}|^2 \approx \left|\tilde{\beta}_{\text{out}}\right|^2 \sim e^{-4\Im S_0(t_*(\xi),\xi)}. \tag{5.85}$$

As with the time-dependent mass in section 3.7 no pairs will be created in the mode with $p=0$ due to the factor of p in the prefactor.

This essentially corresponds to a *locally homogeneous field approximation*: To approximate the result in the presence of the spacetime-dependent mass one solves a time-dependent problem by treating x only as a parameter in the mass. Afterwards, one integrates over the resulting pair production density for all x [see (4.70)] to obtain the number of

[48] Note though that m itself might have other singularities even closer to the real axis that could dominate the result (see section 3.4).

produced pairs. This is an straightforward improvement over the locally constant field approximation where the pair production density for a constant field is integrated over the whole spacetime with E replaced with $E(t, x)$.

However, we now would like to go beyond these approximations. We only get corrections in the exponent if we include terms of order ε^2 as there is no term of order ε^1 in the expansion of S (5.72b). To this order of approximation we would have to include derivatives with respect to r in the equations for α and β. We make the additional assumption that $p = \varepsilon^2 \tilde{p}$ where $\tilde{p} = \mathcal{O}(m)$ (i.e. p is small in comparison to the electron mass) to get rid of some of these terms. We then have

$$R_2 = \mathcal{O}(\varepsilon^2), \quad S_2 = \frac{1}{2} \int \mathrm{d}t \, \frac{(\partial_\xi S_0)^2}{\partial_t S_0} + \mathcal{O}(\varepsilon^2) \tag{5.86}$$

Using these we find

$$\lambda = \lambda_0 + \mathcal{O}(\varepsilon^4) = -\varepsilon^2 \frac{\tilde{p}}{\sqrt{m^2 + p^2}} + \mathcal{O}(\varepsilon^4) \tag{5.87}$$

where actually the square root in the denominator would have to be expanded as well but we expect the approximation to only get better by keeping it like that. With this additional approximation the equations for α and β up to order ε^2 read

$$\begin{aligned} \partial_s \tilde{\alpha} &= -ie^{-2iS} \lambda_0 \left\{ \partial_r \tilde{\beta} - \frac{1}{2} \tilde{\beta} \left[\partial_s \ln \lambda_0 - \partial_r \ln\left(\lambda_0 m^2\right) \right] \right\}, \\ \partial_s \tilde{\beta} &= ie^{2iS} \lambda_0 \left\{ \partial_r \tilde{\alpha} - \frac{1}{2} \tilde{\alpha} \left[\partial_s \ln \lambda_0 - \partial_r \ln \lambda_0 \right] \right\} \end{aligned} \tag{5.88}$$

where $S = S_0 + \varepsilon^2 S_2$ with S_2 from (5.86). This is essentially equivalent to the approximation in (5.58) where we were only keeping terms of order λ: As the next-to-leading order term in λ is already of order ε^2, we can drop terms of order λ^2. Moreover, as we have discussed before, if only very few pairs are created we have $\tilde{\alpha} \gg \tilde{\beta}$ and therefore the first equation reduces to $\partial_s \tilde{\alpha} = 0$ or equivalently $\tilde{\alpha} \approx \tilde{\alpha}(r)$. Thus, $\tilde{\beta}_{\text{out}}$ can be obtained by integrating the second equation in (5.88) over all s while keeping r constant. As with the approximations made $r = \varepsilon \tilde{p} \xi / m_0 + \mathcal{O}(\varepsilon^3)$, keeping r constant is actually the same as keeping ξ (or equivalently x) constant. As before, we assume that the value of the integral is dominated by the pole where λ_0 diverges, i.e. where $m^2(t_*(\xi), \xi) + p^2 = 0$.

Using (4.70) we can approximate the number of produced positrons as

$$N_{e^+}(p) \approx \int \mathrm{d}x\, \Xi(x) \exp\left[-4\Im S(p; t_*, x)\right] \tag{5.89}$$

where $\Xi(x)$ is a slowly-varying prefactor.

As an example, consider a mass of the form

$$m(t, \xi) = m_0 \sqrt{1 + \left[\frac{f(\omega t) g(\omega \xi)}{\gamma}\right]^2}. \tag{5.90}$$

This is essentially the same form as in (3.151) but with an additional space-dependent factor $g(\omega\xi)$. We will again use $f(\tau) = \operatorname{sech} \tau$. Then, the expression for the dominant singularity has the same form as in the time-dependent case,

$$\tau_* = \operatorname{arcosh}\left(\pm \frac{i}{\tilde{\gamma}}\right) = \ln\left[\frac{1}{|\tilde{\gamma}|} + \sqrt{1 + \left(\frac{1}{\tilde{\gamma}^2}\right)}\right] - i\frac{\pi}{2}, \tag{5.91}$$

compare (3.160). The only difference is the expression for $\tilde{\gamma}$ which now includes the space-dependent function $g(\omega\xi)$,

$$\tilde{\gamma} = \frac{\gamma}{g(\omega\xi)} \sqrt{1 + \left(\frac{p}{m_0}\right)^2}. \tag{5.92}$$

Comparing (5.66) and (3.150) we see that S_0 and φ_p are the same up to a sign, i.e. $S_0 = -\varphi_p$. Therefore, we can reuse the analytical result from section 3.7,

$$S_0 = -\frac{m_0}{\omega} \sqrt{1 + \left(\frac{p}{m_0}\right)^2} \left[\phi(\tau) - \phi(-\infty)\right] \tag{5.93}$$

where[49]

$$\phi(\tau) = \frac{1}{\tilde{\gamma}} \arctan\left[\frac{\sinh \tau}{\sqrt{1 + \tilde{\gamma}^2 \cosh^2 \tau}}\right] + \operatorname{artanh}\left[\frac{\tilde{\gamma} \sinh \tau}{\sqrt{1 + \tilde{\gamma}^2 \cosh^2 \tau}}\right]. \tag{5.94}$$

[49] Although this is exactly the same expression as in the time-dependent case, it should be stressed that here $\tilde{\gamma}$ depends on ξ (or x).

The lowest-order contribution to the exponent of the number of produced pairs is thus exactly the same as in the time-dependent case,

$$-4\Im S_0(t_*, x) = -2\pi \frac{m_0}{\omega} \sqrt{1 + \left(\frac{p}{m_0}\right)^2}. \tag{5.95}$$

For the next-order contribution we need to calculate

$$
\begin{aligned}
-4\Im S_2(t_*,x) &= -2\Im\int_{-\infty}^{t_*}\mathrm{d}t\,\frac{\left(\partial_{\tilde{\xi}}S_0\right)^2}{\partial_t S_0}\\
&= 2\frac{m_0}{\omega}\frac{\left[g'(\varepsilon\omega x)\right]^2}{|g(\varepsilon\omega x)|}\frac{1}{\gamma}\Im\int_{-\infty}^{\tau_*}\mathrm{d}\tau\,\frac{\left[\arctan\left(\dfrac{\sinh\tau}{\sqrt{1+\tilde{\gamma}^2\cosh^2\tau}}\right)+\arctan\left(\dfrac{1}{|\tilde{\gamma}|}\right)\right]^2}{\sqrt{\tilde{\gamma}^2+\operatorname{sech}^2\tau}}.
\end{aligned}
\tag{5.96}
$$

Because the integrand is real for real values of τ, the lower limit of integration does not give a contribution to the imaginary part. Therefore, the choice of the lower limit of integration is completely arbitrary and we may parameterise the integral using $\tau = \Re\tau_* + iu\Im\tau_*$ with $u \in [0,1)$. Then, we get

$$
-4\Im S_2(t_*,x) = -\pi\frac{m_0}{\omega}\frac{\left[g'(\varepsilon\omega x)\right]^2}{|g(\varepsilon\omega x)|}\frac{1}{\gamma}h(\tilde{\gamma}) \tag{5.97}
$$

where [50]

$$
\begin{aligned}
h(\tilde{\gamma}) &= \Re\int_0^1 \frac{\mathrm{d}u}{|\tilde{\gamma}|\sqrt{1+\dfrac{1}{\cos(\pi u)+\tilde{\gamma}^2[1+\cos(\pi u)]/2-i\sqrt{1+\tilde{\gamma}^2}\sin(\pi u)}}}\\
&\times\left\{\arctan\left[\frac{1}{|\tilde{\gamma}|}\frac{\cos(\pi u/2)-i\sqrt{1+\tilde{\gamma}^2}\sin(\pi u/2)}{\sqrt{(1+\tilde{\gamma}^2/2)[1+\cos(\pi u)]-i\sqrt{1+\tilde{\gamma}^2}\sin(\pi u)}}\right]+\arctan\left(\frac{1}{|\tilde{\gamma}|}\right)\right\}^2.
\end{aligned}
\tag{5.98}
$$

[50] Note that because $\tilde{\gamma} = \tilde{\gamma}(\xi, p)$, the function $h(\tilde{\gamma})$ still depends on the momentum p and the spatial coordinate ξ.

This integral cannot be solved in terms of elementary functions but we may obtain the asymptotic behaviour of $h(\tilde{\gamma})$. For small $|\tilde{\gamma}| \ll 1$ we find

$$
h(\tilde{\gamma}) = \frac{1}{|\tilde{\gamma}|}\Re\int_0^1\mathrm{d}u\left[\frac{\pi^2}{\sqrt{1+\exp(i\pi u)}}+\mathcal{O}(|\tilde{\gamma}|)\right] = \frac{1}{|\tilde{\gamma}|}\left[\pi^2+\mathcal{O}(|\tilde{\gamma}|)\right]. \tag{5.99}
$$

On the other hand, in the perturbative regime $|\tilde{\gamma}| \gg 1$ the integrand in (5.98) decays as $4/|\tilde{\gamma}|^3$. To check these results, we calculate the function $h(\tilde{\gamma})$ numerically and find very good agreement; see fig. 5.4. Because $h(\tilde{\gamma})$ is positive for all values of $\tilde{\gamma}$, the next-order contribution (5.97) to the pair production density's exponent in (5.89) is always negative, thereby decreasing the pair production density. Thus, the spatial dependence decreases the number of produced pairs in comparison to the locally homogeneous field approximation. This is qualitatively consistent with the numerical findings in [GK05] where it was found

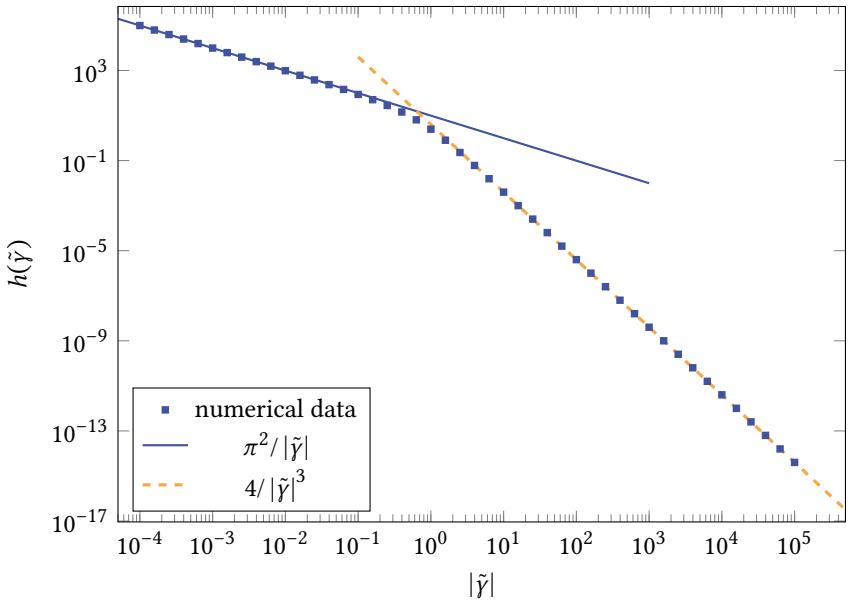

FIGURE 5.4 – Log-log plot of numerically calculated function $h(\tilde{\gamma})$ as given in (5.98). For small $\gamma \ll 1$, $h(\tilde{\gamma})$ approaches $\pi^2/\tilde{\gamma}$, whereas for large $\gamma \gg 1$ it behaves like $4/\tilde{\gamma}^3$.

that the locally constant field approximation overestimates the true pair production probability, at least for the Sauter potential considered.

As a result, we find that, to this order of approximation, the density of produced pairs will be at its maximum where $g'(\omega\xi) = 0$. Therefore, contributions of both minima and maxima of the pulse have to be taken into account when calculating the number of produced pairs (see also [GK05]) because points where $g'(\omega\xi)$ vanishes are saddle points of the spatial integral in (5.89). Still, the exact contribution for each saddle point depends on the value of the prefactor $\Xi(x)$ which we have not computed here.

To obtain our result in (5.97) we assumed that $p = \mathcal{O}(\varepsilon^2)$. However, as the relevant length scale on which mass the mass varies is of order $\mathcal{O}(1/\varepsilon)$, there may be other interesting effects on momentum scales of order $\mathcal{O}(\varepsilon)$. However, this case is much more involved, as R_2 cannot be neglected anymore (see (5.68)). Thus, the coordinates r and x are no longer equivalent and thus keeping r constant is not the same as keeping x constant anymore. Additionally, the parameter λ now is of order $\mathcal{O}(\varepsilon)$ and thus more terms have to be considered to obtain the equations to order $\mathcal{O}(\varepsilon^2)$.

Chapter 6

Spacetime-dependent electric fields

In this chapter we want to outline how the spacetime-dependent WKB method developed in chapter 4 may be applied to the case of a spacetime-dependent electric field. We discuss difficulties and differences to the last chapter and give the relevant equations for α and β in coordinates that are again based on solutions of the eikonal equation.

6.1 Inverse approach to solutions of the eikonal equation

The eikonal equation in the presence of an electromagnetic field,

$$\eta^{\mu\nu}\big(\partial_\mu S_\pm + qA_\mu\big)\big(\partial_\nu S_\pm + qA_\nu\big) = m^2 \tag{6.1}$$

has quite a different structure than the eikonal equation (5.7) with a spacetime-dependent mass. Instead of the scalar potential $m(t,x)$ we now have a vector potential $A_\mu(t,x)$. However, we may still do the same decomposition of the independent solutions of the eikonal equation as in section 5.2,

$$\begin{aligned} S_+ &= R + S, \\ S_- &= R - S, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} R &= \frac{1}{2}(S_+ + S_-), \\ S &= \frac{1}{2}(S_+ - S_-), \end{aligned} \tag{6.2}$$

Inserting these expressions for S_+ and S_- into the eikonal equation (6.1) and taking the sum and the difference of the two equations we obtain

$$\begin{aligned} m^2 &= \eta^{\mu\nu}(\partial_\mu R + qA_\mu)(\partial_\nu R + qA_\nu) + \eta^{\mu\nu}(\partial_\mu S)(\partial_\nu S), \\ 0 &= \eta^{\mu\nu}(\partial_\mu R + qA_\mu)(\partial_\nu S), \end{aligned} \tag{6.3}$$

or in 1+1 dimensions using Cartesian coordinates:

$$m^2 = (\partial_t R + qA_0)^2 - (\partial_x R + qA_1)^2 + (\partial_t S)^2 - (\partial_x S)^2, \tag{6.4a}$$

$$0 = (\partial_t R + qA_0)(\partial_t S) - (\partial_x R + qA_1)(\partial_x S). \tag{6.4b}$$

From the second equation (6.4b) it can be immediately seen that here $S(t,x)$ and $R(t,x)$ are no longer locally orthogonal; compare (5.32b). Therefore, we expect the equations (4.40) for α and β to not simplify considerably when using S and R as coordinates. However, there is another option here.

First, it turns out that R is not related to the physical electromagnetic field as we can always get rid of it by applying a gauge transformation as follows

$$\begin{aligned} qA_\mu &\mapsto qA'_\mu = qA_\mu + \partial_\mu R, \\ S_\pm &\mapsto S'_\pm \;\; = S_\pm - R = \pm S. \end{aligned} \tag{6.5}$$

Thus, the equations (6.4) simplify to

$$(\partial_t S)^2 - (\partial_x S)^2 = m^2 - (qA_0)^2 + (qA_1)^2 = m^2 - qA_\mu qA^\mu, \tag{6.6a}$$

$$qA_0 \partial_t S - qA_1 \partial_x S = 0, \tag{6.6b}$$

where we have dropped the primes. Comparing with the relevant equation (5.32a) in the case of the spacetime-dependent mass we see that upon defining the effective mass

$$m_{\text{eff}}^2 = m^2 - qA_\mu qA^\mu \tag{6.7}$$

the equation (6.6a) is equivalent to the case of a spacetime-dependent mass $m_{\text{eff}}(t,x)$ in the mode $p = 0$. Note though that we excluded this case specifically in section 5.2 as R did vanish identically then. Similarly, here R is not present anymore after the gauge transformation.

As before, analytical solutions to (6.6) are hard to come by in general. However, here an inverse approach (see also [OS15]) can be used as we have two equations that can be solved for the two components of A_μ. If an arbitrary function S is then chosen the associated electric field can be calculated. To derive expressions for qA_0 and qA_1 solely in terms of S or its derivatives we proceed as follows: First, we solve (6.6b) for qA_0:

$$qA_0 = \frac{\partial_x S}{\partial_t S} qA_1. \tag{6.8}$$

Then, inserting this into (6.6a) enables us to calculate $(qA_1)^2$, i.e.

$$(qA_1)^2 = (\partial_t S)^2\left[1 - \frac{m^2}{(\partial_t S)^2 - (\partial_x S)^2}\right], \tag{6.9}$$

which has two solutions. This means that a given solution S is associated with two electromagnetic potentials qA_μ that only differ by a sign. Putting it all together, the Cartesian components of the electromagnetic potential for a known solution of the eikonal equation S are given by

$$\begin{aligned} qA_0 &= \pm|\partial_x S|\sqrt{1 - \frac{m^2}{(\partial_t S)^2 - (\partial_x S)^2}}, \\ qA_1 &= \pm|\partial_t S|\sqrt{1 - \frac{m^2}{(\partial_t S)^2 - (\partial_x S)^2}}. \end{aligned} \tag{6.10}$$

Because we are free to choose the sign of these for a given S, we may instead also use

$$\begin{aligned} qA_0 &= \pm(\partial_x S)\sqrt{1 - \frac{m^2}{(\partial_t S)^2 - (\partial_x S)^2}}, \\ qA_1 &= \pm(\partial_t S)\sqrt{1 - \frac{m^2}{(\partial_t S)^2 - (\partial_x S)^2}}, \end{aligned} \tag{6.11}$$

which has the advantage of being differentiable everywhere. By using (6.6a) and the definition (6.7), this can be written even more concise:

$$qA_\mu = \pm\varepsilon_\mu{}^\nu(\partial_\nu S)\sqrt{1 - \left(\frac{m}{m_{\text{eff}}}\right)^2}. \tag{6.12}$$

This explicit expression allows us to generate solutions of the eikonal equation by just *guessing* some function S and simply calculating the associated electromagnetic potential. However, the problem with such a solution is that in general it is rather difficult to guess a solution where one can change the initial momentum without changing the electric field. Thus, instead of getting a set of solutions for a specific electric field that only differ by their initial momentum one gets a set of solutions for different electric fields.

6.2 Choice of coordinates

To write down the equations (4.40) for α and β with a spacetime-dependent electric field we need to calculate the coefficients $\Pi^{\pm}_{\mu}$, $N_{\pm}$ and κ^{μ}:

$$\Pi^{\pm}_{\mu} = \pm\partial_{\mu}S + qA_{\mu}, \quad N_{\pm} = \frac{1}{\sqrt{2m(m - \partial_t S \mp qA_0)}}, \quad \kappa^{\mu} = \frac{1}{m_{\text{eff}}}\varepsilon^{\mu\nu}\partial_{\nu}S, \tag{6.13}$$

where we used (6.12) to derive the expression for κ^{μ}. As in section 5.2, we use $s = S(t,x)/m$ as our new time coordinate. However, having set $R = 0$ by a gauge transformation we cannot use it to define a new coordinate r as before. Thus, we simply fix r by requiring that it be locally orthogonal to s, i.e.

$$(\partial_t r)(\partial_t s) - (\partial_x r)(\partial_x s) = 0. \tag{6.14}$$

Then, the components of the inverse metric tensor are given by

$$\begin{aligned} g^{ss} &= \left(\frac{m_{\text{eff}}}{m}\right)^2, \quad g^{rr} = -\lambda^2\left(\frac{m_{\text{eff}}}{m}\right)^2, \\ g^{rs} &= g^{sr} = 0, \end{aligned} \tag{6.15}$$

where we have used (6.6a), (6.7) and (6.14). Additionally, as in the case of the spacetime-dependent mass, we use

$$\lambda = \frac{\partial_x R}{\partial_t S} = \frac{\partial_x r}{\partial_t s}, \tag{6.16}$$

see (5.34). Similarly we can calculate the components of the Levi-Civita tensor

$$\varepsilon^{ss} = \varepsilon^{rr} = 0, \qquad \varepsilon^{sr} = -\varepsilon^{rs} = \lambda\left(\frac{m_{\text{eff}}}{m}\right)^2. \tag{6.17}$$

Rewriting (6.6b) using these coordinates, we get

$$g^{ss}qA_s m = 0 \tag{6.18}$$

and thus A_s vanishes. Therefore, the components of $\Pi^{\pm}_{\mu}$ are given by

$$\Pi^{\pm}_{s} = \pm m, \quad \Pi^{\pm}_{r} = qA_r. \tag{6.19}$$

Using these, we can show that

$$\begin{aligned}
&\left(g^{\mu\rho}g^{\nu\lambda} - g^{\mu\nu}g^{\rho\lambda}\right)\Pi^{\pm}_{\rho}\Pi^{\pm}_{\lambda}(\nabla_{\mu}\Pi^{\pm}_{\nu}) = \pm m_{\text{eff}}^2 m\partial_s \ln\lambda + \frac{m^4}{qA_r}\partial_r \ln m_{\text{eff}},\\
&\varepsilon^{\lambda\nu}\varepsilon_{\rho}{}^{\mu}\kappa^{\rho}\Pi^{\pm}_{\lambda}(\nabla_{\mu}\Pi^{\pm}_{\nu}) = \mp\frac{1}{\lambda}\frac{m_{\text{eff}}m^2}{qA_r}\partial_s \ln m_{\text{eff}} - \lambda m_{\text{eff}} m\partial_r \ln m_{\text{eff}}.
\end{aligned} \tag{6.20}$$

Finally, we can write down the equations (4.40) for α and β for the case of spacetime-dependent electric field:

$$\begin{aligned}
&\partial_s\alpha - \frac{1}{2}\alpha\partial_s \ln\lambda - \lambda^2\frac{qA_r}{m}\partial_r\alpha - \frac{1}{2}\alpha\frac{m}{qA_r}\left(\frac{m}{m_{\text{eff}}}\right)^2\partial_r \ln m_{\text{eff}}\\
&= -i\frac{m}{m_{\text{eff}}}\lambda e^{-2iS}\left[\partial_r\beta + \frac{1}{2}\beta\partial_r \ln m_{\text{eff}} - \frac{1}{2\lambda^2}\beta\frac{m}{qA_r}\left(\frac{\partial_t s}{\partial_x r}\right)^2\partial_s \ln m_{\text{eff}}\right],\\
&\partial_s\beta - \frac{1}{2}\beta\partial_s \ln\lambda + \lambda^2\frac{qA_r}{m}\partial_r\beta + \frac{1}{2}\beta\frac{m}{qA_r}\left(\frac{m}{m_{\text{eff}}}\right)^2\partial_r \ln m_{\text{eff}}\\
&= i\frac{m}{m_{\text{eff}}}\lambda e^{2iS}\left[\partial_r\alpha - \frac{1}{2}\alpha\partial_r \ln m_{\text{eff}} - \frac{1}{2\lambda^2}\alpha\frac{m}{qA_r}\partial_s \ln m_{\text{eff}}\right].
\end{aligned} \tag{6.21}$$

These equations are far more involved than the ones we encountered in the case of a spacetime-dependent mass; compare (5.51). Nevertheless, it might be possible to employ suitable approximations to obtain results for interesting physical setups; this work is left for future studies.

Chapter 7

Conclusion

Pair creation by strong external fields, such as in the Sauter-Schwinger effect, is an intriguing phenomenon and still far from being well understood. Of particular interest is understanding the influence of the field's spacetime structure on the pair production probability. This is immensely needed for guiding attempts towards an experimental verification of non-perturbative pair creation such as planned at ELI [Kor+11].

A lot of work has been done on time-dependent electromagnetic fields using a range of different methods. In this thesis, we have reviewed the WKB method (see chapter 3) which has the advantage that it can be used to calculate the momentum spectrum of the produced particles. Various field profiles have already been investigated using the WKB method, but a remarkably interesting configuration is the one used in the dynamically-assisted Sauter-Schwinger effect: The combination of two time-dependent pulses with different time scales and field strengths leads to a pair production probability that is exponentially larger than the naive combination of results for each pulse on its own; see section 3.4. The enhancement only sets in above a certain threshold value γ_{crit} for the combined Keldysh parameter. This threshold, however, is qualitatively different depending on the pulse shape of the weaker pulse, even for visually almost identical pulse shapes like a Sauter pulse and a Gaussian. In section 3.6 we presented a perturbative approach to dynamical assistance that can be used to explain these differences. In this approach, the strong field is treated non-perturbatively while the weak field enters as a perturbation via the interaction picture. The pair production amplitude can then be expanded into a perturbation series for the field strength of the weak field relative to the strong one. This expansion reveals the crucial role of the weak field's Fourier transform in explaining qualitative differences between using a Sauter pulse as the weak field versus using a Gaussian as the weak field. Consequently, weak fields that have similar Fourier transforms give similar results for the pair production probability.

We found that dynamical assistance can be enhanced even further

by adding a high-energy photon to the setup; see section 3.5. This is essentially a combination of two enhancement mechanisms that have been known for some time: The dynamically-assisted Sauter-Schwinger effect [SGD08] and the catalysis mechanism of photon-assisted pair creation [DGS09]. The setup consists of a strong, slowly varying field, a weak, fast varying field and a high-energy photon. Experimentally, for example, the strong field could represent the focus of a high-intensity optical laser while the weak field would be realised by a focused X-ray free electron laser (XFEL) beam with an energy of a few keV. The energy of the photon, however, is in the order of the electron mass, i.e. in the MeV regime. The highest pair production probability is obtained when the photon is incident perpendicular to the electric field and when the produced electron and positron have equal momentum. As a result, we found that the pair production exponent is always (i.e. for any choice of the combined Keldysh parameter γ) larger than in setups where any of the three ingredients is missing. This might be a bit surprising at first as one could expect the effect of the weak field to be negligible in the presence of the photon with much higher frequency. The explanation for this is that, although the oscillations in the photon, electron and positron wavefunctions perpendicular to the electric are too fast to feel the variation of the weak field, the longitudinal oscillations are much slower and thus are influenced by the weak field. Although an interesting setup, the experimental realisation is extremely challenging as each ingredient on its own (high-intensity optical laser, XFEL and gamma ray photon) already requires a massive amount of resources and experimental expertise.

In chapter 4 we proposed a new method for calculating pair production probabilities in general spacetime-dependent fields. The method has been developed to resemble the time-dependent WKB method from chapter 3. It is based on the relativistic eikonal (or Hamilton-Jacobi) equation (4.3) and makes use of a similar decomposition of the spinor into positive-energy solutions and negative-energy solutions. For time-dependent fields, the method reproduces the equations for the Bogoliubov coefficients as in the ordinary WKB method (see section 4.3) and thus is completely equivalent to the latter in that case. For general spacetime-dependent fields the equations for the Bogoliubov coefficients (4.40) are rather convoluted and difficult to solve. However, they probably could be used as the basis for new numerical techniques: Instead of calculating solutions of the (covariant) Dirac equation (4.1) one has to solve the equations (4.40) for the coefficients α and β. For these

one needs solutions of the eikonal equation (4.3) which could also be obtained numerically. This could probably be more efficient than directly solving the Dirac equation numerically as the fast oscillations in the wave function are separated off into the eikonal equation.

Nevertheless, in this thesis we have restricted ourselves to analytic methods. In chapter 5 we used the method developed in chapter 4 to look at pair production due to a spacetime-dependent mass. This is mostly because a scalar potential like the spacetime-dependent mass is considerably simpler to handle than the electromagnetic vector potential. Moreover, as was shown in appendix A and appendix B, a spacetime-dependent mass in a flat 1+1-dimensional spacetime is equivalent to a non-flat 1+1-dimensional spacetime with a constant mass which is able to produce pairs. We demonstrated in section 5.1 that problems occur for truly spacetime-dependent fields due to the nonlinearity of the eikonal equation: Solutions may not be differentiable everywhere because of the crossing of characteristics. For masses where the spatial dependence is much weaker than the temporal dependence these caustics form far away from the region of particle creation and thus should not affect results from our method in that case.

For a general spacetime-dependent mass, the equations (4.40) for the coefficients α and β can be simplified by introducing new coordinates s and r using the solutions of the eikonal equation as coordinate transformations; see section 5.2. In analogy with the time-dependent case, the coordinate s assumes the role of the time coordinate while the orthogonal coordinate r corresponds to the spatial coordinate x. In a low-momentum approximation, where $\lambda \ll 1$, the analogy is even more clear and the final value of the Bogoliubov coefficient β related to pair production may be estimated via an integral over all of the new time coordinate s. Exactly as in the time-dependent case the integral is dominated by the classical turning points where the energy $\sqrt{m^2 + p^2}$ vanishes.

We also looked at a spacetime-dependent mass where the dependence on x was much weaker than the dependence on t; see section 5.3. In this case, we also have $\lambda \ll 1$ and the same steps can be taken. We used an expansion in the relative strength ε of the spatial inhomogeneity. To lowest order we obtained a result that, in analogy to the locally constant field approximation, could be called a locally homogeneous field approximation: For each point x we simply integrate the equation for the Bogoliubov coefficient β over time. For our example, the next-to-leading order contribution always decreases the pair production probability which is consistent with earlier findings for inhomogeneous

electric fields in the Sauter-Schwinger effect (see for example [DS05]). The results for the hyperbolic secant pulse $f(\tau) = \operatorname{sech} \tau$ we looked at were a bit surprising as the lowest-order exponent does not depend on the (Keldysh) parameter γ. This might not be the case for other field profiles. Additionally, it might be interesting to see whether the next-to-leading order correction also decreases the pair production density exponent in these other cases.

We expect that other problems where the background field depends only weakly on one spacetime coordinate may be treated in a similar way; for example, tunnelling through a weakly time-dependent barrier or a pulse depending on x_+ plus a pulse only weakly depending on x_-. Essentially, any problem with a small parameter where the lowest-order problem is highly symmetric and can be solved exactly could be treated in this way. For electromagnetic fields the situation is more involved; see chapter 6. The transformation to new coordinates is more complicated and also the resulting equations for the Bogoliubov coefficients are far more intricate. However, the main strategy is the same as with a spacetime-dependent mass and suitable approximations could lead to further insight into how spacetime-dependent electromagnetic fields influence the pair production probability.

To develop the spacetime WKB method we worked in a 1+1-dimensional spacetime. However, physics in such a low-dimensional spacetime is rather different from the physics in the universe we live in. Nevertheless, we believe that it should be relatively straightforward to extend the spacetime WKB method to work in 3+1 dimensions where the external field only depends on time and a single spatial coordinate, i.e. essentially embedding the method presented in this thesis into a higher-dimensional spacetime. Another possibility, that should be even simpler to realise, is to extend the method to 2+1 dimensions. In that case, we can continue using two-dimensional spinors; see appendix C for a derivation of the coefficients in the equations for α and β in 2+1 dimensions. The 2+1-dimensional setting allows the investigation of transverse electric fields, for example an electric field that depends on t and x but which only has a y-component. This leads to an eikonal equation that is similar to the one for a spacetime-dependent mass in (5.7) in 1+1 dimensions with $m^2(t,x) = m_e^2 + [qA_y(t,x)]^2$. Thus, some techniques used in chapter 5 may be reused in this case. As a first guess, we consequently expect similar behaviour for the exponent whereas the prefactor (and thus the positions of the dominant singularities) may be different.

Appendix A

Metric tensor in 1+1 dimensions

Consider the most general form of a metric in a 1+1 dimensional space-time

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu = a(t,x)\mathrm{d}t^2 - b(t,x)\mathrm{d}x^2 + 2c(t,x)\mathrm{d}t\,\mathrm{d}x\,. \tag{A.1}$$

We first want to show that we can always transform this to a diagonal form. We closely follow [Col77]. We start by going over to coordinates $(\tilde{t}, x)$ with

$$\mathrm{d}\tilde{t} = \tau a\mathrm{d}t + \tau c\mathrm{d}x \tag{A.2}$$

where τ is needed to ensure that the expression on the right-hand side actually is an exact differential. Thus, τ has to be a solution to

$$\partial_x \tau a = \partial_t \tau c. \tag{A.3}$$

Solving (A.2) for $\mathrm{d}t$ yields

$$\mathrm{d}t = \tau^{-1}a^{-1}\mathrm{d}\tilde{t} - a^{-1}c\mathrm{d}x\,. \tag{A.4}$$

Upon inserting this expression into the general form of the metric (A.1) we find the diagonal form

$$\mathrm{d}s^2 = \tau^{-2}a^{-1}\mathrm{d}\tilde{t}^2 - (a^{-1}c^2 - b)\mathrm{d}x^2 = \tilde{a}\mathrm{d}\tilde{t}^2 - \tilde{b}\mathrm{d}x^2 \tag{A.5}$$

where we have used $\tilde{a} = \tau^{-2}a^{-1}$ and $\tilde{b} = a^{-1}c^2 - b$. From now on, we drop the tilde accents:

$$\mathrm{d}s^2 = a\mathrm{d}t^2 - b\mathrm{d}x^2. \tag{A.6}$$

We now want to prove that this can be simplified even further. Consider coordinates (μ, ν) defined by null curves

$$g^{\alpha\beta}(\partial_\alpha\mu)(\partial_\beta\mu) = g^{\alpha\beta}(\partial_\alpha\nu)(\partial_\beta\nu) = 0. \tag{A.7}$$

Using (A.6) these are equivalent to

$$b\dot{\mu}^2 - a\mu'^2 = b\dot{\nu}^2 - a\nu'^2 = 0, \tag{A.8}$$

where as usual $\dot{\mu} = \partial_t \mu$ and $\mu' = \partial_x \mu$ and analogously for ν. Thus,

$$\begin{aligned} \mathrm{d}\mu &= \dot{\mu}\mathrm{d}t + \mu'\mathrm{d}x\,, \\ \mathrm{d}\nu &= \dot{\nu}\mathrm{d}t + \nu'\mathrm{d}x\,. \end{aligned} \tag{A.9}$$

Inverting this relationship we get

$$\begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \end{pmatrix} = \begin{pmatrix} \dot{\mu} & \mu' \\ \dot{\nu} & \nu' \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{d}\mu \\ \mathrm{d}\nu \end{pmatrix} = \frac{1}{\dot{\mu}\nu' - \mu'\dot{\nu}} \begin{pmatrix} \nu' & -\mu' \\ -\dot{\nu} & \dot{\mu} \end{pmatrix} \begin{pmatrix} \mathrm{d}\mu \\ \mathrm{d}\nu \end{pmatrix}. \tag{A.10}$$

Using (A.8) we find the line element

$$\begin{aligned} \mathrm{d}s^2 &= \frac{a}{(\dot{\mu}\nu' - \mu'\dot{\nu})^2}(\nu'\mathrm{d}\mu - \mu'\mathrm{d}\nu)^2 - \frac{b}{(\dot{\mu}\nu' - \mu'\dot{\nu})^2}(-\dot{\nu}\mathrm{d}\mu + \dot{\mu}\mathrm{d}\nu)^2 \\ &= 2\frac{b\dot{\mu}\dot{\nu} - a\mu'\nu'}{(\dot{\mu}\nu' - \mu'\dot{\nu})^2}\mathrm{d}\mu\,\mathrm{d}\nu \\ &= \Phi^2 \mathrm{d}\mu\,\mathrm{d}\nu \end{aligned} \tag{A.11}$$

using the definition $\Phi = \sqrt{2(b\dot{\mu}\dot{\nu} - a\mu'\nu')/(\dot{\mu}\nu' - \mu'\dot{\nu})^2}$. Finally, we employ another set of coordinates (ξ, η) defined as

$$\begin{aligned} \xi &= \frac{\mu + \nu}{2}, \\ \eta &= \frac{\mu - \nu}{2}, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \mu &= \xi + \eta, \\ \nu &= \xi - \eta. \end{aligned} \tag{A.12}$$

Therefore

$$\begin{aligned} \mathrm{d}\mu &= \mathrm{d}\xi + \mathrm{d}\eta\,, \\ \mathrm{d}\nu &= \mathrm{d}\xi - \mathrm{d}\eta \end{aligned} \tag{A.13}$$

and the line element is given by

$$\mathrm{d}s^2 = \Phi^2(\mathrm{d}\xi^2 - \mathrm{d}\eta^2). \tag{A.14}$$

After renaming $\xi \longrightarrow t$ and $\eta \longrightarrow x$ we find

$$\mathrm{d}s^2 = \Phi^2(t, x)(\mathrm{d}t^2 - \mathrm{d}x^2). \tag{A.15}$$

Thus, we have just established that any 1+1 dimensional spacetime is conformally flat, i.e. locally equivalent to the Minkowski spacetime[51].

[51] Note that the Einstein field equations reduce to algebraic equations in 1+1 dimensions and other theories of gravitation have to be used to obtain non-trivial dynamics as in higher dimensions [Col77; MST90; Boo08].

The Christoffel symbols of the second kind

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{A.16}$$

for this metric tensor are

$$\begin{aligned} \Gamma^{t}_{tt} = \Gamma^{t}_{xx} = \Gamma^{x}_{tx} = \Gamma^{x}_{xt} &= \frac{\dot{\Phi}}{\Phi} = \partial_t \ln \Phi, \\ \Gamma^{x}_{xx} = \Gamma^{x}_{tt} = \Gamma^{t}_{tx} = \Gamma^{t}_{xt} &= \frac{\Phi'}{\Phi} = \partial_x \ln \Phi. \end{aligned} \tag{A.17}$$

Appendix B

Dirac equation in 1+1 dimensions

We want to write down the Dirac equation without an electromagnetic potential in a spacetime with a metric as in (A.15). The derivation follows [GSW12, Chapter 12] closely. We start by introducing a vielbein basis on the spacetime's manifold. This means we introduce a set of orthonormal vectors $e^a(\mathrm{x})$, $a = 1, 2$, at each spacetime point x that forms a basis of the tangent space at that point. The components of these basis vectors thus are e^a_μ. We employ the usual convention of Greek indices for components of a vector while the index a only labels the different basis vectors. We may raise the Greek index using the inverse metric tensor, i.e. $e^{a\mu} = g^{\mu\nu} e^a_\nu$, and, because the tangent space is flat, lower the Latin index using the Minkowski metric tensor, i.e. $e_{a\mu} = \eta_{ab} e^b_\mu$. Orthonormality is expressed by $e^a_\mu e^\mu_b = \delta^a_b$ and $e^a_\mu e^\nu_a = \delta^\nu_\mu$. Thus,

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \qquad g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}. \tag{B.1}$$

This means that locally (i.e. in the the neighborhood of each point x where we have defined a vielbein basis $e^a_\mu(\mathrm{x})$) the spacetime can be seen as flat and the global structure of the spacetime is realised using the different vielbeins at different spacetime points. Any tensor may be epxressed in the vielbein basis. For example, the vielbein components of a vector u^μ are $u^a = e^a_\mu u^\mu$.

The choice of the vielbein is not unique. It is only defined up to local Lorentz transformations; the choice $e'^a_\mu = \Lambda^a{}_b e^b_\nu$ would be equally fine. Consequently, a gauge field ω_μ has to be introduced such that our theory is invariant under these local Lorentz transformations. Therefore, the partial derivative ∂_μ has to be replaced by a covariant derivative $\nabla_\mu = \partial_\mu + \omega_\mu$. Applying this to a vector in the vielbein basis is done as follows:

$$\nabla_\mu u^a = \partial_\mu u^a + \omega_\mu{}^a{}_b u^b. \tag{B.2}$$

On the other hand, we already know how to take the covariant derivat-

ive of a vector using the Christoffel symbols:

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda} u^\lambda. \tag{B.3}$$

For these two definitions to be consistent we need

$$\nabla_\mu u^a = e^a_\nu \nabla_\mu v^\nu \tag{B.4}$$

to hold which is equivalent to requiring that

$$0 = \nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_c\, e^c_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda. \tag{B.5}$$

By multiplying by e^ν_b and using the vielbein's orthonormality condition, we are able to find a relationship between our new gauge field (often called the *spin connection*) and the Christoffel symbols:

$$\omega_\mu{}^a{}_b = -e^\nu_b \partial_\mu e^a_\nu + \Gamma^\lambda_{\mu\nu} e^a_\lambda e^\nu_b. \tag{B.6}$$

Using the definition of the Christoffel symbols (A.16) and the relationship between the metric tensor and the vielbein (B.1), we may find an expression for the spin connection solely in terms of the vielbein and its derivatives

$$\omega_\mu{}^{ab} = \frac{1}{2} e^{a\nu}\left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu\right) - \frac{1}{2} e^{b\nu}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right) - \frac{1}{2} e^{a\nu} e^{b\lambda} e^c_\mu \left(\partial_\nu e_{c\lambda} - \partial_\lambda e_{c\nu}\right) \tag{B.7}$$

which is asymmetric under exchange of a and b, hence $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$.

The action of the gauge field ω_μ on spinors is mediated by the generators of the Lorentz group in the spinor representation $\Sigma^{ab} = \left[\gamma^a, \gamma^b\right]/4$ where γ^a are the gamma matrices in a flat spacetime. Thus,

$$\omega_\mu \psi = \frac{1}{2} \omega_\mu{}^{ab} \Sigma_{ab} \psi. \tag{B.8}$$

For the definition of the Dirac equation in a curved spacetime we also need gamma matrices that satisfy the anticommutation relation in the curved spacetime

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{B.9}$$

These may be obtained using the vielbein basis and the gamma matrices in a flat spacetime, i.e.

$$\gamma^\mu = e^\mu_a \gamma^a \tag{B.10}$$

with $\left\{\gamma^a, \gamma^b\right\} = \eta^{ab}$. It is easy to check that these actually satisfy the

anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = e^\mu_a e^\nu_b \left\{\gamma^a, \gamma^b\right\} = 2e^\mu_a e^\nu_b \eta^{ab} = 2g^{\mu\nu}, \tag{B.11}$$

where we have used (B.1) in the last step.

Therefore, we finally find the Dirac equation in a curved spacetime to be

$$\left[ie^\mu_c \gamma^c \left(\partial_\mu + \frac{1}{2}\omega_\mu{}^{ab}\Sigma_{ab}\right) - m\right]\psi = 0. \tag{B.12}$$

In 1+1 dimensions, using the asymmetry of the spin connection this simplifies to

$$\left[ie^\mu_c \gamma^c \left(\partial_\mu + \omega_\mu{}^{01}\Sigma_{01}\right) - m\right]\psi = 0. \tag{B.13}$$

A choice of a vielbein satisfying (B.1) for the conformal form of the metric given in (A.15) is

$$\begin{aligned} e^0_t &= \Phi, \quad & e^0_x &= 0, \\ e^1_t &= 0, \quad & e^1_x &= \Phi. \end{aligned} \tag{B.14}$$

For the spacetime indices (labelled using Greek indices before) we use t and x and for the Lorentz indices (labelled using Latin indices before) we use 0 and 1. Raising the spacetime index we have

$$\begin{aligned} e^{0t} &= \frac{1}{\Phi}, \quad & e^{0x} &= 0, \\ e^{1t} &= 0, \quad & e^{1x} &= -\frac{1}{\Phi}. \end{aligned} \tag{B.15}$$

Now lowering the Lorentz index gives

$$\begin{aligned} e^t_0 &= \frac{1}{\Phi}, \quad & e^x_0 &= 0, \\ e^t_1 &= 0, \quad & e^x_1 &= \frac{1}{\Phi}. \end{aligned} \tag{B.16}$$

Using this vielbein and (B.6) we find

$$\begin{aligned} \omega_t{}^{01} &= -\omega_t{}^{10} = -\Gamma^t_{tx} = -\frac{\Phi'}{\Phi} = -\partial_x \ln\Phi, \\ \omega_x{}^{01} &= -\omega_x{}^{10} = -\Gamma^t_{xx} = -\frac{\dot{\Phi}}{\Phi} = -\partial_t \ln\Phi. \end{aligned} \tag{B.17}$$

Moreover, with the help of the anticommutation relation for the gamma matrices in a flat spacetime it is straightforward to prove

$$\Sigma^{01} = \frac{1}{4}\left[\gamma^0, \gamma^1\right] = \frac{1}{2}\gamma^0\gamma^1. \tag{B.18}$$

Inserting all these expressions for our choice of the vielbein into the Dirac equation (B.13) we get

$$\begin{aligned} 0 &= \left[i\gamma^0(\partial_t + \frac{1}{2}\partial_t \ln\Phi) + i\gamma^1(\partial_x + \frac{1}{2}\partial_x \ln\Phi) - m\Phi \right]\psi \\ &= \left[i\gamma^a(\partial_a + \frac{1}{2}\partial_a \ln\Phi) - m\Phi \right]\psi. \end{aligned} \tag{B.19}$$

Upon rescaling $\psi = \tilde{\psi}/\sqrt{\Phi}$ this further simplifies to

$$\left[i\gamma^a\partial_a - m\Phi \right]\tilde{\psi} = 0. \tag{B.20}$$

This has the exact same form as the ordinary free Dirac equation but instead of a constant mass m we have a (possibly) spacetime-dependent mass $m\Phi(t, x)$. Thus, if the spacetime undergoes an expansion the (effective) electron mass increases whereas if the spacetime is contracted, the mass decreases. In conclusion, instead of working in a curved spacetime with a constant electron mass we may also choose to keep the spacetime flat but then have to use a spacetime-dependent mass.

Appendix C

Spacetime-dependent WKB method in 2+1 dimensions

We want to formulate the spacetime-dependent WKB method developed in chapter 4 for spacetimes of 2+1 dimensions. In that case, we still can use 2×2 matrices for the gamma matrices, for example

$$\gamma^0 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^1 = i\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \gamma^2 = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad \text{(C.1)}$$

Therefore, the operators

$$M_\pm = \begin{pmatrix} -\Pi_t^\pm & -i\Pi_x^\pm - \Pi_y^\pm \\ -i\Pi_x^\pm + \Pi_y^\pm & +\Pi_t^\pm \end{pmatrix}. \quad \text{(C.2)}$$

each only have two eigenvectors again, namely:

$$\begin{aligned} u_+ &= N_+ \begin{pmatrix} m - \Pi_t^+ \\ -i\Pi_x^+ + \Pi_y^+ \end{pmatrix}, \quad u_- = N_+ \begin{pmatrix} i\Pi_x^+ + \Pi_y^+ \\ m - \Pi_t^+ \end{pmatrix} \\ v_+ &= N_- \begin{pmatrix} -i\Pi_x^- - \Pi_y^- \\ m + \Pi_t^- \end{pmatrix}, \quad v_- = N_- \begin{pmatrix} m + \Pi_t^- \\ i\Pi_x^- - \Pi_y^- \end{pmatrix} \end{aligned} \quad \text{(C.3)}$$

with the normalisation constants

$$N_\pm = \frac{1}{\sqrt{2m(m \mp \Pi_t^\pm)}}. \quad \text{(C.4)}$$

We can use the same equations for α and β as in (4.26) which we repeat here:

$$\begin{aligned} (\partial_\mu \alpha)\, \overline{u}_+ \gamma^\mu u_+ + \alpha\, \overline{u}_+ \gamma^\mu \partial_\mu u_+ &= -\overline{u}_+ \gamma^\mu \partial_\mu (\beta v_+)\ e^{-i(S_+ - S_-)}, \\ (\partial_\mu \beta)\, \overline{v}_+ \gamma^\mu v_+ + \beta\, \overline{v}_+ \gamma^\mu \partial_\mu v_+ &= -\overline{v}_+ \gamma^\mu \partial_\mu (\alpha u_+)\ e^{i(S_+ - S_-)}. \end{aligned} \quad \text{(C.5)}$$

We now need to calculate the various inner products involving u_+ and v_+ and their Dirac adjoints. First, using the form of these given in (C.3),

we can explicitly calculate that

$$\begin{aligned}\overline{u}_+\gamma^\mu\gamma^\nu u_+ &= \eta^{\mu\nu} + \frac{i}{m}\varepsilon^{\mu\nu\lambda}\Pi^+_\lambda,\\ \overline{v}_+\gamma^\mu\gamma^\nu v_+ &= -\eta^{\mu\nu} - \frac{i}{m}\varepsilon^{\mu\nu\lambda}\Pi^-_\lambda.\end{aligned} \tag{C.6}$$

Second, after quite a lengthy calculation, the following expressions can be found:

$$\begin{aligned}\overline{u}_+\partial_\mu u_+ &= \frac{i}{2m(m-\Pi^+_t)}\varepsilon^{kj}\Pi^+_j\partial_\mu\Pi^+_k,\\ \overline{v}_+\partial_\mu v_+ &= \frac{i}{2m(m+\Pi^-_t)}\varepsilon^{kj}\Pi^-_j\partial_\mu\Pi^-_k.\end{aligned} \tag{C.7}$$

Inserting these in the expressions from (4.22) and (4.23), we get

$$\begin{aligned}\overline{u}_+\gamma^\mu\partial_\mu u_+ =\ & \frac{1}{2m^3}\left(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\rho\lambda}\right)\Pi^+_\rho\Pi^+_\lambda\partial_\mu\Pi^+_\nu\\ & -\frac{i}{2m^2}\left(\varepsilon^{\mu\nu\lambda} + \frac{1}{m-\Pi^+_t}\varepsilon^{0\nu\lambda}\eta^{\mu\rho}\Pi^+_\rho\right)\Pi^+_\lambda\partial_\mu\Pi^+_\nu,\\ \overline{v}_+\gamma^\mu\partial_\mu v_+ =& -\frac{1}{2m^3}\left(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\rho\lambda}\right)\Pi^-_\rho\Pi^-_\lambda\partial_\mu\Pi^-_\nu\\ & +\frac{i}{2m^2}\left(\varepsilon^{\mu\nu\lambda} - \frac{1}{m+\Pi^-_t}\varepsilon^{0\nu\lambda}\eta^{\mu\rho}\Pi^-_\rho\right)\Pi^-_\lambda\partial_\mu\Pi^-_\nu.\end{aligned} \tag{C.8}$$

Furthermore, with the definition for κ^μ as used before[52]

$$\overline{u}_+\gamma^\mu v_+ = i\kappa^\mu, \quad \overline{v}_+\gamma^\mu u_+ = -i(\kappa^\mu)^*, \tag{C.9}$$

we find

$$\kappa^\mu = N_+N_-\begin{pmatrix}(\Pi^+_x - i\Pi^+_y)(m+\Pi^-_t) - (\Pi^-_x - i\Pi^-_y)(m-\Pi^+_t)\\ (m-\Pi^+_t)(m+\Pi^-_t) - (\Pi^+_x - i\Pi^+_y)(\Pi^-_x - i\Pi^-_y)\\ -i(m-\Pi^+_t)(m+\Pi^-_t) - i(\Pi^+_x - i\Pi^+_y)(\Pi^-_x - i\Pi^-_y)\end{pmatrix}. \tag{C.10}$$

Using $u_- = \gamma^0 C u_+^*$ and $v_- = \gamma^0 C v_+^*$ where C is the charge conjugation operator, we can show that

$$\overline{u}_+\gamma^\mu v_- = \overline{v}_+\gamma^\mu u_- = \zeta^\mu, \tag{C.11}$$

where

$$\zeta^\mu = N_+N_-\begin{pmatrix}(m-\Pi^+_t)(m+\Pi^-_t) - (\Pi^+_x - i\Pi^+_y)(\Pi^-_x + i\Pi^-_y)\\ (\Pi^+_x - i\Pi^+_y)(m+\Pi^-_t) - (\Pi^-_x + i\Pi^-_y)(m-\Pi^+_t)\\ i(\Pi^+_x - i\Pi^+_y)(m+\Pi^-_t) + i(\Pi^-_x + i\Pi^-_y)(m-\Pi^+_t)\end{pmatrix}. \tag{C.12}$$

[52] Note that here the components of κ^μ are not real but complex and thus $(\kappa^\mu)^* = \kappa^\mu$ does not hold anymore.

We expand $\partial_\mu u_+$ using the basis defined by u_+ and u_-,

$$\partial_\mu u_+ = (\overline{u}_+ \partial_\mu u_+) u_+ - (\overline{u}_- \partial_\mu u_+) u_-, \tag{C.13}$$

and similarly we expand $\partial_\mu v_+$ using the basis defined by v_+ and v_-,

$$\partial_\mu v_+ = (\overline{v}_+ \partial_\mu v_+) v_+ - (\overline{v}_- \partial_\mu v_+) v_-. \tag{C.14}$$

Then,

$$\begin{aligned} \overline{u}_+ \gamma^\mu \partial_\mu v_+ &= (\overline{v}_+ \partial_\mu v_+) \overline{u}_+ \gamma^\mu v_+ - (\overline{v}_- \partial_\mu v_+) \overline{u}_+ \gamma^\mu v_-, \\ \overline{v}_+ \gamma^\mu \partial_\mu u_+ &= (\overline{u}_+ \partial_\mu u_+) \overline{v}_+ \gamma^\mu u_+ - (\overline{u}_- \partial_\mu u_+) \overline{v}_+ \gamma^\mu u_-. \end{aligned} \tag{C.15}$$

Using (C.9) and (C.11), we get

$$\begin{aligned} \overline{u}_+ \gamma^\mu \partial_\mu v_+ &= i(\overline{v}_+ \partial_\mu v_+) \kappa^\mu - (\overline{v}_- \partial_\mu v_+) \zeta^\mu, \\ \overline{v}_+ \gamma^\mu \partial_\mu u_+ &= -i(\overline{u}_+ \partial_\mu u_+) (\kappa^\mu)^* - (\overline{u}_- \partial_\mu u_+) \zeta^\mu. \end{aligned} \tag{C.16}$$

From the eigenvalue equations for $u_\pm$ and $v_\pm$ we can derive that

$$\overline{u}_- \partial_\mu u_+ = -\frac{1}{2m} \left(\partial_\mu \Pi_\nu^+\right) \overline{u}_- \gamma^\nu u_+, \quad \overline{v}_- \partial_\mu v_+ = -\frac{1}{2m} \left(\partial_\mu \Pi_\nu^+\right) \overline{v}_- \gamma^\nu v_+. \tag{C.17}$$

In the representation used here, we find

$$\overline{u}_- \gamma^\nu u_+ = \varepsilon^{\nu\lambda\sigma} \frac{\Pi_\lambda^+}{m} \varrho_\sigma^+, \quad \overline{v}_- \gamma^\nu v_+ = \varepsilon^{\nu\lambda\sigma} \frac{\Pi_\lambda^-}{m} \varrho_\sigma^- \tag{C.18}$$

where

$$\varrho_\mu^\pm = \frac{1}{m \mp \Pi_t^\pm} \begin{pmatrix} \Pi_x^\pm \pm i \Pi_y^\pm \\ \mp (m \mp \Pi_t^\pm) \\ -i(m \mp \Pi_t^\pm) \end{pmatrix}. \tag{C.19}$$

Therefore,

$$\begin{aligned} \overline{u}_- \partial_\mu u_+ &= \frac{1}{2m^2} \varepsilon^{\sigma\lambda\nu} \varrho_\sigma^+ \Pi_\lambda^+ \partial_\mu \Pi_\nu^+, \\ \overline{v}_- \partial_\mu v_+ &= \frac{1}{2m^2} \varepsilon^{\sigma\lambda\nu} \varrho_\sigma^- \Pi_\lambda^- \partial_\mu \Pi_\nu^- \end{aligned} \tag{C.20}$$

Now we have expressions for all the coefficients in (C.5): Inserting the expressions from (C.8), (C.16), (C.9), (4.18) and (4.19) into (C.5) yields the equations for α and β in 2+1 dimensions similar to the ones given in (4.40) for 1+1 dimensions. However, the expressions here are more involved than in 1+1 dimensions. Suitable approximations should be applied to reduce the complexity of the equations for physically interesting problems.

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