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Derivation of a refined six-parameter shell model: descent from the three-dimensional Cosserat elasticity using a method of classical shell theory

Mircea Bîrsan

Fakultät für Mathematik, Universität Duisburg-Essen, Essen, Germany;
Department of Mathematics, Alexandru Ioan Cuza University of Iaşi, Iaşi, Romania

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Abstract

Starting from the three-dimensional Cosserat elasticity, we derive a two-dimensional model for isotropic elastic shells. For the dimensional reduction, we employ a derivation method similar to that used in classical shell theory, as presented systematically by Steigmann (Koiter's shell theory from the perspective of three-dimensional nonlinear elasticity. *J Elast* 2013; 111: 91–107). As a result, we obtain a geometrically nonlinear Cosserat shell model with a specific form of the strain energy density, which has a simple expression, with coefficients depending on the initial curvature tensor and on three-dimensional material constants. The explicit forms of the stress–strain relations and the local equilibrium equations are also recorded. Finally, we compare our results with other six-parameter shell models and discuss the relation to the classical Koiter shell model.

Keywords

Shell theory, six-parameter shells, elastic Cosserat material, strain energy density, curvature

1. Introduction

Elastic shell theory is an important branch of the mechanics of deformable bodies, in view of its applications in engineering. It is also a current domain of active research, because scientists are looking for new shell models, with better properties. This task is not easy, since the shell model should be simple enough, on the one hand, to be manageable in practical engineering problems but, on the other hand, it should be complex enough to account for relevant curvature and three-dimensional effects.

The classical shell theory, also called the first-order approximation theory, presents relatively simple shell models (e.g., the well-known Koiter shell model), but it is not applicable to all shell problems. The classical approach can be employed only if the Kirchhoff–Love hypotheses are satisfied; moreover, one can observe the effect of accuracy loss in classical shell theory for certain problems (see, e.g., Berdichevsky and Misyura [1]). Therefore, more refined shell theories are needed.

One of the most general theories of shells, which has been much developed in the last decades, is the so-called six-parameter shell theory. This approach has been initially proposed by Reissner [2]. The theory of six-parameter shells, presented in the books of Libai and Simmonds [3] and Chróścielewski et al. [4], involves

Corresponding author:
Mircea Bîrsan, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany.
Email: mircea.birsan@uni-due.de

two independent kinematic fields: the translation vector (three degrees of freedom) and the rotation tensor (three additional degrees of freedom). Some of the achievements of this general shell theory have been presented in Chróścielewski et al. [5], Eremeyev and Pietraszkiewicz [6] and Pietraszkiewicz [7]. We mention that the kinematic structure of six-parameter shells is identical to the kinematic structure of Cosserat shells, which are regarded as deformable surfaces with a triad of rigid directors describing the orientation of material points. Thus, the rotation tensor in the six-parameter model accounts for the orientation change of the triad of directors. General results concerning the existence of minimizers in the six-parameter shell theory have been presented in Bîrsan and Neff [8].

To be useful in practice, the shell model should present a concrete (specific) form of the constitutive relations and strain energy density. The specific form should satisfy these two requirements: the coefficients of the strain energy density should be determined in terms of the three-dimensional material constants and they should depend on the (initial) curvature tensor $\boldsymbol{b}$ of the reference configuration. In the literature of six-parameter shells, we were not able to find a satisfactory strain energy density for isotropic shells: the available specific forms are either too simple (in the sense that the coefficients are constant, i.e., independent of the initial curvature $\boldsymbol{b}$), or they are general functions of the strain measures, which coefficients are not identified in terms of three-dimensional material constants.

Our present work aims to fill this gap and establishes a specific form for the strain energy density of isotropic six-parameter (Cosserat) elastic shells, together with explicit stress–strain relations, which fulfill the aforementioned requirements. In this model, we retain the terms up to the order $O(h^3)$ with respect to the shell thickness h and derive a relatively simple expression of the strain energy density, which can be used in applications. To obtain the two-dimensional strain energy density (i.e., written as a function of (x_1, x_2), the surface curvilinear coordinates), we descend from a Cosserat three-dimensional elastic model and apply the derivation method from the classical theory of shells, which was systematically presented by Steigmann [9–11]. Thus, in Section 2 we introduce the three-dimensional Cosserat continuum in curvilinear coordinates, with the appropriate strain measures (equations (2) and (3)), equilibrium equations (equation (4)) and constitutive relations (equations (5) to (8)). In Section 3, we describe briefly the geometry of surfaces and the kinematics of six-parameter shells, and define the shell strain tensor and bending curvature tensor (equation (35)).

In the main part of this paper, Section 4, we derive the two-dimensional shell model by performing the integration over the thickness and using the aforementioned derivation method [11], inspired by the classical shell theory. Here, we adopt some assumptions that are common in the shell approaches (such as, for instance, that the stress vectors on the major faces of the shells are of order $O(h^3)$) and are able to neglect some higher-order terms to obtain a simplified form of the strain energy density (equation (68)). For the sake of completeness, we also present the equilibrium equations for six-parameter (Cosserat) shells (equation (90)), which we deduce from the condition that the equilibrium state is a stationary point of the energy functional.

Section 5 is devoted to further remarks and comments on the derived Cosserat shell model. We introduce the fourth-order tensor of elastic moduli for shells (equations (96) and (100)) and present the explicit form of the stress–strain relations (equation (107)). To compare our results with other six-parameter shell models, we write the strain energy density in an alternative useful form (equation (113)). We pay special attention to the comparison with the Cosserat shell model of order $O(h^5)$, which has been presented recently in Bîrsan et al. [12]. Although the derivation methods are different, we obtain the same form of the strain energy density, except for the coefficients of the transverse shear energy, which are unequal. The value of the transverse shear coefficient derived in the present work is confirmed by the results obtained previously through Γ-convergence in Neff et al. [13] for the case of plates.

Finally, we discuss in Subsection 5.3 the relation between our six-parameter shell model and the classical Koiter model. We show that, if we adopt appropriate restrictions (the material is a Cauchy continuum and the Kirchhoff–Love hypotheses are satisfied), we are able to reduce the form of our strain energy density to obtain the classical Koiter energy, see equation (134).

1.1. Notation

Let us present next some useful notation, which will be used throughout this paper. The Latin indices $i, j, k, \ldots$ range over the set $\{1, 2, 3\}$, while the Greek indices $\alpha, \beta, \gamma, \ldots$ range over the set $\{1, 2\}$. The Einstein summation convention over repeated indices is used. A subscript comma preceding an index i (or α) designates partial differentiation with respect to the variable x_i (or x_α, respectively), e.g., $f_{,i} = \partial f / \partial x_i$. We denote by δ_i^j the Kronecker symbol, i.e., $\delta_i^j = 1$ for $i = j$, while $\delta_i^j = 0$ for $i \neq j$.

We employ the direct tensor notation. Thus, $\otimes$ designates the dyadic product, $\mathbb{1}_3 = \boldsymbol{g}_i \otimes \boldsymbol{g}^i$ is the unit second-order tensor in the 3-space, and $\operatorname{axl}(\boldsymbol{W})$ stands for the axial vector of any skew-symmetric tensor $\boldsymbol{W}$.

Let $\operatorname{tr}(\boldsymbol{X})$ denote the trace of any second-order tensor $\boldsymbol{X}$. The symmetric part, skew-symmetric part and deviatoric part of $\boldsymbol{X}$ are defined by

$$\operatorname{sym}\boldsymbol{X} = \frac{1}{2}\big(\boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}}\big), \qquad \operatorname{skew}\boldsymbol{X} = \frac{1}{2}\big(\boldsymbol{X} - \boldsymbol{X}^{\mathrm{T}}\big), \qquad \operatorname{dev}_3\boldsymbol{X} = \boldsymbol{X} - \frac{1}{3}\left(\operatorname{tr}\boldsymbol{X}\right)\mathbb{1}_3 .$$

The scalar product between any second-order tensors $\boldsymbol{A} = A^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = A_{ij}\,\boldsymbol{g}^i \otimes \boldsymbol{g}^j$ and $\boldsymbol{B} = B^{kl}\boldsymbol{g}_k \otimes \boldsymbol{g}_l = B_{kl}\,\boldsymbol{g}^k \otimes \boldsymbol{g}^l$ is denoted by

$$\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}\big(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\big) = A^{ij}B_{ij} = A_{kl}B^{kl} .$$

If $\underline{\boldsymbol{C}} = C^{ijkl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l$ is a fourth-order tensor, then we use the corresponding notation

$$\underline{\boldsymbol{C}} : \boldsymbol{B} = C^{ijkl}B_{kl}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j\,, \qquad \boldsymbol{A} : \underline{\boldsymbol{C}} = C^{ijkl}A_{ij}\,\boldsymbol{g}_k \otimes \boldsymbol{g}_l\,, \qquad \boldsymbol{A} : \underline{\boldsymbol{C}} : \boldsymbol{B} = C^{ijkl}A_{ij}\,B_{kl}\,.$$

For any vector $\boldsymbol{v} = v^i\boldsymbol{g}_i = v_i\,\boldsymbol{g}^i$, we write as usual

$$\boldsymbol{A}\boldsymbol{v} = A^{ij}v_j\,\boldsymbol{g}_i = A_{ij}v^j\,\boldsymbol{g}^i \qquad \text{and} \qquad \boldsymbol{v}\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{v} = A^{ij}v_i\,\boldsymbol{g}_j = A_{ij}v^i\,\boldsymbol{g}^j\,.$$

2. Three-dimensional Cosserat elastic continua

Let us consider a three-dimensional Cosserat body, which occupies the domain $\Omega_\xi \subset \mathbb{R}^3$ in its reference configuration. The deformation is characterized by the vectorial map $\boldsymbol{\varphi}_\xi : \Omega_\xi \to \Omega_c$ (here, $\Omega_c \subset \mathbb{R}^3$ is the deformed configuration) and the microrotation tensor $\boldsymbol{R}_\xi : \Omega_\xi \to \mathrm{SO}(3)$ (the special orthogonal group).

On the reference configuration Ω_ξ, we consider a system of curvilinear coordinates (x_1, x_2, x_3), which are induced by the parametric representation $\boldsymbol{\Theta} : \Omega_h \to \Omega_\xi$ with $(x_1, x_2, x_3) \in \Omega_h$. Using the common notation, we introduce the covariant base vectors $\boldsymbol{g}_i := \partial\boldsymbol{\Theta}/\partial x_i = \boldsymbol{\Theta}_{,i}$ and the contravariant base vectors $\boldsymbol{g}^i$ with $\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j$.

Let

$$\boldsymbol{\varphi} : \Omega_h \to \Omega_c\,, \qquad \boldsymbol{\varphi}(x_1, x_2, x_3) := \boldsymbol{\varphi}_\xi\big(\boldsymbol{\Theta}(x_1, x_2, x_3)\big),$$

be the *deformation function* and

$$\boldsymbol{F}_\xi = \boldsymbol{\varphi}_{,i} \otimes \boldsymbol{g}^i$$

the *deformation gradient*. We refer the domain Ω_h to the orthonormal vector basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, such that $(x_1, x_2, x_3) = x_i\boldsymbol{e}_i$ and $\nabla_x\boldsymbol{\Theta} = \boldsymbol{\Theta}_{,i} \otimes \boldsymbol{e}_i = \boldsymbol{g}_i \otimes \boldsymbol{e}_i$. The microrotation tensor can be represented as

$$\boldsymbol{R}_\xi = \boldsymbol{d}_i \otimes \boldsymbol{d}_i^0\,,$$

where $\{\boldsymbol{d}_1^0, \boldsymbol{d}_2^0, \boldsymbol{d}_3^0\}$ is the orthonormal triad of directors in the reference configuration Ω_ξ and $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ is the orthonormal triad of directors in the deformed configuration Ω_c. We denote by $\boldsymbol{Q}_e$ the *elastic microrotation* given by

$$\boldsymbol{Q}_e : \Omega_h \to \mathrm{SO}(3), \qquad \boldsymbol{Q}_e(x_1, x_2, x_3) := \boldsymbol{R}_\xi\big(\boldsymbol{\Theta}(x_1, x_2, x_3)\big).$$

We choose the initial microrotation tensor $\boldsymbol{Q}_0$, such that

$$\boldsymbol{Q}_0 = \operatorname{polar}(\nabla_x\boldsymbol{\Theta}) \in \mathrm{SO}(3) \qquad \text{and} \qquad \boldsymbol{Q}_0 = \boldsymbol{d}_{\mathrm{i}}^0 \otimes \boldsymbol{e}_{\mathrm{i}}\,. \tag{1}$$

Let

$$\overline{\boldsymbol{E}} := \boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{F}_\xi - \mathbb{1}_3 \tag{2}$$

denote the (non-symmetric) *strain tensor* for nonlinear micropolar media and

$$\boldsymbol{\Gamma} := \operatorname{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{Q}_{e,i}\big) \otimes \boldsymbol{g}^i \tag{3}$$

be the *wryness tensor* (see, e.g., Neff and Münch [14], Pietraszkiewicz and Eremeyev [15] and Bîrsan and Neff [16]), which is a strain measure for curvature (orientation change).

The local equations of equilibrium can be written in the form

$$\operatorname{Div} \boldsymbol{T} + \boldsymbol{f} = \boldsymbol{0}, \qquad \operatorname{Div} \overline{\boldsymbol{M}} - \operatorname{axl}\big(\boldsymbol{F}_\xi \boldsymbol{T}^{\mathrm{T}} - \boldsymbol{T}^{\mathrm{T}} \boldsymbol{F}_\xi\big) + \boldsymbol{c} = \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{T}$ and $\overline{\boldsymbol{M}}$ are the stress tensor and the couple stress tensor (of the first Piola–Kirchhoff type), and $\boldsymbol{f}$ and $\boldsymbol{c}$ are the external body force and couple vectors. To the balance equations (equation (4)), one can adjoin boundary conditions.

Under hyperelasticity assumptions, the stress tensors $\boldsymbol{T}$ and $\overline{\boldsymbol{M}}$ are expressed by the constitutive equations

$$\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{T} = \frac{\partial W}{\partial \overline{\boldsymbol{E}}}\,, \qquad \boldsymbol{Q}_e^{\mathrm{T}} \overline{\boldsymbol{M}} = \frac{\partial W}{\partial \boldsymbol{\Gamma}}\,, \tag{5}$$

where $W = W(\overline{\boldsymbol{E}}, \boldsymbol{\Gamma})$ is the elastically stored energy density. Using the Cosserat model for isotropic materials presented in Bîrsan et al. [12] and Neff et al. [17], we assume the following representation for the energy density:

$$W(\overline{\boldsymbol{E}}, \boldsymbol{\Gamma}) = W_{\mathrm{mp}}(\overline{\boldsymbol{E}}) + W_{\mathrm{curv}}(\boldsymbol{\Gamma}), \tag{6}$$

$$\begin{aligned} W_{\mathrm{mp}}(\overline{\boldsymbol{E}}) &= \mu \,\|\operatorname{dev}_3 \operatorname{sym} \overline{\boldsymbol{E}}\,\|^2 + \mu_c \,\|\operatorname{skew} \overline{\boldsymbol{E}}\,\|^2 + \frac{\kappa}{2}\big(\operatorname{tr} \overline{\boldsymbol{E}}\big)^2 \\ &= \mu \,\|\operatorname{sym} \overline{\boldsymbol{E}}\,\|^2 + \mu_c \,\|\operatorname{skew} \overline{\boldsymbol{E}}\,\|^2 + \frac{\lambda}{2}\big(\operatorname{tr} \overline{\boldsymbol{E}}\big)^2, \end{aligned} \tag{7}$$

$$\begin{aligned} W_{\mathrm{curv}}(\boldsymbol{\Gamma}) &= \mu L_c^2 \Big(b_1 \,\|\operatorname{dev}_3 \operatorname{sym} \boldsymbol{\Gamma}\|^2 + b_2 \,\|\operatorname{skew} \boldsymbol{\Gamma}\|^2 + b_3 \big(\operatorname{tr} \boldsymbol{\Gamma}\big)^2 \Big) \\ &= \mu L_c^2 \left(b_1 \,\|\operatorname{sym} \boldsymbol{\Gamma}\|^2 + b_2 \,\|\operatorname{skew} \boldsymbol{\Gamma}\|^2 + \left(b_3 - \frac{b_1}{3}\right) \big(\operatorname{tr} \boldsymbol{\Gamma}\big)^2 \right), \end{aligned} \tag{8}$$

where $\mu > 0$ is the shear modulus, λ the Lamé constant, $\kappa = (3\lambda + 2\mu)/3$ is the bulk modulus of classical isotropic elasticity, $\mu_c \geq 0$ is the so-called *Cosserat couple modulus*, $b_1, b_2, b_3 > 0$ are dimensionless constitutive coefficients and the parameter $L_c > 0$ introduces an internal length, which is characteristic of the material.

We remark that the model is geometrically nonlinear (since the strain measures $\overline{\boldsymbol{E}}$, $\boldsymbol{\Gamma}$ are nonlinear functions of $\boldsymbol{\varphi}, \boldsymbol{Q}_e$), but it is physically linear in view of equations (5) to (8). Thus, let us denote by

$$\underline{\boldsymbol{C}} = C^{ijkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l \qquad \text{and} \qquad \underline{\boldsymbol{G}} = G^{ijkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l$$

the fourth-order tensors of the elastic moduli, such that

$$\begin{aligned} \boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{T} &= \underline{\boldsymbol{C}} : \overline{\boldsymbol{E}} = 2\mu \operatorname{dev}_3 \operatorname{sym} \overline{\boldsymbol{E}} + 2\mu_c \operatorname{skew} \overline{\boldsymbol{E}} + \kappa (\operatorname{tr} \overline{\boldsymbol{E}}) \mathbb{1}_3 = 2\mu \operatorname{sym} \overline{\boldsymbol{E}} + 2\mu_c \operatorname{skew} \overline{\boldsymbol{E}} + \lambda (\operatorname{tr} \overline{\boldsymbol{E}}) \mathbb{1}_3\,, \\ \boldsymbol{Q}_e^{\mathrm{T}} \overline{\boldsymbol{M}} &= \underline{\boldsymbol{G}} : \boldsymbol{\Gamma} = 2\mu L_c^2 \Big(b_1 \operatorname{dev}_3 \operatorname{sym} \boldsymbol{\Gamma} + b_2 \operatorname{skew} \boldsymbol{\Gamma} + b_3 \big(\operatorname{tr} \boldsymbol{\Gamma}\big) \mathbb{1}_3 \Big). \end{aligned} \tag{9}$$

By virtue of equation (9), we see that the tensor components are

$$\begin{aligned} C^{ijkl} &= \mu \left(g^{ik} g^{jl} + g^{il} g^{jk}\right) + \mu_c \big(g^{ik} g^{jl} - g^{il} g^{jk}\big) + \lambda\, g^{ij} g^{kl}\,, \\ G^{ijkl} &= \mu L_c^2 \left(b_1 \big(g^{ik} g^{jl} + g^{il} g^{jk}\big) + b_2 \big(g^{ik} g^{jl} - g^{il} g^{jk}\big) + 2\left(b_3 - \frac{b_1}{3}\right) g^{ij} g^{kl} \right), \end{aligned} \tag{10}$$

which satisfy the major symmetries $C^{ijkl} = C^{klij}$, $G^{ijkl} = G^{klij}$. Hence, we have

$$W_{\mathrm{mp}}(\overline{\boldsymbol{E}}) = \frac{1}{2}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{T}\big) : \overline{\boldsymbol{E}} = \frac{1}{2}\, \overline{\boldsymbol{E}} : \underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}\,, \qquad W_{\mathrm{curv}}(\boldsymbol{\Gamma}) = \frac{1}{2}\big(\boldsymbol{Q}_e^{\mathrm{T}} \overline{\boldsymbol{M}}\big) : \boldsymbol{\Gamma} = \frac{1}{2}\, \boldsymbol{\Gamma} : \underline{\boldsymbol{G}} : \boldsymbol{\Gamma}\,. \tag{11}$$

Under these assumptions, the deformation function $\boldsymbol{\varphi}$ and microrotation tensor $\boldsymbol{Q}_e$ are the solution of the following minimization problem:

$$I = \int_{\Omega_\xi} W\big(\overline{\boldsymbol{E}}, \boldsymbol{\Gamma}\big)\, \mathrm{d}V \quad \to \quad \min \ \text{w.r.t.}\ (\boldsymbol{\varphi}, \boldsymbol{Q}_e)\,. \tag{12}$$

For the sake of simplicity, we assume here that no external body and surface loads are present. The existence of minimizers to this energy functional has been proved by the direct methods of the calculus of variations (see, e.g., Neff et al. [17] and Neff [18]).

3. Geometry and kinematics of three-dimensional Cosserat shells

For a shell-like three-dimensional Cosserat body, the parametric representation $\boldsymbol{\Theta}$ has the special form (see, e.g., Libai and Simmonds [3], Chróścielewski et al. [4] and Ciarlet [19])

$$\boldsymbol{\Theta}(\boldsymbol{x}) = \boldsymbol{y}_0(x_1, x_2) + x_3\, \boldsymbol{n}_0(x_1, x_2), \tag{13}$$

where

$$\boldsymbol{n}_0 = \frac{\boldsymbol{y}_{0,1} \times \boldsymbol{y}_{0,2}}{\|\boldsymbol{y}_{0,1} \times \boldsymbol{y}_{0,2}\|}$$

is the unit normal vector to the surface ω_ξ , defined by the position vector $\boldsymbol{y}_0(x_1, x_2)$. The parameter domain Ω_h has the special form

$$\Omega_h = \left\{ (x_1, x_2, x_3) \,\middle|\, (x_1, x_2) \in \omega \subset \mathbb{R}^2, \;\; x_3 \in \left(-\frac{h}{2}, \frac{h}{2} \right) \right\},$$

where h is the thickness. Thus, (x_1, x_2) are curvilinear coordinates on the midsurface $\omega_\xi = \boldsymbol{y}_0(\omega)$ and x_3 is the coordinate through the thickness of the shell-like body Ω_ξ .

We denote the covariant and contravariant base vectors in the tangent plane of ω_ξ, as usual, by

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{y}_0}{\partial x_\alpha} = \boldsymbol{y}_{0,\alpha}\,, \qquad \boldsymbol{a}^\beta \cdot \boldsymbol{a}_\alpha = \delta_\alpha^\beta \quad (\alpha, \beta = 1, 2) \qquad \text{and set} \qquad \boldsymbol{a}_3 = \boldsymbol{a}^3 = \boldsymbol{n}_0\,.$$

The surface gradient and surface divergence are then defined by

$$\mathrm{Grad}_s \boldsymbol{f} \; = \; \frac{\partial \boldsymbol{f}}{\partial x_\alpha} \otimes \boldsymbol{a}^\alpha = \boldsymbol{f}_{,\alpha} \otimes \boldsymbol{a}^\alpha\,, \qquad \mathrm{Div}_s\, \boldsymbol{T} \; = \; \boldsymbol{T}_{,\alpha}\, \boldsymbol{a}^\alpha\,.$$

We introduce the first and second fundamental tensors of the surface ω_ξ by

$$\begin{aligned} \boldsymbol{a} &:= \mathrm{Grad}_s\, \boldsymbol{y}_0 = \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\alpha = a_{\alpha\beta} \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta = a^{\alpha\beta} \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta, \\ \boldsymbol{b} &:= -\mathrm{Grad}_s\, \boldsymbol{n}_0 = -\boldsymbol{n}_{0,\alpha} \otimes \boldsymbol{a}^\alpha = b_{\alpha\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta = b^\alpha_\beta\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\beta, \end{aligned} \tag{14}$$

which are symmetric. We shall also need the skew-symmetric tensor $\boldsymbol{c}$, called the alternator tensor in the tangent plane, defined by

$$\boldsymbol{c} := \frac{1}{a}\; \varepsilon_{\alpha\beta}\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta = a\; \varepsilon_{\alpha\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \qquad \text{with} \qquad a := \sqrt{\det\big(a_{\alpha\beta}\big)} \, > 0, \tag{15}$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional alternator ($\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$) and $a(x_1, x_2)$ determines the elemental area of the surface ω_ξ . In view of equations (1) and (13), we can show that (see f. (46) in Bîrsan et al. [12])

$$\boldsymbol{n}_0 = \boldsymbol{d}_3^0 = \boldsymbol{Q}_0 \boldsymbol{e}_3\,. \tag{16}$$

The fundamental tensors satisfy the Cayley–Hamilton type relation,

$$\boldsymbol{b}^2 - 2H\boldsymbol{b} + K\boldsymbol{a} = \boldsymbol{0}, \qquad 2H := \mathrm{tr}\, \boldsymbol{b} = b^\alpha_\alpha\,, \qquad K := \det \boldsymbol{b} = \det\big(b^\alpha_\beta\big), \tag{17}$$

where H and K are the mean curvature and the Gauß curvature of the surface ω_ξ , respectively. We note that $\boldsymbol{a}$ plays the role of the identity tensor in the tangent plane and designate by

$$\boldsymbol{b}^* := -\boldsymbol{b} + 2H\boldsymbol{a} \tag{18}$$

the cofactor of $\boldsymbol{b}$ in the tangent plane, since $\boldsymbol{b}\,\boldsymbol{b}^* = K\boldsymbol{a}$ in view of equation (17). Let us introduce the tensors

$$\boldsymbol{\mu} := \boldsymbol{a} - x_3\, \boldsymbol{b}, \qquad \boldsymbol{\mu}^{-1} := \frac{1}{b}(\boldsymbol{a} - x_3\, \boldsymbol{b}^*), \qquad \text{with} \qquad \boldsymbol{\mu}\, \boldsymbol{\mu}^{-1} = \boldsymbol{\mu}^{-1} \boldsymbol{\mu} = \boldsymbol{a}, \tag{19}$$

where b is the determinant

$$b := \det \boldsymbol{\mu} = 1 - 2H\,x_3 + K\,x_3^2\,. \tag{20}$$

By virtue of $\boldsymbol{g}_i = \boldsymbol{\Theta}_{,i}$ and equations (13) and (19), we find the relations

$$\boldsymbol{g}_\alpha = \boldsymbol{\mu}\,\boldsymbol{a}_\alpha\,, \qquad \boldsymbol{g}^\alpha = \boldsymbol{\mu}^{-1}\boldsymbol{a}^\alpha\,, \qquad \boldsymbol{g}_3 = \boldsymbol{g}^3 = \boldsymbol{n}_0\,, \tag{21}$$

which are well-known in the literature on shells. Hence, we have

$$\boldsymbol{\mu} = \boldsymbol{g}_\alpha \otimes \boldsymbol{a}^\alpha = \boldsymbol{a}^\alpha \otimes \boldsymbol{g}_\alpha\,, \qquad \boldsymbol{\mu}^{-1} = \boldsymbol{g}^\alpha \otimes \boldsymbol{a}_\alpha = \boldsymbol{a}_\alpha \otimes \boldsymbol{g}^\alpha\,. \tag{22}$$

In the derivation of the shell model, we shall employ the expansion of various functions with respect to x_3 about zero. Therefore, we denote the derivative of functions with respect to x_3 with a prime, i.e., $f' := \partial f/\partial x_3$.

We can decompose the deformation gradient as

$$\boldsymbol{F}_\xi = \boldsymbol{F}_\xi\,\mathbb{1}_3 = \boldsymbol{F}_\xi(\boldsymbol{a} + \boldsymbol{n}_0 \otimes \boldsymbol{n}_0) = \boldsymbol{F}_\xi\,\boldsymbol{a} + (\boldsymbol{F}_\xi \boldsymbol{n}_0) \otimes \boldsymbol{n}_0\,, \tag{23}$$

where

$$\boldsymbol{F}_\xi \boldsymbol{n}_0 = (\boldsymbol{\varphi}_{,i} \otimes \boldsymbol{g}^i)\boldsymbol{n}_0 = \boldsymbol{\varphi}_{,3} = \boldsymbol{\varphi}' \qquad \text{and} \tag{24}$$

$$\boldsymbol{F}_\xi\,\boldsymbol{a} = (\mathrm{Grad}_s\,\boldsymbol{\varphi})\boldsymbol{\mu}^{-1}\,. \tag{25}$$

To prove equation (25), we use equations (21) and (22) and write

$$\boldsymbol{F}_\xi\,\boldsymbol{a} = (\boldsymbol{\varphi}_{,i} \otimes \boldsymbol{g}^i)\boldsymbol{a} = \boldsymbol{\varphi}_{,\alpha} \otimes \boldsymbol{g}^\alpha = (\boldsymbol{\varphi}_{,\alpha} \otimes \boldsymbol{a}^\alpha)(\boldsymbol{a}_\beta \otimes \boldsymbol{g}^\beta) = (\mathrm{Grad}_s\,\boldsymbol{\varphi})\boldsymbol{\mu}^{-1}\,.$$

Substituting equations (24) and (25) into equation (23), we get

$$\boldsymbol{F}_\xi = (\mathrm{Grad}_s\,\boldsymbol{\varphi})\,\boldsymbol{\mu}^{-1} + \boldsymbol{\varphi}' \otimes \boldsymbol{n}_0\,. \tag{26}$$

We shall also need the derivatives of $\boldsymbol{F}_\xi$ with respect to x_3 . These are

$$\begin{aligned} \boldsymbol{F}'_\xi &= (\mathrm{Grad}_s\,\boldsymbol{\varphi}')\,\boldsymbol{\mu}^{-1} + (\mathrm{Grad}_s\,\boldsymbol{\varphi})\big(\boldsymbol{\mu}^{-1}\big)' + \boldsymbol{\varphi}'' \otimes \boldsymbol{n}_0\,,\\ \boldsymbol{F}''_\xi &= (\mathrm{Grad}_s\,\boldsymbol{\varphi}'')\,\boldsymbol{\mu}^{-1} + 2(\mathrm{Grad}_s\,\boldsymbol{\varphi}')\big(\boldsymbol{\mu}^{-1}\big)' + (\mathrm{Grad}_s\,\boldsymbol{\varphi})\big(\boldsymbol{\mu}^{-1}\big)'' + \boldsymbol{\varphi}''' \otimes \boldsymbol{n}_0\,. \end{aligned} \tag{27}$$

Differentiating equation (19) with respect to x_3 , we deduce

$$\boldsymbol{\mu}' = -\boldsymbol{b}, \qquad \boldsymbol{\mu}'' = \boldsymbol{0}, \qquad \big(\boldsymbol{\mu}^{-1}\big)' = \boldsymbol{\mu}^{-1}\boldsymbol{b}\,\boldsymbol{\mu}^{-1}, \qquad \big(\boldsymbol{\mu}^{-1}\big)'' = 2\boldsymbol{\mu}^{-1}\boldsymbol{b}\,\boldsymbol{\mu}^{-1}\boldsymbol{b}\,\boldsymbol{\mu}^{-1}. \tag{28}$$

Let us take $x_3 = 0$ in equations (26) to (28). In what follows, we employ the notation $\boldsymbol{f}_0 := \boldsymbol{f}_{|x_3=0}$ for any function $\boldsymbol{f}$. Thus, we have

$$\boldsymbol{\mu}_0 = \boldsymbol{a}, \qquad \big(\boldsymbol{\mu}^{-1}\big)_0 = \boldsymbol{a}, \qquad \big(\boldsymbol{\mu}^{-1}\big)'_0 = \boldsymbol{b}, \qquad \big(\boldsymbol{\mu}^{-1}\big)''_0 = 2\boldsymbol{b}^2 \tag{29}$$

and

$$\begin{aligned} (\boldsymbol{F}_\xi)_0 &= (\mathrm{Grad}_s\,\boldsymbol{\varphi})_0 + \boldsymbol{\varphi}'_0 \otimes \boldsymbol{n}_0\,,\\ (\boldsymbol{F}_\xi)'_0 &= (\mathrm{Grad}_s\,\boldsymbol{\varphi}')_0 + (\mathrm{Grad}_s\,\boldsymbol{\varphi})_0\,\boldsymbol{b} + \boldsymbol{\varphi}''_0 \otimes \boldsymbol{n}_0\,,\\ (\boldsymbol{F}_\xi)''_0 &= (\mathrm{Grad}_s\,\boldsymbol{\varphi}'')_0 + 2(\mathrm{Grad}_s\,\boldsymbol{\varphi}')_0\,\boldsymbol{b} + 2(\mathrm{Grad}_s\,\boldsymbol{\varphi})_0\,\boldsymbol{b}^2 + \boldsymbol{\varphi}'''_0 \otimes \boldsymbol{n}_0\,. \end{aligned} \tag{30}$$

Let us write the Taylor expansion of the deformation function $\boldsymbol{\varphi}(x_1, x_2, x_3)$ with respect to x_3 in the form

$$\boldsymbol{\varphi}(x_1, x_2, x_3) = \boldsymbol{m}(x_1, x_2) + x_3\,\boldsymbol{\alpha}(x_1, x_2) + \frac{x_3^2}{2}\,\boldsymbol{\beta}(x_1, x_2) + \frac{x_3^3}{6}\,\boldsymbol{\gamma}(x_1, x_2) + \cdots\,, \tag{31}$$

where

$$\boldsymbol{m} = \boldsymbol{\varphi}\Big|_{x_3=0} = \boldsymbol{\varphi}_0\,, \qquad \boldsymbol{\alpha} = \boldsymbol{\varphi}'\Big|_{x_3=0} = \boldsymbol{\varphi}'_0\,, \qquad \boldsymbol{\beta} = \boldsymbol{\varphi}''\Big|_{x_3=0} = \boldsymbol{\varphi}''_0\,, \qquad \text{etc.} \tag{32}$$

At the same time, we assume that the microrotation tensor $\boldsymbol{Q}_e$ does not depend on x_3 , i.e.,

$$\boldsymbol{Q}_e(x_i) = \boldsymbol{Q}_e(x_1, x_2). \tag{33}$$

By virtue of equations (30) to (33), we can write the strain tensor $\overline{\boldsymbol{E}} = \boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{F}_\xi - \mathbb{1}_3$ and its derivatives on the midsurface $x_3 = 0$:

$$\begin{aligned}
\overline{\boldsymbol{E}}_0 &= \boldsymbol{Q}_e^{\mathrm{T}} \big(\boldsymbol{F}_\xi\big)_0 - \mathbb{1}_3 = \boldsymbol{Q}_e^{\mathrm{T}} \big(\mathrm{Grad}_s\, \boldsymbol{m} + \boldsymbol{\alpha} \otimes \boldsymbol{n}_0\big) - \mathbb{1}_3\,, \\
\overline{\boldsymbol{E}}_0' &= \boldsymbol{Q}_e^{\mathrm{T}} \big(\boldsymbol{F}_\xi\big)_0' = \boldsymbol{Q}_e^{\mathrm{T}} \big[\mathrm{Grad}_s\, \boldsymbol{\alpha} + \big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b} + \boldsymbol{\beta} \otimes \boldsymbol{n}_0\big]\,, \\
\overline{\boldsymbol{E}}_0'' &= \boldsymbol{Q}_e^{\mathrm{T}} \big(\boldsymbol{F}_\xi\big)_0'' = \boldsymbol{Q}_e^{\mathrm{T}} \big[\mathrm{Grad}_s\, \boldsymbol{\beta} + 2\big(\mathrm{Grad}_s\, \boldsymbol{\alpha}\big)\boldsymbol{b} + 2\big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b}^2 + \boldsymbol{\gamma} \otimes \boldsymbol{n}_0\big]\,.
\end{aligned} \tag{34}$$

We note that the surface ω_ξ (characterized by $x_3 = 0$) is the midsurface of the reference shell Ω_ξ , while $\boldsymbol{m}(x_1, x_2)$ and $\boldsymbol{Q}_e(x_1, x_2)$ represent the deformation vector and microrotation tensor, respectively, for this reference midsurface ω_ξ . Corresponding to $\boldsymbol{m}$ and $\boldsymbol{Q}_e$, we now introduce the *elastic shell strain tensor* $\boldsymbol{E}^e$ and the *elastic shell bending-curvature tensor* $\boldsymbol{K}^e$, which are usually employed in the six-parameter shell theory [3, 4, 8, 20, 21]:

$$\boldsymbol{E}^e := \boldsymbol{Q}_e^{\mathrm{T}} \mathrm{Grad}_s \boldsymbol{m} - \boldsymbol{a}, \qquad \boldsymbol{K}^e := \mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,\alpha}\big) \otimes \boldsymbol{a}^\alpha. \tag{35}$$

These strain measures describe the deformation of the midsurface ω_ξ , see, e.g., Bîrsan and Neff [16, 22]. With the help of equation (35) and the decomposition $\mathbb{1}_3 = \boldsymbol{a} + \boldsymbol{n}_0 \otimes \boldsymbol{n}_0$, we can write equation (34) in the form

$$\overline{\boldsymbol{E}}_0 \;=\; \boldsymbol{E}^e + \big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{\alpha} - \boldsymbol{n}_0\big) \otimes \boldsymbol{n}_0 \;=\; \boldsymbol{E}^e + \boldsymbol{Q}_e^{\mathrm{T}} \big(\boldsymbol{\alpha} - \boldsymbol{d}_3\big) \otimes \boldsymbol{n}_0\,. \tag{36}$$

In the same way, we can compute the wryness tensor $\boldsymbol{\Gamma}$ and its derivatives on the midsurface $x_3 = 0$ in terms of the bending curvature tensor $\boldsymbol{K}^e$. In view of equations (21), (29) and (33), we have

$$\begin{aligned}
\boldsymbol{\Gamma}_0 &= \Big(\mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,i}\big) \otimes \boldsymbol{g}^i\Big)_{x_3=0} \;=\; \mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,\alpha}\big) \otimes \boldsymbol{a}^\alpha = \boldsymbol{K}^e, \\
\boldsymbol{\Gamma}_0' &= \Big(\mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,i}\big) \otimes \boldsymbol{g}^i\Big)'_{x_3=0} \;=\; \mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,\alpha}\big) \otimes \big[(\boldsymbol{\mu}^{-1})_0'\, \boldsymbol{a}^\alpha\big] = \big[\mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,\alpha}\big) \otimes \boldsymbol{a}^\alpha\big]\boldsymbol{b} \;=\boldsymbol{K}^e \boldsymbol{b}, \\
\boldsymbol{\Gamma}_0'' &= \mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,\alpha}\big) \otimes \big[(\boldsymbol{\mu}^{-1})_0''\, \boldsymbol{a}^\alpha\big] = 2\big[\mathrm{axl}\big(\boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{Q}_{e,\alpha}\big) \otimes \boldsymbol{a}^\alpha\big]\boldsymbol{b}^2 \;= 2\boldsymbol{K}^e \boldsymbol{b}^2.
\end{aligned} \tag{37}$$

These expressions will be useful in the following.

4. Derivation of the two-dimensional shell model

To obtain the expression of the elastically stored energy density for the two-dimensional shell model, we shall integrate the strain energy density W over the thickness and then perform some simplifications, suggested by the classical shell theory. Thus, in view of equation (12), the total elastically stored strain energy is

$$I = \int_{\Omega_\xi} W\big(\overline{\boldsymbol{E}}, \boldsymbol{\Gamma}\big)\, \mathrm{d}V = \int_{\omega_\xi} \left(\int_{-h/2}^{h/2} W\big(\overline{\boldsymbol{E}}, \boldsymbol{\Gamma}\big)\, b(x_1, x_2, x_3)\, \mathrm{d}x_3\right) \mathrm{d}a, \tag{38}$$

where $b(x_i)$ is given by equation (20) and $\mathrm{d}a = a(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = \sqrt{\det(a_{\alpha\beta})}\, \mathrm{d}x_1 \mathrm{d}x_2$ is the elemental area of the midsurface ω_ξ .

4.1. Integration over the thickness

With a view towards integrating with respect to x_3 , we expand the integrand from equation (38) in the form

$$Wb = \big(Wb\big)_0 + x_3\, \big(Wb\big)_0' + \frac{1}{2} x_3^2\, \big(Wb\big)_0'' + O(x_3^3)$$

and find

$$\int_{-h/2}^{h/2} Wb\, \mathrm{d}x_3 \;=\; h\, \big(Wb\big)_0 \;+\; \frac{h^3}{24}\, \big(Wb\big)_0'' + o(h^3). \tag{39}$$

By differentiating equation (20), we get $b_0 = 1$, $b_0' = -2H$, $b_0'' = 2K$. Hence, we have

$$\begin{aligned} \big(Wb\big)_0 &= W_0\, b_0 = W_0\,, \\ \big(Wb\big)_0' &= \big(W'b + Wb'\big)_0 = W_0' - 2H\, W_0\,, \\ (Wb)_0'' &= W_0'' - 4H\, W_0' + 2K\, W_0\,. \end{aligned} \tag{40}$$

Inserting equation (40) into equation (39), we obtain the expression

$$\int_{-h/2}^{h/2} Wb\, \mathrm{d}x_3 \;= \Big(h + \frac{h^3}{12}\,K\Big) W_0 + \frac{h^3}{24}\,\big(W_0'' - 4H\, W_0'\big) + o(h^3). \tag{41}$$

According to our constitutive assumptions (equations (6) to (11)), we can write

$$\begin{aligned} W_0 &= W_{\mathrm{mp}}(\overline{\boldsymbol{E}}_0) + W_{\mathrm{curv}}(\boldsymbol{\Gamma}_0) = \frac{1}{2}\,\overline{\boldsymbol{E}}_0 : \underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0 + \frac{1}{2}\,\boldsymbol{\Gamma}_0 : \underline{\boldsymbol{G}} : \boldsymbol{\Gamma}_0 \;= \frac{1}{2}\big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\big) : \overline{\boldsymbol{E}}_0 + \frac{1}{2}\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{\Gamma}_0\,, \\ W_0' &= \overline{\boldsymbol{E}}_0' : \underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0 + \boldsymbol{\Gamma}_0' : \underline{\boldsymbol{G}} : \boldsymbol{\Gamma}_0 \;= \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\big) : \overline{\boldsymbol{E}}_0' + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{\Gamma}_0'\,, \\ W_0'' &= \overline{\boldsymbol{E}}_0'' : \underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0 + \overline{\boldsymbol{E}}_0' : \underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0' + \boldsymbol{\Gamma}_0'' : \underline{\boldsymbol{G}} : \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_0' : \underline{\boldsymbol{G}} : \boldsymbol{\Gamma}_0' \\ &= \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\big) : \overline{\boldsymbol{E}}_0'' + \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0'\big) : \overline{\boldsymbol{E}}_0' + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{\Gamma}_0'' + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0'\big) : \boldsymbol{\Gamma}_0'\,. \end{aligned} \tag{42}$$

If we use equations (34) to (37) in equation (42) and substitute this into equation (41), we deduce the following successive expressions:

$$\begin{aligned} \int_{-h/2}^{h/2} Wb\, \mathrm{d}x_3 = \frac{1}{2}\,\Big(h + \frac{h^3}{12}\,K\Big)\,&\big[\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0 : \big(\boldsymbol{E}^e + \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\alpha} - \boldsymbol{n}_0\big) \otimes \boldsymbol{n}_0\big) + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{K}^e\big] \\ &+ \frac{h^3}{24}\Big\{\boldsymbol{T}_0 : \big[\mathrm{Grad}_s\, \boldsymbol{\beta} + 2\big(\mathrm{Grad}_s\, \boldsymbol{\alpha}\big)\boldsymbol{b} + 2\big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b}^2 + \boldsymbol{\gamma} \otimes \boldsymbol{n}_0\big] \\ &+ \boldsymbol{T}_0' : \big[\mathrm{Grad}_s\, \boldsymbol{\alpha} + \big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b} + \boldsymbol{\beta} \otimes \boldsymbol{n}_0\big] + 2\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \big(\boldsymbol{K}^e\boldsymbol{b}^2\big) + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0'\big) : \big(\boldsymbol{K}^e\boldsymbol{b}\big) \\ &- 4H\, \boldsymbol{T}_0 : \big[\mathrm{Grad}_s\, \boldsymbol{\alpha} + \big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b} + \boldsymbol{\beta} \otimes \boldsymbol{n}_0\big] - 4H\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \big(\boldsymbol{K}^e\boldsymbol{b}\big)\Big\} \;+\; o(h^3) \end{aligned}$$

or, using the decomposition $\boldsymbol{T}_0 = \boldsymbol{T}_0\boldsymbol{a} + \boldsymbol{T}_0\boldsymbol{n}_0 \otimes \boldsymbol{n}_0$,

$$\begin{aligned} \int_{-h/2}^{h/2} Wb\, \mathrm{d}x_3 = \;\frac{1}{2}\,\Big(h + \frac{h^3}{12}\,K\Big)\,&\big[\big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{a}\big) : \boldsymbol{E}^e + \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{n}_0\big) \cdot \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\alpha} - \boldsymbol{n}_0\big) + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{K}^e\big] \\ &+ \frac{h^3}{24}\Big\{\big(\boldsymbol{T}_0\boldsymbol{a}\big) : \big[\mathrm{Grad}_s\, \boldsymbol{\beta} + 2\big(\mathrm{Grad}_s\, \boldsymbol{\alpha}\big)\boldsymbol{b} + 2\big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b}^2\big] + \big(\boldsymbol{T}_0\boldsymbol{n}_0\big) \cdot \boldsymbol{\gamma} \\ &+ \big(\boldsymbol{T}_0'\boldsymbol{a}\big) : \big[\mathrm{Grad}_s\, \boldsymbol{\alpha} + \big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b}\big] + \big(\boldsymbol{T}_0'\boldsymbol{n}_0\big) \cdot \boldsymbol{\beta} + 2\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \big(\boldsymbol{K}^e\boldsymbol{b}^2\big) + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0'\big) : \big(\boldsymbol{K}^e\boldsymbol{b}\big) \\ &- 4H\big(\boldsymbol{T}_0\boldsymbol{a}\big) : \big[\mathrm{Grad}_s\, \boldsymbol{\alpha} + \big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b}\big] - 4H\big(\boldsymbol{T}_0\boldsymbol{n}_0\big) \cdot \boldsymbol{\beta} - 4H\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \big(\boldsymbol{K}^e\boldsymbol{b}\big)\Big\} \;+\; o(h^3). \end{aligned}$$

Making some further calculations using equations (17) and (18), we obtain

$$\begin{aligned} \int_{-h/2}^{h/2} Wb\, \mathrm{d}x_3 = \;\frac{1}{2}\,\Big(h - K\,\frac{h^3}{12}\Big)\,&\big[\big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{a}\big) : \boldsymbol{E}^e + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{K}^e\big] + \frac{1}{2}\,\Big(h + \frac{h^3}{12}\,K\Big)\,\big(\boldsymbol{T}_0\boldsymbol{n}_0\big) \cdot \big(\boldsymbol{\alpha} - \boldsymbol{d}_3\big) \\ &+ \frac{h^3}{24}\Big\{\big(\boldsymbol{T}_0'\boldsymbol{a}\big) : \big[\mathrm{Grad}_s\, \boldsymbol{\alpha} + \big(\mathrm{Grad}_s\, \boldsymbol{m}\big)\boldsymbol{b}\big] + \big(\boldsymbol{T}_0'\boldsymbol{n}_0\big) \cdot \boldsymbol{\beta} + \big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0'\big) : \big(\boldsymbol{K}^e\boldsymbol{b}\big) \\ &+ \big(\boldsymbol{T}_0\boldsymbol{a}\big) : \big[\mathrm{Grad}_s\, \boldsymbol{\beta} - 2\big(\mathrm{Grad}_s\, \boldsymbol{\alpha}\big)\boldsymbol{b}^* - 2K\big(\boldsymbol{Q}_e\boldsymbol{a}\big)\big] + \big(\boldsymbol{T}_0\boldsymbol{n}_0\big) \cdot \big(\boldsymbol{\gamma} - 4H\boldsymbol{\beta}\big)\Big\} \;+\; o(h^3). \end{aligned} \tag{43}$$

4.2. Reduced form of the strain energy density

Equation (43), the expression of the strain energy density per unit area of ω_ξ, can be further reduced, provided we make some assumptions and simplifications that are common in classical shell theory. Thus, let us denote by $\boldsymbol{t}^{\pm}$ the stress vectors on the major faces (upper and lower surfaces) of the shell, given by $x_3 = \pm h/2$. We notice that $\boldsymbol{n}_0$ is orthogonal to the major faces and write

$$\boldsymbol{t}^{+} = \boldsymbol{T}\left(x_\alpha\,,\,\frac{h}{2}\right)\boldsymbol{n}_0 \;=\; \boldsymbol{T}_0\boldsymbol{n}_0 + \frac{h}{2}\,\boldsymbol{T}_0'\boldsymbol{n}_0 + \frac{h^2}{8}\,\boldsymbol{T}_0''\boldsymbol{n}_0 + O(h^3),$$
$$\boldsymbol{t}^{-} = \boldsymbol{T}\left(x_\alpha\,,\,\frac{-h}{2}\right)(-\boldsymbol{n}_0) \;=\; -\boldsymbol{T}_0\boldsymbol{n}_0 + \frac{h}{2}\,\boldsymbol{T}_0'\boldsymbol{n}_0 - \frac{h^2}{8}\,\boldsymbol{T}_0''\boldsymbol{n}_0 + O(h^3),$$

which yields

$$\boldsymbol{t}^{+} + \boldsymbol{t}^{-} \;=\; h\,\boldsymbol{T}_0'\boldsymbol{n}_0 + O(h^3) \qquad \text{and} \qquad \boldsymbol{t}^{+} - \boldsymbol{t}^{-} \;=\; 2\,\boldsymbol{T}_0\boldsymbol{n}_0 + O(h^2). \tag{44}$$

We assume, as in the classical theory, that $\boldsymbol{t}^{\pm}$ are of order $O(h^3)$ and from equation (44) we find

$$\boldsymbol{T}_0\boldsymbol{n}_0 = O(h^2) \qquad \text{and} \qquad \boldsymbol{T}_0'\boldsymbol{n}_0 = O(h^2). \tag{45}$$

On the basis of equation (45), and following the same rational as in the classical shell theory (see, e.g., Steigmann [11]), we shall neglect these quantities and replace

$$\boldsymbol{T}_0\boldsymbol{n}_0 = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{T}_0'\boldsymbol{n}_0 = \boldsymbol{0} \tag{46}$$

in all terms of the energy density (equation (43)). Moreover, we regard relations (46) as two equations for the determination of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in the expansion given by equation (31). Thus, from equations (43) and (46) we obtain

$$\begin{aligned}\int_{-h/2}^{h/2} Wb\,\mathrm{d}x_3 = \frac{1}{2}\left(h - K\,\frac{h^3}{12}\right)&\left[\left(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{a}\right) : \boldsymbol{E}^e + \left(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\right) : \boldsymbol{K}^e\right]\\ &+ \frac{h^3}{24}\Big\{\left(\boldsymbol{T}_0'\boldsymbol{a}\right) : \left[\mathrm{Grad}_s\,\boldsymbol{\alpha} + \left(\mathrm{Grad}_s\,\boldsymbol{m}\right)\boldsymbol{b}\right] + \left(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0'\right) : \left(\boldsymbol{K}^e\boldsymbol{b}\right)\\ &+ \left(\boldsymbol{T}_0\boldsymbol{a}\right) : \left[\mathrm{Grad}_s\,\boldsymbol{\beta} - 2\left(\mathrm{Grad}_s\,\boldsymbol{\alpha}\right)\boldsymbol{b}^* - 2K\left(\boldsymbol{Q}_e\boldsymbol{a}\right)\right]\Big\}.\end{aligned} \tag{47}$$

In view of equations (34) to (36), equation (46) can be written in the form

$$\left[\,\underline{\boldsymbol{C}} : \left(\boldsymbol{E}^e + \left(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\alpha} - \boldsymbol{n}_0\right)\otimes\boldsymbol{n}_0\right)\right]\boldsymbol{n}_0 = \boldsymbol{0}, \tag{48}$$
$$\left[\,\underline{\boldsymbol{C}} : \left(\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\,\boldsymbol{\alpha} + (\boldsymbol{E}^e + \boldsymbol{a})\boldsymbol{b} + \boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\beta}\otimes\boldsymbol{n}_0\right)\right]\boldsymbol{n}_0 = \boldsymbol{0}. \tag{49}$$

Equation (48) can be used to determine the vector $\boldsymbol{\alpha}$: we obtain successively

$$\left[(\mu + \mu_c)\boldsymbol{a} + (\lambda + 2\mu)\,\boldsymbol{n}_0\otimes\boldsymbol{n}_0\right]\left(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\alpha} - \boldsymbol{n}_0\right) = -\left(\underline{\boldsymbol{C}} : \boldsymbol{E}^e\right)\boldsymbol{n}_0\,,$$

or, equivalently,

$$\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\alpha} - \boldsymbol{n}_0 = -\left[\frac{1}{\mu + \mu_c}\,\boldsymbol{a} + \frac{1}{\lambda + 2\mu}\,\boldsymbol{n}_0\otimes\boldsymbol{n}_0\right]\left[(\mu - \mu_c)\left(\boldsymbol{n}_0\boldsymbol{E}^e\right) + \lambda\left(\mathrm{tr}\,\boldsymbol{E}^e\right)\boldsymbol{n}_0\right],$$

which yields (since $\boldsymbol{Q}_e\boldsymbol{n}_0 = \boldsymbol{Q}_e\boldsymbol{d}_3^0 = \boldsymbol{d}_3$)

$$\boldsymbol{\alpha} = \left(1 - \frac{\lambda}{\lambda + 2\mu}\,\mathrm{tr}\,\boldsymbol{E}^e\right)\boldsymbol{d}_3 \;-\; \frac{\mu - \mu_c}{\mu + \mu_c}\,\boldsymbol{Q}_e\left(\boldsymbol{n}_0\boldsymbol{E}^e\right). \tag{50}$$

Further, we solve equation (49) to determine the vector $\boldsymbol{\beta}$. To this aim, we insert $\boldsymbol{\alpha}$, given by equation (50), into equation (49) and (to avoid quadratic terms and derivatives of the strain measures $\boldsymbol{E}^e, \boldsymbol{K}^e$) we use the approximation

$$\boldsymbol{Q}_e^{\mathrm{T}} \mathrm{Grad}_s \boldsymbol{\alpha} \;\simeq\; \boldsymbol{Q}_e^{\mathrm{T}} \mathrm{Grad}_s \boldsymbol{d}_3 \,.$$

Since $\boldsymbol{Q}_e^{\mathrm{T}} \mathrm{Grad}_s \boldsymbol{d}_3 = \boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{b}$ (see f. (70) in Bîrsan et al. [12]), we use

$$\boldsymbol{Q}_e^{\mathrm{T}} \mathrm{Grad}_s \boldsymbol{\alpha} \;=\; \boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{b} \tag{51}$$

and equation (49) becomes

$$\Big[\underline{\boldsymbol{C}} : \big(\boldsymbol{E}^e \boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e + \boldsymbol{Q}_e^{\mathrm{T}} \boldsymbol{\beta} \otimes \boldsymbol{n}_0 \big) \Big] \boldsymbol{n}_0 = \boldsymbol{0},$$

which can be solved similarly to equation (48) and yields

$$\boldsymbol{\beta} = -\frac{\lambda}{\lambda + 2\mu}\, \mathrm{tr}\left(\boldsymbol{E}^e \boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e \right) \boldsymbol{d}_3 \;-\; \frac{\mu - \mu_c}{\mu + \mu_c}\, \boldsymbol{Q}_e \big(\boldsymbol{n}_0 \boldsymbol{E}^e \boldsymbol{b} \big). \tag{52}$$

In view of equations (50) to (52), we can write the tensors $\overline{\boldsymbol{E}}_0$ and $\overline{\boldsymbol{E}}_0'$ in equations (36) and (34) in compact form:

$$\begin{aligned}
\overline{\boldsymbol{E}}_0 &= \boldsymbol{E}^e - \left[\frac{\lambda}{\lambda + 2\mu} \left(\mathrm{tr}\, \boldsymbol{E}^e \right) \boldsymbol{n}_0 + \frac{\mu - \mu_c}{\mu + \mu_c} \left(\boldsymbol{n}_0 \boldsymbol{E}^e \right) \right] \otimes \boldsymbol{n}_0 = L_{n_0}\big(\boldsymbol{E}^e \big), \\
\overline{\boldsymbol{E}}_0' &= \big(\boldsymbol{E}^e \boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e \big) - \left[\frac{\lambda}{\lambda + 2\mu}\, \mathrm{tr}\big(\boldsymbol{E}^e \boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e \big) \boldsymbol{n}_0 + \frac{\mu - \mu_c}{\mu + \mu_c} \big(\boldsymbol{n}_0 \boldsymbol{E}^e \boldsymbol{b} \big) \right] \otimes \boldsymbol{n}_0 = L_{n_0}\big(\boldsymbol{E}^e \boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e \big),
\end{aligned} \tag{53}$$

where we have denoted, for convenience, with L_{n_0} the following linear operator

$$L_{n_0}(\boldsymbol{X}) := \boldsymbol{X} \;-\; \frac{\lambda}{\lambda + 2\mu} \big(\mathrm{tr} \boldsymbol{X} \big) \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 \;-\; \frac{\mu - \mu_c}{\mu + \mu_c} \big(\boldsymbol{n}_0 \boldsymbol{X} \big) \otimes \boldsymbol{n}_0 \qquad \text{for any} \qquad \boldsymbol{X} = X_{i\alpha} \boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha. \tag{54}$$

To write the strain energy density in a condensed form, we designate by

$$\begin{aligned}
W_{\mathrm{mixt}}(\boldsymbol{X}, \boldsymbol{Y}) :=& \mu \, (\mathrm{sym} \boldsymbol{X}) : (\mathrm{sym}\, \boldsymbol{Y}) + \mu_c (\mathrm{skew} \boldsymbol{X}) : (\mathrm{skew}\, \boldsymbol{Y}) + \frac{\lambda\, \mu}{\lambda + 2\mu} \big(\mathrm{tr} \boldsymbol{X} \big) \big(\mathrm{tr} \boldsymbol{Y} \big) \\
=& \mu \, (\mathrm{dev}_3\, \mathrm{sym} \boldsymbol{X}) : (\mathrm{dev}_3\, \mathrm{sym}\, \boldsymbol{Y}) + \mu_c (\mathrm{skew} \boldsymbol{X}) : (\mathrm{skew}\, \boldsymbol{Y}) + \frac{2\mu(2\lambda + \mu)}{3(\lambda + 2\mu)} \big(\mathrm{tr} \boldsymbol{X} \big) \big(\mathrm{tr} \boldsymbol{Y} \big)
\end{aligned} \tag{55}$$

the bilinear form corresponding to the quadratic form

$$\begin{aligned}
W_{\mathrm{mixt}}(\boldsymbol{X}) := W_{\mathrm{mixt}}(\boldsymbol{X}, \boldsymbol{X}) \;=&\; W_{\mathrm{mp}}(\boldsymbol{X}) - \frac{\lambda^2}{2(\lambda + 2\mu)} \big(\mathrm{tr} \boldsymbol{X} \big)^2 \\
=&\; \mu \, \| \, \mathrm{sym} \boldsymbol{X} \, \|^2 + \mu_c \| \, \mathrm{skew} \boldsymbol{X} \, \|^2 + \frac{\lambda\, \mu}{\lambda + 2\mu} \big(\mathrm{tr} \boldsymbol{X} \big)^2.
\end{aligned} \tag{56}$$

For Cosserat shells, it is convenient to introduce the following bilinear form

$$W_{\mathrm{Coss}}(\boldsymbol{X}, \boldsymbol{Y}) := W_{\mathrm{mixt}}(\boldsymbol{X}, \boldsymbol{Y}) - \frac{(\mu - \mu_c)^2}{2(\mu + \mu_c)} \big(\boldsymbol{n}_0 \boldsymbol{X} \big) \cdot \big(\boldsymbol{n}_0 \boldsymbol{Y} \big) \tag{57}$$

for any two tensors of the form $\boldsymbol{X} = X_{i\alpha} \boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, $\boldsymbol{Y} = Y_{i\alpha} \boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, and the corresponding quadratic form

$$W_{\mathrm{Coss}}(\boldsymbol{X}) := W_{\mathrm{Coss}}(\boldsymbol{X}, \boldsymbol{X}) \;=\; W_{\mathrm{mixt}}(\boldsymbol{X}) - \frac{(\mu - \mu_c)^2}{2(\mu + \mu_c)} \| \boldsymbol{n}_0 \boldsymbol{X} \|^2, \tag{58}$$

where $\boldsymbol{n}_0 \boldsymbol{X} = X_{3\alpha} \boldsymbol{a}^\alpha$. We shall prove later that the quadratic form $W_{\mathrm{Coss}}(\boldsymbol{X})$ is positive definite, see equation (106).

With these notations, we can prove, by a straightforward calculation, the following useful relation

$$W_{\mathrm{Coss}}(\boldsymbol{X}) \;=\; \frac{1}{2}\,\boldsymbol{X} : \underline{\boldsymbol{C}} : L_{n_0}(\boldsymbol{X}) \qquad \text{for any} \quad \boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha. \tag{59}$$

Indeed, we have from equations (11), (54), (56) and (58)

$$\begin{aligned}
\boldsymbol{X} : \underline{\boldsymbol{C}} : L_{n_0}(\boldsymbol{X}) &= \boldsymbol{X} : \underline{\boldsymbol{C}} : \boldsymbol{X} - \boldsymbol{X} : \underline{\boldsymbol{C}} : \Big[\frac{\lambda}{\lambda+2\mu}\big(\mathrm{tr}\boldsymbol{X}\big)\,\boldsymbol{n}_0\otimes\boldsymbol{n}_0 + \frac{\mu-\mu_c}{\mu+\mu_c}\big(\boldsymbol{n}_0\boldsymbol{X}\big)\otimes\boldsymbol{n}_0\Big]\\
&= 2W_{\mathrm{mp}}(\boldsymbol{X}) - \boldsymbol{X} : \Big[\frac{\lambda^2}{\lambda+2\mu}\big(\mathrm{tr}\boldsymbol{X}\big)\,\mathbb{1}_3 + (\mu-\mu_c)\big(\boldsymbol{n}_0\boldsymbol{X}\big)\otimes\boldsymbol{n}_0 + \frac{(\mu-\mu_c)^2}{\mu+\mu_c}\,\boldsymbol{n}_0\otimes\big(\boldsymbol{n}_0\boldsymbol{X}\big)\Big]\\
&= 2W_{\mathrm{mp}}(\boldsymbol{X}) - \frac{\lambda^2}{\lambda+2\mu}\big(\mathrm{tr}\boldsymbol{X}\big)^2 - \frac{(\mu-\mu_c)^2}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{X}\|^2\\
&= 2W_{\mathrm{mixt}}(\boldsymbol{X}) - \frac{(\mu-\mu_c)^2}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{X}\|^2 \;=\; 2W_{\mathrm{Coss}}(\boldsymbol{X})
\end{aligned}$$

and equation (59) is proved.

Now, we can simplify the terms appearing in the strain energy density (equation (47)): making use of equations (46), (51), (53) and (59), we find

$$\big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{a}\big) : \boldsymbol{E}^e = \boldsymbol{E}^e : \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\big) = \boldsymbol{E}^e : \big(\underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0\big) = \boldsymbol{E}^e : \underline{\boldsymbol{C}} : L_{n_0}\big(\boldsymbol{E}^e\big) = 2W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\big) \tag{60}$$

and

$$\begin{aligned}
\big(\boldsymbol{T}_0'\boldsymbol{a}\big) : \big[\mathrm{Grad}_s\,\boldsymbol{\alpha} + \big(\mathrm{Grad}_s\,\boldsymbol{m}\big)\boldsymbol{b}\big] &= \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0'\boldsymbol{a}\big) : \big[\big(\boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{b}\big) + \big(\boldsymbol{E}^e + \boldsymbol{a}\big)\boldsymbol{b}\big] = \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0'\big) : \big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big)\\
&= \big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) : \big(\underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0'\big) = \big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) : \underline{\boldsymbol{C}} : L_{n_0}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big)\\
&= 2W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big)
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
&\big(\boldsymbol{T}_0\boldsymbol{a}\big) : \big[\big(\mathrm{Grad}_s\,\boldsymbol{\alpha}\big)\boldsymbol{b}^* + K\big(\boldsymbol{Q}_e\boldsymbol{a}\big)\big]\\
&\quad = \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{a}\big) : \big[\big(\boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{b}\big)\boldsymbol{b}^* + K\boldsymbol{a}\big] = \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{T}_0\big) : \big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big)\\
&\quad = \big(\underline{\boldsymbol{C}} : \overline{\boldsymbol{E}}_0\big) : \big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big) = \big[2\mu\,\mathrm{sym}\,\overline{\boldsymbol{E}}_0 + 2\mu_c\,\mathrm{skew}\,\overline{\boldsymbol{E}}_0 + \lambda(\mathrm{tr}\,\overline{\boldsymbol{E}}_0)\mathbb{1}_3\big] : \big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big)\\
&\quad = 2\mu\,\mathrm{sym}\big(\boldsymbol{E}^e\big) : \mathrm{sym}\,\big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big) + 2\mu_c\,\mathrm{skew}\big(\boldsymbol{E}^e\big) : \mathrm{skew}\,\big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big) + \frac{2\lambda\,\mu}{\lambda+2\mu}\,\mathrm{tr}\big(\boldsymbol{E}^e\big)\,\mathrm{tr}\big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big)\\
&\quad = 2W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\,,\,\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big),
\end{aligned} \tag{62}$$

since

$$\mathrm{tr}\,\overline{\boldsymbol{E}}_0 = \frac{2\,\mu}{\lambda+2\mu}\,\mathrm{tr}\,\boldsymbol{E}^e$$

and the tensor $\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*$ is a planar tensor with basis $\{\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta\}$.

Furthermore, the two terms involving the bending curvature tensor $\boldsymbol{K}^e$ in the strain energy density (equation (47)) can be transformed as follows: by virtue of equations (9), (11) and (37), we have

$$\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0\big) : \boldsymbol{K}^e = \boldsymbol{K}^e : \big(\underline{\boldsymbol{G}} : \boldsymbol{\Gamma}_0\big) = \boldsymbol{K}^e : \underline{\boldsymbol{G}} : \boldsymbol{K}^e = 2W_{\mathrm{curv}}\big(\boldsymbol{K}^e\big) \tag{63}$$

and

$$\big(\boldsymbol{Q}_e^{\mathrm{T}}\overline{\boldsymbol{M}}_0'\big) : \big(\boldsymbol{K}^e\boldsymbol{b}\big) = \big(\boldsymbol{K}^e\boldsymbol{b}\big) : \big(\underline{\boldsymbol{G}} : \boldsymbol{\Gamma}_0'\big) = \big(\boldsymbol{K}^e\boldsymbol{b}\big) : \underline{\boldsymbol{G}} : \big(\boldsymbol{K}^e\boldsymbol{b}\big) = 2W_{\mathrm{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big). \tag{64}$$

Finally, the term $(\boldsymbol{T}_0\boldsymbol{a}) : \mathrm{Grad}_s\boldsymbol{\beta}$ appearing in the strain energy density (equation (47)) can be discarded. To justify this, we proceed as in the classical shell theory, see, e.g., Steigmann [10, 11]: the three-dimensional equilibrium equation $\mathrm{Div}\,\boldsymbol{T} = \boldsymbol{0}$ can be written as $\boldsymbol{T}_{,i}\,\boldsymbol{g}^i = \boldsymbol{0}$ or, equivalently,

$$\boldsymbol{T}_{,\alpha}\,\boldsymbol{g}^\alpha + \boldsymbol{T}'\boldsymbol{n}_0 = \boldsymbol{0}.$$

Therefore, on the midsurface $x_3 = 0$, we have

$$\boldsymbol{T}_{0,\alpha}\boldsymbol{a}^{\alpha} + \boldsymbol{T}_0'\boldsymbol{n}_0 = \boldsymbol{0}. \tag{65}$$

At the same time, we see that

$$\boldsymbol{T}_{0,\alpha}\boldsymbol{a}^{\alpha} = \big(\boldsymbol{T}_0\boldsymbol{a} + \boldsymbol{T}_0\boldsymbol{n}_0\otimes\boldsymbol{n}_0\big)_{,\alpha}\,\boldsymbol{a}^{\alpha} = \big(\boldsymbol{T}_0\boldsymbol{a}\big)_{,\alpha}\,\boldsymbol{a}^{\alpha} + \boldsymbol{T}_0\boldsymbol{n}_0\big(\boldsymbol{n}_{0,\alpha}\cdot\,\boldsymbol{a}^{\alpha}\big) = \mathrm{Div}_s(\boldsymbol{T}_0\boldsymbol{a}) - 2H\,\boldsymbol{T}_0\boldsymbol{n}_0\,.$$

Inserting the last relation into equation (65), we find

$$\mathrm{Div}_s(\boldsymbol{T}_0\boldsymbol{a}) + \boldsymbol{T}_0'\boldsymbol{n}_0 - 2H\,\boldsymbol{T}_0\boldsymbol{n}_0 = \boldsymbol{0}. \tag{66}$$

With the help of equations (46) and (66) and the divergence theorem for surfaces, we get

$$\begin{aligned}\int_{\omega_\xi}\big(\boldsymbol{T}_0\boldsymbol{a}\big) : \big(\mathrm{Grad}_s\,\boldsymbol{\beta}\big)\mathrm{d}a &= \int_{\omega_\xi}\big[\mathrm{Div}_s\big(\boldsymbol{\beta}(\boldsymbol{T}_0\boldsymbol{a})\big) - \boldsymbol{\beta}\cdot\mathrm{Div}_s(\boldsymbol{T}_0\boldsymbol{a})\big]\mathrm{d}a\\ &= \int_{\partial\omega_\xi}\boldsymbol{\beta}\big(\boldsymbol{T}_0\boldsymbol{a}\big)\cdot\boldsymbol{\nu}\,\mathrm{d}\ell - \int_{\omega_\xi}\boldsymbol{\beta}\cdot\big(2H\,\boldsymbol{T}_0\boldsymbol{n}_0 - \boldsymbol{T}_0'\boldsymbol{n}_0\big)\mathrm{d}a = \int_{\partial\omega_\xi}\boldsymbol{\beta}\cdot\big(\boldsymbol{T}_0\boldsymbol{a}\big)\boldsymbol{\nu}\,\mathrm{d}\ell\,,\end{aligned} \tag{67}$$

where $\boldsymbol{\nu}$ is the unit normal to the boundary curve $\partial\omega_\xi$ lying in the tangent plane. The last integral in equation (67) represents a prescribed constant (determined by the boundary data on $\partial\omega_\xi$), which can be omitted, since its variation vanishes identically and thus does not influence the minimizers of the energy functional.

In conclusion, using equations (60) to (64) in equation (47), we obtain the following expression of the areal strain energy density for Cosserat shells:

$$\begin{aligned}W_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) &= \Big(h - K\,\frac{h^3}{12}\Big)\big[W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\big) + W_{\mathrm{curv}}\big(\boldsymbol{K}^e\big)\big]\\ &\quad + \frac{h^3}{12}\big[W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) - 2W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\,,\,\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big) + W_{\mathrm{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\big],\end{aligned} \tag{68}$$

where W_{Coss} is defined by equations (57) and (58) (see also equations (95) and (106)) and W_{curv} is given in equation (8). This is the elastically stored strain energy density for our model, which determines the constitutive equations. In Section 5, we shall present a useful alternative form of the energy $W_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$, together with explicit stress–strain relations (see equations (107) and (113)).

4.3. The field equations for Cosserat shells

For the sake of completeness, we record here the governing field equations of the derived shell model.

We deduce the form of the equilibrium equations for Cosserat shells from the condition that the solution is a stationary point of the energy functional I, i.e., we impose that the variation of the energy functional is zero:

$$\delta I = 0\,,\qquad \text{with}\quad I = \int_{\omega_\xi} W_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)\,\mathrm{d}a. \tag{69}$$

For simplicity, we have assumed in equation (69) that the external body loads are vanishing and the boundary conditions are null. To compute the variation δI, we write

$$\delta\,W_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = \frac{\partial\,W_{\mathrm{shell}}}{\partial\boldsymbol{E}^e} : \big(\delta\boldsymbol{E}^e\big) + \frac{\partial\,W_{\mathrm{shell}}}{\partial\boldsymbol{K}^e} : \big(\delta\boldsymbol{K}^e\big) = \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N}\big) : \big(\delta\boldsymbol{E}^e\big) + \big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M}\big) : \big(\delta\boldsymbol{K}^e\big), \tag{70}$$

where we have introduced the tensors $\boldsymbol{N}$ and $\boldsymbol{M}$, such that

$$\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N} = \frac{\partial\,W_{\mathrm{shell}}}{\partial\boldsymbol{E}^e}\qquad\text{and}\qquad \boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M} = \frac{\partial\,W_{\mathrm{shell}}}{\partial\boldsymbol{K}^e}\,. \tag{71}$$

Let us denote by

$$\boldsymbol{F}_s := \mathrm{Grad}_s\boldsymbol{m} = \boldsymbol{m}_{,\alpha}\otimes\boldsymbol{a}^{\alpha} \tag{72}$$

the *shell deformation gradient* (i.e., the surface gradient of the midsurface deformation $\boldsymbol{m}$). Then, in view of equation (35), we have $\boldsymbol{E}^e = \boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{F}_s - \boldsymbol{a}$ and, hence,

$$\delta\boldsymbol{E}^e = \delta\big(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{F}_s - \boldsymbol{a}\big) = \delta\big(\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{m}\big) = (\delta\boldsymbol{Q}_e)^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{m} + \boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\big(\delta\boldsymbol{m}\big). \tag{73}$$

To compute $\delta\boldsymbol{Q}_e$, we notice that the tensor $(\delta\boldsymbol{Q}_e)\boldsymbol{Q}_e^{\mathrm{T}}$ is skew-symmetric and we denote

$$\boldsymbol{\Omega} := (\delta\boldsymbol{Q}_e)\boldsymbol{Q}_e^{\mathrm{T}}, \qquad \boldsymbol{\omega} := \mathrm{axl}(\boldsymbol{\Omega}), \qquad \text{with} \quad \boldsymbol{\Omega} = \boldsymbol{\omega}\times\mathbb{1}_3\,. \tag{74}$$

In these relations, the axial vector $\boldsymbol{\omega}$ is the virtual rotation vector and $\delta\boldsymbol{m}$ is the virtual translation. From equation (74), we get

$$\delta\boldsymbol{Q}_e = \boldsymbol{\Omega}\,\boldsymbol{Q}_e = -(\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{\Omega})^{\mathrm{T}} \tag{75}$$

and substituting into equation (73) we obtain

$$\delta\boldsymbol{E}^e = \boldsymbol{Q}_e^{\mathrm{T}}\big(\mathrm{Grad}_s(\delta\boldsymbol{m}) - \boldsymbol{\Omega}\,\boldsymbol{F}_s\big). \tag{76}$$

Further, to compute $\delta\boldsymbol{K}^e$, we recall the formula (see f. (63) in Bîrsan and Neff [22])

$$\boldsymbol{K}^e = \frac{1}{2}\left[\boldsymbol{Q}_e^{\mathrm{T}}\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) - \boldsymbol{d}_i^0\times\mathrm{Grad}_s\,\boldsymbol{d}_i^0\,\right] \tag{77}$$

and write (in view of equation (75))

$$\delta\boldsymbol{d}_i = \delta\big(\boldsymbol{Q}_e\boldsymbol{d}_i^0\big) = (\delta\boldsymbol{Q}_e)\boldsymbol{d}_i^0 = \boldsymbol{\Omega}\boldsymbol{Q}_e\boldsymbol{d}_i^0 = \boldsymbol{\Omega}\boldsymbol{d}_i = \boldsymbol{\omega}\times\boldsymbol{d}_i\,. \tag{78}$$

Then, from equation (77), it follows that

$$\begin{aligned}
\delta\boldsymbol{K}^e &= \frac{1}{2}\,\delta\big[\boldsymbol{Q}_e^{\mathrm{T}}\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big)\big]\\
&= \frac{1}{2}\left[(\delta\boldsymbol{Q}_e)^{\mathrm{T}}\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) + \boldsymbol{Q}_e^{\mathrm{T}}\big((\delta\boldsymbol{d}_i)\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) + \boldsymbol{Q}_e^{\mathrm{T}}\big(\boldsymbol{d}_i\times\mathrm{Grad}_s(\delta\boldsymbol{d}_i)\big)\right]\\
&= \frac{1}{2}\,\boldsymbol{Q}_e^{\mathrm{T}}\big[-\boldsymbol{\Omega}\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) + (\boldsymbol{\Omega}\boldsymbol{d}_i)\times\mathrm{Grad}_s\,\boldsymbol{d}_i + \boldsymbol{d}_i\times\mathrm{Grad}_s(\boldsymbol{\Omega}\boldsymbol{d}_i)\big]\\
&= \frac{1}{2}\,\boldsymbol{Q}_e^{\mathrm{T}}\big[-\boldsymbol{\omega}\times\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) + (\boldsymbol{\omega}\times\boldsymbol{d}_i)\times\mathrm{Grad}_s\,\boldsymbol{d}_i + \boldsymbol{d}_i\times\mathrm{Grad}_s(\boldsymbol{\omega}\times\boldsymbol{d}_i)\big].
\end{aligned} \tag{79}$$

By virtue of the Jacobi identity for the cross product, we have

$$-\boldsymbol{\omega}\times\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) + (\boldsymbol{\omega}\times\boldsymbol{d}_i)\times\mathrm{Grad}_s\,\boldsymbol{d}_i = -\boldsymbol{d}_i\times\big(\boldsymbol{\omega}\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big)$$

and inserting this in equation (79) we get

$$\delta\boldsymbol{K}^e = \quad \frac{1}{2}\,\boldsymbol{Q}_e^{\mathrm{T}}\big[\boldsymbol{d}_i\times\big(\mathrm{Grad}_s(\boldsymbol{\omega}\times\boldsymbol{d}_i) - \boldsymbol{\omega}\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big)\big]. \tag{80}$$

For the square brackets in equation (80), we can write

$$\boldsymbol{d}_i\times\big(\mathrm{Grad}_s(\boldsymbol{\omega}\times\boldsymbol{d}_i) - \boldsymbol{\omega}\times\mathrm{Grad}_s\,\boldsymbol{d}_i\big) = -\boldsymbol{d}_i\times\big(\boldsymbol{d}_i\times\mathrm{Grad}_s\,\boldsymbol{\omega}\big) = 2\,\mathrm{Grad}_s\,\boldsymbol{\omega}\,, \tag{81}$$

since

$$-\boldsymbol{d}_i\times\big(\boldsymbol{d}_i\times\boldsymbol{\omega}_{,\alpha}\big) = -(\boldsymbol{d}_i\cdot\boldsymbol{\omega}_{,\alpha})\boldsymbol{d}_i + (\boldsymbol{d}_i\cdot\boldsymbol{d}_i)\boldsymbol{\omega}_{,\alpha} = -\boldsymbol{\omega}_{,\alpha} + 3\,\boldsymbol{\omega}_{,\alpha} = 2\,\boldsymbol{\omega}_{,\alpha}\,.$$

We substitute equation (81) into equation (80) and find

$$\delta\boldsymbol{K}^e = \boldsymbol{Q}_e^{\mathrm{T}}\,\mathrm{Grad}_s\boldsymbol{\omega}. \tag{82}$$

By virtue of equations (76) and (82), equation (70) becomes

$$\delta W_{\text{shell}} = \boldsymbol{N} : \big(\text{Grad}_s(\delta\boldsymbol{m}) - \boldsymbol{\Omega}\,\boldsymbol{F}_s\big) + \boldsymbol{M} : \text{Grad}_s\boldsymbol{\omega}\,. \tag{83}$$

We can rewrite the term $\boldsymbol{N} : (\boldsymbol{\Omega}\boldsymbol{F}_s)$ as

$$\boldsymbol{N} : (\boldsymbol{\Omega}\boldsymbol{F}_s) = -\boldsymbol{\Omega} : (\boldsymbol{F}_s\boldsymbol{N}^{\text{T}}) = -\boldsymbol{\omega}\cdot\text{axl}\big(\boldsymbol{F}_s\boldsymbol{N}^{\text{T}} - \boldsymbol{N}\boldsymbol{F}_s^{\text{T}}\big), \tag{84}$$

since

$$\boldsymbol{\Omega} : \boldsymbol{X} = \text{axl}(\boldsymbol{\Omega})\cdot\text{axl}(\boldsymbol{X} - \boldsymbol{X}^{\text{T}})$$

for any second-order tensor $\boldsymbol{X}$ and any skew-symmetric tensor $\boldsymbol{\Omega}$. We use equation (84) in equation (83) and deduce

$$\delta W_{\text{shell}} = \boldsymbol{N} : \text{Grad}_s(\delta\boldsymbol{m}) + \boldsymbol{M} : \text{Grad}_s\boldsymbol{\omega} + \text{axl}\big(\boldsymbol{F}_s\boldsymbol{N}^{\text{T}} - \boldsymbol{N}\boldsymbol{F}_s^{\text{T}}\big)\cdot\boldsymbol{\omega}\,. \tag{85}$$

For the first two terms in the right-hand side of equation (85), we employ relations of the type

$$\boldsymbol{S} : \text{Grad}_s\boldsymbol{v} = \text{Div}_s(\boldsymbol{S}^{\text{T}}\boldsymbol{v}) - \big(\text{Div}_s\boldsymbol{S}\big)\cdot\boldsymbol{v}\,,$$

together with the divergence theorem on surfaces. Thus, in view of the null boundary conditions on $\partial\omega_\xi$, we derive

$$\int_{\omega_\xi}\boldsymbol{N} : \text{Grad}_s(\delta\boldsymbol{m})\,\text{d}a = \int_{\partial\omega_\xi}(\delta\boldsymbol{m})\cdot(\boldsymbol{N}\boldsymbol{\nu})\,\text{d}\ell - \int_{\omega_\xi}\big(\text{Div}_s\boldsymbol{N}\big)\cdot(\delta\boldsymbol{m})\,\text{d}a = -\int_{\omega_\xi}\big(\text{Div}_s\boldsymbol{N}\big)\cdot(\delta\boldsymbol{m})\,\text{d}a \tag{86}$$

and similarly

$$\int_{\omega_\xi}\boldsymbol{M} : \text{Grad}_s\boldsymbol{\omega}\,\text{d}a = -\int_{\omega_\xi}\big(\text{Div}_s\boldsymbol{M}\big)\cdot\boldsymbol{\omega}\,\text{d}a. \tag{87}$$

Finally, in view of equations (85) to (87), we obtain

$$0 = \delta I = \int_{\omega_\xi}\delta W_{\text{shell}}\,\text{d}a = -\int_{\omega_\xi}\Big[\big(\text{Div}_s\boldsymbol{N}\big)\cdot(\delta\boldsymbol{m}) + \big(\text{Div}_s\boldsymbol{M} + \text{axl}\big(\boldsymbol{N}\boldsymbol{F}_s^{\text{T}} - \boldsymbol{F}_s\boldsymbol{N}^{\text{T}}\big)\big)\cdot\boldsymbol{\omega}\Big]\,\text{d}a, \tag{88}$$

for any virtual translation $\delta\boldsymbol{m}$ and any virtual rotation $\boldsymbol{\omega} = \text{axl}((\delta\boldsymbol{Q}_e)\boldsymbol{Q}_e^{\text{T}})$. Equation (88) yields the following local forms of the equilibrium equations:

$$\text{Div}_s\boldsymbol{N} = \boldsymbol{0} \qquad \text{and} \qquad \text{Div}_s\boldsymbol{M} + \text{axl}\big(\boldsymbol{N}\boldsymbol{F}_s^{\text{T}} - \boldsymbol{F}_s\boldsymbol{N}^{\text{T}}\big) = \boldsymbol{0}. \tag{89}$$

Remark 1. The principle of virtual work for six-parameter shells corresponding to equation (88) has been presented in Bîrsan and Neff [16] and Eremeyev and Pietraszkiewicz [20].

If, now, we consider external body forces $\boldsymbol{f}$ and couples $\boldsymbol{c}$, we can write the equilibrium equations for Cosserat shells in the general form (see, e.g., Bîrsan and Neff [16] and Eremeyev and Pietraszkiewicz [20])

$$\text{Div}_s\boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \qquad \text{Div}_s\boldsymbol{M} + \text{axl}\big(\boldsymbol{N}\boldsymbol{F}_s^{\text{T}} - \boldsymbol{F}_s\boldsymbol{N}^{\text{T}}\big) + \boldsymbol{c} = \boldsymbol{0}. \tag{90}$$

The tensors $\boldsymbol{N}$ and $\boldsymbol{M}$ are the internal surface stress tensor and the internal surface couple tensor (of the first Piola–Kirchhoff type), respectively. They are given by equation (71).

The general form of the boundary conditions of mixed type on $\partial\omega_\xi$ is (see, e.g., Pietraszkiewicz [7, 23] and Bîrsan and Neff [8])

$$\begin{aligned} \boldsymbol{N}\boldsymbol{\nu} &= \boldsymbol{N}^*, \qquad & \boldsymbol{M}\boldsymbol{\nu} &= \boldsymbol{M}^* \quad \text{along } \partial\omega_f\,,\\ \boldsymbol{m} &= \boldsymbol{m}^*, \qquad & \boldsymbol{Q}_e &= \boldsymbol{Q}^* \quad \text{along } \partial\omega_d\,, \end{aligned} \tag{91}$$

where $\partial\omega_f$ and $\partial\omega_d$ build a disjoint partition of the boundary curve $\partial\omega_\xi$. Here, $\boldsymbol{N}^*$ and $\boldsymbol{M}^*$ are the external boundary force and couple vectors, respectively, applied along the deformed boundary curve, but measured per unit length of $\partial\omega_f$. On the portion of the boundary $\partial\omega_d$, we have Dirichlet-type boundary conditions for the deformation vector $\boldsymbol{m}$ and the microrotation tensor $\boldsymbol{Q}_e$.

Using the obtained form of the energy density (equation (68)) and equation (71), we can give the stress–strain relations in explicit form for our shell model. These will be written in the next section.

5. Remarks and discussions on the Cosserat shell model

In this section, we write the strain energy density (equation (68)) in some alternative useful forms and give the explicit expression for the constitutive equations (equation (71)). This allows us to compare the derived shell model with other approaches to six-parameter shells and with the classical Koiter shell model.

We notice that the shell strain measures $\boldsymbol{E}^e$ and $\boldsymbol{K}^e$ (as well as the shell stress tensors $\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N}$ and $\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M}$) are tensors of the form $\boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ (where $\boldsymbol{a}^3 = \boldsymbol{n}_0$). In what follows, we shall decompose any such tensor $\boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ into its 'planar' part $\boldsymbol{aX} = X_{\beta\alpha}\boldsymbol{a}^\beta \otimes \boldsymbol{a}^\alpha$ and its 'transversal' part $\boldsymbol{n}_0\boldsymbol{X} = X_{3\alpha}\boldsymbol{a}^\alpha$, according to

$$\boldsymbol{X} = \mathbb{1}_3\boldsymbol{X} = (\boldsymbol{a} + \boldsymbol{n}_0 \otimes \boldsymbol{n}_0)\boldsymbol{X} = \boldsymbol{aX} + \boldsymbol{n}_0 \otimes (\boldsymbol{n}_0\boldsymbol{X}). \tag{92}$$

Note that $\boldsymbol{aX}$ is a planar tensor in the tangent plane, while $\boldsymbol{n}_0\boldsymbol{X}$ is a vector in the tangent plane. For instance, the decomposition of the shell strain tensor $\boldsymbol{E}^e$ yields

$$\boldsymbol{E}^e = \boldsymbol{aE}^e + \boldsymbol{n}_0 \otimes (\boldsymbol{n}_0\boldsymbol{E}^e), \qquad \boldsymbol{aE}^e = E^e_{\beta\alpha}\boldsymbol{a}^\beta \otimes \boldsymbol{a}^\alpha, \qquad \boldsymbol{n}_0\boldsymbol{E}^e = E^e_{3\alpha}\boldsymbol{a}^\alpha, \tag{93}$$

where $\boldsymbol{n}_0\boldsymbol{E}^e$ describes the transverse shear deformations and $\boldsymbol{aE}^e$ the in-plane deformation of the shell.

With this representation, we can decompose the constitutive equations (equation (71)) in the following way:

$$\boldsymbol{aQ}_e^{\mathrm{T}}\boldsymbol{N} = \frac{\partial W_{\text{shell}}}{\partial(\boldsymbol{aE}^e)}, \quad \boldsymbol{n}_0\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N} = \frac{\partial W_{\text{shell}}}{\partial(\boldsymbol{n}_0\boldsymbol{E}^e)}, \quad \boldsymbol{aQ}_e^{\mathrm{T}}\boldsymbol{M} = \frac{\partial W_{\text{shell}}}{\partial(\boldsymbol{aK}^e)}, \quad \boldsymbol{n}_0\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M} = \frac{\partial W_{\text{shell}}}{\partial(\boldsymbol{n}_0\boldsymbol{K}^e)}. \tag{94}$$

5.1. Explicit stress–strain relations

To write the stress–strain relations explicitly, let us put equations (57) and (58) in the forms

$$\begin{aligned} W_{\text{Coss}}(\boldsymbol{X},\boldsymbol{Y}) &= \mu\,\text{sym}(\boldsymbol{aX}):\text{sym}(\boldsymbol{aY}) + \mu_c\,\text{skew}(\boldsymbol{aX}):\text{skew}(\boldsymbol{aY}) + \frac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\boldsymbol{X}\big)\big(\text{tr}\boldsymbol{Y}\big) \\ &\quad + \frac{2\mu\,\mu_c}{\mu+\mu_c}\big(\boldsymbol{n}_0\boldsymbol{X}\big)\cdot\big(\boldsymbol{n}_0\boldsymbol{Y}\big), \\ W_{\text{Coss}}(\boldsymbol{X}) &= \mu\,\|\text{sym}(\boldsymbol{aX})\|^2 + \mu_c\|\text{skew}(\boldsymbol{aX})\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\boldsymbol{X}\big)^2 + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{X}\|^2 \end{aligned} \tag{95}$$

and note that $\text{tr}\boldsymbol{X} = \text{tr}(\boldsymbol{aX})$. Suggested by equation (95), we introduce the fourth-order planar tensor $\underline{\boldsymbol{C}}_S$ of elastic moduli for the shell

$$\begin{aligned} \underline{\boldsymbol{C}}_S &= C_S^{\alpha\beta\gamma\delta}\boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta \otimes \boldsymbol{a}_\gamma \otimes \boldsymbol{a}_\delta \qquad \text{with} \\ C_S^{\alpha\beta\gamma\delta} &= \mu\left(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma}\right) + \mu_c\left(a^{\alpha\gamma}a^{\beta\delta} - a^{\alpha\delta}a^{\beta\gamma}\right) + \frac{2\lambda\,\mu}{\lambda+2\mu}\,a^{\alpha\beta}a^{\gamma\delta}. \end{aligned} \tag{96}$$

Then, the tensor $\underline{\boldsymbol{C}}_S$ satisfies the major symmetries $C_S^{\alpha\beta\gamma\delta} = C_S^{\gamma\delta\alpha\beta}$ and we have

$$\underline{\boldsymbol{C}}_S : \boldsymbol{T} = 2\mu\,\text{sym}\,\boldsymbol{T} + 2\mu_c\,\text{skew}\,\boldsymbol{T} + \frac{2\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\,\boldsymbol{T}\big)\,\boldsymbol{a}, \tag{97}$$

for any planar tensor $\boldsymbol{T} = T_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$. Owing to the symmetry, equation (95) can be written in a simple way,

$$\begin{aligned} W_{\text{Coss}}(\boldsymbol{X},\boldsymbol{Y}) &= \frac{1}{2}(\boldsymbol{aX}):\underline{\boldsymbol{C}}_S:(\boldsymbol{aY}) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\big(\boldsymbol{n}_0\boldsymbol{X}\big)\cdot\big(\boldsymbol{n}_0\boldsymbol{Y}\big) = \frac{1}{2}\,C_S^{\alpha\beta\gamma\delta}X_{\alpha\beta}Y_{\gamma\delta} + \frac{2\mu\,\mu_c}{\mu+\mu_c}X_{3\alpha}Y_{3\alpha}, \\ W_{\text{Coss}}(\boldsymbol{X}) &= \frac{1}{2}(\boldsymbol{aX}):\underline{\boldsymbol{C}}_S:(\boldsymbol{aX}) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{X}\|^2, \end{aligned} \tag{98}$$

for any tensors $\boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, $\boldsymbol{Y} = Y_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$.

Similarly, the quadratic form W_{curv} defined by equation (8) can be put into the form

$$\begin{aligned} W_{\text{curv}}(\boldsymbol{X}) &= \mu\, L_c^2 \left(b_1 \|\text{sym}(\boldsymbol{aX})\|^2 + b_2 \|\text{skew}(\boldsymbol{aX})\|^2 + \left(b_3 - \frac{b_1}{3}\right) (\text{tr}\boldsymbol{X})^2 + \frac{b_1+b_2}{2}\, \|\boldsymbol{n}_0 \boldsymbol{X}\|^2 \right) \\ &= \frac{1}{2} (\boldsymbol{aX}) : \underline{\boldsymbol{G}}_S : (\boldsymbol{aX}) + \mu\, L_c^2\, \frac{b_1+b_2}{2}\, \|\boldsymbol{n}_0 \boldsymbol{X}\|^2 \\ &= \frac{1}{2}\, G_S^{\alpha\beta\gamma\delta} X_{\alpha\beta} X_{\gamma\delta} + \mu\, L_c^2\, \frac{b_1+b_2}{2}\, X_{3\alpha} X_{3\alpha} \end{aligned} \tag{99}$$

for any tensor $\boldsymbol{X} = X_{i\alpha} \boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, where the fourth-order planar tensor $\underline{\boldsymbol{G}}_S$ is given by

$$\begin{aligned} \underline{\boldsymbol{G}}_S &= G_S^{\alpha\beta\gamma\delta} \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta \otimes \boldsymbol{a}_\gamma \otimes \boldsymbol{a}_\delta \qquad \text{with} \\ G_S^{\alpha\beta\gamma\delta} &= \mu\, L_c^2 \left(b_1 \left(a^{\alpha\gamma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\gamma} \right) + b_2 \left(a^{\alpha\gamma} a^{\beta\delta} - a^{\alpha\delta} a^{\beta\gamma} \right) + \left(b_3 - \frac{b_1}{3} \right) a^{\alpha\beta} a^{\gamma\delta} \right). \end{aligned} \tag{100}$$

We see that $G_S^{\alpha\beta\gamma\delta} = G_S^{\gamma\delta\alpha\beta}$ and for any planar tensor $\boldsymbol{T} = T_{\alpha\beta} \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$ it holds that

$$\underline{\boldsymbol{G}}_S : \boldsymbol{T} = 2\mu\, L_c^2 \left(b_1\, \text{sym}\, \boldsymbol{T} + b_2\, \text{skew}\, \boldsymbol{T} + \left(b_3 - \frac{b_1}{3} \right) (\text{tr}\, \boldsymbol{T})\, \boldsymbol{a} \right). \tag{101}$$

To show that the quadratic forms W_{Coss} and W_{curv} are positive definite, let us introduce the *surface deviator operator* dev_s defined in Bîrsan and Neff [16] by

$$\text{dev}_s \boldsymbol{X} := \boldsymbol{X} - \frac{1}{2} (\text{tr}\boldsymbol{X}) \boldsymbol{a}. \tag{102}$$

According to Lemma 2.1 in Bîrsan and Neff [16], we can decompose any tensor $\boldsymbol{X} = X_{i\alpha} \boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ as a *direct sum* (orthogonal decomposition), as follows:

$$\boldsymbol{X} \;=\; \text{dev}_s \text{sym}\boldsymbol{X} \;+\; \text{skew}\boldsymbol{X} \;+\; \frac{1}{2}\, (\text{tr}\boldsymbol{X})\, \boldsymbol{a}\,. \tag{103}$$

Then, equations (102) and (103) imply

$$\text{sym}\boldsymbol{X} \;=\; \text{dev}_s \text{sym}\boldsymbol{X} \;+\; \frac{1}{2}\, (\text{tr}\boldsymbol{X})\, \boldsymbol{a} \qquad \text{and} \qquad \|\text{sym}\boldsymbol{X}\|^2 = \|\text{dev}_s \text{sym}\boldsymbol{X}\|^2 + \frac{1}{2}\, (\text{tr}\boldsymbol{X})^2. \tag{104}$$

Substituting equation (104) into equations (97) and (101), we get (for any $\boldsymbol{T} = T_{\alpha\beta} \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$)

$$\begin{aligned} \underline{\boldsymbol{C}}_S : \boldsymbol{T} &= 2\mu\, \text{dev}_s \text{sym}\, \boldsymbol{T} + 2\mu_c\, \text{skew}\, \boldsymbol{T} + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu}\, (\text{tr}\, \boldsymbol{T})\, \boldsymbol{a}\,, \\ \underline{\boldsymbol{G}}_S : \boldsymbol{T} &= 2\mu\, L_c^2 \left(b_1\, \text{dev}_s \text{sym}\, \boldsymbol{T} + b_2\, \text{skew}\, \boldsymbol{T} + \left(b_3 + \frac{b_1}{6} \right) (\text{tr}\, \boldsymbol{T})\, \boldsymbol{a} \right) \end{aligned} \tag{105}$$

and the quadratic forms (equations (95) and (99)) become

$$\begin{aligned} W_{\text{Coss}}(\boldsymbol{X}) &= \mu\, \|\text{dev}_s \text{sym}(\boldsymbol{aX})\|^2 + \mu_c \|\text{skew}(\boldsymbol{aX})\|^2 + \frac{\mu(3\lambda + 2\mu)}{2(\lambda + 2\mu)}\, (\text{tr}\boldsymbol{X})^2 + \frac{2\mu\, \mu_c}{\mu + \mu_c}\, \|\boldsymbol{n}_0 \boldsymbol{X}\|^2, \\ W_{\text{curv}}(\boldsymbol{X}) &= \mu\, L_c^2 \left(b_1 \|\text{dev}_s \text{sym}(\boldsymbol{aX})\|^2 + b_2 \|\text{skew}(\boldsymbol{aX})\|^2 + \left(b_3 + \frac{b_1}{6} \right) (\text{tr}\boldsymbol{X})^2 + \frac{b_1+b_2}{2}\, \|\boldsymbol{n}_0 \boldsymbol{X}\|^2 \right). \end{aligned} \tag{106}$$

Under the usual assumptions on the material constants $\mu > 0$, $3\lambda + 2\mu > 0$ (from classical elasticity), together with $\mu_c > 0$ and $b_i > 0$, we now see that the quadratic forms (equations (106)) are positive definite, since all the coefficients are positive.

Finally, we substitute equations (98) and (99) into the strain energy density (equation (68)) and, differentiating according to equation (94), we obtain the following explicit forms of the constitutive equations for the internal surface stress tensor $\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N}$ and the internal surface couple tensor $\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M}$ of Cosserat shells

$$\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N} = \boldsymbol{a}\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N} + \boldsymbol{n}_0 \otimes \big(\boldsymbol{n}_0\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N}\big), \qquad \boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M} = \boldsymbol{a}\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M} + \boldsymbol{n}_0 \otimes \big(\boldsymbol{n}_0\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M}\big)$$

with

$$\begin{aligned}
\boldsymbol{a}\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N} &= \Big(h - K\,\frac{h^3}{12}\Big)\,\underline{\boldsymbol{C}}_S : \big(\boldsymbol{a}\boldsymbol{E}^e\big) + \frac{h^3}{12}\big[\underline{\boldsymbol{C}}_S : \big(\boldsymbol{a}\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big)\big]\boldsymbol{b} - \frac{h^3}{12}\,\underline{\boldsymbol{C}}_S : \big(\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big),\\
\boldsymbol{n}_0\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{N} &= \frac{4\mu\,\mu_c}{\mu+\mu_c}\left[\Big(h - 2K\,\frac{h^3}{12}\Big)\big(\boldsymbol{n}_0\boldsymbol{E}^e\big) + 2H\,\frac{h^3}{12}\big(\boldsymbol{n}_0\boldsymbol{E}^e\boldsymbol{b}\big)\right],\\
\boldsymbol{a}\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M} &= \Big(h - K\,\frac{h^3}{12}\Big)\,\underline{\boldsymbol{G}}_S : \big(\boldsymbol{a}\boldsymbol{K}^e\big) + \frac{h^3}{12}\,\boldsymbol{c}\big[\underline{\boldsymbol{C}}_S : \big(\boldsymbol{a}\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big)\big] - \frac{h^3}{12}\,\boldsymbol{c}\big[\underline{\boldsymbol{C}}_S : \big(\boldsymbol{a}\boldsymbol{E}^e\big)\big]\boldsymbol{b}^*\\
&\quad + \frac{h^3}{12}\,\big[\underline{\boldsymbol{G}}_S : \big(\boldsymbol{a}\boldsymbol{K}^e\boldsymbol{b}\big)\big]\boldsymbol{b},\\
\boldsymbol{n}_0\boldsymbol{Q}_e^{\mathrm{T}}\boldsymbol{M} &= \mu\,L_c^2\,(b_1+b_2)\left[\Big(h - 2K\,\frac{h^3}{12}\Big)\big(\boldsymbol{n}_0\boldsymbol{K}^e\big) + 2H\,\frac{h^3}{12}\big(\boldsymbol{n}_0\boldsymbol{K}^e\boldsymbol{b}\big)\right],
\end{aligned} \tag{107}$$

where the tensors of elastic moduli $\underline{\boldsymbol{C}}_S$ and $\underline{\boldsymbol{G}}_S$ are given in equations (96), (97), (100) and (101).

5.2. Comparison with other six-parameter shell models

We present a detailed comparison with the related shell model of order $O(h^5)$, which has been presented recently in Bîrsan et al. [12]. The Cosserat shell model derived in Bîrsan et al. [12] has many similarities with the present model, but there are also some differences, which we indicate now.

First of all, the derivation method and starting point in Bîrsan et al. [12] is different, since the deformation function $\boldsymbol{\varphi}$ is assumed to be quadratic in x_3 . More precisely, the following ansatz is adopted (see f. (65) in Bîrsan et al. [12])

$$\boldsymbol{\varphi}(x_i) = \boldsymbol{m}(x_1,x_2) + x_3\,\alpha(x_1,x_2)\,\boldsymbol{d}_3 + \frac{x_3^2}{2}\,\beta(x_1,x_2)\,\boldsymbol{d}_3\,. \tag{108}$$

If we compare this ansatz with equation (31), we see that the assumption (108) is more restrictive.

Secondly, the hypotheses (equation (46)) from the classical shell theory were replaced by the weaker requirements (see f. (60) in Bîrsan et al. [12])

$$\boldsymbol{n}_0\cdot\boldsymbol{T}_0\boldsymbol{n}_0 = 0 \qquad \text{and} \qquad \boldsymbol{n}_0\cdot\boldsymbol{T}_0'\boldsymbol{n}_0 = 0, \tag{109}$$

i.e., only the normal components of the stress vectors $\boldsymbol{t}^+$, $\boldsymbol{t}^-$ on the upper and lower surfaces of the shell are assumed to be zero. The two scalar equations (109) are then employed in Bîrsan et al. [12] to determine the two scalar coefficients $\alpha(x_1,x_2)$ and $\beta(x_1,x_2)$ appearing in equation (108). Moreover, we note that the paper by Bîrsan et al. [12] presents a shell model of order $O(h^5)$.

This different approach leads to a slightly different form of the strain energy density. If we retain only the terms up to order $O(h^3)$ in the strain energy density (see f. (104) in Bîrsan et al. [12]), we get

$$\begin{aligned}
\widehat{W}_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) &= \Big(h - K\,\frac{h^3}{12}\Big)\big[W_{\mathrm{mixt}}\big(\boldsymbol{E}^e\big) + W_{\mathrm{curv}}\big(\boldsymbol{K}^e\big)\big]\\
&\quad + \frac{h^3}{12}\big[W_{\mathrm{mixt}}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) - 2W_{\mathrm{mixt}}\big(\boldsymbol{E}^e\,,\,\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big) + W_{\mathrm{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\big],
\end{aligned} \tag{110}$$

where W_{mixt} is given by equation (55). We compare this expression with our energy (equation (68)). Using the decomposition of tensors in planar and transversal parts (equation (92)), we deduce from equations (55) and (57) the relations

$$W_{\text{mixt}}(\boldsymbol{S},\boldsymbol{T}) = W_{\text{mixt}}(\boldsymbol{aS},\boldsymbol{aT}) + \frac{\mu+\mu_c}{2}\big(\boldsymbol{n}_0\boldsymbol{S}\big)\cdot\big(\boldsymbol{n}_0\boldsymbol{T}\big),$$
$$W_{\text{Coss}}(\boldsymbol{S},\boldsymbol{T}) = W_{\text{mixt}}(\boldsymbol{aS},\boldsymbol{aT}) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\big(\boldsymbol{n}_0\boldsymbol{S}\big)\cdot\big(\boldsymbol{n}_0\boldsymbol{T}\big). \tag{111}$$

Thus, using equation (111), the strain energy density (110) (obtained in Bîrsan et al. [12] for order $O(h^3)$) becomes

$$\widehat{W}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = \Big(h - K\,\frac{h^3}{12}\Big)\Big[W_{\text{mixt}}\big(\boldsymbol{aE}^e\big) + \frac{\mu+\mu_c}{2}\,\|\boldsymbol{n}_0\boldsymbol{E}^e\|^2 + W_{\text{curv}}\big(\boldsymbol{K}^e\big)\Big]$$
$$+\frac{h^3}{12}\Big[W_{\text{mixt}}\big(\boldsymbol{aE}^e\boldsymbol{b}+\boldsymbol{cK}^e\big) + \frac{\mu+\mu_c}{2}\,\|\boldsymbol{n}_0\boldsymbol{E}^e\boldsymbol{b}\|^2 - 2W_{\text{mixt}}\big(\boldsymbol{aE}^e\,,\,\boldsymbol{cK}^e\boldsymbol{b}^*\big) + W_{\text{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\Big]. \tag{112}$$

At the same time, our strain energy density (68) can be written with the help of equation (111) in the following alternative form:

$$W_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = \Big(h - K\,\frac{h^3}{12}\Big)\Big[W_{\text{mixt}}\big(\boldsymbol{aE}^e\big) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{E}^e\|^2 + W_{\text{curv}}\big(\boldsymbol{K}^e\big)\Big]$$
$$+\frac{h^3}{12}\Big[W_{\text{mixt}}\big(\boldsymbol{aE}^e\boldsymbol{b}+\boldsymbol{cK}^e\big) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{E}^e\boldsymbol{b}\|^2 - 2W_{\text{mixt}}\big(\boldsymbol{aE}^e\,,\,\boldsymbol{cK}^e\boldsymbol{b}^*\big) + W_{\text{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\Big]. \tag{113}$$

By comparing equations (112) and (113), we see that the only difference between these two strain energy densities resides in the coefficients of the transverse shear deformation terms $\|\boldsymbol{n}_0\boldsymbol{E}^e\|^2$ and $\|\boldsymbol{n}_0\boldsymbol{E}^e\boldsymbol{b}\|^2$. All other terms and coefficients in equations (112) and (113) are identical.

Note that the transverse shear coefficient in the present model (equation (113)) is the harmonic mean $(2\mu\,\mu_c)/(\mu+\mu_c)$, while in the energy density (112) (derived in Bîrsan et al. [12]) it is the arithmetic mean $(\mu+\mu_c)/2$. We mention that the same coefficient $(2\mu\,\mu_c)/(\mu+\mu_c)$ for the transverse shear energy has been obtained using Γ-convergence in Neff et al. [13] in the case of plates. This confirms the result (equation (113)) obtained in our present work. We remind that this coefficient is adjusted in many plate and shell models by a correction factor, the so-called *shear correction factor* (see for instance the discussions in Chróścielewski et al. [5], Altenbach [24] and Vlachoutsis [25]).

5.2.1. Further remarks

1. We remark that the strain energy density (equation (113)) obtained in this paper satisfies the invariance properties required by the local symmetry group of isotropic six-parameter shells. These invariance requirements have been established in a general theoretical framework in Eremeyev and Pietraszkiewicz [20].
2. The form of the constitutive relation (equation (113), equivalent to equation (68)) is remarkable, since one cannot find in the literature on six-parameter shells appropriate expressions of the strain energy density $W_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$ with coefficients depending on the initial curvature $\boldsymbol{b}$ and expressed in terms of the three-dimensional material constants. Indeed, the strain energy densities proposed in the literature are either simple expressions with constant coefficients (see, e.g., Bîrsan and Neff [8, 16] and Chróścielewski et al. [4, 5]), or general quadratic forms of $\boldsymbol{E}^e$, $\boldsymbol{K}^e$ with unidentified coefficients (see, e.g., f. (52) in Eremeyev and Pietraszkiewicz [20]).
3. We mention that the numerical treatment for the related *planar* Cosserat shell model derived in Neff [26, 27] has been presented in Sander et al. [28], using geodesic finite elements.
4. If the thickness h is sufficiently small, one can show that the strain energy density $W_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$ is a coercive and convex function of its arguments. Then, in view of Theorem 6 from Bîrsan and Neff [8], one can prove the existence of minimizers for our nonlinear Cosserat shell model.

5.3. Relation to the classical Koiter shell model

In this section, we discuss the relation to the classical shell theory and show that our strain energy density (equation (113)) can be reduced, in a certain sense, to the strain energy of the classical Koiter model.

Thus, if we consider that the three-dimensional material is a Cauchy continuum (with no microrotation), then the Cosserat couple modulus and the curvature energy W_{curv} are vanishing in the equations (6) and (7):

$$\mu_c = 0, \qquad W_{\text{curv}} \equiv 0. \tag{114}$$

Hence, the fourth-order constitutive tensor for shells (equation (96)) reduces to

$$C_S^{\alpha\beta\gamma\delta} = \mu\left(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma}\right) + \frac{2\lambda\,\mu}{\lambda+2\mu}\,a^{\alpha\beta}a^{\gamma\delta}\,, \tag{115}$$

which coincides with the tensor of linear plane-stress elastic moduli that appears in the Koiter model (see, e.g., Koiter [29], Ciarlet [30] and f. (101) in Steigmann [11]). In view of equations (56) and (114), we notice that in this case

$$W_{\text{mixt}}(\boldsymbol{S}) = W_{\text{Koit}}(\boldsymbol{S}), \tag{116}$$

where

$$W_{\text{Koit}}(\boldsymbol{S}) := \mu\,\|\text{sym}\,\boldsymbol{S}\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\,(\text{tr}\,\boldsymbol{S})^2 \tag{117}$$

is the quadratic form appearing in the Koiter model. We remind that the areal strain energy density for Koiter shells has the expression [11, 29, 30]

$$h\,W_{\text{Koit}}(\boldsymbol{\varepsilon}) + \frac{h^3}{12}\,W_{\text{Koit}}(\boldsymbol{\rho}), \tag{118}$$

where the *change of metric* tensor $\boldsymbol{\varepsilon}$ and the *change of curvature* tensor $\boldsymbol{\rho}$ are the nonlinear shell strain measures, which are given by

$$\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{1}{2}\left(\boldsymbol{m},_\alpha \cdot \boldsymbol{m},_\beta - a_{\alpha\beta}\right)\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta \;=\; \frac{1}{2}\left[\,(\text{Grad}_s\boldsymbol{m})^{\text{T}}(\text{Grad}_s\boldsymbol{m}) - \boldsymbol{a}\,\right],\\
\boldsymbol{\rho} &= \left(\boldsymbol{n}\cdot\boldsymbol{m},_{\alpha\beta} - \boldsymbol{n}_0\cdot\boldsymbol{a}_{\alpha,\beta}\right)\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta \;=\; -(\text{Grad}_s\boldsymbol{m})^{\text{T}}(\text{Grad}_s\boldsymbol{n}) - \boldsymbol{b}\,.
\end{aligned} \tag{119}$$

Here, $\boldsymbol{n}$ designates the unit normal vector to the deformed midsurface and we note that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$ are symmetric planar tensors.

To obtain the classical shell model as a special case of our approach, we adopt the Kirchhoff–Love hypotheses. Thus, we assume that the reference unit normal $\boldsymbol{n}_0$ becomes, after deformation, the unit normal to the deformed midsurface, i.e., $\boldsymbol{n}_0$ transforms to $\boldsymbol{n}$. But since we have $\boldsymbol{Q}_e\boldsymbol{n}_0 = \boldsymbol{Q}_e\boldsymbol{d}_3^0 = \boldsymbol{d}_3$, this assumption means that

$$\boldsymbol{n} = \boldsymbol{d}_3\,. \tag{120}$$

Then, we have $\boldsymbol{d}_3\cdot\boldsymbol{m},_\alpha = \boldsymbol{n}\cdot\boldsymbol{m},_\alpha = 0$ and the transverse shear deformations vanishes, since

$$\boldsymbol{n}_0\boldsymbol{E}^e = \boldsymbol{n}_0\big(\boldsymbol{Q}_e^{\text{T}}\text{Grad}_s\boldsymbol{m} - \boldsymbol{a}\big) = \big(\boldsymbol{n}_0\boldsymbol{Q}_e^{\text{T}}\big)\text{Grad}_s\boldsymbol{m} = \boldsymbol{d}_3\big(\boldsymbol{m},_\alpha \otimes \boldsymbol{a}^\alpha\big) = (\boldsymbol{d}_3\cdot\boldsymbol{m},_\alpha)\boldsymbol{a}^\alpha = \boldsymbol{0}. \tag{121}$$

This shows that the strain shell tensor is a planar tensor in this case, i.e.,

$$\boldsymbol{E}^e = E^e_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta \qquad \text{and} \qquad \boldsymbol{a}\boldsymbol{E}^e = \boldsymbol{E}^e.$$

In view of equations (114) and (121) and $\boldsymbol{b}\,\boldsymbol{b}^* = K\boldsymbol{a}$, we can put the strain energy density (equation (113)) in the following reduced form:

$$\widetilde{W}_{\text{shell}} = \left(h + K\,\frac{h^3}{12}\right)W_{\text{mixt}}\big(\boldsymbol{E}^e\big) + \frac{h^3}{12}\,W_{\text{mixt}}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) - 2\,\frac{h^3}{12}\,W_{\text{mixt}}\big(\boldsymbol{E}^e\,,\,(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e)\boldsymbol{b}^*\big). \tag{122}$$

We see that the right-hand side of equation (122) is a quadratic form of the planar tensors $\boldsymbol{E}^e$ and $\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e$. Let us express these two tensors in terms of the Koiter shell strain measures $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$.

From equations (119) and (35), it follows that

$$\begin{aligned}\boldsymbol{\varepsilon} &= \frac{1}{2}\big[\,(\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{m})^{\mathrm{T}}(\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{m}) - \boldsymbol{a}\,\big] = \frac{1}{2}\big[\,(\boldsymbol{E}^e+\boldsymbol{a})^{\mathrm{T}}(\boldsymbol{E}^e+\boldsymbol{a}) - \boldsymbol{a}\,\big]\\ &= \frac{1}{2}\big(\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{E}^e + \boldsymbol{a}\boldsymbol{E}^e + \boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{a}\big) \;=\; \frac{1}{2}\,\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{E}^e + \mathrm{sym}\big(\boldsymbol{a}\boldsymbol{E}^e\big),\end{aligned} \tag{123}$$

which means that

$$\mathrm{sym}\,\boldsymbol{E}^e \;=\; \boldsymbol{\varepsilon} \;-\; \frac{1}{2}\,\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{E}^e\,. \tag{124}$$

Similarly, using equations (119) and (120) and the relation $\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{d}_3 \;=\; \boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{b}$ (see f. (70) in Bîrsan et al. [12]), we find

$$\begin{aligned}\boldsymbol{\rho} &= -(\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{m})^{\mathrm{T}}(\boldsymbol{Q}_e^{\mathrm{T}}\mathrm{Grad}_s\boldsymbol{d}_3) - \boldsymbol{b} = -(\boldsymbol{E}^e+\boldsymbol{a})^{\mathrm{T}}(\boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{b}) - \boldsymbol{b}\\ &= -\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{c}\boldsymbol{K}^e - \boldsymbol{c}\boldsymbol{K}^e + \boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{b} \;=\; -\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{c}\boldsymbol{K}^e - \big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) + 2\big(\mathrm{sym}\,\boldsymbol{E}^e\big)\boldsymbol{b}\,.\end{aligned} \tag{125}$$

Substituting equation (124) into equation (125), we derive

$$\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e = 2\,\boldsymbol{\varepsilon}\,\boldsymbol{b} - \boldsymbol{\rho} - \boldsymbol{E}^{e,\mathrm{T}}(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e). \tag{126}$$

With the help of equations (124) and (126), we can write now the strain energy (122) as a function of the strain measures $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$: for the first term in equation (122), we obtain (from equations (117) and (123))

$$\begin{aligned}W_{\mathrm{Koit}}(\boldsymbol{\varepsilon}) &= \mu\,\|\,\boldsymbol{\varepsilon}\,\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\,\big(\mathrm{tr}\,\boldsymbol{\varepsilon}\big)^2\\ &= \mu\,\|\,\mathrm{sym}\,\boldsymbol{E}^e + \frac{1}{2}\,\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{E}^e\,\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\left[\mathrm{tr}\left(\mathrm{sym}\,\boldsymbol{E}^e + \frac{1}{2}\,\boldsymbol{E}^{e,\mathrm{T}}\boldsymbol{E}^e\right)\right]^2.\end{aligned} \tag{127}$$

Since our model is physically linear (the strain energy is quadratic in the strain measures), we can neglect the terms in equation (127) that are more than quadratic in $\boldsymbol{E}^e$ and find

$$W_{\mathrm{Koit}}(\boldsymbol{\varepsilon}) = \mu\,\|\,\mathrm{sym}\,\boldsymbol{E}^e\,\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\,\big(\mathrm{tr}\,\boldsymbol{E}^e\big)^2$$

i.e.,

$$hW_{\mathrm{Koit}}(\boldsymbol{\varepsilon}) \;=\; hW_{\mathrm{mixt}}(\boldsymbol{E}^e). \tag{128}$$

Thus, the extensional part of our strain energy density (122) coincides in this case with the extensional part of the Koiter model (equation (118)).

Similarly, we compute the other two terms of the energy (122) and discard the terms that are over-quadratic in the strain measures $\boldsymbol{E}^e$, $\boldsymbol{K}^e$: in view of equations (116) and (126), we have

$$\begin{aligned}W_{\mathrm{Koit}}(\boldsymbol{\rho}) = W_{\mathrm{mixt}}(\boldsymbol{\rho}) &= W_{\mathrm{mixt}}\big(2\,\boldsymbol{\varepsilon}\,\boldsymbol{b} - (\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e) - \boldsymbol{E}^{e,\mathrm{T}}(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e)\big)\\ &= W_{\mathrm{mixt}}\big(2\,\boldsymbol{\varepsilon}\,\boldsymbol{b} - (\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e)\big)\\ &= W_{\mathrm{mixt}}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) + 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\big) - 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\,,\,\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big).\end{aligned}$$

It follows that

$$W_{\mathrm{mixt}}(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e) \;=\; W_{\mathrm{Koit}}(\boldsymbol{\rho}) - 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\big) + 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\,,\,\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big)$$

and inserting equation (126) here we find, for the second term in the energy (122),

$$\begin{aligned}W_{\mathrm{mixt}}(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e) &= W_{\mathrm{Koit}}(\boldsymbol{\rho}) - 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\big) + 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\,,\,2\,\boldsymbol{\varepsilon}\,\boldsymbol{b} - \boldsymbol{\rho}\big)\\ &= W_{\mathrm{Koit}}(\boldsymbol{\rho}) + 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\big) - 4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\,,\,\boldsymbol{\rho}\big).\end{aligned} \tag{129}$$

For the last term in equation (122), we write, with the help of equation (126),

$$(\boldsymbol{E}^e\boldsymbol{b}+c\boldsymbol{K}^e)\boldsymbol{b}^* \;=\; 2K\,\boldsymbol{\varepsilon}-\boldsymbol{\rho}\,\boldsymbol{b}^*-\boldsymbol{E}^{e,\mathrm{T}}(\boldsymbol{E}^e\boldsymbol{b}+c\boldsymbol{K}^e)\boldsymbol{b}^* \tag{130}$$

and derive from equations (124) and (130)

$$\begin{aligned} W_{\mathrm{mixt}}\big(\boldsymbol{E}^e,(\boldsymbol{E}^e\boldsymbol{b}+c\boldsymbol{K}^e)\boldsymbol{b}^*\big) &= W_{\mathrm{mixt}}\big(\mathrm{sym}\,\boldsymbol{E}^e,\,2K\,\boldsymbol{\varepsilon}-\boldsymbol{\rho}\,\boldsymbol{b}^*\big)\\ &= W_{\mathrm{mixt}}\big(\,\boldsymbol{\varepsilon}\,,\,2K\,\boldsymbol{\varepsilon}-\boldsymbol{\rho}\,\boldsymbol{b}^*\big)=2K\,W_{\mathrm{Koit}}(\boldsymbol{\varepsilon})-W_{\mathrm{mixt}}\big(\,\boldsymbol{\varepsilon}\,,\,\boldsymbol{\rho}\,\boldsymbol{b}^*\big), \end{aligned} \tag{131}$$

We substitute equations (128), (129) and (131) into equation (122) and obtain

$$\begin{aligned} \widetilde{W}_{\mathrm{shell}}(\boldsymbol{\varepsilon},\boldsymbol{\rho}) = \Big(h+K\frac{h^3}{12}\Big)\,W_{\mathrm{Koit}}\big(\boldsymbol{\varepsilon}\big)+\frac{h^3}{12}\,\Big(W_{\mathrm{Koit}}(\boldsymbol{\rho})+4W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\boldsymbol{b}\big)-4W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\boldsymbol{b}\,,\,\boldsymbol{\rho}\big)\Big)\\ -2\,\frac{h^3}{12}\,\Big(2K\,W_{\mathrm{Koit}}(\boldsymbol{\varepsilon})-W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,,\,\boldsymbol{\rho}\,\boldsymbol{b}^*\big)\Big), \end{aligned} \tag{132}$$

which can be written in view of equation (116) in the form

$$\widetilde{W}_{\mathrm{shell}}(\boldsymbol{\varepsilon},\boldsymbol{\rho}) \;=\; h\,W_{\mathrm{Koit}}\big(\boldsymbol{\varepsilon}\big)+\frac{h^3}{12}\,W_{\mathrm{Koit}}(\boldsymbol{\rho})+\frac{h^3}{12}\Big[4\,W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,\boldsymbol{b}\,,\,\boldsymbol{\varepsilon}\,\boldsymbol{b}-\boldsymbol{\rho}\big)-\;W_{\mathrm{mixt}}\big(\boldsymbol{\varepsilon}\,,\,3K\,\boldsymbol{\varepsilon}-2\boldsymbol{\rho}\,\boldsymbol{b}^*\big)\Big]. \tag{133}$$

The terms in the square brackets in equation (133) involve the initial curvature of the shell through the tensor $\boldsymbol{b}$, the cofactor $\boldsymbol{b}^* = 2H\boldsymbol{a}-\boldsymbol{b}$ and the determinant $K = \det\boldsymbol{b}$ (Gauß curvature). These terms vanish in the case of plates (since $\boldsymbol{b} = \boldsymbol{0}$); moreover, they can also be neglected for sufficiently thin shells, provided that the midsurface strain is small. We note that the corresponding terms in the classical shell theory have been neglected using similar arguments; see the discussion about the term W_3 in f. (57) in Steigmann [11]. Finally, if we retain only the leading extensional and bending terms in equation (133), we obtained the reduced classical form

$$\widetilde{W}_{\mathrm{shell}}(\boldsymbol{\varepsilon},\boldsymbol{\rho}) \;=\; h\,W_{\mathrm{Koit}}\big(\boldsymbol{\varepsilon}\big)+\frac{h^3}{12}\,W_{\mathrm{Koit}}(\boldsymbol{\rho})\,, \tag{134}$$

in accordance with the Koiter energy density (equation (118)).

In conclusion, our model can be regarded as a generalization of the classical Koiter model in the framework of six-parameter shell theory.

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ORCID iD

Mircea Birsan http://orcid.org/0000-0002-1360-4044

References

[1] Berdichevsky, V, and Misyura, V. Effect of accuracy loss in classical shell theory. *J Appl Mech* 1992; 59: S217–S223.
[2] Reissner, E. Linear and nonlinear theory of shells. In: Fung, Y, and Sechler, E (eds.) *Thin Shell Structures*. Englewood Cliffs, NJ: Prentice-Hall, 1974, 29–44.
[3] Libai, A, and Simmonds, J. *The Nonlinear Theory of Elastic Shells*. 2nd ed. Cambridge: Cambridge University Press, 1998.
[4] Chróścielewski, J, Makowski, J, and Pietraszkiewicz, W. *Statics and Dynamics of Multifold Shells: Nonlinear Theory and Finite Element Method*. Warsaw: Wydawnictwo IPPT PAN, 2004 (in Polish).
[5] Chróścielewski, J, Pietraszkiewicz, W, and Witkowski, W. On shear correction factors in the non-linear theory of elastic shells. *Int J Solids Struct* 2010; 47: 3537–3545.
[6] Eremeyev, V, and Pietraszkiewicz, W. Thermomechanics of shells undergoing phase transition. *J Mech Phys Solids* 2011; 59: 1395–1412.
[7] Pietraszkiewicz, W. Refined resultant thermomechanics of shells. *Int J Eng Science* 2011; 49: 1112–1124.

[8] Bîrsan, M, and Neff, P. Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations. *Math Mech Solids* 2014; 19(4): 376–397.
[9] Steigmann, D. Two-dimensional models for the combined bending and stretching of plates and shells based on three-dimensional linear elasticity. *Int J Eng Sci* 2008; 46: 654–676.
[10] Steigmann, D. Extension of Koiter's linear shell theory to materials exhibiting arbitrary symmetry. *Int J Eng Sci* 2012; 51: 216–232.
[11] Steigmann, D. Koiter's shell theory from the perspective of three-dimensional nonlinear elasticity. *J Elast* 2013; 111: 91–107.
[12] Bîrsan, M, Ghiba, I, Martin, R et al. Refined dimensional reduction for isotropic elastic Cosserat shells with initial curvature. *Math Mech Solids* 2019; 24(12): 4000–4019.
[13] Neff, P, Hong, KI, and Jeong, J. The Reissner–Mindlin plate is the Γ-limit of Cosserat elasticity. *Math Mod Meth Appl Sci* 2010; 20: 1553–1590.
[14] Neff, P, and Münch, I. Curl bounds Grad on SO(3). *ESAIM: Control Optim Calculus Var* 2008; 14: 148–159.
[15] Pietraszkiewicz, W, and Eremeyev, V. On natural strain measures of the non-linear micropolar continuum. *Int J Solids Struct* 2009; 46: 774–787.
[16] Bîrsan, M, and Neff, P. Analysis of the deformation of Cosserat elastic shells using the dislocation density tensor. In: dell'Isola, F et al. (eds.) *Advanced Methods of Continuum Mechanics for Materials and Structures*. Singapore: Springer Nature, 2017, 13–30.
[17] Neff, P, Bîrsan, M, and Osterbrink, F. Existence theorem for geometrically nonlinear Cosserat micropolar model under uniform convexity requirements. *J Elast* 2015; 121: 119–141.
[18] Neff, P. Existence of minimizers for a finite-strain micromorphic elastic solid. *P Roy Soc Edinb A* 2006; 136A: 997–1012.
[19] Ciarlet, P. *Mathematical Elasticity, Vol. III: Theory of Shells*. 1st ed. Amsterdam: North-Holland, 2000.
[20] Eremeyev, V and Pietraszkiewicz, W. Local symmetry group in the general theory of elastic shells. *J Elast* 2006; 85: 125–152.
[21] Bîrsan, M, and Neff, P. Shells without drilling rotations: a representation theorem in the framework of the geometrically nonlinear 6-parameter resultant shell theory. *Int J Eng Sci* 2014; 80: 32–42.
[22] Bîrsan, M, and Neff, P. On the dislocation density tensor in the Cosserat theory of elastic shells. In: Naumenko, K and Aßmus, M (eds.) *Advanced Methods of Continuum Mechanics for Materials and Structures*. Singapore: Springer, 2016, 391–413.
[23] Eremeyev, V, and Pietraszkiewicz, W. The nonlinear theory of elastic shells with phase transitions. *J Elast* 2004; 74: 67–86.
[24] Altenbach, H. An alternative determination of transverse shear stiffnesses for sandwich and laminated plates. *Int J Solids Struct* 2000; 37: 3503–3520.
[25] Vlachoutsis, S. Shear correction factors for plates and shells. *Int J Numer Methods Eng* 1992; 33: 1537–1552.
[26] Neff, P. A geometrically exact Cosserat-shell model including size effects, avoiding degeneracy in the thin shell limit. Part I: formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus. *Continuum Mech Thermodyn* 2004; 16: 577–628.
[27] Neff, P. A geometrically exact planar Cosserat shell-model with microstructure: existence of minimizers for zero Cosserat couple modulus. *Math Models Methods Appl Sci* 2007; 17: 363–392.
[28] Sander, O, Neff, P, and Bîrsan, M. Numerical treatment of a geometrically nonlinear planar Cosserat shell model. *Comput Mechanics* 2016; 57: 817–841.
[29] Koiter, W. A consistent first approximation in the general theory of thin elastic shells. In: Koiter, W (ed.) *The Theory of Thin Elastic Shells*. Amsterdam: North-Holland, 1960, 12–33.
[30] Ciarlet, P. *An Introduction to Differential Geometry with Applications to Elasticity*. Dordrecht: Springer, 2005.

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