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# A commented translation of Hans Richter's early work "The isotropic law of elasticity"

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**Abstract**
We provide a faithful translation of Hans Richter's important 1948 paper "Das isotrope Elastizitätsgesetz" from its original German version into English. Our introduction summarizes Richter's achievements.

**Keywords**
Nonlinear elasticity, isotropic tensor functions, hyperelasticity, logarithmic stretch, volumetric-isochoric split, Hooke's law, finite deformations, isotropy

## Introduction

Shortly after the second World War, in a series of papers [1–4] from 1948–1952, Hans Richter (1912–1978) laid down his general format of isotropic nonlinear elasticity based on a rather modern approach with direct tensor notation. By translating his work "Das isotrope Elastizitätsgesetz" [1], we aim to make his development, which precedes later work in the field by several decades, accessible to the international audience.

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Let us briefly summarize Richter's achievement in this paper. He uses, for the time, rather advanced methods of matrix analysis (including the theory of primary matrix functions [5]) and employs the left polar decomposition [6–8] of the deformation gradient $F = VR$ into a stretch $V \in \mathrm{Sym}^+(3)$ and a rotation $R \in \mathrm{SO}(3)$. For Richter, the "physical stress tensor" is the Cauchy stress tensor $\sigma \in \mathrm{Sym}(3)$. From the coaxiality between $\sigma$ and $V$ for an isotropic response, he deduces the representation formula for isotropic tensor functions (the *Richter representation*, see (2.6) of his text)

$$\sigma = g_1(I_1, I_2, I_3) \cdot \mathbb{1} + g_2(I_1, I_2, I_3) \cdot V + g_3(I_1, I_2, I_3) \cdot V^2 \tag{1}$$

(predating the Rivlin–Ericksen representation theorem [9] by seven years) where $g_i$, $i = 1, 2, 3$ are scalar valued functions of the invariants $I_\nu$, $\nu = 1, 2, 3$, with

$$I_1 = \mathrm{tr}(V), \quad I_2 = \frac{1}{2}\mathrm{tr}(V^2), \quad I_3 = \det V.$$

Alongside, Richter introduces the logarithmic stretch tensor $L = \log V$ without citing the previous work of Hencky [10–18]. He then turns to the question of what happens if the relation (1) is derived from a stored energy $W(I_1, I_2, I_3)$, i.e. when (1) is consistent with hyperelasticity. He obtains the correct representation (see (3.11) in his text)

$$\sigma = \frac{\partial W}{\partial I_3} \cdot \mathbb{1} + \frac{1}{I_3} \cdot \frac{\partial W}{\partial I_1} \cdot V + \frac{1}{I_3} \cdot \frac{\partial W}{\partial I_2} \cdot V^2, \qquad W = W(I_1, I_2, I_3). \tag{2}$$

In the next section, Richter introduces the *multiplicative split* of the elastic stretch $V$ into volume-preserving (isochoric) parts and volume change (see (4.1) in his text)

$$V = \frac{V}{(\det V)^{\frac{1}{3}}} \cdot (\det V)^{\frac{1}{3}} \cdot \mathbb{1} \tag{3}$$

and he observes that the logarithmic stretch tensor additively separates both contributions by using the classical deviator operation (see (4.2) in his text) such that

$$\log V = \mathrm{dev} \log V + \frac{1}{3}\mathrm{tr}(\log V) \cdot \mathbb{1} = \log\left(\frac{V}{(\det V)^{\frac{1}{3}}}\right) + \frac{1}{3}\log\det V \cdot \mathbb{1}, \quad \mathrm{dev}\, X = X - \frac{1}{3}(X) \cdot \mathbb{1}. \tag{4}$$

He also observes that the invariants based on the logarithmic stretch tensor satisfy certain algebraic relations, cf. Criscione et al. [19]. In Richter's fifth section, he introduces the *volumetric-isochoric split*

$$\begin{aligned} W(F) &= W_{\mathrm{iso}}(\mathrm{dev} \log V) + W_{\mathrm{vol}}(\mathrm{tr}(\log V)) \\ &= W_{\mathrm{iso}}\left(\log\left(\frac{V}{(\det V)^{\frac{1}{3}}}\right)\right) + W_{\mathrm{vol}}(\log\det V) = \widetilde{W}_{\mathrm{iso}}\left(\frac{V}{(\det V)^{\frac{1}{3}}}\right) + \widetilde{W}_{\mathrm{vol}}(\det V) \end{aligned}$$

of the stored energy (often erroneously attributed to Flory [20]) and he immediately obtains the important result:

*An isotropic energy is additively split into volumetric and isochoric parts if and only if the mean Cauchy stress* $\frac{1}{3}\mathrm{tr}\sigma$ *is only a function of the relative volume change* $\det V$. *In that case*,

$$\frac{1}{3}\mathrm{tr}\sigma = \frac{1}{\det V} \cdot W'_{\mathrm{vol}}(\log\det V) = \widetilde{W}'_{\mathrm{vol}}(\det V). \tag{5}$$

This result has been rediscovered and rederived multiple times [21–26]. In addition, Richter shows that this property of the volumetric-isochoric split is invariant under a change of the reference temperature. Finally, he poses the question as to whether a linear relation between $\sigma$ and $V$ in the form (the Hooke's law as he perceives it)

$$\sigma = 2\mu(V - \mathbb{1}) + \lambda\,\mathrm{tr}(V - \mathbb{1}) \cdot \mathbb{1}, \tag{6}$$

where $\mu > 0$ is the shear modulus and $\lambda$ is the second Lamé parameter, can be consistent with hyperelasticity. A short calculus reveals that (6) is hyperelastic if and only if $2\mu = \lambda$, i.e. for a Poisson ratio $\nu = \frac{1}{3}$ (which is approximately satisfied for many metals, e.g. aluminum). For all other values of $\nu$, Hooke's law is incompatible with the hyperelastic approach and Richter proposes to use instead (the quadratic Hencky energy [10, 27])

$$W(F) = \mu \|\operatorname{dev} \log V^2\| + \frac{2\mu + 3\lambda}{6} \operatorname{tr}^2(\log V)$$

with the induced stress-strain law

$$\sigma \cdot \det F = \tau = 2\mu \log V + \lambda \operatorname{tr}(\log V) \cdot \mathbb{1}, \tag{7}$$

where $\tau$ is the Kirchhoff stress tensor.

We will briefly discuss the constitutive relation (6). In order to check hyperelasticity of the Cauchy stress-stretch relation in this case,
we use the representation, consistent with (2),

$$\sigma(V) = \frac{2\mu}{J} D_V W(V) \cdot V, \qquad J = \det V, \tag{8}$$

and consider the energy $W(F) = 2\mu \det V[\operatorname{tr}(V) - 4]$. Then $\sigma(V) = 2\mu(V - \mathbb{1}) + 2\mu(V - \mathbb{1}) \cdot \mathbb{1}$.

Since $\operatorname{tr}[\sigma(V)] = 8\mu \operatorname{tr}(V - \mathbb{1})$ and $\operatorname{tr}[\sigma(\alpha\mathbb{1})] = 24\mu(\alpha - 1)$, the Cauchy stress tensor given by (6) with $2\mu = \lambda$ is injective (but not bijective, since for $\operatorname{tr}(\sigma) = -K^+ < -24\mu$ there does not exist a stretch $V \in \operatorname{Sym}^+(3)$ such that $(\sigma(V)) = -K^+$). Furthermore, note that

$$[\operatorname{tr}(V)]^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = \operatorname{tr}(B) + 2\operatorname{tr}(\operatorname{Cof} B), \tag{9}$$

where $\lambda_i$ are the singular values of the deformation gradient $F$. Then

$$\begin{aligned} 2\mu \det V[\operatorname{tr}(V) - 4] &= 2\mu\sqrt{\det(B)}\{\sqrt{\operatorname{tr}(B) + 2\operatorname{tr}(\operatorname{Cof} B)} - 4\} = 2\mu\sqrt{I_3}\{\sqrt{I_1 + 2I_2} - 4\} \\ &= W(I_1, I_2, I_3), \qquad I_1 = \operatorname{tr}(B), \quad I_2 = \operatorname{tr}(\operatorname{Cof} B), \quad I_3 = \det B. \end{aligned} \tag{10}$$

For this energy, the weak empirical inequalities [28] $\frac{\partial W}{\partial I_1} > 0$ and $\frac{\partial W}{\partial I_2} > 0$ are satisfied. The principal Cauchy stresses are given by $\sigma_i = 2\mu \cdot (\lambda_i - 1 + (\lambda_1 + \lambda_2 + \lambda_3 - 3))$, which shows that the tension-extension (TE) inequalities and the Baker–Ericksen (BE) inequalities [29], given by

$$0 < \frac{\partial \sigma_i}{\partial \lambda_i} = 2\mu \cdot (1 + 1) = 4\mu \qquad \text{and} \qquad 0 < (\sigma_i - \sigma_j)(\lambda_i - \lambda_j) = 2\mu(\lambda_i - \lambda_j)^2$$

respectively, are satisfied as well. We also note that $W(V) = 2\mu \cdot \det V \cdot [\operatorname{tr}(V) - 4]$ is the Shield transformation [30] of $W^*(F) = 2\mu \cdot [\operatorname{tr}(V^{-1}) - 4]$, where

$$W^*(F) = 2\mu\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - 4\right) = g(\lambda_1, \lambda_2, \lambda_3) \tag{11}$$

has the Valanis–Landel form[1] [31] and $g$ is convex in $(\lambda_1, \lambda_2, \lambda_3)$; the TE inequalities are satisfied as well.

Richter's paper is not only written in German, but his notation strongly relies on German fraktur letters, which makes reading his original work rather challenging. In our faithful translation of his paper, we have therefore updated the notation to more current conventions; a complete list of notational changes is provided in the Appendix. Richter's original equation numbering has been maintained throughout.

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## Note

1. This calculus shows that the Valanis–Landel form is not invariant under the Shield transformation. In addition, the mapping $U \mapsto T_{\text{Biot}} = D_U W(U)$ of the stretch $U = \sqrt{F^T F}$ to the Biot stress tensor $T_{\text{Biot}}$ is strictly monotone.

## References

[1] Richter, H. Das isotrope Elastizitätsgesetz. *Z Angew Math Mech* 1948; 28(7/8): 205–209.
[2] Richter, H. Verzerrungstensor, Verzerrungsdeviator und Spannungstensor bei endlichen Formänderungen. *Z Angew Math Mech* 1949; 29(3): 65–75.
[3] Richter, H. Zum Logarithmus einer Matrix. *Arch Math* 1949; 2(5): 360–363.
[4] Richter, H. Zur Elastizitätstheorie endlicher Verformungen. *Math Nachr* 1952; 8(1): 65–73.
[5] Martin, R J, and Neff, P. Some remarks on the monotonicity of primary matrix functions on the set of symmetric matrices. *Arch Appl Mech* 2015; 85(12): 1761–1778.
[6] Lankeit, J, Neff, P, and Nakatsukasa, Y. The minimization of matrix logarithms: On a fundamental property of the unitary polar factor. *Linear Algebra Appl* 2014; 449: 28–42.
[7] Neff, P, Nakatsukasa, Y, and Fischle, A. A logarithmic minimization property of the unitary polar factor in the spectral and Frobenius norms. *SIAM J Matrix Anal Appl* 2014; 35(3): 1132–1154.
[8] Neff, P, Lankeit, J, and Madeo, A. On Grioli's minimum property and its relation to Cauchy's polar decomposition. *Int J Eng Sci* 2014; 80: 209–217.
[9] Rivlin, RS, and Ericksen, JL. Stress-deformation relations for isotropic materials. *J Ration Mech Anal* 1955; 4: 323–425.
[10] Hencky, H. Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. *Z Tech Phys* 1928; 9(6): 215–220.
[11] Hencky, H. Welche Umstände bedingen die Verfestigung bei der bildsamen Verformung von festen isotropen Körpern? *Z Phys* 1929; 55(3–4): 145–155.
[12] Hencky, H. Das Superpositionsgesetz eines endlich deformierten relaxationsfähigen elastischen Kontinuums und seine Bedeutung für eine exakte Ableitung der Gleichungen für die zähe Flüssigkeit in der Eulerschen Form. *Ann Phys* 1929; 394(6): 617–630.
[13] Hencky, H. The law of elasticity for isotropic and quasi-isotropic substances by finite deformations. *J Rheol* 1931; 2(2): 169–176.
[14] Neff, P, Eidel, B, and Martin, RJ. Geometry of logarithmic strain measures in solid mechanics. *Arch Ration Mech Anal* 2016; 222(2): 507–572.
[15] Neff, P, Ghiba, ID, and Lankeit, J. The exponentiated Hencky-logarithmic strain energy. Part I: Constitutive issues and rank-one convexity. *J Elast* 2015; 121(2): 143–234.
[16] Neff, P, Lankeit, J, Ghiba, ID, et al. The exponentiated Hencky-logarithmic strain energy. Part II: Coercivity, planar polyconvexity and existence of minimizers. *Z Angew Math Phys* 2015; 66(4): 1671–1693.
[17] Neff, P, and Ghiba, ID. The exponentiated Hencky-logarithmic strain energy. Part III: Coupling with idealized isotropic finite strain plasticity. *Continuum Mech Thermodyn* 2016; 28(1): 477–487.
[18] Neff, P, Münch, I, Martin, RJ., and Rediscovering, G. F. Becker's early axiomatic deduction of a multiaxial nonlinear stress–strain relation based on logarithmic strain. *Math Mech Solids* 2016; 21(7): 856–911.
[19] Criscione, JC, Humphrey, JD, Douglas, AS, et al. An invariant basis for natural strain which yields orthogonal stress response terms in isotropic hyperelasticity. *J Mech Phys Solids* 2000; 48(12): 2445–2465.
[20] Flory, PJ. Thermodynamic relations for high elastic materials. *Trans Faraday Soc* 1961; 57: 829–838.
[21] Charrier, P, Dacorogna, B, Hanouzet, B, et al. An existence theorem for slightly compressible materials in nonlinear elasticity. *SIAM J Math Anal* 1988; 19(1): 70–85.
[22] Murphy, J, and Rogerson, G. Modelling slight compressibility for hyperelastic anisotropic materials. *J Elast* 2018; 131(2): 171–181.
[23] Sansour, C. On the physical assumptions underlying the volumetric-isochoric split and the case of anisotropy. *Eur J Mech Solids* 2008; 27(1): 28–39.
[24] Favrie, N, Gavrilyuk, S, and Ndanou, S. A thermodynamically compatible splitting procedure in hyperelasticity. *J Comput Phys* 2014; 270: 300–324.

[25] Federico, S. Volumetric-distortional decomposition of deformation and elasticity tensor. *Math Mech Solids* 2010; 15(6): 672–690.
[26] Ndanou, S, Favrie, N, and Gavrilyuk, S. The piston problem in hyperelasticity with the stored energy in separable form. *Math Mech Solids* 2017; 22(1): 101–113.
[27] Neff, P, Eidel, B, and Martin, RJ. The axiomatic deduction of the quadratic Hencky strain energy by Heinrich Hencky. arXiv:14024027, 2014.
[28] Thiel, C, Voss, J, Martin, RJ, et al. Do we need Truesdell's empirical inequalities? On the coaxiality of stress and stretch. *Int J Non-Linear Mech*. Epub ahead of print 13 February 2019. DOI: 10.1016/j.ijnonlinmec.2019.02.004.
[29] Baker, M, and Ericksen, JL. Inequalities restricting the form of the stress-deformation relation for isotropic elastic solids and Reiner-Rivlin fluids. *J Washington Acad Sci* 1954; 44(2): 33–35.
[30] Shield, RT. Inverse deformation results in finite elasticity. *Z Angew Math Phys* 1967; 18(4): 490–500.
[31] Valanis, KC, and Landel, RF. The strain-energy function of a hyperelastic material in terms of the extension ratios. *J Appl Phys* 1967; 38(7): 2997–3002.

# The isotropic law of elasticity

By *Hans Richter* in Haltingen (Lörrach)

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**Abstract**
From the demands of isotropy and because of the existence of thermodynamic potentials, a general form of the three-dimensional law of elasticity is stated. In doing so, the logarithmic matrix of relative elongations is used, which permits the separation of the variation of the volume and that of the shape by simply forming the deviator. The resilience energy is exactly the sum of the energy of the variation of the volume and that of the shape, if the average tension depends only on the variation of the volume. For finite deformations, the law of *Hooke* is permissible only in the case $\nu = \frac{1}{3}$.

Aus der Forderung der Isotropie und der Existenz der thermodynamischen Potentiale wird für das räumliche Elastizitätsgesetz eine allgemeine Form angegeben, wobei die logarithmische Dehnungsmatrix verwendet wird, bei der die Trennung in Volum- und Gestaltänderung durch gewöhnliche Deviatorbildung möglich ist. Die elastische Energie ist genau dann die Summe aus Volum- und Gestaltänderungsenergie, wenn die mittlere Spannung nur von der Volumänderung abhängt. Das *Hookesche* Gesetz ist für endliche Verzerrungen nur bei $\nu = \frac{1}{3}$ zulässig.

En supposant l'isotropie et l'existence des potentiels thermodynamiques, on donne une forme générale de la loi de l'élasticité en se servant d'une matrix logarithmique d'allongement. Ce procédé permet une séparation des changements de volume et de forme par une simple formation de déviateur. Si la tension moyenne ne dépend que du changement de volume, l'énergie d'élasticité est la somme des énergies de changement du volume et de la forme. La loi de *Hooke* n'est admissible que pour $\nu = \frac{1}{3}$.

## 1. Definitions

In the generalization of Hooke's law, a material is called *purely elastic* if the Cauchy stresses depend in a uniquely reversible way on the stretches. Strictly speaking, however, it is necessary to discuss the heat transfer that occurs in the tensile test; in particular, it is necessary to distinguish between an adiabatic and isothermal law of elasticity. This choice also clarifies what is meant by *strains*, since strains on the adiabat resp. isotherm can be referred e.g. to the initial state, for which the stresses disappear completely. The strains can also be referred to a stress-free initial state at an arbitrarily chosen initial temperature $\Theta_0$ instead. Then the stress-free state at another temperature $\Theta$ corresponds, in the case of an isotropic material, to uniform stretches in all directions, i.e. the thermal expansion. In this manner the law of thermal expansion is included in the elastic law. Of course, the affected material must be assumed not to change permanently by changes in temperature within the considered temperature range.

Thus, we assume a stress-free state at a temperature $\Theta_0$. Let the deformation of the material into another state be characterized by the matrix $F$ and the related stresses by the stress tensor $\sigma$.[2] We call the material *ideally elastic* if $\sigma$ depends uniquely on $F$ and $\Theta$. The material is said to be *isotropic* if this dependence is invariant under Euclidean rotations.

When solving the problem of finding the most general form of this dependence, one appropriately operates with matrices, where the following abbreviations are used:

$X^{\mathrm{T}}$ is the matrix obtained by reflecting $X$ over its main diagonal. $(X)_{ik}$ is the entry in the $i$th row and the $k$th column of $X$. $\det X$ is the determinant of $X$. $\operatorname{tr} X$ is the sum of the elements on the main diagonal of $X$: called the *trace of* $X$. $\mathbb{1}$ is the identity tensor. If $f(x) = \sum a_n \cdot x^n$, then, assuming convergence, $f(X) = \sum a_n X^n$.

Recall the following simple statements:

$$\operatorname{tr}(X \cdot Y) = \operatorname{tr}(Y \cdot X). \tag{1.1}$$

$$\operatorname{tr}(X \cdot \mathrm{d} \log Y) = \operatorname{tr}(X \cdot Y^{-1} \cdot \mathrm{d}Y) \tag{1.2}$$

if $X$ commutes with $Y$, but not necessarily with $\mathrm{d}Y$.

$$\log(\det X) = \mathrm{tr}(\log X), \tag{1.3}$$

if $\log X$ is well defined.

$$\begin{aligned} &\text{For a pure rotation } R: \quad RR^{\mathrm{T}} = \mathbb{1}. \\ &\text{For a pure stretch } V: \quad V = V^{\mathrm{T}}. \end{aligned} \tag{1.4}$$

Every $X$ can be represented in the form:

$$X = V \cdot R, \tag{1.5}$$

where the multiplication is to be read in its functional notation from right to left.

## 2. Consequence of isotropy

According to (1.5), $F$ can be interpreted as a rotation $R$ followed by a stretch $V$, where the principal stretch directions of the latter are rotated against those of the coordinate axes. For the case of isotropic materials, the application of $R$ must not have any influence on $\sigma$. Therefore, $\sigma$ is a function of $V$ and $\Theta$. For a given $F$, we can find $V$ by using (1.4) and (1.5) by

$$FF^{\mathrm{T}} = VRR^{\mathrm{T}}V^{\mathrm{T}} = V^2. \tag{2.1}$$

The most general coaxial relation between $\sigma$ and $V$ that fulfills the invariance under rotations is now, obviously,

$$\sigma = f(V; I_1, I_2, I_3, \Theta), \tag{2.2}$$

where $I_\nu$ are the invariants[3] of $V$.

Instead of $V$, one can also use a uniquely invertible function of $V$. As we will see later on, it is appropriate to use the *logarithmic stretch*

$$L = \log V, \tag{2.3}$$

which is always defined because of the positive eigenvalues of $V$. We denote the invariants of $L$ by

$$j = \mathrm{tr}(L), \qquad k = \mathrm{tr}(L^2) \qquad \text{and} \qquad l = \mathrm{tr}(L^3). \tag{2.4}$$

Further, from (1.3) and (2.1) we obtain: $j = \frac{1}{2}\mathrm{tr}(\log(FF^{\mathrm{T}})) = \frac{1}{2}\log(\det(FF^{\mathrm{T}})) = \log(\det F)$.

Instead of (2.2), we can now write

$$\sigma = f(L; j, k, l, \Theta). \tag{2.5}$$

Here, $\mathrm{tr}(\sigma)$, $\mathrm{tr}(\sigma L)$ and $\mathrm{tr}(\sigma L^2)$ are functions of $j$, $k$, $l$ and $\Theta$ due to (2.5). If we now define the invariants $f_1, f_2$ and $f_3$ as the solutions to the system of equations

$$\begin{aligned} \mathrm{tr}(\sigma) &= f_1\mathrm{tr}(\mathbb{1}) + f_2\mathrm{tr}(L) + f_3\mathrm{tr}(L^2) \\ \mathrm{tr}(\sigma L) &= f_1\mathrm{tr}(L) + f_2\mathrm{tr}(L^2) + f_3\mathrm{tr}(L^3) \\ \mathrm{tr}(\sigma L^2) &= f_1\mathrm{tr}(L^2) + f_2\mathrm{tr}(L^3) + f_3\mathrm{tr}(L^4) \end{aligned}$$

with, in general, nonvanishing determinant, then we have for

$$X = f_1 \cdot \mathbb{1} + f_2 \cdot L + f_3 \cdot L^2: \qquad \mathrm{tr}(\sigma L^\nu) = \mathrm{tr}(XL^\nu) \quad \text{with} \quad \nu = 0, 1, 2.$$

Since $\sigma$ is coaxial to $L$, it is completely determined by $\mathrm{tr}(\sigma)$, $\mathrm{tr}(\sigma L)$ and $\mathrm{tr}(\sigma L^2)$. Therefore, $\sigma \equiv X$ holds; i.e.

$$\sigma = f_1(j, k, l, \Theta) \cdot \mathbb{1} + f_2(j, k, l, \Theta) \cdot L + f_3(j, k, l, \Theta) \cdot L^2. \tag{2.6}$$

Hence, we have found the most general isotropic relation. Using $V$ instead of $L$, we would correspondingly obtain:

$$\sigma = g_1(I_\nu, \Theta) \cdot \mathbb{1} + g_2(I_\nu, \Theta) \cdot V + g_3(I_\nu, \Theta) \cdot V^2. \tag{2.7}$$

## 3. Consequence of the potential

The internal energy of the material per unit volume in the initial state is denoted by

$$E = E(j, k, l, \Theta); \tag{3.1}$$

the entropy is denoted by

$$S = S(j, k, l, \Theta). \tag{3.2}$$

Then the free energy $W$ takes the form

$$W = E - \Theta \cdot S = W(j, k, l, \Theta). \tag{3.3}$$

If $\mathrm{d}A$ is now the differential of the work done by the element of volume, then

$$\mathrm{d}A = -\mathrm{d}E + \Theta \cdot \mathrm{d}S = -\mathrm{d}W - S \cdot \mathrm{d}\Theta. \tag{3.4}$$

Thus for isothermal elastic changes, we have

$$\mathrm{d}A = -(\mathrm{d}W)_{\Theta = \mathrm{const.}}; \tag{3.5}$$

whereas for adiabatic changes

$$\mathrm{d}A = -(\mathrm{d}E)_{S = \mathrm{const.}}, \tag{3.6}$$

where $\Theta$ has to be eliminated in (3.1) and (3.2), so that $E$ appears as a function of $j$, $k$, $l$ and $S$.

In order to calculate $\mathrm{d}A$, we transition from a deformation $F$ to the neighboring deformation $F + \mathrm{d}F$. Since a pure rotation has no influence on $\mathrm{d}A$, we can assume that $F$ is a pure stretch. Let $e_1$, $e_2$ and $e_3$ be the unit vectors in the principal stretch directions of $V$, which can be interpreted as coordinate vectors. Let $\sigma_1$, $\sigma_2$ and $\sigma_3$ be the components of $\sigma$ in these directions. We can use the rectangular parallelepiped spanned by $Ve_1$, $Ve_2$ and $Ve_3$ as the volume element, which is generated by the stretch $V$ applied to the unit cube. Let us now consider the side that starts from $Ve_1$ and which is spanned by $Ve_2$ and $Ve_3$. Besides an infinitesimal tilting and change of the surface, this side undergoes a displacement in the $e_1$-direction with the magnitude $e_1 \cdot ((V + \mathrm{d}F)e_1 - Ve_1) = e_1 \mathrm{d}Fe_1 = (\mathrm{d}F)_{11}$ in the transition from $V$ to $V + \mathrm{d}F$. The work done on the considered side is therefore

$$-\sigma_1 \cdot (\mathrm{d}F)_{11} \cdot (V)_{22} \cdot (V)_{33} = -\det(V) \cdot \frac{(\mathrm{d}F)_{11} \cdot \sigma_1}{(V)_{11}}.$$

Thus the entire work done by the volume element is

$$\mathrm{d}A = -\det(V) \cdot \sum_{\nu=1}^{3} \frac{(\mathrm{d}F)_{\nu\nu} \cdot \sigma_\nu}{(V)_{\nu\nu}} = -\det(V) \cdot \mathrm{tr}(\sigma V^{-1} F). \tag{3.7}$$

The deformation $V + \mathrm{d}F$ now corresponds to a stretch $V + \mathrm{d}V$, where due to (2.1),

$$(V + \mathrm{d}V)^2 = (V + \mathrm{d}F)(V + \mathrm{d}F^{\mathrm{T}})$$

or

$$V \cdot \mathrm{d}V + \mathrm{d}V \cdot V = V \cdot \mathrm{d}F^{\mathrm{T}} + \mathrm{d}F \cdot V.$$

Multiplying the left side of the equation by $\sigma V^{-2}$, taking the trace and using (1.1), we find

$$2\mathrm{tr}(\sigma V^{-1}\mathrm{d}V) = \mathrm{tr}(\sigma V^{-1}\mathrm{d}F^{\mathrm{T}}) + \mathrm{tr}(\sigma V^{-1}\mathrm{d}F) = 2\mathrm{tr}(\sigma V^{-1}\mathrm{d}F),$$

since $\sigma$ is symmetric and coaxial to $V$. From (3.7) we therefore obtain

$$\mathrm{d}A = -\det(V)\cdot\mathrm{tr}(\sigma\dot{V}^{-1}\mathrm{d}V). \tag{3.8}$$

Hence, due to (1.2), (1.3) and (2.4):

$$\mathrm{d}A = -e^{j}\cdot\mathrm{tr}(\sigma\mathrm{d}L). \tag{3.8*}$$

If we substitute this expression into the isothermal relation (3.5) and use (2.6), then it follows:

$$e^{j}\cdot[f_1\mathrm{tr}(\mathrm{d}L) + f_2(\mathrm{tr}L\mathrm{d}L) + f_3\mathrm{tr}(L^2\mathrm{d}L)] = \frac{\partial W}{\partial j}\mathrm{d}j + \frac{\partial W}{\partial k}\mathrm{d}k + \frac{\partial W}{\partial l}\mathrm{d}l.$$

Since, by (2.4),

$$\mathrm{d}j = \mathrm{tr}(\mathrm{d}L), \qquad \mathrm{d}k = 2\mathrm{tr}(L\mathrm{d}L) \qquad \text{and} \qquad \mathrm{d}l = 3\mathrm{tr}(L^2\mathrm{d}L),$$

we finally conclude that

$$e^{j}f_1 = \frac{\partial W}{\partial j}, \qquad e^{j}f_2 = 2\frac{\partial W}{\partial k}, \qquad e^{j}f_3 = 3\frac{\partial W}{\partial l}$$

and therefore, with (2.6),

$$\sigma e^{j} = \frac{\partial W}{\partial j}\cdot\mathbb{1} + 2\frac{\partial W}{\partial k}\cdot L + 3\frac{\partial W}{\partial l}\cdot L^2, \qquad W = W(j,k,l,\Theta). \tag{3.9}$$

Accordingly, from (3.6) we obtain for the adiabatic law:

$$\sigma e^{j} = \frac{\partial E}{\partial j}\cdot\mathbb{1} + 2\frac{\partial E}{\partial k}\cdot L + 3\frac{\partial E}{\partial l}\cdot L^2, \qquad E = E(j,k,l,S). \tag{3.10}$$

If we want to omit the introduction of $L$ and use $V$ directly when formulating the law of elasticity, then we appropriately use the following as the invariants of $V$:

$$I_1 = \mathrm{tr}(V), \qquad I_2 = \tfrac{1}{2}\mathrm{tr}(V^2), \qquad I_3 = \det(V).$$

Furthermore, according to (2.7), (3.5) and (3.8), an analogous computation leads to the law of elasticity in the form

$$\sigma = \frac{\partial W}{\partial I_3}\cdot\mathbb{1} + \frac{1}{I_3}\cdot\frac{\partial W}{\partial I_1}\cdot V + \frac{1}{I_3}\cdot\frac{\partial W}{\partial I_2}\cdot V^2, \qquad W = W(I_\nu,\Theta) \tag{3.11}$$

and a corresponding formulation with $E(I_1, I_2, I_3, S)$ instead of $W$.

## 4. Transition to the deviators

The introduction of the logarithmic stretch $L$ now proves to be not only appropriate to formulate the law of elasticity as simply as possible, but using $L$ also allows for the decomposition of a deformation into a shape change and volume change by simply taking the deviatoric part, i.e. the same approach as for infinitesimal strains, whereas a corresponding decomposition in terms of $V$ is highly inconvenient. To see this, we decompose the general stretch $V$ into a shape-changing stretch $V_g$ and a volume-changing stretch $V_v$, i.e. we demand:

$$V = V_g\cdot V_v = V_v\cdot V_g \quad \text{with} \quad \det V_g = 1 \quad \text{and} \quad V_v = \beta\cdot\mathbb{1} \quad \text{with} \quad \beta > 0. \tag{4.1}$$

Obviously, (4.1) uniquely determines such a decomposition for each $V$ with $\det V > 0$; namely, for given $V$,

$$\beta = \sqrt[3]{\det V} \qquad \text{and} \qquad V_g = \beta^{-1} \cdot V.$$

Since $V_g$ commutes with $V_v$, we can take the logarithm of (4.1):

$$L = L_g + L_v \qquad \text{with} \qquad L_g = \log V_g \qquad \text{and} \qquad L_v = \log V_v. \tag{4.2}$$

Then, by (1.3), we obtain:

$$\mathrm{tr}(L_g) = \log(\det V_g) = 0, \qquad L_v = \log\beta \cdot \mathbb{1}, \qquad \mathrm{tr}(L_v) = 3\log\beta.$$

If, in general, we denote by $\mathrm{dev}D$ the deviator corresponding to the symmetric matrix $D$, i.e.

$$\mathrm{dev}\, D = D - \frac{1}{3}\mathrm{tr}\, D \cdot \mathbb{1}, \tag{4.3}$$

we can finally write:

$$L_g = \mathrm{dev}\, L \qquad \text{and} \qquad L_v = \frac{1}{3} j \cdot \mathbb{1}. \tag{4.4}$$

Thus the change of shape is indeed characterized by the deviator of $L$. For infinitesimal strains we have $L \approx V - \mathbb{1}$, so that $\mathrm{dev}L$ turns into the usual deformation deviator.

If we now introduce the invariants of $\mathrm{dev}L$:

$$y = \mathrm{tr}((\mathrm{dev}\, L)^2) \qquad \text{and} \qquad z = \mathrm{tr}((\mathrm{dev}\, L)^3), \tag{4.5}$$

then

$$y = k - \frac{1}{3}j^2 \qquad \text{and} \qquad z = l - jk + \frac{2}{9}j^3.$$

We can use $j$, $y$ and $z$ instead of $j$, $k$ and $l$ as variables. Then $j$ characterizes the change of volume, whereas $y$ and $z$ characterize the change of shape. As one can easily calculate, (3.9) leads to the formula

$$\left.\begin{aligned} \frac{1}{3}e^j \mathrm{tr}\sigma &= \frac{\partial W}{\partial j} \\ e^j \cdot \mathrm{dev}\,\sigma &= -y\frac{\partial W}{\partial z} \cdot \mathbb{1} + 2\frac{\partial W}{\partial y} \cdot \mathrm{dev}\, L + 3\frac{\partial W}{\partial z}(\mathrm{dev}\, L)^2 \end{aligned}\right\} \tag{4.6}$$

where, in contrast to (3.9), $W = W(j, y, z, \Theta)$ now holds.

A corresponding formula results from (3.10).

Without proof, let us remark that $y$ and $z$ cannot take on all possible values independently of each other, but are restricted by the condition

$$0 \leqslant \frac{z^2}{y^3} \leqslant \frac{1}{6}.$$

## 5. Decomposable elasticity laws

In the elasticity theory of infinitesimal strains the elastic energy can be interpreted as the sum of the energy of the volume and shape change. Since the change of volume is represented by $j$ and the change of shape is represented by $y$ and $z$, this decomposition is possible for the case of finite strains if and only if

$$\left.\begin{aligned} & W = W_{\mathrm{vol}}(j, \Theta) + W_{\mathrm{iso}}(y, z, \Theta), \\ \text{resp.} \qquad & E = E_{\mathrm{vol}}(j, S) + E_{\mathrm{iso}}(y, z, S) \end{aligned}\right\} \tag{5.1}$$

holds. Then with (4.6):

$$\frac{1}{3}e^{j}\cdot\mathrm{tr}\sigma=\frac{\partial W_{\mathrm{vol}}}{\partial j}(j,\Theta).$$

Thus the average stress depends only on $j$, i.e. on the change of volume. If, vice versa, tr $\sigma$ depends only on $j$, then by (4.6) we obtain

$$\frac{\partial^2 W}{\partial j\partial y}=\frac{\partial^2 W}{\partial j\partial z}=0,$$

which also leads to the form of $W$ in (5.1). Consequently, we can state: *The elastic energy can be decomposed into the energy of change of volume and of change of shape if and only if the mean stress depends only on the change of volume.*

## 6. Transition to a new reference temperature

We referred the deformations to the stress-free state at a certain temperature $\Theta_0$. Now we assume another temperature $\Theta_1$ is to be used as initial temperature instead of $\Theta_0$. For $\sigma=0$, the temperature $\Theta_1$ corresponds to a certain deformation $V_1$ with $\log V_1=L_1$. $V_1$ is a scalar multiple of the identity tensor; thus $\operatorname{dev} L_1=0$, $y_1=z_1=0$. Then with (4.6):

$$\frac{\partial W}{\partial j}(j_1,0,0,\Theta_1)=0,$$

which leads to the law of thermal expansion:

$$j_1=\varphi(\Theta_1). \tag{6.1}$$

Since $\widehat{F}=FV_1^{-1}$ is the matrix corresponding to the deformation $F$ with respect to the new initial state, we thus have $\widehat{V}=VV_1^{-1}$, $\widehat{L}=L-L_1$ and hence

$$\widehat{j}=j-j_1,\qquad \widehat{y}=y,\qquad \widehat{z}=z. \tag{6.2}$$

In formula (4.6), we can now replace $j$ by $\widehat{j}$ if we simultaneously substitute $W$ with

$$\widehat{W}(\widehat{j},y,z,\Theta)=e^{-j_1}\cdot W(\widehat{j}+j_1,y,z,\Theta)=e^{-\varphi(\Theta_1)}\cdot W(\widehat{j}+\varphi(\Theta_1),y,z,\Theta). \tag{6.3}$$

In particular, it follows that the decomposition of the elastic energy, which was discussed in Section 5, is independent of the choice of the reference temperature.

## 7. Validity of Hooke's law

Due to the formulae found previously, one can impose a wide variety of requirements on the law of elasticity, in particular with respect to the dependence on temperature, and verify if these requirements can be satisfied. Let us now consider the question whether the common law by *Hooke* remains valid for finite strains.

Using the Lamé constants, *Hooke*'s law takes the form

$$\sigma=\lambda\cdot\mathrm{tr}(V-\mathbb{1})\cdot\ +2\mu\cdot(V-\mathbb{1}) \tag{7.1}$$

or

$$\sigma=(\lambda\cdot I_1-3\lambda-2\mu)\cdot\mathbb{1}+2\mu\cdot V. \tag{7.2}$$

It is obvious that (7.1) is actually derived from the general formula (3.9) for small $L$.

In order for the isothermal law of elasticity (7.2) to remain valid for finite strains, the following equations must be fulfilled according to (3.11):

$$\lambda I_1-3\lambda-2\mu=\frac{\partial W}{\partial I_3},\qquad 2\mu I_3=\frac{\partial W}{\partial I_1}\qquad\text{and}\qquad 0=\frac{\partial W}{I_2}.$$

This is only possible if $\lambda = 2\dot{\mu}$, which corresponds to the Poisson ratio $\nu = \frac{1}{3}$. For all other values of $\nu$, Hooke's law cannot be used for finite strains. Instead, one can use the corresponding logarithmic law

$$\sigma e^{j} = \lambda j \cdot \mathbb{1} + 2\mu L, \tag{7.3}$$

which, in the isothermal case, corresponds to the decomposable energy

$$W = \frac{\lambda}{2}\dot{j}^2 + \mu k = \left(\frac{\lambda}{2} + \frac{\mu}{3}\right) \cdot j^2 + \mu \cdot y.$$

## Notes

2. $F$ is the Jacobian matrix: $\hat{x} = F\,\mathrm{d}x$. $\sigma$ is the physical stress tensor at the point $\hat{x}$.
3. It is easy to see that here, one of the invariants $I_\nu$ can be omitted, in contrast to the subsequent formula (2.7).

## Appendix

| Our notation | Richter's notation | Meaning |
|---|---|---|
| $X$, $Y$ | $\mathfrak{A}$, $\mathfrak{B}$ | arbitrary $3 \times 3$-matrices |
| $X^{\mathrm{T}}$ | $\overline{\mathfrak{A}}$ | transpose of $X$ |
| $(X)_{ik}$ | $(\mathfrak{A})_{ik}$ | entry in the $i$th row and the $k$th column of $X$ |
| $\det X$ | $\lvert\mathfrak{A}\rvert$ | determinant of $X$ |
| $\mathrm{tr}X$ | $\underline{\mathfrak{A}}$ | trace of $X$ |
| $\mathbb{1}$ | $\mathfrak{E}$ | identity tensor |
| $X^{-1}$ | $\mathfrak{A}^{-1}$ | inverse of $X$ |
| $F$ | $\mathfrak{A}$ | Jacobian matrix (state of strain) |
| $R$ | $\mathfrak{R}$ | pure Euclidean rotation |
| $V$ | $\mathfrak{S}$ | pure stretch |
| $\sigma$ | $\mathfrak{P}$ | stress tensor (state of stress) |
| $\Theta$ | $\Theta$ | temperature |
| $I_1, I_2, I_3$ | $I_1, I_2, I_3$ | invariants of $V$ |
| $L$ | $\mathfrak{L}$ | logarithmic stretch: $L = \log V$ |
| $j, k, l$ | $j, k, l$ | invariants of $L$: $j = \mathrm{tr}(L)$, $k = \mathrm{tr}(L^2)$, $l = \mathrm{tr}(L^3)$ |
| $f_1, f_2, f_3$ | $f_1, f_2, f_3$ | coefficient functions |
| $g_1, g_2, g_3$ | $g_1, g_2, g_3$ | coefficient functions |
| $X$ | $\mathfrak{X}$ | $X = f_1 \cdot \mathbb{1} + f_2 \cdot L + f_3 \cdot L^2$ |
| $E$ | $u$ | internal energy |
| $S$ | $s$ | entropy |
| $W$ | $f$ | free energy |
| $\mathrm{d}A$ | $\mathrm{d}A$ | differential of the work |
| $e_1, e_2, e_3$ | $e_1, e_2, e_3$ | unit vectors in the principal stretch directions of $V$ |
| $\sigma_1, \sigma_2, \sigma_3$ | $\sigma_1, \sigma_2, \sigma_3$ | components of $\sigma$ in the principal stretch directions of $V$ |
| $V_g$, $V_v$ | $\mathfrak{S}_g$, $\mathfrak{S}_v$ | stretch in shape, stretch in volume |
| $\beta$ | $\beta$ | stretch factor of the stretch in volume $V_v$ |
| $L_g$, $L_v$ | $\mathfrak{L}_g$, $\mathfrak{L}_v$ | $L_g = \log V_g$, $L_v = \log V_v$ |
| $D$ | $\underset{\sim}{\mathfrak{D}}$ | arbitrary symmetric matrix |
| $\mathrm{dev}\, D$ | $\widetilde{\mathfrak{D}}$ | common deviator of $D$ |
| $y$, $z$ | $y$, $z$ | invariants of $\mathrm{dev}\,L$: $y = \mathrm{tr}((\mathrm{dev}L)^2)$, $z = \mathrm{tr}((\mathrm{dev}\,L)^3)$ |
| $W_{\mathrm{vol}}$, $E_{\mathrm{vol}}$ | $F$, $U$ | volumetric energies |
| $W_{\mathrm{iso}}$, $E_{\mathrm{iso}}$ | $G$, $V$ | isochoric energies |
| $\Theta_0$ resp. $\Theta_1$ | $\Theta_0$ resp. $\Theta_1$ | reference temperatures |
| $_1$ [index] | $_1$ | indicates the correspondence to the temperature $\Theta_1$ |
| $\underset{\sim}{\varphi}$ | $\varphi$ | logarithmic thermal expansion |
| $\underset{\sim}{F}$ | $\mathfrak{A}'$ | deformation with respect to the initial state at $\Theta_1$ |
| $\frown$ | $'$ | indicates the correspondence to the deformation $\widehat{F}$ |
| $\lambda$, $\mu$ | $\lambda$, $\mu$ | Lamé constants |
| $\nu$ | $m = \frac{1}{\nu}$ | Poisson modulus |

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