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On global solutions of the obstacle problem

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Abstract

In this thesis we investigate global solutions of the classical obstacle problem. We give a partial result towards a conjecture by H. Shahgholian ('92) saying that the coincidence sets of global solutions of the obstacle problem are in the closure of ellipsoids, i.e. ellipsoids, paraboloids, cylinders with one of the two as basis, or half-spaces.

We give a short proof of the known result that bounded coincidence sets of global solutions of the obstacle problem that have non-empty interior are ellipsoids. Our main and new result is that in dimensions greater or equal to 6 coincidence sets of global solutions, that are not constant in any direction and have a blow-down that is independent of exactly one direction, are paraboloids if they have non-empty interior. The proof rests on a careful analysis of the asymptotics of solutions at infinity, the Newton-potential expansion of the solution and a comparison argument that only requires two solutions to be compared on a sufficiently large portion of huge spheres.

Über globale Lösungen des Hindernisproblems

Zusammenfassung

In dieser Arbeit untersuchen wir globale Lösungen des klassischen Hindernisproblems. Wir zeigen einen Teil einer Vermutung von H. Shahgholian ('92), die besagt, dass die Koinzidenzmenge globaler Lösungen des Hindernisproblems im Abschluss von Ellipsoiden liegt, d.h. ein Ellipsoid, ein Paraboloid, ein Zylinder mit einer der beiden Mengen als Basis oder ein Halbraum ist.

Wir geben zunächst einen kurzen Beweis des klassischen Resultats, dass die Koinzidenzmenge globaler Lösungen des Hindernisproblems, wenn sie beschränkt ist und nicht-leeres Inneres hat, ein Ellipsoid sein muss.

Unser neues Resultat und das Hauptresultat dieser Arbeit ist, dass in Dimensionen größer oder gleich sechs die Koinzidenzmenge einer globalen Lösung der Hindernisproblems, die in keine Richtung konstant ist und einen blow-down hat, der in genau eine Richtung verschwindet, wenn sie nicht-leeres Inneres hat, ein Paraboloid ist.

Der Beweis basiert auf einer eingehenden Untersuchung des asymptotischen Verhaltens der Lösung im Unendlichen, der Newton-Potential-Entwicklung der Lösung und einem Vergleich zweier Lösungen, der nur benötigt, dass die beiden Lösungen auf ausreichend großen Bereichen großer Sphären vergleichbar sind.

Contents

To my family and friends who never let me down.

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0

Introduction

Global solutions of the obstacle problem are nonnegative functions $u \geq 0$ that solve

$$\Delta u = \chi_{\{u>0\}} \quad \text{in } \mathbb{R}^N \tag{1}$$

in the sense of distributions. Such global solutions come up naturally e. g. as blow-up or blow-down limits of solutions of the classical obstacle problem. An important question in the field of free boundary problems is the behaviour of the *regular set of the free boundary close to singularities.* A usual ansatz in the study of the free boundary is the analysis and classification of *blow-ups* with moving centre

$$u_k(x) := \frac{u(x^k + r_k x)}{M_k},$$

where x^k is the moving centre and M_k and r_k a rescaling factors. Typically one chooses $M_k \sim r_k^2$. At least in the case $x^k \to x^0$ (as $k \to \infty$) and $x^0 \in \partial\{u > 0\}$ it is known that

$$u_k \to u_0 \quad \text{as } k \to \infty \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N)$$

(for any $\alpha \in (0,1)$, $p \in [1,\infty)$) and that u_0 is a global solution of the obstacle problem, i.e.

$$\Delta u_0 = \chi_{\{u_0>0\}} \quad \text{and} \quad u_0 \geq 0 \quad \text{in } \mathbb{R}^N$$

(see e.g [12, Chapter 5]). In order to proceed from here what is needed is a *classification of global solutions* of the obstacle problem.
Interestingly this problem has been addressed almost 90 years ago from the perspective of *potential theory* and later *null quadrature domains* so that at least a partial classification is already available. In [5] P. Dives showed in 1931 - formulated in the language of potential theory - in *three dimensions* that if $\{u = 0\}$ has non-empty interior and is *bounded* then it is an ellipsoid. This was proved again many years later by H. Lewy in 1979 [11]. In 1981 M. Sakai was even able to give a full classification of global solutions in *two dimensions* using complex analysis ([13]). The higher dimensional analogue to Dive's result, i.e. $\{u = 0\}$ is bounded and has non-empty interior then it is an ellipsoid, was proved shortly after in two steps. First E. DiBenedetto and A. Friedman proved the result in 1986 under the additional assumption that $\{u = 0\}$ is already symmetric with respect to $\{x_j = 0\}$ for all $j \in \{1, \dots, N\}$ ([4]). In the same year A. Friedman and M. Sakai removed this unnecessary symmetry assumption in [7].
Let us make the reader aware of the fact that all of these higher dimensional results ($N \geq 3$) rely heavily on the Newtonian potential of $\{u = 0\}$ being well defined and Newton's *no gravity in the cavity theorem* saying that if the gravitational / Newtonian potential is constant inside the cavity of a homogeneous homoeoid (i.e. $\lambda K \setminus K$ for $\lambda > 1$, K bounded) , then the cavity K is an ellipsoid. Both tools don't easily generalise to unbounded coincidence sets. In chapter 1 we are able to circumvent the use of the theorem by Newton, replacing it with a comparison argument. However in this argument still the Newtonian potential was used in order to control the lower order asymptotics at infinity of the given solution and to match it with the one of the comparison solution. Furthermore the boundedness of the coincidence set was needed several times. The fact that the Newtonian potential ansatz does not generalise to unbounded coincidence sets in a straight forward manner is probably the reason why there has been no progress in the classification of global solutions since the mid 80s.
Yet people have worked on this problem and there is even a conjecture by H. Shahgholian on what the classification will look like.

Conjecture (Shahgholian [14] '92)**.**
Suppose u is a nonnegative global solution of the obstacle problem (1), then $\{u = 0\}$ is an element of the closure of ellipsoids, i.e. there is a sequence $(E_j)_{j\in\mathbb{N}}$ of ellipsoids (in the sense of Definition 0.3) such that

$$\{u = 0\} = \bigcup_{j\in\mathbb{N}} E_j.$$

The conjecture has later been repeated by L. Karp and A. Margulis [10]. The ansatz of L. Karp to attack this problem was to generalize the Newtonian potential to what he called *generalized Newtonian potential* that is well defined also for unbounded sets which are not dense everywhere ([9]). However this new Newtonian potential brings with it its own difficulties. One is that unlike the classical Newtonian potential the generalized Newtonian potential does not have a sign anymore. This makes it harder to get some otherwise trivial a-priori estimates. More unpleasant though is the fact that the generalized Newtonian potential can grow linearly or even quadratically at infinity and is therefore almost useless for the control of lower order asymptotics at infinity.
This is why we are going to work with the classical Newtonian potential that is still well defined on paraboloids *from dimension 6 on.* The classical Newtonian potential of a paraboloid vanishes at infinity everywhere outside a set 'slightly larger' than the paraboloid itself. We were able to work out a comparison argument for which such a control is sufficient. Since we are also able to construct a global solution with a paraboloid as coincidence set that matches the asymptotic behaviour of the given solution u outside the 'slightly larger' set, applying our comparison in a sliding argument yields that $\{u = 0\}$ is a paraboloid. This allows us to add a new partial result towards the full classification of global solutions of the obstacle problem which is the main result of this work.

Theorem I (Unbounded coincidence sets)**.**
Let $N \geq 6$ and let u be a nonnegative *global solution of the obstacle problem* (1) *that has a* blow-down that is independent of only e_N-direction *and is* not 'lower dimensional' *(compare to Definition 0.4). Assume furthermore that the coincidence set $\{u = 0\}$ has* non-empty interior*, then $\{u = 0\}$ is a paraboloid (in the sense of Definition 0.3).*

Before we prove our main result in the second chapter we will, as a warm up, reprove the classical result on bounded coincidence sets on a few pages in the first chapter.

Theorem II (Compact coincidence sets)**.**
Let $N \geq 3$ and u be a nonnegative, global solution of the obstacle problem (1)*. If the coincidence set $\{u = 0\}$ is* bounded *and has* non-empty interior *then it is an ellipsoid (in the sense of Definition 0.3).*

0.1 Clarification of the notation

Let us clarify the notation used in the introduction and in the rest of this thesis.
Throughout this work $\mathbb{R}^N$ will be equipped with the Euclidean inner product $x \cdot y$ and

the induced norm $|x|$. Due to the nature of the problem we will often write $x \in \mathbb{R}^N$ as $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. $B_r(x)$ will be the open N-dimensional ball of centre x and radius r. $B'_r(x')$ will be the open $(N-1)$- dimensional ball of centre $x' \in \mathbb{R}^{N-1}$ and radius r. Whenever the centre is omitted it is assumed to be 0.
When considering a set A, χ_A shall denote the characteristic function of A. $\mathcal{H}^{N-1}$ is the usual $(N-1)$-dimensional Hausdorff measure.
For a function $u : A \subset \mathbb{R}^N \to \mathbb{R}$ we usually mean by u_r the rescaling

$$u_r(x) := \frac{u(rx)}{r^2} \quad \text{for all } x \in \frac{1}{r}A$$

and by u^r

$$u^r(x) := u(rx) \quad \text{for all } x \in \frac{1}{r}A.$$

Definition 0.1 (Coincidence set)**.**
Let u be a nonnegative global solution of the obstacle problem (1). We call the *coincidence set*

$$\mathcal{C} := \{u = 0\}.$$

Remark 0.2. It is known that $\mathcal{C}$ is convex. (See e.g. [12, Theorem 5.1].)

Definition 0.3 (Ellipsoid and paraboloid)**.**
We call a set $E \subset \mathbb{R}^N$ *ellipsoid* if after translation and rotation

$$E = \left\{ x \in \mathbb{R}^N : \sum_{j=1}^{N} \frac{x_j^2}{a_j^2} \leq 1 \right\}$$

for some $a \in (0, \infty)^N$. If after translation and rotation a set $P \subset \mathbb{R}^N$ can be represented as

$$P = \left\{ (x', x_N) \in \mathbb{R}^N : x' \in \sqrt{x_N} E' \right\},$$

where E' is an $(N-1)$-dimensional ellipsoid, we call P a *paraboloid*.

Definition 0.4.
We say that a global solution of the obstacle problem (1)

- has *blow-down that is independent of e_N-direction* if

$$u_r(x) := \frac{u(rx)}{r^2} \to x'^T Q' x' =: p(x') \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \text{ as } r \to \infty, \tag{2}$$

  where $x = (x', x_N)$, $Q' \in \mathbb{R}^{N-1\times N-1}$ is positive definite, symmetric and $\operatorname{tr}(Q) = \frac{1}{2}$. For later reference let us state that this means that there is $c_p > 0$ such that for all $x' \in \mathbb{R}^{N-1}$

$$p(x') \geq c_p |x'|^2, \tag{3}$$

- is *not 'lower dimensional'*, if for all $e \in \partial B_1$ it holds that

$$\nabla u \cdot e \not\equiv 0.$$

Remark 0.5. Note that the claimed strong $W^{2,p}_{loc}$-convergence of the blow-down (2) can be found e.g. in [12, Proposition 3.17 (v)].

Remark 0.6. Every global solution of the obstacle problem (1) has globally bounded (weak) second derivatives, i.e. there is $C > 0$ such that

$$\left\| D^2 u \right\|_{L^\infty(\mathbb{R}^N)} < C.$$

Since we could not find a reference for this known fact in the literature, we have included the proof in Appendix B (see Lemma B.1 therein).

Definition 0.7 (Newton-potential)**.**
Let $N \geq 3$ and $M \subset \mathbb{R}^N$ be a measurable set such that for all $x \in \mathbb{R}^N$

$$\int_M \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y \quad \text{is finite.}$$

Then we call $v^{NP}_M : \mathbb{R}^N \to [0, \infty)$,

$$v^{NP}_M(x) := \alpha(N) \int_M \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y,$$

where $\alpha(N) := \frac{1}{N(N-2)|B_1|}$, the *Newton-potential* or *Newtonian potential of M*.

Salomon saith, There is no new thing upon the earth. So that as Plato had an imagination, that all knowledge was but remembrance; so Salomon giveth his sentence, that all novelty is but oblivion.

Francis Bacon

1

Compact coincidence sets of global solutions are ellipsoids

In this chapter we give a short proof of the classical result Theorem II, that has historically been proved in several steps, on a few pages.

The idea underlying the proof is extremely simple. We will touch the coincidence set $\{u = 0\}$ with a suitable ellipsoid and apply a strong comparison principle to the respective solutions. Of course not any ellipsoid will do. We will need a Lemma that relates ellipsoids to respective global solutions of the obstacle problem. Furthermore we will need to ensure that the ellipsoid-solution shares the asymptotic behaviour of the given solution. This will require $\mathcal{C}$ and the 'matched' ellipsoid to share a 'weighted centre of mass'.

1.1 Preparatory observations and choices

It is known that the blow-down of u is well defined (see e.g. [12, Proposition 3.17]) and is either a half-space solution or a polynomial solution (see e.g. [12, Proposition 5.3]). Using that $\mathcal{C}$ is assumed to be compact in this chapter and that solutions of the obstacle problem are non-degenerate (see e.g. [12, Lemma 3.1]) the blow-down must be a non-degenerate polynomial that is homogeneous of degree 2, i.e.

$$\frac{u(rx)}{r^2} \to x^T Q x =: p(x) \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \text{ as } r \to \infty, \tag{1.1}$$

where $Q \in \mathbb{R}^{N\times N}$ is positive definite, symmetric and $\mathrm{tr}(Q) = \frac{1}{2}$.

Choice of coordinate system. Without loss of generality we assume that the coordinate system is rotated such that Q is a diagonal matrix.

Lemma 1.1 (Existence of ellipsoid solutions).
For any polynomial $p(x) = x^T Q x$, where $Q \in \mathbb{R}^{N\times N}$ is diagonal, positive definite and $\operatorname{tr}(Q) = \frac{1}{2}$, there is an ellipsoid E (symmetric with respect to $\{x_j = 0\}$ for all $j \in \{1, \dots N\}$) and a nonnegative solution of the obstacle problem u^E such that

$$\Delta u^E = \chi_{\{u^E>0\}} \text{ in } \mathbb{R}^N, \left\{u^E = 0\right\} = E \text{ and } \frac{u^E(rx)}{r^2} \to p(x) \text{ in } L^\infty(\partial B_1) \text{ as } r \to \infty.$$

Proof of Lemma 1.1. From [4, see (5.4)] we know that for any polynomial $p(x) = x^T Q x$ there is an ellipsoid $E := \left\{x \in \mathbb{R}^N : x^T A x \le 1\right\}$ ($A \in \mathbb{R}^{N\times N}$ positive definite, diagonal and symmetric) such that its Newtonian potential satisfies

$$v_E^{NP}(x) := \alpha(N) \int\limits_E \frac{1}{|x-y|^{N-2}}\, \mathrm{d}y = v_E^{NP}(0) - p(x) \quad \text{in } E.$$

Here $\alpha(N)$ is given by $\alpha(N) := \frac{1}{N(N-2)|B_1|}$. We now define the solution u^E by

$$u^E(x) := p(x) - v_E^{NP}(0) + v_E^{NP}(x).$$

A direct computation shows that

$$\Delta u^E = 1 - \chi_E = \chi_{\mathbb{R}^N\setminus E}\,,\ u^E = 0 \text{ in } E \text{ and } \frac{u^E(rx)}{r^2} \to p(x) \text{ in } L^\infty(\partial B_1) \text{ as } r \to \infty,$$

where we have used that $v_E^{NP}(x) \to 0$ uniformly as $|x| \to \infty$. So all we need to check is that $u^E > 0$ in $\mathbb{R}^N \setminus E$. From [3, Theorem II] we infer that u^E is nonnegative in $\mathbb{R}^N$ and that the coincidence set $\left\{u^E = 0\right\}$ is convex. Hence if there was $x_0 \notin E$ such that $u^E(x_0) = 0$, then this would imply that $\operatorname{conv}(\{x_0\} \cup E) \subset \left\{u^E = 0\right\}$ but this is impossible since $\Delta u^E = 1$ in $\mathbb{R}^N \setminus E$. □

1.2 Proof of Theorem II

Step 1. *The Newton-potential expansion.*
In order to get a better understanding of the lower order asymptotics of the solution as $|x| \to \infty$, we decompose it into a polynomial and the Newton-potential solution.

First, we modify the solution, such that the Laplacian is supported on a bounded domain. Setting

$$v(x) := u(x) - p(x) \quad \text{ in } \mathbb{R}^N,$$

where p is the blow-down of u as defined in (1.1) we get that v solves

$$\Delta v = -\chi_{\mathcal{C}}.$$

From here on we assume that the coordinate system is translated in such a way that

$$\int_{\mathcal{C}} \frac{y}{|y|^N} \, \mathrm{d}y = 0 \quad \text{ and } 0 \in \operatorname{int} \mathcal{C}.$$

This is possible because $\mathcal{C}$ is bounded and convex with non-empty interior. For details see Appendix A.1.

The Newton-potential solution (for $N \geq 3$) given by

$$v^{NP}(x) := \alpha(N) \int_{\mathcal{C}} \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y \quad , \text{ where } \alpha(N) := \frac{1}{N(N-2)|B_1|} > 0$$

which is a strong solution in $W^{2,p}_{loc}(\mathbb{R}^N)$ of

$$\Delta v^{NP} = -\chi_{\mathcal{C}}.$$

Let us note that $\Delta(v - v^{NP}) = 0$ in $\mathbb{R}^N$, i.e the difference is harmonic.
Since $\frac{(v-v^{NP})(rx)}{r^2} \to 0$ as $r \to \infty$ uniformly on ∂B_1 the difference must by a Liouville-type argument (see Lemma B.4) be a harmonic polynomial of degree at most one, in the following denoted by q.

Recall that $0 \in \operatorname{int} \mathcal{C}$. This allows us to deduce that

$$\nabla q(0) = \nabla v(0) - \nabla v^{NP}(0) = 0 - \alpha(N) \int_{\mathcal{C}} \frac{y}{|y|^N} \, \mathrm{d}y = 0,$$
$$q(0) = v(0) - v^{NP}(0) = 0 - v^{NP}(0) < 0.$$

We infer that $q \equiv q(0) < 0$ and that $v \equiv v^{NP} + q(0) = v^{NP} - v^{NP}(0)$.
Step 2. *The comparison function.*

The idea in the following is to touch $\mathcal{C}$ from the outside with an ellipsoid E satisfying

$$\int_E \frac{y}{|y|^N}\,\mathrm{d}y = \int_{\mathcal{C}} \frac{y}{|y|^N}\,\mathrm{d}y,$$

and to compare u with the respective solution u^E. To do so, let us choose $E \subset \mathbb{R}^N$ as in Lemma 1.1 where we set p in the Lemma to be the blow-down of u as defined in (1.1). The symmetry of E yields that

$$\int_E \frac{y}{|y|^N}\,\mathrm{d}y = 0.$$

Furthermore we define the family of ellipsoids

$$(E_r)_{r>0} \quad , \quad E_r := \frac{1}{r}E$$

and the respective rescalings

$$U_r(x) := \frac{u^E(rx)}{r^2} \quad \text{ in } \mathbb{R}^N.$$

Note that U_r satisfies for all $r > 0$

$$\Delta U_r = \chi_{\mathbb{R}^N \setminus E_r} \quad , \quad U_r \geq 0 \text{ in } \mathbb{R}^N \quad , \quad \{U_r = 0\} = E_r$$

and

$$\frac{U_r(sx)}{s^2} \to p(x) \quad \text{ in } L^\infty(\partial B_1) \text{ as } s \to \infty.$$

Let us set $r_1 > 0$ to be such that $\mathcal{C} \subset \operatorname{int} E_{r_1}$. As for u we modify U_r to

$$V_r := U_r - p.$$

It follows that

$$\Delta V_r = -\chi_{\{U_r=0\}} = -\chi_{E_r}.$$

As before we define the Newton-potential solution

$$V_r^{NP}(x) := \alpha(N) \int_{E_r} \frac{1}{|x-y|^{N-2}}\,\mathrm{d}y.$$

Since E_r is symmetric with respect to all planes $\{x_i = 0\}$ for $i \in \{1, \dots, N\}$ it follows that

$$\nabla V_r^{NP}(0) = \alpha(N) \int\limits_{E_r} \frac{y}{|y|^N} \, \mathrm{d}y = 0.$$

Using again that the difference $V_r - V_r^{NP}$ is harmonic in the entire space, $\frac{(V_r - V_r^{NP})(sx)}{s^2} \to 0$ as $s \to \infty$ uniformly in ∂B_1 and a Liouville-type argument we obtain that the difference must be a polynomial of degree at most one and we denote it by q_r. Calculations similar to the calculations above show that q_r must be of degree zero:

$$\begin{aligned} \nabla q_r(0) &= \nabla V_r(0) - \nabla V_r^{NP}(0) = 0 - \alpha(N) \int\limits_{E_r} \frac{y}{|y|^N} \, \mathrm{d}y = 0 \\ q_r(0) &= V_r(0) - V_r^{NP}(0) = -V_r^{NP}(0) < 0. \end{aligned}$$

It follows that $V_r \equiv V_r^{NP} - V_r^{NP}(0)$ for all $r > 0$. Now we are able to prove the following comparison Lemma.

Step 3. *Comparison principle.*

For all $r > 0$ such that $\mathcal{C} \subset E_r$ and $|E_r \setminus \mathcal{C}| \neq 0$,

$$u \geq U_r \quad \text{in } \mathbb{R}^N \quad \text{and} \quad u > U_r \quad \text{in } \mathbb{R}^N \setminus E_r.$$

For a proof, first note that for all $x \in \mathbb{R}^N$

$$V_r^{NP}(x) = \alpha(N) \int\limits_{E_r} \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y > \alpha(N) \int\limits_{\mathcal{C}} \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y = v^{NP}(x). \quad (1.2)$$

Let us now apply (1.2) to the difference

$$\begin{aligned} (u - U_r)(x) = (v - V_r)(x) &= v^{NP}(x) - V_r^{NP}(x) + V_r^{NP}(0) - v^{NP}(0) \\ &\to V_r^{NP}(0) - v^{NP}(0) > 0 \quad \text{uniformly as } |x| \to \infty. \end{aligned}$$

Hence there is $R > 0$ such that

$$u > U_r \quad \text{in } \mathbb{R}^N \setminus B_R(0). \quad (1.3)$$

As u and U_r solve the same PDE (1) and we have a comparison principle for this nonlinear PDE (for details see Appendix A.2) we infer that

$$u \geq U_r \quad \text{in } \mathbb{R}^N,$$

and furthermore

$$u > U_r \quad \text{in } \mathbb{R}^N \setminus E_r.$$

The strict inequality holds because $u - U_r$ is harmonic in $\mathbb{R}^N \setminus E_r$, and in case of the two graphs touching in $\mathbb{R}^N \setminus E_r$ the strong maximum principle would yield that $u - U_r \equiv 0$ in $\mathbb{R}^N \setminus E_r$, contradicting (1.3).

Step 4. *Applying Hopf's principle to finish the proof.*

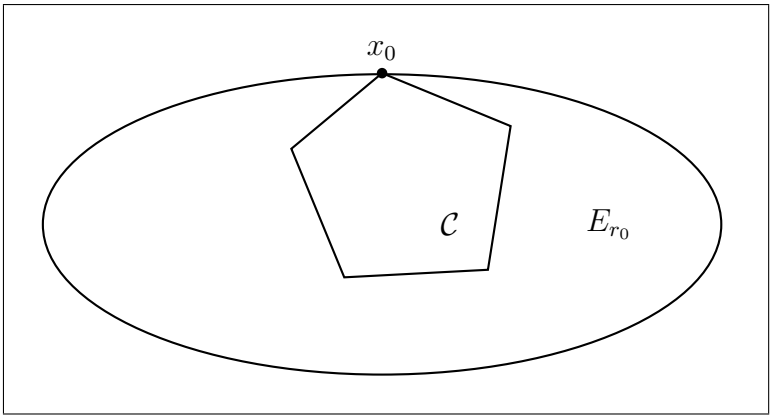

Figure 1.1: Touching the obstacle from outside.

Let us now increase $r > 0$ from $r = r_1$ to $r_0 > 0$ such that the boundaries of E_{r_0} and $\mathcal{C}$ touch for the first time (see Figure 1.1), i.e.

$$\partial E_{r_0} \cap \partial \mathcal{C} \neq \emptyset \text{ and } \partial E_r \cap \partial \mathcal{C} = \emptyset \text{ for all } r < r_0.$$

Let $x_0 \in \partial E_{r_0} \cap \partial \mathcal{C}$ be a touching point. Then either $\mathcal{C} = E_{r_0}$ and Theorem II is proved, or $\mathcal{C} \neq E_{r_0}$. The latter would imply that $|E_{r_0} \setminus \mathcal{C}| > 0$. In order to see this assume that $A, B \subset \mathbb{R}^N$ are two convex, compact sets with non-empty interior such that $A \subset B$ and $|B \setminus A| = 0$. Then since $(\operatorname{int} B) \setminus A$ is open, $|B \setminus A| = 0$ implies that $(\operatorname{int} B) \setminus A = \emptyset$. Thus $\operatorname{int} B \subset A$. If there is $\bar{x} \in \partial B \setminus A$, then since B is convex, also $\operatorname{conv}(\{\bar{x}\} \cup A) \subset B$ and $\operatorname{conv}(\{\bar{x}\} \cup A) \setminus A$ has non-empty interior since $\operatorname{int} B \neq \emptyset$, $\operatorname{int} B \subset A$ and $\operatorname{dist}(\bar{x}, A) > 0$ since A is compact. But this contradicts the fact that $\operatorname{int} B \subset A$ and therefore $\partial B \setminus A = \emptyset$ and we can conclude that $A = B$.

This allows us to apply Step 3 with $r = r_0$ and to obtain that

$$u > U_{r_0} \quad \text{in } \mathbb{R}^N \setminus E_{r_0}.$$

Furthermore

$$\Delta(u - U_{r_0}) = 0 \quad \text{in } \mathbb{R}^N \setminus E_{r_0}.$$

Since the boundary of the ellipsoid E_{r_0} is smooth, there is an open ball $B \subset \mathbb{R}^N \setminus E_{r_0}$ such that

$$\bar{B} \cap E_{r_0} = \{x_0\}.$$

From the classical Hopf principle we infer that

$$\frac{\partial(u - U_{r_0})}{\partial \nu}(x_0) < 0,$$

where ν is the outer unit normal on ∂B at x_0. But this is impossible since x_0 is a free boundary point of both u and U_{r_0}, implying that

$$\nabla u(x_0) = 0 = \nabla U_{r_0}(x_0).$$

Hence the assumption $\mathcal{C} \neq E_{r_0}$ must have been wrong and Theorem II is proved. $\square$

Our doubts are traitors and make us lose the good we oft might win by fearing to attempt.

William Shakespeare

2

Unbounded non-cylindrical coincidence sets in $N \geq 6$ are paraboloids

The flow of the chapter follows the structure of the proof of Theorem I that is gathering enough information about the asymptotics of the given solution u to allow for a squeeze in argument with a constructed paraboloid solution with matched asymptotics. After introducing some notation and making convenient assumptions on the choice of coordinate system in section 2.1, we give a first frequency estimate in section 2.2 and apply the upper estimate on the frequency to show that a normalized (and slightly corrected) rescaling of $u - p$ converges to an affine linear function. In section 2.3 we infer from this preliminary analysis that the coincidence set $\mathcal{C}$ is asymptotically contained in a set that is slightly bigger than a paraboloid but smaller than a cone. This estimate on the coincidence set is shown in section 2.4 to be enough for the Newtonian potential of $\mathcal{C}$ to be well defined. Starting from this realization we decompose the given solution into a quadratic polynomial plus the Newtonian potential of $\mathcal{C}$. The Newton-potential expansion can be understood as lower order asymptotic expansion of the given solution outside a 'small' set. In order to ultimately compare we construct a paraboloid solution, i.e. a solution of the obstacle problem of which the coincidence set is a paraboloid, that has 'matched asymptotics' with the given solution (in the sense that both the quadratic and linear term in its Newton potential expansion coincide with that of the given solution) in section 2.5. We conclude the proof of Theorem I in section 2.6. There we first prove that the Newton-potential expansion can be rigorously understood as asymptotic expansion as $|x| \to \infty$ outside a set that is slightly larger than a paraboloid - for the paraboloid solutions. This allows us to take

the paraboloid solution from the previous section that has 'matched asymptotics' to the given solution and translate it until it is compared to the given solution outside a set slightly larger than the paraboloid on any sufficiently large sphere. We then give a comparison argument that requires only this and yields that the paraboloid solution and the given solution are compared everywhere. Due to this argument we are able to squeeze in the given solution between two translated copies of the paraboloid solution and move these towards each other until they touch.

2.1 Preparatory observations and choices

The claim that u is not 'lower dimensional' implies, as we see in the following lemma, that the coincidence set $\mathcal{C}$ is bounded to one side in e_N-direction.

Lemma 2.1.
If u is a global solution of the obstacle problem as in Theorem I then there is $\lambda \in \mathbb{R}$ such that either $\mathcal{C} \subset \{x_N \geq \lambda\}$ or $\mathcal{C} \subset \{x_N \leq \lambda\}$.

Thus we make for simplicity of notation the following assumptions on the coordinate system for the rest of the chapter.

Choice of coordinate system.
There is no loss of generality in assuming that

- the coordinate system is rotated in x'-coordinates in such a way that Q' is diagonal and
- the coordinate system is translated and (if needed reflected in e_N-direction) in such a way that

$$\mathcal{C} \subset \{x_N \geq 0\} \quad \text{and} \quad 0 \in \mathcal{C}.$$

Let us conclude this section with the proof of Lemma 2.1.

Proof of Lemma 2.1.
Step 1. *$\mathcal{C}$ does not contain a line.*
We first prove that if u is not 'lower dimensional' then $\mathcal{C}$ cannot contain a line. Suppose towards a contradiction that this was not true and $\mathcal{C}$ would contain a line g. Let H be an arbitrary 2-dimensional plane containing g. For the sake of simplicity we identify H with $\mathbb{R}^2$ and g with $\{y_2 = 0\} \subset \mathbb{R}^2$.
Then by convexity of $\mathcal{C} \cap H$ either $\mathcal{C} \cap H$ is all of $\mathbb{R}^2$ or a half-space or the boundary $\partial\mathcal{C} \cap H$ is given by the graphs of two functions. The 'upper boundary' is given by the

graph of a concave function $f^+ : \mathbb{R} \to \mathbb{R}$ and the 'lower boundary' is given by the graph of a convex function $f^- : \mathbb{R} \to \mathbb{R}$ and we have

$$f^+(y_1) \geq 0 \geq f^-(y_1) \quad \text{for all } y_1 \in \mathbb{R}.$$

The aim now is to prove that f^+ and f^- must be constant.
So suppose towards a contradiction that f^+ is not constant. Then there must be $y_1^0 \in \mathbb{R}$ such that the superdifferential $\partial f^+(y_1^0)$ of f^+ in y_1^0 does not contain zero. We understand the superdifferential as usually as

$$\partial f^+(y_1^0) := \big\{c \in \mathbb{R} : f^+(y_1) - f^+(y_1^0) \leq c(y_1 - y_1^0) \text{ for all } y_1 \in \mathbb{R}\big\}.$$

Let $c \in \partial f^+(y_1^0)$ be such that $c \neq 0$. Then the supporting line $c(y_1 - y_1^0)$ does intersect $\{y_2 = 0\} = g$. But this contradicts the assumption that $\mathcal{C}$ and hence $\mathcal{C} \cap H$ contains g. A similar argument applies to f^-.
Since H was an arbitrary 2-dimensional plane containing g this implies that $\mathcal{C}$ is a cylinder in the direction of g. Since $\mathcal{C}$ has non-empty interior and we assume the blow-down of u to be independent only of the e_N-direction (2) g must be parallel to e_N. Hence we know that $\mathcal{C} = \Omega \times \mathbb{R}$ for some convex set $\Omega \subset \mathbb{R}^{N-1}$ that has non-empty interior.
In order to arrive at the desired conclusion that $\partial_N u \equiv 0$ in $\mathbb{R}^N$ we need to combine the cylindrical structure of $\mathcal{C}$ with known facts about solutions of the obstacle problem. First of all we know that

$$\Delta \partial_{NN} u = 0 \ \text{ in } \mathbb{R}^N \setminus \mathcal{C} \tag{2.1}$$

and from the cylindrical structure of the coincidence set $\mathcal{C}$ we infer that

$$\partial_{NN} u = 0 \ \text{ on } \partial\mathcal{C}. \tag{2.2}$$

The well known results by Caffarelli [2, Theorem 2 and Theorem 3] tell us that since $\mathcal{C}$ has non-vanishing Lebesgue-density everywhere the second derivatives of u are continuous up to the free boundary, hence

$$\partial_{NN} u \in C^0(\overline{\mathbb{R}^N \setminus \mathcal{C}}).$$

Furthermore we need the well known fact that global solutions of the obstacle problem are convex and therefore

$$\partial_{NN} u \geq 0 \text{ in } \mathbb{R}^N \tag{2.3}$$

(this can be found e.g. in [12, Theorem 5.1]). From the assumption on the blow-down (2) we obtain that

$$\partial_{NN}u(rx) = \partial_{NN}u_r(x) \to 0 \quad \text{strongly in } L^p_{loc} \text{ as } r \to \infty. \tag{2.4}$$

Let the auxiliary function v be given by

$$v(x) := \begin{cases} \partial_{NN}u(x) & \text{in } \mathbb{R}^N \setminus \mathcal{C}, \\ 0 & \text{in } \mathcal{C}. \end{cases}$$

Then the facts we have collected in (2.1), (2.2) and (2.3) imply that

$$v \text{ is harmonic in } \mathbb{R}^N \setminus \mathcal{C} \quad , \quad v \geq 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad v \in C^0(\mathbb{R}^N).$$

From this we infer that v is subharmonic. This can be seen as follows. For each $\delta > 0$

$$\max\{v - \delta, 0\} \quad \text{is subharmonic in } \mathbb{R}^N$$

as maximum of two harmonic functions. Passing to the limit as $\delta \searrow 0$ we get

$$\max\{v - \delta, 0\} \to \max\{v, 0\} = v \quad \text{in } L^\infty(\mathbb{R}^N) \text{ as } \delta \searrow 0.$$

Since uniform convergence preserves the mean value inequality this implies that v is subharmonic in $\mathbb{R}^N$. Employing the mean value inequality again we get for all $r > 0$ and all $x \in \mathbb{R}^N$

$$0 \leq v(x) \leq \fint_{B_r(x)} v \leq \frac{1}{|B_r|} \int_{B_{2r}} |v| = \frac{1}{|B_1|} \int_{B_2} |v^r|, \tag{2.5}$$

where $v^r(x) := v(rx)$. Taking also (2.4) into account which implies

$$v^r \to 0 \quad \text{in } L^p_{loc}(\mathbb{R}^N) \text{ as } r \to \infty$$

we can make the right hand side in (2.5) arbitrarily small by choosing r sufficiently large. We infer therefore that

$$\partial_{NN}u \equiv 0 \quad \in \mathbb{R}^N \setminus \mathcal{C}$$

and combining this with the cylindrical structure of $\mathcal{C}$ we see that for any $x \in \mathbb{R}^N \setminus \mathcal{C}$

and all $t \in \mathbb{R}$

$$u(x + te_N) = u(x) + t\partial_N u(x).$$

But this contradicts the fact that u is nonnegative if not

$$\partial_N u(x) = 0.$$

Since $x \in \mathbb{R}^N \setminus \mathcal{C}$ was arbitrary this implies that u is independent of x_N and hence 'lower dimensional' and we arrived at the desired contradiction.

Step 2. *$\mathcal{C}$ is bounded in e_N-direction to the left or to the right.*

Assume towards a contradiction that this is not true. Then there are two sequences

$$(x^n)_{n\in\mathbb{N}} \subset \mathcal{C} \text{ and } (y^n)_{n\in\mathbb{N}} \subset \mathcal{C} \quad \text{such that} \quad x_N^n \to +\infty \text{ and } y_N^n \to -\infty \text{ as } n \to \infty$$

and we assume without loss of generality that $x^1 = y^1$. Since $\mathcal{C}$ is convex this implies that the convex combinations

$$\left\{tx^n + (1-t)x^1 : t \in [0,1]\right\} \quad \text{and} \quad \left\{ty^n + (1-t)y^1 : t \in [0,1]\right\} \tag{2.6}$$

are contained in $\mathcal{C}$ for all $n \in \mathbb{N}$. Since ∂B_1 is compact there are $\tilde{x}, \tilde{y} \in \partial B_1$ such that up to taking a subsequence (again labelled the same)

$$\frac{x^n - x^1}{|x^n - x^1|} \to \tilde{x},$$
$$\frac{y^n - y^1}{|y^n - y^1|} \to \tilde{y}$$

as $n \to \infty$. Passing to the limit $n \to \infty$ in (2.6) we get that

$$\left\{x^1 + t\tilde{x} : t \geq 0\right\} \cup \left\{y^1 + t\tilde{y} : t \geq 0\right\} \subset \mathcal{C}. \tag{2.7}$$

In view of (2) this is only possible if $\tilde{x}$ and $\tilde{y}$ are parallel to e_N. So we infer that $\tilde{x} = e_N$ and $\tilde{y} = -e_N$. Inserting this into (2.7) we find that

$$\left\{x^1 + te_N : t \in \mathbb{R}\right\} \subset \mathcal{C}.$$

So we have arrived at a contradiction to step 1 and the lemma is proved. □

2.2 First frequency estimate and first estimate on lower order asymptotics

In this section we derive a first estimate of the asymptotics of the given solution u by studying the blow-down of a normalisation. This analysis is based on a first frequency estimate as introduced below.

Lemma 2.2 (First frequency estimate)**.**
Let $\tilde{v}$ be given as

$$\tilde{v} := u - p,$$

where p is the blow-down limit as introduced in (2) *and let the first frequency functional be defined for all $r > 0$ as*

$$F_1[\tilde{v}](r) := \int_{B_1} |\nabla \tilde{v}_r|^2 - 2 \int_{\partial B_1} \tilde{v}_r^2 \, \mathrm{d}\mathcal{H}^{N-1} = F_1[\tilde{v}_r](1)$$

where $\tilde{v}_r(x) := \frac{\tilde{v}(rx)}{r^2}$.
Then $F_1[\tilde{v}](r)$ is monotone increasing in r and $F_1[\tilde{v}](r)$ is nonpositive for all $r > 0$. Furthermore F_1 is invariant under perturbation by harmonic polynomials that are homogeneous of degree 2, i.e.

$$F_1[\tilde{v} + q](r) = F_1[\tilde{v}](r)$$

for all harmonic polynomials q that are homogeneous of degree 2 and all $r > 0$.

Proof. Note that $\tilde{v}_r$ solves $\Delta \tilde{v}_r = -\chi_{\{u_r = 0\}}$ in $\mathbb{R}^N$. Using this we find that F_1 is monotone increasing since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} F_1[\tilde{v}](r) &= 2 \int_{B_1} \nabla \tilde{v}_r \cdot \nabla \partial_r \tilde{v}_r - 4 \int_{\partial B_1} \tilde{v}_r \partial_r \tilde{v}_r \, \mathrm{d}\mathcal{H}^{N-1} \\
&= 2 \left(\int_{\partial B_1} \nabla \tilde{v}_r \cdot x \, \partial_r \tilde{v}_r \, \mathrm{d}\mathcal{H}^{N-1} - \int_{B_1} \underbrace{\Delta \tilde{v}_r \partial_r \tilde{v}_r}_{=0} - 2 \int_{\partial B_1} \tilde{v}_r \partial_r \tilde{v}_r \, \mathrm{d}\mathcal{H}^{N-1} \right) \\
&= 2 \int_{\partial B_1} (\nabla \tilde{v}_r \cdot x - 2\tilde{v}_r) \partial_r \tilde{v}_r \, \mathrm{d}\mathcal{H}^{N-1} = 2r \int_{\partial B_1} (\partial_r \tilde{v}_r)^2 \, \mathrm{d}\mathcal{H}^{N-1} \geq 0.
\end{aligned}$$

By definition of p we know that

$$\tilde{v}_r \to 0 \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^N) \quad \text{as } r \to \infty.$$

From this we can conclude that

$$\lim_{r\to\infty}\left(\int_{B_1}|\nabla\tilde{v}_r|^2-2\int_{\partial B_1}\tilde{v}_r^2\,\mathrm{d}\mathcal{H}^{N-1}\right)=0.$$

Using that $F_1[\tilde{v}](\cdot)$ is monotone increasing this implies that for all $r>0$

$$\int_{B_1}|\nabla\tilde{v}_r|^2\leq 2\int_{\partial B_1}\tilde{v}_r^2\,\mathrm{d}\mathcal{H}^{N-1}$$

and therefore $F_1[\tilde{v}](\cdot)$ is nonpositive.
It remains to prove the claimed invariance. A direct calculation shows that

$$\begin{aligned}
F_1[\tilde{v}+q](r)&=\int_{B_1}|\nabla(\tilde{v}_r+q)|^2-2\int_{\partial B_1}(\tilde{v}_r+q)^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&=F_1[\tilde{v}](r)+2\left(\int_{B_1}\nabla\tilde{v}_r\cdot\nabla q-2\int_{\partial B_1}\tilde{v}_r q\,\mathrm{d}\mathcal{H}^{N-1}\right)+F_1[q](1)\\
&=F_1[\tilde{v}](r)+2\left(\int_{\partial B_1}\tilde{v}_r\nabla q\cdot x\,\mathrm{d}\mathcal{H}^{N-1}-\int_{B_1}\tilde{v}_r\underbrace{\Delta q}_{=0}-2\int_{\partial B_1}\tilde{v}_r q\,\mathrm{d}\mathcal{H}^{N-1}\right)\\
&\quad+\left(\int_{\partial B_1}q\nabla q\cdot x\,\mathrm{d}\mathcal{H}^{N-1}-\int_{B_1}q\underbrace{\Delta q}_{=0}-2\int_{\partial B_1}q^2\,\mathrm{d}\mathcal{H}^{N-1}\right)\\
&=F_1[\tilde{v}](r)+2\int_{\partial B_1}\tilde{v}_r(\nabla q\cdot x-2q)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial B_1}q(\nabla q\cdot x-2q)\,\mathrm{d}\mathcal{H}^{N-1}\\
&=F_1[\tilde{v}](r).
\end{aligned}$$

In the last step we have used that $\nabla q\cdot x=2q$ for all polynomials q that are homogeneous of degree 2. □

The aim now is to show that the rescaled solution after subtracting the blow-down does only grow linearly in x_N. Unfortunately it has turned out to be inconvenient to work with $\tilde{v}_r$ directly because there are some small error terms that are unpleasant to control. This is why we take them out in the following before passing to the limit $r\to\infty$.

We define for all $r > 0$

$$v_r := u_r - p - h_r, \tag{2.8}$$

where

$$h_r(x') := \Pi'(u_r - p) := L^2(\partial B_1) - \text{ projection of } u_r - p \text{ onto } \mathcal{P}_2', \tag{2.9}$$

and

$$\mathcal{P}_2' := \text{homogeneous, harmonic polynomials of degree 2 depending on } x'.$$

One may understand h_r as correction for discrepancies between u_r and its limit p - but only in x'. Since we are interested in the lower order asymptotics of u in x_N, such a correction is not problematic. Note that v_r solves

$$\Delta v_r = -\chi_{\{u_r=0\}} \quad \text{ in } \mathbb{R}^N \text{ for all } r > 0.$$

Applying Lemma 2.2 with $q = -h_r$ we find that v_r satisfies for all $r > 0$ the following frequency estimate

$$\frac{\int_{B_1} |\nabla v_r|^2}{\int_{\partial B_1} v_r^2 \, \mathrm{d}\mathcal{H}^{N-1}} \leq 2. \tag{2.10}$$

This immediately implies a first estimate on the normalisation

$$w_r := \frac{v_r}{\sqrt{\int_{\partial B_1} v_r^2 \, \mathrm{d}\mathcal{H}^{N-1}}}, \tag{2.11}$$

i.e. invoking Lemma B.2 we see that

$$(w_r)_{r>0} \text{ is bounded in } W^{1,2}(B_1).$$

But this is not enough to pass to the limit $r \to \infty$ in $\mathbb{R}^N$. We need the following lemma.

Lemma 2.3.
Let $N \geq 3$ and let w_r be as defined in (2.11)*. Then*

$$(w_r)_{r>0} \textit{ is bounded in } W^{1,2}_{loc}(\mathbb{R}^N).$$

Proof of Lemma 2.3. A calculation shows that

$$\int\limits_{\partial B_R} v_r^2(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) = \int\limits_{\partial B_1} v_r^2(Ry)\,\mathrm{d}\mathcal{H}^{N-1}(y)R^{N-1} = R^{N+1}\int\limits_{\partial B_1} v_{rR}^2(y)\,\mathrm{d}\mathcal{H}^{N-1}(y).$$

So for any $r > 0$ and any $R > 1$ we get employing Lemma B.2 and (2.10)

$$\begin{aligned}\|w_r\|^2_{W^{1,2}(B_R)} &= \frac{\int_{B_R}|\nabla v_r|^2 + \int_{B_R} v_r^2}{\int_{\partial B_1} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}} \le C_1(R)\frac{\int_{B_R}|\nabla v_r|^2 + \int_{\partial B_R} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}}\\ &\le C_2(R)\frac{\int_{\partial B_R} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}} = C_3(R)\frac{\int_{\partial B_1} v_{rR}^2\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}}.\end{aligned}$$

So the proof is finished if we can show that for each $R > 1$ the following doubling holds

$$\sup_{r>0}\frac{\int_{\partial B_1} v_{rR}^2\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_r^2\,\mathrm{d}\mathcal{H}^{N-1}} < C \tag{2.12}$$

for some $C > 0$. Assume towards a contradiction that this is not true. Then there is $R_0 > 1$ and a sequence of rescalings $(r_k)_{k\in\mathbb{N}} \subset (0,\infty)$, $r_k \to \infty$ as $k \to \infty$ such that

$$\frac{\int_{\partial B_1} v_{r_k}^2\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_{\frac{r_k}{R_0}}^2\,\mathrm{d}\mathcal{H}^{N-1}} \to \infty \quad \text{as } k \to \infty. \tag{2.13}$$

Since we already know that $(w_{r_k})_{k\in\mathbb{N}} = \left(\frac{v_{r_k}}{\sqrt{\int_{\partial B_1} v_{r_k}^2\,\mathrm{d}\mathcal{H}^{N-1}}}\right)_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(B_1)$, for a subsequence (again labelled r_k)

$$w_{r_k} \rightharpoonup w \quad \text{weakly in } W^{1,2}(B_1) \tag{2.14}$$

and due to the compact embeddings $W^{1,2}(B_1) \hookrightarrow L^2(\partial B_1)$ and $W^{1,2}(B_{\frac{1}{R_0}}) \hookrightarrow L^2(\partial B_{\frac{1}{R_0}})$ (recall that $\frac{1}{R_0} < 1$) it follows that

$$\begin{aligned} w_{r_k} &\to w \quad \text{strongly in } L^2(\partial B_1),\\ w_{r_k} &\to w \quad \text{strongly in } L^2(\partial B_{\frac{1}{R_0}}).\end{aligned}$$

By definition for all $k \in \mathbb{N}$ we have that $\int\limits_{\partial B_1} w_{r_k}^2\,\mathrm{d}\mathcal{H}^{N-1} = 1$ and thus it also holds

that

$$\int\limits_{\partial B_1} w^2 \,\mathrm{d}\mathcal{H}^{N-1} = 1. \tag{2.15}$$

On the other hand using (2.13) we obtain that

$$\begin{aligned}
\int\limits_{\partial B_{\frac{1}{R_0}}} w^2 \,\mathrm{d}\mathcal{H}^{N-1} \leftarrow \int\limits_{\partial B_{\frac{1}{R_0}}} w_{r_k}^2 \,\mathrm{d}\mathcal{H}^{N-1} &= \frac{\int_{\partial B_{\frac{1}{R_0}}} v_{r_k}^2 \,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_{r_k}^2 \,\mathrm{d}\mathcal{H}^{N-1}} \\
&= \frac{\int_{\partial B_1} v_{r_k}^2 \left(\frac{1}{R_0}x\right) R_0^{1-N} \,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_{r_k}^2 \,\mathrm{d}\mathcal{H}^{N-1}} \\
&= R_0^{-3-N} \frac{\int_{\partial B_1} v_{\frac{r_k}{R_0}}^2 \,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} v_{r_k}^2 \,\mathrm{d}\mathcal{H}^{N-1}} \to 0 \quad \text{as } k \to \infty
\end{aligned}$$

and this implies that

$$\int\limits_{\partial B_{\frac{1}{R_0}}} w^2 \,\mathrm{d}\mathcal{H}^{N-1} = 0.$$

However w is harmonic in B_1 since it was constructed as the weak limit of $(w_{r_k})_{k\in\mathbb{N}}$ in (2.14) and $\Delta w = 0$ outside a set of 2-capacity zero. Now the maximum principle for harmonic functions implies that

$$w \equiv 0 \quad \text{in } B_{\frac{1}{R_0}}.$$

Since w, being harmonic, is analytic, it already has to vanish in all of B_1. But then w cannot have a nontrivial trace on ∂B_1 and this contradicts (2.15). Therefore the assumption (2.13) must have been false and the Lemma is proved. □

Proposition 2.4.
Let $(w_r)_{r>0}$ be as defined in (2.11). *Then in dimensions $N \geq 3$ there is a subsequence $(w_{r_k})_{k\in\mathbb{N}}$ such that*

$$w_{r_k} \rightharpoonup w \quad \textit{weakly in } W^{1,2}_{loc}(\mathbb{R}^N) \textit{ as } k \to \infty \tag{2.16}$$

and w is harmonic, i.e.

$$\Delta w = 0 \quad \textit{in } \mathbb{R}^N.$$

Proof. From Lemma 2.3 it follows that there is a subsequence $(w_{r_k})_{k\in\mathbb{N}}$

$$w_{r_k} \rightharpoonup w \quad \text{weakly in } W^{1,2}_{loc}(\mathbb{R}^N) \text{ as } k \to \infty.$$

Using the assumption on the blow-down in (2) we obtain that

$$\Delta w = 0 \quad \text{in } \mathbb{R}^N \setminus \left\{te_N \in \mathbb{R}^N : t \in \mathbb{R}\right\}.$$

Since $\left\{te_N \in \mathbb{R}^N : t \in \mathbb{R}\right\}$ is a set of 2-capacity zero in dimensions $N \geq 3$, we infer that in these dimensions

$$\Delta w = 0 \quad \text{in } \mathbb{R}^N.$$

□

Remark 2.5 (2 dimensional case).
If one was be able to prove Lemma 2.3 for $N = 2$ one could derive a 2-dimensional analogue to Proposition 2.4 but with Laplacian of w being a measure supported on the line $\{te_2 \in \mathbb{R}^2 : t \in \mathbb{R}\}$. As a consequence one would not be able to conclude that w is affine linear as we will do for $N \geq 3$.

Lemma 2.6 (w is a quadratic polynomial)**.**
Let $N \geq 3$ and let w be as in Proposition 2.4. Then w is a harmonic polynomial of degree ≤ 2.

Proof of Lemma 2.6. The strategy of the proof of this lemma will be to use the first frequency estimate (2.10) in order to obtain a doubling that allows us to deduce that w has at most quadratic growth at infinity. A Liouville argument implies then that w is a polynomial of degree ≤ 2.
First of all note that since w is a harmonic function it holds that

$$\int_{B_1} |\nabla w|^2 = \int_{\partial B_1} w\, \nabla w \cdot x \,\mathrm{d}\mathcal{H}^{N-1} - \int_{B_1} \underbrace{\Delta w}_{=0}\, w = \int_{\partial B_1} w\, \nabla w \cdot x \,\mathrm{d}\mathcal{H}^{N-1}. \tag{2.17}$$

Let us now define for all $R > 0$ the rescaled average

$$y(R) := \int_{\partial B_1} (w^R)^2 \,\mathrm{d}\mathcal{H}^{N-1} \quad \text{where} \quad w^R(x) := w(Rx) \text{ for all } x \in \mathbb{R}^N.$$

Then the derivative of $y(R)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}y(R) &= \int\limits_{\partial B_1} 2w^R\,\partial_R w^R\,\mathrm{d}\mathcal{H}^{N-1} = 2\int\limits_{\partial B_1} w(Rx)\,\nabla w(Rx)\cdot x\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \frac{2}{R}\int\limits_{\partial B_1} w^R\,\nabla w^R\cdot x\,\mathrm{d}\mathcal{H}^{N-1} = \frac{2}{R}\int\limits_{B_1}\left|\nabla w^R\right|^2,\end{aligned} \tag{2.18}$$

where we have used that w^R is harmonic and (2.17) in the last step. In order to deduce a differential inequality, we use that w^R also obeys the first frequency estimate, i.e.

$$\frac{\int_{B_1}\left|\nabla w^R\right|^2}{\int_{\partial B_1}(w^R)^2\,\mathrm{d}\mathcal{H}^{N-1}} \le 2 \quad \text{for all } R>0.$$

This can be seen as follows: From (2.10) we have that for all $R, r > 0$

$$\begin{aligned}2 &\ge \frac{\int_{B_1}|\nabla v_{rR}|^2}{\int_{\partial B_1} v_{rR}^2\,\mathrm{d}\mathcal{H}^{N-1}} = \frac{\int_{B_1}|\nabla w_{rR}|^2}{\int_{\partial B_1} w_{rR}^2\,\mathrm{d}\mathcal{H}^{N-1}}\\ &= \frac{\int_{B_1}|R^{-1}\,\nabla w_r(Rx)|^2}{\int_{\partial B_1} R^{-4}\,w_r^2(Rx)\,\mathrm{d}\mathcal{H}^{N-1}} = R^2\,\frac{\int_{B_1}|\nabla w_r(Rx)|^2}{\int_{\partial B_1} w_r^2(Rx)\,\mathrm{d}\mathcal{H}^{N-1}}.\end{aligned}$$

Now weak convergence of $(w_{r_k})_{k\in\mathbb{N}}$ as $k\to\infty$ (recall (2.16)) and lower semicontinuity of the Dirichlet-functional and the compactness of the trace embedding imply that for all $R>0$

$$2 \ge\ R^2\,\frac{\int_{B_1}|\nabla w(Rx)|^2}{\int_{\partial B_1} w^2(Rx)\,\mathrm{d}\mathcal{H}^{N-1}} = \frac{\int_{B_1}\left|\nabla w^R\right|^2}{\int_{\partial B_1}(w^R)^2\,\mathrm{d}\mathcal{H}^{N-1}}.$$

Combining this frequency estimate with (2.18) we derive the following differential inequaltiy for y

$$\frac{\mathrm{d}}{\mathrm{d}R}\,y(R) = \frac{2}{R}\int\limits_{B_1}\left|\nabla w^R\right|^2 \le \frac{4}{R}\int\limits_{\partial B_1}(w^R)^2\,\mathrm{d}\mathcal{H}^{N-1} = \frac{4}{R}\,y(R).$$

This implies that for all $R\ge 1$

$$y(R)\le y(1)\,R^4. \tag{2.19}$$

Fubini's Theorem yields that the same growth estimate is also true for the squared

average on the full ball. For each $R \geq 1$ it holds that

$$\begin{aligned}
\int_{B_1} (w^R)^2 &= \int_{\frac{1}{R}}^{1} \int_{\partial B_\varrho} (w^R)^2(x) \,\mathrm{d}\mathcal{H}^{N-1}(x) \,\mathrm{d}\varrho + \int_{B_{\frac{1}{R}}} (w^R)^2 \\
&= \int_{\frac{1}{R}}^{1} \int_{\partial B_1} (w^R)^2(\varrho x) \varrho^{N-1} \,\mathrm{d}\mathcal{H}^{N-1}(x) \,\mathrm{d}\varrho + \int_{B_1} w^2 \; R^{-N} \\
&= \int_{\frac{1}{R}}^{1} \int_{\partial B_1} (w^{R\varrho})^2(x) \varrho^{N-1} \,\mathrm{d}\mathcal{H}^{N-1}(x) \,\mathrm{d}\varrho + \int_{B_1} w^2 \; R^{-N} \\
&\leq \int_{\frac{1}{R}}^{1} \varrho^{N-1} (\varrho R)^4 y(1) \,\mathrm{d}\varrho + \int_{B_1} w^2 \; R^{-N} \\
&= \frac{y(1)}{N+4} \left(1 - \frac{1}{R^{N+4}}\right) R^4 + \int_{B_1} w^2 \; R^{-N} \\
&\leq C_1 R^4,
\end{aligned} \tag{2.20}$$

where we have used that $\varrho R \geq 1$ on $\varrho \in \left[\frac{1}{R}, 1\right]$ and (2.19). Here $C_1 > 0$ is some constant independent of R. We are going to combine this with the mean value property of harmonic functions in order to get a uniform estimate on the second derivatives. Let $x_0 \in B_{\frac{1}{8}}$ be arbitrary. Then for all $i, j \in \{1, \ldots, N\}$ we have by the mean value property

$$\left|\partial_{ij} w^R(x_0)\right| = \left| \fint_{B_{\frac{1}{8}}(x_0)} \partial_{ij} w^R \right| = \frac{1}{\left|B_{\frac{1}{8}}\right|} \left| \int_{\partial B_{\frac{1}{8}}(x_0)} \partial_i w^R \; \nu_j \,\mathrm{d}\mathcal{H}^{N-1} \right| \leq \frac{\left|\partial B_{\frac{1}{8}}\right|}{\left|B_{\frac{1}{8}}\right|} \sup_{\partial B_{\frac{1}{8}}(x_0)} \left|\partial_i w^R\right|, \tag{2.21}$$

where by ν we mean the outer unit normal to $B_{\frac{1}{8}}(x_0)$. The same way we compute for any $x \in B_{\frac{1}{4}}(0)$ (note that for all $x_0 \in B_{\frac{1}{8}}(0) : B_{\frac{1}{8}}(x_0) \subset B_{\frac{1}{4}}(0)$)

$$\left|\partial_i w^R(x)\right| = \left| \fint_{B_{\frac{1}{8}}(x)} \partial_i w^R \right| = \frac{1}{\left|B_{\frac{1}{8}}\right|} \left| \int_{\partial B_{\frac{1}{8}}(x)} w^R \; \nu_i \,\mathrm{d}\mathcal{H}^{N-1} \right| \leq \frac{\left|\partial B_{\frac{1}{8}}\right|}{\left|B_{\frac{1}{8}}\right|} \sup_{\partial B_{\frac{1}{8}}(x)} \left|w^R\right|$$

$$
\begin{aligned}
&\leq \frac{\left|\partial B_{\frac{1}{8}}\right|}{\left|B_{\frac{1}{8}}\right|} \sup_{B_{\frac{3}{8}}(0)} \left|w^R\right| = \frac{\left|\partial B_{\frac{1}{8}}\right|}{\left|B_{\frac{1}{8}}\right|} \sup_{y \in B_{\frac{3}{8}}(0)} \left| \fint_{B_{\frac{1}{8}}(y)} w^R \right| \\
&\leq \frac{\left|\partial B_{\frac{1}{8}}\right|}{\left|B_{\frac{1}{8}}\right|^2} \int_{B_1} \left|w^R\right| \leq \frac{\left|\partial B_{\frac{1}{8}}\right| \sqrt{|B_1|}}{\left|B_{\frac{1}{8}}\right|^2} \sqrt{\int_{B_1} (w^R)^2}, \qquad (2.22)
\end{aligned}
$$

where ν is the outer unit normal to $\partial B_{\frac{1}{8}}(x)$. Combining (2.21) and (2.22) and using (2.20) we obtain

$$
\left\|\partial_{ij} w^R\right\|_{L^\infty(B_{\frac{1}{8}})} \leq \frac{\left|\partial B_{\frac{1}{8}}\right|^2 \sqrt{|B_1|}}{\left|B_{\frac{1}{8}}\right|^3} \sqrt{\int_{B_1} (w^R)^2} \leq C_2 R^2,
$$

where C_2 is again independent of R and $R \geq 1$. Recalling that $\partial_{ij} w^R(x) = \partial_{ij} w(Rx)\, R^2$ this implies that

$$
R^2 \left\|\partial_{ij} w\right\|_{L^\infty(B_{\frac{R}{8}})} \leq C_2 R^2.
$$

Thereby we have arrived at the desired uniform bound

$$
\text{for all } R \geq 1: \quad \left\|D^2 w\right\|_{L^\infty(B_{\frac{R}{8}})} \leq C_2.
$$

A Liouville-type argument now implies that $D^2 w$ (being harmonic) is necessarily constant. Finally this tells us that w is a harmonic polynomial of degree ≤ 2. □

In order to get a nontrivial estimate on the asymptotics of u we need to exclude quadratic growth of w. We do this in the following lemma.

Lemma 2.7 (w is affine linear)**.**
Let $N \geq 3$ and w be as defined in (2.16)*. Then w is (at most) an affine linear function*

$$
w = \ell + c,
$$

where ℓ is the linear part and the constant part c is nonpositive.

Proof of Lemma 2.7. From Lemma 2.6 we already know that w is a harmonic polynomial of degree ≤ 2. Therefore we can write

$$
w = h + \ell + c,
$$

where h is an harmonic, homogeneous polynomial of degree 2, $\ell : \mathbb{R}^N \to \mathbb{R} : x \mapsto b \cdot x$ ($b \in \mathbb{R}^N$) is a linear function and $c \in \mathbb{R}$ is a constant. We prove the claim of the lemma in three steps.

Step 1. *h is independent of x_N.*

This is the main part of the proof. Here the x_N-independence of p is crucial. We claim that as a consequence of the blow-down assumption (2)

$$\partial_N w \geq 0 \text{ in } \mathbb{R}^N \quad \text{or} \quad \partial_N w \leq 0 \text{ in } \mathbb{R}^N,$$

i.e. that w is monotone in e_N-direction. But this is not compatible with the fact that h contains terms of the form x_N^2 and $x_j x_N$, $j \in \{1, \dots, N-1\}$. Let us delay the proof of the claim for a second and let us assume towards a contradiction that there is $d \in \mathbb{R}^N \setminus \{0\}$ such that

$$h(x) = \sum_{j=1}^{N} d_j \, x_j x_N + h'(x'),$$

where h' does only depend on x'. Then the derivative of w in direction e_N is given by

$$\partial_N w(x) = \sum_{j=1}^{N} d_j \, x_j + d_N x_N + b_N$$

for each $x \in \mathbb{R}^N$ and thus we get

$$\partial_N w(\lambda d) = \sum_{j=1}^{N} \lambda d_j^2 + \lambda d_N^2 + b_N \to \pm\infty \quad \text{as } \lambda \to \pm\infty.$$

But this contradicts the claim that w is monotone in e_N-direction. It remains to prove the claim.

We follow a strategy of L. Caffarelli, L. Karp and H. Shahgholian [3, Proof of Case 3 of Theorem II] based on the celebrated Alt-Caffarelli-Friedman monotonicity formula applied to

$$(\partial_N u)^+ := \max\{\partial_N u, 0\} \quad \text{and} \quad (\partial_N u)^- := -\min\{\partial_N u, 0\}.$$

It is known that $(\partial_N u)^+$ and $(\partial_N u)^-$ are subharmonic (see e.g. [12, Remark 2.16]) and therefore we have that

$$(\partial_N u)^\pm \geq 0 \quad , \quad \Delta(\partial_N u)^\pm \geq 0 \quad \text{and} \quad (\partial_N u)^+ (\partial_N u)^- = 0 \quad \text{in } \mathbb{R}^N$$

and hence the functional

$$\phi(r) := \int\limits_{B_1} \frac{|\nabla(\partial_N u_r)^+|^2}{|x|^{N-2}} \int\limits_{B_1} \frac{|\nabla(\partial_N u_r)^-|^2}{|x|^{N-2}},$$

where u_r is the rescaling from (2), is nondecreasing for all $r > 0$ as was proved in the seminal work [1]. Combined with the blow-down assumption (2) this implies that for all $r > 0$

$$\int\limits_{B_1} \frac{|\nabla(\partial_N u_r)^+|^2}{|x|^{N-2}} \int\limits_{B_1} \frac{|\nabla(\partial_N u_r)^-|^2}{|x|^{N-2}} \leq \int\limits_{B_1} \frac{|\nabla(\partial_N p)^+|^2}{|x|^{N-2}} \int\limits_{B_1} \frac{|\nabla(\partial_N p)^-|^2}{|x|^{N-2}} = 0.$$

Thus for each $r > 0$ either $(\partial_N u_r)^+$ or $(\partial_N u_r)^-$ must be constant in B_1. This implies that there is an increasing sequence $(r_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $r_k \nearrow \infty$ as $k \to \infty$ and $(\tilde{c}_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that either

$$\text{for all } k \in \mathbb{N} \quad (\partial_N u_{r_k})^+ \equiv \tilde{c}_k \ \text{ in } B_1 \quad \Longleftrightarrow \quad (\partial_N u)^+ \equiv \tilde{c}_k r_k := c_k \ \text{ in } B_{r_k}$$

or

$$\text{for all } k \in \mathbb{N} \quad (\partial_N u_{r_k})^- \equiv \tilde{c}_k \ \text{ in } B_1 \quad \Longleftrightarrow \quad (\partial_N u)^- \equiv \tilde{c}_k r_k := c_k \ \text{ in } B_{r_k}.$$

Since $B_{r_k} \subset B_{r_{k+1}}$ for all $k \in \mathbb{N}$ we have that $c_k = c_1$ for all $k \in \mathbb{N}$. Therefore either

$$(\partial_N u)^+ \equiv c_1 \quad \text{in} \bigcup_{k\in\mathbb{N}} B_{r_k} = \mathbb{R}^N \quad \text{or} \quad (\partial_N u)^- \equiv c_1 \quad \text{in} \bigcup_{k\in\mathbb{N}} B_{r_k} = \mathbb{R}^N$$

and taking into account that both $(\partial_N u)^+$ and $(\partial_N u)^-$ vanish in $\mathcal{C}$ (which has non-empty interior) the constant c_1 can only be zero. It therefore follows that either $(\partial_N u)^+ \equiv 0$ or $(\partial_N u)^- \equiv 0$ in $\mathbb{R}^N$.

Note that we have constructed h_r in (2.9) to be independent of x_N and p is independent of x_N by assumption (2). Therefore we can transfer this monotonicity over to w_r, i.e. for all $r > 0$

$$\text{either } \ \partial_N w_r \geq 0 \text{ in } \mathbb{R}^N \quad \text{or} \quad \partial_N w_r \leq 0 \text{ in } \mathbb{R}^N.$$

Invoking that

$$\partial_N w_{r_k} \rightharpoonup \partial_N w \quad \text{weakly in } L^2_{loc}(\mathbb{R}^N) \text{ as } k \to \infty$$

this implies that

$$\partial_N w \geq 0 \text{ in } \mathbb{R}^N \quad \text{or} \quad \partial_N w \leq 0 \text{ in } \mathbb{R}^N$$

and thereby the claim is proved.
Step 2. $h \equiv 0$.
By construction for all $r > 0$ we have that

$$\Pi' w_r = 0 \quad \text{i.e.} \int\limits_{\partial B_1} w_r \, q \, \mathrm{d}\mathcal{H}^{N-1} = 0 \quad \text{for all } q \in \mathcal{P}_2'.$$

Employing (2.16) this implies that

$$\int\limits_{\partial B_1} w \, q \, \mathrm{d}\mathcal{H}^{N-1} = 0 \quad \text{for all } q \in \mathcal{P}_2'. \tag{2.23}$$

Furthermore we have for all $q \in \mathcal{P}_2'$

$$\begin{aligned}
\int\limits_{\partial B_1} (\ell + c) q \, \mathrm{d}\mathcal{H}^{N-1} &= \int\limits_{\partial B_1} qb \cdot x \, \mathrm{d}\mathcal{H}^{N-1} + c \int\limits_{\partial B_1} q \, \mathrm{d}\mathcal{H}^{N-1} \\
&= \int\limits_{B_1} \operatorname{div}(qb) + c \int\limits_{\partial B_1} q \, \mathrm{d}\mathcal{H}^{N-1} = \sum_{j=1}^{N-1} b_j \int\limits_{B_1} \partial_j q + c \int\limits_{\partial B_1} q \, \mathrm{d}\mathcal{H}^{N-1} \\
&= \sum_{j=1}^{N-1} b_j |B_1| \partial_j q(0) + c |\partial B_1| q(0) = 0,
\end{aligned}$$

where we have used that q is harmonic, the mean value theorem for harmonic functions and that q is homogeneous of degree 2 and therefore $q(0) = 0$ and $\nabla q(0) = 0$. Since we already know from step 1 that $h \in \mathcal{P}_2'$ it follows from (2.23) that $h \equiv 0$ and hence

$$w = \ell + c.$$

Step 3. $c \leq 0$.
This is actually due to the choice of coordinate system in section 2.1. It can be seen as follows. From the definition of v_r (see (2.8)) we know that

$$\Delta v_r = -\chi_{\mathcal{C}_r} \leq 0, \quad \text{where } \mathcal{C}_r := \tfrac{1}{r}\mathcal{C}.$$

Then the mean value inequality for superharmonic functions tells us that

$$0 = u_r(0) - p(0) - h_r(0) = v_r(0) \geq \fint_{B_1} v_r.$$

This implies that for all $r > 0$

$$0 \geq \fint_{B_1} w_r$$

and therefore

$$0 \geq \fint_{B_1} w = \fint_{B_1} \ell + \fint_{B_1} c = c.$$

□

2.3 A first estimate on the coincidence set $\mathcal{C}$

We are now in the position to derive a first nontrivial growth estimate for $\mathcal{C}$ as $x_N \to \infty$.

Proposition 2.8 (First estimate on $\mathcal{C}$).
Let $\mathcal{C}$ be the coincidence set as defined in Definition 0.4. Then for any $\delta \in (0,1)$ there is $a(\delta) > 0$ such that

i)

$$(\mathcal{C} \cap \{x_N > a\}) \subset \left\{|x'|^2 < x_N^{1+\delta}\right\} \text{ and}$$

ii)

$$\mathcal{C} \cap \{x_N \leq a\} \text{ is bounded.}$$

Proof.

i) The main tool in proving this claim is a quantitative version of the doubling we have already used in (2.12). In order to derive a nontrivial bound on the coincidence set $\mathcal{C}$ from our previous analysis the fact that w is affine linear is essentially needed. Otherwise we could only arrive at the trivial estimate that $\mathcal{C}$ is contained

in a cone. Let us define the following rescaling

$$\begin{aligned} v^r(x) &:= u^r(x) - p^r(x') - h_r^r(x') \\ &= u(rx) - p(rx') - h_r(rx') = r^2 v_r(x) \end{aligned} \tag{2.24}$$

and let us furthermore set for all $r > 0$

$$f(r) := \sqrt{\int_{\partial B_1} (v^r)^2 \,\mathrm{d}\mathcal{H}^{N-1}}.$$

Lemma 2.7 implies the following doubling

$$\begin{aligned} \frac{\int_{\partial B_1} (v^{2r})^2 \,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} (v^r)^2 \,\mathrm{d}\mathcal{H}^{N-1}} &= \frac{\int_{\partial B_2} 2^{1-N} (v^r)^2 \,\mathrm{d}\mathcal{H}^{N-1}}{\int_{\partial B_1} (v^r)^2 \,\mathrm{d}\mathcal{H}^{N-1}} \\ &= 2^{1-N} \int_{\partial B_2} \frac{(v^r)^2}{\int_{\partial B_1} (v^r)^2 \,\mathrm{d}\mathcal{H}^{N-1}} \,\mathrm{d}\mathcal{H}^{N-1} \\ &= 2^{1-N} \int_{\partial B_2} (w_r)^2 \,\mathrm{d}\mathcal{H}^{N-1} \\ &\xrightarrow{r\to\infty} 2^{1-N} \int_{\partial B_2} w^2 \,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial B_1} w^2(2x) \,\mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{\partial B_1} \ell^2(2x) + 2\ell(2x)c + c^2 \,\mathrm{d}\mathcal{H}^{N-1} \\ &= 4 \int_{\partial B_1} \left(\ell^2(x) + 2\ell(x)c + c^2\right) \mathrm{d}\mathcal{H}^{N-1} \\ &\quad - 4 \int_{\partial B_1} \ell(x)c \,\mathrm{d}\mathcal{H}^{N-1} - 3 \int_{\partial B_1} c^2 \,\mathrm{d}\mathcal{H}^{N-1} \\ &\le 4 \int_{\partial B_1} w^2 \,\mathrm{d}\mathcal{H}^{N-1} = 4 \end{aligned}$$

since $\int_{\partial B_1} \ell \,\mathrm{d}\mathcal{H}^{N-1} = 0$. This implies that for any $\varepsilon > 0$ there is $r_0(\varepsilon) > 0$ such that for all $r > r_0(\varepsilon)$

$$\frac{f(2r)}{f(r)} \le 2 + \varepsilon.$$

Iterating this we get for all $k \in \mathbb{N}$ and $r > r_0(\varepsilon)$

$$f\big(2^k r\big) \leq (2+\varepsilon)^k f(r).$$

Let us now choose $\delta \in (0,1)$ arbitrarily small but fixed and $\varepsilon(\delta) > 0$ such that $1 + \frac{\varepsilon}{2} < 2^\delta$. Then we get for all large enough $k \in \mathbb{N}$ and all $r \in \big[2^k r_0, 2^{k+1} r_0\big]$

$$\begin{aligned} f(r) &\leq (2+\varepsilon)^k \sup_{\varrho \in [r_0, 2r_0]} f(\varrho) \leq 2^{\delta k} \cdot 2^k \sup_{\varrho \in [r_0, 2r_0]} f(\varrho) \\ &\leq r^{1+\delta} C(r_0) \sup_{\varrho \in [r_0, 2r_0]} f(\varrho) = r^{1+\delta} C(\delta). \end{aligned} \tag{2.25}$$

This allows us to estimate the asymptotic thickness of the coincidence set $\mathcal{C}$ (as $x_N \to \infty$). For this purpose we need to improve the estimate on the squared average above to a pointwise estimate. We do this using the sup-mean inequality for subharmonic functions from Lemma B.3.
Before we carry this out let us remind the reader that due to the definition of h_r in (2.9) we have that

$$\|h_r\|_{L^2(\partial B_1)} = \|\Pi'(u_r - p)\|_{L^2(\partial B_1)} \leq \|u_r - p\|_{L^2(\partial B_1)} \to 0 \quad \text{as } r \to \infty.$$

Invoking that $\mathcal{P}_2'$ is a finite dimensional vector space where all norms are equivalent all coefficients of h_r must vanish as $r \to \infty$. Since p is non-degenerate in x' we obtain that for all sufficiently large $r > 0$

$$|h_r(x')| \leq \frac{1}{2} p(x') \quad \text{for all } x' \in \mathbb{R}^{N-1}.$$

Setting as in (2.24)

$$p^r(x) + h_r^r(x) - u^r(x) := p(rx) + h_r(rx) - u(rx)$$

we have that

$$\max\{p^r + h_r^r - u^r, 0\}$$

is a nonnegative, subharmonic function and Lemma B.3 tells us that

$$C(N) \sqrt{\int_{\partial B_1} \max\{p^r + h_r^r - u^r, 0\}^2 \, \mathrm{d}\mathcal{H}^{N-1}} \geq \sup_{B_{\frac{1}{2}}} \max\{p^r + h_r^r - u^r, 0\} \tag{2.26}$$

Let now $x \in \mathcal{C}$ be arbitrary up to $r := 4|x|$ being sufficiently large. Combining

(2.25) and (2.26) we get that

$$C(N)C(\delta)r^{1+\delta} \geq \sup_{B_{\frac{r}{2}}} \max\{p + h_r - u, 0\} \geq \max\{p(x') + h_r(x') - u(x), 0\}$$
$$= \max\{p(x') + h_r(x'), 0\} \geq \max\left\{\frac{1}{2}p(x'), 0\right\} = \frac{1}{2}p(x').$$

This means that for all sufficiently large r and arbitrary $x \in \mathcal{C} \cap \left\{|x| = \frac{r}{4}\right\}$ we have that

$$\frac{c_p}{2}|x'|^2 \leq \frac{1}{2}p(x') \leq 4^{1+\delta}C(N)C(\delta)|x|^{1+\delta} \leq 8^{1+\delta}C(N)C(\delta)\Big(|x'|^{1+\delta} + |x_N|^{1+\delta}\Big),$$

where c_p is defined in (3). But for $|x'|$ large, this is only possible if there is a constant $C > 0$ such that

$$|x'|^2 \leq Cx_N^{1+\delta}$$

and this implies i) by absorbing C.

ii) Let us assume towards a contradiction that the claim in ii) is not true. Then there is

$$\left(x^i\right)_{i\in\mathbb{N}} \subset \mathcal{C} \cap \{x_N \leq a\} \quad \text{such that} \quad \left|x^{i'}\right| \to \infty.$$

We define for all $i \in \mathbb{N}$

$$\xi^i := \frac{x^i}{|x^i|} \in \partial B_1.$$

By compactness there is a subsequence (again named $(\xi^i)_{i\in\mathbb{N}}$) such that

$$\xi^i \to \xi \in \partial B_1 \text{ as } i \to \infty.$$

From

$$\left|\xi_N^i\right| = \frac{|x_N^i|}{|x^i|} \leq \frac{a}{\left|x^{i'}\right|} \to 0 \quad \text{as } i \to \infty$$

we conclude that $\xi \neq e_N$. Since $\mathcal{C}$ is convex and closed and since $0 \in \mathcal{C}$ we have

that

$$\xi t \in \mathcal{C} \quad \text{for all } t > 0.$$

This implies that for all $t, r > 0$

$$u_r(\xi t) = \frac{u(\xi t r)}{r^2} = 0.$$

But this is impossible since for all $t > 0$ arbitrary but fixed

$$u_r(\xi t) \to p(\xi t) = t^2 p(\xi') \neq 0 \quad \text{as } r \to \infty$$

by assumption (2).

□

2.4 The Newton-potential expansion of u

In this section we are going to use the growth estimate from Lemma 2.8 in order to show that the Newton-potential of the coincidence set $\mathcal{C}$ is well defined and has subquadratic growth. This yields a Newton-potential expansion of the given solution u. This Newton-potential expansion will allow us to control the asymptotics of the solution up to a constant outside a small region around the coincidence set $\mathcal{C}$.

Lemma 2.9 (Newton-potential of $\mathcal{C}$).
Let $\mathcal{C}$ be as defined in Definition 0.1 and let $N \geq 6$, then

i) for all $x \in \mathbb{R}^N$

$$v_{\mathcal{C}}^{NP}(x) = \alpha(N) \int_{\mathcal{C}} \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y \quad \textit{where } \alpha(N) := \frac{1}{N(N-2)|B_1|}$$

is well defined and

ii) $v_{\mathcal{C}}^{NP}(x)$ grows subquadratically as $|x| \to \infty$, i.e.

$$\frac{v_{\mathcal{C}}^{NP}(rx)}{r^2} \to 0 \quad \textit{strongly in } L^\infty(\partial B_1) \textit{ as } r \to \infty.$$

Proof.

i) Since it is well known that the Newton-potential is well defined on bounded, measurable sets (as can be found e.g. in [8, Theorem 9.9]), all we have to check is that the Newton-potential integral is well defined on $\mathcal{C} \setminus B_R$ for some arbitrarily large $R > 0$.

Let $\delta = \frac{1}{10}$ and $R(x) > 0$ sufficiently large such that

$$\mathcal{C} \setminus B_R \subset \left\{ y \in \mathbb{R}^N : y_N > a(\delta) \text{ and } y_N > 2x_N \right\},$$

where $a(\delta)$ is as in Lemma 2.8, then we have that

$$\begin{aligned}
\int\limits_{\mathcal{C}\setminus B_R} \frac{1}{|x-y|^{N-2}} \,\mathrm{d}y &\leq \int\limits_{\left\{|y'|^2 < y_N^{1+\delta}\right\} \cap \{y_N > \max\{a(\delta), 2x_N\}\}} \frac{1}{|x-y|^{N-2}} \,\mathrm{d}y \\
&\leq \int\limits_{\left\{|y'|^2 < y_N^{1+\delta}\right\} \cap \{y_N > \max\{a(\delta), 2x_N\}\}} \frac{1}{|x_N - y_N|^{N-2}} \,\mathrm{d}y \\
&\leq \int\limits_{a(\delta)}^{\infty} \frac{1}{\left|\frac{y_N}{2}\right|^{N-2}} \left| B'_{y_N^{\frac{1+\delta}{2}}} \right| \,\mathrm{d}y_N \\
&= 2^{N-2} |B'_1| \int\limits_{a(\delta)}^{\infty} y_N^{-N+2+\frac{1+\delta}{2}(N-1)} \,\mathrm{d}y_N.
\end{aligned}$$

This is integrable if and only if

$$-N + 2 + \frac{1+\delta}{2}(N-1) < -1,$$

which is satisfied for $\delta = \frac{1}{10}$ and $N \geq 6$.
(Recall that with $B' := \left\{ x' \in \mathbb{R}^{N-1} : |x'|^2 \leq 1 \right\}$ we mean the $N-1$-dimensional unit ball.)

ii) Let us choose $\delta = \frac{1}{10}$ and let $a(\delta) > 0$ be as in Lemma 2.8 and let $R > 0$ be such that $\mathcal{C} \cap \{y_N < a(\delta)\} \subset B_R$. Since $\mathcal{C}$ is unbounded only in e_N-direction the maximal growth of $v_{\mathcal{C}}^{NP}$ will be achieved for x_N large. For $x_N > 0$ large enough,

we estimate $\mathcal{C}$ as follows

$$\begin{aligned}\mathcal{C} \subset B_R \cup \underbrace{\left\{|y'|^2 < y_N^{1+\delta}; y_N < x_N - x_N^{\frac{23}{24}}\right\}}_{:=I(x_N)} \cup \underbrace{B_{2x_N^{\frac{23}{24}}}(0, x_N)}_{II(x_N)} \\ \cup \underbrace{\left\{|y'|^2 < y_N^{1+\delta}; y_N > x_N + x_N^{\frac{23}{24}}\right\}}_{:=III(x_N)}.\end{aligned} \tag{2.27}$$

This inclusion is satisfied for x_N large enough. Then from (2.27) we get that

$$\begin{aligned}v_{\mathcal{C}}^{NP}(x) &\leq v_{B_R}^{NP}(x) + v_I^{NP}(x) + v_{II}^{NP}(x) + v_{III}^{NP}(x) \\ &\leq v_{B_R}^{NP}(x) + v_I^{NP}((0, x_N)) + v_{II}^{NP}((0, x_N)) + v_{III}^{NP}((0, x_N)).\end{aligned}$$

Since $R > 0$ is independent of x a direct calculation shows that

$$v_{B_R}^{NP}(x) \to 0 \quad \text{as } |x| \to \infty.$$

Another direct calculation shows that

$$\begin{aligned}v_{I(x_N)}^{NP}((0, x_N)) &\leq \alpha(N) \int\limits_{I(x_N)} \frac{1}{|x_N - y_N|^{N-2}} \, \mathrm{d}y \\ &\leq \alpha(N) \left(x_N^{\frac{23}{24}}\right)^{2-N} \int\limits_0^{x_N - x_N^{\frac{23}{24}}} |B'| y_N^{\frac{1+\delta}{2}(N-1)} \, \mathrm{d}y_N \\ &\leq \alpha(N) \left(x_N^{\frac{23}{24}}\right)^{2-N} |B'| \int\limits_0^{x_N} y_N^{\frac{1+\delta}{2}(N-1)} \, \mathrm{d}y_N \\ &= \alpha(N)|B'| \frac{1}{\frac{11}{20}(N-1)+1} \, x_N^{\frac{23}{24}(2-N)+\frac{11}{20}(N-1)+1}\end{aligned}$$

and since for all $N \geq 6$

$$\frac{23}{24}(2-N) + \frac{11}{20}(N-1) + 1 < 0$$

this term vanishes as $x_N \to \infty$. The second part, that contains the singularity of the integrand, can be seen as the Newtonian potential of a ball with increasing

radius evaluated at the centre. Using polar coordinates this equates to

$$v^{NP}_{II(x_N)}(x) \leq v^{NP}_{B_{2x_N^{\frac{23}{24}}}(0,x_N)}((0,x_N)) = \alpha(N) \int\limits_{B_{2x_N^{\frac{23}{24}}}} \frac{1}{|y|^{N-2}}\,\mathrm{d}y$$

$$= \alpha(N) \int\limits_0^{2x_N^{\frac{23}{24}}} \varrho^{2-N}|\partial B_1|\varrho^{N-1}\,\mathrm{d}\varrho = 2\alpha(N)|\partial B_1|x_N^{\frac{23}{12}}$$

which has subquadratic growth and finally

$$v^{NP}_{III(x_N)}(x) \leq \alpha(N) \int\limits_{\left\{|y'|^2<y_N^{1+\delta};y_N>x_N+x_N^{\frac{23}{24}}\right\}} \frac{1}{|x_N-y_N|^{N-2}}\,\mathrm{d}y$$

$$= \int\limits_{x_N^{\frac{23}{24}}}^{x_N} y_N^{2-N}(y_N+x_N)^{(N-1)\frac{1+\delta}{2}}\,\mathrm{d}y_N + \int\limits_{x_N}^{\infty} y_N^{2-N}(y_N+x_N)^{(N-1)\frac{1+\delta}{2}}\,\mathrm{d}y_N$$

$$\leq \int\limits_{x_N^{\frac{23}{24}}}^{x_N} y_N^{2-N}(2x_N)^{(N-1)\frac{1+\delta}{2}}\,\mathrm{d}y_N + \int\limits_{x_N}^{\infty} y_N^{2-N}(2y_N)^{(N-1)\frac{1+\delta}{2}}\,\mathrm{d}y_N$$

$$= (2x_N)^{(N-1)\frac{1+\delta}{2}}\,\frac{x_N^{(3-N)\frac{23}{24}} - x_N^{3-N}}{N-3} + \frac{2^{\frac{1+\delta}{2}(N-1)+1}}{N-5-\delta(N-1)}x_N^{\frac{5-N+\delta(N-1)}{2}}$$

$$\leq Cx_N^{-\frac{1}{8}}$$

for some constant $C>0$, x_N large enough and any $N \geq 6$. This tells us that the growth of the Newton potential is dominated by the middle part II and we have established the subquadratic growth of the Newton-potential of $\mathcal{C}$ as $|x| \to \infty$.

□

We are now in the position to state the Newton-potential expansion of u. As indicated by Lemma 2.13 this can be understood as asymptotic expansion of u for $|x| \to \infty$ (at least outside a set that is slightly larger than paraboloid). And for what follows this will be the main idea behind this expansion. However in order to avoid unnecessary technical complications we will only estimate the asympotics of the comparison solution, where we exactly control the coincidence set, instead of working with the less controlled coincidence set $\mathcal{C}$ of the given solution.

Proposition 2.10 (Newton potential expansion)**.**
Let u be a global solution of the obstacle problem as in Theorem I then u can be expressed as

$$u = p + \ell + c + v_{\mathcal{C}}^{NP}$$

where p is as in (2)*, ℓ is a linear function with $\partial_N \ell < 0$ and $c \in \mathbb{R}$ is a constant.*

Proof. It is well known that $v_{\mathcal{C}}^{NP}$ as defined in Lemma 2.9 solves

$$\Delta v_{\mathcal{C}}^{NP} = -\chi_{\mathcal{C}} \quad \text{strongly in } W^{2,p}_{loc}(\mathbb{R}^N).$$

Let us furthermore set

$$v := u - p \quad \text{in } \mathbb{R}^N.$$

Then v solves the same equation as $v_{\mathcal{C}}^{NP}$, i.e.

$$\Delta v = -\chi_{\mathcal{C}} \quad \text{strongly in } W^{2,p}_{loc}(\mathbb{R}^N).$$

Hence $v - v_{\mathcal{C}}^{NP}$ is harmonic in $\mathbb{R}^N$ and from (2) and Lemma 2.9 ii) we know that $v - v_{\mathcal{C}}^{NP}$ has subquadratic growth. This allows to apply Lemma B.4 and we obtain that

$$v - v_{\mathcal{C}}^{NP} = \ell + c,$$

where ℓ is a linear function and $c \in \mathbb{R}$ is a constant. Thus we have proved the desired Newton-potential expansion of u:

$$u = p + \ell + c + v_{\mathcal{C}}^{NP} \quad \text{in } \mathbb{R}^N. \tag{2.28}$$

What remains to show is that

$$\partial_N \ell < 0. \tag{2.29}$$

This is a direct consequence of u not being lower dimensional. Assume towards a contradiction that this is not be true, i.e. that $\partial_N \ell \geq 0$. Let $x_0 \in \mathcal{C} \cap \{x_N = 0\}$ be arbitrary but fixed. Then then Newton potential expansion (2.28) tells us that

$$0 = u(x_0) = p(x_0') + \ell(x_0) + c + v_{\mathcal{C}}^{NP}(x_0).$$

Let us then choose $\tilde{x} := x_0 - e_N \in \mathbb{R}^N \setminus \mathcal{C}$. For all $y \in \mathcal{C}$ we have that

$$\begin{aligned} |x_0 - y|^2 &= |x_0' - y'|^2 + |(x_0)_N - y_N|^2 = |\tilde{x}' - y'|^2 + |(x_0)_N - y_N|^2 \\ &< |\tilde{x}' - y'|^2 + |\tilde{x}_N - y_N|^2 = |\tilde{x} - y|^2. \end{aligned}$$

Therefore for all $y \in \mathcal{C}$ it holds that

$$\frac{1}{|\tilde{x} - y|^{N-2}} < \frac{1}{|x_0 - y|^{N-2}}$$

and since $\operatorname{int} \mathcal{C} \neq \emptyset$ this implies that

$$v_{\mathcal{C}}^{NP}(\tilde{x}) < v_{\mathcal{C}}^{NP}(x_0).$$

Employing the Newton-potential expansion (2.28) again we obtain that

$$\begin{aligned} u(\tilde{x}) &= p(\tilde{x}') + \ell(\tilde{x}') + \ell(\tilde{x}_N e_N) + c + v_{\mathcal{C}}^{NP}(\tilde{x}) \\ &< p((x_0)') + \ell((x_0)') + \ell((x_0)_N e_N) + c + v_{\mathcal{C}}^{NP}(x_0) = u(x_0) = 0. \end{aligned}$$

But this is impossible since u was assumed to be nonnegative. Hence we must have had $\partial_N \ell < 0$ to begin with. □

2.5 Existence of suitable paraboloid solutions

The objective of this section is the construction of appropriate comparison functions for the comparison argument in section 2.6. As a by-product we also get existence of solutions u with the properties claimed in Theorem I. So this chapter is not concerning itself with the empty set.

Theorem 2.11 (Existence of paraboloid solutions)**.**
Let $N \geq 6$. For any $b \in (0, \infty)^N \times \mathbb{R}$ there is $a \in (0, \infty)^{N-1} \times \mathbb{R}$ such that

$$v_{P_a}^{NP}(x) = -\sum_{j=1}^{N-1} b_j x_j^2 + b_N x_N + b_{N+1} \quad \text{in } P_a,$$

where

$$P_a := \left\{(x', x_N) \in \mathbb{R}^N : x' \in \sqrt{x_N + a_N} E_{a'}\right\} \text{ and } E_{a'} := \left\{x' \in \mathbb{R}^{N-1} : \sum_{j=1}^{N-1} \frac{x_j^2}{a_j^2} \leq 1\right\}.$$

Furthermore

$$u_{P_a}(x) := p_b(x') - b_N x_N - b_{N+1} + v_{P_a}^{NP}(x)$$

solves

$$u_{P_a} \geq 0 \text{ in } \mathbb{R}^N \quad , \quad \Delta u_{P_a} = \chi_{\{u_{P_a}>0\}} \text{ in } \mathbb{R}^N \quad , \quad \{u_{P_a} = 0\} = P_a$$
$$\text{and} \quad \frac{u_{P_a}(rx)}{r^2} \to p_b(x') \text{ strongly in } L^\infty(\partial B_1) \text{ as } r \to \infty,$$

where $p_b(x') := \sum_{j=1}^{N-1} b_j x_j^2$.

The proof will rest on the following lemma that is a corollary of the analysis of the Newton-potential of ellipsoids carried out in [4].

Lemma 2.12 (Existence of suitable ellipsoids)**.**
For any non-degenerate, homogeneous polynomial $q(x) := \sum_{j=1}^N q_j x_j^2$ *with* $q_j > 0$ *for all* $j \in \{1, \dots, N\}$ *and any positive constant* $c > 0$ *we can find a centred ellipsoid* $E = \left\{x \in \mathbb{R}^N : \sum_{j=1}^N \frac{x_j^2}{a_j^2} \leq 1\right\}$ *such that*

$$v_E^{NP}(x) = c - q(x) \quad \text{for all } x \in E.$$

Proof of Lemma 2.12. The main part was proved by E. DiBenedetto and A. Friedman in [4] (see the proof of (5.4) therein), i.e. showing that for any polynomial q as above there is a centred ellipsoid $\tilde{E}$ and some constant $\tilde{c} > 0$ such that

$$v_{\tilde{E}}^{NP}(x) = \tilde{c} - q(x) \quad \text{for all } x \in \tilde{E}.$$

A direct computation shows that the Newton-potential obeys the following scaling

$$\text{for all } \beta > 0 : \quad v_{\beta\tilde{E}}^{NP}(x) = \beta^2 v_{\tilde{E}}^{NP}\left(\frac{x}{\beta}\right)$$

and thus for all $x \in \beta\tilde{E}$ it holds that

$$v_{\beta\tilde{E}}^{NP}(x) = \beta^2\tilde{c} - q(x).$$

Choosing $\beta := \sqrt{\frac{c}{\tilde{c}}}$ and $E := \beta\tilde{E}$ finishes the proof. □

Proof of Theorem 2.11.
Step 1. *Construction of a suitable sequence of ellipsoids.*

Let us define for any $n \in \mathbb{N}$ the following sequence of polynomials and constants

$$q^n(x) := p_b(x') + \frac{1}{n^2}x_N^2 \qquad \text{and} \qquad c_n := \left(\frac{b_N n}{2}\right)^2 > 0$$

Then Lemma 2.12 implies that for all $n \in \mathbb{N}$ there is a centred ellipsoid $\tilde{E}^n$ such that

$$v^{NP}_{\tilde{E}^n} = c_n - q^n \quad \text{on } \tilde{E}^n. \tag{2.30}$$

In order to create a linear term in the Newton-potential expansion we translate $\tilde{E}^n$ by $\tau_n e_N$, where $\tau_n := \frac{b_N}{2}n^2$, i.e.

$$E^n := \tilde{E}^n + \tau_n e_N.$$

We infer from (2.30) that for all $x \in E^n$

$$\begin{aligned} v^{NP}_{E^n}(x) &= v^{NP}_{\tilde{E}^n}(x - \tau_n e_N) = c_n - p_b(x') - \frac{1}{n^2}x_N^2 + \frac{2\tau_n}{n^2}x_N - \frac{1}{n^2}\tau_n^2 \\ &= b_N x_N - q^n(x). \end{aligned} \tag{2.31}$$

Step 2. *Switching to the respective obstacle problems and passing to the limit.*
In order to be able to use known results and technique from the analysis of the obstacle problem we make use of the close relation between null quadrature domains and the obstacle problem. Let us set for all $n \in \mathbb{N}$

$$u_n := q^n - b_N x_N + v^{NP}_{E^n} \quad \text{in } \mathbb{R}^N, \tag{2.32}$$

then u_n is a nonnegative solution of the obstacle problem

$$\Delta u_n = \chi_{\{u_n > 0\}} \text{ in } \mathbb{R}^N \quad \text{and } \{u_n = 0\} = E^n.$$

(This follows e.g. from [3, Theorem II] as in the proof of Lemma 1.1.) Using the non-negativity of the Newton-potential and (2.31) we obtain that for all $x \in E^n$

$$p_b(x') \leq b_N x_N.$$

Note that this estimate is independent of n. Therefore there is a paraboloid $\tilde{P}$ such that for all $n \in \mathbb{N}$

$$E^n \subset \tilde{P}. \tag{2.33}$$

From Lemma 2.9 we know that the Newton-potential of $\tilde{P}$ is well defined in dimensions $N \geq 6$ and we get for all $n \in \mathbb{N}$

$$v_{E^n}^{NP} \leq v_{\tilde{P}}^{NP} \quad \text{in } \mathbb{R}^N.$$

This implies that $v_{E^n}^{NP}$ is uniformly in n bounded in $L^\infty_{loc}(\mathbb{R}^N)$ and therefore we obtain from (2.32) that also u_n must be uniformly in n bounded in $L^\infty_{loc}(\mathbb{R}^N)$. Using well known results from L^p-theory (see e.g. [8, Theorem 9.11] and localize it) this implies that for any $p \in [1, \infty)$

$$(u_n)_{n\in\mathbb{N}} \quad \text{is bounded in } W^{2,p}_{loc}(\mathbb{R}^N).$$

Thus there is a subsequence (again labelled $(u_n)_{n\in\mathbb{N}}$) such that

$$u_n \to u \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^N) \tag{2.34}$$

(for arbitrary $\alpha \in (0, 1)$) and u is a nonnegative solution of the obstacle problem, i.e. u solves

$$\Delta u = \chi_{\{u>0\}} \quad \text{in } \mathbb{R}^N$$

which can be found e.g. in [12, Proposition 3.17].

Step 3. *Identification of the coincidence set of u and switching back to the Newtonian potential expansion*

In order to identify the coincidence set of u we will pass to the limit in the Newton-potential expansion (2.32) of u_n. To this end recall that each ellipsoid E^n is the sublevel set of a polynomial, therefore we can represent E^n as

$$E^n = \left\{ \sum_{j=1}^{N-1} \frac{x_j^2}{B_{j,n}^2} + \frac{(x_N - \tau_n)^2}{B_{N,n}^2} \leq 1 \right\}, \tag{2.35}$$

where $B_{j,n} \in (0, \infty)$ are the semiaxes of E^n and τ_n is the translation in e_N-direction as defined in step 1 (for all $n \in \mathbb{N}$ and $j \in \{1, \dots, N\}$).

Since for all $n \in \mathbb{N}$ we have that E^n is defined by the finitely many coefficients $(B_{1,n}, \dots, B_{N,n}, \tau_n)$ which converge (up to taking a subsequence) in $[0, \infty]^{N+1}$ we infer that

$$\chi_{E^n} \to \chi_M \quad \text{pointwise almost everywhere in } \mathbb{R}^N \text{ as } n \to \infty, \tag{2.36}$$

where $M \subset \tilde{P}$ is some measurable set. This allows to conclude invoking Lebesgue's

dominated convergence theorem and

$$\chi_{E^n}(y)|x-y|^{2-N} \leq \chi_{\tilde{P}}(y)|x-y|^{2-N} \quad \text{for all } x, y \in \mathbb{R}^N$$

that

$$v_{E^n}^{NP} \to v_{\tilde{P}}^{NP} \quad \text{pointwise in } \mathbb{R}^N \text{ as } n \to \infty.$$

Combining this fact with (2.32) and (2.34) we obtain that the Newton-potential expansion

$$u(x) = p_{\mathbf{b}}(x') - b_N x_N + V_M(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{2.37}$$

It remains to identify the set M. First of all form (2.37) we infer that M has non-vanishing Lebesgue-measure, i.e. $|M| > 0$, because otherwise $V_M \equiv 0$ in $\mathbb{R}^N$ and this combined with (2.37) contradicts the fact that u is nonnegative in $\mathbb{R}^N$.

Note that $M \subset \tilde{P}$ implies that for all $n \in \mathbb{N}$ we have that $0 \leq B_{N,n} \leq \tau_N$. Combining this observation with (2.36) and the fact that M has positive measure we obtain that the ellipsoids E^n cannot vanish in e_N-direction and therefore (for a subsequence)

$$\frac{\tau_n}{B_{N,n}} \to 1 \quad \text{as } n \to \infty.$$

Recall that in step 1 we have chosen $\tau_n = \frac{b_N}{2} n^2$. Let us now rewrite (2.35) as

$$E^n = \left\{ \sum_{j=1}^{N-1} \frac{\tau_n}{B_{j,n}^2} x_j^2 + \frac{\tau_n}{B_{N,n}^2} x_N^2 - 2\left(\frac{\tau_n}{B_{N,n}}\right)^2 x_N \leq \left[1 - \left(\frac{\tau_n}{B_{N,n}}\right)^2\right] \tau_n \right\}.$$

We claim that $\left[1 - \left(\frac{\tau_n}{B_{N,n}}\right)^2\right] \tau_n \leq 0$ is bounded in n. Assume towards a contradiction that $\left[1 - \left(\frac{\tau_n}{B_{N,n}}\right)^2\right] \tau_n$ is unbounded, i.e. that there is a subsequence such that $\left[1 - \left(\frac{\tau_n}{B_{N,n}}\right)^2\right] \tau_n \to -\infty$ then

$$E^n \subset \left\{ -2\left(\frac{\tau_n}{B_{N,n}}\right)^2 x_N \leq \left[1 - \left(\frac{\tau_n}{B_{N,n}}\right)^2\right] \tau_n \right\} \to \{x_N \geq \infty\} \quad \text{as } n \to \infty,$$

which implies that $\chi_{E^n} \to 0$ pointwise almost everywhere in $\mathbb{R}^N$ which is incompatible with (2.36) and the fact that $|M| > 0$. Hence (up to taking a subsequence) we have

that $\left[1-\left(\frac{\tau_n}{B_{N,n}}\right)^2\right]\tau_n \to c \in (-\infty, 0]$ as $n \to \infty$.

Up to taking another subsequence we have that $\frac{\tau_n}{B_{j,n}^2} \to B_j \in [0,\infty]$ as $n \to \infty$ for all $j \in \{1,\dots,N-1\}$. We claim now that $B_j \in (0,\infty)$ for all $j \in \{1,\dots,N-1\}$. Assume towards a contradiction that there is $i \in \{1,\dots,N-1\}$ and a subsequence such that $\frac{\tau_n}{B_{i,n}^2} \to \infty$ then

$$E^n \subset \left\{ \frac{\tau_n}{B_{i,n}^2} x_i^2 - 2\left(\frac{\tau_n}{B_{N,n}}\right)^2 x_N \leq \left[1-\left(\frac{\tau_n}{B_{N,n}}\right)^2\right]\tau_n \right\} \to \tilde{E} \subset \{x_i = 0\},$$

which is again a contradiction to (2.36) and the fact that $|M| > 0$. To finish the proof assume towards a contradiction that there is $i \in \{1,\dots,N-1\}$ such that $B_i = 0$. Then for all $n \in \mathbb{N}$

$$\begin{aligned} E^n \supset & \left\{ \frac{\tau_n}{B_{i,n}^2} x_i^2 \leq \min_{n\in\mathbb{N}} \left[1-\left(\frac{\tau_n}{B_{N,n}}\right)^2\right]\tau_n \ , \ x_j = 0, j \neq i \right\} \\ & \to \{x_j = 0, j \neq i\} \quad \text{ as } n \to \infty \end{aligned}$$

But this is impossible since from (2.33) we know that E^n must be contained in a paraboloid $\tilde{P}$ for all $n \in \mathbb{N}$.

Summing it all up we can conclude that (up to taking a subsequence)

$$\chi_{E^n} \to \chi_M \text{ pointwise a.e as } n \to \infty \quad \text{where } M = \left\{ \sum_{j=1}^{N-1} B_j x_j^2 - 2x_N \leq c \right\}$$

and$B_j \in (0,\infty)$ for all $j \in \{1,\dots,N-1\}$ and $c \in (-\infty, 0]$.

Translating the paraboloid in the e_N-direction until the constant part in the expansion agrees with b_{N+1} finishes the proof. □

2.6 A comparison principle with hole / Proof of Theorem I

In this section we are going to finish the proof of Theorem I. The problem is that we only have a very weak control on the coincidence set $\mathcal{C}$ so comparing it with some comparison paraboloid seems hopeless. Instead we will compare u with an appropriate comparison solution on some large sphere but away from the coincidence set $\mathcal{C}$. We will see that such a comparison is possible. For this argument we need to have a good control on the set where the Newton-potential of the comparison solution vanishes. This is established in the following lemma.

Lemma 2.13 (Newton-potential of a paraboloid vanishes outside some set slightly larger than the paraboloid).
Let $N \geq 6$ and $\gamma > 0$ and

$$P := \left\{ (y', y_N) \in \mathbb{R}^N : |y'| < \gamma y_N^{\frac{1}{2}} \right\}$$

and for any $\mu > \frac{25}{72}$ setting

$$P^\mu := \left\{ (y', y_N) \in \mathbb{R}^N : |y'| < \gamma y_N^{\frac{1}{2}+\mu} \right\}$$

it holds that

$$\sup_{x \in (\mathbb{R}^N \setminus P^\mu) \cap \{x_N > k\}} v_P^{NP}(x) \to 0 \quad \text{as } k \to \infty$$

and

$$\sup_{x \in (\mathbb{R}^N \setminus B_k) \cap \left\{x_N \leq \frac{k}{2}\right\}} v_P^{NP}(x) \to 0 \quad \text{as } k \to \infty. \tag{2.38}$$

Proof. As in the proof of Lemma 2.9 we split P up into the part close to x and the part further away from x and we estimate the Newton-potential integral there.

Since P is rotational symmetric and $v_P^{NP}(\lambda x' + x_N e_N)$ is monotone decreasing in $|\lambda|$ we get that

$$\sup_{x \in (\mathbb{R}^N \setminus P^\mu) \cap \{x_N = k\}} v_P^{NP}(x) = v_P^{NP}(\gamma k^{\frac{1}{2}+\mu} e_1 + k e_N)$$

Furthermore we have that

$$P = \underbrace{\left\{ |y'| < \gamma y_N^{\frac{1}{2}} \; ; \; y_N < x_N - x_N^{\frac{8}{9}} \right\}}_{=:I(x_N)} \cup \underbrace{\left\{ |y'| < \gamma y_N^{\frac{1}{2}} \; ; \; |x_N - y_N| \leq x_N^{\frac{8}{9}} \right\}}_{=:II(x_N)}$$
$$\cup \underbrace{\left\{ |y'| < \gamma y_N^{\frac{1}{2}} \; ; \; y_N > x_N + x_N^{\frac{8}{9}} \right\}}_{=:III(x_N)}.$$

Using this splitting we have

$$\begin{aligned} v_P^{NP}(\gamma k^{\frac{1}{2}+\mu} e_1 + k e_N) = {} & v_{I(k)}^{NP}(\gamma k^{\frac{1}{2}+\mu} e_1 + k e_N) + v_{II(k)}^{NP}(\gamma k^{\frac{1}{2}+\mu} e_1 + k e_N) \\ & + v_{III(k)}^{NP}(\gamma k^{\frac{1}{2}+\mu} e_1 + k e_N). \end{aligned}$$

The first term can be estimated as

$$
\begin{aligned}
v^{NP}_{I(k)}(3\gamma k^{\frac{1}{2}+\mu}e_1 + ke_N) &\leq \alpha(N) \int\limits_0^{k-k^{\frac{8}{9}}} (k-y_N)^{2-N}\gamma^{N-1}|B'|\Big(y_N^{\frac{1}{2}}\Big)^{N-1}\,\mathrm{d}y_N \\
&\leq \alpha(N)k^{\frac{8}{9}(2-N)}|B'|\frac{2\gamma^{N-1}|B'|}{N+1}k^{\frac{N+1}{2}} \\
&\leq Ck^{\frac{8}{9}(2-N)+\frac{1}{2}(N+1)}
\end{aligned}
$$

which vanishes as $k \to \infty$ if $N \geq 6$. For the second part we get for large k

$$
\begin{aligned}
v^{NP}_{II(k)}(\gamma k^{\frac{1}{2}+\mu}e_1 + ke_N) &\leq \alpha(N) \int\limits_{II(k)} \frac{1}{\left|\gamma k^{\frac{1}{2}+\mu} - y_1\right|^{N-2}}\,\mathrm{d}y \\
&\leq \alpha(N)\Big(\frac{\gamma}{2}k^{\frac{1}{2}+\mu}\Big)^{2-N} \int\limits_{k-k^{\frac{8}{9}}}^{k+k^{\frac{8}{9}}} |B'|(\gamma\sqrt{y_N})^{N-1}\,\mathrm{d}y_N \\
&\leq \alpha(N)\Big(\frac{\gamma}{2}k^{\frac{1}{2}+\mu}\Big)^{2-N}|B'|\gamma^{N-1}(2k)^{\frac{N-1}{2}}2k^{\frac{8}{9}} \\
&\leq Ck^{\left(\frac{1}{2}+\mu\right)(2-N)+\frac{1}{2}(N-1)+\frac{8}{9}}
\end{aligned}
$$

this vanishes as $k \to \infty$ for any $N \geq 6$ and $\mu > \frac{25}{72}$. For the last term we get

$$
\begin{aligned}
v^{NP}_{III(k)}(\gamma k^{\frac{1}{2}+\mu}e_1 + ke_N) \leq &\int\limits_{k+k^{\frac{8}{9}}}^{2k} |k-y_N|^{2-N}\gamma^{N-1}y_N^{\frac{N-1}{2}}|B'|\,\mathrm{d}y_N \\
&+ \int\limits_{2k}^{\infty} |k-y_N|^{2-N}\gamma^{N-1}y_N^{\frac{N-1}{2}}|B'|\,\mathrm{d}y_N \\
\leq\ & k^{\frac{8}{9}(2-N)} \int\limits_{k+k^{\frac{8}{9}}}^{2k} \gamma^{N-1}y_N^{\frac{N-1}{2}}|B'|\,\mathrm{d}y_N \\
&+ \int\limits_{k}^{\infty} y_N^{2-N}\gamma^{N-1}(y_N+k)^{\frac{N-1}{2}}|B'|\,\mathrm{d}y_N \\
\leq\ & \gamma^{N-1}|B'|\frac{2}{N+1}2^{\frac{N+1}{2}}k^{\frac{8}{9}(2-N)+\frac{1}{2}(N+1)} \\
&+ \frac{\gamma^{N-1}|B'|2^{\frac{N+1}{2}}}{N-5}k^{2-N+\frac{1}{2}(N+1)}
\end{aligned}
$$

and this again vanishes as $k \to \infty$ for any $N \geq 6$.

It remains to prove (2.38). For each $x \in (\mathbb{R}^N \setminus B_k) \cap \left\{x_N \leq \frac{k}{2}\right\}$ and k large enough we have that

$$
\begin{aligned}
v_P^{NP}(x) &= \alpha(N) \int\limits_{P\cap\{y_N\leq k\}} \frac{1}{|x-y|^{N-2}}\,\mathrm{d}y + \alpha(N) \int\limits_{P\cap\{y_N\geq k\}} \frac{1}{|x-y|^{N-2}}\,\mathrm{d}y \\
&\leq \alpha(N) \int\limits_0^k \left(\frac{k}{2}\right)^{2-N} |B_1'| \left(\gamma y_N^{\frac{1}{2}}\right)^{N-1} \mathrm{d}y_N \\
&\quad + \alpha(N) \int\limits_k^\infty \left(y_N - \frac{k}{2}\right)^{2-N} |B_1'| \left(\gamma y_N^{\frac{1}{2}}\right)^{N-1} \mathrm{d}y_N \\
&\leq \alpha(N) \int\limits_0^k \left(\frac{k}{2}\right)^{2-N} |B_1'| \left(\gamma y_N^{\frac{1}{2}}\right)^{N-1} \mathrm{d}y_N \\
&\quad + \alpha(N) \int\limits_k^\infty \left(\frac{y_N}{2}\right)^{2-N} |B_1'| \left(\gamma y_N^{\frac{1}{2}}\right)^{N-1} \mathrm{d}y_N \\
&\leq \alpha(N) \frac{|B_1'|(2\gamma)^{N-1}}{N+1} k^{\frac{5}{2}-\frac{N}{2}} + \alpha(N) \frac{(2\gamma)^{N-1}|B_1'|}{N-5} k^{\frac{5}{2}-\frac{N}{2}} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}
$$

□

Now we have all the tools we need in order to conclude this chapter with the proof of Theorem I.

Proof of Theorem I.

Step 1. *Construction of a comparison solution.*

From (2.28) we have for the given solution u the Newton-potential expansion

$$u = p(x') + \ell(x) + v_{\mathcal{C}}^{NP}(x) + c.$$

Then employing Theorem 2.11 we find a paraboloid $P \subset \{x_N \geq 0\}$, $P \cap \{x_N = 0\} \neq \emptyset$ such that

$$u_P := p(x') + \ell(x) + v_P^{NP}(x) + c_P, \quad c_P \in \mathbb{R}$$

is a solution of the obstacle problem and shares the quadratic and linear term in the Newton-potential expansion of u.

In order to carry this program out let us set for $\lambda \geq 0$:

$$P_\lambda := P - \lambda e_N$$

and

$$u_{P_\lambda}(x) := u_P(x + \lambda e_N).$$

Then it holds that

$$\begin{aligned} u_{P_\lambda}(x) &= p(x') + \ell(x) + v^{NP}_{P_\lambda}(x) + \lambda \ell(e_N) + c_P, \\ u(x) &= p(x') + \ell(x) + v^{NP}_{\mathcal{C}}(x) + c, \end{aligned}$$

and therefore

$$u_{P_\lambda}(x) - u(x) = v^{NP}_{P_\lambda}(x) - v^{NP}_{\mathcal{C}}(x) + \lambda \ell(e_N) + c_P - c \leq v^{NP}_{P_\lambda}(x) + \lambda \ell(e_N) + c_P - c.$$

Step 2. *A first comparison.*
We need to be able to compare u_{P_λ} and u. To this end we want to apply the sup-boundary value inequality from Lemma B.3 to

$$v^r := \max\{u^r_{P_\lambda} - u^r, 0\} \geq 0,$$

where v^r is subharmonic since Δv^r is a nonnegative measure and so from Lemma B.3 ii) we have that

$$\sup_{B_{\frac{1}{2}}} v^r \leq C(N) \fint_{\partial B_1} v^r \, \mathrm{d}\mathcal{H}^{N-1}. \tag{2.39}$$

Let $\gamma > 0$ be such that

$$\tilde{P} := \{(y', y_N) \in \mathbb{R}^N : |y'| \leq \gamma \sqrt{y_N}\} \quad \text{satisfies} \quad P \subset \tilde{P}.$$

This implies that

$$\tilde{P}_\lambda := \tilde{P} - \lambda e_N \supset P_\lambda.$$

Choosing $\mu = \frac{7}{20} > \frac{25}{72}$ and $\tilde{P}^\mu$ as in Lemma 2.13 we set

$$\tilde{P}^\mu_\lambda := \tilde{P}^\mu - \lambda e_N.$$

Let then $\lambda_0 > 0$ be sufficiently large such that

$$c_P - c + \lambda_0 \ell(e_N) < -2.$$

Here we have used (2.29). Lemma 2.13 tells us that there is $r_0 > 0$ such that for all $r > r_0$

$$v^{NP}_{P_{\lambda_0}} \leq v^{NP}_{\tilde{P}_{\lambda_0}} < 1 \quad \text{on } \partial B_r \setminus \tilde{P}^{\mu}_{\lambda_0}.$$

So for $r > r_0$ we have that

$$\max\Big\{u_{P_{\lambda_0}} - u, 0\Big\} = 0 \quad \text{on } \partial B_r \setminus \tilde{P}^{\mu}_{\lambda_0}.$$

This allows us to estimate as follows

$$\begin{aligned}
\fint_{\partial B_1} v^r \,\mathrm{d}\mathcal{H}^{N-1} &\leq \frac{1}{|\partial B_r|} \int_{\partial B_r} v \,\mathrm{d}\mathcal{H}^{N-1} = \frac{1}{|\partial B_r|} \int_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} v \,\mathrm{d}\mathcal{H}^{N-1} \\
&\leq \frac{1}{|\partial B_r|} \int_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} v^{NP}_{P_{\lambda_0}} \,\mathrm{d}\mathcal{H}^{N-1} + \frac{1}{|\partial B_r|} \int_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} c_P - c + \lambda_0 \ell(e_N) \,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}$$

where the last term vanishes because

$$\frac{\left|\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}\right|}{|\partial B_r|} \to 0 \quad \text{as } r \to \infty$$

and therefore all we have to do is estimate the first term. From a direct calculation we get that for sufficiently large r

$$\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0} \subset \big\{r - 5\gamma^2 r^{2\mu} < y_N < r\big\}. \tag{2.40}$$

Using this we are going to estimate $v^{NP}_{\tilde{P}_{\lambda_0}}$ as follows

$$\begin{aligned}
v^{NP}_{\tilde{P}_{\lambda_0}}(x) &= \alpha(N) \int_{\tilde{P}_{\lambda_0}} \frac{1}{|x-y|^{N-2}} \,\mathrm{d}y \\
&\leq \alpha(N) \Bigg(\int_{(\tilde{P}_{\lambda_0})_1} \frac{1}{|x_N - y_N|^{N-2}} \,\mathrm{d}y + \int_{(\tilde{P}_{\lambda_0})_2} \frac{1}{|x' - y'|^{N-2}} \,\mathrm{d}y
\end{aligned}$$

$$+ \int\limits_{(\tilde{P}_{\lambda_0})_3} \frac{1}{|x_N - y_N|^{N-2}} \, \mathrm{d}y \Bigg),$$

where

$$\begin{aligned}
\left(\tilde{P}_{\lambda_0}\right)_1 &:= \tilde{P}_{\lambda_0} \cap \left\{y_N < r - 6\gamma^2 r^{2\mu}\right\}, \\
\left(\tilde{P}_{\lambda_0}\right)_2 &:= \tilde{P}_{\lambda_0} \cap \left\{r - 6\gamma^2 r^{2\mu} \leq y_N \leq r + 6\gamma^2 r^{2\mu}\right\}, \\
\left(\tilde{P}_{\lambda_0}\right)_3 &:= \tilde{P}_{\lambda_0} \cap \left\{y_N > r + 6\gamma^2 r^{2\mu}\right\}.
\end{aligned}$$

In order to avoid unnecessary confusion we will in the following always use y as the variable of integration in the Newton potential integral and x will always be in $\partial B_r \cap \tilde{P}_{\lambda_0^\mu}$ and estimated by (2.40).

Using the scaling and growth properties of the Newton-potential of bounded sets and Fubini's Theorem we obtain for the main part $\left(\tilde{P}_{\lambda_0}\right)_2$

$$\int\limits_{(\tilde{P}_{\lambda_0})_2} \frac{1}{|x' - y'|^{N-2}} \, \mathrm{d}y \leq \frac{1}{\alpha(N-1)} \int\limits_{r-6\gamma^2 r^{2\mu}}^{r+6\gamma^2 r^{2\mu}} v^{NP'}_{2\gamma y_N^{\frac{1}{2}} B_1'}(x') \, \mathrm{d}y_N,$$

where $v_M^{NP'}$ is the $N-1$-dimensional Newton-potential of the set M. A calculation shows that the Newton potential has the following scaling: For all $\beta > 0$ and $M \subset \mathbb{R}^N$ bounded and measurable:

$$v^{NP}_{\beta M}(x) = \beta^2 v^{NP}_M\left(\frac{x}{\beta}\right) \quad \text{for all } x \in \mathbb{R}^N$$

and there is $C(M) > 0$ such that for all $x \in \mathbb{R}^N$

$$v^{NP}_M(x) \leq C(M)|x|^{2-N}.$$

This allows us to estimate

$$v^{NP'}_{2\gamma y_N^{\frac{1}{2}} B_1'}(x') = 4\gamma^2 y_N v^{NP'}_{B_1'}\left(\frac{x'}{2\gamma y_N^{\frac{1}{2}}}\right) \leq 4\gamma^2 y_N C(B_1') \left|\frac{x'}{2\gamma y_N^{\frac{1}{2}}}\right|^{3-N}.$$

This implies for sufficiently large r that

$$\begin{aligned}\int\limits_{(\tilde{P}_{\lambda_0})_2} \frac{1}{|x'-y'|^{N-2}}\,\mathrm{d}y &\leq \frac{C(B_1')(2\gamma)^{N-1}}{\alpha(N-1)} \int\limits_{r-6\gamma^2 r^{2\mu}}^{r+6\gamma^2 r^{2\mu}} y_N^{\frac{N}{2}-\frac{1}{2}}\,\mathrm{d}y_N |x'|^{3-N} \\ &\leq \frac{C(B_1')(2\gamma)^{N-1}}{\alpha(N-1)} |x'|^{3-N} (2r)^{\frac{N-1}{2}} \left(12\gamma^2 r^{2\mu}\right) \\ &= C_1 |x'|^{3-N} r^{\frac{N-1}{2}+2\mu}.\end{aligned} \tag{2.41}$$

Using the graph representation for $\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}$ we get that for any integrable function f and for sufficiently large r

$$\int\limits_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} f(x')\,\mathrm{d}\mathcal{H}^{N-1}(x) = \int\limits_{B'_{\gamma r^{\frac{1}{2}+\mu}}} f(x') \frac{r}{\sqrt{r^2-|x'|^2}}\,\mathrm{d}x' \leq 2 \int\limits_{B'_{\gamma r^{\frac{1}{2}+\mu}}} f(x')\,\mathrm{d}x' \tag{2.42}$$

Hence employing (2.41) we have for large r that

$$\begin{aligned}\int\limits_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} \int\limits_{(\tilde{P}_{\lambda_0})_2} \frac{1}{|x'-y'|^{N-2}}\,\mathrm{d}y\,\mathrm{d}\mathcal{H}^{N-1}(x) &\leq 2 \int\limits_{B'_{\gamma r^{\frac{1}{2}+\mu}}} \int\limits_{(\tilde{P}_{\lambda_0})_2} \frac{1}{|x'-y'|^{N-2}}\,\mathrm{d}y\,\mathrm{d}x' \\ &\leq 2C_1 r^{\frac{N-1}{2}+2\mu} \int\limits_{B'_{\gamma r^{\frac{1}{2}+\mu}}} |x'|^{3-N}\,\mathrm{d}x' \\ &= 2C_1 r^{\frac{N-1}{2}+2\mu} \int\limits_0^{\gamma r^{\frac{1}{2}+\mu}} |\partial B_1'| \varrho^{N-2} \varrho^{3-N}\,\mathrm{d}\varrho \\ &= C_1 |\partial B_1'| \gamma^2 r^{\frac{N+1}{2}+4\mu}.\end{aligned}$$

Thus it follows that

$$\frac{1}{|\partial B_r|} \int\limits_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} \int\limits_{(\tilde{P}_{\lambda_0})_2} \frac{1}{|x'-y'|^{N-2}}\,\mathrm{d}y\,\mathrm{d}\mathcal{H}^{N-1}(x) \leq C_2(N) r^{-\frac{N}{2}+\frac{3}{2}+4\mu}$$

and this vanishes as $r \to \infty$ for $N \geq 6$ and $\mu = \frac{7}{20}$.

For the first part of the Newton-potential we can estimate for sufficiently large r

and for all $x \in \partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}$ (using (2.40))

$$\begin{aligned}
\int\limits_{\left(\tilde{P}_{\lambda_0}\right)_1} \frac{1}{|x_N - y_N|^{N-2}} \,\mathrm{d}y &= \int\limits_{-\lambda_0}^{r-6\gamma^2 r^{2\mu}} \frac{1}{|x_N - y_N|^{N-2}} |B_1'| \Big(\gamma(y_N + \lambda_0)^{\frac{1}{2}}\Big)^{N-1} \,\mathrm{d}y_N \\
&\leq \left(\gamma^2 r^{2\mu}\right)^{2-N} |B_1'| \gamma^{N-1} \int\limits_0^{2r} y_N^{\frac{N-1}{2}} \,\mathrm{d}y_N \\
&\leq C_3 r^{(2\mu)(2-N)+\frac{1}{2}(N+1)}.
\end{aligned}$$

Since this estimate was uniform in $x \in \partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}$ we get

$$\frac{1}{|\partial B_r|} \int\limits_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} \int\limits_{\left(\tilde{P}_{\lambda_0}\right)_1} \frac{1}{|x_N - y_N|^{N-2}} \,\mathrm{d}y \,\mathrm{d}\mathcal{H}^{N-1}(x) \leq C_3 r^{(2\mu)(2-N)+\frac{1}{2}(N+1)} \frac{\left|\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}\right|}{|\partial B_r|}.$$

Estimating as in (2.42) we get

$$\frac{\left|\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}\right|}{|\partial B_r|} \leq 2 \frac{|B_1'| \gamma^{N-1}}{|\partial B_1|} r^{\left(-\frac{1}{2}+\mu\right)(N-1)} \tag{2.43}$$

and therefore

$$\frac{1}{|\partial B_r|} \int\limits_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} \int\limits_{\left(\tilde{P}_{\lambda_0}\right)_1} \frac{1}{|x_N - y_N|^{N-2}} \,\mathrm{d}y \,\mathrm{d}\mathcal{H}^{N-1}(x) \leq C_3 r^{\mu(3-N)+1}$$

and this vanishes as $r \to \infty$ for all $N \geq 6$ and $\mu = \frac{7}{20}$. For the last part of the Newton-potential we can estimate in a similar way for sufficiently large r and for all $x \in \partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}$ (using (2.40))

$$\begin{aligned}
\int\limits_{\left(\tilde{P}_{\lambda_0}\right)_3} \frac{1}{|x_N - y_N|^{N-2}} \,\mathrm{d}y &= \int\limits_{r+6\gamma^2 r^{2\mu}}^{\infty} |x_N - y_N|^{2-N} |B_1'| \Big(\gamma(y_N + \lambda_0)^{\frac{1}{2}}\Big)^{N-1} \,\mathrm{d}y_N \\
&\leq \int\limits_{r+6\gamma^2 r^{2\mu}}^{2r} |x_N - y_N|^{2-N} |B_1'| \Big(\gamma(y_N + \lambda_0)^{\frac{1}{2}}\Big)^{N-1} \,\mathrm{d}y_N \\
&+ \int\limits_{2r}^{\infty} |x_N - y_N|^{2-N} |B_1'| \Big(\gamma(y_N + \lambda_0)^{\frac{1}{2}}\Big)^{N-1} \,\mathrm{d}y_N
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(6\gamma^2 r^{2\mu}\right)^{2-N} |B_1'| 2^{\frac{N-1}{2}} \gamma^{N-1} \int\limits_{r+7\gamma r^{2\mu}}^{2r} y_N^{\frac{N-1}{2}} \, \mathrm{d}y_N \\
&+ \int\limits_r^\infty y_N^{2-N} |B_1'| \left(\gamma (3y_N)^{\frac{1}{2}}\right)^{N-1} \mathrm{d}y_N \\
&\leq C r^{(2\mu)(2-N)+\frac{1}{2}(N+1)}.
\end{aligned}
$$

Since this is estimate is uniform in $x \in \partial B_r \cap \tilde{P}^\mu_{\lambda_0}$ and using again (2.43) we get

$$
\frac{1}{|\partial B_r|} \int\limits_{\partial B_r \cap \tilde{P}^\mu_{\lambda_0}} \int\limits_{\left(\tilde{P}_{\lambda_0}\right)_3} \frac{1}{|x_N - y_N|^{N-2}} \, \mathrm{d}y \, \mathrm{d}\mathcal{H}^{N-1}(x) \leq C(N) r^{\mu(3-N)+1}
$$

and this vanishes as $r \to \infty$ for all $N \geq 6$ and $\mu = \frac{7}{20}$. So the sup-boundary-value inequality (2.39) tells us that for any $\varepsilon > 0$ there is $r_0(\varepsilon) > 0$ such that for all $r > r_0(\varepsilon)$

$$
\sup_{B_{\frac{1}{2}}} v^r \leq \varepsilon \quad \text{and therefore} \quad \sup_{B_{\frac{r}{2}}} v \leq \varepsilon.
$$

Since $\varepsilon > 0$ was arbitrary we conclude that

$$
v \equiv 0 \quad \text{in } \mathbb{R}^N
$$

and we have the desired comparison

$$
u_{P_{\lambda_0}} \leq u \quad \text{in } \mathbb{R}^N.
$$

This implies for the coincidence sets that

$$
\mathcal{C} \subset P_{\lambda_0}.
$$

Step 3. *Sliding the comparison solution back until the constant terms in the expansion are the same and concluding that the two solutions must already coincide.*
For any $\lambda \leq \lambda_0$ we set

$$
c^\lambda := c_P + \lambda \ell(e_N).
$$

Then for the translated paraboloid solution and the given solution we have the Newton-

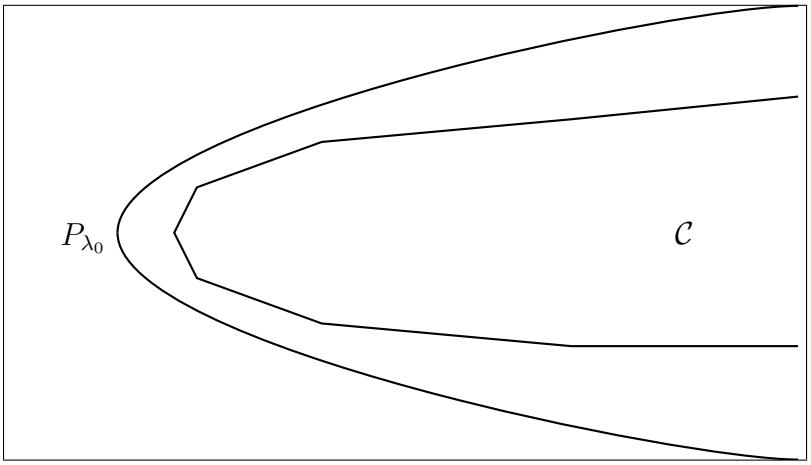

Figure 2.1: $\mathcal{C} \subset P_{\lambda_0}$.

potential expansions

$$\begin{aligned} u_{P_\lambda}(x) &= p(x') + \ell(x) + v^{NP}_{P_\lambda}(x) + c^\lambda, \\ u(x) &= p(x') + \ell(x) + v^{NP}_{\mathcal{C}}(x) + c. \end{aligned} \tag{2.44}$$

We claim now that for any $\lambda \leq \lambda_0$ such that $c^\lambda \leq c$ the comparison from the second step still holds and we therefore have that

$$u_{P_\lambda} \leq u \quad \text{in } \mathbb{R}^N.$$

Recall that for every $\lambda \leq \lambda_0$

$$P_\lambda \subset P_{\lambda_0} \quad , \quad \tilde{P}_\lambda \subset \tilde{P}_{\lambda_0} \quad \text{and} \quad \tilde{P}^\mu_\lambda \subset \tilde{P}^\mu_{\lambda_0}.$$

Let us again set

$$v := \max\{u_{P_\lambda} - u, 0\}.$$

Then from (2.44) we know that

$$v = \max\big\{c^\lambda - c + v^{NP}_{P_\lambda} - v^{NP}_{\mathcal{C}}, 0\big\} \leq 2v^{NP}_{\tilde{P}_{\lambda_0}} \quad \text{in } \mathbb{R}^N.$$

Since v is again a positive subharmonic function we have still the sup-mean-inequality from Lemma B.3, i.e. for any $r > 0$

$$\sup_{B_{\frac{r}{2}}} v \leq C(N) \fint_{\partial B_r} v \, \mathrm{d}\mathcal{H}^{N-1}.$$

From (2.38) we have that

$$\frac{1}{|\partial B_r|} \int\limits_{\partial B_r \setminus \tilde{P}^{\mu}_{\lambda_0}} 2v^{NP}_{\tilde{P}_{\lambda_0}} \, \mathrm{d}\mathcal{H}^{N-1} \to 0 \quad \text{as } r \to \infty$$

and from the calculations in *step 2* we already know that

$$\frac{1}{|\partial B_r|} \int\limits_{\partial B_r \cap \tilde{P}^{\mu}_{\lambda_0}} 2v^{NP}_{\tilde{P}_{\lambda_0}} \, \mathrm{d}\mathcal{H}^{N-1} \to 0 \quad \text{as } r \to \infty.$$

So for any $\varepsilon > 0$ there is $r_0(\varepsilon) > 0$ such that for all $r > r_0(\varepsilon)$

$$\sup_{B_{\frac{r}{2}}} v \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have that

$$v \equiv 0 \quad \text{in } \mathbb{R}^N$$

and hence

$$u_{P_\lambda} \leq u \quad \text{in } \mathbb{R}^N. \tag{2.45}$$

This also implies that for all $\lambda \leq \lambda_0$ such that $c^\lambda \leq c$ we have that

$$\mathcal{C} \subset P_\lambda$$

and therefore

$$v^{NP}_{\mathcal{C}} \leq v^{NP}_{P_\lambda} \quad \text{in } \mathbb{R}^N.$$

Inserting this into (2.44) and using (2.45) we get that

$$u_{P_\lambda} \leq u = u_{P_\lambda} + v^{NP}_{\mathcal{C}} - v^{NP}_{P_\lambda} + c - c^\lambda \leq u_{P_\lambda} + c - c^\lambda.$$

So letting $\lambda \searrow \bar{\lambda}$, where $\bar{\lambda}$ is such that $c^{\bar{\lambda}} = c$, we have that

$$u_{P_{\bar{\lambda}}} \leq u \leq u_{P_{\bar{\lambda}}} \quad \text{in } \mathbb{R}^N$$

and therefore

$$u \equiv u_{P_{\bar{\lambda}}} \quad \text{as well as} \quad \mathcal{C} = P_{\bar{\lambda}}.$$

This finishes the proof of Theorem I. $\square$

A

Appendix A

A.1 The weighted center of gravity of $\mathcal{C}$ can be chosen to be zero by a translation

We will show that there is $x^0 \in \mathcal{C}$ such that

$$\int\limits_{\mathcal{C}-x^0} \frac{y}{|y|^N} \, \mathrm{d}y = 0.$$

Step 1. *There is $x_0 \in \mathbb{R}^N$ with the claimed property.*
First we show that there is $x^0 \in \mathbb{R}^N$ such that

$$\int\limits_{\mathcal{C}-x^0} \frac{y}{|y|^N} \, \mathrm{d}y = 0. \tag{A.1}$$

To do so we choose $R > 0$ such that $\mathcal{C} \subset \bar{B}_R$ and for all $\varepsilon > 0$ define the continuous operator $T^\varepsilon : \bar{B}_R \to \mathbb{R}^N$

$$T^\varepsilon(x) := \frac{\int_{\mathcal{C}} \frac{y}{|y-x|^{N-\varepsilon}} \, \mathrm{d}y}{\int_{\mathcal{C}} \frac{1}{|y-x|^{N-\varepsilon}} \, \mathrm{d}y}.$$

(Note that we have only employed the regularization with ε in order to ensure integrability of all terms involved.) T^ε is a self-map because

$$|T^\varepsilon(x)| \leq \frac{R \int_{\mathcal{C}} \frac{1}{|y-x|^{N-\varepsilon}} \, \mathrm{d}y}{\int_{\mathcal{C}} \frac{1}{|y-x|^{N-\varepsilon}} \, \mathrm{d}y} = R.$$

Now Brouwer's fixed point theorem yields that for all $\varepsilon > 0$ the continuous self-map $T^\varepsilon : \bar{B}_R \to \bar{B}_R$ has a fixed point $x^\varepsilon \in \bar{B}_R$. Therefore there is a subsequence $(x^{\varepsilon_m})_{m\in\mathbb{N}} \subset \bar{B}_R$ such that

$$x^{\varepsilon_m} \to x^0 \in \bar{B}_R \quad \text{as } m \to \infty.$$

Since

$$\chi_{\mathcal{C}}(y)\left|\frac{y - x^{\varepsilon_m}}{|y - x^{\varepsilon_m}|^{N-\varepsilon_m}}\right| \leq \chi_{\mathcal{C}}(y)\left(|y - x^{\varepsilon_m}| + \left|\frac{y - x^{\varepsilon_m}}{|y - x^{\varepsilon_m}|^N}\right|\right)$$

where the right-hand side converges in L^1, Lebesgue's (generalized) convergence theorem implies that

$$\int_{\mathcal{C}} \frac{y - x^{\varepsilon_m}}{|y - x^{\varepsilon_m}|^{N-\varepsilon_m}}\,\mathrm{d}y \to \int_{\mathcal{C}} \frac{y - x^0}{|y - x^0|^N}\,\mathrm{d}y$$

as $m \to \infty$.
(Here we used the boundedness of $\mathcal{C}$.)
Step 2. $x_0 \in \mathcal{C}$.
It remains to show that $x^0 \in \mathcal{C}$. Assume that $x^0 \notin \mathcal{C}$. By the convexity of $\mathcal{C}$ there is a hyperplane separating x^0 and $\mathcal{C}$. Let ν be a unit normal on that hyperplane. By a translation and rotation we may assume that $x^0 = 0$, $\nu = e_1$ and $\mathcal{C} \subset \{x_1 > 0\}$ or $\mathcal{C} \subset \{x_1 < 0\}$, implying that

$$\int_{\mathcal{C}} \frac{y_1}{|y|^N}\,\mathrm{d}y \neq 0,$$

contradicting (A.1).

A.2 Comparison principle for the nonlinear PDE

The following comparison principle for solutions of the obstacle problem is well-known and stated only for the sake of completeness.

Lemma A.1 (Comparison principle for solutions of the obstacle problem)**.**
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz-boundary and let u and v be weak solutions of (1)*, i.e. $u, v \in W^{1,2}(\Omega)$ such that for all $\phi \in W_0^{1,2}(\Omega)$ it holds that*

$$-\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega \chi_{\{u>0\}}\phi \quad \textit{and} \quad -\int_\Omega \nabla v \cdot \nabla\phi = \int_\Omega \chi_{\{v>0\}}\phi. \tag{A.2}$$

If furthermore

$$v \leq u \quad \textit{on } \partial\Omega \tag{A.3}$$

in the sense of traces, then also

$$v \leq u \quad \textit{in } \Omega.$$

Proof. Testing the weak formulation (A.2) with the (admissible) test function $(v - u)^+ := \max\{v - u, 0\}$ and subtracting the equations we obtain

$$-\int_\Omega \left|\nabla(v-u)^+\right|^2 = \int_\Omega \left(\chi_{\{v>0\}} - \chi_{\{u>0\}}\right)(v-u)^+ \geq 0.$$

From this we infer that

$$\nabla(v-u)^+ \equiv 0 \quad \text{a.e. in } \Omega$$

and thus

$$(v-u)^+ \equiv \text{constant} \quad \text{a.e. in } \Omega.$$

Combining this fact with (A.3) we obtain that

$$(v-u)^+ \equiv 0 \quad \text{a.e. in } \Omega$$

which finishes the proof. □

B

Appendix B

Lemma B.1 (Global solutions of the obstacle problem have uniformly bounded (weak) second derivatives)**.** *Let u be a global solution of the obstacle problem* (1)*, then there is $C > 0$ such that*

$$\left\|D^2 u\right\|_{L^\infty(\mathbb{R}^N)} < C.$$

Here $D^2 u$ is the weak second derivative of u.

Proof. The result can be understood as an improvement of the of the classical result that solutions of the obstacle problem grow at most quadratically away from the obstacle (see e.g. [12, Theorem 2.1]) or as a globalization of the classical result that solutions of the obstacle problem are $C^{1,1}_{loc}$ (see e.g. [12, Theorem 2.3]). This is why we will use the first and take ideas from the latter.
First of all note that L^p-theory implies that $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ and therefore $D^2 u = 0$ a.e. in $\{u = 0\}$.
Thus it remains to show that $D^2 u$ is uniformly bounded in $\{u > 0\}$. Let $\tilde{x} \in \{u > 0\}$ be arbitrary but fixed. The fact that $\Delta u \equiv 1$ in $\{u > 0\}$ implies that $u - \frac{1}{2N}|x - \tilde{x}|^2$ is harmonic in $\{u > 0\}$. Using classical estimates for derivatives of harmonic functions (see e.g. [6, § 2.2 Theorem 7]) we get for any $r > 0$ such that $B_r(\tilde{x}) \subset \{u > 0\}$ that

$$\left|D^2 u(\tilde{x})\right| \leq \frac{C_1}{r^2}\left(\|u\|_{L^\infty(B_r(\tilde{x}))} + \frac{1}{2N} r^2\right) + 1. \tag{B.1}$$

Here $C_1 > 0$ only depends on N. In order to finish the proof we use that solutions of the obstacle problem grow at most quadratically away from the free boundary, i.e.

there is $C_2 > 0$ (only depending on N) such that for all $x_0 \in \partial\mathcal{C}$ and all $R > 0$

$$\|u\|_{L^\infty(B_R(x_0))} \leq C_2 R^2. \tag{B.2}$$

This can be found e.g. in [12, Theorem 2.1]. Let now $r_0 > 0$ be such that $\overline{B_{r_0}(\tilde{x})} \cap \partial\mathcal{C} \neq \emptyset$ and $\overline{B_\varrho(\tilde{x})} \cap \partial\mathcal{C} = \emptyset$ for all $\varrho \in (0, r_0)$ and let $x_0 \in \overline{B_{r_0}(\tilde{x})} \cap \partial\mathcal{C}$ (see figure B.1).
Combining (B.1) and (B.2) with $r = r_0$ and $R = 2r_0$ we obtain

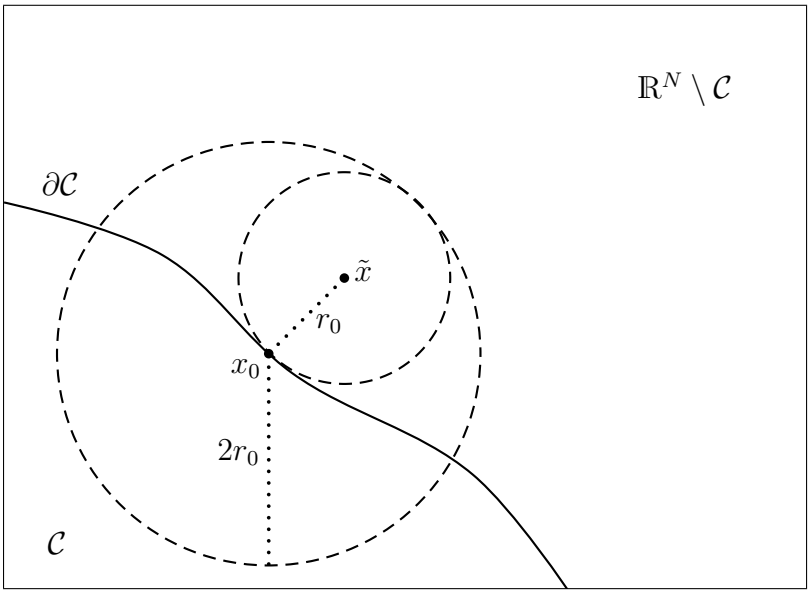

Figure B.1: Choice of r_0.

$$\left|D^2 u(\tilde{x})\right| \leq C_1 \left(4C_2 + \frac{1}{2N}\right) + 1$$

and this estimate is independent of $\tilde{x}$. Since $\tilde{x} \in \mathbb{R}^N \setminus \mathcal{C}$ was arbitrary this finishes the proof. □

Lemma B.2 (Boundary Poincaré).
Let $v \in W^{1,2}_{loc}(B_R)$, then there is $C(R) > 0$ only depending on R such that

$$\|v\|_{L^2(B_R)} \leq C(R)\Big(\|\nabla v\|_{L^2(B_R)} + \|v\|_{L^2(\partial B_R)}\Big).$$

Proof of Lemma B.2 . The idea of the proof is – as always for Poincaré inequalities – to use the fundamental theorem of calculus. We split B_R up into the upper and the lower half ball and set

$$B_R^\pm := \{x \in B_R : \pm x_N > 0\} \quad , \quad (\partial B_R)^\pm := \{x \in \partial B_R : \pm x_N > 0\}.$$

It is well known that $(\partial B_R)^+$ can be written as a graph of the following function

$$g^+ : B'_R \to \mathbb{R} : \quad x' \mapsto \sqrt{R^2 - |x'|^2},$$

where $B'_R := \{x \in \mathbb{R}^{N-1} : |x| < R\}$.

For all $v \in C^1(\bar{B_R})$ we have for all $(x', x_N) \in B_R^+$ that

$$\begin{aligned} v(x', x_N) &= v(x', x_N) - v(x', g^+(x')) + v(x', g^+(x')) \\ &= -\int_{x_N}^{g^+(x')} \partial_{x_N} v(x', t)\,\mathrm{d}t + v(x', g^+(x')). \end{aligned}$$

Hence employing Young's inequality and Cauchy-Schwarz inequality we get

$$\begin{aligned} v^2(x', x_N) &\leq 2\left|\int_{x_N}^{g^+(x')} \partial_{x_N} v(x', t)\,\mathrm{d}t\right|^2 + 2\big|v(x', g^+(x'))\big|^2 \\ &\leq 2R \int_0^{g^+(x')} |\partial_{x_N} v(x', t)|^2\,\mathrm{d}t + 2\big|v(x', g^+(x'))\big|^2. \end{aligned}$$

Since $1 \leq \sqrt{1 + |\nabla g^+|^2}$ it follows that

$$\begin{aligned} \int_{B'_R} v^2(x', x_N)\,\mathrm{d}x' &\leq 2R \int_{B_R^+} |\partial_{x_N} v|^2\,\mathrm{d}x + 2\int_{B'_R} v^2(x', g^+(x'))\sqrt{1 + |\nabla g^+(x')|}\,\mathrm{d}x' \\ &\leq 2R \int_{B_R^+} |\nabla v|^2\,\mathrm{d}x + 2\int_{(\partial B_R)^+} v^2\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Integrating over x_N we get

$$\int_{B_R^+} v^2\,\mathrm{d}x \leq 2R\left(R\int_{B_R^+} |\nabla v|^2 + \int_{(\partial B_R)^+} v^2\,\mathrm{d}\mathcal{H}^{N-1}\right).$$

By approximation this holds for any $v \in W^{1,2}(B_R)$ and the same can be shown for B_R^-. Finally we get what we wanted

$$\|v\|_{L^2(B_R)} \leq 2\sqrt{R}\max\left\{1, \sqrt{R}\right\}\Big(\|\nabla v\|_{L^2(B_R)} + \|v\|_{L^2(\partial B_R)}\Big).$$

□

Lemma B.3 (sup-boundary-value inequality for subharmonic functions).
Let $u \in W^{1,2}_{loc}(\mathbb{R}^N)$, $u \geq 0$, $\Delta u \geq 0$, then

i) for any $x \in \mathbb{R}^N$

$$\fint_{\partial B_R(x)} u \, \mathrm{d}\mathcal{H}^{N-1} \quad \textit{is monotone increasing in } R$$

ii) and for all $R > 0$

$$\sup_{B_{\frac{R}{2}}} u \leq C(N) \fint_{\partial B_R} u \, \mathrm{d}\mathcal{H}^{N-1},$$

where $C(N)$ is independent of R.

Proof.

i) This result is well known for smooth functions and does hold for $u \in W^{1,2}_{loc}(\mathbb{R}^N)$ by approximation. Let ϕ_ε be the standard convolution kernel. Let us set

$$u^\varepsilon := u * \phi_\varepsilon \in C^\infty(\mathbb{R}^N, [0, \infty)) \quad \text{and it hods that } u^\varepsilon \geq 0.$$

This implies that

$$\Delta u^\varepsilon(x) = \Delta(u * \phi_\varepsilon)(x) = \Delta u * \phi_\varepsilon(x) = \int_{\mathbb{R}^N} \phi_\varepsilon(x - y) \, \mathrm{d}\Delta u(y) \geq 0$$

by the assumption that $\Delta u \geq 0$ in measure sense. (We also did understand $\Delta u * \phi_\varepsilon$ as convolution of a measure with a function.) Therefore i) for smooth functions implies that for all $R_1 < R_2$

$$\fint_{\partial B_{R_1}(x)} u^\varepsilon \, \mathrm{d}\mathcal{H}^{N-1} \leq \fint_{\partial B_{R_2}(x)} u^\varepsilon \, \mathrm{d}\mathcal{H}^{N-1}.$$

Since it is well known that $u^\varepsilon \to u$ strongly in $W^{1,2}_{loc}(\mathbb{R}^N)$ as $\varepsilon \to 0$ and from the trace theorem we get the desired result

$$\text{for all } R_1 < R_2 : \quad \fint_{\partial B_{R_1}(x)} u \, \mathrm{d}\mathcal{H}^{N-1} \leq \fint_{\partial B_{R_2}(x)} u \, \mathrm{d}\mathcal{H}^{N-1}.$$

ii) For all Lebesguepoints $x \in B_{\frac{R}{2}}$ of u using i) we get that

$$
\begin{aligned}
u(x) &\leq \fint_{B_{\frac{R}{2}}(x)} u \leq \frac{1}{\left|B_{\frac{R}{2}}\right|} \int_{B_R} u = \frac{1}{\left|B_{\frac{R}{2}}\right|} \int_0^R \int_{\partial B_\varrho} u \,\mathrm{d}\mathcal{H}^{N-1} \,\mathrm{d}\varrho \\
&= \frac{1}{\left|B_{\frac{R}{2}}\right|} \int_0^R |\partial B_1| \varrho^{N-1} \fint_{\partial B_\varrho} u \,\mathrm{d}\mathcal{H}^{N-1} \,\mathrm{d}\varrho \leq \frac{|\partial B_1|}{\left|B_{\frac{R}{2}}\right|} \int_0^R \varrho^{N-1} \fint_{\partial B_R} u \,\mathrm{d}\mathcal{H}^{N-1} \,\mathrm{d}\varrho \\
&= \frac{|\partial B_1|}{\left|B_{\frac{R}{2}}\right|} \frac{R^N}{N} \fint_{\partial B_R} u \,\mathrm{d}\mathcal{H}^{N-1} = 2^N \fint_{\partial B_R} u \,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned}
$$

Taking the $\sup_{B_{\frac{R}{2}}}$ on both sides gives the desired inequality.

□

Lemma B.4 (A Liouville Lemma)**.**
Let $u : \mathbb{R}^N \to \mathbb{R}$ *be* harmonic *with* subquadratic growth. *i.e.*

$$
\frac{u(rx)}{r^2} \to 0 \quad \textit{strongly in } L^\infty(\partial B_1) \textit{ as } r \to \infty, \tag{B.3}
$$

then u is affine linear.

Proof. First of all from (B.3) we infer that there is $C > 0$ such that for all $x \in \mathbb{R}^N$

$$
|u(x)| \leq C\big(1 + |x|^2\big). \tag{B.4}
$$

From [6, §2.2 Theorem 7] we know that there is $C_3 > 0$ (only depending on N) such that for all $r > 0$, $x_0 \in \mathbb{R}^N$, $i, j, k \in \{1, \dots, N\}$

$$
|D_{ijk} u(x_0)| \leq \frac{C_3}{r^{N+3}} \|u\|_{L^1(B_r(x_0))}
$$

This can be estimated using (B.4) as

$$
\begin{aligned}
|D_{ijk} u(x_0)| &\leq \frac{C_3}{r^{N+3}} \|u\|_{L^1(B_r(x_0))} \leq \frac{C_3}{r^{N+3}} |B_r(x_0)| C\big(1 + (|x_0| + r)^2\big) \\
&\leq \frac{C_3}{r^{N+3}} |B_r(x_0)| C\big(1 + 2|x_0|^2 + 2r^2\big) \\
&= \frac{C_3 C |B_1|}{r^3} \big(1 + 2|x_0|^2 + 2r^2\big) \to 0 \quad \text{as } r \to \infty.
\end{aligned}
$$

Since $x_0 \in \mathbb{R}^N$ was arbitrary we obtain that

$$
D^3 u \equiv 0 \quad \text{in } \mathbb{R}^N
$$

and hence u is a quadratic polynomial. So we can write

$$u = q + \ell + c \quad \text{in } \mathbb{R}^N,$$

where q is a homogeneous polynomial of degree 2, ℓ a linear function and $c \in \mathbb{R}$ a constant. Then for all $x \in \partial B_1$ we have that

$$\frac{u(rx)}{r^2} = q(x) + \frac{\ell(x)}{r} + \frac{c}{r^2} \to q(x) \quad \text{as } r \to \infty.$$

Hence (B.3) implies that $q \equiv 0$ and u is affine linear. $\square$

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