A Bloch-Kato formula
for the triple product p-adic L-function

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Abstract

The triple product p-adic L-function interpolates the square root of the central value of the automorphic L-function attached to a tensor product of three suitable modular forms. Certain values of such p-adic L-function, outside its region of interpolation, have been explicitly related to diagonal classes in the motivic cohomology of triple products of Kuga-Sato varieties, notably in the case of ordinary forms. Exploiting recent generalizations of syntomic and finite polynomial cohomology theories to arbitrary varieties over p-adic fields, we are able to extend these results, removing several constraints on the weight and nebentypus of the modular forms considered, as well as allowing for modular forms of finite (bounded) slope.

Zusammenfassung

Das Tripelprodukt p-adischer L-Funktionen interpoliert die Quadratwurzel des zentralen Werts der automorphen L-Funktion eines Tensorprodukts dreier geeigneter Modulformen. Bestimmte Werte einer solchen p-adischen L-Funktion, die außerhalb des Interpolierungsgebiets liegen, können durch Diagonalklassen in der motivischen Kohomologie von Tripelprodukten von Kuga-Sato-Varietäten ausgedrückt werden, insbesondere im Falle gewöhnlicher Modulformen. Wir nutzen neuere Entwicklungen im Bereich syntomischer und polynomineller Kohomologietheorien für allgemeine Varietäten über p-adischen Körpern, um diese Ergebnisse zu verallgemeinern. Hierbei beseitigen wir einige Einschränkungen für das Gewicht und den Nebentypus der betrachteten Modulformen. Zusätzlich erlauben wir Modulformen von endlicher (beschränkter) Steigung.

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Introduction

Let $f \in S_{k+2}(M_f, \psi_f)$, $g \in S_{l+2}(M_g, \psi_g)$, $h \in S_{m+2}(M_h, \psi_h)$ be three cuspidal eigenforms with $k, l, m \geq 0$ and $\psi_f \psi_g \psi_h = 1$. The triple-product L-function $L(f, g, h; s)$ is the automorphic L-function attached to the tensor product $\pi_f \otimes \pi_g \otimes \pi_h$ of the automorphic representations associated to f, g and h. It is defined as an Euler product of local factors on some right half plane and extended to a meromorphic function on the whole of $\mathbf{C}$ by analytic continuation. It satisfies a functional equation whose sign $\varepsilon = \prod \varepsilon_q$, a product of local signs at primes $q \mid M_f M_g M_h \infty$, determines the parity of the order of vanishing of $L(f, g, h; s)$ at the central point $c = \frac{k+l+m+4}{2}$, where the function is known to be holomorphic [**PSR87**]. Let us assume that the local signs at all finite places are $+1$. Then $\varepsilon = \varepsilon_\infty$ is determined by the weights of f, g, h in the following way: it is $+1$ exactly when the triple $(k+2, l+2, m+2)$ is unbalanced, that is if one of the weights is greater or equal than the sum of the other two. In this unbalanced situation, assuming for instance $k+2 \geq l+m+4$, Harris and Kudla [**HK91**] showed that $L(f, g, h; c)$ is proportional by an explicit factor to

$$|(e_{\mathrm{hol}} \delta^t(\breve{g}) \cdot \breve{h}, \breve{f}^c)|^2$$

where:

- f^c is the form whose Fourier coefficients are complex conjugate to those of f;
- $\breve{f}^c \in S_{k+2}(M)[f^c]$, $\breve{g} \in S_{l+2}(M)[g]$, $\breve{h} \in S_{m+2}(M)[h]$ is a choice of test vectors for f^c, g, h, with $M = \mathrm{lcm}(M_f, M_g, M_h)$;
- $\delta = \frac{1}{2\pi i}(\frac{d}{dz} + \frac{\mathrm{weight}}{z - \bar{z}})$ is the Shimura-Maass differential operator on the space of nearly-holomorphic modular forms and $t = \frac{k-l-m-2}{2}$;
- e_{hol} is the holomorphic projection, from nearly-holomorphic modular forms to classical ones;
- $(\,,)$ is the Petersson pairing and $|\,|$ is the complex absolute value;
- the explicit factor mentioned above depends on the choice of test vectors, and there always exists a choice for which it is non-zero.

The formula of Harris and Kudla offered a starting point for the study of the p-adic behaviour of the special value $L(f, g, h; c)$, leading to the definition of a p-adic L-function interpolating these values first in the case of ordinary families of modular forms [**DR14**] and more recently in the case of overconvergent families of finite slope [**AI**]. The aim of the present work is to prove a formula for certain values of these p-adic L-functions originating from geometric input. Let us be more precise. In the case of a balanced triple of weights, when the formula of Harris and Kudla does not hold, the sign of the functional equation is -1, forcing $L(f,g,h;c)=0$. In this situation the conjectures of Bloch-Kato and Beilinson-Bloch predict the existence of a non-trivial class in the motivic cohomology of a certain variety (one whose étale cohomology contains the Galois representation associated to $\pi_f\otimes\pi_g\otimes\pi_h$) which should relate to the value of the derivative $L'(f,g,h;c)$ at the central point. Rather surprisingly, this motivic class is directly connected to the values attained in the balanced region by the p-adic L-functions mentioned above. Such relation has been proven for triples of ordinary modular forms in the case of good reduction [DR14] and later in the case of bad reduction with weights $(2,2,2)$ [DR16]. Our aim is to extend the techniques of Darmon and Rotger and prove such relation under more general assumptions: we allow bad reduction, arbitrary weights and finite (bounded) slope modular forms.

Let us state our main theorem. For the reader's convenience, we gather here all the necessary assumptions, which are otherwise scattered throughout the text. Let $p\geq 5$ be a prime,

$$f\in S_{k+2}(M_f,\psi_f,\bar{\mathbf{Q}}),\quad g\in S_{l+2}(M_g,\psi_g,\bar{\mathbf{Q}}),\quad h\in S_{m+2}(M_h,\psi_h,\bar{\mathbf{Q}})$$

be newforms with weights ≥ 2, assume $(k+2,l+2,m+2)$ is balanced, $k\leq l+m$ (which is redundant if the weights are all even) and $r:=\frac{k+l+m}{2}\in\mathbf{Z}$. Furthermore, assume ψ_f,ψ_g,ψ_h to be primitive at p, that is, for each of the three the p-adic order of the conductors of the form and of its character coincide. Fix a multiple $M=Np^s$ (with $p\nmid N$) of the three levels, big enough for N to satisfy the condition at the beginning of 2.1, Chapter II. If $p\mid M_f$ we write $\alpha_f=a_p(f)$ for the p-th Fourier coefficient of f and $\beta_f=p^{k+1}\psi(p|1)\alpha_f^{-1}$, with $(p|1)$ an integer congruent to p mod N and to 1 mod p^s; if instead $p\nmid M_f$, we write α_f,β_f for the roots of $X^2-a_p(f)X+p^{k+1}\psi(p)$. Similar notations hold for g and h. Assume, for each of the three forms, that $\alpha\neq\beta$ if p does not divide the level of the form; instead, if p does divide the level of the form, we assume $\mathrm{ord}_p(\alpha_f)<k+1$ in the case of f and $\alpha\neq 0$ in the case of g,h. Furthermore, assume that $V_f\oplus V_g\oplus V_h$ (the sum of the p-adic Galois representations attached to f,g,h) becomes crystalline when restricted to some totally ramified extension E of $\mathbf{Q}_p$. Fix also three modular forms

$$\breve{f}\in S_{k+2}(Np^s,\bar{\mathbf{Q}})^{\mathrm{prim}}[f],\quad \breve{g}\in S_{l+2}(Np^s,\bar{\mathbf{Q}})^{\mathrm{prim}}[g],\quad \breve{h}\in S_{m+2}(Np^s,\bar{\mathbf{Q}})^{\mathrm{prim}}[h]$$

where the primitive part is defined as in 2.1, Chapter II, and assume $\breve{g},\breve{h}$ to be eigenvectors for U of eigenvalue α_g,α_h respectively. Our main result is the following (cf. Theorem 13):

Theorem 1. *The formula*

$$\mathrm{AJ}_p([\Delta_s])(\eta^{\alpha}_{\breve{f}^c}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}})=(r-k)!\cdot\left(1-\frac{p^{r+1}}{\alpha_f\alpha_g\alpha_h}\right)^{-1}\cdot J_{\breve{f}^c}(e_{oc}\theta^{k-r-1}(w_s^*\breve{g})^{[p]}\cdot w_s^*\breve{h})$$

holds. □

Let us explain the meaning of the symbols appearing in the statement. On the right-hand-side, the apex $[p]$ denotes p-depletion of p-adic modular forms, that is the operation of erasing all terms in a q-expansion which are indexed by a multiple of p; the operator

θ is Serre's derivative $q\frac{d}{dq}$; the idempotent e_{oc} is the projection from the space of nearly-overconvergent modular forms to that of overconvergent modular forms (cf. Section 3, Chapter II); w_s^* is the inverse of the operator W_{p^s} of [AL78]; finally, $J_{\breve{f}^c}(-)$ is the Hecke compatible extension of the functional $(-, w_s^*\breve{f}^c)/(f^c, f^c)$ to overconvergent p-adic modular forms. The reader should now be convinced that the right-hand-side is an element of $\bar{\mathbf{Q}}_p$, and we turn our attention to the left-hand-side. The scheme $\bar{\mathcal{E}}_s^{\mathbf{r}} = \bar{\mathcal{E}}_s^k \times \bar{\mathcal{E}}_s^l \times \bar{\mathcal{E}}_s^m$, to be introduced in Section 1, Chapter III, is a product of Kuga-Sato varieties containing in its étale cohomology the Galois representation associated to $\pi_f \otimes \pi_g \otimes \pi_h$. In the same section we introduce a cycle Δ_s in $\bar{\mathcal{E}}_s^{\mathbf{r}}$, essentially the diagonal, which can be transformed by means of the cycle class map into a class in the étale cohomology of $\bar{\mathcal{E}}_s^{\mathbf{r}}$. The Hochschild-Serre spectral sequence can then be used to project this class to an element of the Galois cohomology group $H^1(\mathbf{Q}_p, H^{2r+3}_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p}, \mathbf{Q}_p(r+2)))$. Finally, applying a Bloch-Kato logarithm-type map yields an element of $H^{2r+3}_{\mathrm{dR}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p})/\operatorname{Fil}^{r+2}$, which by cup product gives rise to the linear functional

$$\mathrm{AJ}_p([\Delta_s])\colon \operatorname{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p}) \to \bar{\mathbf{Q}}_p$$

appearing in the above formula. We interpret this p-adic Abel-Jacobi map AJ_p (from a Chow group to a de Rham cohomology group) just as a logarithm, the Bloch-Kato logarithm, and consider the previous steps in its construction only as a clean-up of its domain. The element $\eta^{\alpha}_{\breve{f}^c} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}}$ at which it is evaluated is carefully constructed in Section 1, Chapter II: briefly speaking, $\omega_{\breve{g}}$ and $\omega_{\breve{h}}$ are the de Rham cohomology classes obtained considering $\breve{g}$ and $\breve{h}$ as differential forms, while $\eta^{\alpha}_{\breve{f}^c}$ is a class which realizes, via cup product, the linear functional $(-, \breve{f}^c)/(f^c, f^c)$, while also being an eigenvector for the crystalline Frobenius. The proof of Theorem 1 follows the lines of [DR16], mixing in some elements of [DR14] to allow for general weights. The extension to finite slope modular forms is obtained through a more careful study of the filtered (φ, N)-modules associated to such forms. Once all the objects involved have been defined, the heart of the proof is the use of finite-polynomial syntomic cohomology (see [NN16], [BLZ16] and Section 2.3, Chapter III), which allows us to reinterpret all the participants to the game as concrete differential forms, apt to explicit computation.

Let us state explicitly the connection with the triple product p-adic L-function; as the ordinary theory of [DR14] is a particular case of the finite slope one of [AI], we concentrate on the latter. Let $\underline{f}, \underline{g}, \underline{h}$ be three overconvergent families of modular forms as in Section 5.2 of loc. cit., and let $\mathcal{L}_p^{\underline{f}}(\underline{f}, \underline{g}, \underline{h})$ be the associated p-adic L-function. Assume there are points x, y, z in the respective weight spaces such that we have specializations

$$\underline{f}^*_x = w_s^*\breve{f}^c \qquad \underline{g}_y = w_s^*\breve{g} \qquad \underline{h}_z = w_s^*\breve{h}$$

($\underline{f}^*$ is the twist of $\underline{f}$ by the inverse of its tame character) with $\breve{f}^c, \breve{g}, \breve{h}$ as above. The following is then immediate from Theorem 1 and the discussion in 5.2 of loc. cit.:

Theorem 2. *The formula*

$$\mathcal{L}_p^f(\underline{f}, \underline{g}, \underline{h})(x, y, z) = \frac{1}{(r-k)!}\left(1 - \frac{p^{r+1}}{\alpha_f \alpha_g \alpha_h}\right) \mathrm{AJ}_p([\Delta_s])(\eta^{\alpha}_{\breve{f}^c} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}})$$

holds.

This statement fits well in the broadening panorama of p-adic Gross-Zagier type formulas: the resemblance to the main theorem of [**BDP13**], or to the more classical results on circular and elliptic units, is striking. For instance, in our higher-dimensional setting, the diagonal cycle $[\Delta_s]$ takes up the role that Heegner points have on modular curves, the p-adic Abel-Jacobi map AJ_p (or, better, the Bloch-Kato logarithm) replaces the formal logarithm of elliptic curves, and the above formula is then completely analogous to Theorem 1.4 of [**BCD$^+$14**]. Results of this kind are relevant when one approaches the Bloch-Kato conjecture: in [**DR16**], the study of the p-adic variation of the ordinary case of Theorem 1 allowed Darmon and Rotger to prove a certain Galois-equivariant refinement of the Birch and Swinnerton-Dyer conjecture, which leads us to believe that a full study of the p-adic properties of Theorem 1 could well lead to further developments in this direction.

CHAPTER I

Modular forms and étale cohomology

1. Modular curves and Kuga-Sato varieties

In this section, we introduce modular curves, Kuga-Sato varieties, and maps and correspondences between them, which are the geometric incarnations of the arithmetic objects we want to study.

1.1. Let $\bar{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and for all $m > 0$ write $\zeta_m := e^{2\pi i/m} \in \bar{\mathbf{Q}}$ for our favorite m-th root of unity. We fix an embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$, an algebraic closure of $\mathbf{Q}_p$, which will be used throughout the text without further mention. We denote by $\mathbf{C}_p$ the completion of $\bar{\mathbf{Q}}_p$.

1.2. For $M > 4$, let $Y_1(M)$ (resp. $X_1(M)$) be the modular curve over $\mathbf{Q}$ representing the functor which to a $\mathbf{Q}$-scheme S associates the set of S-isomorphism classes of data (E, ι) with E an elliptic curve (resp. a generalized elliptic curve) over S, and $\iota \colon (\mathbf{Z}/M\mathbf{Z})_S \to E^{\mathrm{sm}}$ a closed immersion of S-group schemes (resp. whose image meets all irreducible components in every geometric fiber), where E^{sm} denotes the smooth locus of $E \to S$. It is sometimes more convenient to specify just the S-point $\iota(1)$ rather than the whole closed immersion ι, hence we will use these two equivalent viewpoints interchangeably. To lighten the notation, from now on we omit the base of constant group schemes, as the reader can always conveniently recover it from the context. Let $\bar{\mathcal{E}}_1(M) \to X_1(M)$ be the universal generalized elliptic curve, with $\iota \colon \mathbf{Z}/M\mathbf{Z} \to \bar{\mathcal{E}}_1(M)$ its universal level structure: this is the moduli datum corresponding to the trivial $X_1(M)$-point $\mathrm{id}_{X_1(M)} \colon X_1(M) \to X_1(M)$. The k-th fiber power of $\bar{\mathcal{E}}_1(M)$ over $X_1(M)$ is not a smooth $\mathbf{Q}$-scheme for $k > 1$, but admits a nice desingularization $\bar{\mathcal{E}}_1^k(M)$, the Kuga-Sato variety, which was constructed in [Sch90] (previously, this kind of desingularization had been introduced over finite fields by Deligne [Del69], and in the Appendix to [BDP13] B. Conrad details how it can be extended over $\mathbf{Z}[1/M]$). Similarly, there is a universal datum $(\mathcal{E}, \iota)$ over $Y_1(M)$ and the fibered power $\mathcal{E}_1^k(M)$, which is already smooth, is the open subvariety of $\bar{\mathcal{E}}_1^k(M)$ obtained by base change to $Y_1(M)$.

1.3. Throughout this work, a prime $p \geq 5$ is fixed. When p divides the level, say $M = Np^s$ with $(N, p) = 1$, we set the notations $Y_s := Y_1(Np^s)$, $X_s := X_1(Np^s)$ and similarly $\mathcal{E}_s$, $\mathcal{E}_s^k$, $\bar{\mathcal{E}}_s$ and $\bar{\mathcal{E}}_s^k$ for universal objects and Kuga-Sato varieties. At times, we prefer to split the level structure ι in two parts: a closed immersion $\iota_N \colon \mathbf{Z}/N\mathbf{Z} \to E^{\mathrm{sm}}$ and a closed immersion $\iota_p \colon \mathbf{Z}/p^s\mathbf{Z} \to E^{\mathrm{sm}}$, obtained from $(N + p^s) \cdot \iota$ after using the Chinese remainder theorem. In other words, we have

$$\iota_N(1) = \iota(p^s), \qquad \iota_p(1) = \iota(N).$$

Actually, there are two natural ways of splitting the level structure, and the factor $(N+p^s)$ above picks out one rather than the other (cf. [KM85], §3.5).

1.4. The modular curve $X_1(M)$ is naturally equipped with with an action of the group $(\mathbf{Z}/M\mathbf{Z})^\times/\{\pm1\}$ as follows: given $d \in \mathbf{Z}$, $(d, M) = 1$, the corresponding operator $\langle d\rangle$ sends the class of (E, ι) to that of $(E, d\cdot\iota)$. These are commonly known as diamond operators, from the shape of the brackets used to denote them. Thanks to its universal property, $\bar{\mathcal{E}}_1(M)$ inherits this action: there is a unique endomorphism $\langle d\rangle$ of $\bar{\mathcal{E}}_1(M)$ for which the square

$$\begin{tikzcd} \bar{\mathcal{E}}_1(M) \arrow[r, "\langle d\rangle"] \arrow[d] & \bar{\mathcal{E}}_1(M) \arrow[d] \\ X_1(M) \arrow[r, "\langle d\rangle"] & X_1(M) \end{tikzcd}$$

is cartesian. Indeed, the operator $\langle d\rangle$ sends the $X_1(M)$-point $\mathrm{id}_{X_1(M)}$ to the $X_1(M)$-point $\langle d\rangle$, hence $(\bar{\mathcal{E}}_1(M), d\cdot\iota)$ is isomorphic, over $X_1(M)$, to the couple obtained from $(\bar{\mathcal{E}}_1(M), \iota)$ by base change via $\langle d\rangle$. The action of diamond operators restricts to $\mathcal{E}_1(M)$ and extends to $\mathcal{E}_1^k(M)$ in the obvious way. When $M = Np^s$, it will be convenient to denote by $\langle a|b\rangle$ the diamond operator corresponding to an integer congruent to a mod N and to b mod p^s, that is, sending the class of (E, ι_N, ι_p) to that of $(E, a\cdot\iota_N, b\cdot\iota_p)$.

1.5. For $M > 4$ and l any prime, we denote by $X_1(M; l)$ the modular curve over $\mathbf{Q}$ representing the following functor: to a $\mathbf{Q}$-scheme S, it associates the set of S-isomorphism classes of data (E, ι, C) over S, where E is a generalized elliptic curve, $\iota\colon \mathbf{Z}/M\mathbf{Z} \to E^{\mathrm{sm}}$ is a closed immersion of group schemes and C is a locally free subgroup scheme of E^{sm} of rank l, such that $C \cap \mathrm{Im}\,\iota = 0$ and $C\cdot\mathrm{Im}\,\iota$ meets all irreducible components in every geometric fiber. We denote by $Y_1(M; l)$ the corresponding open modular curve, and by $(\bar{\mathcal{E}}_1(M; l), \iota, \mathcal{C})$ the universal object over $X_1(M; l)$. There are two finite maps $\pi_1, \pi_l\colon X_1(M; l) \to X_1(M)$ described on the open curve $Y_1(M; l)$ by

$$\pi_1(E, \iota, C) = (E/C, \iota\circ\pi), \quad \pi_l(E, \iota, C) = (E, \iota)$$

where π is the isogeny $E \to E/C$. Notice that $X_1(M; l)$ is the quotient of $X_1(Ml)$ by the action of all diamond operators $\langle d\rangle$ with $d \equiv \pm1$ mod M, the quotient map being described over $Y_1(Ml)$ by

$$(E, \iota) \mapsto (E/C_l, \iota \bmod C_l, \ker\phi)$$

with C_l the locally free rank l subgroup scheme of $\mathrm{Im}\,\iota$ and ϕ the isogeny dual to $E \to E/C_l$.

1.6. Given positive integers $M, M' > 4$ and $d > 0$ such that $M' \mid M$, $d \mid \frac{M}{M'}$, there is a map

$$\varpi_d \colon X_1(M) \to X_1(M')$$

described on the open modular curve $Y_1(M)$ by

$$\varpi_d(E, \iota) = (E/C_d, \tfrac{M}{M'd}\iota(1) \bmod C_d)$$

where $C_d = \frac{M}{d}\iota(\mathbf{Z}/M\mathbf{Z})$ is the locally free rank d subgroup scheme of $\operatorname{Im}\iota$. In particular, if $M = Np^{s+1}$ and $M' = Np^s$, the maps $\varpi_1, \varpi_p \colon X_{s+1} \to X_s$ are described over the open modular curve Y_{s+1} by

$$\varpi_1(E, \iota_N, \iota_p) = (E, \iota_N, p\iota_p(1)),$$
$$\varpi_p(E, \iota_N, \iota_p) = (E/C_p, p^{-1}\iota_N(1) \bmod C_p, \iota_p \bmod C_p).$$

Notice also that, if $M = M'l$ with l prime, ϖ_1 and ϖ_l factor through $X_1(M'; l)$, that is, we have a commutative diagram:

$$\begin{tikzcd} & X_1(M) \arrow[dl, "\varpi_1"'] \arrow[d] \arrow[dr, "\varpi_l"] & \\ X_1(M') & X_1(M'; l) \arrow[l, "\pi_1"] \arrow[r, "\pi_l"'] & X_1(M') \end{tikzcd}$$

where the vertical arrow is the quotient described in Section 1.5.

1.7. For any prime l and integer $M > 4$, there is a diagram

$$\begin{tikzcd} \mathcal{E}_1(M) \arrow[d] & \mathcal{E}_1(M; l)/\mathcal{C} \arrow[l, "\pi_1"'] \arrow[d] & \mathcal{E}_1(M; l) \arrow[l, "\phi"'] \arrow[r, "\pi_l"] \arrow[d] & \mathcal{E}_1(M) \arrow[d] \\ Y_1(M) & Y_1(M; l) \arrow[l, "\pi_1"'] \arrow[r, equal] & Y_1(M; l) \arrow[r, "\pi_l"] & Y_1(M) \end{tikzcd}$$

in which the first and last squares are cartesian: to construct them, one repeats the same line of abstract nonsense which we used in 1.4 in the case of diamond operators; the square in the middle, on the other hand, is just the universal object for $Y_1(M; l)$. This diagram defines compatible correspondences on $\mathcal{E}_1(M)$ and $Y_1(M)$: on the cohomology of $\mathcal{E}_1(M)$ this acts as $T_l := \pi_{l*}\phi^*\pi_1^*$, extending the usual Hecke operator T_l on $Y_1(M)$. By iteration on every factor, T_l gives rise to a correspondence on $\mathcal{E}_s^k$ as well.

1.8. We denote by $\operatorname{Tate}(q)$ the universal Tate curve over $\mathbf{Q}((q))$ and by $\operatorname{Tate}(q^n)$, for any $n > 0$, its base change via $\mathbf{Q}((q)) \to \mathbf{Q}((q))$, $q \mapsto q^n$. Both are elliptic curves over $\mathbf{Q}((q))$. For detailed accounts on Tate curves, we refer the reader to ([Hid11], §2.5), or to ([DR73], §VII.1), but in this work we will only need to know the following: for any $n > 0$, μ_n embeds in $\operatorname{Tate}(q)$ canonically as the kernel of an isogeny $\pi_T \colon \operatorname{Tate}(q) \to \operatorname{Tate}(q^n)$; $\operatorname{Tate}(q)$ is endowed with a canonical differential $\omega \in \Omega^1_{\operatorname{Tate}(q)/\mathbf{Q}((q))}$ and the differential on $\operatorname{Tate}(q^n)$ obtained by base change via $q \mapsto q^n$ coincides with $\check{\pi}_T^*\omega$, where $\check{\pi}_T$ is the isogeny dual to π_T. When thinking analytically of $\operatorname{Tate}(q)$ as $\mathbb{G}_m/q^{\mathbf{Z}}$, the isogeny π_T is induced by the n-th power map on $\mathbb{G}_m$, and ω is induced by dt/t on $\mathbb{G}_m$.

1.9. We write $e_n(-,-)$ for the Weil pairing on n-torsion points of the smooth locus of a generalized elliptic curve. There are two possible ways to normalize it: we make our choice by declaring $e_n(\zeta_n, q) = \zeta_n$ on $\mathrm{Tate}(q^n)$.

1.10. There is an Atkin-Lehner operator $w_s\colon X_{s,\mathbf{Q}(\zeta_{p^s})} \to X_{s,\mathbf{Q}(\zeta_{p^s})}$ described on the open curve $Y_{s,\mathbf{Q}(\zeta_{p^s})}$ as follows. Given a triple (E, ι_N, ι_p) over some base-scheme S, let $\pi\colon E \to E'$ be the isogeny whose kernel is the image of ι_p. Locally on S, we can find a p^s-torsion section $R \in E[p^s]$ such that $e_{p^s}(\iota_p(1), R) = \zeta_{p^s}$. Then $\pi(R)$ gives a well defined p^s-torsion section in $E'(S)$ and we can set $w_s(E, \iota_N, \iota_p) := (E', \pi \circ \iota_N, \pi(R))$. This operator satisfies $w_s^2 = \langle p^s | -1\rangle$ and $\langle a|b\rangle w_s = w_s \langle a|b^{-1}\rangle$. We point out that w_s is the inverse of the operator W_{p^s} discussed in [AL78], to which we refer the reader for a much more detailed description.

1.11. We will sometimes make use of the modular curve $Y(M)$ (resp. $X(M)$), which is the fine moduli space over $\mathbf{Q}$ for data (E, α) with E an elliptic curve (resp. generalized elliptic curve) and $\alpha\colon (\mathbf{Z}/M\mathbf{Z})^2 \to E^{\mathrm{sm}}$ a closed immersion of group schemes. $X(M)$ bears a natural action of $\mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z})$ and has a quotient map to $X_1(M)$ which consists in forgetting the second coordinate of the constant group scheme. The base change of $X(M)$ to $\mathbf{Q}(\zeta_M)$ is disconnected: it admits a natural map to μ_M, defined by sending (E, α) to $e_M(\alpha(1,0), \alpha(0,1))$. The fibers of this map are geometrically irreducible and stable under the action of $\mathrm{SL}_2(\mathbf{Z}/M\mathbf{Z})$. We denote by $X(M)_{\mathrm{can}}$ the fiber over ζ_M.

2. Étale cohomology of Kuga-Sato varieties

The étale cohomology of Kuga-Sato varieties notably contains the Galois representations associated to modular forms of higher weights. In this preliminary section, we introduce certain idempotent projectors which cut off some extra data we don't need from these cohomology groups, and try our best to keep the analysis to the level of integral coefficients: while this is not necessary for the present work, it might be useful in future developments.

2.1. We wish to describe to some extent the p-adic étale cohomology of $\mathcal{E}_s^k$. We take as coefficient ring A either $\mathbf{Z}_p$ or $\mathbf{Z}/p^m\mathbf{Z}$ for some positive integer m. Let us write ψ_j for the automorphism of $\mathcal{E}_s^k$ which is multiplication by -1 on the j-th component and the identity on all the others, and define the operator

$$e_2 := \prod_{j=1}^{k} \frac{1 - \psi_j^*}{2}$$

which acts idempotently on the cohomology of $\mathcal{E}_s^k$. Moreover, let $u\colon \mathcal{E}_s \to Y_s$ and $u^k\colon \mathcal{E}_s^k \to Y_s$ be the natural maps.

Proposition 3. *The following identities hold:*

$$e_2 H^k(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = H^0(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*A)^{\otimes k})$$
$$e_2 H^{k+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = H^1(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*A)^{\otimes k})$$
$$e_2 H^j(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = 0 \quad \text{for } j \neq k, k+1,$$

and

$$e_2 H_c^{k+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = H_c^1(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*A)^{\otimes k})$$
$$e_2 H_c^{k+2}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = H_c^2(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*A)^{\otimes k})$$
$$e_2 H_c^j(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = 0 \quad \text{for } j \neq k+1, k+2.$$

Moreover, $e_2 H_c^{k+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Z}_p)$ *is a free* $\mathbf{Z}_p$*-module.*

Proof: The Leray spectral sequences

$$H^p(Y_{s,\bar{\mathbf{Q}}}, R^q u^k_* A) \Rightarrow H^{p+q}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A)$$
$$H_c^p(Y_{s,\bar{\mathbf{Q}}}, R^q u^k_* A) \Rightarrow H_c^{p+q}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A)$$

both degenerate at page two (cf. [Del69], proof of 5.3), thus giving us exact sequences

$$\begin{gathered} 0 \to H^1(Y_{s,\bar{\mathbf{Q}}}, R^i u^k_* A) \to H^{i+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) \to H^0(Y_{s,\bar{\mathbf{Q}}}, R^{i+1} u^k_* A) \to 0 \\ 0 \to \mathrm{Fil}^1\, H_c^i(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) \to H_c^i(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) \to H_c^0(Y_{s,\bar{\mathbf{Q}}}, R^i u^k_* A) \to 0 \\ 0 \to H_c^2(Y_{s,\bar{\mathbf{Q}}}, R^i u^k_* A) \to \mathrm{Fil}^1\, H_c^{i+2}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) \to H_c^1(Y_{s,\bar{\mathbf{Q}}}, R^{i+1} u^k_* A) \to 0 \end{gathered} \tag{1}$$

for all $i \geq 0$. The natural map

$$\bigoplus_{q_1+\cdots+q_k=q} R^{q_1}u_*A \otimes \cdots \otimes R^{q_k}u_*A \xrightarrow{\sim} R^q u^k_* A \tag{2}$$

is an isomorphism, as can be checked on stalks with the aid of the Künneth formula ([Mil80], VI 8.13). Moreover, since u has relative dimension one, $R^q u_* A = 0$ for all $q \geq 3$. We also notice, as in ([Del69], proof of 5.3), that multiplication by -1 in $\mathcal{E}_s$ acts via pullback as $(-1)^q$ on $R^q u_* A$. Combining these two facts with (2), we see that e_2 kills $R^q u^k_* A$ whenever $q \neq k$ and projects $R^k u^k_* A$ onto its direct summand $(R^1 u_* A)^{\otimes k}$. The short exact sequences (1) yield now directly all the equalities in the statement, except for the vanishing of $e_2 H_c^k(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, A) = H_c^0(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*A)^{\otimes k})$, which is deduced by duality since e_2 is self-adjoint for the Poincaré pairing. Moreover, all these cohomology groups are finite when $A = \mathbf{Z}/p^m\mathbf{Z}$ (cf. [Mil80], V 2.1 and VI 11.2), thus we can proceed as in the proof of (loc. cit., V 1.11) to get an exact sequence

$$\begin{aligned} 0 \to H_c^i(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*\mathbf{Z}_p)^{\otimes k})/p^m \to H_c^i(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*\mathbf{Z}/p^m\mathbf{Z})^{\otimes k}) \\ \to H_c^{i+1}(Y_{s,\bar{\mathbf{Q}}}, (R^1u_*\mathbf{Z}_p)^{\otimes k})[p^m] \to 0 \end{aligned}$$

where the suffix $[n]$ stands for the subgroup of n-torsion. Together with the equalities in the statement, this shows that $e_2 H_c^{k+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Z}_p)$ is a free $\mathbf{Z}_p$-module. On the other hand, the analogue short exact sequence for cohomology groups without compact support shows that $e_2 H^{k+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Z}_p)$ can have nontrivial torsion. $\square$

2.2. We turn now our attention to $\bar{\mathcal{E}}_s^k$, and consider étale cohomology with $\mathbf{Q}_p$ coefficients. We define $\mathcal{L} := R^1 u_* \mathbf{Q}_p$ and $\mathcal{L}_k := \mathrm{Sym}^k \mathcal{L}$. Notice that $\mathcal{L}_k$ is a direct summand of $\mathcal{L}^{\otimes k}$: we write $e_{\mathrm{sym}} \colon \mathcal{L}^{\otimes k} \to \mathcal{L}_k$ for the natural projector and ι_{sym} for the section given by $e_1 \cdots e_k \mapsto k!^{-1} \sum_\sigma e_{\sigma(1)} \cdots e_{\sigma(k)}$, where the sum runs over the symmetric group on k elements. We write e_{sym} also for the correspondence $k!^{-1} \sum \sigma$ on $\bar{\mathcal{E}}_s^k$ induced by the action of the symmetric group. Moreover, translation by (universal) sections of order Np^s induces an action of $(\mathbf{Z}/Np^s\mathbf{Z})^k$ on $\bar{\mathcal{E}}_s^k$, and we define a correspondence $e_{\mathrm{t}} := (Np^s)^{-k} \sum a^*$, the sum running over $a \in (\mathbf{Z}/Np^s\mathbf{Z})^k$; set $e_{\mathrm{mod}} := e_{\mathrm{sym}} e_{\mathrm{t}} e_2$. Also, let $j \colon Y_s \hookrightarrow X_s$ be the open immersion of the affine modular curve in the proper one.

Proposition 4. *The following identities hold:*

$$e_{mod} H^i(\bar{\mathcal{E}}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p) = 0 \quad \textit{for } i \neq k+1$$

and

$$\begin{aligned} e_{mod} H^{k+1}(\bar{\mathcal{E}}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p) &= \mathrm{Im}\left(e_{mod} H^{k+1}_c(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p) \to e_{mod} H^{k+1}(\mathcal{E}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p) \right) \\ &= H^1(X_{s,\bar{\mathbf{Q}}}, j_* \mathcal{L}_k). \end{aligned}$$

Proof: This follows from Lemma 2.2 of [BDP13] and therein quoted parts of [Sch85], [Sch90] and [Sch96]. □

3. Recalls of p-adic Hodge Theory

We recall here the basics facts about p-adic Hodge theory that we will need in the following sections: we define Fontaine's period rings and explain how they allow to pass from the world of Galois representations to that of linear algebra.

3.1. We start by defining Fontaine's period rings. Recall that $\mathbf{C}_p$ denotes the completion of $\bar{\mathbf{Q}}_p$, and let $\mathcal{O}_{\mathbf{C}_p}$ be its valuation ring. Let $R := \varprojlim \mathcal{O}_{\mathbf{C}_p}/(p)$ be the ring obtained as inverse limit of $\mathcal{O}_{\mathbf{C}_p}/(p)$ along the maps $x \mapsto x^p$. This is a perfect ring (i.e. $x \mapsto x^p$ is an automorphism of R) and there is a natural multiplicative (but not additive) bijection

$$R \xrightarrow{\sim} \{(x^{(n)})_{n \in \mathbf{N}} \mid x^{(n)} \in \mathcal{O}_{\mathbf{C}_p},\ (x^{(n+1)})^p = x^{(n)}\}$$

sending an element $(x_n)_{n \in \mathbf{N}} \in R$ to the collection $(x^{(n)})_{n \in \mathbf{N}}$ of elements of $\mathcal{O}_{\mathbf{C}_p}$ defined by

$$x^{(n)} := \lim_m x_{n+m}^{p^m}.$$

The ring of Witt vectors over R is denoted A_{inf}: an element $a \in A_{\mathrm{inf}}$ is given by a sequence $a = (a_n)_{n \in \mathbf{N}}$ of elements of R, and sum and multiplication are expressed in terms of Witt polynomials. For $x \in R$ we denote by $[x] = (x, 0, 0, \dots) \in A_{\mathrm{inf}}$ its Teichmüller representative. The ring A_{inf} is equipped with a surjective homomorphism

$$\begin{aligned} \theta \colon A_{\mathrm{inf}} &\twoheadrightarrow \mathcal{O}_{\mathbf{C}_p} \\ a = (a_n)_{n \in \mathbf{N}} &\mapsto \sum_{n \geq 0} p^n a_n^{(n)} \end{aligned}$$

whose kernel is a principal ideal, generated by any element $\xi \in A_{\inf}$ such that $\xi - p = [\varpi]$ is the Teichmüller representative of some $\varpi \in R$ with $\varpi^{(0)} = -p$. Fontaine's ring of de Rham periods is defined as follows:

$$B_{\mathrm{dR}} := \mathrm{Frac}\left(\varprojlim_n A_{\inf}[\tfrac{1}{p}]/(\xi)^n \right)$$

It is a discretely valued field, with valuation ring the complete d.v.r. $B_{\mathrm{dR}}^+ := \varprojlim A_{\inf}[\frac{1}{p}]/(\xi)^n$ with residue field $\mathbf{C}_p$. The natural topology on B_{dR}^+ is the projective limit one induced from the topology of $A_{\inf}$; notice that this is weaker than the d.v.r. topology, for which the residue field is discrete. The natural action of the absolute Galois group $G_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ on $\mathcal{O}_{\mathbf{C}_p}$ induces an action of the same group on B_{dR}, satistying the following: if L is any finite extension of $\mathbf{Q}_p$, one has $B_{\mathrm{dR}}^{G_L} = L$. Finally, there is a somewhat canonical choice for a uniformizer in B_{dR}^+, given as follows. Let $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in R$ (remember that we fixed an embeding of $\bar{\mathbf{Q}}$ in $\bar{\mathbf{Q}}_p$). Then $[\varepsilon] - 1 \in \ker\theta$, so that

$$t := \log[\varepsilon] = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}([\varepsilon]-1)^n$$

is a well defined element of B_{dR}^+, and is in fact a uniformizer. The Galois action on t is easy to compute: for any $\sigma \in G_{\mathbf{Q}_p}$, one has $\sigma(t) = \chi_{\mathrm{cyc}}(\sigma)t$ where χ_{cyc} is the p-adic cyclotomic character.

We now define subrings $B_{\mathrm{cris}}^+ \subset B_{\mathrm{dR}}^+$ and $B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}}$. Firstly, let A_{cris} be the p-adic completion of the divided power envelope of $A_{\inf}$ with respect to $\ker\theta$: in other words, it is the p-adic completion of the ring obtained by adjoining to $A_{\inf}$ all elements of the form $\frac{a^m}{m!}$ with $a \in \ker\theta$ and $m > 0$. Then $B_{\mathrm{cris}}^+ := A_{\mathrm{cris}}[\frac{1}{p}]$ and $B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[\frac{1}{t}]$ (notice that $t \in A_{\mathrm{cris}}$ already). In fact, one can show that $t^{p-1} \in pA_{\mathrm{cris}}$, so that $B_{\mathrm{cris}} = A_{\mathrm{cris}}[\frac{1}{t}]$. All these rings carry naturally the $G_{\mathbf{Q}_p}$-action. By functoriality of Witt vectors, the automorphism $x \mapsto x^p$ of R lifts to an automorphism φ of $A_{\inf}$, which extends naturally to B_{cris} (but not to B_{dR}!). This map is called the Frobenius operator on B_{cris}. In particular, one has $\varphi(t) = pt$.

Let now $\varpi \in R$ be, as before, any element with $\varpi^{(0)} = -p$. Notice that $\frac{[\varpi]}{-p} \in A_{\inf}[\frac{1}{p}]$ lies in the kernel of θ, hence

$$\log[\varpi] := \log\tfrac{[\varpi]}{-p} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(\tfrac{[\varpi]}{-p} - 1)^n$$

is a well defined element of B_{dR}^+. It follows from the definition that $\sigma \log[\varpi] = \log[\varpi] + \chi_{\mathrm{cyc}}(\sigma)t$ for all $\sigma \in G_{\mathbf{Q}_p}$. The ring $B_{\mathrm{st}} := B_{\mathrm{cris}}[\log[\varpi]]$ is the subalgebra of B_{dR} generated by $\log[\varpi]$ over B_{cris}. In fact, $\log[\varpi]$ is transcendental over $\mathrm{Frac}(B_{\mathrm{cris}})$, so that B_{st} is isomorphic to a polynomial ring in one variable over B_{cris}. The Frobenius φ naturally extends to B_{st} by setting $\varphi(\log[\varpi]) = p\log[\varpi]$, and commutes with the $G_{\mathbf{Q}_p}$-action. The ring B_{st} is also equipped naturally with a monodromy operator N: this is the B_{cris}-derivation determined by $N(\log[\varpi]) = -1$. In other words,

$$N\colon \sum_{n\geq 0} b_n(\log[\varpi])^n \mapsto -\sum_{n\geq 0} b_n(\log[\varpi])^{n-1}, \qquad b_n \in B_{\mathrm{cris}}$$

so that we have a short exact sequence

$$0 \to B_{\mathrm{cris}} \to B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}} \to 0$$

The monodromy operator N commutes with the $G_{\mathbf{Q}_p}$-action, and satisfies $N\varphi = p\varphi N$. Moreover, the Galois action satisfies the following: if L is any finite extension of $\mathbf{Q}_p$, and L_0 its maximal unramified subextension, then $(B^+_{\mathrm{cris}})^{G_L} = B^{G_L}_{\mathrm{cris}} = B^{G_L}_{\mathrm{st}} = L_0$. Finally, the inclusion $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ induces a short exact sequence

$$0 \to \mathbf{Q}_p \to B^{\varphi=1}_{\mathrm{cris}} \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0 \tag{3}$$

called the fundamental exact sequence of p-adic Hodge theory.

3.2. Let L be any finite extension of $\mathbf{Q}_p$. We define a category $\underline{\mathrm{Fil}}_L$ as follows: the objects are finite dimensional vector spaces over L together with a filtration, indexed by $\mathbf{Z}$, which is exhausted and separated, and the morphisms are the L-linear maps which respect the filtration, that is, which send the i-th filtration step of the source into the i-th filtration step of the target. If $D_1, D_2 \in \underline{\mathrm{Fil}}_L$, then $D_1 \otimes D_2$ is naturally an object of $\underline{\mathrm{Fil}}_L$, with the filtration given by

$$\mathrm{Fil}^i\, D_1 \otimes D_2 := \sum_{j+k=i} \mathrm{Fil}^j\, D_1 \otimes \mathrm{Fil}^k\, D_2.$$

Also, this category has dual objects: if $D \in \underline{\mathrm{Fil}}_L$, then we let its dual $D^* := \mathrm{Hom}_L(D, L)$ be the space of L-linear functionals on D, together with the filtration

$$\mathrm{Fil}^i\, D^* := (\mathrm{Fil}^{1-i}\, D)^{\perp} = \{f \in D^* \mid f(\mathrm{Fil}^{1-i}\, D) = 0\}.$$

Finally, we want to define exact sequences. If $\eta\colon D_1 \to D_2$ is a morphism in $\underline{\mathrm{Fil}}_L$, we say that η is strict if $\eta(\mathrm{Fil}^i\, D_1) = \mathrm{Fil}^i\, D_2 \cap \mathrm{Im}\,\eta$. Then, a diagram

$$0 \to D_1 \to D_2 \to D_3 \to 0$$

in $\underline{\mathrm{Fil}}_L$ is called a short exact sequence if all maps appearing are strict and it is exact as a sequence of L-vector spaces.

Let now $\mathrm{Rep}_{\mathbf{Q}_p}(G_L)$ be the category of p-adic representation of the absolute Galois group G_L of L, that is, continuous linear representation of G_L on finite dimensional $\mathbf{Q}_p$-vector spaces. Then we can define a functor

$$\begin{aligned} D^L_{\mathrm{dR}}\colon \mathrm{Rep}_{\mathbf{Q}_p}(G_L) &\to \underline{\mathrm{Fil}}_L \\ V &\mapsto (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_L} \end{aligned}$$

where the filtration on $D^L_{\mathrm{dR}}(V)$ is induced by that on B_{dR}, that is,

$$\mathrm{Fil}^i\, D^L_{\mathrm{dR}}(V) = (\mathrm{Fil}^i\, B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_L}$$

with $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$. In general, one has that

$$\dim_L D_{\mathrm{dR}}^L(V) \leq \dim_{\mathbf{Q}_p} V.$$

When equality holds, V is said to be B_{dR}-admissible, or, more simply, a de Rham representation. We should mention that being de Rham is a property insensitive to finite field extensions: if E is a finite extension of L, V is a de Rham representation of G_L if and only if it is a de Rham representation when restricted to G_E. We denote by $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_L)$ the full subcategory of $\mathrm{Rep}_{\mathbf{Q}_p}(G_L)$ whose objects are the de Rham representations. Then the restriction of D_{dR}^L to $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_L)$ is an exact, faithful and tensor functor.

If $V \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_L)$, tensoring (3) with V and taking Galois invariants yields an exact sequence

$$0 \to H^0(L, V) \to D_{\mathrm{cris}}^L(V)^{\varphi=1} \to D_{\mathrm{dR}}^L(V)/\,\mathrm{Fil}^0 \to H_e^1(L, V) \to 0$$

where $H_e^1(L, V) := \ker(H^1(L, V) \to H^1(L, B_{\mathrm{cris}}^{\varphi=1} \otimes_{\mathbf{Q}_p} V))$. The connecting map

$$\exp_{\mathrm{BK}} \colon D_{\mathrm{dR}}^L(V)/\,\mathrm{Fil}^0 \to H_e^1(L, V)$$

is called the Bloch-Kato exponential of V.

3.3. Just as in the case of de Rham representations, we will define now certain categories of linear algebra objects which relate to the rings B_{st} and B_{cris}. Let L_0 be the maximal unramified subextension of L. A filtered (φ, N)-module over L is a L_0-vector space D equipped with the following additional structures:

- a map $\varphi \colon D \to D$ which is L_0-semilinear with respect to the action of Frobenius on L_0;
- a L_0-linear map $N \colon D \to D$ satisfying $N\varphi = p\varphi N$;
- a decreasing, exhausted and separated filtration on $D_L := L \otimes_{L_0} D$.

A morphism of filtered (φ, N)-modules is a L_0-linear map which respects all the extra structure. We denote by $\mathrm{MF}_L(\varphi, N)$ the category of filtered (φ, N)-modules over L.

Assume now that $D \in \mathrm{MF}_L(\varphi, N)$ is finite dimensional, say $\dim_{L_0} D = k$, and that φ is bijective on D. Then the exterior power $\bigwedge_{L_0}^k D \subset D^{\otimes k}$ is a one-dimensional L_0-vector space on which φ acts bijectively. If $0 \neq x \in \bigwedge_{L_0}^k D$, then $\varphi(x) = ax$ for some $a \in L_0$ (depending on the choice x). The Newton number of D is

$$t_N(D) := \mathrm{ord}_p\, a = \mathrm{ord}_p \frac{\varphi(x)}{x}$$

which is in fact independent of the choice of x above. Furthermore, the filtration on D_L induces a filtration on $\bigwedge_L^k D_L$, and we define the Hodge number of D as

$$t_H(D) := \max\{i \in \mathbf{Z} \mid \mathrm{Fil}^i \bigwedge_L^k D_L = \bigwedge_L^k D_L\}.$$

Equivalently, the Hodge number can also be written in the form

$$t_H(D) = \sum_{i \in \mathbf{Z}} i \cdot \dim_L \mathrm{Gr}^i D_L$$

which is more apt to practical computations.

A filtered (φ, N)-module D is said to be admissible if the following conditions are satisfied:

- D is finite dimensional over L_0;
- φ is bijective on D;
- $t_H(D) = t_N(D)$;
- $t_H(D') \le t_N(D')$ for every subobject[1] D' of D in $\mathrm{MF}_L(\varphi, N)$.

We denote by $\mathrm{MF}_L^{\mathrm{ad}}(\varphi, N)$ the full subcategory of $\mathrm{MF}_L(\varphi, N)$ consisting of admissible objects; remarkably, this category is abelian. Furthermore, it has tensor products and dual objects, defined as in the previous section. We should also note that, for any object of $\mathrm{MF}_L^{\mathrm{ad}}(\varphi, N)$, the monodromy operator N is nilpotent (this follows from finite dimensionality and bijectivity of φ).

Let now $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_L)$ and define $D_{\mathrm{st}}^L(V) := (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}$ and $D_{\mathrm{cris}}^L(V) := (B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_L}$. Notice that both $D_{\mathrm{st}}^L(V)$ and $D_{\mathrm{cris}}^L(V)$ are naturally filtered (φ, N)-modules over L (with N the zero map on $D_{\mathrm{cris}}^L(V)$). In general, one has

$$\dim_{L_0} D_{\mathrm{cris}}^L(V) \le \dim_{L_0} D_{\mathrm{st}}^L(V) \le \dim_L D_{\mathrm{dR}}^L(V) \le \dim_{\mathbf{Q}_p} V$$

and we say V is a semistable (resp. crystalline) representation of G_L if $\dim_{L_0} D_{\mathrm{st}}^L(V) = \dim_{\mathbf{Q}_p} V$ (resp. $\dim_{L_0} D_{\mathrm{cris}}^L(V) = \dim_{\mathbf{Q}_p} V$). In particular,

$$\text{crystalline} \Rightarrow \text{semistable} \Rightarrow \text{de Rham}.$$

Being crystalline and being semistable are properties which are stable under finite field extensions: if E is a finite extension of L and V is a semistable (resp. crystalline) representation of G_L, then the restriction of V to G_E is also semistable (resp. crystalline). The converse implication generally fails; if the restriction of $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_L)$ to G_E is semistable (resp. crystalline) for some finite extension E/L, we say that V is a potentially semistable (resp. potentially crystalline) representation of G_L. It is nown that, for any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_L)$,

$$\text{de Rham} \iff \text{potentially semistable}.$$

We denote by $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_L)$ (resp. $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{cris}}(G_L)$) the full subcategory of $\mathrm{Rep}_{\mathbf{Q}_p}(G_L)$ consisting of semistable (resp. crystalline) representations. If $V \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_L)$, then the associated filtered (φ, N)-module $D_{\mathrm{st}}^L(V)$ is admissible. In fact, the functor

$$\begin{aligned} D_{\mathrm{st}}^L \colon \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{st}}(G_L) &\to \mathrm{MF}_L^{\mathrm{ad}}(\varphi, N) \\ V &\mapsto (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L} \end{aligned}$$

is an equivalence of categories, with quasi-inverse

$$V_{\mathrm{st}}^L \colon D \mapsto (B_{\mathrm{st}} \otimes_{L_0} D)^{\varphi=1, N=0} \cap \mathrm{Fil}^0 B_{\mathrm{st}} \otimes_{L_0} D_L.$$

[1] A subobject D' of D is a L_0-vector subspace of D, stable under φ and N, with filtration on D'_L induced by the one on D_L.

Since $B_{\text{cris}} = B_{\text{st}}^{N=0}$, the restriction of D_{st}^L to $\text{Rep}_{\mathbf{Q}_p}^{\text{cris}}(G_L)$ coincides with $D_{\text{cris}}^L\colon V \mapsto (B_{\text{cris}} \otimes_{L_0} V)^{G_L}$ and gives an equivalence with the full subcategory of filtered (φ, N)-modules with trivial monodromy operator.

3.4. Let us recall here some of the comparison theorems involving p-adic Hodge theory.

Theorem 5 (de Rham conjecture C_{dR}). *Let X be a proper and smooth variety over L. There are canonical Galois equivariant isomorphisms of filtered vector spaces*

$$B_{dR} \otimes_{\mathbf{Q}_p} H^m_{\acute{e}t}(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p) \cong B_{dR} \otimes_L H^m_{dR}(X/L) \quad \textit{for all } m \geq 0$$

which are functorial in X and compatible with cup-products.

Theorem 6 (crystalline conjecture C_{cris}). *Let X be a proper and smooth variety over the ring of integers $\mathcal{O}_L$ of L and let Y be its special fiber. There are canonical Galois and Frobenius equivariant isomorphisms*

$$B_{cris} \otimes_{\mathbf{Q}_p} H^m_{\acute{e}t}(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p) \cong B_{cris} \otimes_{L_0} H^m_{cris}(Y/L_0) \quad \textit{for all } m \geq 0$$

which are functorial in X and compatible with cup-products.

Theorem 7 (semistable conjecture C_{st}). *Let X be a proper and smooth variety over L admitting a proper semistable model $\tilde{X}$ over $\mathcal{O}_L$ and let Y be the special fiber of $\tilde{X}$. There are canonical Galois equivariant isomorphisms*

$$B_{st} \otimes_{\mathbf{Q}_p} H^m_{\acute{e}t}(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p) \cong B_{st} \otimes_{L_0} H^m_{log\text{-}cris}(Y/L_0) \quad \textit{for all } m \geq 0$$

with respect to the natural log-structure on Y, which are compatible with Frobenius and monodromy operators, functorial in $\tilde{X}$ and compatible with cup-products.

4. Modular forms and Galois representations

In this section we present the relation between modular forms and the étale cohomology of Kuga-Sato varieties. We pay particular attention to the case of ordinary newforms, and will come back to the more general case of finite slope forms in a later section.

4.1. The cohomology group

$$V_{k+2}(Np^s) := e_{\text{mod}} H^{k+1}_{\text{ét}}(\bar{\mathcal{E}}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p) = H^1_{\text{ét}}(X_{s,\bar{\mathbf{Q}}}, j_*\mathcal{L}_k)$$

contains the Galois representations associated to modular forms of weight $k+2$ and level Np^s, as we explain below in 4.5. As long as s and/or k remain fixed, we lighten the notations as much as possible, writing $V = V_{k+2} = V(Np^s) = V_{k+2}(Np^s)$ for short. As a side remark, notice that by Proposition 4, e_{mod} kills all cohomology groups $H^i_{\text{ét}}(\bar{\mathcal{E}}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ with $i \neq k+1$, thus projecting onto V the whole cohomology of the Kuga-Sato.

4.2. Let us denote by $\mathsf{h}_{\mathrm{GL}_2}$ the $\mathbf{Z}$-algebra of double cosets of 2 by 2 integral matrices with positive determinant modulo $\mathrm{SL}_2(\mathbf{Z})$. As shown in [Shi71], $\mathsf{h}_{\mathrm{GL}_2}$ is isomorphic to the polinomial ring over $\mathbf{Z}$ in the variables T_l and R_l for all primes l, where T_l and R_l represent the double cosets of

$$\begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}$$

respectively. The operators T_n for $n > 0$ are then determined by the following relations: $T_1 = 1$, $T_{l^r} = T_l T_{l^{r-1}} - l R_l T_{l^{r-2}}$ for all $r \geq 2$ and $T_{mn} = T_m T_n$ for $(m, n) = 1$. The natural involution of $\mathsf{h}_{\mathrm{GL}_2}$, induced by the matrix involution $M \mapsto \det(M) M^{-1}$, is denoted by $h \mapsto h^\iota$. For all $k \geq 0$, $\mathsf{h}_{\mathrm{GL}_2}$ maps to the ring of correspondences on $\mathcal{E}_s^k$ as follows:

$$\begin{aligned} T_l &\mapsto T_l && \text{for all primes } l, \\ R_l &\mapsto l^k \langle l \rangle^* && \text{for primes } l \nmid Np, \\ R_l &\mapsto 0 && \text{for primes } l \mid Np. \end{aligned}$$

(Remember we defined a correspondence $T_l = \pi_{l*} \phi^* \pi_1^*$ in 1.7.) We denote by $\mathsf{h}(\mathcal{E}_s^k)$ the image of $\mathsf{h}_{\mathrm{GL}_2}$ in the ring of correspondences on $\mathcal{E}_s^k$. Thanks to Proposition 4, $\mathsf{h}(\mathcal{E}_s^k)$ acts on V.

4.3. For $M \geq 1, k \geq 0$ integers, ψ a mod M Dirichlet character and R a $\mathbf{Q}$-algebra, we denote by $S_{k+2}(M, R)$ (resp. $S_{k+2}(M, \psi, R)$) the space of cusp modular forms of weight $k+2$, level $\Gamma_1(M)$, (resp. nebentypus ψ) and with Fourier coefficients in R, and by $S_{k+2}(M, R)^\vee$ (resp. $S_{k+2}(M, \psi, R)^\vee$) its R-linear dual. Recall that $S_{k+2}(M, R) = S_{k+2}(M, \mathbf{Q}) \otimes R$ ([Kat73], §1.7). For any cusp form f we write

$$f(q) = \sum_{n > 0} a_n(f) q^n$$

for its q-expansion. We refer the reader to ([Shi71], §3.4) for the description of the action of $\mathsf{h}_{\mathrm{GL}_2}$ on $S_{k+2}(M, R)$. Let $\mathsf{h}_{k+2}(M, R)$ (resp. $\mathsf{h}_{k+2}(M, \psi, R)$) be the R-algebra of endomorphisms of $S_{k+2}(M, R)$ (resp. $S_{k+2}(M, \psi, R)$) generated by $\mathsf{h}_{\mathrm{GL}_2}$. If ψ takes values in R, the pairing

$$\begin{aligned} \langle\, ,\, \rangle_{\mathsf{h}} \colon S_{k+2}(M, \psi, R) \times \mathsf{h}_{k+2}(M, \psi, R) &\to R \\ \langle f, h \rangle_{\mathsf{h}} &:= a_1(f|h) \end{aligned} \tag{4}$$

is perfect ([Hid93], §5.3, Theorem 1). Moreover, the idempotent

$$e_\psi := \frac{1}{|(\mathbf{Z}/M)^\times|} \sum_{d \in (\mathbf{Z}/M)^\times} \psi(d)^{-1} \langle d \rangle^* \in \mathsf{h}_{k+2}(M, R)$$

projects $S_{k+2}(M, R)$ onto $S_{k+2}(M, \psi, R)$ and $\mathsf{h}_{k+2}(M, R)$ onto $\mathsf{h}_{k+2}(M, \psi, R)$. Hence (4) extends to a perfect pairing

$$\begin{aligned} \langle\, ,\, \rangle_{\mathsf{h}} \colon S_{k+2}(M, R) \times \mathsf{h}_{k+2}(M, R) &\to R \\ \langle f, h \rangle_{\mathsf{h}} &:= a_1(f|h) \end{aligned} \tag{5}$$

for any R. Recall also Faltings comparison ([Fal87], Theorem 6): there is a canonical isomorphism

$$V \otimes_{\mathbf{Q}_p} \mathbf{C}_p \cong S_{k+2}(Np^s, \mathbf{C}_p)(-1-k) \oplus S_{k+2}(Np^s, \mathbf{C}_p)^\vee \tag{6}$$

for $\mathbf{C}_p$ the completion of an algebraic closure of $\mathbf{Q}_p$, which is $G_{\mathbf{Q}_p}$ and $\mathsf{h}_{\mathrm{GL}_2}$ equivariant. In particular, the action of $\mathsf{h}_{\mathrm{GL}_2}$ on the left-hand-side through $\mathsf{h}(\mathcal{E}_s^k)$ coincides with the action on the right-hand-side through $\mathsf{h}_{k+2}(Np^s, \mathbf{C}_p)$.

4.4. If f is a newform (that is, a new normalized eigenform) in $S_{k+2}(Np^s, \psi, \mathbf{C})$, we write $\mathbf{Q}(f)$, $\mathbf{Q}_p(f)$ for the finite extensions of $\mathbf{Q}$, $\mathbf{Q}_p$ respectively, generated by the Fourier coefficients of f. Since (5) is perfect, the decomposition of $S_{k+2}(Np^s, \mathbf{C})$ as direct sum of the space of oldforms and of eigenspaces attached to newforms reflects into a direct sum decomposition of $\mathsf{h}_{k+2}(Np^s, \mathbf{C})$. One checks easily that this is a direct sum as algebras, not just as vector spaces. Hence, there are idempotents $e_{\mathrm{old}} \in \mathsf{h}_{k+2}(Np^s, \mathbf{Q})$ and $e_f \in \mathsf{h}_{k+2}(Np^s, \mathbf{Q}(f))$ for all newforms f, which project onto the respective direct summands. Note that, as an element of $S_{k+2}(Np^s, \mathbf{Q}(f))^\vee$, e_f coincides with $(-, f)/(f, f)$, where $(\, , \,)$ denotes the Petersson product. We denote by the suffix $[f]$ the image of e_f in whichever space $\mathsf{h}(Np^s, \mathbf{Q}(f))$ acts on. In particular, (5) gives an identification

$$e_f \cdot \mathsf{h}_{k+2}(Np^s, \mathbf{Q}(f)) = S_{k+2}(Np^s, \mathbf{Q}(f))[f]^\vee.$$

4.5. In [Del69], Deligne proved that

$$V_f := V \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)[f]$$

is a Galois representation associated to f in the following sense: it is a two-dimensional, continuous $\mathbf{Q}_p(f)$-representation of $G_{\mathbf{Q}}$, unramified at all primes $l \nmid Np$ and any arithmetic Frobenius element $\mathrm{Frob}_l \in G_{\bar{\mathbf{Q}}}$ at such a prime satisfies $\mathrm{Tr}(\mathrm{Frob}_l^{-1} \mid V_f) = a_l(f)$ and $\det(\mathrm{Frob}_l^{-1} \mid V_f) = l^{k+1}\psi(l)$. In particular, $\det V_f = \chi_{\mathrm{cyc}}^{-1-k}\psi^{-1}$, where χ_{cyc} is the p-adic cyclotomic character and ψ is interpreted as a character of the Galois group $G_{\mathbf{Q}(\zeta_{Np^s})/\mathbf{Q}}$ (cf. [Rib77]).

4.6. The polynomial $X^2 - a_p(f)X + p^{k+1}\psi(p)$ is called the Hecke polynomial of f at p. When one of its roots is a p-adic unit, we say f is ordinary (at p) and we denote such a root by α_f. If f is ordinary and $s > 0$, the restriction of V_f to $G_{\mathbf{Q}_p}$ admits the decomposition

$$0 \to \epsilon_u \to V_f \to \epsilon_{\mathrm{r}} \to 0 \tag{7}$$

where $\epsilon_r = \chi_{\mathrm{cyc}}^{-1-k}\psi^{-1}\epsilon_u^{-1}$ and ϵ_u is the unramified character of $G_{\mathbf{Q}_p}$ given by $\epsilon_u(\mathrm{Frob}_p^{-1}) = \alpha_f$, as shown in [MW86]. We fix the notation $\beta_f = p^{k+1}\psi(p|1){\alpha_f}^{-1}$ where, consistently with our notation for diamond operators, $\psi(a|b)$ stands for the evaluation of ψ at an integer with residue a mod N and b mod p^s. Notice that when $s = 0$, β_f is the other root of the Hecke polynomial, while when $s > 0$ we have $\alpha_f = a_p(f)$.

4.7. Let us analyze the p-adic Hodge theory of V_f. Assume first that $s > 0$ and let $K = \mathbf{Q}_p(\zeta_{p^s})$ (this notation will stick throughout the text). Notice that ϵ_r becomes crystalline when restricted to G_K, hence V_f is a semistable representation of G_K, being a de Rham extension of two semistable representations ([Ber02], Theorem 0.8). We have a short exact sequence of (φ, N)-modules

$$0 \to D^K_{\mathrm{cris}}(\epsilon_u) \to D^K_{\mathrm{st}}(V_{f,\mathbf{Q}_p(f,\alpha_f)}) \to D^K_{\mathrm{cris}}(\epsilon_r) \to 0 \tag{8}$$

where

$$D^K_{\mathrm{cris}}(-) = (- \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})^{G_K} \quad \text{and} \quad D^K_{\mathrm{st}}(-) = (- \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{G_K}$$

with B_{cris} and B_{st} Fontaine's rings of crystalline and semistable periods respectively. The eigenvalues of φ on $D^K_{\mathrm{cris}}(\epsilon_u)$ and $D^K_{\mathrm{cris}}(\epsilon_r)$ (as $\mathbf{Q}_p(f, \alpha_f)$-vector spaces) are α_f and β_f respectively, hence these are also the eigenvalues of φ on $D^K_{\mathrm{st}}(V_f)$.

In the case $s = 0$, V_f is a crystalline representation of $G_{\mathbf{Q}_p}$ because it comes from the cohomology of a variety with good reduction at p (no ordinarity assumption needed). The eigenvalues of φ on $D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)$ are still α_f and β_f ([Sch90], Theorem 1.2.4).

We conclude this section with the following observation.

PROPOSITION 8. *If $f \in S_{k+2}(Np^s, \psi, \bar{\mathbf{Q}})$ is an ordinary newform, with $k \geq 0$ and ψ is primitive at p, that is, p^s divides the conductor of ψ, then V_f is a crystalline representation of G_K.*

Proof: The case $s = 0$ is ok. If $s > 0$, we have to show that on $D^K_{\mathrm{st}}(V_f)$ the monodromy operator N is trivial. Under our assumption on the character, the complex absolute value of $a_p(f)$ is $|a_p(f)|_{\mathbf{C}} = p^{(k+1)/2}$ ([Li75], Theorem 3), hence both the eigenvalues of φ on $D^K_{\mathrm{st}}(V_f)$, as computed above, have complex absolute value $p^{(k+1)/2}$. From the equality $N\varphi = p\varphi N$, we see that N sends a non-trivial φ-eigenspace of eigenvalue α to a (possibly trivial) φ-eigenspace of eigenvalue α/p. But then $|\alpha/p|_{\mathbf{C}} = p^{(k-1)/2} \neq p^{(k+1)/2}$, thus N must be trivial. $\square$

CHAPTER II

Modular forms and rigid differentials

1. Modular forms and de Rham cohomology

There is a natural relation between modular forms and the de Rham cohomology of modular curves and Kuga-Sato varieties, which we describe in this section. We detail as well the link with the étale theory, via Fontaine's de Rham functor. In the last paragraph, we finally deal with the p-adic Hodge theory of finite slope newforms.

1.1. For the rest of this work, we let $K = \mathbf{Q}_p(\zeta_{p^s})$. Let also B_{dR} be Fontaine's ring of de Rham periods and $D_{\mathrm{dR}}^K(-) = (- \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_K}$. The comparison theorem of Faltings and Tsuji gives an isomorphism of filtered K-modules

$$D_{\mathrm{dR}}^K(H_{\text{ét}}^{k+1}(\bar{\mathcal{E}}_{s,\bar{\mathbf{Q}}}^k, \mathbf{Q}_p)) \cong H_{\mathrm{dR}}^{k+1}(\bar{\mathcal{E}}_s^k) \otimes K =: H_{\mathrm{dR}}^{k+1}(\bar{\mathcal{E}}_s^k/K)$$

from which we get

$$D_{\mathrm{dR}}^K(V) = e_{\mathrm{mod}} H_{\mathrm{dR}}^{k+1}(\bar{\mathcal{E}}_s^k/K)$$

and

$$D_{\mathrm{dR}}^K(V_f) = D_{\mathrm{dR}}^K(V) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)[f] \cong e_{\mathrm{mod}} H_{\mathrm{dR}}^{k+1}(\bar{\mathcal{E}}_s^k/K) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)[f].$$

1.2. Using the Leray spectral sequence, one can give a more detailed description of $e_{\mathrm{mod}} H_{\mathrm{dR}}^{k+1}(\bar{\mathcal{E}}_s^k)$, for example as in [Sch85], which we recall here. Let $\boldsymbol{\omega} := e^*\Omega^1_{\mathcal{E}_s/Y_s}$ be the line bundle on Y_s obtained pulling back the sheaf of relative differentials $\Omega^1_{\mathcal{E}_s/Y_s}$ on $\mathcal{E}_s$ through the zero section $e\colon Y_s \to \mathcal{E}_s$ and denote again by $\boldsymbol{\omega}$ its natural extension to X_s (cf. [Kat73], §1.5 or [KM85], §10.13). Let

$$\psi_0 \colon \operatorname{Spec}(\mathbf{Q}(\zeta_{Np^s})((q))) \to X_s$$

be the formal punctured neighborhood of the cusp ∞ given by the datum $(\mathrm{Tate}(q), \zeta_{Np^s})$ and

$$\psi \colon \operatorname{Spec}(\mathbf{Q}(\zeta_{Np^s})[[q]]) \to X_s$$

its extension to the cusp. The line bundle $\psi_0^*\boldsymbol{\omega}$ is trivial, and we fix the generator ω given by the canonical differential on the Tate curve. Let us denote by $\boldsymbol{\mathcal{L}} := R^1u_*\Omega^\bullet_{\mathcal{E}_s/Y_s}$ the first relative de Rham cohomology sheaf of $\mathcal{E}_s$ and write $\boldsymbol{\mathcal{L}}_k := \operatorname{Sym}^k \boldsymbol{\mathcal{L}}$ for its symmetric powers. This sheaf is equipped with a Hodge filtration

$$0 \to \boldsymbol{\omega} \to \boldsymbol{\mathcal{L}} \to \boldsymbol{\omega}^{-1} \to 0 \tag{9}$$

which gives rise to a k-step filtration on $\boldsymbol{\mathcal{L}}_k$ such that $\operatorname{Fil}^0 \boldsymbol{\mathcal{L}}_k = \boldsymbol{\mathcal{L}}_k$, $\operatorname{Fil}^k \boldsymbol{\mathcal{L}}_k = \boldsymbol{\omega}^k$. Around the cusp ∞ we have

$$\psi_0^*\boldsymbol{\mathcal{L}} = \mathbf{Q}(\zeta_{Np^s})((q)) \cdot \omega \oplus \mathbf{Q}(\zeta_{Np^s})((q)) \cdot \eta \tag{10}$$

where η is the basis dual to ω. We can extend $\boldsymbol{\mathcal{L}}$ to a sheaf on X_s by declaring $\{\omega, \eta\}$ to be a basis for $\psi^*\boldsymbol{\mathcal{L}}$ over $\mathbf{Q}(\zeta_{Np^s})[[q]]$ and repeating the above at all other cusps. Then (9) extends to a short exact sequence of sheaves on X_s. The Gauss-Manin connection on Y_s, $\nabla\colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}} \otimes \Omega^1_{Y_s}$, also extends to X_s giving a connection $\nabla\colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}} \otimes \Omega^1_{X_s}(\log)$ with logarithmic poles along the divisor of cusps, with

$$\nabla(\omega) = \eta\frac{dq}{q}, \qquad \nabla(\eta) = 0.$$

around ∞. It extends to a connection on $\boldsymbol{\mathcal{L}}_k$ by taking symmetric powers, satisfies Griffiths transversality

$$\nabla \operatorname{Fil}^{i+1} \boldsymbol{\mathcal{L}}_k \subseteq \operatorname{Fil}^i \boldsymbol{\mathcal{L}}_k \otimes \Omega^1_{X_s}(\log)$$

and induces isomorphisms between the associated graded pieces:

$$\nabla\colon\ \operatorname{Gr}^{i+1} \boldsymbol{\mathcal{L}}_k \xrightarrow{\sim} \operatorname{Gr}^i \boldsymbol{\mathcal{L}}_k \otimes \Omega^1_{X_s}(\log), \quad 0 \le i < k. \tag{11}$$

For any $\mathbf{Q}$-algebra R and $k \geq 0$, we have equalities

$$S_{k+2}(Np^s, R) = H^0(X_s, \boldsymbol{\omega}^k \otimes \Omega^1_{X_s}) \otimes R = H^0(X_s, \boldsymbol{\omega}^k \otimes \Omega^1_{X_s} \otimes R)$$

and for $f \in S_{k+2}(Np^s, R)$ we have

$$\psi^* f = f(q)\omega^k \frac{dq}{q} \in H^0(\operatorname{Spec}(\mathbf{Q}(\zeta_{Np^s})[[q]]), \boldsymbol{\omega}^k \otimes \Omega^1_{X_s} \otimes R). \tag{12}$$

Let $\Omega^\bullet_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k) := \boldsymbol{\mathcal{L}}_k \otimes \Omega^\bullet_{X_s}(\log)$ and define a subcomplex of sheaves $\Omega^\bullet_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)$ as follows:

$$\Omega^0_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k) := \boldsymbol{\mathcal{L}}_k = \Omega^0_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k), \qquad \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k) := \nabla(\boldsymbol{\mathcal{L}}_k) + \boldsymbol{\mathcal{L}}_k \otimes \Omega^1_{X_s} \subseteq \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k).$$

When restricted to Y_s, $\Omega^\bullet_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)$ and $\Omega^\bullet_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)$ coincide, while at the cusp we have

$$\psi^*\Omega^0_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k) = \psi^*\Omega^0_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k) = \bigoplus_{s=0}^{k} \mathbf{Q}(\zeta_{Np^s})[[q]]\omega^{k-s}\eta^s \tag{13}$$

$$\psi^*\Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k) = \mathbf{Q}(\zeta_{Np^s})[[q]]\omega^k dq \oplus \bigoplus_{s=1}^{k} \mathbf{Q}(\zeta_{Np^s})[[q]]\omega^{k-s}\eta^s \frac{dq}{q} \tag{14}$$

We write

$$H^\bullet_{\mathrm{dR}}(X_s, \boldsymbol{\mathcal{L}}_k) = H^\bullet(X_s, \Omega^\bullet_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k))$$
$$H^\bullet_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) = H^\bullet(X_s, \Omega^\bullet_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k))$$

for the hypercohomologies of $\Omega^\bullet_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)$ and $\Omega^\bullet_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)$. Notice that in degree zero they coincide and in degree one the latter is a subgroup of the former. There is a two-step filtration

$$0 \to S_{k+2}(Np^s, \mathbf{Q}) \to H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \to S_{k+2}(Np^s, \mathbf{Q})^\vee \to 0 \tag{15}$$

([Sch85], Theorem 2.7) which allows us to associate to any cusp form $f \in S_{k+2}(Np^s, R)$ a cohomology class $\omega_f \in H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes R$. Moreover, the Leray spectral sequence ([Kat70],

§3.3) together with (6) and (15) yields the equalities

$$(16)\qquad \begin{aligned} D^K_{\mathrm{dR}}(V) &= H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes K \\ D^K_{\mathrm{dR}}(V_f) &= H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)[f] \end{aligned}$$

of filtered K-vector spaces, for all newforms $f \in S_{k+2}(Np^s, \mathbf{C})$.

1.3. As in the previous paragraph we introduced the cohomology class ω_f associated to a newform f, we now want to introduce a cohomology class which is dual to f in a certain sense. Notice that, for f ordinary, $D^K_{\mathrm{dR}}(\epsilon_u)$ is concentrated in the 0-th step of its filtration, while $D^K_{\mathrm{dR}}(\epsilon_r)$ is concentrated in the $(k+1)$-th one. Thus, taking the associated graded of (8) $\otimes_{\mathbf{Q}_p} K$ we get a splitting of the $[f]$-part of (15):

$$(17)\qquad D^K_{\mathrm{dR}}(V_f) = S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))[f] \oplus D^K_{\mathrm{dR}}(V_f)^{\varphi = \alpha_f}$$

where $S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))^\vee[f]$ is identified to $D^K_{\mathrm{dR}}(V_f)^{\varphi=\alpha_f}$.

Recall that if $f \in S_{k+2}(Np^s, \psi, \mathbf{C})$ is a newform, then $\mathbf{Q}(f)$ is either a totally real number field or a quadratic totally imaginary extension of such ([Rib77], Proposition 3.2). Let $f^c \in S_{k+2}(Np^s, \mathbf{Q}(f))$ denote the newform whose Fourier coefficients are complex conjugates to those of f:

$$f^c(q) = \sum_{n>0} \overline{a_n(f)} q^n.$$

One sees readily that

$$e_{f^c} = (e_f)^\iota \quad \text{and} \quad S_{k+2}(Np^s, \mathbf{Q}(f))^\vee[f] = S_{k+2}(Np^s, \mathbf{Q}(f))[f^c]^\vee.$$

The Poincaré pairing on $H^{k+1}_{\text{ét}}(\bar{\mathcal{E}}^k_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ induces a perfect pairing

$$(18)\qquad V_f \times V_{f^c}(k+1) \to \mathbf{Q}_p(f)$$

so that $V_{f^c} = V_f^*(-1-k)$, where V_f^* denotes the dual representation $\mathrm{Hom}_{\mathbf{Q}_p(f)}(V_f, \mathbf{Q}_p(f))$. Hence, if f is ordinary, we deduce from (7) a similar decomposition for V_{f^c}, we see the eigenvalues of φ on $D^K_{\mathrm{st}}(V_{f^c})$ are $\psi(p|1)^{-1}\alpha_f$ and $\psi(p|1)^{-1}\beta_f$, and we have

$$(19)\qquad D^K_{\mathrm{dR}}(V_{f^c}) = S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))[f^c] \oplus D^K_{\mathrm{dR}}(V_{f^c})^{\varphi = \psi(p|1)^{-1}\alpha_f}$$

where $S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))^\vee[f^c]$ is identified to $D^K_{\mathrm{dR}}(V_{f^c})^{\varphi=\psi(p|1)^{-1}\alpha_f}$. The proof of Proposition 8 generalizes as well, showing that if the nebentypus ψ is primitive at p then V_{f^c} is a crystalline representation of G_K.

We eventually define the following cohomology classes. For $f \in S_{k+2}(Np^s, \mathbf{C})$ an ordinary newform, we let $\eta^\alpha_f \in D^K_{\mathrm{dR}}(V_{f^c})^{\varphi=\psi(p|1)^{-1}\alpha_f}$ be the image of the linear functional $(-, f)/(f, f)$ via the splitting (19) and $\eta^\alpha_{f^c} \in D^K_{\mathrm{dR}}(V_f)^{\varphi=\alpha_f}$ be the image of $(-, f^c)/(f^c, f^c)$ via the splitting (17). When thinking of duals in term of the pairing (5), these classes identify with e_f and e_{f^c} respectively.

1.4. Our aim in this paragraph is to generalize the above definitions allowing the newform f to come from a lower level. Recall that for M, d positive integers such that $M \mid Np^s$ and $d \mid \frac{Np^s}{M}$, there are finite maps $\varpi_d \colon X_s \to X_1(M)$. Writing $\pi \colon \mathcal{E} \to \mathcal{E}/C_d$ for the natural projection, we get maps

$$\pi^* \varpi_d^{-1} \colon H^1_{\mathrm{par}}(X_1(M), \boldsymbol{\mathcal{L}}_k) \to H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)$$

which we will also denote by ϖ_d^* to lighten notation. If $f \in S_{k+2}(M, \bar{\mathbf{Q}})$ has q-expansion $f(q) = \sum a_n q^n$, then the q-expansion of $\varpi_d^* f \in S_{k+2}(Np^s, \bar{\mathbf{Q}})$ is $(\varpi_d^* f)(q) = d^{k+1} \sum a_n q^{nd}$. There is a decomposition

$$H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes \bar{\mathbf{Q}} = \bigoplus_f \sum_{d \mid \frac{Np^s}{M_f}} \varpi_d^* \left(H^1_{\mathrm{par}}(X_1(M), \boldsymbol{\mathcal{L}}_k) \otimes \bar{\mathbf{Q}}[f] \right) \tag{20}$$

where the first sum runs over the newforms $f \in S_{k+2}(M_f, \mathbf{C})$ of any level $M_f \mid Np^s$. The action of $\mathsf{h}_{k+2}(Np^s, \bar{\mathbf{Q}})$ respects the first sum, and while diamond operators commute with the ϖ_d^*, in general the operator T_l for l prime does so only when $l \nmid \frac{Np^s}{M}$ ([DS06], proof of Proposition 5.6.2). In particular, the decomposition of $S_{k+2}(Np^s, \bar{\mathbf{Q}})$ through the first sum reflects via (5) into a decomposition of $\mathsf{h}_{k+2}(Np^s, \bar{\mathbf{Q}})$ (the summand of $\mathsf{h}_{k+2}(Np^s, \bar{\mathbf{Q}})$ corresponding to a choice of f consists of the functionals which annihilate all summands of $S_{k+2}(Np^s, \bar{\mathbf{Q}})$ corresponding to newforms $\neq f$). One checks easily that this splits $\mathsf{h}_{k+2}(Np^s, \bar{\mathbf{Q}})$ as a direct sum of algebras, not just of $\bar{\mathbf{Q}}$-vector spaces, hence there are idempotents $e_f \in \mathsf{h}_{k+2}(Np^s, \mathbf{Q}(f))$ corresponding to each summand and, denoting by the suffix $[f]$ the image of e_f in whichever space $\mathsf{h}_{k+2}(Np^s, \mathbf{Q}(f))$ acts on, we can rewrite (20) as

$$H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes \bar{\mathbf{Q}} = \bigoplus_f H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes \bar{\mathbf{Q}}[f]$$

with

$$H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes \mathbf{Q}(f)[f] = \sum_{d \mid \frac{Np^s}{M_f}} \varpi_d^* \left(H^1_{\mathrm{par}}(X_1(M_f), \boldsymbol{\mathcal{L}}_k) \otimes \mathbf{Q}(f)[f] \right).$$

In harmony with the previous situation, we have $D^K_{\mathrm{dR}}(V(Np^s)_f) := H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)[f]$ for $V(Np^s)_f = V(Np^s) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f)[f]$.

Let now f be a newform of level M_f for some $M_f \mid Np^s$ and nebentypus ψ, and assume that f is ordinary. Then the $[f]$-part of the filtration (15) splits:

$$D^K_{\mathrm{dR}}(V(Np^s)_f) = S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))[f] \oplus D^K_{\mathrm{dR}}(V(Np^s)_f)^{\varphi = \alpha_f} =$$

$$= \left(\sum_{d \mid \frac{Np^s}{M_f}} \varpi_d^* S_{k+2}(M_f, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))[f] \right) \oplus \left(\sum_{d \mid \frac{Np^s}{M_f}} \varpi_d^* D^K_{\mathrm{dR}}(V_f)^{\varphi = \alpha_f} \right)$$

where $\varpi_d^* \left(S_{k+2}(M_f, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))^\vee [f] \right)$ is identified to $\varpi_d^* D^K_{\mathrm{dR}}(V_f)^{\varphi = \alpha_f}$. Analogously, we have a splitting

$$D^K_{\mathrm{dR}}(V(Np^s)_{f^c}) = S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))[f^c] \oplus D^K_{\mathrm{dR}}(V(Np^s)_{f^c})^{\varphi = \psi(p|1)^{-1} \alpha_f}$$

where $S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))^\vee[f^c]$ is identified to $D^K_{\mathrm{dR}}(V(Np^s)_{f^c})^{\varphi=\psi(p|1)^{-1}\alpha_f}$. Now for any $\breve{f} \in S_{k+2}(Np^s, K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(f))[f]$ we can define $\eta^\alpha_{\breve{f}} \in D^K_{\mathrm{dR}}(V(Np^s)_{f^c})^{\varphi=\psi(p|1)^{-1}\alpha_f}$ as the image of the functional $(-, \breve{f})/(f, f)$ and $\eta^\alpha_{\breve{f}^c} \in D^K_{\mathrm{dR}}(V(Np^s)_f)^{\varphi=\alpha_f}$ as the image of $(-, \breve{f}^c)/(f^c, f^c)$ (cf. [DR14], Lemma 2.12).

1.5. We will now relax the ordinarity condition and, at the same time, explain how the classes η are actually defined over $\mathbf{Q}_p$, not just K. In order to do so, we need to take a closer look to the p-adic Hodge theory of our modular form. Let f be a newform of weight k, level M_f for some $M_f \mid Np^s$ and nebentypus ψ and let, compatibly with earlier notation, $\alpha_f = a_p(f)$, $\beta_f = p^{k+1}\psi(p|1)\alpha_f^{-1}$ when $p \mid M_f$, and α_f, β_f be the roots of $X^2 - a_p(f)X + p^{k+1}\psi(p)$ with $0 \le \mathrm{ord}_p\, \alpha_f \le \mathrm{ord}_p\, \beta_f$ when $p \nmid M_f$. The local-global compatibility proved in [Sai97] implies that: if $p \mid M_f$ and $\alpha_f \neq 0$, $D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)$ is a 1-dimensional vector space over $\mathbf{Q}_p(f)$ with $\varphi = \alpha_f$, while if $p \nmid M_f$, it has dimension 2 and the eigenvalues of φ are α_f, β_f. Assuming $\beta_f \neq \alpha_f$ if $p \nmid M_f$ and $\alpha_f \neq 0$ if $p \mid M_f$, we see that $D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\alpha_f}$ is a $\mathbf{Q}_p(f)$-line in $D^{\mathbf{Q}_p}_{\mathrm{dR}}(V_f) = H^1_{\mathrm{par}}(X_1(M_f), \boldsymbol{\mathcal{L}}_k) \otimes \mathbf{Q}_p(f)[f]$. The Hodge-Tate weights of $D^{\mathbf{Q}_p}_{\mathrm{dR}}(V_f)$ are $0, k+1$ ([Fal87]) and we bring our attention to the intersection of this line with the $(k+1)$-th step of the filtration: if this intersection is trivial we obtain, as in the ordinary case, a splitting of the filtration.

Lemma 9. *Assume that $\alpha_f \neq \beta_f$ if $p \nmid M_f$ and that $\mathrm{ord}_p(\alpha_f) < k+1$ if $p \mid M_f$. Then we have $\mathrm{Fil}^{k+1}\, D^{\mathbf{Q}_p}_{cris}(V_f)^{\varphi=\alpha_f} = 0$.*

Proof: Let $E/\mathbf{Q}_p$ be a finite extension over which V_f becomes semistable and E_0 its maximal unramified subextension. If $\mathrm{Fil}^{k+1}\, D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\alpha_f} \neq 0$, then it must be the whole $D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\alpha_f}$ as both are $\mathbf{Q}_p(f)$-vector spaces and the bigger one has dimension one. Then $D := D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\alpha_f} \otimes_{\mathbf{Q}_p} E_0$ is a subobject of $D^E_{\mathrm{st}}(V_f)$ in the category of filtered (φ, N)-modules over E. Its Hodge and Newton numbers are easily seen to be

$$t_H(D) = (k+1) \cdot \dim_{E_0} D = (k+1) \cdot \dim_{\mathbf{Q}_p} \mathbf{Q}_p(f),$$
$$t_N(D) = \mathrm{ord}_p(N^{\mathbf{Q}_p(f)}_{\mathbf{Q}_p} \alpha_f) = \mathrm{ord}_p(\alpha_f) \cdot \dim_{\mathbf{Q}_p} \mathbf{Q}_p(f).$$

Since V_f is semistable over E, $D^E_{\mathrm{st}}(V_f)$ is admissible, hence $t_H(D) \le t_N(D)$, but this contradicts our assumptions. $\square$

Under the assumptions of the above lemma, we obtain a splitting of the filtration on $D^{\mathbf{Q}_p}_{\mathrm{dR}}(V_f)$,

$$D^{\mathbf{Q}_p}_{\mathrm{dR}}(V_f) = S_{k+2}(M_f, \mathbf{Q}_p(f))[f] \oplus D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\alpha_f}$$

where $S_{k+2}(M_f, \mathbf{Q}_p(f))^\vee[f]$ is identified to $D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\alpha_f}$, hence a splitting

$$D^{\mathbf{Q}_p}_{\mathrm{dR}}(V(Np^s)_f) = S_{k+2}(Np^s, \mathbf{Q}_p(f))[f] \oplus D^{\mathbf{Q}_p}_{\mathrm{cris}}(V(Np^s)_f)^{\varphi=\alpha_f}$$

which allows to define classes $\eta^\alpha_{\breve{f}^c}$ for every $\breve{f} \in S_{k+2}(Np^s, \mathbf{Q}_p(f))[f]$. Of course the same arguments produce a splitting of $D^{\mathbf{Q}_p}_{\mathrm{dR}}(V(Np^s)_{f^c})$, hence we also have classes $\eta^\alpha_{\breve{f}}$. We remark that in the case $p \nmid M_f$, $\mathrm{ord}_p(\beta_f) < k+1$ the proof of the above lemma shows also that $\mathrm{Fil}^{k+1}\, D^{\mathbf{Q}_p}_{\mathrm{cris}}(V_f)^{\varphi=\beta_f} = 0$, thus providing an alternative splitting which identifies

$S_{k+2}(M_f, \mathbf{Q}_p(f))^\vee[f]$ to $D_{\mathrm{cris}}^{\mathbf{Q}_p}(V_f)^{\varphi=\beta_f}$, giving rise to alternative classes $\eta^{\beta}_{\breve{f}^c}, \eta^{\beta}_{\breve{f}}$. This is still true even if $\mathrm{ord}_p(\beta_f) = k+1$: indeed, in that case we first obtain the statement for f^c and then take duals. Finally, taking a look back to the proof of Proposition 8, we see that if ψ is primitive at p, then V_f is potentially crystalline.

2. The rigid analytic viewpoint

One of the key ingredients to this work is an explicit description of the action of crystalline Frobenius on the cohomology of Kuga-Sato varieties. In order to obtain it, we need to compare the de Rham theory with its rigid counterpart, where the more flexible topology allows one to cut out a part of the variety whose geometry naturally offers such description.

2.1. We start by introducing the rigid spaces whose cohomology we are interested in. Most of the material in this paragraph can be found in [Col97] and ([BE05], §4.4). Let $\mathcal{X}_s$ be the proper, flat, regular model for $X_{s,\mathbf{Q}_p(\zeta_{p^s})}$ over $\mathcal{O}_K = \mathbf{Z}_p[\zeta_{p^s}]$ defined in [KM85], representing the moduli problem ([$\Gamma_1(N)$],[bal.$\Gamma_1(p^s)^{\mathrm{can}}$]). Its special fiber is a union of Igusa curves crossing at supersingular points. Exactly two of the components are isomorphic to $\mathrm{Ig}(Np^s)$, the curve representing the moduli problem ([$\Gamma_1(N)$],[$\mathrm{Ig}(p^s)$]) over $\mathbf{F}_p$. One of the two, which we denote by Ig_∞, contains the reduction of the cusp ∞, and the other of the two is denoted Ig_0. Assume now that N is big enough, so to guarantee that X_s and all irreducible components of the special fiber of $\mathcal{X}_s$ have genus at least 2. Let L be a finite extension of K over which X_s admits a stable model, let $\mathcal{O}_L$ be its ring of integers, and let $\mathcal{X}$ be such a stable model for $X_{s,L}$ with a birational map $\mathcal{X} \to \mathcal{X}_{s,\mathcal{O}_L}$. This map identifies two irreducible components of the special fiber of $\mathcal{X}$ with Ig_∞ and Ig_0, so we will call them by the same name. Let W_∞ and W_0 be the rigid spaces over L inverse images of respectively Ig_∞ and Ig_0 under the specialization map for $\mathcal{X}$. Then, for $i = 0, \infty$, we have L-vector spaces

$$H^1_{\mathrm{dR}}(W_i, \boldsymbol{\mathcal{L}}_k) = \frac{H^0(W_i, \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k))}{\nabla H^0(W_i, \Omega^0_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k))}, \quad H^1_{\mathrm{par}}(W_i, \boldsymbol{\mathcal{L}}_k) = \frac{H^0(W_i, \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k))}{\nabla H^0(W_i, \Omega^0_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k))}$$

and restriction maps

$$\mathrm{res}_i\colon H^\bullet_{\mathrm{dR}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes L \to H^\bullet_{\mathrm{dR}}(W_i, \boldsymbol{\mathcal{L}}_k),$$
$$\mathrm{res}_i\colon H^\bullet_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k) \otimes L \to H^\bullet_{\mathrm{par}}(W_i, \boldsymbol{\mathcal{L}}_k).$$

Let $H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}}$ be the primitive part of $H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)$ in the sense of [Col97]: explicitly, this is the $\mathbf{Q}$-subspace whose extension to $\bar{\mathbf{Q}}$ is spanned by

$$\varpi_d^*\left(H^1_{\mathrm{par}}(X_1(M_f), \boldsymbol{\mathcal{L}}_k) \otimes \bar{\mathbf{Q}}[f]\right)$$

where f runs over all newforms of level $M_f \mid Np^s$ with nebentypus ψ_f primitive at p (recall this means that the p-adic order of M_f equals that of the conductor of ψ_f) and $d \mid \frac{Np^s}{M_f}$

satisfies: $p \nmid d$ if $p \mid M_f$ and $\operatorname{ord}_p d \leq 1$ if $p \nmid M_f$. Then it is a theorem of Coleman (loc. cit., Theorem 2.1) that the map

$$\mathrm{res}_\infty \oplus \mathrm{res}_0 \colon H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}} \otimes L \xrightarrow{\sim} H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \oplus H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k)^*$$

is an isomorphism, where the apex $*$ denotes the pure subspaces, consisting of those classes whose restriction to all supersingular annuli (i.e. reduction inverses of singular points of the special fiber of $\mathcal{X}$) is trivial. Diamond operators and T_l operators for $l \nmid Np$ respect this decomposition. Observe that primitive classes are clearly trivial on all supersingular annuli: for old primitive classes coming from level N this is true because $X_1(N)$ has good reduction at p and for all those coming from a level multiple of p this is follows from the observation that p-part diamond operators act trivially on our cohomology groups restricted to supersingular annuli. Finally, we recall that w_s exchanges W_∞ and W_0, and induces an isomorphism $w_s^* \colon H^1_{\mathrm{dR}}(W_\infty, \boldsymbol{\mathcal{L}}_k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(W_0, \boldsymbol{\mathcal{L}}_k)$.

2.2. Let us introduce the ordinary subspaces $\mathcal{A}_\infty, \mathcal{A}_0 \subseteq X_s^{\mathrm{rig}}$, obtained from W_∞ and W_0 respectively by removing all supersingular annuli: the space $W_\infty \smallsetminus \mathcal{A}_\infty$ is a disjoint union of open rigid annuli, each one the reduction inverse of a singular point of the special fiber of $\mathcal{X}$. Choosing a local parameter in each of these annuli, one can build a system of wide open neighborhoods of $\mathcal{A}_\infty$: for $\varepsilon \in |L|_p \cap (0,1)$, $W_\infty[\varepsilon]$ is defined imposing the norm of each local parameter to be $> \varepsilon$. For ε close enough[1] to 1, $W_\infty[\varepsilon]$ does not depend on the choice of the local parameters, and henceforth we will only consider $W_\infty[\varepsilon]$ defined for such ε. We define $W_0[\varepsilon]$ analogously. We have cohomology groups

$$H^1_{\mathrm{dR}}(W_i[\varepsilon], \boldsymbol{\mathcal{L}}_k) = \frac{H^0(W_i[\varepsilon], \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k))}{\nabla H^0(W_i[\varepsilon], \Omega^0_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k))}, \quad H^1_{\mathrm{par}}(W_i[\varepsilon], \boldsymbol{\mathcal{L}}_k) = \frac{H^0(W_i[\varepsilon], \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k))}{\nabla H^0(W_i[\varepsilon], \Omega^0_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k))}$$

and we remark that, by the same argument as in ([Col95], §8), which relies on ([BC94], Theorem 2.4), the restriction maps

$$H^1_{\mathrm{dR}}(W_i, \boldsymbol{\mathcal{L}}_k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(W_i[\varepsilon], \boldsymbol{\mathcal{L}}_k), \quad H^1_{\mathrm{par}}(W_i, \boldsymbol{\mathcal{L}}_k) \xrightarrow{\sim} H^1_{\mathrm{par}}(W_i[\varepsilon], \boldsymbol{\mathcal{L}}_k) \tag{21}$$

are isomorphisms. For ε close enough to 1, there is a finite étale rank p map

$$\mathrm{Frob} \colon W_\infty[\varepsilon^p] \to W_\infty[\varepsilon]$$

which, when restricted to the formal neighborhood of ∞ given by the Tate curve, is the classical operator $V \colon f(q) \mapsto f(q^p)$ on q-expansions (cf. [Gou88], Proposition II.3.2). The map Frob admits a description in terms of moduli problem, which allows one to extend V to an endomorphism of $H^1_{\mathrm{dR}}(W_\infty, \boldsymbol{\mathcal{L}}_k)$; we will detail this in 2.4 below.

2.3. To obtain the desired description for crystalline Frobenius, we need to make use of the theory of canonical subgroups. Let R be a p-adically complete $\mathbf{Z}_p$-algebra, E an elliptic curve over R, E_{p-1} be the Eisenstein series of weight $p-1$, ω a basis of $\Omega^1_{E/R}$, and $r, Y \in R$ such that $Y \cdot E_{p-1}(E, \omega) = r$, thinking of E_{p-1} as a modular form à la Katz. Assuming $\operatorname{ord}_p r < \frac{p}{p+1}$, ([Kat73], Theorem 3.1) defines a finite flat rank p

[1] One can always replace L by a finite extension, so to be able to pick ε as close to 1 as needed.

subgroup scheme of E, called the canonical subgroup, which we denote by $\mathcal{K}$. Let us write $E^{(0)} = E$ and inductively, for $n > 0$, $E^{(n)} = E^{(n-1)}/\mathcal{K}$, assuming $E^{(n-1)}$ admits a canonical subgroup. We will denote by $F^{(n)}$ the isogeny $E \to E^{(n)}$, whenever $E^{(n)}$ is defined, that is, when $\operatorname{ord}_p r < 1/(p+1)p^{n-2}$. When R is the valuation ring of a complete discretely valued subfield of $\mathbf{C}_p$, one can set $Y = 1$ and $r = E_{p-1}(E,\omega)$, so that $E^{(n)}$ and $F^{(n)}$ are defined if $|E_{p-1}(E,\omega)|_p > p^{-1/(p+1)p^{n-2}}$.

2.4. We discuss now local Frobenius maps on X_s^{rig}/K (cf. [Col97], §1). For $i = 1,2$, let $W_i(p)$ be the connected component of ∞ in the rigid subspace of X_1^{rig} defined by $|E_{p-1}|_p > p^{-p^{2-i}/(p+1)}$ (cf. [Col95], §1 and §2). The Deligne-Tate map $\Phi_\infty\colon W_2(p) \to W_1(p)$ sends a point (E,ι_N,ι_p) to $(E^{(1)}, F^{(1)}\circ\iota_N, \iota_p')$ where $\iota_p'(1)$ is the image of any point in $\ker F^{(2)}$ whose p-th multiple is $\iota_p(1)$. Let $W_i(p^s)$ be the rigid subspace of X_s^{rig} inverse image of $\Phi_\infty^{1-s} W_i(p)$ via the map $\varpi_{p^{s-1}}\colon X_s \to X_1$. The points of $W_1(p^s)$ are classes (E,ι_N,ι_p) such that $E^{(s)}$ is defined and $\operatorname{Im}\iota_p = \ker F^{(s)}$. The Deligne-Tate map lifts to $\Phi_\infty\colon W_2(p^s) \to W_1(p^s)$ sending the class of a triple (E,ι_N,ι_p) to that of $(E^{(1)}, F^{(1)}\circ\iota_N,\iota_p')$ where $\iota_p'(1)$ is the image of any point in $\ker F^{(s+1)}$ whose p-th multiple is $\iota_p(1)$. Testing it on the universal Tate curve we get

$$\Phi_\infty(\mathrm{Tate}(q),\zeta_N,\zeta_{p^s}) = (\mathrm{Tate}(q)/\mu_p, \zeta_N \bmod \mu_p, \zeta_{p^{s+1}} \bmod \mu_p) = (\mathrm{Tate}(q^p), \zeta_N^p, \zeta_{p^s}),$$

thus $\langle p|1\rangle^{-1}\Phi_\infty$ restricts, on the formal neighborhood $\operatorname{Spec}(\mathbf{Q}(\zeta_{Np^s})((q)))$ of the cusp ∞, to the endomorphism $q \mapsto q^p$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_s \arrow[r, "\pi"] \arrow[d] & \mathcal{E}_s/\mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_s \arrow[d] \\
W_2(p^s) \arrow[r, equal] & W_2(p^s) \arrow[r, "\Phi_\infty"] & W_1(p^s)
\end{tikzcd}$$

of rigid spaces over K in which the rightmost square is cartesian. Pulling back through the diagram yields a map between relative de Rham cohomologies $\pi^*\Phi_\infty^{-1}\colon \mathcal{H}^1(\mathcal{E}_s/W_1(p^s)) \to \mathcal{H}^1(\mathcal{E}_s/W_2(p^s))$, hence a map

$$\tilde{\varphi}_\infty := \pi^*\Phi_\infty^{-1}\colon H^0(W_1(p^s), \Omega^\bullet_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)) \to H^0(W_2(p^s), \Omega^\bullet_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k))$$

which in turn induces

$$\varphi_\infty\colon H^1_{\mathrm{par}}(W_1(p^s), \boldsymbol{\mathcal{L}}_k) \to H^1_{\mathrm{par}}(W_2(p^s), \boldsymbol{\mathcal{L}}_k)$$

which is exactly the K-linear extension of the Frobenius from the $\mathbf{Q}_p$-structure induced by log-crystalline cohomology (cf. [CI10]). By the discussion in 2.2,

$$H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k) \xrightarrow{\sim} H^1_{\mathrm{par}}(W_1(p^s), \boldsymbol{\mathcal{L}}_k) \otimes_K L \xrightarrow{\sim} H^1_{\mathrm{par}}(W_2(p^s)), \boldsymbol{\mathcal{L}}_k) \otimes_K L$$

so that φ_∞ induces an L-linear endomorphism of $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)$. We can describe somewhat explicitly $\tilde{\varphi}_\infty \circ \langle p^{-1}|1\rangle^*$ around the cusp ∞:

$$\tilde{\varphi}_\infty \langle p^{-1}|1\rangle^* \left(f(q)\frac{dq}{q}\omega^i\eta^{k-i}\right) = \pi_T^* \langle p^{-1}|1\rangle^* \Phi_\infty^* \left(f(q)\frac{dq}{q}\omega^i\eta^{k-i}\right) =$$
$$= \pi_T^* \left(f(q^p)\frac{dq^p}{q^p}(\check{\pi}_T^*\omega)^i(\frac{1}{p}\check{\pi}_T^*\eta)^{k-i}\right) = \left(pf(q^p)\frac{dq}{q}(p\omega)^i\eta^{k-i}\right) =$$
$$= p^{i+1}(f(q^p)\frac{dq}{q}\omega^i\eta^{k-i}) = p^{i+1}V(f(q)\frac{dq}{q}\omega^i\eta^{k-i})$$

where $V\colon f(q) \mapsto f(q^p)$ is the usual operator on q-expansions, and the equality

$$\langle p^{-1}|1\rangle^*\Phi_\infty^*\eta = \tfrac{1}{p}\check{\pi}_T^*\eta$$

is deduced by Poincaré duality from $\langle p^{-1}|1\rangle^*\Phi_\infty^*\omega = \check{\pi}_T^*\omega$ (cf. 1.8, Chapter I). In particular, the action of $\tilde{\varphi}_\infty$ on p-adic modular forms coincides with $p^{k+1}\langle p|1\rangle^* V$, so that it makes sense to write V for the endomorphism $p^{-k-1}\varphi_\infty\langle p^{-1}|1\rangle^*$ of $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)$. We obtain a similar construction on W_0 simply putting $\tilde{\varphi}_0 := (w_s^*)^{-1}\tilde{\varphi}_\infty w_s^*$, and we write $\tilde{\varphi} = \tilde{\varphi}_\infty \sqcup \tilde{\varphi}_0$ for the disjoint union of the two maps, which is meaningful on appropriate neighborhoods of $\mathcal{A} := \mathcal{A}_\infty \sqcup \mathcal{A}_0$. Observe that comparing the moduli descriptions of Φ_∞ and Frob (see [Gou88], II.2), it is clear that Frob $= \langle p^{-1}|1\rangle\Phi_\infty$. Moreover, the U operator on $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)$ defined in [Col97], which acts on p-adic modular forms by the usual rule, satisfies then $UV = 1$. In particular, U and V are inverses of each other on $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^*$.

2.5. In some cases, the eigenspaces for the crystalline Frobenius can be very conveniently described. Let $f \in S_{k+2}(M_f, \psi, \mathbf{Q}(f))$ be a newform, with $M_f \mid Np^s$, and ψ is primitive at p. Assume also $N \mid M_f$, the general situation being easily deduced from this one.

We consider first the case $p \mid M_f$. Then on $H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}} \otimes \bar{\mathbf{Q}}_p[f]$ the operator U acts by $\alpha_f = a_p(f) \neq 0$. The above formula for φ reveals that $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$ is the φ-eigenspace of eigenvalue β_f. It follows that $H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$ is the φ-eigenspace of eigenvalue α_f. In particular,

$$H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f] \cong D_{\mathrm{cris}}^{\mathbf{Q}_p}(V_f)^{\varphi=\alpha_f} \otimes_{\mathbf{Q}_p(f)} \bar{\mathbf{Q}}_p$$

where the right hand side is identified with a subspace of $H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}}$ via ϖ_1^*.

If $p \nmid M_f$ the situation is a bit different, as $H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}} \otimes \bar{\mathbf{Q}}_p[f]$ is now a $\bar{\mathbf{Q}}_p$-vector space of dimension 4, rather than 2. Let us assume $\alpha_f \neq \beta_f$. The operator U acts on this space with eigenvalues α_f, β_f, decomposing it into two summands of dimension 2. Since they are obtained as image of linear combinations of ϖ_1 and ϖ_p, these eigenspaces are stable under Frobenius. We claim that they intersect $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$ in one-dimensional subspaces. Indeed, if this were not the case, since $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$ is 2-dimensional and stable under U, it would coincide with one of the two eigenspaces for U. Thus, by the formula of the previous number, it would be a Frobenius eigenspace for the other eigenvalue. But it would also contain a modular form, that is, intersect non-trivially the smallest piece of the Hodge filtration, thus contradicting Lemma 9 (or the discussion following it). Hence, $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$ splits in two one-dimensional pieces, each belonging to a different U-eigenspace, and each stable by Frobenius, with eigenvalue different from the U-eigenvalue. Now, each U-eigenspace is direct sum of two Frobenius eigenspaces, one of which, as we just saw, lies in $H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$. This means that the two remaining ones must lie in $H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]$. Hence, we conclude that

$$H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}} \otimes \bar{\mathbf{Q}}_p[f]^{U=\alpha_f} = \\ = H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]^{\varphi=\beta_f} \oplus H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]^{\varphi=\alpha_f}$$

and

$$H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_k)^{\mathrm{prim}} \otimes \bar{\mathbf{Q}}_p[f]^{U=\beta_f} = \\ = H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]^{\varphi=\alpha_f} \oplus H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k)^* \otimes_L \bar{\mathbf{Q}}_p[f]^{\varphi=\beta_f}$$

where all summands on the right-hand-side are one-dimensional.

3. Nearly-overconvergent p-adic modular forms

Our computations will lead us to consider nearly-overconvergent p-adic modular forms. We recall here their definition and basic properties.

3.1. The direct limit along restriction maps

$$S^{\mathrm{oc}}_{k+2}(Np^s, \mathbf{C}_p) = \varinjlim_{\varepsilon<1} H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \boldsymbol{\omega}^k \otimes \Omega^1_{X_s})$$

is the space of overconvergent p-adic cusp forms of weight $k+2$, level Np^s, with coefficients in $\mathbf{C}_p$. Restriction to $\mathcal{A}_\infty$ gives an embedding of overconvergent modular forms into p-adic modular forms which preserves q-expansions. If L is any subfield of $\mathbf{C}_p$, then $S^{\mathrm{oc}}_{k+2}(Np^s, L)$ is the subspace of overconvergent forms whose q-expansion (defined via (12)) takes coefficients in L. It follows from (21) that we have a map

$$S^{\mathrm{oc}}_k(Np^s, \mathbf{C}_p) \to H^1_{\mathrm{par}}(W_\infty, \boldsymbol{\mathcal{L}}_k)$$

associating to an overconvergent form its cohomology class.

3.2. The direct limit

$$S^{\text{n-oc}}_{k+2}(Np^s, \mathbf{C}_p) := \varinjlim_{\varepsilon<1} H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}^k))$$

is the space of nearly-overconvergent modular forms. They admit q-expansion via (13). While Φ_∞ "expands" strict neighborhoods of $\mathcal{A}_\infty$, it restricts to an endomorphism of $\mathcal{A}_\infty$ itself. Thus, $\tilde{\varphi}_\infty$ produces a slope decomposition of $\boldsymbol{\mathcal{L}}$ over $\mathcal{A}_\infty$, that is, a splitting of (9). In particular, we get a projection $\boldsymbol{\mathcal{L}}_k \to \boldsymbol{\omega}^k$, which after restriction induces a map

$$e_{\text{n-oc}} \colon S^{\text{n-oc}}_{k+2}(Np^s, \mathbf{C}_p) \hookrightarrow H^0(\mathcal{A}_\infty, \boldsymbol{\omega}^k \otimes \Omega^1_{X_s})$$

which embeds nearly-overconvergent modular forms into the space of p-adic modular forms. On the level of q-expansions, this map is simply given by $\eta \mapsto 0$. For further details on the theory of nearly-overconvergent modular forms, we refer the reader to [Urb14].

3.3. Using inductively (11), it is easy to see that

$$H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)) = H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \boldsymbol{\omega}^k \otimes \Omega^1_{X_s}) + \nabla H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \boldsymbol{\mathcal{L}}_k)$$

and a quick check on q-expansions shows that Hida's ordinary projector $e_{\mathrm{ord}} = \varinjlim T_p^{n!}$ kills the space $e_{\text{n-oc}}\nabla H^0(W_\infty[\varepsilon], \boldsymbol{\mathcal{L}}_k)$ (cf. [DR14], Lemma 2.7). Combining this with Coleman's classicality result ([Col97], Theorem 1.1), we get a well defined map

$$e_{\mathrm{ord}} \colon S^{\text{n-oc}}_{k+2}(Np^s, \mathbf{C}_p) \to S_{k+2}(Np^s, \mathbf{C}_p)$$

or, to put it in other words, ordinary nearly-overconvergent cusp forms are classical.

More generally, in [Urb14] Urban defines an overconvergent projector

$$e_{\mathrm{oc}} \colon S^{\text{n-oc}}_{k+2}(Np^s, \mathbf{C}_p) \to S^{\mathrm{oc}}_{k+2}(Np^s, \mathbf{C}_p) \tag{22}$$

mimicking the construction of the holomorphic projection in the classic case of nearly-holomorphic modular forms. Let us see how this is done. Let

$$N_i := \operatorname{coker}\left(\operatorname{Fil}^{i+1} \boldsymbol{\mathcal{L}}_k \xrightarrow{\nabla} \operatorname{Fil}^i \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)\right)$$

and consider the following diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Fil}^{i+2} \boldsymbol{\mathcal{L}}_k & \longrightarrow & \operatorname{Fil}^{i+1} \boldsymbol{\mathcal{L}}_k & \longrightarrow & \operatorname{Gr}^{i+1} \boldsymbol{\mathcal{L}}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} & & \\
0 & \longrightarrow & \operatorname{Fil}^{i+1} \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k) & \longrightarrow & \operatorname{Fil}^{i} \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k) & \longrightarrow & \operatorname{Gr}^{i} \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k) & \longrightarrow & 0
\end{array}$$

For all $0 \leq i < k$, the rightmost vertical arrow is an isomorphism by (11), hence the snake lemma yields $N_i \cong N_{i+1}$. Notice that

$$N_0 = \frac{\Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)}{\nabla \operatorname{Fil}^1 \boldsymbol{\mathcal{L}}_k} = \frac{\Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)}{\nabla \boldsymbol{\mathcal{L}}_k}$$

because $\nabla \operatorname{Gr}^0 \boldsymbol{\mathcal{L}}_k = 0$ by Griffiths transversality, and

$$N_k = \frac{\operatorname{Fil}^k \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)}{\nabla \operatorname{Fil}^{k+1} \boldsymbol{\mathcal{L}}_k} = \operatorname{Fil}^k \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k)$$

because $\operatorname{Fil}^{k+1} \boldsymbol{\mathcal{L}}_k = 0$. If we replace Ω^1_{dR} with Ω^1_{par} everything works in the same way, because $\operatorname{Gr}^i \Omega^1_{\mathrm{dR}}(\boldsymbol{\mathcal{L}}_k) = \operatorname{Gr}^i \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)$ as long as $i \neq k$. Hence, we get a canonical surjection

$$\Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k) \twoheadrightarrow \operatorname{Fil}^k \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k) = \boldsymbol{\omega}^k \otimes \Omega^1_{X_s}$$

with kernel $\nabla \boldsymbol{\mathcal{L}}_k$, which in turn induces a map

$$H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)) \to H^0(W_{\infty,\mathbf{C}_p}[\varepsilon], \boldsymbol{\omega}^k \otimes \Omega^1_{X_s}).$$

Taking the direct limit over $\varepsilon < 1$, we obtain the overconvergent projector e_{oc} of (22). Notice that a nearly-overconvergent modular form and its overconvergent projection give rise to the same cohomology class, in particular they are interchangeable when computing cohomological pairings.

CHAPTER III

A formula for the p-adic L-function

1. The diagonal cycle and the p-adic Abel-Jacobi map

In this section we introduce the diagonal cycle appearing in our main statement, and construct the p-adic Abel-Jacobi map out of the Bloch-Kato logarithm of our Galois representations.

1.1. Let us fix a triple $\mathbf{r} = (k, l, m)$ of non-negative integers such that $r := \frac{k+l+m}{2} \in \mathbf{Z}$, and consider three subsets A, B, C of $\{1, \ldots, r\}$ of cardinalities k, l, m respectively, and such that every element of $\{1, \ldots, r\}$ belongs to exactly two of them; this is possible exactly when the triple of weights $(k+2, l+2, m+2)$ is balanced, that is, when each of the three weights is strictly less than the sum of the other two. We define a map $\varphi_A : \bar{\mathcal{E}}_s^r \to \bar{\mathcal{E}}_s^k$ which simply "forgets" the copies of $\mathcal{E}_s$ with index outside of A over the open modular curve, and extends to the cusps by the construction of $\bar{\mathcal{E}}_s^k$ and the universal property of blow-ups. Similarly, define maps φ_B and φ_C. Their product

$$\varphi_{ABC} : \bar{\mathcal{E}}_s^r \to \bar{\mathcal{E}}_s^{\mathbf{r}} := \bar{\mathcal{E}}_s^k \times \bar{\mathcal{E}}_s^l \times \bar{\mathcal{E}}_s^m$$

is then a closed immersion, whose image we denote by Δ_s and call the diagonal cycle.

1.2. Let us write $\mathrm{CH}^i(X)$ for the Chow group of rational equivalence classes of cycles of codimension i of a variety X, so that $[\Delta_s] \in \mathrm{CH}^{r+2}(\bar{\mathcal{E}}_s^{\mathbf{r}})$. The results of 2, Chapter I, together with the Künneth decomposition show that the projector $e_{\mathrm{mod}}^{\otimes 3}$ kills the cohomology groups $H^i_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ for all $i \neq 2r+3$. Hence, cutting the Hochschild-Serre spectral sequence of [Jan88] for $\bar{\mathcal{E}}_s^{\mathbf{r}}$ with $e_{\mathrm{mod}}^{\otimes 3}$ gives (as in loc. cit., 6.15.c) cohomology classes

$$\mathrm{cl}_{\text{ét}}([\Delta_s]) \in \mathrm{Fil}^1\, e_{\mathrm{mod}}^{\otimes 3} H^{2r+4}_{\mathrm{cont}}(\bar{\mathcal{E}}_s^{\mathbf{r}}, \mathbf{Q}_p(r+2))$$
$$\mathrm{AJ}_{\text{ét}}([\Delta_s]) \in H^1(\mathbf{Q}, e_{\mathrm{mod}}^{\otimes 3} H^{2r+3}_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2)))$$

which map one to the other under the quotient map $\mathrm{Fil}^1 \to \mathrm{Gr}^1$ induced by the spectral sequence. If we restrict from $\mathbf{Q}$ to $\mathbf{Q}_p$, Theorem A of [NN16] says the cycle class and Abel-Jacobi maps factor through syntomic cohomology:

$$\begin{tikzcd}
\mathrm{CH}^{r+2}(\bar{\mathcal{E}}_s^{\mathbf{r}}) \arrow[d, "\mathrm{cl}_{\mathrm{syn}}"] \arrow[dr, "\mathrm{AJ}_{\mathrm{syn}}"] & \\
\mathrm{Fil}^1\, e_{\mathrm{mod}}^{\otimes 3} H^{2r+4}_{\mathrm{syn}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\mathbf{Q}_p}, r+2) \arrow[r] \arrow[d, "\rho_{\mathrm{syn}}"] & H^1_{\mathrm{st}}(\mathbf{Q}_p, e_{\mathrm{mod}}^{\otimes 3} H^{2r+3}_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p}, \mathbf{Q}_p(r+2))) \arrow[d, hook] \\
\mathrm{Fil}^1\, e_{\mathrm{mod}}^{\otimes 3} H^{2r+4}_{\mathrm{cont}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\mathbf{Q}_p}, \mathbf{Q}_p(r+2)) \arrow[r] & H^1(\mathbf{Q}_p, e_{\mathrm{mod}}^{\otimes 3} H^{2r+3}_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p}, \mathbf{Q}_p(r+2)))
\end{tikzcd}$$

where $H^1_{\mathrm{st}}(\mathbf{Q}_p, W) = \ker(H^1(\mathbf{Q}_p, W) \to H^1(\mathbf{Q}_p, W \otimes_{\mathbf{Q}_p} B_{\mathrm{st}}))$ for W any p-adic representation of the absolute Galois group of $\mathbf{Q}_p$.

Let now

$$f \in S_{k+2}(M_f, \psi_f, \bar{\mathbf{Q}}_p), \quad g \in S_{l+2}(M_g, \psi_g, \bar{\mathbf{Q}}_p), \quad h \in S_{m+2}(M_h, \psi_h, \bar{\mathbf{Q}}_p)$$

be newforms with $M_f, M_g, M_h \mid Np^s$, ψ_f, ψ_g, ψ_h primitive at p and assume $V_f \oplus V_g \oplus V_h$ becomes crystalline over a totally ramified extension E/K. Then we have an idempotent $e_{f^c} \otimes e_{g^c} \otimes e_{h^c}$ acting on $e_{\mathrm{mod}}^{\otimes 3} H^{2r+3}_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p$ thanks to the Künneth decomposition and we know its image to be a crystalline representation of G_E. Better, we let $e_{\mathrm{cris}}^{\times 3}$ be the sum of these idempotents as f, g, h runs through the triples of modular forms satisfying the above hypotheses. Notice that $e_{\mathrm{cris}}^{\times 3}$ belongs to the Hecke algebra with coefficients in $\mathbf{Q}_p$ already. Applying this projector to the target of the syntomic Abel-Jacobi map, and setting $V_{\mathbf{r}}^{\mathrm{cris}} := e_{\mathrm{cris}}^{\times 3} e_{\mathrm{mod}}^{\otimes 3} H^{2r+3}_{\text{ét}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\bar{\mathbf{Q}}_p}, \mathbf{Q}_p)$, we get a class

$$\kappa_s := e_{\mathrm{cris}}^{\times 3} \operatorname{AJ}_{\mathrm{syn}}([\Delta_s]) \in H^1_{\mathrm{st}}(\mathbf{Q}_p, V_{\mathbf{r}}^{\mathrm{cris}}(r+2)). \tag{23}$$

Notice that the complex absolute value of the eigenvalues of crystalline Frobenius acting on $V_{\mathbf{r}}^{\mathrm{cris}}(r+2)$ is $p^{-1/2}$: this follows from the Weil conjectures for the part of cohomology coming from level coprime with p ([Del74], Theorem 8.2) and from the primitivity assumption on the other part. In particular, no such eigenvalue can equal 1. Moreover, using (18) we see that this representation is self-dual, in the sense that $(V_{\mathbf{r}}^{\mathrm{cris}}(r+2))^\vee(1) = V_{\mathbf{r}}^{\mathrm{cris}}(r+2)$. Hence, Corollaries 1.16 and 1.18 of [Nek93] yield

$$H^1_{\mathrm{st}}(\mathbf{Q}_p, V_{\mathbf{r}}^{\mathrm{cris}}(r+2)) = H^1_e(\mathbf{Q}_p, V_{\mathbf{r}}^{\mathrm{cris}}(r+2))$$

where $H^1_e(\mathbf{Q}_p, W) = \ker(H^1(\mathbf{Q}_p, W) \to H^1(\mathbf{Q}_p, W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}^{\varphi=1}))$ for W any p-adic representation of the absolute Galois group of $\mathbf{Q}_p$. The Bloch-Kato exponential map for $V_{\mathbf{r}}^{\mathrm{cris}}(r+2)$ is an isomorphism (cf. loc. cit., Theorem 1.15) and we write $\log_{\mathrm{BK}}$ for its inverse, the Bloch-Kato logarithm:

$$\log_{\mathrm{BK}} \colon H^1_e(\mathbf{Q}_p, V_{\mathbf{r}}^{\mathrm{cris}}(r+2)) \to D_{\mathrm{dR}}^{\mathbf{Q}_p}(V_{\mathbf{r}}^{\mathrm{cris}}) / \operatorname{Fil}^{r+2}.$$

Poincaré duality identifies the target with the $\mathbf{Q}_p$-linear dual of $\operatorname{Fil}^{r+2} D_{\mathrm{dR}}^{\mathbf{Q}_p}(V_{\mathbf{r}}^{\mathrm{cris}})$ and composing the logarithm map with the syntomic Abel-Jacobi, we get a p-adic Abel-Jacobi map

$$\operatorname{AJ}_p \colon \operatorname{CH}^{r+2}(\bar{\mathcal{E}}^{\mathbf{r}}_s) \to \left(\operatorname{Fil}^{r+2} D_{\mathrm{dR}}^{\mathbf{Q}_p}(V_{\mathbf{r}}^{\mathrm{cris}})\right)^\vee$$

and hence a functional $\log_{\mathrm{BK}}(\kappa_s) = \operatorname{AJ}_p([\Delta_s])$ on

$$\operatorname{Fil}^{r+2} D_{\mathrm{dR}}^{\mathbf{Q}_p}(V_{\mathbf{r}}^{\mathrm{cris}}) \cong e_{\mathrm{cris}}^{\times 3} e_{\mathrm{mod}}^{\otimes 3} \operatorname{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,\mathbf{Q}_p}).$$

1.3. We close the section describing a sheaf pairing induced by Δ_s, which will be of use later on. Let us write $\mathcal{H}^\bullet(X/S)$ for the relative de Rham cohomology sheaves of a morphism $X \to S$. Consider the sequence of maps:

$$\mathcal{H}^\bullet(\bar{\mathcal{E}}_s^l/X_s) \otimes \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^m/X_s) \xrightarrow{\varphi_B^* \otimes \varphi_C^*} \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^r/X_s) \otimes \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^r/X_s) \xrightarrow{\sqcup} \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^r/X_s) \xrightarrow{\varphi_{A*}} \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^k/X_s).$$

The first arrow is induced by pullback along φ_B and φ_C, the second one by cup product, and the last one by the pushforward along φ_A. The first term of the sequence receives a map

$$\mathcal{L}_l \otimes \mathcal{L}_m \xrightarrow{\iota_{\mathrm{sym}} \otimes \iota_{\mathrm{sym}}} \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^l/X_s) \otimes \mathcal{H}^\bullet(\bar{\mathcal{E}}_s^m/X_s)$$

while the last term is the source of the projection $\mathcal{H}^\bullet(\bar{\mathcal{E}}_s^k/X_s) \to \mathcal{L}_k$. Composing all the maps, we obtain a non-trivial pairing

$$\Psi\colon \mathcal{L}_l \otimes \mathcal{L}_m \to \mathcal{L}_k. \tag{24}$$

Let us write $\varphi_{BC} := (\varphi_B, \varphi_C)$. It is clear from the construction that:

PROPOSITION 10. *The map $H^1_{par}(X_s, \mathcal{L}_l) \otimes H^1_{par}(X_s, \mathcal{L}_m) \xrightarrow{\Psi} H^1_{par}(X_s, \mathcal{L}_k)$ induced by the above pairing identifies with $\varphi_{A*}\varphi_{BC}^*\colon D^K_{dR}(V_{l+2}) \otimes_K D^K_{dR}(V_{m+2}) \to D^K_{dR}(V_{k+2})$ through the comparison (16).* □

We remark finally that Ψ induces the map $\varphi_{A*}\varphi_{BC}^*$ already at the level of complexes.

2. Besser's integration theory

The ideas involved in the explicit computation of the p-adic Abel-Jacobi map originate in work of Coleman [Col85] and Besser [Bes00]. Although their original theories do not apply in our bad reduction situation, and we will need to introduce the generalization developed in [BLZ16], building on [NN16], a review of their line of thought might shed some light on the technical steps of the computation in the next section.

2.1. We start by recalling Coleman's integration of differential 1-forms of the second kind, which was first developed in [Col85]. We follow here [Bre00], to which we refer the reader for a broader introduction to the theory. Let X be an algebraic variety over $\mathbf{C}_p$, admitting a proper smooth model $\mathcal{X}$ over the valuation ring $\mathcal{O}_{\mathbf{C}_p}$ and denote by Y its special fiber over $\bar{\mathbf{F}}_p$. Since Y is of finite type over $\bar{\mathbf{F}}_p$ it must come via base change from a scheme over some finite extension $\mathbf{F}_{p^r}$ of $\mathbf{F}_p$. The r-th power of the absolute Frobenius of such a scheme extends then linearly to a map $\phi\colon Y \to Y$ of $\bar{\mathbf{F}}_p$-schemes; any such map is called a Frobenius morphism. Thanks to the comparison theorem of Berthelot and Ogus [BO83]

$$H^\bullet_{\mathrm{dR}}(X/\mathbf{C}_p) \cong H^\bullet_{\mathrm{cris}}(Y/W(\bar{\mathbf{F}}_p)) \otimes \mathbf{C}_p$$

any Frobenius morphism ϕ induces a $\mathbf{C}_p$-linear endomorphism of $H^1_{\mathrm{dR}}(X/\mathbf{C}_p)$, whose characteristic polynomial we denote by P_ϕ. Given an affine open $U \subseteq Y$, we denote by $]U[$ its inverse image under the specialization map of $\mathcal{X}$, which is an open affinoid in the rigid analytic space X^{rig} associated to X. Any Frobenius morphism $\phi_U\colon U \to U$ can be lifted to a map of rigid spaces $\varphi_{]U[}\colon]U[\to]U[$. Fix now a rational 1-form of the second kind ω

over X, that is, a regular differential on some open dense subscheme U_ω of X with $d\omega = 0$ and such that, Zariski locally on X, it can be written as the sum of a regular differential plus the differential of a rational function. One can show that there exists a finite cover of Y by open affine subschemes $Y_i \subseteq Y$ such that $\omega|_{]Y_i[} = \omega_i + df_i|_{]Y_i[}$ where the f_i are rational functions on X and each ω_i is a closed rigid 1-form on $]Y_i[$. Assume there exists a Frobenius morphism ϕ of Y which restricts to each Y_i, and fix lifts $\phi_i \colon]Y_i[\to]Y_i[$.

THEOREM 11. *Up to addition by a constant, there exists a unique locally analytic function f_ω on $U_\omega(\mathbf{C}_p)$ such that the following are satisfied:*

(1) *$df_\omega = \omega$;*
(2) *for each i, the function $f_\omega - f_i$ on $U_\omega \cap]Y_i[$ extends to a locally analytic function g_i on $]Y_i[$ and $P_\phi(\phi_i^*)(g_i)$ is a rigid analytic function on $]Y_i[$.*

Moreover, f_ω is independent of all the choices above.

A function f_ω as in the theorem is called a Coleman primitive of ω. Given two points $P, Q \in X(\mathbf{C}_p)$, it is then possible to define the line integral of ω from P to Q as

$$\int_P^Q \omega := f_\omega(Q) - f_\omega(Q)$$

and, more generally, the integral of ω against any zero-cycle $Z = \sum n_j P_j$ with $\sum n_j = 0$ as

$$\int_Z \omega := \sum n_j f_\omega(P_j).$$

The map which associates to Z the linear functional $[\omega] \mapsto \int_Z \omega$ on $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/\mathbf{C}_p)$ is, in fact, the syntomic Abel-Jacobi map for zero-cycles. This is perhaps not surprising, when we recall the analogous situation in classical complex geometry: if we replace X by a variety over $\mathbf{C}$, ω by a holomorphic 1-form on X, $\int_Z$ by the usual complex line integral and f_ω by a primitive of ω (defined only locally), then the same association as above yields the complex Abel-Jacobi map on zero-cycles. It is now clear that the key to computing such map lies in the non-trivial task of integrating the differential forms ω. Hence, in order to describe the Abel-Jacobi of cycles of arbitrary dimension, one needs first to extend the theory of Coleman integration to forms of arbitrary degree.

2.2. In [Bes00], Besser was able to extend Coleman's integration theory to forms of degree higher than one, through the introduction of a variant of syntomic cohomology called finite polinomial cohomology. Given a scheme X smooth and of finite type over the ring of integers $\mathcal{O}_E$ of a complete subfield E of $\mathbf{C}_p$, and integers $i, n \in \mathbf{Z}$, Besser defines E-vector spaces $H^i_{\mathrm{fp}}(X, n)$, functorial in X and admitting a cup product (additive in both i and n). When X is proper and E is discretely valued, there are short exact sequences

$$0 \to H^{i-1}_{\mathrm{dR}}(X_E/E)/\mathrm{Fil}^n \xrightarrow{i} H^i_{\mathrm{fp}}(X,n) \xrightarrow{p} \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_E/E) \to 0 \tag{25}$$

where $\mathrm{Fil}^\bullet$ is the Hodge filtration; in particular, if X is irreducible, of relative dimension d, then the map $i \colon H^{2d}_{\mathrm{dR}}(X_E/E)/\mathrm{Fil}^{d+1} \to H^{2d+1}_{\mathrm{fp}}(X, d+1)$ is an isomorphism, hence the trace map for de Rham cohomology induces a trace map

$$\mathrm{Tr}\colon H^{2d+1}_{\mathrm{fp}}(X, d+1) \to E$$

for finite polynomial cohomology. Combined with cup product, this yields a pairing

$$H^{i}_{\mathrm{fp}}(X, n) \times H^{2d+1-i}_{\mathrm{fp}}(X, d+1-n) \to E$$

which is in fact perfect; as a consequence, one obtains pushforwards for the finite polynomial theory. When the base field E is discretely valued, there is also a theory of Chern classes, which produces cycle class maps to finite polynomial cohomology, compatible with the de Rham ones via the map p of (25).

Since we are omitting the definition of the groups H_{fp}, the reader might wonder in which sense these are a variant of syntomic cohomology, and also be curious about the name "polynomial". To address the first issue, we can at least say that, when X is projective and E a finite extension of $\mathbf{Q}_p$, the exact sequence (25) with $i = 2n$ fits into a diagram

$$\begin{tikzcd}
H^{2n-1}_{\mathrm{dR}}(X_E/E)/\operatorname{Fil}^n \arrow[r, "i_{\mathrm{syn}}"] \arrow[d, equal] & H^{2n}_{\mathrm{syn}}(X, n) \arrow[r, "p_{\mathrm{syn}}"] \arrow[d] & \begin{matrix}\operatorname{Fil}^n H^{2n}_{\mathrm{dR}}(X_E/E) \cap \\ (H^{2n}_{\mathrm{cris}}(X_{k_E}/W(k_E)) \otimes E_0)^{\phi = p^n}\end{matrix} \arrow[d, hook] \\
H^{2n-1}_{\mathrm{dR}}(X_E/E)/\operatorname{Fil}^n \arrow[r, "i"] & H^{2n}_{\mathrm{fp}}(X, n) \arrow[r, "p"] & \operatorname{Fil}^n H^{2n}_{\mathrm{dR}}(X_E/E)
\end{tikzcd}$$

where k_E is the residue field of E, $W(k_E)$ the ring of Witt vectors of k_E and $E_0 = \mathrm{Frac}(W(k_E))$ is the maximal unramified subextension of E. On the other hand, the name "polynomial" is at least partly justified by the relation with Coleman integrals, whose construction requires the use of polynomials in a Frobenius morphism of X. Let us explain this relation. Consider the sequence (25) in the case $n = i = 1$: it says that the class $[\omega] \in \operatorname{Fil}^1 H^1_{\mathrm{dR}}(X_E/E)$ can be lifted to an element $\tilde{\omega} \in H^1_{\mathrm{fp}}(X, 1)$, and the lift is unique up to an element of $H^0_{\mathrm{dR}}(X_E/E) \cong E$, that is, up to a constant. Since X is proper, a point $x \in X(E)$ can be seen as a section $x\colon \mathrm{Spec}(\mathcal{O}_E) \to X$. Pulling back $\tilde{\omega}$ by x yields then an element of the group $H^1_{\mathrm{fp}}(\mathrm{Spec}(\mathcal{O}_E), 1)$, which is isomorphic to $H^0_{\mathrm{dR}}(\mathrm{Spec}(E)/E) \cong E$ again by (25). We thus obtain a map $f_{\tilde{\omega}}\colon X(E) \to E$, $x \mapsto x^*\tilde{\omega}$ which is in fact a Coleman primitive of ω; moreover, the association $\tilde{\omega} \mapsto f_{\tilde{\omega}}$ gives a bijection between lifts of ω to $H^1_{\mathrm{fp}}(X, 1)$ and Coleman primitives of ω. In particular, we recover the syntomic Abel-Jacobi map on zero-cycles: a cycle $Z = \sum n_j P_j$ with $\sum n_j = 0$ is sent to the functional

$$[\omega] \mapsto \sum n_j P_j^* \tilde{\omega}.$$

It is now clear how one gets the desired generalization of Coleman integration: given a class $[\omega] \in \operatorname{Fil}^{i+1} H^{2i+1}_{\mathrm{dR}}(X_E/E)$ and a cycle $Z = \sum n_j Z_j$ with the Z_j smooth of relative dimension i, we let

$$\int_Z \omega := \sum n_j \,\mathrm{Tr}(\iota_j^* \tilde{\omega})$$

where $\tilde{\omega} \in H^{2i+1}_{\mathrm{fp}}(X, i+1)$ is a lift of $[\omega]$, and ι_j denotes the closed immersion $Z_j \to X$. Besser shows that, when Z is homologically equivalent to zero, the sum is indeed independent of the choice of $\tilde{\omega}$, and that the functional $[\omega] \mapsto \int_Z \omega$ coincides with the syntomic Abel-Jacobi of the cycle Z. Up to replacing Besser's original finite polynomial cohomology with the more high-tech version [BLZ16], the reader will see that the computation in the next section follows exactly the same lines.

2.3. We recall very briefly the definition of the finite polynomial variant of semistable Galois cohomology from ([BLZ16], Paragraph 1.2) in the cases we are interested in, that is, for semistable modules. The spectral sequence (28) illustrates how this is tied to the finite polynomial syntomic theory also introduced in loc. cit., which is the generalization of Besser's original theory to bad reduction situations.

Let, just for this number, E be any fixed finite extension of $\mathbf{Q}_p$, M any intermediate extension between $\mathbf{Q}_p$ and E, M_0 be the maximal unramified subfield of M and write $d := [M_0 : \mathbf{Q}_p]$, $q := p^d$. If D is a filtered (φ, N)-module over E, we set $D_M := D \otimes_{M_0} M$ and write Φ for its endomorphism obtained as M-linear extension of φ^d. For any polynomial $P(T) \in 1 + TM[T]$, $H^\bullet_{\mathrm{st},M,P}(D)$ is the cohomology of the complex

$$D_M \oplus \mathrm{Fil}^0 D_E \to D_M \oplus D_M \oplus D_E \to D_M \tag{26}$$

with maps $(u, v) \mapsto (P(\Phi)u, Nu, u - v)$ and $(\xi, x, y) \mapsto (N\xi - P(q\Phi)x)$. If W is any semistable representation of G_E, then

$$H^0_{\mathrm{st},\mathbf{Q}_p,1-T}(D^E_{\mathrm{st}}(W)) = H^0(E, W) \quad \text{and} \quad H^1_{\mathrm{st},\mathbf{Q}_p,1-T}(D^E_{\mathrm{st}}(W)) = H^1_{\mathrm{st}}(E, W). \tag{27}$$

Suppose now that D is convenient in the sense of loc. cit. with respect to M and P; this means that $N = 0$ and that $P(\Phi)$ and $P(q\Phi)$ are bijective on D_M. A quick computation shows that under this assumption $H^2_{\mathrm{st},M,P}(D) = 0$ and $H^1_{\mathrm{st},M,P}(D) \cong D_E/\mathrm{Fil}^0$: every class in H^1 is represented by an element $(\xi, 0, y) \in D_M \oplus D_M \oplus D_E$, and such an element is identified with $y - P(\Phi)^{-1}\xi \bmod \mathrm{Fil}^0$. The simplest example of such a convenient module is $D^E_{\mathrm{st}}(\mathbf{Q}_p(1))$, for $P(1) \neq 0 \neq P(q^{-1})$, in which case we get $H^1_{\mathrm{st},M,P}(D^E_{\mathrm{st}}(\mathbf{Q}_p(1))) \cong E$. This has the following useful application. Let X be any variety over E and consider its finite polynomial syntomic cohomology, as introduced in loc. cit.: this is the generalization of Besser's finite polynomial theory presented in 2.2 to varietyes with bad reduction. The spectral sequence (loc. cit., Proposition 2.3.2)

$$E_2^{i,j} = H^i_{\mathrm{st},M,P}(D^E_{\mathrm{st}}(H^j_{\text{ét}}(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)(r))) \Rightarrow H^{i+j}_{\mathrm{syn},M,P}(X, r) \tag{28}$$

gives a surjection

$$\begin{aligned} H^{2\dim(X)+1}_{\mathrm{syn},M,P}(X, \dim(X)+1) \longrightarrow \mathrm{Fil}^1 / \mathrm{Fil}^2 = \\ = H^1_{\mathrm{st},M,P}(D^E_{\mathrm{st}}(H^{2\dim(X)}_{\text{ét}}(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)(\dim(X)+1))). \end{aligned}$$

Using the trace map for étale cohomology, and assuming $P(1) \neq 0 \neq P(q^{-1})$ as above, we obtain a trace map

$$\mathrm{Tr}\colon H^{2\dim(X)+1}_{\mathrm{syn},M,P}(X, \dim(X)+1) \twoheadrightarrow E$$

for finite polynomial syntomic cohomology. This map is always surjective, but it need not be injective, and is compatible with pushforwards when they exist.

3. The explicit formula for the p-adic Abel-Jacobi

We finally dedicate ourselves to the proof of our main result.

3.1. Let

$$f \in S_{k+2}(M_f, \psi_f, \bar{\mathbf{Q}}), \quad g \in S_{l+2}(M_g, \psi_g, \bar{\mathbf{Q}}), \quad h \in S_{m+2}(M_h, \psi_h, \bar{\mathbf{Q}})$$

be newforms with $M_f, M_g, M_h \mid Np^s$ and ψ_f, ψ_g, ψ_h primitive at p, assume f satisfies the hypotheses of Lemma 9 and assume $\alpha_g \neq \beta_g$ (resp. $\alpha_h \neq \beta_h$) if $p \nmid M_g$ (resp. M_h) and $\alpha_g \neq 0$ (resp. $\alpha_h \neq 0$) if $p \mid M_g$ (resp. $p \mid M_h$). Furthermore, assume that $V_f \oplus V_g \oplus V_h$ becomes crystalline over a totally ramified extension E/K. Fix also three modular forms

$$\breve{f} \in S_{k+2}(Np^s, \bar{\mathbf{Q}})^{\mathrm{prim}}[f], \quad \breve{g} \in S_{l+2}(Np^s, \bar{\mathbf{Q}})^{\mathrm{prim}}[g], \quad \breve{h} \in S_{m+2}(Np^s, \bar{\mathbf{Q}})^{\mathrm{prim}}[h]$$

where the primitive part is defined as in 2.1, Chapter II, and assume $\breve{g}, \breve{h}$ to be eigenvectors for U of eigenvalue α_g, α_h respectively[1]. Our aim in the next two sections is to provide a formula for the value $\mathrm{AJ}_p([\Delta_s])(\eta^{\alpha}_{\breve{f}^c} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}})$ in terms of p-adic modular forms. The ideas involved in this computation arise from work of Darmon and Rotger ([DR14], [DR16]). The main tool we need is the finite polynomial generalization of syntomic cohomology, described in [BLZ16]. Consider at first the spectral sequence (28) for the variety $\bar{\mathcal{E}}^{\mathbf{r}}_{s,\mathbf{Q}_p}$. The projector $e^{\otimes 3}_{\mathrm{mod}}$ kills everything at page two except the $(2r+3)$-th row, hence the resulting sequence degenerates right away and we have, for any polynomial $P(T) \in 1 + T\mathbf{Q}_p[T]$ and any $j \in \mathbf{Z}$,

$$e^{\times 3}_{\mathrm{cris}} e^{\otimes 3}_{\mathrm{mod}} H^{\bullet}_{\mathrm{syn},\mathbf{Q}_p,P}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,E}, j) = H^{\bullet-2r-3}_{\mathrm{st},\mathbf{Q}_p,P}(D^E_{\mathrm{st}}(V^{\mathrm{cris}}_{\mathbf{r}}(j))).$$

In particular, by (23) and (27), we have a class $\mathrm{res}_{G_E}(\kappa_s) \in H^1_{\mathrm{st},\mathbf{Q}_p,1-T}(D^E_{\mathrm{st}}(V^{\mathrm{cris}}_{\mathbf{r}}(r+2)))$ which is just the image of $\mathrm{cl}_{\mathrm{syn}}([\Delta_s]) \in e^{\times 3}_{\mathrm{cris}} e^{\otimes 3}_{\mathrm{mod}} H^{2r+4}_{\mathrm{syn},\mathbf{Q}_p,1-T}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,E}, r+2)$. Analogously, we have lifts

$$\tilde{\eta}^{\alpha}_{\breve{f}^c} \in H^0_{\mathrm{st},E,P_f}(D^E_{\mathrm{st}}(V_{k+2}) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p) = e_{\mathrm{mod}} H^{k+1}_{\mathrm{syn},E,P_f}(\bar{\mathcal{E}}^k_{s,E}, 0) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p$$

and

$$\begin{aligned} \tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}} \in H^0_{\mathrm{st},E,P_{gh}}(D^E_{\mathrm{st}}(V_{l+2} \otimes_{\mathbf{Q}_p} V_{m+2}(r+2)) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p) = \\ = e^{\otimes 2}_{\mathrm{mod}} H^{l+m+2}_{\mathrm{syn},E,P_{gh}}(\bar{\mathcal{E}}^l_{s,E} \times \bar{\mathcal{E}}^m_{s,E}, r+2) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p \end{aligned}$$

(cf. Section 2.5, Chapter II) of $\eta^{\alpha}_{\breve{f}^c}$ and $\omega_{\breve{g}} \otimes \omega_{\breve{h}}$ to syntomic cohomology groups, where $P_f(T), P_{gh}(T) \in 1 + T\mathbf{Q}_p[T]$ are the polynomials of minimal degree such that α_f is a zero of P_f and

$$\frac{\alpha_g \alpha_h}{p^{r+2}} \qquad \frac{\alpha_g \beta_h}{p^{r+2}} \qquad \frac{\beta_g \alpha_h}{p^{r+2}} \qquad \frac{\beta_g \beta_h}{p^{r+2}}$$

[1] Of course, this assumption is automatically satisfied if $p \mid M_g, M_h$ respectively.

are zeros of P_{gh}. Here our choice of twists requires $l+m+2 \geq r+2$, which we remark is slightly stronger than the balancedness condition $k+2 < l+m+4$. In particular

$$\tilde{\eta}^{\alpha}_{\breve{f}c} \otimes \tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}} \in H^0_{\mathrm{st},E,P_f \star P_{gh}}(D^E_{\mathrm{st}}(V^{\mathrm{cris}}_{\mathbf{r}}(r+2))) = e^{\times 3}_{\mathrm{cris}} e^{\otimes 3}_{\mathrm{mod}} H^{2r+3}_{\mathrm{syn},E,P_f \star P_{gh}}(\bar{\mathcal{E}}^{\mathbf{r}}_{s,E}, r+2)$$

(given P with roots $\{\lambda_i\}$ and Q with roots $\{\mu_j\}$, $P \star Q$ is the polynomial with roots $\{\lambda_i \mu_j\}$). It now follows that

$$\mathrm{AJ}_p([\Delta_s])(\eta^{\alpha}_{\breve{f}c} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}}) = -\operatorname{Tr}(\mathrm{cl}_{\mathrm{syn}}([\Delta_s]) \cup \tilde{\eta}^{\alpha}_{\breve{f}c} \otimes \tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}})$$

(cf. loc. cit., Remark 3.1.1). We are slightly abusing the notations here, in that before taking cup product one needs to use change-polynomial and change-field properties to bring $\mathrm{cl}_{\mathrm{syn}}([\Delta_s])$ into $H^{2r+4}_{\mathrm{syn},E,1-T}$. A quick check on the complex absolute values of the roots shows that the condition

$$P_f \star P_{gh}(1) \neq 0 \neq P_f \star P_{gh}(p^{-1})$$

which is necessary to define the trace is always satisfied. Now by definition $\mathrm{cl}_{\mathrm{syn}}([\Delta_s]) = \varphi_{ABC*}(1)$ where $1 \in \mathbf{Q}_p = H^0_{\mathrm{syn},E,1-T}(\bar{\mathcal{E}}^r_{s,E}, 0)$, so that

$$\begin{aligned}
\operatorname{Tr}(\mathrm{cl}_{\mathrm{syn}}([\Delta_s]) \cup \tilde{\eta}^{\alpha}_{\breve{f}c} \otimes \tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}}) &= \operatorname{Tr}(\varphi^*_{ABC}(\tilde{\eta}^{\alpha}_{\breve{f}c} \otimes \tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}})) \\
&= \operatorname{Tr}(\varphi^*_A \tilde{\eta}^{\alpha}_{\breve{f}c} \otimes \varphi^*_{BC}(\tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}})) \\
&= \operatorname{Tr}(\tilde{\eta}^{\alpha}_{\breve{f}c} \cup \varphi_{A*}\varphi^*_{BC}(\tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}})) = \dots
\end{aligned}$$

where for the first and third equalities we used the product formula (loc. cit., Theorem 2.5.3) and invariance of the trace under pushforwards. Continuing the chain of equalities, we have

$$\begin{aligned}
\dots &= \operatorname{Tr}(e_{\mathrm{mod}} \tilde{\eta}^{\alpha}_{\breve{f}c} \cup \varphi_{A*}\varphi^*_{BC}(\tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}})) \\
&= \operatorname{Tr}(\tilde{\eta}^{\alpha}_{\breve{f}c} \cup e_{\mathrm{mod}} \varphi_{A*}\varphi^*_{BC}(\tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}})) = \dots
\end{aligned}$$

because e_{mod} is self-adjoint. Now,

$$\begin{aligned}
e_{\mathrm{mod}} \varphi_{A*}\varphi^*_{BC}(\tilde{\omega}_{\breve{g}} \otimes \tilde{\omega}_{\breve{h}}) \in H^1_{\mathrm{st},E,P_{gh}}(D^E_{\mathrm{st}}(V_{k+2}(k+2)) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p) = \\
= e_{\mathrm{mod}} H^{k+2}_{\mathrm{syn},E,P_{gh}}(\bar{\mathcal{E}}^k_{s,E}, k+2) \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p
\end{aligned}$$

thus it is represented by some triple (ξ, x, y) as in (26). Its explicit form can be read from the description of syntomic cohomology given in (loc. cit., Proposition 2.2.6): ξ is the cohomology class of the differential obtained applying $\varphi_{A*}\varphi^*_{BC}$ to a primitive of $P_{gh}(\tilde{\varphi} \otimes \tilde{\varphi}/p^{r+2})(\omega_{\breve{g}} \otimes \omega_{\breve{h}})$, and both x and y are zero. Then the cup product rules (loc. cit., Table 1) and the definition of the trace map yield

$$\dots = -(P_f \star P_{gh})(p^{-1})^{-1} \left(b(\Phi, \Phi)(\eta^{\alpha}_{\breve{f}c} \otimes \xi) \right) = \dots$$

where $b(T_1, T_2) \in \mathbf{Q}_p[T_1, T_2]$ is any polynomial such that $(P_f \star P_{gh})(T_1 T_2) - b(T_1, T_2) P_{gh}(T_2)$ is multiple of $P_f(T_1)$, and the first and second Φ in $b(\Phi, \Phi)$ act respectively on $\eta^{\alpha}_{\breve{f}c}$ and on ξ. Here our notation is a bit sloppy: we omit the cup product pairing (18), which allows us to see $b(\Phi, \Phi)(\eta^{\alpha}_{\breve{f}c} \otimes \xi)$ as an element of E; we will denote the pairing with angle brackets in the following lines. Although not strictly necessary, we can pick a choice for $b(T_1, T_2)$: we use the equality

$$(29)\quad 1-\frac{p^{r+2}}{\alpha_f\alpha_g\alpha_h}T_1T_2 = \frac{1}{2}\left(1+\alpha_f^{-1}T_1\right)\left(1-\frac{p^{r+2}}{\alpha_g\alpha_h}T_2\right)+\frac{1}{2}\left(1-\alpha_f^{-1}T_1\right)\left(1+\frac{p^{r+2}}{\alpha_g\alpha_h}T_2\right)$$

and analogous ones for the other linear factors of $P_f \star P_{gh}$ to get

$$(P_f \star P_{gh})(T_1T_2) = b(T_1, T_2)P_{gh}(T_2) + P_f(T_1) \cdot (\dots)$$

where the first summand collects all the terms which are not multiples of $P_f(T_1)$ in virtue of the rightmost terms of (29). Our chain of equalities continues with

$$\dots = -(P_f \star P_{gh})(p^{-1})^{-1}b(\alpha_f, p^{-1}\alpha_f^{-1})\langle\eta^{\alpha}_{\breve{f}^c}, \xi\rangle$$

because Φ acts as $\varphi = \alpha_f$ on $\eta^{\alpha}_{\breve{f}^c}$ and as $p^{-1}\alpha_f^{-1}$ on the part of cohomology which pairs non-trivially with $\eta^{\alpha}_{\breve{f}^c}$ (remember the trace is built out of the convenient module $D^E_{\mathrm{st}}(\mathbf{Q}_p(1))$, whose Frobenius acts as p^{-1}). The defining relation of $b(T_1, T_2)$ yields $P_f \star P_{gh}(p^{-1}) = b(\alpha_f, p^{-1}\alpha_f^{-1})P_{gh}(p^{-1}\alpha_f^{-1})$, hence we obtained the equality

$$(30)\qquad \mathrm{AJ}_p([\Delta_s])(\eta^{\alpha}_{\breve{f}^c} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}}) = P_{gh}(p^{-1}\alpha_f^{-1})^{-1}\langle\eta^{\alpha}_{\breve{f}^c}, \xi\rangle.$$

In the next section we reinterpret this in terms of rigid cohomology and complete our computation.

3.2. In order to make ξ explicit, recall that we can see the differentials $\omega_{\breve{g}}, \omega_{\breve{h}}$ as elements of

$$H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_l)^{\mathrm{prim}} \otimes \bar{\mathbf{Q}}, \quad H^1_{\mathrm{par}}(X_s, \boldsymbol{\mathcal{L}}_m)^{\mathrm{prim}} \otimes \bar{\mathbf{Q}}$$

respectively. Then

$$P_{gh}(\Phi)\cdot\omega_{\breve{g}}\otimes\omega_{\breve{h}} = P_{gh}\left(\frac{\varphi\otimes\varphi}{p^{r+2}}\right)\cdot\omega_{\breve{g}}\otimes\omega_{\breve{h}} = 0 \quad \text{in} \quad H^2(X_s\times X_s, \boldsymbol{\mathcal{L}}_l\boxtimes\boldsymbol{\mathcal{L}}_m\otimes\Omega^2_{X_s\times X_s}(\log))\otimes\bar{\mathbf{Q}}$$

so that $P_{gh}\left(\tilde{\varphi}\otimes\tilde{\varphi}/p^{r+2}\right)\cdot\omega_{\breve{g}}\otimes\omega_{\breve{h}}$ is a rigid differential form on some neighborhood $\mathcal{A}\times\mathcal{A} \subseteq \mathcal{U}\times\mathcal{U} \subseteq X_s^{\mathrm{rig}} \times X_s^{\mathrm{rig}}$ (over L, cf. 2.1, Chapter II) whose associated cohomology class is trivial, hence has a primitive:

$$\tilde{\xi} \in H^0(\mathcal{U}\times\mathcal{U}, \boldsymbol{\mathcal{L}}_l \boxtimes \boldsymbol{\mathcal{L}}_m \otimes \Omega^1_{X_s\times X_s}(\log)) \otimes_L \bar{\mathbf{Q}}_p, \quad \nabla\tilde{\xi} = P_{gh}\left(\tilde{\varphi}\otimes\tilde{\varphi}/p^{r+2}\right)\cdot\omega_{\breve{g}}\otimes\omega_{\breve{h}}.$$

Restricting to the diagonal and applying the pairing (24), we obtain a differential

$$\Psi(\tilde{\xi}|_{\mathcal{U}}) \in H^0(\mathcal{U}, \Omega^1_{\mathrm{par}}(\boldsymbol{\mathcal{L}}_k)) \otimes_L \bar{\mathbf{Q}}_p$$

and take its cohomology class $\xi \in H^1(X_s, \boldsymbol{\mathcal{L}}_k) \otimes \bar{\mathbf{Q}}_p$. Since we have to pair it with $\eta^{\alpha}_{\breve{f}^c}$, we might as well project it to $H^1_{\mathrm{par}}(W_0, \boldsymbol{\mathcal{L}}_k) \otimes_L \bar{\mathbf{Q}}_p[f^c]$. Writing $\omega_{\breve{g}} \otimes \omega_{\breve{h}}$ as the sum of four differentials $\omega_{\breve{g},i} \otimes \omega_{\breve{h},j}$, $i, j \in \{0, \infty\}$, obtained by restriction to $W_i \times W_j$, we get a similar decomposition for ξ and we see that the only term to survive after this projection is the one coming from $\omega_{\breve{g},0}\otimes\omega_{\breve{h},0}$. Indeed, the mixed terms are killed by φ^*_{BC} (that is, by restricting to the diagonal) and the term supported on W_∞ is killed by pairing with $\eta^{\alpha}_{\breve{f}^c}$. Thus, we replace ξ with the class obtained from $\omega_{\breve{g},0} \otimes \omega_{\breve{h},0}$. Now,

$$P_{gh}(T) = \left(1 - \frac{p^{r+2}}{\alpha_g \alpha_h} T\right) \cdot Q(T)$$

and the first factor suffices to kill the cohomology class of $\omega_{\breve{g},0} \otimes \omega_{\breve{h},0}$, hence we will just carry over $Q(\Phi)$ until the end of the computation. We can rewrite the first factor of P_{gh} above in the following, more convenient form:

$$\left(1 - \frac{\tilde{\varphi} \otimes \tilde{\varphi}}{\alpha_g \alpha_h}\right) = \frac{1}{2}\left(1 - \frac{\tilde{\varphi}}{\alpha_g}\right) \otimes \left(1 + \frac{\tilde{\varphi}}{\alpha_h}\right) + \frac{1}{2}\left(1 + \frac{\tilde{\varphi}}{\alpha_g}\right) \otimes \left(1 - \frac{\tilde{\varphi}}{\alpha_h}\right)$$

From the definitions, we have:

$$\tilde{\varphi}\omega_{\breve{g},0} = \tilde{\varphi}_0 \omega_{\breve{g},0} = (w_s^*)^{-1} p^{l+1} \langle p|1\rangle^* V \omega_{w_s^* \breve{g},\infty} = p^{l+1} \psi_g(p|1)(w_s^*)^{-1} V \omega_{w_s^* \breve{g},\infty}$$

so that

$$\left(1 \pm \frac{\tilde{\varphi}}{\alpha_g}\right) \omega_{\breve{g},0} = (w_s^*)^{-1} \left(1 \pm \beta_g V\right) \omega_{w_s^* \breve{g},\infty} = (w_s^*)^{-1} \left(1 \pm VU\right) \omega_{w_s^* \breve{g},\infty}$$

and similarly for h, where in the last equality we used ([**AL78**], equation (1.1)). Let us write

$$(w_s^* \breve{g})(q) = \sum_{n>0} b_n(\breve{g}) q^n, \quad (w_s^* \breve{h})(q) = \sum_{n>0} b_n(\breve{h}) q^n.$$

The operator $1 - VU$ is also called p-depletion and denoted by the apex $^{[p]}$, and acts on q-expansions by erasing all terms in which the exponent of q is multiple of p. In particular, around the cusp ∞ we have

$$(w_s^* \breve{g})^{[p]}(q) = (1 - VU)(w_s^* \breve{g})(q) = \sum_{p \nmid n} b_n(\breve{g}) q^n \omega^l \frac{dq}{q} = \left(q \frac{d}{dq} \sum_{p \nmid n} \frac{b_n(\breve{g})}{n} q^n\right) \omega^l \frac{dq}{q}$$

where $\theta := q\frac{d}{dq}$ is Serre's differential operator (cf. [**Ser73**], Theorem 5). For $i > 0$, let us denote by θ^{-i} the operator $\sum_{p \nmid n} n^{-i} a_n q^n$, also described in (loc. cit.). Then we define two rigid functions $G \in H^0(\mathcal{A}_0, \boldsymbol{\mathcal{L}}_l) \otimes_L \bar{\mathbf{Q}}_p$ and $H \in H^0(\mathcal{A}_0, \boldsymbol{\mathcal{L}}_m) \otimes_L \bar{\mathbf{Q}}_p$ imposing the following.

$$w_s^* G(q) = \sum_{i=0}^{l} (-1)^{l-i} \frac{l!}{i!} \theta^{i-l-1} (w_s^* \breve{g}(q))^{[p]} \omega^i \eta^{l-i}$$
$$w_s^* H(q) = \sum_{i=0}^{m} (-1)^{m-i} \frac{m!}{i!} \theta^{i-m-1} (w_s^* \breve{h}(q))^{[p]} \omega^i \eta^{m-i}$$

Assume for a moment that these are, as claimed, sections over $\mathcal{A}_0$ and not just around the cusp. Using (10), it is easy to see that

$$\nabla(w_s^* G) = (\omega_{w_s^* \breve{g},\infty})^{[p]}, \quad \nabla(w_s^* H) = (\omega_{w_s^* \breve{h},\infty})^{[p]}.$$

In particular,

$$\nabla(G) = (w_s^*)^{-1}\nabla(w_s^* G) = (w_s^*)^{-1}(\omega_{w_s^*\breve{g},\infty})^{[p]} = \left(1 - \frac{\tilde{\varphi}}{\alpha_g}\right)\omega_{\breve{g},0},$$
$$\nabla(H) = (w_s^*)^{-1}\nabla(w_s^* H) = (w_s^*)^{-1}(\omega_{w_s^*\breve{h},\infty})^{[p]} = \left(1 - \frac{\tilde{\varphi}}{\alpha_h}\right)\omega_{\breve{h},0},$$

hence, since the rightmost terms are trivial in cohomology, there are indeed rigid sections G, H over $\mathcal{A}_0$ with the q-expansions prescribed above. Notice, in fact, that G and H are actually defined on some wide open neighborhood of $\mathcal{A}_0$: $(\omega_{w_s^*\breve{g},\infty})^{[p]}$ and $(\omega_{w_s^*\breve{h},\infty})^{[p]}$ are overconvergent differentials which vanish in the cohomology of W_∞, thus they must have overconvergent primitives. Let us define

$$\tilde{\xi} := Q(\tilde{\varphi}\otimes\tilde{\varphi}/p^{r+2})\cdot\left[\frac{1}{2}G\otimes\left(1+\frac{\tilde{\varphi}}{\alpha_h}\right)\omega_{\breve{h},0} - \frac{1}{2}\left(1+\frac{\tilde{\varphi}}{\alpha_g}\right)\omega_{\breve{g},0}\otimes H\right]$$

so that $\nabla\tilde{\xi} = P_{gh}\left(\tilde{\varphi}\otimes\tilde{\varphi}/p^{r+2}\right)\cdot\omega_{\breve{g},0}\otimes\omega_{\breve{h},0}$ as wished. Now we restrict to the diagonal, apply the pairing (24) and analyze the four summands separately: we start off with

$$\Psi\left(G\otimes\frac{\tilde{\varphi}}{\alpha_h}\omega_{\breve{h},0}\right) = (w_s^*)^{-1}\Psi(w_s^* G\otimes VU\omega_{w_s^*\breve{h},\infty}) =$$
$$= (w_s^*)^{-1}\sum_{i=0}^{l}\sum_{n>0}\left((-1)^{l-i}\frac{l!}{i!}\theta^{i-l-1}(w_s^*\breve{g}(q))^{[p]}b_{pn}(\breve{h})q^{pn}\right)\cdot\Psi(\omega^i\eta^{l-i}\otimes\omega^m)\cdot\frac{dq}{q}. \tag{31}$$

Notice that in the q-expansion all summands with exponent of q multiple of p have trivial coefficient. This implies that the operator $(w_s^*)^{-1}Uw_s^*$ kills (31). But the same operator acts invertibly on $H^1(W_0,\boldsymbol{\mathcal{L}}_k)[f^c]$, on which pairing with $\eta^{\alpha}_{\breve{f}^c}$ is supported, thus (31) pairs to zero with $\eta^{\alpha}_{\breve{f}^c}$ (cf. 2.5, Chapter II). The analogous argument applies to the summand symmetric to this one, so we are left with

$$\frac{1}{2}\Psi(G\otimes\omega_{\breve{h},0} - \omega_{\breve{g},0}\otimes H)$$

to work with. Observe that

$$\nabla\Psi(G\otimes H)|e_{f^c} = \Psi\left((1-\frac{\tilde{\varphi}}{\alpha_g})\omega_{\breve{g},0}\otimes H + G\otimes(1-\frac{\tilde{\varphi}}{\alpha_h})\omega_{\breve{h},0}\right)|e_{f^c}$$
$$= \Psi(\omega_{\breve{g},0}\otimes H + G\otimes\omega_{\breve{h},0})|e_{f^c}$$

where the second equality holds because, as above, e_{f^c} kills (31). Thus, once we take cohomology,

$$\Psi(G\otimes\omega_{\breve{h},0})|e_{f^c} = -\Psi(\omega_{\breve{g},0}\otimes H)|e_{f^c}\in H^1(X_s,\boldsymbol{\mathcal{L}}_k)\otimes\bar{\mathbf{Q}}_p.$$

In particular, we have shown that

$$\langle\eta^{\alpha}_{\breve{f}^c},\xi\rangle = \langle\eta^{\alpha}_{\breve{f}^c}, Q(\Phi)\cdot\Psi(G\otimes\omega_{\breve{h},0})\rangle = Q(p^{-1}\alpha_f^{-1})\cdot\langle\eta^{\alpha}_{\breve{f}^c},\Psi(G\otimes\omega_{\breve{h},0})\rangle$$

because, once again, Φ acts as $p^{-1}\alpha_f^{-1}$ on the eigenspace that pairs nontrivially with $\eta^{\alpha}_{\breve{f}^c}$. Now,

$$\Psi(G\otimes\omega_{\breve{h},0})|e_{f^c} = (w_s^*)^{-1}\left(\Psi(w_s^* G\otimes\omega_{w_s^*\breve{h},\infty})|e_{w_s^* f^c/a_1(w_s^* f^c)}\right).$$

and

$$\Psi(w_s^*G\otimes\omega_{w_s^*\breve{h},\infty})(q)=\sum_{i=0}^{l}\left((-1)^{l-i}\frac{l!}{i!}\theta^{i-l-1}(w_s^*\breve{g}(q))^{[p]}\sum_{n>0}b_n(\breve{h})q^n\right)\Psi(\omega^i\eta^{l-i}\otimes\omega^m)\frac{dq}{q}$$

and going through the definition of Ψ, we directly compute that

$$\Psi(\omega^i\eta^{l-i}\otimes\omega^m)=\begin{cases}(-1)^{r-k}\dfrac{(r-k)!(r-m)!}{l!}\omega^k+\eta\cdot(\dots) & \text{for } i=r-m\\ \eta\cdot(\dots) & \text{otherwise}\end{cases}$$

hence the p-adic modular form associated to $\Psi(w_s^*G\otimes\omega_{w_s^*\breve{h},\infty})$ as in (3.2, Chapter II) is

$$e_{\text{n-oc}}\Psi(w_s^*G\otimes\omega_{w_s^*\breve{h},\infty})(q)=(r-k)!\cdot\theta^{k-r-1}(w_s^*\breve{g}(q))^{[p]}\cdot(w_s^*\breve{h})(q)\cdot\omega^k\frac{dq}{q}.$$

Since we can replace the nearly-overconvergent $\Psi(w_s^*G\otimes\omega_{w_s^*\breve{h},\infty})$ with its projection via e_{oc} and the value of the pairing will not change (cf. 3.3, Chapter II), this concludes the proof of the following.

PROPOSITION 12. *We have*

$$\langle\eta^{\alpha}_{\breve{f}^c},\xi\rangle=(r-k)!\cdot Q(p^{-1}\alpha_f^{-1})\cdot\langle\eta^{\alpha}_{\breve{f}^c},(w_s^*)^{-1}e_{oc}\diamondsuit\rangle$$

where $\diamondsuit$ is the nearly-overconvergent form with q-expansion $\theta^{k-r-1}(w_s^*\breve{g}(q))^{[p]}\cdot w_s^*\breve{h}(q)$.

Combining the above with (30), we obtain

THEOREM 13. *We have*

$$\mathrm{AJ}_p([\Delta_s])(\eta^{\alpha}_{\breve{f}^c}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}})=(r-k)!\cdot\left(1-\frac{p^{r+1}}{\alpha_f\alpha_g\alpha_h}\right)^{-1}\cdot\langle\eta^{\alpha}_{\breve{f}^c},(w_s^*)^{-1}e_{oc}\diamondsuit\rangle$$

where $\diamondsuit$ is the nearly-overconvergent form with q-expansion $\theta^{k-r-1}(w_s^*\breve{g}(q))^{[p]}\cdot w_s^*\breve{h}(q)$.

We remark that for each of the three modular forms, if p does not divide its conductor, we can switch the roles of α and β in the above proof and nothing changes. In particular, taking linear combinations, this allows to obtain an explicit formula also when $\breve{g},\breve{h}$ are not eigenvectors for U.

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