The Deninger-Werner correspondence for rigid analytic varieties

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Abstract

In this thesis we construct a fully faithful functor from the category of vector bundles with numerically flat reduction on a proper rigid analytic variety X over $\mathbb{C}_p$ to the category of continuous p-adic representations of the étale fundamental group. This generalizes results obtained by Deninger and Werner for smooth algebraic varieties.

Our approach uses fundamental results of Scholze on the pro-étale site of a rigid analytic variety. In particular we prove that the p-adic completion of modules over the integral structure sheaf $\mathcal{O}_X^+$ on the pro-étale site, whose mod p reduction is Frobenius-trivial, are trivialized by a profinite étale covering, and that such profinite étale trivializable modules give rise to representations. We then show that the pullback of a formal vector bundle with numerically flat reduction, defined on a formal model of X, to the pro-étale site is a Frobenius-trivial module.

We go on to show that all line bundles for which a tensor power is a connected deformation of the trivial bundle give rise to a p-adic local system. Lastly, we show that the cohomology of a local system, which is associated to a vector bundle with numerically flat reduction, admits a Hodge-Tate filtration.

Zusammenfassung

In dieser Arbeit konstruieren wir einen volltreuen Funktor von der Kategorie der Vektorbündel mit numerisch flacher Reduktion auf einer rigid analytischen Varietät X über $\mathbb{C}_p$ in die Kategorie der stetigen p-adischen Darstellungen der étalen Fundamentalgruppe von X. Damit verallgemeinern wir Arbeiten von Deninger und Werner für glatte algebraische Varietäten.

Unser Zugang benutzt fundamentale Resultate von Scholze über den pro-étalen Situs einer rigid analytischen Varietät. Hierbei zeigen wir zunächst, dass die p-adische Vervollständigung von Moduln über der Strukturgarbe $\mathcal{O}_X^+$ der ganzen Elemente auf dem pro-étalen Situs, welche modulo p durch eine Frobenius-Potenz trivialisiert werden, auf einer proendlichen étalen Überlagerung trivialisierbar sind, und dass proendlich étale trivialisierbaren Moduln eine Darstellung zugeordnet werden kann. Dann beweisen wir, dass man durch Zurückziehen von formalen Vektorbündeln mit numerisch flacher Reduktion zum pro-étalen Situs Frobenius-trivialisierbare Moduln erhält.

Ferner zeigen wir noch, dass alle Geradenbündel, für die eine Tensorpotenz eine Deformation des trivialen Bündels über einer zusammenhängenden Basis ist, von einem $\mathbb{C}_p$-lokalen System kommen. Letztlich zeigen wir noch, dass die Kohomologiegruppen der zu Vektorbündeln mit numerisch flacher Reduktion zugeordneten lokalen Systeme eine Hodge-Tate Filtrierung besitzen.

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0.1 Introduction

In complex geometry there is a beautiful theory relating linear representations of the topological fundamental group of a compact Kähler manifold to a certain class of holomorphic objects. The story begins with a theorem of Narasimhan-Seshadri ([NS65]). Let X be a compact Riemann surface. Then there is an equivalence of categories between irreducible unitary representations of the topological fundamental group of X and so-called stable holomorphic vector bundles of degree 0. This remarkably links complex/algebro-geometric objects with purely topological ones. The theorem was then extended to the higher dimensional case by Donaldson and Mehta-Ramanathan for projective algebraic varieties (see [MR84, Theorem 5.1]), and later by Uhlenbeck-Yau to compact Kähler manifolds (see [UY86, §8]). In these cases the theorem states that a vector bundle E comes from an irreducible unitary representation if and only if E is stable and satisfies $c_1(E) \cdot H^{dim X-1} = c_2(E) \cdot H^{dim X-2} = 0$, where H is either an ample divisor (in the Mehta-Ramanathan case) or a Kähler class. Vector bundles which are semistable and satisfy

$$c_1(E) \cdot H^{dim X-1} = c_2(E) \cdot H^{dim X-2} = 0$$

can moreover be characterized as so called numerically flat vector bundles (see [DPS94, Theorem 1.18]), so that we can view the Uhlenbeck-Yau theorem as an equivalence between stable numerically flat bundles and irreducible unitary representations.

By the work of Hitchin, Corlette and Simpson (see [Sim92]) this correspondence has been vastly extended to a correspondence between irreducible general linear representations and stable Higgs-bundles.

In the p-adic case Faltings (in [Fal05]) has introduced an analogue of the Simpson-correspondence. For a p-adic algebraic variety X, defined over the completed algebraic closure of a discretely valued p-adic field he provides a correspondence between so-called (small) generalized representations (a category containing the continuous p-adic representations of the étale fundamental group) and (small) Higgs-bundles (see in particular [Fal05, Theorem 6]). This has been further extended in the monograph [AGT16].

In the p-adic case, if X lives over a discretely valued extension of $\mathbb{Q}_p$, there are of course different kinds of local systems one may look at. Studying local systems on X means studying representations of the arithmetic fundamental group. A p-adic Simpsons functor for such arithmetic local systems on a rigid analytic variety X over a finite extension K of $\mathbb{Q}_p$ has been introduced by Liu and Zhu in [LZ16] (see in particular Theorem 2.1 in loc. cit) using the tools introduced by Scholze in [Sch13a]. They construct a functor from the category of arithmetic local systems to the category of Higgs-bundles on $X_{\hat{\overline{K}}}$ (see remark 2.1.12 for a comparison of our work with this functor).

What is missing in all these approaches is the other direction. Namely to find a category of Higgs-bundles which come from actual representations. One may expect that, as in the complex case, there should be a semistability condition which would give the desired characterization. For the Higgs field 0 case such a semistability condition has been found through the work of Deninger and Werner ([DW05b], [DW17]).

For X a smooth proper algebraic variety over $\bar{\mathbb{Q}}_p$ they construct a functor from the category of vector bundles E on $X_{\mathbb{C}_p}$ for which there exists an integral model, i.e. a flat proper scheme $\mathcal{X}$ over $Spec(\bar{\mathbb{Z}}_p)$ together with a vector bundle $\mathcal{E}$ on $\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}$ with generic fiber X (resp. E) such that the special fiber $\mathcal{E} \times Spec(\overline{\mathbb{F}}_p)$ is numerically flat, into the category of p-adic representations of the geometric fundamental group $\pi_1^{ét}(X_{\mathbb{C}_p})$. So the condition one is familiar with from complex geometry shows up here as a condition on the special fiber of an integral model.
The main goal of the present thesis is to give a new approach to the theory of Deninger and Werner, and generalize it to the case of proper rigid analytic varieties over $\mathbb{C}_p$. Our main theorem is the following:

Theorem 0.1.1. *Let X be a connected proper rigid analytic variety over $\mathbb{C}_p$. Denote by $\mathcal{B}^s(X)$ the category of vector bundles on X with numerically flat reduction. Then there is a functor, compatible with tensor products, duals, inner homs and exterior products*

$$\rho : \mathcal{B}^s(X) \to Rep_{\pi_1^{ét}(X,x)}(\mathbb{C}_p).$$

If X is seminormal, this functor is fully faithful. Here $Rep_{\pi_1^{ét}(X,x)}(\mathbb{C}_p)$ denotes the category of continuous representations of the profinite fundamental group on finite dimensional $\mathbb{C}_p$ vector spaces.

This may be seen as a step towards an analogue of the Uhlenbeck-Yau theorem in rigid analytic geometry. Note however that it is very complicated to find vector bundles (apart from line bundles) which have numerically flat reduction, and we cannot offer anything new on this problem. One may of course hope that all vector bundles which are numerically flat on X or (semi-)stable of degree 0 (in a suitable sense) possess a model with numerically flat reduction.
The recent preprint [HW19] presents some new cases of vector bundles for which numerically flat reduction can be proved.

We make some comments on what is new in the above theorem: Apart from generalizing everything to the analytic category we also get rid of any smoothness assumption, as well as the assumption that X should be defined over a discretely valued p-adic field. Moreover full faithfulness of the functor could not be seen from the approach in [DW17]. The representations we construct give back the representations constructed in [DW17] for smooth algebraic varieties.
We remark that the full faithfulness has been established for curves in the work [Xu17], in which the constructions of Deninger-Werner (from [DW05b]) are analysed via the so-called Faltings topos. Our work bears some parallels to [Xu17], which we will try to make apparent.
The work we present rests crucially on the fundamental results on the pro-étale site of a rigid analytic variety introduced by Scholze in [Sch13a].
Let us sketch how we go about proving the above theorem. The category $Rep_{\pi_1(X)}(\mathcal{O}_{\mathbb{C}_p})$ of continuous representations on finite free $\mathcal{O}_{\mathbb{C}_p}$-modules is equivalent to the category of locally free $\hat{\mathcal{O}}_{\mathbb{C}_p}$-sheaves (i.e. local systems with coefficients in $\mathcal{O}_{\mathbb{C}_p}$). There is a functor of the latter category to the category of locally free $\hat{\mathcal{O}}_X^+$-modules, where $\hat{\mathcal{O}}_X^+$ is the completed integral structure sheaf on the pro-étale site of X, given by

$\mathbb{L} \mapsto \mathbb{L} \otimes \hat{\mathcal{O}}_X^+$ (If X is smooth, by the so called primitive comparison theorem of Scholze (see theorem 1.1.72) this becomes fully faithful after passing to $\hat{\mathcal{O}}_X^{+a}$-modules, where $\hat{\mathcal{O}}_X^{+a}$ denotes the almost version of the completed integral structure sheaf.).[1] We note here that the analogous statement on the Faltings topos is also the starting point of Faltings's p-adic Simpson correspondence.
One can then show (see theorem 2.1.8, lemma 2.1.9) that the essential image of the above functor is given by the $\hat{\mathcal{O}}_X^+$-modules which become trivial on a profinite étale covering of X. If $\mathcal{E}$ is a vector bundle on a formal scheme $\mathcal{X}$ over $Spf(\mathcal{O}_{\mathbb{C}_p})$, with generic fiber X, we can form its pullback $\mathcal{E}^+$ to the pro-étale site of X. The p-adic completion $\hat{\mathcal{E}}^+$ will then be an $\hat{\mathcal{O}}_X^+$-module and we can prove the following:

Theorem 0.1.2. *Let $\mathcal{X}$ be a proper flat connected formal scheme over $Spf(\mathcal{O}_{\mathbb{C}_p})$ and $\mathcal{E}$ a vector bundle on $\mathcal{X}$ with numerically flat reduction. Then $\hat{\mathcal{E}}^+$ is trivialized by a profinite étale cover.*

In the terminology of [Xu17] (see definition 2.1.11) this proves that all vector bundles with numerically flat reduction are Weil-Tate. This was shown in loc. cit. for the case of curves using the constructions from [DW05b].
This theorem will also be used to construct étale parallel transport on $\mathcal{E}$, as in [DW17].
The proof of the theorem follows very much the path laid out in [DW17]. In particular one shows that a vector bundle $\mathcal{E}$ with numerically flat reduction is trivialized modulo p after pullback along a composition of a finite étale cover and some power of the absolute Frobenius map. One is then faced with two problems: One is dealing with the Frobenius pullback and the other is to inductively get rid of obstructions preventing the bundle in question to be trivial modulo p^n. Both problems become much simpler after pulling back to the pro-étale site (theorem 2.2.3). In particular we can avoid the complications that arise in [DW17] in the study of integral models.

Let us give an overview of the different chapters. In the first chapter we first put together basic results from the theory of rigid analytic spaces and their formal models within Huber's theory of adic spaces. We then introduce the pro-étale site and discuss p-adic local systems and the main comparison theorem of Scholze. Then we put together some results from the theory of vector bundles in positive characteristic. In particular we recall v-descent for vector bundles on perfect schemes after Bhatt-Scholze, and then discuss numerically flat vector bundles on algebraic varieties in positive characteristic. In a small side remark we moreover show how perfect schemes can be used to capture the notion of strong-semistability. Apart from said side remark, the first chapter contains no original material.
Chapters 2 and 3 then form the main part of our work. Chapter 2 is devoted to the study of locally free $\mathcal{O}_X^+$-modules on the pro-étale site whose p-adic completion is trivialized on a profinite étale covering. To these types of modules we can associate continuous $\mathcal{O}_{\mathbb{C}_p}$-representations of the fundamental group, and show that this association gives a fully faithful functor after inverting p. We then show that any $\mathcal{O}_X^+$-module, whose mod p reduction is trivialized by a Frobenius pullback, can be

[1] We remark that locally free $\hat{\mathcal{O}}_X^{+a}$-modules are analogous to the objects called generalized representations studied in [Fal05] and [AGT16].

trivialized on a profinite étale cover.
Chapter 3 then contains the main results. Using v-descent for vector bundles on perfect schemes as established by Bhatt-Scholze we first generalize a theorem of Deninger-Werner on numerically flat vector bundles on projective schemes over a finite field to the non-projective case. Namely we show that any vector bundle E on a proper scheme over $Spec(\mathcal{O}_{\mathbb{C}_p}/p)$, for which the induced bundle on the underlying reduced scheme is numerically flat, can be trivialized by a composition of a finite étale cover with a purely inseparable map. This result is then combined with the results of chapter 2 to prove theorem 0.1.1. We will then treat the case of line bundles, and in particular show that connected deformations of the trivial bundle give rise to continuous representations. Then lastly we show that the cohomology of the local systems, which are associated to Deninger-Werner representations, comes with a Hodge-Tate filtration, which splits in case X and E are defined over a finite extension of $\mathbb{Q}_p$.

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Chapter 1

Preliminaries

1.1 Adic spaces and rigid analytic varieties

In this section we wish to recall the needed background material on rigid analytic varieties within the language of adic spaces. For adic spaces we mostly use Huber's book [Hub96] as the main reference. But see also [Sch12].
A non-archimedean field K is a topological field whose topology is induced by a non-trivial multiplicative valuation $|\cdot| : K^\times \to \mathbb{R}_{>0}$ of rank 1; i.e. $|\cdot|$ satisfies

1. $|1| = 1$
2. $|xy| = |x||y|$
3. $|x + y| \leq max(|x|, |y|)$.

Moreover we will always assume that K is complete with respect to this topology. By setting $|0| = 0$ we get an extension $K \to \mathbb{R}_{\geq 0}$ of $|\cdot|$ to K, which is also called a non-archimedean absolute value. We then denote by $\mathcal{O}_K := \{x \in K : |x| \leq 1\}$ the ring of integers of K. It is a valuation ring of height 1 with maximal ideal given by $\mathfrak{m} = \{x \in K : |x| < 1\}$. The field $\mathcal{O}_K/\mathfrak{m}$ is called the residue field of K. Moreover we will usually assume that K has mixed characteristic, i.e. K will be of characteristic 0, and $\mathcal{O}_K/\mathfrak{m}$ of characteristic p, for some prime p.
The topology on such a field is always totally disconnected, so it is not clear at all that it should be possible to set up a theory of analytic geometry over K that is worthy of the name. The first major breakthrough was Tate's introduction of so called rigid analytic varieties (which will henceforth be referred to as classical rigid spaces). Refinements of this theory were then proposed by Berkovich and later by Huber. It is the theory of Huber's so called adic spaces that we wish to employ here.

1.1.1 Adic spaces

Any form of analytic geometry is based on the idea that geometric objects should locally be defined by convergent power series (as opposed to, say, just polynomials as in algebraic geometry). In the non-archimedean context the so obtained function algebras are called affinoid Tate-algebras.

Definition 1.1.1. Define by

$$K\langle T_1, \cdots, T_n\rangle = \{\sum x_{i_1,\cdots,i_n} T_{i_1}^{m_{i_1}} \cdots T_{i_n}^{m_{i_n}} | x_{i_1,\cdots,i_n} \in K, \ |x_{i_1,\cdots,i_n}| \to 0\}$$

the ring of convergent power series on the unit ball over K. Then a Tate K-algebra A of topologically finite type (tft) is a quotient $A = K\langle T_1, \cdots, T_n\rangle/I$ for some ideal $I \subset K\langle T_1, \cdots, T_n\rangle$.

Remark 1.1.2. One can define a non-archimedean norm on $K\langle T_1, \cdots, T_n\rangle$ by setting

$$|\sum x_{i_1,\cdots,i_n} T_{i_1}^{m_{i_1}} \cdots T_{i_n}^{m_{i_n}}|_{Gauss} := max|x_{i_1,\cdots,i_n}|.$$

This norm is called Gauss-norm. One can check that $K\langle T_1, \cdots, T_n\rangle$ is complete with respect to this norm, making it a Banach K-algebra.
If A is a tft Tate K-algebra the Gauss norm on $K\langle T_1, \cdots, T_n\rangle$ induces a residue norm on A via the quotient map $K\langle T_1, \cdots, T_n\rangle \to A$ and A is again complete with respect to this norm.

Classically, the only algebras showing up in geometric situations are the topologically finite type Tate-algebras. But we will later see that Scholze's approach to p-adic Hodge theory of rigid analytic varieties rests crucially on the fact that one can define a certain topology even on varieties of topologically finite type whose generating objects are very far from being of topologically finite type themselves. These kind of spaces are not captured by classical rigid analytic geometry.

Definition 1.1.3. A topological K-algebra A is called Tate if there exists an open subring $A_0 \subset A$ together with an element $t \in A_0$ such that t is a unit in A and the induced subspace topology on A_0 is the t-adic topology. Then (A_0, t) is called couple of definition.

Example 1.1.4. Any topologically finite type Tate K-algebra A, is a Tate K-algebra. A couple of definition is given by $A_0 := \{f \in A : |f|_{Gauss} \leq 1\}$, where $|f|_{Gauss} = max|x_{i_0,\cdots,i_n}|$ denotes the Gauss norm, together with any pseudouniformizer $t \in \mathcal{O}_K$. Note that

$$A_0 = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle/(I \cap \mathcal{O}_K\langle T_1, \cdots, T_n\rangle)$$

if $A = K\langle T_1, \cdots, T_n\rangle/I$
More generally, any K-Banach algebra is Tate.

For any couple of definition (A_0, t) one checks that $A_0[\frac{1}{t}] = A$.

Definition 1.1.5. Let A be a Tate algebra, then a set $S \subset A$ is called bounded if $S \subset t^{-n}A_0$ for some couple of definition (A_0, t) of A.
An element $a \in A$ is called power bounded if the set $\{a^n : n \in \mathbb{N}\}$ is bounded. The set of power bounded elements in A is denoted by A°.

Remark 1.1.6.
- The set $A^\circ \subset A$ is actually a subring, which is open and bounded. Moreover any ring of definition is contained in A°. Also, if (A_0, t) is a couple of definition, then so is (A°, t).

- If A is topologically of finite type then the power bounded elements are again given as the elements f satisfying $|f|_{Gauss} \leq 1$.

- Clearly the element t is always topologically nilpotent. I.e. $t^n \to 0$ for $n \to \infty$.

Definition 1.1.7. Let A be a Tate algebra over K. An affinoid K-algebra is a pair (A, A^+), where $A^+ \subset A^\circ$ is an open and integrally closed subring. A^+ is then called ring of integral elements of A.
A morphism $(A, A^+) \to (B, B^+)$ of affinoid K-algebras is a continuos morphism $A \to B$ taking A^+ into B^+.
An affinoid K-algebra is called of topologically finite type if A is a topologically finite type Tate K-algebra and $A^+ = A^\circ$.

Tate's theory of rigid analytic spaces uses as its starting point the set $Sp(A)$ of maximal ideals of a topologically finite type Tate K-algebra A. In Huber's theory one has additional points.
Any maximal ideal $\mathfrak{m}$ of A defines a so called rank 1 valuation $A \to A/\mathfrak{m} \xrightarrow{|\cdot|} \mathbb{R}$, where $|\cdot|$ denotes the unique extension of the absolute value of K to the finite extension $A/\mathfrak{m}$. The idea is now to replace $Sp(A)$ by a set of valuations:

Definition 1.1.8. Let A be a ring. A valuation of A is a map $|\cdot| : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered, abelian, multiplicative group such that

- $|0| = 0$, $|1| = 1$

- $|xy| = |x||y|$ (where one defines $0 \cdot \gamma := 0$, for all $\gamma \in \Gamma$)

- $|x + y| \leq max\{|x|, |y|\}$, for all $x, y \in A$.

If A is a topological ring, a valuation $|\cdot|$ is called continuous if the set $\{a \in A : |a| < \gamma\} \subset A$ is open in A for all $\gamma \in \Gamma$.
Two valuations $|\cdot|$, $|\cdot|'$ are called equivalent if for all $a, b \in A$

$$|a| \leq |b| \iff |a|' \leq |b|'.$$

Remark 1.1.9. For any valuation $|\cdot|$ the support $supp(|\cdot|) = \{x \in A : |x| = 0\}$ is a prime ideal in A as can easily be seen from the multiplicativity of $|\cdot|$. If Q denotes the quotient field of $A/supp(|\cdot|)$ then the valuation factors as $A \to Q \to \Gamma \cup \{0\}$. Then $A(|\cdot|) := \{a \in Q : |a| \leq 1\}$ is a valuation ring.
One can then check that two valuations $|\cdot|$, $|\cdot|'$ are equivalent if and only if their supports and their valuation rings agree.

We can now come to the definition of the adic spectrum.

Definition 1.1.10. Let (A, A^+) be an affinoid K-algebra. Then define

$$\begin{aligned} Spa(A, A^+) = \{ |\cdot| : A \to \Gamma \cup \{0\} \text{ continuous valuations s.th. } |f| \leq 1, \\ \forall f \in A^+\}/equiv \end{aligned}$$

For any $x \in Spa(A, A^+)$ one writes $f \mapsto |f(x)|$ for the corresponding valuation.
We define a topology on $Spa(A, A^+)$ as the topology generated by the so called rational subsets

$$Spa(A, A^+)(\frac{f_1, \cdots, f_n}{g}) = \{x : |f_i(x)| \leq |g(x)|, \forall i\}$$

where $f_1, \cdots, f_n$ generate A, and $g \in A$.

Remark 1.1.11.

- The topological spaces $Spa(A, A^+)$ are spectral spaces, i.e. they are (abstractly) homeomorphic to the topological space underlying $Spec(B)$ for some ring B. In particular there exist generic points.
- By a fundamental result of Huber, the rational subsets are quasi-compact.
- One has a canonical isomorphism $Spa(A, A^+) \cong Spa(\hat{A}, \hat{A}^+)$ of topological spaces which takes rational subsets to rational subsets (see below).
- Let K be a complete non-archimedean field, then the space $Spa(K, \mathcal{O}_K)$ consists of a single point, corresponding to the valuation defining the topology on K.
- The classical Tate algebra gives rise to the closed unit ball

  $$\mathbb{B}^n_K = Spa(K\langle T_1, \cdots, T_n\rangle, \mathcal{O}_K\langle T_1, \cdots, T_n\rangle).$$

  Then the Gauss-norm defined above defines a non-classical point of $\mathbb{B}^n$. We refer to [Sch12, Example 2.20] for a nice description of all points occuring in the closed unit disc.

One now constructs a structure presheaf $\mathcal{O}_{Spa(A,A^+)_{an}}$ on $Spa(A, A^+)$ in the following way: Fix a couple of definition (A_0, t). Let $U = Spa(A, A^+)(\frac{f_1, \cdots f_n}{g})$ be a rational subset. We now equip $A[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]$ with the topology which makes $tA_0[\frac{f_1}{g}, \cdots, \frac{f_n}{g}] \subset A[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]$ a system of open neighbourhoods ($(A_0[\frac{f_1}{g}, \cdots, \frac{f_n}{g}], t)$ is then a couple of definition). Then let $B \subset A[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]$ be the integral closure of A^+. And let $(A\langle\frac{f_1, \cdots, f_n}{g}\rangle, \hat{B})$ be the completion. One then defines:

$$\mathcal{O}_{Spa(A,A^+)_{an}}(U) := A\langle\frac{f_1, \cdots, f_n}{g}\rangle$$

One can similarly define $\mathcal{O}^+_{Spa(A,A^+)_{an}}(U) := B$. We have the following

Proposition 1.1.12. *[Hub94, Proposition 1.3] For any map of affinoid K-algebras $f : (A, A^+) \to (S, S^+)$ sucht that $Spa(S, S^+) \to Spa(A, A^+)$ factors through U, there is a unique map $g : (A\langle\frac{f_1, \cdots, f_n}{g}\rangle, \hat{B}) \to (S, S^+)$, such that $f = g \circ \phi$, where ϕ is the canonical map $(A, A^+) \to (A\langle\frac{f_1, \cdots, f_n}{g}\rangle, \hat{B})$.*

From this one sees that $(A\langle\frac{f_1, \cdots, f_n}{g}\rangle, \hat{B})$ only depends on U. Moreover the proposition ensures that there is a unique continuous map $\mathcal{O}_{Spa(A,A^+)_{an}}(U) \to \mathcal{O}_{Spa(A,A^+)_{an}}(V)$, whenever $V \subset U$ are rational subsets.
Now for an arbitrary open $V \subset Spa(A, A^+)$ one defines

$$\mathcal{O}_{Spa(A,A^+)_{an}}(V) := \varprojlim_{U \subset V} \mathcal{O}_{Spa(A,A^+)_{an}}(U),$$

where the U run over rational subsets contained in V. Similarly, one defines $\mathcal{O}^+_{Spa(A,A^+)_{an}}(V)$. Note that one can check that

$$\mathcal{O}^+_{Spa(A,A^+)_{an}}(V) = \{f \in \mathcal{O}_{Spa(A,A^+)_{an}}(V) : |f(x)| \leq 1, \forall x \in V\}.$$

There is a natural isomorphism $\mathcal{O}_{Spa(A,A^+)_{an}}(Spa(A, A^+)) \cong (\hat{A}, \hat{A}^+)$.
One subtlety in the theory is that $\mathcal{O}_{Spa(A,A^+)_{an}}$ is not automatically a sheaf (and might indeed not be). One does however have the following, which is enough for us:

Definition 1.1.13. A Tate K-algebra R is called strongly noetherian if

$$R\langle T_1, \cdots T_n\rangle := \{ \text{ convergent power series } \sum x_{i_1,\cdots,i_n} T_{i_1}^{m_{i_1}} \cdots T_{i_n}^{m_{i_n}} \text{ with } x_{i_1,\cdots,i_n} \in \hat{R}\}$$

is noetherian for all $n \geq 0$.

In particular any tft Tate K-algebra is strongly noetherian.

Theorem 1.1.14. *[Hub94, Theorem 2.2] Let (A, A^+) be an affinoid Tate K-algebra, such that A is strongly noetherian. Then $\mathcal{O}_{Spa(A,A^+)_{an}}$ is a sheaf.*

Remark 1.1.15.

- It is easy to see that $\mathcal{O}^+_{Spa(A,A^+)_{an}}$ is a sheaf whenever $\mathcal{O}_{Spa(A,A^+)_{an}}$ is a sheaf as it is characterized by the pointwise conditions

  $$f \in \mathcal{O}^+_{Spa(A,A^+)_{an}}(U) \iff |f(x)| \leq 1, \forall x \in U$$

  for some open U and $f \in \mathcal{O}_{Spa(A,A^+)_{an}}(U)$.

- Let $X = Spa(A, A^+)$. Then by [Hub94, Lemma 1.5] the stalks $\mathcal{O}_{X_{an},x}$ and $\mathcal{O}^+_{X_{an},x}$ are local rings, and the valuation x extends to $\mathcal{O}_{X_{an},x}$ (remark however that one does not put a topology on $\mathcal{O}_{X_{an},x}$). One has

  $$\mathcal{O}^+_{X_{an},x} = \{f \in \mathcal{O}_{X_{an},x} : |f(x)| \leq 1\}.$$

- If $U \subset X$ is a rational subset one gets $U \cong Spa(\mathcal{O}_{X_{an}}(U), \mathcal{O}^+_{X_{an}}(U))$.

Definition 1.1.16. Consider the category $\mathcal{V}$ of triples $(Y, \mathcal{O}_Y, |(\cdot(y)|, y \in Y)$, where $(Y, \mathcal{O}_Y)$ is a locally ringed space, where moreover $\mathcal{O}_Y$ is a sheaf of complete topological K-algebras and $|\cdot(y)|$ is an equivalence class of valuations on the stalk $\mathcal{O}_{Y,y}$ at y.
An adic space X over K is an object in the category $\mathcal{V}$ which is locally isomorphic to $(Spa(A, A^+), \mathcal{O}_{Spa(A,A^+)_{an}}, v_x)$, where v_x is the valuation induced by x on the stalk.

Remark 1.1.17.

- One can show that for any adic space X, and affinoid adic space $Spa(A, A^+)$ one has a bijection

  $$Hom(X, Spa(A, A^+)) = Hom((\hat{A}, \hat{A}^+), (\mathcal{O}_{X_{an}}(X), \mathcal{O}^+_{X_{an}}(X)).$$

Definition 1.1.18. A rigid analytic variety over K is a quasi-separated adic space X over K, such that X is locally isomorphic to $Spa(A, A^+)$, where (A, A^+) is an affinoid K-algebra of topologically finite type.

Remark 1.1.19.

- As stated before, classical rigid analytic varieties over K are built by gluing maximal spectra $Sp(A)$ of topologically finite type Tate K-algebras. There is a functor $Sp(A) \to Spa(A, A^\circ)$ which extends to the category of rigid analytic varieties, and provides a full embedding in the quasi-separated (qs) case

$$r : (\text{ qs classical rigid spaces } /K) \hookrightarrow (\text{ qs adic spaces } /Spa(K, \mathcal{O}_K))$$

which induces an equivalence of ringed topoi $(X, \mathcal{O}_X) \cong (|r(X)|, \tilde{\mathcal{O}}_{r(X)})$, where on the right we mean the topos associated to the ringed space $r(X)$. Recall that on a classical rigid space there is only a Grothendieck topology, which is why we need to take the associated topos also on $r(X)$. This embedding justifies our terminology.

- Taking a maximal ideal $\mathfrak{m}$ to the valuation $A \to A/\mathfrak{m} \to \mathbb{R}_{\geq 0}$ realizes $Sp(A)$ as a subspace of $Spa(A, A^\circ)$. The corresponding points are called classical points.

Definition 1.1.20. [Hub94, Proposition 3.8] Let X be a scheme of finite type over $Spec(K)$. We then define the analytification X^{an} of X as the fiber product in the category of locally ringed spaces

$$\begin{tikzcd} X^{an} \arrow[r] \arrow[d] & X \arrow[d] \\ Spa(K, \mathcal{O}_K) \arrow[r] & Spec(K) \end{tikzcd}$$

.

Example 1.1.21. Consider the affine line $\mathbb{A}^1_K = Spec(K[T])$ over K. Then $(\mathbb{A}^1_K)^{an} = \bigcup_{n\geq 1} Spa(K\langle r^{-n}T\rangle, \mathcal{O}_K\langle r^{-n}T\rangle)$ (where r is some element in K, such that $|r| > 1$). Indeed if $Spa(A, A^+)$ is any K-affinoid adic space, we need to show that any map $Spa(A, A^+) \to Spec(K[T])$ factors uniquely through $\bigcup_{n\geq 1} Spa(K\langle r^{-n}T\rangle, \mathcal{O}_K\langle r^{-n}T\rangle)$. Any such map corresponds to a map $K[T] \to A$. Then this extends uniquely to the completion $K\langle T\rangle \to \hat{A}$. As $r^{-n}T$ tends to zero, and $\hat{A}^+ \subset \hat{A}$ is open, for some large enough $N >> 0$ the induced map $\mathcal{O}_K\langle r^{-N}T\rangle \to K\langle T\rangle \to \hat{A}$ has image in $\hat{A}^+$. This gives a map $(K\langle r^{-N}T\rangle, \mathcal{O}_K\langle r^{-N}T\rangle) \to (\hat{A}, \hat{A}^+)$ of affinoid K-algebras. Then

$$Spa(\hat{A}, \hat{A}^+) \to Spa(K\langle r^{-N}T\rangle, \mathcal{O}_K\langle r^{-N}T\rangle) \to \bigcup_{n\geq 1} Spa(K\langle r^{-n}T\rangle, \mathcal{O}_K\langle r^{-n}T\rangle)$$

is the desired map.

Definition 1.1.22. A separated, topologically finite type adic space $X \to Spa(K, \mathcal{O}_K)$ is called partially proper, if for all non-archimedean fields $K \subset L$, with any bounded valuation subring $L^+ \subset L$, and all morphisms $\phi : Spa(L, \mathcal{O}_L) \to X$, there exists a unique morphism $\tilde{\phi} : Spa(L, L^+) \to X$ such that

$$\begin{tikzcd} Spa(L, \mathcal{O}_L) \arrow[r] \arrow[d, "\phi"] & Spa(L, L^+) \arrow[dl, "\tilde{\phi}"] \\ X & \end{tikzcd}$$

is commutative.
X is called proper, if it is partially proper and quasi-compact.

Remark 1.1.23. The way of defining partial properness is reminiscent of the valuative criterion in algebraic geometry. Remark here that in the definition above L^+ may be a higher rank valuation ring, in which case $Spa(L, L^+)$ contains more than one point.
There is of course also a relative notion or (partial) properness (see [Hub96, Definition 1.3.3]).

1.1.2 Formal models

In this subsection we wish to recall a few notions of Raynaud's approach to rigid analytic geometry via formal schemes (see [BL93]), within the language of adic spaces.
Fix again a non-archimedean field and let $\mathcal{O}_K$ be its ring of integers. This is a valuation ring of rank 1. Fix a pseudouniformizer π of K, i.e. π is some non-zero element in the maximal ideal. Then the topology on $\mathcal{O}_K$ is the π-adic topology. Similar to before, we call an $\mathcal{O}_K$-algebra A of topologically of finite type if $A = \mathcal{O}_K\langle T_1, \cdots T_n\rangle/I$, for some ideal I.

Definition 1.1.24. An $\mathcal{O}_K$-algebra A is called admissible, if $A = \mathcal{O}_K\langle T_1, \cdots T_n\rangle/I$ is topologically of finite type, I is finitely generated and A is $\mathcal{O}_K$-torsion free.

Note that in our case admissible $\mathcal{O}_K$-algebras are topologically finite type algebras that are flat over $\mathcal{O}_K$.

Definition 1.1.25. A formal scheme $\mathcal{X}$ over $Spf(\mathcal{O}_K)$ is called admissible, if it is locally isomorphic to $Spf(A)$ for an admissible $\mathcal{O}_K$-algebra A.

There is a functor from admissible formal schemes over $\mathcal{O}_K$ to the category of rigid analytic varieties over K, given locally (in Huber's language) by $Spf(A) \to Spa(A.A[\frac{1}{t}])$, where t is any pseudouniformizer in $\mathcal{O}_K$. More precisely one has the follwoing

Proposition 1.1.26. *[Hub96, Proposition 1.9.1] Let $\mathcal{X}$ be an admissible formal scheme over $Spf(\mathcal{O}_K)$. Then there exists a rigid analytic variety X over $Spa(K, \mathcal{O}_K)$, together with a morphism $\phi : (X, \mathcal{O}_{X_{an}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of topologically ringed spaces such that*

1. *$im(\mathcal{O}_{X_{an}} \to \phi_*\mathcal{O}_{X_{an}}) \subset \phi_*\mathcal{O}^+_{X_{an}}$ and the induced morphism $sp : (X, \mathcal{O}^+_{X_{an}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a morphism of locally ringed spaces.*
2. *For any adic space over Z over $Spa(K, \mathcal{O}_K)$ with a morphism $f : Z \to \mathcal{X}$ of topologically ringed spaces satisfying property 1, there is a unique morphism $\tau : Z \to X$ of adic spaces, such that $f = \phi \circ \tau$.*

This association induces a functor

$$d : (\textit{Admissible formal schemes } /Spf(\mathcal{O}_K) \to (\textit{Rigid analytic varieties over } Spa(K, \mathcal{O}_K)).$$

Remark 1.1.27. In the case where $\mathcal{O}_K$ is noetherian (i.e. when K is discretely valued) one can embed the category of admissible formal schemes over $Spf(\mathcal{O}_K)$ into the category of adic spaces (locally the functor is given by $Spf(A) \mapsto Spa(A, A)$). Then the above construction can be carried out entirely within this big category (see [Hub94]). In the non-noetherian case one runs into the problem of proving that the structure presheaf on $Spa(A, A)$ is a sheaf.

The construction of the above map follows from the following lemma which can be found in the proof of [Hub94, Proposition 4.1].

Lemma 1.1.28. *[Hub94] Let Y be any adic space, and $(Z, \mathcal{O}_Z)$ be any locally topologically ringed space. Then there is a bijection between the set of morphisms $(Y, \mathcal{O}^+_{Y_{an}}) \to (Z, \mathcal{O}_Z)$ of locally topologically ringed spaces, and the set of continuous ring homomorphisms $\mathcal{O}_Z(Z) \to \mathcal{O}^+_{Y_{an}}(Y)$.*

Using the above statement, the identity map $A \to A$ gives rise to a morphism of locally ringed spaces

$$sp : (Spa(A[\tfrac{1}{t}], A), \mathcal{O}^+) \to (Spf(A), \mathcal{O}_{SpfA}),$$

Definition 1.1.29. Let $\mathcal{X}$ be an admissible formal scheme over $Spf(\mathcal{O}_K)$. Then the rigid analytic space associated to it in proposition 1.1.26 is called the generic fiber of $\mathcal{X}$. The morphism

$$sp : (X, \mathcal{O}^+_{X_{an}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

is called specialization map.

Remark 1.1.30. For any continuous valuation $x \in Spa(A[\frac{1}{t}], A)$, the image $sp(x)$ is given by the open prime ideal $\{f \in A | |f(x)| < 1\} \subset A$ (this is by construction of the bijection in the above lemma in [Hub94]).

It is important to note (see [Hub96, Example 1.9.2]) that this generic fiber construction is compatible with the classical construction of Raynaud (see [BL93, §4]). More precisely, if

$$r : (\text{qs classical rigid varieties}/\ K \text{ in the sense of Tate }) \hookrightarrow (\text{ Adic spaces over } Spa(K, \mathcal{O}_K))$$

denotes the full embedding, and $\mathcal{X}_{rig}$ denotes the rigid generic fiber construcion of Raynaud from [BL93, §4], then there is a functorial isomorphism

$$r(\mathcal{X}_{rig}) \cong d(\mathcal{X})$$

where $d(\mathcal{X}) = X$ denotes the adic space constructed in proposition 1.1.26.

Remark 1.1.31. One can show ([Hub96, Remark 1.3.18]) that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of admissible formal schemes is proper if and only if the induced map f^{ad} on the generic fiber is proper.

One of the main results of Raynaud is that a formal model of a rigid analytic variety is unique up to a blowup in the special fiber. More precisely:
Let $\mathcal{X}$ be an admissible formal scheme over $Spf(\mathcal{O}_K)$ and let $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ be an open ideal. Then define:

$$\mathcal{X}' := \varinjlim_n Proj(\bigoplus_{n=0}^{\infty}(\mathcal{I}^n \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}/\pi^{n+1}).$$

There is a natural projection $\phi : \mathcal{X}' \to \mathcal{X}$. ϕ is called admissible formal blowup. The ideal $\mathcal{I}$ defines a formal subscheme $\mathcal{Y} \subset \mathcal{X}$, which is called the center of the blowup.

Proposition 1.1.32. *[BL93, Proposition 2.1] If $\mathcal{X}' \to \mathcal{X}$ is an admissible formal blowup, then $\mathcal{X}'$ is an admissible formal scheme.*

As $\mathcal{I}$ is assumed to be an open ideal, ϕ induces an isomorphism on the generic fiber. The main result of Raynaud is then the following:

Theorem 1.1.33. *[BL93, Theorem 4.1] The functor d from proposition 1.1.26 induces an equivalence of categories between*

- *The category of quasi-compact admissible formal schemes over $Spf(\mathcal{O}_K)$, localized by admissible formal blowups, and*
- *The category of quasi-compact and quasi-separated rigid analytic varieties over $Spa(K, \mathcal{O}_K)$.*

For any admissible formal scheme $\mathcal{X}$ over $Spf(\mathcal{O}_K)$ we call the associated rigid analytic variety X the generic fiber of $\mathcal{X}$ and also denote it by $\mathcal{X}_K$.

Example 1.1.34.

- If we take the affine line over $\mathcal{O}_K$ and complete it along its special fiber we get the formal scheme $Spf(\mathcal{O}_K\langle T\rangle)$. The rigid space generic fiber is then the unit disc $Spa(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$.
- We will mostly only be interested in proper rigid analytic varieties. In the case where $\mathcal{X}$ is a proper scheme over $Spec(\mathcal{O}_K)$, let $\hat{\mathcal{X}}$ denote the completion along its special fiber. Then the generic fiber $\hat{\mathcal{X}}_K$ of the formal scheme is canonically isomorphic to the analytification of $\mathcal{X}_K$.
  In the non-proper case this is not the case, as can be seen from the previous example.

Using the formal model approach to rigid geometry one can find many examples of non-algebraic rigid analytic varieties by looking at non-algebraizable lifts of algebraic varieties of characteristic p to characteristic 0. Such lifts already exist for abelian varieties or K3-surfaces.

Example 1.1.35. In complex geometry one of the first examples of a non-Kähler manifold is the so called Hopf-surface. One can mimick its construction in the non-archimedean case: Fix a non-zero element $q \in K^*$ such that $|q| < 1$. Then consider the action $(T_1, T_2) \mapsto (q^n T_1, q^n T_2)$, for $n \in \mathbb{Z}$ on the punctured affine space $(\mathbb{A}^2_K)^{an} \backslash (\{0, 0\})$. Then the condition $|q| < 1$ ensures that the action is properly discontinuous. Hence the quotient $H = (\mathbb{A}^2_K)^{an} \backslash (\{0, 0\})/q^{\mathbb{Z}}$ exists and is called a Hopf-surface. It was recently shown in [Li17] that H does not possess a formal model with projective special fiber.

1.1.2.1 Vector bundles on rigid analytic varieties and their formal models

Let X be a rigid analytic variety. A vector bundle E on X is a locally free $\mathcal{O}_{X_{an}}$-module. We remark again, that the embedding $X \mapsto r(X)$ of classical quasi-separated rigid spaces into the category of adic spaces is compatible with admissible open covers, and one gets an equivalence of ringed topoi $(X, \mathcal{O}_X) \cong (|r(X)|, \tilde{\mathcal{O}}_{r(X)})$, where here again $(|r(X)|, \tilde{\mathcal{O}}_{r(X)})$ denotes the topos associated to the ringed topological space $(r(X), \mathcal{O}_{r(X)})$. Using this we are again at liberty to use all the results proved on locally free (or coherent) sheaves within the framework of classical rigid geometry.

Remark 1.1.36. One can prove very generally (see [KL15, §2.7]) that if $X = Spa(A, A^+)$ is an affinoid adic space, then vector bundles on X correspond to projective A-modules. Note that this is false for $\mathcal{O}^+_{X_{an}}$ (as this sheaf is not locally acyclic).

Let $\mathcal{X}$ be an admissible formal scheme over $\mathcal{O}_K$, with generic fiber X. Recall from the last section that we have a morphism of locally ringed spaces $sp : (X, \mathcal{O}^+_{X_{an}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. So if $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$, then we can associate to it an $\mathcal{O}^+_{X_{an}}$-module $sp^{-1}\mathcal{F} \otimes_{sp^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}^+_{X_{an}}$. We call $\mathcal{F}^{ad} := (sp^{-1}\mathcal{F} \otimes_{sp^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}^+_{X_{an}}) \otimes K$ the generic fiber of $\mathcal{F}$. $\mathcal{F}^{ad}$ is then a coherent sheaf on X.

Definition 1.1.37. Let E be a vector bundle on X. A formal model $\mathcal{E}$ of E is a locally free sheaf on a formal model $\mathcal{X}$ of X, such that there is an isomorphism $E \cong \mathcal{E}^{ad}$.

Definition 1.1.38. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be an admissible formal blowup with respect to the open ideal $\mathcal{I}$ and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. We then call $(\pi^*\mathcal{F})/\mathcal{N}$ the strict transform of $\mathcal{F}$ with respect to π. Here $\mathcal{N} = Ann_{\pi^*\mathcal{F}}(\mathcal{I}\mathcal{O}_{\mathcal{X}'})$.

It is always possible to extend a coherent sheaf on the generic fiber to the formal model:

Lemma 1.1.39. *[BL93, Proposition 5.6] Let $\mathcal{X}$ be an admissible formal scheme over $Spf(\mathcal{O}_K)$, with generic fiber X, and let F be a coherent sheaf on X. Then there exists a coherent sheaf on $\mathcal{X}$ such that $\mathcal{F}^{ad} \cong F$.*

Formal models of locally free sheaves then always exist by the flattening techniques in [BLR95] (note that here we cannot fix a model for X) :

Proposition 1.1.40. *(see [BLR95, Theorem 4.1]) Let X be a quasi-compact rigid analytic variety, and let E be a vector bundle on X. Then there exists a formal model $\mathcal{X}$ of X and a locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ whose generic fiber is isomorphic to E.*

Proof. Let $\mathcal{X}$ be a formal model of X. By the previous lemma one can always find a coherent sheaf model $\mathcal{F}$. Then by [BLR95, Theorem 4.1] there is an admissible formal blowup $\mathcal{X}' \to \mathcal{X}$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is flat over $\mathcal{X}'$. But then $\mathcal{F}'$ is locally free. □

1.1.3 The étale fundamental group

In this section we will first recall a few notions on the étale topology of an adic space. As in the case of schemes there are many equivalent characterizations of étale maps (via the functional determinant, differential module, infinitesimal lifting..., see [Hub96, §1.6 and 1.7]), at least in the topologically finite type situation. We opt here for the quickest definition:

Definition 1.1.41. [Sch12, Definition 7.1] A morphism $Spa(A, A^+) \to Spa(B, B^+)$ is called finite étale, if $B \to A$ is a finite étale map of K-algebras and A^+ is the integral closure of the image of B^+.
A morphism $X \to Y$ of adic spaces is called finite étale if it is finite étale locally, i.e. if there exists an open affinoid cover $\{U_i\}$ of Y, such that $U_i \times_Y X \to U_i$ is affinoid and $U_i \times_Y X \to U_i \to U_i$ is finite étale.
A morphism $X \to Y$ is called étale for every point $x \in X$ there exists an open neighbourhood $x \in U$, and an open $V \subset Y$ such that $f(U) \subset V$, and such that $f|_U$ factors as $U \xrightarrow{i} W \xrightarrow{f} V$, where i is an open embedding, and f is finite étale.

Remark 1.1.42. One nice aspect of the above definition is that it also works in non topologically finite type situations where there is no differential calculus available.

Assume for the rest of the section that X is a rigid analytic variety.

Definition 1.1.43. The (small) étale site of X is the site whose objects are the étale maps $Y \to X$ and covers are given by families $(Y_i \xrightarrow{f_i} Y)$ of morphisms of étale adic spaces over X, such that $\bigcup_i f_i(|Y_i|) = |Y|$.

Remark 1.1.44. There is a structure sheaf on the étale site of an adic space, given by $\mathcal{O}_{X_{ét}}(U) = \mathcal{O}_{U_{an}}(U)$ for $U \in X_{ét}$ (see [Hub96, p. 2.2.5]).
Similarly one also has the integral version

$$\mathcal{O}^+_{X_{ét}}(U) = \{f \in \mathcal{O}_{X_{ét}}(U) : |f(x)| \leq 1, \forall x \in U\}$$

There are different types of fundamental groups on non-archimedean analytic spaces one may consider, see [DJ95].

Remark 1.1.45. The article [DJ95] uses the language of classical rigid geometry. However, by comparison results of Huber the étale topology behaves well with respect to the embedding of classical rigid varieties into the category of adic spaces. More precisely, if X is a classical Tate rigid variety and $r(X)$ is the associated adic space (a rigid analytic variety in our language), one has a natural equivalence of sites $X_{qsét} \leftrightarrow r(X)_{qsét}$, where the objects are étale objects which are quasi-separated. For abstract reasons this induces an equivalence of the full étale topoi $X^{\sim}_{ét} \leftrightarrow r(X)^{\sim}_{ét}$. (see [Hub96, Proposition 2.1.4])
Moreover, if $f : X \to Y$ is a morphism of classical rigid analytic varieties, then by [Hub96, Lemma 1.4.5] f is finite if and only if $r(f) : r(X) \to r(Y)$ is finite. So one always has a canonical equivalence of sites $X_{fét} \leftrightarrow r(X)_{fét}$.

We will only be concerned with the profinite fundamental group. Keep assuming that X is a connected rigid analytic variety.
Define by $Cov^{f\acute{e}t}(X)$ the category of finite étale surjective maps (such maps will also be refered to as finite étale covers). Then we fix a classical geometric point $\bar{x} : Spa(K(x), K(x)^{\circ}) \to X$, i.e. $K(x)$ is an algebraically closed non-archimedean field with ring of integers $K(x)^{\circ}$ (one could actually also consider geometric points with values in $(K(x), K(x)^{+})$, where $K(x)^{+}$ is a more general bounded valuation subring), we define the fiber functor $F_{\bar{x}} : Cov^{f\acute{e}t}(X) \to (Set)$ by letting

$$F_{\bar{x}}(f : Y \to X) = \{y : Spa(K(x), K(x)^{\circ}) \to Y : f(y) = \bar{x}\}.$$

Assume from now on that X is connected. The étale fundamental group with respect to a base point $\bar{x}$ is then defined as

$$\pi_1^{\acute{e}t}(X, \bar{x}) := Aut(F_{\bar{x}})$$

The group $\pi_1^{\acute{e}t}(X, \bar{x})$ is endowed with the profinite topology as usual. We remark that $\pi_1^{\acute{e}t}(X, \bar{x})$ is called algebraic fundamental group (and denoted by π_1^{alg}) in [DJ95]. For any topological group G we define $G - fset$ as the category of finite sets (with the discrete topology) with a continuous G action. As in the case of schemes one can prove

Proposition 1.1.46. *[DJ95, Theorem 2.10] $F_{\bar{x}}$ may be seen as a functor*

$$Cov^{f\acute{e}t}(X) \to \pi_1^{\acute{e}t}(X) - fset$$

which is an equivalence of categories.

We will now define the fundamental groupoid of a rigid analytic variety over K.

Definition 1.1.47. [DW05b, §3] A topological groupoid is a groupoid $\mathcal{C}$ such that for all objects $x, y \in \mathcal{C}$ the set of morphisms $Mor(x, y)$ is a topological space.
A morphism $F : \mathcal{C} \to \mathcal{D}$ of topological groupoids is a functor such that the map on morphisms $Mor(x, y) \to Mor(F(x), F(y))$ is continuous.

Recall the following construction from [DW05b]: Let X be connected. Denote by $\Pi_1(X)$ the category whose objects are given by the (classical) geometric points of X, and for any x, y geometric points, we set $Mor(x, y) := Isom(F_x, F_y)$. Then the sets of morphisms again can be endowed with the profinite topology. This makes $\Pi_1(X)$ a topological groupoid, called the fundamental groupoid.

Remark 1.1.48. Assume that K is a non-archimedean field of characteristic 0. Let X be a quasi-separated scheme of finite type over $Spec(K)$. Denote by X^{an} be the analytification (definition 1.1.20). Then there is the following important theorem

Theorem 1.1.49. *[Lüt93, Theorem 3.1] Assume that $\tilde{f} : \tilde{Y} \to X^{an}$ is a finite étale morphism. Then there exists a finite étale morphism of schemes $f : Y \to X$ such that $Y^{an} \cong \tilde{Y}$ and $f^{an} = \tilde{f}$.*

From this one sees that $\Pi_1(X) \cong \Pi_1(X^{an})$.
Remark here that one can check that any morphism $Spa(R, R^{+}) \to X$, where X is a scheme, factors uniquely through the map $Spa(R, R^{+}) \to Spec(R)$ which takes a valuation v to its support. So in particular $Spa(L, \mathcal{O}_L) \to X$ factors uniquely as $Spa(L, \mathcal{O}_L) \to Spec(L) \to X$, which shows that $X(Spec(L)) = X^{an}(Spa(L, \mathcal{O}_L))$.

1.1.4 The pro-étale site of a rigid analytic space

The pro-étale site of a rigid analytic variety was introduced in [Sch13a]. The main idea is to make the results on perfectoid spaces applicable in more geometric situations. Consider the standard example of an affinoid perfectoid space, which is given by

$$Spa(K\langle T_1^{\pm 1/p^\infty}, \cdots, T_n^{\pm 1/p^\infty}\rangle^\wedge, \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \cdots, T_n^{\pm 1/p^\infty}\rangle^\wedge),$$

for a perfectoid field K. Then this is obtained by taking an infinite tower of finite étale maps over $\mathbb{T}_n$, and then taking the p-adic completion. This motivates the following definition:

Definition 1.1.50. [BMS18, §5] Let X be a locally noetherian adic space. Denote by $pro - X_{\acute{e}t}$ the category of pro-objects over the étale site. We write the objects in $pro - X_{\acute{e}t}$ as $''\varprojlim''_{i\in I} U_i$, where I is a small cofiltered category with a functor $I \to X_{\acute{e}t}$. $U \in X_{pro\acute{e}t}$ is called pro-étale if it is isomorphic to some $''\varprojlim''_{i\in I} U_i$, where all transition maps $U_i \to U_j$ are finite étale.
The pro-étale site $X_{pro\acute{e}t}$ is the full subcategory of pro-étale objects over X, and covers are given by collection of maps $(f_i : V^i \to U)$, such that $\bigcup_{i\in I} f_i(|V^i|) = |U|$, where the topological space for some $U ='' \varprojlim''_{i\in I} U_i$ is defined as the inverse limit of topological spaces $|U| := \varprojlim |U_i|$. And moreover every $V^i \to U$ can be written as an inverse limit $V^i = \varprojlim_{\mu<\lambda} V_\mu$, with $V_\mu \in X_{pro\acute{e}t}$, for some ordinal λ, such that $V^0 \to U$ is étale (i.e. a pullback of a morphism in $X_{\acute{e}t}$), and for all $\mu > 0$ the morphism $V_\mu \to \varprojlim_{\mu'<\mu} V_{\mu'}$ is finite étale surjective (i.e a pullback of a finite étale surjective morphism in $X_{\acute{e}t}$).
The pro-finite étale site $X_{prof\acute{e}t}$ is the site with underlying category $pro - X_{f\acute{e}t}$ with covers defined as above, but where we now ask that $V_\mu \in X_{prof\acute{e}t}$.

In practice, if $\varprojlim_i U_i$ is some presentation of $U \in X_{pro\acute{e}t}$ we will usually drop the quotation marks and sloppily write $U = \varprojlim U_i$.

Example 1.1.51.
1. Assume that X is defined over $Spa(K, \mathcal{O}_K)$ for some non-archimedean field K, and let $K \subset L$ be a separable field extension. Then the pro-system $(X_{L_i})_{K\subset L_i\subset L}$, where L_i runs over all finite intermediate extensions, defines an object in $X_{pro\acute{e}t}$. In fact one can check that the localized site $X_{pro\acute{e}t}/(X_{L_i})_{K\subset L_i\subset L}$ is equivalent to $(X_L)_{pro\acute{e}t}$.

2. Let K be a perfectoid field. The perfectoid tower

$$\varprojlim Spa(K\langle T_1^{\pm 1/p^n}, \cdots, T_n^{\pm 1/p^n}\rangle, \mathcal{O}_K\langle T_1^{\pm 1/p^n}, \cdots, T_n^{\pm 1/p^n}\rangle) \to \\ Spa(K\langle T_1^{\pm}, \cdots, T_n^{\pm}\rangle, \mathcal{O}_K\langle T_1^{\pm}, \cdots, T_n^{\pm}\rangle)$$

is a standard example of a profinite étale morphism. It is a profinite torsor under the group $\mathbb{Z}_p(1)^n$.

It is shown in [Sch13a, Lemma 3.10] that $X_{pro\acute{e}t}$ is indeed a site. Any étale map $U \to X$ is in particular also pro-étale. This gives a canonical projection

$$\nu : X_{pro\acute{e}t} \to X_{\acute{e}t}$$

One has the following:

Lemma 1.1.52. *[Sch13a, Lemma 3.16 + Corollary 3.17]*

- *Assume that X is qcqs and let $\mathcal{F}$ be any abelian sheaf on $X_{\acute{e}t}$. Then $H^i(U, \nu^*\mathcal{F}) = \varinjlim H^i(U_i, \mathcal{F})$, for all qcqs objects $U = \varprojlim U_i \in X_{pro\acute{e}t}$.*
- *For any abelian sheaf $\mathcal{F}$ on $X_{\acute{e}t}$ the adjuntion morphism $R\nu_*\nu^*\mathcal{F} \to \mathcal{F}$ is an isomorphism. In particular the pullback functor on abelian sheaves*

$$\nu^* : Ab(X_{\acute{e}t}) \to Ab(X_{pro\acute{e}t})$$

is fully faithful.

Remark 1.1.53. The most important feature of the pro-étale site is that it is locally perfectoid in the following sense: Let X be a locally noetherian adic space over $Spa(K, \mathcal{O}_K)$, where $(K, \mathcal{O}_K)$ is a perfectoid field of characteristic 0. An affinoid perfectoid object $U \in X_{pro\acute{e}t}$ is an object that has a presentation $U = \varprojlim Spa(R_i, R_i^+)$, such that if R^+ denotes the p-adic completion of $\varinjlim_i R_i^+$, and $R := R^+[\frac{1}{p}]$, the pair (R, R^+) is an affinoid perfectoid $(K, \mathcal{O}_K)$-algebra.
Then [Sch13a, Proposition4.8] states that $X_{pro\acute{e}t}$ is generated by such objects. In the case where X is smooth this can be seen in the following way: By smoothness one has étale local coordinates, i.e. locally there is an étale map which factors as a composition of rational embeddings and finite étale maps

$$X \to \mathbb{T}_n = Spa(K\langle T_1^{\pm}, \cdots, T_n^{\pm}\rangle, \mathcal{O}_K\langle T_1^{\pm}, \cdots, T_n^{\pm}\rangle)$$

Now the tower $\mathbb{T}_n^{\infty} = \varprojlim Spa(K\langle T_1^{\pm 1/p^n}, \cdots, T_n^{\pm 1/p^n}\rangle, \mathcal{O}_K\langle T_1^{\pm 1/p^n}, \cdots, T_n^{\pm 1/p^n}\rangle)$ is affinoid perfectoid. One then proves ([Sch13a, Lemma 4.5]) that being perfectoid behaves well with pullbacks (with respect to finiteétale maps and rational embeddings).
If $U = \varprojlim Spa(R_i, R_i^+)$ is affinoid perfectoid, then $\hat{U} := Spa(R, R^+)$ is an affinoid perfectoid space, and one has $\hat{U} \sim \varprojlim Spa(R_i, R_i^+)$ in the sense of [Sch12].
Note that, since by example 1.1.51 the base change to any field extension is an object in $X_{pro\acute{e}t}$, one gets that $X_{pro\acute{e}t}$ is locally perfectoid for any locally noetherian adic space over $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$.

A convenient feature of the pro-étale site is that inverse systems of sheaves are often well behaved.

Definition 1.1.54. [Sch13a, Definition 4.1]
Let X be a locally noetherian adic space over $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$. We have the following structure sheaves

- $\mathcal{O}_X^+ := \nu^{-1}\mathcal{O}_{X_{\acute{e}t}}^+$, $\mathcal{O}_X = \nu^{-1}\mathcal{O}_{X_{\acute{e}t}}$

- $\hat{\mathcal{O}}_X^+ = \varprojlim_n \mathcal{O}_X^+/p^n$, $\hat{\mathcal{O}}_X = \hat{\mathcal{O}}_X^+[\frac{1}{p}]$ (completed structure sheaves)

- $\hat{\mathcal{O}}_{X^\flat}^+ = \varprojlim_\Phi \mathcal{O}_X^+/p$ (tilted structure sheaf)

where ϕ denotes the (surjective) Frobenius on $\mathcal{O}_X^+/p$.

Remark 1.1.55. Assume that X is defined over $Spa(K, \mathcal{O}_K)$ for some perfectoid field K of characteristic 0. We will sometimes refer to the almost version of the integral structure sheaf $\mathcal{O}_X^{+a}$. This may simply be defined by setting $\mathcal{O}_X^{+a}(U) = \mathcal{O}_X^+(U)^a$, where here we mean the almost module associated to $\mathcal{O}_X^+(U)$ (over $\mathcal{O}_K$, where almost mathematics is done with respect to the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$). $\mathcal{O}_X^{+a}$ is then a sheaf of almost algebras over $\mathcal{O}_K^a$. This last point will not be relevant for us.

One has the following basic properties

Lemma 1.1.56. *[Sch13a, Lemma 4.2] Let X be a locally noetherian adic space over $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$. And let $U \in X_{proét}$. Then*

1. *Every $x \in |U|$ defines a continuous valuation $f \mapsto |f(x)|$ on $\mathcal{O}_X(U)$.*
2. *One has $\mathcal{O}_X^+(U) = \{x \in \mathcal{O}_X(U) : |x| \leq 1\}$.*
3. *For all $n \geq 1$, the natural morphism of sheaves $\mathcal{O}_X^+/p^n \cong \hat{\mathcal{O}}_X^+/p^n$ is an isomorphism, and $\hat{\mathcal{O}}_X^+(U)$ is flat over $\mathbb{Z}_p$.*
4. *For all $x \in |U|$, the valuation $f \mapsto |f(x)|$ extends to $\hat{\mathcal{O}}_X(U)$.*
5. *One has $\hat{\mathcal{O}}_X^+(U) = \{x \in \hat{\mathcal{O}}_X(U) : |x| \leq 1\}$.*

Remark 1.1.57.
- It would be quite difficult to work with the pro-étale site, if not for the fact that there is a basis of $X_{proét}$ on which the behavior of the introduced structure sheaves is (almost) well understood. This is of course the basis introduced in remark 1.1.53. Namely if $\varprojlim Spa(R_i, R_i^+) = U \in X_{proét}$ is affinoid perfectoid, with (R, R^+) the p-adic completion, one has $(\hat{\mathcal{O}}_X(U), \hat{\mathcal{O}}^+(U)) = (R, R^+)$ and the higher cohomology groups $H^i(U, \mathcal{O}_X^+)$ are almost zero (see [Sch13a, Lemma 4.10]).
- Assume that X is defined over $Spa(K, \mathcal{O}_K)$ for some perfectoid field K of characteristic 0. Then there exists a pseudouniformizer in the tilt $t \in \mathcal{O}_{K^\flat}$ such that $|t^\sharp| = |p|$ (see the Appendix). One then has $\mathcal{O}_X^+/p = \hat{\mathcal{O}}_{X^\flat}^+/t$.

Composing the projection $\nu : X_{proét} \to X_{ét}$ with the projection $X_{ét} \to X_{an}$ gives a projection

$$\lambda : X_{proét} \to X_{an}.$$

The following lemma shows that vector bundles on rigid analytic varieties can be seen as locally free sheaves over $\mathcal{O}_X$ on the pro-étale site:

Lemma 1.1.58. *[Sch13a, Lemma 7.3] Let X be a rigid analytic variety over a p-adic field K. Then pullback along the natural projections $X_{proét} \xrightarrow{\nu} X_{ét} \to X_{an}$ induces equivalences of categories between the categories of finite locally free modules over $\mathcal{O}_X$, resp. $\mathcal{O}_{X_{ét}}$, resp. $\mathcal{O}_{X_{an}}$.*

Note that in [Sch13a] X is assumed to be smooth, but the proof given there works also for the general case. (see also [KL15, Theorem 8.2.22])
We will also need the following result of Kedlaya-Liu

Theorem 1.1.59. *[KL16, Theorem 8.2.3] Let X be a semi-normal rigid analytic variety over K. Then there is a natural isomorphism $\nu_* \hat{\mathcal{O}}_X \cong \mathcal{O}_{X_{\acute{e}t}}$.*

Remark 1.1.60. The condition on X is necessary. The reason for this is that all perfectoid rings are semi-normal. So since $\hat{\mathcal{O}}_X$ takes values in perfectoid rings on a basis of $X_{pro\acute{e}t}$ it cannot distinguish between X and its seminormalization.

Corollary 1.1.61. *[KL16, Corollary 8.2.4] For any locally free $\mathcal{O}_{X_{\acute{e}t}}$-module E, the adjunction map $E \to \nu_*(\nu^* E \otimes \hat{\mathcal{O}}_X)$ is an isomorphism. In particular the functor $E \mapsto \lambda^* E \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X$ gives a fully faithful embedding of the category of vector bundles on X into the category of locally free $\hat{\mathcal{O}}_X$-modules.*

Assume now that X is a rigid analytic variety over a mixed characteristic perfectoid field K. And assume that $\mathcal{X}$ is a formal model of X, i.e. an admissible formal scheme over $Spf(\mathcal{O}_K)$ with generic fiber X. By composing the canonical projection $X_{pro\acute{e}t} \to X_{an}$ with the specialization map we get a projection map of ringed sites

$$\mu : (X_{pro\acute{e}t}, \mathcal{O}_X^+) \to (\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}}).$$

For any coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, we define the associated $\mathcal{O}_X^+$-module as the pullback: $\mathcal{E}^+ := \mu^{-1}\mathcal{E} \otimes_{\mu^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X^+$. One then easily checks the following:

Lemma 1.1.62. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of admissible formal schemes over $Spf(\mathcal{O}_K)$. Then for any coherent $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{E}$ there is a canonical isomorphism $f^*_{pro\acute{e}t}(\mathcal{E}^+) \cong (f^*\mathcal{E})^+$.*

1.1.5 p-adic local systems on rigid analytic varieties

In this section we will recall some results on local systems. Let X be a qcqs rigid analytic variety, or a qcqs noetherian adic space, defined over a characteristic 0 non-archimedean field.

Definition 1.1.63. [Sch13a, Defn. 8.1] Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$, or the ring of integers of $\mathbb{C}_p$. Let π be a (pseudo-)uniformizer of $\mathcal{O}$. An $\mathcal{O}/\pi^n$-local system is a finite locally free $\mathcal{O}/\pi^n$-module. where $\mathcal{O}/\pi^n$, denotes the constant sheaf on $X_{pro\acute{e}t}$. We denote the category of $\mathcal{O}/\pi^n$-local system by $LS_{\mathcal{O}/\pi^n}(X)$.
Define $\hat{\mathcal{O}} := \varprojlim \mathcal{O}/\pi^n$ as sheaves on $X_{pro\acute{e}t}$. Then an $\hat{\mathcal{O}}$-local system is a finite locally free $\hat{\mathcal{O}}$-module on $X_{pro\acute{e}t}$. The category of $\hat{\mathcal{O}}$-local systems will be denoted by $LS_{\hat{\mathcal{O}}}(X)$.

Remark 1.1.64. The reason for defining $\hat{\mathcal{O}}$ not simply as the constant sheaf, is that one wants to keep track of the π-adic topology. One could also define $\hat{\mathcal{O}}$ as the sheaf taking any $U \in X_{pro\acute{e}t}$ to the set of continuous maps $Map_{cont}(|U|, \mathcal{O})$ (see [KL15, §9.1]).

Most of the time we will only be interested in the cases $\mathcal{O} = \mathbb{Z}_p$, or $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$ with $\pi = p$.

Remark 1.1.65. The category of $\mathcal{O}/\pi^n$-local systems is equivalent to the category of finite locally free $\mathcal{O}/\pi^n$-sheaves on $X_{ét}$, i.e. they alle lie in the essential image of ν^*. This follows from the fact that $(\mathcal{O}/\pi^n)_{proét} = \nu^*(\mathcal{O}/\pi^n)_{ét}$, and $\nu_*\nu^*(\mathcal{O}/\pi^n)_{ét} = (\mathcal{O}/\pi^n)_{ét}$, by lemma 1.1.52. Here by $(\mathcal{O}/\pi^n)_{ét}$ (resp. $(\mathcal{O}/\pi^n)_{proét}$) we mean the constant sheaf on the étale (resp. pro-étale) site with values in $\mathcal{O}/\pi^n$.

Lemma 1.1.66. *There is an equivalence of categories*

$$LS_{\mathcal{O}_{\mathbb{C}_p}/p^n}(X) \leftrightarrow colimLS_{\mathcal{O}_K/p^n}(X),$$

where K runs over all finite extensions of $\mathbb{Q}_p$.

Proof. This follows from the fact that $\mathcal{O}_{\mathbb{C}_p}/p^n = \varinjlim \mathcal{O}_K/p^n$: For full faithfulness assume that $\{\mathbb{L}_K\}_{K\subset\overline{\mathbb{Q}}_p}$, $\{\mathbb{L}'_K\}_{K\subset\overline{\mathbb{Q}}_p}$ are two direct systems, with direct limits $\tilde{\mathbb{L}}$ and $\tilde{\mathbb{L}}'$. Then using inner homs:

$$Hom(\tilde{\mathbb{L}}, \tilde{\mathbb{L}}') = \mathcal{H}om(\tilde{\mathbb{L}}, \tilde{\mathbb{L}}')(X) = \varinjlim_{K\subset\overline{\mathbb{Q}}_p} \mathcal{H}om(\mathbb{L}_K, \mathbb{L}'_K)(X) = \varinjlim_{K\subset\overline{\mathbb{Q}}_p} Hom(\mathbb{L}_K, \mathbb{L}'_K).$$

Note here that the last equality follows from the fact that X is quasi-compact; see[Sta19, Tag 0738].
For essential surjectivity note that, since X is qcqs, for any $\mathbb{L} \in LS_{\mathcal{O}_{\mathbb{C}_p}/p^n}(X)$, there is a quasi-compact cover $Y \to X$ with quasi-compact overlaps, on which $\mathbb{L}$ becomes trivial. So $\mathbb{L}|_Y$ descends to $\mathbb{Z}_p/p^n$. But then, as $Y \times_X Y$ is quasi-compact, the gluing data will descend to some $\mathcal{O}_K/p^n$ by the full faithfulness proved above. $\square$

To see that Scholze's notion of local systems is compatible with the usual one (where one considers inverse systems of $\mathcal{O}/\pi^n$-modules on $X_{ét}$) one shows that inverse limits of certain sheaves on $X_{proét}$ are well behaved. This rests on the following general lemma:

Lemma 1.1.67. *[Sch13a, Lemma 3.18] Let $\mathcal{T}$ be a site, and $\mathcal{B}$ a basis of $\mathcal{T}$. Let $\mathcal{F}_i$ be an inverse system of abelian sheaves on $\mathcal{T}$ such that for all $U \in \mathcal{B}$:*

- $R^1 \varprojlim \mathcal{F}_i(U) = 0$
- $H^j((U, \mathcal{F}_i) = 0$, *for all $j > 0$.*

Then $R^j \varprojlim \mathcal{F}_i = 0$, for all $j > 0$.

Proposition 1.1.68. *Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$ or $\mathcal{O} = \mathbb{Z}_p$. Any inverse system $\mathbb{L}_n$ of finite locally free $\mathcal{O}/p^n$-modules on $X_{proét}$ satisfies the conditions of lemma 1.1.67.*
Also, any inverse system $\mathcal{E}_n$ of finite locally free $\mathcal{O}_X^{+a}/p^n$-modules satisfies the conditions from lemma 1.1.67.

Proof. The first condition in lemma 1.1.67 is of course always satisfied, as the inverse systems evaluated on a perfectoid basis $Spa(R, R^+)$, where the objects in questions are free, are given by the modules $\mathcal{O}^r/p^n$, resp. $(R^{+a}/p^n)^r$, where r denotes the rank. But here the transition maps are surjective.
The second condition follows from the proof of [Sch13a, Lemma 4.10 (i)] for $\mathcal{O}_X^{+a}/p^n$. For $\mathcal{O}/p^n$ the local vanishing of the cohomology is shown in the proof of [Sch13a, Theorem 4.9] in the mod p case. The rest then follows by induction. $\square$

The above lemma gives the following:

Proposition 1.1.69. *[Sch13a, Proposition 8.2] The category $LS_{\hat{\mathbb{Z}}_p}(X)$ is equivalent to the category of inverse systems $\mathbb{L}_n$ of finite locally free $\mathbb{Z}_p/p^n$-modules on $X_{\acute{e}t}$. Similarly $LS_{\hat{\mathcal{O}}_{\mathbb{C}_p}}(X)$ is equivalent to the category of inverse systems of locally free $\mathcal{O}_{\mathbb{C}_p}/p^n$-modules on $X_{\acute{e}t}$.*

We finish this section by recalling the relation of local systems with continuous representations of the fundamental group. Assume that X is connected and fix a geometric point $\bar{x}$ of X. For simplicity we write $\pi_1(X) = \pi_1^{\acute{e}t}(X, \bar{x})$. For any topological ring R we denote by $Rep_{\pi_1(X)}(R)$ the category of continuous representations of $\pi_1(X)$ on finite free R-modules.

Proposition 1.1.70. *There is an equivalence of categories*

$$LS_{\mathcal{O}_{\mathbb{C}_p}}(X) \leftrightarrow Rep_{\pi_1^{\acute{e}t}(X,\bar{x})}(\mathcal{O}_{\mathbb{C}_p})$$

Proof. The following arguments are well known. First fix $n \geq 1$. As usual, finite $\pi_1(X)$-sets correspond to finite étale covers, so $\mathbb{L} \mapsto \mathbb{L}_{\bar{x}}$ gives an equivalence of categories

$$LS_{\mathcal{O}_K/p^n}(X) \leftrightarrow Rep_{\pi_1^{\acute{e}t}(X,\bar{x})}(\mathcal{O}_K/p^n),$$

for all finite extensions $K/\mathbb{Q}_p$. Clearly, these equivalences are compatible with base extensions $\mathcal{O}_K/p^n \to \mathcal{O}_{K'}/p^n$, for $K \subset K'$. So we get

$$colim_{K \subset \overline{\mathbb{Q}}_p} LS_{\mathcal{O}_K/p^n}(X) \leftrightarrow colim_{K \subset \overline{\mathbb{Q}}_p} Rep_{\pi_1^{\acute{e}t}(X,\bar{x})}(\mathcal{O}_K/p^n).$$

Now by lemma 1.1.66, we know that $colim_{K \subset \overline{\mathbb{Q}}_p} LS_{\mathcal{O}_K/p^n}$ is equivalent to $LS_{\mathcal{O}_{\mathbb{C}_p}/p^n}(X)$. On the other hand, that $colim_{K \subset \overline{\mathbb{Q}}_p} Rep_{\pi_1^{\acute{e}t}(X,\bar{x})}(\mathcal{O}_K/p^n)$ is equivalent to $Rep_{\pi_1^{\acute{e}t}(X,\bar{x})}(\mathcal{O}_{\mathbb{C}_p}/p^n)$ follows from the fact that $\pi_1(X)$ is compact: since $GL_n(\mathcal{O}_{\mathbb{C}_p}/p^n)$ carries the discrete topology the image $\rho(\pi_1(X))$ will be finite for any continuous representation $\rho : \pi_1(X) \to GL_n(\mathcal{O}_{\mathbb{C}_p}/p^n)$.
Now passing to the p-adic completion, using proposition 1.1.69, gives the claim. $\square$

1.1.6 The primitive comparison theorem

Theorem 1.1.71. *[Sch13a, Theorem 5.1] Let X be a proper rigid analytic space over $Spa(\boldsymbol{C}, \mathcal{O}_{\boldsymbol{C}})$, where $\boldsymbol{C}$ is an algebraically closed perfectoid field of characteristic 0 with ring of integers $\mathcal{O}_{\boldsymbol{C}}$. Then the canonical maps*

$$H^i(X_{et}, \mathbb{Z}_p/p^n) \otimes \mathcal{O}_{\boldsymbol{C}}/p^n \to H^i(X, \mathcal{O}_X^+/p^n)$$
$$H^i(X_{et}, \mathbb{Z}_p) \otimes \mathcal{O}_{\boldsymbol{C}} \to H^i(X, \hat{\mathcal{O}}_X^+)$$

are almost isomorphisms for all $i \geq 0$.
If X is in addition smooth or the analytification of a proper scheme over Spec($\boldsymbol{C}$), then the canonical maps

$$H^i(X_{et}, \mathbb{L}/p^n) \otimes \mathcal{O}_{\boldsymbol{C}}/p^n \to H^i(X, \mathcal{O}_X^+/p^n \otimes \mathbb{L}/p^n)$$
$$H^i(X_{et}, \mathbb{L}) \otimes \mathcal{O}_{\boldsymbol{C}} \to H^i(X, \mathbb{L} \otimes \hat{\mathcal{O}}_X^+)$$

are almost isomorphisms for any lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$.

Proof. The mod p statements can be found in [Sch13a, Theorem 5.1] (smooth case), [Sch13b, Theorme 3.13] (algebraic case) and [Sch13b, Theorem 3.17] (for general proper rigid analytic varieties). The full statements all then follow by induction combined with [Sch13a, Lemma 3.18].
We will sketch the induction argument for the constant local system:
For any $n \geq 1$ one has a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "="] & p^n\mathbb{Z}_p/p^{n+1} \arrow[r] \arrow[d] & \mathbb{Z}_p/p^{n+1} \arrow[r] \arrow[d] & \mathbb{Z}_p/p^n \arrow[r] \arrow[d] & 0 \arrow[d, "="] \\ 0 \arrow[r] & p^n\mathcal{O}_X^+/p^{n+1} \arrow[r] & \mathcal{O}_X^+/p^{n+1} \arrow[r] & \mathcal{O}_X^+/p^n \arrow[r] & 0 \end{tikzcd}$$

. As $\mathcal{O}^+$ is flat over $\mathbb{Z}_p$ one has $p^n\mathcal{O}_X^+/p^{n+1} \cong \mathcal{O}_X^+/p^n$. Taking cohomology for all $i \geq 0$ one gets a commutative diagram with exact rows

$$\begin{tikzcd} H^{i-1}(\mathbb{Z}_p/p^n)_{\mathcal{O}_{\mathbf{C}}} \arrow[r] \arrow[d] & H^i(\mathbb{Z}_p/p^n)_{\mathcal{O}_{\mathbf{C}}} \arrow[r] \arrow[d] & H^i(\mathbb{Z}_p/p^{n+1})_{\mathcal{O}_{\mathbf{C}}} \arrow[r] \arrow[d] & H^i(\mathbb{Z}_p/p^n)_{\mathcal{O}_{\mathbf{C}}} \arrow[r] \arrow[d] & H^{i+1}(\mathbb{Z}_p/p^n)_{\mathcal{O}_{\mathbf{C}}} \arrow[d] \\ H^{i-1}(\mathcal{O}_X^+/p^n) \arrow[r] & H^i(\mathcal{O}_X^+/p^n) \arrow[r] & H^i(\mathcal{O}_X^+/p^{n+1}) \arrow[r] & H^i(\mathcal{O}_X^+/p^n) \arrow[r] & H^{i+1}(\mathcal{O}_X^+/p^n). \end{tikzcd}$$

Now using the induction hypothesis and the 5-lemma, one gets that $H^i(\mathbb{Z}_p/p^n) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{C}} \to H^i(\mathcal{O}_X^+/p^n)$ is an almost isomorphism for all $i, n \geq 0$. Now one can use the fact (proposition 1.1.68) that the inverse systems $\mathbb{Z}_p/p^n$ (as sheaves on $X_{pro\acute{e}t}$) and $\mathcal{O}_X^{+a}/p^n$ (here $\mathcal{O}_X^{+a}$ denotes the almost version of the structure sheaf) satisfy the conditions from lemma 1.1.67. Hence taking the inverse limit commutes with taking cohomology, so that one arrives at the almost isomorphism
$H^i(X_{pro\acute{e}t}, \mathbb{Z}_p) \cong^a H^i(\hat{\mathcal{O}}_X^+)$. □

One can then also generalize the primitive comparison theorem to the case of $\mathcal{O}_{\mathbb{C}_p}$-coefficients.

Theorem 1.1.72. *Let X be a proper rigid analytic space over $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$, which is either smooth or the analytification of a proper scheme of finite type over Spec($\mathbb{C}_p$). And let $\mathbb{L}$ be an $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system on X. The canonical map*

$$H^i(X_{pro\acute{e}t}, \mathbb{L}) \to H^i(X_{pro\acute{e}t}, \mathbb{L} \otimes_{\hat{\mathcal{O}}_{\mathbb{C}_p}} \hat{\mathcal{O}}_X^+)$$

is an almost isomorphism, for all $i \geq 0$.

Proof. By the above remark we see that $\mathbb{L}/p \cong \mathbb{L}' \otimes \mathcal{O}_{\mathbb{C}_p}/p$ where $\mathbb{L}'$ is defined over $\mathcal{O}_K/p$ for $K/\mathbb{Q}_p$ a finite extension. Let π be a uniformizer of K. Then $\mathbb{L}'/\pi$ is an $\mathbb{F}_p$ local system. Hence we get $H^i(X_{\acute{e}t}, \mathbb{L}'/\pi) \otimes \mathcal{O}_{\mathbb{C}_p}/p \cong^a H^i(X_{\acute{e}t}, \mathbb{L}'/\pi \otimes \mathcal{O}^+/p)$ by theorem 1.1.71. But then by induction along the exact sequences

$$0 \to \pi^{n-1}\mathbb{L}'/\pi^n \to \mathbb{L}'/\pi^n \to \mathbb{L}'/\pi^{n-1} \to 0$$

we find that

$$H^i(\mathbb{L}/p) = H^i(\mathbb{L}') \otimes_{\mathcal{O}_K/p} \mathcal{O}_{\mathbb{C}_p}/p \to H^i(\mathbb{L}' \otimes_{\mathcal{O}_K/p} \hat{\mathcal{O}}^+/p) = H^i(\mathbb{L} \otimes_{\hat{\mathcal{O}}_{\mathbb{C}_p}} \hat{\mathcal{O}}^+/p)$$

is an almost isomorphism. But then the full statement follows again by induction and using [Sch13a, Lemma 3.18] as in the case of $\hat{\mathbb{Z}}_p$-local systems (see the proof of theorem 1.1.71). □

Remark 1.1.73. In the last proof we have used the fact that if $\mathbb{L}$ is an $\mathcal{O}_K/p^n$ local system, where $K/\mathbb{Q}_p$ is finite, one always has $H^i(X_{\acute{e}t}, \mathbb{L} \otimes_{\mathcal{O}_K/p^n} \mathcal{O}_{\mathbb{C}_p}/p^n) \cong H^i(X_{\acute{e}t}, \mathbb{L}) \otimes_{\mathcal{O}_K/p^n} \mathcal{O}_{\mathbb{C}_p}/p^n$, as $\mathcal{O}_K \subset \mathcal{O}_{\mathbb{C}_p}$ is flat.

There is a functor

$$LS_{\mathcal{O}_{\mathbb{C}_p}}(X) \to LF(\hat{\mathcal{O}}_X^+)$$

which takes $\mathbb{L}$ to $\mathbb{L} \otimes \hat{\mathcal{O}}_X^+$. Assume now that X is proper over $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$.

Proposition 1.1.74. *Let X be proper smooth or the analytification of a proper scheme over $Spec(\mathbb{C}_p)$. Then the induced functor*

$$Rep_{\pi_1(X)}(\mathcal{O}_{\mathbb{C}_p}) \otimes \mathbb{Q} \cong LS_{\mathcal{O}_{\mathbb{C}_p}}(X) \otimes \mathbb{Q} \to LF(\hat{\mathcal{O}}_X),$$

taking ρ to $\mathbb{L}_\rho \otimes \hat{\mathcal{O}}_X$, is fully faithful. Here for a representation ρ, we denote by $\mathbb{L}_\rho$ the associated local system.

Proof. This follows from theorem 1.1.72. □

Remark 1.1.75. At the integral level theorem 1.1.72 shows that one has a fully faithful embedding of $\mathcal{O}_{\mathbb{C}_p}$-local systems into the category of finite locally free almost $\hat{\mathcal{O}}_X^+$-modules.

1.2 Vector bundles on algebraic varieties in positive chracteristic

1.2.1 Vector bundles on perfect schemes

All the material presented here is from [BS17]. For any $\mathbb{F}_p$-scheme Z we denote by F_Z the absolute Frobenius, which is the identity on topological spaces and given by $x \mapsto x^p$ on sections.

Definition 1.2.1. An $\mathbb{F}_p$-scheme Z is called perfect if the absolute Frobenius morphism F_Z is an automorphism. We denote by $Perf$ the category of perfect schemes.

For an arbitrary $\mathbb{F}_p$-scheme Z one can form the inverse limit $Z_{perf} := \varprojlim_{F_Z} Z$ along the absolute Frobenius map. Then Z_{perf} is perfect scheme, called the perfection of Z. It comes with a natural projection $\pi_Z : Z_{perf} \to Z$.
Since the absolute Frobenius commutes with all morphisms of $\mathbb{F}_p$-schemes, the association $Z \mapsto Z_{perf}$ defines a functor $Sch_{\mathbb{F}_p} \to Perf$.

Remark 1.2.2. It is easy to see that Z_{perf} is the initial object of perfect schemes over Z, i.e. for any perfect scheme P and morphism $f : P \to Z$, there exists a unique morphism $\tilde{f} : P \to Z_{perf}$ such that $f = \pi_Z \circ \tilde{f}$.

A suitable topology on $Perf$ is given by morphisms which are surjective on points valued in valuation rings.

Definition 1.2.3. Let $f : X \to Y$ be a morphism of quasi-compact, quasi-separated schemes. Then f is called a v-cover if for every map $Spec(V) \to Y$, where V is a valuation ring, there is a commutative diagram

$$\begin{array}{ccc} Spec(W) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Spec(V) & \longrightarrow & Y, \end{array}$$

where $V \subset W$ is an extension of valuation rings.

Remark 1.2.4. There is a naive functor $X \mapsto X^{ad}$ from the category of schemes to the category of (discrete) adic spaces, given locally by $Spec(A) \mapsto Spa(A, A)$ (beware that this is not to be confused with the analytification if X is defined over a non-archimedean field). One checks immediately, that $f : X \to Y$ is a v-cover if and only if $f^{ad} : X^{ad} \to Y^{ad}$ is surjective on topological spaces.

Example 1.2.5.
- Every proper surjective map $X \to Y$ is a v-cover. For this note that one can lift the generic point of a valuation ring V with $Spec(V) \to Y$ to X and then use the valuative criterion.
- If $f : X \to Y$ is a v-cover, then so is the perfection $f_{perf} : X_{perf} \to Y_{perf}$. Indeed if $Spec(V) \to Y_{perf}$ is a morphism, where V is a valuation ring, we can first find a valuation ring $V \subset W$ with a map $Spec(W) \to X$, such that

$$\begin{tikzcd}
Spec(W) \arrow[rr] \arrow[d] & & X \arrow[d, "f"] \\
Spec(V) \arrow[r] & Y_{perf} \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

commutes. Now the perfection $W_{perf} = \varinjlim_{Frob} W$ is again a valuation ring, and one has $W \subset W_{perf}$ (as W is a domain), hence $V \subset W_{perf}$. Then, applying the perfection functor we get a map $Spec(W_{perf}) \to X_{perf}$ and the following diagram commutes:

$$\begin{tikzcd}
Spec(W_{perf}) \arrow[r] \arrow[d] & X_{perf} \arrow[d, "f_{perf}"] \\
Spec(V) \arrow[r] & Y_{perf}.
\end{tikzcd}$$

The main result for us is the following descent result, which in finite type situation ensures effective descent of vector bundles along proper surjective covers up to pullback along an inseparable map:

Theorem 1.2.6. *[BS17, Theorem 4.1] Let $Vect_r(-)$ denote the groupoid of vector bundles of rank r. The association $Z \mapsto Vect_n(Z)$ is a v-stack on $Perf$.*

Remark 1.2.7.
- The main technical fact which makes such a descent result possible is the fact that any two maps of perfect rings $A \to B$ and $A \to C$ are always Tor-independent, i.e. $Tor_i^A(B, C) = 0$ for all $i > 0$ (this is [BS17, Lemma 3.16]).
- Note that local freeness is a necessary condition for theorem 1.2.6.

1.2.1.1 An aside: Strongly semistable vector bundles via perfections

In a brief excursion we wish to show how one can use perfect schemes to capture strong semistability of vector bundles, these remarks will not be used anywhere else. Let X be a normal projective algebraic variety of dimension n over a perfect field k of characteristic p, with a polarization H. Then we can define slope semistability with respect to the polarization H:

Definition 1.2.8. For any vector bundle E on X define the slope as $\mu(E) = \frac{c_1(E)\cdot H^{n-1}}{rank(E)}$. Then E is called semistable (wrt H) if for all sub-vector bundles $E' \subset E$, one has $\mu(E') \leq \mu(E)$.
E is called strongly semistable if $F_X^{n*}E$ is semistable for all $n \geq 0$.

Remark 1.2.9. A standard example of a semistable vector bundle in positive characteristic which becomes unstable after Frobenius pullback is given by $F_{C*}\mathcal{O}_C$ for a smooth projective curve C of genus ≥ 2. In this case a direct computation of the degree shows that $F_C^*F_{C*}\mathcal{O}_C \to \mathcal{O}_C$ is destabilizing.

Remark 1.2.10. One could more generally define semistability with respect to a numerical effective divisor, or, even more generally, with respect to a movable curve class. We only consider the standard case, where H is ample.

We call the number $deg(E) := c_1(E)\cdot H^{n-1}$ the degree of E (of course it depends on H). One has $deg(E) = deg(det(E))$.
For every vector bundle there exists a canonical filtration whose graded pieces are semistable:

Lemma 1.2.11. *[HL10, Theorem 1.6.7] Let E be a vector bundle on X. Then there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that E_i/E_{i-1} is a semistable vector bundle, for all i, and one has $\mu(E_1) > \cdots > \mu(E/E_{l-1})$.

The filtration is called Harder-Narasimhan filtration.
We want to define a slope function on the perfection of X. Note first that for the Picard group one has the following:

Lemma 1.2.12. *[BS17, Lemma 3.5] Let X be as above. Then one has for the Picard group of the perfection*

$Pic(X_{perf}) \cong Pic(X)[\frac{1}{p}]$ as abelian groups, where the isomorphism is induced by pullback.

Proof. This is because $Pic(X_{perf})$ is just the colimit along Frobenius pullbacks, and one has $F_X^*L = L^{\otimes p}$ for all $L \in Pic(X)$. □

Using this one sees that for any invertible sheaf L on X_{perf} there exists some n such that $L^{\otimes p^n} = \pi_X^*\tilde{L}$, where $\pi : X_{perf} \to X$ is the canonical map.

Definition 1.2.13. Let E be a vector bundle on X_{perf}. Define the degree of E as

$$deg(E) := \frac{deg(\tilde{L})}{p^n}$$

where $\tilde{L}$ is a line bundle on X for which $det(E)^{\otimes p^n} = \pi_X^*\tilde{L}$.
Define then the slope of E to be $\mu_p(E) = \frac{deg(E)}{rank(E)}$. Again we call E semistable, if $\mu_p(E') \leq \mu_p(E)$ for all subbundles $E' \subset E$.

Remark 1.2.14. Note that the definition of the degree does not depend on the pair $(\tilde{L}, n)$.
Another way to define the degree is the following: The category of vector bundles on X_{perf} is equivalent to $colim_{F_X^*}(Vect(X))$. Hence if E is a vector bundle on X_{perf}, then there is a vector bundle $\tilde{E}$ on X, such that for some $n \geq 0$, $E = \pi_n^*\tilde{E}$, where $\pi_n : X_{perf} \to X$ denotes the structure map to the n-th copy of X. We then have $deg(E) = \frac{deg(\tilde{E})}{p^n}$. For this note that we have $\pi_n \circ F_X^n = \pi_X$, and if $\tilde{L}$ is a line bundle on X, one has $\tilde{L}^{\otimes p^n} = F_X^{n*}\tilde{L}$. Hence, if $L = \pi_n^*\tilde{L}$ we get $L^{\otimes p^n} = \pi_n^* F_X^{n*}\tilde{L} = \pi_X^*\tilde{L}$, so the two notions of degree coincide.

The slope defines a map $\mu_p : Vect_n(X_{perf}) \to \mathbb{Q}$. A general theory of slope functions and associated Harder-Narasimhan filtrations has been developed in [And09].

Lemma 1.2.15. *μ_p is a slope function in the sense of [And09, Def. 1.3.1].*

Proof. As $Vect_n(X_{perf})$ is an abelian category we only need to check condition (2) in loc. cit., i.e. that the degree is additive on exact sequences. But again, any exact sequence of locally free sheaves on X_{perf} will descend to an exact sequence on some copy of X. Then the additivity follows from the additivity of the degree there. □

Proposition 1.2.16. *A vector bundle E on X is strongly semistable if and only if the pullback π_X^*E is semistable with respect to μ_p.*

Proof. Assume that E is strongly semistable. If $E' \subset \pi_X^*E$ is destabilizing, then this means that, for some $n \geq 0$, we have $E' = \pi_n^*\tilde{E}'$ and $\frac{\mu(\tilde{E}')}{p^n} > \mu(E)$. But (after possibly enlarging n) we can assume that $\tilde{E}' \subset F_X^{n*}E$. But, since $\mu(F_X^{n*}E) = p^n\mu(E)$, this contradicts the strong semistability of E.
Conversely, any destabilizing $E' \subset F_X^{n*}E$ gives rise to a destabilizing subsheaf $\pi_n^*E' \subset \pi_X^*E$. □

In particular for any locally free sheaf on X_{perf} there exists a Harder-Narasimhan filtration for μ_p. The following important technical result of Langer is then a formal consequence:

Theorem 1.2.17. *[Lan04, Theorem 2.7] For any vector bundle E on X there is some $n \geq 0$ such that the Harder-Narasimhan filtration of $F_X^{n*}E$ has strongly semistable quotients.*

Proof. Again as the category of vector bundles on X_{perf} is the colimit of copies of the category $Vect(X)$ along Frobenius pullbacks, one sees that the Harder-Narasimhan filtration for μ_p on X_{perf} descends to a filtration $Fil^\bullet \subset F_X^{n*}E$ for some $n >> 0$. By proposition 1.2.16 all the graded pieces are strongly semistable with descending slopes. As the Harder-Narasimhan filtration is unique it must then coincide with $Fil^\bullet$. □

Remark 1.2.18. With a bit more care it is actually also possible to treat the case of torsion-free sheaves.

1.2.2 Numerically flat vector bundles

We will recollect some facts about numerically flat vector bundles. For most of the material we refer to [Lan11]. For the rest of this section k denotes a perfect field.

Definition 1.2.19. Let X be a proper scheme over k. A vector bundle E on X is called numerically effective (nef) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on the associated projective bundle $\mathbb{P}(E)$.
E is called numerically flat if both E and its dual $E^\vee$ are nef.

Remark 1.2.20.
- Let E be a vector bundle on a proper scheme X over k. The property of being nef or numerically flat is invariant under pullbacks along surjective maps. Precisely, E is nef (numerically flat) if and only if f^*E is nef (numerically flat) for a surjective map $f : Y \to X$ of proper k-schemes.

Definition 1.2.21. A vector bundle E on a proper scheme X over k is called Nori-semistable if for all morphisms $f : C \to X$, where C is a smooth and proper curve, the pullback f^*E is semistable of degree 0.

Proposition 1.2.22. *Let X be a proper scheme over k and E a vector bundle on X. Then E is numerically flat if and only if it is Nori-semistable.*

As we did not find a good reference for the proof we will provide it here.

Proof. We prove the following

Claim. *[Lan11, §1.2] A vector bundle E is nef if and only if $\mu_{min}(f^*E) \geq 0$ for all morphisms $f : C \to X$ from smooth projective curves to X. Recall that $\mu_{min}(f^*E)$ denotes the minimum of the slopes of quotients of f^*E.*

For this, recall that $\mathbb{P}(E)$ represents the functor which associates to an X-scheme $g : T \to X$ the set of isomorphism classes of line bundle quotients $g^*E \twoheadrightarrow L$, and $\mathcal{O}_{\mathbb{P}}(1)$ is the universal quotient.
Assume that $\mu_{min}(f^*E) \geq 0$ for all maps $f : C \to X$ from projective smooth curves. Then in particular, for any curve $C \subset \mathbb{P}(E)$ the pullback of $\mathcal{O}(1)$ to the normalization $\bar{C} \to C$ must have non-zero degree. This implies that $\mathcal{O}(1)$ is nef.
Now assume that E is nef. Then in particular f^*E is nef for any smooth projective curve $f : C \to X$. So one has to show that any nef vector bundle on a smooth projective curve has no quotients with negative slope:
So assume X is a smooth projective curve and E nef on X, and $E \twoheadrightarrow Q$ is a quotient bundle. Then note first of all that there is a natural embedding $\mathbb{P}(Q) \subset \mathbb{P}(E)$ of projective bundles, such that $\mathcal{O}_{\mathbb{P}(E)}(1)$ restricts to $\mathcal{O}_{\mathbb{P}(Q)}(1)$. From this it follows that Q is nef. So we are left to show that any nef vector bundle on a smooth projective curve has non-zero degree. But this is clear.
Applying the claim to E and its dual $E^\vee$ everything follows. $\square$

Remark 1.2.23. Using the characterization via Nori-semistability one can easily prove that a vector bundle E is numerically flat if and only if $E_{k'}$ is numerically flat on $X_{k'}$ for any field extension k' of k. Namely, this follows from the invariance of semistablity with respect to arbitrary field extensions (see [HL10, Theorem 1.3.7]).

Definition 1.2.24. Let X be a proper scheme over a perfect field k. We denote the category of numerically flat vector bundles by $NF(X)$.

Proposition 1.2.25. *The category $NF(X)$ is closed under taking duals, internal homs, tensor products and extensions.*
If $\bar{x} : Spec(\bar{k}) \to X$ is a geometric point of X, and $Fib_{\bar{x}} : NF(X) \to Vec_{\bar{k}}$ is the fiber functor taking a numerically flat bundle E to the fiber $E_{\bar{x}}$ over $\bar{x}$, then the pair $(NF(X), Fib_{\bar{x}})$ is a neutral Tannakian category.

Proof. All the statements can be found in [Lan11], or can be easily deduced by using the previous proposition. $\square$

By the Tannakian formalism the pair $(NF(X), Fib_{\bar{x}})$ gives rise to a group scheme $\pi_1^S(X, \bar{x})$, which in the positive characteristic case has been extensively studied in [Lan11]. Moreover it is shown in loc. cit. that in the smooth projective case numerical flatness is equivalent to strong semistability of degree 0. More precisely:

Proposition 1.2.26. *[Lan11, Proposition 5.1] Let X be smooth and projective over a perfect field k of characteristic p, and assume that X is of dimension n. Then a vector bundle E is numerically flat, if and only if E is strongly semistable with respect to some ample divisor H, and $c_1(E) \cdot H^{n-1} = c_2(E) \cdot H^{n-2} = 0$.*

Note that the last property does not depend on the polarization H.

Remark 1.2.27. In characteristic 0 the numerically flat vector bundles are the semistable vector bundles with vanishing Chern classes. In positive characteristic this latter category is not as well behaved, in particular it is not a tensor category.

Definition 1.2.28. A family $\{F_i\}$ of coherent sheaves on a scheme X over some base S is called bounded, if there exists a flat family F of coherent sheaves over $X \times_S T$, such that T is of finite type over S, and all F_i arise as fibers of F.

One can then use the fundamental results from [Lan04] to show that the isomorphism classes of numerically flat vector bundles on a smooth projective variety is a bounded family. More generally one has:

Proposition 1.2.29. *[Lan12, Theorem 1.1] The family of numerically flat bundles on a normal projective variety is bounded.*

Chapter 2

Pro-étale trivializable modules and representations

Let X be a connected proper rigid analytic variety over a complete algebraically closed non-archimedean extension $\mathbf{C}$ of $\mathbb{Q}_p$. In this chapter we will introduce a category of locally free $\mathcal{O}_X^+$-modules to which one can functorially attach continuous representations of $\pi_1(X)$. This will be the category of modules whose p-adic completion is trivialized on a profinite étale cover of X. We then single out a class of modules (which we call Frobenius-trivial) for which such a trivializing cover always exists.

Notation Let Γ be the value group of $\mathbf{C}$ and denote by $log\Gamma \subset \mathbb{R}$ the induced subset given by taking the logarithm with base $|p|$. Then for any $\epsilon \in log\Gamma$ we choose an element $p^\epsilon \in \mathbf{C}$, which satisfies $|p^\epsilon| = |p|^\epsilon$.
Moreover we fix a pseudouniformizer t in the tilt $\mathcal{O}_{\mathbf{C}^\flat}$ such that $t^\sharp = p$.
Whenever we speak about almost mathematics we mean almost mathematics with respect to the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{\mathbf{C}}$.
We denote by $\mathbb{C}_p$ the completion of a fixed algebraic closure $\bar{\mathbb{Q}}_p$ of the field of p-adic numbers.

2.1 Representations attached to $\mathcal{O}_X^+$-modules

Fix a $Spa(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$-valued point x of X. We will show how to attach a continuous $\mathcal{O}_{\mathbf{C}}$-representation of $\pi_1(X) := \pi_1(X, x)$ to an $\mathcal{O}_X^+$-module $\mathcal{E}^+$ for which the p-adic completion $\hat{\mathcal{E}}^+ = \varprojlim \mathcal{E}^+/p^n$ is trivialized on a profinite étale cover.
Let X be a proper and connected rigid analytic space over $Spa(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. Then the only global functions are the constant ones, i.e. $\Gamma(X, \mathcal{O}_X) = \mathbf{C}$. As $\Gamma(X, \mathcal{O}_X^+)$ consists of the functions f for which $\left|f(x)\right| \leq 1$ for all $x \in X$, we see that $\Gamma(X, \mathcal{O}_X^+) = \mathcal{O}_{\mathbf{C}}$. Similarly one has $\Gamma(X, \hat{\mathcal{O}}_X^+) \subset \Gamma(X, \hat{\mathcal{O}}_X)$. And at the same time $\Gamma(X, \hat{\mathcal{O}}_X^+)$ is almost isomorphic to $\mathcal{O}_{\mathbf{C}}$ by theorem 1.1.71. So we must have $\Gamma(X, \hat{\mathcal{O}}_X^+) = \mathcal{O}_{\mathbf{C}}$.
We first record the following

Lemma 2.1.1. *Let $\tilde{Y} = \varprojlim_i Y_i \to X$ be profinite étale, and let $\mathcal{E}^+$ be a locally free $\mathcal{O}^+$-module, such that $\hat{\mathcal{E}}^+|_{\tilde{Y}}$ is trivial. Then for any $n \geq 1$ there exists some i, such* *that $\mathcal{E}^+/p^n$ becomes trivial on Y_i.*

Proof. Let $\nu : X_{proét} \to X_{ét}$ denote the canonical projection. There exists a locally free $\mathcal{O}^+_{X_{ét}}$-module $\mathcal{F}$, such that $\nu^*\mathcal{F} = \mathcal{E}^+$. Hence we also have $\hat{\mathcal{E}}^+/p^n = \nu^*(\mathcal{F}/p^n)$. But then by [Sch13a, Lemma 3.16] $\hat{\mathcal{E}}^+/p^n$ is the sheaf given by $\hat{\mathcal{E}}^+/p^n(V) = \varinjlim_j \mathcal{F}/p^n(V_j)$ for any qcqs object $V = \varprojlim_j V_j \in X_{proét}$.

Claim. *The category of finite locally free $\mathcal{O}^+_{\tilde{Y}}/p^n$-modules is the colimit over the categories of finite locally free $\mathcal{O}^+_{Y_i}/p^n$-modules.*

As all Y_i are quasi-compact and quasi-separated we have that $\tilde{Y}$ is qcqs by [Sch13a, Lemma3.12] (v). Now we can use the standard arguments: For one direction note that if $\mathcal{E}, \mathcal{E}'$ are locally free $\mathcal{O}^+_{Y_i}/p^n$-modules for some i, then by what we said above

$$Hom(\mathcal{E}_{\tilde{Y}}, \mathcal{E}'_{\tilde{Y}}) = \mathcal{H}om(\mathcal{E}, \mathcal{E}')(\tilde{Y}) = \varinjlim_{j \geq i} \mathcal{H}om(\mathcal{E}, \mathcal{E}')(Y_j) = \varinjlim_{j \geq i} Hom(\mathcal{E}_{Y_j}, \mathcal{E}'_{Y_j}).$$

By this we get that the pullback functor $colimLF(\mathcal{O}^+_{Y_i}/p^n) \to LF(\mathcal{O}^+_{\tilde{Y}}/p^n)$ is fully faithful. Essential surjectivity follows because we can descend glueing data on proétale covers because $\tilde{Y}$ is qcqs. We leave out the details.
Now from the above we see that the isomorphism $(\hat{\mathcal{E}}^+/p^n)|_{\tilde{Y}} \cong (\hat{\mathcal{O}}^+_{\tilde{Y}}/p^n)^r$ descends to an isomorphism $(\hat{\mathcal{E}}^+/p^n)|_{Y_i} \cong \hat{\mathcal{O}}^+_{Y_i}/p^n$ for some large enough i. □

Lemma 2.1.2. *Let X be proper, connected and $\tilde{Y} = \varprojlim_i Y_i \to X$ be a profinite étale cover where each Y_i is connected. Then $\Gamma(\tilde{Y}, \hat{\mathcal{O}}^+_X) = \mathcal{O}_{\mathbf{C}}$*

Proof. Let $\mathcal{O}^{+a}_X$ be the almost version of the integral structure sheaf on $X_{proét}$. theorem 1.1.71 gives $\mathcal{O}^{+a}_X/p^n(\tilde{Y}) = \varinjlim_i \mathcal{O}^{+a}/p^n(Y_i) = \mathcal{O}^a_{\mathbf{C}}/p^n$ (As the Y_i are all connected the direct limit is taken along isomorphisms). As the inverse limit of sheaves agrees with the inverse limit of presheaves, we get $\hat{\mathcal{O}}^{+a}_X(\tilde{Y}) = \mathcal{O}^a_{\mathbf{C}}$. But then again, as $\hat{\mathcal{O}}^+_X(\tilde{Y}) \subset \hat{\mathcal{O}}_X(\tilde{Y}) = \mathbf{C}$, we see that $\hat{\mathcal{O}}^+_X(\tilde{Y}) = \mathcal{O}_{\mathbf{C}}$ □

Remark 2.1.3. We wish to remark that if the Y_i are connected, then so is $\tilde{Y}$. For this note that $\tilde{Y}$ is quasi-compact by [Sch13a, 3.12 (v)]. Now suppose that $|\tilde{Y}| = V_1 \cup V_2$ for some open and closed V_1, V_2. Then V_1 and V_2 descend to closed subsets $V_1^{i_0}, V_2^{i_0} \subset Y_{i_0}$, for some i_0, which cover $|Y_{i_0}|$. But then, since all Y_i are connected, $f^{-1}_{ji_0}(V_1^{i_0}) \cap f^{-1}_{ji_0}(V_2^{i_0})$ is non-empty for all $j \geq i_0$. But $V_1 \cap V_2 = \varprojlim_{j \geq i_0} f^{-1}_{ji_0}(V_1^{i_0}) \cap f^{-1}_{ji_0}(V_1^{i_0})$ and an inverse limit of non-empty spectral spaces along spectral maps is non-empty.

We will now adjust the exposition in [DW17, §4] to our setting: Assume $\mathcal{E}^+$ is as above, with p-adic completion $\hat{\mathcal{E}}^+$ trivialized on some connected profinite étale cover $f : \tilde{Y} = \varprojlim_i Y_i \to X$. In particular by the above lemma we get $\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \cong (\mathcal{O}_{\mathbf{C}})^r$. Now pick a point $y : Spa(\mathbf{C}, \mathcal{O}_{\mathbf{C}}) \to \tilde{Y}$ lying over x. As $\hat{\mathcal{E}}^+$ is trivial on $\tilde{Y}$ we have an isomorphism $y^* : \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \xrightarrow{\cong} \Gamma(x^*\hat{\mathcal{E}}^+) = \hat{\mathcal{E}}^+_x$ by pullback. For any $g \in \pi_1(X)$ we get another point gy lying above x. We can then define an automorphism on $\hat{\mathcal{E}}^+_x$ by

$$\hat{\mathcal{E}}^+_x \xrightarrow{(y^*)^{-1}} \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \xrightarrow{(gy)^*} \hat{\mathcal{E}}^+_x.$$

By this we get a representation

$$\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} : \pi_1(X) \to GL_r(\hat{\mathcal{E}}^+_x).$$

We need to show that this representation is continuous and independent of the choices.

Lemma 2.1.4. *Let $\tilde{Y}$, y be as above. Let $\phi : \tilde{Z} \to \tilde{Y}$ be a morphism of connected objects in $X_{prof\acute{e}t}$, and let z be a point lying above y. Then $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} = \rho_{\mathcal{E}^+}^{(\tilde{Z},z)}$.*

Proof. For any $g \in \pi_1(X)$, gz lies above gy. Then there is a commutative diagram

$$\begin{tikzcd}
\hat{\mathcal{E}}^+_x \arrow[r, "(y^*)^{-1}"] \arrow[d, "="] & \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \arrow[r, "(gy)^*"] \arrow[d, "\phi^*"] & \hat{\mathcal{E}}^+_x \arrow[d, "="] \\
\hat{\mathcal{E}}^+_x \arrow[r, "(z^*)^{-1}"] & \Gamma(\tilde{Z}, \hat{\mathcal{E}}^+) \arrow[r, "(gz)^*"] & \hat{\mathcal{E}}^+_x
\end{tikzcd}$$

which gives the claim. $\square$

Proposition 2.1.5. *The representation $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)}$ attached to $\mathcal{E}^+$ is continuous.*

Proof. We need to show that, for all $n \geq 1$, $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} \otimes \mathcal{O}_{\mathbf{C}}/p^n$ factors through some finite quotient of $\pi_1(X)$. Let $\tilde{Y} = \varprojlim_i Y_i$ be a presentation where each $Y_i \to X$ is connected finite étale. Then y corresponds to a compatible system y_i of points of Y_i.
By lemma 2.1.1 $\hat{\mathcal{E}}^+/p^n$ is trivialized on some $Y_i \to X$. On the almost level, we then get $\Gamma(\tilde{Y}, \mathcal{E}^+/p^n)^a \cong \Gamma(Y_i, \mathcal{E}^+/p^n)^a \cong (\mathcal{O}^a_{\mathbf{C}}/p^n)^r$, where r denotes the rank of $\mathcal{E}^+$. The reason here is again, that $\mathcal{O}^{+a}_{Y_j}/p^n(Y_j) \to \mathcal{O}^{+a}_{Y_{j'}}/p^n(Y_{j'})$ is an isomorphism for any transition map $Y_{j'} \to Y_j$.
We thus get a representation $\alpha_n^{(Y_i,y_i)}$ of $\pi_1(X)$ on the $\mathcal{O}_{\mathbf{C}}/p^n$-module of almost elements $(\hat{\mathcal{E}}^+/p^n)_*$ via

$$(\hat{\mathcal{E}}^+_x/p^n)_* \xrightarrow{(y_i^*)^{-1}} \Gamma(Y_i, \hat{\mathcal{E}}^+/p^n)_* \xrightarrow{(gy_i)^*} ((\hat{\mathcal{E}}^+_x/p^n)_*.$$

But now the natural map $\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+)/p^n \to \Gamma(\tilde{Y}, \mathcal{E}^+/p^n)_* = \Gamma(Y_i, \mathcal{E}^+/p^n)_*$ is injective (after fixing a basis it is just given by the embedding $(\mathcal{O}_{\mathbf{C}}/p^n)^r \hookrightarrow (\mathcal{O}_{\mathbf{C}}/p^n)^r_*$).
This realizes $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} \otimes \mathcal{O}_{\mathbf{C}}/p^n$ as a subrepresentation of $\alpha_n^{(Y_i,y_i)}$. But now there are of course only finitely many points of Y_i lying over x, so $\alpha_n^{(Y_i,y_i)}$ has finite image, hence so has $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} \otimes \mathcal{O}_{\mathbf{C}}/p^n$. $\square$

Lemma 2.1.6. *The representation $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)}$ does not depend on $(\tilde{Y}, y)$.*

Proof. We only need to show that the representation is independent of the point y, the rest then follows from lemma 2.1.4.
Moreover it is enough to show that $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} \otimes \mathcal{O}_{\mathbf{C}}/p^n$ is independent of y for any $n \geq 1$. Let $\tilde{Y} = \varprojlim Y_i$ be a presentation as before, and y correspond to a compatible system of points y_i of Y_i. Assume that $\hat{\mathcal{E}}^+/p^n$ becomes trivial on Y_i. Then by the proof of the previous proposition, $\rho_{\mathcal{E}^+}^{(\tilde{Y},y)} \otimes \mathcal{O}_{\mathbf{C}}/p^n$ is a subrepresentation of $\alpha_n^{(Y_i,y_i)}$, so it is

enough to show that $\alpha_n^{(Y_i,y_i)}$ is independent of the point y_i above x. Now consider the Galois closure $Y_i' \to X$ of Y_i. Again, as any point y_i has a point y_i' of Y_i' lying above it, by lemma 2.1.4 it is enough to show that $\alpha_n^{(Y_i',y_i')}$ is independent of the point y_i' above x. But now the Galois group G of $Y_i \to X$ acts simply transitively on the points lying above x. Now one can immediately check as in the proof of [DW17, lemma 4.5] that the representation is independent of the point y_i'. □

We thus see that we get a well defined continuous representation $\rho_{\mathcal{E}^+}$ associated to $\mathcal{E}^+$.
We denote by $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$ the category of finite locally free $\mathcal{O}_X^+$-modules, whose p-adic completion is trivialized on a profinite étale cover of X. One easily checks:

Lemma 2.1.7. *The category $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$ is closed under taking tensor products, duals, internal homs, exterior products and extensions.*

Theorem 2.1.8. *The association $\mathcal{E}^+ \mapsto \rho_{\mathcal{E}^+}$ defines an exact functor*

$$\rho_{\mathcal{O}} : \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+) \to Rep_{\pi_1}(\mathcal{O}_{\boldsymbol{C}}).$$

Moreover $\rho_{\mathcal{O}}$ is compatible with tensor products, duals, inner homs and exterior products, and for every morphism $f : X' \to X$ of connected proper rigid analytic spaces over $Spa(\boldsymbol{C}, \mathcal{O}_{\boldsymbol{C}})$ we have $\rho_{f^*\mathcal{E}^+} = f^*\rho_{\mathcal{E}^+}$, where $f^*\rho_{\mathcal{E}^+}$, denotes the composition

$$\pi_1(X', x') \xrightarrow{f_*} \pi_1(X, f(x')) \xrightarrow{\rho_{\mathcal{E}^+}} GL_r(\mathcal{O}_{\boldsymbol{C}})$$

and $x' : Spa(\boldsymbol{C}, \mathcal{O}_{\boldsymbol{C}}) \to X'$ is a point of X'.

Define by $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X) := \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+) \otimes \mathbb{Q}$ the category of finite locally free $\mathcal{O}_X$-modules $\mathcal{E}$ for which there exists a locally free $\mathcal{O}_X^+$-module $\mathcal{E}^+$ with $\mathcal{E}^+[\frac{1}{p}] = \mathcal{E}$, and $\mathcal{E}^+ \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$.
Passing to isogeny classes induces a functor

$$\rho : \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X) \to Rep_{\pi_1(X)}(\boldsymbol{C}),$$

which is compatible with tensor products, duals, inner homs and exterior products.

Proof. The functoriality of the construction is clear. Let $\mathcal{E}_1^+$, $\mathcal{E}_2^+ \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$ be such that $\hat{\mathcal{E}}_1^+$ is trivialized on $\tilde{Y}^1 = \varprojlim_i Y_i^2$ and $\hat{\mathcal{E}}_2^+$ is trivialized on $\tilde{Y}^2 = \varprojlim_i Y_i^2$. Then the tensor product $\hat{\mathcal{E}}_1^+ \otimes_{\hat{\mathcal{O}}_X^+} \hat{\mathcal{E}}_2^+$ is trivialized on $\tilde{Y}^1 \times_X \tilde{Y}^2$. Note that here $\hat{\mathcal{E}}_1^+ \otimes_{\hat{\mathcal{O}}_X^+} \hat{\mathcal{E}}_2^+$ is indeed the p-adic completion of $\mathcal{E}_1^+ \otimes_{\mathcal{O}_X^+} \mathcal{E}_2^+$. Now $\hat{\mathcal{E}}_1^+ \otimes_{\hat{\mathcal{O}}_X^+} \hat{\mathcal{E}}_2^+$ is trivial on $\tilde{Y}^1 \times_X \tilde{Y}^2$. But then the canonical map

$$\Gamma(\tilde{Y}^1 \times_X \tilde{Y}^2, \hat{\mathcal{E}}_1^+) \otimes_{\mathcal{O}_{\mathbf{C}}} \Gamma(\tilde{Y}^1 \times_X \tilde{Y}^2, \hat{\mathcal{E}}_2^+) \to \Gamma(\tilde{Y}^1 \times_X \tilde{Y}^2, \hat{\mathcal{E}}_1^+ \otimes_{\hat{\mathcal{O}}_X^+} \hat{\mathcal{E}}_2^+)$$

is an isomorphism. Using lemma 2.1.4 we get the claim. The compatibility with the other operations also follows easily.
For compatibility with pullbacks, note that if $\hat{\mathcal{E}}^+$ is trivial on a profinite étale cover $\tilde{Y} \to X$, then $\widehat{(f^*\mathcal{E}^+)}$ becomes trivial on the pullback $\tilde{Y}_{X'}$. By pullback one has an isomorphism

$$\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \xrightarrow{\cong} \Gamma(\tilde{Y}_{X'}, \widehat{f^*\mathcal{E}^+}).$$

Denote by $pr_{\tilde{Y}} : \tilde{Y}_{X'} \to \tilde{Y}$ the projection. Then for any $g \in \pi_1(X', x')$ and point y' of $\tilde{Y}_{X'}$ above x', one has $pr_{\tilde{Y}}(gy') = f'_*(g)y'$ and there is a commutative diagram

$$\begin{tikzcd}
\hat{\mathcal{E}}^+_{f(x')} \arrow[r, "(pr_{\tilde{Y}}(y')^*)^{-1}"] \arrow[d, "f^*"] & \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \arrow[r, "pr_{\tilde{Y}}(gy')^*"] \arrow[d, "pr^*_{\tilde{Y}}"] & \hat{\mathcal{E}}^+_{f(x')} \arrow[d, "f^*"] \\
\hat{\mathcal{E}}^+_{x'} \arrow[r, "(y'^*)^{-1}"] & \Gamma(\tilde{Y}_{X'}, f^*\hat{\mathcal{E}}^+) \arrow[r, "(gy')^*"] & \hat{\mathcal{E}}^+_{x'}.
\end{tikzcd}$$

□

2.1.1 $\mathcal{O}_{\mathbb{C}_p}$-local systems and full faithfulness

For this section we assume that $(\mathbf{C}, \mathcal{O}_{\mathbf{C}}) = (\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Let again X denote a proper, connected rigid analytic variety over $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Recall that we have a functor $\mathbb{L} \mapsto \mathbb{L} \otimes \hat{\mathcal{O}}^+_X$ from $LS_{\mathcal{O}_{\mathbb{C}_p}}(X)$ to $LF(\hat{\mathcal{O}}^+_X)$.

Lemma 2.1.9. *Let $\mathcal{E}^+ \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}^+)$, and let $\mathbb{L}$ be the $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system corresponding to $\rho_{\mathcal{E}^+}$. Then $\hat{\mathcal{E}}^+ \cong \hat{\mathcal{O}}^+_X \otimes_{\hat{\mathcal{O}}_{\mathbb{C}_p}} \mathbb{L}$.*

Proof. We can compare the gluing data: Assume that $\hat{\mathcal{E}}^+$ is trivialized on $\tilde{Y} \to X$, a connected profinite étale cover. By the discussion above we then have a $\pi_1(X, x)$-equivariant isomorphism $\underline{\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+)} \xrightarrow{\cong} \underline{\hat{\mathcal{E}}^+_x} = \mathbb{L}|_{\tilde{Y}}$, where $\underline{\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+)} := \varprojlim \underline{\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+/p^n)}$, where $\underline{\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+/p^n)}$ denotes the constant sheaf on $X_{pro\acute{e}t}/\tilde{Y}$ associated to $\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+/p^n)$ and similarly $\underline{\hat{\mathcal{E}}^+_x} := \varprojlim \underline{\hat{\mathcal{E}}^+_x}/p^n$. From this one also gets a $\pi_1(X)$-equivariant isomorphism

$$\phi : \hat{\mathcal{E}}^+|_{\tilde{Y}} = \underline{\Gamma(\tilde{Y}, \hat{\mathcal{E}}^+)} \otimes \hat{\mathcal{O}}^+_{\tilde{Y}} \xrightarrow{\cong} (\mathbb{L} \otimes \hat{\mathcal{O}}^+_X)|_{\tilde{Y}}.$$

Then modulo p^n this equivariant isomorphism descends to some Y_i. Pulling back to the Galois closure $Y'_i \to X$ we get a $\pi_1(X)$-equivariant isomorphism

$$\hat{\mathcal{E}}^+/p^n|_{Y'_i} \cong (\hat{\mathcal{O}}^+ \otimes \mathbb{L})/p^n|_{Y'_i}.$$

But here the $\pi_1(X)$-action is through the quotient G^{op}, where G denotes the Galois group of $Y_i \to X$. But then the isomorphism desends to an isomorphism $\hat{\mathcal{E}}^+/p^n \cong (\hat{\mathcal{O}}^+ \otimes \mathbb{L})/p^n$.
So ϕ mod p^n descends to X for all n, hence so does ϕ. □

Theorem 2.1.10. *Assume that X is seminormal.Then the functor*

$$\rho : \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X) \to Rep_{\pi_1(X)}(\mathbb{C}_p)$$

is fully faithful.

Proof. Let $E \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X)$. By lemma 2.1.9 we see that $\lambda^*E \otimes \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}_E$. Since X is seminormal, by corollary 1.1.61 we have $\lambda_*(\lambda^*E \otimes \hat{\mathcal{O}}_X) \cong E$. Thus $\mathbb{L} \mapsto \lambda_*(\hat{\mathcal{O}}_X \otimes \mathbb{L})$ gives a quasi-inverse to ρ. □

As already remarked in the last proof, we see that the vector bundles E in $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X)$ are precisely the vector bundles for which there exists an $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system $\mathbb{L}$ such that $E \otimes \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}$. We borrow the following terminology from [Xu17, Definition 10.3].

Definition 2.1.11. A vector bundle E is called Weil-Tate if there exists an $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system $\mathbb{L}$ such that $E \otimes \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}$.
Similarly the $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local systems $\mathbb{L}$ that are associated to vector bundles in this sense, are also called Weil-Tate.

Remark 2.1.12. Let X be a smooth rigid analytic variety over $Spa(K, \mathcal{O}_K)$ for K a finite extension of $\mathbb{Q}_p$. In [LZ16] Liu and Zhu have introduced a functor from the category of arithmetic $\hat{\mathbb{Z}}_p$-local systems on X to the category of nilpotent Higgs bundles on $X_{\hat{\overline{K}}}$. Recall that a Higgs-bundle is a vector bundle E on $X_{\hat{\overline{K}}}$ together with an endomorphism valued 1-form, i.e. a section $\phi \in H^0(\mathcal{E}nd(E) \otimes \Omega^1)$, satisfying the integrability condition $\phi \wedge \phi = 0$.
We give a quick recollection of their construction. Let $\mathcal{O}\mathbb{B}_{dR}$ be the de Rham structure sheaf defined in [Sch16]. It carries a flat connection ∇ (acting trivially on $\mathbb{B}_{dR}$) and a filtration coming from the usual filtration on $\mathbb{B}_{dR}$. Then define $\mathcal{O}\mathbb{C} = gr^0(\mathcal{O}\mathbb{B}_{dR})$.
Let $\nu' : X_{pro\acute{e}t}/X_{\hat{\overline{K}}} \to (X_{\hat{\overline{K}}})_{\acute{e}t}$ denote the natural projection. Then for any $\hat{\mathbb{Z}}_p$-local system $\mathbb{L}$ the associated Higgs bundle is defined to be $\mathcal{H}(\mathbb{L}) = \nu'_*(\mathcal{O}\mathbb{C} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{L})$, with Higgs field $\theta_{\mathbb{L}}$ coming from the Higgs field on $\mathcal{O}\mathbb{C}$. Note that $\mathcal{O}\mathbb{C}$ carries a Higgs field coming from the associated graded of ∇.
One may try to write down this functor for geometric local systems, i.e. we will consider the functor $\mathcal{H}(\mathbb{L}) = \nu'_*(\mathcal{O}\mathbb{C} \otimes_{\hat{\mathcal{O}}_{\mathbb{C}_p}} \mathbb{L})$ for $\mathcal{O}_{\mathbb{C}_p}$-local systems on $X_{\hat{\overline{K}}}$.
We wish to show that our constructions are compatible with this functor. In general, proving that this functor actually gives a Higgs bundle is a complicated endevour, and is what occupies the large part of §2 in [LZ16] (for arithmetic local systems). In our case however this is immediate, as the local system is already attached to a vector bundle.
More precisely we have the following result (compare also with [Xu17, Proposition 11.7]):

Proposition 2.1.13. *Let X be a smooth proper rigid analytic variety over $Spa(K, \mathcal{O}_K)$. Let E be a Weil-Tate vector bundle on $X_{\hat{\overline{K}}}$, viewed as a sheaf on the étale site. Let $\mathbb{L}$ be the $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system associated to E.*
Then $\nu'_*(\mathcal{O}\mathbb{C} \otimes \mathbb{L}) \cong E$, with vanishing Higgs field $\theta_{\mathbb{L}} = 0$.

Proof. We have $\widehat{(\nu'^* E)} \cong \hat{\mathcal{O}}_X \otimes \mathbb{L}$. We have $gr^0\mathbb{B}_{dR} = \hat{\mathcal{O}}_{X_{\hat{\overline{K}}}}$, hence $\hat{\mathcal{O}}_{X_{\hat{\overline{K}}}} \subset \mathcal{O}\mathbb{C}$ (see [Sch13a, Prop. 6.7]). Then $\mathcal{O}\mathbb{C} \otimes \mathbb{L} = \widehat{(\nu'^* E)} \otimes_{\hat{\mathcal{O}}_{X_{\hat{\overline{K}}}}} \mathcal{O}\mathbb{C}$. Since $\nu_* \mathcal{O}\mathbb{C} = \mathcal{O}_{X_{\acute{e}t}}$ we get $\nu'_*(\mathcal{O}\mathbb{C} \otimes \mathbb{L}) \cong E$.
For the claim $\theta_{\mathbb{L}} = 0$, note that the Higgs field on $\mathcal{O}\mathbb{C}$ is trivial on $\hat{\mathcal{O}}_{X_{\hat{\overline{K}}}}$ (as it comes from the connection ∇ on $\mathcal{O}\mathbb{B}_{dR}$ which is trivial on $\mathbb{B}_{dR}$). But then the induced Higgs field on $\mathcal{H}(\mathbb{L}) = \nu'_*(\widehat{(\nu'^* E)} \otimes_{\hat{\mathcal{O}}_{X_{\hat{\overline{K}}}}} \mathcal{O}\mathbb{C})$ is simply obtained by trivially extending the Higgs field from $\nu'_* \mathcal{O}\mathbb{C} = \mathcal{O}_{X_{\acute{e}t}}$. But this is the zero Higgs field. $\square$

2.2 Frobenius-trivial modules

In this section we will deal with $\mathcal{O}_X^+$-modules whose mod p reduction is trivialized by some Frobenius pullback. Let $\mathcal{E}$ be an $\mathcal{O}_X^+/p$-module. Then $\mathcal{E}$ is called F^m-trivial, if $\Phi^{m*}\mathcal{E} \cong (\mathcal{O}_X^+/p)^r$. The goal of this section is to show that all such modules lie in $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$.
We have the following (Recall that $t \in \mathcal{O}_{\mathbf{C}}^\flat$, such that $|t^\sharp| = |p|$):

Lemma 2.2.1. *The Frobenius induces an equivalence of categories*

$$\{F^m\text{-trivial locally free } \mathcal{O}_X^+/p\text{-modules }\}$$
$$\longleftrightarrow$$
$$\{\text{locally free } \hat{\mathcal{O}}_{X^\flat}^+/t^{p^m}\text{-modules trivial mod } t\}$$

Proof. There is a commutative diagram

$$\begin{tikzcd} \hat{\mathcal{O}}_{X^\flat}^+ \arrow[r, "\tilde{\Phi}"] \arrow[d, "\theta"] & \hat{\mathcal{O}}_{X^\flat}^+ \arrow[d, "\theta"] \\ \hat{\mathcal{O}}_{X^\flat}^+/t \arrow[r, "\Phi"] & \hat{\mathcal{O}}_{X^\flat}^+/t. \end{tikzcd}$$

As θ is surjective, we see that $\theta^*\theta_*\mathcal{M} = \mathcal{M}$ for any $\hat{\mathcal{O}}_{X^\flat}^+/t$-module $\mathcal{M}$. Here θ_* denotes the restriction of scalars. But then using the commutativity of the diagram above we see that $(\tilde{\Phi}^m)^*$ defines the desired functor with quasi-inverse given by $(\tilde{\Phi}^{-m})^*$. $\square$

Remark 2.2.2. One also sees in the same manner that we also have an equivalence of categories

$$\{\text{ mod } p\ F^m\text{-trivial locally free } \hat{\mathcal{O}}_X^+\text{-modules }\}$$
$$\longleftrightarrow$$
$$\{\text{ locally free } \mathbb{A}_{inf,X}/\tilde{\Phi}(\xi)\text{-modules trivial mod } (\xi, p)\}$$

where $\mathbb{A}_{inf,X} = W(\hat{\mathcal{O}}_{X^\flat}^+)$, and now $\tilde{\Phi}$ denotes the Frobenius on $\mathbb{A}_{inf,X}$.

Theorem 2.2.3. *Let $\mathcal{E}^+$ be a locally free $\mathcal{O}_X^+$-module of rank r such that $\mathcal{E}^+/p$ is F^m-trivial. Then there exists a profinite étale cover $\tilde{Y} = \varprojlim_n Y_n \to X$ such that $\hat{\mathcal{E}}^+|_{\tilde{Y}} \cong (\hat{\mathcal{O}}_X^+|_{\tilde{Y}})^r$.*

Proof. Assume first that $m = 0$, so that $\mathcal{E}^+/p$ is trivial. We want to show that $\mathcal{E}^+/p^n$ becomes trivial after passing to a further finite étale cover. We have an exact sequence:

$$0 \to M_n(\mathcal{I}) \to GL_r(\mathcal{O}_X^+/p^2) \to GL_r(\mathcal{O}_X^+/p) \to 1.$$

where $\mathcal{I} = (p)\mathcal{O}_X^+/p^2$ and the first map is given by $A \mapsto 1 + A$. Taking cohomology we get an exact sequence

$$H^1(M_n(\mathcal{I})) \to H^1(GL_r(\mathcal{O}_X^+/p^2)) \to H^1(GL_r(\mathcal{O}_X^+/p))$$

Pick a pro-étale cover $\{U_i\}$ of X on which $\mathcal{E}^+$ becomes trivial. Using the exact sequence above plus the fact that $\mathcal{E}^+/p$ is trivial we see that $\mathcal{E}^+/p^2$ is (after possibly refining the cover $\{U_i\}$) defined by a cocycle of the form $(id + g_{ij})_{ij}$ on the overlaps $U_i \times_X U_j$, where (g_{ij}) defines a class in $H^1(M_n(\mathcal{I}))$. Since $\mathcal{O}_X^+$ is p-torsion free we have an isomorphism of pro-étale sheaves $\mathcal{I} \cong \mathcal{O}_X^+/p$. Hence by the primitive comparison theorem 1.1.71 we see that $H^1(M_n(\mathcal{I})) = M_n(H^1(\mathcal{I}))$ is almost isomorphic to $M_n(H^1(X_{\acute{e}t}, \mathbb{F}_p) \otimes \mathcal{O}_{\mathbf{C}}/p)$. But the classes in the latter cohomology group become zero on suitable finite étale covers. Hence we can assume that the class defined by (g_{ij}) is almost trivial, which means that $p^\epsilon g_{ij}$ becomes a coboundary for any $\epsilon \in log\Gamma$. Write $p^\epsilon g_{ij} = p(\gamma_j - \gamma_i)$, where the γ_i are matrices with entries in $\mathcal{O}^+/p^2(U_i)$. Then $p^{1-\epsilon}(\gamma_j - \gamma_i) - g_{ij}$ is divisible by $p^{2-\epsilon}$. Hence g_{ij} is given by a coboundary modulo $p^{2-\epsilon}$, so $\mathcal{E}^+/p^{2-\epsilon}$ is trivial. Inductively we see that $\mathcal{E}^+$ can be trivialized on suitable finite étale covers modulo $p^{n-\sum \epsilon_n}$, where we can choose the sequence ϵ_n in such a way that $\epsilon_n \to 0$ as n goes to infinity, hence giving the claim.

Now assume that $\Phi^{m*}(\mathcal{E}^+/p)$ is trivial for some $m > 0$. Using lemma 2.2.1 we see that $\mathcal{F} = \tilde{\Phi}^{m*}(\mathcal{E}^+/p)$ is a locally free $\hat{\mathcal{O}}^+_{X^\flat}/t^{p^m}$-module trivial mod t. The obstruction for triviality of $\mathcal{F}/t^2$ lies again in $M_n(H^1(\hat{\mathcal{O}}^+_{X^\flat}/t)) = M_n(H^1(\mathcal{O}_X^+/p))$. Now applying the same arguments as in the first part of the proof, we see that $\mathcal{F}$ becomes trivial on a finite étale cover $Y \to X$. But then, applying $(\tilde{\Phi}^{-m})^*$, we see that $\mathcal{E}^+/p$ becomes trivial on Y as well. □

We found out, that the idea for the first part of the proof of the last theorem is essentially already contained in [Fal05, §5].

Remark 2.2.4. For generalities about non-abelian cohomology on sites, which were used in the last proof, we refer to [Gir71] (in particular [Gir71, III Proposition 3.3.1]).

Chapter 3

The Deninger-Werner correspondence

We will use the results from the previous section to give a new approach to the Deninger-Werner correspondence, which works for general (seminormal) proper rigid analytic varieties.

3.1 Numerically flat vector bundles over finite fields

We want to generalize the results from [DW17, §2] on numerically flat vector bundles over finite fields to the non-projective case. For any $\mathbb{F}_p$-scheme Y we will denote by F_Y the absolute Frobenius morphism of Y. For a vector bundle E on Y we denote its dual by $E^\vee$. In particular we see that, if E is numerically flat, $F_Y^{n*}E$ is also numerically flat for all $n \geq 0$.
The main goal of this section is to generalize a structure theorem for numerically flat bundles ([DW17, Theorem 2.2]) to non-projective proper schemes. As we will later study formal models over $Spf(\mathcal{O}_{\mathbb{C}_p})$ we will actually immediately deal with the situation of a proper scheme over $\mathcal{O}_{\mathbb{C}_p}/p$.

Theorem 3.1.1. *Let Y be a proper connected scheme over $Spec(\mathcal{O}_{\mathbb{C}_p}/p)$ and let E be a vector bundle on Y.*
Then $E \otimes \bar{\mathbb{F}}_p$ is numerically flat on $Y \times_{Spec(\mathcal{O}_{\mathbb{C}_p}/p)} Spec(\bar{\mathbb{F}}_p)$ if and only if there exists a finite étale cover $f : Y' \to Y$, and an $e \geq 0$ such that $F_{Y'}^{e*} f^* E \cong \mathcal{O}_{Y'}^r$.

Assume first that we are on a normal projective scheme over a finite field $\mathbb{F}_q$. Then recall from proposition 1.2.29 that the numerically flat sheaves on a normal projective variety is bounded. Hence they all arise as fibers of a family over a finite type scheme T. But then over a finite field there are only finitely many rational points of a finite type scheme, so $T(\mathbb{F}_q)$ is finite. This is generalized by Deninger-Werner to the following:

Proposition 3.1.2. *[DW17, Theorem 2.4] Let Y be a projective connected scheme over $\mathbb{F}_q$. Then the set of isomorphism classes of numerically flat vector bundles of fixed rank r on Y is finite.*

From this it follows that for any numerically flat bundle E there exist numbers $r > s \geq 0$, such that $F_Y^{r*}E \cong F_Y^{s*}E$, i.e. one obtains a Fobenius structure on some Frobenius pullback of E. One then concludes with the following

Theorem 3.1.3. *[LS77, Satz 1.4][Kat73, Proposition 4.1] Let Z be any $\mathbb{F}_p$-scheme and G a vector bundle on Z for which there exists an isomorphism $F_Z^{n*}G \cong G$ for some $n > 0$. Then there is a finite étale cover of Z on which G becomes trivial.*

Applying the above theorem to $F_Y^{(r-s)*}E$ one obtains a finite étale cover $\phi : Y' \to Y$, such that $F_{Y'}^{(r-s)*}\phi^*E \cong \phi^*F_Y^{(r-s)*}E$ is trivial.
As the author knows of no way to bound vector bundles on non-projective schemes, we are not able to show finiteness of isomorphism classes as in proposition 3.1.2.
The proof of theorem 3.1.1 will instead be an application of v-descent for perfect schemes as established in §1.2.2:

Proof of theorem 3.1.1. Let Y be proper connected over $\mathcal{O}_{\mathbb{C}_p}/p$ and E a vector bundle on Y. By standard descent results for finitely presented modules, we can assume that Y and E descend to (Y', E') over $\mathcal{O}_K/p$ for some finite extension $K/\mathbb{Q}_p$. Let $\pi \in \mathcal{O}_K$ be a uniformizer and $\mathcal{O}_K/\pi = \mathbb{F}_q$. Assume first that Y' is projective. Then the following argument is essentially already contained in the proof of Theorem 2.2 in [DW17]:
By proposition 3.1.2 there are only finitely many numerically flat bundles on $Y' \times Spec(\mathbb{F}_q)$ up to isomorphism. But Y' is an infinitesimal thickening of $Y' \times Spec(\mathbb{F}_q)$ (as $\mathcal{O}_K/p$ is an Artin ring). But then the lifts of a fixed vector bundle G on $Y' \times Spec(\mathbb{F}_q)$ to Y' are parametrized by a finite dimensional vector space over $\mathbb{F}_q$. This means that there are only finitely many vector bundles of rank r on Y', whose reduction mod π is numerically flat. As $F_{Y'}^{n*}E'$ lies in this set for all $n \geq 0$, we find some natural numbers $r > s \geq 0$ such that $F_{Y'}^{r*}E' \cong F_{Y'}^{s*}E'$.
Now assume that Y' is proper but not projective. By Chow's lemma we can find a proper surjective cover $f : Z \to Y'$, where Z is projective over $\mathcal{O}_K/p$. Then f^*E' is a numerically flat vector bundle. We have the canonical gluing datum $\phi_{can} : pr_1^*(f^*E') \to pr_2^*(f^*E')$, where $pr_1, pr_2 : Z \times_{Y'} Z \to Z$ denote the canonical projections.

Claim. *The set $M = \{$ descent data (G, ϕ) wrt f, where G is numerically flat of rk r $\}/Iso$ is finite.*

The claim follows from the fact that G runs through finitely many isomorphism classes, and ϕ lies in the finite $\mathbb{F}_q$-vector space $Hom_{Z\times_{Y'}Z}(pr_1^*G, pr_2^*G)$.
As Frobenius commutes with all maps, F_Z acts on M. Hence we get an isomorphism

$$\Psi : F_Z^{r*}(f^*E', \phi_{can}) \cong F_Z^{s*}(f^*E', \phi_{can})$$

for some natural numbers $r > s \geq 0$. Now of course Ψ will in general not descend to an isomorphism between $F_{Y'}^{r*}E'$ and $F_{Y'}^{s*}E'$. But by theorem 1.2.6, after passing to the perfection, we see that $f_{perf} : Z_{perf} \to Y'_{perf}$ satisfies effective descent for vector bundles. Hence $\pi_Z^*(\Psi)$ descends to an isomorphism $\pi_{Y'}^*(F_{Y'}^{r*}E') \cong \pi_{Y'}^*(F_{Y'}^{r*}E')$, where $\pi_Z : Z_{perf} \to Z$ and $\pi_{Y'} : Y'_{perf} \to Y'$ denote the canonical projections.
But the category of (descent data of) vector bundles on Z_{perf} is the colimit of the categories of (descent data of) vector bundles on copies of Z along Frobenius

pullbacks. But then we see that Ψ already becomes effective after a high enough Frobenius pullback. This gives an isomorphism $F_{Y'}^{n*}F_{Y'}^{r*}E' \cong F_{Y'}^{n*}F_{Y'}^{s*}E'$ for some $n >> 0$. $\square$

3.2 The Deninger-Werner correspondence for rigid analytic varieties

In this section we will prove our main result, which is the construction of p-adic representations attached to vector bundles with numerically flat reduction on an arbitrary proper connected (seminormal) rigid analytic variety X. Moreover we will later show that our representations coincide with the ones constructed in [DW17] whenever X is the analytification of a smooth algebraic variety over $\overline{\mathbb{Q}}_p$.
We will treat the integral and rational case simultaneously. So let $\mathcal{X}$ be a proper connected admissible formal scheme over $Spf(\mathcal{O}_{\mathbb{C}_p})$ with generic fiber X.

Definition 3.2.1. Define $\mathcal{B}^s(\mathcal{X})$ to be the category of vector bundles $\mathcal{E}$ on $\mathcal{X}$ for which $\mathcal{E} \otimes \overline{\mathbb{F}}_p$ is a numerically flat vector bundle.
Similarly, we define $\mathcal{B}^s(X)$ to be the category of vector bundles on X for which there exists an integral formal model with numerically flat reduction.

We remark the following (compare with [DW17, §9]):

Lemma 3.2.2. *The categories $\mathcal{B}^s(X)$ and $\mathcal{B}^s(\mathcal{X})$ are closed under tensor products, extensions, duals, internal homs and exterior powers.*

Proof. The claim for $\mathcal{B}^s(\mathcal{X})$ follows from the analogous statement for numerically flat vector bundles on a proper scheme, which is well known (alternatively it can be deduced from theorem 3.1.1).
For $\mathcal{B}^s(X)$, we just note that by the fundamental results of Raynaud, for any two formal models $\mathcal{X}, \mathcal{Y}$ of X, there exists an admissible blowup $\mathcal{X}' \to \mathcal{X}$, together with a morphism $\mathcal{X}' \to \mathcal{Y}$ (which can actually also be assumed to be an admissible blowup), which also induces an isomorphism on the generic fiber.
Using this the claim follows. $\square$

Recall that we have a canonical projection $\mu : X_{pro\acute{e}t} \to \mathcal{X}_{Zar}$. For any $\mathcal{E} \in Vect(\mathcal{X})$ we again denote by $\mathcal{E}^+ := \mu^{-1}\mathcal{E} \otimes_{\mu^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X^+$ its pullback to the pro-étale site.

Proposition 3.2.3. *For any $\mathcal{E} \in \mathcal{B}^s(\mathcal{X})$, the pullback to the pro-étale site $\mathcal{E}^+$ is contained in $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$.*

Proof. By theorem 3.1.1 there exists a finite étale cover $f : Y_0 \to \mathcal{X} \times Spec(\mathcal{O}_{\mathbb{C}_p}/p)$ such that $F_{Y_0}^{e*}f^*(\mathcal{E}/p)$ is trivial for some $e \geq 0$. We can lift f to a finite étale cover $\mathcal{Y} \to \mathcal{X}$. Denote by Y the generic fiber of $\mathcal{Y}$. Then $\mathcal{E}^+|_Y$ is F^e-trivial. Hence by theorem 2.2.3 we have $\mathcal{E}^+ \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$. $\square$

So pullback to the pro-étale site gives a functor $\mathcal{B}^s(\mathcal{X}) \to \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$. Using theorem 2.1.8 we get our main result:

Theorem 3.2.4. *The composition* $\mathcal{B}^s(\mathcal{X}) \xrightarrow{\mu^*} \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+) \xrightarrow{\rho_{\mathcal{O}}} Rep_{\pi_1(X)}(\mathcal{O}_{\mathbb{C}_p})$ *is an exact functor of tensor categories*

$$DW : \mathcal{B}^s(\mathcal{X}) \to Rep_{\pi_1(X)}(\mathcal{O}_{\mathbb{C}_p})$$

compatible with duals, internal homs and exterior products. Moreover, for any morphism $f : \mathcal{Y} \to \mathcal{X}$ *of proper admissible formal schemes over* $Spf(\mathcal{O}_{\mathbb{C}_p})$*, with generic fiber* $f_{\mathbb{C}_p} : X \to Y$*, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{B}^s(\mathcal{X}) \arrow[r, "DW"] \arrow[d, "f^*_{\mathbb{C}_p}"] & Rep_{\pi_1(X)}(\mathcal{O}_{\mathbb{C}_p}) \arrow[d, "f^*_{\mathbb{C}_p}"] \\
\mathcal{B}^s(\mathcal{Y}) \arrow[r, "DW"] & Rep_{\pi_1(Y)}(\mathcal{O}_{\mathbb{C}_p}).
\end{tikzcd}$$

Proof. Everything follows from theorem 2.1.8 and lemma 1.1.62. □

Using theorem 2.1.10 we see that:

Corollary 3.2.5. *The tensor functor*

$$DW_{\mathbb{Q}} : \mathcal{B}^s(X) \to Rep_{\pi_1(X)}(\mathbb{C}_p)$$

is compatible with duals, internal homs and exterior products, and commutes with pullbacks along arbitrary morphisms of proper connected rigid analytic varieties. Moreover, $DW_{\mathbb{Q}}$ *is fully faithful.*

In the language of [Xu17], proposition 3.2.3 implies that all vector bundles with numerically flat reduction are Weil-Tate (compare definition 2.1.11). The case of curves has already been dealt with in [Xu17, Corollaire 14.5] (using the Faltings topos). All Weil-Tate vector bundles are semistable in the following sense: (compare with [DW17, Theorem 9.7])

Proposition 3.2.6. *Let* E *be a vector bundle on* X *such that* $\lambda^* E \otimes \hat{\mathcal{O}}_X$ *is trivialized by a profinite étale cover. Then* f^*E *is semistable of degree* 0 *for every morphism* $f : C \to X$ *where* C *is s smooth projective curve.*

Proof. We have to show that any vector bundle E on a smooth projective curve for which $\lambda^* E \otimes \hat{\mathcal{O}}_X$ is profinite étale trivializable, is semistable of degree 0. So assume that X is a curve of genus g, and that $\tilde{Y} = \varprojlim Y_i \to X$ is a profinite étale cover trivializing $\lambda^* E \otimes \hat{\mathcal{O}}_X$. If $\tilde{Y}$ stabilizes, i.e. if $\lambda^* E \otimes \hat{\mathcal{O}}_X$ becomes trivial on some finite étale cover $f_i : Y_i \to X$, then the pullback $f_i^* E$ to Y_i is also trivial (as $\lambda_{Y_i *} \hat{\mathcal{O}}_{Y_i} = \mathcal{O}_{(Y_i)_{an}}$), so in particular semistable of degree 0. But then E is also semistable of degree 0. So assume that $\tilde{Y}$ does not stabilize. We can assume that $deg(E) \geq 0$ (otherwise pass to the dual bundle). Let r be the rank of E. Assume that $L \subset E$ is destabilizing. By passing to exterior powers we can assume that L is a line bundle, and by passing to tensor powers we can assume that $deg(L) \geq g$. Then by Riemann-Roch $dim_{\mathbb{C}_p} \Gamma(Y_i, f_i^* L) \geq deg(f_i)$, where f_i is the finite étale map $Y_i \to X$. As $\tilde{Y}$ does not stabilize, $deg(f_i)$ must grow to infinity. But $\varinjlim \Gamma(Y_i, f_i^* L) = \Gamma(\tilde{Y}, L) \subset \Gamma(\tilde{Y}, E) \subset \Gamma(\tilde{Y}, E \otimes \hat{\mathcal{O}}_X) = \mathbb{C}_p^r$.
In the same way one may check that the degree of E must be 0. □

If X is an algebraic variety the proposition says that E is numerically flat. In general, if X is not algebraic, one may expect that E is semistable with respect to any polarization on the special fiber of a formal model in the sense of [Li17]. We do not pursue this question here.

3.2.1 The case of line bundles

We will show that line bundles on a rigid analytic variety X over $\mathbb{C}_p$ which are deformations of the trivial bundle over a connected quasi-compact base are Weil-Tate. In case the Picard functor is representable and its identity component Pic^0_X is bounded, this implies that all line bundles in $Pic^0_X(\mathbb{C}_p)$ are Weil-Tate. General representability results for the Picard functor have been announced in [War17] (see in particular Theorem 1.0.2 in loc. cit.). If Pic_X is representable and in addition X is proper smooth and admits a formal model whose special fiber is projective, then the properness of Pic^0_X is ensured by [Li17, Theorem 1.1].
More generally we will prove that any line bundle L for which some tensor power is a qc connected deformation of the trivial bundle gives rise to a local system.

Definition 3.2.7. Let X be a proper connected rigid analytic variety over $\mathbb{C}_p$. Then a line bundle L on X is said to be analytically equivalent to $\mathcal{O}_X$ over T, where T is a connected rigid-analytic variety over $\mathbb{C}_p$, if there exists a line bundle $\tilde{L}$ on $X \times T$, such that both L and $\mathcal{O}_X$ occur as fibers of $\tilde{L}$.
We call a line bundle L τ-equivalent to $\mathcal{O}_X$ if some tensor-power of L is analytically equivalent to $\mathcal{O}_X$.

Remark 3.2.8. If Pic_X is representable, one may check that the line bundles, which are analytically equivalent to the trivial bundle, are precisely the ones in $Pic^0_X(\mathbb{C}_p)$. In this case let $\phi_n : Pic_X \to Pic_X$ denote the n-th power map. The group adic space $Pic^\tau_X := \bigcup_{n\leq 1} \phi_n^{-1}(Pic^0_X) \subset Pic_X$, is called the torsion component of the identity. As in the case of schemes (see [Kle14, §6]) one sees that, if Pic_X is representable, Pic^τ_X is also representable by an open subgroup adic space.
We do not need this here.

Lemma 3.2.9. *Let Z be a proper, reduced and connected rigid analytic variety over $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$, and let L be a line bundle on Z which is analytically equivalent to $\mathcal{O}_Z$ over a quasi-compact base T, then L has a formal model $\mathcal{L}$ on some $\mathcal{Z}$, such that $\mathcal{L}_s$ is numerically flat.*

In case Pic^0_Z exists and is proper, the lemma is equivalent to the claim that all $L \in Pic^0_Z(\mathbb{C}_p)$ have numerically flat reduction.

Proof. Let P be said family on $Z \times T$. By assumption there exist points $v : Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to T$ and $v_0 : Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to T$ such that $v^*P = L$ and $v_0^*P \cong \mathcal{O}_Z$. Using the reduced fiber theorem from [BLR95] we can find a connected formal model $\mathcal{Z} \times \mathcal{T}$ of $Z \times T$, such that $\mathcal{Z}$ is proper connected and has reduced special fiber, together with a locally free model $\mathcal{P}$ of P (by proposition 1.1.40).
Let $\tilde{v}, \tilde{v_0} : Spf(\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{T}$ be the specializations of v and v_0. Moreover we can

replace $\mathcal{P}$ by another model $\tilde{\mathcal{P}}$, for which some isomorphism $(\tilde{v}_0^*\tilde{\mathcal{P}})_{\mathbb{C}_p} \cong \mathcal{O}_Z$ extends to an isomorphism $\tilde{v}_0^*\tilde{\mathcal{P}} \cong \mathcal{O}_{\mathcal{Z}}$: Namely, simply consider the pullback $pr_{\mathcal{Z}}^*((\tilde{v}_0^*\mathcal{P})^{-1})$. Then $\mathcal{P}' = pr_{\mathcal{Z}}^*((\tilde{v}_0^*\mathcal{P})^{-1}) \otimes \mathcal{P}$ does the job.
But then the special fiber of $\mathcal{L} = \tilde{v}^*\mathcal{P}$ is a deformation of the trivial line bundle over a connected base. Using Chow's lemma there is a proper surjective map $\pi : Y \to \mathcal{Z}_s$, where Y is a normal projective variety. Then Pic_Y^0 exists by classical results, and all line bundles in $Pic_Y^0(\bar{\mathbb{F}}_p)$ are numerically flat. Pulling back the family $\mathcal{P}_s$ along π shows that $\pi^*\mathcal{L}_s$ lies in $Pic_Y^0(\bar{\mathbb{F}}_p)$ and hence is numerically flat. But then $\mathcal{L}_s$ is numerically flat as well. □

The following proposition shows that every line bundle L on a smooth proper rigid analytic space, which lies in the torsion component, is associated to a $\mathbb{C}_p$-local system. We define $\hat{\mathbb{C}}_p := \hat{\mathcal{O}}_{\mathbb{C}_p}[\frac{1}{p}]$, as sheaves on the proétale site, and say that a vector bundle E is associated to a $\mathbb{C}_p$-local system, if there exists a locally free $\hat{\mathbb{C}}_p$-module $\mathbb{L}$ such that $\mathbb{L} \otimes_{\hat{\mathbb{C}}_p} \hat{\mathcal{O}}_X \cong \hat{\mathcal{O}}_X \otimes \lambda^* E$.

Proposition 3.2.10. *Assume that X is proper smooth or the analytification of a proper scheme over $Spec(\mathbb{C}_p)$, and that L is a line bundle such that $L^{\otimes n}$ is Weil-Tate. Then L is associated to a $\hat{\mathbb{C}}_p$ local system $\mathbb{M}$.*

Proof. We use Kummer-theory on the pro-étale site. Let $\mathbb{L}$ be the $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system associated to $L^{\otimes n}$.

Claim. *There is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mu_n \arrow[r] \arrow[d, "="] & \hat{\mathcal{O}}_{\mathbb{C}_p}^\times \arrow[r, "x \to x^n"] \arrow[d] & \hat{\mathcal{O}}_{\mathbb{C}_p}^\times \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mu_n \arrow[r] & \hat{\mathcal{O}}_X^\times \arrow[r, "x \to x^n"] & \hat{\mathcal{O}}_X^\times \arrow[r] & 0
\end{tikzcd}$$

of Kummer exact sequences.

The commutativity of the diagram is clear. Moreover so is exactness of the upper sequence. For the lower sequence, note that as usual the only non-trivial part is showing right exactness. This follows in the same way as exactness of the Artin-Schreier sequence (for the tilted structure sheaf) proved in the proof of [Sch13a, Theorem 5.1]: If $U = \varprojlim Spa(A_i, A_i^+) \in X_{pro\acute{e}t}$ is affinoid perfectoid with associated perfectoid space $\hat{U} = Spa(A, A^+)$ (i.e. (A, A^+) is the completion of the direct limit of the (A_i, A_i^+)), then $\hat{\mathcal{O}}_X(U) = A$, and so any equation $x^n - a = 0$ for fixed $a \in \hat{\mathcal{O}}_X(U)$, can be solved by passing to a finite étale extension of A, which gives a finite étale map of affinoid perfectoid spaces $\tilde{V} \to \hat{U}$, say $\tilde{V} = Spa(B, B^+)$. But now by [Sch12, Lemma 7.5] any finite étale cover of $\hat{U}$ comes from a finite étale cover $V \to U$ in $X_{pro\acute{e}t}$, where then V is affinoid perfectoid (as an object in $X_{pro\acute{e}t}$) with $\hat{V} = \tilde{V}$. Hence $\hat{\mathcal{O}}_X(V) = B$, so we can find an n-th root of a by passing to the étale cover $V \to U$.
Now, taking pro-étale cohomology gives a commutative diagram

$$\begin{tikzcd}
H^1(\mu_n) \arrow[r] \arrow[d, "="] & H^1(\hat{\mathcal{O}}_{\mathbb{C}_p}^{\times}) \arrow[r] \arrow[d] & H^1(\hat{\mathcal{O}}_{\mathbb{C}_p}^{\times}) \arrow[r] \arrow[d] & H^2(\mu_n) \arrow[d, "="] \\
H^1(\mu_n) \arrow[r] & H^1(\hat{\mathcal{O}}_X^{\times}) \arrow[r] & H^1(\hat{\mathcal{O}}_X^{\times}) \arrow[r] & H^2(\mu_n),
\end{tikzcd}$$

where the map $H^1(\hat{\mathcal{O}}_{\mathbb{C}_p}^{\times}) \to H^1(\hat{\mathcal{O}}_X^{\times})$ is given by taking a rank 1 local system $\mathbb{L}$ to $\hat{\mathcal{O}}_X \otimes \mathbb{L}$.

By some diagram chasing one gets from this that there exists an $\mathcal{O}_{\mathbb{C}_p}$-local system $\mathbb{H}$ such that $\mathbb{H}^{\otimes n} \cong \mathbb{L}$. From this one sees that

$$(\mathbb{H} \otimes \hat{\mathcal{O}}_X)^{\otimes n} \cong \mathbb{L} \otimes \hat{\mathcal{O}}_X \cong (\lambda^* L \otimes \hat{\mathcal{O}}_X)^{\otimes n}.$$

But then these sheaves become isomorphic on some finite étale Kummer cover $\pi : X' \to X$, so that $\lambda_{X'}^*(\pi^* L) \otimes \hat{\mathcal{O}}_{X'} = \pi_{pro\acute{e}t}^*(\lambda_X^* L \otimes \hat{\mathcal{O}}_X) \cong \pi_{pro\acute{e}t}^* \mathbb{H} \otimes \hat{\mathcal{O}}_{X'}$. In particular $\pi^* L$ is Weil-Tate.

We conclude with the following lemma:

Lemma 3.2.11. *Let $\pi : X' \to X$ be a finite étale cover and let E be a vector bundle on X such that $\pi^* E$ is associated to a $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system $\mathbb{K}$. Then E is associated to a $\hat{\mathbb{C}}_p$-local system.*

For this assume that π is Galois with Galois group G. We then have a canonical Galois descent datum on $\pi_{pro\acute{e}t}^*(E \otimes \hat{\mathcal{O}}_X) \cong \mathbb{K} \otimes \hat{\mathcal{O}}_{X'}$. This induces a descent datum on $\mathbb{K}$, as $\mathbb{K} \mapsto \mathbb{K} \otimes \hat{\mathcal{O}}_{X'}$ is fully faithful (by proposition 1.1.74), hence it descends to some $\mathbb{M}$ on X and then, as the glueing datum is compatible with the one on $\pi_{pro\acute{e}t}^*(E \otimes \hat{\mathcal{O}}_X)$, one has $\mathbb{M} \otimes \hat{\mathcal{O}}_X \cong E \otimes \hat{\mathcal{O}}_X$. □

Corollary 3.2.12. *If L is a line bundle on X which is τ-equivalent to $\mathcal{O}_X$ over a quasi-compact base, then L is associated to a $\hat{\mathbb{C}}_p$-local system.*

If X is a normal algebraic variety, this $\mathbb{C}_p$-local system admits a lattice, in which case L is Weil-Tate.

We recall that if X is a projective scheme, then the line bundles in $Pic^{\tau}(X)$ are precisely the numerically flat line bundles.

3.3 Étale parallel transport

In this section we wish to show that the discussion in section 3.1 can be upgraded to construct étale parallel transport on pro-étale trivializable vector bundles. We will then compare our construction to the one from [DW17]. This is in some sense close to the discussion in [Xu17, §8], where the results from [DW05b] for the curve case are recast in light of the Faltings topos. Note however that, in contrast to loc. cit., even though we need to pass through almost mathematics, our final statement (theorem 3.3.5) will be an honest isomorphism even at the integral level.

Recall that to any connected rigid analytic variety X over $\mathbb{C}_p$ we can associate a topological groupoid $\Pi_1(X)$, whose objects are the $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$-valued points of X, and $Hom(x, x') = Isom(F_x, F_{x'})$, where F_x denotes the finite étale fiber functor.

Let $Free_r(\mathcal{O}_{C_p})$, resp $Free_r(\mathbb{C}_p)$ denote the topological groupoid of free rank r

modules over $\mathcal{O}_{\mathbb{C}_p}$, resp $\mathbb{C}_p$. Let E be a vector bundle on X. We say that E has étale parallel transport if the association $x \mapsto E_x$, can be extended to a functor $\Pi_1(X) \to Free_r(\mathbb{C}_p)$. Similarly, for a locally free $\mathcal{O}^+_{X_{an}}$-module E^+, we say that E^+ has étale parallel transport if $x \mapsto \Gamma(x^* E^+)$ can be extended to a functor $\Pi_1(X) \to Free_r(\mathcal{O}_{\mathbb{C}_p})$.
Let $\mathcal{E}^+ \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}^+_X)$, such that $\hat{\mathcal{E}}^+$ is trivial on $\tilde{Y}$. Let r denote the rank of $\mathcal{E}^+$. We can define a functor

$$\alpha_{\hat{\mathcal{E}}^+}\Pi_1(X) \to Free_r(\mathcal{O}_{C_p})$$

of pro-étale parallel transport on $\hat{\mathcal{E}}^+$ as in [DW17, §4] (see also section 3): On objects $\alpha_{\hat{\mathcal{E}}^+}$ takes $x \in X(\mathbb{C}_p)$ to $\hat{\mathcal{E}}^+_x$. On morphisms, if we have an étale path $\gamma \in Mor_{\Pi(X)}(x, x')$, we can pick a $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$-valued point y of $\tilde{Y}$ lying over x , then γ induces a point $y' = \gamma y$ over x'. Using triviality of $\mathcal{E}^+$ on $\tilde{Y}$ pulling back global sections along y, y' will then give isomorphisms

$$\hat{\mathcal{E}}^+_x \xleftarrow{y^*} \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+) \xrightarrow{y'^*} \hat{\mathcal{E}}^+_{x'},$$

and so we let γ map to $y'^* \circ (y^*)^{-1}$.
As in section 3.1 one can check that this is independent of the trivializing cover $\tilde{Y}$ and the chosen point y. Furthermore one can check in a similar fashion as in section 3.1 that this is a continuous functor of topological groupoids. Fixing a base point then gives back the representation from theorem 2.1.8.
Now, if E is a vector bundle on X such that the pullback $\lambda^* E$ to the pro-étale site lies in $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X)$ we can also define étale parallel transport on E: For any point $x : Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to X$ there is a canonical isomorphism $\Gamma(x^* E) \cong \Gamma(x^*(\lambda^* E)) \cong \Gamma(x^*(\lambda^* E \otimes \hat{\mathcal{O}}_X))$. Hence, using étale parallel transport on $E \otimes \hat{\mathcal{O}}_X$, we get an isomorphism $E_x \xrightarrow{\cong} E_y$ for any étale path $x \mapsto y$. Moreover this construction is compatible with composition of étale paths.
The same construction also works for locally free $\mathcal{O}^+_{X_{an}}$-modules E^+ whose pullback to the pro-étale site lies in $\mathcal{B}^{p\acute{e}t}(\mathcal{O}^+_X)$. We arrive at:

Proposition 3.3.1. *Let E be a vector bundle on X (resp. let E^+ be a locally free $\mathcal{O}^+_{X_{an}}$-module) for which the pullback to the pro-étale site $\lambda^* E$ (resp. $\lambda^* E^+$) lies in $\mathcal{B}^{p\acute{e}t}(\mathcal{O}_X)$ (resp. $\mathcal{B}^{p\acute{e}t}(\mathcal{O}^+_X)$). Then E (resp. E^+) has étale parallel transport.
In particular we get functors*

$$\alpha : \mathcal{B}^{p\acute{e}t}(\mathcal{O}_{X_{an}}) \to Rep_{\Pi_1(X)}(\mathbb{C}_p)$$
$$\alpha_{\mathcal{O}_{\mathbb{C}_p}} : \mathcal{B}^{p\acute{e}t}(\mathcal{O}^+_{X_{an}}) \to Rep_{\Pi_1(X)}(\mathcal{O}_{\mathbb{C}_p}).$$

Remark 3.3.2. Here $Rep_{\Pi_1(X)}(\mathcal{O}_{\mathbb{C}_p})$ denotes the category of continuous functors from $\Pi_1(X)$ to $Free_r(\mathcal{O}_{\mathbb{C}_p})$. A functor $F : \Pi_1(X) \to Free_r(\mathcal{O}_{\mathbb{C}_p})$ is called continuous if the induced maps on morphisms $Mor(x, x') \to Mor(F(x), F(x'))$ are continuous maps for all $x, x' \in \Pi_1(X)$.

To avoid overloading our notation even more we will also always write $\alpha(-)$ to denote parallel transport for any $\mathcal{O}^+_{X_{an}}$ or $\mathcal{O}^+_X$-module.
Note that whenever $E^+ = \mathcal{O}^+_{X_{an}} \otimes_{sp^{-1}\mathcal{O}_{\mathcal{X}}} sp^{-1}(\mathcal{E})$ comes from an integral model $(\mathcal{X}, \mathcal{E})$ of (X, E) the canonical isomorphisms $\Gamma(sp(x)^*\mathcal{E}) \cong \Gamma(x^* E^+)$ also allow us

to define parallel transport on $\mathcal{E}$.
For the comparison with the Deninger-Werner construction we need to work modulo p^n. As we have less control here, we need to pass through the almost setting.
So denote by $Free_r((\mathcal{O}_{\mathbb{C}_p}/p^n)_*)$ the groupoid (endowed with the discrete topology) of free $(\mathcal{O}_{\mathbb{C}_p}/p^n)_*$-modules of rank r, where $(\mathcal{O}_{\mathbb{C}_p}/p^n)_*$ again denotes the almost elements. We can then similarly define a mod p^n almost parallel transport for some $\mathcal{E}^+ \in \mathcal{B}^{p\acute{e}t}(\mathcal{O}_X^+)$:

$$\alpha_n^a(\mathcal{E}^+/p^n) : \Pi_1(X) \to Free_r((\mathcal{O}_{\mathbb{C}_p}/p^n)_*).$$

Now, as in the proof of proposition 2.1.5 we see that $(\mathcal{O}_{\mathbb{C}_p}/p^n)^r = \Gamma(\tilde{Y}, \hat{\mathcal{E}}^+)/p^n \hookrightarrow \Gamma(\tilde{Y}, \mathcal{E}^+/p^n)_*$ realizes $\alpha(\mathcal{E}^+)(\gamma)/p^n$ as a subobject of $\alpha_n^a(\mathcal{E}^+/p^n)(\gamma)$ for any étale path. Here by subobject we mean that there is a commutative diagram

$$\begin{tikzcd} \hat{\mathcal{E}}_x^+/p^n \arrow[r, "\alpha(\mathcal{E})(\gamma)/p^n"] \arrow[d, hook] & \hat{\mathcal{E}}_{x'}^+/p^n \arrow[d, hook] \\ (\hat{\mathcal{E}}_x^+/p^n)_* \arrow[r, "\alpha_n^a(\mathcal{E}^+/p^n)(\gamma)"] & (\hat{\mathcal{E}}_{x'}^+/p^n)_* \end{tikzcd} \tag{3.3.1}$$

for each étale path γ from x to x', and this association is compatible with composition of paths.
Assume now that X is a proper smooth connected algebraic variety over $\overline{\mathbb{Q}}_p$ with a flat proper integral model $\mathcal{X}$ over $\overline{\mathbb{Z}}_p$. Assume further, that we have a vector bundle $\mathcal{E}$ on $\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}$ with numerically flat reduction. Then one of the main results in [DW17] is the following:

Proposition 3.3.3. *[DW17, Theorem 7.1] Fix $n \geq 1$. Then there exists an open cover $\{U_i\}$ of X, such that for every i there is a proper surjective map $\tilde{f}_i : \mathcal{Y}_i \to \mathcal{X}$ which is finite étale over U_i and is such that $f_i := \tilde{f}_{i\mathcal{O}_{\mathbb{C}_p}}^* \mathcal{E}$ is trivial mod p^n.*

Using this result they construct a parallel transport functor $\alpha_{n,i}^{DW}(\mathcal{E}) : \Pi_1(U_i) \to Free_r(\mathcal{O}_{\mathbb{C}_p})$ for every i, which is then shown to glue to a functor $\alpha_n^{DW}(\mathcal{E})$ from $\Pi_1(X)$, which is again functorial in $\mathcal{E}$.
The construction of $\alpha_{n,i}^{DW}(\mathcal{E})$ is of course as above: For any étale path γ from x to $x' \in U_i(\mathbb{C}_p)$ one gets an isomorphism

$$\Gamma(x_n^*\mathcal{E}/p^n) = \Gamma(y_n^* f_i^*(\mathcal{E}/p^n) \xrightarrow{\cong} \Gamma(f_i^*(\mathcal{E}/p^n)) \xrightarrow{\cong} \Gamma((\gamma y)_n^* f_i^*(\mathcal{E}/p^n) = \Gamma((x')_n^*\mathcal{E}/p^n),$$

where $x_n, \cdots$ denotes the specializations with respect to the integral model $\mathcal{X}$ (resp. $\mathcal{Y}_i$). Taking the projective limit, one gets $\alpha_{\mathcal{O}_{\mathbb{C}_p}}^{DW}(\mathcal{E}) : \Pi_1(X) \to Free_r(\mathcal{O}_{\mathbb{C}_p})$. Now let $\hat{\mathcal{X}}$ be the admissible formal scheme obtained by completing $\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}$ along its special fiber. We denote by X^{an} the adic space generic fiber of $\hat{\mathcal{X}}$. X^{an} then coincides with the analytification of $X_{\mathbb{C}_p}$. We have $\Pi_1(X) \cong \Pi_1(X^{an})$ by remark 1.1.48. Let $\mathcal{E} \in \mathcal{B}^s(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}})$. Pulling back $\mathcal{E}$ to $\hat{\mathcal{X}}$ gives an object $\tilde{\mathcal{E}}$ in $\mathcal{B}^s(\hat{\mathcal{X}})$.

Remark 3.3.4. It is easy to see that the parallel transport defined on the formal vector bundle $\tilde{\mathcal{E}}/p^n$ coincides with the parallel transport on $\mathcal{E}/p^n$. Indeed one always has a canonical commutative diagram

$$\begin{array}{ccccc}
\Gamma(y_n^* f_i^*(\mathcal{E}/p^n)) & \xrightarrow{\cong} & \Gamma(f_i^*(\mathcal{E}/p^n)) & \xrightarrow{\cong} & \Gamma((\gamma y)_n^* f_i^*(\mathcal{E}/p^n)) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\Gamma(\tilde{y}^* \tilde{f}_i^*(\tilde{\mathcal{E}}/p^n)) & \xrightarrow{\cong} & \Gamma(\tilde{f}_i^*(\tilde{\mathcal{E}}/p^n)) & \xrightarrow{\cong} & \Gamma((\gamma\tilde{y})^* \tilde{f}_i^*(\tilde{\mathcal{E}}/p^n))
\end{array}$$

induced by pullback along the map $\hat{\mathcal{X}} \to \mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}$, resp. $\hat{\mathcal{Y}}_i \to (\mathcal{Y}_i)_{\mathcal{O}_{\mathbb{C}_p}}$. Here $\tilde{f}_i$, $\tilde{y}$, etc. denote the associated formal objects.
Note here also that one always has $\Gamma(y_n^* f_i^*(\mathcal{E}/p^n)) = \Gamma(\bar{y}^* f_i^*(\mathcal{E}/p^n)$, if now $\bar{y} \in \tilde{Y}_i(\mathcal{O}_{\mathbb{C}_p})$ is the point corresponding to y.

Pulling back $\tilde{\mathcal{E}}$ further to the pro-étale site gives an object $\mathcal{E}^+$ of $\mathcal{B}^{pét}(\mathcal{O}^+_{X^{an}})$. Denote by $\tilde{\alpha}_{\mathcal{O}_{\mathbb{C}_p}}$ the composition

$$\tilde{\alpha}_{\mathcal{O}_{\mathbb{C}_p}} : \mathcal{B}^s(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}) \to \mathcal{B}^s(\hat{\mathcal{X}}) \xrightarrow{sp^*} \mathcal{B}^{pét}(\mathcal{O}^+_{(X^{an})_{an}}) \xrightarrow{\alpha_{\mathcal{O}_{\mathbb{C}_p}}} Rep_{\Pi_1(X)}(\mathcal{O}_{\mathbb{C}_p}),$$

where $sp^*(\mathcal{E}) = sp^{-1}(\mathcal{E}) \otimes \mathcal{O}^+_{(X^{an})_{an}}$, and $sp : (X^{an})_{an} \to \hat{\mathcal{X}}_{Zar}$ denotes the specialization.

Theorem 3.3.5. *The functors $\tilde{\alpha}_{\mathcal{O}_{\mathbb{C}_p}}$ and $\alpha^{DW}_{\mathcal{O}_{\mathbb{C}_p}}$ are naturally isomorphic.*

Proof. We will show that $\tilde{\alpha}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{E})$ is isomorphic to $\alpha^{DW}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{E})$. Functoriality in $\mathcal{E}$ will be left to the reader.
First fix $n \geq 1$, and let $\{U_i\}$ be an open cover of X and $\{\mathcal{Y}_i\}$ the associated proper, surjective covers trivializing $\mathcal{E}/p^n$ as in proposition 3.3.3. Denote by Y_i the generic fiber of $\mathcal{Y}_i$. Let $Z_n \to X_{\mathbb{C}_p}$ be a finite étale cover trivializing $\mathcal{E}^+/p^n$. Let $\mathcal{Z}_n \to \mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}$ be a good model for Z_n in the sense of [DW17, Definition 3.5]. The cover $f_i : \mathcal{Z}_n \times_{\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}} \mathcal{Y}_{i\mathcal{O}_{\mathbb{C}_p}} \to \mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}$ is still a trivializing cover for $\mathcal{E}/p^n$ and finite étale over U_i. Denote by f_i^{an} the analytification of $f_i \otimes \mathbb{C}_p$. Then f_i^{an} trivializes $\mathcal{E}^+/p^n$ and $(\tilde{\alpha}_{\mathcal{O}_{\mathbb{C}_p}}/p^n)|_U$ can be realized on $f_i^{an*}\mathcal{E}^+/p^n$. Let $x, x' \in U(\mathbb{C}_p)$ and let $\gamma \in Mor(x, x')$ be an étale path. Consider the following commutative diagram of $\mathcal{O}_{\mathbb{C}_p}/p^n$-modules:

$$\begin{array}{ccccccccc}
(\mathcal{E}/p^n)_{x_n} & \xrightarrow{=} & \Gamma(y_n^* f_i^*(\mathcal{E}/p^n)) & \xrightarrow{\cong} & \Gamma(f_i^*(\mathcal{E}/p^n)) & \xrightarrow{\cong} & \Gamma((\gamma y)_n^* f_i^*(\mathcal{E}/p^n)) & \xrightarrow{=} & (\mathcal{E}/p^n)_{x_n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathcal{E}^+_x/p^n)_* & \xrightarrow{=} & \Gamma(y^* f_i^{an*}(\mathcal{E}^+/p^n))_* & \xrightarrow{\cong} & \Gamma(f_i^{an*}(\mathcal{E}^+/p^n))_* & \xrightarrow{\cong} & \Gamma((\gamma y)^* f_i^{an*}(\mathcal{E}^+/p^n))_* & \xrightarrow{=} & (\mathcal{E}^+_x/p^n)_*,
\end{array}$$

where the upper row is $(\alpha^{DW}_{n,i}(\mathcal{E})(\gamma))$ and the lower row is $\alpha^a_n(\mathcal{E}^+/p^n)(\gamma)$ and y is a point of $(\mathcal{Z}_n \times_{\mathcal{X}} \mathcal{Y}_i)_{\mathbb{C}_p}$ above x, and x_n is the mod p^n specialization of x. Recall also that $(\mathcal{E}/p^n)_{x_n}$ can always be identified with $(\mathcal{E}/p^n)_{\bar{x}} = \mathcal{E}_{\bar{x}}/p^n$, where $\bar{x} \in \mathcal{X}(\mathcal{O}_{\mathbb{C}_p})$ is the point associated to x.
For the construction of the vertical maps use remark 3.3.4 and note that if $\mathcal{Z}$ is a proper scheme with associated formal scheme $\hat{\mathcal{Z}}$, one has a composition of morphisms of ringed sites

$$(Z_{proét}, \mathcal{O}^+_Z) \to (\hat{\mathcal{Z}}_{Zar}, \mathcal{O}_{\hat{\mathcal{Z}}}) \to (\mathcal{Z}_{Zar}, \mathcal{O}_{\mathcal{Z}})$$

The vertical maps are then given by pulling back global sections along this and embedding into almost elements at the end. As everything is functorial, the diagram commutes. All vertical arrows are canonical almost isomorphisms.
More precisely, going through all identifications, one checks that this takes $\alpha_{n,i}^{DW}(\mathcal{E})(\gamma)$ isomorphically to $(\tilde{\alpha}/p^n)(\mathcal{E})(\gamma) \hookrightarrow \alpha_n^a(\mathcal{E}^+/p^n)(\gamma)$ (see diagram (3.3.1)): Indeed, the map on the fiber $(\mathcal{E}/p^n)_{x_n} \hookrightarrow (\mathcal{E}_x^+/p^n)_*$ factors of course through $(\mathcal{E}_x^+/p^n) \hookrightarrow (\mathcal{E}_x^+/p^n)_*$. Also, everything is compatible with the composition of paths.
From this we see that we get an isomorphism $(\alpha_{n,i}^{DW}(\mathcal{E})(\gamma)) \cong (\tilde{\alpha}/p^n)|_{U_i}$, where $(\tilde{\alpha}/p^n)|_{U_i}$ just denotes the restriction to $\Pi_1(U_i)$. But then, using the Seifert-van-Kampen result from [DW17, Theorem 4.1], we see that $(\alpha_n^{DW}) \cong (\tilde{\alpha}/p^n)$ are isomorphic.
Passing to the p-adic completion we get the desired isomorphism. □

3.4 The Hodge-Tate spectral sequence

Assume that X is a proper smooth rigid analytic variety over $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. We will see that the Deninger-Werner local systems satisfy a Hodge-Tate decomposition.

Theorem 3.4.1. *[Sch13b, Theorem 3.20] Let E be a vector bundle on X associated to a $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system $\mathbb{L}$. Then there is a Hodge-Tate spectral sequence:*

$$E_2^{ij} = H^i(X, E \otimes \Omega^j(-j)) \Longrightarrow H^{i+j}(X_{ét}, \mathbb{L})[\tfrac{1}{p}]$$

Recall that $\Omega^j(-j) := \Omega^j \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathbb{Z}}_p(-1)^{\otimes j}$, where $\hat{\mathbb{Z}}_p(1) := \varprojlim_n \mu_{p^n}$ as pro-étale sheaves, and $\hat{\mathbb{Z}}_p(-1)$ denotes the dual of $\hat{\mathbb{Z}}_p(1)$.

Proof. By assumption $\hat{E} = \hat{\mathcal{O}}_X \otimes \mathbb{L}$. Let again $\nu : X_{proét} \to X_{ét}$ denote the canonical projection. Then the Cartan-Leray spectral sequence reads

$$H^i(X_{ét}, R^j\nu_*(\hat{\mathcal{O}}_X \otimes \mathbb{L})) \Longrightarrow H^{i+j}(X_{proét}, \hat{\mathcal{O}}_X \otimes \mathbb{L}).$$

By the primitive comparison theorem we have $H^{i+j}(X_{proét}, \hat{\mathcal{O}}_X \otimes \mathbb{L}) = H^{i+j}(X_{proét}, \mathbb{L})[\frac{1}{p}]$. But now, by [Sch13b, Proposition 3.23] there is an isomorphism $R^j\nu_*\hat{\mathcal{O}}_X \cong \Omega^j(-j)$, and hence

$$R^j\nu_*(\hat{\mathcal{O}}_X \otimes \mathbb{L}) = R^j\nu_*(\nu^*E \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X) \cong E \otimes R^j\nu_*\hat{\mathcal{O}}_X \cong E \otimes \Omega^j(-j).$$

□

As usual one does not get a canonical splitting of this spectral sequence in general. However one does have the following:

Proposition 3.4.2. *Assume that X is a proper smooth rigid analytic space over $Spa(K, \mathcal{O}_K)$, where $K/\mathbb{Q}_p$ is a finite extension. Let further E be a vector bundle on X such that $E_{\hat{\bar{K}}}$ is associated to an $\hat{\mathcal{O}}_{\mathbb{C}_p}$-local system $\mathbb{L}$. Then the Hodge-Tate spectral sequence degenerates canonically at E_2.*

Note that the local system must not be defined over a finite extension of K, as indeed will generally not be the case for Deninger-Werner local systems.

Proof. Now there is a $Gal(\bar{K}/K)$-action on the cohomology groups in theorem 3.4.1, and in particular the differentials in the Cartan-Leray spectral sequence will be invariant under this action. But then as
$H^i(X_{\hat{\bar{K}}}, E_{\hat{\bar{K}}} \otimes \Omega^j_{X_{\hat{\bar{K}}}}(-j)) = H^i(X, E \otimes \Omega^j_X) \otimes_K \mathbb{C}_p(-j)$ by base change for cohomology, one gets that all differentials are zero, as $Hom(\mathbb{C}_p(-j), \mathbb{C}_p(-j')) = 0$ for $j \neq j'$ by Tate's theorem. □

We do not know whether there is non-canonical degeneration in general (i.e. when X is not defined over a finite extension of $\mathbb{Q}_p$).
For the constant local system, consider the map $\alpha : H^1(X, \mathcal{O}_{X_{\hat{\bar{K}}}}) \to H^1_{\acute{e}t}(X, \mathbb{Z}_p) \otimes \hat{\bar{K}}$ from the Hodge-Tate decomposition. In [DW05a] Deninger and Werner showed that if $X = A$ is an abelian variety with good reduction, there is a commutative diagram

$$\begin{CD}
H^1(A, \mathcal{O}_{A_{\hat{\bar{K}}}}) @>{\alpha}>> H^1_{\acute{e}t}(A, \mathbb{Z}_p) \otimes \hat{\bar{K}} \\
@V{\cong}VV @V{\cong}VV \\
Ext^1(\mathcal{O}_{A_{\hat{\bar{K}}}}, \mathcal{O}_{A_{\hat{\bar{K}}}}) @>{DW}>> Ext^1(\hat{\mathbb{C}}_p, \hat{\mathbb{C}}_p),
\end{CD}$$

where the map below means applying the Deninger-Werner functor to a unipotent rank 2 vector bundle, which gives a unipotent rank 2 local system.
We can show that this generalizes to arbitrary extensions on any proper smooth rigid analytic variety. Namely, let X be any proper and smooth rigid analytic variety over $Spa(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Then for any $E \in \mathcal{B}^s(X)$ the Hodge-Tate spectral sequence gives an injection $H^1(X, E) \to H^1_{\acute{e}t}(X, \mathbb{L}_E)$. We then get the following

Proposition 3.4.3. *For any $E, E' \in \mathcal{B}^s(X)$ there is a commutative diagram*

$$\begin{CD}
H^1(X, \mathcal{H}om(E, E')) @>{\alpha}>> H^1_{\acute{e}t}(X, \mathcal{H}om(\mathbb{L}_E, \mathbb{L}_{E'})) \\
@V{\cong}VV @V{\cong}VV \\
Ext^1(E, E') @>{DW}>> Ext^1(\mathbb{L}_E, \mathbb{L}_{E'}).
\end{CD}$$

where for any $F \in \mathcal{B}^s(X)$ we denote by $\mathbb{L}_F$ the local system obtained by applying the functor DW, and α denotes the map coming from the Hodge-Tate spectral sequence.

Proof. The map α is given by the composition

$$H^1(X, \mathcal{H}om(E, E')) \xrightarrow{\beta} H^1(X, \mathcal{H}om(E, E') \otimes \hat{\mathcal{O}}_X) \xrightarrow{com} H^1_{\acute{e}t}(X, \mathcal{H}om(\mathbb{L}_E, \mathbb{L}_{E'})),$$

where com is the comparison isomorphism from theorem 1.1.72 and β is simply given as the map on H^1 associated to the injection of pro-étale sheaves $\nu^*\nu_*(\hat{\mathcal{O}}_X \otimes \mathcal{H}om(\mathbb{L}_E, \mathbb{L}_{E'})) \hookrightarrow \hat{\mathcal{O}}_X \otimes \mathcal{H}om(\mathbb{L}_E, \mathbb{L}_{E'})$.

Now take an extension

$$e = (0 \to E' \to E \to E'' \to 0)$$

of vector bundles on X. By lemma 2.1.9 the functors $DW(-) \otimes \hat{\mathcal{O}}_X$ and $\nu^*(-) \otimes \hat{\mathcal{O}}_X$ are canonically isomorphic. So one only has to check that the map β takes the extension e to

$$0 \to \nu^* E' \otimes \hat{\mathcal{O}}_X \to \nu^* E \otimes \hat{\mathcal{O}}_X \to \nu^* E'' \otimes \hat{\mathcal{O}}_X \to 0.$$

But this is clear. $\square$

Appendix A

Basics on perfectoid rings and almost mathematics

We will to put together some basic results on perfectoid algebras and almost mathematics following [Sch12].

Definition A.0.1. A perfectoid field K is a non-archimedean field whose residue field is of characteristic p, and such that

- The value group $|K^\times| \subset \mathbb{R}_{>0}$ is not discrete.
- The Frobenius map $x \mapsto x^p$ is surjective on $\mathcal{O}_K/p$.

Example A.0.2.
- The field of p-adic numbers $\mathbb{Q}_p$ is not perfectoid, as it is discretely valued.
- Denote by $\mathbb{Q}_p(\mu_{p^\infty}) := \bigcup_{n\geq 1} \mathbb{Q}_p(\mu^{p^n})$ the extension obtained by adjoining all p^n-th roots of unity. Then the completion of $\mathbb{Q}_p(\mu_{p^\infty})$ is a perfectoid field.
- Any complete algebraically closed extension $\mathbb{Q}_p \subset \mathbf{C}$ (for example $\mathbf{C} = \hat{\bar{\mathbb{Q}}}_p$) is perfectoid. Indeed, the value group always contains $|p|^{\mathbb{Q}}$ and as $\mathbf{C}$ is algebraically closed, any $x \in \mathcal{O}_{\mathbf{C}}$ has a p-th root.

Definition A.0.3. Let K be a perfectoid field with ring of integers $\mathcal{O}_K$. Let ϕ be the Frobenius on $\mathcal{O}_K/p$. Then the ring $\mathcal{O}_K^\flat := \varprojlim_\phi \mathcal{O}_K/p$ is called the tilt of $\mathcal{O}_K$.

By construction $\mathcal{O}_K^\flat$ is a perfect ring, i.e. the absolute Frobenius is an isomorphism. There is a bijective multiplicative map $\mathcal{O}_K^\flat \to \varprojlim_{x\mapsto x^p} \mathcal{O}_K$ (see [Sch12, Lemma 3.4]). Composing this with the projection to the first factor gives a map

$$\sharp : \mathcal{O}_K^\flat \to \mathcal{O}_K.$$

If $a \in \mathcal{O}_K^\flat$ one usually writes $a^\sharp \in \mathcal{O}_K$ for the image of a under this map.

Lemma A.0.4. *[Sch12, Lemma 3.4 (ii)] There exists an element $t \in \mathcal{O}_K^\flat$ for which $|t^\sharp| = |p|$. Moreover, t maps to zero in $\mathcal{O}_K/p$ which induces an isomorphism $\mathcal{O}_K^\flat/t \cong \mathcal{O}_K/p$.*

Remark A.0.5. The above lemma holds true more generally for any pseudouniformizer $\pi \in \mathfrak{m}$ with $|p| \leq |\pi|$.

From the above lemma one also gets that $\mathcal{O}_K^\flat$ is t-adically complete. One can moreover check that it is a valuation ring of rank 1.

Definition A.0.6. The field $K^\flat := \mathcal{O}_K^\flat[\frac{1}{t}]$ is called the tilt of K.

Definition A.0.7. Let K be a perfectoid field, with pseudo-uniformizer π, with $|p| \leq |\pi|$. A Banach K-algebra R is called perfectoid if the ring of power bounded elements $R^\circ \subset R$ is open and bounded, and the Frobenius on $R^\circ/\pi \to R^\circ/\pi$ is surjective.
An affinoid $(K, \mathcal{O}_K)$-algebra (R, R^+) is called perfectoid, if R is a perfectoid K-algebra.

Remark A.0.8. For geometric considerations the most important perfectoid algebra is given by $(\mathcal{O}_K[T_1^{\pm\frac{1}{p^\infty}}, \cdots, T_n^{\pm\frac{1}{p^\infty}}]\hat{\ })[\frac{1}{\pi}]$, where $\hat{\ }$ means taking completion.

If R is perfectoid, then one defines $A := \varprojlim_\phi R^\circ/\pi$, and calls it the tilt of R°. It is a perfect algebra over $\mathcal{O}_K^\flat$. Letting $t \in \mathcal{O}_K^\flat$ be a pseudouniformizer with $|t^\sharp| = |\pi|$, one sets $R^\flat := A[t^{-1}]$, and calls it the tilt of R. One can then prove that $(R^\flat)^\circ = A$ and $R^{\flat\circ}/t = R^\circ/\pi$. One has

Theorem A.0.9. *[Sch12, Theorem 5.2] Tilting induces an equivalence of categories between perfectoid K-algebras and perfectoid $K^\flat$-algebras.*

This theorem is upgraded in [Sch12] to a homeomorphism of the associated adic spaces, and then further to an equivalence of the étale sites.

A.0.1 Almost mathematics

Let K be a perfectoid field, with ring of integers $\mathcal{O}_K$ and maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$. As $\mathfrak{m}$ is a torsionfree module over a valuation ring it is flat, Moreover it is easy to show that $\mathfrak{m}^2 = \mathfrak{m}$ (in fact one can show that $\mathfrak{m}$ is a union of principal ideals).

Definition A.0.10. An $\mathcal{O}_K$-module M is called almost zero if $\mathfrak{m}M = 0$. A homomorphism $f : M \to N$ of $\mathcal{O}_K$-modules is called almost surjective (resp. almost injective) if $coker(f)$ (resp. $ker(f)$) is almost zero. f is called almost isomorphism if it is almost surjective and almost injective.

Remark A.0.11. Consider the restriction of scalars functor $i_* : Mod_{\mathcal{O}_K/\mathfrak{m}} \to Mod_{\mathcal{O}_K}$. Then the condition $\mathfrak{m}^2 = \mathfrak{m}$ ensures that this realizes $Mod_{\mathcal{O}_K/\mathfrak{m}}$ as a Serre-subcategory of $Mod_{\mathcal{O}_K}$. The quotient category $Mod^a_{\mathcal{O}_K} = Mod_{\mathcal{O}_K}/Mod_{\mathcal{O}_K/\mathfrak{m}}$ is called category of almost modules. It allows for a systematic study of the phenomena occuring in almost mathematics. We will not need these constructions.

Definition A.0.12. Let M be an $\mathcal{O}_K$-module. Then the module $M_* = Hom_{\mathcal{O}_K}(\mathfrak{m}, M)$ is called module of almost elements of M.

Remark A.0.13.

- Clearly, if M is almost zero one has $M_* = 0$.

- If M is torsion-free one can check that $M_* = \{m \in M \otimes_{\mathcal{O}_K} K | \epsilon m \in M, \forall \epsilon \in \mathfrak{m}\}$. In particular, one has $(\mathcal{O}_K)_* = \mathcal{O}_K$.

For any $\mathcal{O}_K$-module M there is a canonical map $M \to M_*$ (by applying $Hom(-, M)$ to $\mathfrak{m} \subset \mathcal{O}_K$). This map becomes an isomorphism after passing to the almost category.
One has the following general fact:

Lemma A.0.14. *[Sch12, Lemma 5.3] Assume that M is (almost) flat, then for any $t \in \mathcal{O}_K$ one has $tM_* = (tM)_*$ and the canonical map $M_*/tM_* \to (M/tM)_*$ is injective.*

Remark A.0.15. The only thing that we actually use is that $(\mathcal{O}_{\mathbf{C}})_* = \mathcal{O}_{\mathbf{C}}$ and that the canonical map $\mathcal{O}_{\mathbf{C}}/t \to (\mathcal{O}_{\mathbf{C}}/t)_*$ is injective. This can of course be checked directly.

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