RATIONAL AND p-LOCAL MOTIVIC HOMOTOPY THEORY

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Contents

Abstract

Let F be a perfect field and k an arbitrary field. The main goal of this thesis is to investigate algebraic models for the Morel-Voevodsky unstable motivic homotopy category $\mathcal{H}(F)$ after $\mathbf{H}^{\mathbb{A}^1}k$ localization. More specifically, we extend results of Goerss to the $\mathbb{A}^1$-algebraic topology setting: we study the homotopy theory of the category $s\mathrm{coCAlg}_k(Sm_F)$ of presheaves of simplicial coalgebras over a field k and their τ and $\mathbb{A}^1$-localizations. For k algebraically closed, we show that the unit of the adjunction $k^\delta[-] \dashv (-)^{gp}$ determines the $\mathbf{H}^{\mathbb{A}^1}k$ homotopy type, where $k^\delta[-]$ is the canonical coalgebra functor induced by the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$. Furthermore, for $k = \mathbb{Q}$, we introduce the notion of $\mathbb{A}^1$-nilpotent space and we provide an explicit formula for the $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$ localization functor for such spaces. For this goal, we study the category of G-discrete motivic spaces, with G a profinite group.

On the other hand, we show that the category of coalgebra objects in $\mathrm{PST}(Sm_F, k)$ is locally presentable, where $\mathrm{PST}(Sm_F, k)$ is the category of presheaves with Voevodsky transfers and the monoidal structure is given by a Day convolution product.

Zusammenfassung

Es sei F ein perfekter und k ein beliebiger Körper. Das Hauptziel dieser Dissertation besteht darin algebraische Modelle zu untersuchen, für die Morel-Voevodsky instabile motivische Homotopiekategorie $\mathcal{H}(F)$ nach $\mathbf{H}^{\mathbb{A}^1}k$-Lokalisation. Genauer gesagt verallgemeinern wir die folgenden Ergebnisse von Goerss für die algebraische $\mathbb{A}^1$-Topologie: Wir studieren die Homotopietheorie der Kategorie $s\mathrm{coCAlg}_k(Sm_F)$ von Prägarben simplizialer Koalgebren über einem Körper K und deren $\mathbb{A}^1$-Lokalisation. Ist k algebraisch abgeschlossen, so zeigen wir, dass die einheit der Adjunktion $k^\delta[-] \dashv (-)^{gp}$ den $\mathbf{H}^{\mathbb{A}^1}k$-Homotopietyp bestimmt, wobei $k^\delta[-]$ die kanonische Funktor von Koalgebren induziert durch die Diagonalabbildung $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ ist. Insbesondere führen wir für den Fall $k = Q$ den Begriff eines $\mathbb{A}^1$-nilpotenten Raumes ein und geben eine explizite Formel für den $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$-Lokalisationsfunktor solcher Räume an. Dazu studieren wir die Kategorie von G-diskreten motivischen Räumen, wobei G eine proendliche Gruppe ist.

Außerdem zeigen wir, dass die Kategorie von Koalgebren in $\mathrm{PST}(Sm_F, k)$ lokal darstellbar ist, wobei $\mathrm{PST}(Sm_F, k)$ die Kategorie von Prägarben mit Transfers und der monoidalen Struktur ist, die durch ein Day-Konvolutionsprodukt gegeben ist.

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Introduction

Motivation: One of the problems in classical algebraic topology is to find good algebraic invariants. Even in classical algebraic topology, homotopy groups $\pi_*(X)$, homology groups $H_*(X,k)$, cohomology rings $H^*(X,k)$ and Steenrod operations $Sq^i : H^n(X,\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X,\mathbb{Z}/2\mathbb{Z})$ are not sufficient to distinguish homotopy types. It is possible to construct examples where two spaces have isomorphic singular homology groups, but their cohomology rings are not isomorphic, or two spaces with isomorphic cohomology rings but with different Steenrod operations. Finally, it is possible to construct spaces with isomorphic cohomology rings and Steenrod operations but with different Massey products. Massey products are a consequence of the existence of chain level multiplication on $C^*(X,\mathbb{Z})$, this suggests that Differential Graded Algebras or dually Differential Graded Coalgebras are finer algebraic invariants.

Formally the problem is, for detecting the homology localization, whether there exists a category $\mathcal{D}$ of algebraic nature, *e.g;* group objects, ring objects, algebra objects or coalgebra objects, and a functor

$$F : \mathrm{Ho}(\mathcal{S}) \to \mathcal{D}$$

such that F is fully faithful.

More concretely, this problem was studied by Quillen [**Qui69**] and Sullivan [**Sul77**], [**BG76**] for rational coefficients, and later by Goerss [**Goe95b**] for coefficients in an arbitrary field. Specifically, Quillen showed that there exists an equivalence among the categories of simply connected spaces, 1-reduced differential graded Lie algebras and 2-reduced Differential Graded Coalgebras.

$$\mathrm{L}_{H_*\mathbb{Q}}\mathrm{Ho}(\mathcal{S}_{\geq 1}) \to \mathrm{Ho}(DGL_{\mathbb{Q},\geq 1}) \to \mathrm{Ho}(DGC_{\mathbb{Q},\geq 2}).$$

On the other hand, Sullivan proved that there is an equivalence between the category of nilpotent spaces of finite $\mathbb{Q}$-type, *i.e* nilpotent spaces such that $H^*(X,\mathbb{Q})$ is of finite dimension, and the subcategory of $DGA_{\mathbb{Q}}$ such that A^n is a $\mathbb{Q}$-vector space of finite dimension.

$$A_{PL} : \mathrm{L}_{H^*\mathbb{Q}}\mathrm{Ho}(\mathcal{S}^{fin,Nil}) \to \mathrm{Ho}(DGA^{fin}_{\mathbb{Q}}).$$

Goerss avoided those conditions and used the category of simplicial coalgebras $s\mathrm{coCAlg}_k$ to show that the canonical chain complex functor at the simplicial level, and with a coalgebra structure induced by the diagonal map $\Delta : X \to X \times X$, induces, for k an algebraically closed field, a fully faithful functor

$$\mathrm{L}_{H_*k}\mathcal{S} \to s\mathrm{coCAlg}_k. \tag{0.1}$$

The key ingredient in Goerss' approach is a good understanding of the category of coalgebras over an algebraically closed field, provided in [**Swe69**]. The condition of k being an algebraically closed field is a strong condition. Goerss refine this functor to the category of simplicial coalgebras over an arbitrary field k. For that, he uses an intermediate category of spaces with a group action of the absolute Galois group $G = Gal(\bar{k}/k)$. Here it becomes fundamental to understand the homotopy fixed points functor, which was studied by Goerss in [**Goe95a**].

Let us now turn to the algebraic geometry setting and fix F a perfect field. Motivic homotopy theory was introduced by Morel and Voevodsky in their foundational paper [**MV99**], where topological spaces are replaced by presheaves of spaces on the category of smooth schemes of finite type over F. They constructed the unstable motivic homotopy category as the $\mathbb{A}^1$-localization of the Jardine model structure $\mathrm{L}_{Nis}s\mathrm{PSh}(Sm_F)$. In [**Mor12**], the $\mathbb{A}^1$-homology sheaves $\mathbf{H}^{\mathbb{A}^1}(\mathcal{X})$ were introduced; this homology theory could be a priori naive, but actually computes a wealth of important information.

Suslin and Voevodsky in [**SV96**] introduced a singular homology theory for algebraic varieties known as Suslin homology. The idea of their definition is based on the Dold-Thom theorem. Specifically, given X a scheme of finite type over a field F, the Suslin homology group $H_i^{Sus}(X,\mathbb{Z})$ is defined as π_i of the simplicial abelian group

$$Hom(\Delta_k^\bullet, \coprod_{d=0}^{\infty} S^d(X))^+.$$

Furthermore, they note that after inverting p, with p being the exponential characteristic of F, $H_*^{Sus}(X)$ coincides with

$$\pi_*(C_*^{Sus}(X)) = H_*(C_*^{Sus}(X), d = \sum(-1)^i\delta_i),$$

where $C_n^{Sus}(X)$ is the simplicial abelian group generated by closed integral subschemes $Z \subset \Delta^n \times X$ such that $Z \to \Delta^n$ is finite and surjective. In other words, Suslin homology is the homology of the global sections of the Suslin complex.

For each $\mathcal{X} \in Spc_\bullet(k)$, we can define the Suslin homology sheaves as

$$\mathbf{H}_i^{Sus}(\mathcal{X}) := H_i(\mathrm{L}_{\mathbb{A}^1} k_{tr}(\phi_{rep}(\mathcal{X})_\bullet),$$

where $\phi_{rep}(\mathcal{X})_n$ is a coproduct of representable sheaves and $\phi_{rep}(\mathcal{X})_\bullet \to \mathcal{X}$ is a simplicial weak equivalence, which is also a local fibration.

The canonical map adding transfers induces a morphism between the homology sheaves

$$\mathbf{H}_*^{\mathbb{A}^1}(\mathcal{X}) \to \mathbf{H}_*^{Sus}(\mathcal{X}).$$

In general, this map is not an isomorphism. By results of Morel [**Mor12**, Theorem 6.40], $\mathbf{H}_*^{\mathbb{A}^1}(\mathbb{G}_m)^{\wedge n}$ is related to Milnor-Witt K-theory and by results of Suslin and Voevodsky [**SV00**, Theorem 3.4] $\mathbf{H}_*^{Sus}(\mathbb{G}_m)^{\wedge n}$ is closely related to Milnor K-theory.

Furthermore, by [**CD09**, Corollary 16.2.22] we have an equivalence of symmetric monoidal triangulated categories

$$\mathbf{D}_{\mathbb{A}^1,\acute{e}t}(F,\mathbb{Q}) \simeq \mathbf{DM}_{\acute{e}t}(F,\mathbb{Q})$$

where $\mathbf{D}_{\mathbb{A}^1,\acute{e}t}(F,\mathbb{Q})$ is the étale $\mathbb{A}^1$-derived category with $\mathbb{Q}$-coefficients and $\mathbf{DM}_{\acute{e}t}(F,\mathbb{Q})$ is Voevodsky's triangulated category of motives with $\mathbb{Q}$-coefficients. Then, after taking the version of $\mathbf{H}_*^{\mathbb{A}^1}(\mathcal{X})$ for the étale topology and using $\mathbb{Q}$-coefficients, the $\mathbb{A}^1$-homology is the same as rational Suslin homology.

Main Content: For k and F perfect fields, we investigate algebraic models for $\mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathcal{H}_\bullet(F)$ in terms of homotopy categories of coalgebras, where $\mathbf{H}^{\mathbb{A}^1}k$ is the $\mathbb{A}^1$-homology theory of Morel with coefficients in k and $\mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathcal{H}_\bullet(F)$ is the localization of the unstable motivic homotopy category with respect to these homology theory. The results are quite parallel to the results of Goerss. The diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ induces a coalgebra structure in $C_\bullet^{\mathbb{A}^1}(\mathcal{X}, k)$, which is a presheaf of coalgebras. By results of [**Rap13**], we know that the category of presheaves of simplicial coalgebras $s\mathrm{coCAlg}(Sm_F)$ is endowed with a left proper, simplicial, cofibrantly generated model category structure, which is left induced from a combinatorial model structure in $s\mathrm{Mod}_k(Sm_F)$. We extend Goerss' results and we show:

Theorem 1 (2.18). *Let k be an algebraically closed field. The functor $k^\delta[-]$ induces a fully faithful functor in the homotopy categories:*

$$\mathrm{L}k^\delta[-] : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathcal{H}_\bullet(F) \to \mathrm{Ho}(\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}} s\mathrm{coCAlg}_k(Sm_F)).$$

Furthermore, for every motivic space $\mathcal{X}$, the derived unit map

$$\mathcal{X} \to (k^\delta[\mathcal{X}]^{fib})^{gp}$$

exhibits the target as the $\mathbf{H}^{\mathbb{A}^1}k$ localization of $\mathcal{X}$.

If k is not algebraically closed, we define the category of discrete G-motivic spaces as:

$$Spc_\bullet^G(F) := \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{G\times\mathrm{Nis}} s\mathrm{PSh}_{inj}(\mathrm{Orb}(G) \times \mathrm{Sm}_F).$$

Sending each representable sheaf G/H to the constant simplicial presheaf of coalgebras $(\bar{k}^H)^\vee$ placed on simplicial degree 0, where $(\bar{k}^H)^\vee$ is the k dual of $\bar{k}^H$, we define the functor

$$\bar{k}^\vee[-]_G : Spc_\bullet^G(F) \to s\mathrm{coCAlg}_k(Sm_F).$$

The functor defined above has a right adjoint $R : s\mathrm{coCAlg}_k(Sm_F) \to Spc_\bullet^G(F)$, such that each $RC_n(U)$ indexes all the embeddings of the coalgebra $(\bar{k}^H)^\vee$ in $C(U)$.

THEOREM 2 (2.37). *Let k be a perfect field. The functor $k^{\vee}[-]_G$ induces a fully faithful functor in the homotopy categories:*

$$\mathrm{L}\bar{k}^{\vee}[-]_G : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathcal{H}((Spc^G_{\bullet,}(F)) \to \mathrm{Ho}(\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{coCAlg}_k(\mathrm{Sm}_F)).$$

Furthermore, for every motivic space $\mathcal{X}$, the derived unit map

$$\mathcal{X} \to R(\bar{k}^{\vee}[\mathcal{X}]_G)^{fib})$$

exhibits the target as the $\mathbf{H}^{\mathbb{A}^1}k$ localization of $\mathcal{X}$ in discrete G-motivic spaces.

Furthermore, for $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$-localization we have the following result:

THEOREM 3 (2.33). *Let $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$ be the rational $\mathbb{A}^1$-homology and $\mathcal{X}$ an $\mathbb{A}^1$-nilpotent space. Let $\mathcal{Y}$ be the $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$-localization of $\mathcal{X}$ as described in Theorem 2. Then Y^{hG} is the $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$-localization of $\mathcal{X}$ as a motivic space, where $\mathcal{Y}^{hG}$ is the homotopy fixed point space.*

The previous results are not restricted to the Nisnevich topology. Using the étale version of $\mathbb{A}^1$-homology with rational coefficients $\mathbf{H}^{\mathbb{A}^1}_{*,\acute{e}t}\mathbb{Q}$, we can interpret our results as giving a corresponding localization functor for the rational Suslin homology localization (see below for more details).

For the Suslin homology theory $\mathbf{H}^{Sus}k$ we get partial results. Our first result is about the local presentability of the category of coalgebras with transfers. More precisely, consider the category of presheaves with transfers $\mathrm{PST}(Sm_F, k)$ (see [**MVW06**, Definition 2.1]), this category is endowed with a monoidal structure, which is given by a Day convolution product. We denote by $s\mathrm{coCAlg}^{tr}_k(Sm_F)$ the category of coalgebra objects in $\mathrm{PST}(Sm_F, k)$

THEOREM 4 (Corollary 1.31 and Theorem1.32). *The category of coalgebras $\mathrm{coCAlg}^{tr}_k(Sm_F)$ is locally presentable, with strong generators given by $\{F \in \mathrm{coCAlg}^{tr}_k(Sm_F) : \#(F) \leq max(\#(C), \aleph_0)\}$. Furthermore, the underlying functor $U : \mathrm{coCAlg}^{tr}_k(Sm_F) \to \mathrm{PST}(Sm_F, k)$ has a right adjoint and is comonadic.*

We will use this result in a future paper to study the homotopy type of motivic spaces after Suslin homology localization.

Outline of this thesis: In Chapter 1 we discuss the structure theory of coalgebras over a field following [**Swe69**] and the category of presheaves of coalgebras. We discuss the category of presheaves of coalgebras for the Day convolution product, here we prove Theorem 4. In Chapter 2 we recall some well-known properties of the category of motivic spaces. We discuss the homotopy theory for presheaves of simplicial coalgebras, and rely on ideas from [**Rap13**] to study the $\mathbb{A}^1$-localization of $s\mathrm{coCAlg}_k(Sm_F)$; here we proved Theorem 1. We introduce the notion of G-discrete motivic spaces and we study the notion of homotopy fixed points and here we prove Theorem 2. Finally, in this chapter, we introduce the notion of $\mathbb{A}$-nilpotent space and we prove Theorem 3.

To the memory of my mom,
who was always enthusiastic about my idea of studying mathematics.

CHAPTER 1

Structure Theory of Coalgebras

Let $\mathcal{R}$ be a commutative ring and $(\mathcal{M}_k\Delta_C R, \otimes_{\mathcal{R}}, 1_{\mathcal{M}})$ an $\mathcal{R}$-linear symmetric monoidal category. To an $\mathcal{R}$-linear symmetric monoidal category we associate the category of cocommutative, coassociative, counital $\mathcal{R}$-coalgebras with respect to the monoidal pairing, we denote this category as $\mathrm{coCAlg}(\mathcal{M}_{\mathcal{R}})$. More explicitly an $\mathcal{R}$-coalgebra $(C, \Delta_C, \varepsilon_C)$ is an object $C \in \mathcal{M}_{\mathcal{R}}$ together with $\mathcal{R}$-linear morphisms

$$\Delta_C : C \to \mathcal{C} \otimes_{\mathcal{R}} C$$
$$\varepsilon_C : C \to \mathcal{R}$$

such that the following diagrams commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[dr, "\Delta_C"'] & C \otimes_{\mathcal{R}} C \arrow[d, "\tau"] \\ & C \otimes_{\mathcal{R}} C \end{tikzcd}$$

Cocommutativity

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & C \otimes_{\mathcal{R}} C \arrow[d, "\Delta_C \otimes 1"] \\ C \otimes_{\mathcal{R}} C \arrow[r, "1\otimes\Delta_C"'] & C \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} C \end{tikzcd}$$

Coassociativity

$$\begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[d, "\Delta_C"] \arrow[dr, "\simeq"] & \\ C \otimes_{\mathcal{R}} \mathcal{R} & C \otimes_{\mathcal{R}} C \arrow[l, "1\otimes\varepsilon_C"] \arrow[r, "\varepsilon_C\otimes 1"'] & \mathcal{R} \otimes_{\mathcal{R}} C \end{tikzcd}$$

Counitality

A morphism between $\mathcal{R}$-coalgebras $C, D \in \mathrm{coCAlg}(\mathcal{M}_k)$ is a $\mathcal{R}$-linear morphism that it is compatible with the structure maps Δ and ε, *i.e* the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes_{\mathcal{R}} C \arrow[r, "\varphi\otimes\varphi"] & D \otimes_{\mathcal{R}} D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & k & \end{tikzcd}$$

Example 1.0.1.
(1) For $\mathcal{M} = \mathcal{M}\mathrm{od}_{\mathcal{R}}$ we get $\mathrm{coCAlg}_{\mathcal{R}} = \mathrm{coCAlg}(\mathcal{M}\mathrm{od}_{\mathcal{R}})$
(2) Let $\mathcal{C}$ be a small category and $\mathcal{M} = \mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C}) := \mathrm{Fun}(C^{op}, \mathcal{R}\text{-}\mathcal{M}\mathrm{od})$, the category of presheaves of modules. This category is endowed with the *sectionwise tensor product* which induces a symmetric monoidal structures. Let us denoted by $\mathrm{coCAlg}_k(\mathcal{C})$ the category of coalgebras.
(3) A particular case of the previous example is formed by considering the category Δ of finite ordered sets and the section-wise tensor product. This so-called category of simplicial coalgebras is denoted by $s\mathrm{coCAlg}_k$.
(4) Let $\mathcal{C}$ be a small category and $s\mathcal{M} = s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C}) := \mathrm{Fun}(\Delta^{op}, \mathcal{R}\text{-}\mathcal{M}\mathrm{od}(\mathcal{C}))$, the category of simplicial presheaves of modules we write $s\mathrm{coCAlg}_{\mathcal{R}}(\mathcal{C})$ for the category of presheaves of simplicial coalgebras.

1.1. The category of coalgebras over a field coCAlg_k

We now discuss the structure of the category of coalgebras over a field k. These results are well known, a good reference is [**Swe69**] and Theorem 1.15 is proved in [**Goe95b**] for the case of an algebraically closed field. In *loc.cit.* Goerss claims that the theorem is true for k a perfect field, an explicit proof is given in [**Nik16**].

First lets recall the definition of a locally finitely presentable category.

Definition 1.1.1. Let $\mathcal{C}$ be a category. An object $C \in \mathcal{C}$ is said to be compact or finitely presentable if the representable functor $\mathcal{C}(C, -)$ preserves filtered colimits.

Definition 1.1.2. Let $\mathcal{C}$ be a category. It is said to be locally finitely presentable if $\mathcal{C}$ has all small colimits and the subcategory of compact objects $\mathcal{C}_{fp}$ is essentially small and every object $C \in \mathcal{C}$ is a filtered colimit of the canonical diagram of the finitely presentable objects mapping in to it.

Proposition 1.1 (Fundamental Theorem of coalgebras)**.** *Let C be a coalgebra over a field k and $x \in C$. Then there exists a finite dimensional subcoalgebra $D \subset C$ with $x \in D$.*

Proof. Given a basis for C we can express $\Delta_C(x) = \sum x_i \otimes c_i$ with c_i elements in the basis, and $(\Delta_C \otimes id)(\Delta_C(x)) = \sum \Delta_C(x_i) \otimes c_i = \sum a_j \otimes b_{i,j} \otimes c_i$ with a_j and c_j linear independent. Then define $D = \langle b_{i,j} \rangle$ as the subspace generated by $b_{i,j}$. By the properties of the coassociativity, counitality and cocommutativity of the coalgebra structure, it is not difficult to prove that D is a subcoalgebra. $\square$

As consequence of Proposition 1.1 we have

Proposition 1.2. *Every coalgebra over a field k is the filtered colimit of its finite dimensional sub coalgebras.*

Proof. From 1.1 we have that the underlying vector space C is a the colimit of a diagram C_α, where each C_α is a finite dimensional coalgebra. It remains to show that the colimits of a diagram of coalgebras is again a coalgebra. This last statement is straightforward. $\square$

Theorem 1.3. *The category* coCAlg_k *for k a field is finitely presentable and the forgetful functor $U : \mathrm{coCAlg}_k \to \mathcal{M}\mathrm{od}_k$ has the right adjoint CF. For a k-vector space V, $CF(V)$ is the so-called cofree coalgebra on V.*

Proof. Since $- \otimes -$ preserves colimits, the forgetful functor creates colimits. Then, considering 1.2, we can conclude that coCAlg_k is finite presentable. In order to prove that the forgetful functor has a right adjoint, we need to apply the dual version of the Special Adjoint Functor Theorem [**ML98**]. It suffices to verify that the category coCAlg_k is well-copowered, which means that the collection of quotients is a set. This last conditions follows from the fact that $\mathcal{M}\mathrm{od}_K$ is well copowered. Then the forgetful functor has a right adjoint; the cofree algebra functor. $\square$

We have already mentioned the existence of colimits. Since the category of coalgebras is finitely presentable, small limits also exist, because every finitely presentable subcategory is a reflective subcategory of a category of presheaves. [**AR94**, Corollary 1.28].

Definition 1.1.3. A coalgebra C over a field k is called simple if it has no non-trivial subcoalgebras.

Given K/k a finite field extension. Then $K^\vee = \mathrm{Hom}_k(K, k)$ is a finite dimensional coalgebra over k and it is simple because subcoalgebras of $K^\vee$ corresponds to quotients of K, and only the trivial quotient K exists.

Proposition 1.4. *Let D a simple coalgebra over k. There exists a finite field extension K over k such that $D \cong K^\vee$.*

Proof. By Proposition 1.1 every non-finite dimensional coalgebra over k contains a non-trivial subcoalgebra. Then every simple subcoalgebra D is finite dimensional, and $D^\vee$ is a finite-dimensional commutative algebra which has no non-trivial quotients. Then $D^\vee$ has only the trivial ideals, thus it is isomorphic to a field. Then it is a finite field extension of k. Furthermore, again since D is finite dimensional, $D \cong D^{\vee\vee}$. $\square$

Remark 1.5. In particular if k is algebraically closed there are no simple coalgebras over k besides k itself.

Definition 1.1.4. Let C be a coalgebra over k, the *étale part* $\acute{E}t(C)$ of C is the direct sum $\oplus_{C_\alpha \subset C} C_\alpha$ where C_α runs through all the simple subcoalgebras of C.

A coalgebra C over a field k is called *irreducible* if it contains a unique simple subcoalgebra, *i.e.* if the étale part consists only of one single summand. A coalgebra is called an *irreducible component* if it is a maximal irreducible subcoalgebra of C.

Lemma 1.6. *Let $C = \sum_{i \in I} C_\alpha$ be a (not necessarily direct) sum of subcoalgebras $C_\alpha \subset C$. Any simple subcoalgebra of C lies in one of the summands C_α.*

Proof. Let $D \subset C$ be a simple coalgebra. Since D is finite dimensional, it lies in the sum of a finite number of summands. By induction on n, it suffices to prove that if $D \subset C_{\alpha_1} + C_{\alpha_2}$ then $D \subset C_{\alpha_1}$ or $D \subset C_{\alpha_2}$. Suppose that D is not contained in C_{α_1}. Since D is simple, then $D \cap C_{\alpha_1} = 0$. We can choose a linear map $f : C \to k$ such that $f|_D = \varepsilon_D$ and $f|_{C_{\alpha_1}} = 0$. Every $d \in D$ satisfies

$$(f \otimes 1)(\Delta_D(d)) = (\varepsilon_D \otimes 1)(\Delta_D(d)) = d \tag{1.1}$$

but $\Delta_D(D) \subset C_{\alpha_1} \otimes C_{\alpha_1} + C_{\alpha_2} \otimes C_{\alpha_2}$. Since $f|_{C_{\alpha_1}} = 0$, we conclude that for every $d \in D$, $d \in C_{\alpha_2}$. □

Lemma 1.7. *The canonical morphism $\acute{E}t(C) \to C$ is an injective morphism of coalgebras and in fact it defines an endofunctor:*

$$\acute{E}t : \mathrm{coCAlg}_k \to \mathrm{coCAlg}_k$$
$$C \mapsto \acute{E}t(C)$$

such that the canonical morphism is natural.

Proof. We claim that the sum of simple coalgebras $\sum C_\alpha \subset C$ is a direct sum. To prove that we have to show that $C_{\alpha_0} \cap \sum_{\alpha \neq \alpha_0} C_\alpha = 0$ for every $\alpha \neq 0$. Suppose that $C_{\alpha_0} \cap \sum_{\alpha \neq \alpha_0} C_\alpha \neq 0$, since C_{α_0} is a simple coalgebra it follows by Lemma 1.6 that $C_{\alpha_0} = C_{\alpha_1}$ for some α_1, which is a contradiction.

It remains to prove the functoriality and the naturality. We claim that given $f : C \to D$ a morphism of coalgebras the image $f(C_\alpha) \subset D$ is simple for every simple subcoalgebra $C_\alpha \subset C$. Since the image is a quotient of C_α, it suffices to showing that the quotient of a simple coalgebra is simple. This is equivalent to showing that the subalgebra of a finite field extension is finite field extension, which is true. □

Remark 1.8. Let $\mathcal{R}$ be a ring. For a set X, we let $\mathcal{R}[X]$ denote the free $\mathcal{R}$-module on X. The coalgebra $R^\delta[X]$ is the $\mathcal{R}$-module $\mathcal{R}[X]$ with coproduct induced by the diagonal map $\Delta : X \to X \times X$ and the isomorphism $\mathcal{R}[X \times X] \cong \mathcal{R}[X] \otimes_{\mathcal{R}} \mathcal{R}[X]$. Note that the coalgebra functor

$$\mathcal{R}^\delta[-] : Sets \to \mathrm{coCAlg}_{\mathcal{R}}$$

admits a right adjoint given by

$$(-)^{gp} : \mathrm{coCAlg}_{\mathcal{R}} \to Sets \qquad C \mapsto \mathrm{Hom}_{\mathrm{coCAlg}_{\mathcal{R}}}(\mathcal{R}, C).$$

The right adjoint $(-)^{gp}$ can be given more explicitly, a morphism of coalgebras $\mathcal{R} \to C$ sends 1 to an element $c \in C$ such that $\Delta_C(c) = c \otimes c$ and $\varepsilon_C(c) = 1$, such elements are known as *group like* (and throughout the text) elements in C. Every group-like element determines a unique morphism $\mathcal{R} \to C$ of coalgebras. Thus the right adjoint is given by sending the coalgebra C to the subset C^{gp}.

Proposition 1.9. *Let k be an algebraically closed field and C a coalgebra over k. Then the counit of the adjunction $k^\delta[C^{gp}] \to C$ factors through the inclusion $\acute{E}t(C) \subset C$ and induces a natural isomorphism*

$$k^\delta[C^{gp}] \cong \acute{E}t(C).$$

Proof. Let X be a set and consider $k^\delta[X] = \sum_X k$ with the coalgebra structure induced by the diagonal map. We have that $\acute{E}t(k^\delta[X]) = k^\delta[X]$. Then by Lemma 1.7 the counit map $k^\delta[C^{gp}] \to C$ factors through the étale part of $\acute{E}t(C)$. Now since k is algebraically closed a simple subcoalgebra of C is given by k and then the étale part is given by the direct sum over all morphisms of coalgebras $k \to C$, which is the description of the group like elements. □

Lemma 1.10. *Every coalgebra over a field k is the direct sum of its irreducible components. In other words for every simple coalgebra C_α there is a unique irreducible component $\overline{C}_\alpha \subset C$ such that $C_\alpha \subset \overline{C}_\alpha$ and the canonical morphism*

$$\bigoplus_\alpha \overline{C}_\alpha \to C$$

is an isomorphism of coalgebras.

Proof. The sum of all irreducible coalgebras which contains C_α is also irreducible: if another simple subcoalgebra C_β is a subcoalgebra of this sum, then it is contained in one of the summands but this is a contradiction because each summand is an irreducible containing C_α. By construction the sum contains C_α and is maximal then is an irreducible component.

We show now that he sum of the irreducible components is a direct sum. Suppose that there is a non-trivial intersection $\overline{C}_{\alpha_0} \cap \sum_{\alpha \neq \alpha_0} \overline{C}_\alpha$ for some α_0. Then this intersection contains a simple subcoalgebra which has to be $\overline{C}_{\alpha_0}$, since it is a subcoalgebra of $\overline{C}_{\alpha_0}$, but again Lemma 1.6 shows that C_{α_0} is a subcoalgebra of C_α for some $\alpha \neq \alpha_0$ which is a contradiction.

We have that $\sum_\alpha C_\alpha \subset C$, then it is enough to show that each element $c \in C$ lies in a sum of irreducible coalgebras. Take $\overline{\{c\}}$ the subcoalgebra generated by c, which is finite dimensional by the fundamental theorem of coalgebras. The $A = C^\vee$ is an artinian algebra and we have that $A \cong A_1 \oplus \cdots \oplus A_n$ with each A_i a local artinian subalgebra, but since A_i is local $A_i^\vee$ is irreducible. $\square$

Lemma 1.11. *Let $f : C \to D$ be a morphism of coalgebras over a field k. Then f restricts to a morphism of irreducible components $\overline{C_\alpha} \to \overline{f(C_\alpha)}$ for every simple subcoalgebra C_α of C.*

Proof. We first prove the result in case C is irreducible and f is surjective. Let us show that D is irreducible. By Proposition 1.2 $C = \mathrm{colim}_\alpha C_\alpha$ where C_α runs over all the finite dimensional subcoalgebras and thus $D = \mathrm{colim}_\alpha f(C_\alpha)$.

Claim 1: A coalgebra D is irreducible if and only if every element lies in some irreducible subcoalgebra.

Claim 2: Let C be an irreducible finite dimensional coalgebra and $f : C \to D$ a surjective morphism of coalgebras. Let C_0 be the unique simple subcoalgebra of C. Then $f(C_0)$ is the unique simple subcoalgebra of D.

Granting Claim 1 we can assume that C is finite dimensional and f surjective, then the proposition for C irreducible and f surjective follows from Claim 2.

Proof of claim 1: One direction is obvious. For the converse suppose that D is not irreducible. Then there exists E and E' two distinct irreducible components. We know that $E + E'$ is a direct sum. Choose $e \in E$ and $e' \in E'$ and consider $e + e' \in E \oplus E'$. Let F be the subcoalgebra generated by $e + e'$. By hypothesis F must be contained in an irreducible subcoalgebra, so it is irreducible as well. Let E_0 be the simple subcoalgebra of E, E_0' the simple subcoalgebra of E', F_0 the simple subcoalgebra of F. Since, F is contained in $E \oplus E'$, F_0 is either E_0 or E_0', say $F_0 = E_0$. Then $E + F$ is also irreducible and by maximality, $E + F = E$ so $F \subset E$. Thus $e + e'$ is in E and thus e' is in E. But then the subcoalgebra generated by e' is contained in $E \cap E'$ and must contain both E_0 and E_0', so $E_0' \subset E$, contrary to the assumption that D is irreducible.

Proof of claim2:

The exact sequence of coalgebras $C \to D \to 0$ induces an exact sequence of finite dimensional algebras $0 \to D^\vee \to C^\vee$. Since C is irreducible $C^\vee$ is a finite dimensional local algebra. Let $\mathfrak{m}$ the maximal ideal. By the Nakayama Lemma there exists $n \in \mathbb{N}$ such that $\mathfrak{m}^n = 0$. Denote $\mathfrak{n} = D^\vee \cap \mathfrak{m}$, it is an ideal such that $\mathfrak{n}^n = 0$, thus lies in the Jacobson radical of $D^\vee$. And all maximal ideals of $D^\vee$ contain $\mathfrak{n}$. Then all simple subcoalgebras of D are contained in $\mathfrak{n}^\perp$. But an explicit computation shows that $\mathfrak{n}^\perp = D^\vee \cap C_0^\perp = f(C_0)$ where C_0 is the unique simple subcoalgebra of C. Since $f(C_0)$ is a simple subcoalgebra of D it follows that is the only one. Thus D is irreducible.

We now discuss the general case. By the first part we have that $f(\overline{C_\alpha})$ is irreducible, then it lies in an irreducible component which contains $f(C_\alpha)$. But the former is a simple subcoalgebra. Thus $f(\overline{C_\alpha}) \subset \overline{f(C_\alpha)}$.

$\square$

Corollary 1.12. *Let k be a field. The decomposition of Lemma 1.10 is functorial.*

Proof. Let $f : C \to D$ be a morphism of coalgebras. Combining Lemma 1.10 and Lemma 1.11 we get:

$$\begin{tikzcd}
\bigoplus_\alpha \overline{C_\alpha} \arrow[r, "\cong"] \arrow[d, "f_*"'] & C \arrow[d, "f"] \\
\bigoplus_\alpha \overline{D_\alpha} \arrow[r, "\cong"] & D
\end{tikzcd}$$

where f_* on the summand $\overline{C_\alpha}$ agrees with $f|_{\overline{C_\alpha}}$. $\square$

Lemma 1.13. *Let C be an irreducible coalgebra over a perfect field. Then there is a unique retract of the inclusion $\acute{E}t(C) \subset C$ that is a map of coalgebras. Moreover, the retraction is natural in C.*

Proof. Since C is an irreducible coalgebra, $\acute{E}t(C) = K^\vee$ for K a finite field extension of k. By the Fundamental Theorem of Coalgebras $C = \mathrm{colim}_i C_i$ where $C_i \subset C$ are finite dimensional subcoalgebras. Every C_i is an irreducible coalgebra and contains $\acute{E}t(C)$. Thus, it is enough to show the Lemma for the finite dimensional case. We reduce to show the dual statement: Let A be a finite dimensional local algebra over a perfect field k, with $\mathfrak{m}$ the unique maximal ideal. Then there exists a unique subfield $K \subset A$ such that $A = K \oplus \mathfrak{m}$

Since k is perfect, the field extension $A/\mathfrak{m}$ is a separable extension over k. By the Primitive Element Theorem, $A/\mathfrak{m}$ corresponds to the form $k(\alpha)$, with $\alpha \in A/\mathfrak{m}$. Let $p(x) \in k[x]$ be the minimal polynomial of α, since it is a separable polynomial over k, $p'(\alpha) \neq 0$. Since A is of finite dimension,we have that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$, for some $n \in \mathbb{N}$. Nakayama lemma implies that $\mathfrak{m}^n = 0$. Thus A is complete with respect to the $\mathfrak{m}$-adic topology. By Hensel's Lemma there exists a unique element $x \in \alpha \subset A$ such that $p(x) = 0$. Set $K = k(x) \subset A$, this field has the required property since the composition $K \to A \to A/\mathfrak{m}$ is an isomorphism by construction.

It remains to show that the retract is natural. If $C \to D$ is a morphism of irreducible coalgebras over k, then for the commutative diagram

$$\begin{tikzcd} \acute{E}t(C) \arrow[r] \arrow[d] & \acute{E}t(D) \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

we need to show that the diagram of retracts $C \to \acute{E}t(C)$ and $D \to \acute{E}t(D)$ commutes. Recall that any morphism of coalgebras $C \to D$ can be factored into an epimorphism followed by a monomorphism $C \twoheadrightarrow F \hookrightarrow D$. Since C and D are irreducible, Lemma 1.11 implies that F is irreducible as well. Thus $\acute{E}t(F) \simeq \acute{E}t(D)$ and we get the following diagram

$$\begin{tikzcd} \acute{E}t(C) \arrow[r, two heads] \arrow[d, shift left] & \acute{E}t(F) \arrow[r, "\cong"] \arrow[d, shift left] & \acute{E}t(D) \arrow[d, shift left] \\ C \arrow[r, two heads] \arrow[u, bend left] & F \arrow[r, hook] \arrow[u, bend left] & D \arrow[u, bend left] \end{tikzcd}$$

By the uniqueness of the retraction $F \to \acute{E}t(F)$, we have that the right-hand side square, formed with the retractions, commutes.

It remains to show the the left-hand side diagram commutes. It suffices to reduce again to a finite dimensional case. Thus we have an inclusion of finite dimensional local algebras $A \hookrightarrow B$ and we have to show that the following diagram commutes

$$\begin{tikzcd} A/\mathfrak{m} \arrow[r] \arrow[d] & B/\mathfrak{n} \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

Since $\mathfrak{m} = \mathfrak{n} \cap A$ from a similar argument for the uniqueness of the retraction,we get that

$$A = B/\mathfrak{m} \cap A \oplus \mathfrak{m} \cap A = k \oplus A$$

which implies the claim. $\square$

Remark 1.14. Note that for any D simple coalgebra over an algebraically closed field k, the counit map $\varepsilon : D \to k$ is an isomorphism.

$$\begin{tikzcd} k \cong \acute{E}t(D) \arrow[r, hook] \arrow[d, "\cong"'] & D \arrow[d, "\varepsilon_D"] \\ k \arrow[r, equal] & k \end{tikzcd}$$

thus the natural splitting is given by the counit map.

THEOREM 1.15. *Let k be a perfect field. Then for every coalgebra C the inclusion $\acute{E}t(C) \to C$ has a unique and natural splitting which is a map of coalgebras*

PROOF. If C is an irreducible coalgebra the result follows by Lemma 1.13. For the general case by Lemma 1.10 we have that $D \cong \oplus_\alpha \overline{D_\alpha}$ and $\acute{E}t(D) \cong \oplus_\alpha D_\alpha$, where $\{D_\alpha\}$ is the collection of simple subcoalgebras of D. We define the splitting as the direct sum of the splittings for the irreducible components. By Lemma 1.10 every simple subcoalgebra is contained in a unique irreducible component. Then there can not be a splitting mixing the components and the naturality follows from Lemma 1.11.

□

Corollary 1.16. *Let k an algebraically closed field. Then the counit of the adjunction*

$$k^{\delta}[-] : \mathrm{Sets} \rightleftarrows \mathrm{coCAlg}_k : (-)^{gp}$$

given by $k^{\delta}[C^{gp}] \to C$ has a natural retraction.

Proof. This follows from Proposition 1.9 and Theorem 1.15. □

Remark 1.17. In proposition 1.9 the condition that the field is algebraically closed is necessary and it is needed for the proof of Theorem 2.18. We will introduce a treatment for k non-algebraically closed in chapter 2, we will require an auxiliary category of discrete G-sets.

1.1.0.1. *Discrete* G-Sets.

Definition 1.1.5. Let G be a profinite group. Then a G-set is *discrete* if the action is continuous when X is given with the discrete topology.

This is equivalent to ask that the isotropy groups $H_x \subset G$ of x be open, or equivalently that

$$X = \bigcup_{H \subset G} X^H$$

where H runs over all the open subgroups of G and X^H is the set of fixed points for H. In particular the orbit of any $x \in X$ is finite.

We can actually describe the category of discrete G-sets in a more sophisticated language. Let us denote by G-Sets_{fd} the full subcategory of finite discrete sets in G-Sets_d, the category G-Sets_{fd} has a pretopology defined by the covering families $U_i \to X$ such that $\coprod_i U_i \to X$ are surjections. The associated Grothendieck topos is called the *classifying topos* for the profinite group G, and is denoted as BG, *i.e* an object in BG is a sheaf of sets on the site G-Sets_{fd}.

For each presheaf on G-Sets_{fd}, we can define a G-set LF as follows:

$$LF := \mathrm{colim}_{i \in I} F(G_i)$$

Right multiplication by elements of G_i induces a left G_i-action on $F(G_i)$ and so there is an induced left G-action on LF.

Definition 1.1.6. Let $F\acute{E}t/k$ be the full subcategory of Sm_k consisting of all the schemes of finite type over k which are smooth of dimension zero. Every object $S \in F\acute{E}t/k$ is a finite disjoint union of $\mathrm{Spec}(k')$ for k' a finite separable field extension of k.

Let $U \in F\acute{E}t/k$, the functor defined by

$$U \mapsto \hom_k(Spec(k_{sep}), U)$$

defines an isomorphism of sites

$$\mathcal{F} : F\acute{E}t_k \to G\text{-}\mathrm{Sets}_{fd},$$

existance of this isomorphism is just Galois theory and observe for F a sheaf over G-Sets_{fd} the associated G-set LF correspondes to $F_{Spec(k_{sep})}$, the stalk at the geometric point.

Proposition 1.18. *Let k be a separable field and $G = Gal(k_{sep}/k)$ the absolute Galois group. Then the following categories are equivalent:*

(1) The category of discrete G-sets.
(2) The category of sheaves of sets on G-$Sets_{fd}$
(3) The category of sheaves of sets on $F\acute{E}t_k$

Proof. The equivalence between *(1)* and *(2)* is given for example in [**Jar10**, Proposition 6.20]. The equivalence between *(1)* and *(3)* follows from the Galois correspondence.

□

Let us make some remarks about the topos G-Sets_d of discrete G-sets

- The topos of discrete G-sets has enought points *i.e* there is a functor

$$u^* : G\text{-Sets}_d \to Sets$$

  which is defined by forgetting the group structure. Colimits and finite limits in the category of discrete G-sets are formed in the category of sets, then the functor u^* is faithful and exact. It is enough to check isomorphism between discrete G-sets $F \to G$ at only one stalk, the underlying set.

In particular if $G = Gal(k_{sep}/k)$ for k a field, then we have the following well known identification between the finite étale site $F\acute{E}t/k$ defined below and the site $G\text{-Sets}_{fd}$ associated to the profinite group G.

1.1.0.2. *The category of coalgebras over a non-algebraically closed field k.*

Proposition 1.19. *There exists a left adjoint functor* $\bar{k}^\vee[-]_G : G-Sets_d \to \mathrm{coCAlg}_{\mathcal{R}}$ *with right adjoint given by*

$$RC : Orb(G)^{op} \to Sets$$

$$G/H \mapsto Hom_{\mathrm{coCAlg}_k}((\bar{k}^H)^\vee, C).$$

Furthermore, the functor $\bar{k}^\vee[-]_G$ *is fully faithful, and the counit of the adjunction is given by the embedding*

$$\acute{E}t(C) \to C$$

Proof. The functor is defined as the left Kan extension of the functor

$$Orb(G) \to Sets$$

$$G/H \mapsto (\bar{k}^H)^\vee$$

along the Yonneda embedding *i.e.*,

$$\bar{k}^\vee[X]_G := \mathrm{colim}_{G/H\to X}(\bar{k}^H)^\vee.$$

Let us prove that R is its right adjoint. It is enough to construct the unit and counit maps. Note that for every representable sheaf G/H we have the following isomorphism

$$\mathrm{Hom}_{coCAlg_k}((\bar{k}^{H'})^\vee, (\bar{k}^H)^\vee) \simeq Hom(G/H', G/H).$$

Then we have an isomorphism $X \to R(\bar{k}^\vee[X]_G)$ for each $X \in G\text{-}Sets_d$. On the other hand, note that for each $C \in \mathrm{coCAlg}$ $RC(G/H)$ index all the embeddings of $(\bar{k}^H)^\vee$ in C, then:

$$\bar{k}^\vee[RC]_G = \mathrm{colim}_{G/H\to RC}(\bar{k}^H)^\vee = \acute{E}t(C).$$

Furthermore, we have a natural embedding $\acute{E}t(C) \to C$. A computation shows that the identity transformation $X \to R(\bar{k}^\vee[X]_G)$ and the inclusion $\acute{E}t(C) \to C$ defines the unit and counit maps. $\square$

1.2. Category of presheaves of coalgebras $\mathrm{coCAlg}_k(\mathcal{C})$

For this section let us denote $\mathcal{C}$ an small category and k a field. We extend the theorem 1.3 to the category of presheaves with the sectionwise tensor product as monoidal structure. Here the argument is taken from [**Rap13**]. For completeness we reproduce the proof here.

Proposition 1.20. *The category of presheaves of coalgebras* $\mathrm{coCAlg}_k(\mathcal{C})$ *is locally finitely presentable and the forgetful functor* $\mathrm{coCAlg}_k(\mathcal{C}) \to \mathrm{Mod}_k(\mathcal{C})$ *has a right adjoint,* $CF : \mathrm{Mod}_k(\mathcal{C}) \to \mathrm{coCAlg}_k(\mathcal{C})$.

Proof. First assume that $\mathcal{C}$ is a discrete category, i.e. for each object U the hom sets $\mathrm{Hom}(U,U) = \{id_U\}$ and empty otherwise. By 1.3 the category of presheaves of coalgebras over $\mathcal{C}$, is finitely presentable with strong small set of generators $\mathcal{D}$ given by the collection of presheaves of coalgebras D such that the sections of D are zero except for just one object $U \in Ob(\mathcal{C})$, and $D(U)$ is a finite dimensional coalgebra. For the general case let $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be the canonical inclusion of the discrete subcategory. This inclusion induces an adjunction in the categories of presheaves:

$$i_! : \mathrm{coCAlg}_k(\mathcal{C}_0) \rightleftarrows \mathrm{coCAlg}_k(\mathcal{C}) : i^*.$$

The functor $i_!$ is given explicitly as:

$$i_!(\mathcal{D})(V) = \bigoplus_{V\to U} \mathcal{D}(U).$$

We want to show that the category coCAlg_k is finitely presentable, by [**AR94**, Theorem 1.1] it is enough to show that coCAlg_k is cocomplete and has a strong set of generators. The counit of the adjunction

$i_!i^*C \to C$, with $C \in \mathrm{coCAlg}_k$, is a surjective map and by the discrete case $i^*C = \mathrm{colim}_\alpha D_\alpha$ with $D_\alpha \in \mathcal{D}$. Because $i_!$ is a left adjoint, it commutes with colimits and we get a surjective map $\mathrm{colim}_\alpha i_!(D_\alpha) \to D$, from this surjection it follows that given two different maps $f, g : D \rightrightarrows E$ there exists a map $i_!(D_\beta) \to D$ such that the composition with f and g is different and for every proper subobject K of D, there exists $i_!(D_\gamma) \to D$ a map such that it does not factorize trough K, i.e. $\mathrm{colim}_\alpha i_!(D_\alpha)$ is an strong generator. Furthermore coCAlg_k is cocomplete, colimits are created by the forgetful functor because the section-wise tensor product preserves colimits in both variables, the category is locally finitely presentable. By the dual of the special adjoint functor theorem we have the existence of the cofree algebra functor CF. ☐

Remark 1.21. In *loc.cit* the previous statement is proved for $\mathcal{R}$ a presheaf of rings on $\mathcal{C}$, using the local presentability of the category of coalgebras over a ring $\mathrm{coCAlg}_{\mathcal{R}}$ proved in [**Bar74**, Theorem 3.1]. We do not need that generality in this work. Moreover, only in the case of algebraically closed fields we have a good description of the category of coalgebras.

Definition 1.2.1. Let C be a coalgebra object in $\mathrm{Mod}_k(\mathcal{C})$ under the section-wise monoidal structure. The étale subpresheaf is defined as $U \mapsto \acute{E}t(C(U))$.

The étale presheaf is well-defined since for each $U \in \mathcal{C}$, $C(U)$ is a coalgebra over k. Furthermore, the functor $k^\delta[-]$ extends to a functor of presheaves:

$$k^\delta[-] : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Mod}_k(\mathcal{C})$$

and the right adjoint $(-)^{gp}$ is given section-wise. As a consequence Proposition 1.9 extends to the categories of presheaves.

Proposition 1.22. *Let k an algebraically closed field and C a presheaf of coalgebras. Then the unit of the adjunction $k^\delta[C^{gp}] \to C$ factors through the inclusion of presheaves $\acute{E}t(C) \subset C$ and induces a natural equivalence*

$$k^\delta[C^{gp}] \cong \acute{E}t(C).$$

Proof. This is trivial from 1.9 ☐

Theorem 1.23. *Let k be a perfect field. Then for every presheaf of coalgebras C the inclusion $\acute{E}t(C) \to C$ has a unique and natural splitting, which is a map of presheaves of coalgebras*

Proof. This is a direct consequence of Theorem 1.15. Since the splitting is natural, this defines a morphism of presheaves of coalgebras $C \to \acute{E}t(C) \to C$. ☐

Corollary 1.24. *Let k an algebraically closed field. Then the counit of the adjunction given by $k[C^{gp}] \to C$ has a natural retraction.*

Proof. This follows from the previous theorem, noting that the isomorphism in 1.9 is natural. ☐

1.3. Category of presheaves of coalgebras for the Day convolution product $\mathrm{coCAlg}_k^{Day}(\mathcal{C})$

For this section we fix the following notation. Let $\mathcal{R}$ be a commutative ring and $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ a small $\mathrm{Mod}_{\mathcal{R}}$-enriched $\mathcal{R}$-linear symmetric monoidal category.

1.3.1. Day convolution Product. Consider $\mathrm{Psh}(\mathcal{C}, \mathrm{Mod}_{\mathcal{R}})$ the category of presheaves, there is a natural extension of the symmetric monoidal structure on $\mathcal{C}$ to the category of presheaves $\mathrm{Psh}(\mathcal{C}, \mathrm{Mod}_{\mathcal{R}})$, introduced by Day in [**Day70**]. The Day convolution product of two presheaves F and G is given by the coend formula.

$$F \otimes^{Day} G = \int^{X,Y} \mathcal{C}(-, X \otimes_{\mathcal{C}} Y) \otimes_k F(X) \otimes_{\mathcal{R}} G(Y)$$

in other words it is the left Kan extension of the external tensor product of F and G, denoted by $F \overline{\otimes} G$, along $- \otimes_{\mathcal{C}} -$. Explicitly it is the coequalizer of the two maps:

(1.2)

$$\bigoplus_{\substack{(X,Y)\in\mathcal{C}\times\mathcal{C} \\ (X',X')\in\mathcal{C}\times\mathcal{C}}} \mathcal{C}(U, X'\otimes Y')\otimes_{\mathcal{R}}\mathcal{C}\times\mathcal{C}((X',Y'),(X,Y))\otimes_{\mathcal{R}} F(X)\otimes_{\mathcal{R}} G(Y) \rightrightarrows \bigoplus_{(X,Y)\in\mathcal{C}\times\mathcal{C}} \mathcal{C}(U, X\otimes Y)\otimes_{\mathcal{R}} F(X)\otimes_{\mathcal{R}} G(X)$$

given $\phi \in \mathcal{C}(U, X' \otimes Y')$, $(\alpha, \beta) \in \mathcal{C} \times \mathcal{C}((X', Y'), (X, Y))$ and $s \otimes t \in F(X) \otimes_{\mathcal{R}} G(X)$, then the top map is given by:

$$\phi \otimes (\alpha, \beta) \otimes (s \otimes t) \mapsto ((\alpha, \beta) \circ \phi) \otimes (s \otimes t)$$

and the bottom map is given by:

$$\phi \otimes (\alpha, \beta) \otimes (s \otimes t) \mapsto \phi \otimes (\alpha^*(s) \otimes \beta^*(t)))$$

equivalently

$$(F \otimes^{Day} G)(U) = \mathrm{colim}_{U \to X \otimes Y} F(X) \otimes G(Y)$$

in particular for two representable sheaves $\mathbf{h}_X$ and $\mathbf{h}_Y$ the Day convolution product is given by the representable sheaf $\mathbf{h}_{X \otimes Y}$. In other words the Yoneda embedding $y : \mathcal{C}^{op} \to \mathcal{D}$ is a symmetric monoidal functor.

Proposition 1.25. *Let $\mathcal{R}$ be a commutative ring and $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ a small $\mathrm{Mod}_{\mathcal{R}}$-enriched $\mathcal{R}$-linear symmetric monoidal category. The monoidal category with the Day convolution product $(Psh(\mathcal{C}, \mathcal{M}), \otimes_{Day}, h_{1_{\mathcal{C}}})$ is a closed symmetric monoidal category. The internal hom is given by the end formula:*

$$[F, G]_{Day} = \int_{X,Y} Hom_{\mathcal{M}}(Hom_{\mathcal{C}}(- \otimes_{\mathcal{C}} X, Y), Hom_{\mathcal{M}}(F(X), G(Y))$$

PROOF. [**IK86**, Proposition 4.1] □

As a consequence the category of presheaves with the Day convolution product is monoidal and cocomplete, i.e. all the endofunctors $F \otimes^{Day} -$, $- \otimes^{Day} G$ for $F, G \in \mathrm{Psh}(\mathcal{C}, \mathcal{M})$ are cocontinuous. In [**IK86**] it is observed that the monoidal structure given by the Day convolution is the free monoidal cocompletion of $\mathcal{C}$, in the sense that:

Proposition 1.26. *Let $\mathcal{D}$ be a monoidal cocomplete category and assume the condition in Proposition 1.25. Then the functor $[Psh(\mathcal{C}, \mathcal{M}), \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$ given by the composition with the Yoneda embedding induces an equivalence of categories between the cocontinuous monoidal functors $\Phi : Psh(\mathcal{C}, \mathcal{M}) \to \mathcal{D}$ and the monoidal functors $\phi : \mathcal{C} \to \mathcal{D}$. This equivalence restricts to the corresponding subcategories of strong monoidal functors. Furthermore the strong monoidal functor $\phi : Psh(\mathcal{C}, \mathcal{M}) Psh(\mathcal{C}, \mathcal{M}) \to \mathcal{D}$ are exactly the monoidal functors which are left adjoint.*

PROOF. [**IK86**, Proposition 5.1] □

1.3.2. Local presentability of $\mathrm{coCAlg}_k^{Day}(\mathcal{C})$. Let k be a field and $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ a small Mod_k-enriched k-linear symmetric monoidal category.

We extend the theorem 1.20 to the category of presheaves of coalgebras for the Day convolution product. The proof in 1.3 relies on the duality of finite dimensional vector spaces and the proof uses the discrete category $\mathcal{C}_0$ associated to $\mathcal{C}$, but the discrete category is not monoidal anymore. Here we have to work a bit more in the argument, which is inspired on the argument in [**Bar74**] (see Theorems [3.1, 3.2] in *loc.cit*).

Lemma 1.27. *Let $F \in \mathrm{Mod}_k(\mathcal{C})$ be a sheaf of k-vector spaces and suppose that we are given for every object $x \in Ob(\mathcal{C})$ a subspace $G_0(x) \subset F(x)$. There exists a subpresheaf G' of vector spaces, such that for every object $x \in Ob(\mathcal{C})$ $G_0(x) \subset G'(x)$ and $\#(G') \leq \{\#(G), \aleph_0\}$.*

PROOF. By induction we define a collection of subspaces $G_n(x) \subset G(x)$ for every $x \in Ob(\mathcal{C})$. For every $y \in \mathcal{C}$ and $f \in \coprod_{x \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(y, x)$ define the subspace $G_1(y)$ generated by $\langle f^*(G_0(x)), G_0(y) \rangle$, by definition $G_0(y) \subset G_1(y)$ but the restrictions maps f^* are not yet defined. Assume $G_n(x)$ is given then define G_{n+1} as $G_{n+1}(x) := \langle f^*(G_n(x)), G_n(y) \rangle$ and take $G' := \mathrm{colim}_n G_n(x)$, which is the desired sub-presheaf. □

Proposition 1.28. *Let M and N be two presheaves of vector spaces and $M_0 \subset M$ sub-presheaf of vector spaces. Then there exists M' a presheaf of vector spaces such that $M_0 \subset M'$ and $M' \otimes^{Day} N \to M \otimes^{Day} N$ is a monomorphism. Furthermore $\#(M') \leq \{\#(M_0), \#(\mathcal{C}), \aleph_0\}$.*

PROOF. For simplicity, we write $- \otimes -$ instead of $- \otimes^{Day} -$. Let $G_0 = ker(M_0 \otimes N \to M \otimes N)$ be the kernel of the canonical map. Then for each $x \in \mathcal{C}$ consider $\{m_{i,x}\}$ a basis for $G_0(x)$. There exists a finite number of objects $(u_{i,j,x}, v_{i,j,x}) \in Ob(\mathcal{C} \times \mathcal{C})$ and there are maps $x \to u_{i,j,x} \otimes v_{i,j,x}$ such that $m_{i,x} = \sum_{i,j,x} r_{i,j,x} \otimes s_{i,j,x}$ with $r_{i,j,x} \in M_0(u_{i,j,x})$ and $s_{i,j,x} \in N(v_{i,j,x})$. Since $m_{i,x}$ goes to zero in $(M \otimes N)(x)$, there exists a finite number of maps $(f_{i,j,x}, g_{i,j,x}) \in Mor(\mathcal{C} \times \mathcal{C})$ such that each pair fits in a commutative diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & u' \otimes v' \arrow[d, "f\otimes g"] \\ & u \otimes v \end{tikzcd} .$$

and

$$\sum_{i,j,x} (f^*_{i,j,x} \otimes g^*_{i,j,x})(r_{i,j,x} \otimes s_{i,j,x}) = 0 \tag{1.3}$$

Applying Lemma 1.27 we can construct a subpresheaf M_1 such that for each $x \in Ob(\mathcal{C})$, $M_1(x)$ contains the subspace generated by

$$M_0(x) \coprod (\coprod_{\substack{x=u_{i,j,y} \\ x=v_{i,j,z}}} \{r_{i,j,y}, s_{i,j,z}\}),$$

i.e if x happen to be equal to $u_{i,j,y}$ for some $y \in \mathcal{C}$ or $v_{i,j,z}$ for some $z \in \mathcal{C}$.

By construction $G_0(x)$ goes to zero in $(M_1 \otimes N)(x)$. Assume that M_{n-1} is constructed then we repeat the argument. We construct M_n taking $G_{n-1}(x) = ker(M_{n-1} \otimes N)(x) \to (M \otimes N)(x)$ and the composition is equal to zero

$$\begin{tikzcd} G_{n-1}(x) \arrow[r, hook] \arrow[dr] & (M_{n-1} \otimes N)(x) \arrow[d] \\ & (M_n \otimes N)(x) \end{tikzcd}$$

we get a chain of subpresheaves $M_0 \subset M_1 \subset \cdots \subset M_n$. Set $M := \mathrm{colim}_n M_n$, this is the desired presheaf. $\square$

Definition 1.3.1. Let M be two presheaf of vector spaces and $M' \subset M$ a sub-presheaf of vector spaces. We say that M' is a *pure sub-presheaf* if $M' \otimes^{Day} N \to M \otimes^{Day} N$ is a monomorphism

The presheaf constructed in 1.28 is a pure sub-presheaf.

Let $F \in \mathrm{coCAlg}^{Day}_K(\mathcal{C})$ be a presheaf of coalgebras. A subpresheaf of vector spaces $M \subset F$ is called invariant if $\Delta_F(M) \subset Im(M \otimes^{Day} M \to F \otimes^{Day} F)$.

Proposition 1.29. *Given $F \in coCAlg^{Day}_K(\mathcal{C})$ and M_0 a subpresheaf of vector spaces. There exists a invariant sub-presheaf M' of vector spaces such that $M_0 \subset M'$ and $\#(M') \le \{\#(M_0), \#(\mathcal{C}), \aleph_0\}$.*

Proof. For each $x \in Ob(\mathcal{C})$ let us fix a basis $\{e_{x,i}\}$ for the subspace $M_0(x) \subset F(x)$. For each $e_{x,i}$ there exists a finite number of object $u_{x,i,j}, v_{x,i,j} \in \mathcal{C}$ and a finite number of elements $m_{u,i,j} \in F(u_{x,i,j})$ and $n_{u,i,j} \in F(v_{x,i,j})$ such that $\Delta_F(F(x)) \in (F \otimes F)(x)$ is contained in the subspace generated by $\{m_{u,i,j} \otimes n_{v,i,j}\}$. Applying Lemma 1.27 we construct M_1 the subsheaf of F generated by $M_0(x)$ and $\coprod_{i,j,x}\{m_{i,j,x}, n_{i,j,x}\}$ for all $x \in Ob(\mathcal{C})$. By construction $\Delta_F(M_0)(x) \subset (M_1 \otimes M_1)(x)$, and F_0 is a subsheaf of M_1, inductively we construct M_n such that $\Delta_F(M_{n-1})(x) \subset (M_n \otimes M_n)(x)$ and taking $M' = \mathrm{colim}_n M_n$ we get a presheaf which is invariant under the diagonal map.

$\square$

Theorem 1.30. *Let $F \in coCAlg^{Day}_R(\mathcal{C})$ and M_0 a presheaf of vector spaces of $\mathcal{C}$. Then there exists a subcoalgebra F' such that $M_0 \subset F'$ and $\#(F') \le \{\#(M_0), \#(\mathcal{C}), \aleph_0\}$*

Proof. Let F_1 be the pure sub-presheaf of vector spaces given by Lemma 1.28 which contains F_0 and let F_2 be the invariant sub-presheaf of vector spaces which contains F_1, iterating both lemmas F_n is defined as the pure sub-presheaf associated to F_{n-1} when n is odd and F_n the invariant sub-presheaf associated F_{n-1} when n is even. Set $F' = \mathrm{colim}_n F_n$, it is clear that F' is a pure and invariant. It remains to show that it is a subcoalgebra. By Lemma 1.28 $F' \otimes F' \to F \otimes F$ is injective. Thus the comultiplication map is induced by the comultiplication on F and $F' \otimes F' \otimes F' \to F \otimes F \otimes F$ is injective, which give us the coassociativity. A similar observation is enough for the cocommutativity. The counit map is just the composition $F' \hookrightarrow F \xrightarrow{\varepsilon} k$. $\square$

Corollary 1.31. *Let $\mathcal{C}$ be a small category. The category of coalgebras $coCAlg^{Day}_k(\mathcal{C})$ is locally presentable, with strong generators given by $\{F \in coCAlg^{Day}_k(\mathcal{C}) : \#(F) \le max(\#(C), \aleph_0)\}$.*

PROOF. Let $F_1 \underset{\psi}{\overset{\eta}{\rightrightarrows}} F_2$ be the maps of presheaves of coalgebras with $\eta \neq \psi$. Then there is some $x \in Ob(C)$ and $s \in F(x)$ with $\eta(x) \neq \psi(x)$. Let $M(x)$ be the presheaf vector spaces associated to the subspace of dimension one $\langle s \rangle$ then by the previous theorem exits $F_0 \subset F_1$ a presheaf of coalgebras such that $M(x) \subset F_0(x)$ then the restriction of η and ψ to F_0 are to different maps. The generators are strong because for ever $K \in \mathrm{coCAlg}_K^{Day}(\mathcal{C})$ and $L \subset K$ a proper subpresheaf there exists $x \in Ob(\mathcal{C})$ such that $L(x) \subset K(x)$ is a proper subspace. Let $s \in K(x) \setminus L(x)$ then take $G_s \subset K$ the subsheaf associated to the vector space generated by s and this map does not factorize through K. □

THEOREM 1.32. *The underlying functor* $U : \mathrm{coCAlg}^{Day}(\mathcal{C}) \to \mathcal{M}\mathrm{od}_k(\mathcal{C})$ *has a right adjoint and is comonadic.*

PROOF. The proof follows word by word [**BW85**, Theorem 4.1] □

Remark 1.33. The category of coalgebras is cartesian closed, with the product given by the Day convolution product of two coalgebras.

We would like to get a more refined description of category $\mathrm{coCAlg}^{Day}(\mathcal{C})$. Since the coalgebra objects in this category are not section-wise coalgebras, we can not extend the results from Section 1.1 easily. As well, the existence of a functor $k^{\delta,Day}[-] : \mathrm{PSh}(\mathcal{C}') \to \mathrm{coCAlg}_k^{Day}(\mathcal{C})$ which admits a right adjoint is not totally straight forward. Nevertheless, if we restrict to the category of coalgebra objects in $\mathrm{PST}(Sm_F, k)$ (the category of presheaves with transfers of Voevodsky) we are able to get a nice description. We will discuss these problem in a future paper.

CHAPTER 2

Motivic Spaces and $\mathbb{A}^1$-Goerss Theorem

The theory of motivic spaces and unstable motivic homotopy theory was set up by Morel and Voevodsky in [**MV99**]. Roughly speaking a motivic space (resp. an étale motivic) is a simplicial presheaf which is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich descent (resp. étale descent). This has been widely discussed in the literature. In the following section we recall the local and global model structures for simplicial presheaves studied in [**Jar87**] and [**MV99**]. Furthermore, we state results by Dugger [**DHI04**] and , which allows to see Jardine's model structure as a left Bousfield localization the local objects in Jardine's model structures satisfy hyperdescent.

2.1. Recollection in motivic model structures

2.1.1. Generalities and basic definitions. Denote by $\mathcal{S}$ the category of simplicial sets endowed with the Kan model structure; where a map f is a weak equivalence if the induced map of geometric realizations $|f| : |X| \to |Y|$ is an homotopy equivalence of topological spaces, it is a cofibrations if it is a monomorphism levelwise and is a fibrations if it is a Kan fibration (see [**Lur09**, Example A.2.7.3])

Before introduce the motivic model structure, let us give the definition of right and left induced model structures. Left induced model structures are particularly interesting for us in order to induce a model structure in the category of coalgebras.

2.1.1.1. *Right and left induced model structures.* Let $\mathcal{M}$ and $\mathcal{N}$ be complete and cocomplete categories. Assume that either $\mathcal{M}$ or $\mathcal{N}$ is a model category and let $\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N}$ be a pair of adjoint functors. A standard question is wether we can use R or L to build a model structure on $\mathcal{N}$ or $\mathcal{M}$. More specifically we can introduce the notion of *left* or *right induced model category.*

Definition 2.1.1. Let $\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N}$ be an adjoint pair of functors.

(1) Assume that $\mathcal{M}$ is a model category and $\mathcal{N}$ is a complete and cocomplete category. If the the classes of morphisms $R^{-1}(Fib)$ and $R^{-1}(W)$ satisfy the axioms of a model category. Then it is called the *right-induced model structure.*
(2) Assume that $\mathcal{N}$ is a model category and $\mathcal{M}$ is a complete and cocomplete category. If the the classes of morphisms $L^{-1}(Cof)$ and $L^{-1}(W)$ satisfy the axioms of a model category. Then it is called the *left-induced model structure.*

Example 2.1.1. The category of simplicial k-modules $s\mathrm{Mod}_k$ is endowed with the right-induced model structure from the Kan model structure on $\mathcal{S}$ by the forgetful functor $U : s\mathrm{Mod}_k \to \mathcal{S}$.

In 2.2 we are going to study an example of left induced model structure, the homotopy category of coalgebras.

Via the Dold-Kan equivalence $N : s\mathrm{Mod}_k \rightleftarrows Cplx(k)_{\geq 0} : \Gamma$ we give $Cplx(k)_{\geq 0}$ a model structure; in this model structure, the weak equivalences are the quasi-isomorphisms.

Let $\mathcal{C}$ be a small category, by Theorem A.11 it is possible to endow the category $s\mathrm{PSh}(\mathcal{C})$ and $s\mathrm{Mod}_k(\mathcal{C})$ with the injective and the projective model structures. Moreover, we get the following commutative diagram.

$$\begin{tikzcd} s\mathrm{PSh}(\mathcal{C})_{proj} \arrow[r, shift left, "k"] \arrow[d, shift right] & s\mathrm{Mod}_k(\mathcal{C})_{proj} \arrow[l, shift left, "U"] \arrow[d, shift right] \\ s\mathrm{PSh}(\mathcal{C})_{inj} \arrow[r, shift left, "k"] \arrow[u, shift right] & s\mathrm{Mod}_k(\mathcal{C})_{inj}. \arrow[l, shift left, "U"] \arrow[u, shift right] \end{tikzcd}$$

Since $\mathcal{S}$ and $s\mathrm{Mod}_{\mathcal{R}}$ are combinatorial model categories [**Lur09**, A.2.7], both the *projective* and *injective* model structures over $s\mathrm{Mod}_{\mathcal{R}}(\mathcal{C})$ are combinatorial model structures and then, by definition, cofibrantly generated. For the projective model structure, a set of generators is given by

$$\mathcal{I} = \{\mathcal{R}[id_X] \otimes \mathcal{R}[\iota_n] : \mathcal{R}[h_X] \otimes \mathcal{R}[\partial\Delta^n] \to \mathcal{R}[h_X] \otimes \mathcal{R}[\Delta^n], \;\; for\, n \geq 0,\; X \in \mathcal{C}\}$$
$$\mathcal{J} = \{\mathcal{R}[id_X] \otimes \mathcal{R}[j^n_k] : \mathcal{R}[h_X] \otimes \mathcal{R}[\partial\Lambda^n_k] \to \mathcal{R}[h_X] \otimes \mathcal{R}[\Delta^n], \;\; for\, n \geq 1,\;\; 0 \leq k \leq n \;\; X \in \mathcal{C}\}$$

from this description immediately follows that every representable simplicial presheaf is cofibrant for the projective model structure.

The *injective model structure* is also cofibrantly generated, we would like to make some remarks about the generators since we are particularly interested in this model structure. The proof of Theorem A.11 follows applying Theorem A.6. Since $s\mathrm{Mod}_{\mathcal{R}}(\mathcal{M})$ is a Grothendieck abelian category by [**Bek00**] the class of monomorphisms is cofibrantly generated by the set of monomorphisms

$$\mathcal{I} = \{f : A \to B : \; B \text{ is } \kappa\text{-bounded}\}.$$

Since the categories of simplicial presheaves and presheaves of simplicial modules are presentable categories, the set

$$\mathcal{J} = \{f : A \to B : \; B \text{ is } \kappa\text{-bounded and } f \text{ is section-wise weak equivalence}\}$$

generates the trivial cofibrations.

This fact is also observed in the axioms given by Jardine, for any monomorphism $f : X \to Y$ which is weak equivalence, and any A κ-bounded, there exists B κ-bounded subobject $B \subset Y$ such that $B \cup X \to B$ is a weak equivalence.

$$\begin{tikzcd}
 & & X \arrow[dd, tail] \\
 & X \cap B \arrow[ur, tail] \arrow[dd] & \\
A \arrow[rr, tail] \arrow[dr, tail] & & Y \\
 & B \arrow[ur, tail] &
\end{tikzcd}$$

2.1.2. Local model structures. Let $(\mathcal{C}, \tau)$ be a small Grothendieck site. Following Jardine (see [**Jar87**]) we can study local model structures on the category of simplicial presheaves $s\mathrm{PSh}(\mathcal{C}, \tau)$. We denote $\pi_n(\mathcal{X})$ the n-homotopy presheaf of $\mathcal{X}$. For a presheaf F on $(\mathcal{C}, \tau)$, we write $\widetilde{F}$ for the associated sheaf.

DEFINITION 2.1.2. A map $\mathcal{X} \to \mathcal{Y}$ of simplicial presheaves is called a local weak equivalence if

(1) the map $\widetilde{\pi_0} \to \widetilde{\pi_0}$ is an isomorphism of sheaves/
(2) for any object U of $\mathcal{C}$, any $x \in \mathcal{X}(U)$ and any $n > 1$ the morphism of associated sheaves
$$\widetilde{\pi_n}(\mathcal{X}, x) \to \widetilde{\pi_n}(\mathcal{Y}, f(x))$$
on the overcategory $\mathcal{C}/U$ defined by f is an isomorphism.

DEFINITION 2.1.3. Let $(\mathcal{C}, \tau)$ be a small Grothendieck site and $f : \mathcal{X} \to \mathcal{Y}$ a morphism of simplicial presheaves:

(1) f is called a simplicial weak equivalence if for any point x of the site $(\mathcal{C}, \tau)$ the morphism of simplicial sets $x^*(f) : x^*(\mathcal{X}) \to y^*(\mathcal{X})$ is a weak equivalence.
(2) f is called a cofibration if it is a monomorphism;
(3) f is called a fibration if it has the right lifting property with respect to any cofibration which is a weak equivalence.

Denote by $\mathbf{W}_s$ (resp. $\mathbf{C}$, $\mathbf{F}_s$) the class of simplicial weak equivalences (resp. cofibrations, simplicial fibrations).

Remark 2.1. If $(\mathcal{C}, \tau)$ is a small Gorthendieck site with enough points. Then the notion of simplicial weak equivalence and local weak equivalence coincides (see [**Jar87**, Section 2]).

THEOREM 2.2. *Let $(\mathcal{C}, \tau)$ be a small Grothendieck site with enough points. Then the triple $(\mathbf{W}_s, \mathbf{C}, \mathbf{F}_s)$ defines a model structure in the category $s\mathrm{PSh}(\mathcal{C})$.*

PROOF. This is proved in [**Jar87**, Theorem 2.3]. □

This model structure is known as Jardine model structure.

In general it is hard to describe the fibrations or give an explicit fibrant replacement. In [**MV99**] they construct an explicit resolution fibrant replacement. Before describe this fibrant replacement lets provided the following definition of local fibration.

DEFINITION 2.1.4. A morphism of simplicial presheaves $f : \mathcal{X} \to \mathcal{Y}$ is called a *local fibration* (*resp.* trivial local fibration) if for any point x of τ the corresponding morphism of simplicial sets $x^*(\mathcal{X}) \to y^*(\mathcal{Y})$ is a Kan fibration (*resp.* a Kan fibration and a weak equivalence). A morphism of simplicial presheaves $f : \mathcal{X} \to \mathcal{Y}$ is called *local cofibration* if it satisfies the left lifting property with respect to all trivial fibrations.

Remark 2.3. Every simplicial fibrations is a local fibration, but not every local fibration is a simplicial fibration. Eilenberg-Mac Lane spaces fails to be simplicial fibrant. In [**Jar87**] it is called the global and local theory respectively.

2.1.3. Hyperdescent and Čech descent. Note that class of section-wise weak equivalences $\mathbf{W}$ is contained in the class of simplicial weak equivalences $\mathbf{W}_s$ and the cofibrations are the injective cofibrations. As a consequence Jardine model structure is a left Bousfield localization of the injective model structure. The local objects are the simplicial presheaves satisfying τ-hyperdescent i.e. descent for arbitrary hypercovers. Recall that a hypercover of X is a simplicial presheaf with an augmentation $U \to X$, such that each U_n is a coproduct of representable, and $U \to X$ is a local acyclic fibration.

DEFINITION 2.1.5. An object-fibrant simplicial presheaf F satisfies descent for a hypercover $U \to X$ if the natural map from $F(X)$ to the homotopy limit of the diagram

$$\prod_a F(U_0^a) \rightrightarrows \prod_a F(U_1^a) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

is a weak equivalence, where the product runs over all the representable summands of each U_n. If F is not objectwise-fibrant we say the F satisfies descent if some object-wise fibrant replacement satisfies descent.

THEOREM 2.4. *Let $(\mathcal{C}, \tau)$ be an small Grothendieck site and S the collection of hypercovers. Then the Bousfield localization at S exists and coincides with the Jardine's model structure. We denote this category as $s\mathrm{PSh}(\mathcal{C})_{\tau,inj}$.*

PROOF. This is proved in [**DHI04**, Corollary 7.1] □

DEFINITION 2.1.6. Let $(\mathcal{C}, \tau)$ be a small Grothendieck site and S_τ the set of covering sieves for τ, which is given by the set of monomorphisms $R \hookrightarrow X$ with X the representable presheaves. We say that a simplicial presheaf on $\mathcal{C}$ satisfies τ-Čech descent or is τ-Čech local or τ- Čech fibrant if it is $\mathcal{S}_\tau$-local and we say that a morphism of simplicial presheaves is a τ-local weak equivalence if it is an $\mathcal{S}_\tau$-equivalence.

Remark 2.5. Since $(\mathcal{C}, \tau)$ is an small Grothendieck site, S_τ is actually a set. And since $s\mathrm{PSh}(\mathcal{C})_{inj}$ is a combinatorial model structure, the Bousfield localization with respect to S_τ exists. We denote this model structure as $s\mathrm{PSh}(\mathcal{C})_{\check{\mathcal{C}},\tau,inj}$

If $\mathcal{U} = \{U_i \to X\}_{i\in I}$ is a family of maps in $\mathcal{C}$, we denote by $\check{C}(\mathcal{U})$ its Čech nerve, which is a simplicial presheaf on $\mathcal{C}$ with an augmentation to h_X. The following result gives a characterization of τ-local objects in terms of Čech descent.

Lemma 2.6. *Let $(\mathcal{C}, \tau)$ be a small Grothendieck site. For every $X \in \mathcal{C}$, let $Cov(X)$ be a set whose elements are families of maps $\{U_i \to X\}_{i\in I}$ which are coverings for X in the Grothendieck topology τ. A simplicial presheaf F on $\mathcal{C}$ is τ-local if and only if it is $\mathcal{S}$-local, i.e. if and only if for every $X \in \mathcal{C}$ and every $\{U_i \to X\}_{i\in I} \in Cov(X)$, the canonical map*

$$F(X) \to holim_{n\in\Delta} \prod_{i_0,\dots,i_n\in I} F(U_{i_0 \times_X \cdots \times_X U_{i_n}})$$

is a weak equivalence.

PROOF. [**AHW17**, Lemma 3.1.3] □

In general the Jardine model structure $s\mathrm{PSh}(\mathcal{C})_{\tau,inj}$ is a left Bousfield localization of the Čech τ-local model structure $s\mathrm{PSh}(\mathcal{C})_{\check{\mathcal{C}},\tau,inj}$. Since we are inverting arbitrary hypercovers and not just Čech hypercovers, and the two model structures may be different, as it is exhibited in [**DHI04**, Example A.9].

However, if S is a Noetherian scheme of finite type and finite Krull dimension and Sm_S is provided with the Nisnevich topology both localizations coincide [**DHI04**, Example A.10].

2.1.4. Nisnevich and étale-descent. From now we will be particularly interested in the standard topologies for algebraic geometry, the Zariski, Nisnevich and the étale topology.

DEFINITION 2.1.7. Let $\mathcal{C}$ be a small category with initial object 0. A *cd-structure* P on $\mathcal{C}$ is a collection of commutative squares in $\mathcal{C}$. We say that P forms a cd-structure on $\mathcal{C}$ if whenever $\mathcal{Q} \in P$ and $\mathcal{Q}'$ is isomorphic to $\mathcal{Q}$, then $\mathcal{Q}'$ is also in P. The squares of the collection P are called distinguished squares.

DEFINITION 2.1.8. The *cd-topology,* τ_p, associated to a *cd-structure* P is the Grothendieck topology on $\mathcal{C}$ generated by the coverings sieves of the following form:

(1) The empty sieve is a covering sieve of the initial object 0,
(2) The sieve generated by morphisms of the form $\{A \to X, Y \to X\}$ where $A \to X$ and $Y \to X$ are two sides of the squares in P of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"'] \\ A \arrow[r, "e"'] & X \end{tikzcd}$$

EXAMPLE 2.1.2.

(1) The *Zariski cd-structure* on Sm_S consisting of Cartesian squares

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"'] \\ U \arrow[r, "e"'] & X \end{tikzcd}$$

where $U, V \subset X$ are open subschemes such that $U \cap V = X$.

(2) The *Nisnevich cd-structure* on Sm_S consists of Cartesian squares

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where j is an open immersion and π is étale and furthermore π induces an isomorphism $V \times_X Z \cong Z$, where Z is the reduced closed complement of j.

Remark 2.7. Since the schemes on Sm_S are quasi-compact and quasi-separated, the topology generated by the Zariski *cd-structure* is the usual Zariski topology. The proof in [**MV99**, Proposition 1.4] shows that the topology generated by the *Nisnevich cd-structure* is the Nisnevich topology. This fact is not true for the *étale topology* but in [**Isa04**] they discuss the behavior of the étale topology in terms of distinguished squares.

2.1.5. Excision and descent.

DEFINITION 2.1.9. Let $\mathcal{C}$ be a small category with an initial object 0 and let P be a *cd-structure* on $\mathcal{C}$. A simplicial presheaves F on $\mathcal{C}$ satisfies P-excision if

(1) F(0) is weakly contractible;
(2) for every square Q in P, F(Q) is homotopy Cartesian.

Proposition 2.8. *Let $\mathcal{C}$ be a small category with a strictly initial object let P be a cd-structure* on $\mathcal{C}$ such that

(1) Every square in P is Cartesian;
(2) pullbacks of squares in P exists and belongs to P;
(3) for every square in P $e : A \to X$ is a monomorphism;

(4) for every square in P, the square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ W \times_U W \arrow[r, "e"'] & V \times_X V \end{tikzcd}$$

is also in P.

Then a simplicial presheaves F on $\mathcal{C}$ satisfies P-excision if and only if satisfies τ-Čech descent.

2.1.6. Derived categories of presheaves of $\mathcal{R}$-modules.

DEFINITION 2.1.10. A morphism $F \to G$ in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C})$ is called τ-local weak equivalence if for any point of the site $(\mathcal{C}, \tau)$ the morphism of simplicial sets $x^*(f) : x^*(F) \to x^*(G)$ is a weak equivalence in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}$.

THEOREM 2.9. *Let $\mathcal{R}$ be a commutative ring and $(\mathcal{C}, \tau)$ a small Groethendieck site with enough points. There is a simplicial, cofibrantly generated model category structure on $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C})$ where the cofibrations are the monomorphisms and the weak equivalences are object-wise weak equivalences (resp. τ-local weak equivalence) of the underlying simplicial sets.*

PROOF. The proof follows as [**Rap13**, Theorem 5.7]. Let us denote $W_{\mathcal{R}}$ (resp. $W_{\tau,\mathcal{R}}$) the class of object-wise weak equivalences (resp. τ-local weak equivalence) in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C})$. The forgetful functor $U : s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C}) \to s\mathrm{PSh}(\mathcal{C})$ is a right adjoint hence preserves filtered colimits, as a consequence it is an accessible functor. Then the condition *(3)* from the theorem A.6 is satisfied. Condition *(4)* is trivial.

Since $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C})$ is a Grothendieck abelian category by [**Bek00**] the class of monomorphisms is cofibrantly generated by a set of monomorphisms. Thus condition *(1)* is satisfied. It remains to show conditions *(2)* and *(5)*.

For *(2)* the class of monomorphisms which are object-wise weak equivalences (resp. $\mathcal{W}_{\tau,\mathcal{R}}$) are closed under transfinite composition and push-outs. The closure under transfinite composition follows because it is true in $s\mathrm{PSh}(\mathcal{C})$, and it transfinite composition can be computed in $s\mathrm{PSh}(\mathcal{C})$.

So is enough to show that for every push-out square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where $j \in (mono) \cap \mathcal{W}_{\mathcal{R}}$ (resp. $(mono) \cap \mathcal{W}_{\tau,\mathcal{R}}$) then $f \in \mathcal{W}_{\mathcal{R}}$ (resp. $\mathcal{W}_{\tau,\mathcal{R}}$) .

The case $j \in (mono) \cap \mathcal{W}_{\mathcal{R}}$ is obvious because local weak equivalences in $\mathcal{S}$ which are monomorphisms are closed under push-outs.

For the case $j \in (mono) \cap \mathcal{W}_{\tau,\mathcal{R}}$: Since push-outs are preserved under colimits we have the following push-out diagram

$$\begin{tikzcd} x^*(A) \arrow[r] \arrow[d, "j"] & x^*(F) \arrow[d, "f"] \\ x^*(B) \arrow[r] & x^*(G) \end{tikzcd} \tag{2.1}$$

$x^*(j) : x^*(A) \to x^*(B)$ is a monomorphism and also a weak equivalence in $\mathcal{S}$. Since local weak equivalences in $\mathcal{S}$ which are monomorphisms are closed under push-outs.

Condition *(5)* is verified because we can consider

$$\mathcal{R}[\mathcal{I}^{proj}] := \{\mathcal{R}[h_U] \otimes \mathcal{R}[\partial\Delta_n] \longrightarrow \mathcal{R}[h_U] \otimes \mathcal{R}[\Delta_n] \ \ : \ U \in \mathcal{C}, \ n \in \mathbb{N}\}$$

the set of generating cofibrations on $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(\mathcal{C})_{proj}$ which are also monomorphisms then $mono - inj \subset (\mathcal{R}_{tr}[\mathcal{I}^{proj}]) - inj$ but a morphism in the former set is also a section-wise weak equivalence and thus a simplicial weak equivalence W_s. We have therefore defined a cofibrantly generated model category.

□

Remark 2.10. The class of object-wise weak equivalences $\mathcal{W}_{\mathcal{R}}$ and the class of monomorphisms actually define the injective model structure in the category $Fun(\mathcal{C}, s\mathcal{M}\mathrm{od}_{\mathcal{R}})$. Furthermore, the model structure defined by the class $\mathcal{W}_{\tau,\mathcal{R}}$ and the class of monomorphisms is the Bousfield localization of the injective model structure with respect to the class

$$\{\mathcal{R}[h_{\mathcal{U}}] \to \mathcal{R}[h_X] \ : \ \mathcal{U} \to X \text{ is an hypercover of X and } X \in \mathcal{C}\}$$

This follows because a morphism $A \to B$ is an element in $\mathcal{W}_{\mathcal{R}}$ if and only of $x^*A \to x^*B$ is a weak equivalence of simplicial sets. Then by the Whitehead Theorem the morphism on stalks is a quasi-isomorphism in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}$ for all points x. Then $A \to B$ induces an isomorphism in $a_\tau H_*(A) \to a_\tau H_*(B)$.

2.1.7. $\mathbb{A}^1$-localization. We now introduce the $\mathbb{A}^1$-localization. We will consider Sm_F the category of smooth schemes of finite type over F with a Grothendieck topology τ, which is either the Nisnevich topology or the étale topology.

Definition 2.1.11. A presheaf of simplicial $\mathcal{R}$-modules M is $\mathbb{A}^1$-local if for any $X \in Sm_F$ the induced map

$$\mathrm{Map}_{s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)}(\mathcal{R}[h_X], M) \to \mathrm{Map}_{s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)}(\mathcal{R}[h_{X\times\mathbb{A}^1}], M)$$

is a weak equivalence. A morphism $M \to N \in s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$ is a $\mathbb{A}^1$-weak equivalence if for any $P \in s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$ $\mathbb{A}^1$-local, the induced map $\mathrm{Map}_{s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)}(N, P) \to \mathrm{Map}_{s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)}(M, P)$ is a weak equivalence. Since $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$ is endowed with the local cofibrantly generated model category $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj,\tau}$, then the Bousfield localization exists with respect to the set

$$\{\mathcal{R}[h_{X\times\mathbb{A}^1}] \to \mathcal{R}[h_X] \ : \ X \in Sm_F\}$$

we denote this model structure as $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)^{\mathbb{A}^1}_{inj,\tau}$.

Note that the model category $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)^{\mathbb{A}^1}_{inj,\tau}$ has as cofibrations the monomorphisms since cofibrations do not change under Bousfield localization.

Proposition 2.11. *If we endow the category* $s\mathrm{PSh}(Sm_F)$ *with the injective (resp. τ-local and τ-$\mathbb{A}^1$-inj) and* $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$ *with the injective (resp. τ-inj and τ-$\mathbb{A}^1$-inj) 2.12. Then the adjunction (free-forgetful adjunction)*

$$\mathcal{R}[-] : s\mathrm{PSh}(Sm_F) \rightleftarrows s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F) : u$$

is a Quillen adjunction.

Proof. The cofibrations in the injective (resp. τ-local and τ-$\mathbb{A}^1$-local) model structure are the monomorphisms, which clearly are preserved by $\mathcal{R}[-]$

We claim that $\mathcal{R}[-]$ preserves trivial cofibrations in the injective (resp. τ-local and τ-$\mathbb{A}^1$-local) model structure. For the injective model structure, this follows because every sectionwise weak equivalence induces a sectionwise homology equivalence by the Whitehead Theorem.

For the τ-local model structure, let x be a point in the Grothendieck site $(Sm_F)_\tau$. Since $\mathcal{R}[-]$ commutes with filtered colimits $\mathcal{R}[x^*(\mathcal{X})] = x^*(\mathcal{R}[\mathcal{X}])$, and then by the Whitehead theorem again it follows that $\mathcal{R}[-]$ preserves τ-local weak equivalences.

For the τ-$\mathbb{A}^1$-local model structure, we claim that $\mathcal{R}[-]$ preserves $\mathbb{A}^1$-weak-equivalences. To show this, let $N : s\mathrm{Mod}(Sm_F) \to Cplx(Sm_F, \mathcal{M}\mathrm{od}_{\mathcal{R}})$ be the functor of taking the normalizing chain complex of simplicial presheaves of modules and let $C_*(-,\mathcal{R}) = N \circ \mathcal{R}[-]$.

$$\begin{tikzcd} s\mathrm{PSh}(Sm_F) \arrow[r, "\mathcal{R}[-]"] \arrow[rr, bend left, "C_*(-{,}\mathcal{R})"] & s\mathrm{Mod}(Sm_F) \arrow[r, "N"] & Cplx(Sm_F, \mathcal{M}\mathrm{od}_{\mathcal{R}}) \end{tikzcd}$$

Notice that $A_* \to B_*$ is an $\mathbb{A}^1$-weak equivalence in $s\mathrm{Mod}(Sm_F)$ when $N(A_*) \to N(B_*)$ is an $\mathbb{A}^1$-weak equivalence of $Cplx(Sm_F, \mathcal{M}\mathrm{od}_{\mathcal{R}})$. We need to show that $C_*(-,\mathcal{R})$ transforms $\mathbb{A}^1$-weak equivalences to $\mathbb{A}^1$-weak equivalences. Recall that $C_*(-,\mathcal{R})$ has a right adjoint

$$K : Cplx(Sm_F, \mathcal{M}\mathrm{od}_{\mathcal{R}}) \to s\mathrm{PSh}(Sm_F)$$

called the Eilenberg-MacLane space functor. If C_* is an $\mathbb{A}^1$-local complex then $K(C_*)$ is an $\mathbb{A}^1$-local space. We claim that $C_*(-,\mathcal{R})$ preserves $\mathbb{A}^1$-weak equivalence in $s\mathrm{PSh}(Sm_F)$. Let $\mathcal{X} \to \mathcal{Y}$ be an $\mathbb{A}^1$-weak equivalence and C_* be an $\mathbb{A}^1$-local complex C_*, since $K(C_*)$ is $\mathbb{A}^1$-local and by adjointness

$$\begin{tikzcd}
Hom(\mathcal{Y}, K(C_*)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & Hom(\mathcal{X}, K(C_*)) \arrow[d, "\cong"'] \\
Hom(C_*(\mathcal{Y},\mathcal{R}), C_*) \arrow[r] & Hom(C_*(\mathcal{X},\mathcal{R}), C_*)
\end{tikzcd}$$

then the claim follows and $\mathcal{R}[-]$ preserves $\mathbb{A}^1$-weak equivalences.

□

2.2. Homotopy theory for simplicial coalgebras

Since $\mathrm{coCAlg}_R(Sm_F)$ is complete and cocomplete by [**GJ99**, Theorem 2.5] the category $s\mathrm{coCAlg}_R(Sm_F)$ is a simplicial category, where the simplicial structure is given by:

$$-\otimes - : \mathcal{S} \times s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F) \to s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$$

defined on sections by

$$(K \otimes F)_n(U) := \mathcal{R}[K_n] \otimes_{\mathcal{R}} F_n(U)$$

for $F \in s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ and $K \in \mathcal{S}$. Giving $\phi : [n] \to [m]$ the induced map ϕ^* is giving by the composition

$$\mathcal{R}[K_m] \otimes_{\mathcal{R}} F_m(U) \to \mathcal{R}[K_m] \otimes_{\mathcal{R}} F_n(U) \to \mathcal{R}[K_n] \otimes_{\mathcal{R}} F_n(U)$$

and remember that the colimits on $\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ are created by the forgetful functor then the coalgebra structure on $K \otimes F$ is given by the tensor of the coalgebra structure on F and the coalgebra structure in $\mathcal{R}[K]$ induced by the diagonal map $\Delta_K : K \to K \times K$. The mapping space functor is given by

$$\mathrm{Map}_{s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)}(F,G)_n := \mathrm{Hom}_{s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)}(\mathcal{R}[\Delta^n] \otimes F, G).$$

Given the adjunction

$$U : s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F) \rightleftarrows s\mathrm{Mod}_{\mathcal{R}}(Sm_F) : CF$$

we would like to lift a given model structure on $s\mathrm{Mod}_R(Sm_F)$ along the right adjoint functor in order that the adjunction becomes a Quillen adjunction and that the category s $\mathrm{coCAlg}_R(Sm_F)$ satisfies the axioms of a simplicial model structure. This is dual to the usual Quillen's argument of transfer a model structure along a left adjoint, but close to the arguments provided for the existence of the Bousfield Localization. The strategy is essentially provided a class of weak equivalence and cofibrations and prove the existence of the model structure using Theorem A.6.

This proof is essentially given in [**Rap13**]. We reproduced the proof here for completeness.

Theorem 2.12. *Let $\mathcal{R}$ be a commutative ring and $s\mathcal{M}\mathrm{od}_{\mathcal{R}}\mathcal{R}(Sm_F)$ the category of simplicial modules endowed with the injective (resp. τ-local and τ-$\mathbb{A}^1$-inj). There is simplicial, cofibrantly generated model category structure on $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ left induced by the forgetful functor*

$$U : s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F) \to s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$$

where the class of weak equivalences in $\mathcal{W}_{\mathcal{R}}$ (resp. $\mathcal{W}_{\mathcal{R},\tau}$ and $\mathcal{W}^{\mathbb{A}^1}_{\mathcal{R},\tau}$) is given by the maps of coalgebras $f : C \to D$ such that $U(f) : U(C) \to U(D)$ is an object-wise (resp. τ, τ-$\mathbb{A}^1$) weak equivalence and the set of U-monomorphisms between κ-presentable objects is the generating set of cofibrations (for any choice of a large enough regular cardinal κ).

Proof. It is enough to show that the class of weak equivalences $\mathcal{W}_{\mathcal{R}}$ (resp. $\mathcal{W}_{\mathcal{R},\tau}$, $\mathcal{W}^{\mathbb{A}^1}_{\mathcal{R},\tau}$) and class of U-monomorphisms satisfies the conditions *(1)-(5)* from Theorem A.6. Condition *(3)* is satisfied because $\mathcal{W}_{\mathcal{R}}$ (resp. $\mathcal{W}_{\mathcal{R},\tau}$, $\mathcal{W}^{\mathbb{A}^1}_{\mathcal{R},\tau}$) is the inverse image of $\mathcal{W}$ (resp. $\mathcal{W}_s$,$\mathcal{W}^{\mathbb{A}^1}_{\tau}$) under the composition $u \circ U$ where $U : s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F) \to s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$ and $u : s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F) \to s\mathrm{PSh}(Sm_F)$. Since U is a left adjoint it preserves all colimits and u is a right adjoint then it preserves filtered colimits it follows that $u \circ U$ is an accessible functor. Then by proposition A.8 the class of weak equivalences $\mathcal{W}_{\mathcal{R}}$ (resp. $\mathcal{W}_{\mathcal{R},\tau}$, $\mathcal{W}^{\mathbb{A}^1}_{\mathcal{R},\tau}$) is accessible embedded in the category of arrows $Arrow(s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F))$. Condition *(4)* is obvious. Conditions *(1)* and *(2)* are satisfied because colimits in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ are created by the forgetful functor and Theorem 2.9. It remains to show condition *(5)* , this will be a consequence of Lemma 2.14 below. □

We call *left injective* , *left τ-local* and *left τ-$\mathbb{A}^1$-local* model structures for the left model structures induced in $s\mathrm{coCAlg}_k(Sm_F)$ in the previous theorem.

Lemma 2.13. *Any morphism $f : F \to G$ in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ can be factored as*

$$F \xrightarrow{i} D \xrightarrow{q} G$$

where $U(i)$ is a cofibration in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$ and $U(q)$ is a trivial fibration in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{proj}$.

Proof. The proof of this result is the mapping cylinder construction this argument goes back to Goerss, Jardine and Raptis in the setting of coalgebras with the section-wise monoidal structure.

Choose the cylinder object of F in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ given by:

$$\begin{tikzcd} F \oplus F \arrow[r, "\nabla"] \arrow[d, "(i_0,i_1)"'] & F \\ F \otimes \mathcal{R}[\Delta^1] \arrow[ur, "p"'] & \end{tikzcd}$$

here the coalgebra structure in $\mathcal{R}[\Delta^1]$ is given by the diagonal map $\Delta^1 \to \Delta^1 \times \Delta^1$. The mapping cylinder $Cyl(f)$, is constructed by the push-out diagram in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$.

$$\begin{tikzcd} F \otimes \mathcal{R} \arrow[r, "f"] \arrow[d, "i_0"] & G \arrow[d, "j"] \\ F \otimes \mathcal{R}[\Delta^1] \arrow[r, "j"] & Cyl(f) \end{tikzcd} .$$

Since $(f \circ p) \circ i_0 = f$ by the universal property of the push-out diagram there exists a unique map $q : Cyl(f) \to B$ such that $f \circ p = q \circ j$ and $G \to Cyl(f) \to G$ is the identity map. Then we can construct the factorization of $f : F \to G$ given by:

$$F \xrightarrow{i_1} F \otimes \mathcal{R}[\Delta^1] \xrightarrow{j} Cyl(f) \xrightarrow{p} G$$

the composition of the first two maps is a section-wise monomorphism in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$, in other words it is a cofibration in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$. The map $U(i_0)$ is a trivial cofibration in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$, and trivial cofibrations are preserved under push-outs, then $U(j)$ is a section-wise weak equivalence, then $U(q)$ is also a section-wise weak equivalence, in other words a trivial fibration in $s\mathrm{Mod}_{\mathcal{R}}(Sm_F)_{proj}$. □

We want to prove that the maps which has the right lifting property with respect to all the cofibrations are weak equivalences; this is the remaining condition to check in the proposition. In suffices to show the following proposition.

Proposition 2.14. *Let $\mathcal{I}$ the set of monomorphisms between κ-presentable objects. Then the $\mathcal{I}$-inj is contained in the set of weak equivalences $\mathcal{W}_{\mathcal{R}}$ (resp. $\mathcal{W}_{\mathcal{R},\tau}$ and $\mathcal{W}^{\mathbb{A}^1}_{\mathcal{R},\tau}$)*

Proof. Let $f : F \to G$ be a map in $\mathcal{I}$-inj, in order to show that f is a weak equivalence in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ it is enough to show that it is a weak equivalence in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{proj}$. Then it suffices to show that $U(f) : U(F) \to U(G)$ has the right lifting property with respect to the set of generating cofibrations in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{proj}$ given by:

$$\{\mathcal{R}[h_X] \otimes \mathcal{R}[\partial\Delta^n] \to \mathcal{R}[h_X] \otimes \mathcal{R}[\Delta^n] : \ X \in Ob(Sm_F), \ n \geq 1\}.$$

The map $f : F \to G$ is a κ-directed colimit of κ-presentable objects in $\mathrm{Arrw}(\mathrm{coCAlg}_{\mathcal{R}}(Sm_F))$, i.e. f is of the form $\mathrm{colim}_\alpha f_\alpha : \mathrm{colim}_\alpha F_\alpha \to \mathrm{colim}_\alpha G_\alpha$ where F_α and G_α are κ-presentable. Since the forgetful functor U preserves colimits then there exists a factorization:

$$\begin{tikzcd} h_X \otimes \mathcal{R}[\partial\Delta^n] \arrow[r] \arrow[d] & U(F_\alpha) \arrow[r] \arrow[d] & U(F) \arrow[d] \\ h_X \otimes \mathcal{R}[\Delta^n] \arrow[r] & U(G_\alpha) \arrow[r] & U(G) \end{tikzcd}$$

By the Lemma 2.13, there exists a factorization in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$

$$F_\alpha \xrightarrow{i_\alpha} D_\alpha \xrightarrow{p_\alpha} G_\alpha$$

where $U(p_\alpha)$ is a trivial fibration in $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{proj}$, then there exists a morphism

$$g_{1,X} : \mathcal{R}[h_X] \otimes \mathcal{R}[\Delta^n] \dashrightarrow U(D_\alpha)$$

such that the diagram commutes:

$$\begin{tikzcd}
 & U(F_\alpha) \arrow[r] \arrow[d] & U(F) \arrow[dd] \\
\mathcal{R}[h_X] \otimes \mathcal{R}[\partial\Delta^n] \arrow[r] \arrow[ur] \arrow[d] & U(D_\alpha) \arrow[ur, dashed, "g_2"] \arrow[d] & \\
\mathcal{R}[h_X] \otimes \mathcal{R}[\Delta^n] \arrow[r] \arrow[ur, dashed, "g_{1,X}"] & U(G_\alpha) \arrow[r] & U(G)
\end{tikzcd}$$

by the construction of the cylinder object D_α is also κ-presentable then the map $i_\alpha : F_\alpha \to D_\alpha$ is an element in $\mathcal{I}$ then by assumption there exists a morphism g_2 in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ which makes the diagram commutes:

$$\begin{tikzcd}
F_\alpha \arrow[r] \arrow[d, leftrightarrow, "h_2"'] & F \arrow[d] \\
D_\alpha \arrow[r] & G
\end{tikzcd} .$$

Then the composition $U(g_2) \circ g_1$ provides the lift and then $U(f)$ is a weak equivalence. $\square$

Remark 2.15. Let $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)$ be the category of presheaves of simplicial modules over Sm_F, let $\mathrm{L}s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$ and $\mathrm{L}s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{proj}$ be the Bousfield localizations of the injective and projective model structure with the same class of weak equivalences $\mathcal{W}'$. Recall that under the left Bousfield localization the class of cofibrations remains the same, then the class of trivial fibrations is preserved, because the former are the maps which satisfies the right lifting property with respect to all the cofibration. Therefore Lemma 2.13 says that every morphism $f : F \to G$ in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ can be factored as

$$F \xrightarrow{i} D \xrightarrow{q} G$$

where $U(i)$ is a cofibration in $\mathrm{L}s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$ and $U(q)$ is a trivial fibration in $\mathrm{L}s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{proj}$ and Proposition 2.14 tells that $\mathcal{I}-inj \subset \mathcal{W}'$. Then by Theorem A.6 there exists a model category structure in $s\mathrm{coCAlg}_{\mathcal{R}}(Sm_F)$ where the weak equivalences are $\mathcal{W}'$. Furthermore, note that this model structure is the left induced model structure from $\mathrm{L}s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$. Thus cofibrations in the former model structure remains the same than cofibration in the left induced model structure from $s\mathcal{M}\mathrm{od}_{\mathcal{R}}(Sm_F)_{inj}$. Then it is a left Bousfield localization.

In particular the *left τ-local* and *left τ-$\mathbb{A}^1$-local* model structures induced in $s\mathrm{coCAlg}_k(Sm_F)$ are left Bousfield localizations of the *left injective* model structure in $s\mathrm{coCAlg}_k(Sm_F)$.

2.3. Homology Localization

Definition 2.3.1. The $\mathbb{A}^1$-singular chain complex of $\mathcal{X}$ with $\mathcal{R}$ coefficients, denoted by $C_*^{\mathbb{A}^1}(\mathcal{X}, \mathcal{R})$, is defined to be the τ-$\mathbb{A}^1$-localization $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau(C_*(\mathcal{X}, \mathcal{R}))$. We just write $C_*(\mathcal{X})$ for $C_*(\mathcal{X}, \mathbb{Z})$. The $\mathbb{A}^1$-homology sheaves of $\mathcal{X}$ with coefficients in $\mathcal{R}$ are defined by $\mathbf{H}_{\tau,i}^{\mathbb{A}^1}(\mathcal{X}, \mathcal{R}) := a_\tau H_i(C_*^{\mathbb{A}^1}(\mathcal{X}, \mathcal{R}))$.

Definition 2.3.2. A morphism $\mathcal{X} \to \mathcal{Y}$ of motivic spaces is called an $\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}$-homology equivalence if the induced morphism $\mathbf{H}_{\tau,*}^{\mathbb{A}^1}(\mathcal{X}, \mathcal{R}) \to \mathbf{H}_{\tau,*}^{\mathbb{A}^1}(\mathcal{Y}, \mathcal{R})$ is an isomorphism. A space $\mathcal{Z}$ is called $\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}$-local if fo every $\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}$-equivalence $\mathcal{X} \to \mathcal{Y}$ the induced map

$$Map(\mathcal{Y}, \mathcal{Z}) \to Map(\mathcal{Y}, \mathcal{Z})$$

is a weak equivalence. A map $\mathcal{X} \to \mathcal{X}_{\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}}$ is called $\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}$-localization if it is an $\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}$-equivalence and $\mathcal{X}_{\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}}$ is $\mathbf{H}_\tau^{\mathbb{A}^1}\mathcal{R}$-local.

If $\tau = \mathrm{Nis}$, $\mathbf{H}_{\tau,i}^{\mathbb{A}^1}(\mathcal{X}, \mathcal{R})$ are the $\mathbb{A}^1$-homology groups defined by Morel [**Mor12**].

Remark 2.16. Let $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F)$ be the category of simplicial presheaves endowed with the injective-τ-$\mathbb{A}^1$-local model structure, by [**MV99**] this model structure is proper and by [**Lur09**, A.2.7] it is combinatorial. Then by the methods in [**GJ98**] the Bousfield localization with respect to the class of $\mathbf{H}^{\mathbb{A}^1}\mathcal{R}$-homology equivalence should exist.

2.4. $\mathbb{A}^1$-Goerss Theorem for k algebraically closed

Let k be an algebraically closed field. In this section we will generalize Theorem C in [**Goe95b**], *i.e* the diagonal coalgebra $k[\mathcal{X}]$ determines the homotopy type of $\mathcal{X}$ up to Bousfield Localization with respect to $\mathbb{A}^1$-homology with coefficients in k.

Proposition 2.17. *Let $\mathcal{R}$ be a commutative ring. Endow the category $s\mathrm{PSh}(Sm_F)$ with the injective (resp. τ-local and τ-$\mathbb{A}^1$-local) model structure and $s\mathrm{coCAlg}_k(Sm_F)$ with the respective model structure from 2.12. Then the adjunction*

$$\mathcal{R}^\delta[-] : s\mathrm{PSh}(Sm_F) \rightleftarrows s\mathrm{coCAlg}_k(Sm_F) : (-)^{gp}$$

is a Quillen adjunction.

Proof. It is enough to show that $k^\delta[-]$ preserves cofibrations and acyclic cofibrations. Since the model structure in $s\mathrm{coCAlg}_k(Sm_F)$ is left induced by the forgetful functor $U : s\mathrm{coCAlg}_k(Sm_F) \to s\mathcal{M}\mathrm{od}(Sm_F)$, where we endow the category $s\mathcal{M}\mathrm{od}(Sm_F)$ with the injective (resp. τ-local and τ-$\mathbb{A}^1$-local) model structure, it is enough to show that $k[-]$ preserves cofibrations and acyclic cofibrations. This follows from proposition 2.11.

$$\begin{tikzcd} s\mathrm{PSh}(Sm_F) \arrow[r, "k^\delta[-]"] \arrow[dr, "k[-]"'] & s\mathrm{coCAlg}_k(Sm_F) \arrow[d, "U"] \\ & s\mathcal{M}\mathrm{od}(Sm_F) \end{tikzcd} .$$

$\square$

Theorem 2.18. *Let k be an algebraically closed field. The functor $k^\delta[-]$ induces a fully faithful functor in the homotopy categories:*

$$\mathrm{L}k^\delta[-] : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathcal{H}_\bullet(F) \to \mathrm{Ho}(\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{coCAlg}_k(Sm_F)).$$

Furthermore, for every motivic space $\mathcal{X}$, the derived unit map

$$\mathcal{X} \to (k^\delta[\mathcal{X}]^{fib})^{gp}$$

exhibits the target as the $\mathbf{H}^{\mathbb{A}^1}k$ localization of $\mathcal{X}$.

Proof. First let us show that $k^\delta[-] : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F) \to \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{coCAlg}_k(Sm_F)$ is a left Quillen functor. $k^\delta[-]$ preserves cofibrations because the cofibrations in $\mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F)$ are the same than cofibrations in $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F)$. It also preservers trivial cofibrations by the definition of $\mathbf{H}^{\mathbb{A}^1}k$-weak equivalences (resp. $\mathbf{H}k$, $\mathbf{H}_\tau k$). Thus $(-)^{gp}$ is a right Quillen functor in particular preserves fibrations and trivial fibrations.

Let us show now that given $\mathcal{X}$ a motivic space (respectively $inj - fib$, τ-fib), the derived unit $\mathcal{X} \to (k^\delta[\mathcal{X}]^{fib})^{gp}$ exhibits the target as a the $\mathbf{H}k$-Bousfield Localization.

Claim: $(-)^{gp}$ sends injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalences to $\mathbf{H}_\tau^{\mathbb{A}^1}k$ (resp. $\mathbf{H}k$, $\mathbf{H}_\tau k$) weak equivalences.

Granting this, it follows immediately that given $\mathcal{X} \in s\mathrm{PSh}(Sm_F)$

$$\mathcal{X} = (k^\delta[\mathcal{X}])^{gp} \to (k[\mathcal{X}]^{fib})^{gp} \qquad \text{(resp. } \tau\text{-fib, } \tau\text{-}\mathbb{A}^1\text{-fib)}$$

is an $\mathbf{H}k$- injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalences in $s\mathrm{PSh}(Sm_F)$.

It remains to show that $(k^\delta[\mathcal{X}]^{\mathbb{A}^1\text{-fib}})^{gp}$ is an $\mathbf{H}k$-local space. We factor the counit map $\epsilon : k[\mathcal{X}] \to k$ by a trivial-cofibration followed by fibration $k[\mathcal{X}] \to k[\mathcal{X}]^{fib} \to k$. Applying the group-like functor $(-)^{gp}$ we get

$$\mathcal{X} \longrightarrow (k[\mathcal{X}]^{fib})^{gp} \longrightarrow * \qquad \text{(resp. } \tau\text{-fib, } \tau\text{-}\mathbb{A}^1\text{-fib).}$$

We already show that the first map is an $\mathbf{H}k$-injective (resp. $\mathbf{H}_\tau k$, $\mathbf{H}_\tau^{\mathbb{A}^1} k$) weak equivalence. Furthermore, since $(-)^{gp}$ is a right Quillen functor, it preserves fibrations. Then $(k[\mathcal{X}]^{fib})^{gp}$ is $\mathbf{H}k$ (resp. $\mathbf{H}_\tau k$, $\mathbf{H}_\tau^{\mathbb{A}^1} k$) local space.

Proof of the claim: Let $\alpha : C \to D$ be an injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalence in $s\mathrm{coCAlg}_k(Sm_F)$. By definition it suffices to show that $k[C^{gp}] \to k[D^{gp}]$ is a an injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalence. Since every element in $s\mathrm{coCAlg}_k(Sm_F)$ is an objectwise coalgebra, by the Theorem 1.15 $\acute{E}t$ induces a functor in $s\mathrm{coCAlg}_k(Sm_F)$, which has a natural splitting. This implies that $\acute{E}t(\alpha)$ is a retract of α. Then $\acute{E}t(\alpha)$ is an injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalence in $s\mathrm{coCAlg}_k(Sm_F)$.

$$\begin{tikzcd}
\acute{E}t(C) \arrow[r] \arrow[d, "\acute{E}t(\alpha)"'] & C \arrow[d, "\alpha"] \\
\acute{E}t(D) \arrow[r] & D
\end{tikzcd}$$

Since we are assuming k to be an algebraically closed field by Proposition 1.22

$$k^\delta[(C)^{gp}] \cong \acute{E}t(C) \xrightarrow{\acute{E}t(\alpha)} \acute{E}t(D) \cong k^\delta[(D)^{gp}] \ .$$

Then $k[C^{gp}] \to k[D^{gp}]$ is an injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalence.

□

Remark 2.19. Note that it was enough to show that $k[C^{gp}] \to k[(C^{fib})^{gp}]$ is an injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalence. In general it is hard to give an explicit fibrant replacement for coalgebras. At least up to our knowledge, we could not construct an explicit one. But since we have a good knowledge of the category of coalgebras over an algebraically closed field k, coCAlg_k, Proposition 1.9 allows us to show that $(-)^{gp}$ sends injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalences between coalgebras to $\mathbf{H}_\tau^{\mathbb{A}^1} k$ (resp. $\mathbf{H}k$, $\mathbf{H}_\tau k$) weak equivalences.

2.5. Discrete G-objects and $\mathbb{A}^1$-Goerss theorem for k non-algebraically closed

Let G be a profinite group. In this section we will define the notion of discrete G-objects for the categories of abelian groups, sheaves of sets, sheaves of abelian groups, simplicial sets $\mathcal{S}$ and simplicial sheaves. The aim is to understand the notion of discret G-motivic spaces.

2.5.1. Discrete G-Spaces. Let us take the category of simplicial discrete G-Sets

$$sG\text{-Sets}_d \simeq s\mathrm{Sh}(G\text{-Sets}_{fd}) \tag{2.2}$$

Definition 2.5.1. Let $f : F \to G$ be a morphism between simplicial discrete G-sets:

- f is a weak equivalence if and only if it is a weak equivalence between the underlying simplicial sets.
- f is a cofibration if and only if it is a injection.
- f is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.

Proposition 2.20. *The class of weak equivalence, fibrations and cofibrations from Definition 2.5.1 defines a simplicial model model structure in sG-Sets$_d$.*

Proof. This is proved in [**Goe95a**]. Since the category of simplicial discrete G-sets is the category of simplicial sheaves $s\mathrm{Sh}(G\text{-Sets}_{fd})$, an alternative proof follows from [**Jar87**]. Recall that there is only one stalk, the forgetful functor. □

Definition 2.5.2. Let sG-Sets$_d$ be the category of simplicial discrete G-spaces endowed with the Jardine model structure. Consider its homotopy category $\mathrm{H}(sG\text{-Sets}_d)$ an element in the homotopy category is called a discrete G-space.

2.5.2. Discrete G-Motivic spaces. Let us consider the category of simplicial presheaves on the product site G-$\mathrm{Sets}_{fd} \times Sm_F$, where Sm_F is endowed with a Grothendieck topology τ and G-Sets_{fd} is endowed with the cofinite topology, by abuse of notation we denote this topology with G.

By [**Jar87**], the category $s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd})$ is endowed with injective model structure. Note that, if we consider the trivial Grothendieck topology in Sm_F, the fibrant objects in the $triv \times G\text{-}inj$ model structure are elements in $s\mathrm{PSh}(Sm_F, sG\text{-}\mathrm{Sets}_d)$.

Furthermore, as in the definition of discrete G-spaces, we can define a discrete G-simplicial sheaf over the site (Sm_F, τ) as an element in the homotopy category:

As usual we introduce the $\mathbb{A}^1$-localization in the homotopy category

$$\mathrm{Ho}(\mathrm{L}_{\tau\times G} s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd}))$$

Definition 2.5.3. A presheaf of simplicial discrete G-sets $\mathcal{X} \in \mathrm{PSh}(Sm_F, sG\text{-}\mathrm{Sets}_{fd})$ is $\mathbb{A}^1$-local if for any $X \in Sm_F$ the induced

$$\mathrm{Map}_{\mathrm{PSh}(Sm_F, sG\text{-}\mathrm{Sets}_d)}(h_U, \mathcal{X}) \to \mathrm{Map}_{\mathrm{PSh}(Sm_F, sG\text{-}\mathrm{Sets}_d)}(h_{U\times\mathbb{A}^1}, \mathcal{X})$$

is a weak equivalence, where h_U and $h_{U\times\mathbb{A}^1}$ are consider as presheaves with the trivial action by G.

Definition 2.5.4. Discrete G-motivic spaces are fibrant objects in the model category

$$\mathrm{L}_{\mathbb{A}^1} L_{\tau\times G}\mathrm{PSh}(Sm_F, sG\text{-}\mathrm{Sets}_{fd})$$

. We will denote the category of discrete G-τ-motivic spaces as

$$Spc^G_{\tau,\bullet}(F)$$

2.5.3. Homotopy Fixed points for discrete G-Motivic Spaces. Here we define and discuss elementary properties of homotopy fixed points for discrete G-motivic spaces, this depends in the properties of discrete G-spaces.

Let $G = Gal(k_{sep}/k)$ be the absolute Galois group of a field k and F a perfect field, consider (Sm_F, τ) with τ a Grothendieck topology over Sm_F. We have a canonical functor given by the constant sheaf, in other words we can endowed every simplicial presheaf over Sm_F with the trivial action.

$$s\mathrm{PSh}(Sm_F)_{\tau-inj} \underset{(-)^G}{\overset{constant}{\rightleftarrows}} s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd}). \tag{2.3}$$

The right adjoint is given by the sections $\mathcal{X}(\mathrm{Spec}(k))$ in other words; it is given by the fixed points $\mathcal{X}^G$. Since the cofibrations in $s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd})$ are section-wise and the constant sheaf functor preserves weak equivalences, the adjunction 2.3 induces a Quillen adjunction. Then we have a well-defined adjoint pair on the homotopy categories

$$\mathrm{Ho}(s\mathrm{PSh}(Sm_F)_{\tau-inj}) \underset{(-)^G}{\overset{constant}{\rightleftarrows}} \mathrm{Ho}(s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd})). \tag{2.4}$$

Definition 2.5.5. Let $\mathcal{X} \in Ho(s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd}))$ be a G-discrete object. Define the homotopy fixed objects as:

$$\mathcal{X}^{hG} := (\mathcal{X}^{fib})^G$$

where the fibrant replacement is taken in the category $s\mathrm{PSh}(Sm_F \times G\text{-}\mathrm{Sets}_{fd})_{\tau\times G\text{-}inj}$.

2.5.4. Homotopy fixed points for G-spaces. Here we will recall some results from [**Goe95a**]. It is worthy to note that Proposition 4.7 and Theorem 8.1 (in *loc.cit*) extend to nilpotent spaces, although the statements in *loc. cit.* are given for simple and simply connected spaces respectively. We do not claim any originality here, the proof follows from the same argument provided there and it was confirmed personal communication. To be more precise in both cases we need the convergence of particular Postnikov towers and [**MV99**, Theorem 1.37] provides evidence to this be true for nilpotent spaces. Here we state those results as Proposition 2.25 and Theorem 2.33.

Let G be a fixed profinite group. The main purpose of this section is to understand the homotopy fixed points for Eilenberg-MacLane spaces and under some restrictions for nilpotent spaces.

The following lemma identifies a class of fibrant objects:

Lemma 2.21. *Let $f : K \to L$ be a morphism of discrete G-simplicial abelian groups sAb^G with the property that after apply the normalization functor, the map $Nf : NK_n \to NL_n$ satisfies the following conditions:*

(1) $Nf : NK_n \to NL_n$ is split surjective for $n \geq 1$;
(2) $J_n = ker(Nf : NK_n \to NL_n)$ is an injective discrete G-abelian group for $n \geq 1$;
(3) there exists a $k \geq 1$ so that $J_n = 0$ for all $n \geq k$. Then f is a fibration in sG-Sets$_d$.

PROOF. [**Goe95a**, Lemma 3.1] □

As we noted before in particular $K(M,n)$ is not fibrant unless M is injective or $n = 0$. Given an injective resolution of M as a discrete G-module:

$$0 \to M \to J^0 \to J^1 \to J^2 \to \cdots$$

let $K^{fib}(M,n) \in s\mathbb{Z}[G]$ such that

$$NK^{fib}(M,n)_k = \begin{cases} J^{n-k} & 1 \geq k \geq n \\ K^n = ker(\partial : J^n \to J^{n+1}) & k = 0 \\ 0 & k > 0 \end{cases}$$

the augmentation ϵ defines a weak equivalence and $K^{fib}(M,n)$ is fibrant by Lemma 2.21. Furthermore, there is a fibration in sG-Sets$_d$

$$PK^{fib}(M,n) \to K^{fib}(M,n)$$

with contractible total space and fiber $K^{fib}(M,n-1)$.

2.5.4.1. *Homotopy fixed points for Eilenberg-MacLane spaces.* The following proposition provides a nice computation of the homotopy fixed points for an Eilenberg-MacLane space.

Proposition 2.22. *Let M be a discrete group G. Then*

$$\pi_k K(M,n)^{hG} = H^{n-k}_{Gal}(G,M)$$

for $0 \leq k \leq n$ and zero otherwise.

PROOF. This is given in [**Goe95a**]. By definition $K(M,n)^{hG} = K^{fib}(K,m)^G$ and since the normalization functor is functorial

$$\pi_* K(M,n)^{hG} \cong \prod_{k=0}^{n} K(H^{n-k}_{Gal}(G,M),k)$$

If $k > n$, $\pi_k K(M,n)^{hG} = 0$ and for $1 \leq k \leq n$, by the definition of the Galois cohomology

$$\pi_k K(M,n)^{hG} = H^{n-k}_{Gal}(G,M).$$

For $k = 0$, we should compute the cokernel of $(J^{n-1})^G \to (K^n)^G$. Since $0 \to K^n \to J^n \to J^{n+1}$ is exact, the

$$0 \to (K^n)^G \to (J^n)^G \to (J^{n+1})^G$$

is also exact, and the cokernel is $H^n_{Gal}(G,M)$.

□

Remark 2.23. Assume that G has cohomological dimension d, *i.e* $H^q_{Gal}(G,M) = 0$ for all discrete G-modules M and $q > d$ we can refine the fibrant model, with the property that $K_0(M,n)_k = 0$ for $k < n - d$ when $n > d$. In particular we are interested when $G = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, then $G = \hat{\mathbb{Z}}$, which has cohomological dimension 1.

To be more precise, let

$$0 \to M \to J^0 \to J^1 \to J^2 \to \cdots$$

be an injective resolutions as discrete G-modules and

$$0 \to M \to J^0 \to J^1 \to J^2 \to \cdots \to J^{d-1} \to K^d \to 0$$

the truncation with $K^d := ker(\partial : J^d \to J^{d+1})$.

For $n \leq d$, define $K_0(M,n) := K^{fib}(M,n)$. If $n > d$ define $K_0(M,n)$ by the requirement that

$$NK^{fib}(M,n)_k = \begin{cases} J^{n-k} & n-d < k \geq n \\ K^d = ker(\delta : J^d \to J^{d+1}) & k = n-d \\ 0 & \text{otherwise} \end{cases}$$

Claim: $K_0(M,n)$ is fibrant in sG-Sets$_d$, in fact $j : K_0(M,n) \to K^{fib}(M,n)$ has a retraction in sG-Sets$_d$. Furthermore, there is a fibration in sG-Sets$_d$

$$PK_0(M,n) \to K_0(M,n)$$

with contractible total space and fiber $K_0(M,n-1)$, this follows from [**Goe95a**, Lemma 3.6]

2.5.5. Postnikov Towers of discrete G-spaces. Recall that studying the homotopy category of discrete G-spaces is equivalent to study the homotopy category of simplicial presheaves

$$s\mathrm{PSh}(G - Sets_{fd}) \simeq s\mathrm{PSh}(F\acute{E}t_k)$$

endowed with the Jardine model structure. Postnikov towers on an arbitrary site were studied in [**MV99**]. Here we recall the construction for G-discrete spaces.

If X is a nilpotent discrete G-space then in particular is a nilpotent spaces in $\mathcal{S}$. By [**GJ99**, Proposition 6.1] the Postnikov tower $\{P_n(X)\}$ can be refined to a tower of principal fibrations in $\mathcal{S}$. This Postnikov tower is built out of k-invariants. We refine this construction in discrete G-spaces. The proof essentially follows [**Goe95a**]. Consider the refined Postnikov tower $\{P_{n,i}(X)\}$ given in [**GJ99**, Proposition 6.1]. Factor $p_{n,i} : P_{n,i}(X) \to P_{n,i-1}$ as

$$P_{n,i}(X) \to Z \to P_{n,i-1}$$

where $P_{n,i}(X) \to Z$ is a cofibration and $Z \to P_{n,i-1}(X)$ is a weak equivalence. By the long exact sequence in homotopy groups we have that the action of $\pi_1(X)$ on $\pi_{n+1}(Z, P_{n,i-1}(X))$ is trivial. Then, the Hurewicz homomorphism

$$p : \pi_{n+1}(Z, P_{n,i-1}(X)) \to H_{n+1}(Z, P_{n,i-1}(X))$$

is an isomorphism. And the following composition is an isomorphism:

$$H_{n+1}(Z, P_{n,i}(X)) \simeq \pi_{n+1}(Z, P_{n,i}(X)) \to F_{i-1}/F_i \simeq gr^{i-1}\pi_n. \tag{2.5}$$

Since the factorization of $p_{n,i}$ is given in discrete G-spaces; then 2.8 is an isomorphism of G-discrete modules. By construction $H_k(Z, P_{n,i}(X)) = 0$ for $k < n+1$, then by the Universal coefficient spectral sequence in discrete G-spaces

$$H^{n+1}_G(Z, P_n(X); gr^{i-1}\pi_n) \simeq \mathrm{Hom}_G(H_{n+1}(Z, P_{n,i}(X)), gr^{i-1}\pi_n)$$

so the isomorphism 2.8 provides a class $\alpha_{n,i} \in H^{n+1}_G(Z, P_{n,i}(X); gr^{i-1}\pi_n)$. Using the long exact sequence in cohomology and the fact that $Z \to P_{n-1}(X)$ is a weak equivalence; the image of $\alpha_{n,i}$ in $H^{n+1}(Z,\pi_n) \simeq H^{n+1}(X_{n-1}; gr^{i-1}\pi_n)$ is denoted by $[k^G_{n,i}]$. This equivariant k-invariant is represented by a map in sG-Sets$_d$

$$k^G_n : P_{n-1}(X) \to K^{fib}(gr^{i-1}\pi_n, n+1)$$

where $K^{fib}(gr^{i-1}\pi_n, n+1)$ is the fibrant replacement for $K(gr^{i-1}\pi_n, n+1)$ in sG-Sets$_d$. Since $[k^G_n]$ arises as a class on $H^{n+1}(Z, P_n(X); gr^{i-1}\pi_n)$, then $[k^G_n]$ goes to zero in $H^{n+1}(P_n(X), gr^{i-1}\pi_n)$ and the map

$$P_{n,i}(X) \to P_{n,i-1}(X) \xrightarrow{k^G_{n,i}} K^{fib}(gr^{i-1}\pi_n, n+1)$$

is null homotopic. This provides a weak equivalence in sG-Sets$_d$

$$P_{n,i}(X) \to P_{n,i-1}(X) \times_{K^{fib}(gr^{i-1}\pi_n,n+1)} PK^{fib}(gr^{i-1}\pi_n, n+1). \tag{2.6}$$

We now define recursively the discrete G-spaces $P'_n(X)$ and weak equivalences with $P_n(X) \to P'_n(X)$ using the G-equivariant k-invariants. Let $j_1 : P_1(X) \to P'_1(X)$ be a fibrant replacement, with j_1 a cofibration. If $P'_{n,i-1}(X)$ and $j_{n,i-1} : P_{n,i-1}(X) \to P'_{n,i-1}(X)$ have been defined, then there is a factoring up to homotopy in sG-Sets$_d$

$$\begin{tikzcd} P_{n,i-1}X \arrow[r] \arrow[d] & K^{fib}(gr^{i-1}\pi_n, n+1) \\ P'_{n,i-1}X \arrow[ur] & \end{tikzcd}$$

and hence a weak equivalence

$$P_{n,i-1}X \times_{K^{fib}(gr^{i-1}\pi_n,n+1)} PK^{fib}(gr^{i-1}\pi_n, n+1) \to P'_{n,i-1}X \times_{K^{fib}(gr^{i-1}\pi_n,n+1)} PK^{fib}(gr^{i-1}\pi_n, n+1)$$

since $j_{n,i-1} : P_{n,i-1}(X) \to P'_{n,i-1}(X)$ is a trivial cofibration. We define

$$P'_{n,i}X := P'_{n,i-1}X \times_{K^{fib}(gr^{i-1}\pi_n,n+1)} PK^{fib}(gr^{i-1}\pi_n, n+1),$$

precomposing with 2.6 we obtain a weak equivalence $j_{n,i} : P_{n,i}(X) \to P'_{n,i}(X)$ in sG-Sets$_d$.

Having constructed this Postnikov tower it is not true in general for an arbitrary G-profinite space that $X \to \mathrm{holim}_{Q_n(X)}$ is a weak equivalence. We need to impose some condition on the profinite group G.The fact that the G-discrete Postnikov tower does not converge was studied in more generality in [**MV99**] (for simplicial sheaves in an arbitrary Grothendieck topology). In this work we are interested in two particular cases $G = Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ and $G = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. In the first case, the group is of finite cohomological dimension. Although the second case is not a group of finite cohomological dimension we will be able to treated separately after $\mathbb{Q}$-localization of the discrete G-spaces.

2.5.6. Nilpotent discrete G-spaces.

Definition 2.5.6. Let G be a profinite group. A discrete G-space X is nilpotent if the following conditions are satisfied:

- $\pi_1(X)$ is a discrete G-nilpoten group *i.e.*, it has a continuos action by G and there exists a finite normal central series
$$\{1\} = F_r \subset F_{r-1} \subset \cdots \subset F_0 = \pi(X)$$
by normal discrete G-subgroups F_i such that
$$1 \to F_{i-1}/F_i \to \pi_1(X)/F_{i-1} \to \pi_1(X)/F_i \to 1$$
is a central extension.
- Every $\pi_n(X)$ for $n > 1$ has a finite filtration
$$\{1\} = M_s \subset M_{s-1} \subset \cdots \subset M_0 = \pi(X)$$
by $\pi_1(X)$-modules which furthermore are discrete G-abelian groups so that $\pi_1(X)$ acts trivially on the successive quotients.

Note that $\pi_1(X)$ and $\pi_n(X)$ are discrete G-groups with respect to the natural action of G induced from the action of G on X.

Proposition 2.24. *Let X be a pointed connected nilpotent discrete G-space. Assume that G has finite cohomological dimension. Let $\{P_n(X)\}$ be an sG-Sets$_d$-Postnikov tower. Then*

$$X \to \lim_{sG\text{-Sets}_d} P_n(X)$$

is a weak equivalence.

Proof. Let d be the cohomological dimension of G. This by proposition **??** there is a firbrant model $K^{fib}(M,n)$ of $K(M,n)$ in sG-Sets$_d$, with the property that

$$K^{fib}(M,n)_k = 0$$

in simplicial degree $k < n - d$. Using this model for the Eilenberg-Mac Lane spaces; we can build a sG-Sets$_d$-Postnikov tower out of equivariant k-invariants. Then we one has that

$$p'_{n,j} : P'_{n,i}(X) \to P''_{n,i-1}(X)$$

is an equality in simplicial degrees $k \geq n-d$. In particular the canonical map $\lim_{\mathcal{S}} P'_{n,i}(X) \to P'_{n,i-1}(X)$ is an isomorphism in simplicial degrees $k \geq n - d$, then $\lim_{\mathcal{S}} P'_{n,i}(X)$ is a discrete G-space, so

$$\lim_{\mathcal{S}} P'_n(X) \simeq \lim_{sG\text{-Sets}_d} P'_n(X).$$

□

Proposition 2.25. *Let G be a profinite group of finite cohomological dimension and X pointed connected nilpotent discrete G-space. Then there is a sequence of G-discrete spaces $\{P^G_n X\}$, morphisms $p_n : X \to P^G_n X$, and morphisms $f_n : P^G_{n+1}X \to P_n X$ such that:*

1. *$P^G_n X$ has the property that $\pi_k(P^G_n X) = 0$ for $k > n$;*
2. *the morphism p_n induces an isomorphism on G-discrete homotopy groups in degree lower or equal to n;*

(3) *the morphism f_n is an fibration in sG-Sets$_d$-, and the homotopy fiber of f_n is*
$$K(\pi_{n+1}(P_n^G X), n+1)$$
(4) *the induced morphism $X \to \operatorname{holim}_n P_n X$ is a weak equivalence in sG-Sets$_d$, where the homotopy limit is taken in sG-Sets$_d$.*

PROOF. Take $\{P_n X\}$ the usual Postnikov tower. Since X is nilpotent each of the maps $p_n : P_n X \to P_{n-1}X$ can be refined to a finite composition
$$X_n = Y_k \to Y_{k-1} \to \cdots \to Y_1 \to Y_0 = X_{n-1}$$
such that each of the maps $Y_i \to Y_{i-1}$ fits in a homotopy pullback diagram in sG-Sets$_d$.
$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & PK^{fib}(F_{i-1}/F_{i-1}, i+1) \arrow[d] \\ Y_{i-1} \arrow[r] & K^{fib}(F_{i-1}/F_{i-1}, i+1) \end{tikzcd}$$
where $k^G_{n,i-1} : X_{n-1} \to K^{fib}(\pi_n(X), n+1)$ represents an equivariant k-invariant
$$[k^G_{n,i-1}] \in H^{n+1}_G(X_{n-1}, F_{i-1}/F_{i-1})$$
In particular we use the fibrant replacement $K_0(M, n)$, with the property that
$$K_0(M, n)_k = 0$$
in simplicial degree $k < n - d$. Then we have that the map in $\mathcal{S}$
$$(Y_i)_k \to (Y_{i-1})_k$$
is an equality in simplicial degrees $k \geq n - d$. In particular the morphism
$$\lim_{\mathcal{S}}(Y_i) \to (Y_i)$$
is an isomorphism in simplicial degrees $k \geq n - d$. But note that $\lim_{\mathcal{S}} Y_i$ is a discrete G-space, so $\lim_{sG\text{-Sets}_d} Y_i = \lim_{\mathcal{S}} Y_i$.

□

2.5.7. Strictly and strongly $\mathbb{A}^1$-invariant sheaves of discrete G-groups. We first recall the definition of strictly $\mathbb{A}^1$-invariant sheaves.

DEFINITION 2.5.7. (1) Let S be a sheaf of sets on (Sm_F, τ), with τ a Grothendieck topology. It is said that S is τ-$\mathbb{A}^1$-invariant if for any $X \in Sm_F$ the map
$$S(X) \to S(\mathbb{A}^1 \times X),$$
induced by the projection $p_X : X \times \mathbb{A}^1 \to X$, is a bijection.
(2) Let $\mathcal{M}$ be a sheaf of groups on (Sm_F, τ), with τ a Grothendieck topology. We say that $\mathcal{M}$ is strongly τ-$\mathbb{A}^1$-invariant if for every $X \in Sm_F$, the canonical map
$$p_X^* : H^i_\tau(X, \mathcal{M}) \to H^i_\tau(X \times \mathbb{A}^1, \mathcal{M})$$
induced by the projection $p_X : X \times \mathbb{A}^1 \to X$ is a bijection for $i = 0, 1$.
(3) Let $\mathcal{M}$ be a sheaf of abelian groups on (Sm_F, τ), with τ a Grothendieck topology. We said that $\mathcal{M}$ is strictly τ-$\mathbb{A}^1$-invariant if for every $X \in Sm_F$, the canonical map
$$p_X^* : H^i_\tau(X, \mathcal{M}) \to H^i_\tau(X \times \mathbb{A}^1, \mathcal{M})$$
induced by the projection $p_X : X \times \mathbb{A}^1 \to X$ is a bijection for $i \in \mathbb{N}$.

Strong $\mathbb{A}^1$-invariance for the small Nisnevich site was extensively studied by Morel in [**Mor12**] and it was introduced in [**VSF00**]. The $\mathbb{A}^1$-invariant presheaves with transfers are the basic example of strictly $\mathbb{A}^1$-invariant sheaves.

Let us now introduce the notions of stricly and strongly $\mathbb{A}^1$-invariant discrete G-sheaves.

DEFINITION 2.5.8. (1) Let S be a sheaf of discrete G-sets on (Sm_F, τ), with τ a Grothendieck topology. It is said that S is G-τ-$\mathbb{A}^1$-invariant if for any $X \in Sm_F$ the map
$$S(X) \to S(\mathbb{A}^1 \times X),$$
induced by the projection $p_X : X \times \mathbb{A}^1 \to X$, is an imorphism of discrete G-sets.

(2) Let $\mathcal{M}$ be a sheaf of discrete G-groups on (Sm_F, τ), with τ a Grothendieck topology. We say that $\mathcal{M}$ is strongly G-τ-$\mathbb{A}^1$-invariant if for every $X \in Sm_F$, the canonical map

$$p_X^* : H^i_{\tau\times G}(X, \mathcal{M}) \to H^i_{\tau\times G}(X \times \mathbb{A}^1, \mathcal{M})$$

induced by the projection $p_X : X \times \mathbb{A}^1 \to X$ is a bijection for $i = 0, 1$.

(3) Let $\mathcal{M}$ be a sheaf of discrete G-abelian groups on (Sm_F, τ), with τ a Grothendieck topology. We said that $\mathcal{M}$ is strictly τ-$\mathbb{A}^1$-invariant if for every $X \in Sm_F$, the canonical map

$$p_X^* : H^i_{\tau\times G}(X, \mathcal{M}) \to H^i_{\tau\times G}(X \times \times \mathbb{A}^1, \mathcal{M})$$

induced by the projection $p_X : X \times \mathbb{A}^1 \to X$ is a bijection for $i \in \mathbb{N}$.

Where $H^*_{\tau\times G}(X, M)$ denotes the sheaf cohomology groups over the site $(Sm_F, \tau) \times G\text{-}Sets_{fd}$

2.5.8. $\mathbb{A}^1$-Postnikov towers. Postnikov towers for motivic spaces are discussed in [**AF14**, Theorem 6.1] based on results from [**MV99**, Proposition 1.27] and [**Mor12**, Appendix B].

2.5.8.1. Let $\mathcal{X}$ be a locally fibrant simplicial sheaf, $P_n\mathcal{X}$ is the associated simplicial sheaf to the presheaf $U \mapsto P_n(X(U))$, where $P_n(X(U))$ is the usual Postnikov section for the simplicial set $\mathcal{X}(U)$. By construction this tower is a tower of local fibrations and the stalks at every point x of the site τ gives the Postnikov tower for the $x^*\mathcal{X}$.

2.5.8.2. Given the tower of local fibrations $\{P_n\mathcal{X}\}$ from 2.5.8.1 we can construct inductively a tower of τ-fibrations $\{\mathcal{Y}_n\}$. Let $Ex_\tau(-)$ be a τ-fibrant replacement functor. We set $\mathcal{Y}_1 = Ex_\tau P_1\mathcal{X}$ for the τ-fibrant replacement of $P_1\mathcal{X}$. We factor the composition $P_2\mathcal{X} \to P_1\mathcal{X} \to Ex_\tau(P_1\mathcal{X})$ by a trivial cofibration followed by a fibration

$$P_2\mathcal{X} \to \mathcal{Y}_2 \to Ex_\tau(P_1\mathcal{X})$$

then $\mathcal{Y}_2$ is fibrant.

The functors P_n do not take τ-fibrant simplicial presheaves to τ-fibrant simplicial presheaves, as a consequence the homotopy limit $\operatorname{holim}_n Ex_\tau P_n\mathcal{X}$ of the tower of fibrant objects associated to the Postnikov tower of $\mathcal{X}$ is not in general weakly equivalent to $\mathcal{X}$ as it is shown in [**MV99**, Example 1.30]. In the same nature for the category of discrete spaces $sG\text{-Sets}_d$, with G a profinite group, the Postnikov tower of fibrant objects does not necessarily converge to $\mathcal{X}$. We need to impose finite cohomological dimension conditions in the site.

In this work we are particularly interested in the Nisnevich and étale site and in $G = Gal(\bar{k}/k)$, with $k = \mathbb{F}_p$ a finite field or $k = \mathbb{Q}$.

2.5.8.3. By [**MV99**, Theorem 1.37] the Nisnevich site $(Sm_F)_{Nis}$ is a site of finite type in the sense of [Definition 1.3 *loc.cit*], *i.e*

$$\mathcal{X} \to \operatorname{holim}_n Ex(P_n\mathcal{X}) \tag{2.7}$$

is a weak equivalence.

2.5.8.4. The étale site is not of finite type but [**Jar87**] provides enough conditions for the tower of fibrant objects to converge. Assume that k is of finite p-torsion Galois cohomology, where p is prime to the characteristic of k, and let $\mathcal{X}$ be a locally fibrant simplicial sheaf on $(Sm_F, \acute{e}t)$, which satisfies that there exists $K \in \mathbb{N}$, such that the homotopy sheaves $\pi_n(\mathcal{X}|_U, x)$, $U \to \operatorname{Spec}(k)$ étale are l-torsion for $n \geq K$. Then 2.7 is a weak equivalence.

2.5.8.5. Let G be the absolute Galois group of k , with $k = \mathbb{F}_p$ a finite field or $k = \mathbb{Q}$. The finite étale site $F\acute{E}t/k$ is not of finite type, but if G is of finite cohomological dimension in [**Goe95a**], they provided convergent Postnikov towers if $X \in \mathcal{S}$ is nilpotent, as we recall in 2.5.4.

In the first two previous cases, the argument uses the Godement resolution for Eilenberg Mac Lane spaces, and they describe how it does behave under finite cohomological dimension conditions.

2.5.8.6. Let $\mathcal{L}$ be any set of points of (Sm_F, τ) and define the functor from sheaves to cosimplicial sheaves

$$\mathcal{G}^\bullet_{\mathcal{L}} : \mathrm{Sh}_\tau(Sm_F) \to \mathrm{Sh}_\tau(Sm_F)^\Delta$$

as follows, a point in (Sm_F, τ) is a morphism of sites $Sets \to (Sm_F, \tau)$, and a set of points is a morphism of sites $p : \prod_{\mathcal{L}} Sets \to (Sm_F, \tau)$, consider the corresponding adjunction

$$p^* : \prod_{\mathcal{L}} Sets \rightleftarrows (Sm_F, \tau) : p_*$$

then given $F \in \mathrm{Sh}_\tau(\mathcal{C})$ we construct the cosimplicial sheaf, with terms given by $(p_*p^*)^{n+1}(F)$.

Remark 2.26. The composition functor p_*p^* preserves pointwise fibrations of simplicial presheaves, and also it can be showed that the composite p_*p^* also preserves and reflects finite limits of sheaves and exactness of sequences of abelian sheaves [**Jar10**].

Recall that local fibrations are not simplicial fibrations, but the Godement resolution sends local fibrations to simplicial fibrations [**MV99**, Proposition 1.59]. By [**MV99**, Theorem 1.66] for $(\mathcal{C}, \tau)$ a site of finite type the fibrant resolution functor is given by

$$Ex^{\mathcal{G}}(\mathcal{X}) = \operatorname{holim}_{\Delta} \mathcal{G}^{\bullet}(Ex^{Sets}\mathcal{X})$$

where $Ex^{Sets}(\mathcal{X})$ is the sheaf associated to the simplicial presheaf of the form $U \mapsto Ex^{\infty}(\mathcal{X}(U))$, where Ex^{∞} is a resolution functor on the category of simplicial sets [**MV99**, Lemma 1.67].

2.5.8.7. On the other hand given a resolution functor $Ex^{\mathcal{G}}$ for $s\mathrm{Sh}_{Nis}(Sm_F)$, for example we can take the Godement resolution, the functor

$$Ex_{\mathbb{A}^1} = Ex^{\mathcal{G}} \circ (\operatorname{holim}_n (Ex^{\mathcal{G}} \circ Sing_*)^n) \circ Ex^{\mathcal{G}}$$

is an $\mathbb{A}^1$-resolution functor (see [**MV99**, Lemma 2.6]). Using this resolution functor and proceeding as in 2.5.8.2, we get the following result.

THEOREM 2.27. *Let $(\mathcal{X}, x)$ be any pointed $\mathbb{A}^1$-connected space, then there is a sequence of pointed $\mathbb{A}^1$-connected spaces $\{P_n\mathcal{X}, x\}$, morphisms $p_n : \mathcal{X} \to P_n\mathcal{X}$, and morphisms $f_n : \mathcal{X}_{n+1} \to \mathcal{X}_n$ such that:*

(1) $P_n\mathcal{Y}$ has the property that $\pi_k^{\mathbb{A}^1}(P_n\mathcal{X}) = 0$ for $k > n$;
(2) the morphism p_n induces an isomorphism on $\mathbb{A}^1$-homotopy sheaves in degree lower or equal to i;
(3) the morphism f_n is an $\mathbb{A}^1$-fibration, and the $\mathbb{A}^1$-homotopy fiber of f_n is a $K(\pi_{n+1}^{\mathbb{A}^1}(P_n\mathcal{X}), n+1)$;
(4) the induced morphism $X \to \operatorname{holim}_n P_n\mathcal{X}$ is an $\mathbb{A}^1$-Nis-weak equivalence. Furthermore, f_n is a twisted $\mathbb{A}^1$-principal fibration; that is there is a unique up to $\mathbb{A}^1$-homotopy morphism

$$k_{n+1} : P_n\mathcal{X} \to K^{\pi_1^{\mathbb{A}^1}(\mathcal{X})}(\pi_{n+1}^{\mathbb{A}^1}(P_n\mathcal{X}), n+2)$$

called a k-invariant, such that k_{n+1} fits in a $\mathbb{A}^1$-homotopy pullback square of the form

$$\begin{tikzcd} \mathcal{X}^{n+1} \arrow[r] \arrow[d] & B\pi_1^{\mathbb{A}^1}(\mathcal{X}_n) \arrow[d] \\ \mathcal{X}_i \arrow[r, "k_{n+1}"] & K^{\pi_1^{\mathbb{A}^1}(\mathcal{X})}(\pi_{n+1}^{\mathbb{A}^1}(\mathcal{X}_n), n+2) \end{tikzcd}$$

PROOF. [**AF14**, Theorem 6.1] ☐

2.5.9. $\mathbb{A}^1$-Nilpotent spaces.

DEFINITION 2.5.9. Let $\mathcal{G}$ be an strongly $\mathbb{A}^1$-invariant sheaf of groups. We say that $\mathcal{G}$ is $\mathbb{A}^1$-nilpotent sheaf if there exists a finite $\mathbb{A}^1$-normal central series, *i.e* finite series

$$\{1\} = \mathcal{F}_r \subset \mathcal{F}_{r-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{G}$$

by normal sheaves of subgroups of $\mathcal{G}$ such that:

(1) $\mathcal{F}_i$ are strongly $\mathbb{A}^1$-invariant,
(2) $\mathcal{F}_i/\mathcal{F}_{i+1}$ is strongly $\mathbb{A}^1$-invariant and

$$1 \to \mathcal{F}_i/\mathcal{F}_{i+1} \to \mathcal{G}/\mathcal{F}_{i+1} \to \mathcal{G}/\mathcal{F}_i \to 1$$

is a central extension.

DEFINITION 2.5.10. Let $\mathcal{G}$ be an strongly $\mathbb{A}^1$-invariant sheaf and $M \in Ab^{\mathbb{A}^1}(k)$ be a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups with an action by $\mathcal{G}$. We said that $\mathcal{G}$ acts nilpotently if there exists a sequence

$$\{1\} = \mathcal{F}_r \subset \mathcal{F}_{r-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = M$$

by strictly $\mathbb{A}^1$-invariant sheaves of abelian groups, such that $\mathcal{G}$ acts trivially in $\mathcal{F}_i/\mathcal{F}_{i+1}$.

DEFINITION 2.5.11. A motivic space $\mathcal{X} \in \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{PSh}(Sm_F)$ is $\mathbb{A}^1$-nilpotent if $\mathcal{X}$ is $\mathbb{A}^1$-connected, $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-nilpotent sheaf of group and it acts nilpotently over $\pi_n^{\mathbb{A}^1}(\mathcal{X})$

EXAMPLE 2.5.1. It is showed in [**Mor12**] that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ fits in a central extension given by:

$$1 \to \mathbf{K}_2^{MW} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m \to 1$$

Remark 2.28. Let $\mathcal{X} \in \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{PSh}(Sm_F)$ be an $\mathbb{A}^1$-nilpotent space and $\{P_n\mathcal{X}, x\}$ the Postnikov tower from Theorem 2.27. Each of the maps $p_n : P_n\mathcal{X} \to P_{n-1}\mathcal{X}$ can be refined to a finite composition

$$\mathcal{X}_n = \mathcal{Y}_k \to \mathcal{Y}_{k-1} \to \cdots \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{X}_{n-1}$$

such that each of the maps $\mathcal{Y}_i \to \mathcal{Y}_{i-1}$ fits in a $\mathbb{A}^1$-homotopy pullback square

$$\begin{tikzcd} \mathcal{Y}_i \arrow[r] \arrow[d] & PK(\mathcal{F}_{i-1}/\mathcal{F}_{i-1}, i+1) \arrow[d] \\ \mathcal{Y}_{i-1} \arrow[r] & K(\mathcal{F}_{i-1}/\mathcal{F}_{i-1}, i+1) \end{tikzcd} .$$

We will give a more detail proof of this fact for the case of discrete G-motivic spaces.

2.5.10. Discrete G-equivariant $\mathbb{A}^1$-Nilpotent spaces. In this subsection we will define the notion of discrete G-τ-$\mathbb{A}^1$-Nilpotent spaces.

DEFINITION 2.5.12. Let $\mathcal{G}$ be an strongly τ-$\mathbb{A}^1$ invariant sheaf of groups. We say that $\mathcal{G}$ is discrete G-τ-$\mathbb{A}^1$-nilpotent sheaf if $\mathcal{G}$ is sheaf on the site $G\text{-}Sets_{fd} \times Sm_F$, endowed with the product topology $G \times \tau$ and there exists a finite discrete G-equivariant $\mathbb{A}^1$-normal central series, *i.e* finite series

$$\{1\} = \mathcal{F}_r \subset \mathcal{F}_{r-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{G}$$

by normal sheaves on $G\text{-}Sets_{fd} \times Sm_F$ of subgroups of $\mathcal{G}$ such that:

(1) $\mathcal{F}_i$ are τ strongly $\mathbb{A}^1$-invariant.
(2) $\mathcal{F}_i/\mathcal{F}_{i+1}$ is τ strongly $\mathbb{A}^1$-invariant and

$$1 \to \mathcal{F}_i/\mathcal{F}_{i+1} \to \mathcal{G}/\mathcal{F}_{i+1} \to \mathcal{G}/\mathcal{F}_i \to 1$$

is a central extension.

DEFINITION 2.5.13. Let $\mathcal{G}$ be a discret G strongly τ-$\mathbb{A}^1$-invariant sheaf and $\mathcal{M} \in Ab^{\mathbb{A}^1}(k)$ be a discrete G strictly τ-$\mathbb{A}^1$-invariant sheaf of abelian groups with an action by $\mathcal{G}$. We said that $\mathcal{G}$ acts nilpotently if there exists a sequence

$$\{1\} = \mathcal{F}_r \subset \mathcal{F}_{r-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{M}$$

by discrete G strongly τ-$\mathbb{A}^1$-invariant sheaves of G-discrete groups, such that $\mathcal{G}$ acts trivially in $\mathcal{F}_i/\mathcal{F}_{i+1}$.

DEFINITION 2.5.14. A discrete G-τ-motivic space $\mathcal{X} \in \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F)$ is discrete G-τ-$\mathbb{A}^1$-nilpotent if $\mathcal{X}$ is $\mathbb{A}^1$-connected, $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is a discrete G-τ-$\mathbb{A}^1$-nilpotent sheaf and it acts nilpotently over $\pi_n^{\mathbb{A}^1}(\mathcal{X})$

Remark 2.29. Let $\mathcal{X}$ be a discrete G-τ-$\mathbb{A}^1$-nilpotent space and $\{P_n\mathcal{X}, x\}$ the Postnikov tower from Theorem 2.27. By abuse of notation we omit the Grothendieck topology τ in this remark. Each of the maps $p_n : P_n\mathcal{X} \to P_{n-1}\mathcal{X}$ can be refined to a finite composition

$$P_n\mathcal{X} = P_{n,k}\mathcal{X} \to P_{n,k}\mathcal{X} \to \cdots \to P_{n,1}\mathcal{X} \to \mathcal{Y}_0 = P_{n-1}\mathcal{X}$$

such that each of the maps $P_{n,i}\mathcal{X} \to P_{n,i-1}\mathcal{X}$ fits in a $\mathbb{A}^1$-homotopy pullback square

$$\begin{tikzcd} \mathcal{Y}_i \arrow[r] \arrow[d] & PK(\mathcal{F}_{i-1}/\mathcal{F}_{i-1}, i+1) \arrow[d] \\ \mathcal{Y}_{i-1} \arrow[r] & K(\mathcal{F}_{i-1}/\mathcal{F}_{i-1}, i+1) \end{tikzcd} .$$

The construction of the refinement is similar to 2.5.5. Let $\{P_n\mathcal{X}, x\}$ be the Postnikov tower form Theorem 2.27. Since $\mathcal{X}$ is discrete G-$\mathbb{A}^1$-nilpotent space, then for each $U \in Sm_F$, $\mathcal{X}(U)$ is in particular a nilpotent space. For each $U \in Sm_F$, consider the refined Postnikov tower $\{P_{n,i}(\mathcal{X}(U))\}$ given in [**GJ99**, Proposition 6.1]. By abuse of notation let us denote as $P_{n,i}(\mathcal{X})$ the tower of associated sheaves. Factor $p_{n,i} : P_{n,i}(\mathcal{X}) \to P_{n,i-1}(\mathcal{X})$ as

$$P_{n,i}(\mathcal{X}) \to \mathcal{Z} \to P_{n,i-1}(\mathcal{X})$$

where $P_{n,i}(\mathcal{X}) \to \mathcal{Z}$ is a cofibration and $\mathcal{Z} \to P_{n,i-1}(\mathcal{X})$ is a weak equivalence in

$$\mathrm{L}_{\tau \times G}(s\mathrm{PSh}(Sm_F \times G\text{-}Sets_{fd}))_{inj}.$$

By the long exact sequence in homotopy groups we have that the action of $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ on $\pi_{n+1}(\mathcal{Z}, P_{n,i-1}(\mathcal{X}))$ is trivial, where $\pi_{n+1}(\mathcal{Z}, P_{n,i-1}(\mathcal{X}))$ denotes the sheaf associated to the presheaf

$$U \mapsto \pi_{n+1}(\mathcal{Z}(U), P_{n,i-1}(\mathcal{X}(U))).$$

Then, the Hurewicz homomorphism

$$p : \pi_{n+1}(\mathcal{Z}, P_{n,i-1}(\mathcal{X})) \to H_{n+1}(\mathcal{Z}, P_{n,i-1}(\mathcal{X}))$$

is an isomorphism. And the following composition is an isomorphism of sheaves on $Sm_F \times G\text{-}Sets_{fd}$

$$H_{n+1}(\mathcal{Z}, P_{n,i}(\mathcal{X})) \simeq \pi_{n+1}(\mathcal{Z}, P_{n,i}(\mathcal{X})) \to gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}). \tag{2.8}$$

By construction $H_k(\mathcal{Z}, P_{n,i}(\mathcal{X})) = 0$ for $k < n+1$, then by the spectral sequence

$$E^{p,q} = Ext^p_{\tau\times G}(H_q(\mathcal{X}), \mathcal{M}) \Rightarrow H^{p+q}_{\tau\times G}(\mathcal{X}, \mathcal{M})$$

we get

$$H^{n+1}_{\tau\times G}(Z, P_n(\mathcal{X}); gr^{i-1}\pi_n) \simeq \mathrm{Hom}_G(H_{n+1}(\mathcal{Z}, P_{n,i}(\mathcal{X})), gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}))$$

so the isomorphism 2.8 provides a class $\alpha_{n,i} \in H^{n+1}_G(\mathcal{Z}, P_{n,i}(\mathcal{X}); gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}))$. Using the long exact sequence in cohomology and the fact that $\mathcal{Z} \to P_{n-1}(\mathcal{X})$ is a weak equivalence; the image of $\alpha_{n,i}$ in $H^{n+1}(\mathcal{Z}, \pi_n) \simeq H^{n+1}(\mathcal{X}_{n-1}; gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}))$ is denoted by $[k^G_{n,i}]$. This k-invariant is represented by a map in

$$\mathrm{L}_{\tau\times G}(s\mathrm{PSh}(Sm_F \times G\text{-}Sets_{fd}))_{inj}.$$

$$k_n^G : P_{n-1}(\mathcal{X}) \to K^{\tau\times G-fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1)$$

where $K^{\tau\times G-fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1)$ is the fibrant replacement for $K(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1)$ in sG-Sets$_d$. Since $[k_n^G]$ arises as a class on $H^{n+1}(Z, P_n(X); gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}))$, then $[k_n^G]$ goes to zero in $H^{n+1}(P_n(\mathcal{X}), gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}))$ and the map

$$P_{n,i}(\mathcal{X}) \to P_{n,i-1}(\mathcal{X}) \xrightarrow{k^G_{n,i}} K^{\tau\times G-fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1)$$

is null homotopic. This provides a weak equivalence in $\mathrm{L}_{\tau\times G}(s\mathrm{PSh}(Sm_F \times G\text{-}Sets_{fd}))_{inj}$

$$P_{n,i}(\mathcal{X}) \to P_{n,i-1}(\mathcal{X}) \times_{K^{\tau\times G-fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}),n+1)} PK^{\tau\times G-fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1). \tag{2.9}$$

We now define recursively the discrete G- motivic spaces $P'_n(\mathcal{X})$ and weak equivalences with $P_n(\mathcal{X}) \to P'_n(\mathcal{X})$ using the k-invariants constructed above. Let $j_1 : P_1(\mathcal{X}) \to P'_1(\mathcal{X})$ be a fibrant replacement, with j_1 a cofibration. If $P'_{n,i-1}(\mathcal{X})$ and $j_{n,i-1} : P_{n,i-1}(\mathcal{X}) \to P'_{n,i-1}(\mathcal{X})$ have been defined, then there is a factoring up to homotopy in sG-Sets$_d$

$$\begin{tikzcd}
P_{n,i-1}\mathcal{X} \arrow[r] \arrow[d] & K^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1) \\
P'_{n,i-1}\mathcal{X} \arrow[ur] &
\end{tikzcd}$$

and hence a weak equivalence

$$P_{n,i-1}\mathcal{X}\times_{K^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}),n+1)} PK^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1) \to P'_{n,i-1}\mathcal{X}\times_{K^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}),n+1)} PK^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1)$$

since $j_{n,i-1} : P_{n,i-1}(\mathcal{X}) \to P'_{n,i-1}(\mathcal{X})$ is a trivial cofibration. We define

$$P'_{n,i}\mathcal{X} := P'_{n,i-1}\mathcal{X} \times_{K^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}),n+1)} PK^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1),$$

precomposing with 2.9 we obtain a weak equivalence $j_{n,i} : P_{n,i}(X) \to P'_{n,i}(X)$. Note that since $gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is τ-G-$\mathbb{A}^1$-strictly invariant, the spaces $K^{fib}(gr^{i-1}\pi_n^{\mathbb{A}^1}(\mathcal{X}), n+1)$ are $\mathbb{A}^1$-local. Then $P'_{n,i}\mathcal{X}$ is also $\mathbb{A}$-fibrant.

We were able to refine the $\mathbb{A}^1$-Postnikov tower, but this Postnikow tower does not always converge. We expect a similar result as in 2.25 but it has to be treated more carefully.

2.5.11. Rational localization. We need some preliminary results about discrete G-modules

Lemma 2.30. *Let M be a rational discrete G-module, then M is injective as discrete G-module.*

Proof. Let E(-) be the right adjoint to the forgetful functor from discrete G-modules to abelian groups. Then the unit of the adjunction $M \to E(M)$ has a natural retraction. $E(M)$ is given by the continuos maps $\varphi : G \to M$. Let H_φ be the isotropy subgroup of φ, set $n = [G : H_\varphi]$ the index and $g_1, \cdots, g_n$ coset representatives. The retraction is given by

$$r(\varphi) = \frac{1}{n}\sum_{n=1}^{n} g_i^{-1}\varphi(g_i)$$

□

As a consequence is not difficult to see the following corollary

Corollary 2.31. *Let $f : K \to L$ be a morphism of simplicial rational discrete G-abelian groups with the property that after apply the normalization functor, the map $NK_n \to NL_n$ is surjective for $n \geq 1$ and an isomorphism for n sufficiently large. Then f is an $H_*(-,\mathbb{Q})$-fibration in sG-Sets$_d$.*

Proof. Using the previous lemma, the proof in 2.21 can be adapted to this case. □

Corollary 2.32. *Let M be a discrete G-abelian group and $n \geq 1$. The induced map $K(M, n) \to K(M \otimes \mathbb{Q}, n)$ is the $\mathbb{Q}$-localization of $K(M, n)$ in sG-Sets$_d$.*

Proof. This follows from Corollary 2.31 □

Theorem 2.33. *Let $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$ the $\mathbb{A}^1$-rational homology spectrum and $\mathcal{X} \in Spc^G_\bullet(Sm_F)$ an $\mathbb{A}^1$-nilpotent discrete G-space for G a profinite group. Assume that $\mathcal{X} \to \mathcal{Y}$ is the $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$-localization, then the localization morphism induces for $n \geq 2$ an isomorphism of sheaves:*

$$\pi_n^{\mathbb{A}^1}(\mathcal{Y}^G) \simeq (\pi_n^{\mathbb{A}^1}(\mathcal{X}) \otimes \mathbb{Q})^G$$

In particular, if $\mathcal{X}$ has the trivial action by G, then Y^G is the $\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}$-localization.

Proof. *Claim: As in 2.25 the proposition follows if we set $H_* = H_*(-,\mathbb{Q})$ to be the rational homology and $X \in sG\text{-}Sets_d$ a nilpotent space.* Let P_nX be the Postnikov tower in sG-Sets$_d$ constructed from equivariant k_i-invariants. Since X is a nilpotent space we can refine this tower to a tower of principal fibrations $\{Y_i\}$, *i.e* each $Y_i \to Y_{i-1}$ fits in an homotopy pullback

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & PK(A_i, i+1) \arrow[d] \\ Y_{i-1} \arrow[r, "k_{i+1}"] & K(A_i, i+1) \end{tikzcd}$$

Inductively we construct the tower of $\mathbb{Q}$-localizations $j_i : Y_i \to Z_i$ as follows. First Z_1 is the localization given in 2.32. Having constructed Z_i, consider the diagram

$$\begin{tikzcd} Y_i \arrow[r, "k_i^G"] \arrow[d] & PK(A_i, i+1) \arrow[d, "i_n"] \\ Z_{i-1} \arrow[r, "k_i^{\mathbb{Q}}"] & K(A_i \otimes \mathbb{Q}, i+1) \end{tikzcd}$$

where k_i^G is an equivariant k-invariant. By the universal property of the Bousfield Localization, the equivariant map $k_i^{\mathbb{Q}}$ exists and is unique up to homotopy. We define:

$$Z_i = Y_{i-1} \times_{K(A_i \otimes \mathbb{Q}, i+1)} PK(A_i \otimes \mathbb{Q}, i+1).$$

By Lemma 2.31, we have that Z_i is a $\mathbb{Q}$-local space. By Serre spectral sequence, the induced map $Y_i \to Z_i$ is a $\mathbb{Q}$-localization. Notice that $X \to \lim Z_i$ is a $\mathbb{Q}$-localization. The argument for motivic spaces follows from the definition of $\mathbb{A}^1$-nilpotent discrete G-spaces.

□

Remark 2.34. Note that $\pi_n^{\mathbb{A}^1}(\mathcal{Y}^G)$ has a finite filtration

$$\mathcal{G}_n(\mathcal{Y}^G) \subset \cdots \subset \mathcal{G}_1(\mathcal{Y}^G) \subset \pi_n^{\mathbb{A}^1}(\mathcal{Y}^G)$$

of strictly $\mathbb{A}^1$-invariant sheaves of groups, such that

$$\mathcal{G}_i(\mathcal{Y}^G)/\mathcal{G}_{i+1}(\mathcal{Y}^G) \simeq \mathcal{G}_i(\mathcal{X})/\mathcal{G}_{i+1}(\mathcal{X}) \otimes \mathbb{Q}$$

where

$$\mathcal{G}_n(\mathcal{X}) \subset \cdots \subset \mathcal{G}_1(\mathcal{X}) \subset \pi_n^{\mathbb{A}^1}(\mathcal{X})$$

is the finite filtration of strictly $\mathbb{A}^1$-invariant sheaves of groups form Definition 2.5.14.

2.5.12. $\mathbb{A}^1$-Goerss Theorem for k non-algebraically closed.

Proposition 2.35. *There is a left adjoint functor $\bar{k}^\vee[-]_G : \mathrm{PSh}(Sm_F, sG\text{-Sets}_d) \to s\mathrm{coCAlg}_k(Sm_F)$ which sends G/H to the constant presheaf of coalgebras $(\bar{k}^H)^\vee$ placed in simplicial degree zero.*

Notation: Here the dual k-vector spaces $(L)^\vee$ is regarded as coalgebra with the comultiplication induced by the dual of the multiplication map.

Proof. Let $\mathcal{X} \in \mathrm{PSh}(Sm_F, sG\text{-Sets}_d)$ for every $U \in Sm_F$, from 1.19 we define the presheaf of coalgebras $\bar{k}^\vee[\mathcal{X}]_G$ section-wise.

$$\bar{k}^\vee[\mathcal{X}]_G(U) := \bar{k}^\vee[\mathcal{X}(U)]_G.$$

□

Proposition 2.36. *Let k be a perfect field. Endow the category $s\mathrm{PSh}(Sm_F, sG\text{-Sets}_d)$ with the injective (resp.$G \times \tau$-local and $G \times \tau$-$\mathbb{A}^1$-local) model structure and $s\mathrm{coCAlg}_k$ with the injective (resp.τ-local and τ-$\mathbb{A}^1$-local) from 2.12. Then the adjunction*

$$\bar{k}^\vee[-]_G : s\mathrm{PSh}(Sm_F, sG\text{-Sets}_d) \rightleftarrows s\mathrm{coCAlg}_k : R$$

is a Quillen adjunction.

Proof. It is enough to show that $\bar{k}^\vee[-]_G$ preserves cofibrations and acyclic cofibrations. Since the model structure in $s\mathrm{coCAlg}_k(Sm_F)$ is left induced by the forgetful functor $U : s\mathrm{coCAlg}_k(Sm_F) \to s\mathcal{M}\mathrm{od}(Sm_F)$, where we endow the category $s\mathcal{M}\mathrm{od}(Sm_F)$ with the injective (resp. τ-local and τ-$\mathbb{A}^1$-local) model structure, it is enough to show that $\bar{k}^\vee[-]_G \circ U$ preserves cofibrations and acyclic cofibrations. Note that for X a discrete G-Set the underlying vector space $\bar{k}^\vee[X]_G \circ U$ is isomorphic to $k[X]$. Then the proposition follows from proposition 2.11.

□

Theorem 2.37. *The functor $\bar{k}^\vee[-]_G : \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F, sG\text{-Sets}_d) \to s\mathrm{coCAlg}_k$ sends $\mathbf{H}_\tau^{\mathbb{A}^1}k$-equivalences to $\mathbb{A}^1$-Nis-weak equivalences of coalgebras and thus induces a functor:*

$$\mathrm{L}\bar{k}^\vee[-]_G : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathrm{Ho}((Spc_\bullet^G(F)) \to \mathrm{Ho}((\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{coCAlg}_k(\mathrm{Sm}_F)).$$

This functor is fully faithful.

Furthermore, for every motivic space $\mathcal{X}$ the derived unit map

$$\mathcal{X} \to R((\bar{k}^\vee[\mathcal{X}]_G)^{fib})$$

exhibits the target as the $\mathbf{H}^{\mathbb{A}^1}k$ localization of $\mathcal{X}$ in discrete G-motivic spaces.

Proof. First let us show that $\bar{k}^\vee[-]_G : \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F, sG\text{-Sets}_d) \to s\mathrm{coCAlg}_k$ is a left Quillen functor. From Proposition 2.35 we notice that the functor is given section-wise; $\bar{k}^\vee[-]_G$ preserves cofibrations because the cofibrations in $\mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F, sG\text{-Sets}_d)$ are the same than cofibrations in $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{PSh}(Sm_F, sG\text{-Sets}_d)$. It also preserves trivial cofibrations, since by the proof in 2.36 we know that $\bar{k}^\vee[\mathcal{X}]_G \simeq k[\mathcal{X}]$ as presheaves of vector spaces.Then it follows from the definition of $\mathbf{H}^{\mathbb{A}^1}k$-weak equivalences (resp. $\mathbf{H}k$, $\mathbf{H}_\tau k$). Thus R is a right Quillen functor in particular preserves fibrations and trivial fibrations. To show that the functor

$$\mathrm{L}\bar{k}^\vee[-]_G : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}k}\mathrm{Ho}((Spc_\bullet^G(F)) \to \mathrm{Ho}((\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_\tau s\mathrm{coCAlg}_k(\mathrm{Sm}_F))$$

is fully faithful Quillen functor, we have to prove that

$$\mathcal{X} \mapsto R((\bar{k}^\vee[\mathcal{X}]_G)) \mapsto R((\bar{k}^\vee[\mathcal{X}]_G)^{fib}) \tag{2.10}$$

is a weak equivalence.

By Proposition 1.19 the first morphism is an isomorphism, it remains to show that R sends injective (resp. τ, $\mathbb{A}^1$-τ) weak equivalences to $\mathbf{H}_\tau^{\mathbb{A}^1}k$ (resp. $\mathbf{H}k$, $\mathbf{H}_\tau k$) weak equivalences. Again by 1.19 the counit of the adjunction is given by $\acute{E}t(C) \to C$. Then the proof follows as in 2.18

□

Corollary 2.38. *Let* $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{PSh}(Sm_F)^{Nil}$ *be the subcategory of* $\mathbb{A}^1$*-nilpotent spaces. Then the functor*

$$\mathbb{Q}^\delta[-] : \mathrm{L}_{\mathbf{H}^{\mathbb{A}^1}\mathbb{Q}}\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{PSh}(Sm_F)^{Nil} \to \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{coCAlg}_{\mathbb{Q}}(Sm_F)$$

is fully faithful.

Proof. This follows because the functor $\mathbb{Q}^\delta[-]$ factors as the embedding

$$\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{PSh}(Sm_F) \xrightarrow{trivial} Spc_\bullet^G(Sm_F)$$

followed by the functor $\mathbb{Q}^{G,\delta}[-] : Spc_\bullet^G(Sm_F) \to \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}s\mathrm{coCAlg}_{\mathbb{Q}}(Sm_F)$. Then by Theorems 2.33 and 2.37 we conclude the result. □

APPENDIX A

Combinatorial model categories

In this appendix we recall several results about combinatorial model categories. We essentially follow [**Lur09**, Appendix A.2].

A.1. Compactness, Presentability and Accessible Categories

Definition A.1.1. Let κ be a regular cardinal and $\mathcal{J}$ a κ-filtered partially order set. Let $\mathcal{C}$ be a category which admits small colimits and let X be an object of $\mathcal{C}$. Let $\{Y_{\alpha\in\mathcal{J}}\}$ be a diagram in $\mathcal{C}$ indexed by $\mathcal{J}$. Let $Y = \operatorname{colim}_{\alpha\in\mathcal{J}} Y_\alpha$ be the colimit of this diagram. There is an associated map of sets

$$\phi : \operatorname{colim}_\alpha \hom_{\mathcal{C}}(X, Y_\alpha) \to \hom_{\mathcal{C}}(X, Y)$$

We say that X is *κ-compact* if ϕ is a bijective map of sets for every κ-filtered partial order set $\mathcal{J}$ and every diagram $\{Y_\alpha\}$ indexed by $\mathcal{J}$. We say that X is *small* if it is κ-compact for some small regular cardinal κ.

Definition A.1.2. A category $\mathcal{C}$ is locally presentable if it satisfies the following conditions:
(1) The category $\mathcal{C}$ admits all small colimits.
(2) There exists a small set S of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits in other words every object of C may be obtained as the colimit of small diagrams taking values in S.
(3) Every object in $\mathcal{C}$ is small. This is equivalent to say that every object in S is small.
(4) For any pair of objects $X, Y \in \mathcal{C}$ the set $\hom_{\mathcal{C}}(X, Y)$ is small.

Remark A.1. A no-trivial consequence of presentability for a category $\mathcal{C}$ is that it also admits all small limits.

A.2. Model categories

Definition A.2.1. A *model category* is a category $\mathcal{C}$ which is equipped with three distinguished classes of morphisms in $\mathcal{C}$, called *cofibrations*, *fibrations* and *weak equivalences, in which the following axioms are satisfied:*
(1) The category $\mathcal{C}$ admits small limits and colimits.
(2) Given a composable pair of maps $X \to Y \to Z$, if any two of $g\circ f$, f and g are weak equivalences, then so is the third.
(3) Suppose $f \to Y$ is a retract of $g : X' \to Y'$, that is suppose that there exists a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left] & X' \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "i'"] \arrow[rr, bend right] & Y' \arrow[r, "r'"] & Y \end{tikzcd}$$

(4) Given a solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a dotted arrow can be found making the diagram commute if either:
(a) The map i is a cofibration and the map p is both a fibration and a weak equivalence.
(b) The map i is both a cofibration and a weak equivalence, and the map p is a fibration.

(5) Any map $X \to Z$ in $\mathcal{C}$ admits factorizations

$$X \to Y \to Z$$
$$X \to Y \to Z$$

where f is a cofibration, g is a fibration and a weak equivalence, f' is a cofibration and a weak equivalence and g' is a fibration.

Definition A.2.2. The homotopy category is defined as follows:

(1) The objects of $h\mathcal{C}$ are the fibrant-cofibrant objects of $\mathcal{C}$.
(2) For $X, Y \in h\mathcal{C}$, the set $\hom_{h\mathcal{C}}(X, Y)$ is the set of homotopy equivalences classes on $\hom(X, Y)$

A.2.1. Properness and Homotopy Push out squares.

Definition A.2.3. A model category $\mathcal{C}$ is left proper if weak equivalences are stable under push-out along cofibrations and it is called right proper if weak equivalences are stable under pullbacks along fibrations.

A.2.2. Quillen adjunctions and Quillen equivalences. Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories and suppose we are given a pair of adjoint functors

$$L : \mathcal{M} \rightleftarrows \mathcal{N} : R$$

with L a left adjoint and R a right adjoint. The following conditions are equivalent:

- The functor L preserves cofibrations and trivial cofibrations
- The functor R preserves fibrations and trivial fibrations
- The functor L preserves cofibrations and the functor R preserves fibrations
- The functor L preserves trivial fibrations and the functor R preserves trivial fibrations.

Definition A.2.4. Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories and suppose we are given a pair of adjoint functors

$$L : \mathcal{M} \rightleftarrows \mathcal{N} : R.$$

If any of the equivalent conditions above is satisfied. Then we say that the pair (F, G) is a *Quillen adjunction* between $\mathcal{M}$ and $\mathcal{N}$.

A.2.3. Combinatorial model categories.

Definition A.2.5. A model category $\mathcal{M}$ is called combinatorial if it is locally presentable and it is cofibrantly generated *i.e.*:

- There exists a set $\mathcal{I}$ of generating cofibrations, *i.e.* the collection of all cofibrations in $\mathcal{M}$ is the smallest weakly saturated class of morphisms containing $\mathcal{I}$
- There exists a set $\mathcal{J}$ of generating trivial cofibrations, *i.e.* the collection of all trivial cofibrations in $\mathcal{M}$ is the smallest weakly saturated class of morphisms containing $\mathcal{J}$.

A combinatorial model structure is uniquely determined by the generating cofibrations and generating trivial cofibrations. In [**Lur09**, §A.2.6] the definition is reformulated in order to emphasize in the class of weak equivalences which are easier to describe. More concretely they prove the following proposition.

Recall that given a presentable category and κ a regular cardinal, it is said that full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is *κ-accessible subcategory* of $\mathcal{C}$ if satisfies the following conditions:

(1) $\mathcal{C}_0$ is stable under κ-filtered colimits.
(2) There exists a small subset of objects of C_0 which generates C_0 under κ-filtered colimits.

If the subcategory $\mathcal{C}_0$ satisfies (1), this second condition is equivalent to say that:

(2_τ) Let A be a τ-filtered partially ordered set and $\{X_\alpha\}_{\alpha \in A}$ a diagram of τ-compact objects of $\mathcal{C}$ indexed by A. For every κ-filtered subset $B \subset A$ we let X_B denote the (κ-filtered) colimit of the diagram $\{X_\alpha\}_{\alpha \in B}$. Furthermore suppose that X_A belongs to $\mathcal{C}_0$. Then for every τ-small subset $C \subset A$, there exists a τ-small κ-filtered subset $C \subset B \subset A$, such that X_B belongs to C_0.

Remark A.2. This characterization gives the immediate consequence that for every κ-filtered colimit preserving functor $f : \mathcal{C} \to \mathcal{D}$ between presentable categories and let $\mathcal{D}_0 \subset \mathcal{D}$ be a κ-accessible subcategory. Then $f^{-1}(\mathcal{D}_0)$ is a κ-accessible subcategory.

Proposition A.3. *[Bousfield, Smith, Lurie] Let $\mathcal{M}$ be a presentable category endowed with a model structure. Assume that there exists an small set which generates the collection of cofibrations in $\mathcal{C}$. Then the following are equivalent:*

(1) The model category $\mathcal{M}$ is combinatorial.
(2) The collection of weak equivalences in $\mathcal{M}$ determines an accessible subcategory if $\mathcal{M}^{[1]}$ (the category of morphisms on $\mathcal{M}$).

Proof. This is proved in [**Lur09**, Corollary A.2.6.9]. □

Observe that the subcategory of weak equivalences $W \subset \mathcal{M}^{[1]}$ in a combinatorial model category $\mathcal{M}$ is an accessible category. Then we have the following proposition:

Theorem A.4. *(Bousfield, Smith, Lurie) Let S be a class of morphisms in a combinatorial model category $\mathcal{C}$ with corresponding full subcategory $\mathcal{C}^0 \subset \mathcal{C}$ of S-local objects. Then the following conditions are equivalent:*

(1) $\mathcal{C}^0 \subset \mathcal{C}$ is a colocalization and C^0 presentable.
(2) $\mathcal{C}^0 \subset \mathcal{C}$ is a colocalization and the inclusion preserves κ-filtered colimits for some regular cardinal κ.
(3) There exists a small set $S^0 \subset S$ such that an object in $\mathcal{C}$ is S-local precisely if it is S^0-local, equivalently $\bar{S^0} \subset \bar{S}$.
(4) There exists a colimit preserving functor $F : \mathcal{C} \to \mathcal{D}$ to a combinatorial model category $\mathcal{D}$ such that $\bar{S}$ consist of those morphisms which are sent to equivalences by F.

Remark A.5. If we can construct a functor $F : \mathcal{C} \to \mathcal{D}$ which satisfies the conditions of the situation (4) then we can guarantee the existence of the Bousfield localization, but we can not characterize the local objects in terms of such functor.

For doing that lets recall the following proposition from [**Lur09**]:

Theorem A.6. *Let $\mathcal{M}$ a locally presentable category and let W and C be classes of morphisms in $\mathcal{M}$ with the following properties:*

(1) The collection C is a weakly saturated class of morphisms of $\mathcal{M}$ of morphisms of $\mathcal{M}$, and there exists a small subset $C_0 \subset C$ which generates C as a weakly saturated class of morphisms.
(2) The intersection $C \cap W$ is a weakly saturated class of morphisms of $\mathcal{M}$
(3) The full subcategory $W \in \mathcal{M}^{[1]}$ is an accessible subcategory of $\mathcal{M}^{[1]}$.
(4) The class W has the two-out-of-three property.
(5) If f is a morphism in $\mathcal{M}$ which has the right lifting property with respect to each element of C, then $f \in W$.

Then $\mathcal{M}$ admits a combinatorial model structure, where the weak equivalences are the elements of C and the weak equivalences in $\mathcal{M}$ are the elements of W and a morphism is a fibration if and only if it has the right lifting property with respect to every morphisms in $C \cap W$.

Proof. [**Lur09**, Proposition A.2.6.8] □

Remark A.7. The theorem A.6 is useful to create new model structures on a locally presentable categories. As is pointed out in [**Rap13**] this theorem does not assume have a given explicit set of generating trivial cofibrations but the existence depends on the accessibility of the class of weak equivalences. Usually, the condition of the accessibility of the weak equivalences is not that easy to verify but for our purposes the following proposition will be useful.

Proposition A.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be an accessible functor and D' an accessible and accessible embedding subcategory of $\mathcal{D}$. Then $F^{-1}(\mathcal{D}')$ is an accessible and accessible embedding subcategory of $\mathcal{C}$.*

Proof. [**Lur09**, Remark 2.50] □

Remark A.9. In particular if $\mathcal{D}$ is a combinatorial model category and let $\mathcal{D}^{[1]}$ be the category of morphisms in $\mathcal{D}$, then by [**Lur09**, Corollary A.2.6.6] the full subcategory spanned by the weak equivalences $W \in A^{[1]}$, the full subcategory spanned by the fibrations $F \in A^{[1]}$ and $F \cup W$ are accessible subcategories of $\mathcal{D}^{[1]}$. By [**AR94**, Proposition 2.23] each left or right adjoint between accessible categories is an accessible functor, this will be useful to induce model structures by left or right adjoint.

Definition A.2.6. Let $\mathcal{M}$ and $\mathcal{M}_{loc}$ be two model structures in the same underlying category. We say that $\mathcal{M}_{loc}$ is a *left Bousfield localization* of $\mathcal{M}$, if the following conditions are satisfied:

- A morphism f is a cofibration in $\mathcal{M}$ if and only if f is a cofibration in $\mathcal{M}_{loc}$.
- If a morphism f is a weak equivalence on $\mathcal{M}$, then f is a weak equivalence in $\mathcal{M}_{loc}$.

Proposition A.10. *Let $\mathcal{M}$ be a left proper combinatorial simplicial model category. Then, every combinatorial Bousfield localization of $\mathcal{M}$ has the form $S^{-1}\mathcal{M}$, where S is some small set of cofibrations in $\mathcal{M}$.*

PROOF. This proof is giving in [**Lur09**, Proposition A.3.7.4]. □

A.3. Diagram Categories and Homotopy Limits and Colimits

DEFINITION A.3.1. Let $\mathcal{I}$ be an small category and $\mathcal{M}$ a combinatorial model category. We will that a natural transformation $\alpha : F \to G$ is:

- a *level-wise weak equivalence* if $f(C) : \mathcal{X}(C) \to \mathcal{Y}(C)$ is a weak equivalence in $\mathcal{M}$ for each $C \in \mathcal{I}$.
- an *injective cofibration* if $\alpha(C) : F(C) \to G(C)$ is a cofibration in $\mathcal{M}$ for each $C \in \mathcal{I}$.
- a *projective fibration* if $\alpha(C) : F(C) \to G(C)$ is a fibration in $\mathcal{M}$ for each $C \in \mathcal{I}$.
- an *injective fibration* if it has the right lifting property with respect to every morphism $\alpha \in \mathrm{Fun}(\mathcal{I}, \mathcal{M})$ which that is both a level weak equivalence and an injective cofibration.
- a *projective cofibration* it it has the left lifting property with respect to every morphism $\alpha \in \mathrm{Fun}(\mathcal{I}, \mathcal{M})$ which is both a level weak equivalence and a projective fibration.

Proposition A.11. *Let $\mathcal{I}$ be a small category and $\mathcal{M}$ a combinatorial model category. There exist two combinatorial model structures on Fun$(\mathcal{I}, \mathcal{M})$.*

- *The projective model structure determined by the level-wise weak equivalences, projective fibrations and projective cofibrations.*
- *The injective model structure determined by the level-wise weak equivalence, injective cofibrations, injective fibrations.*

PROOF. This is proved in [**Lur09**, Proposition A.2.8.2]. □

Remark A.12.

- Let $\mathcal{I}$ be an essentially small category and $\mathcal{M}$ a combinatorial model category. Then the proposition above is valid for the functor category $\mathrm{Fun}(I, M)$ [**Lur09**, Proposition A.2.8.2]. These model structures are useful to define *homotopy limits* and *homotopy colimits.*
- It follows from the definitions that the class of projective cofibrations is contained in the class of injective cofibrations and dually the class of injective fibrations is contained in the class of projective fibrations. Then it induces a Quillen adjunction given by the identity maps.

$$id\colon \mathrm{Fun}(\mathcal{I}, \mathcal{M})_{proj} \leftrightarrows \mathrm{Fun}(\mathcal{I}, \mathcal{M})_{inj} \colon id,$$

  which is a Quillen equivalent because both model structures have the same weak equivalences.
- Given a Quillen adjunction between combinatorial model structures $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ and a small category $\mathcal{I}$ the adjunction induced in the categories of functors $F^{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{M}) \leftrightarrows \mathrm{Fun}(\mathcal{I}, \mathcal{N})\colon \mathrm{G}^{\mathcal{I}}$ is a Quillen adjunction with respect to either the projective or the injective model structures. Furthermore if the (F, G) is a Quillen equivalence, then so is $(F^{\mathcal{I}}, G^{\mathcal{I}})$.
- Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between small categories. Then the composition with f induces a pullback functor $f^*\colon \mathrm{Fun}(\mathcal{I}, \mathcal{M}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{M})$. Since $\mathcal{M}$ admits small limits and colimits, f^* has a right adjoint, which is denoted as f_*, and a left adjoint which is denoted as $f_!$.

Proposition A.13. *Let $\mathcal{M}$ be a combinatorial model category and $f\colon \mathcal{I} \to \mathcal{J}$ functor between small categories. Then*

- *The pair $(f_!, f^*)$ is a Quillen adjunction between the projective model structures on Fun$(\mathcal{I}, \mathcal{M})$ and Fun$(\mathcal{J}, \mathcal{M})$.*
- *The pair (f^*, f_*) is a Quillen adjunction between the injective model structures on Fun$(\mathcal{I}, \mathcal{M})$ and Fun$(\mathcal{J}, \mathcal{M})$.*

PROOF. This follows from the fact that f^* preserves level-wise weak equivalences, projective fibrations and injective cofibrations. □

A.3.1. Homotopy limits and homotopy colimits. Let $[0]$ be the category with one object and one identity morphism. Let $f\colon \mathcal{I} \to [0]$ be the unique functor and $\mathcal{M}$ a category which admits small limits and colimits. the pullback functor $\Delta := f^*\colon \mathcal{M} \to \mathrm{Fun}(\mathcal{I}, \mathcal{M})$, which sends every object $m \in \mathcal{M}$ to the constant functor admits left and right adjoint.

Definition A.3.2. The right adjoint to Δ is called the *limit functor* $\lim_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{M})_{inj} \to \mathcal{M}$ and the left adjoint is called the *colimit functor* $\mathrm{colim}_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{M})_{proj} \to \mathcal{M}$

Remark A.14. By the Proposition A.13 the pair of adjoint functors $\mathrm{colim}_I\colon \mathrm{Fun}(\mathcal{I}, \mathcal{M})_{proj} \leftrightarrows \mathcal{M}\colon \Delta$ and $\Delta\colon \mathcal{M} \leftrightarrows \mathrm{Fun}(\mathcal{I}, \mathcal{M})_{inj}\colon \lim_I$ are Quillen pairs.

Definition A.3.3. Let $\mathcal{I}$ be an small category and $\mathcal{M}$ a combinatorial model category. The *homotopy limit functor* is the right derived functor $\mathbf{R}\lim_{\mathcal{I}}$ and the *homotopy colimit functor* is the left derived functor $\mathbf{L}\mathrm{colim}_I$

Definition A.3.4. A monoidal model structure on a closed symmetric monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ is model stricture on $\mathcal{M}$ such that the following properties are fulfilled.

- *Push-out product axiom.* For every pair of cofibrations $i : A \hookrightarrow B$ and $j : C \hookrightarrow D$ their push-out product
$$i \square j : B \otimes C \coprod_{A \otimes C} A \otimes D \to B \otimes D$$
is also a cofibration. If in addition one of the former morphisms is a *weak-equivalence* , so is the later morphism.
- *Unit Axiom.* For every cofibrant object A and for a cofibrant replacement $Q(\{*\})$, the induced map $Q(\{*\}) \otimes A \to \{*\} \otimes A$ is a weak equivalence.

Remark A.15.
This is equivalent to saying that for every symmetric monoidal model category and a cofibrant object A the adjunction $(- \otimes A, \underline{\mathrm{hom}}(A, -))$ is a Quillen adjunction.

Bibliography

[AF14] Aravind Asok and Jean Fasel. A cohomological classification of vector bundles on smooth affine threefolds. *Duke Math. J.*, 163(14):2561–2601, 2014.

[AHW17] Aravind Asok, Marc Hoyois, and Matthias Wendt. Affine representability results in ∂^1 -homotopy theory, i: Vector bundles. *Duke Math. J.*, 166(10):1923–1953, 07 2017.

[AR94] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories.* London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[Bar74] Michael Barr. Coalgebras over a commutative ring. *Journal of Algebra*, 32(3):600 – 610, 1974.

[Bek00] Tibor Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.

[BG76] A. K. Bousfield and V. K. A. M. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 8(179):ix+94, 1976.

[BW85] Michael Barr and Charles Wells. *Toposes, triples and theories.* Springer, New York [u.a.], 1985.

[CD09] Denis-Charles Cisinski and Frédéric Déglise. Triangulated categories of mixed motives. *arXiv e-prints*, page arXiv:0912.2110, Dec 2009.

[Day70] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, Lecture Notes in Mathematics, Vol. 137, pages 1–38. Springer, Berlin, 1970.

[DHI04] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. *Math. Proc. Cambridge Philos. Soc.*, 136(1):9–51, 2004.

[GJ98] P. G. Goerss and J. F. Jardine. Localization theories for simplicial presheaves. *Canadian Journal of Mathematics*, 50(5):1048?1089, 1998.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Birkhuser, Basel [u.a.], 1999.

[Goe95a] Paul G. Goerss. Homotopy fixed points for Galois groups. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 187–224. Amer. Math. Soc., Providence, RI, 1995.

[Goe95b] Paul G. Goerss. Simplicial chains over a field and p-local homotopy theory. *Math. Z.*, 220(4):523–544, 1995.

[IK86] Geun Bin Im and G. M. Kelly. A universal property of the convolution monoidal structure. *J. Pure Appl. Algebra*, 43(1):75–88, 1986.

[Isa04] Daniel C. Isaksen. Etale realization on the a1-homotopy theory of schemes. *Advances in Mathematics*, 184(1):37 – 63, 2004.

[Jar87] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[Jar10] J.F. Jardine. *Generalized Etale Cohomology Theories.* Modern Birkhäuser Classics. Springer Basel, 2010.

[Lur09] J. Lurie. *Higher Topos Theory (AM-170).* Annals of Mathematics Studies. Princeton University Press, 2009.

[ML98] Saunders Mac Lane. *Categories for the working mathematician.* Springer, New York, NY [u.a.], 2. ed. edition, 1998.

[Mor12] Fabien Morel. *$\mathbb{A}^1$-Algebraic topology over a field.* Springer, Berlin [u.a.], 2012.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, page 45–143 (2001), 1999.

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs.* American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[Nik16] Thomas Nikolaus. Rational and p-adic homotopy type, 2016. URL: `http://people.mpim-bonn.mpg.de/thoni/Papers/Padic.pdf`.

[Qui69] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[Rap13] George Raptis. Simplicial presheaves of coalgebras. *Algebr. Geom. Topol.*, 13(4):1967–2000, 2013.

[Sul77] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.

[SV96] Andrei Suslin and Vladimir Voevodsky. Singular homology of abstract algebraic varieties. *Invent. Math.*, 123(1):61–94, 1996.

[SV00] Andrei Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[Swe69] M.E. Sweedler. *Hopf algebras.* Mathematics lecture note series. W. A. Benjamin, 1969.

[VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2000.