PEER INSTRUCTION IN ELEMENTARY MATHEMATICS WITH THE PYTHAGOREAN THEOREM

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Peer instruction is an active learning method which was primarily invented by Eric Mazur for the needs of university level physics. The effectiveness of this method stands primarily on the group discussion that was raised by the conceptual question of the so-called ConcepTest. In this paper we will introduce peer instruction and then we will see how to implement this strategy in the teaching of elementary level mathematics using the Pythagorean Theorem. The following text will be also supplemented by examples of specific ConcepTests and it will be accompanied by statements of pupils about peer instruction.

Keywords: peer instruction, Pythagorean Theorem, Socrative, mathematical concepts

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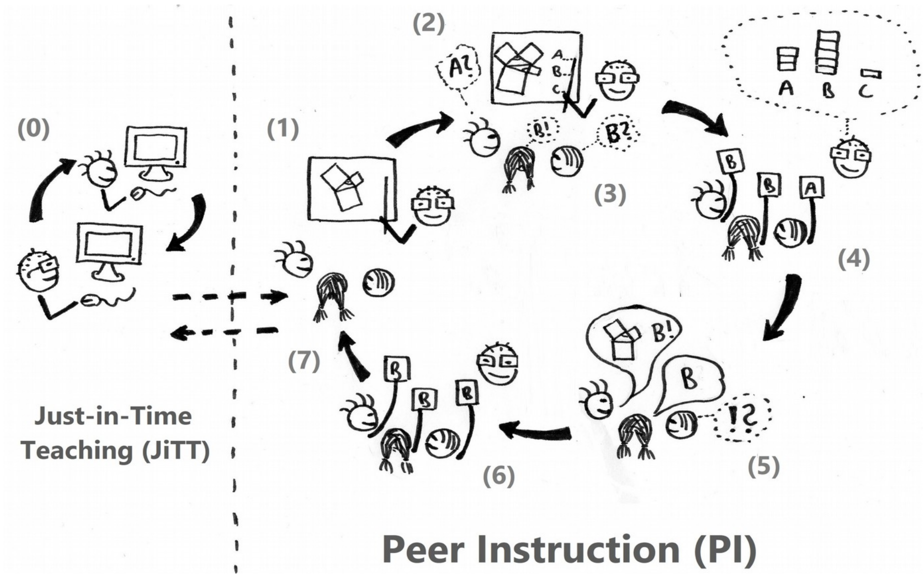


Figure 1. One block of peer instruction with just-in-time teaching

Please note in the introduction that most of the presented material and data was tested and collected in a class of 8th grade students at a Czech grammar school.

PEER INSTRUCTION

In 1984 Eric Mazur started teaching Harvard's introductory physics courses for medics. His lectures were evaluated quiete positively, and his students achieved very good results in terms of classic tests and exams. Based on these indications Mazur considered himself to be a very good lecturer. However, after about seven years of “successful teaching,” he read the article from Hestenes and Hallouin referring to introductory physics courses. The courses change practically nothing on students’ input misconceptions about Newtonian mechanics. Mazur's first response to the article was simply a statement: “Not my students – not Harvard's students!” However, as a scientist he knew he needed data to assert his claim. Therefore, students were given a simple test which was discussed in the mentioned article (this test is known as Force Concept Inventory Test – FCI). The test was aimed at the conceptual understanding of three of Newton's laws. The results he received shocked him completely. Some students succeeded little better than a gorilla randomly pressing the keys on a keyboard. This finding led Mazur to change his teaching approach completely and he developed peer instruction (Mazur, 1997).

The lessons taught by peer instruction are usually divided into several blocks. The schematic structure of one such block can be seen in *Figure 1*. Each block starts with a short presentation of the selected concept (1). In his presentation an instructor tries to avoid formulas or other mnemonics that mislead students from the true meaning. After the presentation, the instructor provides a ConcepTest aimed at deepening the understanding of the presented concept (2). Students are given a short amount of time to think individually. Subsequently they are called to vote by voting cards, clickers or smart devices (3). Based on the distribution of pupils' responses, the instructor briefly explains the correct answer (more than 70% for the correct answer), tries to explain the problem once more (less than 30% for the correct answer), or goes to a group discussion phase (between 30% and 70% for the correct answer). At the stage of group discussions, students try to persuade their colleagues about the correctness of their answers, and they are encouraged by the instructor to justify them – not just make mere statements (4).

Research shows that a student is often able to understand the concept more easily through his classmates' interpretations than from his instructor's interpretations. Students who have understood the discussed concept remember the obstacles they had to overcome and the steps they had to make. On the other hand, the instructor often suffers from the so-called "curse of knowledge" because he understands the discussed concept very well and he is no longer able to see students' difficulties. Group discussions end with a revised student vote (5) and a brief explanation of the correct answer (6). There will usually be a significant increase in votes in favour of the right answer (Mazur, 1997; Vickrey et al., 2015).

The described block takes approximately 10 to 15 minutes. We are able to discuss three to four concepts during the 45-minute lesson. Therefore, it is obvious that in order to achieve the same amount of curriculum in the classic classwork design, we have to place some work on the students. For example, we can do this by submitting preparatory self-study materials before the lessons. After the lessons, the students will have the necessary knowledge to master them (0). In his book, Eric Mazur (1997) recommends a Just-in-Time Teaching strategy. Just-in-time Teaching is a feedback strategy based on a feedback loop between the online preparation environment and follow-up in the classroom. In short, the instructor provides preparatory materials to the students via the Internet. Preparatory materials are accompanied by tasks and questions that students must work on and submit before the beginning of the next lesson. Based on the feedback provided by the students' answers, the instructor will appropriately adjust the content of the next lesson. The instructor will also adjust the content of the preparatory materials that have been adapted to the events of the previous lesson (Novak, 1999).

THEORETICAL FRAMEWORK

Although peer instruction is one of the most surveyed teaching methods (Vickrey, 2015) not many studies were aimed on peer instruction in mathematics neither in elementary school education.

In his study (2001) Scott Pilzer showed that we could teach calculus by peer instruction with similar learning gains (Hake, 1998) which are usually obtained in physics (Mazur, 1997; Vickerey 2015). Specifically, success rates for conceptual questions of Pilzers students taught by peer instruction were in average three times higher than success rates of his students taught classically (54% versus 17%). An experimental group also had a slightly better average of success rates for conventional problems than a control group (73% versus 63%). In addition to mentioned benefits Pilzer also pointed out several difficulties associated with the use of peer instruction in teaching of mathematics. Students usually do not have any preconceptions connected to concepts that are newly introduced to them. Mathematics is also more abstract than physics and included ideas are harder to imagine for students. In other words, it is more difficult for the instructor to prepare appropriate ConcepTests and it is more difficult for students to take their part in group discussions.

Similar results as Pilzers but in university algebra courses were reported by (Teixtera et. al., 2015).

Another study (Weurlander, 2016) showed an improvement in students' attitudes towards calculus due to the use of peer instruction.

A positive attitude of pre-service mathematics teachers towards peer instruction itself was shown in Turkish study (Olpak et. al., 2018). This study also showed significant increase of success rates between pre and post test which was aimed on understanding of statistics and probability.

Again by comparing pre and post test successes rates Kenyan study (Aurah & Ouko, 2015) showed that high school students taught by peer instruction achieved on average considerably higher understanding to vectors than their peers taught classically. In parallel to this was found a positive relationship of participants to peer instruction itself.

In this study (Yu-Fen Chen et. al., 2005) was tested an effectiveness of using peer instruction in teaching of elementary school physics. Parallel to typical benefits it was pointed out that pupils had troubles with visualisations of presented concepts and that they also had a pretty low social skills necessery for group discussions. In other words pupils needed to be helped by an instructor.

Based on the presented review we could clearly say that more studies of implementation of peer instruction in elementary school mathematics are needed.

TWO BASIC WAYS OF VOTING

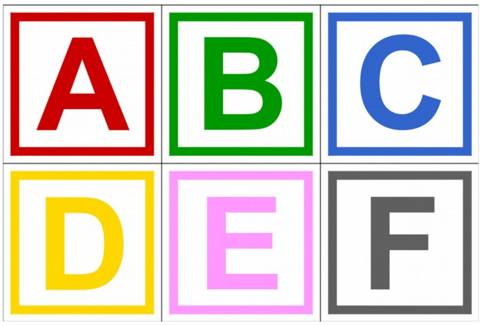


Figure 2. Flash cards

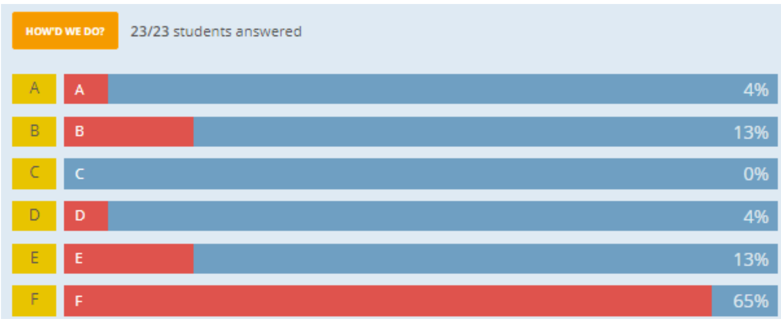


Figure 3. Socratevite - the instructor 's point of view

As we could see in the previous section, voting is really important for peer instruction and there are two ways we can deal with it. We can use simple flash cards (see *Figure 2*) or we can use wireless devices. Because almost everyone has a smartphone, we will choose the application Socrative (see *Figure 3*) as an example.

There were studies (Vickrey et al., 2015) which indicated that peer instruction can be effectively implemented with wireless devices or flash cards without statistical differences in learning gains. Therefore, I have asked my pupils if there is a difference between voting by phone (via Socrative) or by flash cards. My pupils have responded with these statements:

Feedback01: It seems to me that when I'm on the phone it's more anonymous.

Feedback02: We are not supposed to check answers of the others but everyone is so curious…

Feedback03: It is embarrassing for me if my answer isn't correct and anyone can see it because the cards are transparent and the others are always turning around.

Feedback04: It has an influence on me when I see different answers of the others.

Feedback05: The others usually turn to me and try to persuade me if they have a different answer from me.

Based on the pupils' statements, we could say that using a phone for voting is more comfortable because it is more anonymous and they are not affected by their peers' answers. Using Socrative is also really comfortable for the instructor because we can see the distribution of pupils' responses in real time (see *Figure 3*). This application is also really useful for future statistics.

MATHEMATICAL CONCEPTS

Point (2) in *Figure 1* can be a viable according to David Tall's approach to mathematical concepts (Tall & Vinner, 1981). In this approach, there is both the definition of a mathematical concept and its image. Both of these components interact with each other. In a particular situation, a specific part of the concept's image can be evoked. However, the image of the concept may be subject to misconceptions.

For example, if we are to determine the height of the staircase at the known length of the railing, the number of stairs and the known length of one stair will usually recall a right-angled triangle. In other words, we get an evoked image of a right-angle triangle, the Pythagorean Theorem and "the Pythagorean formula for computing lengths in a right-angle triangle: $c^2=a^2+b^2$". Without a deeper understanding of the Pythagorean Theorem, it is possible that the formula could lead us to completely absurd results. We could calculate the length of the hypotenuse instead the length of the leg, or we could simple forget to calculate the root of the obtained result.

As for the ConcepTest, we can work in two planes. We can focus on understanding the particular definition of a mathematical concept or refine the relevant concept's image. In the second of these cases, we purposefully selected or designed questions that could cause pupils to evoke the wrong image of the chosen concept in order to cause a cognitive conflict.

It is also possible to design our ConcepTests to target pupils' ability to apply their knowledge in a non-traditional context.

On a full sheet of paper, perform the steps below.

1. Construct a scalane right-angle triangle in the middle of your paper (hypotenuse down). Label it so that the hypotenuse is AB and the longer leg is BC.
2. Construct a square on each side of the triangle. Label the square on the longer leg $BCDE$. Label the square on the smaller leg $AGFC$. Label the square on the hypotenuse $ABIH$.
3. Locate the center of $BCDE$. Label the point O.
4. Through point O construct line j perpendicular to the hypotenuse.
5. Through point O construct line k parallel to the hypotenuse. Lines j and k divide $BCDE$ into four parts.
6. Cut out the smaller square $AGFC$ and the four parts of square $BCDE$. Arrange them to exactly cover the square $ABIH$ on the hypotenuse.

Figure 4. Group activity

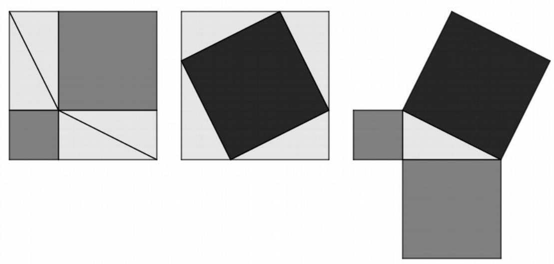


Figure 5. Picture for discussion

TEACHING THE PYTHAGOREAN THEOREM VIA PEER INSTRUCTION

As mentioned in the first section, we have to make a short presentation of a chosen mathematical concept without using formulas or other mnemonics. For the Pythagorean Theorem, we used an activity shown in *Figure 4*. It seems that for pupils who are not used to following a sequence of instructions on their own, it is really hard to accomplish this activity without any help. That is the reason why we usually let pupils work in groups of three or four. After each group is finished, we discuss the outcomes. Since there is just a small probability that two or more groups have drawn the same triangle, it leads us to the formulation of the Pythagorean hypothesis. After the Pythagorean hypothesis, we discuss the picture on *Figure 5* to turn the hypothesis into the Pythagorean Theorem.

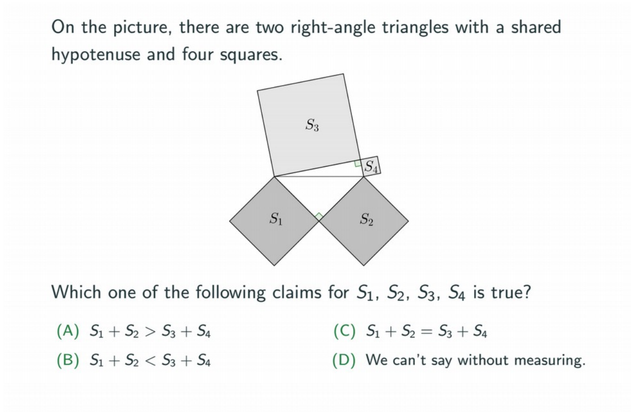


Figure 6. The first ConcepTest

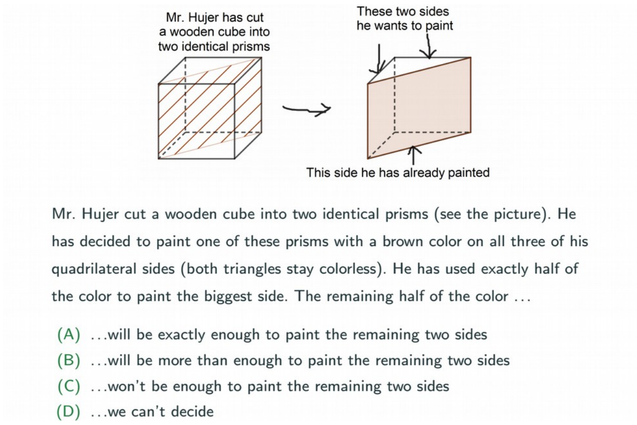


Figure 7. The second ConcepTest

After formulation of the Pythagorean Theorem, it is time for the first and second ConcepTests (see *Figures 6 and 7*). The first ConcepTest usually has really good results (almost 100% for the correct answer (C) at first voting). On the other hand, the second ConcepTest usually has a success rate around 40%–55% at first voting and around 80%–95% at second voting. Pupils know that they need to use the Pythagorean Theorem, but the biggest obstacle is the knowledge that the painted quadrilateral is a rectangle, not a square. There were pupils who used the voting card to demonstrate the fact that the diagonal of the square is longer than its side.

After the first two ConcepTests, there was time for an application question (see *Figure 8*). This question was independently submitted to three groups (see *Figure 9*). The first group S1 was the class of 30 eighth grade pupils. The second group S2 was formed by 18 future mathematics teachers in their first year of university. The third group S3 was formed by 28 university students majoring in mathematics or physics.

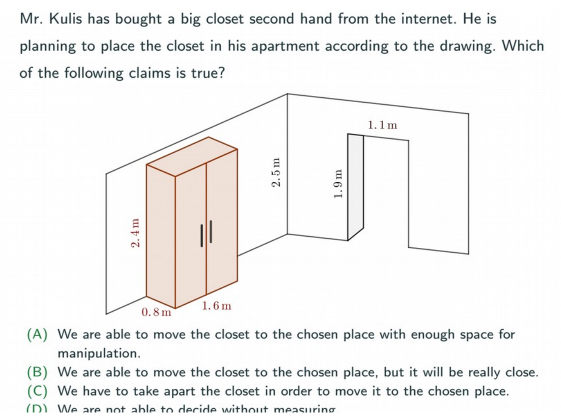


Figure 8. The third ConcepTest

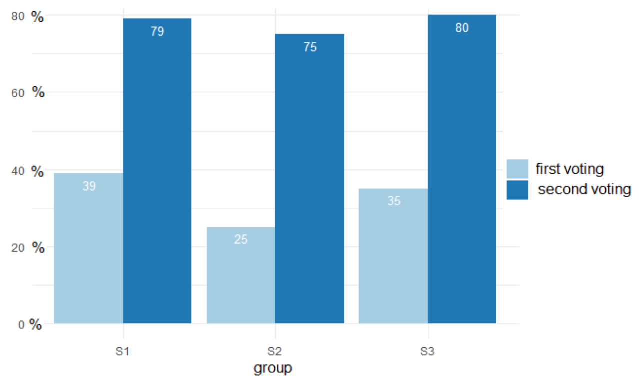


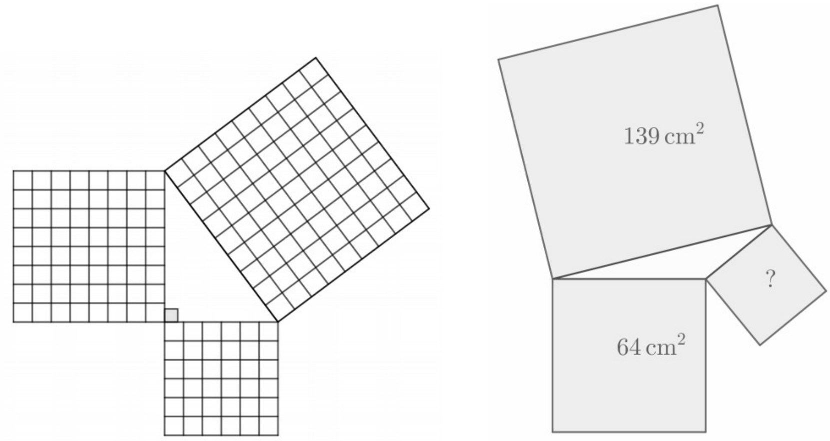
Figure 9. Results of the third ConcepTest

In *Figure 9* we can see different results after the first voting, but really similar results of revised voting following the group discussion. For this particular ConcepTest, the group discussions lead to similar learning gains regardless of the input levels of participating students. The right answer for this question is option (C). After the closet has been moved into the room through the door, we have to stand it up against the ceiling height, which is less than the length of the diagonal of the closet's marginal side. My pupils have given me following feedback about this questiondifficulties:

Feedback07: They kept focusing on that door. It was really hard to tell them it was not in that door, but it was in that ceiling.

Feedback08: As she was talking, she was drawing it. And as I looked at it and then I heard it I figured it out too.

Group discussions about this question have shown that although most of the students have realized it is necessary to lay the closet down in order to move it through the door, more than half of them have not realized it is necessary to stand it up again in the room – which is impossible.



(a) (b)

1. On figure (a) is a right-angle triangle, three big squares and many little squares. Are all the little squares identical? Briefly explain your answer.

2. Decide which of the following statements for $S_?$ on figure (b) is true. Briefly explain your answer.
(A) $S_? < 75\,\text{cm}^2$, (B) $S_? = 75\,\text{cm}^2$, (C) $S_? > 75\,\text{cm}^2$, (D) Can't decide.

3. For right-angle triangle ABC with a right angle at vertex C, calculate the length of side c if $a = 12\,\text{cm}$ and $b = 9\,\text{cm}$.

4. Grandpa David doesn't like pigeons. He lives on the third floor exactly above the entrance door. Pigeons are usually fed by pensioners from a bench that is located at a direct distance of 10 steps from the entrance door. Grandpa David has already decided to solve the pigeon problem with a crossbow. Determine the minimal effective range of the crossbow, which he needs in order to permanently solve this problem.

Figure 10. Post-test

Pupils' understanding of the Pythagorean Theorem was tested approximately one month after finishing the theme in *Figure 10*. The first question had a success rate of approximately 83% (25 out of 30 pupils were able to answer correctly and justify their answer). The second question had a success rate of approximately 67% (20 out of 30 pupils were able to answer correctly and justify their answer). The third question had a success rate of exactly 80% (24 out of 30 pupils were able to answer correctly). The fourth question had a success rate of approximately 73% (22 out of 30 pupils were able to answer correctly). The majority of incorrect solutions on the fourth question originated from an incorrect estimation of the necessary distances. Obtained results clearly imply that most pupils understand the formal part of the Pythagorean Theorem and they are able to adapt the corresponding formula accordingly. A typical misconception related to the Pythagorean Theorem is an inability to transfer the formula from a right-angle triangle ABC with a right angle at vertex C to another right-angle triangle without a right angle at vertex C.

PUPILS' ATTITUDES TOWARDS PEER INSTRUCTION

To find out what popularity and usefulness pupils assign to peer's instructions and other learning activities, a simple questionnaire was submitted. In this questionnaire pupils had to mark 10 (one for each 10 evaluated activities – see *Figure 11*) points on each two 147 millimetres length segments of each line. The first line with the two opposite measures of USELESS and USEFUL was for persuaded usefulness, and the second line with the two opposite measures of UNPOPULAR and POPULAR was for perceived popularity. The results of the questionnaire can be seen in *Figure 11*, where coordinates of each point are the arithmetic averages of the given scales.

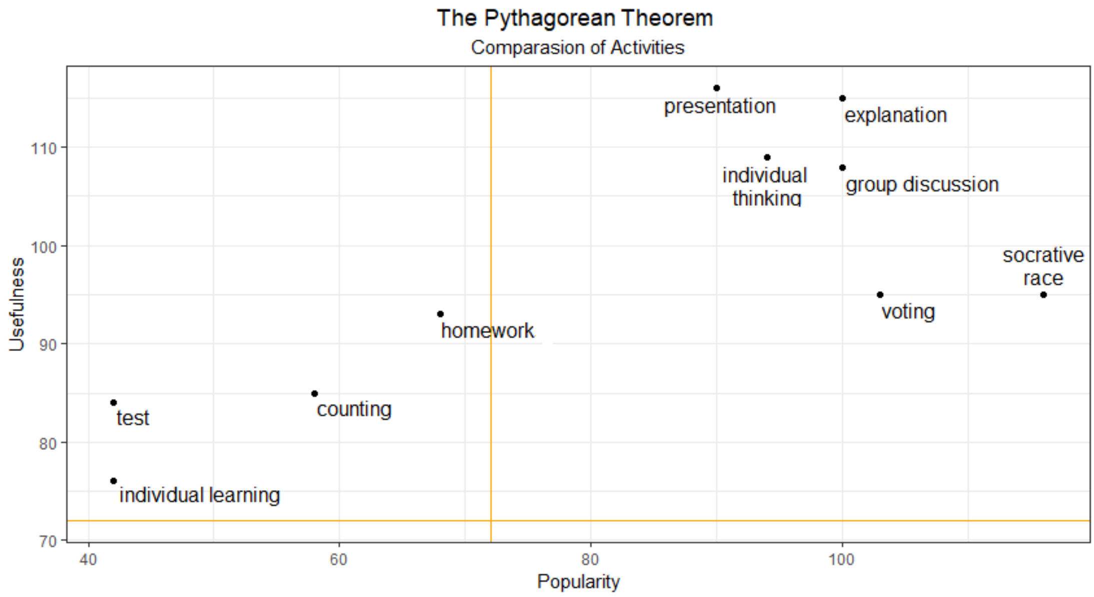


Figure 11. Results of the questionnaire

Based on the chart in *Figure 11*, it can be said that pupils enjoy peer's instruction activities and that they see them as useful in comparison to classic learning activities. Peer's instruction activities have also been discussed with a group of eight chosen pupils. This discussion was recorded on an audio-recorder.

A few pupils' interesting answers are listed below.

Question: What benefits have group discussions provided to you?

Feedback08: We have to think more carefully about what the others say than if it is explained by you because we don't know whether their arguments are true or not. You do it always well and so over your arguments we do not think so much.

Feedback09: I usually remember it longer when it was discussed.

Question: Does it make sense to discuss even easier questions with high success during the first voting?

Feedback08: Well, it makes sense. At least I can confirm that my ideas were good and that I understand it well.

Feedback09: It does because we can practise it on something easier so we can get ready for the harder ones. An athlete will also get warmer before he goes running.

Feedback10: It does make sense. We will not always solve it in the way we should, or we solve it as it happens and we don't know why our answer is right.

CONCLUSION

In the introduction, a peer instruction teaching strategy was presented. It was mentioned that this method is based mainly on conceptual questions, so-called ConcepTests, and group discussions. It has been shown that it is possible to implement peer instruction to teach mathematics in elementary school using the Pythagorean Theorem. It has also been shown that pupils prefer voting with wireless devices over flash cards because cards are less anonymous, leading to higher levels of discomfort. The paper was also guided by examples of particular ConcepTests that were used in real classwork. The last part of the paper was devoted to pupils' attitudes towards peer's instruction activities, especially group discussions. It was shown that these activities are perceived useful and popular, and that the group discussions are perceived as an opportunity to think and verify the truth of their understanding of discussed mathematical concepts. Finally, it is necessary to mention that collections of mathematical ConcepTests at the elementary or high school level are needed.

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