EVALUATING *CAS* AND *DGS* AT THE MATHS CLASSROOM: A PROPOSAL FOR AN UNBIASSED EXPERIMENTAL STUDY ABOUT THE IMPACT OF THE COMPUTATIONAL ROLE OF THE STUDENTS IN THEIR LEARNING

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There are didactic theories about teaching mathematics with Computer Algebra Systems (CAS), like the 'White-Box/Black-Box Principle' or the 'Scaffolding Method'. There are also many experiences on implementing specific mathematical topics with the aid of CAS or Dynamic Geometry Systems (DGS) in the maths classroom and even of courses proposals using these technologies. Nevertheless, we know of no proposal of evaluation of the impact of the computational role of the students in their learning. We therefore propose a design for such an evaluation at Secondary Education with three scenarios. The three scenarios should use the same methods (in order the results to be unbiased): oral presentation, interrogative method and enquiry based learning. The three scenarios incorporate an increasing computational role of the students. We plan to implement it along the second semester of the academic year 2019-2020 at different high schools.

Keywords: Active methodological strategies; computer algebra systems; dynamic geometry systems; educational evaluation.

1 INTRODUCTION: CAS AND DGS

In our opinion, Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS) are the key software for mathematics teaching.

1.1 Some brief notes about CAS

CAS were initially developed for performing calculations in high energy physics and astronomy at the end of the '60s, and were later spread among mathematicians (Wester, 1999). While its adoption at university levels is widespread, its use at Secondary Education level is not very frequent (with exceptions, like Austria). For instance, they are not much more popular at this level in Spain than twenty years ago (Burrel, Cabezas, Roanes-Lozano and Roanes-Macías, 1997).

As CAS use exact arithmetic by default (Figure 1), the user can trust the numerical results obtained (what is very important when the students work without supervision). Moreover, CAS can handle non assigned variables, what allows these systems to deal with polynomial computations, symbolic differentiation and integration, symbolic matrices, etc. For instance, Sarrus' rule in its usual form can be obtained just asking the CAS for the determinant of a 3×3 matrix which elements are a_{ij}, $i=1,\ldots,3$, $j=1..3$ (Figure 2).

```
> 200!;
```

78865786736479050355236321393218506229513597768717326329474253324435944996340334292030428401198462390417721213891963883025764279024263710506192662495282993111346285727076331723739698894392244562145166424025403329186413122742829485327752424240757390324032125740557956866022603190417032406235170085879617892222278962370389737472000000000000000000000000000000000000000000000000000

```
> evalf(%);
```

$$0.7886578674\ 10^{375}$$

```
> 1/7+1/3;
```

$$\frac{10}{21}$$

```
> evalf(%);
```

$$0.4761904762$$

Figure 1: Two examples of computations in exact arithmetic in the CAS Maple[1] and the corresponding values in floating point arithmetic.

```
> with(LinearAlgebra):
> Matrix([seq([seq(x[i,j] , j=1..3)] , i=1..3)]);
```

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

```
> Determinant(%);
```

$$x_{1,1}x_{2,2}x_{3,3} - x_{1,1}x_{2,3}x_{3,2} - x_{1,2}x_{2,1}x_{3,3} + x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,1}x_{3,2} - x_{1,3}x_{2,2}x_{3,1}$$

Figure 2: Obtaining Sarrus' rule in Maple.

1.2 Some brief notes about DGS

Meanwhile, the first DGS were developed at the end of the '80s (Figure 3), but they have become much more popular at Secondary Education level after the release of GeoGebra[2] (that is now really a DGS with CAS capabilities, as the earlier Geometry Expressions[3] –see Figure 4). Nevertheless, its classroom use is many times restricted to function plotting and little more.

The main application of DGS is the exploration of geometry. The user can check results or make guesses just drawing the geometric construction with the mouse and dragging the initial objects (points). In case it holds when dragging, a formal proof is not obtained, but that it holds (at least) in most cases becomes a certainty.

1.3 Some brief notes about CAS and education

The relatively sparse use of CAS and DGS at Secondary Education is surprising, as they cover almost all its curricula.

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[1] Maple is a trademark of Waterloo Maple Inc.
[2] GeoGebra is a trademark of GeoGebra Inc.
[3] Geometry Expressions is a registered trademark of Saltire Software Inc.

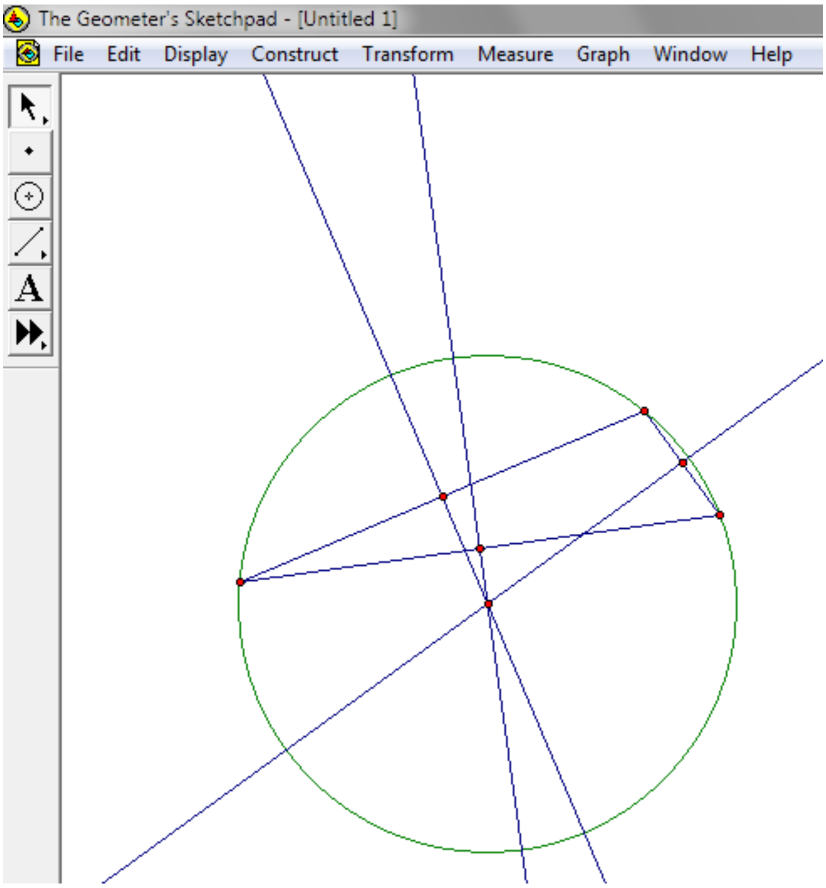

Figure 3: Exploring the circumcentre of a triangle with the DGS The Geometer's Sketchpad[4].

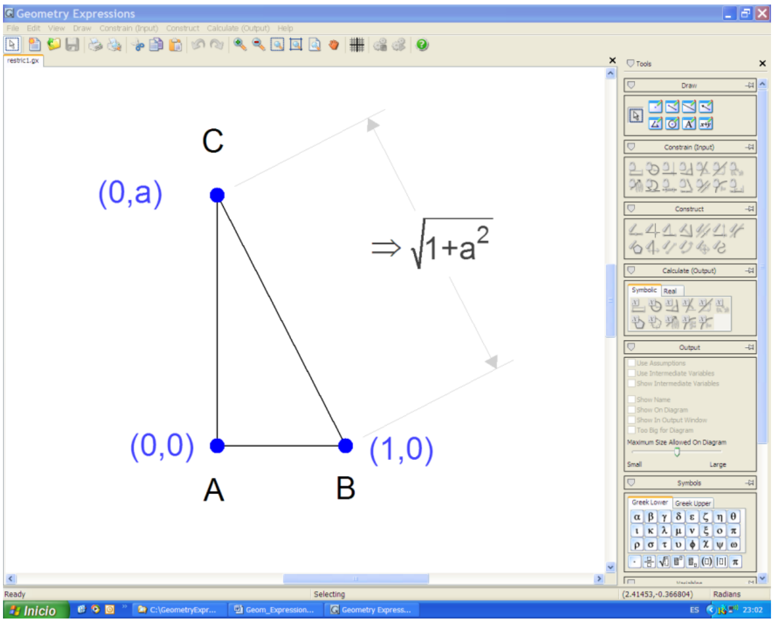

Figure 4: The DGS Geometry Expressions showing how it can handle non assigned variables.

[4] The Geometer's Sketchpad is a registered trademark of Key Curriculum Press.

There are specific didactic theories about teaching mathematics with computer algebra systems (CAS), like:

- the 'White-Box/Black-Box Principle' (Buchberger, 1990; Drijvers, 1995)
- the 'Scaffolding Method' (Kutzler, 1996; Kutzler, 1998),
- the 'Elevator Principle' (Cabezas & Roanes-Lozano, 2015).

There are also many experiences on implementing specific topics with the aid of CAS or DGS in the maths classroom and even of courses proposals using these technologies. Nevertheless, we know of no proposal of evaluation of the impact of the computational role of the students in their learning.

1.4 Some brief notes about DGS and education

The main application of DGS is the exploration of geometry. The user can check results or make guesses just drawing the geometric construction with the mouse and dragging the initial objects (points). In case it holds when dragging, a formal proof is not obtained, but that it holds (at least) in most cases becomes a certainty.

This allows to easily change the classic geometry master class into a class where the students explore possible results (as challenges).

1.5 The spark for this work

Due to the success of this previous research, we have considered that it would be interesting to implement a research that compared different levels of use of technology.

More precisely, we would like to implement a related study that evaluated the impact of the computational role of students of the last years of Secondary Education in their learning. It would be necessary to experiment it in the classroom and to evaluate the impact of applying the same methods with different uses of the technological utensils.

This new research proposal is focused on students of a different level (Secondary Education), with different educational needs and objectives.

2 DESIGN OF THE EXPERIMENTAL STUDY

2.1 General description

The authors have a long experience in educational software development, its implementation in the classroom and continuing education. After using different methodologies supported by technology, we would like to carry out a comparative analysis of the goodness of different levels of use of technology at a certain educational stage.

The idea is to carry out the experience in three scenarios, where the students have different computational roles. The chosen scenarios try to represent the possible levels of use of the technological utensils for mathematics learning, from inexistent to intensive:

- Scenario I: the students don't use technology.
- Scenario II: the students use some previously developed specific purpose resources (simulations). This approach has the 'advantage' that students wouldn't have to learn how to use the computational systems. The approach would be traditional, 'White-Box/Black-Box'.
- Scenario III: the students use the CAS and DGS as 'symbolic calculators' or 'geometric calculators' (what requires some knowledge about the computational tool) and sometimes

even develop small applications. For instance, if we suppose that f(a)=f(b), a Maple procedure that checks Rolle's Theorem for any given function, *f*, is one line of code long!:

```
rolle:=proc(f,a,b)
  fsolve(diff(f,x)=0,x,a,b);
 end:
```

(what can be implemented by students). On the other hand, complex implementations such as Roanes-Lozano (2017b) could be used as simulations. Again, the approach would be traditional, 'White-Box/Black-Box' (in all cases).

2.2 Pedagogical details

The key for developing an unbiased experiment lies, in our opinion, in using the same theory and methodological strategies (Viladot, 2002) in the three scenarios.

We believe that an active learning based on pure discovery, as proposed in Logo microworlds (Papert, 1980), without teacher intervention or clear curricular objectives, slows down the learning process and doesn't focus on optimized goals. A similar point of view can be found in the last lines of the first paragraph of Mayer (2004):

> *In each case, guided discovery was more effective than pure discovery in helping students learn and transfer. Overall, the constructivist view of learning may be best supported by methods of instruction that involve cognitive activity rather than behavioral activity, instructional guidance rather than pure discovery, and curricular focus rather than unstructured exploration.*

Therefore, we propose to choose:

- theory: constructivism, a pedagogic theory that intends the student to develop skills (Perrenoud, 1999),
- methodological strategies: both expository instruction and active teaching, with strong emphasis in the latter.

The expository instruction would use:

- oral presentations (technique: master class)
- interrogative methods (technique: open questions)

in all the three scenarios.

The active teaching would apply the techniques of the method:

- enquiry based learning (EBL)

to mathematical problems in all the three scenarios.

Regarding active teaching, we would also use:

- the simulation method (technique: computer simulations) in Scenarios II and III.

Concerning EBL:

- techniques borrowed from the EBL method would be applied to mathematical problems in Scenarios I, II and III.

Moreover:

– techniques borrowed from the EBL method would be applied to software development in Scenario III.

The situation is clarified in (the non-exhaustive) Tables 1, 2 and 3. These classifications are inspired by Bernal, Fernández-Salinero and Pineda (2019), Fernández-Salinero (2013), Fernández-Salinero (2004).

Scenario I Theory	Methodological strategy	Method	Technique
Constructivism	**Expository Instruction**	**Oral presentation**	**Master class** Round table
		Interrogative method	**Open questions**
		Self learning	Individualized learning
	Demonstrative Teaching	Demonstration	Mentoring Training Within Industry (TWI) Coaching …
	Active Teaching	Simulation	Role playing Computer simulation
		Projects based learning Challenge based learning Problems based learning **Enquiry based learning** Guided discovery …	Design thinking Visual thinking Phillips 66 SCAMPER **applied to:** *** Mathematical problems** * Software development

Table 1. Classification of methodological strategies, methods and techniques (all related to constructivism) corresponding to Scenario I.

Banchi and Bell (2008) introduce a hierarchy of levels in EBL:

- Level 1: *Confirmation Inquiry*. The students confirm (check or prove, according to the case) a known result.
  Example: Draw function $f(x)=1/x$ and check that its one-sided limits at 0 are +/- infinity, respectively.
- Level 2: *Structured Inquiry*. The students investigate about a problem proposed by the teacher in a way proposed by the teacher.

<table>
<tr><th>Scenario II
Theory</th><th>Methodological strategy</th><th>Method</th><th>Technique</th></tr>
<tr><td rowspan="6">Constructivism</td><td rowspan="3">Expository Instruction</td><td>Oral presentation</td><td>Master class
Round table</td></tr>
<tr><td>Interrogative method</td><td>Open questions</td></tr>
<tr><td>Self learning</td><td>Individualized learning</td></tr>
<tr><td>Demonstrative Teaching</td><td>Demonstration</td><td>Mentoring
Training Within Industry (TWI)
Coaching
…</td></tr>
<tr><td rowspan="2">Active Teaching</td><td>Simulation</td><td>Role playing
Computer simulation</td></tr>
<tr><td>Projects based learning
Challenge based learning
Problems based learning
Enquiry based learning
Guided discovery
…</td><td>Design thinking
Visual thinking
Phillips 66
SCAMPER
applied to:
* Mathematical problems
* Software development</td></tr>
</table>

Table 2. Classification of methodological strategies, methods and techniques (all related to constructivism) corresponding to Scenario II.

Example: Napoleon's theorem is presented as a research to be carried out using a DGS. The DGS has to be used to determine the kind of triangle the resulting one is. The teacher gives as clues to use GeoGebra's 'regular polygon tool' and to measure the sides or angles of the resulting triangle.

- Level 3: *Guided Inquiry.* The teacher only proposes the problem. The procedures to be followed in the research are designed or chosen by the students.
  Example: Napoleon's theorem presented as a research to be carried out using a DGS (with no clues).
- Level 4: *Open Inquiry.* The students investigate problems proposed by themselves (using procedures designed or chosen by themselves too).
- Example: the students try to discover (or, more probably, rediscover) geometric theorems with the help of a DGS.

In the experiment proposed, Level 2 (Structured Enquiry) and Level 3 (Guided Enquiry) would be normally used. Level 1 (Confirmation Enquiry) would be sparsely used.

<table>
<tr><th>Scenario III
Theory</th><th>Methodological strategy</th><th>Method</th><th>Technique</th></tr>
<tr><td rowspan="7"><u>Constructivism</u></td><td rowspan="3"><u>Expository Instruction</u></td><td><u>Oral presentation</u></td><td><u>Master class</u>
Round table</td></tr>
<tr><td><u>Interrogative method</u></td><td><u>Open questions</u></td></tr>
<tr><td>Self learning</td><td>Individualized learning</td></tr>
<tr><td>Demonstrative Teaching</td><td>Demonstration</td><td>Mentoring
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…</td></tr>
<tr><td rowspan="2"><u>Active Teaching</u></td><td><u>Simulation</u></td><td>Role playing
<u>Computer simulation</u></td></tr>
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<u>Enquiry based learning</u>
Guided discovery
…</td><td>Design thinking
Visual thinking
Phillips 66
SCAMPER
applied to:
<u>* Mathematical problems</u>
<u>* Software development</u></td></tr>
</table>

Table 3. Classification of methodological strategies, methods and techniques (all related to constructivism) corresponding to Scenario III.

2.3 Computational details

Let us note that the educational digital resources (simulations) of Scenario II (and III) should be developed in advance, prior to the experimentation (as isolated applications of specific purpose, but making use of the power of CAS and DGS, not from a standard programming language).

The authors have a long experience in developing, implementing and using this kind of resources (Roanes-Lozano, 1987; Roanes-Lozano, 1993; Roanes-Lozano, 2017a; Roanes-Lozano, 2017b; Roanes-Macías & Roanes-Lozano, 1992; Roanes-Macías & Roanes-Lozano, 1994; Roanes-Macías & Roanes-Lozano, 2016;). Some of them can be adapted to the present proposal and others will be developed in collaboration with the Secondary School teachers. Moreover, simulations based on the use of CAS could benefit from the GUI detailed in Roanes-Lozano & Hernando (2014).

Which issues could be developed by the students with a CAS or a DGS (Scenario III) should be determined.

2.4 Evaluation of the experience

The last step prior to experimentation would be to design ad hoc evaluation tests on the issues addressed in the three ways.

Subsequently, the results obtained by the students in the math tests designed ad hoc would be evaluated and compared.

The assessment will take place in collaboration with the Secondary School teachers in the research project.

2.5 Implementation if the experience

We plan to implement the experiment during the second semester of the academic year 2019-2020 in Secondary Education centres.

There are Secondary School teachers that collaborate with our university. Among those most experimented, we shall choose some for this study. It will be applied to whole classes without selecting the students. The assessment will take place in collaboration with the Secondary School teachers.

A key issue is that exactly the same hours of face-to-face and homework should be dedicated by students in the three scenarios (what would imply that more time could be spent doing ‘mathematical exercises’ in Scenarios I and II than in Scenario III).

As the research is at a very early stage, the amount of hours in face-to-face and for homework haven’t been detailed yet.

3 CONCLUSSIONS

This is so far only a theoretical development, but we believe that the hypothesis that Scenario III will produce the best learning will be confirmed.

In order to verify that hypothesis we plan to implement the experiment along the second semester of the academic year 2019-2020 at different Secondary Schools in Madrid area.

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