CONSTRAINED SKETCHING ON A GRID: A LENS FOR ONLINE ASSESSMENT OF DERIVATIVE SKETCHING

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In this study we aim to characterize the challenge related to sketching functions by constrained sketching on a grid. Our research question is as follows: What are the characteristics of sketching graphs by dragging points vertically in assessment e-tasks? Specifically, which functionality does this design support? Towards this goal we explore how students construct a sketch of f'(x) based on a given graphic representation of f(x), and vice versa, *how they construct a sketch of a function based on the graphic representation of its derivative. The analysis of 114 submissions of high-school students, support the formation of assessment tasks' design principles for drawing a sketch. The sketch that students construct makes it possible to characterize each submission with respect to the actions the students took at critical or non-critical points; to whether they attend to certain points separately or to domain and to conjecture about various concept images.*

Keywords: automatic, assessment, derivative, design, principles.

INTRODUCTION AND THEORETICAL FRAMEWORK

Multiple representations are often important components of rich tasks. The ability to identify and represent the same element numerically, graphically, and algebraically has been one of the goals of reform in calculus education (Berry & Nyman, 2003). Tasks designed within multiple representation environments have the potential to support the formative assessment of problem-solving processes, to catalyze ideas, and to provide indications of students' perceptions and concept images (as described by Tall & Vinner, 1981). Tracking and studying choices of representation made by students during problem solving are likely to inform us about the students' interests, preferences, and difficulties (Ainsworth 1999). Solving interactive tasks that involve multiple representations supports understanding by the teacher of the student's concept image because it invites students to consider mathematical concepts in relation to their properties (e.g., Even 1990). Interactive diagrams are a primary means of designing such problems (Naftaliev & Yerushalmy 2013).The concept of derivative is one of the main ones in school mathematics. It is an epistemologically and psychologically difficult to understand, and can benefit from work with an especially designed interactive multiple linked representations (MLR) learning environment. Construction e-tasks (Nagari-Haddif & Yerushalmy 2018; Yerushalmy, Nagari-Haddif, & Olsher, 2017) require students to construct examples that satisfy given conditions by the optional use of technological affordances, such as symbolic expressions and sketches. Using the Seeing the Entire Picture (STEP[1]) online assessment platform, in our design of technology-based rich assessment tasks, we provide means for generating examples in multiple representations, to support calculus problem-solving, and to mirror reasoning processes. STEP enables students to submit examples of mathematical objects constructed as interactive diagrams (based on Geogebra). One of our efforts focuses on the design of tasks that require freehand construction of the graph of a function in ways that facilitate automatic analysis of submissions. We found that a design that allows students to choose in which representation to submit the answer, either by freehand sketching or through a symbolic expression that is automatically graphed, offers a mathematically appropriate means of expression. Most often, such design encourages students to start by "sensing" the problem using freehand sketching, and provides informative indication of the mathematical knowledge of the student (Yerushalmy et al., 2017). Accordingly, calculus tasks in

STEP are designed to include several tools that provide different ways of expressing mathematical ideas. This design principle assumes that for many students, symbolic expressions are not the only choice of communication, that it is important to retain the natural communication of mathematical ideas through freehand drawing, and that such tasks increase the ability to make informed assessment decisions about students' work. Sketches, which are less accurate mathematical representations than formal drawings constructed using symbolic expressions, in other words, than a neat graph, are often part of problem solving and can support the analysis of problem-solving reasoning. Free-hand sketching provides students with great flexibility in answering questions (Yerushalmy et al. 2017), and poses challenges when teachers attempt to interpret the sketches as mathematical objects that are part of the expected solution to the task (tangency points, functions, asymptotes, etc.). In the current report we extend our view on sketching functions and focus on constrained sketching on a grid. When drawing a freehand sketch, students might ignore certain points for various reasons, and sketch the entire graph as a curve made up of infinite many points. In the current design, however, students drag a finite number of equally spaced, connected horizontally. In this respect, the design is more restrictive, but it requires students to pay attention to the accuracy of the points, and makes it possible to assess students' perceptions. Figure 1 shows an example of sketching a function by dragging points vertically. The connected red points are placed horizontally at the upper edge of the screen (Figure 1 (a)). To construct a sketch, it is necessary to drag the red points vertically (Figure 1 (b)). (One can imagine a series of beads constrained to move along parallel vertical paths on a Cartesian grid). It is possible to evaluate the locations of the dragged points using the coordinate system and the zoom in and out tools (the coordinates are marked numerically).

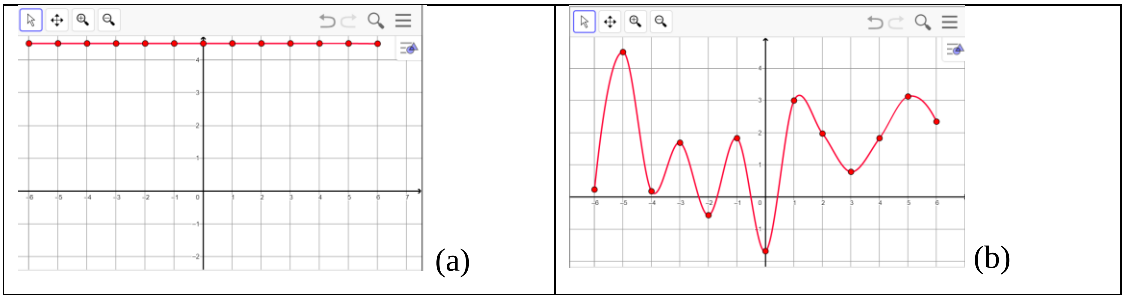


Figure 1. Example of sketching a function by dragging points vertically.

Asking students to construct a sketch by dragging vertically points should allow to learn about their intentions regarding the place of each point (up to a certain deviation).

CONTENT OF THE E-TASK

The relationship between a function and its derivative is a central theme of calculus. High-school students tend to assume that there are partial resemblances between a function and its derivative. These assumptions lead them to ascribe similar global features to both (such as increasing/ decreasing sign), generally focusing on either the function or the derivative rather than on the relation between them (Nemirovsky & Rubin 1992). One of the most useful and instructive applications of derivative is to aid in determining the maximum and minimum values of a function. Problems of optimization and can be formulated as problems of finding the maximum, minimum, and inflection points of a function. Focusing on such significant or critical points of the function can lead to ignoring other non-critical points, which may be a sign of a problem in understanding the derivative as a function that represents a collection of slopes of lines at various points, not necessarily critical ones. This example illustrates why we are interested in formulating design principles for drawing a sketch that **assigns the same weight to critical and non-critical points.** Traditionally, when analyzing function characteristics in calculus, we calculate several critical components: extremum points, intersection points with the axis, increasing and decreasing domains, inflection points, concave upward and downward domains. While sketching, these critical characteristics are preserved. The design of the task that requires creating sketch should allow analyzing and assessing these and other characteristics that students perceive to be important (including mistakes).

RESEARCH GOAL AND QUESTIONS

We aim to characterize the challenge related to constrained sketching on a grid[2]. Our research question is as follows: What are the characteristics of sketching graphs by dragging points vertically in assessment e-tasks? Specifically, which functionality does this design support? Towards this goal we explore how students construct a sketch of f'(x) based on a given graphic representation of f(x), and *vice versa*, how they construct a sketch of a function based on the graphic representation of its derivative. The study seeks to determine whether students follow the (approximately) correct value of critical and non-critical points and analyze other submission characteristics.

METHODOLOGY, RESEARCH TOOLS, AND DATA ANALYSIS

Design-based research (DBR), the methodology for studying the innovative principles of assessment, is characterized by an iterative cycle of design, implementation, analysis, and redesign. The present study is part of a larger research project on the principles of innovative assessment designs in an MLR environment. The data consisted of **several cycles** of students' submissions, each cycle examined a design pattern that was refined and re-examined in the next cycle. Using a DBR methodology, we conducted a study focusing on an activity called "The relationship between the graphs of a function and its derivative." We use this activity as a research tool, consists of three tasks. Data for this report consist of submissions by 114 high school Israeli students aged 16-17, who volunteered to anonymously solve the tasks. All students were enrolled in the most advanced high school calculus course. They all used the same curricular resources, but were taught by different teachers in different schools. They were all conversant with the basic graphing technology. Before the experiment, students participated in a preparatory session to familiarize themselves with the STEP environment. Each student submitted an answer for the first two tasks, and 109 out of the 114 students submitted an answer for the third task. Below we describe each task, its design considerations, and possible solutions. For each task we present findings and data analysis.

TASK 1: EXPERIMENTATION WITH THE MEASURING MECHANISM

Description and design rationale

Task 1 includes a dynamic diagram of function f(x) (Figure 2). The requirement of the task is to drag the red tangency point and place it so that the slope value of the tangent line is approximately 1.5, and to attach an appropriate screenshot. The two possible correct answers are shown in Figure 2.

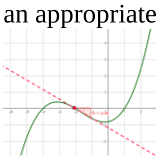
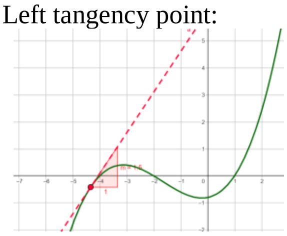
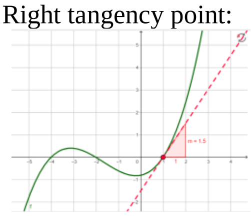
The task:	**Two possible correct answers:**	
The function f(x) is described in the dynamic diagram. Drag the red tangency point and place it so that the slope of the tangent line is approximately 1.5. Attach an appropriate screenshot. 	(a) Left tangency point: 	(b) Right tangency point: 

Figure 2: Task 1 and its possible correct submissions.

Adding grids and the numeric value of the tangent slope (measuring mechanism) to the task convey the message that the required answer needs accuracy. This is a preparatory task aimed mainly to familiarize students with the dynamic diagram that appears also in task 2 (see Nagari-Haddif 2017).

Findings and data analysis

All the students submitted correct answers for task 1. Most students (74%) chose to submit a right tangency point (Figure 2 (b)), which may imply that students have a preference to drag points from left to right. The result raises questions about further research regarding students' general and personal tendency for a specific side of a graph. To answer this question, it is necessary to analyze additional similar tasks, following the process by which students reached their solutions, in addition to their final submissions. Tasks 2 and 3 required submitting a sketch drawn by dragging 13 equidistant points vertically in a closed domain (Figure 1). In task 2, students were asked to produce a sketch that represents the derivative of the given function in the presence of measuring mechanism (Figure 3), and in task 3, they were required sketch a function whose derivative was given in a dynamic diagram (no symbolic expression was provided in any of the three tasks) using approximate measurements (Figure 6).

TASK 2: CONSTRAINED SKETCHING IN THE PRESENCE OF MEASURING MECHANISM

Description and design rationale

In this task, the function f(x) is described in the dynamic diagram (Figure 3) in the domain $-6 \leq x \leq 6$. Students were asked to construct, as accurately as possible, the derivative function by dragging each of the 13 red points vertically to sketch the derivative of f(x). The task supports the requirement for accuracy by providing the numeric value of the tangent slope in each point (Figure 2), by adding the grids, and by marking the scale values in the coordinate system. Sketching the derivative requires dragging each point to the y value measured by the slope tool. The construction is discrete, and therefore it may encourage students to be accurate in the numeric value of each dragged point. The measuring mechanism may help students be more precise. Therefore, incorrect values of the locations of the points in the sketch suggest that students may have difficulty in relating the slope of the function to the derivative function. We placed 13 points in the given domain $-6 \leq x \leq 6$ at intervals of one unit apart: $x = -6, -5 \ldots, 5, 6$. The purpose of this design is to help students calculate the slope between two adjacent points $a, a+1$ as the difference of values of y $(f(a+1) - f(a))$ rather than as the quotient $\frac{\Delta f}{\Delta x}$.

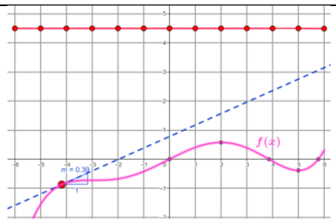
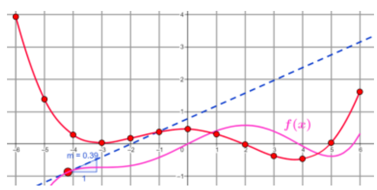
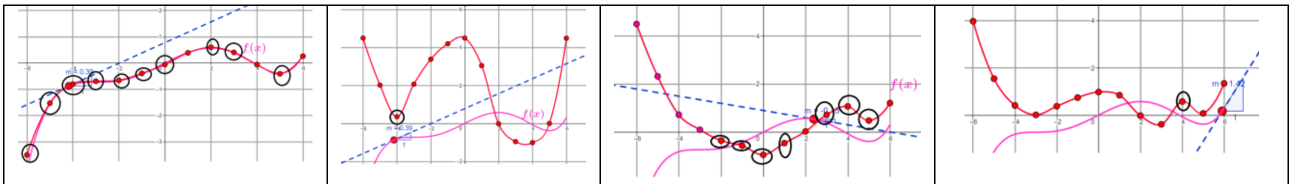
The function f(x) is described in the dynamic diagram (pink). Drag the red points so that the resulting red curve is the derivative of the pink function f(x). Try to be as accurate as possible. Attach an appropriate screenshot.		The correct answer: 

Figure 3. Task 2 and possible correct submissions.

By analyzing the sketches submitted by the students it is expected to characterize students' submissions. We were able to derive the following types of information from the submissions: (a) setting critical points to zero points only suggests that students do not make the connection between points with non-zero slope and the derivative; (b) sketches that represent incorrect values but with the correct sign (negative or positive) suggest that students connect between the increasing (decreasing) domains of the function and the positive (negative) domain of the derivative, but might ignore concavity upward and downward of the function domain; (c) sketches that follow correct increasing and decreasing domains but contain mistakes in values suggest that students make the connection between the concavity upward and downward of the function domain, and the increasing and decreasing domains of the derivative. These characteristics can be analyzed automatically.

Findings and data analysis

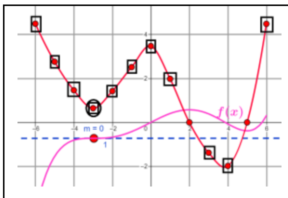
Findings and data analysis: The vast majority of the submissions have the correct signs (see Figure 4 (b), (d)). This is not surprising, because the relationship between the function and its derivative is studied in Israeli schools, with especial emphasis on the relationship between increasing (decreasing) domains of f(x) and positive (or negative) domains of f'(x). Most students knew that a critical point of f(x) is a zero point of f'(x): 93 (81.6%) students submitted a zero point at x=-3, which is an inflection point of f(x) with a zero slope (see Figure 5 (BON53)); 107 (93.9%) students submitted a zero point at x=2, which is a maximum point of f(x) (see Figure 5 (BON53, DALII20)); 95 (83.3%) students submitted a zero point at x=5, which is a minimum point of f(x) (see Figure 5 (BON53, DALII20)). Most of the students identified the correct sign, and although they could deviate by 0.3 from the exact value of each non-critical point, fewer students by far were able to find its correct value. For example, students BON53, DALII20 (Figure 5) submitted sketches with correct sign of 10 non-critical points (out of 10) and incorrect values of f'(x).



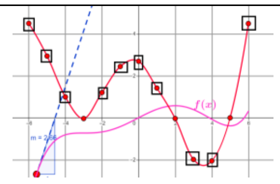
(a) IRIN5 | (b) BON29 | (c) GH8 | (d) GH30

Figure 4. Task 2: Examples of submissions with incorrect signs of the y value (circled in black).

From the perspective of a single submission, most students (84 students, 74%) submitted 9-13 correct y values per submission. The students DALII20, BON53 (Figure 5) submitted correct signs for most dragged points, but the values of these points were not precise at all. Their explanations reinforce the impression that the mistakes are not incidental. Students appear to have recalled a "recipe" regarding to the connection between the sign of the derivative and the increasing or decreasing domain. When focusing on each point separately, however, without referring to other points, it is possible to miss other important characteristics of the solution, such as the increasing and decreasing domains of the derivative function (which are affected by the concaving upward and downward domains of the original function, and vice versa). Students' mistakes in the value or sign of any point may affect the correctness of the increasing and decreasing domains of the derivative function. Student GH8 (Figure 4) appears to have ignored the increasing and decreasing domains of the functions. Student IRIN5 (Figure 4 (a)) constructed a sketch that follows the original graph of f(x) instead of constructing f'(x). This result is consistent with the findings of Nemirovsky and Rubin (1992), according to which students tend to assume resemblances between the behaviour or appearance of a function and its derivative. In submissions with only one or two incorrect signs, in some cases the shape of the constructed derivative function is similar to the correct shape (for example, the sketch of Figure 4 (b)), but in other cases it is completely different (Figure 4 (d),). Generally, students constructed a sketch that decreases in the $-6<x<-3$ domain (94.7%), increases in the $-3<x<0$ (88.6%) domain, decreases in the $0<x<4$ domain (83.3%), and increases in the $4<x<6$ domain (86.8%). Other students submitted decreasing domains instead of increasing ones, increasing domains instead of decreasing ones, combined domains instead of decreasing/increasing ones (GH30 at 0<x<4, 4<x<6, Figure 4).



DALII20: 11 dragged points with incorrect values, 1 dragged point with incorrect sign.



BON53: 10 dragged points with incorrect values, 0 dragged points with incorrect sign

Figure 5. Task 2: Examples of submissions. Dragged points with an incorrect sign for the y value are circled, and those with incorrect y values are marked by a square

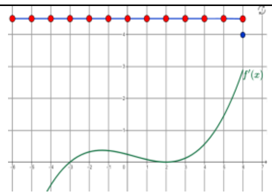
In sum, most students submitted a fairly accurate graph, with almost all points dragged to the correct place of the coordinate system. This confirms that in some cases we can expect accuracy, and students can sketch relatively accurately. Some students, however, do not appear to have been concerned with the precise value of the derivative, but only with its sign. They did not seem to have perceived the tangent as a construct that can help them measure the location of the points and is useful for defining the derivative function in any given point on the graph (where it was provided). These students referred only to the critical points, and in non-critical points, only to the sign of the derivative, although their slope tool allowed them to read the slope at any point, and although they had available a coordinate system with marked ticks that provided accuracy. This strategy resulted in different shapes of the derivative sketch, including incorrectly increasing or decreasing the domains of the derivative function (for example Figure 4 (b), (d)). Although it is possible to construct the sketch as a collection of precise points, these points should not be analyzed only separately. To assess additional characteristics of students' submissions, such as increasing or decreasing domains, the points should be analyzed in relation to all the other points (because if one point is out of place, it changes the shape of the entire graph).

TASK 3: CONSTRAINED SKETCHING USING APPROXIMATE MEASUREMENTS

In this task, students were asked to drag the red points vertically to sketch the function f(x) that passes through the blue point (6,4), and whose derivative f'(x) is described as the green graph in the dynamic diagram (Figure 6). They are asked to be accurate as much as possible. We support his requirement for accuracy by adding the grids and by marking the scale values in the coordinate system. Because the expression of f' (x) was hidden from the students' eyes, we cannot expect the same accuracy as in task 2. Indeed, we expected students to construct the function f(x) based on the values of the derivative function at each of the 13 points. Therefore, the approximate derivative should be based on the "neighbors" of each point. To avoid analyzing errors resulting from technical inaccuracies, we defined a more lenient requirements for the endpoints: (a) for x=6, the slope is bigger than 2: f(6)-f(5)>2; (b) for x=-6, the slope is smaller than -2.5: f(-5)-f(-6)<-2.5. By analyzing the submitted sketches, we expected to distinguish between students who paid attention to (a) domains (the slopes of each endpoints); (b) critical points (x=0,2,-3); (c) domains for which the function is increasing and decreasing; and (d) concaving upward and downward.

The task:

With reference to function f'(x) (in green), drag the red points so that the red curve produces function f(x), which passes through the blue point (6,4). Attach an appropriate screenshot.



Possible correct answer

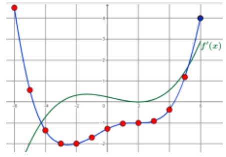


Figure 6. Task 3 and its possible correct answer.

Findings and data analysis: A hundred and nine out of the 114 participants submitted an answer for this task. Some students mistakenly focused on the zero points of the derivative and interpreted them as extremum points, apparently arbitrarily (Figure 7). An unusual and surprising finding, which raises some concern, was that about 10 students (9%) interpreted the intersection point of the derivative with the y axis as an extremum point (Figure 7 (c), (d)). This finding is particularly worrisome because there is no connection between the point of intersection of the derivative with the y-axis and the function, and it may indicate a real difficulty students have understanding the basic meaning of a slope of a function and its relation to the first derivative. Some students submitted sketches with a W shape, which had at least two extrema with similar y values (Figure 7 (c)). This may explain some of the submissions in which an extremum point occurs at x=0. All the W-shaped sketches had an extremum point at x=0.

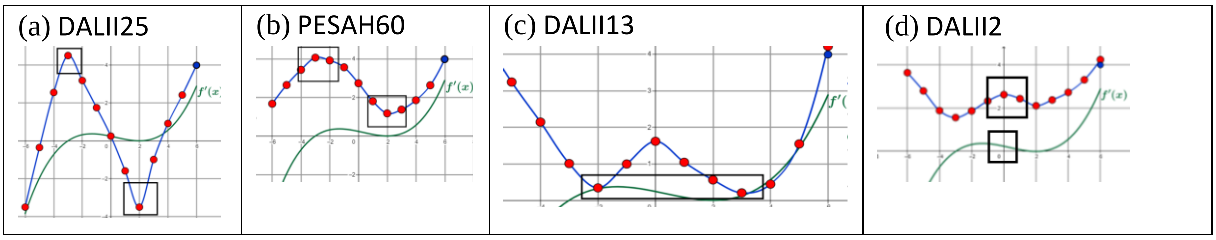


Figure 7. Sample of submissions.

In sum, in this task, dragging the points on a coordinate system with marked ticks that provide accuracy, enabled students to construct a relatively accurate function for the given derivative (albeit less so than in task 2). We analyzed a large number of attributes. Teachers, however, may choose to ignore some of the attributes, such as domains concaving upward and downward, and focus on others, such as decreasing and increasing domains. The task illustrates the problematic concept of "correctness" and the importance of analyzing the attributes of the submissions, which provide a rich picture of the student's knowledge and have the potential to promote learning and teaching.

DISCUSSION

This design of tasks requiring students to sketch functions using a vertically draggable points supports the following functionality: (1) Helps students focus on the required place of each dragged point, to make possible the accurate assessment of students' perceptions. Asking to drag a finite number of points on a coordinate system with marked ticks, together with other optional measurements, helps students drag the points accurately, up to a certain standard deviation. The required accuracy depends on pedagogical and technical factors, such as the content and the given measuring mechanisms. (2) The sketch that students construct this way makes it possible to characterize each submission with respect to the actions the students took at critical or non-critical points, as they considered certain points separately or a collection of points (a domain). The design of this type of sketching tool makes it possible to characterize the submissions along various concept images and to conjecture whether they were concerned with the value of each dragged point, or with the value of only the critical points. Their choice of value for each dragged point may suggest which points they perceive to be critical (if, for example, students submit only a few points with precise values, it may indicate that they perceive these points to be more important or critical than the others). It is recommended that task designers make draggable the important points, which students may perceive as critical, such as the intersection point with the y axis, as demonstrated in tasks 2 and 3. In addition to assessing the correctness of the solution and the accuracy of each point, it is possible to evaluate other attributes of the solution, such as the domains in which the function is increasing and decreasing, and the ones in which it is concaving upward and downward. It is also possible to identify misconceptions described in the literature. When comparing the shape of the sketches with that of the given function by assessing the difference between the y value of each dragged point and the corresponding y value of the given function, phenomena that are familiar from previous research (of Nemirovsky and Rubin, 1992, (Figure 4 (a)) and new surprising phenomena (Figure 7 (c), (d)) were revealed. When dragging two non-adjacent points, students may accidently create unintended extremum points between them, which might change unintentionally other characteristics of the functions such as increasing and decreasing domains. Finally, trying to generalize, this sketching tool, which was demonstrated here being lens to assess students understanding of the relationship between a function and its derivative, is suitable for other tasks that have a single or a few possible continuous solutions; for tasks where the accuracy of the solution is important for assessing the students' knowledge; for constructing a function in relation to another function, for example, constructing f(ax), f(x+a), f(x)+a for a given f(x); and for tasks that require to create a sketch of a continuous function that meets given requirements. Analyzing task characteristics, as demonstrated above, affects teaching practices by providing feedback to students and teachers, but this is beyond the scope of this article.

NOTES

1. Seeing the Entire Picture (STEP) is a formative assessment platform developed at the Mathematics Education Research and Innovation Center, at the University of Haifa. Details about this platform are available at www.visustep.com.

2. This study is part of a doctoral research carried at the MERI center, the University of Haifa and supervised by Prof. Michal Yerushalmy.

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