On the Itô-Alekseev-Gröbner Formula for Stochastic Differential Equations

Von der Fakultät für Mathematik der Universität Duisburg-Essen
zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte Dissertation

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Datum der mündlichen Prüfung: 31.Mai 2019

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DOI: 10.17185/duepublico/70569
URN: urn:nbn:de:hbz:464-20191001-103121-8

Contents

Abstract

In this thesis we establish the Itô-Alekseev-Gröbner formula which can be regarded as a generalization of the Alekseev-Gröbner lemma and Itô's lemma.

Analogous to the well-known deterministic case of the Alekseev-Gröbner lemma, this formula allows us to estimate the global error between the exact solution of a stochastic differential equation (SDE) and a general Itô process in terms of the local characteristics. In particular, our Itô-Alekseev-Gröbner formula can be applied to derive strong approximation rates for implementable approximations of SDEs. To apply the Itô-Alekseev-Gröbner formula, we need to ensure that there exists a version of the solution processes of the SDE which is twice continuously differentiable in the starting point.

In the last part of this thesis, we derive conditions on the coefficient functions which ensure existence of solution versions.

Zusammenfassung in deutscher Sprache

In dieser Doktorarbeit beweisen wir die Itô-Alekseev-Gröbner-Formel, welche wir als eine Verallgemeinerung sowohl der Itô-Formel als auch des Alekseev-Gröbner-Lemmas ansehen. Analog zum wohlbekannten deterministischen Fall des Alekseev-Gröbner-Lemmas erlaubt uns diese Formel, den globalen Fehler von einer exakten Lösung einer stochastischen Differentialgleichung und einem allgemeinen Itô-Prozess mithilfe der lokalen Charakteristiken darzustellen. Insbesondere kann unsere Itô-Alekseev-Gröbner-Formel angewendet werden, um starke Approximationsraten für implementierbare Approximationen von stochastischen Differentialgleichungen herzuleiten. Um die Itô-Alekseev-Gröbner-Formel anzuwenden müssen wir sicherstellen, dass eine Version der Lösung der stochastischen Differentialgleichung existiert, welche im Anfangswert zweimal stetig differenzierbar ist. Im letzten Teil der Doktorarbeit widmen wir uns dieser Frage.

Acknowledgement

Lieber Martin, ich habe Dir viel zu verdanken! Es war eine großartige Erfahrung, mit Dir zusammzuarbeiten. Ich bin sehr dankbar für die vielen Ideen, die Inspiration, und für Deine stete Unterstützung. Besonders Dein Überblick des Forschungsgebiets, Deine schnelle Auffassungsgabe und Deine Fähigkeit, immer wieder Lösungen zu finden, haben mich sehr beeindruckt.

Dear Sara (Mazzonetto), I really enjoyed working with you. You became very important for our project and spent many thoughts on the right frame to apply our theory, especially for the last chapter. Your door was always open for me and I felt strongly supported by you – It was a pleasure to work with you!

Liebe Mitglieder der *Research training group 2131 - High-Dimensional Phenomena in Probability*, ich bin sehr froh, dass ich Teil dieses Graduiertenkollegs sein durfte. Ich hatte dort hervorragende Arbeitsbedingungen und es hat mich sehr gefreut, regelmäßigen Austausch mit so vielen hochkaraätigen Mathematikern, seien es Doktoranden, PostDocs oder Professoren, zu haben. Besonders gerne habe ich an den wöchentlichen RTG-Nachmittagen, den RTG Feedback-Treffen und den RTG Summer Schools teilgenommen. Weiterhin danke ich für die Unterstützung bei Reisen zu weiteren interessanten Konferenzen und Summer sowie Winter Schools.

Liebe Mitglieder des Organisationsteams des Doktorandinnentreffens 2018 und alle weiteren, die daran mitgewirkt haben, es hat wirklich Spaß gemacht, dies mit Euch zu organisieren, ich habe nicht nur viel dabei gelernt, sondern war auch froh, dass wir damit unsere Fakultät und das RTG repräsentieren konnten.

Dear Professor Yano, I would like to thank you for your invitation to Kyoto. It was there that I started to learn about stochastic analysis which finally brought me to write this thesis. I learned very much under your supervision and I alway feel gratitude for the great time at your faculty.

Lieber Herr Professor Rüschendorf, durch Vorlesungen, Seminare, beim Verfassen der Masterarbeit und als wissenschaftlicher Mitarbeiter habe ich bei Ihnen viel gelernt. Die Kombination aus Freiraum und steter Unterstützung haben mich ideal auf die Promotionsphase vorbereitet.

Lieber Bernd, liebe Anna, lieber Angel, liebe Kollegen im Bereich IAC der Union Investment, vielen Dank für die gute Einarbeitungsphase und die Unterstützung bei der Fertigstellung meiner Dissertation. Ihr wart mir eine große Hilfe!

Notation

The following notation is used throughout this thesis:

We use the convention that $0 \notin \mathbb{N}$ and $0 \in \mathbb{N}_0$.

For all $c \in (0, \infty)$ let 0^0, $\frac{0}{0}$, $\frac{c}{0}$, $\frac{-c}{0}$, $0 \cdot \infty$, $0 \cdot (-\infty)$, ∞^c denote the extended real numbers $0^0 = 1$, $\frac{0}{0} = 0$, $\frac{c}{0} = \infty$, $\frac{-c}{0} = -\infty$, $0 \cdot \infty = 0$, $0 \cdot (-\infty) = 0$, and $\infty^c = \infty$.

For two real numbers $a, b \in \mathbb{R}$ we denote $[a, b] = \{\ x \in \mathbb{R} \colon a \leq x \text{ and } \ x \leq b\}$, $[a, b) = \{x \in \mathbb{R} \colon a \leq x \text{ and } x < b\}$, $(a, b] = \{x \in \mathbb{R} \colon a < x \text{ and } \ x \leq b\}$ and $(a, b) = \{x \in \mathbb{R} \colon a < x \text{ and } x < b\}$. Note that we do not require that $a \leq b$ holds.

For all $T \in [0, \infty)$ let $\Delta_T \subseteq [0, T]^2$ denote the subset with the property that $\Delta_T = \{(s, t) \in [0, T]^2 \colon s \leq t\}$.

For all $S, T \in \mathbb{R}$ with the property that $S \leq T$ let $\Delta_{S,T} \subseteq [S, T]^2$ denote the subset with the property that $\Delta_{S,T} = \{(s, t) \in [S, T]^2 : s \leq t\}$.

For all $h \in (0, \infty)$, $r \in [0, \infty)$ let $\lceil r \rceil_h, \lfloor r \rfloor_h, \lceil r \rceil_0, \lfloor r \rfloor_0 \in [0, \infty)$ be the real numbers with the properties that $\lceil r \rceil_h = \inf\{nh \in [r, \infty) \colon n \in \mathbb{N}_0\}$, $\lfloor r \rfloor_h = \sup\{nh \in [0, r] \colon n \in \mathbb{N}_0\}$, $\lceil r \rceil_0 = r$, and $\lfloor r \rfloor_0 = r$. We call these the *generalized Gaussian Brackets*.

For all $T \in (0, \infty)$ we denote by $T/\mathbb{N}$ the set $T/\mathbb{N} = \{\ T/n \colon n \in \mathbb{N}\}$.

For a real vector space V and a subset $S \subseteq V$ let $\operatorname{span}(S) \subseteq V$ denote the set with the property that $\operatorname{span}(S) = \{\sum_{i=1}^n r_i v_i \colon n \in \mathbb{N}, r_1, \ldots, r_n \in \mathbb{R}, v_1, \ldots, v_n \in S\}$.

For all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ we write $\|x\|_{\mathbb{R}^d}$ for the Euclidean norm of x and for all $i \in \{1, \ldots, d\}$ let $e_i^{(d)}$ denote the i-th unit vector in $\mathbb{R}^d$.

For two separable real Hilbert spaces U and H and a linear mapping $A \colon U \to H$ we denote the operator norm of A as $\|A\|_{L(U,H)}$, the Hilbert-Schmidt norm of A as $\|A\|_{\mathrm{HS}(U,H)}$, the space of linear operators from U to H with finite operator norm as $L(U, H) = L(U, H)$, the space of linear operators from U to H with finite Hilbert-Schmidt norm as $\mathrm{HS}(U, H)$, and we write $L^{(2)}(U, L(U, H))$.

For all $d, m \in \mathbb{N}$ and all $A \in \mathbb{R}^{d \times m}$ we denote by $A^\top$ the transpose of A.

For a set S we denote by $\mathcal{P}(S)$ the power set of S.

For every normed vector space $(V, \|\cdot\|_V)$ and every subset $A \subseteq V$ let $\mathcal{B}(V)$ denote the Borel algebra on V. For a natural number $d \in \mathbb{N}$ and a Borel measurable set $A \in \mathcal{B}_{\mathbb{R}^d}$ we denote by $\lambda_A^d \colon \mathcal{B}_{\mathbb{R}^d}(A) \to [0, \infty]$ the Lebesgue-Borel measure on A. For a Borel measurable set $A \in \mathcal{B}_{\mathbb{R}}$ we often write λ_A instead of λ_A^1.

For all measurable spaces $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{B})$ let $\mathcal{M}(\mathcal{F}, \mathcal{B})$ be the set

$$\mathcal{M}(\mathcal{A}, \mathcal{B}) = \big\{ f \colon \Omega \to \Omega' \colon f \text{ is } \mathcal{F}/\mathcal{B}\text{-measurable} \big\}. \tag{1}$$

For every measure space $(\Omega, \mathcal{F}, \mu)$ and every measurable space $(\Omega', \mathcal{B})$ let $L^0(\mathbb{P}, \mathcal{B})$ be the set with the property that

$$L^0(\mathbb{P}, \mathcal{B}) = \big\{ \{ f \in \mathcal{M}(\mathcal{A}, \mathcal{B}) \colon f = g \ \mathbb{P}\text{-a.s.} \} \colon g \in \mathcal{M}(\mathcal{A}, \mathcal{B}) \big\}. \tag{2}$$

For every measure space $(\Omega, \mathcal{F}, \mu)$, every normed vector space $(V, \| \cdot \|_V)$, and all $p \in [1, \infty)$ let

$$\| \cdot \|_{L^p(\mu; V)} \colon \big(\mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \cup L^0(\mathbb{P}, \mathcal{B}(V)) \big) \to [0, \infty] \tag{3}$$

be the mapping which satisfies for all $f \in \big(\mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \cup L^0(\mathbb{P}, \mathcal{B}(V)) \big)$ that
$\|f\|_{L^p(\mu; V)} = \big(\int_\Omega \|f\|_V^p d\mu \big)^{\frac{1}{p}}$, let $\mathcal{L}^p(\mu; V)$ be the set with the property that

$$\mathcal{L}^p(\mu; V) = \{ f \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \colon \|f\|_{L^p(\mu; V)} < \infty \}, \tag{4}$$

and let $L^p(\mu; V)$ be the set with the property that

$$L^p(\mu, V) = \big\{ \{ f \in \mathcal{L}^p(\mu, V) \colon f = g \ \mu\text{-a.e.} \} \colon g \in \mathcal{L}^p(\mu, V) \big\}. \tag{5}$$

For every measure space $(\Omega, \mathcal{F}, \mu)$ and every normed vector space $(V, \| \cdot \|_V)$ let

$$\langle \cdot, \cdot \rangle_{L^2(\mu; \mathbb{V})} \colon \big(\mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \cup L^0(\mathbb{P}, \mathcal{B}(V)) \big)^2 \to [0, \infty] \tag{6}$$

be the function with the property that for all $f, g \in \big(\mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \cup L^0(\mathbb{P}, \mathcal{B}(V)) \big)$ it holds that

$$\langle f, g \rangle_{L^2(\mu; \mathbb{V})} = \int\limits_\Omega \langle f, g \rangle_V \, d\mu. \tag{7}$$

For every measure space $(\Omega, \mathcal{F}, \mu)$ and every normed vector space $(V, \| \cdot \|_V)$ let

$$\| \cdot \|_{L^\infty(\mu; V)} \colon \big(\mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \cup L^0(\mathbb{P}, \mathcal{B}(V)) \big) \to [0, \infty] \tag{8}$$

the mapping with the property that for all $f \in \big(\mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \cup L^0(\mathbb{P}, \mathcal{B}(V)) \big)$ it holds that

$$\|f\|_{\mathcal{L}^\infty(\mu; V)} = \operatorname{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|_V, \tag{9}$$

let $\mathcal{L}^\infty(\mu; V)$ be the set with the property that

$$\mathcal{L}^\infty(\mu; V) = \big\{ f \in \mathcal{M}(\mathcal{A}, \mathcal{B}_V) \colon \|f\|_{\mathcal{L}^\infty(\mu; V)} < \infty \big\}, \tag{10}$$

and let $L^\infty(\mu; V)$ be the set with the property that

$$L^\infty(\mu, V) = \big\{\{f \in \mathcal{L}^\infty(\mu, V) \colon f = g \ \mu\text{-a.e.}\} \colon g \in \mathcal{L}^p(\mu, V)\big\}. \tag{11}$$

For normed real vector spaces V_1, V_2 and $k \in \mathbb{N} \cup \{0, \infty\}$ we denote by $C^k(V_1, V_2)$ the set of functions $f : V_1 \to V_2$ that are k times continuously differentiable, we denote by

$$|\cdot|_{C^k_{\mathrm{b}}(V_1,V_2)} \colon C^k(V_1, V_2) \to [0, \infty] \tag{12}$$

and

$$\|\cdot\|_{C^k_{\mathrm{b}}(V_1,V_2)} \colon C^k(V_1, V_2) \to [0, \infty] \tag{13}$$

the mappings with the properties that it holds for all $f \in C^k(V_1, V_2)$ that

$$|f|_{C^k_{\mathrm{b}}(V_1,V_2)} = \sup_{x \in V_1} \|f^{(k)}(x)\|_{L^{(k)}(V_1,V_2)} \tag{14}$$

and

$$\|f\|_{C^k_{\mathrm{b}}(V_1,V_2)} = \|f(0)\|_{V_2} + \sum_{l=1}^{k} |f|_{C^l_{\mathrm{b}}(V_1,V_2)}, \tag{15}$$

and we denote by $C^k_{\mathrm{b}}(V_1, V_2)$ the set given by

$$C^k_{\mathrm{b}}(V_1, V_2) = \{f \in C^k(V_1, V_2) \colon \ \|f\|_{C^k_{\mathrm{b}}(V_1,V_2)} < \infty\}. \tag{16}$$

For every measurable space $(\Omega, \mathcal{F})$ and every $n \in \mathbb{N}$ let $C_b^{\infty,\mathcal{F}}(\mathbb{R}^n \times \Omega, \mathbb{R})$ be the set which satisfies

$$C_b^{\infty,\mathcal{F}}(\mathbb{R}^n \times \Omega, \mathbb{R}) = \left\{ \begin{matrix} f \colon \mathbb{R}^n \times \Omega \to \mathbb{R} \colon & \forall \omega \in \Omega \colon f(\cdot, \omega) \in C_b^\infty(\mathbb{R}^n, \mathbb{R}), \\ & \forall x \in \mathbb{R}^d \colon f(x, \cdot) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(\mathbb{R})) \end{matrix} \right\}. \tag{17}$$

For all finite dimensional Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ with the property that $H \neq \{0\}$, all open subsets $O \subseteq H$, and all functions $\mu \colon O \to H$ and $\sigma \colon O \to \mathrm{HS}(U, H)$, let $\mathcal{G}_{\mu,\sigma} \colon C^2(O, \mathbb{R}) \to \mathcal{M}(O, \mathcal{B}(\mathbb{R}))$ be a linear operator with the property that for all $x \in O$, $\phi \in C^2(O, \mathbb{R})$ it holds that

$$(\mathcal{G}_{\mu,\sigma}\phi)(x) := \phi'(x)\,\mu(x) + \tfrac{1}{2}\,\mathrm{tr}\big(\sigma(x)\sigma(x)^*(\mathrm{Hess}\,\phi)(x)\big). \tag{18}$$

We refer to this linear operator $\mathcal{G}_{\mu,\sigma}$ as generator.

Chapter 1

Introduction

The linear integration-by-parts formula states in the simplest case that for all $a, b \in \mathbb{R}$, $t \in [0, \infty)$ it holds that

$$e^{at} - e^{bt} = -\int_0^t \tfrac{d}{ds}\big(e^{a(t-s)}e^{bs}\big)\,ds = \int_0^t e^{a(t-s)}(a-b)e^{bs}\,ds. \tag{1.1}$$

The nonlinear integration-by-parts formula, which is usually referred to as *Alekseev-Gröbner formula* or nonlinear variation-of-constants formula, generalizes this relation to nonlinear ordinary differential equations and has been established in Alekseev 1961 and in Gröbner 1960. More formally, the Alekseev-Gröbner formula (cf., e.g., Hairer, Nørsett, and Wanner 1993, Theorem I.14.5) asserts the following:

Theorem 1.1 (Alekseev-Gröbner formula)**.** *Let $T > 0$, let $\mu \in C^{0,1}([0,1] \times \mathbb{R}, \mathbb{R})$ let $X \in C^{0,1}(\Delta_T \times \mathbb{R}, \mathbb{R})$ be a mapping such that for all $s \in [0, T]$, $x \in \mathbb{R}$ it holds that $X^x_{s,s} = x$ and such that for all $(s,t) \in \Delta_T$, $x \in \mathbb{R}$ it holds that*

$$X^x_{s,t} = x + \int_s^t \mu(r, X^x_{s,r})\,dr, \tag{1.2}$$

and let $Y \in C^1(\mathbb{R}, \mathbb{R})$. Then it holds that

$$X^{Y_0}_{0,T} - Y_T = \int_0^T \big(\tfrac{\partial}{\partial x} X^x_{r,T}\big)\big|_{x=Y_r}\big(\mu(r, Y_r) - \tfrac{d}{dr}Y_r\big)\,dr. \tag{1.3}$$

The Alekseev-Gröbner formula expresses the global error $X^{Y_0}_{0,T} - Y_T$ in terms of the local error $\mu(r, Y_r) - \frac{d}{dr}Y_r$ which corresponds to the difference of time derivatives. For this reason, the Alekseev-Gröbner formula is a powerful tool for studying perturbations of ordinary differential equations; see, e.g., Nørsett and Wanner 1979, Theorem 3, Lie and Norsett 1989, Theorem 1, Iserles and Söderlind 1993, Theorem 1, and Iserles 2009.

In this thesis we generalize the Alekseev-Gröbner lemma to a (non-pathwise) stochastic setting and derive the following perturbation formula for not necessarily differentiable perturbation processes:

Theorem 1.2 (Itô-Alekseev-Gröbner formula). *Let* $d, m \in \mathbb{N}$, $T \in (0, \infty)$, $p \in (4, \infty)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ *be a standard Brownian motion with continuous sample paths on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *let* $\mathcal{N} = \{A \in \mathcal{F}\colon \mathbb{P}(A) = 0\}$ *denote the* $\mathbb{P}$*-null sets, let* $\mu\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon [0,T] \times \mathbb{R}^d \to \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)$ *be continuous mappings, let* $X_{\cdot,\cdot}^{\cdot}(\cdot)\colon \Delta_T \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ *be a continuous random field such that for all* $s \in [0,T]$, $\omega \in \Omega$ *the mapping* $\mathbb{R}^d \ni x \mapsto X_{s,T}^x(\omega) \in \mathbb{R}^d$ *is in* $C^2(\mathbb{R}^d, \mathbb{R}^d)$, *such that for all* $s \in [0,T]$, $x \in \mathbb{R}^d$ *the stochastic process* $[s,T] \times \Omega \ni (t, \omega) \mapsto X_{s,t}^x(\omega) \in \mathbb{R}^d$ *is* $(\sigma(\mathcal{N} \cup \sigma(W_r - W_s\colon r \in [s,t])))_{t \in [s,T]}$*-adapted and for all* $t \in [s,T]$ *it holds* $\mathbb{P}$*-almost surely that*

$$X_{s,t}^x = x + \int\limits_s^t \mu(r, X_{s,r}^x)dr + \int\limits_s^t \sigma(r, X_{s,r}^x)dW_r, \tag{1.4}$$

such that for all $(s,t) \in \Delta_T$, $x \in \mathbb{R}^d$ *it holds* $\mathbb{P}$*-almost surely that* $X_{t,T}^{X_{s,t}^x} = X_{s,T}^x$, *let* $A, Y\colon [0,T] \times \Omega \to \mathbb{R}^d$, $B\colon [0,T] \times \Omega \to \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)$ *be stochastic processes such that* Y *has continuous sample paths, such that* Y *and* B *are* $(\sigma(\mathcal{N} \cup \sigma(W_r\colon r \in [0,t])))_{t \in [0,T]}$*-progressively measurable, such that* $\int_0^T \mathbb{E}\Big[\|A_s\|_{\mathbb{R}^d}^p + \|Y_s\|_{\mathbb{R}^d}^p + \|B_s\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}^p\Big]ds < \infty$, *and such that for all* $t \in [0,T]$ *it holds* $\mathbb{P}$*-almost surely that*

$$Y_t = Y_0 + \int\limits_0^t A_s ds + \int\limits_0^t B_s dW_s, \tag{1.5}$$

assume that

$$\sup_{h \in T/\mathbb{N}} \mathbb{E}\left[\int\limits_0^T \left\|\mu\Big(t, X_{\lfloor t \rfloor_h, t}^{Y_{\lfloor t \rfloor_h}}\Big)\right\|_{\mathbb{R}^d}^p + \left\|\sigma\Big(t, X_{\lfloor t \rfloor_h, t}^{Y_{\lfloor t \rfloor_h}}\Big)\right\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}^p dt\right] < \infty, \tag{1.6}$$

assume that

$$\begin{split}\sup_{\substack{r,s,t \in [0,T]\\ r \le s \le t}} \mathbb{E}\Bigg[\left\|X_{t,T}^{X_{r,s}^{Y_r}}\right\|_{\mathbb{R}^d}^p + \left\|\Big(\tfrac{\partial}{\partial x} X_{t,T}^x\Big)\Big|_{x = X_{r,s}^{Y_r}}\right\|_{L(\mathbb{R}^d,\mathbb{R}^d)}^{\frac{4p}{p-4}} \\ + \left\|\Big(\tfrac{\partial^2}{\partial x^2} X_{t,T}^x\Big)\Big|_{x = X_{r,s}^{Y_r}}\right\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}^{\frac{2p}{p-4}}\Bigg] < \infty\end{split} \tag{1.7}$$

and let $k \in \mathbb{N}$, $c_0 \in [0, \infty)$, *and let* $f \in C^2(O, \mathbb{R}^k)$ *satisfy that for all* $x \in O$ *it holds that*

$$\max\left\{\tfrac{\|f(x)\|_{\mathbb{R}^k}}{1 + \|x\|_{\mathbb{R}^d}}, \|f'(x)\|_{L(\mathbb{R}^d,\mathbb{R}^k)}, \|f''(x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^k)}\right\} \le c_0(1 + \|x\|_{\mathbb{R}^d}^q). \tag{1.8}$$

Then

(i) *the stochastic process* $\big(f'\big(X_{r,T}^{Y_r}\big)X_{r,T}^{1,Y_r}(\sigma(r, Y_r) - B_r)\big)_{r \in [0,T]}$ *is Skorohod-integrable and* *(ii) it holds $\mathbb{P}$-almost surely that*

$$\begin{aligned}
&f\big(X^{Y_0}_{0,T}\big) - f(Y_T)\\
&= \int\limits_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\Big(\mu(r,Y_r) - A_r\Big)dr + \int\limits_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\Big(\sigma(r,Y_r) - B_r\Big)\delta(\mathbb{P},\mathbb{F}_0,W;\mathbb{R}^k)W_r\\
&+ \tfrac{1}{2}\sum_{l,j=1}^{d}\int\limits_0^T \Big(\sigma(r,Y_r)\sigma(r,Y_r)^\top - B_rB_r^\top\Big)_{l,j}\\
&\qquad\cdot\Big(f''\big(X^{Y_r}_{r,T}\big)\big(X^{1,Y_r}_{r,T}, X^{1,Y_r}_{r,T}\big) + f'\big(X^{Y_r}_{r,T}\big)X^{2,Y_r}_{r,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)dr.
\end{aligned} \tag{1.9}$$

We give the main idea of the proof for $f = \mathrm{Id}_{\mathbb{R}^d}$: We use a telescoping sum argument to conclude that it holds $\mathbb{P}$-almost surely that

$$X^{Y_0}_{0,T} - Y_T = \sum_{i=0}^{n-1}\Big(X^{Y_{ih}}_{ih,T} - X^{Y_{ih}}_{(i+1)h,T}\Big) - \sum_{i=0}^{n-1}\Big(X^{Y_{(i+1)h}}_{(i+1)h,T} - X^{Y_{ih}}_{(i+1)h,T}\Big). \tag{1.10}$$

Then, we go on applying the Itô formula to both summands of the right-hand side of equation (1.10). We demonstrate this on the second summand and conclude that it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\sum_{i=0}^{n-1}\Big(X^{Y_{(i+1)h}}_{(i+1)h,T} - X^{Y_{ih}}_{(i+1)h,T}\Big)\\
&= \sum_{i=0}^{n-1}\int\limits_{ih}^{(i+1)h} X^{1,Y_r}_{\lceil r\rceil_{T/n},T}A_r dr + \sum_{i=0}^{n-1}\int\limits_{ih}^{(i+1)h} X^{1,Y_r}_{\lceil r\rceil_{T/n},T}B_r dW_r\\
&\quad + \tfrac{1}{2}\sum_{i=0}^{n-1}\sum_{l,j=1}^{d}\int\limits_{ih}^{(i+1)h}\big(B_rB_r^\top\big)_{l,j}\\
&\qquad\cdot\Big(f''\big(X^{Y_r}_{\lceil r\rceil_{T/n},T}\big)\big(X^{1,Y_r}_{\lceil r\rceil_{T/n},T}, X^{1,Y_r}_{\lceil r\rceil_{T/n},T}\big) + f'\big(X^{Y_r}_{\lceil r\rceil_{T/n},T}\big)X^{2,Y_r}_{\lceil r\rceil_{T/n},T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)dr.
\end{aligned} \tag{1.11}$$

Heuristically, the next step would be to set

$$\sum_{i=0}^{n-1}\int\limits_{ih}^{(i+1)h} = \int\limits_0^T \tag{1.12}$$

and to let $n \to \infty$. To this end, we take a closer look on the sum of Itô integrals

$$\sum_{i=0}^{n-1}\int\limits_{ih}^{(i+1)h} f'\big(X^{Y_r}_{(i+1)h,T}\big)X^{1,Y_r}_{(i+1)h,T}B_r dW_r. \tag{1.13}$$

Here, the stochastic processes $(f'\big(X^{Y_r}_{(i+1)h,T}\big)X^{1,Y_r}_{(i+1)h,T}B_r)_{r\in[ih,(i+1)h]}$ are predictable with respect to the filtration generated by the Brownian motion $(W_r)_{r\in[ih,(i+1)h]}$, which is an essentially necessary condition for the process to be Itô integrable. But the expression

$$\int_0^T f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big)X^{1,Y_r}_{\lceil r\rceil_h,T}B_r dW_r \tag{1.14}$$

is not defined in the Itô sense since the stochastic process $(f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big)X^{1,Y_r}_{\lceil r\rceil_h,T}B_r)_{r\in[0,T]}$ is not predictable with respect to the filtration generated by the Brownian motion $(W_r)_{r\in[0,T]}$ (see the Notation section for the definition of the generalized Gaussian brackets). The intuition is that this process is "looking into the future". An idea to overcome this difficulty is to use the classical Skorohod integral. But here, the problem is that for all $i \in \{1, \ldots, n-1\}$ the Skorohod integral

$$\int_{ih}^{(i+1)h} f'\big(X^{Y_r}_{(i+1)h,T}\big)X^{1,Y_r}_{(i+1)h,T}B_r\delta W_r \tag{1.15}$$

is not defined in the classical Skorohod sense because the process $(f'\big(X^{Y_r}_{(i+1)h,T}\big)X^{1,Y_r}_{(i+1)h,T}B_r)_{r\in[ih,(i+1)h]}$ is not measurable with respect to the σ-algebra generated by the Brownian motion $(W_r - W_{ih})_{r\in[ih,(i+1)h]}$. For this reason, we need a generalization of the Mallavin Calculus to larger σ-algebras, analogous to the enlargement of filtration of the Itô calculus. In the second chapter, we construct the Mallavin Calculus, as it is done in Nualart 1995 or in Kruse 2014, but in a measure-theoretically broader setting.

In the third chapter, we then prove the Itô-Alekseev-Gröbner formula, 1.2 which follows immediately from Theorem 3.3 (applied with $\mathbb{F}_0 = \sigma(\mathcal{N})$, $O = \mathbb{R}^d$, $k = d$, $q = 0$, $c_0 = 1$, $f = \mathrm{Id}_{\mathbb{R}^d}$ in the notation of Theorem 3.3).

Theorem 1.2 together with the fact that for all $a, b \in \mathbb{R}$ it holds that $(a+b)^2 \leq 2a^2 + 2b^2$ immediately implies an L^2-estimate. For example, the L^2-norm of the right-hand side of equation (1.9) can be bounded by the triangle inequality. The L^2-norm of the Skorohod integral on the right-hand side of the estimate can then be calculated by applying the Itô isometry for Skorohod integrals (see, e.g., Lemma 4 in Alòs and Nualart 1998 which does not require the approximation process to be Malliavin differentiable). Another approach for obtaining L^2-estimates is to apply the Itô formula for Skorohod processes to the squared norm of the right-hand side of equation (1.9). However, this seems to require additional regularity. To demonstrate the applicability of Theorem 1.2, we apply Theorem 1.2 to the stochastic van-der-Pol oscillator in Section 3.3 below and obtain in Lemma 3.10 that the stochastic van-der-Pol oscillator can be approximated with L^2-rate $\frac{1}{2}$.

Theorem 1.2 can be applied to any approximation of an SDE which is an Itô process with respect to the same Wiener process driving the SDE. Possible applications include, in the notation of Theorem 1.2,

(i) strong convergence rates for *time-discrete numerical approximations of SODEs* (e.g., the Euler-Maruyama approximation with $N \in \mathbb{N}$ time discretization steps is given by $A_t = \mu(\frac{kT}{N}, Y_{\frac{kT}{N}})$ and $B_t = \sigma(\frac{kT}{N}, Y_{\frac{kT}{N}})$ for all $t \in [\frac{kT}{N}, \frac{(k+1)T}{N})$, $k \in \mathbb{N}$),

(ii) strong convergence rates for *Galerkin approximations for stochastic evolution equations (SEEs)* (choose $A_t = P(\mu(t, Y_t))$ and $B_t u = P(\sigma(t, Y_t)u)$ for all $u \in \mathbb{R}^m$, $t \in [0, T]$ and some suitable projection operator $P \in L(\mathbb{R}^d)$ where $d, m \in \mathbb{N}$; Theorem 1.2 is applied to a finite-dimensional approximation of the exact solution of the SPDE of which convergence in probability is known), and

(iii) strong convergence rates for *small noise perturbations* of solutions of deterministic differential equations (choose $\sigma = 0$, $A_t = \mu(t, Y_t)$ and $B_t = \varepsilon\, \tilde{\sigma}(t, Y_t)$ for all $t \in [0, T]$ where $\tilde{\sigma} \colon [0, T] \times \mathbb{R}^d \to \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)$ is a suitable Borel measurable function and where $\varepsilon > 0$ is a sufficiently small parameter).

In the literature, most estimates of perturbation errors (for exceptions see Hutzenthaler and Jentzen 2014 and the references therein) exploit the popular *global monotonicity* assumption, which, in the notation of Theorem 1.2, assumes existence of a real number $c \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\langle x - y, \mu(t, x) - \mu(t, y)\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\|\sigma(t, x) - \sigma(t, y)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)} \leq c\|x - y\|^2_{\mathbb{R}^d}. \tag{1.16}$$

We emphasize that many SDEs from the literature do not satisfy inequality (1.16) and that Theorem 1.2 does not require that the global monotonicity assumption is fulfilled.

Our main motivation for a stochastic Alekseev-Gröbner lemma are *strong convergence rates* for *time-discrete numerical approximations of SPDEs.* In the literature, positive strong convergence rates have been established for SDEs with monotone nonlinearity; see, e.g., Gyöngy and Millet 2007; Kovács, Larsson, and Lindgren 2015; Jentzen and Pušnik 2015; Becker and Jentzen 2019; Becker, Gess, Jentzen, and Kloeden 2017; Bréhier and Goudenège 2018; Bréhier, Cui, and Hong 2018; Liu and Qiao 2018b; Wang 2018 for the case of additive noise and Majee and Prohl 2018; Liu and Qiao 2018a for the case of multiplicative noise. To the best of our knowledge, strong convergence rates for time-discrete approximations of SDEs with non-monotone superlinearly growing nonlinearities remain an open problem. This problem now becomes feasible by applying our perturbation result in Theorem 1.2. Clearly the classical Euler-Maruyama approximations cannot be used as temporal discretizations in the case of super-linearly growing coefficients, since these approximations diverge in the strong and weak sense; see Hutzenthaler, Jentzen, and Kloeden 2011; Hutzenthaler, Jentzen, and Kloeden 2013; Beccari, Hutzenthaler, Jentzen, Kurniawan, Lindner, and Salimova 2019. Summarizing, we believe that Theorem 1.2 is an appropriate tool to analyze temporal approximations of semilinear SDEs.

A crucial assumption in Theorem 1.2 is the existence of a solution of the SDE (1.4) which is twice continuously differentiable in the starting point, since in the proof of Theorem 1.2 we apply Itô's formula for independent random fields to the random functions $\mathbb{R}^d \ni x \mapsto X^x_{t,T} \in \mathbb{R}^d$, $t \in [0, T]$. This assumption is not satisfied in a number of cases. For example, Li and Scheutzow 2011 construct a two-dimensional example with smooth and globally bounded coefficient functions which is not even strongly complete (that is, the exceptional subset of Ω where (1.4) does not hold cannot be chosen independently of the starting point); cf. also Theorem 1.2 in Hairer, Hutzenthaler, and Jentzen 2015. Under suitable assumptions on the coefficients, however, strong completeness and existence of a solution of (1.4) which is continuous in the starting point can be ensured; see, e.g., Cox, Hutzenthaler, and Jentzen 2013; Zhang 2010; Li 1994. Existence of a solution of (1.4) which satisfies the assumptions of Theorem 1.2 is currently known essentially only in the case of twice continuously differentiable coefficient functions whose derivatives up to second order are bounded; see, e.g., Theorem 1.4.1 in Kunita 1986. The last part of the thesis strives to relax the conditions on the coefficient functions such that the assumptions of Theorem 1.2 are fulfilled. The main contribution here is the following theorem, which follows immediately from Proposition 4.34 below:

Theorem 1.3 (Spatially differentiable solution)**.** *Let $T \in (0,\infty)$, let $(H, \langle \cdot,\cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot,\cdot \rangle_U, \|\cdot\|_U)$ be finite-dimensional real Hilbert spaces, assume $H \neq \{0\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0,T] \times \Omega \to U$ be a standard $(\Omega, \mathcal{F}, (\mathbb{F}_t)_{t\in[0,T]}, \mathbb{P})$-Brownian motion, for all $(s,t) \in \Delta_T$ let $\mathcal{F}_t^s \subseteq \mathbb{F}_t$ be a subset such that for all $s \in [0,T]$ the filtration $(\mathcal{F}_t^s)_{t\in[s,T]}$ is a normal filtration, assume that for all $(s,t) \in \Delta_T$ it holds that $\sigma(W_r - W_s \colon r \in [s,t]) \subseteq \mathcal{F}_t^s$, let $D, O \subseteq H$ be non-empty open sets, let $B \subseteq H$ be a Borel measurable set with the property that $D \subseteq B \subseteq O$, let $\mu \in C^1(O,H)$, $\sigma \in C^1(O, \mathrm{HS}(U,H))$, for all $s \in [0,T]$, $x \in B$ let $X_s^x \colon [s,T] \times \Omega \to B$ be an $(\mathcal{F}_t^s)_{t\in[s,T]}$-adapted stochastic process with continuous sample paths such that for all $x \in B$, $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that*

$$X_{s,t}^x = x + \int\limits_s^t \mu(X_{s,r}^x)\,dr + \int\limits_s^t \sigma(X_{s,r}^x)\,dW_r, \tag{1.17}$$

let $\alpha_0, \alpha_1 \in [0,\infty)$, $\beta_0, \beta_1 \in \mathbb{R}$, $U_0, U_1 \in C^2(O,[0,\infty))$, let $\overline{U}\colon [0,T] \times B \to \mathbb{R}$ be a Borel measurable function with the property that for all $s \in [0,T]$, $x \in B$ it holds $\mathbb{P}$-almost surely that $\int_s^T |\overline{U}(r, X_{s,r}^x)|\,dr < \infty$ and such that for all $i \in \{0,1\}$, $s \in [0,T]$,

$$(t,x) \in \cup_{\omega\in\Omega} \cup_{z\in B} \cup_{r\in[s,T]} \{(r, X_{s,r}^z(\omega))\} \tag{1.18}$$

it holds that

$$(\mathcal{G}_{\mu,\sigma} U_i)(x) + \tfrac{1}{2\exp(\alpha_i t)} \|\sigma(x)^*(\nabla U_i)(x)\|_U^2 + \mathbb{1}_{\{1\}}(i) \cdot \overline{U}(t,x) \leq \alpha_i U_i(x) + \beta_i, \tag{1.19}$$

let $\phi\colon [0,T] \to \mathbb{R}$ be a Borel measurable function with the property that $\int_0^T \max\{0, \phi(r)\}\,dr < \infty$, and let $V \in C^2(O,[0,\infty))$ be a function and $\alpha \in [0,\infty)$ be a real number such that for all $x \in B$, $s \in [0,T]$, $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that $(\mathcal{G}_{\mu,\sigma} V)(X_{s,t}^x) \leq \alpha V(X_{s,t}^x)$, for all $s \in [0,T]$ let $Z_s^v(x,y)\colon [s,T] \times \Omega \to H$, $x \in B$, $v \in H$, $y \in \{z \in \mathbb{R}\colon x + vz \in B\}$, be $(\mathcal{F}_t^s)_{t\in[s,T]}$-adapted stochastic processes with continuous sample paths satisfying for all $t \in [s,T]$, $x \in B$, $v \in H$,

$y \in \mathbb{R} \setminus \{0\}$ with $x + vy \in B$ that

$$Z^v_{s,t}(x,y) = \frac{X^{x+yv}_{s,t} - X^x_{s,t}}{y} \tag{1.20}$$

and for all $t \in [s,T]$, $x \in B$, $v \in H$ that it holds $\mathbb{P}$-almost surely that

$$Z^v_{s,t}(x,0) = v + \int_s^t \mu'(X^x_{s,r}) Z^v_{s,r}(x,0)\, dr + \int_s^t \sigma'(X^x_{s,r}) Z^v_{s,r}(x,0)\, dW_r. \tag{1.21}$$

let $q, p_1 \in [1,\infty)$, $p \in (3+\sqrt{10}, \infty)$, $\theta \in \big[\frac{p^2-1}{p^2-6p-1}, \infty\big)$, $q_0, q_1 \in (0,\infty]$ such that $\sum_{i=0}^1 \frac{1}{q_i} = \frac{1}{(p+1)q}$ and that $q(p+1) \geq 2p$, let $\gamma \in (0,\infty)$, assume that for all $s \in [0,T]$, $x_1, x_2 \in B$ it holds $\mathbb{P}$-almost surely for all $t \in [s,T]$ that

$$\begin{aligned} &2 \max\Big\{0, \sup_{u \in H \setminus \{0\}} \tfrac{\langle u, \int_0^1 \mu'(X^{x_1}_{s,t} + \lambda(X^{x_2}_{s,t} - X^{x_1}_{s,t}))\, d\lambda\, u\rangle_H}{\|u\|^2_H}\Big\} + 2\gamma \\ &\quad + (qp+q+1) \Big\| \int_0^1 \sigma'(X^{x_1}_{s,t} + \lambda(X^{x_2}_{s,t} - X^{x_1}_{s,t}))(\cdot)\, d\lambda \Big\|^2_{L(H,\mathrm{HS}(U,H))} \\ &\leq \Big(\phi(t) + \sum_{j=1}^2 \tfrac{U_0(X^{x_j}_{s,t})}{2q_0(T-s)\exp(\alpha_0 t)} + \sum_{j=1}^2 \tfrac{\overline{U}(t, X^{x_j}_{s,t})}{2q_1 \exp(\alpha_1 t)}\Big) \end{aligned} \tag{1.22}$$

let $V \in C^2(O, [0,\infty))$ be a function and $\alpha \in [0,\infty)$ be a real number which satisfy for all $s \in [0,T]$, $x, x_1, x_2, x_3, x_4 \in B$, not all equal, that for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that $(\mathcal{G}_{\mu,\sigma} V)(X^x_{s,t}) \leq \alpha V(X^x_{s,t})$, and that

$$\begin{aligned} &\max\Big\{ \tfrac{1}{\gamma^{1/2}} \int_0^1 \tfrac{\big\|\mu'\big((1-\lambda)X^{x_1}_{s,t} + \lambda X^{x_2}_{s,t}\big) - \mu'\big((1-\lambda)X^{x_3}_{s,t} + \lambda X^{x_4}_{s,t}\big)\big\|_{L(H)}}{\|(1-\lambda)(X^{x_1}_{s,t} - X^{x_3}_{s,t}) + \lambda(X^{x_2}_{s,t} - X^{x_4}_{s,t})\|_H} d\lambda, \\ &\qquad 2 \int_0^1 \tfrac{\big\|\sigma'\big((1-\lambda)X^{x_1}_{s,t} + \lambda X^{x_2}_{s,t}\big) - \sigma'\big((1-\lambda)X^{x_3}_{s,t} + \lambda X^{x_4}_{s,t}\big)\big\|_{L(H,HS(U,H))}}{\|(1-\lambda)(X^{x_1}_{s,t} - X^{x_3}_{s,t}) + \lambda(X^{x_2}_{s,t} - X^{x_4}_{s,t})\|_H} d\lambda \Big\} \\ &\leq \Big(\textstyle\sum_{j=1}^4 V(X^{x_j}_{s,t})\Big)^{(p-1)/(2pp_1 q\theta)}, \end{aligned} \tag{1.23}$$

let $p_2 \in [1,\infty)$, $\vartheta \in \big[2\frac{p^2-1}{p^2-p+1}\frac{p^2-3p+1}{p^2-5p+1}, \infty\big)$, and assume that for all $(s,t) \in \Delta_T$, $x \in B$, $v \in H$, $y \in \{z \in \mathbb{R} \colon x + vz \in B\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} &\sup_{\lambda \in [0,1]} \max\big\{\|\mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\|_{L(H,H)}, \|\sigma'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\|_{L(H,\mathrm{HS}(U,H))}\big\} \\ &\leq \big(V(X^x_{s,t}) + V(X^{x+vy}_{s,t})\big)^{(p-1)/(2\vartheta qp)} \end{aligned} \tag{1.24}$$

and

$$\max\big\{\|\mu(X^x_{s,t})\|_H, \|\sigma(X^x_{s,t})\|_{\mathrm{HS}(U,H)}\big\} \leq \big(V(X^x_{s,t})\big)^{(p^2-p+1)/(4p(p+1)qp_2)}, \tag{1.25}$$

assume that $\inf_{(s,t)\in\Delta_T}\inf_{\omega\in\Omega}\inf_{z\in B}\overline{U}(t,X_{s,t}^z(\omega))\in\mathbb{R}$*, and assume that for all* $k\in\mathbb{N}$ *it holds that*

$$\sup_{x\in B\colon \|x\|_H\le k} V(x)+U_0(x)+U_1(x)<\infty. \tag{1.26}$$

let $\mathbb{H}\subseteq H$ *be an orthonormal basis of* H*, and assume that* $q>2(\dim(H)+2)$*.*
Then there is a probability space $(\Omega,\mathcal{F},\mathbb{P})$*, a standard Brownian motion* $W\colon[0,\infty)\times\Omega\to U$*, and a random field* $X\colon\{(s,t)\in[0,\infty)\times s\le t\}\times H\times\Omega\to H$ *such that for all* $s,t\in[0,\infty)$*,* $x\in H$ *with the property that* $s\le t$ *it holds* $\mathbb{P}$*-almost surely that*

$$X_{s,t}^x=x+\int_s^t\mu(X_{s,u}^x)\,du+\int_s^t\sigma(X_{s,u}^x)dW_u \tag{1.27}$$

and such that for all $\omega\in\Omega$ *the spatial derivative* $\{(s,t)\in[0,\infty)\times s\le t\}\times H\ni(s,t,x)\mapsto\frac{\partial}{\partial x}X_{s,t}^x(\omega)\in L^2(H,H)$ *exists and is continuous.*

A major difficulty in the proof of Theorem 1.3 is to show that there is a flow solving the SDE (1.27) which is strongly complete. This means that there exists a flow which is defined on a subset of Ω with measure 1 such that the subset is independent of the starting point x. Strong completeness does not hold in general. In fact, there exist counterexample SDEs with smooth and globally bounded coefficients; see Li and Scheutzow 2011. Under suitable general assumptions on the coefficient functions, however, strong completeness of the SDE (1.27) holds; see Cox, Hutzenthaler, and Jentzen 2013 and also, e.g., Zhang 2010; Attanasio 2010; Fang, Imkeller, and Zhang 2007; Fang and Zhang 2005; Schmalfuß 1997; Schenk-Hoppé 1996; Li 1994.

Chapter 2

The extended Malliavin Calculus

The classical Malliavin Calculus introduced for example in Nualart 1995 or in Kruse 2014 is defined for stochastic processes that are measurable with respect to the filtration generated by the underlying Brownian motion. Analogous to the enlargement of filtration of the Itô integral, we enlarge the underyling σ-algebra of the Skorohod integral. We do this by constructing the Malliavin Calculus from scratch with greater σ-algebra, extending the ideas of Nualart 1995 and Kruse 2014. The main setting of this chapter is the following:

Setting 2.1. *Let* $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be two separable real Hilbert spaces, let* $S, T \in \mathbb{R}$ *satisfy that* $S < T$, *let* $(\Omega, \mathcal{A}, \mathbb{P})$ *be a probability space, let* $W\colon [S,T] \times \Omega \to U$ *be a standard cylindrical Brownian motion with continuous sample paths on the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *(as defined in Prévôt and Röckner 2007), let* $\mathcal{F}_0 \subseteq \mathcal{A}$ *be a* σ*-algebra which is independent of the Brownian motion* W, *let* $\mathcal{N} \subseteq \mathcal{A}$ *denote the null sets in* $(\Omega, \mathcal{A}, \mathbb{P})$, *and for all* $(s,t) \in \Delta_{S,T}$ *let* $\mathcal{F}_{[S,s]\cup[t,T]} \subseteq \mathcal{A}$ *be a* σ*-algebra with the property that* $\mathcal{F}_{[S,s]\cup[t,T]} = \mathcal{F}_0 \vee \sigma(W_r \colon r \in [S,s]) \vee \sigma(W_r - W_t \colon r \in [t,T]) \vee \mathcal{N}$.

2.1 The Malliavin Differential Operator

In the first section of this chapter we introduce the Malliavin Differential Operator and discuss some of its basic properties.

Definition 2.2. *Assume Setting 2.1 and let* $\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H) \subseteq L^2(\mathbb{P}; H)$ *be the subset with the property that*

$$\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H) = \left\{ \begin{array}{c} F \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H) \colon \exists n \in \mathbb{N}, \exists \phi_1, \ldots, \phi_n \in \mathcal{L}^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R})), \\ \exists f \in C_b^{\infty, \sigma(\mathcal{F}_0 \cup \mathcal{N})}(\mathbb{R}^n \times \Omega, \mathbb{R}), \exists h \in H \text{ such that it holds } \mathbb{P}\text{-a.s. that} \\ F = f\big(\textstyle\int_S^T \phi_1(r) dW_r, \ldots, \int_S^T \phi_n(r) dW_r \big) h \end{array} \right\}. \tag{2.1}$$

The simple Malliavin differential operator

$$\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)\colon \operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\big) \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})}; L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))) \tag{2.2}$$

is a linear operator with the property that for all $F \in \mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)$ *with the property that* $\exists n \in \mathbb{N}$, $\exists \phi_1,\dots,\phi_n \in \mathcal{L}^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$, $\exists f \in C_b^{\infty,\sigma(\mathcal{F}_0\cup\mathcal{N})}(\mathbb{R}^n\times\Omega,\mathbb{R})$, $\exists h \in H$ *such that it holds* $\mathbb{P}$*-almost surely that*

$$F = f\Big(\int_S^T \phi_1(s)dW_s,\dots,\int_S^T \phi_n(s)dW_s\Big)h \tag{2.3}$$

it holds $\lambda_{[S,T]}\otimes\mathbb{P}$*-almost everywhere that*

$$\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(\int_S^T \phi_1(s)dW_s,\dots,\int_S^T \phi_n(s)dW_s\Big)\phi_i h. \tag{2.4}$$

Lemma 2.3. Assume Setting 2.1. Then the simple Malliavin differential operator

$$\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)\colon \operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\big) \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})}; L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))) \tag{2.5}$$

is well-defined.

Proof of Lemma 2.3. Fix $F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\big)$ for the rest of the proof. Since the set $\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H) \subseteq L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ is closed under scalar multiplication, it suffices to consider the case where there are $k,l\in\mathbb{N}$, $n_1,\dots,n_{k+l}\in\mathbb{N}$, for each $i\in\{1,\dots,k+l\}$ there is an $f_i \in \mathcal{M}(\mathcal{B}_{\mathbb{R}^{n_i}}\otimes\sigma(\mathcal{F}_0\cup\mathcal{N}),\mathcal{B}_{\mathbb{R}})$ such that for all $\omega\in\Omega$ it holds that $f_i(\cdot)(\omega)\in C_b^\infty(\mathbb{R}^{n_i},\mathbb{R})$, for every $i\in\{1,\dots,k+l\}$ and for every $j\in\{1,\dots,n_i\}$ there is a $\phi_{ij}\in L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$, and where there are $h_1,\dots,h_{k+l}\in H$ such that it holds $\mathbb{P}$-almost surely that

$$F = \sum_{i=1}^k f_i\Big(\int_S^T \phi_{i1}(s)dW_s,\dots,\int_S^T \phi_{in_i}(s)dW_s\Big)h_i = \sum_{i=k+1}^{k+l} f_i\Big(\int_S^T \phi_{i1}(s)dW_s,\dots,\int_S^T \phi_{in_i}(s)dW_s\Big)h_i. \tag{2.6}$$

Let $d\in\mathbb{N}$ be the dimension of the subspace

$$\operatorname{span}\Big(\big\{\phi_{ij}\colon i\in\{1,\dots,k+l\}, j\in\{1,\dots,n_i\}\big\}\Big) \subseteq L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R})). \tag{2.7}$$

Let $\psi_1,\dots,\psi_d \in L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$ be such that $\{\psi_1,\dots,\psi_d\}$ is an orthonormal basis of $\operatorname{span}(\{\phi_{ij}\colon i\in\{1,\dots,k+l\}, j\in\{1,\dots,n_i\}\}) \subseteq L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$. Then it holds for all $i\in\{1,\dots,k\}$ and all $j\in\{1,\dots,n_i\}$ $\lambda_{[S,T]}$-almost everywhere that

$$\phi_{ij} = \sum_{p=1}^d \Big(\int_S^T \langle\phi_{ij}(s),\psi_p(s)\rangle_{\operatorname{HS}(U,\mathbb{R})}ds\Big)\psi_p. \tag{2.8}$$

Since the Itô integral is linear it holds for all $i \in \{1, \dots, k+l\}$ and all $j \in \{1, \dots, n_i\}$ that

$$\int_S^T \phi_{ij}(s)dW_s = \sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{ij}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) \int_S^T \psi_p(s)dW_s. \tag{2.9}$$

For every $i \in \{1, \dots, k+l\}$ let $g_i \in C_b^{\infty, \sigma(\mathcal{F}_0 \cup \mathcal{N})}(\mathbb{R}^d \times \Omega, \mathbb{R})$ be a function with the property that for all $(y_1, \dots, y_d) \in \mathbb{R}^d$ it holds $\mathbb{P}$-almost surely that

$$g_i(y_1, \dots, y_d) = f_i\Big(\sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{i1}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) y_p, \dots, \sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{in_i}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) y_p \Big). \tag{2.10}$$

It follows that for all $i \in \{1, \dots, k+l\}$ and all $(x_1, \dots, x_d), (y_1, \dots y_d) \in \mathbb{R}^d$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\sum_{r=1}^{d} \frac{\partial g_i}{\partial y_r}(y_1, \dots, y_d) x_r \qquad (2.11) \\
&= \sum_{r=1}^{d} \frac{\partial f_i}{\partial y_r} \Big(\sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{i1}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) y_p, \dots, \sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{in_i}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) y_p \Big) x_r \\
&= \sum_{r=1}^{d} \sum_{j=1}^{n_i} \frac{\partial f_i}{\partial x_j} \Big(\sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{i1}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) y_p, \dots, \sum_{p=1}^{d} \Big(\int_S^T \langle \phi_{in_i}(t), \psi_p(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) y_p \Big) \\
&\qquad \cdot \Big(\int_S^T \langle \phi_{ij}(t), \psi_r(t) \rangle_{\mathrm{HS}(U,\mathbb{R})} dt \Big) x_r.
\end{aligned}$$

It follows from equations (2.6), (2.9) and (2.10) that it holds $\mathbb{P}$-almost surely that

$$F = \sum_{i=1}^{k} g_i\Big(\int_S^T \psi_1(s)dW_s, \dots, \int_S^T \psi_d(s)dW_s \Big) h_i = \sum_{i=k+1}^{k+l} g_i\Big(\int_S^T \psi_1(s)dW_s, \dots, \int_S^T \psi_d(s)dW_s \Big) h_i. \tag{2.12}$$

For all $i \in \{1, \dots, k+l\}$ let $Z_i \in \mathcal{M}(\sigma(\sigma(W) \cup \mathcal{N}), \mathcal{B}_{\mathbb{R}})$ be a random variable such that it holds $\mathbb{P}$-almost surely that $Z_i = \int_S^T \psi_i(s)dW_s$. We conclude that it holds $\mathbb{P}$-almost surely that

$$\sum_{i=1}^{k} g_i(Z_1, \dots, Z_d) h_i = \sum_{i=k+1}^{k+l} g_i(Z_1, \dots, Z_d) h_i. \tag{2.13}$$

Let

$$f : \mathbb{R}^d \times C_b^\infty(\mathbb{R}^d, \mathbb{R}) \to \mathbb{R} \tag{2.14}$$

be a mapping with the property that for all $(y_1, \dots, y_d) \in \mathbb{R}^d$ and all $l \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$ it holds that

$$f\big((y_1, \dots, y_d), l\big) = l(y_1, \dots, y_d). \tag{2.15}$$

Let $Z : \Omega \to \mathbb{R}^d$ be a random variable with the property that it holds $\mathbb{P}$-almost surely that $Z = (Z_1, \dots, Z_d)$ and let $Y : \Omega \to C_b^\infty(\mathbb{R}^d, \mathbb{R})$ be a random variable with the property that for all $\omega \in \Omega$ it holds that

$$Y(\omega) = \sum_{i=1}^{k} g_i(\cdot)(\omega) h_i - \sum_{i=k+1}^{k+l} g_i(\cdot)(\omega) h_i. \tag{2.16}$$

Then it holds $\mathbb{P}$-almost surely that

$$f(Z, Y) = 0. \tag{2.17}$$

Consider the random variables

$$\tilde{Z} : \Omega \times \Omega \to \mathbb{R}^d; (\omega, \tilde{\omega}) \mapsto Z(\omega) \tag{2.18}$$

and

$$\tilde{Y} : \Omega \times \Omega \to \mathbb{R}^d; (\omega, \tilde{\omega}) \mapsto Y(\tilde{\omega}). \tag{2.19}$$

Since the σ-algebras $\sigma(\sigma(W) \cup \mathcal{N})$ and $\sigma(\mathcal{F}_0 \cup \mathcal{N})$ are independent, it follows that the random variables Z and Y are independent. We conclude that

$$\mathbb{P} \otimes \mathbb{P}_{(\tilde{Z}, \tilde{Y})} = \mathbb{P}_{(Z,Y)}. \tag{2.20}$$

From

$$\begin{aligned}
&\mathbb{P} \otimes \mathbb{P}\big(\big\{(\omega, \tilde{\omega}) \in \Omega \times \Omega \colon f(\tilde{Z}, \tilde{Y})(\omega, \tilde{\omega}) = 0\big\}\big) \\
&= \mathbb{P} \otimes \mathbb{P}_{(\tilde{Z}, \tilde{Y})}\big(\big\{(z, y) \in \mathbb{R}^d \times C_b^\infty(\mathbb{R}^d, \mathbb{R}) \colon f(z, y) = 0\big\}\big) \\
&= \mathbb{P}_{(Z,Y)}\big(\big\{(z, y) \in \mathbb{R}^d \times C_b^\infty(\mathbb{R}^d, \mathbb{R}) : f(z, y) = 0\big\}\big) \\
&= \mathbb{P}\big(\big\{\omega \in \Omega \colon f(Z(\omega), Y(\omega)) = 0\big\}\big) = 1
\end{aligned} \tag{2.21}$$

it follows that for $\mathbb{P} \otimes \mathbb{P}$-almost all $(\omega, \tilde{\omega}) \in \Omega \times \Omega$ it holds that $f(\tilde{Z}, \tilde{Y})(\omega, \tilde{\omega}) = 0$. It follows that for $\mathbb{P} \otimes \mathbb{P}$-almost all $(\omega, \tilde{\omega}) \in \Omega \times \Omega$ it holds that

$$\sum_{i=1}^{k} g_i(Z_1(\omega), \dots, Z_d(\omega))(\tilde{\omega}) h_i = \sum_{i=k+1}^{k+l} g_i(Z_1(\omega), \dots, Z_d(\omega))(\tilde{\omega}) h_i. \tag{2.22}$$

It follows that for $\mathbb{P}_Z$-almost all $(y_1, \dots, y_d) \in (Z_1, \dots, Z_d)(\Omega)$ and $\mathbb{P}$-almost all $\tilde{\omega} \in \Omega$ it holds that

$$\sum_{i=1}^{k} g_i(y_1, \dots, y_d)(\tilde{\omega}) h_i = \sum_{i=k+1}^{k+l} g_i(y_1, \dots, y_d)(\tilde{\omega}) h_i. \tag{2.23}$$

Since $(Z_1, \dots, Z_d)(\Omega)$ is dense in $\mathbb{R}^d$ and due to the smoothness of $g_1, \dots, g_{k+l}$, we conclude that for all $(y_1,\ldots,y_d)\in\mathbb{R}^d$ and $\mathbb{P}$-almost all $\tilde{\omega}\in\Omega$ it holds that

$$\sum_{i=1}^{k} g_i(y_1,\ldots,y_d)(\tilde{\omega})h_i=\sum_{i=k+1}^{k+l} g_i(y_1,\ldots,y_d)(\tilde{\omega})h_i. \tag{2.24}$$

Consequently, we obtain that for all $j\in\{1,\ldots,d\}$, all $(y_1,\ldots,y_d)\in\mathbb{R}^d$ and $\mathbb{P}$-almost all $\tilde{\omega}\in\Omega$ it holds that

$$\sum_{i=1}^{k}\frac{\partial g_i}{\partial y_j}(y_1,\ldots,y_d)(\tilde{\omega})h_i=\sum_{i=k+1}^{k+l}\frac{\partial g_i}{\partial y_j}(y_1,\ldots,y_d)(\tilde{\omega})h_i. \tag{2.25}$$

From equation (2.11) and equation (2.25) we infer that it holds $\mathbb{P}\times\lambda_{[S,T]}$-almost everywhere that

$$\begin{aligned}
\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)F &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\partial f_i}{\partial x_j}\Big(\int_S^T \phi_{i1}(s)dW_s,\ldots,\int_S^T \phi_{in_i}(s)dW_s\Big)\phi_{ij}h_i\\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\partial f_i}{\partial x_j}\Big(\sum_{p=1}^{d}\Big(\int_S^T\langle\phi_{i1}(t),\psi_p(t)\rangle_{\mathrm{HS}(U,\mathbb{R})}dt\int_S^T\psi_p(s)dW_s\Big),\ldots,\\
&\qquad \sum_{p=1}^{d}\Big(\int_S^T\langle\phi_{in_i}(t),\psi_p(t)\rangle_{\mathrm{HS}(U,\mathbb{R})}dt\int_S^T\psi_p(s)dW_s\Big)\Big)\\
&\qquad \cdot\Big(\sum_{r=1}^{d}\int_S^T\langle\phi_{ij}(s),\psi_r(s)\rangle_{\mathrm{HS}(U,\mathbb{R})}ds\;\psi_r\Big)h_i\\
&= \sum_{i=1}^{k}\sum_{r=1}^{d}\sum_{j=1}^{n_i}\frac{\partial f_i}{\partial x_j}\Big(\sum_{p=1}^{d}\Big(\int_S^T\langle\phi_{i1}(t),\psi_p(t)\rangle_{\mathrm{HS}(U,\mathbb{R})}dt\int_S^T\psi_p(s)dW_s\Big),\ldots,\\
&\qquad \sum_{p=1}^{d}\Big(\int_S^T\langle\phi_{in_i}(t),\psi_p(t)\rangle_{\mathrm{HS}(U,\mathbb{R})}dt\int_S^T\psi_p(s)dW_s\Big)\Big)\\
&\qquad \cdot\Big(\int_S^T\langle\phi_{ij}(s),\psi_r(s)\rangle_{\mathrm{HS}(U,\mathbb{R})}ds\;\psi_r\Big)h_i\\
&= \sum_{i=1}^{k}\sum_{r=1}^{d}\frac{\partial g_i}{\partial x_k}\Big(\int_S^T\psi_1(s)dW_s,\ldots,\int_S^T\psi_d(s)dW_s\Big)\psi_r h_i\\
&= \Big(\sum_{r=1}^{d}\psi_r\Big)\sum_{i=1}^{k}\frac{\partial g_i}{\partial x_k}\Big(\int_S^T\psi_1(s)dW_s,\ldots,\int_S^T\psi_d(s)dW_s\Big)h_i\\
&= \Big(\sum_{r=1}^{d}\psi_r\Big)\sum_{i=k+1}^{k+l}\frac{\partial g_i}{\partial x_k}\Big(\int_S^T\psi_1(s)dW_s,\ldots,\int_S^T\psi_d(s)dW_s\Big)h_i\\
&= \sum_{i=k+1}^{k+l}\sum_{j=1}^{n_i}\frac{\partial f_i}{\partial x_j}\Big(\int_S^T \phi_{i1}(s)dW_s,\ldots,\int_S^T \phi_{in_i}(s)dW_s\Big)\phi_{ij}h_i.
\end{aligned} \tag{2.26}$$

The proof of Lemma 2.3 is thus completed as $F\in\operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\big)$ was arbitrary. $\square$

Remark 2.4. Assume Setting 2.1. Then it holds for all $F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ that

$$\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \operatorname{HS}(U, H)))\big) \tag{2.27}$$

and it holds for all $k \in \mathbb{N}$, $F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]^k}; (\operatorname{HS}(U, H))^k))\big)$ that

$$\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]^k}; (\operatorname{HS}(U, H))^k))F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]^{k+1}}; (\operatorname{HS}(U, H))^{k+1}))\big). \tag{2.28}$$

The previous remark ensures that the multiple Malliavin differential operator is well-defined:

Definition 2.5. *Assume Setting 2.1 and let $k \in \mathbb{N}$. If $k = 0$, the k-th simple Malliavin differential operator is an operator*

$$\widetilde{\mathcal{D}}^0(\mathbb{P}, \mathcal{F}_0, W; H)\colon \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big) \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H) \tag{2.29}$$

with the property that for all $F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ it holds $\mathbb{P}$-almost surely that $\widetilde{\mathcal{D}}^0(\mathbb{P}, \mathcal{F}_0, W; H)F = F$. If $k = 1$, the k-th simple Malliavin differential operator is the linear operator with the property that $\widetilde{\mathcal{D}^1}(\mathbb{P}, \mathcal{F}_0, W; H) = \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)$. If $k \geq 2$, the k-th simple Malliavin differential operator

$$\begin{aligned}\widetilde{\mathcal{D}^k}(\mathbb{P}, \mathcal{F}_0, W; H)\colon \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]^{k-1}}; (\operatorname{HS}(U, H))^{k-1}))\big) \\ \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]^k}; (\operatorname{HS}(U, H))^k))\end{aligned} \tag{2.30}$$

is the linear operator with the property that for all $F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ it holds that

$$\widetilde{\mathcal{D}^k}(\mathbb{P}, \mathcal{F}_0, W; H)F = \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]^{k-1}}; (\operatorname{HS}(U, H))^{k-1})\big(\widetilde{\mathcal{D}^{k-1}}(\mathbb{P}, \mathcal{F}_0, W; H)F\big). \tag{2.31}$$

The following two lemmata are needed to prove Lemma 2.8.

Lemma 2.6. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $F \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R})$ be a random variable which is bounded $\mathbb{P}$-almost surely. Then for every $i \in \mathbb{N}$ there is an $\alpha_i \in \mathbb{R}$ and a $C_i \in \mathcal{A}$ such that

$$\sum_{i=1}^{\infty} |\alpha_i| = \operatorname{ess\,sup}_{\omega \in \Omega} F(\omega) \mathbb{1}_{\{\bar\omega \in \Omega\colon F(\bar\omega) \geq 0\}} - \operatorname{ess\,inf}_{\omega \in \Omega} F(\omega) \mathbb{1}_{\{\bar\omega \in \Omega\colon F(\bar\omega) \leq 0\}} \tag{2.32}$$

and such that it holds $\mathbb{P}$-almost surely that $F = \sum_{i=1}^{\infty} \alpha_i \mathbb{1}_{C_i}$.

Proof of Lemma 2.6. Let $F^+ \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R})$ be a random variable with the property that for all $\omega \in \Omega$ it holds that

$$F^+(\omega) = F(\omega) \mathbb{1}_{\{\bar\omega \in \Omega\colon F(\bar\omega) \geq 0\}} \tag{2.33}$$

and let $s^+ \in [0, \infty)$ be a real number with the property that

$$s^+ = \operatorname{ess\,sup}_{\omega\in\Omega} F^+(\omega). \tag{2.34}$$

Let $A_1 \in \mathcal{A}$ be a set with the property that

$$A_1 = \left\{\omega \in \Omega \colon F^+(\omega) \geq \tfrac{1}{2}s^+\right\} \tag{2.35}$$

and for all $i \in \{2, 3, \dots\}$ let $A_i \in \mathcal{A}$ be a set with the property that

$$A_i = \Big\{w \in \Omega \colon F^+(\omega) \geq \sum_{j=1}^{i} \tfrac{1}{2^j} \operatorname{ess\,sup}_{\omega\in\Omega} F^+(\omega)\mathbb{1}_{A_i}(\omega) + \tfrac{1}{2^i}s^+\Big\}. \tag{2.36}$$

Moreover, let $F^- \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R})$ be a random variable with the property that for all $\omega \in \Omega$ it holds that

$$F^-(\omega) = F(\omega)\mathbb{1}_{\{\bar{\omega}\in\Omega\colon F(\bar{\omega})\leq 0\}} \tag{2.37}$$

and let $s^- \in (-\infty, 0]$ be a real number with the property that

$$s^- = \operatorname{ess\,inf}_{\omega\in\Omega} F^-(\omega). \tag{2.38}$$

Let $B_1 \in \mathcal{A}$ be a set with the property that

$$B_1 = \left\{\omega \in \Omega \colon F^-(\omega) \leq \tfrac{1}{2}s^-\right\} \tag{2.39}$$

and for all $i \in \{2, 3, \dots\}$ let $B_i \in \mathcal{A}$ be a set with the property that

$$B_i = \Big\{w \in \Omega \colon F^-(\omega) \leq \sum_{j=1}^{i} \tfrac{1}{2^j} \operatorname{ess\,inf}_{\omega\in\Omega} F^-(\omega)\mathbb{1}_{A_i}(\omega) + \tfrac{1}{2^i}s^-\Big\}. \tag{2.40}$$

Finally, for all $i \in \mathbb{N}$ let $C_{2i} \in \mathcal{A}$ be a set with the property that $C_{2i} = A_i$ and let $C_{2i-1} \in \mathcal{A}$ be a set with the property that $C_{2i-1} = B_i$. For all $i \in \mathbb{N}$ let $\alpha_{2i} \in [0, \infty)$ be a real number with the property that $\alpha_{2i} = \frac{1}{2^i}s^+$ and let $\alpha_{2i-1} \in (-\infty, 0]$ be a real number with the property that $\alpha_{2i-1} = \frac{1}{2^i}s^-$. Then we conclude that it holds $\mathbb{P}$-almost surely that $F = \sum_{i=1}^{\infty} \alpha_i \mathbb{1}_{C_i}$. The proof of Lemma 2.6 is thus completed. ☐

Lemma 2.7. Let Ω be a set, let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\Omega)$ be algebras, and let $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ be a subset with the property that

$$\mathcal{M} = \{A \cap B \colon A \in \mathcal{A}, B \in \mathcal{B}\}. \tag{2.41}$$

Then $\mathcal{M}$ is a semiring that generates the σ-algebra $\sigma(\mathcal{A} \cup \mathcal{B})$, i.e. $\sigma(\mathcal{M}) = \sigma(\mathcal{A} \cup \mathcal{B})$.

Proof of Lemma 2.7. Clearly, $\emptyset \in \mathcal{M}$ and $\mathcal{M}$ is $\cap$-closed. Let $M, \bar{M} \in \mathcal{M}$. Then there are $A, \bar{A} \in \mathcal{A}$ and $B, \bar{B} \in \mathcal{B}$ such that $M = A \cap B$ and $\bar{M} = \bar{A} \cap \bar{B}$. The following calculation shows that $M \setminus \bar{M}$ can be written as a disjoint union of two sets in $\mathcal{M}$.

$$\begin{aligned} M \setminus \bar{M} &= (A \cap B) \setminus (\bar{A} \cap \bar{B}) \\ &= (A \cap B) \cap (\bar{A}^c \cup \bar{B}^c) \\ &= (A \cap B \cap \bar{A}^c) \cup (A \cap B \cap \bar{B}^c) \\ &= (A \cap B \cap \bar{A}^c \setminus \bar{B}^c) \dot{\cup} (A \cap B \cap \bar{B}^c) \\ &= ((A \cap \bar{A}^c) \cap (B \cap \bar{B})) \dot{\cup} (A \cap (B \cap \bar{B}^c)). \end{aligned} \tag{2.42}$$

Therefore, the set $\mathcal{M}$ is a semiring of sets. The proof of Lemma 2.7 is thus completed. □

The set $\operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ is sufficiently rich, as the following Lemma shows:

Lemma 2.8. Assume Setting 2.1 and let

$$\begin{aligned} \bar{\mathcal{S}}(\mathbb{P}, \mathcal{F}_0, W; H) &= \left\{ \begin{array}{c} F \in L^\infty(\mathbb{P}; H) \colon \exists \phi \in \mathcal{L}^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R})), \exists g \in \mathrm{C}_b^\infty(\mathbb{R}, \mathbb{R})), \\ \exists X_0 \in L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \mathcal{N})}; \mathbb{R}), \exists h \in H \text{ such that it holds } \mathbb{P}\text{-a.s. that} \\ F = g\Big(\int_S^T \phi(s) dW_s \Big) X_0 h \end{array} \right\} \\ &\subseteq \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H). \end{aligned} \tag{2.43}$$

Then it holds for all $p \in [1, \infty)$ that the set $\operatorname{span}\big(\bar{\mathcal{S}}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ is dense in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$ and that the set $\operatorname{span}\big(\bar{\mathcal{S}}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ is dense in $L^0(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; \mathcal{B}_H)$ with respect to $\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}$-almost sure convergence.

Proof of Lemma 2.8. **In the first step** we prove that the set $\operatorname{span}(\bar{\mathcal{S}}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})) \subseteq L^\infty(\mathbb{P}; \mathbb{R})$ is dense in $L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; \mathbb{R})$ with respect to convergence in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; \mathbb{R})$.
Fix $F \in L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; \mathbb{R})$ throughout the first step. From Lemma 2.6 we infer that for every $i \in \mathbb{N}$ there is an $\alpha_i \in \mathbb{R}$ and a $D_i \in \sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})$ such that

$$\sum_{i=1}^\infty |\alpha_i| = \operatorname{ess\,sup}_{\omega \in \Omega} F(\omega) \mathbb{1}_{\{\bar{\omega} \in \Omega \colon F(\bar{\omega}) \geq 0\}} - \operatorname{ess\,inf}_{\omega \in \Omega} F(\omega) \mathbb{1}_{\{\bar{\omega} \in \Omega \colon F(\bar{\omega}) \leq 0\}} \tag{2.44}$$

and such that it holds $\mathbb{P}$-almost surely that

$$F = \sum_{i=1}^\infty \alpha_i \mathbb{1}_{D_i}. \tag{2.45}$$

Consider the set

$$\mathcal{M} = \{A \cap B \colon A \in \sigma(W), B \in \mathcal{F}_0\}. \tag{2.46}$$

It follows from Lemma 2.7 that $\mathcal{M}$ is a semiring. Moreover, $\mathcal{M}$ generates the σ-algebra $\sigma(\mathcal{F}_0 \cup \sigma(W))$. It follows from the Approximation Theorem for Measures (Theorem 1.65(ii) in Klenke 2008) that for every $n \in \mathbb{N}$ and every $i \in \mathbb{N}$ there is a $k_i^n \in \mathbb{N}$ and sets $\bar{A}_{i1}^n, \ldots, \bar{A}_{ik_i^n}^n \in$

$\sigma(W)$ and $C^n_{i1}, \dots, C^n_{ik^n_i} \in \mathcal{F}_0$ such that the sets $\bar{A}^n_{i1} \cap C^n_{i1}, \dots, \bar{A}^n_{ik^n_i} \cap C^n_{ik^n_i}$ are pairwise disjoint and such that

$$\mathbb{P}\Big(D_i \triangle \cup_{j=1}^{k^n_i} (\bar{A}^n_{ij} \cap C^n_{ij})\Big) \le \frac{1}{n^{ip}}. \tag{2.47}$$

Note that for all $i, n \in \mathbb{N}$ it holds that $|\mathbb{1}_{D_i} - \sum_{j=1}^{k^n_i} \mathbb{1}_{\bar{A}^n_{ij}} \mathbb{1}_{C^n_{ij}}|^p \in \{0,1\}$. For all $i, n \in \mathbb{N}$ it therefore holds that

$$\Big\| \mathbb{1}_{D_i} - \sum_{j=1}^{k^n_i} \mathbb{1}_{\bar{A}^n_{ij}} \mathbb{1}_{C^n_{ij}} \Big\|^p_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} = \mathbb{E}\Big[\Big| \mathbb{1}_{D_i} - \sum_{j=1}^{k^n_i} \mathbb{1}_{\bar{A}^n_{ij}} \mathbb{1}_{C^n_{ij}} \Big|^p \Big] \le \frac{1}{n^{ip}}. \tag{2.48}$$

We have

$$\begin{aligned} \sigma(W) &= \left\{ \begin{matrix} Y^{-1}(B) \colon B \in \mathcal{B}_\mathbb{R}, Y \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R}) \text{ such that there is a } \phi \in \mathcal{L}^2(\lambda_{[S,T]}; \operatorname{HS}(U,\mathbb{R})) \\ \text{with the property that it holds } \mathbb{P}\text{-a.s. that } Y = \textstyle\int_S^T \phi(s) dW_s \end{matrix} \right\} \\ &= \sigma\left(\left\{ \begin{matrix} Y^{-1}([a,b]) \colon a, b \in \mathbb{R}, Y \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R}) \text{ such that there is a } \phi \in \mathcal{L}^2(\lambda_{[S,T]}; \operatorname{HS}(U,\mathbb{R})) \\ \text{with the property that it holds } \mathbb{P}\text{-a.s. that } Y = \textstyle\int_S^T \phi(s) dW_s \end{matrix} \right\} \right). \end{aligned} \tag{2.49}$$

The set

$$\left\{ \begin{matrix} Y^{-1}([a,b]) \colon a, b \in \mathbb{R}, Y \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R}) \text{ such that there is a } \phi \in \mathcal{L}^2(\lambda_{[S,T]}; \operatorname{HS}(U,\mathbb{R})) \\ \text{with the property that it holds } \mathbb{P}\text{-a.s. that } Y = \textstyle\int_S^T \phi(s) dW_s \end{matrix} \right\} \tag{2.50}$$

is a semiring of sets. We apply the Approximation Theorem for measures (Theorem 1.65(ii) in Klenke 2008) again and conclude that for all $i \in \mathbb{N}$ and all $j \in \{1, \dots, k^n_i\}$ there is a $p^n_{ij} \in \mathbb{N}$ and pairwise sets

$$\begin{aligned} & A^n_{ij1}, \dots, A^n_{ijp^n_{ij}} \in \\ & \left\{ \begin{matrix} Y^{-1}([a,b]) \colon a, b \in \mathbb{R}, Y \in \mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R}) \text{ such that there is a } \phi \in \mathcal{L}^2(\lambda_{[S,T]}; \operatorname{HS}(U,\mathbb{R})) \\ \text{with the property that it holds } \mathbb{P}\text{-a.s. that } Y = \textstyle\int_S^T \phi(s) dW_s \end{matrix} \right\} \end{aligned} \tag{2.51}$$

with the properties that $A^n_{ij1}, \dots, A^n_{ijp^n_{ij}}$ is a pairwise disjoint family and that

$$P\Big(\bar{A}^n_{ij} \triangle \cup_{l=1}^{p^n_{ij}} A^n_{ijl}\Big) \le \frac{1}{k^n_i n^{ip}}. \tag{2.52}$$

Note again, that for all $i, n \in \mathbb{N}$ it holds that $|\mathbb{1}_{\bar{A}^n_{ij}} - \sum_{l=1}^{p^n_{ij}} \mathbb{1}_{A^n_{ijl}}|^p \in \{0,1\}$. Therefore, it holds for all $i, n \in \mathbb{N}$ that

$$\begin{aligned}
&\Big\| \sum_{j=1}^{k_i^n} \mathbb{1}_{\bar{A}_{ij}^n} \mathbb{1}_{C_{ij}^n} - \sum_{j=1}^{k_i^n} \sum_{l=1}^{p_{ij}^n} \mathbb{1}_{A_{ijl}^n} \mathbb{1}_{C_{ij}^n} \Big\|^p_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} \\
&\leq \sum_{j=1}^{k_i^n} \|\mathbb{1}_{C_{ij}^n}\|^p_{L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} \Big\| \mathbb{1}_{\bar{A}_{ij}^n} - \sum_{l=1}^{p_{ij}^n} \mathbb{1}_{A_{ijl}^n} \Big\|^p_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} \\
&\leq \sum_{j=1}^{k_i^n} \mathbb{E}\Big[\Big| \mathbb{1}_{\bar{A}_{ij}^n} - \sum_{l=1}^{p_{ij}^n} \mathbb{1}_{A_{ijl}^n} \Big|^p \Big] \leq \sum_{j=1}^{k_i^n} \frac{1}{k_i^n n^{ip}} = \frac{1}{n^{ip}}.
\end{aligned} \tag{2.53}$$

Together with inequality (2.48) this leads to

$$\Big\| \mathbb{1}_{D_i} - \sum_{j=1}^{k_i^n} \sum_{l=1}^{p_{ij}^n} \mathbb{1}_{A_{ijl}^n} \mathbb{1}_{C_{ij}^n} \Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} < \frac{2}{n^i}. \tag{2.54}$$

It follows from (2.51) that for all $i, n \in \mathbb{N}$, all $j \in \{1, \ldots, k_i^n\}$ and all $l \in \{1, \ldots, p_{ij}^n\}$ there are $a_{ijl}^n, \bar{a}_{ijl}^n \in \mathbb{R}$ and $\phi_{ijl}^n \in \mathcal{L}^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}))$ such that it holds $\mathbb{P}$-almost surely that

$$\mathbb{1}_{A_{ijl}^n} = \mathbb{1}_{[a_{ijl}^n, \bar{a}_{ijl}^n]} \Big(\int\limits_S^T \phi_{ijl}^n(s) dW_s \Big). \tag{2.55}$$

From inequaltiy (2.54) it follows that for all $i \in \mathbb{N}$ it holds that

$$\Big\| \mathbb{1}_{D_i} - \sum_{j=1}^{k_i^n} \sum_{l=1}^{p_{ij}^n} \mathbb{1}_{[a_{ijl}^n, \bar{a}_{ijl}^n]} \Big(\int\limits_S^T \phi_{ijl}^n(s) dW_s \Big) \mathbb{1}_{C_{ij}^n} \Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} \leq \frac{2}{n^i}. \tag{2.56}$$

For all $i, n \in \mathbb{N}$, all $j \in \{1, \ldots, k_i^n\}$ and all $l \in \{1, \ldots, p_{ij}^n\}$ let $f_{ijl}^n(A) \in C^\infty_{\mathrm{cpt}}(\mathbb{R}, \mathbb{R})$ be a smooth function with values in $[0, 1]$ such that

$$f_{ijl}^n(A) \big| [a_{ijl}^n, \bar{a}_{ijl}^n] \equiv 1 \tag{2.57}$$

and such that

$$\mathbb{E}\Big[\Big| \mathbb{1}_{[a_{ijl}^n, \bar{a}_{ijl}^n]} \Big(\int\limits_S^T \phi_{ijl}^n(s) dW_s \Big) - f_{ijl}^n(A) \Big(\int\limits_S^T \phi_{ijl}^n(s) dW_s \Big) \Big| \Big] \leq \Big(\frac{1}{k_i^n p_{ij}^n n^i} \Big)^p. \tag{2.58}$$

Since it holds that

$$\Big\| \mathbb{1}_{[a_{ijl}^n, \bar{a}_{ijl}^n]} \Big(\int\limits_S^T \phi_{ijl}^n(s) dW_s \Big) - f_{ijl}^n(A) \Big(\int\limits_S^T \phi_{ijl}^n(s) dW_s \Big) \Big\|_{L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})};\mathbb{R})} \leq 1 \tag{2.59}$$

we have

$$\Big\|\mathbb{1}_{[a^n_{ijl},\bar{a}^n_{ijl}]}\Big(\int\limits_S^T \phi^n_{ijl}(s)dW_s\Big) - f^n_{ijl}(A)\Big(\int\limits_S^T \phi^n_{ijl}(s)dW_s\Big)\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})} \le \frac{1}{k^n_i p^n_{ij} n^i}. \tag{2.60}$$

Therefore, we have

$$\Big\|\mathbb{1}_{D_i} - \sum_{j=1}^{k^n_i}\sum_{l=1}^{k^n_{ij}} f^n_{ijl}(A)\Big(\int\limits_S^T \phi^n_{ijl}(s)dW_s\Big)\mathbb{1}_{C^n_{ij}}\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})} \le \frac{3}{n^i}. \tag{2.61}$$

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{span}(\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R}))$ with the property that for all $n\in\mathbb{N}$ it holds $\mathbb{P}$-almost surely that

$$F_n = \sum_{i=1}^{n}\alpha_i\sum_{j=1}^{k^n_i}\sum_{l=1}^{k^n_{ij}} f^n_{ijl}(A)\Big(\int\limits_S^T \phi^n_{ijl}(s)dW_s\Big)\mathbb{1}_{C^n_{ij}}. \tag{2.62}$$

Then it holds for all $n\in\{2,3,\dots\}$ that

$$\begin{aligned}
&\Big\|\sum_{i=1}^{n}\alpha_i\mathbb{1}_{D_i} - F_n\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})}\\
&= \Big\|\sum_{i=1}^{n}\alpha_i\mathbb{1}_{D_i} - \sum_{i=1}^{n}\alpha_i\sum_{j=1}^{k^n_i}\sum_{l=1}^{k^n_{ij}}[f^n_{ijl}(A)\Big(\int\limits_S^T \phi^n_{ijl}(s)dW_s\Big)\mathbb{1}_{C^n_{ij}}\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})}\\
&\le \sum_{i=1}^{n}|\alpha_i|\Big\|\mathbb{1}_{D_i} - \sum_{j=1}^{k^n_i}\sum_{l=1}^{k^n_{ij}} f^n_{ijl}(A)\Big(\int\limits_S^T \phi^n_{ijl}(s)dW_s\Big)\mathbb{1}_{C^n_{ij}}\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})}\\
&\le \sum_{i=1}^{n}|\alpha_i|\frac{3}{n^i}\\
&\le \tfrac{3}{n-1}\cdot\max\{\operatorname{ess\,sup}_{\omega\in\Omega}F(\omega), -\operatorname{ess\,inf}_{\omega\in\Omega}F(\omega)\}
\end{aligned} \tag{2.63}$$

and

$$\Big\|F - \sum_{i=1}^{n}\alpha_i\mathbb{1}_{D_i}\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})} = \Big\|\sum_{i=n+1}^{\infty}\alpha_i\mathbb{1}_{D_i}\Big\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})} \le \sum_{n+1}^{\infty}|\alpha_i|. \tag{2.64}$$

This proves that the sequence $(F_n)_{n\in\mathbb{N}}$ converges to $F\in L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})$ with respect to convergence in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})$ as $n\to\infty$. This finishes the first step as $F\in L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})$ was arbitrary.

In the second step we prove that the space $\operatorname{span}(\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;H))$ is dense in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$. Fix $F\in L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ throughout the second step. Let $(F^n)_{n\in\mathbb{N}}$ be a sequence in

$L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ such that for all $n\in\mathbb{N}$ it holds $\mathbb{P}$-almost surely that

$$F^n = F\mathbb{1}_{\{\|F\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)}\leq n\}}. \tag{2.65}$$

The sequence $(F^n)_{n\in\mathbb{N}}$ converges to F in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ as $n\to\infty$. Let $(h_i)_{i\in I}$, $I\in\{\{1,\dots,k\}\colon k\in\mathbb{N}\}\cup\{\mathbb{N}\}$, be an orthonormal basis of H. For all $n\in\mathbb{N}$ it holds that

$$F^n = \sum_{i\in I}\langle F^n,h_i\rangle_H h_i. \tag{2.66}$$

For all $n\in\mathbb{N}$ and all $i\in I$ let $(F_k^{n,i})_{k\in\mathbb{N}}$ be a sequence in $\operatorname{span}(\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R}))$ with the property that for all $k\in\mathbb{N}$ it holds that

$$\|F_k^{n,i}-\langle F^n,h_i\rangle_H\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})}\leq\frac{1}{k^i}. \tag{2.67}$$

For all $n\in\mathbb{N}$ let $(F_k^n)_{k\in\mathbb{N}}$ be a sequence in $\operatorname{span}(\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;H))$ such that for all $k\in\mathbb{N}$ it holds that $F_k^n=\sum_{i\in I}F_k^{n,i}h_i$. We conclude that for all $n,k\in\mathbb{N}$ it holds that

$$\begin{aligned}\|F_k^n-F^n\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)}&\leq\sum_{i\in I}\|F_k^{n,i}-\langle F^n,h_i\rangle_H\|_{L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})}\\&\leq\sum_{i\in I}\frac{1}{k^i}\leq\frac{1}{k-1}.\end{aligned} \tag{2.68}$$

The sequence $(F_n^n)_{n\in\mathbb{N}}$ converges therefore to F with respect to convergence in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ as $n\to\infty$. The space $\operatorname{span}(\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;H))$ is therefore dense in $L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$. This finishes the second step of the proof as $F\in L^p(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ was arbitrary.

In the third and last step we finally show that the set $\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;H)$ is dense in $L^0(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathcal{B}_H)$ with respect to $\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})}$-almost sure convergence. Fix $F\in L^0(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathcal{B}_\mathbb{R})$ for the rest of the proof. Let $(h_i)_{i\in I}$, $I\in\{\{1,\dots,k\}\colon k\in\mathbb{N}\}\cup\{\mathbb{N}\}$, be an orthonormal basis of H.

Note that for all $i\in I$ and all $n\in\mathbb{N}$ it holds that $\langle F,h_i\rangle_H\mathbb{1}_{\{\langle F,h_i\rangle_H^2\leq n\}}\in L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};\mathbb{R})$. From the second step of the proof it follows that for all $i\in I$ there is a sequence $(F_{in})_{n\in\mathbb{N}}$ in $\bar{\mathcal{S}}(\mathbb{P},\mathcal{F}_0,W;H)$ such that for all $\varepsilon\in(0,\infty)$ there is an $n_0(\varepsilon)\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ such that $n\geq n_0(\varepsilon)$ and such that $n\geq i$ it holds that

$$\mathbb{P}\left[\left|\langle F,h_i\rangle_H\mathbb{1}_{\{\langle F,h_i\rangle_H^2\leq n\}}-F_{in}\right|\geq\tfrac{\varepsilon}{\sqrt{n}}\right]\leq\frac{1}{n^2}. \tag{2.69}$$

Consequently, it holds for all $n \in \mathbb{N}$ and all $\varepsilon \in (0, \infty)$ that

$$\begin{aligned} &\mathbb{P}\Big[\Big\|F - \sum_{i\in I, i\leq n} F_{in} h_i\Big\|_H \geq 3\varepsilon\Big] \\ &\leq \mathbb{P}\Big[\Big\|F - \sum_{i\in I, i\leq n} \langle F, h_i\rangle_H h_i\Big\|_H \geq \varepsilon\Big] \\ &\qquad + \mathbb{P}\Big[\Big\| \sum_{i\in I, i\leq n} \Big(\langle F, h_i\rangle_H - \langle F, h_i\rangle_H \mathbb{1}_{\{\langle F, h_i\rangle_H^2 \leq n\}}\Big) h_i\Big\|_H \geq \varepsilon\Big] \\ &= \mathbb{P}\Big[\sum_{i\in I, i\geq n+1} \langle F, h_i\rangle_H^2 \geq \varepsilon^2\Big] + \mathbb{P}\Big[\sum_{i\in I, i\leq n} \langle F, h_i\rangle_H^2 \mathbb{1}_{\{\langle F, h_i\rangle_H^2 > n\}} \geq \varepsilon^2\Big] \\ &\qquad + \mathbb{P}\Big[\sum_{i\in I, i\leq n} \Big(\langle F, h_i\rangle_H \mathbb{1}_{\{\langle F, h_i\rangle_H^2 \leq n\}} - F_{in}\Big)^2 \geq \varepsilon^2\Big]. \end{aligned} \tag{2.70}$$

From equations (2.69) and (2.70) it follows that the sequence $\big(\sum_{i\in I, i\leq n} F_{in} h_i\big)_{n\in\mathbb{N}}$ in $\operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H))$ converges to F as $n \to \infty$ in $\mathbb{P}$-probability. As convergence in probability implies almost sure convergence along subsequences, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that the sequence $\big(\sum_{i\in I, i\leq n_k} F_{in_k} h_i\big)_{k\in\mathbb{N}}$ in $\operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H))$ converges to F as $k \to \infty$ $\mathbb{P}$-almost surely. This proof of Lemma 2.8 is thus completed as $F \in L^0(\mathbb{P}; \mathcal{B}_H)$ was arbitrary. □

The following result is an immediate corollary of Lemma 2.8:

Corollary 2.9. Assume Setting 2.1, let

$$\begin{aligned} &\bar{\mathcal{S}}(\mathbb{P}, \mathcal{F}_0, W; H) = \\ &\left\{ \begin{array}{c} F \in L^\infty(\mathbb{P}; H)\colon \exists \phi \in \mathcal{L}^2(\lambda_{[S,T]}; \operatorname{HS}(U, \mathbb{R})), \exists g \in C_b^\infty(\mathbb{R}, \mathbb{R}), \exists X_0 \in L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \mathcal{N})}; \mathbb{R}), \\ \exists h \in H \text{ with the property that it holds } \mathbb{P}\text{-a.s. that } F = g\Big(\int_S^T \phi(s) dW_s\Big) X_0 h \end{array} \right\} \end{aligned} \tag{2.71}$$

and let $X \in L^2(\mathbb{P}; H)$. If for all $F \in \bar{\mathcal{S}}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds that $\mathbb{E}[\langle X, F\rangle_H] = 0$, then $X = 0$.

Lemma 2.10 (Product rule)**.** Assume Setting 2.1 and let H_1 and H_2 be two separable real Hilbert spaces. For all $F \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_1, H_2)))$ and all $G \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_0, H_1)))$ it holds that $FG \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_0, H_2)))$ and for all $u \in U$ it holds that

$$\begin{aligned} &\widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_0, H_2)\big)(FG)(u) \\ &= \widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_1, H_2)\big)(F)(u) G + F \widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_0, H_1)\big)(G)(u). \end{aligned} \tag{2.72}$$

Proof of Lemma 2.10. Due to linearity it suffices to prove that for all $u \in U$, $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_1, H_2))$ and all $G \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_0, H_1))$ equation (2.72) holds true. Let $u \in U$, $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_1, H_2))$ and $G \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \operatorname{HS}(H_0, H_1))$. Then there are $n, l \in \mathbb{N}$, $\phi_1, \ldots, \phi_{n+l} \in \mathcal{L}^2(\lambda_{[S,T]}; \operatorname{HS}(U, \mathbb{R}))$, $\psi \in \operatorname{HS}(H_1, H_2)$, $\chi \in \operatorname{HS}(H_0, H_1)$ and $f \in \mathcal{M}(\mathcal{B}_{\mathbb{R}^n} \otimes \sigma(\mathcal{F}_0 \cup \mathcal{N}), \mathcal{B}_{\mathbb{R}})$, $g \in \mathcal{M}(\mathcal{B}_{\mathbb{R}^n} \otimes \sigma(\mathcal{F}_0 \cup \mathcal{N}), \mathcal{B}_{\mathbb{R}})$ such that for all $\omega \in \Omega$ it holds that $f(\cdot)(\omega) \in C_b^\infty(\mathbb{R}^n, L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\mathcal{N})};\mathbb{R}))$ and $g(\cdot)(\omega) \in C_b^\infty(\mathbb{R}^m, L^\infty(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\mathcal{N})};\mathbb{R}))$ such that it holds $\mathbb{P}$-almost surely that

$$F = f\Big(\int_S^T \phi_1(s)dW_s, \ldots, \int_S^T \phi_n(s)dW_s\Big)\psi \tag{2.73}$$

and

$$G = g\Big(\int_S^T \phi_{n+1}(s)dW_s, \ldots, \int_S^T \phi_{n+l}(s)dW_s\Big)\chi. \tag{2.74}$$

Then $FG \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H_0, H_2)))$ and it holds $\mathbb{P}\otimes\lambda_{[S,T]}$-almost everywhere that

$$\begin{aligned}
&\widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H_0, H_2)\big)(FG)(u)\\
&= \widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H_0, H_2)\big)\\
&\quad \Big(fg\Big(\int_S^T \phi_1(s)dW_s, \ldots, \int_S^T \phi_n(s)dW_s, \int_S^T \phi_{n+1}(s)dW_s, \ldots, \int_S^T \phi_{n+l}(s)dW_s\Big)(\psi\chi)\Big)(u)\\
&= \sum_{i=1}^{n+l}\Big(\frac{\partial}{\partial x_i} fg\Big(\int_S^T \phi_1(s)dW_s, \ldots, \int_S^T \phi_n(s)dW_s, \int_S^T \phi_{n+1}(s)dW_s, \ldots, \int_S^T \phi_{n+l}(s)dW_s\Big)\Big)\\
&\quad \phi_i(u)\psi\chi\\
&= \left(\sum_{i=1}^{n} \tfrac{\partial}{\partial x_i} f\Big(\int_S^T \phi_1(s)dW_s, \ldots, \int_S^T \phi_n(s)dW_s\Big)\phi_i(u)\psi\right)\\
&\quad \Big(g\Big(\int_S^T \phi_{n+1}(s)dW_s, \ldots, \int_S^T \phi_{n+l}(s)dW_s\Big)\chi\Big)\\
&\quad + \Big(f\Big(\int_S^T \phi_1(s)dW_s, \ldots, \int_S^T \phi_n(s)dW_s\Big)\psi\Big)\\
&\quad \left(\sum_{j=1}^{l} \tfrac{\partial}{\partial x_i} g\Big(\int_S^T \phi_{n+1}(s)dW_s, \ldots, \int_S^T \phi_{n+l}(s)dW_s\Big)\phi_{n+j}(u)\chi\right)\\
&= \widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H_1, H_2)\big)F(u)G + F\widetilde{\mathcal{D}}\big(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H_0, H_1)\big)G(u).
\end{aligned} \tag{2.75}$$

The proof of Lemma 2.10 is thus completed. □

The simple Malliavin derivative satisfies an integration-by-parts formula.

Lemma 2.11. Assume Setting 2.1. Then it holds for all $F \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ and all $\psi \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H))$ that

$$\mathbb{E}\Big[\big\langle\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F, \psi\big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\Big] = \mathbb{E}\Big[\Big\langle F, \int_S^T \psi(s)dW_s\Big\rangle_H\Big]. \tag{2.76}$$

Proof of Lemma 2.11. Due to linearity, it suffices to prove that for all $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)$ and all $\psi \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H))$ equation (2.76) holds true. Fix $\psi \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H))$ and $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)$ for the rest of the proof. Then there are $n \in \mathbb{N}$, $\phi_1, \ldots, \phi_n \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}))$, $f \in C_b^{\infty,\sigma(\mathcal{F}_0\cup\mathcal{N})}(\mathbb{R}^n \times \Omega, \mathbb{R})$, $h \in H$ such that it holds $\mathbb{P}$-almost surely that

$$F = f\Big(\int_S^T \phi_1(s)dW_s, \dots, \int_S^T \phi_n(s)dW_s\Big)h. \tag{2.77}$$

Let $d \in \mathbb{N}$ be the dimension of

$$\operatorname{span}(\{\phi_1, \dots, \phi_n\}) \subseteq L^2(\lambda_{[S,T]}; \mathbb{R}^m). \tag{2.78}$$

Moreover, let $\{\chi_1, \dots, \chi_d\} \subseteq L^2(\lambda_{[S,T]}; \mathbb{R}^m)$ be an orthonormal base of $\operatorname{span}(\{\phi_1, \dots, \phi_n\})$. We conclude that for all $i \in \{1, \dots, n\}$ it holds $\lambda_{[S,T]}$-almost everywhere that

$$\phi_i = \sum_{l=1}^{d} \Big(\int_S^T \langle \phi_i(s), \chi_l(s)\rangle_{\mathrm{HS}(U,H)} ds\Big)\chi_l. \tag{2.79}$$

Since the Itô integral is linear it holds for all $i \in \{1, \dots, n\}$ $\mathbb{P}$-almost surely that

$$\int_S^T \phi_i(s)dW_s = \sum_{l=1}^{d} \Big(\int_S^T \langle \phi_i(t), \chi_l(t)\rangle_{\mathrm{HS}(U,H)} dt\Big) \int_S^T \chi_l(s)dW_s. \tag{2.80}$$

Let $g \in \mathcal{M}(\mathcal{B}_H \otimes \sigma(\mathcal{F}_0 \cup \mathcal{N}), \mathcal{B}_{\mathbb{R}})$ be a random function such that for all $\omega \in \Omega$ it holds that $g(\cdot)(\omega) \in \mathrm{C}_b^\infty(\mathbb{R}^d, \mathbb{R})$ and such that for all $(y_1, \dots, y_d) \in \mathbb{R}^d$ it holds that

$$g(y_1, \dots, y_d) = f\Big(\sum_{l=1}^{d} \Big(\int_S^T \langle \phi_1(t), \chi_l(t)\rangle_{\mathrm{HS}(U,\mathbb{R})} dt\Big) y_l, \dots, \sum_{l=1}^{d} \Big(\int_S^T \langle \phi_n(t), \chi_l(t)\rangle_{\mathrm{HS}(U,\mathbb{R})} dt\Big) y_l\Big). \tag{2.81}$$

From equation (2.80) we conclude that it holds $\mathbb{P}$-almost surely that

$$F = g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)h. \tag{2.82}$$

We have

$$\begin{aligned} \mathbb{E}\left[\left\langle F, \int_S^T \psi(s)dW_s\right\rangle_H\right] &= \mathbb{E}\left[\left\langle g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \psi_d(s)dW_s\Big)h, \int_S^T \psi(s)dW_s\right\rangle_H\right] \\ &= \mathbb{E}\left[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \psi_d(s)dW_s\Big) \int_S^T \langle h, \psi(s)\rangle_H dW_s\right]. \end{aligned} \tag{2.83}$$

Next, let $\psi^\perp \in \mathcal{L}^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))$ be such that it holds $\mathbb{P} \otimes \lambda_{[S,T]}$-almost everywhere that

$$\langle h, \psi^\perp\rangle_H = \langle h, \psi\rangle_H - \sum_{l=1}^{d} \Big(\int_S^T \chi_l(s)\langle h, \psi(s)\rangle_H ds\Big)\chi_l. \tag{2.84}$$

For all $l \in \{1, \dots, d\}$ it holds that $\chi_l \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U,\mathbb{R}))$ is orthogonal to $\langle h, \psi^\perp\rangle_H \in L^2(\mathbb{P}|_{\mathcal{B}_{\mathbb{R}}([S,T])}; \mathrm{HS}(U,\mathbb{R}))$. We conclude that for all $l \in \{1, \dots, d\}$ the random variables $\int_S^T \chi_i(s)dW_s$

and $\int_S^T \langle h, \psi^\perp(s)\rangle_H dW_s$ are uncorrelated. Consequently, it follows form the linearity of the Itô integral that

$$\begin{aligned}
&\mathbb{E}\left[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\int_S^T \langle h, \psi(s)\rangle_H dW_s\right] \\
&= E\left[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\right. \\
&\qquad\left.\Big(\Big(\sum_{l=1}^{d}\Big(\int_S^T \chi_l(s)\langle h, \psi(s)\rangle_H ds\Big)\int_S^T \chi_l(s)dW_s\Big) + \int_S^T \langle h, \psi^\perp(s)\rangle_H dW_s\Big)\right] \\
&= \sum_{l=1}^{d}\Big(\int_S^T \chi_l(s)\langle h, \psi(s)\rangle_H ds\Big)\mathbb{E}\Big[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\int_S^T \chi_l(s)dW_s\Big] \\
&\quad + \mathbb{E}\Big[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\Big]\mathbb{E}\Big[\int_S^T \langle h, \psi^\perp(s)\rangle_H dW_s\Big] \\
&= \sum_{l=1}^{d}\Big(\int_S^T \chi_l(s)\langle h, \psi(s)\rangle_H ds\Big)\mathbb{E}\Big[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\int_S^T \chi_l(s)dW_s\Big].
\end{aligned} \tag{2.85}$$

Since $\{\chi_1, \dots, \chi_d\} \subseteq L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}))$ is an orthonormal family, it follows that the distribution of $\big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\big)$ is an d-dimensional standard normal distribution. For all $l \in \{1, \dots, d\}$ the real-valued integration by parts formula yields that

$$\begin{aligned}
\mathbb{E}\Big[g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\int_S^T \chi_l(s)dW_s\Big] &= \int_{\mathbb{R}^d} g(x)x_l(2\pi)^{-\frac{d}{2}}\exp(-\tfrac{1}{2}\|x\|_{\mathbb{R}^d}^2)\,dx \\
&= -\int_{\mathbb{R}^d} g(x)(2\pi)^{-\frac{d}{2}}\tfrac{\partial}{\partial x_l}\exp(-\tfrac{1}{2}\|x\|_{\mathbb{R}^d}^2)\,dx \\
&= \int_{\mathbb{R}^d} \big(\tfrac{\partial}{\partial x_l}g\big)(x)(2\pi)^{-\frac{d}{2}}\exp(-\tfrac{1}{2}\|x\|_{\mathbb{R}^d}^2)\,dx \\
&= \mathbb{E}\left[\tfrac{\partial}{\partial x_l}g\Big(\int_S^T \chi_1(s)dW_s, \dots, \int_S^T \chi_d(s)dW_s\Big)\right].
\end{aligned} \tag{2.86}$$

Inserting this into equation (2.85) and inserting equation (2.85) into equation (2.83) results in

$$\begin{aligned} &E\Big[\Big\langle F, \int\limits_S^T \psi(s)dW_s\Big\rangle_H\Big] \\ &= \sum_{l=1}^d \Big(\int\limits_S^T \chi_l(s)\langle h, \psi(s)\rangle_H ds\Big)\mathbb{E}\Big[g\Big(\int\limits_S^T \chi_1(s)dW_s, \ldots, \int\limits_S^T \chi_d(s)dW_s\Big)\int\limits_S^T \chi_i(s)dW_s\Big] \\ &= \sum_{l=1}^d \Big(\int\limits_S^T \langle \chi_l(s)h, \psi(s)\rangle_H ds\Big)\mathbb{E}\Big[\tfrac{\partial}{\partial x_l} g\Big(\int\limits_S^T \chi_1(s)dW_s, \ldots, \int\limits_S^T \chi_d(s)dW_s\Big)\Big] \\ &= \sum_{l=1}^d \Big(\int\limits_S^T \Big\langle \chi_l(s)h, \psi(s)\Big\rangle_H ds\Big)\mathbb{E}\Big[\tfrac{\partial}{\partial x_l} g\Big(\int\limits_S^T \chi_1(s)dW_s, \ldots, \int\limits_S^T \chi_d(s)dW_s\Big)\Big] \\ &= \mathbb{E}\Big[\int\limits_S^T \Big\langle \sum_{l=1}^d \tfrac{\partial}{\partial x_l} g\Big(\int\limits_S^T \chi_1(s)dW_s, \ldots, \int\limits_S^T \chi_d(s)dW_s\Big)\chi_l(s)h, \psi(s)\Big\rangle_H ds\Big] \\ &= \mathbb{E}\left[\Big\langle \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F, \psi\Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\right]. \end{aligned} \tag{2.87}$$

The proof of Lemma 2.11 is thus completed. □

Lemma 2.12. Assume Setting 2.1. The simple Malliavin derivative

$$\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)\colon \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big) \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H))) \tag{2.88}$$

is a closable operator.

Proof of Lemma 2.12. Let $X \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H)))$ and let $(F_n)_{n\in\mathbb{N}}$ be a sequence such that for each $n \in \mathbb{N}$ it holds that $F_n \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H))$ and with the properties that $F_n \to 0$ in $L^2(\mathbb{P}; H)$ as $n \to \infty$ and that $\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F_n \to X$ in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H)))$ as $n \to \infty$. Then Lemma 2.10 and Lemma 2.11 imply that $F_n G \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)\big)$ and that for all $n \in \mathbb{N}$, all $\psi \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H))$ and all $G \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}))$ it holds that

$$\begin{aligned} &\mathbb{E}\Big[\Big\langle \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F_n, G\psi\Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\Big] \\ &= \mathbb{E}\Big[\Big\langle G(\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F_n), \psi\Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\Big] \\ &= \mathbb{E}\Big[\Big\langle \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)\,(GF_n) - (\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})G)F_n, \psi\Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\Big] \\ &= \mathbb{E}\Big[\Big\langle GF_n, \int\limits_S^T \psi(s)dW_s\Big\rangle_H\Big] - \mathbb{E}\Big[\Big\langle (\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})G)F_n, \psi\Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\Big]. \end{aligned} \tag{2.89}$$

Therefore, $F_n \to 0$ in $L^2(\mathbb{P}; H)$ as $n \to \infty$ and $\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H)F_n \to X$ in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H)))$ as $n \to \infty$ together with Hölder's inequality implies that for all $\psi \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))$ and all $G \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}))$ it holds that

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\langle X, G\psi \rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))} \Big]\Big| \\
&= \Big| \lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; H) F_n, G\psi \Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))} \Big]\Big| \qquad (2.90)\\
&\leq \limsup_{n\to\infty} \mathbb{E}\Big[\Big| \Big\langle G F_n, \int_S^T \psi(s) dW_s \Big\rangle_H \Big|\Big] + \limsup_{n\to\infty} \mathbb{E}\Big[\Big| \Big\langle (\widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}) G) F_n, \psi \Big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))} \Big|\Big] \\
&\leq \limsup_{n\to\infty} \|F_n\|_{L^2(\mathbb{P};H)} \Big(\|G\|_{L^\infty(\mathbb{P};\mathbb{R})} \Big\| \int_S^T \psi(s) dW_s \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\qquad\qquad + \Big\| \widetilde{\mathcal{D}}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}) G \Big\|_{L^\infty(\mathbb{P}; L^2(\lambda_{[S,T]};\mathrm{HS}(U,\mathbb{R})))} \|\psi\|_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))} \Big) = 0.
\end{aligned}$$

Together with linearity, this implies that for all

$$Y \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))\big) \qquad (2.91)$$

it holds that $\mathbb{E}[\langle X, Y \rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}] = 0$. Finally we conclude from Corollary 2.9 that $X = 0$. The proof of Lemma 2.12 is thus completed. ☐

Now, we can finally define the Malliavin differential operator.

Definition 2.13. *Assume Setting 2.1 and let $k \in \mathbb{N}$. Let*

$$\|\cdot\|^{(k,2)}_{\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H))} \colon \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)) \to [0,\infty) \qquad (2.92)$$

be a function with the property that for all $F \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H))$ it holds that

$$\|F\|^{(k,2)}_{\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H))} = \mathbb{E}\Big[\|F\|_H^2 + \sum_{i=1}^k \|\widetilde{\mathcal{D}^i}(\mathbb{P}, \mathcal{F}_0, W; H) F\|^2_{L^2(\lambda_{[S,T]^i};(\mathrm{HS}(U,H))^i)} \Big]^{1/2}. \qquad (2.93)$$

Denote by

$$\mathbb{D}^{(k,2)}(\mathbb{P}, \mathcal{F}_0, W; H) = (\mathbb{D}^{(k,2)}(\mathbb{P}, \mathcal{F}_0, W; H), \|\cdot\|_{\mathbb{D}^{(k,2)}(\mathbb{P},\mathcal{F}_0,W;H)}) \qquad (2.94)$$

the closure of the normed space

$$(\operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)), \|\cdot\|^{(k,2)}_{\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H))}) \qquad (2.95)$$

in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$. The k-th Malliavin differential operator

$$\mathcal{D}^k(\mathbb{P}, \mathcal{F}_0, W; H) \colon \mathbb{D}^{(k,2)}(\mathbb{P}, \mathcal{F}_0, W; H) \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))) \qquad (2.96)$$

is the closure of the simple k-th Malliavin differential operator in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$. If

$k=1$ *we call*

$$\mathcal{D}^1(\mathbb{P},\mathcal{F}_0,W;H)\colon \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)\to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))) \tag{2.97}$$

the Malliavin differential operator and write

$$\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)=\mathcal{D}^1(\mathbb{P},\mathcal{F}_0,W;H). \tag{2.98}$$

A simple corollary of Lemma 2.10 and Lemma 2.12 is the product rule for the Malliavin Differential operator.

Corollary 2.14 (Product rule)**.** Assume Setting 2.1. For all $F\in\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})$ and all $G\in\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)$ it holds that $FG\in\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)$ and

$$\mathcal{D}\big(\mathbb{P},\mathcal{F}_0,W;H\big)(FG)=\mathcal{D}\big(\mathbb{P},\mathcal{F}_0,W;\mathbb{R}\big)(F)G+F\mathcal{D}\big(\mathbb{P},\mathcal{F}_0,W;H\big)(G). \tag{2.99}$$

The following definition is needed to formulate Lemma 2.16.

Definition 2.15. *Assume Setting 2.1 and let* $(s,t)\in\Delta_{S,T}$*. Then*

$$\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)|_{[s,t]}\colon \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)\to L^2(\mathbb{P};L^2(\lambda_{[s,t]};H)) \tag{2.100}$$

is the operator with the property that for all $F\in\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)$ *with the property that* $\exists n\in\mathbb{N}$, $\exists\phi_1,\dots,\phi_n\in\mathcal{L}^2(\lambda_{[S,T]};\mathrm{HS}(U,\mathbb{R}))$, $\exists f\in C_b^{\infty,\sigma(\mathcal{F}_0\cup\mathcal{N})}(\mathbb{R}^n\times\Omega,\mathbb{R})$, $\exists h\in H$ *such that it holds* $\mathbb{P}$*-almost surely that*

$$F=f\Big(\int_S^T\phi_1(s)dW_s,\dots,\int_S^T\phi_n(s)dW_s\Big)h \tag{2.101}$$

it holds $\mathbb{P}\otimes\lambda_{[s,t]}$*-almost everywhere that*

$$\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)F=\sum_{i=1}^n\frac{\partial f}{\partial x_i}\Big(\int_S^T\phi_1(s)dW_s,\dots,\int_S^T\phi_n(s)dW_s\Big)\phi_i\big|_{[s,t]}h. \tag{2.102}$$

The following lemma shows how the Malliavin derivative depends on the underlying filtration.

Lemma 2.16. Assume Setting 2.1, let $\tilde{\mathcal{F}}_0\subseteq\mathcal{F}_0$ be a σ-algebra, let $(s,t)\in\Delta_{S,T}$, and let $\mathcal{F}_{[S,s]\cup[t,T]}\subseteq\mathcal{A}$ be a σ-algebra with the property that
$\mathcal{F}_{[S,s]\cup[t,T]}=\sigma(\sigma(\{W_r\colon r\in[0,s]\})\cup\sigma(\{W_r\colon r\in[t,T]\})\cup\mathcal{N})$. Then

$$\mathbb{D}^{(1,2)}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)\subseteq\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)\subseteq\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_{[S,s]\cup[t,T]}\vee\mathcal{F}_0,W|_{[s,t]};H) \tag{2.103}$$

and for all $F\in\mathbb{D}^{(1,2)}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)$ it holds that

$$\mathcal{D}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)F=\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)|_{[s,t]}F \tag{2.104}$$

and for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)|_{[s,t]} F = \mathcal{D}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]} \vee \mathcal{F}_0, W|_{[s,t]}; H)F. \tag{2.105}$$

Proof of Lemma 2.16. **For the first part of the proof** let $F \in \mathcal{S}(\mathbb{P}, \tilde{\mathcal{F}}_0, W|_{[s,t]}; H)$. Then there are $n \in \mathbb{N}$, $\phi_1, \dots, \phi_n \in \mathcal{L}^2(\lambda_{[s,t]}; \mathbb{R}^m)$, $f \in \mathcal{C}_b^{\infty,\sigma(\tilde{\mathcal{F}}_0 \cup \mathcal{N})}(\mathbb{R}^n \times \Omega, \mathbb{R}) \subseteq \mathcal{C}_b^{\infty,\sigma(\mathcal{F}_0 \cup \mathcal{N})}(\mathbb{R}^n \times \Omega, \mathbb{R})$ such that for all $\omega \in \Omega$ it holds that $f(\cdot)(\omega) \in \mathrm{C}_b^{\infty}(\mathbb{R}^n, \mathbb{R})$, and $h \in H$ such that it holds $\mathbb{P}$-almost surely that

$$F = f\Big(\int_s^t \phi_1(r)dW_r, \dots, \int_s^t \phi_n(r)dW_r\Big)h. \tag{2.106}$$

Let $\psi_1, \dots, \psi_n \in \mathcal{L}^2(\lambda_{[S,T]}; \mathbb{R}^m)$ be functions such that for all $i \in \{1, \dots, n\}$ and all $r \in [S,T]$ it holds that

$$\psi_i(r) = \begin{cases} \phi_i(r) & \text{if } r \in [s,t], \\ 0 & \text{else.} \end{cases} \tag{2.107}$$

Then it holds $\mathbb{P}$-almost surely that $F = f(\int_S^T \psi_1(r)dW_r, \dots, \int_S^T \psi_n(r)dW_r)h$. Therefore, we have

$$F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H). \tag{2.108}$$

It follows that it holds $\lambda_{[s,t]} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned} \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)|_{[s,t]} F &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(\int_S^T \psi_1(r)dW_r, \dots, \int_S^T \psi_n(r)dW_r\Big)\psi_i|_{[s,t]} h \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(\int_s^t \phi_1(r)dW_r, \dots, \int_s^t \phi_n(r)dW_r\Big)\phi_i h \\ &= \mathcal{D}(\mathbb{P}, \tilde{\mathcal{F}}_0, W|_{[s,t]}; H)F. \end{aligned} \tag{2.109}$$

Furthermore, it holds $\lambda_{[S,s]\cup[t,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)|_{[S,s]\cup[t,T]} F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(\int_S^T \psi_1(r)dW_r, \dots, \int_S^T \psi_n(r)dW_r\Big)\psi_i|_{[S,s]\cup[t,T]} h = 0. \tag{2.110}$$

From equations (2.109) and (2.110) we conclude that

$$\begin{aligned}\|F\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)} &= \mathbb{E}\big[\|F\|^2_H + \|\mathcal{D}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)F\|^2_{L^2(\lambda_{[s,t]};H)}\big]\\ &= \mathbb{E}\big[\|F\|^2_H + \int_s^t \|\mathcal{D}_r(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)F\|^2_H dr\big]\\ &= \mathbb{E}\big[\|F\|^2_H + \int_s^t \|\mathcal{D}_r(\mathbb{P},\mathcal{F}_0,W;H)|_{[s,t]}F\|^2_H dr\big]\\ &= \mathbb{E}\big[\|F\|^2_H + \int_s^T \|\mathcal{D}_r(\mathbb{P},\mathcal{F}_0,W;H)F\|^2_H dr\big]\\ &= \|F\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)}.\end{aligned} \tag{2.111}$$

Since $F \in \mathcal{S}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)$ was chosen arbitrarily, it follows from (2.108) and equation (2.111) that

$$\begin{aligned}&\big(\operatorname{span}(\mathcal{S}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)), \|\cdot\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)}|\operatorname{span}(\mathcal{S}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H))\big)\\ &= \big(\operatorname{span}(\mathcal{S}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)), \|\cdot\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)}|\operatorname{span}(\mathcal{S}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H))\big)\\ &\subseteq \big(\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)), \|\cdot\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)}|\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H))\big).\end{aligned} \tag{2.112}$$

Thus the first inclusion in (2.103) follows by taking the closures of the normed spaces in (2.112) as subsets of $L^2(\mathbb{P};H)$ and from equation (2.109) we conclude that for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P},\tilde{\mathcal{F}}_0,W|_{[s,t]};H)$ equation (2.104) holds true.

For the second part of the proof let $F \in \mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)$. Then there are $n \in \mathbb{N}$, $\phi_1,\dots,\phi_n \in \mathcal{L}^2(\lambda_{[S,T]};\mathrm{HS}(U,\mathbb{R}))$, $f \in \mathcal{M}(\mathcal{B}_{\mathbb{R}^n} \otimes \sigma(\mathcal{F}_0 \cup \mathcal{N}),\mathcal{B}_{\mathbb{R}})$ such that for all $\omega \in \Omega$ it holds that $f(\cdot)(\omega) \in \mathrm{C}^\infty_b(\mathbb{R}^n,\mathbb{R})$ and $h \in H$ such that it holds $\mathbb{P}$-almost surely that

$$F = f\Big(\int_S^T \phi_1(s)dW_s,\dots,\int_S^T \phi_n(s)dW_s\Big)h. \tag{2.113}$$

Let $g \in \mathcal{M}(\mathcal{B}_{\mathbb{R}^n} \otimes \mathcal{F}_{[S,s]\cup[t,T]} \vee \mathcal{F}_0,\mathcal{B}_{\mathbb{R}})$ be such that for all $\omega \in \Omega$ it holds that $g(\cdot)(\omega) \in \mathrm{C}^\infty_b(\mathbb{R}^n,\mathbb{R})$ is a random function such that for all $(x_1,\dots,x_n) \in \mathbb{R}^n$ and all $\omega \in \Omega$ it holds $\mathbb{P}$-almost surely that

$$g = f\Big(x_1 + \int_S^s \phi_1(s)dW_s + \int_t^T \phi_1(s)dW_s,\dots,x_n + \int_S^s \phi_n(s)dW_s + \int_t^T \phi_n(s)dW_s\Big)h. \tag{2.114}$$

Then it holds $\mathbb{P}$-almost surely that

$$F = g\Big(\int_s^t \phi_1|_{[s,t]}(s)dW_s,\dots,\int_s^t \phi_n|_{[s,t]}(s)dW_s\Big)h. \tag{2.115}$$

For all $i \in \{1,\dots,n\}$ it holds $\mathbb{P}$-almost surely that

$$\frac{\partial f}{\partial x_i}\Big(\int_S^T \phi_1(s)dW_s,\dots,\int_S^T \phi_n(s)dW_s\Big) = \frac{\partial g}{\partial x_i}\Big(\int_s^t \phi_1|_{[s,t]}(s)dW_s,\dots,\int_s^t \phi_n|_{[s,t]}(s)dW_s\Big). \tag{2.116}$$

We conclude that it holds $\lambda_{[s,t]} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned} \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)|_{[s,t]} F &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Big(\int_S^T \phi_1(s) dW_s, \ldots, \int_S^T \phi_n(s) dW_s \Big) \phi_i|_{[s,t]} h \\ &= \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \Big(\int_s^t \phi_1|_{[s,t]}(s) dW_s, \ldots, \int_s^t \phi_n|_{[s,t]}(s) dW_s \Big) \phi_i|_{[s,t]} h \\ &= \mathcal{D}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]} \vee \mathcal{F}_0, W|_{[s,t]}; H) F. \end{aligned} \tag{2.117}$$

It follows directly that

$$\begin{aligned} \|F\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]} \vee \mathcal{F}_0, W|_{[s,t]}; H)} &= \mathbb{E}\Big[\|F\|_H^2 + \int_s^t \|\mathcal{D}_s(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]} \vee \mathcal{F}_0, W|_{[s,t]}; H) F\|_H^2 ds \Big] \\ &\leq \mathbb{E}\Big[\|F\|_H^2 + \int_S^T \|\mathcal{D}_s(\mathbb{P}, \mathcal{F}_0, W; H) F\|_H^2 ds \Big] \\ &= \|F\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)}. \end{aligned} \tag{2.118}$$

Since $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)$ was chosen arbitrarily it follows that

$$\operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)) \subseteq \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]} \vee \mathcal{F}_0, W|_{[s,t]}; H)). \tag{2.119}$$

From inequality (2.118) finally follows the second inclusion in (2.103), and it follows that for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ equation (2.105) holds true. The proof of Lemma 2.16 is thus completed. □

The Malliavin differential operator fulfills a commutativity relation.

Lemma 2.17. Assume Setting 2.1, let $F \in \operatorname{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H))$, let $\mathrm{id} = (\mathrm{id}_1, \mathrm{id}_2) \colon [S,T]^2 \to [S,T]^2$ a mapping with the property that for $\lambda_{[S,T]^2}$-almost all $(s,t) \in [S,T]^2$ it holds that $\mathrm{id}(s,t) = (s,t)$ and let $u_1, u_2 \in U$. Then it holds for $\lambda_{[S,T]^2} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned} &\mathcal{D}_{\mathrm{id}_1}(\mathbb{P}, \mathcal{F}_0, W; H) \big((\mathcal{D}_{\mathrm{id}_2}(\mathbb{P}, \mathcal{F}_0, W; H) F)(u_1) \big)(u_2) \\ &= \mathcal{D}_{\mathrm{id}_2}(\mathbb{P}, \mathcal{F}_0, W; H) \big((\mathcal{D}_{\mathrm{id}_1}(\mathbb{P}, \mathcal{F}_0, W; H) F)(u_2) \big)(u_1). \end{aligned} \tag{2.120}$$

Proof of Lemma 2.17. Let $n \in \mathbb{N}$, $\phi_1, \ldots, \phi_n \in L^2(U, \mathbb{R})$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$, $X_0 \in L^\infty(\mathcal{F}_0; \mathbb{R})$ and $h \in H$ with the property that it holds $\mathbb{P}$-almost surely that $F = f(\int_S^T \phi_1(r) dW_r, \ldots, \int_S^T \phi_n(r) dW_r) X_0 h$. From the commutativity of the partial derivatives of f it follows that it holds $\lambda_{[S,T]^2} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned}
&\mathcal{D}_{\mathrm{id}_1}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\big((\mathcal{D}_{\mathrm{id}_2}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)F)\,(u_1)\big)(u_2)\\
&= \mathcal{D}_{\mathrm{id}_1}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\left(\sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\left(\int_S^T \phi_1(r)dW_r, \ldots, \int_S^T \phi_n(r)dW_r\right)\phi_k(t,u_1)X_0 h\right)(u_2)\\
&= \sum_{k=1}^{n} \mathcal{D}_{\mathrm{id}_1}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\left(\frac{\partial f}{\partial x_k}\left(\int_S^T \phi_1(r)dW_r, \ldots, \int_S^T \phi_n(r)dW_r\right)X_0 h\right)(u_2)\phi_k(t,u_1)\\
&= \sum_{k=1}^{n}\left(\sum_{l=1}^{n} \frac{\partial^2 f}{\partial x_l \partial x_k}\left(\int_S^T \phi_1(r)dW_r, \ldots, \int_S^T \phi_n(r)dW_r\right)\phi_l(s,u_2)X_0 h\right)\phi_k(t,u_1)\\
&= \sum_{l=1}^{n}\left(\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_l}\left(\int_S^T \phi_1(r)dW_r, \ldots, \int_S^T \phi_n(r)dW_r\right)\phi_k(t,u_1)X_0 h\right)\phi_l(s,u_2)\\
&= \mathcal{D}_{\mathrm{id}_2}(\mathbb{P},\mathcal{F}_0,W;H)\big((\mathcal{D}_{\mathrm{id}_1}(\mathbb{P},\mathcal{F}_0,W;H)F)(u_2)\big)(u_1).
\end{aligned} \tag{2.121}$$

Equation (2.121) and Lemma 2.12 complete the proof of Lemma 2.17. □

In the case of a deterministic integral, the Malliavin differential operator can be "pulled inside the integral". For a special case of the following lemma see Proposition 4.8 in Kruse 2014.

Lemma 2.18. Assume Setting 2.1, let $(E,\mathcal{E},\nu)$ be a finite measure space, and let $F \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))$. Then $\int_E F d\nu \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)$ and

$$\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\big(\int_E F d\nu\big) = \int_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))F d\nu. \tag{2.122}$$

Proof of Lemma 2.18. From the definition of the simple Malliavin differential operator 2.2 it follows that it holds for all $F_0 \in \mathcal{S}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})$, $\psi \in L^2(\nu;H)$ that

$$\begin{aligned}
\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\big(\int_E F_0\psi d\nu\big) &= \widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)\big(F_0 \cdot \int_E \psi(y) d\nu\big)\\
&= \big(\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})F_0\big) \cdot \int_E \psi d\nu\\
&= \int_E \big(\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})F_0\big) \cdot \psi d\nu\\
&= \int_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))(F_0\psi) d\nu.
\end{aligned} \tag{2.123}$$

We conclude that for all $\bar{F} \in \mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))$ it holds that

$$\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\big(\int_E \bar{F} d\nu\big) = \int_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))\bar{F} d\nu. \tag{2.124}$$

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H)))$ which converges to F in $\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))$ as $n \to \infty$. It follows from equation (2.124) that for all $n \in \mathbb{N}$ it holds

that

$$
\begin{aligned}
&\big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\big(\textstyle\int\limits_E F_n d\nu\big) - \textstyle\int\limits_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))F d\nu\big\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))} \\
&= \big\| \textstyle\int\limits_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;HL^2(\nu;H)F_n d\nu - \textstyle\int\limits_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))F d\nu\big\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))} \\
&= \big\| \textstyle\int\limits_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))(F_n - F) d\nu\big\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))} \\
&= \Big\| \textstyle\int\limits_S^T \big\| \textstyle\int\limits_E \mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))(F_n - F) d\nu\big\|^2_{\mathrm{HS}(U,H)} ds\Big\|_{L^1(\mathbb{P};\mathbb{R})} \qquad (2.125)\\
&\leq \nu(E)\Big\| \textstyle\int\limits_S^T\int\limits_E \big\|\mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))(F_n - F)\big\|^2_{\mathrm{HS}(U,H)} d\nu ds\Big\|_{L^1(\mathbb{P};\mathbb{R})} \\
&= \nu(E)\big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))(F_n - F)\big\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,L^2(\nu;H))))} \\
&\leq \nu(E)\, \|F_n - F\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))}\, .
\end{aligned}
$$

It follows from Lemma 2.12, inequality (2.125), and the assumption that ν is a finite measure that it holds that

$$
\begin{aligned}
&\big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)(\textstyle\int\limits_E F d\nu) - \textstyle\int\limits_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))F d\nu\big\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))} \\
&= \lim_{n\to\infty} \big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\big(\textstyle\int\limits_E F_n d\nu\big) - \textstyle\int\limits_E \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))F d\nu\big\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))} \\
&\leq \nu(E) \lim_{n\to\infty} \|F_n - F\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))} = 0. \qquad (2.126)
\end{aligned}
$$

We conclude that equation (2.122) holds true. The proof of Lemma 2.18 is thus completed. □

For the following Lemma, see Lemma 4.7 in Kruse 2014 which is based on the finite-dimensional version of Proposition 1.2.2 in Nualart 1995.

Lemma 2.19 (Chain rule)**.** Assume Setting 2.1, let $(H_1, \|\cdot\|_{H_1}, \langle\cdot,\cdot\rangle_{H_1})$ be a real separable Hilbert space with Hilbert space basis $(e_i)_{i\in I}$, $I \subseteq \mathbb{N}$, let $\psi\colon H \to H_1$ be a continously Frèchet-differentiable function with the property that

$$\|\psi\|_{L(H,H_1)} + \sup_{h\in H} \|\psi'(h)\|_{L(H,H_1)} < \infty. \tag{2.127}$$

Then it holds for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)$ that $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H_1)$ and $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H_1)(\psi(F)) = \psi'(F)\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)F. \tag{2.128}$$

Proof of Lemma 2.19. **For the first part of the proof,** let $F \in \mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)$, $n \in \mathbb{N}$, $\phi_1,\dots,\phi_n \in \mathcal{L}^2(\lambda_{[S,T]};\mathrm{HS}(U,\mathbb{R}))$, $f \in C_b^{\infty,\sigma(\mathcal{F}_0\cup\mathcal{N})}(\mathbb{R}^n \times \Omega,\mathbb{R})$, $h \in H$ be such that it holds

$\mathbb{P}$-almost surely that

$$F = f\Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) h. \tag{2.129}$$

For all $i \in I$ let $g_i \in C_b^{\infty,\sigma(\mathcal{F}_0 \cup \mathcal{N})}(\mathbb{R}^n \times \Omega, \mathbb{R})$ be a mapping with the property that for all $x \in \mathbb{R}^n$ it holds $\mathbb{P}$-almost surely that

$$g_i(x) = \big\langle \psi\big(f(x)h\big), e_i \big\rangle_{H_1}. \tag{2.130}$$

Then it holds $\mathbb{P}$-almost surely that

$$\psi(F) = \sum_{i=1}^{\infty} g_i \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) e_i. \tag{2.131}$$

From condition (2.128) it follows that $\psi(F) \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H_1)$. For all $k \in \mathbb{N}$ let $G_k \in \operatorname{span}\big(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H_1)\big)$ be a simple random variable with the property that it holds $\mathbb{P}$-almost surely that

$$G_k = \sum_{i=1}^{k} g_i \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) e_i. \tag{2.132}$$

Note that G_k converges to $\psi(F) \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H_1)$ $\mathbb{P}$-almost surely as $k \to \infty$. Together with the dominated convergence theorem, we conclude that G_k converges to $\psi(F)$ in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H_1)$ as $k \to \infty$. For all $k \in \mathbb{N}$ it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned}
\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H_1) G_k &= \sum_{i=1}^{k} \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}) \Big(g_i \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) \Big) e_i \\
&= \sum_{i=1}^{k} \sum_{j=1}^{n} \tfrac{\partial}{\partial x_j} g_i \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) \phi_j e_i \\
&= \sum_{j=1}^{n} \sum_{i=1}^{k} \Big\langle \tfrac{\partial}{\partial x_j} \Big(\psi \Big(f \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) h \Big) \Big), e_i \Big\rangle_{H_1} \phi_j e_i \\
&= \sum_{j=1}^{n} \sum_{i=1}^{k} \Big\langle \psi'(F) \tfrac{\partial}{\partial x_j} f \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) h \Big), e_i \Big\rangle_{H_1} \phi_j e_i \\
&= \sum_{i=1}^{k} \Big\langle \Big(\psi'(F) \sum_{j=1}^{n} \tfrac{\partial}{\partial x_j} f \Big(\int\limits_S^T \phi_1(r)dW_r, \ldots, \int\limits_S^T \phi_n(r)dW_r \Big) h \Big) \phi_j, e_i \Big\rangle_{H_1} e_i \\
&= \psi'(F) \sum_{i=1}^{k} \big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) F, e_i \big\rangle_{H_1} e_i.
\end{aligned} \tag{2.133}$$

We conclude that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H_1) G_k \xrightarrow{k \to \infty} \psi'(F) \sum_{i=1}^{\infty} \big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) F, e_i \big\rangle_{H_1} e_i = \psi'(F) \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) F, \tag{2.134}$$

in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H_1)))$. By linearity and from Lemma 2.12 it follows that for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ equation (2.128) holds true $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere. The proof of Lemma 2.19 is thus completed. □

Proposition 1.2.3 in Nualart 1995 shows that $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \sigma(\{\emptyset, \Omega\}), W; \mathbb{R})$ if ψ is globally Lipschitz continuous. The following lemma is needed in the Hilbert space setting.

Lemma 2.20 (Chain rule)**.** Assume Setting 2.1, let $(H_1, \langle\cdot,\cdot\rangle_{H_1})$ be a separable real Hilbert space, let $\psi\colon H \to H_1$ be a globally Lipschitz continuous function, and let $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$. Then it holds that $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)$.

Proof of Lemma 2.20. **In the first step, we consider the finite-dimensional case**. Let $k \in \mathbb{N}$ and assume that $H = \mathbb{R}^k$. Let $\phi \in C_b^\infty(\mathbb{R}^k, [0,\infty))$ be a function with support in $\{x \in \mathbb{R}^k\colon \|x\|_{\mathbb{R}^k} \leq 1\}$ and with the property that $\int_{\mathbb{R}^k} \phi(x)\, dx = 1$. For all $n \in \mathbb{N}$ let $\phi_n \in C_b^\infty(\mathbb{R}^k, [0,\infty))$ be a function with the property that for all $x \in \mathbb{R}^k$ it holds that $\phi_n(x) = n^k \phi(nx)$. For all $n \in \mathbb{N}$ let $\eta_n\colon \mathbb{R}^k \to H_1$ be a function with the property that for all $z \in \mathbb{R}^k$ it holds that $\eta_n(z) = \int_{\mathbb{R}^k} \psi(x)\phi_n(x-z)\, dx$. Integration by substitution implies that for all $n \in \mathbb{N}$, $z \in \mathbb{R}^k$ it holds that

$$\int_{\mathbb{R}^k} \psi(x)\phi_n(x-z)\, dx = \int_{\mathbb{R}^k} \psi(y+z)\phi_n(y) dy = \int_{\mathbb{R}^k} \psi(y+z)\phi(ny)n^{-k} dy = \int_{\mathbb{R}^k} \psi\left(\tfrac{x}{n} + z\right)\phi(x)\, dx. \tag{2.135}$$

We conclude that for all $n \in \mathbb{N}$, $x \in \mathbb{R}^k$ it holds that

$$\left\|\psi\left(\tfrac{x}{n} + F\right) - \psi(F)\right\|_{H_1} \leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|x\|_{\mathbb{R}^k}. \tag{2.136}$$

It follows from the dominated convergence theorem that it holds that

$$\begin{aligned} \lim_{n\to\infty} \|\eta_n(F) - \psi(F)\|_{L^2(\mathbb{P};H_1)} &= \lim_{n\to\infty} \left\| \int_{\mathbb{R}^k} \left(\psi\left(\tfrac{x}{n} + F\right) - \psi(F)\right)\phi(x)\, dx \right\|_{L^2(\mathbb{P};H_1)} \\ &= \left\| \int_{\mathbb{R}^k} \lim_{n\to\infty} \left(\psi\left(\tfrac{x}{n} + F\right) - \psi(F)\right)\phi(x)\, dx \right\|_{L^2(\mathbb{P};H_1)} = 0. \end{aligned} \tag{2.137}$$

Observe that for all $n \in \mathbb{N}$ it holds that $\eta_n \in C^\infty(\mathbb{R}^k, H_1)$. For all $n \in \mathbb{N}$, $y, z \in \mathbb{R}^k$ it holds that

$$\begin{aligned} \|\eta_n(z) - \eta_n(y)\|_{H_1} &= \left\| \int_{\mathbb{R}^k} (\psi(\tfrac{x}{n} + z) - \psi(\tfrac{x}{n} + y))\phi(x)\, dx \right\|_{H_1} \\ &\leq \int_{\mathbb{R}^k} \left\|\psi(\tfrac{x}{n} + z) - \psi(\tfrac{x}{n} + y)\right\|_{H_1} \phi(x)\, dx \\ &\leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|z - y\|_{\mathbb{R}^k} \int_{\mathbb{R}^k} \phi(x)\, dx \\ &= \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|z - y\|_{\mathbb{R}^k}. \end{aligned} \tag{2.138}$$

From inequality (2.138) follows that for all $n \in \mathbb{N}$ it holds that

$$\sup_{x \in \mathbb{R}^k} \|\eta_n'(x)\|_{L(\mathbb{R}^k, H_1)} \leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}}. \tag{2.139}$$

Therefore, Lemma 2.19 yields that for all $n \in \mathbb{N}$ it holds that $\eta_n(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)$ and that for all $n \in \mathbb{N}$ it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H_1)(\eta_n(F)) = \eta_n'(F)\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)F. \tag{2.140}$$

In addition, it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \|\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H_1)(\eta_n(F))\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H_1)))} \\ &= \sup_{n \in \mathbb{N}} \left\|\eta_n'(F)\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)F\right\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H_1)))} \\ &\leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \left\|\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)F\right\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}^k)))}. \end{aligned} \tag{2.141}$$

The right-hand side is finite as $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)$ and as ψ is globally Lipschitz continuous. Now Corollary 2.34 together with equation (2.137) and inequality (2.141) imply that $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)$. This together with

$$\|\psi(F)\|_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)} \leq \|\psi(0)\|_{H_1} + \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|F\|_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)} \tag{2.142}$$

results in $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)$. Since $k \in \mathbb{N}$ was chosen arbitrarily, this proves the assertion of the lemma whenever $\dim(H) < \infty$ and finishes the first step.

In the second step, let H be a general separable Hilbert space. Let $(h_i)_{i \in \mathbb{N}}$ be an orthonormal Hilbert space base of H. Lemma 2.19 yields that for all $k \in \mathbb{N}$ it holds that $(\langle F, h_1 \rangle_H, \ldots, \langle F, h_k \rangle_H) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)$. For all $k \in \mathbb{N}$ and all $x = (x_1, \ldots, x_k), z = (z_1, \ldots, z_k) \in \mathbb{R}^k$ it holds that

$$\begin{aligned} \Big\|\psi\Big(\sum_{i=1}^k x_i h_i\Big) - \psi\Big(\sum_{i=1}^k z_i h_i\Big)\Big\|_{H_1} &\leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \Big\|\sum_{i=1}^k (x_i - z_i) h_i\Big\|_H \\ &= \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|x - z\|_{\mathbb{R}^k}. \end{aligned} \tag{2.143}$$

Therefore, the first step implies that for all $k \in \mathbb{N}$ it holds that

$$\psi\left(\sum_{i=1}^{k} \langle F, h_i \rangle_H h_i\right) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H) \tag{2.144}$$

and

$$\begin{aligned}
&\Big\| \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\Big(\sum_{i=1}^{k} \langle F, h_i \rangle_H h_i\Big)\Big\|_{L^2(\lambda_{[S,T]} \otimes \mathbb{P}, \mathrm{HS}(U,H))} \\
&\leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \big\|\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}^k)\big(\langle F, h_1 \rangle_H, \ldots, \langle F, h_k \rangle_H\big)\big\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}^k)))} \\
&= \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \\
&\qquad \cdot \Big\|\Big(\langle h_1, \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) F \rangle_H, \ldots, \langle h_k, \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) F \rangle_H\Big)\Big\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}^k)))} \\
&\leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) F\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H)))}.
\end{aligned} \tag{2.145}$$

Note that for all $k \in \mathbb{N}$ it holds that

$$\Big\|\psi(F) - \psi\Big(\sum_{i=1}^{k} \langle F, h_i \rangle_H h_i\Big)\Big\|_{L^2(\mathbb{P}; H_1)} \leq \sup_{\substack{y_1, y_2 \in \mathbb{R}^k \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \Big\| \sum_{i=k+1}^{\infty} \langle F, h_i \rangle_H h_i \Big\|_{L^2(\mathbb{P}; H)}. \tag{2.146}$$

Since $F \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$ it follows that the right-hand side of inequality (2.146) converges to 0 as $k \to \infty$. Finally Corollary 2.34 together with inequalities (2.146) and (2.145) implies that $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)$. This together with

$$\|\psi(F)\|_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)} \leq \|\psi(0)\|_{H_1} + \sup_{\substack{y_1, y_2 \in H, \\ y_1 \neq y_2}} \frac{\|\psi(y_1) - \psi(y_2)\|_{H_1}}{\|y_1 - y_2\|_{\mathbb{R}^k}} \|F\|_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)} \tag{2.147}$$

results in $\psi(F) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H_1)$. The proof of Lemma 2.20 is thus completed. □

The following lemma is needed in the sequel.

Lemma 2.21. Let $S, T \in \mathbb{R}$ satisfy that $S < T$, let $(H_1, \langle \cdot, \cdot \rangle_{H_1})$ and $(H_2, \langle \cdot, \cdot \rangle_{H_2})$ be separable real Hilbert spaces, let $f \in C^0([S,T] \times H_1, H_2)$ be a function with the property that

$$\sup_{s \in [S,T]} \sup_{x, y \in H_1 \colon x \neq y} \frac{\|f(s,x) - f(s,y)\|_{H_2}}{\|x - y\|_{H_1}} < \infty, \tag{2.148}$$

and let $\psi\colon L^2(\lambda_{[S,T]}; H_1) \to L^2(\lambda_{[S,T]}; H_2)$ be the function with the property that for all $s \in$

$[S,T]$, $\eta \in L^2(\lambda_{[S,T]}; H_1)$ it holds that

$$(\psi(\eta))(s) = f(s, \eta(s)). \tag{2.149}$$

Then ψ is well defined and globally Lipschitz continuous.

Proof of Lemma 2.21. For all $\eta \in L^2(\lambda_{[S,T]}; H_1)$ it holds that

$$\begin{aligned}
\|\psi(\eta)\|^2_{L^2(\lambda_{[S,T]};H_2)} &= \int_S^T \|(\psi(\eta))(s)\|^2_{H_2} ds \\
&= \int_S^T \|f(s,\eta(s)) - f(s,0) + f(s,0)\|^2_{H_2} ds \\
&\leq 2 \int_S^T \|f(s,\eta(s)) - f(s,0)\|^2_{H_2}\, ds + 2\int_S^T \|f(s,0)\|^2_{H_2} ds \\
&\leq 2 \sup_{u\in[S,T]} \sup_{x\in H_1\setminus\{0\}} \frac{\|f(u,x) - f(u,0)\|^2_{H_2}}{\|x\|^2_{H_1}} \int_S^T \|\eta(s)\|^2_{H_1} ds + 2T \sup_{u\in[S,T]} \|f(u,0)\|^2_{H_2}.
\end{aligned} \tag{2.150}$$

The right-hand side of inequality (2.150) is finite due to assumption (2.148) and as the continuous function $[S,T] \ni s \mapsto f(s,0) \in H_2$ is bounded. Moreover, assumption (2.148) implies that for all $\eta, \xi \in L^2(\lambda_{[S,T]}; H_1)$ it holds that

$$\begin{aligned}
\|\psi(\xi) - \psi(\eta)\|^2_{L^2(\lambda_{[S,T]};H_2)} &= \|f(\cdot,\xi) - f(\cdot,\eta)\|^2_{L^2(\lambda_{[S,T]};H_2)} \\
&= \int_S^T \|f(s,\xi_s) - f(s,\eta_s)\|^2_{H_2} ds \\
&\leq \sup_{u\in[S,T]} \sup_{x,y\in H_1\colon x\neq y} \frac{\|f(u,x) - f(u,y)\|_{H_2}}{\|x-y\|_{H_1}} \int_S^T \|\xi_s - \eta_s\|^2_{H_1} ds \\
&= \sup_{u\in[S,T]} \sup_{x\neq y\in H_1} \frac{\|f(u,x) - f(u,y)\|_{H_2}}{\|x-y\|_{H_1}} \|\xi - \eta\|^2_{L^2(\lambda_{[S,T]};H_1)}.
\end{aligned} \tag{2.151}$$

This proves that ψ is globally Lipschitz continuous. The proof of Lemma 2.21 is thus completed. □

Lemma 2.22. Assume Setting 2.1, let $\phi \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))$, and let $X_0 \in L^\infty(\mathbb{P}|_{\mathcal{F}_0}; \mathbb{R})$. Then $X_0(\int_S^T \phi(s) dW_s) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ and it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\Big(X_0\Big(\int_S^T \phi(s) dW_s\Big)\Big) = X_0 \phi. \tag{2.152}$$

Proof of Lemma 2.22. **In the first step, assume that** $H = \mathbb{R}$**.** Let $(f_n)_{n\in\mathbb{N}}$ be the sequence in $C^0(\mathbb{R},\mathbb{R})$ with the property that for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ it holds that $f_n(x) = x(1 - \exp(-nx^2))$. Moreover, let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\mathcal{A}, \mathcal{B}_\mathbb{R})$ with the property that for all $n \in \mathbb{N}$ it holds $\mathbb{P}$-almost surely that

$$F_n = f_n\Big(\int_S^T \phi(s) dW_s\Big) X_0. \tag{2.153}$$

Note that for all $n \in \mathbb{N}$ it holds that $F_n \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})$. Moreover, it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}
&\Big\| F_n - X_0\Big(\int_S^T \phi(s)dW_s\Big)\Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&= \Big\| f_n\Big(\int_S^T \phi(s)dW_s\Big)X_0 - X_0\Big(\int_S^T \phi(s)dW_s\Big)\Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&= \Big\| \int_S^T \phi(s)dW_s \exp\Big(-n\Big(\int_S^T \phi(s)dW_s\Big)^2\Big)X_0\Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&= \Big\| \int_S^T \phi(s)dW_s \exp\Big(-n\Big(\int_S^T \phi(s)dW_s\Big)^2\Big)\Big\|_{L^2(\mathbb{P};\mathbb{R})} \|X_0\|_{L^2(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{2.154}$$

The right-hand side of equation (2.154) converges to 0 as $n \to \infty$ by the dominated convergence theorem. Similarly, it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}
&\|\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})F_n - \phi X_0\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,\mathbb{R})))} \\
&= \Big\| f_n'\Big(\int_S^T \phi(s)dW_s\Big)\phi X_0 - \phi X_0\Big\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,\mathbb{R})))} \\
&= \Big\| 1 - f_n'\Big(\int_S^T \phi(s)dW_s\Big)\Big\|_{L^2(\mathbb{P};\mathbb{R})} \|\phi\|_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,\mathbb{R}))} \|X_0\|_{L^2(\mathbb{P};\mathbb{R})} \\
&= \Big\| \exp\Big(-n\Big(\int_S^T \phi(s)dW_s\Big)^2\Big)\Big(1 - 2n\Big(\int_S^T \phi(s)dW_s\Big)^2\Big)\Big\|_{L^2(\mathbb{P};\mathbb{R})} \|\phi\|_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,\mathbb{R}))} \|X_0\|_{L^2(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{2.155}$$

The right-hand side of equation (2.155) converges to 0 as $n \to \infty$ by the dominated convergence theorem. From the definition of the Malliavin derivative it follows that $X_0\big(\int_S^T \phi(s)dW_s\big) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})$ and that it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})\Big(X_0\Big(\int_S^T \phi(s)dW_s\Big)\Big) = \phi X_0. \tag{2.156}$$

In the second and last step, let H be an arbitrary real separable Hilbert space. Let $(h_i)_{i\in\mathbb{N}}$ be an orthonormal Hilbert space base of H. For all $i \in \mathbb{N}$ let $\tilde{\phi}_i \in L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathbb{R}))$ be a mapping with the property that for all $t \in [S, T]$, $u \in U$ it holds that $\tilde{\phi}_i(t, u) = \langle \phi(t, u), h_i\rangle_H$. Then the first step implies that $\int_S^T \tilde{\phi}_i(s)dW_s X_0 \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})$ and that for all $i \in \mathbb{N}$ it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that $\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})(\int_S^T \tilde{\phi}_i(s)dW_s X_0) = \tilde{\phi}_i X_0$. By definition of the H-valued Itô integral it follows that it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
\sum_{i=1}^n h_i \int_S^T \tilde{\phi}_i(s)dW_s X_0 &= \sum_{i=1}^n h_i \int_S^T \langle \phi(t), h_i\rangle_H \, dW_t X_0 = \sum_{i=1}^n h_i \Big\langle \int_S^T \phi(t)dW_t, h_i\Big\rangle_H X_0 \\
&= \sum_{i=1}^n h_i \Big\langle \int_S^T \phi(s)dW_s, h_i\Big\rangle_H X_0.
\end{aligned} \tag{2.157}$$

We conclude that the sequence $(\sum_{i=1}^n h_i \int_S^T \tilde{\phi}_i(s)dW_s X_0)_{n\in\mathbb{N}}$ converges to $\sum_{i=1}^\infty h_i \langle \int_S^T \phi(s)dW_s, h_i\rangle_H X_0 = X_0\big(\int_S^T \phi(s)dW_s\big)$ in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$ and $\mathbb{P}$-almost surely as $n \to 0$. Moreover, linearity of the Malliavin derivative and the first step imply that it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned} \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\Big(\sum_{i=1}^{n} h_i \int\limits_S^T \tilde{\phi}_i(s) dW_s X_0\Big) &= \sum_{i=1}^{n} h_i \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\Big(\int\limits_S^T \tilde{\phi}_i(s) dW_s X_0\Big) \\ &= \sum_{i=1}^{n} h_i \tilde{\phi}_i X_0 = \sum_{i=1}^{n} h_i \langle \phi, h_i \rangle_H X_0. \end{aligned} \tag{2.158}$$

It follows that the sequence $(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)(\sum_{i=1}^{n} h_i \int_S^T \tilde{\phi}_i(s) dW_s X_0))_{n \in \mathbb{N}}$ converges to $\sum_{i=1}^{\infty} h_i \langle \phi, h_i \rangle_H X_0 = \phi X_0$ in $L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H)))$ as $n \to \infty$.
By definition of the Malliavin derivative, we conclude that $X_0\big(\int_S^T \phi(s) dW_s\big) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ and that it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\Big(X_0\Big(\int\limits_S^T \phi(s) dW_s\Big)\Big) = \phi X_0. \tag{2.159}$$

The proof of Lemma 2.22 is thus completed. □

Lemma 2.23. Assume Setting 2.1, let $n \in \mathbb{N}$, $f \in C^1(\mathbb{R}^n, \mathbb{R})$ be a function whose derivative grows at most polynomially, let $\phi_1, \dots, \phi_n \in L^2(\lambda_{[S,T]}; \mathbb{R})$, let $X_0 \in L^2(\mathbb{P}|_{\mathcal{F}_0}; \mathbb{R})$, let $h \in H$, and let $F \in L^0(\mathbb{P}; \mathbb{R})$ be such that it holds $\mathbb{P}$-almost surely that

$$F = f\left(\int\limits_S^T \phi_1(r) dW_r, \dots, \int\limits_S^T \phi_n(r) dW_r\right) X_0 h. \tag{2.160}$$

Then $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ and it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big(\int\limits_S^T \phi_1(r) dW_r, \dots, \int\limits_S^T \phi_n(r) dW_r\Big)\phi_i X_0 h. \tag{2.161}$$

Proof of Lemma 2.23. Analogous to previous lemma. □

2.2 The Skorohod Integral

In the second part of this chapter, we introduce the Skorohod integral which is the tool we will be using in the second chapter. The Skorohod integral is defined as the adjoint operator of the Malliavin derivative and can be viewed as a generalization of the Itô integral for L^2-processes. The classical Skorohod integral, as introduced in Nualart 2006, is solely defind for processes which are measurable with respect to the filtration which is generated by the underlying Brownian motion. We generalize the Skorohod integral to more general stochastic processes, analoguous to the enlargement for filtration of the Itô integral.

Definition 2.24. *Let $H_1 = (H_1, \langle \cdot, \cdot \rangle_{H_1}, \| \cdot \|_{H_1})$ and $H_2 = (H_2, \langle \cdot, \cdot \rangle_{H_2}, \| \cdot \|_{H_2})$ be two separable real Hilbert spaces with $H_1 \neq \{0\}$ and $H_2 \neq \{0\}$. Let $A\colon \mathrm{Dom}_A \subseteq H_1 \to H_2$ be a closed linear operator such that Dom_A is dense in H_1. The adjoint operator*

$$A^*\colon \mathrm{Dom}_{A^*} \subseteq H_2 \to H_1 \tag{2.162}$$

is the operator with domain

$$\mathrm{Dom}_{A^*} = \{h_2 \in H_2 \colon \exists c \in [0, \infty) \forall h_1 \in H_1 \colon \langle A h_1, h_2 \rangle_{H_2} \leq c \|h_1\|_{H_1}\} \tag{2.163}$$

and with the property that for all $h_1 \in \mathrm{Dom}_A$ and all $h_2 \in \mathrm{Dom}_{A^*}$ it holds that

$$\langle A h_1, h_2 \rangle_{H_2} = \langle h_1, A^* h_2 \rangle_{H_1}. \tag{2.164}$$

Proposition 2.25. *Let the setting from Definition 2.24 be given. Then A^* is a well-defined linear operator.*

Proof of Proposition 2.25. Let $h_2 \in \mathrm{Dom}_{A^*}$. Then the operator $\mathrm{Dom}_A \to \mathbb{R};\ h_1 \mapsto \langle A h_1, h_2 \rangle_{H_2}$ is continuous and can therefore be continuously extended to H_1. From the Riesz representation theorem we conclude that there is a $v \in H_2$ such that for all $h_1 \in \mathrm{Dom}_A$ it holds that

$$\langle A h_1, h_2 \rangle_{H_2} = \langle h_1, v \rangle_{H_2} \tag{2.165}$$

and by setting $A^* h_2 = v$, we see that A^* is well-defined. The proof of Proposition 2.25 is thus completed. ☐

Proposition 2.26. *Let the setting from Definition 2.24 be given and let $U_1 \subseteq \mathrm{Dom}_A$ be a subset with the property that $\mathrm{span}(U_1)$ is dense in the space H_1. Then it holds that*

$$\mathrm{Dom}_{A^*} = \big\{h_2 \in H_2 \colon \exists c \in [0, \infty) \forall h_1 \in U_1 \colon \langle A h_1, h_2 \rangle_{H_2} \leq c \|h_1\|_{H_1}\big\}. \tag{2.166}$$

Proof of Proposition 2.26. Let $h_2 \in \mathrm{Dom}_{A^*}$. Then the operator $\mathrm{Dom}_A \to \mathbb{R};\ h_1 \mapsto \langle A h_1, h_2 \rangle_{H_2}$ is continuous and can therefore be continuously extended to H_1. From the Riesz representation theorem we conclude that there is a $v \in H_2$ such that for all $h_1 \in \mathrm{Dom}_A$ it holds that

$$\langle A h_1, h_2 \rangle_{H_2} = \langle h_1, v \rangle_{H_2} \tag{2.167}$$

and by setting $A^* h_2 = v$ we see, that A^* is well-defined. (2.166) then follows from the closedness of the operator A. The proof of Proposition 2.26 is thus completed. ☐

Proposition 2.27. *Let the setting from Definition 2.24 be given, let $U_1 \subseteq \mathrm{Dom}_A$ be a subset with the property that $\mathrm{span}(U_1)$ is dense in the space H_1, and let $U_2 \subseteq \mathrm{Dom}_{A^*}$ be a subset with the property that $\mathrm{span}(U_2)$ is dense in the space H_2.*

For all $h_1 \in H_1$ *and* $h_2 \in H_2$ *that have the property that for all* $u \in U_1$ *it follows that*

$$\langle Au, h_2\rangle_{H_2} = \langle u, h_1\rangle_{H_1}, \tag{2.168}$$

it holds that $h_2 \in \mathrm{Dom}_{A^*}$ *and* $A^* h_2 = h_1$.
For all $h_1 \in H_1$ *and* $h_2 \in H_2$ *that have the property that for all* $u \in U_2$ *it holds that*

$$\langle h_1, A^* u\rangle_{H_1} = \langle h_2, u\rangle_{H_2}, \tag{2.169}$$

it holds that $h_1 \in \mathrm{Dom}_A$ *and* $Ah_1 = h_2$.
For all $h_2 \in \mathrm{Dom}_{A^*}$ *it holds that*

$$\|A^* h_2\|_{H_1} = \sup_{\{u \in \mathrm{span}(U_1)\colon \|u\|_{H_1} > 0\}} \frac{|\langle h_1, A^* u\rangle_{H_1}|}{\|u\|_{H_2}}. \tag{2.170}$$

For all $h_1 \in \mathrm{Dom}_A$ *it holds that*

$$\|Ah_1\|_{H_2} = \sup_{\{u \in \mathrm{span}(U_2)\colon \|u\|_{H_2} > 0\}} \frac{|\langle h_2, Au\rangle_{H_2}|}{\|u\|_{H_1}}. \tag{2.171}$$

For all $h_2 \in H_2$ *and all sequences* $(h_2^{(n)})$ *in* $\mathrm{Dom}(A^*)$ *with the properties that* $h_2^{(n)}$ *converges to* h_2 *in the weak topology of* H_2 *as* $n \to \infty$ *and that* $\sup_{n\in\mathbb{N}} \|A^* h_2^{(n)}\|_{H_1} < \infty$, *it holds that* $h_2 \in \mathrm{Dom}_{A^*}$ *and that* $A^* h_2^{(n)}$ *converges to* $A^* h_2$ *in the weak topology of* H_1.
For all $h_1 \in H_1$ *and all sequences* $(h_1^{(n)})$ *in* $\mathrm{Dom}(A)$ *with the properties that* $h_1^{(n)}$ *converges to* h_1 *in the weak topology of* H_1 *as* $n \to \infty$ *and that* $\sup_{n\in\mathbb{N}} \|Ah_1^{(n)}\|_{H_2} < \infty$, *it holds that* $h_1 \in \mathrm{Dom}_{A^*}$ *and that* $A^* h_1^{(n)}$ *converges to* $A^* h_1$ *in the weak topology of* H_2.

Definition 2.28. *Assume Setting 2.1. The unbounded operator*

$$\delta(\mathbb{P}, \mathcal{F}_0, W; H)\colon \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H) \subseteq L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))) \to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H) \tag{2.172}$$

is the adjoint of the operator

$$\begin{aligned}\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H) \subseteq L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)\colon &\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H) \\ &\to L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))).\end{aligned} \tag{2.173}$$

In particular, the domain $\mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$ *of* δ *is the set*

$$\mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H) = \left\{ \begin{array}{c} X \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))\colon \exists c \in [0,\infty) \\ \text{s.t. for all } F \in \mathbb{D}^{1,2}(\mathbb{P}, \mathcal{F}_0, W; H) \text{ it holds that} \\ \mathbb{E}[\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, X\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))}] \le c\|F\|_{L^2(\mathbb{P}; H)} \end{array} \right\}. \tag{2.174}$$

If $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$, *then* $\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)$ *is the element in* $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$

with the property that for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ *it holds that*

$$E\big[\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)\rangle_H\big] = \mathbb{E}\big[\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, X\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\big]. \tag{2.175}$$

For all mappings $Y\colon \Omega \times [S,T] \to \mathrm{HS}(U,H)$ *and for all* $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$ *such that it holds* $\lambda_{[S,T]} \otimes \mathbb{P}$*-almost everywhere that* $Y = X$ *we say that* Y *is* $(\mathbb{P}, \mathcal{F}_0, W; H)$*-Skorohod integrable and we write*

$$\int_S^T Y_s \, \delta(\mathbb{P}, \mathcal{F}_0, W; H)W_s = \delta(\mathbb{P}, \mathcal{F}_0, W; H)(Y) = \delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) = \int_S^T X_s \, \delta(\mathbb{P}, \mathcal{F}_0, W; H)W_s. \tag{2.176}$$

If $\mathcal{F}_0 = \{\emptyset, \Omega\}$ *and if* Y *is a* $(\mathbb{P}, \mathcal{F}_0, W; H)$*-Skorohod integrable mapping* $Y\colon \Omega \times [S,T] \to \mathrm{HS}(U,H)$ *we write*

$$\int_S^T Y_s \delta W_s = \delta(\mathbb{P}, \{\emptyset, \Omega\}, W; H)(X). \tag{2.177}$$

The operator δ *is also called* divergence operator*.*

Remark 2.29. It follows from Lemma 2.8 that the set $\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ is dense in the space $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$. Therefore, it follows from Proposition 2.26, that the Skorohod integral is well-defined.

Lemma 2.30. Assume Setting 2.1 and let $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$. Then it holds that

$$E[\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)] = 0. \tag{2.178}$$

Proof of Lemma 2.30. Since the Malliavin derivative of a constant mapping vanishes, we have that

$$\begin{aligned}
\big\|\mathbb{E}[\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)]\big\|_H^2 &= \big\langle \mathbb{E}[\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)], \mathbb{E}[\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)]\big\rangle_H \\
&= \mathbb{E}\big[\big\langle \delta(\mathbb{P}, \mathcal{F}_0, W; H)(X), \delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)\big\rangle_H\big] \\
&= \mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\big(E[\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)]\big), X\big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\big] \\
&= 0.
\end{aligned} \tag{2.179}$$

The proof Lemma 2.30 is thus completed. □

The following corollaries follow from Propostion 2.26:

Corollary 2.31. Assume Setting 2.1 and let $X \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$, $Y \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$ such that for all $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds that

$$E\big[\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, X\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\big] = E\big[\langle F, Y\rangle_H\big]. \tag{2.180}$$

Then $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$ and it holds $\mathbb{P}$-almost surely that $\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) = Y$.

Corollary 2.32. Assume Setting 2.1 and let $F \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$, $X \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ be such that for all $Y \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds that

$$\mathbb{E}[\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; H)(Y) \rangle_H] = \mathbb{E}[\langle X, Y \rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))}]. \tag{2.181}$$

Then $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ and it holds that $\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F = X$.

Now we are able to characterize the domain of the Malliavin differential operator and we can show, that this operator is closed under weak $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$-limits.

Corollary 2.33. Assume Setting 2.1 and let $F \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$. Then $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ if and only if

$$\sup_{Y \in \mathrm{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(U,H))) \setminus \{0\}} \frac{|\mathbb{E}[\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; H)(Y) \rangle_H]|}{\|Y\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))}} < \infty \tag{2.182}$$

and for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds that

$$\begin{aligned} \|F\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)} &= \|F\|^2_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))} \\ &\quad + \sup_{Y \in \mathrm{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(U,H))) \setminus \{0\}} \frac{|\mathbb{E}[\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; H)(Y) \rangle_H]|}{\|Y\|_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))}}. \end{aligned} \tag{2.183}$$

Corollary 2.34. Assume Setting 2.1, let $F \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$, and let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ which converges to F in the weak topology of $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$. Then $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ and the sequence $(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F_n)_{n \in \mathbb{N}}$ converges to $\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F$ it the weak topology of $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$.

Corollary 2.35. Assume Setting 2.1, let $X \in L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$, and let $(X_n)_{n \in \mathbb{N}}$ be a sequence with the property that $\sup_{n \in \mathbb{N}} \|\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X_n)\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)} < \infty$ which converges to X in the weak topology of $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$. Then $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$ and $(\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X_n))_{n \in \mathbb{N}}$ converges to $\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)$ in the weak topology of $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; H)$.

The following result is a simple but central property of the Skorohod integral, which is crucial for the proof of the Itô-Alekseev-Gröbner formula 3.3.

Lemma 2.36. Assume Setting 2.1, let $(s,t) \in \Delta_{S,T}$, let $\mathcal{F}_{[S,s] \cup [t,T]} \subseteq \mathcal{A}$ be a σ-algebra with the property that $\mathcal{F}_{[S,s] \cup [t,T]} = \sigma(\mathcal{F}_0 \cup \{W_r \colon r \in [S,s] \cup [t,T]\} \cup \mathcal{N})$, and let $X \in L^0(\mathbb{P}_{\sigma(\mathcal{F}_0 \cup \sigma(W) \cup \mathcal{N})}; \mathcal{B}_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))})$. Then

$$X|_{[s,t]} \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_{[S,s] \cup [t,T]}, W|_{[s,t]}; H) \tag{2.184}$$

if and only if

$$\mathbb{1}_{[s,t]} X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H) \tag{2.185}$$

and, in that case,

$$\int_s^t X_u \delta(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H) W_u = \int_S^T \mathbb{1}_{[s,t]}(u) X_u \, \delta(\mathbb{P}, \mathcal{F}_0, W; H) W_u. \tag{2.186}$$

Proof of Lemma 2.36. **We start with the "only if" part.** Assume that (2.185) holds true. In order to show, that (2.185) is true, we have to show that there exists a $c \in [0, \infty)$ such that for all $F \in \mathbb{D}^{1,2}(\mathbb{P}, \mathcal{F}_0, W; H)$ such that $\|F\|_{L^2(\mathbb{P};H)} \leq 1$ it holds that

$$E\big[\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, \mathbb{1}_{[s,t]} X \rangle_{L^2(\lambda_{[S,T]}, \mathrm{HS}(U,H))}\big] < c. \tag{2.187}$$

Let $F \in \mathbb{D}^{1,2}(\mathbb{P}, \mathcal{F}_0, W; H)$ be such that $\|F\|_{L^2(\mathbb{P};H)} \leq 1$. Then Lemma 2.16 implies that $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)$. From Lemma 2.16, the defintion of the Skorohod integral and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, \mathbb{1}_{[s,t]} X \big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))}\big] \\
&= \mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)|_{[s,t]} F, X|_{[s,t]} \big\rangle_{L^2(\lambda_{[s,t]}; H)}\big] \\
&= \mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)F, X|_{[s,t]} \big\rangle_{L^2(\lambda_{[s,t]}; H)}\big] \\
&= \mathbb{E}\big[\big\langle F, \delta(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)(X|_{[s,t]}) \big\rangle_H\big] \\
&\leq \big\|\delta(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)(X|_{[s,t]})\big\|_{L^2(\mathbb{P};H)} < \infty.
\end{aligned} \tag{2.188}$$

We conclude that (2.185) holds true.

Now we show the "if" part. Assume, that (2.185) holds true. Let $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ for the remainder of the proof. It follows from Lemma 2.16 that $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)$. From Lemma 2.16 and the defintion of the Skorohod integral Definition 2.28 it follows that

$$\begin{aligned}
&\mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)F, X|_{[s,t]} \big\rangle_{L^2(\lambda_{[s,t]}; H)}\big] \\
&= \mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)|_{[s,t]} F, X|_{[s,t]} \big\rangle_{L^2(\lambda_{[s,t]}; H)}\big] \\
&= \mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, \mathbb{1}_{[s,t]} X \big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))}\big] \\
&= \mathbb{E}\big[\big\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; H)(\mathbb{1}_{[s,t]} X) \big\rangle_H\big].
\end{aligned} \tag{2.189}$$

Since $\mathrm{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)) \subseteq \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ and $\mathrm{span}(\mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H))$ is dense in $L^2(\mathbb{P}; H)$ it follows that $\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ is dense in $\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)$ with respect to the topology induced by $\|\cdot\|_{L^2(\mathbb{P};H)}$. Therefore, we conclude from Lemma 2.26 that (2.184) holds true and that

$$\delta(\mathbb{P}, \mathcal{F}_{[S,s]\cup[t,T]}, W|_{[s,t]}; H)(X|_{[s,t]}) = \delta(\mathbb{P}, \mathcal{F}_0, W; H)(\mathbb{1}_{[s,t]} X). \tag{2.190}$$

The proof of Lemma 2.36 is thus completed. □

The following Lemma is a generalization of Lemma 2.9 in Grorud and Pardoux 1992.

Lemma 2.37. Assume Setting 2.1. Then

$$\operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\operatorname{HS}(U,H)))) \subseteq \operatorname{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;\operatorname{HS}(U,H)) \tag{2.191}$$

and for all $G \in \operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R}))$, $\eta \in L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$, $h \in H$ it holds that

$$\int_S^T G\eta_t h\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_t = G\Big(\int_S^T \eta_t dW_t\Big)h - \Big(\int_S^T \langle\widetilde{\mathcal{D}}_t(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G,\eta_t\rangle_{\operatorname{HS}(U,\mathbb{R})}dt\Big)h. \tag{2.192}$$

Proof of Lemma 2.37. Let $G \in \mathcal{S}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})$, $\eta \in L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$ and $h \in H$. Then $G\eta h \in \operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))))$. The product rule Lemma 2.10 and the integration by parts formula Lemma 2.11 imply that for all $F \in \operatorname{span}(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H))$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[\langle\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)F,G\eta h\rangle_{L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))}\big]\\
&=\mathbb{E}\big[\langle G\big(\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;H)F\big),\eta h\rangle_{L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))}\big]\\
&=\mathbb{E}\big[\langle\big(\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G\big)F,\eta h\rangle_{L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))}\big]\\
&=\mathbb{E}\Big[\Big\langle GF,\int_S^T \eta_t h dW_t\Big\rangle_H\Big]-\mathbb{E}\Big[\Big\langle F,(\widetilde{\mathcal{D}}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G)\eta h\Big\rangle_{L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))}\Big]\\
&=\mathbb{E}\Big[\Big\langle GF,\int_S^T \eta_t h dW_t\Big\rangle_H\Big]-\mathbb{E}\Big[\Big\langle F,\int_S^T(\widetilde{\mathcal{D}}_t(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G)\eta_t h dt\Big\rangle_{\operatorname{HS}(U,H)}\Big]\\
&=\mathbb{E}\Big[\Big\langle F,\Big(\int_S^T \eta_t dW_t\Big)Gh-\Big(\int_S^T(\widetilde{\mathcal{D}}_t(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G)\eta_t dt\Big)h\Big\rangle_{\operatorname{HS}(U,H)}\Big].
\end{aligned} \tag{2.193}$$

Therefore, the assertion follows from Lemma 2.31. The proof of Lemma 2.37 is thus completed. □

The following is a generalization of Exercise 3.2.8 in Nualart 1995.

Lemma 2.38 (A stochastic analogon of Fubini's theorem)**.** Assume Setting 2.1, let $(E,\mathcal{E},\nu)$ be a finite measure space, and let
$X \in \operatorname{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))$. Then

$$\int_E X_t(y)d\nu(y) \in \operatorname{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;H), \tag{2.194}$$

and it holds $\nu\otimes\mathbb{P}$-almost everywhere that

$$\int_S^T\int_E X_t(y)d\nu(y)\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_t = \int_E\left(\int_S^T X_t\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_t\right)(y)d\nu(y). \tag{2.195}$$

Proof of Lemma 2.38. From $X \in \operatorname{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))$ it follows that
$\delta(\mathbb{P},\mathcal{F}_0,W;L^2(\nu;H))(X) \in L^2(\mathbb{P};L^2(\nu;H))$. This together with $\nu(E) < \infty$ and together with

Fubini's theorem implies that $\int_E \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\nu; H))(X)(y) d\nu(y) \in L^2(\mathbb{P}; H)$ and that $\int_E X(y) d\nu(y) \in L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$. Consequently, Definition 2.28 of the Skorohod integral and Fubini's theorem imply that for all $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\Big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, \int_E X(y) d\nu(y) \Big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))}\Big] \\
&= \mathbb{E}\Big[\int_E \Big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)F, X(y) \Big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))} d\nu(y)\Big] \\
&= \mathbb{E}\Big[\Big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\nu; H))F, X \Big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U, L^2(E;H)))}\Big] \\
&= \mathbb{E}\Big[\big\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\nu; H))(X) \big\rangle_{L^2(E;H)}\Big] \\
&= \mathbb{E}\Big[\int_E \big\langle F, \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\nu; H))(X)(y) \big\rangle_H d\nu(y)\Big] \\
&= \mathbb{E}\Big[\Big\langle F, \int_E \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\nu; H))(X)(y) d\nu(y) \Big\rangle_H\Big].
\end{aligned} \tag{2.196}$$

Therefore, Lemma 2.31 yields that the stochastic process $\{\int X_t(y) d\nu(y) \colon t \in [S,T]\}$ is Skorohod integrable and that (2.195) holds true. The proof of Lemma 2.38 is thus completed. □

2.3 Derivative of an adapted process

The following lemma follows from Lemma 2.6 in León and Nualart 1998.

Lemma 2.39. Assume Setting 2.1, let $s \in [S,T]$, $r \in [S,s]$, let $\mathcal{F}_{[S,r]\cup[s,T]} \subseteq \mathcal{A}$ be the σ-algebra with the property that

$$\mathcal{F}_{[S,r]\cup[s,T]} = \sigma\big(\{W_t - W_S \colon t \in [S,r]\} \cup \{W_t - W_s \colon t \in [s,T]\} \cup \mathcal{F}_0 \cup \mathcal{N}\big), \tag{2.197}$$

and let $F \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ be $\mathcal{F}_{[S,r]\cup[s,T]}/\mathcal{B}_H$-measurable. Then it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)(\mathbb{1}_{(r,s)} F) = 0. \tag{2.198}$$

Proof of Lemma 2.39. **In the first step**, let $\widetilde{F} \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H)$. As $\widetilde{F}$ is $\mathcal{F}_{[S,r]\cup[s,T]}/\mathcal{B}_H$-measurable, it follows that there are with the property that $n \in \mathbb{N}$, $\phi_1, \ldots, \phi_n \in \mathcal{L}^2(\lambda_{[S,T]}; \mathrm{HS}(U,\mathbb{R}))$, $f \in C_b^{\infty,\sigma(\mathcal{F}_0 \cup \mathcal{N})}(\mathbb{R}^n \times \Omega, \mathbb{R})$, $h \in H$ such that it holds $\mathbb{P}$-almost surely that

$$F = f\Big(\int_S^T \phi_1(s) dW_s, \ldots, \int_S^T \phi_n(s) dW_s\Big) h \tag{2.199}$$

such that for all $i \in \{1, \ldots, n\}$ it holds $\lambda_{[S,T]}$-almost everywhere that $\phi_i \mathbb{1}_{(r,s)} = 0$. It follows from the product rule Corollary 2.14 and and from the fact, that the Malliavin derivative of a deterministic function vanishes, that it holds $\lambda_{[S,T]} \otimes \mathbb{P}$-almost everywhere that

$$\begin{aligned}\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)(\mathbb{1}_{(r,s)}\widetilde{F}) &= \mathbb{1}_{(r,s)}\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\widetilde{F}\\ &= \mathbb{1}_{(r,s)}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Big(\int\limits_S^T \phi_1(s)dW_s,\dots,\int\limits_S^T \phi_n(s)dW_s\Big)\phi_i h\\ &= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Big(\int\limits_S^T \phi_1(s)dW_s,\dots,\int\limits_S^T \phi_n(s)dW_s\Big)\big(\mathbb{1}_{(r,s)}\phi_i\big)h\\ &= g'\Big(\int\limits_S^T \phi(s)dW_s\Big)\phi(\mathbb{1}_{(r,s)}h) = 0.\end{aligned}\tag{2.200}$$

In the second step, let $\widetilde{F} \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)$. Then there exists a sequence $(\widetilde{F}_n)_{n\in\mathbb{N}}$ in $\operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;H)\big)$ such that $\|\widetilde{F}-\widetilde{F}_n\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)} \to 0$ as $n\to\infty$. Therefore, the first step together with linearity implies that for all $n\in\mathbb{N}$ it holds that

$$\begin{aligned}&\big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\widetilde{F}\mathbb{1}_{(r,s)}\big\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))}\\ &= \big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\widetilde{F}\mathbb{1}_{(r,s)}\\ &\qquad - \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\widetilde{F}_n\mathbb{1}_{(r,s)}\big\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))}\\ &\le \big\|\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\widetilde{F}-\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\widetilde{F}_n\big\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))}\\ &= \big\|\widetilde{F}-\widetilde{F}_n\big\|_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)}.\end{aligned}\tag{2.201}$$

The right-hand side converges to 0 as $n\to\infty$. The proof of Lemma 2.39 is thus completed. □

2.4 Skorohod integral generalizes the Itô integral for L^2-integrands

It is well-known that the classical Skorohod-Integral generalizes the Itô-integral restricted to square-integrable integrands which are adapted to the Brownian filtration. The following result, Proposition 2.40, generalizes this.

Proposition 2.40. *Assume Setting 2.1, let $s,t\in[S,T]$ satisfy $s<t$, let $\mathcal{F}=(\mathcal{F}_r)_{r\in[s,t]}$ be the filtration with the property that for all $r\in[s,t]$ it holds that*

$$\mathcal{F}_r = \sigma(W_u - W_s\colon u\in[s,r]) \vee \sigma(W_u\colon u\in[S,s]) \vee \sigma(W_u - W_t\colon u\in[t,T]) \vee \mathcal{F}_0,\tag{2.202}$$

and let $X\in L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))$ satisfy that $X|_{[s,t]}$ is $\mathcal{F}$-adapted. Then $X|_{[s,t]}$ is $(\mathbb{P},\sigma(W_u\colon u\in[S,s])\vee\sigma(\{W_r-W_t\colon r\in[t,T]\})\vee\mathcal{F}_0,W|_{[s,t]};H)$-Skorohod integrable and it holds

$\mathbb{P}$*-almost surely that*

$$\int_s^t X_r \delta(\mathbb{P}, \sigma(W_u \colon u \in [S,s]) \vee \sigma(\{W_r - W_t \colon r \in [t,T]\}) \vee \mathcal{F}_0, W|_{[s,t]} - W_s; H) W_r = \int_s^t X_r dW_r. \tag{2.203}$$

Proof of Proposition 2.40. **For the first step of the proof,** let $r_1, r_2 \in [s,t]$ with the property that $r_1 < r_2$, let $G \in \mathcal{M}(\sigma(\sigma(\{W_u - W_s \colon u \in [s, r_1]\}) \cup \mathcal{F}_0 \cup \mathcal{N}), \mathcal{B}_\mathbb{R})$, $h \in \mathrm{HS}(U,H)$, and let $Y \in \mathcal{S}(\mathbb{P}, \mathcal{G}, W; L^2(\lambda_{[s,t]}; \mathrm{HS}(U,H)))$ be such that it holds $\lambda_{[s,t]} \otimes \mathbb{P}$-almost everywhere that $Y = G \mathbb{1}_{(r_1,r_2]} h$. Therefore, Lemma 2.37 implies that it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\int_s^t Y_u \delta(\mathbb{P}, \sigma(\sigma(\{W_u - W_s \colon u \in [s, r_1]\}) \cup \mathcal{F}_0), W; H) W_u \\
&= G\Big(\int_s^t \mathbb{1}_{(r_1,r_2]}(u) dW_u\Big) h - \Big(\int_s^t \langle \mathcal{D}_u(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R}) G, \mathbb{1}_{(r_1,r_2]}(u) \rangle_{\mathrm{HS}(U,\mathbb{R})} du\Big) h \\
&= G\Big(\int_{r_1}^{r_2} \mathbb{1}_{(r_1,r_2]}(u) dW_u\Big) h \\
&= \int_s^t Y_u dW_u.
\end{aligned} \tag{2.204}$$

Linearity implies that (2.203) holds for all elementary $(\mathcal{F}_t)_{t \in [s,t]}$-adapted stochastic processes. Moreover, every square integrable $(\mathcal{F}_t)_{t \in [s,t]}$-predictable process is the $L^2(\lambda_{[s,t]} \otimes \mathbb{P}; \mathrm{HS}(U,H))$-limit of a sequence of elementary $(\mathcal{F}_t)_{t \in [s,t]}$-predictable stochastic processes. The fact that both the Skorohod integral and the Itô integral are closed linear operators together with equation (2.204) implies that $X|_{[s,t]}$ is $(\mathbb{P}, \sigma(W_u \colon u \in [S,s]) \vee \sigma(\{W_r - W_t \colon r \in [t,T]\}) \vee \mathcal{F}_0, W|_{[s,t]} - W_s; H)$-Skorohod integrable and that equation (2.203) holds true $\mathbb{P}$-almost surely. The proof of Lemma 2.40 is thus completed. □

2.5 Multiplication with a random variable

The following is a generalization of Display (1.49) in Nualart 1995.

Lemma 2.41. Assume Setting 2.1, let $(u_i)_{i \in \mathbb{N}}$ be a Hilbert space base of U, let H_1 be a separable real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{H_1}$, let $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H)$, and let $G \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H, H_1))$. Then $GX \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H_1)$ and it holds that

$$\begin{aligned}
&G \int_S^T X_t \, \delta(\mathbb{P}, \mathcal{F}_0, W; H) W_t \\
&= \int_S^T G X_t \delta(\mathbb{P}, \mathcal{F}_0, W; H_1) W_t + \int_S^T \sum_{i=1}^\infty \big(\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(U, H_1)) G\big)(u_i) \big(X_t(u_i)\big) dt.
\end{aligned} \tag{2.205}$$

Proof of Lemma 2.41. **For the first part of the proof**, assume that $G \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H, H_1))$. For the rest of the proof, let $G_0 \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathbb{R})$ and $\phi \in \mathrm{HS}(H, H_1)$ be such that holds $\mathbb{P}$-almost surely that $G = G_0 \phi$, let $\phi^* \in \mathrm{HS}(H_1, H)$ be the adjoint operator of ϕ, and let $G^* \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H_1, H))$ have the property that it holds $\mathbb{P}$-almost surely that $G^* = G_0\phi^*$. It follows that for all $x \in H_1$, $y \in H$ it holds $\mathbb{P}$-almost surely that

$$\langle x, G(y)\rangle_{H_1} = G_0\langle x, \phi(y)\rangle_{H_1} = G_0\langle \phi^*(x), y\rangle_H = \langle G^*(x), y\rangle_H. \tag{2.206}$$

From Fubini's Theorem, equation (2.206), and the product rule Corollary 2.14 it follows that for all $F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; H_1)$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H_1)F, GX\big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H_1))}\Big] \\
&= \int_S^T \mathbb{E}\Big[\big\langle \mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)F, GX_t\big\rangle_{\mathrm{HS}(U,H_1)}\Big]\,dt \\
&= \int_S^T \mathbb{E}\Big[\big\langle G^*\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)F, X_t\big\rangle_{\mathrm{HS}(U,H)}\Big]\,dt \\
&= \int_S^T \mathbb{E}\Big[\big\langle \mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H)(G^*F) - (\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)G^*)F, X_t\big\rangle_{\mathrm{HS}(U,H)}\Big]\,dt \\
&= \mathbb{E}\Big[\big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)(G^*F), X\big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\Big] \\
&\qquad\qquad - \sum_{i=1}^{\infty}\int_S^T \mathbb{E}\Big[\big\langle (\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)G^*)(u_i)F, X_t(u_i)\big\rangle_H\Big]\,dt \\
&= \mathbb{E}\Big[\big\langle G^*F, \int_S^T X_t\,\delta(\mathbb{P}, \mathcal{F}_0, W; H)W_t\big\rangle_H\Big] - \sum_{i=1}^{\infty}\mathbb{E}\Big[\int_S^T \big\langle F, (\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)G)(u_i)(X_t(u_i))\big\rangle_{H_1}dt\Big] \\
&= \mathbb{E}\Big[\Big\langle F, G\int_S^T X_t\,\delta(\mathbb{P}, \mathcal{F}_0, W; H)W_t - \int_S^T\sum_{i=1}^{\infty}(\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)G)(u_i)(X_t(u_i))dt\Big\rangle_{H_1}\Big].
\end{aligned} \tag{2.207}$$

Thus, Lemma 2.31 implies that

$$GX \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; H_1)(L^2(\lambda_{[S,T]}; \mathrm{HS}(U, H_1))) \tag{2.208}$$

and that it holds $\mathbb{P}$-almost surely that

$$G\int_S^T X_t\,\delta(\mathbb{P}, \mathcal{F}_0, W; H)W_t = \int_S^T GX_t\delta(\mathbb{P}, \mathcal{F}_0, W; H_1)W_t - \int_S^T\sum_{i=1}^{\infty}(\mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; H_1)G)(u_i)(X_t(u_i))dt). \tag{2.209}$$

For the second part of the proof, we observe, that by linearity, (2.208) and equation (2.209) hold true for all $G \in \operatorname{span}\big(\mathcal{S}(\mathrm{HS}(H, H_1))\big)$, $n \in \mathbb{N}$. From Lemma 2.12 we conclude that (2.208) and equation (2.209) holds true for all $G \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; \mathrm{HS}(H, H_1))$. The proof of Lemma 2.41 is thus completed. □

2.6 Derivative of an elementary Skorohod integral

Lemma 2.42. Assume Setting 2.1. Then

$$\operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\operatorname{HS}(U,H)))\big)\subseteq \operatorname{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;H) \tag{2.210}$$

and for all $X\in\operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\operatorname{HS}(U,H)))\big)$ it holds that $\int_S^T X_s\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_s\in\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)$ and it holds $\lambda_{[S,T]}\otimes\mathbb{P}$-almost everywhere that

$$\begin{aligned}&\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\Big(\int\limits_S^T X_s\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_s\Big)\\&=X+\int\limits_S^T\mathcal{D}\big(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\operatorname{HS}(U,H))\big)X_s\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_s.\end{aligned}\tag{2.211}$$

Proof of Lemma 2.42. It follows from Lemma 2.41, the product rule Corollary 2.14, Lemma 2.41, the commutativity relation Lemma 2.17, and Lemma 2.41 again that for all $G\in\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})$, $h\in H$, and all $\eta\in L^2(\lambda_{[S,T]};\operatorname{HS}(U,\mathbb{R}))$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}&\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\Big(\int\limits_S^T G\eta_t h\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_t\Big)\\
&=\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\Big(G\int\limits_S^T\eta_t\delta(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})W_th+\int\limits_S^T\sum_{i=1}^{\infty}\big(\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;\operatorname{HS}(U,\mathbb{R}))G\big)(u_i)\big(\eta_t(u_i)h\big)dt\Big)\\
&=G\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})\Big(\int\limits_S^T\eta_t\delta(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})W_t\Big)h+\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})(G)\Big(\int\limits_S^T\eta_t\delta(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})W_t\Big)h\\
&\quad+\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})\Big(\int\limits_S^T\sum_{i=1}^{\infty}\big(\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;\operatorname{HS}(U,\mathbb{R}))G\big)(u_i)\big(\eta_t(u_i)\big)dt\Big)h\\
&=G\eta h+\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})(G)\Big(\int\limits_S^T\eta_t\delta(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})W_t\Big)h\\
&\quad-\int\limits_S^T\sum_{i=1}^{\infty}\big(\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})\big(\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G\big)\big)\,(e_i)h\eta_r(e_i)dt\\
&=G\eta h+\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})(G)\Big(\int\limits_S^T\eta_t\delta(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})W_t\Big)h\\
&\quad-\int\limits_S^T\sum_{i=1}^{\infty}\big(\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})\big(\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;\mathbb{R})G\big)\big)\,(e_i)h\eta_r(e_i)dt\\
&=G\eta h+\int\limits_S^T\mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)G\eta_t h\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_t.\end{aligned}\tag{2.212}$$

Linearity of the Malliavin derivative and of the Skorohod integral implies that for all $X\in\operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\operatorname{HS}(U,H)))\big)$, equation (2.211) holds true. The proof of Lemma 2.42 is thus completed. ☐

2.7 Analogon of the Itô isometry

The following analogon of the Itô isometry is a generalization of (1.48) in Nualart 1995.

Lemma 2.43. Assume Setting 2.1. Then

$$\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))) \subseteq \mathrm{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;H), \tag{2.213}$$

for all $X, Y \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))$ it holds that

$$\|\delta(\mathbb{P},\mathcal{F}_0,W;H)(X)\|^2_{L^2(\mathbb{P};H)} \leq \|X\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))}, \tag{2.214}$$

and for all $X, Y \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))$ it holds that

$$\begin{aligned}
&\big\langle \delta(\mathbb{P},\mathcal{F}_0,W;H)(X), \delta(\mathbb{P},\mathcal{F}_0,W;H)(Y)\big\rangle_{L^2(\mathbb{P};H)} \\
&= \int_S^T \mathbb{E}\big[\langle X_t, Y_t\rangle_{\mathrm{HS}(U,H)}\big]\,dt + \int_S^T\int_S^T \mathbb{E}\big[\langle \mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;H)(X_s), \mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;H)(Y_t)\rangle_{\mathrm{HS}(U,\mathrm{HS}(U,H))}\big]\,dsdt.
\end{aligned} \tag{2.215}$$

Proof of Lemma 2.43. The definition of the Skorohod integral and Lemma 2.42 imply that for all $X, Y \in \mathrm{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))\big)$ it holds that

$$\delta(\mathbb{P},\mathcal{F}_0,W;H)(X), \delta(\mathbb{P},\mathcal{F}_0,W;H)(Y) \in \mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))) \tag{2.216}$$

and

$$\begin{aligned}
&\big\langle \delta(\mathbb{P},\mathcal{F}_0,W;H)(X), \delta(\mathbb{P},\mathcal{F}_0,W;H)(Y)\big\rangle_{L^2(\mathbb{P};H)} \\
&= \mathbb{E}\big[\big\langle \mathcal{D}(\mathbb{P},\mathcal{F}_0,W;H)\left(\delta(\mathbb{P},\mathcal{F}_0,W;H)(X)\right), Y\big\rangle_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\big] \\
&= \mathbb{E}\big[\int_S^T \langle \mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;H)\left(\delta(\mathbb{P},\mathcal{F}_0,W;H)(X)\right), Y_t\rangle_{\mathrm{HS}(U,H)}\,dt\big] \\
&= \mathbb{E}\big[\int_S^T \langle X_t + \int_S^T \mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))X_s\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_s, Y_t\rangle_{\mathrm{HS}(U,H)}\,dt\big] \\
&= \int_S^T \mathbb{E}\big[\langle X_t, Y_t\rangle_{\mathrm{HS}(U,H)}\big]\,dt \\
&\quad + \int_S^T \mathbb{E}\big[\langle \int_S^T \mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))X_s\,\delta(\mathbb{P},\mathcal{F}_0,W;H)W_s, Y_t\rangle_{\mathrm{HS}(U,H)}\big]\,dt \\
&= \int_S^T \mathbb{E}\big[\langle X_t, Y_t\rangle_{\mathrm{HS}(U,H)}\big]\,dt \\
&\quad + \int_S^T\int_S^T \mathbb{E}\big[\langle \mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))X_s, \\
&\qquad\qquad \mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))Y_t\rangle_{\mathrm{HS}(U,\mathrm{HS}(U,H))}\big]\,dsdt.
\end{aligned} \tag{2.217}$$

From inequality (2.217) and the Cauchy-Schwarz inequality it follows that for all

$X \in \operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))\big)$ it holds that

$$\begin{aligned}
&\|\delta(\mathbb{P},\mathcal{F}_0,W;H)(X)\|^2_{L^2(\mathbb{P};H)}\\
&=\mathbb{E}\Big[\int\limits_S^T\|X_t\|^2_{\mathrm{HS}(U,H)}dt\Big]+\int\limits_S^T\int\limits_S^T\mathbb{E}\big[\langle\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;H)(X_s),\mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;H)(X_t)\rangle_{\mathrm{HS}(U,\mathrm{HS}(U,H))}\big]dsdt\\
&\leq\mathbb{E}\big[\|X\|^2_{L^2(\lambda_{[S,T]};\mathrm{HS}(U,H))}\big] \qquad (2.218)\\
&+\int\limits_S^T\int\limits_S^T\mathbb{E}\Big[\tfrac{1}{2}\|\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;H)(X_s)\|^2_{\mathrm{HS}(U,\mathrm{HS}(U,H))}+\tfrac{1}{2}\|\mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;H)(X_t)\|^2_{\mathrm{HS}(U,\mathrm{HS}(U,H))}\Big]dsdt\\
&=\|X\|^2_{L^2(\mathbb{P};L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))}+\int\limits_S^T\int\limits_S^T\mathbb{E}\Big[\|\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;H)(X_s)\|^2_{\mathrm{HS}(U,\mathrm{HS}(U,H))}\Big]dsdt\\
&=\|X\|^2_{\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;H)}.
\end{aligned}$$

For the rest of the proof let $X,Y\in\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))$. Let $(X^{(n)})_{n\in\mathbb{N}}$ and $(Y^{(n)})_{n\in\mathbb{N}}$ be sequences in $\operatorname{span}\big(\mathcal{S}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))\big)$ which converge to X and Y in $\mathbb{D}^{(1,2)}(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))\big)$, respectively. It follows that the sequences $(X^{(n)})_{n\in\mathbb{N}}$ and $(Y^{(n)})_{n\in\mathbb{N}}$ also converge to X and Y, respectively, in the topology of $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$ as $n\to\infty$. Moreover, it follows that $\sup_{n\in\mathbb{N}}\|\delta(\mathbb{P},\mathcal{F}_0,W;H)(X_n)\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)}<\infty$ and $\sup_{n\in\mathbb{N}}\|\delta(\mathbb{P},\mathcal{F}_0,W;H)(Y_n)\|_{L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)}<\infty$ hold true. Together with Lemma 2.35 we conclude that $X,Y\in\mathrm{Dom}_\delta(\mathbb{P},\mathcal{F}_0,W;H)$ and that the sequences $(\delta(\mathbb{P},\mathcal{F}_0,W;H)(X_n))_{n\in\mathbb{N}}$ and $(\delta(\mathbb{P},\mathcal{F}_0,W;H)(Y_n))_{n\in\mathbb{N}}$ converge to $(\delta(\mathbb{P},\mathcal{F}_0,W;H)(X))_{n\in\mathbb{N}}$ and $(\delta(\mathbb{P},\mathcal{F}_0,W;H)(Y))_{n\in\mathbb{N}}$, respectively, in the weak topology of $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})};H)$, as $n\to\infty$. Together with equation (2.217) we infer that

$$\begin{aligned}
&\big\langle\delta(\mathbb{P},\mathcal{F}_0,W;H)(X),\delta(\mathbb{P},\mathcal{F}_0,W;H)(Y)\big\rangle_{L^2(\mathbb{P};H)}\\
&=\lim_{n\to\infty}\mathbb{E}\big[\langle\delta(\mathbb{P},\mathcal{F}_0,W;H)(X^{(n)}),\delta(\mathbb{P},\mathcal{F}_0,W;H)(Y)\rangle_H\big]\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}\big[\langle\delta(\mathbb{P},\mathcal{F}_0,W;H)(X^{(n)}),\delta(\mathbb{P},\mathcal{F}_0,W;H)(Y^{(m)})\rangle_H\big] \qquad (2.219)\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\Big(\int\limits_S^T\mathbb{E}\big[\langle X_t^{(n)},Y_t^{(m)}\rangle_{\mathrm{HS}(U,H)}\big]dt+\int\limits_S^T\int\limits_S^T\mathbb{E}\big[\langle\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))X_s^{(n)},\\
&\qquad\qquad\mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))Y_t^{(m)}\rangle_{\mathrm{HS}(U,H)}\big]dsdt\Big)\\
&=\int\limits_S^T\mathbb{E}\big[\langle X_t,Y_t\rangle_{\mathrm{HS}(U,H)}\big]dt+\int\limits_S^T\int\limits_S^T\mathbb{E}\big[\langle\mathcal{D}_t(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))X_s,\\
&\qquad\qquad\mathcal{D}_s(\mathbb{P},\mathcal{F}_0,W;L^2(\lambda_{[S,T]};\mathrm{HS}(U,H)))Y_t\rangle_{\mathrm{HS}(U,H)}\big]dsdt.
\end{aligned}$$

The proof of Lemma 2.43 is thus completed. □

2.8 The Skorohod integral for differentiable processes is closed under strong L^2-limits

Lemma 2.44. Assume Setting 2.1, let $X \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$, let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$, with the property that $X^{(n)}$ converges to X in $\mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ as $n \to \infty$.
Then $X \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ and $\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X^{(n)})$ converges to $\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X)$ in $L^2(\mathbb{P}|_{\sigma(\mathcal{F}_0\cup\sigma(W)\cup\mathcal{N})}; H)$ as $n \to \infty$.

Proof of Lemma 2.44. This follows immediately from inequality (2.214) in Lemma 2.43. □

2.9 Derivative of a Skorohod integral

The following lemma is a generalization of Proposition 3.4 in Grorud and Pardoux 1992.

Lemma 2.45. Assume Setting 2.1. For all $X \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ it holds that

$$\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H) \tag{2.220}$$

if and only if

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))) \tag{2.221}$$

and for all $X \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ with the property that $\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}&\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\big(\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)\big)\\&= X + \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))X_s).\end{aligned} \tag{2.222}$$

Proof of Lemma 2.45. For the **only if**-part, let $X \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ be such that $\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; H)$. We conclude that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)(\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)) - X \in L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))) \tag{2.223}$$

and

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \in L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))))). \tag{2.224}$$

From Lemma 2.43 and the definition of the Skorohod integral it follows that for all

$F \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ we have

$$\begin{aligned}
&\Big\langle \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(F), \\
&\qquad\qquad \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \Big\rangle_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))))} \\
&= \int_S^T \int_S^T \mathbb{E}\Big[\langle \mathcal{D}_s(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(F_t), \qquad (2.225) \\
&\qquad\qquad \mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X_s) \rangle_{\mathrm{HS}(U, \mathrm{HS}(U,H))} \Big] ds dt \\
&= \mathbb{E}\Big[\big\langle \delta(\mathbb{P}, \mathcal{F}_0, W; H)(F), \delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) \big\rangle_H \Big] - \int_S^T \mathbb{E}[\langle F_t, X_t \rangle_{\mathrm{HS}(U,H)}] dt \\
&= \mathbb{E}\big[\langle F, \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\big(\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)\big) \rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))} \big] - \int_S^T \mathbb{E}[\langle F_t, X_t \rangle_{\mathrm{HS}(U,H)}] dt \\
&= \mathbb{E}\Big[\big\langle F, \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\big(\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)\big) - X \big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))} \Big].
\end{aligned}$$

Therefore, Lemma 2.31 implies that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))) \qquad (2.226)$$

and that it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; H)\big(\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X)\big) \\
&\quad = X + \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))X_s).
\end{aligned} \qquad (2.227)$$

For the **if**-part let $X \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ with the property that

$$\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \in \mathrm{Dom}_\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))). \qquad (2.228)$$

From Lemma 2.43 and the definition of the Skorohod integral it follows that for all

$Y \in \mathcal{S}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X), \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(Y)\big\rangle_H\Big] \\
&= \int_S^T \mathbb{E}[\langle X_t, Y_t\rangle_{\mathrm{HS}(U,H)}]dt \\
&+ \int_S^T \int_S^T \mathbb{E}\Big[\big\langle \mathcal{D}_s(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X_t), \\
&\qquad\qquad \mathcal{D}_t(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(Y_s)\big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U, \mathrm{HS}(U,H)))}\Big] dt \qquad (2.229) \\
&= \int_S^T \mathbb{E}[\langle X_t, Y_t\rangle_{\mathrm{HS}(U,H)}]dt \\
&+ \big\langle D(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))X, \\
&\qquad\qquad \mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))Y\big\rangle_{L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U, L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))))} \\
&= \langle X, Y\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))} \\
&\quad + \big\langle \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))X), Y\big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))} \\
&= \big\langle X + \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))X), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Y\big\rangle_{L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))}.
\end{aligned}$$

Lemma 2.32 and the facts that $\delta(\mathbb{P}, \mathcal{F}_0, W; H)(X) \in L^2(\mathbb{P}; H)$ and that

$$\begin{aligned}
X + \delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(\mathcal{D}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))X) \\
\in L^2(\mathbb{P}; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))
\end{aligned} \qquad (2.230)$$

imply that it holds that

$$\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H))). \qquad (2.231)$$

Thus the if-part of the above assertion implies that
$\delta(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))(X) \in \mathbb{D}^{(1,2)}(\mathbb{P}, \mathcal{F}_0, W; L^2(\lambda_{[S,T]}; \mathrm{HS}(U,H)))$. The proof of Lemma 2.45 is thus completed. ☐

Chapter 3

Perturbation of SDEs

We are now equipped with the prerequisites to focus on the main problem of the thesis: How can we estimate the L^2-distance of an Itô process and a stochastic flow? On this behalf, we introduce two formulas: The first formula, the deterministic Itô-Alekseev-Gröbner formula which generalizes the Alekseev-Gröbner Theorem 1.1 from the introduction allows us to express the difference of a pathwise defined stochastic flow and a pathwise defined stochastic process with continuously differentiably paths.
The second formula, which is the main result of this thesis, is a non-deterministic, non-pathwise generalization of the Itô- and the Alekseev-Gröbner-formula and allows us to express the difference of a general stochastic flow which is twice continuously differentiable in the starting point and a general L^p-Itô process.
Finally, we demonstrate how the Itô-Alekseev-Gröbner formula can be applied to prove strong L^2-convergence rates of approximation schemes of SDEs using the example of the stochastic van-der-Pol oscillator.
Section 3.1 and Section 3.2 are the result of joint work with Martin Hutzenthaler, Arnulf Jentzen and Sara Mazzonetto which is available online as arXiv preprint Hudde, Hutzenthaler, Jentzen, and Mazzonetto 2018.

3.1 The deterministic Itô-Alekseev-Gröbner formula

The following proposition, Proposition 3.1, generalizes the Alekseev-Gröbner formula (cf., e.g., Hairer, Nørsett, and Wanner 1993, Theorem I.14.5) (which is the special case $k = d$, $f = \mathrm{Id}_{\mathbb{R}^d}$ of Proposition 3.1) to general test functions.

Proposition 3.1 (Deterministic Itô-Alekseev-Gröbner formula)**.** *Let $d, k \in \mathbb{N}$, $T \in (0, \infty)$, let $O \subseteq \mathbb{R}^d$ be a non-empty open set, let $\mu \in C^0([0,T] \times O, \mathbb{R}^d)$, $Y \in C^1([0,T], O)$, $X^{(\cdot)}_{\cdot,\cdot} = (X^x_{s,t})_{s\in[0,T],t\in[s,T],x\in O} \in C^0(\Delta_T \times O, O)$, $f \in C^1(O, \mathbb{R}^k)$, assume for all $s \in [0,T]$ that $(O \ni x \mapsto \mu(s,x) \in \mathbb{R}^d) \in C^1(O, \mathbb{R}^d)$, assume that $([0,T] \times O \ni (s,x) \mapsto (\tfrac{\partial}{\partial x}\mu)(s,x) \in L(\mathbb{R}^d, \mathbb{R}^d)) \in$*

$C^0([0,T] \times O, L(\mathbb{R}^d, \mathbb{R}^d))$, and assume for all $s \in [0,T]$, $t \in [s,T]$, $x \in O$ that $X^x_{s,t} = x + \int_s^t \mu(r, X^x_{s,r})\, dr$. Then

$$f(X^{Y_0}_{0,T}) - f(Y_T) = \int_0^T f'(X^{Y_s}_{s,T}) \tfrac{\partial}{\partial x} X^{Y_s}_{s,T} \Big(\mu(s, Y_s) - \tfrac{d}{ds} Y_s \Big)\, ds. \tag{3.1}$$

Proof of Proposition 3.1. The assumptions and the fundamental theorem of calculus imply for all $s \in [0,T)$, $t \in [s,T]$, $x \in O$ that $([s,T] \ni u \mapsto X^x_{s,u} \in O) \in C^1(O,O)$ and that $\frac{\partial}{\partial t} X^x_{s,t} = \mu(t, X^x_{s,t})$. This, the assumptions, and Hairer, Nørsett, and Wanner 1993, Theorem I.14.3) prove that for all $s \in [0,T]$, $t \in [s,T]$ it holds that $(O \ni x \mapsto X^x_{s,t} \in O) \in C^1(O,O)$ and that $\frac{\partial}{\partial x} X^{\cdot}_{\cdot,\cdot} \in C^0(\Delta_T \times O, L(\mathbb{R}^d, \mathbb{R}^d))$. Moreover, the assumptions, and Hairer, Nørsett, and Wanner 1993, Theorem I.14.4) show that for all $x \in O$ it holds that $([0,T] \ni s \mapsto X^x_{s,T} \in O) \in C^1([0,T], O)$, that $\frac{\partial}{\partial s} X^{\cdot}_{\cdot,T} \in C^0([0,T] \times O, \mathbb{R}^d)$, and that for all $s \in [0,T]$, $x \in O$ it holds that

$$\tfrac{\partial}{\partial s} X^x_{s,T} = -\tfrac{\partial}{\partial x} X^x_{s,T} \mu(s,x). \tag{3.2}$$

Therefore, the chain rule implies that $([0,T] \ni s \mapsto X^{Y_s}_{s,T} \in O \in C^1([0,T], O)$. Moreover, the fundamental theorem of calculus, the chain rule, and (3.2) yield that

$$\begin{aligned} f(X^{Y_0}_{0,T}) - f(Y_T) &= -\int_0^T \tfrac{d}{ds} \Big(f\big(X^{Y_s}_{s,T}\big) \Big)\, ds \\ &= -\int_0^T f'(X^{Y_s}_{s,T}) \left(\left(\tfrac{\partial}{\partial s} X^x_{s,T} \right)\Big|_{x=Y_s} + \tfrac{\partial}{\partial x} X^{Y_s}_{s,T} \tfrac{d}{ds} Y_s \right)\, ds \\ &= -\int_0^T f'(X^{Y_s}_{s,T}) \Big(-\tfrac{\partial}{\partial x} X^{Y_s}_{s,T} \mu(s, Y_s) + \tfrac{\partial}{\partial x} X^{Y_s}_{s,T} \tfrac{d}{ds} Y_s \Big)\, ds \\ &= \int_0^T f'(X^{Y_s}_{s,T}) \tfrac{\partial}{\partial x} X^{Y_s}_{s,T} \Big(\mu(s, Y_s) - \tfrac{d}{ds} Y_s \Big)\, ds. \end{aligned} \tag{3.3}$$

The proof of Proposition 3.1 is thus completed. □

3.2 The Itô-Alekseev-Gröbner formula

Setting 3.2. *Let $d, m \in \mathbb{N}$, $T \in (0,\infty)$, $p \in (4,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{N} = \{A \in \mathcal{F} \colon \mathbb{P}(A) = 0\}$ denote the $\mathbb{P}$-null sets, let $\mathbb{F} = (\mathbb{F}_t)_{t \in [0,T]}$ be a filtration on $(\Omega, \mathcal{F})$ which satisfies that $\mathbb{F}_0$ and $\sigma(W_s \colon s \in [0,T])$ are independent and which satisfies for all $t \in [0,T]$ that $\mathbb{F}_t = \sigma(\mathbb{F}_0 \cup \sigma(W_s \colon s \in [0,t]) \cup \mathcal{N})$, let $O \subseteq \mathbb{R}^d$ be a non-empty open set, let $\mu\colon [0,T] \times O \to \mathbb{R}^d$, $\sigma\colon [0,T] \times O \to \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)$ be continuous mappings, let $X^{\cdot}_{\cdot,\cdot}(\cdot)\colon \Delta_T \times O \times \Omega \to O$, $X^{1,\cdot}_{\cdot,\cdot}(\cdot)\colon \Delta_T \times O \times \Omega \to L(\mathbb{R}^d, \mathbb{R}^d)$, and $X^{2,\cdot}_{\cdot,\cdot}(\cdot)\colon \Delta_T \times O \times \Omega \to L^{(2)}(\mathbb{R}^d, \mathbb{R}^d)$ be continuous random fields such that for all $s \in [0,T]$, $\omega \in \Omega$ the mapping $O \ni x \mapsto X^x_{s,T}(\omega) \in O$ is in $C^2(O,O)$, such that for all $s \in [0,T]$, $x \in O$ the stochastic process $[s,T] \times \Omega \ni (t,\omega) \mapsto X^x_{s,t} \in O$*

is $(\sigma(\mathcal{N} \cup \sigma(W_r - W_s \colon r \in [s,t])))_{t\in[s,T]}$*-adapted and for all* $t \in [s,T]$ *it holds* $\mathbb{P}$*-almost surely that*

$$X^x_{s,t} = x + \int\limits_s^t \mu(r, X^x_{s,r})dr + \int\limits_s^t \sigma(r, X^x_{s,r})dW_r, \tag{3.4}$$

such that for all $(s,t) \in \Delta_T$*,* $x \in O$ *it holds* $\mathbb{P}$*-almost surely that* $X^{X^x_{s,t}}_{t,T} = X^x_{s,T}$*, and such that for all* $(s,t,x,\omega) \in \Delta_T \times O \times \Omega$ *it holds that* $X^{1,x}_{s,t}(\omega) = \frac{\partial}{\partial x}\big(X^x_{s,t}(\omega)\big)$ *and* $X^{2,x}_{s,t}(\omega) = \frac{\partial^2}{\partial x^2}\big(X^x_{s,t}(\omega)\big)$*, and let* $Y \in \mathcal{L}^p(\lambda_{[0,T]} \otimes \mathbb{P}; O)$*,* $A \in \mathcal{L}^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathbb{R}^d)$*,* $B \in \mathcal{L}^p(\lambda_{[0,T]} \otimes \mathbb{P}; \operatorname{HS}(\mathbb{R}^m, \mathbb{R}^d))$ *be stochastic processes such that* Y *has continuous sample paths, such that* Y *and* B *are* $\mathbb{F}$*-progressively measurable and such that for all* $t \in [0,T]$ *it holds* $\mathbb{P}$*-almost surely that*

$$Y_t = Y_0 + \int\limits_0^t A_s ds + \int\limits_0^t B_s dW_s. \tag{3.5}$$

Theorem 3.3 (The Itô-Alekseev-Gröbner formula)**.** *Assume Setting 3.2, let* $q \in [0, \frac{p}{2} - 2)$*, assume that*

$$\sup_{h\in T/\mathbb{N}} \mathbb{E}\Bigg[\int\limits_0^T \Big\|\mu\Big(t, X^{Y_{\lfloor t\rfloor_h}}_{\lfloor t\rfloor_h,t}\Big)\Big\|^p_{\mathbb{R}^d} + \Big\|\sigma\Big(t, X^{Y_{\lfloor t\rfloor_h}}_{\lfloor t\rfloor_h,t}\Big)\Big\|^p_{\operatorname{HS}(\mathbb{R}^m,\mathbb{R}^d)} dt\Bigg] < \infty \tag{3.6}$$

and

$$\sup_{\substack{r,s,t\in[0,T]\\ r\le s\le t}} \mathbb{E}\Bigg[\Big\|X^{X^{Y_r}_{r,s}}_{t,T}\Big\|^p_{\mathbb{R}^d} + \Big\|X^{1,X^{Y_r}_{r,s}}_{t,T}\Big\|^{\frac{4p}{p-2(q+2)}}_{L(\mathbb{R}^d,\mathbb{R}^d)} + \Big\|X^{2,X^{Y_r}_{r,s}}_{t,T}\Big\|^{\frac{2p}{p-2(q+2)}}_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}\Bigg] < \infty, \tag{3.7}$$

let $k \in \mathbb{N}$*,* $c_0 \in [0,\infty)$*, and let* $f \in C^2(O, \mathbb{R}^k)$ *satisfy that for all* $x \in O$ *it holds that*

$$\max\Big\{\tfrac{\|f(x)\|_{\mathbb{R}^k}}{1+\|x\|_{\mathbb{R}^d}}, \|f'(x)\|_{L(\mathbb{R}^d,\mathbb{R}^k)}, \|f''(x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^k)}\Big\} \le c_0(1 + \|x\|^q_{\mathbb{R}^d}). \tag{3.8}$$

Then the stochastic process $\big(f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}(\sigma(r,Y_r) - B_r)\big)_{r\in[0,T]}$ *is* $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)$*-Skorohod-integrable and it holds* $\mathbb{P}$*-almost surely that*

$$\begin{aligned}
&f\big(X^{Y_0}_{0,T}\big) - f(Y_T) \\
&= \int\limits_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\Big(\mu(r,Y_r) - A_r\Big)dr + \int\limits_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\Big(\sigma(r,Y_r) - B_r\Big)\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)W_r \\
&+ \tfrac{1}{2}\sum_{l,j=1}^d \int\limits_0^T \Big(\sigma(r,Y_r)\sigma(r,Y_r)^\top - B_r B_r^\top\Big)_{l,j} \\
&\qquad\qquad \cdot \Big(f''\big(X^{Y_r}_{r,T}\big)\big(X^{1,Y_r}_{r,T}, X^{1,Y_r}_{r,T}\big) + f'\big(X^{Y_r}_{r,T}\big)X^{2,Y_r}_{r,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)dr.
\end{aligned} \tag{3.9}$$

Proof of Theorem 3.3. The fact that for all $\omega \in \Omega$ the mapping $O \ni x \mapsto X^x_{T,T}(\omega) \in O$ is continuous and equation (3.4) imply that it holds $\mathbb{P}$-almost surely that $X^{Y_T}_{T,T} = Y_T$. Moreover, we rewrite the left-hand side of equation (3.9) as telescoping sum and obtain that for all $n \in \mathbb{N}$,

$h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
f\big(X^{Y_0}_{0,T}\big) - f\big(Y_T\big) &= f\big(X^{Y_{0h}}_{0h,T}\big) - f\big(X^{Y_{nh}}_{nh,T}\big) \\
&= \sum_{i=0}^{n-1} \Big(f\big(X^{Y_{ih}}_{ih,T}\big) - f\big(X^{Y_{(i+1)h}}_{(i+1)h,T}\big)\Big) \qquad (3.10) \\
&= \sum_{i=0}^{n-1} \Big(f\big(X^{Y_{ih}}_{ih,T}\big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big)\Big) - \sum_{i=0}^{n-1} \Big(f\big(X^{Y_{(i+1)h}}_{(i+1)h,T}\big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big)\Big).
\end{aligned}$$

First, we analyze the second sum on the right-hand side of equation (3.10)**.** For all $t \in [0,T]$, $x \in O$, $i \in \{1,2\}$ the mappings $\Omega \ni \omega \mapsto X^x_{t,T}(\omega) \in O$, $\Omega \ni \omega \mapsto X^{i,x}_{t,T}(\omega) \in L^{(i)}(\mathbb{R}^d, \mathbb{R}^d)$ are $\sigma(\mathcal{N} \cup \sigma(W_s - W_t \colon s \in [t,T]))$-measurable. This together with the fact that for all $\omega \in \Omega$, $t \in [0,T]$ it holds that $\Big(O \ni x \mapsto f(X^x_{t,T}(\omega)) \in \mathbb{R}^k\Big) \in C^2(O, \mathbb{R}^k)$ implies that for all $t \in [0,T]$ the mapping $\Omega \ni \omega \mapsto \Big(O \ni x \mapsto f(X^x_{t,T}(\omega)) \in \mathbb{R}^k\Big) \in C^2(O, \mathbb{R}^k)$ is independent of the σ-algebra $\mathbb{F}_t$. Itô's formula for independent random fields (e.g., Theorem 25.30 and Remark 25.26 in Klenke 2008) (applied with the mappings $\Omega \ni \omega \mapsto \Big(O \ni x \mapsto f(X^x_{(i+1)h,T}(\omega)) \in \mathbb{R}^k\Big) \in C^2(O, \mathbb{R}^k)$ for $n \in \mathbb{N}$, $i \in \{0,1,\dots,n-1\}$, $h \in \{\frac{T}{n}\}$) yields that for all $n \in \mathbb{N}$, $i \in \{0,1,\dots,n-1\}$, $h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&f\big(X^{Y_{(i+1)h}}_{(i+1)h,T}\big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big) \\
&= \int_{ih}^{(i+1)h} \tfrac{\partial}{\partial x}\big(f(X^x_{(i+1)h,T})\big)\big|_{x=Y_r} dY_r + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_{ih}^{(i+1)h} \tfrac{\partial^2}{\partial x^2}\big(f(X^x_{(i+1)h,T})\big)\big|_{x=Y_r} (e^{(d)}_l, e^{(d)}_j) d\left(\langle Y\rangle_r\right)_{l,j} \\
&= \int_{ih}^{(i+1)h} f'\big(X^{Y_r}_{(i+1)h,T}\big) X^{1,Y_r}_{(i+1)h,T} A_r dr + \int_{ih}^{(i+1)h} f'\big(X^{Y_r}_{(i+1)h,T}\big) X^{1,Y_r}_{(i+1)h,T} B_r dW_r \qquad (3.11) \\
&\quad + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_{ih}^{(i+1)h} \big(B_r B_r^\top\big)_{l,j} \\
&\qquad \cdot \Big(f''\big(X^{Y_r}_{(i+1)h,T}\big)\big(X^{1,Y_r}_{(i+1)h,T}, X^{1,Y_r}_{(i+1)h,T}\big) + f'\big(X^{Y_r}_{(i+1)h,T}\big) X^{2,Y_r}_{(i+1)h,T}\Big)(e^{(d)}_l, e^{(d)}_j) dr.
\end{aligned}$$

Inequalities (3.8) and (3.7) imply for all $i \in \{1,2\}$ that

$$\begin{aligned}
\sup_{\substack{r,s,t \in [0,T] \\ r \le s \le t}} \Big\| f^{(i)}\big(X^{X^{Y_r}_{r,s}}_{t,T}\big)\Big\|_{L^{\frac{p}{q}}(\mathbb{P}; L^{(i)}(\mathbb{R}^d, \mathbb{R}^k))} &\le c_0 \sup_{\substack{r,s,t \in [0,T] \\ r \le s \le t}} \Big\| 1 + \Big\| X^{X^{Y_r}_{r,s}}_{t,T} \Big\|^q_{\mathbb{R}^d} \Big\|_{L^{\frac{p}{q}}(\mathbb{P}; \mathbb{R})} \\
&\le c_0 \left(1 + \sup_{\substack{r,s,t \in [0,T] \\ r \le s \le t}} \Big\| X^{X^{Y_r}_{r,s}}_{t,T} \Big\|^q_{L^p(\mathbb{P}; \mathbb{R}^d)} \right) < \infty.
\end{aligned} \qquad (3.12)$$

Hölder's inequality, inequalities (3.7), (3.12), and the assumption $B \in \mathcal{L}^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d))$

imply that for all $n \in \mathbb{N}$, $i \in \{0, 1, \dots, n-1\}$, $h \in \{\frac{T}{n}\}$ it holds that

$$\begin{aligned}
&\Big\| f'\big(X^{Y_\cdot}_{(i+1)h,T}\big) X^{1,Y_\cdot}_{(i+1)h,T} B_\cdot \Big\|_{L^2(\mathbb{P}; L^2(\lambda_{[ih,(i+1)h]}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k)))} \\
&\leq \Big\| f'\big(X^{Y_\cdot}_{\lceil \cdot \rceil_h, T}\big) X^{1,Y_\cdot}_{\lceil \cdot \rceil_h, T} B_\cdot \Big\|_{L^2(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k))} \\
&\leq \Big\| \| f'\big(X^{Y_\cdot}_{\lceil \cdot \rceil_h, T}\big) \|_{L(\mathbb{R}^d, \mathbb{R}^k)} \| X^{1,Y_\cdot}_{\lceil \cdot \rceil_h, T} \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \| B_\cdot \|_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)} \Big\|_{L^2(\lambda_{[0,T]} \otimes \mathbb{P}; \mathbb{R})} \qquad (3.13) \\
&\leq \| f'\big(X^{Y_\cdot}_{\lceil \cdot \rceil_h, T}\big) \|_{L^{\frac{p}{q}}(\lambda_{[0,T]} \otimes \mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^k))} \| X^{1,Y_\cdot}_{\lceil \cdot \rceil_h, T} \|_{L^{\frac{2p}{p-2(q+1)}}(\lambda_{[0,T]} \otimes \mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^d))} \| B \|_{L^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d))} \\
&\leq T^{\frac{p-2}{2p}} \Big(\sup_{(r,s) \in \Delta_T} \| f'\big(X^{Y_r}_{s,T}\big) \|_{L^{\frac{p}{q}}(\mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^k))} \Big) \Big(\sup_{(r,s) \in \Delta_T} \| X^{1,Y_r}_{s,T} \|_{L^{\frac{2p}{p-2(q+1)}}(\mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^d))} \Big) \\
&\qquad \cdot \| B \|_{L^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d))} < \infty.
\end{aligned}$$

For all $n \in \mathbb{N}$, $i \in \{0, 1, \dots, n-1\}$, $h \in \{\frac{T}{n}\}$ the stochastic process

$$\big(f'\big(X^{Y_r}_{(i+1)h,T}\big) X^{1,Y_r}_{(i+1)h,T} B_r \big)_{r \in [ih,(i+1)h]} \tag{3.14}$$

is progressively measurable with respect to the filtration

$$\big(\sigma\big(\mathbb{F}_r \cup \sigma\big(\{ W_s - W_{(i+1)h} \colon s \in [(i+1)h, T] \} \big) \big) \big)_{r \in [ih,(i+1)h]}. \tag{3.15}$$

Proposition 2.40 together with inequality (3.13), Proposition 2.36, and linearity of the Skorohod integral yield that for all $h \in T/\mathbb{N}$ it holds that $(f'\big(X^{Y_r}_{\lceil r \rceil_h, T}\big) X^{1,Y_r}_{\lceil r \rceil_h, T} B_r)_{r \in [0,T]}$ is $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)$-Skorohod integrable and that for all $n \in \mathbb{N}$, $h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} f'\big(X^{Y_r}_{(i+1)h,T}\big) X^{1,Y_r}_{(i+1)h,T} B_r dW_r \\
&= \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} f'\big(X^{Y_r}_{(i+1)h,T}\big) X^{1,Y_r}_{(i+1)h,T} B_r \\
&\qquad\qquad \delta(\mathbb{P}, \sigma(\mathbb{F}_{ih} \cup \sigma(\{ W_s - W_{(i+1)h} \colon s \in [(i+1)h, T] \})), W|_{[ih,(i+1)h] \times \Omega}; \mathbb{R}^k) W_r \qquad (3.16) \\
&= \sum_{i=0}^{n-1} \int_0^T \mathbb{1}_{[ih,(i+1)h]}(r) f'\big(X^{Y_r}_{\lceil r \rceil_h, T}\big) X^{1,Y_r}_{\lceil r \rceil_h, T} B_r \, \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r \\
&= \int_0^T f'\big(X^{Y_r}_{\lceil r \rceil_h, T}\big) X^{1,Y_r}_{\lceil r \rceil_h, T} B_r \, \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r.
\end{aligned}$$

Equations (3.11) and (3.16) imply that for all $n \in \mathbb{N}$, $h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \Big(f\big(X^{Y_{(i+1)h}}_{(i+1)h,T}\big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big)\Big) \\
&= \int_0^T f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{1,Y_r}_{\lceil r\rceil_h,T} A_r dr + \int_0^T f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{1,Y_r}_{\lceil r\rceil_h,T} B_r \delta(\mathbb{P},\mathbb{F}_0,W;\mathbb{R}^k) W_r \\
&\quad + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \big(B_r B_r^\top\big)_{l,j} \Big(f''\big(X^{Y_r}_{\lceil r\rceil_h,T}\big)\big(X^{1,Y_r}_{\lceil r\rceil_h,T}, X^{1,Y_r}_{\lceil r\rceil_h,T}\big) + f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{2,Y_r}_{\lceil r\rceil_h,T}\Big)(e^{(d)}_l, e^{(d)}_j) dr.
\end{aligned} \tag{3.17}$$

Next, we analyze the first sum on the right-hand side of equation (3.10)**.** For all $(s,t) \in \Delta_T$, $x \in O$ it holds that $\mathbb{P}\big(X^x_{s,T} = X^{X^x_{s,t}}_{t,T}\big) = 1$. This and the fact that X is a continuous random field imply for all $(s,t) \in \Delta_T$ that $\mathbb{P}\Big(X^{Y_s}_{s,T} = X^{X^{Y_s}_{s,t}}_{t,T}\Big) = 1$. For all $t \in [0,T]$, $x \in O$, $i \in \{1,2\}$, the mappings $\Omega \ni \omega \mapsto X^x_{t,T}(\omega) \in O$, $\Omega \ni \omega \mapsto X^{i,x}_{t,T}(\omega) \in L^{(i)}(\mathbb{R}^d,\mathbb{R}^d)$ are $\sigma(\mathcal{N} \cup \sigma(W_s - W_t \colon s \in [t,T]))$-measurable. This together with the fact that for all $\omega \in \Omega$, $t \in [0,T]$, it holds that $\Big(O \ni x \mapsto f(X^x_{t,T}(\omega)) \in \mathbb{R}^k\Big) \in C^2(O,\mathbb{R}^k)$ implies that for all $t \in [0,T]$ the mapping $\Omega \ni \omega \mapsto \Big(O \ni x \mapsto f(X^x_{t,T}(\omega)) \in \mathbb{R}^k\Big) \in C^2(O,\mathbb{R}^k)$ is independent of the σ-algebra $\mathbb{F}_t$. Itô's formula for independent random fields (e.g., Theorem 25.30 and Remark 25.26 in Klenke 2008) (applied with the mappings $\Omega \ni \omega \mapsto \Big(O \ni x \mapsto f(X^x_{(i+1)h,T}(\omega)) \in \mathbb{R}^k\Big) \in C^2(O,\mathbb{R}^k)$ for $n \in \mathbb{N}$, $i \in \{0,1,\ldots,n-1\}$, $h \in \{\frac{T}{n}\}$) yields that for all $n \in \mathbb{N}$, $i \in \{0,1,\ldots,n-1\}$, $h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&f\big(X^{Y_{ih}}_{ih,T}\big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big) \\
&= f\Big(X^{X^{Y_{ih}}_{ih,(i+1)h}}_{(i+1)h,T}\Big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big) \\
&= \int_{ih}^{(i+1)h} f'\Big(X^{X^{Y_{ih}}_{ih,r}}_{(i+1)h,T}\Big) X^{1,X^{Y_{ih}}_{ih,r}}_{(i+1)h,T} dX^{Y_{ih}}_{ih,r} \\
&\quad + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_{ih}^{(i+1)h} \Big(f''\Big(X^{X^{Y_{ih}}_{ih,r}}_{(i+1)h,T}\Big)\Big(X^{1,X^{Y_{ih}}_{ih,r}}_{(i+1)h,T}, X^{1,X^{Y_{ih}}_{ih,r}}_{(i+1)h,T}\Big) \\
&\quad + f'\Big(X^{X^{Y_{ih}}_{ih,r}}_{(i+1)h,T}\Big) X^{2,X^{Y_{ih}}_{ih,r}}_{(i+1)h,T}\Big)(e^{(d)}_l, e^{(d)}_j) d\big(\langle X^{Y_{ih}}_{ih,\cdot}\rangle_r\big)_{l,j} \\
&= \int_{ih}^{(i+1)h} f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \mu\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big) dr + \int_{ih}^{(i+1)h} f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big) dW_r \\
&\quad + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_{ih}^{(i+1)h} \Big(\sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big)\sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big)^\top\Big)_{l,j} \\
&\qquad \cdot \Big(f''\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big)\Big(X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}, X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) + f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{2,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big)(e^{(d)}_l, e^{(d)}_j) dr.
\end{aligned} \tag{3.18}$$

Hölder's inequality and inequalities (3.12), (3.7), and (3.6) imply that for all $n \in \mathbb{N}$, $i \in$

$\{0, 1, \ldots, n-1\}$, $h \in \{\frac{T}{n}\}$ it holds that

$$\begin{aligned}
&\left\| f'\left(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \right) X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big) \right\|_{L^2(\mathbb{P};L^2(\lambda_{[ih,(i+1)h]};\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^k)))} \\
&\leq \left\| f'\left(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \right) X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big) \right\|_{L^2(\lambda_{[0,T]}\otimes\mathbb{P};\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^k))} \\
&\leq \left\| \left\| f'\left(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \right) \right\|_{L(\mathbb{R}^d,\mathbb{R}^k)} \left\| X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \right\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \left\| \sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big) \right\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \right\|_{L^2(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \qquad (3.19) \\
&\leq \left\| f'\left(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \right) \right\|_{L^{\frac{p}{q}}(\lambda_{[0,T]}\otimes\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^k))} \left\| X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \right\|_{L^{\frac{2p}{p-2(q+1)}}(\lambda_{[0,T]}\otimes\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^d))} \\
&\qquad \cdot \left\| \sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big) \right\|_{L^p(\lambda_{[0,T]}\otimes\mathbb{P};\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \\
&\leq T^{\frac{p-2}{2p}} \left(\sup_{\substack{r,s,t\in[0,T] \\ r\leq s\leq t}} \left\| f'\left(X^{X^{Y_r}_{r,s}}_{t,T} \right) \right\|_{L^{\frac{p}{q}}(\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^k))} \right) \left(\sup_{\substack{r,s,t\in[0,T] \\ r\leq s\leq t}} \left\| X^{1,X^{Y_r}_{r,s}}_{t,T} \right\|_{L^{\frac{2p}{p-2(q+1)}}(\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^d))} \right) \\
&\qquad \cdot \left(\sup_{\kappa\in T/N} \left\| \sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_\kappa}}_{\lfloor\cdot\rfloor_\kappa,\cdot}\Big) \right\|_{L^p(\lambda_{[0,T]}\otimes\mathbb{P};\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \right) < \infty.
\end{aligned}$$

For all $n \in \mathbb{N}$, $i \in \{0, 1, \ldots, n-1\}$, $h \in \{\frac{T}{n}\}$ the process

$$\left(f'\Big(X^{X^{Y_{ih}}_{ih,r}}_{(i+1)h,T} \Big) X^{1,X^{Y_{ih}}_{ih,r}}_{(i+1)h,T} \sigma\Big(r, X^{Y_{ih}}_{ih,r}\Big) \right)_{r\in[ih,(i+1)h]} \qquad (3.20)$$

is progressively measurable with respect to the filtration (3.15). Proposition 2.40 together with inequality (3.19), Proposition 2.36, and linearity of the $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)$-Skorohod integral assert that the process $f'\Big(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \Big) X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T} \sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big)$ is $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)$-Skorohod integrable and that for all $n \in \mathbb{N}$, $h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} f'\left(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \right) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) dW_r \\
&= \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} f'\left(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \right) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) \qquad (3.21) \\
&\qquad\qquad \delta(\mathbb{P}, \sigma(\mathbb{F}_{ih} \cup \sigma(W_s - W_{(i+1)h} \colon s \in [(i+1)h, T])), W|_{[ih,(i+1)h]\times\Omega}; \mathbb{R}^k) W_r \\
&= \sum_{i=0}^{n-1} \int_0^T \mathbb{1}_{[ih,(i+1)h]}(r) f'\left(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \right) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) \, \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r \\
&= \int_0^T f'\left(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \right) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r.
\end{aligned}$$

Equations (3.18) and (3.21) imply that for all $n \in \mathbb{N}$, $h \in \{\frac{T}{n}\}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \Big(f\big(X^{Y_{ih}}_{ih,T}\big) - f\big(X^{Y_{ih}}_{(i+1)h,T}\big)\Big) \\
&= \int_0^T f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \mu\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) dr \\
&\quad + \int_0^T f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r \\
&\quad + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \Big(\sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big)\sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big)^{\top}\Big)_{l,j} \\
&\qquad \cdot \Bigg(f''\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big)\Big(X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}, X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) + f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{2,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \Bigg) \big(e^{(d)}_l, e^{(d)}_j\big) dr.
\end{aligned} \tag{3.22}$$

Equations (3.10), (3.22), and (3.17) imply that for all $h \in {}^{T}\!/\mathbb{N}$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&f\big(X^{Y_0}_{0,T}\big) - f\big(Y_T\big) \\
&= \int_0^T f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \mu\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big) - f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{1,Y_r}_{\lceil r\rceil_h,T} A_r dr \\
&\quad + \int_0^T f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\Big) - f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{1,Y_r}_{\lceil r\rceil_h,T} B_r \, \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r \\
&\quad + \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \Big(\sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big)\sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big)^{\top}\Big)_{l,j} \\
&\qquad \cdot \Bigg(f''\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big)\Big(X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}, X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) + f'\Big(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Big) X^{2,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \Bigg) \big(e^{(d)}_l, e^{(d)}_j\big) dr \\
&\quad - \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \big(B_r B_r^{\top}\big)_{l,j} \Big(f''\big(X^{Y_r}_{\lceil r\rceil_h,T}\big)\big(X^{1,Y_r}_{\lceil r\rceil_h,T}, X^{1,Y_r}_{\lceil r\rceil_h,T}\big) + f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{2,Y_r}_{\lceil r\rceil_h,T}\Big) \big(e^{(d)}_l, e^{(d)}_j\big) dr.
\end{aligned} \tag{3.23}$$

Next, we want to let ${}^{T}\!/\mathbb{N} \ni h \to 0$ in (3.23) in a suitable sense and first justify this. Hölder's inequality, inequalities (3.12), (3.7), (3.6), and the fact that $A \in \mathcal{L}^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathbb{R}^d)$ imply that

$$\begin{aligned}
&\sup_{h \in {}^{T}\!/\mathbb{N}} \Bigg\| f'\Big(X^{X^{Y_{\lfloor \cdot\rfloor_h}}_{\lfloor \cdot\rfloor_h,\cdot}}_{\lceil \cdot\rceil_h,T}\Big) X^{1,X^{Y_{\lfloor \cdot\rfloor_h}}_{\lfloor \cdot\rfloor_h,\cdot}}_{\lceil \cdot\rceil_h,T} \mu\Big(\cdot, X^{Y_{\lfloor \cdot\rfloor_h}}_{\lfloor \cdot\rfloor_h,\cdot}\Big) - f'\big(X^{Y_\cdot}_{\lceil \cdot\rceil_h,T}\big) X^{1,Y_\cdot}_{\lceil \cdot\rceil_h,T} A_\cdot \Bigg\|_{L^2(\lambda_{[0,T]} \otimes \mathbb{P}; \mathbb{R}^k)} \\
&\quad \le T^{\frac{p-2}{2p}} \Big(\sup_{\substack{r,s,t \in [0,T] \\ r \le s \le t}} \Big\| f'\Big(X^{X^{Y_r}_{r,s}}_{t,T}\Big)\Big\|_{L^{\frac{p}{q}}(\mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^k))}\Big)\Big(\sup_{\substack{r,s,t \in [0,T] \\ r \le s \le t}} \Big\| X^{1,X^{Y_r}_{r,s}}_{t,T}\Big\|_{L^{\frac{2p}{p-2(q+1)}}(\mathbb{P}; L(\mathbb{R}^d, \mathbb{R}^d))}\Big) \\
&\qquad \cdot \Big(\sup_{h \in {}^{T}\!/\mathbb{N}} \Big\| \mu\big(\cdot, X^{Y_{\lfloor \cdot\rfloor_h}}_{\lfloor \cdot\rfloor_h,\cdot}\big)\Big\|_{L^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathbb{R}^d)} + \|A\|_{L^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathbb{R}^d)}\Big) < \infty.
\end{aligned} \tag{3.24}$$

Hölder's inequality and inequalities (3.12) and (3.7) imply that for all $l, j \in \{1, \ldots, d\}$ it holds

that

$$
\begin{aligned}
&\sup_{h\in T/\mathbb{N}} \Bigg\| \Bigg(f''\Big(X_{\lceil\cdot\rceil_h,T}^{X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) \Big(X_{\lceil\cdot\rceil_h,T}^{1,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}}, X_{\lceil\cdot\rceil_h,T}^{1,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) \\
&\qquad\qquad + f'\Big(X_{\lceil\cdot\rceil_h,T}^{X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) X_{\lceil\cdot\rceil_h,T}^{2,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Bigg) \big(e_l^{(d)}, e_j^{(d)}\big) \Bigg\|_{L^{\frac{2p}{p-4}}(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)} \\
&\leq \sup_{h\in T/\mathbb{N}} \Bigg\| \Bigg\| f''\Big(X_{\lceil\cdot\rceil_h,T}^{X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) \Bigg\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^k)} \Bigg\| X_{\lceil\cdot\rceil_h,T}^{1,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Bigg\|_{L(\mathbb{R}^d,\mathbb{R}^d)}^2 \\
&\qquad\qquad + \Bigg\| f'\Big(X_{\lceil\cdot\rceil_h,T}^{X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) \Bigg\|_{L(\mathbb{R}^d,\mathbb{R}^k)} \Bigg\| X_{\lceil\cdot\rceil_h,T}^{2,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Bigg\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \Bigg\|_{L^{\frac{2p}{p-4}}(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\leq \sup_{h\in T/\mathbb{N}} \Bigg(\Bigg\| f''\Big(X_{\lceil\cdot\rceil_h,T}^{X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) \Bigg\|_{L^{\frac{p}{q}}(\lambda_{[0,T]}\otimes\mathbb{P};L^{(2)}(\mathbb{R}^d,\mathbb{R}^k))} \Bigg\| X_{\lceil\cdot\rceil_h,T}^{1,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Bigg\|_{L^{\frac{4p}{p-2(q+2)}}(\lambda_{[0,T]}\otimes\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^d))}^2 \\
&\qquad + \Bigg\| f'\Big(X_{\lceil\cdot\rceil_h,T}^{X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Big) \Bigg\|_{L^{\frac{p}{q}}(\lambda_{[0,T]}\otimes\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^k))} \Bigg\| X_{\lceil\cdot\rceil_h,T}^{2,X_{\lfloor\cdot\rfloor_h,\cdot}^{Y_{\lfloor\cdot\rfloor_h}}} \Bigg\|_{L^{\frac{2p}{p-2(q+2)}}(\lambda_{[0,T]}\otimes\mathbb{P};L^{(2)}(\mathbb{R}^d,\mathbb{R}^d))} \Bigg) \\
&\leq T^{\frac{p-4}{2p}} \Bigg(\Bigg(\sup_{\substack{r,s,t\in[0,T] \\ r\leq s\leq t}} \Big\| f''\Big(X_{t,T}^{X_{r,s}^{Y_r}} \Big) \Big\|_{L^{\frac{p}{q}}(\mathbb{P};L^{(2)}(\mathbb{R}^d,\mathbb{R}^k))} \Bigg) \Bigg(\sup_{\substack{r,s,t\in[0,T] \\ r\leq s\leq t}} \Big\| X_{t,T}^{1,X_{r,s}^{Y_r}} \Big\|_{L^{\frac{4p}{p-2(q+2)}}(\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^d))}^2 \Bigg) \\
&\qquad + \Bigg(\sup_{\substack{r,s,t\in[0,T] \\ r\leq s\leq t}} \Big\| f'\Big(X_{t,T}^{X_{r,s}^{Y_r}} \Big) \Big\|_{L^{\frac{p}{q}}(\mathbb{P};L(\mathbb{R}^d,\mathbb{R}^k))} \Bigg) \Bigg(\sup_{\substack{r,s,t\in[0,T] \\ r\leq s\leq t}} \Big\| X_{t,T}^{2,X_{r,s}^{Y_r}} \Big\|_{L^{\frac{2p}{p-2(q+2)}}(\mathbb{P};L^{(2)}(\mathbb{R}^d,\mathbb{R}^d))} \Bigg) \Bigg) < \infty
\end{aligned}
\tag{3.25}
$$

and, analogously, that for all $l, j \in \{1, \ldots, d\}$ it holds that

$$
\sup_{h\in T/\mathbb{N}} \Big\| \Big(f''(X_{\lceil\cdot\rceil_h,T}^{Y_\cdot}) \big(X_{\lceil\cdot\rceil_h,T}^{1,Y_\cdot}, X_{\lceil\cdot\rceil_h,T}^{1,Y_\cdot} \big) + f'(X_{\lceil\cdot\rceil_h,T}^{Y_\cdot}) X_{\lceil\cdot\rceil_h,T}^{2,Y_\cdot} \Big) \big(e_l^{(d)}, e_j^{(d)}\big) \Big\|_{L^{\frac{2p}{p-4}}(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)} < \infty.
\tag{3.26}
$$

The fact that for all $C \in \mathbb{R}^{d\times m}$ it holds that $\sum_{i,j=1}^{d} |(CC^\top)_{i,j}| \leq d\|C\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}^2$, Hölder's inequality, assumption (3.6) and inequality (3.25) imply that

$$\begin{aligned}
&\sup_{h\in T/\mathbb{N}} \tfrac{1}{2}\Bigg\| \sum_{l,j=1}^{d} \Big(\sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big)\sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big)^{\top}\Big)_{l,j} \\
&\cdot \Bigg(f''\Bigg(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)\Bigg(X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}, X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg) + f'\Bigg(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)X^{2,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)\big(e^{(d)}_l, e^{(d)}_j\big)\Bigg\|_{L^2(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)} \\
&\leq \sup_{h\in T/\mathbb{N}} \tfrac{1}{2}\Bigg\| d\,\Big\|\sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big)\Big\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \\
&\cdot \sum_{l,j=1}^{d} \Bigg| f''\Bigg(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)\Bigg(X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}, X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg) + f'\Bigg(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)X^{2,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg|\big(e^{(d)}_l, e^{(d)}_j\big)\Bigg\|_{L^2(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)} \\
&\leq \sup_{h\in T/\mathbb{N}} \tfrac{d}{2}\Bigg(\Big\|\sigma\Big(\cdot, X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}\Big)\Big\|^2_{L^p(\lambda_{[0,T]}\otimes\mathbb{P};\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \\
&\qquad\cdot \sum_{l,j\in\{1,\dots,d\}} \Bigg\| f''\Bigg(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)\Bigg(X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}, X^{1,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg) \\
&\qquad\qquad + f'\Bigg(X^{X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)X^{2,X^{Y_{\lfloor\cdot\rfloor_h}}_{\lfloor\cdot\rfloor_h,\cdot}}_{\lceil\cdot\rceil_h,T}\Bigg)\big(e^{(d)}_l, e^{(d)}_j\big)\Bigg\|_{L^{\frac{2p}{p-4}}(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)}\Bigg) < \infty.
\end{aligned}\tag{3.27}$$

Analogously, the fact that for all $C \in \mathbb{R}^{d\times m}$ it holds that $\sum_{i,j=1}^{d} |(CC^{\top})_{i,j}| \leq d\|C\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$, Hölder's inequality, the assumption $B \in \mathcal{L}^p(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d))$, and inequality (3.26) yield that

$$\begin{aligned}
&\sup_{h\in T/\mathbb{N}} \tfrac{1}{2}\Bigg\| \sum_{l,j=1}^{d} \big(B_\cdot B_\cdot^{\top}\big)_{l,j} \\
&\qquad\cdot \Big(f''(X^{Y_\cdot}_{\lceil\cdot\rceil_h,T})\big(X^{1,Y_\cdot}_{\lceil\cdot\rceil_h,T}, X^{1,Y_\cdot}_{\lceil\cdot\rceil_h,T}\big) + f'(X^{Y_\cdot}_{\lceil\cdot\rceil_h,T})X^{2,Y_\cdot}_{\lceil\cdot\rceil_h,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)\Bigg\|_{L^2(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)} \\
&\leq \frac{d}{2}\|B\|^2_{L^p(\lambda_{[0,T]}\otimes\mathbb{P};\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \\
&\cdot \sum_{l,j\in\{1,\dots,d\}} \sup_{h\in T/\mathbb{N}} \Bigg\|\Big(f''(X^{Y_\cdot}_{\lceil\cdot\rceil_h,T})\big(X^{1,Y_\cdot}_{\lceil\cdot\rceil_h,T}, X^{1,Y_\cdot}_{\lceil\cdot\rceil_h,T}\big) + f'(X^{Y_\cdot}_{\lceil\cdot\rceil_h,T})X^{2,Y_\cdot}_{\lceil\cdot\rceil_h,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)\Bigg\|_{L^{\frac{2p}{p-4}}(\lambda_{[0,T]}\otimes\mathbb{P};\mathbb{R}^k)} \\
&< \infty.
\end{aligned}\tag{3.28}$$

Next, Corollary 6.21 and Theorem 6.25 in Klenke 2008 together with the uniform L^2-bounds in (3.24), (3.27), and (3.28), continuity of f' and of f'', path continuity of Y and of $\Delta_T \times O \ni$

$(s,t,x) \mapsto X^x_{s,t} \in O$, and $\inf_{r\in[0,T]} \mathbb{P}(X^{Y_r}_{r,r} = Y_r) = 1$ imply that

$$\begin{aligned}
&\lim_{T/\mathbb{N}\ni h\searrow 0} \Bigg\| \int_0^T f'\Bigg(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Bigg) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T} \mu(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}) - f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{1,Y_r}_{\lceil r\rceil_h,T} A_r\,dr \\
&- \int_0^T f'\big(X^{Y_r}_{r,T}\big) X^{1,Y_r}_{r,T} \Big(\mu\Big(r, Y_r\Big) - A_r\Big)dr \\
&+ \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \Big(\sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big)\sigma\big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}\big)^{\top}\Big)_{l,j} \qquad (3.29) \\
&\quad \cdot \Bigg(f''\Bigg(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Bigg)\Bigg(X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}, X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Bigg) + f'\Bigg(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Bigg) X^{2,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h,r}}_{\lceil r\rceil_h,T}\Bigg)\big(e^{(d)}_l, e^{(d)}_j\big)dr \\
&- \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \big(B_r B_r^{\top}\big)_{l,j} \Big(f''\big(X^{Y_r}_{\lceil r\rceil_h,T}\big)\big(X^{1,Y_r}_{\lceil r\rceil_h,T}, X^{1,Y_r}_{\lceil r\rceil_h,T}\big) + f'\big(X^{Y_r}_{\lceil r\rceil_h,T}\big) X^{2,Y_r}_{\lceil r\rceil_h,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)dr \\
&- \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \big(\sigma(r,Y_r)\sigma(r,Y_r)^{\top} - B_r B_r^{\top}\big)_{l,j} \\
&\qquad\qquad \cdot \Big(f''\big(X^{Y_r}_{r,T}\big)\big(X^{1,Y_r}_{r,T}, X^{1,Y_r}_{r,T}\big) + f'\big(X^{Y_r}_{r,T}\big) X^{2,Y_r}_{r,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)dr \Bigg\|_{L^1(\mathbb{P};\mathbb{R}^k)} = 0.
\end{aligned}$$

Inequality (3.8) implies that for all $x, y \in O$ it holds that

$$\begin{aligned}
\|f(x) - f(y)\|_{\mathbb{R}^k} &\le \|\, f(x)\|_{\mathbb{R}^k} + \|f(y)\|_{\mathbb{R}^k} \\
&\le c_0(1 + \|x\|_{\mathbb{R}^d})(1 + \|x\|_{\mathbb{R}^d})^q + c_0(1 + \|y\|_{\mathbb{R}^d})(1 + \|y\|_{\mathbb{R}^d})^q.
\end{aligned} \tag{3.30}$$

Inequality (3.30), Hölder's inequality, the fact that $2q + 2 < p$, the fact that

$$\mathbb{P}\Big(X^{Y_0}_{0,T} = X^{X^{Y_0}_{0,0}}_{0,T}\Big) = 1 = \mathbb{P}\Big(Y_T = X^{X^{Y_T}_{T,T}}_{T,T}\Big), \tag{3.31}$$

and inequality (3.7) show that

$$\begin{aligned}
\big\|f\big(X^{Y_0}_{0,T}\big) - f\big(Y_T\big)\big\|_{L^2(\mathbb{P};\mathbb{R}^k)} &\le c_0\big\|\big(1 + \|X^{Y_0}_{0,T}\|_{\mathbb{R}^d}\big)^{1+q}\big\|_{L^2(\mathbb{P};\mathbb{R})} + c_0\big\|\big(1 + \|Y_T\|_{\mathbb{R}^d}\big)^{1+q}\big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\le c_0\big(1 + \|X^{Y_0}_{0,T}\|_{L^{2q+2}(\mathbb{P};\mathbb{R}^d)}\big)^{q+1} + c_0\big(1 + \|Y_T\|_{L^{2q+2}(\mathbb{P};\mathbb{R}^d)}\big)^{q+1} \\
&\le \sup_{\substack{r,s,t\in[0,T]\\ r\le s\le t}} 2c_0\Big(1 + \Big\|X^{X^{Y_r}_{r,s}}_{t,T}\Big\|_{L^p(\mathbb{P};\mathbb{R}^d)}\Big)^{q+1} < \infty.
\end{aligned} \tag{3.32}$$

Equation (3.23) and inequalities (3.32), (3.24), (3.27), and (3.28) imply that there exists a constant $K \in [0, \infty)$ such that for all $h \in {}^{T}\!/\mathbb{N}$ it holds that

$$\begin{aligned}
&\left\| \int_0^T f'\!\left(X_{\lceil r \rceil_h, T}^{X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) X_{\lceil r \rceil_h, T}^{1, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \sigma\Big(r, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}\Big) - f'\Big(X_{\lceil r \rceil_h, T}^{Y_r}\Big) X_{\lceil r \rceil_h, T}^{1, Y_r} B_r \, \delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k) W_r \right\|_{L^2(\mathbb{P}; \mathbb{R}^k)} \\
&\leq \left\| f\big(X_{0,T}^{Y_0}\big) - f\,(Y_T) \right\|_{L^2(\mathbb{P}; \mathbb{R}^k)} \\
&\quad + \left\| \int_0^T f'\!\left(X_{\lceil r \rceil_h, T}^{X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) X_{\lceil r \rceil_h, T}^{1, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \mu(r, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}) - f'\big(X_{\lceil r \rceil_h, T}^{Y_r}\big) X_{\lceil r \rceil_h, T}^{1, Y_r} A_r \, dr \right\|_{L^2(\mathbb{P}; \mathbb{R}^k)} \qquad (3.33) \\
&+ \Bigg\| \tfrac{1}{2} \sum_{l,j=1}^{d} \int_0^T \Big(\sigma\big(r, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}\big) \sigma\big(r, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}\big)^{\top} \Big)_{l,j} \\
&\quad \cdot \left(f''\!\left(X_{\lceil r \rceil_h, T}^{X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) \left(X_{\lceil r \rceil_h, T}^{1, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}}, X_{\lceil r \rceil_h, T}^{1, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) + f'\!\left(X_{\lceil r \rceil_h, T}^{X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) X_{\lceil r \rceil_h, T}^{2, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) (e_l^{(d)}, e_j^{(d)}) \\
&- \big(B_r B_r^{\top}\big)_{l,j} \left(f''\big(X_{\lceil r \rceil_h, T}^{Y_r}\big) \big(X_{\lceil r \rceil_h, T}^{1, Y_r}, X_{\lceil r \rceil_h, T}^{1, Y_r}\big) + f'\big(X_{\lceil r \rceil_h, T}^{Y_r}\big) X_{\lceil r \rceil_h, T}^{2, Y_r} \right) (e_l^{(d)}, e_j^{(d)}) \, dr \Bigg\|_{L^2(\mathbb{P}; \mathbb{R}^k)} < K.
\end{aligned}$$

The fact that Y, X, X^1 are continuous random fields, continuity of f', and the fact that

$$\inf_{r \in [0,T]} \mathbb{P}(X_{r,r}^{Y_r} = Y_r) = 1 \tag{3.34}$$

yield that for all $r \in [0, T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\lim_{{}^{T}\!/\mathbb{N} \ni h \searrow 0} \left(f'\!\left(X_{\lceil r \rceil_h, T}^{X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \right) X_{\lceil r \rceil_h, T}^{1, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \sigma\Big(r, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}\Big) - f'\Big(X_{\lceil r \rceil_h, T}^{Y_r}\Big) X_{\lceil r \rceil_h, T}^{1, Y_r} B_r \right) \\
&= f'\big(X_{r,T}^{Y_r}\big) X_{r,T}^{1, Y r} (\sigma(r, Y_r) - B_r).
\end{aligned} \tag{3.35}$$

This, Fatou's lemma, and the inequalities (3.19) and (3.13) yield that the sequence

$$\left(f'\!\left(X_{\lceil \cdot \rceil_h, T}^{X_{\lfloor \cdot \rfloor_h, \cdot}^{Y_{\lfloor \cdot \rfloor_h}}} \right) X_{\lceil \cdot \rceil_h, T}^{1, X_{\lfloor \cdot \rfloor_h, \cdot}^{Y_{\lfloor \cdot \rfloor_h}}} \sigma\Big(\cdot, X_{\lfloor \cdot \rfloor_h, \cdot}^{Y_{\lfloor \cdot \rfloor_h}}\Big) - f'\Big(X_{\lceil \cdot \rceil_h, T}^{Y_\cdot}\Big) X_{\lceil \cdot \rceil_h, T}^{1, Y_\cdot} B_\cdot - f'\big(X_{\cdot, T}^{Y_\cdot}\big) X_{\cdot, T}^{1, Y_\cdot} (\sigma(\cdot, Y_\cdot) - B_\cdot) \right)_{h \in {}^{T}\!/\mathbb{N}} \tag{3.36}$$

is bounded in $L^2(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k))$. This, the fact that every bounded sequence in the separable Hilbert space $L^2(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k))$ has a weakly converging subsequence (e.g., Lemma 5.1.4 in Kato 1980), and the convergence (3.35) ensure that the sequence (3.36) converges to 0 in the weak topology of $L^2(\lambda_{[0,T]} \otimes \mathbb{P}; \mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k))$ as ${}^{T}\!/\mathbb{N} \ni h \searrow 0$. This, the fact that for all $h \in T/\mathbb{N}$ the process

$$\left(f'\big(X_{\lceil r \rceil_h, T}^{X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \big) X_{\lceil r \rceil_h, T}^{1, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}} \sigma\Big(r, X_{\lfloor r \rfloor_h, r}^{Y_{\lfloor r \rfloor_h}}\Big) - f'\big(X_{\lceil r \rceil_h, T}^{Y_r}\big) X_{\lceil r \rceil_h, T}^{1, Y_r} B_r \right)_{r \in [0,T]} \tag{3.37}$$

is $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)$-Skorohod-integrable, (3.33), and Lemma 2.35 imply that the stochastic process

$$\big(f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}(\sigma(r,Y_r) - B_r)\big)_{r\in[0,T]} \tag{3.38}$$

is $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)$-Skorohod-integrable and that for every $\mathbb{F}_T/\mathcal{B}([-1,1]^k)$-measurable random variable $Z\colon \Omega \to [-1,1]^k$ it holds that

$$\begin{aligned}\lim_{T/\mathbb{N}\ni h\searrow 0} \mathbb{E}\Bigg[\Bigg\langle Z, \int_0^T f'\Bigg(X^{X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h, r}}_{\lceil r\rceil_h, T}\Bigg) X^{1,X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h, r}}_{\lceil r\rceil_h, T}\sigma\Big(r, X^{Y_{\lfloor r\rfloor_h}}_{\lfloor r\rfloor_h, r}\Big)& \\ - f'\Big(X^{Y_r}_{\lceil r\rceil_h, T}\Big)X^{1,Y_r}_{\lceil r\rceil_h, T}B_r\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)W_r& \\ - \int_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\sigma(r,Y_r) - f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}B_r\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)W_r \Bigg\rangle_{\mathbb{R}^k}\Bigg] &= 0.\end{aligned} \tag{3.39}$$

Equation (3.23) and the convergences (3.29) and (3.39) imply that for every $\mathbb{F}_T/\mathcal{B}([-1,1]^k)$-measurable random variable $Z\colon \Omega \to [-1,1]^k$ it holds that

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg\langle Z, &\int_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\Big(\mu(r,Y_r) - A_r\Big)dr + \int_0^T f'\big(X^{Y_r}_{r,T}\big)X^{1,Y_r}_{r,T}\Big(\sigma(r,Y_r) - B_r\Big)\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^k)W_r \\ &+ \tfrac{1}{2}\sum_{l,j=1}^{d}\int_0^T \Big(\sigma(r,Y_r)\sigma(r,Y_r)^\top - B_rB_r^\top\Big)_{l,j}\Big(f''\big(X^{Y_r}_{r,T}\big)\big(X^{1,Y_r}_{r,T}, X^{1,Y_r}_{r,T}\big) + f'\big(X^{Y_r}_{r,T}\big)X^{2,Y_r}_{r,T}\Big)\big(e^{(d)}_l, e^{(d)}_j\big)dr \\ &- f\big(X^{Y_0}_{0,T}\big) + f(Y_T)\Bigg\rangle_{\mathbb{R}^k}\Bigg] = 0.\end{aligned} \tag{3.40}$$

This implies equation (3.9). The proof of Theorem 3.3 is thus completed. □

A direct corollary of Theorem 3.3 is the following:

Corollary 3.4. Assume Setting 3.2 and assume that $\mathbb{F}_0 = \sigma(\mathcal{N})$. Then the stochastic process $\big(X^{1,Y_r}_{r,T}(\sigma(r,Y_r) - B_r)\big)_{r\in[0,T]}$ is $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d)$-Skorohod-integrable and it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}X^{Y_0}_{0,T} - Y_T = &\int_0^T X^{1,Y_r}_{r,T}(\mu(r,Y_r) - A_r)dr + \int_0^T X^{1,Y_r}_{r,T}(\sigma(r,Y_r) - B_r)\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d)W_r \\ &+ \tfrac{1}{2}\sum_{l,j=1}^{d}\int_0^T \big(\sigma(r,Y_r)\sigma(r,Y_r)^\top - B_rB_r^\top\big)_{l,j}X^{2,Y_r}_{r,T}(e^{(d)}_l, e^{(d)}_j)dr.\end{aligned} \tag{3.41}$$

Corollary 3.5. Assume Setting 3.2. Then the stochastic process $\big(X^{1,Y_r}_{r,T}(\sigma(r,Y_r) - B_r)\big)_{r\in[0,T]}$

is $(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d)$-Skorohod-integrable and it holds that

$$\begin{aligned}
&\tfrac{1}{2}\mathbb{E}\Big[\big\|X_{0,T}^{Y_0} - Y_T\big\|_{\mathbb{R}^d}^2\Big] \\
&\leq \mathbb{E}\Big[\Big\| \int_0^T X_{r,T}^{1,Y_r}(\mu(r,Y_r) - A_r)dr \\
&\qquad + \tfrac{1}{2}\sum_{l,j=1}^{d} \int_0^T \big(\sigma(r,Y_r)\sigma(r,Y_r)^\top - B_r B_r^\top\big)_{l,j} X_{r,T}^{2,Y_r}(e_l^{(d)}, e_j^{(d)})dr\Big\|_{\mathbb{R}^d}^2\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \|X_{r,T}^{1,Y_r}(\sigma(r,Y_r) - B_r)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}^2 dr \\
&\qquad + \int_0^T\int_0^T \Big\langle \mathcal{D}_s(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d) X_{r,T}^{1,Y_r}(\sigma(r,Y_r) - B_r), \\
&\qquad\qquad \mathcal{D}_r(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d) X_{s,T}^{1,Y_s}(\sigma(s,Y_s) - B_s)\Big\rangle_{\mathrm{HS}(\mathbb{R}^m,\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} drds\Big].
\end{aligned} \tag{3.42}$$

Proof of Corollary 3.5. Corollary 3.4 and the fact that for all $a, b \in \mathbb{R}$ it holds that $\frac{1}{2}(a+b)^2 \leq a^2 + b^2$ imply that

$$\begin{aligned}
\tfrac{1}{2}\mathbb{E}\Big[\big\|X_{0,T}^{Y_0} - Y_T\big\|_{\mathbb{R}^d}^2\Big] &\leq \mathbb{E}\Big[\Big\| \int_0^T X_{r,T}^{1,Y_r}(\mu(r,Y_r) - A_r)dr \\
&\qquad + \tfrac{1}{2}\sum_{l,j=1}^{d} \int_0^T \big(\sigma(r,Y_r)\sigma(r,Y_r)^\top - B_r B_r^\top\big)_{l,j} X_{r,T}^{2,Y_r}(e_l^{(d)}, e_j^{(d)})dr\Big\|_{\mathbb{R}^d}^2\Big] \\
&\quad + \mathbb{E}\Big[\Big\| \int_0^T X_{r,T}^{1,Y_r}(\sigma(r,Y_r) - B_r)\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d)W_r\Big\|_{\mathbb{R}^d}^2\Big].
\end{aligned} \tag{3.43}$$

By the analogon of the Itô isometry, Lemma 2.43 it follows that

$$\begin{aligned}
&\mathbb{E}\Big[\Big\| \int_0^T X_{r,T}^{1,Y_r}(\sigma(r,Y_r) - B_r)\delta(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d)W_r\Big\|_{\mathbb{R}^d}^2\Big] \\
&= \mathbb{E}\Big[\int_0^T \|X_{r,T}^{1,Y_r}(\sigma(r,Y_r) - B_r)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}^2 dr \\
&\qquad + \int_0^T\int_0^T \Big\langle \mathcal{D}_s(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d) X_{r,T}^{1,Y_r}(\sigma(r,Y_r) - B_r), \\
&\qquad\qquad \mathcal{D}_r(\mathbb{P}, \mathbb{F}_0, W; \mathbb{R}^d) X_{s,T}^{1,Y_s}(\sigma(s,Y_s) - B_s)\Big\rangle_{\mathrm{HS}(\mathbb{R}^m,\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} drds\Big].
\end{aligned} \tag{3.44}$$

The proof of Corollary 3.5 is thus completed. □

3.3 An Application to the stochastic van-der-Pol oscillator with additive noise

In this chapter we give an example application of our perturbation theory and prove that the approximation processes (3.45) converge to the exact solution of the stochastic van-der-Pol oscillator with L^2-rate $1/2$ which is given by the SDE (3.46). Our proof is considerably shorter than the analysis in Hutzenthaler and Jentzen 2014; Hutzenthaler, Jentzen, and Wang 2018 which is needed to prove the analogous statement with the approach of Hutzenthaler and Jentzen 2014. Here, we assume for simplicity of exposition that the diffusion coefficient is constant (but this is not the reason for the much simpler analysis compared to Hutzenthaler and Jentzen 2014). First, we introduce the setting for the stochastic van-der-Pol oscillator with additive noise, then we provide three auxiliary results (Lemma 3.7, Lemma 3.8, and Lemma 3.9), and finally we prove Lemma 3.10, the main result of this section, by an application of Theorem 3.3.

Setting 3.6. *Let $T \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard Brownian motion with continuous sample paths on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $\mathcal{N} = \{A \in \mathcal{F}\colon \mathbb{P}(A) = 0\}$ denote the $\mathbb{P}$-null sets, let $a,b,c \in \mathbb{R}$ with the property that $2b^2c \neq 1$, let $\alpha,\beta,\gamma,\delta \in [0,\infty)$, let $\mu\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping such that for all $(x_1,x_2) \in \mathbb{R}^2$ it holds that $\mu(x_1,x_2) = (x_2, (\gamma - \alpha x_1^2)x_2 - \delta x_1)^\top$, let $\xi = (\xi_1,\xi_2)^\top \in \mathbb{R}^2$, for all $N \in \mathbb{N}$ let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^2$, be a stochastic process such that for all $k \in \{0,\ldots,N-1\}$, $\varepsilon \in [0,\frac{T}{N}]$ it holds $\mathbb{P}$-almost surely that $Y_0^N = \xi$ and that*

$$Y^N_{\frac{kT}{N}+\varepsilon} = Y^N_{\frac{kT}{N}} + \varepsilon\mathbb{1}_{\{\|\mu(Y^N_{\frac{kT}{N}})\|^2_{\mathbb{R}^2} < \frac{N}{T}\}}\mu(Y^N_{\frac{kT}{N}}) + (W_{\frac{kT}{N}+\varepsilon} - W_{\frac{kT}{N}})\left(\begin{smallmatrix}0\\ \beta\end{smallmatrix}\right), \tag{3.45}$$

and let $X_{\cdot,\cdot}^{\cdot}(\cdot)\colon \Delta_T\times\mathbb{R}^2\times\Omega \to \mathbb{R}^2$, $X_{\cdot,\cdot}^{1,\cdot}(\cdot)\colon \Delta_T\times\mathbb{R}^2\times\Omega \to L(\mathbb{R}^2,\mathbb{R}^2)$, and $X_{\cdot,\cdot}^{2,\cdot}(\cdot)\colon \Delta_T\times\mathbb{R}^2\times\Omega \to L^{(2)}(\mathbb{R}^2,\mathbb{R}^2)$ be continuous random fields such that for all $s \in [0,T]$, $\omega \in \Omega$ the mapping $\mathbb{R}^2 \ni x \mapsto X^x_{s,T}(\omega) \in \mathbb{R}^2$ is in $C^2(\mathbb{R}^2,\mathbb{R}^2)$, such that for all $(s,x) \in [0,T]\times\mathbb{R}^2$, $i \in \{1,2\}$ the stochastic processes $[s,T]\times\Omega \ni (t,\omega) \mapsto X^x_{s,t} \in \mathbb{R}^2$, $[s,T]\times\Omega \ni (t,\omega) \mapsto X^{i,x}_{s,t} \in L^{(i)}(\mathbb{R}^2,\mathbb{R}^2)$ are $(\sigma(\mathcal{N}\cup\sigma(W_r - W_s\colon r \in [s,t])))_{t\in[s,T]}$-adapted and for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that

$$X^x_{s,t} = x + \int_s^t \mu(X^x_{s,r})dr + \int_s^t \left(\begin{smallmatrix}0\\ \beta\end{smallmatrix}\right)dW_r, \tag{3.46}$$

such that for all $r,s,t \in [0,T]$, $x \in \mathbb{R}^2$ with the property that $r \le s \le t$ it holds $\mathbb{P}$-almost surely that $X_{s,t}^{X^x_{r,s}} = X^x_{r,t}$ and such that for all $(s,t,x,\omega) \in \Delta_T\times\mathbb{R}^2\times\Omega$ it holds that $X^{1,x}_{s,t}(\omega) = \frac{\partial}{\partial x}X^x_{s,t}(\omega)$ and $X^{2,x}_{s,t}(\omega) = \frac{\partial^2}{\partial x^2}X^x_{s,t}(\omega)$.

The assumption in Setting 3.6 of existence of continuous random fields which solve equation (3.46) and which are twice continuously differentiable in the space variable can be deferred from the third chapter.

Lemma 3.7. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $X\colon\Omega\to\mathbb{R}$ be a standard normally distributed $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping, and let $a,b,c \in \mathbb{R}$ satisfy that $2b^2c < 1$. Then it holds that

$$\mathbb{E}\Big[\exp\Big(c(a+bX)^2\Big)\Big] = \frac{1}{\sqrt{1-2b^2c}}\exp\Big(a^2\Big(c + \frac{2(bc)^2}{1-2b^2c}\Big)\Big). \tag{3.47}$$

Proof of Lemma 3.7. For all $y \in \mathbb{R}$ it holds that

$$c(a+by)^2 - \tfrac{y^2}{2} = ca^2 + 2abcy - y^2\big(\tfrac{1-2b^2c}{2}\big) = a^2\big(c + \tfrac{2(bc)^2}{1-2b^2c}\big) - \big(y - \tfrac{2abc}{1-2b^2c}\big)^2\big(\tfrac{1-2b^2c}{2}\big). \tag{3.48}$$

This, the definition of the standard normal distribution, equation (3.48), and the substitution rule imply that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(c(a+bX)^2\Big)\Big] &= \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\Big(c(a+by)^2 - \tfrac{y^2}{2}\Big)\,dy \\
&= \exp\Big(a^2\Big(c + \tfrac{2(bc)^2}{1-2b^2c}\Big)\Big) \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{\big(y - \frac{2abc}{1-2b^2c}\big)^2}{\big(\frac{2}{1-2b^2c}\big)}\right) dy \\
&= \tfrac{1}{\sqrt{1-2b^2c}} \exp\Big(a^2\Big(c + \tfrac{2(bc)^2}{1-2b^2c}\Big)\Big) \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\Big(\tfrac{-y^2}{2}\Big)\,dy \\
&= \tfrac{1}{\sqrt{1-2b^2c}} \exp\Big(a^2\Big(c + \tfrac{2(bc)^2}{1-2b^2c}\Big)\Big).
\end{aligned} \tag{3.49}$$

The proof of Lemma 3.7 is thus completed. ☐

Lemma 3.8. Assume Setting 3.6, let $c \in (0, \exp(-T(1+3\beta^2+\delta+2\gamma))]$, and let $N \in \mathbb{N}$ satisfy that $N \geq \max\{6\beta^2 T, T\}$. Then it holds for all $r \in [0,T]$ that

$$\mathbb{E}\Big[\ \exp\Big(c\big\|Y_r^N\big\|_{\mathbb{R}^2}^2\Big)\Big] \leq \exp\Big((2\beta^2+1)T + \|\xi\|_{\mathbb{R}^2}^2\Big). \tag{3.50}$$

Proof of Lemma 3.8. Throughout the proof of Lemma 3.8 let $(\theta_k)_{k\in\{0,\dots,N\}} \subseteq (0,\infty)$ satisfy that for all $k \in \{0,\dots,N-1\}$ it holds that $\theta_N = c$ and $\theta_k = \theta_{k+1}(1 + 3\beta^2\frac{T}{N})\big(1 + \frac{T}{N}(1+\delta+2\gamma)\big)$. Equation (3.45) and the fact that for all $k \in \{0,\dots,N-1\}$, $\varepsilon \in [0,\frac{T}{N}]$ the σ-algebras $\sigma\big(Y_{\frac{kT}{N}}^N\big)$ and $\sigma\big(W_{\frac{kT}{N}+\varepsilon} - W_{\frac{kT}{N}}\big)$ are independent and the random variables $(W_{\frac{kT}{N}+\varepsilon} - W_{\frac{kT}{N}})$, $\sqrt{\varepsilon}\frac{W_T}{\sqrt{T}}$ are identically distributed yield that for all $k \in \{0,\dots,N-1\}$, $\varepsilon \in [0,\frac{T}{N}]$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\theta_{k+1}\big\|Y_{\frac{kT}{N}+\varepsilon}^N\big\|_{\mathbb{R}^2}^2\Big)\Big] \\
&= \mathbb{E}\left[\mathbb{E}\Big[\exp\Big(\theta_{k+1}\big\|Y_{\frac{kT}{N}}^N + \varepsilon\mathbb{1}_{\{\|\mu(Y_{kT/N}^N)\|_{\mathbb{R}^2}^2 < \frac{N}{T}\}}\mu(Y_{\frac{kT}{N}}^N) + (W_{\frac{kT}{N}+\varepsilon} - W_{\frac{kT}{N}})\big(\begin{smallmatrix}0\\ \beta\end{smallmatrix}\big)\big\|_{\mathbb{R}^2}^2\Big)\Big|\sigma\big(Y_{\frac{kT}{N}}^N\big)\Big]\right] \\
&= \mathbb{E}\left[\mathbb{E}\Big[\ \exp\big(\theta_{k+1}\big\|v + \big(W_{\frac{kT}{N}+\varepsilon} - W_{\frac{kT}{N}}\big)\big(\begin{smallmatrix}0\\ \beta\end{smallmatrix}\big)\big\|_{\mathbb{R}^2}^2\big)\Big]\Big|_{v = Y_{\frac{kT}{N}}^N + \varepsilon\mathbb{1}_{\{\|\mu(Y_{kT/N}^N)\|_{\mathbb{R}^2}^2 < \frac{N}{T}\}}\mu(Y_{\frac{kT}{N}}^N)}\right] \\
&= \mathbb{E}\left[\exp(\theta_{k+1}v_1^2)\mathbb{E}\Big[\ \exp\Big(\theta_{k+1}\big(v_2 + \sqrt{\varepsilon}\beta\tfrac{W_T}{\sqrt{T}}\big)^2\Big)\Big]\Big|_{(v_1,v_2)^\top = Y_{\frac{kT}{N}}^N + \varepsilon\mathbb{1}_{\{\|\mu(Y_{kT/N}^N)\|_{\mathbb{R}^2}^2 < \frac{N}{T}\}}\mu(Y_{\frac{kT}{N}}^N)}\right].
\end{aligned} \tag{3.51}$$

Induction, the fact that $\theta_N = c$, the fact that for all $x \in [0,\infty)$ it holds that $1 + x \leq \exp(x)$, and the assumption $c \leq \exp(-T(1+3\beta^2+\delta+2\gamma))$ yield that for all $k \in \{0,\ldots,N\}$ it holds that

$$\begin{aligned}\theta_k &= \theta_N \Big((1+3\beta^2 \tfrac{T}{N})\big(1+\tfrac{T}{N}(1+\delta+2\gamma)\big)\Big)^{N-k} \\ &\leq c\exp\big((N-k)\tfrac{T}{N}(1+3\beta^2+\delta+2\gamma)\big) \\ &\leq \exp\big(-T(1+3\beta^2+\delta+2\gamma)\big)\exp\big(T(1+3\beta^2+\delta+2\gamma)\big) = 1.\end{aligned} \tag{3.52}$$

Inequality (3.52), the fact that for all $\varepsilon \in \big[0, \frac{T}{N}\big]$, $k \in \{0,\ldots,N-1\}$ it holds that $2(\sqrt{\varepsilon}\beta)^2\theta_{k+1} \leq \frac{2T\beta^2}{N} \leq \frac{1}{3}$, Lemma 3.7, the fact that $\frac{W_T}{\sqrt{T}}$ is standard normally distributed, and the fact that for all $x \in \big[0, \frac{1}{2}\big]$ it holds that $\frac{1}{1-x} \leq \exp(2x)$ imply that for all $k \in \{0,\ldots,N-1\}$, $\varepsilon \in [0, \frac{T}{N}]$, $v = (v_1, v_2)^\top \in \mathbb{R}^2$ it holds that

$$\begin{aligned}&\exp(\theta_{k+1}v_1^2)\mathbb{E}\Big[\ \exp\Big(\theta_{k+1}\big(v_2+\sqrt{\varepsilon}\beta\tfrac{W_T}{\sqrt{T}}\big)^2\Big)\Big] \\ &= \tfrac{1}{\sqrt{1-2\varepsilon\beta^2\theta_{k+1}}}\exp\Big(\theta_{k+1}v_1^2 + v_2^2\Big(\theta_{k+1}+\tfrac{2\varepsilon\beta^2\theta_{k+1}^2}{1-2\varepsilon\beta^2\theta_{k+1}}\Big)\Big) \\ &\leq \sqrt{\exp(4\varepsilon\beta^2\theta_{k+1})}\exp\Big(\theta_{k+1}\big(1+\tfrac{2\varepsilon\beta^2}{1-2\varepsilon\beta^2\theta_{k+1}}\big)\|v\|_{\mathbb{R}^2}^2\Big) \\ &\leq \exp\Big(2\tfrac{T}{N}\beta^2\Big)\exp\Big(\theta_{k+1}(1+3\beta^2\tfrac{T}{N})\|v\|_{\mathbb{R}^2}^2\Big).\end{aligned} \tag{3.53}$$

Equation (3.51) and inequality (3.53) imply that for all $k \in \{0,\ldots,N-1\}$, $\varepsilon \in [0, \frac{T}{N}]$ it holds that

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(\theta_{k+1}\big\|Y^N_{\frac{kT}{N}+\varepsilon}\big\|_{\mathbb{R}^2}^2\Big)\Big] \\ &\leq \exp\big(2\tfrac{T}{N}\beta^2\big)\ \mathbb{E}\left[\exp\left(\theta_{k+1}(1+3\beta^2\tfrac{T}{N})\Big\|Y^N_{\frac{kT}{N}} + \varepsilon\mathbb{1}_{\big\{\|\mu(Y^N_{\frac{kT}{N}})\|_{\mathbb{R}^2}^2<\frac{N}{T}\big\}}\mu(Y^N_{\frac{kT}{N}})\Big\|_{\mathbb{R}^2}^2\right)\right].\end{aligned} \tag{3.54}$$

Young's inequality shows that for all $x = (x_1, x_2)^\top \in \mathbb{R}^2$ it holds that

$$\langle x, \mu(x)\rangle_{\mathbb{R}^2} = x_1x_2 + x_2\big((\gamma-\alpha x_1^2)x_2 - \delta x_1\big) \leq x_1x_2 + \gamma x_2^2 - \delta x_1x_2 \leq \tfrac{1}{2}\big(1+\delta+2\gamma\big)\|x\|_{\mathbb{R}^2}^2. \tag{3.55}$$

This implies that for all $k \in \{0,\ldots,N-1\}$, $\varepsilon \in (0, \frac{T}{N}]$ it holds that

$$\begin{aligned}&\Big\|Y^N_{\frac{kT}{N}} + \varepsilon\mathbb{1}_{\{\|\mu(Y^N_{\frac{kT}{N}})\|_{\mathbb{R}^2}^2<\frac{N}{T}\}}\mu(Y^N_{\frac{kT}{N}})\Big\|_{\mathbb{R}^2}^2 \\ &= \big\|Y^N_{\frac{kT}{N}}\big\|_{\mathbb{R}^2}^2 + 2\varepsilon\mathbb{1}_{\{\|\mu(Y^N_{\frac{kT}{N}})\|_{\mathbb{R}^2}^2<\frac{N}{T}\}}\big\langle Y^N_{\frac{kT}{N}}, \mu\big(Y^N_{\frac{kT}{N}}\big)\big\rangle_{\mathbb{R}^2} + \varepsilon^2\mathbb{1}_{\{\|\mu(Y^N_{\frac{kT}{N}})\|_{\mathbb{R}^2}^2<\frac{N}{T}\}}\big\|\mu(Y^N_{\frac{kT}{N}})\big\|_{\mathbb{R}^2}^2 \\ &\leq \big\|Y^N_{\frac{kT}{N}}\big\|_{\mathbb{R}^2}^2 + 2\varepsilon\mathbb{1}_{\{\|\mu(Y^N_{\frac{kT}{N}})\|_{\mathbb{R}^2}^2<\frac{N}{T}\}}\tfrac{(1+\delta+2\gamma)}{2}\big\|Y^N_{\frac{kT}{N}}\big\|_{\mathbb{R}^2}^2 + \varepsilon^2\mathbb{1}_{\{\|\mu(Y^N_{\frac{kT}{N}})\|_{\mathbb{R}^2}^2<\frac{N}{T}\}}\tfrac{N}{T} \\ &\leq \big(1+\tfrac{T}{N}\big(1+\delta+2\gamma\big)\big)\big\|Y^N_{\frac{kT}{N}}\big\|_{\mathbb{R}^2}^2 + \tfrac{T}{N}.\end{aligned} \tag{3.56}$$

Inequalities (3.54), (3.56), and (3.52) imply that for all $k \in \{0,\ldots,N-1\}$, $\varepsilon \in [0, \frac{T}{N}]$ it holds

that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\theta_{k+1}\|Y^N_{\frac{kT}{N}+\varepsilon}\|^2_{\mathbb{R}^2}\Big)\Big]\\
&\leq \exp(2\tfrac{T}{N}\beta^2)\mathbb{E}\Big[\exp\Big(\theta_{k+1}\Big(1+3\beta^2\tfrac{T}{N}\Big)\Big(\big(1+\tfrac{T}{N}(1+\delta+2\gamma)\big)\|Y^N_{\frac{kT}{N}}\|^2_{\mathbb{R}^2}+\tfrac{T}{N}\Big)\Big)\Big]\\
&= \exp\Big((2\beta^2+\theta_{k+1}(1+3\beta^2\tfrac{T}{N}))\tfrac{T}{N}\Big)\mathbb{E}\Big[\exp\Big(\theta_k\|Y^N_{\frac{kT}{N}}\|^2_{\mathbb{R}^2}\Big)\Big]\\
&\leq \exp\Big((2\beta^2+1)\tfrac{T}{N}\Big)\mathbb{E}\Big[\exp\Big(\theta_k\|Y^N_{\frac{kT}{N}}\|^2_{\mathbb{R}^2}\Big)\Big].
\end{aligned} \tag{3.57}$$

Next, we prove by induction on $k \in \{0,\dots,N-1\}$ that for all $k \in \{0,\dots,N-1\}$, $r \in [\frac{kT}{N}, \frac{(k+1)T}{N}]$ it holds that

$$\mathbb{E}\Big[\exp\Big(\theta_{k+1}\|Y^N_r\|^2_{\mathbb{R}^2}\Big)\Big] \leq \exp\Big((2\beta^2+1)\tfrac{(k+1)T}{N}\Big)\exp\big(\|\xi\|^2_{\mathbb{R}^2}\big). \tag{3.58}$$

Inequality (3.57) and $\theta_0 \leq 1$ imply the base case. For the induction step $\{0,\dots,N-2\} \ni k \mapsto k+1 \in \{0,\dots,N-1\}$ note that inequality (3.57) and the induction hypothesis imply that for all $r \in [\frac{(k+1)T}{N}, \frac{(k+2)T}{N}]$ it holds that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(\theta_{k+2}\|Y^N_r\|^2_{\mathbb{R}^2}\Big)\Big] &\leq \exp\big((2\beta^2+1)\tfrac{T}{N}\big)\mathbb{E}\Big[\exp\Big(\theta_{k+1}\Big\|Y^N_{\frac{(k+1)T}{N}}\Big\|^2_{\mathbb{R}^2}\Big)\Big]\\
&\leq \exp\Big((2\beta^2+1)\tfrac{T}{N}\Big)\exp\Big((2\beta^2+1)\tfrac{(k+1)T}{N}\Big)\exp\big(\|\xi\|^2_{\mathbb{R}^2}\big)\\
&= \exp\big(2(\beta^2+1)\tfrac{(k+2)T}{N}\big)\exp\big(\|\xi\|^2_{\mathbb{R}^2}\big).
\end{aligned} \tag{3.59}$$

This finishes the induction step. Induction thus establishes inequality (3.58). Finally, inequalities (3.52) and (3.58) yield that for all $k \in \{0,\dots,N-1\}$, $r \in [\frac{kT}{N}, \frac{(k+1)T}{N}]$ it holds that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(c\|Y^N_r\|^2_{\mathbb{R}^2}\Big)\Big] &\leq \mathbb{E}\Big[\exp\Big(\theta_{k+1}\|Y^N_r\|^2_{\mathbb{R}^2}\Big)\Big]\\
&\leq \exp\big((2\beta^2+1)\tfrac{(k+1)T}{N}\big)\exp\big(\|\xi\|^2_{\mathbb{R}^2}\big)\\
&\leq \exp\Big((2\beta^2+1)T+\|\xi\|^2_{\mathbb{R}^2}\Big).
\end{aligned} \tag{3.60}$$

The proof of Lemma 3.8 is thus completed. □

Lemma 3.9. Assume Setting 3.6, let $p \in [1,\infty)$, $r \in [0,T]$, $q \in (0,\infty)$, and let $Z\colon \Omega \to \mathbb{R}^2$ be a $\sigma(\mathcal{N} \cup \sigma(W_s\colon s \in [0,r]))/\mathcal{B}(\mathbb{R}^2)$-measurable mapping. Then it holds that

$$\begin{aligned}
&\sup_{i\in\{1,2\}}\sup_{t\in[r,T]}\mathbb{E}\Big[\|X^{i,Z}_{r,t}\|^p_{L^{(i)}(\mathbb{R}^2,\mathbb{R}^2)}\Big]\\
&\leq \Bigg(\sup_{y\in(0,\infty)}\tfrac{(16\alpha^2Ty)^p}{\exp(q\exp(-(|\delta-1|+2\gamma+4\beta^2q)T)y)}+\exp\Big(6pT\big(1+\tfrac{\alpha^2 6p\exp((|\delta-1|+2\gamma+4\beta^2q)T)}{8\alpha q}+\gamma+\tfrac{\delta}{2}\big)\Big)\Bigg)\\
&\quad\cdot\mathbb{E}\Big[\exp(\tfrac{1}{4}+q\|Z\|^2_{\mathbb{R}^2})\Big].
\end{aligned} \tag{3.61}$$

Proof of Lemma 3.9. First, equation (3.46), the dominated convergence theorem together with continuity of the functions $[r,T]\times\mathbb{R}^2 \ni (s,x) \mapsto \frac{\partial}{\partial x}\big(\mu\big(X^x_{r,s}(\omega)\big)\big) \in L(\mathbb{R}^2,\mathbb{R}^2)$, $\omega \in \Omega$, and the chain rule imply that for all $t \in [r,T]$, $x, v \in \mathbb{R}^2$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} X^{1,x}_{r,t} v &= \Big(\tfrac{\partial}{\partial x} X^x_{r,t}\Big) v \\ &= \tfrac{\partial}{\partial x}\Big(x + \int_r^t \mu(X^x_{r,s})\,ds + \int_r^t \big(\begin{smallmatrix}0\\ \beta\end{smallmatrix}\big)\,dW_s\Big)(v) \\ &= v + \int_r^t \tfrac{\partial}{\partial x}\Big(\mu(X^x_{r,s})\Big) v\, ds = v + \int_r^t \mu'\big(X^x_{r,s}\big) X^{1,x}_{r,s} v\, ds. \end{aligned} \tag{3.62}$$

This, the fundamental theorem of calculus together with path continuity and the chain rule imply that for all $t \in [r,T]$, $v \in \mathbb{R}^2$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} \big\|X^{1,Z}_{r,t} v\big\|^2_{\mathbb{R}^2} &= \|v\|^2_{\mathbb{R}^2} + \int_r^t 2\big\langle X^{1,Z}_{r,s} v, \mu'(X^Z_{r,s}) X^{1,Z}_{r,s} v\big\rangle_{\mathbb{R}^2} ds \\ &= \|v\|^2_{\mathbb{R}^2} + \int_r^t \tfrac{2\left\langle X^{1,Z}_{r,s} v, \mu'(X^Z_{r,s}) X^{1,Z}_{r,s} v\right\rangle_{\mathbb{R}^2}}{\|X^{1,Z}_{r,s} v\|^2_{\mathbb{R}^2}} \|X^{1,Z}_{r,s} v\|^2_{\mathbb{R}^2} ds. \end{aligned} \tag{3.63}$$

This and Gronwall's inequality together with path continuity imply that for all $t \in [r,T]$, $v \in \mathbb{R}^2$ it holds that

$$\big\|X^{1,Z}_{r,t} v\big\|^2_{\mathbb{R}^2} \leq \|v\|^2_{\mathbb{R}^2} \exp\Big(\int_r^t \tfrac{2\left\langle X^{1,Z}_{r,s} v, \mu'(X^Z_{r,s}) X^{1,Z}_{r,s} v\right\rangle_{\mathbb{R}^2}}{\|X^{1,Z}_{r,s} v\|^2_{\mathbb{R}^2}} ds\Big). \tag{3.64}$$

For all $(u,v)^\top, (x,y)^\top \in \mathbb{R}^2$, $\varepsilon \in (0,\infty)$ with the property that $x^2 + y^2 = 1$ it holds that

$$\begin{aligned} \big\langle \big(\begin{smallmatrix}x\\ y\end{smallmatrix}\big), \mu'(u,v)\big(\begin{smallmatrix}x\\ y\end{smallmatrix}\big)\big\rangle_{\mathbb{R}^2} &= \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -2\alpha u v - \delta & \gamma - \alpha u^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\ &= (1 - 2\alpha u v - \delta) x y + (\gamma - \alpha u^2) y^2 \\ &\leq \big(\tfrac{1}{2} + \alpha |uv| + \tfrac{\delta}{2}\big) + \gamma \leq \tfrac{1}{2} + \big(\tfrac{\alpha^2}{4\varepsilon} + \varepsilon (uv)^2\big) + \tfrac{\delta}{2} + \gamma \\ &= \big(\tfrac{1}{2} + \tfrac{\alpha^2}{4\varepsilon} + \gamma + \tfrac{\delta}{2}\big) + \varepsilon (uv)^2. \end{aligned} \tag{3.65}$$

Inequalities (3.64) and (3.65) imply that for all $t \in [r,T]$, $\varepsilon \in (0,\infty)$, $v \in \mathbb{R}^2$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} \big\|X^{1,Z}_{r,t} v\big\|^p_{\mathbb{R}^2} &\leq \|v\|^p_{\mathbb{R}^2} \exp\left(p \int_r^t \tfrac{\left\langle X^{1,Z}_{r,s} v, \mu'(X^Z_{r,s}) X^{1,Z}_{r,s} v\right\rangle_{\mathbb{R}^2}}{\|X^{1,Z}_{r,s} v\|^2_{\mathbb{R}^2}} ds\right) \\ &\leq \|v\|^p_{\mathbb{R}^2} \exp\Big(p \int_r^t \big(\tfrac{1}{2} + \tfrac{\alpha^2}{4\varepsilon} + \gamma + \tfrac{\delta}{2}\big) + \varepsilon \Big(\Big\langle X^Z_{r,s}, e^{(2)}_1 \Big\rangle_{\mathbb{R}^2} \Big\langle X^Z_{r,s}, e^{(2)}_2 \Big\rangle_{\mathbb{R}^2}\Big)^2 ds\Big) \\ &\leq \|v\|^p_{\mathbb{R}^2} \exp\Big(pt\big(\tfrac{1}{2} + \tfrac{\alpha^2}{4\varepsilon} + \gamma + \tfrac{\delta}{2}\big)\Big) \exp\Big(p\varepsilon \int_r^t \Big(\Big\langle X^Z_{r,s}, e^{(2)}_1 \Big\rangle_{\mathbb{R}^2} \Big\langle X^Z_{r,s}, e^{(2)}_2 \Big\rangle_{\mathbb{R}^2}\Big)^2 ds\Big). \end{aligned} \tag{3.66}$$

Observe that

$$\min_{r \in (0,\infty)} \max\Big\{ \Big[\tfrac{|\delta - 1|}{r}\Big], \big[r|\delta - 1| + 2\gamma + 4\beta^2 q\big] \Big\} \leq |\delta - 1| + 2\gamma + 4\beta^2 q. \tag{3.67}$$

Inequality (3.66) (applied with $\varepsilon = \frac{2\alpha q}{p\exp((|\delta-1|+2\gamma+4\beta^2 q)t)}$ in the notation of inequality (3.66)), inequality (3.67), and equation (4.4) in Cox, Hutzenthaler, and Jentzen 2013 imply that for all $t \in [r,T]$ it holds that

$$\mathbb{E}\Big[\big\|X^{1,Z}_{r,t}\big\|^p_{L(\mathbb{R}^2,\mathbb{R}^2)}\Big] \leq \exp\Big(pt\big(\tfrac{1}{2} + \tfrac{\alpha^2 p\exp((|\delta-1|+2\gamma+4\beta^2 q)t)}{8\alpha q} + \gamma + \tfrac{\delta}{2}\big)\Big)\mathbb{E}\Big[\exp\big(\tfrac{1}{4} + q\|Z\|^2_{\mathbb{R}^2}\big)\Big]. \tag{3.68}$$

Next, equation (3.62), the dominated convergence theorem together with continuity of the functions $[r,T] \times \mathbb{R}^2 \ni (s,x) \mapsto \frac{\partial^2}{\partial x^2}\big(\mu(X^x_{r,s}(\omega))\big) \in L^{(2)}(\mathbb{R}^2,\mathbb{R}^2)$, $\omega \in \Omega$, and the chain rule imply that for all $t \in [r,T]$, $x,v,w \in \mathbb{R}^2$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
X^{2,x}_{r,t}(v,w) &= \Big(\tfrac{\partial}{\partial x} X^{1,x}_{r,t} v\Big) w \\
&= \tfrac{\partial}{\partial x}\Big(v + \int_r^t \mu'(X^x_{r,s}) X^{1,x}_{r,s}\; v\, ds\Big)(w) \\
&= \int_r^t \tfrac{\partial}{\partial x}\Big(\mu'(X^x_{r,s}) X^{1,x}_{r,s} v\Big)(w)\, ds \\
&= \int_r^t \mu''\big(X^x_{r,s}\big)\big(X^{1,x}_{r,s} v, X^{1,x}_{r,s} w\big) + \mu'\big(X^x_{r,s}\big) X^{2,x}_{r,s}(v,w)\, ds.
\end{aligned} \tag{3.69}$$

Equation (3.69), the fundamental theorem of calculus together with path continuity, the chain rule, the Cauchy-Schwarz inequality, Young's inequality, and inequality (3.65) imply that for all $t \in [r,T]$, $v,w \in \mathbb{R}^2$, $\varepsilon \in (0,\infty)$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\big\|X^{2,Z}_{r,t}(v,w)\big\|^2_{\mathbb{R}^2} \\
&= 2\int_r^t \big\langle X^{2,Z}_{r,s}(v,w), \mu''\big(X^Z_{r,s}\big)\big(X^{1,Z}_{r,s} v, X^{1,Z}_{r,s} w\big) + \mu'\big(X^Z_{r,s}\big) X^{2,Z}_{r,s}(v,w)\big\rangle_{\mathbb{R}^2} ds \\
&\leq \int_r^t 2\big\|X^{2,Z}_{r,s}\; (v,w)\big\|_{\mathbb{R}^2}\big\|\mu''(X^Z_{r,s})(X^{1,Z}_{r,s}\; v, X^{1,Z}_{r,s} w)\;\big\|_{\mathbb{R}^2} ds \\
&\quad + 2\int_r^t \big\langle X^{2,Z}_{r,s}\; (v,w), \mu'\big(X^Z_{r,s}\big) X^{2,Z}_{r,s}(v,w)\;\big\rangle_{\mathbb{R}^2} ds \\
&\leq \int_r^t \big\|X^{2,Z}_{r,s}(v,w)\big\|^2_{\mathbb{R}^2} + \big\|\mu''\big(X^Z_{r,s}\big)\big(X^{1,Z}_{r,s} v, X^{1,Z}_{r,s} w\big)\big\|^2_{\mathbb{R}^2}\, ds \\
&\quad + 2\int_r^t \big\langle X^{2,Z}_{r,s}\; (v,w), \mu'\big(X^Z_{r,s}\big) X^{2,Z}_{r,s}(v,w)\;\big\rangle_{\mathbb{R}^2} ds \\
&\leq \int_r^t \big\|\mu''\big(X^Z_{r,s}\big)\big(X^{1,Z}_{r,s} v, X^{1,Z}_{r,s} w\big)\big\|^2_{\mathbb{R}^2}\, ds \\
&\quad + 2\int_r^t \Big(\big(\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{\alpha^2}{4\varepsilon} + \gamma + \tfrac{\delta}{2}\big) + \varepsilon\Big(\big\langle X^Z_{r,s}, e^{(2)}_1\big\rangle_{\mathbb{R}^2}\big\langle X^Z_{r,s}, e^{(2)}_2\big\rangle_{\mathbb{R}^2}\Big)^2\Big)\big\|X^{2,Z}_{r,s}(v,w)\big\|^2_{\mathbb{R}^2} ds.
\end{aligned} \tag{3.70}$$

This and Gronwall's inequality together with path continuity imply that for all $t \in [r,T]$, $v,w \in$

$\mathbb{R}^2$, $\varepsilon \in (0,\infty)$ it holds that

$$\begin{aligned}\big\|X^{2,Z}_{r,t}(v,w)\big\|^2_{\mathbb{R}^2} \leq \int_r^t \big\|\mu''\big(X^Z_{r,s}\big)\big(X^{1,Z}_{r,s}v, X^{1,Z}_{r,s}w\big)\big\|^2_{\mathbb{R}^2}\,ds \\ \cdot \exp\Big(2t\big(1+\tfrac{\alpha^2}{4\varepsilon}+\gamma+\tfrac{\delta}{2}\big)\Big)\exp\Big(2\varepsilon\int_r^t\Big(\big\langle X^Z_{r,s}, e^{(2)}_1\big\rangle_{\mathbb{R}^2}\big\langle X^Z_{r,s}, e^{(2)}_2\big\rangle_{\mathbb{R}^2}\Big)^2\,ds\Big).\end{aligned} \tag{3.71}$$

For all $(u,v)^\top, (x,y)^\top, (w,z)^\top \in \mathbb{R}^2$ with the property that $x^2+y^2 = 1 = w^2+z^2$ it holds that

$$\begin{aligned}\Big\|\mu''(u,v)\Big(\big(\begin{smallmatrix}x\\y\end{smallmatrix}\big),\big(\begin{smallmatrix}w\\z\end{smallmatrix}\big)\Big)\Big\|_{\mathbb{R}^2} &= \left\|\begin{pmatrix}0 & 0\\ -2\alpha vx-2\alpha uy & -2\alpha ux\end{pmatrix}\begin{pmatrix}w\\z\end{pmatrix}\right\|_{\mathbb{R}^2} \\ &= 2\alpha|(vx+uy)w+uxz| \leq 4\alpha\,\big\|\big(\begin{smallmatrix}u\\v\end{smallmatrix}\big)\big\|_{\mathbb{R}^2}.\end{aligned} \tag{3.72}$$

This, inequality (3.71), and inequality (3.66) yield that for all $t \in [r,T]$, $\varepsilon \in (0,\infty)$ it holds that

$$\begin{aligned}\big\|X^{2,Z}_{r,t}\big\|^p_{L^{(2)}(\mathbb{R}^2,\mathbb{R}^2)} &\leq \Big(\int_r^t 16\alpha^2\,\big\|X^Z_{r,s}\big\|^2_{\mathbb{R}^2}\,\big\|X^{1,Z}_{r,s}\big\|^4_{L(\mathbb{R}^2,\mathbb{R}^2)}\,ds\Big)^{\frac{p}{2}}\exp\Big(pt\big(1+\tfrac{\alpha^2}{4\varepsilon}+\gamma+\tfrac{\delta}{2}\big)\Big) \\ &\quad\cdot\exp\Big(p\varepsilon\int_r^t\Big(\Big\langle X^Z_{r,s}, e^{(2)}_1\Big\rangle_{\mathbb{R}^2}\Big\langle X^Z_{r,s}, e^{(2)}_2\Big\rangle_{\mathbb{R}^2}\Big)^2\,ds\Big) \\ &\leq \Big(\int_r^T 16\alpha^2\,\big\|X^Z_{r,s}\big\|^2_{\mathbb{R}^2}\,ds\Big)^{\frac{p}{2}}\exp\Big(3pt\big(1+\tfrac{\alpha^2}{4\varepsilon}+\gamma+\tfrac{\delta}{2}\big)\Big) \\ &\quad\cdot\exp\Big(3p\varepsilon\int_r^t\Big(\Big\langle X^Z_{r,s}, e^{(2)}_1\Big\rangle_{\mathbb{R}^2}\Big\langle X^Z_{r,s}, e^{(2)}_2\Big\rangle_{\mathbb{R}^2}\Big)^2\,ds\Big).\end{aligned} \tag{3.73}$$

The triangle inequality yields that

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_r^T 16\alpha^2\,\big\|X^Z_{r,s}\big\|^2_{\mathbb{R}^2}\,ds\Big)^p\Big] &\leq \left(\int_r^T 16\alpha^2\Big(\mathbb{E}\Big[\,\big\|X^Z_{r,s}\big\|^{2p}_{\mathbb{R}^2}\Big]\Big)^{\frac{1}{p}}ds\right)^p \\ &\leq (16\alpha^2T)^p\sup_{u\in[r,T]}\mathbb{E}\Big[\,\big\|X^Z_{r,u}\big\|^{2p}_{\mathbb{R}^2}\Big] \\ &\leq (16\alpha^2T)^p\sup_{y\in(0,\infty)}\tfrac{y^p}{\exp(q\exp(-(|\delta-1|+2\gamma+4\beta^2q)T)y)} \\ &\quad\cdot\sup_{u\in[r,T]}\mathbb{E}\Big[\exp\Big(\tfrac{q}{\exp((|\delta-1|+2\gamma+4\beta^2q)u)}\,\big\|X^Z_{r,u}\big\|^2_{\mathbb{R}^2}\Big)\Big].\end{aligned} \tag{3.74}$$

Inequality (3.73) (applied with $\varepsilon = \frac{2\alpha q}{6p\exp((|\delta-1|+2\gamma+4\beta^2q)t)}$ for $t \in [r,T]$ in the notation of inequality (3.73)), the fact that for all $a,b \in \mathbb{R}$ it holds that $ab \leq a^2+b^2$, inequality (3.74), and equation (4.4) in Cox, Hutzenthaler, and Jentzen 2013 imply that for all $t \in [r,T]$ it holds that

$$
\begin{aligned}
&\mathbb{E}\Big[\|X^{2,Z}_{r,t}\|^p_{L^{(2)}(\mathbb{R}^2,\mathbb{R}^2)}\Big]\\
&\leq \mathbb{E}\Big[\Big(\int_r^T 16\alpha^2 \left\|X^Z_{r,s}\right\|^2_{\mathbb{R}^2} ds\Big)^p\Big] + \exp\Big(2\cdot 3pt\big(1 + \tfrac{\alpha^2 6p\exp((|\delta-1|+2\gamma+4\beta^2 q)t)}{8\alpha q} + \gamma + \tfrac{\delta}{2}\big)\Big)\\
&\quad \cdot \mathbb{E}\Big[\exp\Big(2\cdot 3p\tfrac{2\alpha q}{6p\exp((|\delta-1|+2\gamma+4\beta^2 q)t)}\int_r^t \Big(\Big\langle X^Z_{r,s}, e^{(2)}_1\Big\rangle_{\mathbb{R}^2}\Big\langle X^Z_{r,s}, e^{(2)}_2\Big\rangle_{\mathbb{R}^2}\Big)^2 ds\Big)\Big)\Big] \qquad (3.75)\\
&\leq \Big(\sup_{y\in(0,\infty)} \tfrac{(16\alpha^2 Ty)^p}{\exp(q\exp(-(|\delta-1|+2\gamma+4\beta^2 q)T)y)} + \exp\Big(6pt\big(1+\tfrac{\alpha^2 6p\exp((|\delta-1|+2\gamma+4\beta^2 q)t)}{8\alpha q} + \gamma + \tfrac{\delta}{2}\big)\Big)\Big)\\
&\quad \cdot \sup_{u\in[r,T]} \mathbb{E}\Big[\exp\Big(\tfrac{q}{\exp((|\delta-1|+2\gamma+4\beta^2 q)u)}\left\|X^Z_{r,u}\right\|^2_{\mathbb{R}^2}\\
&\qquad\qquad + \int_r^u \tfrac{2\alpha q}{\exp((|\delta-1|+2\gamma+4\beta^2 q)s)}\Big(\Big\langle X^Z_{r,s}, e^{(2)}_1\Big\rangle_{\mathbb{R}^2}\Big\langle X^Z_{r,s}, e^{(2)}_2\Big\rangle_{\mathbb{R}^2}\Big)^2 ds\Big)\Big]\\
&\leq \Big(\sup_{y\in(0,\infty)} \tfrac{(16\alpha^2 Ty)^p}{\exp(q\exp(-(|\delta-1|+2\gamma+4\beta^2 q)T)y)} + \exp\Big(6pT\big(1+\tfrac{\alpha^2 6p\exp((|\delta-1|+2\gamma+4\beta^2 q)T)}{8\alpha q} + \gamma + \tfrac{\delta}{2}\big)\Big)\Big)\\
&\quad \cdot \mathbb{E}\Big[\exp(\tfrac{1}{4} + q\|Z\|^2_{\mathbb{R}^2})\Big].
\end{aligned}
$$

Combining inequalities (3.68) and (3.75) proves inequality (3.61). The proof of Lemma 3.9 is thus completed. □

Lemma 3.10. Assume Setting 3.6. Then there exists a constant $\kappa \in (0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$
\Big(\mathbb{E}\Big[\Big\|X^{\xi}_{0,T} - Y^N_T\Big\|^2_{\mathbb{R}^2}\Big]\Big)^{1/2} \leq \tfrac{\kappa}{\sqrt{N}}. \qquad (3.76)
$$

Proof of Lemma 3.10. For the rest of the proof let $p \in [5 + \exp(T(4+\delta+2\gamma)), \infty)$ be a real number with the property that for all $x, y \in \mathbb{R}^2$ it holds that

$$
\|\mu(x) - \mu(y)\|_{\mathbb{R}^2} \leq p\|x-y\|_{\mathbb{R}^2}(1 + \|x\|^p_{\mathbb{R}^2} + \|y\|^p_{\mathbb{R}^2}) \qquad (3.77)
$$

and

$$
\|\mu(x)\|_{\mathbb{R}^2} \leq p(1 + \|x\|^p_{\mathbb{R}^2}). \qquad (3.78)
$$

Such a real number exists because μ is a polynomial. Lemma 3.9 (applied with $p = 5$, $r = s$, $q = q\exp(-(|\delta-1| + 2\gamma + 4\beta^2 q)s)$, $Z = X^{Y^N_r}_{r,s}$ for $r, s \in [0,T]$, $N \in \mathbb{N}$ with $r \leq s$ in the notation of Lemma 3.9), equation (4.4) in Cox, Hutzenthaler, and Jentzen 2013, and Lemma 3.8 (applied with $c = \exp(-T(1 + 3\beta^2 + \delta + 2\gamma))$ in the notation of Lemma 3.8) imply that there exists a constant $C \in [1,\infty)$ such that for all $N \in \mathbb{N}$, $q \in \{\exp(-T(1 + 3\beta^2 + \delta + 2\gamma))\}$, $r, s, t \in [0,T]$

with the property that $r \leq s \leq t$ and that $N \geq \max\{6\beta^2 T, T\}$ it holds that

$$\begin{aligned}
&\max\left\{\mathbb{E}\left[\left\|X_{s,t}^{X_{r,s}^{Y_r^N}}\right\|_{\mathbb{R}^2}^{20p}\right], \sup_{i\in\{1,2\}} \mathbb{E}\left[\left\|X_{s,t}^{i,X_{r,s}^{Y_r^N}}\right\|_{L^{(i)}(\mathbb{R}^2,\mathbb{R}^2)}^{\frac{4\cdot 20}{20-4}}\right]\right\}\\
&\leq C\mathbb{E}\left[\exp\left(\tfrac{q}{\exp((|\delta-1|+2\gamma+4\beta^2 q)s)}\left\|X_{r,s}^{Y_r^N}\right\|_{\mathbb{R}^2}^2\right)\right]\\
&\leq C\exp\left(\tfrac{1}{4}\right)\mathbb{E}\left[\exp\left(q\left\|Y_r^N\right\|_{\mathbb{R}^2}^2\right)\right]\\
&\leq C\exp\left(\tfrac{1}{4}\right)\exp\left((2\beta^2+1)T + \|\xi\|_{\mathbb{R}^2}^2\right) < \infty.
\end{aligned} \tag{3.79}$$

This together with inequality (3.78) implies that the assumptions of Theorem 3.3 are satisfied. Then, the perturbation formula in Theorem 3.3 (applied with $d = 2$, $m = 1$, $p = 20$, $\mu(r,x) = \mu(x)$, $\sigma(r,x) = \binom{0}{\beta}$, $A_r = \mu\big(Y_{\lfloor r\rfloor_{T/N}}^N\big)\mathbb{1}_{\{\|\mu(Y_{\lfloor r\rfloor_{T/N}}^N)\|_{\mathbb{R}^2}^2 < \frac{N}{T}\}}$, $B_r = \binom{0}{\beta}$ for all $r \in [0,T]$, $x \in \mathbb{R}^2$ in the notation of Theorem 3.3), Jensen's inequality, and Hölder's inequality imply that for all $N \in \mathbb{N}$ with the property $N \geq \max\{6\beta^2 T, T\}$ it holds that

$$\begin{aligned}
&\mathbb{E}\left[\left\|X_{0,T}^{\xi} - Y_T^N\right\|_{\mathbb{R}^2}^2\right]\\
&= \mathbb{E}\left[\left\|\int_0^T X_{r,T}^{1,Y_r^N}\left(\mu(Y_r^N) - \mu\big(Y_{\lfloor r\rfloor_{T/N}}^N\big)\mathbb{1}_{\{\|\mu(Y_{\lfloor r\rfloor_{T/N}}^N)\|_{\mathbb{R}^2}^2 < \frac{N}{T}\}}\right)dr\right\|_{\mathbb{R}^2}^2\right]\\
&\leq T\left(\int_0^T \mathbb{E}\left[\left\|X_{r,T}^{1,Y_r^N}\right\|_{L(\mathbb{R}^2,\mathbb{R}^2)}^4\right]dr\right)^{\frac{1}{2}}\left(\int_0^T \mathbb{E}\left[\left\|\mu(Y_r^N) - \mu\big(Y_{\lfloor r\rfloor_{T/N}}^N\big)\mathbb{1}_{\{\|\mu(Y_{\lfloor r\rfloor_{T/N}}^N)\|_{\mathbb{R}^2}^2 < \frac{N}{T}\}}\right\|_{\mathbb{R}^2}^4\right]dr\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.80}$$

Moreover, inequality (3.77), Hölder's inequality, equation (3.45), the scaling property of the Brownian motion, and the fact that $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}x^8\exp(-\frac{x^2}{2})dx = 105$ yield that for all $N \in \mathbb{N}$, $k \in \{0,\dots,N-1\}$, $r \in [\frac{kT}{N}, \frac{(k+1)T}{N}]$ it holds that

$$\begin{aligned}
&\left\|\big(\mu(Y_r^N) - \mu(Y_{\frac{kT}{N}}^N)\big)\mathbb{1}_{\left\{\left\|\mu\left(Y_{\frac{kT}{N}}^N\right)\right\|_{\mathbb{R}^2}^2 < \frac{N}{T}\right\}}\right\|_{L^4(\mathbb{P};\mathbb{R}^2)}\\
&\leq p\left\|\left\|Y_r^N - Y_{\frac{kT}{N}}^N\right\|_{\mathbb{R}^2}\big(1 + \|Y_r^N\|_{\mathbb{R}^2}^p + \|Y_{\frac{kT}{N}}^N\|_{\mathbb{R}^2}^p\big)\mathbb{1}_{\left\{\left\|\mu\left(Y_{\frac{kT}{N}}^N\right)\right\|_{\mathbb{R}^2}^2 < \frac{N}{T}\right\}}\right\|_{L^4(\mathbb{P};\mathbb{R})}\\
&\leq p\left\|\big(Y_r^N - Y_{\frac{kT}{N}}^N\big)\mathbb{1}_{\left\{\left\|\mu\left(Y_{\frac{kT}{N}}^N\right)\right\|_{\mathbb{R}^2}^2 < \frac{N}{T}\right\}}\right\|_{L^8(\mathbb{P};\mathbb{R}^2)}\left\|(1 + \|Y_r^N\|_{\mathbb{R}^2}^p + \|Y_{\frac{kT}{N}}^N\|_{\mathbb{R}^2}^p)\right\|_{L^8(\mathbb{P};\mathbb{R})}\\
&\leq p\left(\tfrac{T}{N}\left\|\mu\big(Y_{\frac{kT}{N}}^N\big)\mathbb{1}_{\left\{\left\|\mu\left(Y_{\frac{kT}{N}}^N\right)\right\|_{\mathbb{R}^2}^2 < \frac{N}{T}\right\}}\right\|_{L^8(\mathbb{P};\mathbb{R}^2)} + \left\|\big(W_r - W_{\frac{kT}{N}}\big)\tbinom{0}{\beta}\right\|_{L^8(\mathbb{P};\mathbb{R}^2)}\right)\\
&\quad\cdot\left(1 + 2\sup_{s\in[0,T]}\left\|Y_s^N\right\|_{L^{8p}(\mathbb{P};\mathbb{R}^2)}^p\right)\\
&\leq p\left(\tfrac{T}{N}\sqrt{\tfrac{N}{T}} + (105)^{\frac{1}{8}}\beta\sqrt{\tfrac{T}{N}}\right)\left(1 + 2\sup_{s\in[0,T]}\left\|Y_s^N\right\|_{L^{20p}(\mathbb{P};\mathbb{R}^2)}^p\right) < \infty.
\end{aligned} \tag{3.81}$$

This, inequality (3.80), Hölder's inequality, and inequality (3.79) yield that for all $N \in \mathbb{N}$ with

$N \geq \max\{6\beta^2 T, T\}$ it holds that

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\big\|X^{\xi}_{0,T} - Y^N_T\big\|^2_{\mathbb{R}^2}\Big]\Big)^{\frac{1}{2}} \\
&\leq \sqrt{T}\Big(\int_0^T \mathbb{E}\Big[\Big\|X^{1,Y^N_r}_{r,T}\Big\|^4_{L(\mathbb{R}^2,\mathbb{R}^2)}\Big]dr\Big)^{\frac{1}{4}} \cdot T^{\frac{1}{4}}\sqrt{\tfrac{T}{N}}p(1+2\beta)\Big(1 + 2\sup_{s\in[0,T]}\Big(\mathbb{E}\Big[\big\|Y^N_s\big\|^{20p}_{\mathbb{R}^2}\Big]\Big)^{\frac{1}{20}}\Big) \qquad (3.82) \\
&\leq \sqrt{\tfrac{T}{N}}T\left(C\exp\left(\tfrac{1}{4}\right)\exp\Big((2\beta^2+1)T + \|\xi\|^2_{\mathbb{R}^2}\Big)\right)^{\frac{1}{5}+\frac{1}{20}} p(1+2\beta)3.
\end{aligned}$$

This together with $\max_{N\in\mathbb{N}\cap[0,\max\{6\beta^2T,T\}+1]}\mathbb{E}\big[\big\|X^{\xi}_{0,T} - Y^N_T\big\|^2_{\mathbb{R}^2}\big] < \infty$ implies inequality (3.76). The proof of Lemma 3.10 is thus completed. □

Chapter 4

Existence of continuously differentiable solutions of stochastic differential equations with non-globally Lipschitz coefficients

In the last chapter, we investigate whether there are conditions on the coefficients of the SDE under which the assumptions of Setting 3.2 are fulfilled and finally establish the existence of a continuously differentiable flow. This chapter is the result of joint work with Martin Hutzenthaler and Sara Mazzonetto which is available online as arXiv preprints Hudde, Hutzenthaler, and Mazzonetto 2019a and Hudde, Hutzenthaler, and Mazzonetto 2019b.

4.1 Uniform moment estimates for affine-linear SDEs

Setting 4.1. *Let* $T \in (0, \infty)$*, let* $(U, \langle \cdot, \cdot \rangle_U, \| \cdot \|_U)$ *be a separable real Hilbert space with orthonormal basis* $\mathbb{U}$*, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t \in [0,T]}$*, and let* $(W_t)_{t \in [0,T]}$ *be an* Id_U*-cylindrical Wiener process.*

4.1.1 Uniform exponential moment estimates for Itô processses

The following lemma, Lemma 4.2, is a well-known exponential moment estimate for Itô processes and is included here for the reader's convenience.

Lemma 4.2. Assume Setting 4.1, let $\tau \colon \Omega \to [0, T]$ be an $(\mathbb{F}_t)_{t \in [0,T]}$-stopping time, let $\alpha \colon [0, T] \times \Omega \to \mathbb{R}$ be a $\mathcal{B}([0, T]) \otimes \mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable stochastic process, let $R \colon [0, T] \times \Omega \to \mathrm{HS}(U, \mathbb{R})$ be a $\mathcal{B}([0, T]) \otimes \mathcal{F}/\mathcal{B}(\mathrm{HS}(U, \mathbb{R}))$-measurable stochastic process which is $(\mathbb{F}_t)_{t \in [0,T]}$-adapted and such that it holds $\mathbb{P}$-almost surey that

$$\int_0^\tau \max\{0, \alpha_r\} + \|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr < \infty, \tag{4.1}$$

and let $p \in (1, \infty)$. Then it holds that

$$\mathbb{E}\Big[\exp\big(\int_0^\tau \alpha_r \, dr + \int_0^\tau R_r \, dW_r\big)\Big] \leq \Big(\mathbb{E}\Big[\exp\big(\int_0^\tau \tfrac{p}{(p-1)}\alpha_r + \tfrac{p^2}{2(p-1)}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr\big)\Big]\Big)^{(p-1)/p}. \tag{4.2}$$

Proof of Lemma 4.2. Throughout the proof of Lemma 4.2 let $\mathcal{E} \colon [0,T] \times \Omega \to (0,\infty)$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-almost surely that

$$\mathcal{E}_t = \exp\Big(p \int_0^{t\wedge\tau} R_r \, dW_r - \tfrac{1}{2}p^2 \int_0^{t\wedge\tau} \|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr\Big). \tag{4.3}$$

It follows from, e.g., Gawarecki and Mandrekar 2010, Exercise 2.15(b), the optional stopping theorem and Lemma 18.21 in Kallenberg 2002 that $\mathcal{E}$ is a positive local martingale with respect to the filtration $(\mathbb{F}_t)_{t\in[0,T]}$. This together with the fact that $\mathbb{E}[\mathcal{E}_0] < \infty$ assures that $\mathcal{E}$ is an $(\mathbb{F}_t)_{t\in[0,T]}$-supermartingale, and the optional stopping theorem implies that $\mathbb{E}[\mathcal{E}_\tau] \leq \mathbb{E}[\mathcal{E}_0] = 1$. This and Hölder's inequality yield that

$$\begin{aligned}
\mathbb{E}\Big[\exp\big(\int_0^\tau \alpha_r \, dr + \int_0^\tau R_r \, dW_r\big)\Big] &= \mathbb{E}\Big[\exp\big(\int_0^\tau \alpha_r + \tfrac{p}{2}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr\big) \, (\mathcal{E}_\tau)^{1/p}\Big] \\
&\leq \Big(\mathbb{E}\Big[\exp\big(\tfrac{p}{p-1}\int_0^\tau \alpha_r + \tfrac{p}{2}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr\big)\Big]\Big)^{(p-1)/p} \big(\mathbb{E}[\mathcal{E}_\tau]\big)^{1/p} \\
&\leq \Big(\mathbb{E}\Big[\exp\big(\tfrac{p}{p-1}\int_0^\tau \alpha_r + \tfrac{p}{2}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr\big)\Big]\Big)^{(p-1)/p}.
\end{aligned} \tag{4.4}$$

The proof of Lemma 4.2 is thus completed. □

The following lemma, Lemma 4.3, provides exponential moment estimates for Itô processes and generalizes Lemma 2.7 in Cox, Hutzenthaler, and Jentzen 2013.

Lemma 4.3. Assume Setting 4.1, let $\alpha \colon [0,T] \times \Omega \to \mathbb{R}$ be a $\mathcal{B}([0,T]) \otimes \mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process, and let $R \colon [0,T] \times \Omega \to \mathrm{HS}(U,\mathbb{R})$ be a $\mathcal{B}([0,T]) \otimes \mathcal{F}/\mathcal{B}(\mathrm{HS}(U,\mathbb{R}))$-measurable $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process such that it holds $\mathbb{P}$-almost surely that

$$\int_0^T \max\{0, \alpha_r\} + \|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr < \infty. \tag{4.5}$$

Then it holds that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\Big(\int_0^t \alpha_r\,dr + \int_0^t R_r\,dW_r\Big)\Big] \\ &\leq \inf_{p,q\in(1,\infty)} \bigg(\big(\tfrac{q}{q-1}\big)^q\Big(\mathbb{E}\Big[\exp\Big(\tfrac{p}{(p-1)}\int_0^T \max\big\{\alpha_r, \tfrac{(-1)}{2q}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\big\} + \tfrac{p}{2}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} dr\Big)\Big]\Big)^{(p-1)/p}\bigg).\end{aligned} \tag{4.6}$$

Proof of Lemma 4.3. Throughout the proof let $p, q \in (1,\infty)$, and for all $n \in \mathbb{N}$ let $\tau_n \colon \Omega \to [0,T]$ be a mapping with the property that it holds $\mathbb{P}$-almost surely that

$$\tau_n = \inf\Big(\Big\{t\in[0,T]\colon \int_0^t \|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})} + \max\big\{0, \tfrac{1}{q}\big(\alpha_r + \tfrac{1}{2q}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\big)\big\}\,dr \geq n\Big\} \cup \{T\}\Big). \tag{4.7}$$

The fact that it holds $\mathbb{P}$-almost surely that $\int_0^T \max\{0,\alpha_r\} + \|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr < \infty$ implies that it holds $\mathbb{P}$-almost surely that $\lim_{n\to\infty} \tau_n = T$. It holds for all $n \in \mathbb{N}$ that τ_n is an $(\mathbb{F}_t)_{t\in[0,T]}$-stopping time. Itô's formula (e.g., Gawarecki and Mandrekar 2010, Theorem 2.10) implies that the process

$$\Big(\exp\Big(\int_0^s \max\big\{0, \tfrac{\alpha_r}{q} + \tfrac{\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2q^2}\big\}dr + \tfrac{1}{q}\int_0^s R_r dW_r - \int_0^s \tfrac{\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2q^2} dr\Big)\Big)_{s\in[0,T]} \tag{4.8}$$

is a non-negative $(\mathbb{F}_t)_{t\in[0,T]}$-submartingale. Then, Doob's inequality (see, e.g., Revuz and Yor 2013, Proposition II.1.7) and Lemma 4.2 (applied with $\tau = \tau_n$ and $\alpha_. = \max\big\{-\frac{1}{2q}\|R_.\|^2_{\mathrm{HS}(U,\mathbb{R})}, \alpha_.\big\}$ in the notation of Lemma 4.2) show that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,\tau_n]\cap\mathbb{Q}} \exp\Big(\int_0^t \alpha_r\,dr + \int_0^t R_r\,dW_r\Big)\Big] \\ &\leq \mathbb{E}\Big[\sup_{t\in[0,\tau_n]\cap\mathbb{Q}} \Big(\exp\Big(\int_0^t \max\Big\{0, \tfrac{\alpha_r}{q} + \tfrac{\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2q^2}\Big\}\,dr + \tfrac{1}{q}\int_0^t R_r\,dW_r - \int_0^t \tfrac{\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2q^2}\,dr\Big)\Big)^q\Big] \\ &\leq \big(\tfrac{q}{q-1}\big)^q\,\mathbb{E}\Big[\Big(\exp\Big(\int_0^{\tau_n} \max\Big\{0, \tfrac{\alpha_r}{q} + \tfrac{\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2q^2}\Big\} - \tfrac{\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2q^2}\,dr + \tfrac{1}{q}\int_0^{\tau_n} R_r\,dW_r\Big)\Big)^q\Big] \\ &= \big(\tfrac{q}{q-1}\big)^q\,\mathbb{E}\Big[\exp\Big(\int_0^{\tau_n} \max\Big\{-\tfrac{1}{2q}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}, \alpha_r\Big\}\,dr + \int_0^{\tau_n} R_r\,dW_r\Big)\Big] \\ &\leq \big(\tfrac{q}{q-1}\big)^q\,\Big(\mathbb{E}\Big[\exp\Big(\int_0^{\tau_n} \tfrac{p}{(p-1)} \max\Big\{-\tfrac{1}{2q}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}, \alpha_r\Big\} + \tfrac{p^2}{2(p-1)}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big]\Big)^{(p-1)/p} \\ &\leq \big(\tfrac{q}{q-1}\big)^q\,\Big(\mathbb{E}\Big[\exp\Big(\tfrac{p}{(p-1)}\int_0^T \max\Big\{-\tfrac{1}{2q}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}, \alpha_r\Big\} + \tfrac{p}{2}\|R_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big]\Big)^{(p-1)/p}.\end{aligned} \tag{4.9}$$

Inequality (4.6) then follows from inequality (4.9) and the monotone convergence theorem. The proof of Lemma 4.3 is thus completed. □

4.1.2 Uniform moment estimates for SDEs of affine-linear type

The following lemma, Lemma 4.4, provides uniform moment estimates for one-dimensional SDEs of affine-linear type. Informally speaking, this means that the drift is bounded from above by an affine-linear adapted function of the process and the diffusion term is an affine-linear adapted function of the process.

Lemma 4.4. Assume Setting 4.1, let $X\colon [0,T]\times\Omega\to[0,\infty)$ be an $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths, let $\mu,\beta\colon[0,T]\times\Omega\to\mathbb{R}$ be $\mathcal{B}([0,T])\otimes\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable stochastic processes, let $\alpha\colon[0,T]\times\Omega\to[-\infty,\infty]$ be a $\mathcal{B}([0,T])\otimes\mathcal{F}/\mathcal{B}([-\infty,\infty])$-measurable $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process, let $\eta,\zeta\colon[0,T]\times\Omega\to\mathrm{HS}(U,\mathbb{R})$ be $\mathcal{B}([0,T])\otimes\mathcal{F}/\mathcal{B}(\mathrm{HS}(U,\mathbb{R}))$-measurable $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic processes with the properties that for all $t\in[0,T]$ it holds $\mathbb{P}$-almost surely that $\mu_t - X_t\,\alpha_t - \langle\eta_t,\zeta_t\rangle_{\mathrm{HS}(U,\mathbb{R})}\leq\beta_t$, that it holds $\mathbb{P}$-almost surely that

$$\int_0^T |\mu_r| + \max\{\alpha_r,0\} + \|X_r\eta_r+\zeta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr<\infty, \tag{4.10}$$

that for all $t\in[0,T]$ it holds $\mathbb{P}$-almost surely that

$$X_t = X_0 + \int_0^t \mu_r\,dr + \int_0^t X_r\,\eta_r + \zeta_r\,dW_r, \tag{4.11}$$

let $p,q,u,u'\in[1,\infty)$, $p'\in[2/q,\infty)\cap(1,\infty)$, $c_1,c_2,c_3,c_4,\theta\in(1,\infty)$ be real numbers, and let $\psi\colon(0,\infty)\times(1,\infty)\to(0,\infty]$ be the function which satisfies for all $\lambda\in(0,\infty)$, $c\in(1,\infty)$ that

$$\psi(\lambda,c) = \left(\tfrac{\theta}{\theta-1}\right)^{\frac{\theta}{\lambda}}\left\|\exp\left(\int_0^T \max\left\{\alpha_r + \tfrac{\frac{c\lambda}{c-1}-1}{2}\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}, \tfrac{\frac{c\lambda}{c-1}-\frac{\lambda}{\theta}}{2}\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\right\}dr\right)\right\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})}. \tag{4.12}$$

Then it holds that

$$\begin{aligned}\Big\|\sup_{t\in[0,T]} X_t\Big\|_{L^q(\mathbb{P};\mathbb{R})} &\leq \psi(pq,c_1)\|X_0\|_{L^{(pq)/(p-1)}(\mathbb{P};\mathbb{R})} + \psi(uq,c_2)\|\beta\|_{L^1(\lambda_{[0,T]};L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R}))}\\ &\quad + p'q\,\psi\Big(\tfrac{(p'q)}{(p'-1)},c_3\Big)\|\zeta\|_{L^2(\lambda_{[0,T]};L^{\frac{u'p'q}{(u'-1)}}(\mathbb{P};\mathrm{HS}(U,\mathbb{R})))}\\ &\quad\cdot\Big\|\exp\Big(\int_0^T \tfrac{u'p'q}{2}\big(\tfrac{c_4}{c_4-1}+\tfrac{1}{\theta}\big)\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big\|_{L^{u'p'qc_4}(\mathbb{P};\mathbb{R})}.\end{aligned} \tag{4.13}$$

Proof of Lemma 4.4. Throughout the proof of Lemma 4.4 let $\hat{\alpha},\Psi\colon[0,T]\times\Omega\to\mathbb{R}$ be mappings with the properties that Ψ has continuous sample paths, that for all $t\in[0,T]$ it holds that $\hat{\alpha}_s=\max\{\alpha_s,(\frac{1}{2}-\frac{\lambda}{2\theta})\|\eta_s\|^2_{\mathrm{HS}(U,\mathbb{R})}\}$, and that for all $t\in[0,T]$ it holds $\mathbb{P}$-almost surely that

$$\Psi_t = \exp\Big(\int_0^t \hat{\alpha}_r - \tfrac{1}{2}\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr + \int_0^t \eta_r\,dW_r\Big). \tag{4.14}$$

From Lemma 4.2 (applied with $\tau=t$, $\alpha_\cdot = -\lambda\big(\hat{\alpha}_\cdot - \frac{1}{2}\|\eta_\cdot\|^2_{\mathrm{HS}(U,\mathbb{R})}\big)$, $R_\cdot = -\lambda\eta_\cdot$, $p=\frac{c}{c-1}$ in the notation of Lemma 4.2) together with inequality (4.10) and the fact that for all $r\in[0,T]$ it holds that $-\hat{\alpha}_r \leq \frac{1}{2}(\frac{\lambda}{\theta} - 1)\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}$ it follows that for all $t \in [0,T]$, $\lambda \in [1,\infty)$, $c \in (1,\infty)$ it holds that

$$\begin{aligned}
&\big\|(\Psi_t)^{-1}\big\|_{L^{\lambda}(\mathbb{P};\mathbb{R})} \\
&= \Big(\mathbb{E}\Big[\exp\Big(-\int_0^t \lambda(\hat{\alpha}_r - \tfrac{1}{2}\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})})\,dr - \int_0^t \lambda\eta_r\,dW_r\Big)\Big]\Big)^{1/\lambda} \\
&\leq \Big(\mathbb{E}\Big[\exp\Big(\int_0^t (-1)\tfrac{c/(c-1)}{c/(c-1)-1}\lambda\big(\hat{\alpha}_r - \tfrac{1}{2}\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\big) + \tfrac{(c/(c-1))^2}{2(c/(c-1)-1)}\|\lambda\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big]\Big)^{1/(\lambda c)} \qquad (4.15) \\
&= \Big\|\exp\Big(\int_0^t (-1)\hat{\alpha}_r + \tfrac{1}{2}\big(\tfrac{\lambda c}{c-1} + 1\big)\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})} \\
&\leq \Big\|\exp\Big(\int_0^t \tfrac{1}{2}\big(\tfrac{\lambda c}{c-1} + \tfrac{\lambda}{\theta}\big)\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})}.
\end{aligned}$$

From inequality (4.10), Lemma 4.3 (applied with $T = T - s$, $\mathbb{F}_{\cdot} = \mathbb{F}_{\cdot+s}$, $W_{\cdot} = W_{\cdot+s} - W_s$, $\alpha_{\cdot} = \lambda\hat{\alpha}_{\cdot+s} - \frac{\lambda}{2}\|\eta_{\cdot+s}\|^2_{\mathrm{HS}(U,\mathbb{R})}$, $R_{\cdot} = \lambda\eta_{\cdot+s}$ in the notation of Lemma 4.3), the fact that for all $r \in [0,T]$ it holds that $\hat{\alpha}_r \geq (\frac{1}{2} - \frac{\lambda}{2\theta})\|\eta_r\|^2_{\mathrm{HS}(U,R)}$, and the fact that for all $c \in (1,\infty)$ it holds that $\frac{c\lambda}{c-1} \geq \frac{\lambda}{\theta}$ it follows that for all $s \in [0,T)$, $\lambda \in [1,\infty)$, $c \in (1,\infty)$ it holds that

$$\begin{aligned}
&\Big\|(\Psi_s)^{-1}\sup_{t\in[s,T]}\Psi_t\Big\|_{L^{\lambda}(\mathbb{P};\mathbb{R})} \\
&= \Big\|\sup_{t\in[0,T-s]\cap\mathbb{Q}}\Big((\Psi_s)^{-1}\Psi_{s+t}\Big)\Big\|_{L^{\lambda}(\mathbb{P};\mathbb{R})} \\
&= \Big(\mathbb{E}\Big[\sup_{t\in[0,T-s]\cap\mathbb{Q}}\exp\Big(\int_0^t \lambda\hat{\alpha}_{r+s} - \tfrac{\lambda}{2}\|\eta_{r+s}\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr + \int_0^t \lambda\eta_{r+s}\,dW_{r+s}\Big)\Big]\Big)^{1/\lambda} \\
&\leq \Bigg(\big(\tfrac{\theta}{\theta-1}\big)^{\theta}\Bigg(\mathbb{E}\Big[\exp\Big(c\int_0^{T-s}\max\Big\{\lambda\hat{\alpha}_{r+s} - \tfrac{\lambda}{2}\|\eta_{r+s}\|^2_{\mathrm{HS}(U,\mathbb{R})}, -\tfrac{1}{2\theta}\|\lambda\eta_{r+s}\|^2_{\mathrm{HS}(U,\mathbb{R})}\Big\} \qquad (4.16) \\
&\qquad\qquad + \tfrac{c/(c-1)}{2}\|\lambda\eta_{r+s}\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big]\Bigg)^{1/c}\Bigg)^{1/\lambda} \\
&= \big(\tfrac{\theta}{\theta-1}\big)^{\frac{\theta}{\lambda}}\Big\|\exp\Big(\int_0^{T-s}\hat{\alpha}_{r+s} + \tfrac{\frac{c\lambda}{c-1}-1}{2}\|\eta_{r+s}\|^2_{\mathrm{HS}(U,\mathbb{R})}\,dr\Big)\Big\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})} \\
&= \big(\tfrac{\theta}{\theta-1}\big)^{\frac{\theta}{\lambda}}\Big\|\exp\Big(\int_s^T\max\Big\{\alpha_r + \tfrac{1}{2}(\tfrac{c\lambda}{c-1} - 1)\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}, \tfrac{1}{2}(\tfrac{c\lambda}{c-1} - \tfrac{\lambda}{\theta})\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\Big\}\,dr\Big)\Big\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})} \\
&\leq \psi(\lambda,c).
\end{aligned}$$

From Itô's formula (e.g., Gawarecki and Mandrekar 2010, Theorem 2.10), equation (4.14), and the fact that for all $r \in [0,T]$ it holds $\mathbb{P}$-almost surely that $\mu_r - X_r\alpha_r - \langle\eta_r,\zeta_r\rangle_{\mathrm{HS}(U,\mathbb{R})} \leq \beta_r$ it follows that for all $t \in [0,T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} X_t(\Psi_t)^{-1} &= X_0(\Psi_0)^{-1} + \int_0^t (\Psi_r)^{-1}\left(X_r\eta_r + \zeta_r - X_r\eta_r\right) dW_r \\ &\quad + \int_0^t \Big(\mu_r - X_r\hat{\alpha}_r + (1-2+1)\,\tfrac{1}{2}X_r\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})} - \langle \eta_r, \zeta_r\rangle_{\mathrm{HS}(U,\mathbb{R})}\Big)(\Psi_r)^{-1}\, dr \\ &\leq X_0 + \int_0^t \beta_r(\Psi_r)^{-1}\, dr + \int_0^t (\Psi_r)^{-1}\,\zeta_r\, dW_r. \end{aligned} \tag{4.17}$$

Hence, we obtain that for all $t \in [0,T]$ it holds $\mathbb{P}$-almost surely that

$$X_t \leq \Psi_t X_0 + \int_0^t \beta_r(\Psi_r)^{-1}\Psi_t\, dr + \Psi_t \int_0^t (\Psi_r)^{-1}\zeta_r\, dW_r. \tag{4.18}$$

The triangle inequality therefore yields that

$$\begin{aligned} \Big\|\sup_{t\in[0,T]} X_t\Big\|_{L^q(\mathbb{P};\mathbb{R})} &\leq \Big\|\big(\sup_{t\in[0,T]}\Psi_t\big)\cdot|X_0|\Big\|_{L^q(\mathbb{P};\mathbb{R})} + \int_0^T \big\|\beta_r(\Psi_r)^{-1}\sup_{t\in[r,T]}\Psi_t\big\|_{L^q(\mathbb{P};\mathbb{R})} dr \\ &\quad + \Big\|\big(\sup_{t\in[0,T]}\Psi_t\big)\cdot\sup_{t\in[0,T]}\Big|\int_0^t (\Psi_r)^{-1}\zeta_r\, dW_r\Big|\Big\|_{L^q(\mathbb{P};\mathbb{R})}. \end{aligned} \tag{4.19}$$

This and Hölder's inequality imply that

$$\begin{aligned} \Big\|\sup_{t\in[0,T]} X_t\Big\|_{L^q(\mathbb{P};\mathbb{R})} &\leq \Big\|\sup_{t\in[0,T]}\Psi_t\Big\|_{L^{pq}(\mathbb{P};\mathbb{R})}\|X_0\|_{L^{(pq)/(p-1)}(\mathbb{P};\mathbb{R})} \\ &\quad + \Big[\sup_{r\in[0,T]}\Big\|(\Psi_r)^{-1}\sup_{t\in[r,T]}\Psi_t\Big\|_{L^{uq}(\mathbb{P};\mathbb{R})}\Big]\Big\|\beta\Big\|_{L^1(\lambda_{[0,T]};L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R}))} \\ &\quad + \Big\|\sup_{t\in[0,T]}\Psi_t\Big\|_{L^{(p'q)/(p'-1)}(\mathbb{P};\mathbb{R})}\Big\|\sup_{t\in[0,T]}\Big|\int_0^t (\Psi_r)^{-1}\zeta_r dW_r\Big|\Big\|_{L^{p'q}(\mathbb{P};\mathbb{R})}. \end{aligned} \tag{4.20}$$

Next, observe that the fact that $p'q \geq 2$, the Burkholder-Davis-Gundy type inequality Da Prato and Zabczyk 1992, Lemmas 7.2 and 7.7, Hölder's inequality, and equation (4.15) yield that

$$\begin{aligned} &\tfrac{1}{p'q}\Big\|\sup_{t\in[0,T]}\Big|\int_0^t (\Psi_r)^{-1}\zeta_r dW_r\Big|\Big\|_{L^{p'q}(\mathbb{P};\mathbb{R})} \\ &\leq \Big(\int_0^T \Big\|(\Psi_r)^{-1}\|\zeta_r\|_{\mathrm{HS}(U,\mathbb{R})}\Big\|^2_{L^{p'q}(\mathbb{P};\mathbb{R})} dr\Big)^{1/2} \\ &\leq \Big(\int_0^T \|(\Psi_r)^{-1}\|^2_{L^{u'p'q}(\mathbb{P};\mathbb{R})}\|\zeta_r\|^2_{L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathrm{HS}(U,\mathbb{R}))} dr\Big)^{1/2} \\ &\leq \Big[\sup_{t\in[0,T]}\Big\|(\Psi_t)^{-1}\Big\|_{L^{u'p'q}(\mathbb{P};\mathbb{R})}\Big]\|\zeta_\cdot\|_{L^2\left(\lambda_{[0,T]};L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathrm{HS}(U,\mathbb{R}))\right)} \\ &\leq \Big\|\exp\Big(\int_0^T \tfrac{u'p'q}{2}\big(\tfrac{c_4}{c_4-1}+\tfrac{1}{\theta}\big)\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\, dr\Big)\Big\|_{L^{u'p'qc_4}(\mathbb{P};\mathbb{R})}\|\zeta_\cdot\|_{L^2\left(\lambda_{[0,T]};L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathrm{HS}(U,\mathbb{R}))\right)}. \end{aligned} \tag{4.21}$$

Finally, inequalities (4.20), (4.16), (4.21), and the fact that $\mathbb{P}(\Psi_0 = 1) = 1$ imply that

$$\begin{aligned}\Big\| \sup_{t\in[0,T]} X_t \Big\|_{L^q(\mathbb{P};\mathbb{R})} &\leq \psi(pq, c_1)\, \|X_0\|_{L^{(pq)/(p-1)}(\mathbb{P};\mathbb{R})} + \psi(uq, c_2)\|\beta\|_{L^1\left(\lambda_{[0,T]};L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R})\right)} \\ &\quad + p'q\,\psi(\tfrac{(p'q)}{(p'-1)}, c_3)\|\zeta\|_{L^2\left(\lambda_{[0,T]};L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathrm{HS}(U,\mathbb{R}))\right)} \\ &\quad \cdot \left\|\exp\left(\int\limits_0^T \tfrac{u'p'q}{2}(\tfrac{c_4}{c_4-1} + \tfrac{1}{\theta})\|\eta_r\|^2_{\mathrm{HS}(U,\mathbb{R})}\, dr\right)\right\|_{L^{u'p'qc_4}(\mathbb{P};\mathbb{R})}.\end{aligned} \tag{4.22}$$

The proof of Lemma 4.4 is thus completed. □

The following lemma, Lemma 4.5, is a generalization of Richard's inequality (see Richard 1972 or, e.g., Section 5.13 in Cerone and Dragomir 2010) which is a special case of Lemma 4.5 for $\|u\|_H = 1$ and is a generalization of the Cauchy-Schwarz inequality which is a special case of Lemma 4.5 for $\|u\|_H = 0$.

Lemma 4.5. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a real Hilbert space and let $u, v, w \in H$ satisfy that $\|u\|_H \leq 1$. Then it holds that

$$|\langle v, w\rangle_H - 2\langle v, u\rangle_H \langle u, w\rangle_H| \leq \|v\|_H \|w\|_H. \tag{4.23}$$

Proof of Lemma 4.5. The Cauchy-Schwarz inequality yields that

$$\begin{aligned}|\langle v, w\rangle_H \|u\|_H^2 - 2\langle v, u\rangle_H \langle u, w\rangle_H|^2 &= |\langle \|u\|_H^2 v - 2\langle v, u\rangle_H u, w\rangle_H|^2 \\ &\leq \Big\|\|u\|_H^2 v - 2\langle v, u\rangle_H u\Big\|_H^2 \|w\|_H^2 \\ &= (\|v\|_H^2\|u\|_H^4 - 4(\langle v, u\rangle_H)^2\|u\|_H^2 + 4(\langle v, u\rangle_H)^2\|u\|_H^2)\|w\|_H^2 \\ &= \|v\|_H^2 \|w\|_H^2 \|u\|_H^4.\end{aligned} \tag{4.24}$$

This together with the triangle inequality and with the Cauchy-Schwarz inequality yields that

$$\begin{aligned}|\langle v, w\rangle_H - 2\langle v, u\rangle_H \langle u, w\rangle_H| &\leq |\langle v, w\rangle_H \|u\|_H^2 - 2\langle v, u\rangle_H \langle u, w\rangle_H| + |\langle v, w\rangle_H|(1 - \|u\|_H^2) \\ &\leq \|v\|_H\|w\|_H\|u\|_H^2 + \|v\|_H\|w\|_H(1 - \|u\|_H^2) \\ &= \|v\|_H \|w\|_H.\end{aligned} \tag{4.25}$$

This finishes the proof of Lemma 4.5. □

The following proposition, Proposition 4.6, provides uniform moment estimates for Hilbert-space-valued SDEs of affine-linear type. Informally speaking, that means that the scalar product of drift and process is bounded from above by an affine-linear adapted function of the squared norm of the process and the diffusion term is an affine-linear adapted function of the process.

Proposition 4.6 (Uniform moment estimates for affine-linear type SDEs)**.** *Assume Setting 4.1, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be a separable real Hilbert space, let* $s \in [0, T]$*, let* $\beta\colon [s, T] \times \Omega \to \mathbb{R}$ *be a*

$\mathcal{B}([s,T])\otimes\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable stochastic process, let $\eta,\zeta\colon [s,T]\times\Omega\to HS(U,H)$, $\alpha\colon [s,T]\times\Omega\to\mathbb{R}$, and $\mu\colon [s,T]\times\Omega\to H$ be product measurable $(\mathbb{F}_t)_{t\in[s,T]}$-adapted stochastic processes, let $X\colon [s,T]\times\Omega\to H$ be an $(\mathbb{F}_t)_{t\in[s,T]}$-adapted stochastic process with continuous sample paths with the property that for all $t\in[s,T]$ it holds $\mathbb{P}$-almost surely that $\langle X_t,\mu_t\rangle_H\le\alpha_t\|X_t\|_H^2+\beta_t$ and that for all $t\in[s,T]$ it holds $\mathbb{P}$-almost surely that $\int_s^T\|\mu_r\|_H+\max\{\alpha_r,0\}+\|\eta_r+\zeta_r\|^2_{\mathrm{HS}(U,H)}\,dr<\infty$ and

$$X_t = X_s + \int_s^t \mu_r\,dr + \int_s^t (\eta_r+\zeta_r)\,dW_r, \tag{4.26}$$

and let $\psi\colon [1,\infty)\times(1,\infty)\times(1,\infty)\times(0,\infty)\to(0,\infty]$ be the function with the property that for all $\lambda\in[1,\infty)$, $c,\theta\in(1,\infty)$, $\delta\in[0,\infty)$ it holds that

$$\begin{aligned}&\psi(\lambda,c,\theta,\delta) \qquad\qquad (4.27)\\ &= \left(\tfrac{\theta}{\theta-1}\right)^{\frac{\theta}{\lambda}}\left\|\exp\left(\int_s^T \max\left\{\alpha_r+\tfrac{1+\delta}{2}\tfrac{\|\eta_r\|^2_{\mathrm{HS}(U,H)}}{\|X_r\|_H^2},0\right\}+\tfrac{\frac{c\lambda}{c-1}-\min\{2,\frac{\lambda}{\theta}\}}{2}\tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|_H^4}\,dr\right)\right\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})}.\end{aligned}$$

Then it holds

(i) for all $q,p,u,u'\in[1,\infty)$, $p'\in(1,\infty)\cap[2/q,\infty)$, $c_1,c_2,c_3,c_4,\theta\in(1,\infty)$, $\delta\in[0,\infty)$ that

$$\begin{aligned}&\left\|\sup_{t\in[s,T]}\|X_t\|_H\right\|_{L^q(\mathbb{P};\mathbb{R})}\le\psi(pq,c_1,\theta,\delta)\|X_s\|_{L^{\frac{(pq)}{(p-1)}}(\mathbb{P};H)}\\ &+2\Big(\psi(pq,c_1,\theta,\delta)\psi(uq,c_2,\theta,\delta)\Big)^{\frac12}\left(\int_s^T\left\|\max\left\{\beta_r+\tfrac{1+\delta}{2\delta}\|\zeta_r\|^2_{\mathrm{HS}(U,H)},0\right\}\right\|_{L^{\frac{(uq)}{(u-1)}}(\mathbb{P};\mathbb{R})}dr\right)^{\frac12}\\ &+p'q\,\psi\!\left(\tfrac{(p'q)}{(p'-1)},c_3,\theta,\delta\right)\left\|\tfrac{\|\langle X,\zeta\rangle_H\|_{\mathrm{HS}(U,\mathbb{R})}}{\|X\|_H}\right\|_{L^2(\lambda_{[s,T]};L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathbb{R}))} \qquad (4.28)\\ &\quad\cdot\left\|\exp\left(\int_s^T\tfrac{u'p'q}{2}\left(\tfrac{c_4}{c_4-1}-\tfrac{1}{\theta}\right)\tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|_H^4}dr\right)\right\|_{L^{u'p'qc_4}(\mathbb{P};\mathbb{R})}\end{aligned}$$

and

(ii) for all $q,u\in[1,\infty)$, $p\in[1,u]$ with $(u+1)q\ge 2u$ that

$$\begin{aligned}&\left\|\sup_{t\in[s,T]}\|X_t\|_H\right\|_{L^q(\mathbb{P};\mathbb{R})}\\ &\le\exp(\tfrac{1}{pq})\left\|\exp\left(\int_s^T\left\{\alpha_r+\tfrac{u}{2}\tfrac{\|\eta_r\|^2_{\mathrm{HS}(U,H)}}{\|X_r\|_H^2},0\right\}+\tfrac{c\lambda}{2c-2}\tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|_H^4}\,dr\right)\right\|_{L^{cuq}(\mathbb{P};\mathbb{R})} \qquad (4.29)\\ &\cdot\Bigg(\|X_s\|_{L^{\frac{(pq)}{(p-1)}}(\mathbb{P};H)}+2\left\||\beta|^{\frac12}\right\|_{L^2(\lambda_{[s,T]};L^{\frac{(2uq)}{(u-1)}}(\mathbb{P};\mathbb{R}))}+\tfrac{u(2+q)}{u-1}\|\zeta\|_{L^2\left(\lambda_{[s,T]};L^{\frac{(2u^2q)}{(u-1)^2}}(\mathbb{P};\mathrm{HS}(U,H))\right)}\\ &\qquad\cdot\left\|\exp\left(\int_s^T\tfrac{u^2q}{u-1}\tfrac{c}{c-1}\tfrac{\|\langle X_r,\eta_r(X_r)\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|_H^4}\,dr\right)\right\|_{L^{cu^2q/(u-1)}(\mathbb{P};\mathbb{R})}\Bigg).\end{aligned}$$

Proof of Proposition 4.6. Throughout the proof of item (i) let $q,p,u,u'\in[1,\infty)$, $p'\in(1,\infty)\cap[2/q,\infty)$, $c_1,c_2,c_3,c_4,\theta\in(1,\infty)$, $\delta\in[0,\infty)$, for all $\varepsilon\in(0,\infty)$ let $\phi_\varepsilon\colon H\to(0,\infty)$ be the function with the property that for all $v \in H$ it holds that $\phi_\varepsilon(v) = \big(\varepsilon^2 + \|v\|_H^2\big)^{1/2}$, let $\hat{\alpha}, \hat{\beta} \colon [s,T] \times \Omega \to [0,\infty]$ be functions with the properties that for all $r \in [s,T]$ it holds that $\hat{\alpha}_r = \max\big\{\alpha_r + \frac{1+\delta}{2} \frac{\|\eta_r\|^2_{\mathrm{HS}(U,H)}}{\|X_r\|_H^2}, 0\big\}$ and that $\hat{\beta}_r = \max\big\{\beta_r + \frac{1+\delta}{2\delta}\|\zeta_r\|^2_{\mathrm{HS}(U,H)}, 0\big\}$, and let $\gamma \colon [s,T] \times \Omega \mapsto \mathbb{R}$ be the stochastic process with the property that for all $t \in [s,T]$ it holds that

$$\gamma_t = \tfrac{2\langle X_t, \mu_t\rangle_H + \|\eta_t + \zeta_t\|^2_{\mathrm{HS}(U,H)}}{2\phi_\varepsilon(X_t)} - \tfrac{1}{2\phi_\varepsilon(X_t)} \left\| \tfrac{\langle X_t, \eta_t + \zeta_t\rangle_H}{\phi_\varepsilon(X_t)} \right\|^2_{\mathrm{HS}(U,\mathbb{R})}. \tag{4.30}$$

For all $\varepsilon \in (0,\infty)$, $v, w, z \in H$ it holds that $\phi_\varepsilon \in C^2(H, [0,\infty))$ and that

$$\phi_\varepsilon'(v)(w) = \tfrac{\langle v, w\rangle_H}{\phi_\varepsilon(v)} \text{ and } \phi_\varepsilon''(v)(w,z) = \tfrac{\langle w, z\rangle_H}{\phi_\varepsilon(v)} - \tfrac{\langle v, w\rangle_H \langle v, z\rangle_H}{(\phi_\varepsilon(v))^3}. \tag{4.31}$$

Next, we observe that equation (4.26) implies that for all $t \in [0, T-s]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} X_{t+s} &= X_s + \int_s^{t+s} \mu_r \, dr + \int_s^{t+s} (\eta_r + \zeta_r) \, dW_r \\ &= X_s + \int_0^t \mu_{r+s} \, dr + \int_0^t (\eta_{r+s} + \zeta_{r+s}) \, d(W_{r+s} - W_s) \end{aligned} \tag{4.32}$$

and Itô's formula Gawarecki and Mandrekar 2010, Theorem 2.10 imply that for all $\varepsilon \in (0,\infty)$, $t \in [0, T-s]$ it holds $\mathbb{P}$-almost surely that

$$\int_s^T |\gamma_r| + \left\| \tfrac{\langle X_r, \eta_r\rangle_H}{(\phi_\varepsilon(X_r))^2} \, \phi_\varepsilon(X_r) + \tfrac{\langle X_r, \zeta_r\rangle_H}{\phi_\varepsilon(X_r)} \right\|^2_{\mathrm{HS}(U,\mathbb{R})} dr < \infty \tag{4.33}$$

and that

$$\begin{aligned} &\phi_\varepsilon(X_{t+s}) - \phi_\varepsilon(X_s) - \int_0^t \Big(\tfrac{\langle X_{r+s}, \eta_{r+s}\rangle_H}{(\phi_\varepsilon(X_{r+s}))^2} \, \phi_\varepsilon(X_{r+s}) + \tfrac{\langle X_{r+s}, \zeta_{r+s}\rangle_H}{\phi_\varepsilon(X_{r+s})} \Big) \, d(W_{r+s} - W_s) \\ &= \int_s^{t+s} \tfrac{2\langle X_r, \mu_r\rangle_H + \|\eta_r + \zeta_r\|^2_{\mathrm{HS}(U,H)}}{2\phi_\varepsilon(X_r)} - \tfrac{1}{2\phi_\varepsilon(X_r)} \left\| \tfrac{\langle X_r, \eta_r + \zeta_r\rangle_H}{\phi_\varepsilon(X_r)} \right\|^2_{\mathrm{HS}(U,\mathbb{R})} dr \\ &= \int_0^t \gamma_{s+r} \, dr. \end{aligned} \tag{4.34}$$

The assumption that for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that $\langle X_t, \mu_t\rangle_H \leq \alpha_t \|X_t\|_H^2 + \beta_t$, Lemma 4.5, Young's inequality, nonnegativity of $\hat{\alpha}, \hat{\beta}$, and the fact that for all $\varepsilon \in (0,\infty)$, $v \in H$ it holds that $\phi_\varepsilon(v) \geq \|v\|_H$ and $\phi_\varepsilon(v) \geq \varepsilon$ imply that for all $\varepsilon \in (0,\infty)$, $t \in [s,T]$ it holds

$\mathbb{P}$-almost surely that

$$\begin{aligned}
&\gamma_t - \Big\langle \tfrac{\langle X_t,\eta_t\rangle_H}{(\phi_\varepsilon(X_t))^2}, \tfrac{\langle X_t,\zeta_t\rangle_H}{\phi_\varepsilon(X_t)} \Big\rangle_{\mathrm{HS}(U,\mathbb{R})} \\
&= \tfrac{2\langle X_t,\mu_t\rangle_H + \|\eta_t\|^2_{\mathrm{HS}(U,H)} + \|\zeta_t\|^2_{\mathrm{HS}(U,H)}}{2\phi_\varepsilon(X_t)} - \tfrac{1}{2\phi_\varepsilon(X_t)} \Big(\Big\| \tfrac{\langle X_t,\eta_t\rangle_H}{\phi_\varepsilon(X_t)} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})} + \Big\| \tfrac{\langle X_t,\zeta_t\rangle_H}{\phi_\varepsilon(X_t)} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})} \Big) \\
&\quad + \tfrac{2}{2\phi_\varepsilon(X_t)} \textstyle\sum_{u\in\mathbb{U}} \Big(\langle \eta_t u, \zeta_t u\rangle_H - 2\big\langle \eta_t u, \tfrac{X_t}{\phi_\varepsilon(X_t)} \big\rangle_H \big\langle \tfrac{X_t}{\phi_\varepsilon(X_t)}, \zeta_t u \big\rangle_H \Big) \\
&\leq \frac{2(\alpha_t\|X_t\|^2_H + \beta_t) + \|\eta_t\|^2_{\mathrm{HS}(U,H)} + \|\zeta_t\|^2_{\mathrm{HS}(U,H)} - \Big\| \frac{\langle X_t,\eta_t\rangle_H}{\phi_\varepsilon(X_t)} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})} + 2\sum_{u\in\mathbb{U}} \|\eta_t u\|_H \|\zeta_t u\|_H}{2\,\phi_\varepsilon(X_t)} \\
&\leq \frac{2(\alpha_t\|X_t\|^2_H + \beta_t) + (1+\delta)\|\eta_t\|^2_{\mathrm{HS}(U,H)} + (1+\frac{1}{\delta})\|\zeta_t\|^2_{\mathrm{HS}(U,H)} - \Big\| \frac{\langle X_t,\eta_t\rangle_H}{\phi_\varepsilon(X_t)} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2\,\phi_\varepsilon(X_t)} \\
&\leq \frac{2\hat\alpha_t\|X_t\|^2_H + 2\hat\beta_t - \Big\| \frac{\langle X_t,\eta_t\rangle_H}{\phi_\varepsilon(X_t)} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})}}{2\,\phi_\varepsilon(X_t)} \\
&\leq \Big(\hat\alpha_t - \tfrac{1}{2} \Big\| \tfrac{\langle X_t,\eta_t\rangle_H}{(\phi_\varepsilon(X_t))^2} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})} \Big) \phi_\varepsilon(X_t) + \tfrac{\hat\beta_t}{\varepsilon}.
\end{aligned} \tag{4.35}$$

The fact that for all $\varepsilon \in (0,\infty)$, $v \in H$ it holds that $\phi_\varepsilon(v) \geq \|v\|_H$ implies that for all $\lambda \in [1,\infty)$, $c \in (1,\infty)$, $\varepsilon, \eta \in (0,\infty)$ it holds that

$$\begin{aligned}
&\Big\| \exp\Big(\int_s^T \max\Big\{ \hat\alpha_r + \tfrac{\frac{c\lambda}{c-1} - 2}{2} \Big\| \tfrac{\langle X_r,\eta_r\rangle_H}{(\phi_\varepsilon(X_r))^2} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})}, \tfrac{\frac{c\lambda}{c-1} - \frac{\lambda}{\theta}}{2} \Big\| \tfrac{\langle X_r,\eta_r\rangle_H}{(\phi_\varepsilon(X_r))^2} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})} \Big\} \, dr \Big\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})} \Big) \\
&\leq \Big\| \exp\Big(\int_s^T \hat\alpha_r + \tfrac{\frac{c\lambda}{c-1} - \min\{2,\frac{\lambda}{\theta}\}}{2} \tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|^4_H} \, dr \Big) \Big\|_{L^{\lambda c}(\mathbb{P};\mathbb{R})} \\
&= \big(\tfrac{\theta}{\theta-1}\big)^{-\frac{\theta}{\lambda}} \psi(\lambda, c, \theta, \delta).
\end{aligned} \tag{4.36}$$

From equation (4.34), inequality (4.35), and Lemma 4.4 (applied with $T = T - s$, $\mathbb{F}_r = \mathbb{F}_{t+s}$, $W_t = W_{t+s} - W_s$, $X_t = \phi_\varepsilon(X_{t+s})$, $\mu_t = \gamma_{t+s}$, $\eta_t = \frac{\langle X_{t+s},\eta_{t+s}\rangle_H}{(\phi_\varepsilon(X_{t+s}))^2}$, $\zeta_t = \frac{\langle X_{t+s},\zeta_{t+s}\rangle_H}{\phi_\varepsilon(X_{t+s})}$, $\alpha_t = \hat\alpha_{t+s} - \frac{1}{2} \Big\| \frac{\langle X_{t+s},\eta_{t+s}\rangle_H}{(\phi_\varepsilon(X_{t+s}))^2} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})}$, and $\beta_t = \frac{\hat\beta_{t+s}}{\varepsilon}$ in the notation of Lemma 4.4) it follows that for all $\varepsilon \in (0,\infty)$ it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[s,T]} \|X_t\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq \Big\| \sup_{t\in[0,T-s]} \phi_\varepsilon(X_{t+s}) \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq \psi(pq, c_1, \theta, \delta) \, \|\phi_\varepsilon(X_s)\|_{L^{(pq)/(p-1)}(\mathbb{P};\mathbb{R})} + \psi(uq, c_2, \theta, \delta) \Big\| \tfrac{\hat\beta_t}{\varepsilon} \Big\|_{L^1(\lambda_{[s,T]};L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R}))} \\
&\quad + p'q\,\psi\Big(\tfrac{(p'q)}{(p'-1)}, c_3, \theta, \delta \Big) \Big(\int_s^T \Big\| \tfrac{\langle X_r,\zeta_r\rangle_H}{\phi_\varepsilon(X_r)} \Big\|^2_{L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathrm{HS}(U,\mathbb{R}))} \, dr \Big)^{1/2} \\
&\qquad \cdot \Big\| \exp\Big(\int_s^T \tfrac{u'p'q}{2} \Big(\tfrac{c_4}{c_4-1} + \tfrac{1}{\theta} \Big) \Big\| \tfrac{\langle X_r,\eta_r\rangle_H}{(\phi_\varepsilon(X_r))^2} \Big\|^2_{\mathrm{HS}(U,\mathbb{R})} \, dr \Big) \Big\|_{L^{u'p'qc_4}(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{4.37}$$

From inequality (4.37), subadditivity of the square root, the triangle inequality, and the fact that for all $\varepsilon \in (0,\infty), v \in H$ it holds that $\phi_\varepsilon(v) \geq \|v\|_H$ it follows that

$$\begin{aligned}
&\Big\| \sup_{t\in[s,T]} \|X_t\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq \psi(pq, c_1, \theta, \delta) \|X_s\|_{L^{(pq)/(p-1)}(\mathbb{P};H)} \\
&+ \inf_{\varepsilon\in(0,\infty)} \Big(\varepsilon\, \psi(pq, c_1, \theta, \delta) + \varepsilon^{-1} \psi(uq, c_2, \theta, \delta) \int_s^T \big\| \max\{\beta_r, \tfrac{1+\delta}{\delta} \|\zeta_r\|^2_{\mathrm{HS}(U,H)}, 0\} \big\|_{L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R})} dr \Big) \\
&\quad + p' q\, \psi({}^{(p'q)}/{}_{(p'-1)}, c_3, \theta, \delta) \Big\| \tfrac{\|\langle X,\zeta\rangle_H\|_{\mathrm{HS}(U,\mathbb{R})}}{\|X\|_H} \Big\|_{L^2\left(\lambda_{[s,T]}; L^{(u'p'q)/(u'-1)}(\mathbb{P};\mathbb{R})\right)} \qquad (4.38) \\
&\qquad \cdot \Big\| \exp\Big(\int_s^T \tfrac{u'p'q}{2} \Big(\tfrac{c_4}{c_4-1} + \tfrac{1}{\theta} \Big) \tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|^4_H} \, dr \Big) \Big\|_{L^{u'p'qc_4}(\mathbb{P};\mathbb{R})} .
\end{aligned}$$

Combining this and the fact that for all $a, b \in [0,\infty)$ it holds that $\inf_{x\in(0,\infty)}(ax+bx^{-1}) = 2\,(ab)^{1/2}$ establishes item (i).

Next, we prove item (ii). Throughout this step let $q \in [1,\infty)$, $u \in [1,\infty)$, $p \in [1,u]$ with the property that $uq \geq 2(u-1)$ be fixed. From item (i) (applied with $p' = \frac{u}{u-1}$, $u' = u$, $c_1 = c$, $c_2 = c$, $c_3 = c$, $c_4 = c$ for all $c \in (1,\infty)$ in the notation of item (i)) and the fact that $\frac{\frac{u+1}{u} q}{\frac{u+1}{u} - 1} = (u+1)q$ it follows that for all $\delta \in [0,\infty)$, $c, \theta \in (1,\infty)$ it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[s,T]} \|X_t\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq \psi(pq, c, \theta, \delta) \|X_s\|_{L^{(pq)/(p-1)}(\mathbb{P};H)} \qquad (4.39) \\
&\quad + 2\, \big(\psi(pq, c, \theta, \delta) \psi(uq, c, \theta, \delta)\big)^{1/2} \Big(\int_s^T \big\| \max\{\beta_r + \tfrac{1+\delta}{\delta} \|\zeta_r\|^2_{\mathrm{HS}(U,H)}, 0\} \big\|_{L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R})} dr \Big)^{1/2} \\
&\quad + \tfrac{uq}{u-1} \psi(uq, c, \theta, \delta) \Big\| \tfrac{\|\langle X,\zeta\rangle_H\|_{\mathrm{HS}(U,\mathbb{R})}}{\|X\|_H} \Big\|_{L^2\left(\lambda_{[s,T]}; L^{(u^2q)/(u-1)^2}(\mathbb{P};\mathbb{R})\right)} \\
&\qquad \cdot \Big\| \exp\Big(\int_s^T \tfrac{u^2 q}{2(u-1)} \Big(\tfrac{c}{c-1} + \tfrac{1}{\theta} \Big) \tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|^4_H} \, dr \Big) \Big\|_{L^{cu^2q/(u-1)}(\mathbb{P};\mathbb{R})} .
\end{aligned}$$

Next, we observe that equation (4.27) assures that for all $\lambda \in \{pq, uq\}$, $\theta \in (1,\infty)$ it holds that

$$\begin{aligned}
&\psi(\lambda, c, \theta, u-1) \\
&\leq \big(\tfrac{\theta}{\theta-1}\big)^{\frac{\theta}{pq}} \Big\| \exp\Big(\int_s^T \max\big\{\alpha_r + \tfrac{u}{2} \tfrac{\|\eta_r\|^2_{\mathrm{HS}(U,H)}}{\|X_r\|^2_H}, 0\big\} + \tfrac{cuq}{2c-2} \tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|^4_H} \, dr \Big) \Big\|_{L^{cuq}(\mathbb{P};\mathbb{R})} .
\end{aligned} \qquad (4.40)$$

Furthermore, the Cauchy-Schwarz inequality implies that for all $c \in [1,\infty)$ it holds that

$$\Big\| \tfrac{\|\langle X,\zeta\rangle_H\|_{\mathrm{HS}(U,\mathbb{R})}}{\|X\|_H} \Big\|_{L^2(\lambda_{[s,T]}; L^c(\mathbb{P};\mathbb{R}))} \leq \|\zeta\|_{L^2(\lambda_{[s,T]}; L^c(\mathbb{P};\mathrm{HS}(U,H)))} \,. \qquad (4.41)$$

In addition, note that the triangle inequality ensures that

$$\begin{aligned}
&\Big(\int_s^T \big\| \max\{\beta_r + \tfrac{u}{u-1}\|\zeta_r\|^2_{\mathrm{HS}(U,H)}, 0\}\big\|_{L^{(uq)/(u-1)}(\mathbb{P};\mathbb{R})} dr\Big)^{1/2} \\
&\leq \Big(\int_s^T \big\||\beta_r|^{\frac{1}{2}}\big\|^2_{L^{(2uq)/(u-1)}(\mathbb{P};\mathbb{R})} + \tfrac{u}{u-1}\big\|\zeta_r\big\|^2_{L^{(2uq)/(u-1)}(\mathbb{P};\mathrm{HS}(U,H))}\, dr\Big)^{1/2} \\
&\leq \big\||\beta|^{1/2}\big\|_{L^2(\lambda_{[s,T]};L^{(2uq)/(u-1)}(\mathbb{P};\mathbb{R}))} + \tfrac{u}{u-1}\big\|\zeta\big\|_{L^2(\lambda_{[s,T]};L^{(2u^2q)/(u-1)^2}(\mathbb{P};\mathrm{HS}(U,H)))}.
\end{aligned} \tag{4.42}$$

This, inequalities (4.39) and (4.40), the fact that $\inf_{\theta\in(1,\infty)}\big(\frac{\theta}{\theta-1}\big)^\theta = e^1$, and inequality (4.41) imply that

$$\begin{aligned}
&\Big\|\sup_{t\in[s,T]}\|X_t\|_H\Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq \exp(\tfrac{1}{pq})\left\|\exp\left(\int_s^T \left\{\alpha_r + \tfrac{u}{2}\tfrac{\|\eta_r\|^2_{\mathrm{HS}(U,H)}}{\|X_r\|^2_H}, 0\right\} + \tfrac{cuq}{2c-2}\tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|^4_H}\, dr\right)\right\|_{L^{cuq}(\mathbb{P};\mathbb{R})} \\
&\cdot \Bigg(\|X_s\|_{L^{(pq)/(p-1)}(\mathbb{P};H)} + 2\big\||\beta|^{1/2}\big\|_{L^2(\lambda_{[s,T]};L^{(2uq)/(u-1)}(\mathbb{P};\mathbb{R}))} \\
&+ \tfrac{u(2+q)}{u-1}\,\|\zeta\|_{L^2\left(\lambda_{[s,T]};L^{(2u^2q)/(u-1)^2}(\mathbb{P};\mathrm{HS}(U,H))\right)}\left\|\exp\left(\int_s^T \tfrac{u^2q}{u-1}\tfrac{c}{c-1}\tfrac{\|\langle X_r,\eta_r\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X_r\|^4_H}\, dr\right)\right\|_{L^{\frac{cu^2q}{(u-1)}}(\mathbb{P};\mathbb{R})}\Bigg).
\end{aligned} \tag{4.43}$$

This proves item (ii). The proof of Proposition 4.6 is thus completed. ☐

4.2 Inferring a differentiable version from sufficient regularity of difference quotients

4.2.1 Inferring a differentiable version from continuity of difference quotients

The following lemma shows, informally speaking, that if the difference quotient of a random field with continuous sample paths has a continuous version, then the random field has differentiable sample paths.

Lemma 4.7 (A continuous version of the difference quotient implies differentiability)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable real Hilbert spaces, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, let $O \subseteq H$ be an open subset, let $D \subseteq O$ be a countable dense subset, let T be a set, and let $\mathcal{X}\colon T \times O \times \Omega \to U$ and $\mathcal{Z}\colon T \times \cap_{h\in\mathbb{H}}\{(x,p) \in O \times \mathbb{R}\colon x + hp \in O\} \times \Omega \to \mathrm{HS}(H,U)$ be random fields such that for all $x \in D$, $p \in \mathbb{Q}\setminus\{0\}$, $h \in \mathbb{H}$ with the property that $x + ph \in O$ and for all $t \in T$ it holds $\mathbb{P}$-almost surely that

$$\mathcal{Z}_t(x,p)h = \tfrac{\mathcal{X}_t^{x+ph} - \mathcal{X}_t^x}{p} \tag{4.44}$$

and such that for $\mathbb{P}$-almost all $\omega \in \Omega$ and all $t \in T$ it holds that the function $O \ni x \mapsto \mathcal{X}_t^x(\omega) \in U$ and $\cap_{h \in \mathbb{H}} \{(y,q) \in O \times \mathbb{R} \colon y + hq \in O\} \ni (x,p) \mapsto \mathcal{Z}_t(x,p,\omega) \in \mathrm{HS}(H,U)$ are continuous. Then there is a set $\Omega_0 \in \mathcal{F}$ such that

(i) $\mathbb{P}(\Omega_0) = 1$, and

(ii) it holds for all $\omega \in \Omega_0$ and all $t \in T$ that the mapping $O \ni x \mapsto \mathcal{X}_t^x(\omega) \in U$ is continuously Fréchet-differentiable and it holds for all $x \in O$ that $\frac{d}{dx}\mathcal{X}_t^x(\omega) = \mathcal{Z}_t(x,0,\omega)$.

Proof of Lemma 4.7. We prove the infinite-dimensional version. Assume, that the orthonormal vector space base of $\mathbb{H}$ is the form $(h_i)_{i \in \mathbb{N}}$. Let $O^{\mathbb{R}} \subseteq H \times \mathbb{R}$ be the set $O^{\mathbb{R}} = \cap_{h \in \mathbb{H}} \{(x,p) \in O \times \mathbb{R} \colon x + hp \in O\}$. By assumption there exists a set $\Omega_1 \in \mathcal{F}$ satisfying $\mathbb{P}(\Omega_1) = 1$ and that for all $\omega \in \Omega_1$, $t \in T$ the functions $O \ni x \mapsto \mathcal{X}_t^x(\omega) \in U$ and $\cap_{h \in \mathbb{H}} \{(y,q) \in O \times \mathbb{R} \colon y + hq \in O\} \ni (x,p) \mapsto \mathcal{Z}_t(x,p,\omega) \in \mathrm{HS}(H,U)$ are continuous. Let $\Omega_0 \in \mathcal{F}$ be the set satisfying that

$$\Omega_0 = \Omega_1 \cap \bigcap_{(x,p) \in O^{\mathbb{R}} \cap (D \times \mathbb{Q})} \bigcap_{h \in \mathbb{H}} \big\{ \forall t \in T \colon \mathcal{X}_t^{x+ph} - \mathcal{X}^x = p\mathcal{Z}_t(x,p)h \big\}. \tag{4.45}$$

The fact that for all $(x,p) \in O^{\mathbb{R}} \cap (D \times \mathbb{Q})$, $h \in \mathbb{H}$, $t \in T$ it holds $\mathbb{P}$-almost surely that $\mathcal{Z}_t(x,p)h = \frac{\mathcal{X}_t^{x+ph} - \mathcal{X}_t^x}{p}$, the fact that $(O^{\mathbb{R}} \cap (D \times \mathbb{Q})) \times \mathbb{H}$ is a countable set, the fact that $\mathcal{X}, \mathcal{Z}$ are random fields, and the fact that $\mathbb{P}(\Omega_1) = 1$ imply that $\Omega_0 \in \mathcal{F}$ and that $\mathbb{P}(\Omega_0) = 1$. This proves item (i).

Next, we prove item (ii)**.** For the rest of the proof, let $\omega \in \Omega_0$, $t \in T$, $x \in O$, and $r \in (0,\infty)$ with the property that $\{y \in O \colon \|y - x\|_H < 2r\} \subseteq O$. The fact that $\omega \in \Omega_0$, the fact that the functions $O \ni y \mapsto \mathcal{X}_t^y(\omega) \in U$ and $\cap_{h \in \mathbb{H}} \{(z,q) \in O \times \mathbb{R} \colon z + hq \in O\} \ni (y,p) \mapsto \mathcal{Z}_t(y,p,\omega) \in \mathrm{HS}(H,U)$ are continuous, and the fact that $\{(y,p) \in O \times \mathbb{R} \colon \|y - x\|_H^2 + |p|^2 < r^2\} \cap (D \times \mathbb{Q})$ is dense in $\{(y,p) \in O \times \mathbb{R} \colon \|y - x\|_H^2 + |p|^2 < r^2\} \subseteq O^{\mathbb{R}}$ imply that for all $(y,p) \in O \times \mathbb{R}$, $h \in \mathbb{H}$ with $\|y - x\|_H^2 + |p|^2 < r^2$ it holds that

$$\mathcal{X}_t^{y+ph}(\omega) - \mathcal{X}_t^y(\omega) = p\mathcal{Z}_t(y,p,\omega)h. \tag{4.46}$$

For all $v \in H$, $i \in \mathbb{N}$ with the property that $\|v\|_H < r$ it holds that

$$\Big\| x + \sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j - x \Big\|_H^2 + |\langle v, h_i \rangle_H|^2 = \sum_{j=1}^{i} \left(\langle v, h_j \rangle_H\right)^2 \leq \|v\|_H^2 < r^2. \tag{4.47}$$

From equation (4.47), a telescoping sum argument and equation (4.46) it follows that for all

$v \in H$ with the property that $\|v\|_H < r$ it holds that

$$\begin{aligned}
&\mathcal{X}_t^{x+v}(\omega) - \mathcal{X}_t^{x}(\omega) - \mathcal{Z}_t(x,0,\omega)v \\
&= \sum_{i=1}^{\infty} \Big(\mathcal{X}_t^{x+\sum_{j=1}^{i}\langle v,h_j\rangle_H h_j}(\omega) - \mathcal{X}_t^{x+\sum_{j=1}^{i-1}\langle v,h_j\rangle_H h_j}(\omega) \Big) - \mathcal{Z}_t(x,0,\omega)\sum_{i=1}^{\infty}\langle v,h_i\rangle_H h_i \\
&= \sum_{i=1}^{\infty}\langle v,h_i\rangle_H \mathcal{Z}_t\Big(x+\textstyle\sum_{j=1}^{i-1}\langle v,h_j\rangle_H h_j, \langle v,h_i\rangle_H, \omega\Big)h_i - \displaystyle\sum_{i=1}^{\infty}\langle v,h_i\rangle_H \mathcal{Z}_t(x,0,\omega)h_i.
\end{aligned} \tag{4.48}$$

This, the triangle inequality, and the Cauchy-Schwarz inequality show that for all $v \in H$ with $\|v\|_H \in (0,r)$ it holds that

$$\begin{aligned}
&\tfrac{\left\|\mathcal{X}_t^{x+v}(\omega) - \mathcal{X}_t^{x}(\omega) - \mathcal{Z}_t(x,0,\omega)v\right\|_U}{\|v\|_H} \\
&= \tfrac{1}{\|v\|_H}\left\| \sum_{i=1}^{\infty}\langle v,h_i\rangle_H \Big(\mathcal{Z}_t\Big(x+\textstyle\sum_{j=1}^{i-1}\langle v,h_j\rangle_H h_j, \langle v,h_i\rangle_H, \omega\Big)h_i - \mathcal{Z}_t(x,0,\omega)h_i \Big) \right\|_U \\
&\leq \tfrac{1}{\|v\|_H}\left(\sum_{i=1}^{\infty} \left(\langle v,h_i\rangle_H\right)^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \left\| \mathcal{Z}_t\Big(x+\textstyle\sum_{j=1}^{i-1}\langle v,h_j\rangle_H h_j, \langle v,h_i\rangle_H, \omega\Big)h_i - \mathcal{Z}_t(x,0,\omega)h_i \right\|_U^2 \right)^{1/2} \\
&= \tfrac{\|v\|_H}{\|v\|_H}\left\| \mathcal{Z}_t\Big(x+\textstyle\sum_{j=1}^{i-1}\langle v,h_j\rangle_H h_j, \langle v,h_i\rangle_H, \omega\Big) - \mathcal{Z}_t(x,0,\omega) \right\|_{\mathrm{HS}(H,U)}.
\end{aligned} \tag{4.49}$$

This and continuity of the function $\cap_{h \in \mathbb{H}}\{(z,q) \in O \times \mathbb{R} \colon z + hq \in O\} \ni (y,p) \mapsto \mathcal{Z}(y,p,\omega) \in \mathrm{HS}(H,U)$ in the point $(x,0)$ yield that

$$\lim_{H \ni v \to 0} \tfrac{\left\|\mathcal{X}_t^{x+v}(\omega) - \mathcal{X}_t^{x}(\omega) - \mathcal{Z}_t(x,0,\omega)v\right\|_U}{\|v\|_H} = 0. \tag{4.50}$$

This together with continuity of the function $O \ni x \mapsto \mathcal{Z}_t(x,0,\omega) \in \mathrm{HS}(H,U)$ proves item (ii) and thus completes the proof of Lemma 4.7. □

The following proposition, Proposition 4.8, provides a method which allows to obtain a continuous version of difference quotients. A central assumption of Proposition 4.8 is that difference quotients are locally Hölder continuous with respect to the L^p-norm and sufficiently high p. Proposition 4.8 is a generalization of the Kolmogorov-Chentsov continuity theorem. In the case where $F = E$, $H = \mathbb{R}^d$ and inequality (4.51) below holds for $n = \infty$, the proof of Proposition 4.8 is provided in Theorem 2.1 in Mittmann and Steinwart 2003. In the case where inequality (4.51) below holds for $n = \infty$ (and $H = \mathbb{R}^d$), the proof of Proposition 4.8 is provided in Theorem 3.5 in Cox, Hutzenthaler, and Jentzen 2013.

Proposition 4.8 (Existence of a continuous version)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $d \in \mathbb{N}$, $p \in (d,\infty)$, $\alpha \in (\frac{d}{p}, 1]$, *let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be a d-dimensional real Hilbert space, let* $(E, \|\cdot\|_E)$ *be a Banach space, let* $F \subseteq E$ *be a closed subset, let* $D \subseteq H$ *be a non-empty set, for all* $n \in \mathbb{N}$ *let* $B_n = \{x \in H \colon \|x\|_H \leq n\}$, *and let* $X \colon D \times \Omega \to F$ *be a random field which satisfies for all*

$n \in \mathbb{N}$ that

$$\sup_{x \in D \cap B_n} \mathbb{E}[\|X(x)\|_E^p] + \sup_{\substack{x,y \in D \cap B_n \\ x \neq y}} \tfrac{\mathbb{E}[\|X(x)-X(y)\|_E^p]}{\|x-y\|_H^{\alpha p}} < \infty. \tag{4.51}$$

Then there is a $\mathcal{B}(\overline{D}) \otimes \mathcal{F}/\mathcal{B}(F)$-measurable function $\mathcal{X} \colon \overline{D} \times \Omega \to F$ such that for all $\omega \in \Omega$, $n \in \mathbb{N}$, $x \in D$ it holds that $\mathcal{X}|_{(\overline{D} \cap B_n) \times \Omega} \in \bigcap_{\beta \in (0, \alpha - \frac{d}{p})} \mathcal{L}^p(\mathbb{P}; C_b^\beta(\overline{D} \cap B_n, F))$, such that for all $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C(\overline{D}, F)$, and such that it holds $\mathbb{P}$-almost surely that $\mathcal{X}(x) = X(x)$.

Proof of Proposition 4.8. Throughout this proof and for all $n \in \mathbb{N}$, let $D_n \subseteq H$ be the set satisfying that $D_n = D \cap B_n$, let $\{h_1, \ldots, h_d\} = \mathbb{H} \subseteq H$ be an orthonormal basis of H, and let $\mathcal{D} \subseteq D$ be the set $\mathcal{D} = \{x \in H \colon (\langle x, h_i \rangle_H)_{i=1}^d \in \mathbb{Q}^d\}$.
By assumption it holds for all $n \in \mathbb{N}$ that $X|_{D_n} \in C_b^\alpha(D_n, \mathcal{L}^p(\mathbb{P}; F))$. Then Theorem 3.5 in Cox, Hutzenthaler, and Jentzen 2013 shows that for all $n \in \mathbb{N}$ there is an
$\mathcal{X}^n \in \bigcap_{\beta \in (0, \alpha - \frac{d}{p})} \mathcal{L}^p(\mathbb{P}; C_b^\beta(\overline{D_n}, F))$ such that for all $x \in D_n$ it holds $\mathbb{P}$-almost surely that $\mathcal{X}^n(x) = X(x)$. Let $\Omega_0 \subseteq \Omega$ be the set satisfying that $\Omega_0 = \cap_{n \in \mathbb{N}} \cap_{x \in D_n \cap \mathcal{D}} \cap_{m \in \mathbb{N} \cap [n, \infty)} \{\mathcal{X}^m(x) = X(x)\}$. Then this, the fact that X and $(\mathcal{X}^n)_{n \in \mathbb{N}}$ are random fields, and the fact that $\mathbb{N} \times \mathcal{D} \times \mathbb{N}$ is a countable set imply that $\Omega_0 \in \mathcal{F}$ and that $\mathbb{P}(\Omega_0) = 1$. Continuity yields that for all $\omega \in \Omega_0$, $n, m \in \mathbb{N}$ with $m \geq n$ it holds that $\mathcal{X}^m(\omega)|_{\overline{D_n}} = \mathcal{X}^n(\omega)$. Now let $\mathcal{X} \colon \overline{D} \times \Omega \to F$ be the function satisfying for all $x \in \overline{D}$, $\omega \in \Omega$ that $\mathcal{X}(x, \omega) = \mathbb{1}_{\Omega_0}(\omega) \lim_{n \to \infty} \mathcal{X}^n(x, \omega)$. Then it holds for all $n \in \mathbb{N}$ that $\mathcal{X}|_{\overline{D_n} \times \Omega} = \mathbb{1}_{\Omega_0} \mathcal{X}^n \in \bigcap_{\beta \in (0, \alpha - \frac{d}{p})} \mathcal{L}^p(\mathbb{P}; C_b^\beta(\overline{D_n}, F))$ and consequently that for all $x \in D$ it holds $\mathbb{P}$-almost surely that $\mathcal{X}(x) = X(x)$. The proof of Proposition 4.8 is thus completed. ☐

4.2.2 Exponential integrability estimates

In this subsection we collect two results from the literature which formalize a Lyapunov-method to derive (exponential) moment estimates. We will use these estimates to prove condition (4.51) for suitable difference quotients.
In this subsection we frequently use the following setting.

Setting 4.9. *Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be finite-dimensional real Hilbert spaces, assume that $H \neq \{0\}$, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t \in [0,T]}$, let $W \colon [0,T] \times \Omega \to U$ be a standard $(\Omega, \mathcal{F}, (\mathbb{F}_t)_{t \in [0,T]}, \mathbb{P})$-Brownian motion, let $O \subseteq H$ be an open set, let $\mu \colon O \to H$ and $\sigma \colon O \to \mathrm{HS}(U, H)$ be Borel measurable functions, and let $X \colon [0,T] \times \Omega \to O$ be an $(\mathbb{F}_t)_{t \in [0,T]}$-adapted stochastic process with continuous sample paths with the property that it holds $\mathbb{P}$-almost surely that $\int_0^T \|\mu(X_s)\|_H + \|\sigma(X_s)\|_{\mathrm{HS}(U,H)}^2 ds < \infty$ and that $X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s$.*

For the convenience of the reader, we recall the following well-known Lyapunov estimate consequence of e.g., Lemma 2.2 in Cox, Hutzenthaler, and Jentzen 2013 or the proof of Lemma 2.2 in Gyöngy & Krylov Gyöngy and Krylov 1996).

Lemma 4.10 (A Lyapunov estimate)**.** Assume Setting 4.9 and let $V \in C^2(O, [0, \infty))$ and a real number $\alpha \in [0, \infty)$ satisfy that for all $t \in [0, T]$ it holds $\mathbb{P}$-almost surely that $(\mathcal{G}_{\mu,\sigma}V)(X_t) \leq \alpha V(X_t)$. Then it holds that $\mathbb{E}\big[V(X_T)\big] \leq \exp(\alpha T)\,\mathbb{E}\big[V(X_0)\big]$.

For the convenience of the reader, we recall the following well-known exponential integrability bound from Corollary 2.4 in Cox, Hutzenthaler, and Jentzen 2013.

Lemma 4.11 (Exponential integrability bound)**.** Assume Setting 4.9, let $\alpha \in \mathbb{R}$, $U_0 \in C^2(O, \mathbb{R})$, and let $\overline{U}\colon [0,T] \times O \to \mathbb{R}$ be a Borel measurable function with the property that it holds $\mathbb{P}$-almost surely that $\int_0^T |\overline{U}(s, X_s)|\, ds < \infty$ and that for all $(t, x) \in \cup_{\omega\in\Omega} \cup_{s\in[0,T]} \{(s, X_s(\omega)) \in [0, T] \times O\}$ it holds that

$$(\mathcal{G}_{\mu,\sigma}U_0)(x) + \tfrac{1}{2\exp(\alpha t)} \left\|\sigma(x)^* \left(\nabla U_0\right)(x)\right\|_U^2 + \overline{U}(t, x) \leq \alpha U_0(x). \tag{4.52}$$

Then

$$\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_T)}{\exp(\alpha T)} + \int\limits_0^T \tfrac{\overline{U}(s,X_s)}{\exp(\alpha s)}\, ds\Big)\Big] \leq \mathbb{E}\big[\exp\big(U_0(X_0)\big)\big] \in [0, \infty]. \tag{4.53}$$

The next result, Corollary 4.12, states Lemma 4.11 for positive initial times.

Corollary 4.12 (Exponential integrability estimates)**.** Assume Setting 4.9, let $\alpha \in \mathbb{R}$, $U_0 \in C^2(O, \mathbb{R})$, $s \in [0, T]$, and let $\overline{U}\colon [s, T] \times O \to \mathbb{R}$ be a Borel measurable function with the properties that it holds $\mathbb{P}$-almost surely that $\int_s^T |\overline{U}(r, X_r)|\, dr < \infty$ and that for all $(t, x) \in \cup_{\omega\in\Omega} \cup_{r\in[s,T]}$ $\{(r, X_r(\omega)) \in [s, T] \times O\}$ it holds that

$$(\mathcal{G}_{\mu,\sigma}U_0)(x) + \tfrac{1}{2\exp(\alpha t)} \left\|\sigma(x)^* \left(\nabla U_0\right)(x)\right\|_U^2 + \overline{U}(t, x) \leq \alpha U_0(x). \tag{4.54}$$

Then

$$\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_T)}{\exp(\alpha T)} + \int\limits_s^T \tfrac{\overline{U}(r,X_r)}{\exp(\alpha r)}\, dr\Big)\Big] \leq \mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_s)}{\exp(\alpha s)}\Big)\Big] \in [0, \infty]; \tag{4.55}$$

moreover, if there exists a $K \in \mathbb{R}$ such that $K = \inf_{t\in[s,T]} \inf_{\omega\in\Omega} \overline{U}(t, X_t(\omega))$, then

$$\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_T)}{\exp(\alpha T)}\Big)\Big] \leq \exp\Big(\int\limits_s^T \tfrac{-K}{\exp(\alpha r)} dr\Big)\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_s)}{\exp(\alpha s)}\Big)\Big] \in [0, \infty]. \tag{4.56}$$

Proof of Corollary 4.12. Lemma 4.11 (applied with $T = T - s$, $(\mathbb{F}_t)_{t\in[0,T-s]} = (\mathbb{F}_{t+s})_{t\in[0,T-s]}$, $(W_t)_{t\in[0,T-s]} = (W_{s+t} - W_s)_{t\in[0,T-s]}$, $(X_t)_{t\in[0,T-s]} = (X_{t+s})_{t\in[0,T-s]}$,
$U_0 = (O \ni x \mapsto \exp(-\alpha s)U_0(x) \in [0, \infty))$,
$\overline{U} = \big([0, T - s] \times O \ni (t, x) \mapsto \exp(-\alpha s)\overline{U}(t + s, x) \in \mathbb{R}\big)$ in the notation of Lemma 4.11) implies inequality (4.55). Deriving inequality (4.56) from inequality (4.55) is straightforward. The proof of Corollary 4.12 is thus completed. ☐

The following result, Lemma 4.13, generalizes Cox, Hutzenthaler, and Jentzen 2013, Lemma 2.23 with $k = 1$ which is a special case of Lemma 4.13 with $H = \mathbb{R}^d$, $s = 0$, $\overline{U}(t, \cdot) = \overline{U}(\cdot)$ for all $t \in [0, T]$, $X^{(1)} = X^x$, $X^{(3)} = X^x$, $X^{(2)} = X^y$, and $X^{(4)} = X^y$ for $d \in \mathbb{N}$, $x, y \in O$.

Lemma 4.13. Assume Setting 4.9, let $s \in [0, T]$ let $X^{(j)} \colon [s, T] \times \Omega \to O$, $j \in \{1, \dots, 4\}$ be $(\mathbb{F}_t)_{t \in [s,T]}$-adapted stochastic processes with continuous sample paths satisfying for all $j \in \{1, \dots, 4\}$, $t \in [s, T]$ that it holds $\mathbb{P}$-almost surely that $\int_s^T \|\mu(X_r^{(j)})\|_H + \|\sigma(X_r^{(j)})\|^2_{\mathrm{HS}(U,H)}\, dr < \infty$ and that $X_t^{(j)} = X_s^{(j)} + \int_s^t \mu(X_r^{(j)})\, dr + \int_s^t \sigma(X_r^{(j)})\, dW_r$, let α_0, α_1, β_0, $\beta_1 \in \mathbb{R}$, let U_0, $U_1 \in C^2(O, [0, \infty))$, let a Borel measurable function $\overline{U} \colon [s, T] \times O \to \mathbb{R}$ satisfy for all $j \in \{1, \dots, 4\}$ that it holds $\mathbb{P}$-almost surely that $\int_s^T |\overline{U}(r, X_r^{(j)})|\, dr < \infty$ and that for all $i \in \{0, 1\}$, $(t, x) \in \cup_{\omega \in \Omega} \cup_{r \in [s,T]} \cup_{j=1}^4 \{(r, X_r^{(j)}(\omega)) \in [s, T] \times O\}$ it holds that

$$(\mathcal{G}_{\mu,\sigma} U_i)(x) + \tfrac{1}{2 \exp(\alpha_i t)} \|\sigma(x)^* (\nabla U_i)(x)\|_U^2 + \mathbb{1}_{\{1\}}(i) \cdot \overline{U}(t, x) \le \alpha_i U_i(x) + \beta_i, \tag{4.57}$$

and let p, p_0, $p_1 \in (0, \infty]$ satisfy $\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p}$, let $\phi \colon [s, T] \to \mathbb{R}$ be a Borel measurable function which satisfies that it holds $\mathbb{P}$-almost surely that $\int_s^T \max(\phi(r), 0)\, dr < \infty$. Then it holds that

$$\begin{aligned} &\Big\| \exp\Big(\int\limits_s^T \phi(r) + \sum_{j=1}^4 \Big(\tfrac{U_0(X_r^{(j)})}{4 p_0 (T-s) \exp(\alpha_0 r)} + \tfrac{\overline{U}(r, X_r^{(j)})}{4 p_1 \exp(\alpha_1 r)} \Big)\, dr \Big) \Big\|_{L^p(\mathbb{P};\mathbb{R})} \\ &\le \exp\Big(\int\limits_s^T \phi(r) + \tfrac{\beta_0\, (1 - \frac{r-s}{T-s})}{p_0 \exp(\alpha_0 r)} + \tfrac{\beta_1}{p_1 \exp(\alpha_1 r)} dr \Big) \prod_{i=0}^1 \prod_{j=1}^4 \Big\| \exp\Big(\tfrac{U_i(X_s^{(j)})}{4 p_i \exp(\alpha_i s)} \Big) \Big\|_{L^{4p_i}(\mathbb{P};\mathbb{R})}. \end{aligned} \tag{4.58}$$

Proof of Lemma 4.13. Without loss of generality we assume for the rest of the proof that $\phi \equiv 0$, otherwise divide by $\exp(\int_s^T \phi(r)\, dr) \in (0, \infty)$. Hölder's inequality together with $\frac{1}{p} = 4 \frac{1}{4p_0} + 4 \frac{1}{4p_1}$, Jensen's inequality, nonnegativity of U_1, Corollary 4.12 (applied with $T = t$, $\overline{U} = -\beta_0$, $\alpha = \alpha_0$, $X = X^{(j)}$ for $j \in \{1, \dots, 4\}$ and $t \in [s, T]$ and applied with $U_0 = U_1$, $\overline{U} = \overline{U} - \beta_1$, $\alpha = \alpha_1$, $X = X^{(j)}$ for $j \in \{1, \dots, 4\}$ in the notation of Corollary 4.12), and the fact that

$\int_s^T \int_s^r \exp(-\alpha_0 u)\,du\,dr = \int_s^T (T-r)\exp(-\alpha_0 r)\,dr$ show that

$$\begin{aligned}
&\Big\| \exp\Big(\int_s^T \sum_{j=1}^4 \Big(\tfrac{U_0(X_r^{(j)})}{4p_0(T-s)\exp(\alpha_0 r)} + \tfrac{\overline{U}(r, X_r^{(j)})}{4p_1\exp(\alpha_1 r)} \Big) dr \Big) \Big\|_{L^p(\mathbb{P};\mathbb{R})} \exp\Big(-\int_s^T\int_s^r \tfrac{\beta_0}{p_0(T-s)\exp(\alpha_0 u)}\,du\,dr \Big) \\
&\le \prod_{j=1}^4 \Big[\Big\| \exp\Big(\int_s^T \tfrac{U_0(X_r^{(j)})}{4p_0(T-s)\exp(\alpha_0 r)} - \int_s^r \tfrac{\beta_0}{4p_0(T-s)\exp(\alpha_0 u)}\,du\,dr \Big) \Big\|_{L^{4p_0}(\mathbb{P};\mathbb{R})} \\
&\qquad\qquad \cdot \Big\| \exp\Big(\int_s^T \tfrac{\overline{U}(r, X_r^{(j)})}{4p_1\exp(\alpha_1 r)}\,dr \Big) \Big\|_{L^{4p_1}(\mathbb{P};\mathbb{R})} \Big] \\
&\le \prod_{j=1}^4 \Big(\sup_{t\in[s,T]} \Big(\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_t^{(j)})}{\exp(\alpha_0 t)} - \int_s^t \tfrac{\beta_0}{\exp(\alpha_0 u)}\,du \Big) \Big] \Big)^{1/4p_0} \\
&\qquad \cdot \Big(\mathbb{E}\Big[\exp\Big(\tfrac{U_1(X_T^{(j)})}{\exp(-\alpha_1 T)} + \int_s^T \tfrac{\overline{U}(r, X_r^{(j)})}{\exp(\alpha_1 r)}\,dr \Big) \Big] \Big)^{1/4p_1} \Big) \\
&\le \prod_{j=1}^4 \Big(\Big(\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X_s^{(j)})}{\exp(\alpha_0 s)} \Big) \Big] \Big)^{1/4p_0} \Big(\mathbb{E}\Big[\exp\Big(\tfrac{U_1(X_s^{(j)})}{\exp(\alpha_1 s)} + \int_s^T \tfrac{\beta_1}{\exp(\alpha_1 r)}\,dr \Big) \Big] \Big)^{1/4p_1} \Big) \\
&= \exp\Big(\int_s^T \tfrac{\beta_0\,(1-\frac{r-s}{T-s})}{p_0\exp(\alpha_0 r)} + \tfrac{\beta_1}{p_1\exp(\alpha_1 r)}\,dr \Big) \exp\Big(-\int_s^T\int_s^r \tfrac{\beta_0}{p_0(T-s)\exp(\alpha_0 u)}\,du\,dr \Big) \\
&\qquad \cdot \prod_{i=0}^1 \prod_{j=1}^4 \Big\| \exp\Big(\tfrac{U_i(X_s^{(j)})}{4p_i\exp(\alpha_i s)} \Big) \Big\|_{L^{4p_i}(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{4.59}$$

This implies inequality (4.64). The proof of Lemma 4.13 is thus completed. □

4.3 Strong completeness for SDEs

Setting 4.14. *Let $T \in (0,\infty)$, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be finite-dimensional real Hilbert spaces, assume $H \neq \{0\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0,T]\times\Omega \to U$ be a standard $(\Omega, \mathcal{F}, (\mathbb{F}_t)_{t\in[0,T]}, \mathbb{P})$-Brownian motion, let $D, O \subseteq H$ be non -empty open sets, let $B \subseteq H$ be a Borel measurable set with the property that $D \subseteq B \subseteq O$, let $\mu \in C(O,H)$ $\sigma \in C(O, \mathrm{HS}(U,H))$ be locally Lipschitz continuous, for all $s \in [0,T]$, $x \in B$, let $X_{s,\cdot}^x\colon [s,T]\times\Omega \to B$ be an $(\mathbb{F}_t)_{t\in[s,T]}$-adapted stochastic process with continuous sample paths and with the property that for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that*

$$X_{s,t}^x = x + \int_s^t \mu(X_{s,r}^x)\,dr + \int_s^t \sigma(X_{s,r}^x)\,dW_r, \tag{4.60}$$

let α_0, $\alpha_1 \in [0,\infty)$, β_0, $\beta_1 \in \mathbb{R}$, U_0, $U_1 \in C^2(O,[0,\infty))$, let $\overline{U}\colon [0,T]\times B \to \mathbb{R}$ be a Borel measurable function satisfying that for all $s \in [0,T]$, $x \in B$ it holds $\mathbb{P}$-almost surely that $\int_s^T |\overline{U}(r, X_{s,r}^x)|dr < \infty$ and that for all $i \in \{0,1\}$, $s \in [0,T]$, $t \in [s,T]$, $x \in B$ it holds that

$$(\mathcal{G}_{\mu,\sigma}U_i)(x) + \tfrac{1}{2\exp(\alpha_i t)}\|\sigma(x)^*(\nabla U_i)(x)\|_U^2 + \mathbb{1}_{\{1\}}(i)\cdot\overline{U}(t,x) \le \alpha_i U_i(x) + \beta_i, \tag{4.61}$$

let $\phi\colon [0,T] \to \mathbb{R}$ *be a Borel measurable function satisfying that* $\int_0^T \max\{0, \phi(r)\}\, dr < \infty$*, let* $q \in [1,\infty)$*,* $c \in (1,\infty)$*,* $q_0, q_1 \in [1,\infty)$ *satisfy that* $\frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{q}$*, and assume that for all* $t \in [0,T]$*,* $x, y \in B$ *it holds that*

$$\begin{aligned}&\big\langle x-y, \mu(x)-\mu(y)\big\rangle_H + \tfrac{1}{2}\big\|\sigma(x)-\sigma(y)\big\|^2_{\mathrm{HS}(U,H)}\\&+\tfrac{1}{2}\Big(\tfrac{cq}{c-1} - \min\{2, \tfrac{(c-1)q}{c}\}\Big)\Big\|\big\langle x-y, \sigma(x)-\sigma(y)\big\rangle_H\Big\|_{\mathrm{HS}(U,\mathbb{R})}\\&\leq \|x-y\|^2_H \cdot \Big(\phi(t) + \tfrac{U_0(x)+U_0(y)}{2q_0(T-t)\exp(\alpha_0 t)} + \tfrac{\bar{U}(x)+\bar{U}(y)}{2q_1(T-t)\exp(\alpha_1 t)}\Big).\end{aligned} \tag{4.62}$$

The next lemma follows from pathwise uniqueness which is guaranteed by the fact that drift and diffusion coefficients are locally Lipschitz (see e.g. Theorem 2.5 in Section 5.2.B in Karatzas and Shreve 2012).

Lemma 4.15 (Flow property)**.** Assume Setting 4.14. Then for all $(s_1, s_2) \in \Delta_T$, $t \in [s_2, T]$, $x \in B$ it holds $\mathbb{P}$-almost surely that $X^x_{s_1,t} = X^{X^x_{s_1,s_2}}_{s_2,t}$.

Lemma 4.16. Assume Setting 4.14, let $(s_1, s_2) \in \Delta_T$, $x_1, x_2 \in B$, let $p, p_0, p_1 \in (0,\infty]$ satisfy $\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p}$, and assume that there is a $K \in \mathbb{R}$ such that

$$K = \inf_{t\in(s_1,s_2)} \inf_{\omega\in\Omega} \inf_{z\in B} \overline{U}(t, X^z_{s_1,t}(\omega)). \tag{4.63}$$

Then it holds that

$$\begin{aligned}&\Bigg\|\exp\Bigg(\int\limits_{s_2}^{T} \phi(r) + \sum_{j=1}^{2}\Big(\tfrac{U_0(X^{x_j}_{s_1,r})}{4p_0(T-s_2)\exp(\alpha_0 r)} + \tfrac{\overline{U}(r, X^{x_j}_{s_1,r})}{4p_1\exp(\alpha_1 r)}\Big)dr\Bigg)\Bigg\|_{L^p(\mathbb{P};\mathbb{R})}\\&\leq \exp\Bigg(\int\limits_{s_2}^{T} \phi(r) + \tfrac{\beta_0\,(1-\frac{r-s_2}{T-s_2})}{p_0\exp(\alpha_0 r)} + \tfrac{\beta_1}{p_1\exp(\alpha_1 r)}dr + \int\limits_{s_1}^{s_2} \tfrac{\beta_0}{2p_0\exp(\alpha_0 r)} + \tfrac{\beta_1-K}{2p_1\exp(\alpha_1 r)}\,dr + \sum_{i=0}^{1}\tfrac{U_i(x_1)+U_i(x_2)}{2p_i\exp(\alpha_i s_1)}\Bigg).\end{aligned} \tag{4.64}$$

Proof of Lemma 4.16. Lemma 4.15 and Lemma 4.13 (applied with $s = s_2$, $X^{(1)}_\cdot = X^{X^{x_1}_{s_1,s_2}}_{s_2,\cdot}$, $X^{(2)}_\cdot = X^{X^{x_2}_{s_1,s_2}}_{s_2,\cdot}$, $X^{(3)}_\cdot = X^{X^{x_1}_{s_1,s_2}}_{s_2,\cdot}$, $X^{(2)}_\cdot = X^{X^{x_2}_{s_1,s_2}}_{s_2,\cdot}$ in the notation of Lemma 4.13) yield

$$\begin{aligned}&\Bigg\|\exp\Bigg(\int\limits_{s_2}^{T} \phi(r) + \sum_{j=1}^{2}\Big(\tfrac{U_0(X^{x_j}_{s_1,r})}{4p_0(T-s_2)\exp(\alpha_0 r)} + \tfrac{\overline{U}(r, X^{x_j}_{s_1,r})}{4p_1\exp(\alpha_1 r)}\Big]dr\Bigg)\Bigg\|_{L^p(\mathbb{P};\mathbb{R})}\\&\leq \exp\Bigg(\int\limits_{s_2}^{T} \phi(r) + \tfrac{\beta_0\,(1-\frac{r-s_2}{T-s_2})}{p_0\exp(\alpha_0 r)} + \tfrac{\beta_1}{p_1\exp(\alpha_1 r)}\,dr\Bigg)\prod_{i=0}^{1}\prod_{j=1}^{2}\mathbb{E}\Big[\exp\Big(\tfrac{U_i(X^{x_j}_{s_1,s_2})}{\exp(\alpha_i s_2)}\Big)\Big]^{1/2p_i}.\end{aligned} \tag{4.65}$$

Applying Corollary 4.12 (with $T = s_2$, $s = s_1$, $X = X^{x_2}_{s_1,\cdot}$, $U_0 = U_0$ (resp. $U_0 = U_1$), $\overline{U} = -\beta_0$ (resp. $\overline{U} = \overline{U} - \beta_1$), $\alpha = \alpha_0$ (resp. $\alpha = \alpha_1$) in the notation of Corollary 4.12) completes the proof of Lemma 4.16. $\square$

Lemma 4.17 (Strong local Lipschitz continuity in the initial value)**.** Assume Setting 4.14, let

$s \in [0,T]$, $x, y \in B$. Then it holds that

$$\begin{aligned}&\Big\| \sup_{t\in[s,T]} \|X^x_{s,t} - X^y_{s,t}\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\ &\leq \|x-y\|_H \exp\left(\tfrac{1}{q}\right) \exp\left(\int_s^T \phi(r) + \tfrac{\beta_0\,(1-\frac{r-s}{T-s})}{q_0\exp(\alpha_0 r)} + \tfrac{\beta_1}{q_1\exp(\alpha_1 r)}\,dr + \sum_{i=0}^{1} \tfrac{U_i(x)+U_i(y)}{2q_i\exp(\alpha_i s)} \right).\end{aligned} \tag{4.66}$$

Proof of Lemma 4.17. It follows from equation (4.60) that for all $t \in [0, T-s]$ it holds $\mathbb{P}$-almost surely that

$$X^x_{s,s+t} - X^y_{s,s+t} = x - y + \int_0^t \mu(X^x_{s,s+r}) - \mu(X^y_{s,s+r})\,dr + \int_0^t \sigma(X^x_{s,s+r}) - \sigma(X^y_{s,s+r})\,dW_r. \tag{4.67}$$

Then item (i) in Proposition 4.6 (applied with $X_t = X^x_{s,s+t} - X^y_{s,s+t}$, $\mu_t = \mu(X^x_{s,s+t}) - \mu(X^y_{s,s+t})$, $\alpha_t = \frac{\langle X^x_{s,s+t}-X^y_{s,s+t}, \mu(X^x_{s,s+t})-\mu(X^y_{s,s+t})\rangle_H}{\|X^x_{s,s+t}-X^y_{s,s+t}\|^2_H}$, $\beta_t = 0$, $\eta_t = \sigma(X^x_{s,s+t}) - \sigma(X^y_{s,s+t})$, $\zeta_t = 0$, $p = 1$, $u = 1$, $u' = 2$, $p' = 2$, $c_1 = c$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $\theta = \frac{c}{c-1}$, $\delta = 0$ for all $t \in [0, T-s]$, $c \in (1,\infty)$ in the notation of Proposition 4.6), the fact that $c \leq \exp(c-1)$, inequality (4.62), and Lemma 4.16 imply that

$$\begin{aligned}&\Big\| \sup_{t\in[s,T]} \|X^x_{s,s+t} - X^y_{s,s+t}\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \leq \|x-y\|_H c^{\frac{c}{(c-1)q}} \\ &\cdot \Bigg\| \exp\Bigg(\int_s^T \max\left\{ \tfrac{\langle X^x_{s,s+r}-X^y_{s,s+r}, \mu(X^x_{s,s+r})-\mu(X^y_{s,s+r})\rangle_H}{\|X^x_{s,s+r}-X^y_{s,s+r}\|^2_H} + \tfrac{1}{2}\tfrac{\|\sigma(X^x_{s,s+t})-\sigma(X^y_{s,s+t})\|^2_{HS(U,H)}}{\|X^x_{s,s+r}-X^y_{s,s+r}\|^2_H}, 0 \right\} \\ &\qquad + \tfrac{\frac{cq}{c-1} - \frac{(c-1)q}{c}}{2} \tfrac{\|\langle X^x_{s,s+r}-X^y_{s,s+r}, \sigma(X^x_{s,s+t})-\sigma(X^y_{s,s+t})\rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})}}{\|X^x_{s,s+r}-X^y_{s,s+r}\|^4_H}\,dr \Bigg) \Bigg\|_{L^{qc}(\mathbb{P};\mathbb{R})} \\ &\leq \|x-y\|_H \exp\left(\tfrac{1}{q}\right) \Bigg\| \exp\Bigg(\int_s^T \phi(r) + \sum_{j=1}^{2} \Big(\tfrac{U_0(X^x_{s,r})+U_0(X^y_{s,r})}{4q_0(T-s)\exp(\alpha_0 r)} + \tfrac{\overline{U}(r,X^x_{s,r})+\overline{U}(r,X^y_{s,r})}{4q_1\exp(\alpha_1 r)} \Big) dr \Bigg) \Bigg\|_{L^q(\mathbb{P};\mathbb{R})} \\ &\leq \|x-y\|_H \exp\left(\tfrac{1}{q}\right) \exp\left(\int_s^T \phi(r) + \tfrac{\beta_0\,(1-\frac{r-s}{T-s})}{q_0\exp(\alpha_0 r)} + \tfrac{\beta_1}{q_1\exp(\alpha_1 r)}\,dr + \sum_{i=0}^{1} \tfrac{U_i(x)+U_i(y)}{2q_i\exp(\alpha_i s)} \right).\end{aligned} \tag{4.68}$$

The proof of Lemma 4.17 is thus completed. □

Lemma 4.18 (Moment estimate)**.** Assume Setting 4.14, let $\alpha \in [0,\infty)$, $p \in [2,\infty)$, let $V \in C^2(O,[0,\infty))$ satisfy for all $x \in B$ that $(\mathcal{G}_{\mu,\sigma}V)(x) \leq \alpha V(x)$ and

$$\max\left\{ \|\mu(x)\|^p_H, \|\sigma(x)\|^p_{\mathrm{HS}(U,H)} \right\} \leq V(x), \tag{4.69}$$

and let $s \in [0,T]$, $x_1, x_2 \in B$. Then it holds that

$$\Big\| \sup_{t\in[0,T]} \|X^{x_1}_{s,(t+s)\wedge T} - x_2\|_H \Big\|_{L^p(\mathbb{P};\mathbb{R})} \leq \|x_1 - x_2\|_H + \big(\exp(\alpha T)V(x)\big)^{\frac{1}{p}} \big(\sqrt{T} + p\big)\sqrt{T-s}. \tag{4.70}$$

Proof of Lemma 4.18. Note that equation (4.60), the triangle inequality, the Burkholder-Davis-

Gundy type inequality and Lemmas 7.2 and 7.7 in Da Prato and Zabczyk 1992 ensure that for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^{x_1}_{s,(t+s)\wedge T} - x_2\|_H \Big\|_{L^p(\mathbb{P};\mathbb{R})} \\
&\leq \|x_1 - x_2\|_H + \Big\| \sup_{t\in[0,T]} \int_s^t \big\|\mu(X^{x_1}_{s,r})\big\|_H \, dr \Big\|_{L^p(\mathbb{P};\mathbb{R})} + \Big\| \sup_{t\in[0,T]} \Big\| \int_s^t \sigma(X^{x_1}_{s,r})\, dW_r \Big\|_H \Big\|_{L^p(\mathbb{P};\mathbb{R})} \\
&\leq \|x_1 - x_2\|_H + \int_s^T \|\mu(X^{x_1}_{s,r})\|_{L^p(\mathbb{P};H)} dr + p\Big(\int_s^T \|\sigma(X^{x_1}_{s,r})\|^2_{L^p(\mathbb{P};\mathbb{R})} \, dr \Big)^{\frac{1}{2}} \\
&\leq \|x_1 - x_2\|_H + \int_s^T \big(\mathbb{E}\big[V(X^{x_1}_{s,r})\big]\big)^{\frac{1}{p}} \, dr + p\Big(\int_s^T \big(\mathbb{E}\big[V(X^{x_1}_{s,r})\big]\big)^{\frac{2}{p}} \, dr \Big)^{\frac{1}{2}} \\
&\leq \|x_1 - x_2\|_H + \big(\exp(\alpha T)V(x)\big)^{\frac{1}{p}} \big(\sqrt{T} + p\big)\sqrt{T-s}.
\end{aligned} \tag{4.71}$$

The proof of Lemma 4.18 is thus completed. □

The following lemma, Lemma 4.19, is well-known and is included for the convenience of the reader.

Lemma 4.19 (Temporal regularity)**.** Assume Setting 4.14, let $\alpha \in [0,\infty)$, $p \in [2,\infty)$, let $V \in C^2(O,[0,\infty))$ satisfy for all $x \in B$ that $(\mathcal{G}_{\mu,\sigma}V)(x) \leq \alpha V(x)$ and

$$\max\big\{\|\mu(x)\|^p_H, \|\sigma(x)\|^p_{\mathrm{HS}(U,H)}\big\} \leq V(x), \tag{4.72}$$

and let $s_1, s_2 \in [0,T]$, $x \in B$ satisfy $s_1 \leq s_2$. Then it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^x_{s_1,(t+s_1)\wedge T} - X^x_{s_1,(t+s_2)\wedge T}\|_H \Big\|_{L^p(\mathbb{P};\mathbb{R})} \\
&\qquad\qquad \leq \|x_1 - x_2\|_H + \big(\exp(\alpha T)V(x)\big)^{\frac{1}{p}} \big(\sqrt{T} + p\big)\sqrt{T-s}.
\end{aligned} \tag{4.73}$$

Proof of Lemma 4.19. Note that equation (4.60), the triangle inequality, the Burkholder-Davis-Gundy type inequality and Lemmas 7.2 and 7.7 in Da Prato and Zabczyk 1992 ensure that for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^x_{s_1,(t+s_1)\wedge T} - X^x_{s_1,(t+s_2)\wedge T}\|_H \Big\|_{L^p(\mathbb{P};\mathbb{R})} \\
&\leq \Big\| \sup_{t\in[0,T]} \int_s^t \big\|\mu(X^{x_1}_{s_1,r})\big\|_H \, dr \Big\|_{L^p(\mathbb{P};\mathbb{R})} + \Big\| \sup_{t\in[0,T]} \Big\| \int_s^t \sigma(X^{x_1}_{s_1,r})\, dW_r \Big\|_H \Big\|_{L^p(\mathbb{P};\mathbb{R})} \\
&\leq \int_s^T \big\|\mu(X^{x_1}_{s_1,r})\big\|_{L^p(\mathbb{P};H)} \, dr + p\Big(\int_s^T \big\|\sigma(X^{x_1}_{s_1,r})\big\|^2_{L^p(\mathbb{P};\mathbb{R})} \, dr \Big)^{\frac{1}{2}} \\
&\leq \int_s^T \Big(\mathbb{E}\big[V(X^{x_1}_{s_1,r})\big]\Big)^{\frac{1}{p}} dr + p\Big(\int_s^T \big(\mathbb{E}\big[V(X^{x_1}_{s_1,r})\big]\big)^{\frac{2}{p}} \, dr \Big)^{\frac{1}{2}} \\
&\leq \Big(\exp(\alpha T)V(x)\Big)^{\frac{1}{p}} \big(\sqrt{T} + p\big)\sqrt{T-s}.
\end{aligned} \tag{4.74}$$

The proof of Lemma 4.19 is thus completed. □

Lemma 4.20 (Strong local Hölder estimate)**.** Assume Setting 4.14, let $\alpha \in [0,\infty)$, let $V \in C^2(O,[0,\infty))$ satisfy for all $x \in B$ that $(\mathcal{G}_{\mu,\sigma}V)(x) \leq \alpha V(x)$ and

$$\max\Big\{\|\mu(x)\|_H^{\max\{q,2\}}, \|\sigma(x)\|_{\mathrm{HS}(U,H)}^{\max\{q,2\}}\Big\} \leq V(x), \tag{4.75}$$

and let $s \in [0,T]$, $x_1, x_2 \in B$. Then it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^{x_2}_{s_2,(t+s_2)\wedge T} - X^{x_1}_{s_1,(t+s_1)\wedge T}\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq c^{\frac{c}{q(c-1)}} \exp\left(\int\limits_{s_2}^{T} \phi(r)\,dr + \sum_{i=0}^{1} \int\limits_{s_2}^{T} \frac{\beta_i\Big(1-\frac{r-s_2}{(T-s_2)}\Big)^{1-i}}{q_i \exp(\alpha i r)}\,dr \right) \cdot 3^{\frac{2cq}{2c-1}} \exp(s_2-s_1) \exp(\tfrac{(2c-1)}{2qc}) \\
&\quad \cdot \Big[\|x_2-x_1\|_H^{1/2} (2k)^{1/2} + 2\Big(V(x_1)\exp\big(\alpha(s_2-s_1)\big)\Big)^{(2c-1)/(2u(2qc))} (s_2-s_1)^{1/2} \Big] \\
&\quad \cdot \exp\left(\frac{U_0(x_1)}{2q_0\exp(\alpha_0 s_1)} + \frac{\beta_0}{2q_0} \int\limits_{s_1}^{s_2} \exp(-\alpha_0 r)\,dr + \frac{U_1(x_1)}{2q_1\exp(\alpha_1 s_1)} + \frac{1}{2q_1} \int\limits_{s_1}^{s_2} \frac{\beta_1 - K}{\exp(\alpha_1 r)}\,dr \right) \\
&\quad + 6q\exp(\tfrac{1}{q}) \exp\big((1+\alpha)(s_2-s_1)\big) (s_2-s_1)^{1/2} \Big(\exp(\alpha T) V(x_1) \Big)^{(2c-1)/(4uqc)}.
\end{aligned} \tag{4.76}$$

Proof of Lemma 4.20. The triangle inequality, Lemma 4.17 (applied with $T = (t+s_2)\wedge T$, $\mathcal{F} = \mathbb{F}_{s_2}$, $\mathbb{P} = \mathbb{P}|_{\mathbb{F}_{s_2}}$, $x = X^{x_2}_{s_1,s_2}$, $y = 1$, $v = x_1 - X^{x_2}_{s_1,s_2}$, $s = s_2$, for $(s_1,s_2) \in \Delta_T$, $x_1, x_2 \in B$ in the notation of Lemma 4.26), and Lemma 4.18 (applied with $T = (t+s_2)\wedge T$, $\mathcal{F} = \mathbb{F}_{(t+s_1)\wedge T}$, $\mathbb{P} = \mathbb{P}|_{\mathbb{F}_{(t+s_1)\wedge T}}$, $x_1 = X^{x_2}_{s_1,(t+s_1)\wedge T}$, $x_2 = X^{x_2}_{s_1,(t+s_1)\wedge T}$, $s = (t+s_1)\wedge T$, $p = \frac{2c}{2c-1}$ for $x_2 \in B$, $(s_1,s_2) \in \Delta_T$, $t \in [0, T-s_1)$ in the notation of Lemma 4.18) show that for all $x_1, x_2 \in B$, $(s_1,s_2) \in \Delta_T$ it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \big\|X^{x_2}_{s_2,(t+s_2)\wedge T} - X^{x_1}_{s_1,(t+s_1)\wedge T}\big\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq \Big(\mathbb{E}\Big[\mathbb{E}\Big[\sup_{t\in[s_2,T]} \big\|X^{x_2}_{s_2,t} - X^{X^{x_1}_{s_1,s_2}}_{s_2,t}\big\|_H^q \Big| \mathbb{F}_{s_2} \Big]\Big] \Big)^{1/q} \\
&\quad + \Big(\mathbb{E}\Big[\mathbb{E}\Big[\sup_{t\in[0,T-s_1]} \|X^{x_1}_{s_1,(t+s_2)\wedge T} - X^{x_1}_{s_1,t+s_1}\|_H^{\max\{q,2\}} \Big| \mathbb{F}_{t+s_1} \Big]\Big] \Big)^{1/\max\{q,2\}} \\
&\leq \exp\big(\tfrac{1}{q}\big) \exp\Big(\int\limits_{s_2}^{T} \phi(r) + \sum_{i=0}^{1} \frac{\beta_i\Big(1-\frac{r-s_2}{(T-s_2)}\Big)^{1-i}}{q_i\exp(\alpha i r)}\,dr + \sum_{i=0}^{1} \frac{U_i(x_2)}{2q_i\exp(\alpha_i s_2)} \Big) \\
&\quad \cdot \Big(\mathbb{E}\Big[\|x_2 - X^{x_1}_{s_1,s_2}\|_H^q \exp\Big(q \sum_{i=0}^{1} \frac{U_i(X^{x_1}_{s_1,s_2})}{2q_i\exp(\alpha_i s_2)} \Big) \Big] \Big)^{1/q} \\
&\quad + 6q\exp(\tfrac{1}{q}) \sup_{t\in[0,T-s_1)} \left(\frac{((t+s_2)\wedge T - (t+s_1))^{1/2}}{\exp\big(-\alpha((t+s_2)\wedge T-(t+s_1))(1+{(2c-1)}/{(4uqc)})\big)} \Big(\mathbb{E}\Big[\big(V(X^{x_1}_{s_1,t+s_1})\big)^{(2c-1)/(4uc)} \Big] \Big)^{1/q} \right).
\end{aligned} \tag{4.77}$$

Combining this and the fact that $\frac{1}{q} = \frac{(2c-1)}{2qc} + \frac{1}{2q_0} + \frac{1}{2q_1}$ with Hölder's inequality implies that for all $x_1, x_2 \in B$, $(s_1, s_2) \in \Delta_T$ it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^{x_2}_{s_2,(t+s_2)\wedge T} - X^{x_1}_{s_1,(t+s_1)\wedge T}\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq c^{\frac{c}{q(c-1)}} \exp\Big(\int_{s_2}^{T} \phi(r)\,dr + \sum_{i=0}^{1} \int_{s_2}^{T} \tfrac{\beta_i \left(1-\frac{r-s_2}{(T-s_2)}\right)^{1-i}}{q_i \exp(\alpha i r)} dr \Big) \\
&\quad \cdot \big\| x_2 - X^{x_1}_{s_1,s_2} \big\|_{L^{(2qc)/(2c-1)}(\mathbb{P};H)} \Big\| \exp\Big(\tfrac{U_0(X^{x_1}_{s_1,s_2})}{2q_0 \exp(\alpha_0 s_2)} \Big) \Big\|_{L^{2q_0}(\mathbb{P};\mathbb{R})} \Big\| \exp\Big(\tfrac{U_1(X^{x_1}_{s_1,s_2})}{2q_1 \exp(\alpha_1 s_2)} \Big) \Big\|_{L^{2q_1}(\mathbb{P};\mathbb{R})} \\
&\quad + 6q \exp(\tfrac{1}{q}) \exp\big((1+\alpha)(s_2-s_1)\big)\, (s_2-s_1)^{1/2} \Big(\sup_{t\in[0,T-s_1)} \mathbb{E}\big[V(X^{x_1}_{s_1,t+s_1})\big]\Big)^{(2c-1)/(4uqc)}.
\end{aligned} \tag{4.78}$$

This, Lemma 4.18 (applied with $q = \frac{2c}{(2c-1)} q$, $T = s_2$, $s = s_1$, $x_2 = x_1$, $x_1 = x_2$, for $x_1, x_2 \in B$, $(s_1, s_2) \in \Delta_T$ in the notation of Lemma 4.18) and Lemma 4.10 (applied with $T = t$, $(X^x_r)_{r\in[0,t]} = (X^x_{s_1,r+s_1})_{r\in[0,t]}$, $\mathcal{F} = \bigvee_{r\in[0,t]} \mathbb{F}_{r+s_1}$, $(\mathbb{F}_r)_{r\in[0,t]} = (\mathbb{F}_{r+s_1})_{r\in[0,t]}$, $(W_r)_{r\in[0,t]} = (W_{r+s_1} - W_r)_{r\in[0,t]}$ for $t \in [0, T-s_1)$ in the notation of Lemma 4.10) show that for all $x_1, x_2 \in B$, $(s_1, s_2) \in \Delta_T$ it holds that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^{x_2}_{s_2,(t+s_2)\wedge T} - X^{x_1}_{s_1,(t+s_1)\wedge T}\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq c^{\frac{c}{q(c-1)}} \exp\Big(\int_{s_2}^{T} \phi(r)\,dr + \sum_{i=0}^{1} \int_{s_2}^{T} \tfrac{\beta_i \left(1-\frac{r-s_2}{(T-s_2)}\right)^{1-i}}{q_i \exp(\alpha i r)} dr \Big) \cdot 3^{\frac{2cq}{2c-1}} \exp(s_2-s_1) \exp\big(\tfrac{(2c-1)}{2qc}\big) \\
&\cdot \Big(\|x_2 - x_1\|_H + 2\big(V(x_1)\exp(\alpha(s_2-s_1))\big)^{(2c-1)/(2u(2qc))} (s_2-s_1)^{1/2} \Big) \\
&\quad \cdot \Big(\mathbb{E}\Big[\exp\Big(\tfrac{U_0(X^{x_1}_{s_1,s_2})}{\exp(\alpha_0 s_2)} - \int_{s_1}^{s_2} \tfrac{\beta_0}{\exp(\alpha_0 r)}\,dr \Big) \Big] \Big)^{1/2q_0} \exp\Big(\tfrac{\beta_0}{2q_0} \int_{s_1}^{s_2} \exp(-\alpha_0 r)\,dr \Big) \\
&\quad \cdot \Big(\mathbb{E}\Big[\exp\Big(\tfrac{U_1(X^{x_1}_{s_1,s_2})}{\exp(\alpha_1 s_2)} + \int_{s_1}^{s_2} \tfrac{K-\beta_1}{\exp(\alpha_1 r)}\,dr \Big) \Big] \Big)^{1/2q_1} \exp\Big(\tfrac{1}{2q_1} \int_{s_1}^{s_2} \tfrac{\beta_1 - K}{\exp(\alpha_1 r)}\,dr \Big) \\
&\quad + 6q \exp(\tfrac{1}{q}) \exp\big((1+\alpha)(s_2-s_1)\big)(s_2-s_1)^{1/2} \Big(\sup_{t\in[0,T-s_1)} \exp(\alpha t) V(x_1) \Big)^{(2c-1)/(4uqc)}.
\end{aligned} \tag{4.79}$$

Corollary 4.12 (with $T = s_2$, $s = s_1$, $X = X^{x_1}_{s_1,\cdot}$, $U_0 = U_0$ (resp. $U_0 = U_1$), $\overline{U} = -\beta_0$ (resp. $\overline{U} = \overline{U} - \beta_1$), $\alpha = \alpha_0$ (resp. $\alpha = \alpha_1$) for $x_1 \in B$, $(s_1, s_2) \in \Delta_T$ in the notation of Corollary 4.12) hence yields for all $k \in \mathbb{N}$, $x_1, x_2 \in B \cap B_k$, $(s_1, s_2) \in \Delta_T$ that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|X^{x_2}_{s_2,(t+s_2)\wedge T} - X^{x_1}_{s_1,(t+s_1)\wedge T}\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq c^{\frac{c}{q(c-1)}} \exp\left(\int_{s_2}^{T} \phi(r)\,dr + \sum_{i=0}^{1} \int_{s_2}^{T} \frac{\beta_i \left(1 - \frac{r-s_2}{(T-s_2)}\right)^{1-i}}{q_i \exp(\alpha i r)}\,dr \right) \cdot 3\tfrac{2cq}{2c-1} \exp(s_2 - s_1) \exp(\tfrac{(2c-1)}{2qc}) \\
&\quad \cdot \left[\|x_2 - x_1\|_H^{1/2} (2k)^{1/2} + 2\big(V(x_1) \exp\big(\alpha(s_2 - s_1)\big)\big)^{(2c-1)/(2u(2qc))} (s_2 - s_1)^{1/2} \right] \qquad (4.80) \\
&\quad \cdot \exp\left(\tfrac{U_0(x_1)}{2q_0 \exp(\alpha_0 s_1)} + \tfrac{\beta_0}{2q_0} \int_{s_1}^{s_2} \exp(-\alpha_0 r)\,dr + \tfrac{U_1(x_1)}{2q_1 \exp(\alpha_1 s_1)} + \tfrac{1}{2q_1} \int_{s_1}^{s_2} \tfrac{\beta_1 - K}{\exp(\alpha_1 r)}\,dr \right) \\
&\quad + 6q \exp(\tfrac{1}{q}) \exp\big((1+\alpha)(s_2 - s_1)\big) (s_2 - s_1)^{1/2} \Big(\exp(\alpha T) V(x_1) \Big)^{(2c-1)/(4uqc)}.
\end{aligned}$$

The proof of Lemma 4.20 is thus completed. □

Lemma 4.21 (Existence of a C^0-flow)**.** Assume Setting 4.14, let $\alpha \in [0, \infty)$, $p \in [2, \infty)$, let $V \in C^2(O, [0, \infty))$ satisfy for all $x \in B$ that $(\mathcal{G}_{\mu,\sigma} V)(x) \leq \alpha V(x)$, assume that for all $k \in \mathbb{N}$ it holds that

$$\sup_{x\in B\colon \|x\|_H \leq k} V(x) < \infty, \tag{4.81}$$

and

$$\max\left\{ \|\mu(x)\|_H^p, \|\sigma(x)\|_{\mathrm{HS}(U,H)}^p \right\} \leq V(x), \tag{4.82}$$

and assume that $q > 2(\dim(H) + 1)$.

Then there exists a $\mathcal{B}(\Delta_T \times \overline{B}) \otimes \mathcal{F}/\mathcal{B}(\overline{B})$-measurable function $\mathcal{X}\colon \Delta_T \times \overline{B} \times \Omega \to \overline{B}$ such that

1. for all $x \in B$, $s \in [0, T]$ it holds $\mathbb{P}$-almost surely that $(\mathcal{X}^x_{s,t})_{t\in[s,T]} = (X^x_{s,t})_{t\in[s,T]}$,
2. it holds for all $\omega \in \Omega$ that $\mathcal{X}(\omega) \in C(\Delta_T \times \overline{B}, \overline{B})$,
3. it holds $\mathbb{P}$-almost surely for all $x \in B$, $r, s, t \in [0, T]$ with $r \leq s \leq t$ that $\mathcal{X}^x_{r,t} = \mathcal{X}^{\mathcal{X}^x_{r,s}}_{s,t}$, and
4. it holds for all $x \in B$, $s \in [0, T]$ that $\mathcal{X}^x_{s,\cdot}\colon [0, T] \times \Omega \to \overline{B}$ is an $(\mathcal{F}^s_t)_{t\in[s,T]}$-adapted stochastic process.

Proof of Lemma 4.21. For all $k \in \mathbb{N}$ let $B_k \subseteq H$ be a subset with the property that $B_k = \{v \in H\colon \|v\|_H \leq k\}$ and let $K \in \mathbb{R}$ be the real number with the property that $K = \inf_{(s,t)\in\Delta_T} \inf_{\omega\in\Omega} \inf_{z\in B} \overline{U}(t, X^z_{s,t}(\omega))$. From Lemma 4.18 (applied with $p = \frac{2c}{2c-1}$, $x_1 = x$,

$x_2 = x$) it follows that for all $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\sup_{s\in[0,T]} \sup_{x\in B\cap B_k} \Big\| \sup_{t\in[0,T]} \|X^x_{s,(t+s)\wedge T} - x\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\le \sup_{s\in[0,T]} \sup_{x\in B\cap B_k} 6q \exp(T-s) \exp(\tfrac{1}{q}) \big(V(x)\exp(\alpha(T-s))\big)^{(2c-1)/(2uq(2c))} (T-s)^{1/2} \qquad (4.83)\\
&\le 6qT^{1/2} \exp\Big(T\Big(1+\alpha\tfrac{(2c-1)}{(4uqc)}\Big)\Big) \exp(\tfrac{1}{q}) \sup_{x\in B\cap B_k} (V(x))^{(2c-1)/(4uqc)}.
\end{aligned}$$

From inequalities (4.81) and (4.83) it follows that for all $k \in \mathbb{N}$ it holds that

$$\sup_{(s,x)\in[0,T]\times(B\cap B_k)} \mathbb{E}\Big[\sup_{t\in[0,T]} \|X^x_{s,(t+s)\wedge T}\|_H^q\Big] < \infty. \tag{4.84}$$

From Lemma 4.15 it follows that for all $(s_1,s_2) \in \Delta_T$, $t \in [0,T]$, $x_1, x_2 \in B$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
X^{x_1}_{s_2,(t+s_2)\wedge T} - X^{x_2}_{s_1,(t+s_1)\wedge T} &= X^{x_1}_{s_2,(t+s_2)\wedge T} - X^{x_2}_{s_1,(t+s_2)\wedge T} + X^{x_2}_{s_1,(t+s_2)\wedge T} - X^{x_2}_{s_1,(t+s_1)\wedge T} \qquad (4.85)\\
&= X^{x_1}_{s_2,(t+s_2)\wedge T} - X^{X^{x_2}_{s_1,s_2}}_{s_2,(t+s_2)\wedge T} + X^{X^{x_2}_{s_1,(t+s_1)\wedge T}}_{(t+s_1)\wedge T,(t+s_2)\wedge T} - X^{x_2}_{s_1,(t+s_1)\wedge T}.
\end{aligned}$$

Observe that for all $(s_1,s_2) \in \Delta_T$, $t \in [T-s_1,T]$, $x_1, x_2 \in B$ it holds that $(t+s_1)\wedge T = (t+s_2)\wedge T$ and it holds $\mathbb{P}$-almost surely that $X^{x_1}_{s_2,(t+s_2)\wedge T} - X^{x_2}_{s_1,(t+s_1)\wedge T} = X^{x_1}_{s_2,T} - X^{X^{x_2}_{s_1,s_2}}_{s_2,T}$. From Lemma 4.20 and the fact that $\int_0^T \max\{0,\phi(r)\}\,dr < \infty$ it follows that

$$\sup_{\substack{(s_2,x_1),(s_1,x_2)\in[0,T]\times(B\cap B_k):\\ (s_2,x_1)\neq(s_1,x_2),\ s_1<s_2}} \frac{\mathbb{E}\Big[\sup_{t\in[0,T]} \|X^{x_1}_{s_2,(t+s_2)\wedge T} - X^{x_2}_{s_1,(t+s_1)\wedge T}\|_H^q\Big]}{\big(\|x_1-x_2\|_H^2 + |s_2-s_1|^2\big)^{q/4}} < \infty. \tag{4.86}$$

Proposition 4.8 (applied with $p = q$, $\alpha = 1/2$, $D = [0,T]\times B$, $H = \mathbb{R}\oplus H$, $X = ([0,T]\times B \ni (s,x) \mapsto X^x_{s,(\cdot+s)\wedge T} \in C([0,T],B))$, $E = C([0,T],H)$, $F = C([0,T],\overline{B})$ in the notation of Proposition 4.8) establishes the existence of a $\mathcal{B}([0,T]\times\overline{B})\otimes\mathcal{F}/\mathcal{B}(C([0,T],\overline{B}))$ -measurable function $Y\colon [0,T]\times\overline{B}\times[0,T]\times\Omega \to \overline{B}$ such that for all $\omega \in \Omega$ it holds that $Y(\omega) \in C([0,T]\times\overline{B}, C([0,T],\overline{B}))$ and for all $(s,x) \in [0,T]\times\overline{B}$ it holds $\mathbb{P}$-almost surely that $(Y(s,x,t))_{t\in[0,T]} = (X^x_{s,(t+s)\wedge T})_{t\in[0,T]}$. Let us consider $\mathcal{X}\colon \Delta_T\times\overline{B}\times\Omega \to \overline{B}$ which satisfies for all $x \in \overline{B}$, $(s,t) \in \Delta_T$, $\omega \in \Omega$ that $\mathcal{X}^x_{s,t}(\omega) = Y(s,t-s,x,\omega)$. The proof of Lemma 4.21 is thus completed. $\square$

4.4 Existence of a C^1-solution

Setting 4.22. *Let* $T \in (0,\infty)$, *let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *and* $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ *be finite-dimensional real Hilbert spaces, assume* $H \neq \{0\}$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$, *let* $W\colon [0,T]\times\Omega \to U$ *be a standard* $(\Omega, \mathcal{F}, (\mathbb{F}_t)_{t\in[0,T]}, \mathbb{P})$*-Brownian motion, for all* $(s,t) \in \Delta_T$ *let* $\mathcal{F}_t^s \subseteq \mathbb{F}_t$ *be a subset such that for all* $s \in [0,T]$ *the filtration* $(\mathcal{F}_t^s)_{t\in[s,T]}$ *is a normal filtration, assume that for all* $(s,t) \in \Delta_T$ *it holds that* $\sigma\left(W_r - W_s \colon r \in [s,t]\right) \subseteq \mathcal{F}_t^s$, *let* $D, O \subseteq H$ *be non-empty open sets, let* $B \subseteq H$ *be a Borel measurable set with the property that* $D \subseteq B \subseteq O$, *let* $\mu \in C^1(O,H)$, $\sigma \in C^1(O, \mathrm{HS}(U,H))$, *for all* $s \in [0,T]$, $x \in B$ *let* $X_s^x\colon [s,T]\times\Omega \to B$ *be an* $(\mathcal{F}_t^s)_{t\in[s,T]}$*-adapted stochastic process with continuous sample paths such that for all* $x \in B$, $t \in [s,T]$ *it holds* $\mathbb{P}$*-almost surely that*

$$X_{s,t}^x = x + \int_s^t \mu(X_{s,r}^x)\,dr + \int_s^t \sigma(X_{s,r}^x)\,dW_r, \tag{4.87}$$

let $\alpha_0, \alpha_1 \in [0,\infty)$, $\beta_0, \beta_1 \in \mathbb{R}$, $U_0, U_1 \in C^2(O,[0,\infty))$, *let* $\overline{U}\colon [0,T]\times B \to \mathbb{R}$ *be a Borel measurable function with the property that for all* $s \in [0,T]$, $x \in B$ *it holds* $\mathbb{P}$*-almost surely that* $\int_s^T |\overline{U}(r, X_{s,r}^x)|\,dr < \infty$ *and such that for all* $i \in \{0,1\}$, $s \in [0,T]$, *and all*

$$(t,x) \in \cup_{\omega\in\Omega} \cup_{z\in B} \cup_{r\in[s,T]} \{(r, X_{s,r}^z(\omega))\} \tag{4.88}$$

it holds that

$$(\mathcal{G}_{\mu,\sigma}U_i)(x) + \tfrac{1}{2\exp(\alpha_i t)} \|\sigma(x)^*(\nabla U_i)(x)\|_U^2 + \mathbb{1}_{\{1\}}(i)\cdot\overline{U}(t,x) \leq \alpha_i U_i(x) + \beta_i, \tag{4.89}$$

let $\phi\colon [0,T] \to \mathbb{R}$ *be a Borel measurable function with the property that* $\int_0^T \max\{0, \phi(r)\}\,dr < \infty$, *and let* $V \in C^2(O,[0,\infty))$ *be a function and* $\alpha \in [0,\infty)$ *be a real number such that for all* $x \in B$, $s \in [0,T]$, $t \in [s,T]$ *it holds* $\mathbb{P}$*-almost surely that* $(\mathcal{G}_{\mu,\sigma}V)(X_{s,t}^x) \leq \alpha V(X_{s,t}^x)$.

4.4.1 Moment estimates for the first derivative process

Setting 4.23. *Assume Setting 4.22, for all* $s \in [0,T]$ *let* $Z_s^v(x,y)\colon [s,T]\times\Omega \to H$, $x \in B$, $v \in H$, $y \in \{z \in \mathbb{R}\colon x + vz \in B\}$, *be* $(\mathcal{F}_t^s)_{t\in[s,T]}$*-adapted stochastic processes with continuous sample paths satisfying for all* $t \in [s,T]$, $x \in B$, $v \in H$, $y \in \mathbb{R}\setminus\{0\}$ *with* $x + vy \in B$ *that*

$$Z_{s,t}^v(x,y) = \frac{X_{s,t}^{x+yv} - X_{s,t}^x}{y} \tag{4.90}$$

and for all $t \in [s,T]$, $x \in B$, $v \in H$ *that it holds* $\mathbb{P}$*-almost surely that*

$$Z_{s,t}^v(x,0) = v + \int_s^t \mu'(X_{s,r}^x) Z_{s,r}^v(x,0)\,dr + \int_s^t \sigma'(X_{s,r}^x) Z_{s,r}^v(x,0)\,dW_r. \tag{4.91}$$

Lemma 4.24 (Difference process satisfies linear SDE and one-sided local Lipschitz inequality)**.** Assume Setting 4.23, let $x \in B$, $v \in H$, $y \in \mathbb{R}$ with the property that $x + vy \in B$ and let $(s,t) \in \Delta_T$. Then it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} Z^v_{s,t}(x,y) = v &+ \int_s^t \int_0^1 \mu'(X^x_{s,r} + \lambda(X^{x+vy}_{s,r} - X^x_{s,r})) Z^v_{s,r}(x,y)\, d\lambda\, dr \\ &+ \int_s^t \int_0^1 \sigma'(X^x_{s,r} + \lambda(X^{x+vy}_{s,r} - X^x_{s,r})) Z^v_{s,r}(x,y)\, d\lambda\, dW_r \end{aligned} \tag{4.92}$$

and it holds $\mathbb{P}$-almost surely that

$$\begin{aligned} &\max\Big\{ \Big\langle Z^v_{s,t}(x,y), \int_0^1 \mu'\Big(X_s, t^x + \lambda(X^{x+vy}_{s,t} - X^x_{s,t})\Big)\, d\lambda Z^v_{s,t}(x,y) \Big\rangle_H \\ &\qquad + \tfrac{1}{2} \Big\| \int_0^1 \sigma'\Big(X^x_{s,t} + \lambda(X^{x+\lambda y}_{s,t} - X^x_{s,t})\Big)\, d\lambda Z^v_{s,t}(x,y) \Big\|^2_{\mathrm{HS}(U,H)}, 0 \Big\} \\ &+ \tfrac{1}{2}\Big(\tfrac{cq}{c-1} - \min\{2, \tfrac{(c-1)q}{c}\} \Big) \Big\| \Big\langle Z^v_{s,t}(x,y), \int_0^1 \sigma'\Big(X^x_{s,t} + \lambda\big(X^{x+yv}_{s,t} - X^x_{s,t}\big)\Big)\, d\lambda Z^v_{s,t}(x,y) \Big\rangle_H \Big\|_{\mathrm{HS}(U,\mathbb{R})} \\ &\le \|Z^v_{s,t}(x,y)\|^2_H \Big(\phi(t) + \tfrac{U_0(X^x_{s,t}) + U_0(X^{x+yv}_{s,t})}{2q_0(T-s)\exp(\alpha_0 t)} + \tfrac{\overline{U}(t, X^x_{s,t}) + \overline{U}(t, X^{x+yv}_{s,t})}{2q_1 \exp(\alpha_1 t)} \Big). \end{aligned} \tag{4.93}$$

Proof of Lemma 4.24. If $y = 0$, then equation (4.92) follows from equation (4.91). If $y \neq 0$, then equations (4.90) and (4.60) imply that it holds $\mathbb{P}$-almost surely that

$$Z^v_{s,t}(x,y) = v + \int_s^t \tfrac{\mu(X^{x+vy}_{s,r}) - \mu(X^x_{s,r})}{y}\, dr + \int_s^t \tfrac{\sigma(X^{x+vy}_{s,r}) - \sigma(X^x_{s,r})}{y}\, dW_r. \tag{4.94}$$

The fundamental theorem of calculus hence yields equation (4.92). The proof of Lemma 4.24 is thus completed. ☐

Setting 4.25. *Assume Setting 4.23, let* $q \in [1,\infty)$, $c \in (1,\infty)$, $q_0, q_1 \in (0,\infty]$ *such that* $\sum_{i=0}^1 \frac{1}{q_i} = \frac{1}{qc}$, *and assume that for all* $(s,t) \in \Delta_T$ $x_1, x_2 \in B$ *it holds* $\mathbb{P}$*-almost surely that*

$$\begin{aligned} &\max\Big\{ 0, \sup_{u \in H\setminus\{0\}} \tfrac{\langle u, \int_0^1 \mu'(X^{x_1}_{s,t} + \lambda(X^{x_2}_{s,t} - X^{x_1}_{s,t})) u\, d\lambda \rangle_H}{\|u\|^2_H} \Big\} \\ &\quad + \tfrac{\max\{1, \frac{qc}{c-1} - \frac{q(c-1)}{c}, \frac{qc}{c-1} - 1\}}{2} \Big\| \int_0^1 \sigma'(X^{x_1}_{s,t} + \lambda(X^{x_2}_{s,t} - X^{x_1}_{s,t}))(\cdot)\, d\lambda \Big\|^2_{L(H,\mathrm{HS}(U,H))} \\ &\le \Big(\phi(t) + \tfrac{U_0(X^{x_1}_{s,t}) + U_0(X^{x_2}_{s,t})}{2q_0(T-s)\exp(\alpha_0 t)} + \tfrac{\overline{U}(t, X^{x_1}_{s,t}) + \overline{U}(t, X^{x_2}_{s,t})}{2q_1 \exp(\alpha_1 t)} \Big). \end{aligned} \tag{4.95}$$

The following lemma, Lemma 4.26 essentially follows from Lemma 2.4 in Cox, Hutzenthaler, and Jentzen 2013. We include a proof of Lemma 4.26 based on Proposition 4.6 for convenience of the reader.

Lemma 4.26 (Moment estimates for the first derivative process)**.** Assume Setting 4.25 and let $s \in [0,T]$, $x \in B$, $v \in H$, $y \in \{z \in \mathbb{R} \colon x + vz \in B\}$.

Then it holds that

$$\begin{aligned}&\Big\| \sup_{t\in[0,T]} \|Z^v_{s,(t+s)\wedge T}(x,y)\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\ &\leq c^{\frac{c}{q(c-1)}} \|v\|_H \exp\Big(\int_s^T \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i \left(1-\frac{(r-s)}{(T-s)}\right)^{1-i}}{q_i \exp(\alpha i r)} \, dr + \sum_{i=0}^{1} \tfrac{U_i(x+vy)+U_i(x)}{2q_i \exp(\alpha_i s)} \Big).\end{aligned} \tag{4.96}$$

Proof of Lemma 4.26. Throughout this proof let $\alpha\colon [0,T]\times\Omega \to [0,\infty)$ and $\eta\colon [0,T]\times\Omega \to L(H)$ be the measurable stochastic process satisfying for all $t \in [0,T]$ that

$$\alpha_t = \max\Big\{0, \mathbb{1}_{\{Z^v_t(x,y)\neq 0\}} \tfrac{\left\langle Z^v_t(x,y), \int_0^1 \mu'(X^x_t + \lambda(X^{x+vy}_t - X^x_t)) Z^v_t(x,y)\, d\lambda \right\rangle_H}{\|Z^v_t(x,y)\|^2_H} \Big\} \tag{4.97}$$

and $\eta_t(\cdot) = \int_0^1 \sigma'(X^x_t + \lambda(X^{x+vy}_t - X^x_t))(\cdot)\, d\lambda$. From Lemma 4.24 it follows that for all $t \in [s,T]$ equation (4.92) holds true $\mathbb{P}$-almost surely. item (i) in Proposition 4.6 (applied with $s = s$, $X_t = Z^v_{s,t}(x,y)$, $\alpha_t = \max\Big\{0, \sup_{u\in H\setminus\{0\}} \frac{\left\langle u, \int_0^1 \mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t})) u\, d\lambda \right\rangle_H}{\|u\|^2_H} \Big\}$, $\beta_t = 0$, $\zeta_t = 0_{\mathrm{HS}(H,U)}$, $\eta_t = \int_0^1 \sigma'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))(\cdot)\, d\lambda$, $\mu_t = \int_0^1 \mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t})) Z^v_{s,t}(x,y)\, d\lambda$, $p = 1$, $p' = u = u' = 2$, $c_1 = c_2 = c_3 = c_4 = c$, $\delta = 0$ in the notation of Proposition 4.6), inequality (4.95), and Lemma 4.13 (applied with $p_0 = q_0$, $p_1 = q_1$, $p = qc$, $X^{(1)}_\cdot = X^x_{s,\cdot}$, $X^{(2)}_\cdot = X^x_{s,\cdot}$, $X^{(3)}_\cdot = X^{x+vy}_{s,\cdot}$, $X^{(4)}_\cdot = X^{x+vy}_{s,\cdot}$ in the notation of Lemma 4.13) show that

$$\begin{aligned}&\Big\| \sup_{t\in[0,T]} \|Z^v_{s,(t+s)\wedge T}(x,y)\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\ &\leq c^{\frac{c}{q(c-1)}} \|v\|_H \Big\| \exp\Big(\int_s^T \max\Big\{0, \sup_{u\in H\setminus\{0\}} \tfrac{\langle u, \int_0^1 \mu'(X^x_{s,r} + \lambda(X^{x+vy}_{s,r} - X^x_{s,r})) u\, d\lambda \rangle_H}{\|u\|^2_H} \Big\} \\ &\qquad + \tfrac{\max\{1, \frac{qc}{c-1} - \frac{q(c-1)}{c}, \frac{qc}{c-1} - 1\}}{2} \Big\| \int_0^1 \sigma'(X^x_{s,r} + \lambda(X^{x+vy}_{s,r} - X^x_{s,r}))(\cdot)\, d\lambda, \Big\|^2_{L(H,\mathrm{HS}(U,H))} dr \Big) \Big\|_{L^{qc}(\mathbb{P};\mathbb{R})} \\ &\leq c^{\frac{c}{q(c-1)}} \|v\|_H \Big\| \exp\Big(\int_s^T \phi(r) + \tfrac{U_0(X^x_{s,r}) + U_0(X^{x+vy}_{s,r})}{2q_0(T-s)\exp(\alpha_0 r)} + \tfrac{\overline{U}(r, X^x_{s,r}) + \overline{U}(r, X^{x+vy}_{s,r})}{2q_1 \exp(\alpha_1 r)} \, dr \Big) \Big\|_{L^{qc}(\mathbb{P};\mathbb{R})} \\ &\leq c^{\frac{c}{q(c-1)}} \|v\|_H \exp\Big(\int_s^T \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i \left(1-\frac{(r-s)}{(T-s)}\right)^{1-i}}{q_i \exp(\alpha i r)} \, dr + \sum_{i=0}^{1} \tfrac{U_i(x+vy)+U_i(x)}{2q_i \exp(\alpha_i s)} \Big).\end{aligned} \tag{4.98}$$

This proves inequality (4.96). The proof of Lemma 4.26 is thus completed. □

4.4.2 Properties of the first derivative process

The proof of the next lemma is similar to the proof of Lemma 4.26.

Lemma 4.27. *Assume Setting 4.25, let $(s_1, s_2) \in \Delta_T$, and let $v \in H$, $x \in B$, $y \in \mathbb{R}$ such that $x + vy \in B$. Then it holds that*

$$\mathbb{P}\Big(Z^v_{s_1,t}(x,y) = Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y) \text{ for all } t \in [s_2, T] \Big) = 1. \tag{4.99}$$

Proof of Lemma 4.27. First, note that Lemma 4.24 implies that for all $t \in [s_2, T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&Z^v_{s_1,t}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y) \\
&= Z^v_{s_1,t}(x,y) - Z^v_{s_1,s_2}(x,y) + Z^v_{s_1,s_2}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y) \\
&= \int\limits_{s_2}^{t}\int\limits_{0}^{1} \mu'(X^x_{s_1,r} + \lambda(X^{x+vy}_{s_1,r} - X^x_{s_1,r}))Z^v_{s_1,r}(x,y)\, d\lambda\, dr \\
&\quad + \int\limits_{s_2}^{t}\int\limits_{0}^{1} \sigma'(X^x_{s_1,r} + \lambda(X^{x+vy}_{s_1,r} - X^x_{s_1,r}))Z^v_{s_1,r}(x,y)\, d\lambda\, dW_r \\
&\quad + Z^v_{s_1,s_2}(x,y) - Z^v_{s_1,s_2}(x,y) \\
&\quad - \int\limits_{s_2}^{t}\int\limits_{0}^{1} \mu'(X^{X^x_{s_1,s_2}}_{s_2,r} + \lambda(X^{X^x_{s_1,s_2}+yZ^v_{s_1,s_2}(x,y)}_{s_2,r} - X^{X^x_{s_1,s_2}}_{s_2,r}))Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,r}(X^x_{s_1,s_2}, y)\, d\lambda\, dr \\
&\quad - \int\limits_{s_2}^{t}\int\limits_{0}^{1} \sigma'(X^{X^x_{s_1,s_2}}_{s_2,r} + \lambda(X^{X^x_{s_1,s_2}+yZ^v_{s_1,s_2}(x,y)}_{s_2,r} - X^{X^x_{s_1,s_2}}_{s_2,r}))Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,r}(X^x_{s_1,s_2}, y)\, d\lambda\, dW_r.
\end{aligned} \tag{4.100}$$

Next, observe that for all $t \in [s_2, T]$ it holds that $X^{X^x_{s_1,s_2}+yZ^v_{s_1,s_2}(x,y)}_{s_2,t} = X^{X^{x+vy}_{s_1,s_2}}_{s_2,t}$ (trivial for $y = 0$, and ensured by equation (4.90) if $y \neq 0$) and note that Lemma 4.15 assures that it holds $\mathbb{P}$-almost surely for all $z \in B$, $t \in [s_2, T]$ that $X^{X^z_{s_1,s_2}}_{s_2,t} = X^z_{s_1,t}$. Hence, for all $t \in [s_2, T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&Z^v_{s_1,t}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y) \\
&= \int\limits_{s_2}^{t}\int\limits_{0}^{1} \mu'(X^x_{s_1,r} + \lambda(X^{x+vy}_{s_1,r} - X^x_{s_1,r}))(Z^v_{s_1,r}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,r}(X^x_{s_1,s_2}, y))\, d\lambda\, dr \\
&\quad + \int\limits_{s_2}^{t}\int\limits_{0}^{1} \sigma'(X^x_{s_1,r} + \lambda(X^{x+vy}_{s_1,r} - X^x_{s_1,r}))(Z^v_{s_1,r}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,r}(X^x_{s_1,s_2}, y))\, d\lambda\, dW_r.
\end{aligned} \tag{4.101}$$

This and item (i) in Proposition 4.6 (applied with $s = s_2$, $X_t = Z^v_{s_1,t}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y)$, $\alpha_t = \max\Big\{0, \sup_{u \in H\setminus\{0\}} \frac{\langle u, \int_0^1 \mu'(X^x_{s_1,t} + \lambda(X^{x+vy}_{s_1,t} - X^x_{s_1,t}))u\, d\lambda\rangle_H}{\|u\|^2_H}\Big\}$, $\beta_t = 0$, $\zeta_t = 0_{\mathrm{HS}(H,U)}$, $\eta_t = \int_0^1 \sigma'(X^x_{s_1,t} + \lambda(X^{x+vy}_{s_1,t} - X^x_{s_1,t}))(\cdot)\, d\lambda$, $\mu_t = \int_0^1 \mu'(X^x_{s_1,t} + \lambda(X^{x+vy}_{s_1,t} - X^x_{s_1,t}))(Z^v_{s_1,t}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y))\, d\lambda$, $p = 1$, $p' = u = u' = 2$, $c_1 = c_2 = c_3 = c_4 = c$, $\delta = 0$ for all $t \in [s_2, T]$ in the notation of Proposition 4.6), inequality (4.95), Lemma 4.15, Lemma 4.13 (applied with $s = s_2$, $p_0 = q_0$, $p_1 = q_1$, $p = qc$, $X^{(1)}_{\cdot} = X^{X^x_{s_1,s_2}}_{s_1,\cdot}$, $X^{(2)}_{\cdot} = X^{X^x_{s_1,s_2}}_{s_1,\cdot}$, $X^{(3)}_{\cdot} = X^{X^{x+vy}_{s_1,s_2}}_{s_1,\cdot}$, $X^{(4)}_{\cdot} = X^{X^{x+vy}_{s_1,s_2}}_{s_1,\cdot}$ in the notation of Lemma 4.13), and the fact that $\int_0^T \max\{0, \phi(r)\}\, dr < \infty$ show that

$$\begin{aligned}
&\Big\| \sup_{t \in [s_2,T]} \big\| Z^v_{s_1,t}(x,y) - Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2}, y) \big\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq c^{\frac{c}{q(c-1)}} \cdot 0 \cdot \exp\Big(\int\limits_{s}^{T} \phi(r) + \sum_{i=0}^{1} \frac{\beta_i \big(1 - \frac{(r-s)}{(T-s)}\big)^{1-i}}{q_i \exp(\alpha i r)}\, dr + \sum_{i=0}^{1} \frac{U_i(x+vy) + U_i(x)}{2 q_i \exp(\alpha_i s)} \Big) = 0.
\end{aligned} \tag{4.102}$$

This implies that it holds $\mathbb{P}$-almost surely that $\sup_{t\in[s_2,T]}\|Z^v_{s_1,t}(x,y)-Z^{Z^v_{s_1,s_2}(x,y)}_{s_2,t}(X^x_{s_1,s_2},y)\|_H=0$. The proof of Lemma 4.27 is thus completed. ☐

Setting 4.28. *Assume Setting 4.23, let $p,q,\vartheta\in[1,\infty)$, $q_0,q_1\in(0,\infty]$ such that $(p+1)q\geq 2p$ and $\sum_{i=0}^1\frac{1}{q_i}=\frac{1}{(p+1)q}$ and assume that for all $(s,t)\in\Delta_T$, $x\in B$, $v\in H$, $y\in\{z\in\mathbb{R}\colon x+vz\in B\}$ it holds $\mathbb{P}$-almost surely that*

$$\begin{aligned}\sup_{\lambda\in[0,1]}\max\big\{\|\mu'(X^x_{s,t}+\lambda(X^{x+vy}_{s,t}-X^x_{s,t}))\|_{L(H,H)},\|\sigma'(X^x_{s,t}+\lambda(X^{x+vy}_{s,t}-X^x_{s,t}))\|_{L(H,\mathrm{HS}(U,H))}\big\}&\\ \leq\big(V(X^x_{s,t})+V(X^{x+vy}_{s,t})\big)^{(p-1)/(2\vartheta qp)},&\qquad(4.103)\end{aligned}$$

let $\gamma\in(0,\infty)$, and assume that for all $(s,t)\in\Delta_T$, $x_1,x_2\in B$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}&2\max\Big\{0,\sup_{u\in H\setminus\{0\}}\tfrac{\langle u,\int_0^1\mu'(X^{x_1}_{s,t}+\lambda(X^{x_2}_{s,t}-X^{x_1}_{s,t}))\,d\lambda\,u\rangle_H}{\|u\|_H^2}\Big\}+2\gamma\\ &\quad+(qp+q+1)\Big\|\int_0^1\sigma'(X^{x_1}_{s,t}+\lambda(X^{x_2}_{s,t}-X^{x_1}_{s,t}))(\cdot)\,d\lambda\Big\|^2_{L(H,\mathrm{HS}(U,H))}\qquad(4.104)\\ &\leq\Big(\phi(t)+\sum_{j=1}^2\tfrac{U_0(X^{x_j}_{s,t})}{2q_0(T-s)\exp(\alpha_0 t)}+\sum_{j=1}^2\tfrac{\overline{U}(t,X^{x_j}_{s,t})}{2q_1\exp(\alpha_1 t)}\Big).\end{aligned}$$

Lemma 4.29. Assume Setting 4.28 and let $s\in[0,T]$, $x\in B$, $v,w\in H$, $y\in\{z\in\mathbb{R}\colon x+vz\in B\}$. Then it holds that

$$\begin{aligned}&\Big\|\sup_{t\in[0,T]}\|Z^v_{s,(t+s)\wedge T}(x,y)-w\|_H\Big\|_{L^q(\mathbb{P};\mathbb{R})}\\ &\leq 3\,q\,\exp\big(\tfrac{1}{q}\big)\exp\Big(\int_s^T\phi(r)+\sum_{i=0}^1\tfrac{\beta_i\left(1-\frac{(r-s)}{(T-s)}\right)^{1-i}}{q_i\exp(\alpha ir)}\,dr+\sum_{i=0}^1\tfrac{U_i(x+vy)+U_i(x)}{2q_i\exp(\alpha_i s)}\Big)\qquad(4.105)\\ &\quad\cdot\Big(\|v-w\|_H+\|w\|_H\big(\tfrac{1}{2\sqrt{\gamma}}+1\big)\big(V(x)+V(x+vy)\big)^{(p-1)/(2\vartheta qp)}\Big(\int_0^{T-s}\exp(\alpha\tfrac{(p-1)}{(\vartheta qp)}r)dr\Big)^{1/2}\Big).\end{aligned}$$

Proof of Lemma 4.29. First, note that Lemma 4.24 ensures that for all $t\in[s,T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}Z^v_{s,t}(x,y)-w&=v-w+\int_s^t\int_0^1\mu'(X^x_{s,r}+\lambda(X^{x+vy}_{s,r}-X^x_{s,r}))Z^v_{s,r}(x,y)\,d\lambda\,dr\\ &\quad+\int_s^t\int_0^1\sigma'(X^x_{s,r}+\lambda(X^{x+vy}_{s,r}-X^x_{s,r}))Z^v_{s,r}(x,y)\,d\lambda\,dW_r\\ &=v-w+\int_s^t\int_0^1\mu'(X^x_{s,r}+\lambda(X^{x+vy}_{s,r}-X^x_{s,r}))(Z^v_{s,r}(x,y)-w+w)\,d\lambda\,dr\qquad(4.106)\\ &\quad+\int_s^t\int_0^1\sigma'(X^x_{s,r}+\lambda(X^{x+vy}_{s,r}-X^x_{s,r}))(Z^v_{s,r}(x,y)-w+w)\,d\lambda\,dW_r.\end{aligned}$$

The continuity of $[s,T]\ni t\mapsto Z^v_{s,t}(x,y)\in H$ and the Cauchy-Schwarz inequality yield that for all $t \in [s,T]$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}&\Big\langle Z^v_{s,t}(x,y) - w, \int_0^1 \mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\,(Z^v_{s,t}(x,y) - w + w)\,d\lambda\Big\rangle_H \\ &\leq \Big(\max\Big\{0, \sup_{u\in H} \tfrac{\big\langle u, \int_0^1 \mu'(X^x_{s,t}+\lambda(X^{x+vy}_{s,t}-X^x_{s,t}))\,d\lambda\,u\big\rangle_H}{\|u\|^2_H}\Big\} + \gamma\Big)\|Z^v_{s,t}(x,y) - w\|^2_H \\ &\quad + \tfrac{1}{4\gamma}\Big\|\int_0^1 \mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\,w\,d\lambda\Big\|^2_H.\end{aligned} \tag{4.107}$$

item (ii) in Proposition 4.6 (applied with $X_t = Z^v_{s,t}(x,y) - w$, $\alpha_t = \max\{0, \sup_{u\in H\setminus\{0\}} \frac{\langle u, \int_0^1 \mu'(X^x_{s,t}+\lambda(X^{x+vy}_{s,t}-X^x_{s,t}))\,d\lambda\,u\rangle_H}{\|z\|^2_H}\} + \gamma$, $\beta_t = \frac{1}{4\gamma}\Big\|\int_0^1 \mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\,w\,d\lambda\Big\|^2_H$, $\zeta_t = \int_0^1 \sigma'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\,w\,d\lambda$, $\eta_t = \int_0^1 \sigma'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\,(\cdot)\,d\lambda$, $\mu_t = \int_0^1 \mu'(X^x_{s,t} + \lambda(X^{x+vy}_{s,t} - X^x_{s,t}))\,Z^v_{s,t}(x,y)\,d\lambda$, $p = 1$, $u = p$ for all $t \in [s,T]$ in the notation of Proposition 4.6) shows that

$$\begin{aligned}&\Big\|\sup_{t\in[0,T]} \|Z^v_{s,(t+s)\wedge T}(x,y) - w\|_H\Big\|_{L^q(\mathbb{P};\mathbb{R})} \\ &= \Big\|\sup_{t\in[s,T]} \|Z^v_{s,t}(x,y) - w\|_H\Big\|_{L^q(\mathbb{P};\mathbb{R})} \\ &\leq 3\,q\,\exp\big(\tfrac{1}{q}\big)\Big\|\exp\Big(\int_s^T 2\max\Big\{0, \sup_{u\in H\setminus\{0\}} \tfrac{\big\langle u, \int_0^1 \mu'(X^x_{s,r}+\lambda(X^{x+vy}_{s,r}-X^x_{s,r}))\,d\lambda\,u\big\rangle_H}{\|u\|^2_H}\Big\} + 2\gamma\,dr\Big) \\ &\qquad \cdot \exp\Big((qp + q + 1)\Big\|\int_0^1 \sigma'(X^x_{s,r} + \lambda(X^{x+vy}_{s,r} - X^x_{s,r}))\,(\cdot)\,d\lambda\Big\|^2_{L(H,\mathrm{HS}(U,H))}\,dr\Big)\|_{L^{(p+1)q}(\mathbb{P};\mathbb{R})} \\ &\quad \cdot \Big(\|v - w\|_H + \tfrac{1}{2\sqrt{\gamma}}\Big\|\int_0^1 \mu'(X^x_{s,\cdot} + \lambda(X^{x+vy}_{s,\cdot} - X^x_{s,\cdot}))\,w\,d\lambda\Big\|_{L^2(\lambda_{[s,T]};L^{(2pq)/(p-1)}(\mathbb{P};H))} \\ &\qquad + \Big\|\int_0^1 \sigma'(X^x_{s,\cdot} + \lambda(X^{x+vy}_{s,\cdot} - X^x_{s,\cdot}))\,w\,d\lambda\Big\|_{L^2(\lambda_{[s,T]};L^{(2pq)/(p-1)}(\mathbb{P};\mathrm{HS}(U,H)))}\Big).\end{aligned} \tag{4.108}$$

Combining this, equations (4.103) and (4.104) with Lemma 4.13 (applied with $p = (p+1)q$, $p_0 = q_0$, $p_1 = q_1$, $X^{(1)}_\cdot = X^x_{s,\cdot}$, $X^{(3)}_\cdot = X^x_{s,\cdot}$, $X^{(2)}_\cdot = X^{x+vy}_{s,\cdot}$, $X^{(4)}_\cdot = X^{x+vy}_{s,\cdot}$ for $s \in [0,T]$ in the notation of Lemma 4.13), Hölder's inequality, and Lemma 4.10 (applied with $T = r - s$, $(X^x_t)_{t\in[0,r-s]} = (X^{x+vy}_{s,t+s})_{t\in[0,r-s]}$ (resp. $(X^x_t)_{t\in[0,r-s]} = (X^x_{s,t+s})_{t\in[0,r-s]}$), $\mathcal{F} = \bigvee_{t\in[0,r-s]} \mathbb{F}_{t+s}$, $(\mathbb{F}_t)_{t\in[0,r-s]} = (\mathbb{F}_{t+s})_{t\in[0,r-s]}$, $(W_t)_{t\in[0,r-s]} = (W_{t+s} - W_s)_{t\in[0,r-s]}$ for $r \in [s,T]$ in the notation of Lemma 4.10) establishes that

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|Z^v_{s,(t+s)\wedge T}(x,y) - w\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&= \Big\| \sup_{t\in[s,T]} \|Z^v_{s,t}(x,y) - w\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq 3\,q\,\exp\left(\tfrac{1}{q}\right) \left\| \exp\left(\int_s^T \phi(r) + \tfrac{U_0(X^x_{s,r}) + U_0(X^{x+vy}_{s,r})}{2q_0(T-s)\exp(\alpha_0 r)} + \tfrac{\overline{U}(r,X^x_{s,r}) + \overline{U}(r,X^{x+vy}_{s,r})}{2q_1 \exp(\alpha_1 r)}\, dr \right) \right\|_{L^{(p+1)q}(\mathbb{P};\mathbb{R})} \\
&\quad \cdot \left(\|v-w\|_H + \big(\tfrac{1}{2\sqrt{\gamma}} + 1\big) \|w\|_H \left(\int_0^1 \left(\mathbb{E}\Big[\big(V(X^x_{s,r}) + V(X^{x+vy}_{s,r})\big)^{1/p} \Big] \right)^{(p-1)/(pq)} d\lambda \right)^{1/2} \right) \\
&\leq 3\,q\,\exp\left(\tfrac{1}{q}\right) \exp\left(\int_s^T \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i \left(1 - \frac{(r-s)}{(T-s)}\right)^{1-i}}{q_i \exp(\alpha i r)}\, dr + \sum_{i=0}^{1} \tfrac{U_i(x+vy) + U_i(x)}{2q_i \exp(\alpha_i s)} \right) \\
&\quad \cdot \left(\|v-w\|_H + \|w\|_H \big(\tfrac{1}{2\sqrt{\gamma}} + 1\big) \big(V(x) + V(x+vy)\big)^{(p-1)/(2\vartheta q p)} \left(\int_0^{T-s} \exp\big(\alpha \tfrac{(p-1)}{(\vartheta q p)} r\big) dr \right)^{1/2} \right).
\end{aligned}
\tag{4.109}
$$

This proves inequality (4.105). The proof of Lemma 4.29 is thus completed. ☐

Setting 4.30. *Assume Setting 4.23, let* $q, p_1 \in [1,\infty)$, $p \in (3+\sqrt{10}, \infty)$, $\theta \in \big[\frac{p^2-1}{p^2-6p-1}, \infty\big)$, $q_0, q_1 \in (0,\infty]$ *such that* $\sum_{i=0}^1 \frac{1}{q_i} = \frac{1}{(p+1)q}$ *and that* $q(p+1) \geq 2p$, *let* $\gamma \in (0,\infty)$, *assume that for all* $s \in [0,T]$, $x_1, x_2 \in B$ *it holds* $\mathbb{P}$*-almost surely for all* $t \in [s,T]$ *that*

$$
\begin{aligned}
&2\max\left\{0, \sup_{u\in H\setminus\{0\}} \tfrac{\langle u, \int_0^1 \mu'(X^{x_1}_{s,t} + \lambda(X^{x_2}_{s,t} - X^{x_1}_{s,t}))\, d\lambda\, u\rangle_H}{\|u\|_H^2} \right\} + 2\gamma \\
&\quad + (qp+q+1) \left\| \int_0^1 \sigma'(X^{x_1}_{s,t} + \lambda(X^{x_2}_{s,t} - X^{x_1}_{s,t}))(\cdot)\, d\lambda \right\|^2_{L(H,\mathrm{HS}(U,H))} \\
&\leq \left(\phi(t) + \sum_{j=1}^{2} \tfrac{U_0(X^{x_j}_{s,t})}{2q_0(T-s)\exp(\alpha_0 t)} + \sum_{j=1}^{2} \tfrac{\overline{U}(t,X^{x_j}_{s,t})}{2q_1\exp(\alpha_1 t)} \right),
\end{aligned}
\tag{4.110}
$$

and let $V \in C^2(O,[0,\infty))$ *be a function and* $\alpha \in [0,\infty)$ *be a real number which satisfy for all* $s \in [0,T]$, $x, x_1, x_2, x_3, x_4 \in B$, *not all equal, that for all* $t \in [s,T]$ *it holds* $\mathbb{P}$*-almost surely that* $(\mathcal{G}_{\mu,\sigma}V)(X^x_{s,t}) \leq \alpha V(X^x_{s,t})$, *and that*

$$
\begin{aligned}
&\max\left\{ \tfrac{1}{\gamma^{1/2}} \int_0^1 \tfrac{\left\| \mu'\left((1-\lambda)X^{x_1}_{s,t} + \lambda X^{x_2}_{s,t}\right) - \mu'\left((1-\lambda)X^{x_3}_{s,t} + \lambda X^{x_4}_{s,t}\right) \right\|_{L(H)}}{\|(1-\lambda)(X^{x_1}_{s,t} - X^{x_3}_{s,t}) + \lambda(X^{x_2}_{s,t} - X^{x_4}_{s,t})\|_H} d\lambda, \right. \\
&\qquad \left. 2\int_0^1 \tfrac{\left\| \sigma'\left((1-\lambda)X^{x_1}_{s,t} + \lambda X^{x_2}_{s,t}\right) - \sigma'\left((1-\lambda)X^{x_3}_{s,t} + \lambda X^{x_4}_{s,t}\right) \right\|_{L(H,HS(U,H))}}{\|(1-\lambda)(X^{x_1}_{s,t} - X^{x_3}_{s,t}) + \lambda(X^{x_2}_{s,t} - X^{x_4}_{s,t})\|_H} d\lambda \right\} \\
&\leq \left(\textstyle\sum_{j=1}^4 V(X^{x_j}_{s,t}) \right)^{(p-1)/(2pp_1q\theta)}.
\end{aligned}
\tag{4.111}
$$

Lemma 4.31 (Partial strong local Lipschitz property of the first derivative process)**.** Assume Setting 4.30, let $v_1, v_2 \in H$, $s \in [0,T]$, and let $(x_1,y_1), (x_2,y_2) \in \{(x,y) \in B \times \mathbb{R} \colon x + v_i y \in B\}$.

Then it holds that

$$
\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|Z^{v_1}_{s,(s+t)\wedge T}(x_1,y_1) - Z^{v_2}_{s,(s+t)\wedge T}(x_2,y_2)\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})} \\
&\leq 6q\exp(\tfrac{1}{q}) \left(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)}\right)^{6/q} \left(\|x_1-x_2\|_H + \|y_1v_1 - y_2v_2\|_H\right)\|v_2\|_H \\
&\cdot \Big(\sum_{j=1}^{2}(V(x_j)+V(x_j+v_jy_j))\Big)^{(p-1)/(2pp_1q\theta)} \Big(\int\limits_0^{T-s} \exp\Big(r\,\tfrac{\alpha(p-1)}{(pp_1q\theta)}\Big)\,dr\Big)^{1/2} \\
&\quad\cdot \exp\Big(3\int\limits_s^T \phi(r) + \Big(\tfrac{\beta_0\left(1-\frac{(r-s)}{(T-s)}\right)}{q_0\exp(\alpha_0 r)} + \tfrac{\beta_1}{q_1\exp(\alpha_1 r)}\Big)dr\Big) \\
&\quad\cdot \exp\Big(\sum_{i=0}^{1}\Big(3\tfrac{U_i(x_1+v_1y_1)+U_i(x_1)}{2q_i\exp(\alpha_i s)} + 4\,\tfrac{U_i(x_2+v_2y_2)+U_i(x_2)}{2q_i\exp(\alpha_i s)}\Big)\Big).
\end{aligned}
\tag{4.112}
$$

Proof of Lemma 4.31. If $v_1 = 0_H$, $v_2 = 0$ then $Z^{v_1}(x_1,y_1) = 0$, $Z^{v_2}(x_2,y_2) = 0$ and the statement is trivial. If $x_1 = x_2$, $y_1 = y_2$, $v_1 = v_2$ the statement is trivial as well. Without loss of generality, assume that $v_1 \neq 0_H$ and that $(x_1,y_1,v_1) \neq (x_2,y_2,v_2)$ for the rest of the proof. Throughout this proof let $\nu\colon [s,T]\times\Omega \to H$, $\eta\colon [s,T]\times\Omega \to L(H,\mathrm{HS}(U,H))$, $\zeta\colon [s,T]\times\Omega \to \mathrm{HS}(U,H)$, $A,B\colon [s,T]\times\Omega\to[0,\infty)$ be processes which satisfy for all $t\in[s,T]$ that

$$\nu_t = \sum_{j=1}^{2}(-1)^{j+1}\int\limits_0^1 \mu'\Big(\lambda X^{x_j+v_jy_j}_{s,t} + (1-\lambda)X^{x_j}_{s,t}\Big)Z^{v}_t(x_j,y_j)\,d\lambda, \tag{4.113}$$

$$\eta_t = \int\limits_0^1 \sigma'\big(\lambda X^{x_1+v_1y_1}_{s,t} + (1-\lambda)X^{x_1}_{s,t}\big)\,d\lambda, \tag{4.114}$$

$$\zeta_t = \int\limits_0^1 \Big(\sum_{j=1}^{2}(-1)^j\sigma'(\lambda X^{x_j+v_jy_j}_{s,t} + (1-\lambda)X^{x_j}_{s,t})\Big)Z^{v_2}_{s,t}(x_2,y_2)\,d\lambda, \tag{4.115}$$

$$A_t = \max\Big\{0, \sup_{u\in H\setminus\{0\}} \tfrac{\langle u, \int_0^1 \mu'(X^{x_1}_{s,t}+\lambda(X^{x_1+v_1y_1}_{s,t}-X^{x_1}_{s,t}))\,d\lambda u\rangle_H}{\|u\|_H^2}\Big\} + \gamma, \tag{4.116}$$

$$B_t = \tfrac{1}{4\gamma}\Big\|\int\limits_0^1\sum_{j=1}^{2}(-1)^j\mu'(\lambda X^{x_j+v_jy_j}_{s,t} + (1-\lambda)X^{x_j}_{s,t})\,d\lambda\, Z^{v_2}_{s,t}(x_2,y_2)\Big\|_H^2. \tag{4.117}$$

First, Lemma 4.24 ensures that for all $t\in[s,T]$ it holds $\mathbb{P}$-almost surely that

$$Z^{v_1}_{s,t}(x_1,y_1) - Z^{v_2}_{s,t}(x_2,y_2) = \int\limits_s^t \nu_r\,dr + \int\limits_s^t \eta_r\left(Z^{v_1}_{s,r}(x_1,y_1) - Z^{v_2}_{s,r}(x_2,y_2)\right) + \zeta_r\,dW_r. \tag{4.118}$$

Next, note that subtracting and adding a term in equation (4.113) implies for all $t\in[s,T]$ that

$$
\begin{aligned}
\nu_t &= \int\limits_0^1 \mu'\Big(\lambda X^{x_1+v_1y_1}_{s,t} + (1-\lambda)X^{x_1}_{s,t}\Big)\,d\lambda\left(Z^{v_1}_{s,t}(x_1,y_1) - Z^{v_2}_{s,t}(x_2,y_2)\right) \\
&\quad + \int\limits_0^1 \mu'\Big(\lambda X^{x_1+v_1y_1}_{s,t} + (1-\lambda)X^{x_1}_{s,t}\Big) - \mu'\Big(\lambda X^{x_2+v_2y_2}_{s,t} + (1-\lambda)X^{x_2}_{s,t}\Big)\,d\lambda Z^{v_2}_{s,t}(x_2,y_2).
\end{aligned}
\tag{4.119}
$$

This, the Cauchy-Schwarz inequality, the fact that for all $a, b \in \mathbb{R}$ it holds that $ab \leq \gamma a^2 + \frac{1}{4\gamma} b^2$, and equations (4.116) and (4.117) ensure that for all $t \in [s, T]$ it holds that

$$\begin{aligned}
&\langle Z^{v_1}_{s,t}(x_1, y_1) - Z^{v_2}_{s,t}(x_2, y_2), \nu_t \rangle_H \\
&\leq \Big(\max \Big\{ 0, \sup_{u \in H \setminus \{0\}} \tfrac{\langle u, \int_0^1 \mu'(X^{x_1}_{s,t} + \lambda(X^{x_1 + v_1 y_1}_{s,t} - X^{x_1}_{s,t}))\, d\lambda u \rangle_H}{\|u\|_H^2} \Big\} + \gamma \Big) \| Z^{v_1}_{s,t}(x_1, y_1) - Z^{v_2}_{s,t}(x_2, y_2) \|_H^2 \\
&\quad + \tfrac{1}{4\gamma} \left\| \int\limits_0^1 \mu'\Big(\lambda X^{x_1+v_1 y_1}_{s,t} + (1-\lambda) X^{x_1}_{s,t}\Big) - \mu'\Big(\lambda X^{x_2+v_2 y_2}_{s,t} + (1-\lambda) X^{x_2}_{s,t}\Big)\, d\lambda \, Z^{v_2}_{s,t}(x_2, y_2) \right\|_H^2 \\
&= A_t \| Z^{v_1}_{s,t}(x_1, y_1) - Z^{v_2}_{s,t}(x_2, y_2) \|_H^2 + B_t.
\end{aligned} \tag{4.120}$$

Now, item (ii) in Proposition 4.6 (applied with $X_\cdot = Z^{v_1}_{s,\cdot}(x_1, y_1) - Z^{v_2}_{s,\cdot}(x_2, y_2)$, $\mu = \nu$, $u = p$, $p = 1$ in the notation of in Proposition 4.6) together with equation (4.118) and inequality (4.120) show that

$$\begin{aligned}
&\Big\| \sup_{t \in [s,T]} \| Z^{v_1}_{s,t}(x_1, y_1) - Z^{v_2}_{s,t}(x_2, y_2) \|_H \Big\|_{L^q(\mathbb{P}; \mathbb{R})} \\
&\leq 3q \exp(\tfrac{1}{q}) \Big\| \exp\Big(\int\limits_s^T 2A_r + (pq + q + 1) \|\eta_r\|^2_{L(H, \mathrm{HS}(U,H))}\, dr \Big) \Big\|_{L^{(p+1)q}(\mathbb{P}; \mathbb{R})} \\
&\cdot \Big(\|v_1 - v_2\|_H + \|B_\cdot^{1/2}\|_{L^2(\lambda_{[s,T]}; L^{(2pq)/(p-1)}(\mathbb{P}; \mathbb{R}))} + \|\zeta_\cdot\|_{L^2(\lambda_{[s,T]}; L^{(2pq)/(p-1)}(\mathbb{P}; \mathrm{HS}(U,H)))} \Big).
\end{aligned} \tag{4.121}$$

Next, inequality (4.110) and Lemma 4.13 (applied with $p = (p+1)q$, $p_0 = q_0$, $p_1 = q_1$, $X^{(1)}_\cdot = X^{x_1}_{s,\cdot}$, $X^{(3)}_\cdot = X^{x_1}_{s,\cdot}$, $X^{(2)}_\cdot = X^{x_1 + v_1 y_1}_{s,\cdot}$, $X^{(4)}_\cdot = X^{x_1 + v_1 y_1}_{s,\cdot}$ in the notation of Lemma 4.13) yield that

$$\begin{aligned}
&\Big\| \exp\Big(\int\limits_s^T 2A_r + (pq + q + 1) \|\eta_r\|^2_{L(H, \mathrm{HS}(U,H))}\, dr \Big) \Big\|_{L^{(p+1)q}(\mathbb{P}; \mathbb{R})} \\
&\leq \Big\| \exp\Big(\int\limits_s^T \phi(r) + \tfrac{U_0(X^{x_1}_{s,r}) + U_0(X^{x_1 + v_1 y_1}_{s,r})}{2 q_0 (T-s) \exp(\alpha_0 r)} + \tfrac{\overline{U}(r, X^{x_1}_{s,r}) + \overline{U}(r, X^{x_1 + v_1 y_1}_{s,r})}{2 q_1 \exp(\alpha_1 r)}\, dr \Big) \Big\|_{L^{(p+1)q}(\mathbb{P}; \mathbb{R})} \\
&\leq \exp\Big(\int\limits_s^T \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i \big(1 - \frac{(r-s)}{(T-s)}\big)^{1-i}}{q_i \exp(\alpha i r)}\, dr + \sum_{i=0}^{1} \tfrac{U_i(x_1 + v_1 y_1) + U_i(x_1)}{2 q_i \exp(\alpha_i s)} \Big).
\end{aligned} \tag{4.122}$$

Moreover, Hölder's inequality together with $\frac{p-1}{2pq} = \frac{p-1}{2pq\theta} + \frac{(p-1)(\theta-1)}{4pq\theta} + \frac{(p-1)(\theta-1)}{4pq\theta}$ yields that

$$\begin{aligned}
&\left\| B_{\cdot}^{1/2} \right\|^2_{L^2(\lambda_{[s,T]};L^{(2pq)/(p-1)}(\mathbb{P};\mathbb{R}))} \\
&= \int_s^T \left\| (B_t)^{1/2} \right\|^2_{L^{(2pq)/(p-1)}(\mathbb{P};\mathbb{R})} \, dt \\
&= \tfrac{1}{4\gamma} \int_s^T \left\| \int_0^1 \sum_{j=1}^2 (-1)^j \mu'(\lambda X_{s,t}^{x_j+v_j y_j} + (1-\lambda) X_{s,t}^{x_j}) \, d\lambda \, Z_{s,t}^{v_2}(x_2,y_2) \right\|^2_{L^{(2pq)/(p-1)}(\mathbb{P};H)} dt \\
&\leq \tfrac{1}{4\gamma} \int_s^T \Bigg\| \int_0^1 \frac{\left\| \mu'\left(\lambda X_{s,t}^{x_1+v_1y_1} + (1-\lambda) X_{s,t}^{x_1}\right) - \mu'\left(\lambda X_{s,t}^{x_2+v_2y_2} + (1-\lambda) X_{s,t}^{x_2}\right) \right\|_{L(H)}}{\left\| \lambda\left(X_{s,t}^{x_1+v_1y_1} - X_{s,t}^{x_2+v_2y_2}\right) + (1-\lambda)\left(X_{s,t}^{x_1} - X_{s,t}^{x_2}\right) \right\|_H} d\lambda \qquad (4.123) \\
&\quad \left(\left\| X_{s,t}^{x_1+v_1y_1} - X_{s,t}^{x_2+v_2y_2} \right\|_H + \left\| X_{s,t}^{x_1} - X_{s,t}^{x_2} \right\|_H \right) \| Z_{s,t}^{v_2}(x_2,y_2) \|_H \Bigg\|^2_{L^{(2pq)/(p-1)}(\mathbb{P};\mathbb{R})} dt \\
&\leq \tfrac{1}{4\gamma} \int_s^T \left\| \int_0^1 \frac{\left\| \mu'\left(\lambda X_{s,t}^{x_1+v_1y_1} + (1-\lambda) X_{s,t}^{x_1}\right) - \mu'\left(\lambda X_{s,t}^{x_2+v_2y_2} + (1-\lambda) X_{s,t}^{x_2}\right) \right\|_{L(H)}}{\left\| \lambda\left(X_{s,t}^{x_1+v_1y_1} - X_{s,t}^{x_2+v_2y_2}\right) + (1-\lambda)\left(X_{s,t}^{x_1} - X_{s,t}^{x_2}\right) \right\|_H} \, d\lambda \right\|^2_{L^{(2pq\theta)/(p-1)}(\mathbb{P};\mathbb{R})} dt \\
&\quad \cdot 4 \sup_{t\in[0,T]} \Bigg(\max_{h\in\{0,1\}} \left\| X_{s,(s+t)\wedge T}^{x_1+hv_1y_1} - X_{s,(s+t)\wedge T}^{x_2+hv_2y_2} \right\|^2_{L^{(4pq\theta)/((p-1)(\theta-1))}(\mathbb{P};H)} \\
&\qquad\qquad \cdot \| Z_{s,(s+t)\wedge T}^{v_2}(x_2,y_2) \|^2_{L^{(4pq\theta)/((p-1)(\theta-1))}(\mathbb{P};H)} \Bigg).
\end{aligned}$$

The fact that $\theta \geq \frac{p^2-1}{(p^2-6p-1)}$ implies that $\frac{\theta}{\theta-1} = \frac{1}{1-\frac{1}{\theta}} \leq \frac{(p^2-1)}{6p}$. This yields that

$$\frac{(4p\theta)}{(p^2-1)(\theta-1)} \leq \frac{2}{3} \qquad (4.124)$$

and that $\frac{1}{2}\frac{4pq\theta}{(p-1)(\theta-1)} \leq \frac{1}{3}q(p+1)$. Therefore, it holds that

$$\frac{1}{2}\frac{4pq\theta}{(p-1)(\theta-1)} \frac{\frac{(p^2-1)(\theta-1)}{(4p\theta)}}{\frac{(p^2-1)(\theta-1)}{(4p\theta)}-1} = \frac{1}{2}\frac{4pq\theta}{(p-1)(\theta-1)} \left(1 - \frac{(4p\theta)}{(p^2-1)(\theta-1)}\right)^{-1} \leq \frac{1}{3}q(p+1)3 \leq qp+q+1. \qquad (4.125)$$

From equation (4.125) and assumption (4.110) it follows that for all $t \in [s,T]$, $z_1, z_2 \in B$ it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\max\left\{0, \sup_{u\in H\setminus\{0\}} \frac{\left\langle u, \int_0^1 \mu'(X_{s,t}^{z_1} + \lambda(X_{s,t}^{z_2} - X_{s,t}^{z_1})) \, d\lambda \, u \right\rangle_H}{\|u\|_H^2} \right\} \\
&\quad + \frac{1}{2}\frac{4pq\theta}{(p-1)(\theta-1)} \frac{\frac{(p^2-1)(\theta-1)}{(4p\theta)}}{\frac{(p^2-1)(\theta-1)}{(4p\theta)}-1} \left\| \int_0^1 \sigma'(X_{s,t}^{z_1} + \lambda(X_{s,t}^{z_2} - X_{s,t}^{z_1}))(\cdot) \, d\lambda \right\|^2_{L(H,\mathrm{HS}(U,H))} \qquad (4.126) \\
&\leq \left(\phi(t) + \sum_{j=1}^2 \frac{U_0(X_{s,t}^{z_j})}{2q_0(T-s)\exp(\alpha_0 t)} + \sum_{j=1}^2 \frac{\overline{U}(t, X_{s,t}^{z_j})}{2q_1 \exp(\alpha_1 t)} \right)
\end{aligned}$$

and that $\frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{q(p+1)} = \frac{(p-1)(\theta-1)}{4pq\theta} \frac{4p\theta}{(p^2-1)(\theta-1)}$. This, inequalities (4.124) and (4.126) and

Lemma 4.26 (applied with $q = \frac{(4pq\theta)}{(p-1)(\theta-1)}$, $c = \frac{(p^2-1)(\theta-1)}{4p\theta}$, $x = x_2$ (resp. $x = x_2 + hv_2y_2$), $y = y_2$ (resp. $y = 1$), $v = v$ (resp. $v = x_1 - x_2 + h(v_1y_1 - v_2y_2)$) for $h \in \{0,1\}$ in the notation of Lemma 4.26) show that for all $t \in [s,T]$, $h \in \{0,1\}$ it holds that

$$\begin{aligned}
&\left\| X_{s,t}^{x_1+hv_1y_1} - X_{s,t}^{x_2+hv_2y_2} \right\|_{L^{(4pq\theta)/((p-1)(\theta-1))}(\mathbb{P};H)} \left\| Z_{s,t}^{v_2}(x_2,y_2) \right\|_{L^{(4pq\theta)/((p-1)(\theta-1))}(\mathbb{P};H)} \\
&= \left\| Z_{s,t}^{x_1-x_2+h(y_1v_1-y_2v_2)}(x_2+hv_2y_2,1) \right\|_{L^{(4pq\theta)/((p-1)(\theta-1))}(\mathbb{P};H)} \left\| Z_{s,t}^{v_2}(x_2,y_2) \right\|_{L^{(4pq\theta)/((p-1)(\theta-1))}(\mathbb{P};H)} \\
&\leq \left(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)} \right)^{\frac{2}{q}\left(1-\frac{(4p\theta)}{(p^2-1)(\theta-1)}\right)^{-1}} \left\| x_1 - x_2 + h(y_1v_1 - y_2v_2) \right\|_H \left\| v_2 \right\|_H \qquad (4.127) \\
&\quad \cdot \exp\left(2 \int_s^T \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i \left(1-\frac{(r-s)}{(T-s)}\right)^{1-i}}{q_i \exp(\alpha i r)} \, dr + 2\sum_{i=0}^{1} \tfrac{U_i(x_2+v_2y_2)+U_i(x_2)+U_i(x_1+hv_1y_1)+U_i(x_2+hv_2y_2)}{2q_i \exp(\alpha_i s)} \right).
\end{aligned}$$

Furthermore, e.g. Lemma 4.10 (applied with $T = r - s$, $(X_t^x)_{t\in[0,r-s]} = (X_{s,t+s}^x)_{t\in[0,r-s]}$, $\mathcal{F} = \bigvee_{t\in[0,r-s]} \mathbb{F}_{t+s}$, $(\mathbb{F}_t)_{t\in[0,r-s]} = (\mathbb{F}_{t+s})_{t\in[0,r-s]}$, $(W_t)_{t\in[0,r-s]} = (W_{t+s} - W_s)_{t\in[0,r-s]}$ for $r \in [s,T]$, $x \in B$ in the notation of Lemma 4.10) assures that for all $t \in [0, T-s]$, $x \in B$ it holds that

$$\mathbb{E}[V(X_{s,t+s}^x)] \leq \exp(\alpha t) V(x). \qquad (4.128)$$

Assumption (4.111) and Hölder's inequality hence imply that

$$\begin{aligned}
&\tfrac{1}{\gamma} \int_s^T \left\| \int_0^1 \tfrac{\left\| \mu'\left(\lambda X_{s,t}^{x_1+v_1y_1} + (1-\lambda) X_{s,t}^{x_1}\right) - \mu'\left(\lambda X_{s,t}^{x_2+v_2y_2} + (1-\lambda) X_{s,t}^{x_2}\right) \right\|_{L(H)}}{\left\| \lambda\left(X_{s,t}^{x_1+v_1y_1} - X_{s,t}^{x_2+v_2y_2}\right) + (1-\lambda)\left(X_{s,t}^{x_1} - X_{s,t}^{x_2}\right) \right\|_H} \, d\lambda \right\|_{L^{\frac{(2pq\theta)}{(p-1)}}(\mathbb{P};\mathbb{R})}^2 dt \\
&\leq \int_s^T \left(\mathbb{E}\left[\textstyle\sum_{j=1}^2 (V(X_{s,t}^{x_j+v_jy_j}) + V(X_{s,t}^{x_j})) \right] \right)^{(p-1)/(pp_1q\theta)} dt \qquad (4.129) \\
&\leq \left(\textstyle\sum_{j=1}^2 (V(x_j + v_jy_j) + V(x_j)) \right)^{(p-1)/(pp_1q\theta)} \int_0^{T-s} \exp\left(t \, \tfrac{\alpha(p-1)}{(pp_1q\theta)} \right) dt.
\end{aligned}$$

This and inequalities (4.123), (4.124), and (4.127) show that

$$\begin{aligned}
&\left\| B_{\cdot}^{1/2} \right\|_{L^2(\lambda_{[s,T]}; L^{(2pq)/(p-1)}(\mathbb{P};\mathbb{R}))} \\
&\leq \left(\textstyle\sum_{j=1}^2 (V(x_j + v_jy_j) + V(x_j)) \right)^{(p-1)/(2pp_1q\theta)} \left(\int_0^{T-s} \exp\left(t \, \tfrac{\alpha(p-1)}{(pp_1q\theta)} \right) dt \right)^{1/2} \\
&\quad \cdot \left(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)} \right)^{6/q} \left(\|x_1 - x_2\|_H + \|y_1v_1 - y_2v_2\|_H \right) \|v_2\|_H \qquad (4.130) \\
&\quad \cdot \exp\left(2 \int_s^T \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i \left(1-\frac{(r-s)}{(T-s)}\right)^{1-i}}{q_i \exp(\alpha i r)} \, dr + 2\sum_{i=0}^{1} \tfrac{2U_i(x_2+v_2y_2)+2U_i(x_2)+U_i(x_1)+U_i(x_1+v_1y_1)}{2q_i \exp(\alpha_i s)} \right).
\end{aligned}$$

Analogously to inequalities (4.123) - (4.130) (with μ replaced by σ and suitable norms), we obtain

that

$$\begin{aligned}
&\left\|\zeta_\cdot\right\|_{L^2(\lambda_{[s,T]};L^{(2pq)/(p-1)}(\mathbb{P};\mathrm{HS}(U,H)))}\\
&\leq \Big(\textstyle\sum_{j=1}^2 (V(x_j+v_jy_j)+V(x_j))\Big)^{(p-1)/(2pp_1q\theta)} \left(\int\limits_0^{T-s} \exp\Big(t\,\tfrac{\alpha(p-1)}{(pp_1q\theta)}\Big)\,dt\right)^{1/2}\\
&\quad\cdot \Big(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)}\Big)^{6/q} \left(\|x_1-x_2\|_H+\|v_1y_1-v_2y_2\|_H\right)\|v_2\|_H \qquad (4.131)\\
&\quad\cdot \exp\Big(2\int\limits_s^T \phi(r)+\sum_{i=0}^{1} \tfrac{\beta_i\left(1-\frac{r-s}{T-s}\right)^{1-i}}{q_i\exp(\alpha i r)}\,dr+2\sum_{i=0}^{1}\tfrac{2U_i(x_2+v_jy_2)+2U_i(x_2)+U_i(x_1)+U_i(x_1+v_jy_1)}{2q_i\exp(\alpha_i s)}\Big).
\end{aligned}$$

Combining this and inequalities (4.122) and (4.130) with inequality (4.121) establishes inequality (4.112). The proof of Lemma 4.31 is thus completed. □

Setting 4.32. *Assume Setting 4.30, let* $p_2\in[1,\infty)$, $\vartheta\in\big[2\frac{p^2-1}{p^2-p+1}\frac{p^2-3p+1}{p^2-5p+1},\infty\big)$, *and assume that for all* $(s,t)\in\Delta_T$, $x\in B$, $v\in H$, $y\in\{z\in\mathbb{R}\colon x+vz\in B\}$ *it holds* $\mathbb{P}$*-almost surely that*

$$\begin{aligned}
&\sup_{\lambda\in[0,1]}\max\left\{\|\mu'(X^x_{s,t}+\lambda(X^{x+vy}_{s,t}-X^x_{s,t}))\|_{L(H,H)},\|\sigma'(X^x_{s,t}+\lambda(X^{x+vy}_{s,t}-X^x_{s,t}))\|_{L(H,\mathrm{HS}(U,H))}\right\}\\
&\leq \left(V(X^x_{s,t})+V(X^{x+vy}_{s,t})\right)^{(p-1)/(2\vartheta qp)} \qquad (4.132)
\end{aligned}$$

and

$$\max\left\{\|\mu(X^x_{s,t})\|_H,\|\sigma(X^x_{s,t})\|_{\mathrm{HS}(U,H)}\right\}\leq\left(V(X^x_{s,t})\right)^{(p^2-p+1)/(4p(p+1)qp_2)}, \qquad (4.133)$$

assume that $\inf_{(s,t)\in\Delta_T}\inf_{\omega\in\Omega}\inf_{z\in B}\overline{U}(t,X^z_{s,t}(\omega))\in\mathbb{R}$, *and assume that for all* $k\in\mathbb{N}$ *it holds that*

$$\sup_{x\in B\colon\|x\|_H\leq k} V(x)+U_0(x)+U_1(x)<\infty. \qquad (4.134)$$

Lemma 4.33 (Local Hölder property of the first derivative process)**.** Assume Setting 4.32. Then it holds for all $k,h\in\mathbb{N}$, $v\in H$ that

$$\sup_{\substack{(s_1,x_1,y_1)\neq(s_2,x_2,y_2)\in[0,T]\times B\times[0,h]\\ x_1+vy_1,x_2+vy_2\in B\ \|x_1\|_H\leq k,\|x_2\|_H\leq k}}\frac{\left\|\sup_{t\in[0,T]}\|Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1)-Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2)\|_H\right\|_{L^q(\mathbb{P};\mathbb{R})}}{\left(\|x_1-x_2\|_H^2+|y_1-y_2|^2+|s_1-s_2|^2\right)^{1/4}}<\infty. \qquad (4.135)$$

Proof. Throughout this proof let $B_k=\{z\in H\colon\|z\|_H\leq k\}$, $k\in\mathbb{N}$, and let $K\in\mathbb{R}$ be the real number satisfying that $K=\inf_{(s,t)\in\Delta_T}\inf_{\omega\in\Omega}\inf_{z\in B}\overline{U}(t,X^z_{s,t}(\omega))$.

First, note that Lemma 4.27 ensures that for all $(s_1,s_2)\in\Delta_T$, $x_1,x_2\in B$, $v\in H$, $y_i\in\{z\in$

$\mathbb{R}\colon x_i + vy_i \in B\}$ it holds $\mathbb{P}$-almost surely for all $t \in [0,T]$ that

$$\begin{aligned}
&Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2)\\
&= Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^v_{s_1,(t+s_2)\wedge T}(x_2,y_2) + Z^v_{s_1,(t+s_2)\wedge T}(x_2,y_2) - Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2)\\
&= Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^{Z^v_{s_1,s_2}(x_2,y_2)}_{s_2,(t+s_2)\wedge T}(X^{x_2}_{s_1,s_2},y_2)\\
&\quad + Z^{Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2)}_{(t+s_1)\wedge T,(t+s_2)\wedge T}(X^{x_2}_{s_1,(t+s_1)\wedge T},y_2) - Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2).
\end{aligned} \tag{4.136}$$

Observe that in particular for all $(s_1,s_2) \in \Delta_T$, $x_1,x_2 \in B$, $v \in H$, $y_i \in \{z \in \mathbb{R}\colon x_i + vy_i \in B\}$, $j \in \{1,2\}$, it holds $\mathbb{P}$-almost surely for all $t \in [T-s_1,T]$ that $(t+s_1)\wedge T = (t+s_2)\wedge T$ and $Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2) = Z^v_{s_2,T}(x_1,y_1) - Z^{Z^v_{s_1,s_2}(x_2,y_2)}_{s_2,T}(X^{x_2}_{s_1,s_2},y_2)$.
Note that by applying the triangle inequality we obtain for all $x_1,x_2 \in B$, $(s_1,s_2) \in \Delta_T$, $v \in H$, $y_i \in \{z \in \mathbb{R}\colon x_i + vy_i \in B\}$, $j \in \{1,2\}$, that

$$\begin{aligned}
&\Big\| \sup_{t\in[0,T]} \|Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^v_{s_1,(t+s_1)\wedge T}(x_2,y_2)\|_H \Big\|_{L^q(\mathbb{P};\mathbb{R})}\\
&\le \Big(\mathbb{E}\Big[\sup_{t\in[0,T]} \|Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^{Z^v_{s_1,s_2}(x_2,y_2)}_{s_2,(t+s_2)\wedge T}(X^{x_2}_{s_1,s_2},y_2)\|^q_H\Big]\Big)^{1/q}\\
&\quad + \sup_{t\in[0,T-s_1)} \Big(\mathbb{E}\Big[\|Z^{Z^v_{s_1,t+s_1}(x_2,y_2)}_{t+s_1,(t+s_2)\wedge T}(X^{x_2}_{s_1,t+s_1},y_2) - Z^v_{s_1,t+s_1}(x_2,y_2)\|^q_H\Big]\Big)^{1/q}\\
&\le \Big(\mathbb{E}\Big[\mathbb{E}\Big[\sup_{t\in[0,T]} \|Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^{Z^v_{s_1,s_2}(x_2,y_2)}_{s_2,(t+s_2)\wedge T}(X^{x_2}_{s_1,s_2},y_2)\|^q_H \Big| \mathbb{F}_{s_2}\Big]\Big]\Big)^{1/q}\\
&\quad + \sup_{t\in[0,T-s_1)} \Big(\mathbb{E}\Big[\mathbb{E}\Big[\|Z^{Z^v_{s_1,t+s_1}(x_2,y_2)}_{t+s_1,(t+s_2)\wedge T}(X^{x_2}_{s_1,t+s_1},y_2) - Z^v_{s_1,t+s_1}(x_2,y_2)\|^q_H \Big| \mathbb{F}_{t+s_1}\Big]\Big]\Big)^{1/q}.
\end{aligned} \tag{4.137}$$

Let us consider the two latter terms separately.
Observe that Lemma 4.31 (applied with $\mathcal{F} = \mathbb{F}_{s_2}$, $\mathbb{P} = \mathbb{P}|_{\mathbb{F}_{s_2}}$, $x_1 = X^{x_2}_{s_1,s_2}$, $x_2 = x_1$, $y_1 = y_2$, $y_2 = y_1$, $v_2 = v$, $v_1 = Z^v_{s_1,s_2}(x_2,y_2)$, and $s = s_2$, in the notation of Lemma 4.31), Hölder's inequality, and the fact that

$$\tfrac{1}{q} = \tfrac{3}{(p+1)q} + \tfrac{p^2-3p-1}{2p(p+1)q} + \tfrac{p^2-p+1}{2p(p+1)q} \tag{4.138}$$

show that for all $x_1,x_2 \in B$, $(s_1,s_2) \in \Delta_T$, $v \in H$, $y_i \in \{z \in \mathbb{R}\colon x_i + vy_i \in B\}$, $j \in \{1,2\}$, it holds that

$$
\begin{aligned}
&\left(\mathbb{E}\left[\mathbb{E}\Big[\sup_{t\in[0,T]}\|Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1)-Z^{Z^v_{s_1,s_2}(x_2,y_2)}_{s_2,(t+s_2)\wedge T}(X^{x_2}_{s_1,s_2},y_2)\|^q_H\Big|\mathbb{F}_{s_2}\Big]\right]\right)^{1/q}\\
&\leq 6q\exp(\tfrac{1}{q})\left(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)}\right)^{6/q}\|v\|_H\left(\int_0^{T-s_2}\exp\left(r\,\tfrac{\alpha(p-1)}{(pp_1q\theta)}\right)dr\right)^{1/2}\\
&\quad\cdot\Big\|\big(\|x_1-X^{x_2}_{s_1,s_2}\|_H+\|y_1v-y_2Z^v_{s_1,s_2}(x_2,y_2)\|_H\big)\\
&\qquad\cdot\big(V(x_1)+V(x_1+vy_1)+V(X^{x_2}_{s_1,s_2})+V(X^{x_2+vy_2}_{s_1,s_2})\big)^{(p-1)/(2pp_1q\theta)}\\
&\qquad\cdot\exp\left(\sum_{i=0}^{1}\left(3\tfrac{U_i(X^{x_2}_{s_1,s_2})+U_i(X^{x_2+vy_2}_{s_1,s_2})}{2q_i\exp(\alpha_is_2)}+4\,\tfrac{U_i(x_1+vy_1)+U_i(x_1)}{2q_i\exp(\alpha_is_2)}\right)\right)\Big\|_{L^q(\mathbb{P};\mathbb{R})}\\
&\qquad\cdot\exp\left(3\int_{s_2}^{T}\phi(r)+\left(\tfrac{\beta_0\left(1-\frac{(r-s_2)}{(T-s_2)}\right)}{q_0\exp(\alpha_0r)}+\tfrac{\beta_1}{q_1\exp(\alpha_1r)}\right)dr\right)\\
&\leq 6q\exp(\tfrac{1}{q})\left(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)}\right)^{6/q}\|v\|_H\left(\int_0^{T-s_2}\exp\left(r\,\tfrac{\alpha(p-1)}{(pp_1q\theta)}\right)dr\right)^{1/2}\\
&\quad\cdot\Big(\big\|x_1-X^{x_2}_{s_1,s_2}\big\|_{L^{(2p(p+1)q)/(p^2-p+1)}(\mathbb{P};H)}\\
&\qquad\quad+|y_1-y_2|\|v\|_H+|y_2|\,\big\|v-Z^v_{s_1,s_2}(x_2,y_2)\big\|_{L^{(2p(p+1)q)/(p^2-p+1)}(\mathbb{P};H)}\Big)\\
&\qquad\cdot\Big(\mathbb{E}\big[\big(V(x_1)+V(x_1+vy_1)+V(X^{x_2}_{s_1,s_2})+V(X^{x_2+vy_2}_{s_1,s_2})\big)^{(p^2-1)/(\theta p_1(p^2-3p-1))}\big]\Big)^{(p^2-3p-1)/(2p(p+1)q)}\\
&\qquad\cdot\exp\left(4\sum_{i=0}^{1}\tfrac{U_i(x_1+vy_1)+U_i(x_1)}{2q_i\exp(\alpha_is_2)}\right)\Big\|\exp\left(\sum_{i=0}^{1}\tfrac{U_i(X^{x_2}_{s_1,s_2})+U_i(X^{x_2+vy_2}_{s_1,s_2})}{2q_i\exp(\alpha_is_2)}\right)\Big\|^3_{L^{(p+1)q}(\mathbb{P};\mathbb{R})}\\
&\qquad\cdot\exp\left(3\int_{s_2}^{T}\phi(r)+\left(\tfrac{\beta_0\left(1-\frac{(r-s_2)}{(T-s_2)}\right)}{q_0\exp(\alpha_0r)}+\tfrac{\beta_1}{q_1\exp(\alpha_1r)}\right)dr\right).
\end{aligned}
\tag{4.139}
$$

Note that $(p+1)q=\frac{2p(p+1)}{(p^2-p+1)}\cdot q\cdot\big(\frac{p^2-3p+1}{2p}+1\big)$ and the fact that $p\geq 3+\sqrt{10}$ ensures that $\frac{p^2-3p+1}{2p}\geq 1$. Lemma 4.18 (applied with $T=s_2$, $s=s_1$, $q=\frac{2p(p+1)}{p^2-p+1}q$, $p=1$, $u=p_2$, $x_1=x_2$ for $(s_1,s_2)\in\Delta_T$, $x_1,x_2\in B$ in the notation of Lemma 4.18), Lemma 4.29 (applied with $T=s_2$, $s=s_1$, $q=\frac{2p(p+1)}{p^2-p+1}q$, $p=\frac{p^2-3p+1}{2p}$, $\vartheta=\frac{(p^2-p+1)(p^2-5p+1)}{2(p^2-1)(p^2-3p+1)}\vartheta$, $w=v$, $x=x_2$, $y=y_2$ for $(s_1,s_2)\in\Delta_T$, $v\in H$, $x_2\in B$, $y_2\in\{y\in\mathbb{R}\colon x_2+vy\in B\}$ in the notation of Lemma 4.29), the fact that $\frac{\theta p_1(p^2-3p-1)}{p^2-1}\geq\frac{p_1(p^2-3p-1)}{p^2-6p-1}\geq p_1\big(1+\frac{3p}{p^2-6p-1}\big)\geq 1$, Hölder's inequality, and Lemma 4.10 (applied with $T=s_2-s_1$, $(\mathbb{F}_t)_{t\in[0,s_2-s_1]}=(\mathbb{F}_{t+s_1})_{t\in[0,s_2-s_1]}$, $(W_t)_{t\in[0,s_2-s_1]}=(W_{t+s_1}-W_{s_1})_{t\in[0,s_2-s_1]}$, $X_\cdot=X^{x_2}_{s_1,s_1+\cdot}$ (resp. with $X_\cdot=X^{x_2+vy_2}_{s_1,s_1+\cdot}$) with $(s_1,s_2)\in\Delta_T$, $v\in H$, $x_2\in B$, $y_2\in\{y\in\mathbb{R}\colon x_2+yv\in B\}$ in the notation of Lemma 4.10) yield for all $x_1,x_2\in B$, $(s_1,s_2)\in\Delta_T$, $v\in H$,

$y_i \in \{z \in \mathbb{R} \colon x_i + v y_i \in B\}$, $j \in \{1,2\}$, that

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\mathbb{E}\Big[\sup_{t\in[0,T]} \Big\| Z^v_{s_2,(t+s_2)\wedge T}(x_1,y_1) - Z^{Z^v_{s_1,s_2}(x_2,y_2)}_{s_2,(t+s_2)\wedge T}(X^{x_2}_{s_1,s_2}, y_2)\Big\|^q_H \Big| \mathbb{F}_{s_2}\Big]\Big]\Big)^{1/q} \\
&\leq 6q \exp(\tfrac{1}{q}) \Big(\tfrac{(p^2-1)(\theta-1)}{(4p\theta)}\Big)^{6/q} \|v\|_H \Big(\int\limits_0^{T-s_2} \exp\Big(r\, \tfrac{\alpha(p-1)}{(pp_1q\theta)}\Big) dr\Big)^{1/2} \\
&\quad \cdot \Big(3^{\frac{2p(p+1)q}{p^2-p+1}} \exp(s_2 - s_1) \exp(\tfrac{p^2-p+1}{2p(p+1)q}) \\
&\Big[\|x_1 - x_2\|_H + 2\big(V(x_2) \exp\big(\alpha(s_2 - s_1)\big)\big)^{(p^2-p+1)/(4p(p+1)qp_2)} (s_2 - s_1)^{1/2} \\
&\qquad + |y_1 - y_2| \|v\|_H + |y_2| 3^{\frac{2p(p+1)q}{p^2-p+1}} \exp(\tfrac{p^2-p+1}{2p(p+1)q})\Big] \\
&\qquad \cdot \exp\Big(\int\limits_{s_1}^{s_2} \phi(r) + \sum_{i=0}^{1} \tfrac{\beta_i (1 - \frac{(r-s_1)}{(s_2-s_1)})^{1-i}}{q_i \exp(\alpha i r)}\, dr + \sum_{i=0}^{1} \tfrac{U_i(x_2+vy_2)+U_i(x_2)}{2q_i \exp(\alpha_i s_1)}\Big) \qquad (4.140) \\
&\qquad \cdot \|v\|_H \big(\tfrac{1}{2\sqrt{\gamma}} + 1\big)\big(V(x_2) + V(x_2 + vy_2)\big)^{\frac{(p-1)}{(2p\vartheta q)}} \Big(\int\limits_0^{s_2-s_1} \exp(\alpha \tfrac{(p-1)}{(p\vartheta q)} r) dr\Big)^{1/2}\Big) \\
&\quad \cdot \Big(V(x_1) + V(x_1 + vy_1) + (V(x_2) + V(x_2 + vy_2)) \exp\big(\alpha(s_2 - s_1)\big)\Big)^{(p-1)/(2pp_1q\theta)} \\
&\quad \cdot \exp\Big(2 \sum_{i=0}^{1} \tfrac{U_i(x_1+vy_1)+U_i(x_1)}{q_i \exp(\alpha_i s_2)}\Big) \prod_{i=0}^{1} \Big\| \exp\Big(\tfrac{U_i(X^{x_2}_{s_1,s_2})+U_i(X^{x_2+vy_2}_{s_1,s_2})}{2q_i \exp(\alpha_i s_2)}\Big)\Big\|^3_{L^{q_i}(\mathbb{P};\mathbb{R})} \\
&\quad \cdot \exp\Big(3 \int\limits_{s_2}^{T} \phi(r) + \Big(\tfrac{\beta_0 (1 - \frac{(r-s_2)}{(T-s_2)})}{q_0 \exp(\alpha_0 r)} + \tfrac{\beta_1}{q_1 \exp(\alpha_1 r)}\Big) dr\Big).
\end{aligned}$$

Let us consider the last term of inequality (4.137). Lemma 4.29 (applied with $T = (t + s_2) \wedge T$, $\mathcal{F} = \mathbb{F}_{t+s_1}$, $\mathbb{P} = \mathbb{P}|_{\mathbb{F}_{t+s_1}}$, $x = X^{x_2}_{s_1,t+s_1}$, $y = y_2$, $v = Z^v_{s_1,t+s_1}(x_2,y_2)$, $w = Z^v_{s_1,t+s_1}(x_2,y_2)$, $s = t + s_1$ for $v \in H$, $x_2 \in B$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in B\}$, $(s_1,s_2) \in \Delta_T$, $t \in [0, T - s_1)$ in the notation of Lemma 4.29), the fact that

$$\tfrac{1}{q} = \tfrac{1}{(p+1)q} + \tfrac{p^2-p-1}{2p(p+1)q} + \tfrac{p^2+p+1}{2p(p+1)q}, \tag{4.141}$$

and Hölder's inequality demonstrate for all $x_2 \in B$, $(s_1,s_2) \in \Delta_T$, $v \in H$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in$

$B\}$, that

$$
\begin{aligned}
&\sup_{t\in[0,T-s_1)} \Big(\mathbb{E}\Big[\mathbb{E}\Big[\Big\| Z_{t+s_1,(t+s_2)\wedge T}^{Z^v_{s_1,t+s_1}(x_2,y_2)}(X^{x_2}_{s_1,t+s_1}, y_2) - Z^v_{s_1,t+s_1}(x_2,y_2)\Big\|_H^q \Big| \mathbb{F}_{t+s_1}\Big]\Big]\Big)^{1/q} \\
&\le 3q\exp\big(\tfrac{1}{q}\big) \sup_{t\in[0,T-s_1)} \Big\| \exp\Big(\int\limits_{t+s_1}^{(t+s_2)\wedge T} \phi(r) \\
&\qquad\qquad + \sum_{i=0}^{1} \frac{\beta_i\big(1-\frac{(r-t-s_1)}{((t+s_2)\wedge T-t-s_1)}\big)^{1-i}}{q_i\exp(\alpha ir)}\, dr + \sum_{i=0}^{1} \frac{U_i(X^{x_2+vy_2}_{s_1,t+s_1})+U_i(X^{x_2}_{s_1,t+s_1})}{2q_i\exp(\alpha_i(t+s_1))}\Big) \\
&\quad \cdot \|Z^v_{s_1,t+s_1}(x_2,y_2)\|_H \big(\tfrac{1}{2\sqrt{\gamma}}+1\big)\Big(\int\limits_0^{(t+s_2)\wedge T-t-s_1} \exp(\alpha\tfrac{(p-1)}{(\vartheta qp)}r)dr\Big)^{1/2} \\
&\quad \cdot \big(V(X^{x_2}_{s_1,t+s_1}) + V(X^{x_2+vy_2}_{s_1,t+s_1})\big)^{(p-1)/(2\vartheta qp)}\Big\|_{L^q(\mathbb{P};\mathbb{R})} \qquad (4.142)\\
&\le 3q\exp\big(\tfrac{1}{q}\big)\big(\tfrac{1}{2\sqrt{\gamma}}+1\big) \sup_{t\in[0,T-s_1)} \exp\Big(\int\limits_{t+s_1}^{(t+s_2)\wedge T}\phi(r) + \sum_{i=0}^{1}\frac{\beta_i(1-\frac{(r-t-s_1)}{((t+s_2)\wedge T-t-s_1)})^{1-i}}{q_i\exp(\alpha ir)}\, dr\Big) \\
&\quad \cdot \sup_{t\in[0,T-s_1]}\Big\|\exp\Big(\sum_{i=0}^{1}\frac{U_i(X^{x_2+vy_2}_{s_1,t+s_1})+U_i(X^{x_2}_{s_1,t+s_1})}{2q_i\exp(\alpha_i(t+s_1))}\Big)\Big\|_{L^{(p+1)q}(\mathbb{P};\mathbb{R})} \\
&\quad \cdot \Big\|\sup_{t\in[0,T-s_1]}\|Z^v_{s_1,t+s_1}(x_2,y_2)\|_H\Big\|_{L^{(2p(p+1)q)/(p^2+p+1)}(\mathbb{P};\mathbb{R})} \Big(\sup_{s\in[0,T-s_1)}((s+s_2)\wedge T-(s+s_1))\Big)^{1/2} \\
&\quad \cdot \exp(\alpha\tfrac{(p-1)}{(p\vartheta q)}r) \sup_{t\in[0,T-s_1]}\Big\|\big(V(X^{x_2}_{s_1,t+s_1})+V(X^{x_2+vy_2}_{s_1,t+s_1})\big)^{(p-1)/(2\vartheta qp)}\Big\|_{L^{(2p(p+1)q)/(p^2-p-1)}(\mathbb{P};\mathbb{R})}.
\end{aligned}
$$

Note that the fact that for all $a\in\mathbb{R}$ it holds that $a^2-a+1>0$ and $(a-1)^2\ge 0$ ensures that $\frac{p}{p^2-p+1}\le 1$ and that $\frac{p^2+p+1}{2p}\ge 1$. This implies that $\frac{1}{2}\frac{2p(p+1)}{p^2+p+1}\cdot q\cdot\frac{\frac{p^2+p+1}{2p}}{\frac{p^2+p+1}{2p}-1} = \frac{(p+1)pq}{p^2-p+1}\le (p+1)q < (p+1)q+1$. This and the fact that $(p+1)q = \frac{2p(p+1)}{p^2+p+1}\cdot q\cdot\frac{p^2+p+1}{2p}$ ensure that the assumptions of Lemma 4.26 are satisfied. Lemma 4.26 (applied with $q=\frac{2p(p+1)}{p^2+p+1}q$, $c=\frac{p^2+p+1}{2p}$, $s=s_1$, $x=x_2$, $y=y_2$ for $s_1\in[0,T]$, $t\in[0,T-s_1]$, $v\in H$, $x_2\in B$, $y_2\in\{y\in\mathbb{R}\colon x_2+vy\in B\}$ in the notation of Lemma 4.26), the fact that $\frac{p-1}{2\vartheta pq}\frac{2p(p+1)q}{p^2-p-1}\le 1$, Hölder's inequality, and Lemma 4.10 (applied with $T=t$, $(\mathbb{F}_s)_{s\in[0,t]}=(\mathbb{F}_{s+s_1})_{s\in[0,t]}$, $(W_s)_{s\in[0,t]}=(W_{s+s_1}-W_{s_1})_{s\in[0,t]}$, $(X_s)_{s\in[0,t]}=(X^{x_2}_{s_1,s_1+s})_{s\in[0,t]}$ (resp. with $(X_s)_{s\in[0,t]}=(X^{x_2+vy_2}_{s_1,s_1+s})_{s\in[0,t]}$) for $t\in[0,T-s_1)$, $x_2\in B$, $v\in H$, $y_2\in\{y\in\mathbb{R}\colon x_2+vy\in B\}$ in the notation of Lemma 4.10) show that for all $x_2\in B$, $(s_1,s_2)\in\Delta_T$,

$v \in H$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in B\}$ that

$$
\begin{aligned}
&\sup_{t\in[0,T-s_1)} \left(\mathbb{E}\left[\mathbb{E}\left[\| Z^{Z^v_{s_1,t+s_1}(x_2,y_2)}_{t+s_1,(t+s_2)\wedge T}(X^{x_2}_{s_1,t+s_1}, y_2) - Z^v_{s_1,t+s_1}(x_2,y_2) \|_H^q \middle| \mathbb{F}_{t+s_1} \right] \right] \right)^{1/q} \\
&\leq 3q \exp\left(\tfrac{1}{q}\right)\left(\tfrac{1}{2\sqrt{\gamma}} + 1\right) \sup_{t\in[0,T-s_1)} \exp\left(\int\limits_{t+s_1}^{(t+s_2)\wedge T} \phi(r) + \sum_{i=0}^{1} \frac{\beta_i \left(1 - \frac{(r-t-s_1)}{((t+s_2)\wedge T - t - s_1)}\right)^{1-i}}{q_i \exp(\alpha i r)} \, dr \right) \\
&\quad \cdot \sup_{t\in[0,T-s_1]} \prod_{i=0}^{1} \left\| \exp\left(\tfrac{U_i(X^{x_2+vy_2}_{s_1,t+s_1}) + U_i(X^{x_2}_{s_1,t+s_1})}{2q_i \exp(\alpha_i(t+s_1))} \right) \right\|_{L^{q_i}(\mathbb{P};\mathbb{R})} \qquad (4.143) \\
&\quad \cdot \left(\tfrac{p^2+p+1}{2p} \right)^{\frac{(p^2+p+1)^2}{(2p(p+1)(p^2-p+1)q)}} \|v\|_H \exp\left(\int\limits_{s_1}^{T} \phi(r) + \sum_{i=0}^{1} \frac{\beta_i\left(1 - \frac{(r-s_1)}{(T-s_1)}\right)^{1-i}}{q_i \exp(\alpha i r)} \, dr + \sum_{i=0}^{1} \frac{U_i(x_2+vy_2) + U_i(x_2)}{2q_i \exp(\alpha_i s_1)} \right) \\
&\quad \cdot (s_2 - s_1)^{1/2} \exp(\alpha \tfrac{(p-1)}{(p\vartheta q)} r) \sup_{t\in[0,T-s_1]} \exp(\alpha \tfrac{(p-1)}{(2pq\vartheta)} t) \big(V(x_2) + V(x_2+vy_2) \big)^{(p-1)/(2\vartheta q p)}.
\end{aligned}
$$

Note that Hölder's inequality and Corollary 4.12 (applied with $T = s_1 + t$, $s = s_1$, $X = X^{x_2+jvy_2}_{s_1,}$, $U_0 = U_i$, $\overline{U} = i\overline{U} - \beta_i$, $\alpha = \alpha_i$ for $i, j \in \{0,1\}$, $x_2 \in B$, $s_1 \in [0,T]$, $t \in [0, T-s_1]$, $v \in H$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in B\}$ in the notation of Corollary 4.12) yield for all $x_2 \in B$, $s_1 \in [0,T]$, $t \in [0, T-s_1]$, $v \in H$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in B\}$ that

$$
\begin{aligned}
&\sup_{t\in[0,T-s_1]} \prod_{i=0}^{1} \left\| \exp\left(\tfrac{U_i(X^{x_2+vy_2}_{s_1,t+s_1}) + U_i(X^{x_2}_{s_1,t+s_1})}{2q_i \exp(\alpha_i(t+s_1))} \right) \right\|_{L^{q_i}(\mathbb{P};\mathbb{R})} \\
&\leq \sup_{t\in[0,T-s_1]} \prod_{i=0}^{1} \left\| \exp\left(\tfrac{U_i(X^{x_2+vy_2}_{s_1,t+s_1})}{2q_i \exp(\alpha_i(t+s_1))} \right) \right\|_{L^{2q_i}(\mathbb{P};\mathbb{R})} \left\| \exp\left(\tfrac{U_i(X^{x_2}_{s_1,t+s_1})}{2q_i \exp(\alpha_i(t+s_1))} \right) \right\|_{L^{2q_i}(\mathbb{P};\mathbb{R})} \qquad (4.144) \\
&\leq \sup_{t\in[0,T-s_1]} \prod_{j=0}^{1} \prod_{i=0}^{1} \mathbb{E}\left[\exp\left(\tfrac{U_i(X^{x_2+jvy_2}_{s_1,t+s_1})}{\exp(\alpha_i(t+s_1))} + \int\limits_{s_1}^{t+s_1} \tfrac{i\overline{U}(X^{x_2+jvy_2}_{s_1,r}) - \beta_i}{\exp(\alpha_i r)} dr \right) \exp\left(\int\limits_{s_1}^{t+s_1} -\tfrac{iK-\beta_i}{\exp(\alpha_i r)} dr \right) \right]^{1/2q_i} \\
&\leq \prod_{j=0}^{1} \prod_{i=0}^{1} \exp\left(\tfrac{U_i(x_2+jvy_2)}{2q_i \exp(\alpha_i s_1)} \right) \left(\sup_{t\in[0,T-s_1]} \exp\left(\int\limits_{s_1}^{t+s_1} -\tfrac{iK-\beta_i}{2q_i \exp(\alpha_i r)} dr \right) \right) \\
&\leq \prod_{i=0}^{1} \exp\left(\tfrac{U_i(x_2)+U_i(x_2+vy_2)}{2q_i \exp(\alpha_i s_1)} \right) \exp\left(\tfrac{\max\{-iK+\beta_i, 0\} T}{2q_i \exp(\alpha_i s_1)} \right).
\end{aligned}
$$

Analogously, applying Corollary 4.12 (with $T = s_2$, $s = s_1$, $X = X^{x_2+jvy_2}_{s_1,}$, $U_0 = U_i$, $\overline{U} = i\overline{U} - \beta_i$, $\alpha = \alpha_i$ for $i, j \in \{0,1\}$, $x_2 \in B$, $(s_1, s_2) \in \Delta_T$, $v \in H$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in B\}$ in the notation of Corollary 4.12) yields for all $x_2 \in B$, $(s_1, s_2) \in \Delta_T$, $v \in H$, $y_2 \in \{y \in \mathbb{R} \colon x_2 + vy \in B\}$ it holds that

$$
\prod_{i=0}^{1} \left\| \exp\left(\tfrac{U_i(X^{x_2}_{s_1,s_2}) + U_i(X^{x_2+vy_2}_{s_1,s_2})}{2q_i \exp(\alpha_i s_2)} \right) \right\|^3_{L^{q_i}(\mathbb{P};\mathbb{R})} \leq \prod_{i=0}^{1} \exp\left(\tfrac{3(U_i(x_2) + U_i(x_2+vy_2) + \max\{-iK+\beta_i, 0\}(s_2 - s_1))}{2q_i \exp(\alpha_i s_1)} \right). \qquad (4.145)
$$

Combining this and inequalities (4.140), (4.144), (4.143), and (4.134) with inequality (4.137) establishes inequality (4.135). The proof of Lemma 4.33 is thus completed. $\square$

4.4.3 Existence of a continuously differentiable flow

Proposition 4.34 (Existence of a C^1-solution)**.** *Assume Setting 4.32, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, and assume that $q > 2(\dim(H) + 2)$, Then there exists a $\mathcal{B}(\Delta_T \times \overline{B}) \otimes \mathcal{F}/\mathcal{B}(\overline{B})$-measurable function $\mathcal{X}\colon \Delta_T \times \overline{B} \times \Omega \to \overline{B}$ and a $\mathcal{B}(\Delta_T \times \overline{B}) \otimes \mathcal{F}/\mathcal{B}(L(H))$-measurable function $\mathcal{X}^1\colon \Delta_T \times \overline{B} \times \Omega \to L(H)$ satisfying that*

1. *for all $x \in B$, $(s,t) \in \Delta_T$ it holds $\mathbb{P}$-almost surely that $\mathcal{X}^x_{s,t} = X^x_{s,t}$,*
2. *for all $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C^0(\Delta_T \times \overline{B}, \overline{B})$,*
3. *for all $x \in D$, $s \in [0,T]$, $h \in \mathbb{H}$ it holds $\mathbb{P}$-almost surely that $(\mathcal{X}^{1,x}_{s,t} h)_{t\in[s,T]} = (Z^h_{s,t}(x,0))_{t\in[s,T]}$,*
4. *for all $\omega \in \Omega$ it holds that $\mathcal{X}^1(\omega) \in C^0(\Delta_T \times \overline{B}, L(H))$,*
5. *for all $(s,t) \in \Delta_T$ it holds that $D \ni x \mapsto X^x_{s,t}(\omega) \in H$ is differentiable and that for all $\omega \in \Omega$, $x \in D$ it holds that $\frac{\partial}{\partial x}\mathcal{X}^x_{s,t}(\omega) = \mathcal{X}^{1,x}_{s,t}(\omega)$.*

Proof of Proposition 4.34. For all $k \in \mathbb{N}$ let $B_k = \{v \in H\colon \|v\|_H \leq k\}$, for all $h \in H$ let $B^h = \{(x,y) \in B \times \mathbb{R}\colon x + yh \in B\}$, let $B^{\mathbb{H}} = \cap_{h\in\mathbb{H}} \overline{B^h}$, let $D^{\mathbb{H}} = \cap_{h\in\mathbb{H}}\{(x,y) \in D \times \mathbb{R}\colon x + hy \in D\}$, let $\mathcal{D} = D \cap \{v \in H\colon (\langle h, v\rangle_H)_{h\in\mathbb{H}} \in \mathbb{Q}^{\dim(\mathbb{H})}\}$, and let $\mathcal{D}^{\mathbb{H}} = D^{\mathbb{H}} \cap (\mathcal{D} \times \mathbb{Q})$.
Lemma 4.21 (with $q = 2q$, $c = \frac{p+1}{2}$, $u = \frac{p^2 p_2}{p^2-p+1}$ in the notation of Lemma 4.21) assures the existence of a $\mathcal{B}(\Delta_T \times \overline{B}) \otimes \mathcal{F}/\mathcal{B}(\overline{B})$-measurable function $Y\colon \Delta_T \times \overline{B} \times \Omega \to \overline{B}$ with the property that for all $\omega \in \Omega$ it holds that $Y(\omega) \in C(\Delta_T \times \overline{B}, \overline{B})$ and that for all $x \in B$, $s \in [0,T]$ it holds $\mathbb{P}$-almost surely that

$$(Y^x_{s,t})_{t\in[s,T]} = (X^x_{s,t})_{t\in[s,T]}. \tag{4.146}$$

This and the fact that $\mathcal{D}$ is countable implies that for all $s \in [0,T]$, $t \in [s,T]$, $x \in \mathcal{D}$ it holds $\mathbb{P}$-almost surely that $Y^x_{s,t} = X^x_{s,t}$.
Moreover, note that for all $v \in H$ Lemma 4.26 (with $q = 2q$, $c = \frac{p+1}{2}$) ensures that the function

$$[0,T] \times B^v \ni (s,(x,y)) \mapsto Z^v_{s,(\cdot+s)\wedge T}(x,y) \in L^q(\mathbb{P}; C([0,T]; H)) \tag{4.147}$$

is locally bounded and Lemma 4.33 implies that the latter function is locally $1/2$-Hölder continuous. Hence, Proposition 4.8 (applied with $D = [0,T] \times B^v$, $H = \mathbb{R} \otimes H \otimes \mathbb{R}$, $d = \dim(H) + 2$, $p = q$, $\alpha = 1/2$, $X = \Big([0,T] \times B^v \ni (s,(x,y)) \mapsto Z^v_{s,(\cdot+s)\wedge T}(x,y) \in C([0,T],B)\Big)$, $E = F = C([0,T],H)$ for $v \in H$ in the notation of Proposition 4.8) proves that for all $v \in H$ there exists a $\mathcal{B}(\overline{[0,T] \times B^v}) \otimes \mathcal{F}/\mathcal{B}(C([0,T],H))$-measurable function $\mathcal{Z}^v\colon [0,T] \times \overline{B^v} \times \Omega \to C([0,T],H)$ such that for all $\omega \in \Omega$ it holds that $\mathcal{Z}^v(\cdot,\cdot,\cdot,\omega) \in C(\overline{[0,T] \times B^v}, C([0,T],H))$ and such that for all $s \in [0,T]$, $(x,y) \in B^v$ it holds $\mathbb{P}$-almost surely that $(\mathcal{Z}^v(s,x,y)(t))_{t\in[0,T]} = (Z^v_{s,(t+s)\wedge T}(x,y))_{t\in[0,T]}$. Let $\mathcal{Z}\colon \Delta_T \times B^{\mathbb{H}} \times \Omega \to L(H)$ be the unique function satisfying for all $\omega \in \Omega$, $(s,t) \in \Delta_T$, $(x,y) \in B^{\mathbb{H}}$, $v \in H$ that $\mathcal{Z}(s,t,x,y,\omega)(v) = \sum_{h\in\mathbb{H}} \langle v, h\rangle_H \mathcal{Z}^h(s,x,y,\omega)(t-s)$

and let $Z\colon \Delta_T \times \cap_{h\in\mathbb{H}} B^h \times \Omega \to L(H)$ be the unique function satisfying for all $\omega \in \Omega$, $(s,t) \in \Delta_T$, $(x,y) \in \cap_{h\in\mathbb{H}} B^h$, $v \in H$ that $Z(s,t,x,y,\omega)(v) = \sum_{h\in\mathbb{H}} \langle v,h\rangle_H Z^h_{s,t}(x,y,\omega)$. Therefore, it holds that $\mathcal{Z}\colon \Delta_T \times B^{\mathbb{H}} \times \Omega \to L(H)$ is $\mathcal{B}(\Delta_T \times B^{\mathbb{H}}) \otimes \mathcal{F}/\mathcal{B}(L(H))$-measurable, that for all $\omega \in \Omega$ it holds that $\mathcal{Z}(\cdot,\cdot,\cdot,\omega) \in C(\Delta_T \times B^{\mathbb{H}}, L(H))$, and that for all $s \in [0,T]$, $(x,y) \in \cap_{h\in\mathbb{H}} B^h$ it holds $\mathbb{P}$-almost surely that

$$(\mathcal{Z}(s,t,x,y))_{t\in[s,T]} = (Z(s,t,x,y))_{t\in[s,T]}. \tag{4.148}$$

It follows from this and equation (4.90) that there exists an $\Omega_0 \in \mathcal{F}$ which satisfies that $\mathbb{P}(\Omega_0) = 1$ and that it holds for all $\omega \in \Omega_0$, $(s,t) \in \Delta_T$, $h \in \mathbb{H}$, $(x,y) \in \mathcal{D}^{\mathbb{H}} \setminus (\mathcal{D} \times \{0\})$ that

$$\mathcal{Z}(s,t,x,y,\omega)h = \frac{Y^{x+hy}_{s,t}(\omega) - Y^x_{s,t}(\omega)}{y}. \tag{4.149}$$

The assumptions of Lemma 4.7 (applied with $U = H$, $T = \Delta_T$, $O = D$, $D = \mathcal{D}$, $(\mathcal{X}_t)_{t\in\Delta_T} = (Y_t|_D)_{t\in\Delta_T}$, $(\mathcal{Z}(t))_{t\in\Delta_T} = (\mathcal{Z}(t)|_{D^{\mathbb{H}}})_{t\in\Delta_T}$ in the notation of Lemma 4.7) are satisfied. Then item (i) and item (ii) of Lemma 4.7 ensure that there exists an $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$, $(s,t) \in \Delta_T$ it holds that the mapping $D \ni x \mapsto Y^x_{s,t}(\omega) \in B$ is continuously Fréchet-differentiable and it holds for all $(s,t) \in \Delta_T$, $x \in D$ that $\frac{d}{dx} Y^x_{s,t}(\omega) = \mathcal{Z}(s,t,x,0,\omega)$. Moreover, observe that the fact that $D \times \{0\} \subseteq \cap_{h\in\mathbb{H}} B^h$ and equation (4.148) imply that for all $s \in [0,T]$, $x \in D$, $h \in \mathbb{H}$ it holds $\mathbb{P}$-almost surely that $(\mathcal{Z}(s,t,x,0)h)_{t\in[s,T]} = (Z^h_{s,t}(x,0))_{t\in[s,T]}$. Let $\mathcal{X}\colon \Delta_T \times \overline{B} \times \Omega \to \overline{B}$, $\mathcal{X}^{1,\cdot}_{\cdot}\colon \Delta_T \times \overline{B} \times \Omega \to L(H)$ be the functions satisfying for all $(s,t) \in \Delta_T$, $\omega \in \Omega$, $x \in \overline{B}$ that

$$\mathcal{X}^x_{s,t}(\omega) = \begin{cases} Y^x_{s,t}(\omega) & : \omega \in \Omega_1 \\ 0_H & : \omega \in \Omega \setminus \Omega_1 \end{cases} \tag{4.150}$$

and

$$\mathcal{X}^{1,x}_{s,t}(\omega) = \begin{cases} \mathcal{Z}(s,t,x,0,\omega) & : \omega \in \Omega_1 \\ 0_{L(H)} & : \omega \in \Omega \setminus \Omega_1. \end{cases} \tag{4.151}$$

The fact that for all $h \in \mathbb{H}$ it holds that $\overline{B} \times \{0\} \subseteq \overline{B^h}$ ensures that $\overline{B} \times \{0\} \subseteq B^{\mathbb{H}}$ and therefore the latter function is well defined. The choice of these two functions completes the proof of Proposition 4.34. $\square$

Bibliography

Alekseev, V.M. (1961). *An estimate for the perturbations of the solution of ordinary differential equations (Russian).* Vestn. Mosk.Univ., Ser.I, Math. Meh, 2, pp. 28–36.

Alòs, Elisa and David Nualart (1998). "An extension of Itô's formula for anticipating processes". In: *Journal of Theoretical Probability* 11.2, pp. 493–514.

Attanasio, Stefano (2010). "Stochastic flows of diffeomorphisms for one-dimensional SDE with discontinuous drift". In: *Electronic Communications in Probability* 15, pp. 213–226.

Beccari, Matteo, Martin Hutzenthaler, Arnulf Jentzen, Ryan Kurniawan, Felix Lindner, and Diyora Salimova (2019). "Strong and weak divergence of exponential and linear-implicit Euler approximations for stochastic partial differential equations with superlinearly growing nonlinearities". In: *arXiv:1903.06066*, p. 65.

Becker, Sebastian, Benjamin Gess, Arnulf Jentzen, and Peter E Kloeden (2017). "Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen–Cahn equations". In: *arXiv preprint arXiv:1711.02423.*

Becker, Sebastian and Arnulf Jentzen (2019). "Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg–Landau equations". In: *Stochastic Processes and their Applications* 129.1, pp. 28–69.

Bréhier, Charles-Edouard, Jianbo Cui, and Jialin Hong (2018). "Strong convergence rates of semi-discrete splitting approximations for stochastic Allen–Cahn equation". In: *arXiv preprint arXiv:1802.06372.*

Bréhier, Charles-Edouard and Ludovic Goudenège (2018). "Analysis of Some Splitting Schemes for the Stochastic Allen–Cahn Equation". In: *arXiv preprint arXiv:1801.06455.*

Cerone, Pietro and Silvestru Sever Dragomir (2010). *Mathematical inequalities: a perspective.* Chapman and Hall/CRC.

Cox, Sonja G., Martin Hutzenthaler, and Arnulf Jentzen (2013). "Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations". In: *arXiv:1309.5595v1*, pp. 1–54.

Da Prato, Giuseppe and Jerzy Zabczyk (1992). *Stochastic equations in infinite dimensions.* Vol. 44. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, pp. xviii+454. ISBN: 0-521-38529-6.

Fang, Shizan, Peter Imkeller, and Tusheng Zhang (2007). "Global flows for stochastic differential equations without global Lipschitz conditions". In: *Ann. Probab.* 35.1, pp. 180–205. ISSN: 0091-1798. DOI: `10.1214/009117906000000412`. URL: `http://dx.doi.org/10.1214/009117906000000412`.

Fang, Shizan and Tusheng Zhang (2005). "A study of a class of stochastic differential equations with non-Lipschitzian coefficients". In: *Probab. Theory Related Fields* 132.3, pp. 356–390. ISSN: 0178-8051. DOI: `10.1007/s00440-004-0398-z`. URL: `http://dx.doi.org/10.1007/s00440-004-0398-z`.

Gawarecki, Leszek and Vidyadhar Mandrekar (2010). *Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations.* Springer Science & Business Media.

Gröbner, W. (1960). *Die Liereihen und ihre Anwendungen.* second edition(1967). Berlin: VEB Deutscher Verlag der Wiss.

Grorud, A. and É. Pardoux (1992). "Intégrales hilbertiennes anticipantes par rapport à un processus de Wiener cylindrique et calcul stochastique associé". In: *Appl. Math. Optim.* 25.1, pp. 31–49. ISSN: 0095-4616. DOI: `10.1007/BF01184155`. URL: `http://dx.doi.org/10.1007/BF01184155`.

Gyöngy, István and Nicolai Krylov (1996). "Existence of strong solutions for Itô's stochastic equations via approximations". In: *Probab. Theory Related Fields* 105.2, pp. 143–158. ISSN: 0178-8051. DOI: `10.1007/BF01203833`. URL: `http://dx.doi.org/10.1007/BF01203833`.

Gyöngy, István and Annie Millet (2007). "Rate of convergence of implicit approximations for stochastic evolution equations". In: *Stochastic differential equations: theory and applications.* Vol. 2. Interdiscip. Math. Sci. World Sci. Publ., Hackensack, NJ, pp. 281–310.

Hairer, Ernst, Syvert P. Nørsett, and Gerhard Wanner (1993). *Solving Ordinary Differential Equations I.* second edition. Berlin Heidelberg New York: Springer-Verlag.

Hairer, Martin, Martin Hutzenthaler, and Arnulf Jentzen (2015). "Loss of regularity for Kolmogorov equations". In: *Ann. Probab.* 43.2, pp. 468–527. ISSN: 0091-1798. DOI: `10.1214/13-AOP838`. URL: `http://dx.doi.org/10.1214/13-AOP838`.

Hudde, Anselm, Martin Hutzenthaler, Arnulf Jentzen, and Sara Mazzonetto (2018). "On the Itô-Alekseev-Gröbner formula for stochastic differential equations". In: *arXiv:1812.09857*, p. 29.

Hudde, Anselm, Martin Hutzenthaler, and Sara Mazzonetto (2019a). "A stochastic Gronwall inequality and applications to moments, strong completeness, strong local Lipschitz continuity, perturbations". In: *arXiv:1903.08727*, p. 21.

— (2019b). "Existence of spatially differentiable solutions of stochastic differential equations with non-globally monotone coefficient functions". In: *arXiv:1903.09707*, p. 22.

Hutzenthaler, Martin and Arnulf Jentzen (2014). "On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients". In: *arXiv:1401.0295*, pp. 1–41.

Hutzenthaler, Martin, Arnulf Jentzen, and Peter E. Kloeden (2011). "Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients". In: *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 467, pp. 1563–1576. DOI: `10.1098/rspa.2010.0348`. URL: `http://rspa.royalsocietypublishing.org/content/early/2010/12/08/rspa.2010.0348.abstract`.

— (2013). "Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations". In: *Ann. Appl. Probab.* 23.5, pp. 1913–1966.

Hutzenthaler, Martin, Arnulf Jentzen, and Xiaojie Wang (2018). "Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations". In: *Math. Comp.* 87.311, pp. 1353–1413.

Iserles, Arieh (2009). *A first course in the numerical analysis of differential equations.* Second. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, pp. xx+459. ISBN: 978-0-521-73490-5.

Iserles, Arieh and Gustaf Söderlind (Mar. 1993). "Global Bounds on Numerical Error for Ordinary Differential Equations". In: *Journal of Complexity* 9, pp. 97–112.

Jentzen, A. and P. Pušnik (2015). "Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities". In: *arXiv:1504.03523*, pp. 1–38.

Kallenberg, Olav (2002). *Foundations of Modern Probability.* Applied probability. Springer. ISBN: 9780387953137. URL: `https://books.google.de/books?id=L6fhXh13OyMC`.

Karatzas, Ioannis and Steven Shreve (2012). *Brownian motion and stochastic calculus.* Vol. 113. Springer Science & Business Media.

Kato, Tosio (1980). *Perturbation theory for linear operators.* Vol. 132. Springer Science & Business Media.

Klenke, Achim (2008). *Probability theory.* Universitext. A comprehensive course, Translated from the 2006 German original. London: Springer-Verlag London Ltd., pp. xii+616. ISBN: 978-1-84800-047-6. DOI: `10.1007/978-1-84800-048-3`. URL: `http://dx.doi.org/10.1007/978-1-84800-048-3`.

Kovács, Mihály, Stig Larsson, and Fredrik Lindgren (2015). "On the backward Euler approximation of the stochastic Allen-Cahn equation". In: *J. Appl. Probab.* 52.2, pp. 323–338. ISSN: 0021-9002. DOI: `10.1239/jap/1437658601`. URL: `http://dx.doi.org/10.1239/jap/1437658601`.

Kruse, Raphael (2014). *Strong and weak approximation of semilinear stochastic evolution equations.* Vol. 2093. Lecture Notes in Mathematics. Springer, Cham, pp. xiv+177. ISBN: 978-3-319-02230-7; 978-3-319-02231-4. DOI: `10.1007/978-3-319-02231-4`. URL: `http://dx.doi.org/10.1007/978-3-319-02231-4`.

Kunita, H. (1986). *Lectures on stochastic flows and applications.* Vol. 78. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Published for the Tata Institute of Fundamental Research, Bombay, pp. v+121. ISBN: 3-540-12878-6.

León, Jorge A. and David Nualart (1998). "Stochastic evolution equations with random generators". In: *Ann. Probab.* 26.1, pp. 149–186. ISSN: 0091-1798. DOI: `10.1214/aop/1022855415`. URL: `http://dx.doi.org/10.1214/aop/1022855415`.

Li, Xue-Mei (1994). "Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds". In: *Probab. Theory Related Fields* 100.4, pp. 485–511. ISSN: 0178-8051. DOI: `10.1007/BF01268991`. URL: `http://dx.doi.org/10.1007/BF01268991`.

Li, Xue-Mei and Michael Scheutzow (2011). "Lack of strong completeness for stochastic flows". In: *Ann. Probab.* 39.4, pp. 1407–1421.

Lie, Ivar and Syvert Norsett (Jan. 1989). "Superconvergence for Multistep Collocation". In: 52, pp. 65–65.

Liu, Zhihui and Zhonghua Qiao (2018a). "Strong Approximation of Monotone Stochastic Partial Different Equations Driven by Multiplicative Noise". In: *arXiv preprint arXiv:1811.05392*.

— (2018b). "Strong approximation of stochastic Allen-Cahn equation with white noise". In: *arXiv preprint arXiv:1801.09348*.

Majee, Ananta K and Andreas Prohl (2018). "Optimal Strong Rates of Convergence for a Space-Time Discretization of the Stochastic Allen–Cahn Equation with Multiplicative Noise". In: *Computational Methods in Applied Mathematics* 18.2, pp. 297–311.

Mittmann, Katrin and Ingo Steinwart (2003). "On the existence of continuous modifications of vector-valued random fields". In: *Georgian Mathematical Journal* 10.2, pp. 311–317.

Nørsett, S. P. and G. Wanner (1979). "The real-pole sandwich for rational approximations and oscillation equations". In: *BIT Numerical Mathematics* 19.1, pp. 79–94.

Nualart, David (1995). *The Malliavin calculus and related topics*. Probability and its Applications (New York). New York: Springer-Verlag, pp. xii+266. ISBN: 0-387-94432-X.

— (2006). *The Malliavin Caculus and Related Topics*. second edition. Berlin Heidelberg: Springer-Verlag.

Prévôt, Claudia and Michael Röckner (2007). *A concise course on stochastic partial differential equations*. Vol. 1905. Lecture Notes in Mathematics. 144 pages. Berlin: Springer, pp. vi+144. ISBN: 978-3-540-70780-6; 3-540-70780-8.

Revuz, Dominique and Marc Yor (2013). *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg. ISBN: 9783662064009. URL: `https://books.google.de/books?id=OYbnCAAAQBAJ`.

Richard, U (1972). "Sur des inégalités du type Wirtinger et leurs application aux équationes différentielles ordinaires". In: *Colloquium of Analysis held in Rio de Janeiro*. Vol. 2334244.

Schenk-Hoppé, Klaus Reiner (1996). "Deterministic and stochastic Duffing-van der Pol oscillators are non-explosive". In: *Z. Angew. Math. Phys.* 47.5, pp. 740–759. ISSN: 0044-2275. DOI: `10.1007/BF00915273`. URL: `http://dx.doi.org/10.1007/BF00915273`.

Schmalfuß, Björn (1997). "The random attractor of the stochastic Lorenz system". In: *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* 48 (6), pp. 951–975.

Wang, Xiaojie (2018). “An efficient explicit full discrete scheme for strong approximation of stochastic Allen–Cahn equation”. In: *arXiv preprint arXiv:1802.09413*.

Zhang, Xicheng (2010). “Stochastic flows and Bismut formulas for stochastic Hamiltonian systems”. In: *Stochastic Process. Appl.* 120.10, pp. 1929–1949. ISSN: 0304-4149. DOI: `10.1016/j.spa.2010.05.015`. URL: `http://dx.doi.org/10.1016/j.spa.2010.05.015`.