# Local models, Mustafin varieties and semi-stable resolutions

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**Abstract**

In this thesis we will analyse singularities of local models. More precisely we will attack the question of existence of semi-stable resolutions. We will discuss an approach mentioned in [Gen00]. In this approach a candidate for a semi-stable resolution was given as the blow-up of a Grassmannian variety in Schubert varieties of its special fiber. Explicit calculations with Sage described in Appendix D show that this approach is not working in general. Starting from the proof of flatness of the local models in [Gör01], we describe these local models as Mustafin varieties over Grassmannian varieties. We are combining several results on the structure of Mustafin varieties over projective spaces (cf. [CHSW11],[AL17]) with the Plücker embedding to be able to construct a candidate for a semi-stable resolution of local models. Under some additional assumptions this candidate is generalising the approach suggested by Genestier. Furthermore under the same assumptions the new candidate agrees with the semi-stable resolution constructed in [Gör04] for small dimensions.

**Zusammenfassung**

In dieser Arbeit beschäftigen wir uns mit den Singularitäten von lokalen Modellen. Genauer untersuchen wir die Existenz semi-stabiler Auflösungen. Wir werden einen in [Gen00] erwähnten Ansatz diskutieren. Dieser Ansatz beschreibt einen Kandidaten für eine Auflösung als Aufblasung von Grassmannschen Varietäten in den Schubert Varietäten der speziellen Faser. Explizite Berechnungen in Appendix D zeigen, dass dieser Ansatz nicht zum Ziel führt. Ausgehend von dem Beweis der Flachheit der lokalen Modelle in [Gör01] lassen sich die lokalen Modelle auch als Mustafin-Varietäten über Grassmannschen Varietäten auffassen. Unter anderem in [CHSW11] und [AL17] wurden Mustafin-Varietäten über projektiven Räumen ausgiebig untersucht, was uns ermöglicht mit Hilfe der Plücker Einbettung einen Kandidaten für eine semi-stabile Auflösung zu konstruieren. Unter gewissen Voraussetzungen verallgemeinert dieser den Kandidaten von Genestier und stimmt mit der in [Gör04] konstruierten Auflösung für kleine Dimensionen überein.

## Contents

## Introduction

**Motivation:** In the study of Shimura varieties it is of great interest to construct models over the ring of integers $\mathcal{O}$ of the completion of the reflex field at a prime with finite residue characteristic $p$. These models should at least be flat and ideally have mild singularities. The special case of Shimura varieties of PEL type are moduli spaces of abelian varieties with some extra structures (polarisation, endomorphism and level structure). For parahoric level structures candidates for such models are constructed by Rapoport and Zink in [RZ96] by posing the moduli problem over $\mathcal{O}$.

In the attempt to analyse the occurring singularities they define so called *local models*. These models are constructed as projective varieties over $\mathcal{O}$ and are supposed to model singularities of the integral models. More precisely every point in an integral model has an étale neighbourhood isomorphic to an étale neighbourhood of the corresponding local model. The advantage of local models is that they are cut out in a product of Grassmannian varieties by equations arising from linear algebra and hence are easier to handle. Although these models are not flat in general as pointed out by Pappas in [Pap00], it was proven by Görtz that local models in the so called *linear case* and the *symplectic case* are flat (cf. [Gör01] and [Gör03]).

In the present thesis we will focus on local models in the linear case for $F = \mathbb{Q}_p$ with Iwahori level structure and study their singularities in this case. Below we will give a precise definition of the local model corresponding to this data.

In the case as above the local model over the complete discrete valuation ring $\mathcal{O}$ with uniformizer $\pi$ and quotient field K is constructed as follows. Fix two natural numbers $k < n$ and the canonical basis $\{e_i\}_i$ of $\mathrm{K}^n$. For $i \leq n-1$ we denote by $\Lambda_i$ the lattice generated by the elements $\pi^{-1}e_0, \dots, \pi^{-1}e_{i-1}, e_i, \dots, e_{n-1}$ and define the standard lattice chain $\Gamma^{\mathrm{st}}$ to be

$$\dots \to \Lambda_0 \to \Lambda_1 \to \dots \to \Lambda_n = \pi^{-1}\Lambda_0 \to \dots$$

For an $\mathcal{O}$-scheme $S$ we write $\Lambda_{i,S}$ for $\Lambda_i \otimes_{\mathcal{O}} \mathcal{O}_S$ and define the $S$-valued points of the functor $\mathcal{M}^{\mathrm{loc}}$ to be diagrams of the form

$$\begin{tikzcd} \Lambda_{0,S} \arrow[r] & \Lambda_{1,S} \arrow[r] & \dots \arrow[r] & \Lambda_{n-1,S} \arrow[r, "\pi"] & \Lambda_{0,S} \\ \mathcal{F}_0 \arrow[r] \arrow[u, hook] & \mathcal{F}_1 \arrow[r] \arrow[u, hook] & \dots \arrow[r] & \mathcal{F}_{n-1} \arrow[r] \arrow[u, hook] & \mathcal{F}_0 \arrow[u, hook] \end{tikzcd}$$

where the $\mathcal{F}_i$'s are locally free $\mathcal{O}_S$-submodules of $\Lambda_{i,S}$ of rank $k$ that are Zariski-locally direct summands.

We can easily identify the generic fiber $\mathcal{M}^{\mathrm{loc}}_{\mathrm{K}}$ with the Grassmannian $\mathrm{Gr}^{n,r}_{\mathrm{K}}$, but the special fiber is much more complicated.

Let us illustrate some of the behaviour in the case $n = 2$, $k = 1$ and $\mathcal{O} = \mathbb{Z}_p$ (cf. [Hai05, Section 4.4]). In this case $\mathcal{M}^{\mathrm{loc}}$ is modelling the singularities of the modular curve endowed with $\Gamma_0(p)$-level structure.

Fix a $\mathbb{Z}_p$-algebra $R$. To simplify the notation let us identify $\Lambda_{0,R}$ and $\Lambda_{1,R}$ with $R \oplus R$. The $R$-valued points $\mathcal{M}^{\mathrm{loc}}(R)$ are now given by commutative diagrams of the form

$$\begin{tikzcd} R \oplus R \arrow[r, "\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}"] & R \oplus R \arrow[r, "\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}"] & R \oplus R \\ \mathcal{F}_0 \arrow[r] \arrow[u, hook] & \mathcal{F}_1 \arrow[r] \arrow[u, hook] & \mathcal{F}_0 \arrow[u, hook] \end{tikzcd}$$

where $\mathcal{F}_i$ is an element in $\mathbb{P}^1(R)$ for $i = 1, 2$. Let us fix local coordinates and take a pair

$(\mathcal{F}_0, \mathcal{F}_1) \in \mathbb{P}^1(R) \times \mathbb{P}^1(R)$. First we note that if $\mathcal{F}_0$ is represented by a homogeneous column vector $[x:1]^{\mathrm{t}}$ then the image $[px:1]^{\mathrm{t}}$ again represents an element in $\mathbb{P}^1(R)$ hence has to coincide with $\mathcal{F}_1$. In particular this chart of the local model can be identified with a chart in $\mathbb{P}^1_{\mathbb{Z}_p}$. Now let us assume that $\mathcal{F}_0$ is represented by a homogeneous vector of the form $[1:x]^{\mathrm{t}}$ and $\mathcal{F}_1$ is represented by $[y:1]^{\mathrm{t}}$. In particular we see that the pair $(\mathcal{F}_0, \mathcal{F}_1)$ is in $\mathcal{M}^{\mathrm{loc}}(R)$ precisely when $xy = p$. Hence in these charts we can describe the local model as $\mathrm{Spec}\,(\mathbb{Z}_p[x,y]/(xy-p))$. Gluing the charts lets us identify $\mathcal{M}^{\mathrm{loc}}$ with the blow-up of $\mathbb{P}^1_{\mathbb{Z}_p}$ in the origin of the special fiber. In particular the special fiber $\mathcal{M}^{\mathrm{loc}}_{\mathbb{F}_p}$ is described by two projective spaces $\mathbb{P}^1_{\mathbb{F}_p}$ intersecting transversally in one point.

Generalising the type of singularities of the example above leads to the notion of semi-stability defined below. More detailed discussions of this definition can be found for example in [dJ96] or [Har01].

**Definition.** *For a complete discrete valuation ring $\mathcal{O}$ with uniformizer $\pi$ we call an $\mathcal{O}$-variety $X$ semi-stable if étale locally $X$ is of the form*

$$\mathrm{Spec}\left(\mathcal{O}[x_0,\ldots,x_r]/(\prod_{i\le m} x_i - \pi)\right)$$

*for some $r$ and $m$.*

As a generalisation of the example above it is well known that the local models $\mathcal{M}^{\mathrm{loc}}$ in the so called *Drinfeld case* (cf. [RZ96]), i.e for $k=1$ and $n$ arbitrary, are semi-stable (see [RZ96, Section 3.69] cf. also [Fal01] or [Mus78]). In [Fal01] Faltings also constructs toroidal resolutions for $k=2$.
We can also define a symplectic version of the local model cf. [Gen00]. This version is obtained by imposing a certain self-duality condition in the moduli description above. In loc. cit. a semi-stable resolution of the local model in the symplectic case for $n \le 6$ was constructed. It was also suggested that a similar construction produces a semi-stable resolution $\mathcal{G} \to \mathcal{M}^{\mathrm{loc}}$ in the linear case.
A semi-stable resolution $\widetilde{\mathcal{G}} \to \mathcal{M}^{\mathrm{loc}}$ for $n \le 5$ at least for an open neighbourhood of the "most singular point" was given in [Gör04].

**Main results:** The goal of this thesis is to construct a candidate for a semi-stable resolution for arbitrary $n$ and $k$. Starting with the observation that the candidate $\mathcal{G}$ in [Gen00] does not factor over $\mathcal{M}^{\mathrm{loc}}$ for $n=5$ and $k=2$ (see Appendix D for explicit Sage calculations), we consider the strict transform $\mathcal{S}$ of the projection $\mathcal{M}^{\mathrm{loc}} \to \mathrm{Gr}^k(\Lambda_0)$ under the blow-up $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$. For this strict transform we can show the following theorem:

**Theorem.** *For $n=5$ and $k=2$ the blow-up $\mathcal{S}$ is a semi-stable resolution of $\mathcal{M}^{loc}$. By passing to a neighbourhood of the worst singularity of $\mathcal{M}^{loc}$ one recovers the local semi-stable resolution defined in* [Gör04].

If $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{M}^{\mathrm{loc}}$, the projection $\mathcal{S} \to \mathcal{G}$ is an isomorphism. We thus have shown, that $\mathcal{S}$ is a better candidate for a semi-stable resolution.
But since it is hard to show the semi-stability of $\mathcal{S}$ (cf. [Gör04]), we will adapt the idea of blowing up $\mathrm{Gr}^k(\Lambda_0)$ and construct a candidate $\mathcal{M}$ for a semi-stable resolution as a blow-up $\mathcal{M} \to \mathrm{Gr}^k(\Lambda_0)$ but with slightly different centers. The advantage of this approach is, that in some cases we might be able to use a lemma in [Gen00] showing that semi-stability is preserved under blow-ups provided that the centers of the blow-ups are sufficiently nice.
In contrast to loc. cit. we will use the language of Mustafin varieties and the recent results

on their behaviour (cf. [CHSW11], [AL17]). Since $\mathcal{M}^{\mathrm{loc}}$ is flat (see [Gör01]), we can identify it with the closure of the generic fiber embedded into $\prod_{\Lambda_i \in \Gamma^{\mathrm{st}}} \mathrm{Gr}^k(\Lambda_i)$, which is by definition the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})$. The main idea for the construction of $\mathcal{M}$ is to use the Plücker embedding $\mathrm{Gr}^k(\Lambda_0) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ and a compatible embedding of $\mathcal{M}^{\mathrm{loc}}$ into the Mustafin variety $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma^{\mathrm{st}})$ where $\bigwedge^k \Gamma^{\mathrm{st}}$ denotes the set $\{\bigwedge^k \Lambda_i | \Lambda_i \in \Gamma^{\mathrm{st}}\}$. We expect that these two Mustafin varieties have the same number of irreducible components of their special fibers. In this case describe an explicit bijection of the sets of irreducible components in Conjecture 4.1 (see Section 4.2 for a more detailed discussion). Assuming the conjecture, we can show the following behaviour of this embedding above.

**Proposition.** *Assume Conjecture 4.1. Then we have a bijection*

$$\left\{ C \,\middle|\, C \text{ irr. component of } \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p} \right\} \longrightarrow \left\{ C \,\middle|\, C \text{ irr. component of } \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p} \right\}$$
$$C \longmapsto C \cap \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$$

*between the sets of irreducible components of the special fibres.*

We will discuss that a Mustafin variety $\mathcal{M}_{\mathbb{P}}(\Gamma)$ is semi-stable if $\Gamma$ is convex (cf. [Fal01]). In particular if we denote by $\overline{\bigwedge^k \Gamma^{\mathrm{st}}}$ the convex closure of $\bigwedge^k \Gamma^{\mathrm{st}}$, then $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ is a semi-stable resolution of $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma^{\mathrm{st}})$. Moreover $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ is given by a sequence of blow-ups $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ (cf. [Fal01]). The candidate $\mathcal{M}$ is now defined as the strict transform $\mathcal{M} \to \mathrm{Gr}^k(\Lambda_0)$ of the Plücker embedding $\mathrm{Gr}^k(\Lambda_0) \subseteq \mathbb{P}(\bigwedge^k \Lambda_0)$ under this sequence of blow-ups. Although we are not able to show the semi-stability of $\mathcal{M}$, we can still show the theorem below under some technical conditions. Let us denote with $\mathcal{S}^{\mathrm{pl}}$ the blow-up of $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma^{\mathrm{st}})$ constructed similarly to the blow-up $\mathcal{S} \to \mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})$ (see Section 5 for a more detailed discussion).

**Proposition.** *Assume Conjecture 4.1 and that $\mathcal{S}$ is semi-stable. Then $\mathcal{S}$ and $\mathcal{M}$ coincide in both of the following cases:*

*(i) $\mathcal{S}^{\mathrm{pl}}$ is semi-stable*

*(ii) for every irreducible component $C$ of $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ the intersection $C \cap \mathcal{S}$ is a union of irreducible components*

We should also remark that it is not hard to show that for $n \leq 4$ all the candidates $\mathcal{M}$, $\mathcal{S}$ and $\mathcal{G}$ are isomorphic and give semi-stable resolutions of $\mathcal{M}^{\mathrm{loc}}$. This computations can be done using for example the program in Appendix D.

**Structure of the thesis:** This thesis is divided into five chapters. In Chapter 1 we will start by providing some background on blow-ups. Besides getting a short overview on some basic constructions related to blow-ups, the construction of the blow-up as a $\mathbb{G}_m$-quotient found in [Gen00] will be explicitly used later on. Due to the lack of references for this particular point of view we will provide some proofs.

The aim of Chapter 2 is to define the local model and to describe some of its combinatorial behaviour. Therefore we will start by giving some background on alcoves and extended affine Weyl groups. Then we will go on with a brief overview on affine flag varieties. In the last part of the chapter we will define local models and use the preceding sections to get some insight on the combinatorics of their special fibers.

We will start Chapter 3 by defining Mustafin varieties $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma)$ for general sets $\Gamma$ of lattice classes. Then we describe the local model as the special case $\mathcal{M}^{\mathrm{loc}} = \mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})$ where $\Gamma^{\mathrm{st}}$ is the set of homothety classes of the lattices in the standard lattice chain. In the second section we will prove in Proposition 3.22 that if $\Gamma$ is convex the Mustafin variety $\mathcal{M}_{\mathbb{P}}(\Gamma)$ for projective spaces can be described as a blow-up of $\mathbb{P}(\Lambda)$ for any class $[\Lambda]$ in $\Gamma$. This was stated and proven before in [Fal01]. Then we will use this sequence of blow-ups and a result stating that in certain situations semi-stability is preserved under blow-ups (cf. [Gen00, Lemma 3.2.1]), to show that for $\Gamma$ convex the Mustafin variety $\mathcal{M}_{\mathbb{P}}(\Gamma)$ is semi-stable. Also this result was stated in [Fal01] but proven in a different way using the moduli description available for Mustafin varieties over projective spaces.

In Chapter 4 we are dealing with the combinatorics of Mustafin varieties in the special case of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$, where $\bigwedge^k \Gamma^{\mathrm{st}}$ is the set $\{[\bigwedge^k \Lambda_i] | [\Lambda_i] \in \Gamma^{\mathrm{st}}\}$ of homothety classes of lattices in $\bigwedge^k \mathbb{Q}^n$. In the first section we will start by analysing irreducible components of the special fiber $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ of the Mustafin variety for the convex closure $\overline{\bigwedge^k \Gamma^{\mathrm{st}}}$ of $\bigwedge^k \Gamma^{\mathrm{st}}$. In particular we will define linear subspaces $\mathbb{P}(V_I) \subseteq \mathbb{P}(\bigwedge^k \Lambda_0)_{\mathbb{F}_p}$ for every $I \subseteq \{0, \dots, n-1\}$ with $k$ elements and in Lemma 4.6 identify these spaces with the images of irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$. Moreover we will show in Proposition 4.15 that an irreducible component is uniquely determined by its image.
In the second section we will discuss the relation of irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ and irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ and then formulate Conjecture 4.1. This conjecture states that the image in $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ of an irreducible component of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ is an irreducible component and we get a bijection on the sets of irreducible components in this way. We will prove the conjecture in Proposition 4.17 for $k = 2$ and refer to Appendix C for the Sage calculations confirming the conjecture for $n \leq 7$.
In the last section we then will introduce the Plücker embedding $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \to \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ and show that Conjecture 4.1 implies that we have an explicit bijection between the set of irreducible components of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ and that of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$.

The main goal of Chapter 5 is to construct a candidate $\mathcal{M}$ for a semi-stable resolution. This will be done by taking the strict transform of the Plücker embedding $\mathrm{Gr}^k(\Lambda_0) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ under the blow-ups $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ constructed in Chapter 3. Using Conjecture 4.1 we can show that under some technical assumptions this generalises the candidate $\mathcal{G}$ mentioned in [Gen00], i.e. if $\mathcal{G}$ is a semi-stable resolution of $\mathcal{M}^{\mathrm{loc}}$ then it coincides with $\mathcal{M}$.
Moreover we are able to show that for $n = 5$ and $k = 2$ the candidate $\mathcal{S}$ restricts to the semi-stable resolution constructed in [Gör04].
Although the results presented in the last two chapters are stated over the ring of $p$-adic integers $\mathbb{Z}_p$, they should be generalisable to every complete discrete valuation ring.

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## 1. Blow-up in different perspectives

We will start by collecting some well known results on blow-ups. As references for omitted proofs and the theory in general we recommend [Hau06] for a first introduction. A more detailed discussion can be found for example in [GW10] or [EH00].

**Definition 1.1** (Blow-up via universal property)**.** Fix a scheme $X$ and a closed subscheme $Z \subseteq X$. An $X$-scheme $(\tilde{X}, \pi)$ is called *blow-up* of $X$ with center $Z$, if $E := \pi^{-1}(Z)$ is an effective Cartier divisor in $\tilde{X}$ and every $X$-scheme $(X', \tau)$, such that $E' := \tau^{-1}(Z)$ is an effective Cartier divisor in $X'$, factors uniquely through $(\tilde{X}, \pi)$.
When we want to be more precise about the center we write $\mathrm{BL}_Z(X)$ for the blow-up.

**Lemma 1.2.** *If the closed subscheme $Z$ is a disjoint union of two closed subschemes $Z_1, Z_2$, then the blow-up $\pi\colon \mathrm{Bl}_Z(X) \to X$ can be described as the blow-up of $\mathrm{Bl}_{Z_1}(X) \xrightarrow{\pi_1} X$ in the inverse image $\pi_1^{-1}(Z_2)$. More concisely we have $\pi_1 \circ \pi_2\colon \mathrm{BL}_Z(X) = \mathrm{BL}_{\pi_1^{-1}(Z_2)}(\mathrm{BL}_{Z_1}(X)) \to X$.*

*Proof.* Looking at the diagram

$$\begin{tikzcd}
 & & \mathrm{BL}_{\pi_1^{-1}(Z_2)}(\widetilde{X}) \arrow[d, "\pi_2"] \\
\mathrm{BL}_Z(X) \arrow[urr, dotted, "\varphi_2"] \arrow[rr, dotted, "\varphi_1"] \arrow[dr, "\pi"'] & & \mathrm{BL}_{Z_1}(X) = \widetilde{X} \arrow[dl, "\pi_1"] \\
 & X &
\end{tikzcd}$$

showing the three blow-ups of the statement, we need to construct an isomorphism $\varphi_2$.
We first get a morphism $\varphi_1$ from the universal property of $\mathrm{BL}_{Z_1}(X)$, because $\pi^{-1}(Z_1)$ is clearly an effective Cartier divisor. Since we have $\pi_1 \circ \varphi_1 = \pi$ we have a effective Cartier divisor $\varphi_1^{-1}(\pi_1^{-1}(Z_2))$ and the universal property of $\mathrm{BL}_{\pi_1^{-1}(Z_2)}(\widetilde{X})$ gives us a morphism $\varphi_2$. On the other side $(\pi_1 \circ \pi_2)^{-1}(Z)$ is an effective Cartier divisor and the universal property of $\mathrm{BL}_Z(X)$ gives us an inverse of $\varphi_2$. ☐

**Proposition 1.3** (Functoriality of the blow-up)**.** [GW10, Proposition 13.91] *Let $f\colon X' \to X$ be a morphism of schemes. For a closed subscheme $Z \subseteq X$ there exists a unique morphism* $\mathrm{BL}(f)$ *making the diagram*

$$\begin{tikzcd}
\mathrm{BL}_{f^{-1}(Z)}(X') \arrow[r, dotted, "\exists!\ \mathrm{Bl}(f)"] \arrow[d, "\pi'"] & \mathrm{BL}_Z(X) \arrow[d, "\pi"] \\
X' \arrow[r, "f"] & X \\
f^{-1}(Z) \arrow[u, hook] \arrow[r, "f|_{f^{-1}(Z)}"] & Z \arrow[u, hook]
\end{tikzcd}$$

*commutative.*

**Definition 1.4.** For a morphism of schemes $f\colon X' \to X$ and a closed subscheme $Z \subseteq X$ the blow-up $\mathrm{BL}_{f^{-1}(Z)}(X') \to X'$ will be called the strict transform of the map $f$ under the blow-up $\mathrm{Bl}_X(Z)$.

In the rest of this chapter we will describe several constructions of the blow-up. Most of them will not be used explicitly in the rest of this thesis, but since blow-ups will play an important roll, it might be helpful to feel comfortable with them.

**Definition 1.5.** For a ring $R$ and an ideal $I \trianglelefteq R$ we define the *Rees algebra*

$$\operatorname{Rees}(I) \coloneqq \bigoplus_{i \in \mathbb{N}} I^i \cdot t^i \subseteq R[t],$$

where we denote by $I^0$ the unit ideal in $R$. We obtain an inclusion of $R$ in $\operatorname{Rees}(I)$.

**Proposition 1.6** (Blow-up via Rees Algebra)**.** [GW10, Proposition 13.92] *Let $X = \operatorname{Spec}(R)$ be an affine scheme and $Z = \operatorname{Spec}(R/I)$ a closed subscheme. Then we can describe the blow-up $\operatorname{BL}_Z(X)$ as the scheme $\tilde{X} \coloneqq \operatorname{Proj}(\operatorname{Rees}(I))$ together with the morphism $\pi\colon \tilde{X} \to X$ induced by the inclusion $R \subseteq \operatorname{Rees}(I)$.*

For an affine scheme $X = \operatorname{Spec}(R)$ and an ideal $I = (g_0, \dots, g_k)$ we get a morphism

$$\begin{aligned} \mathcal{O}_X^n &\longrightarrow \mathcal{O}_X \\ (a_0, \dots, a_k) &\longmapsto \sum a_i g_i \end{aligned}$$

which restricts to an epimorphism on $X \setminus \operatorname{V}(g_0, \dots, g_k)$. In particular we get a map

$$\gamma_{g_0,\dots,g_k}\colon X \setminus \operatorname{V}(g_0, \dots, g_k) \longrightarrow \mathbb{P}_A^k.$$

**Proposition 1.7** (Blow-up via closure of a graph)**.** [EH00, Proposition IV-22] *Let $X = \operatorname{Spec}(R)$ be an affine scheme with a closed subscheme $Z = \operatorname{Spec}(R/I)$. Fix a set $\{g_0, \dots, g_k\}$ of generators for $I$. Then we can construct the blow-up $\operatorname{BL}_Z(X)$ as the closure $\tilde{X}$ of the graph $\Gamma_{\gamma_{g_0,\dots,g_k}}$ in $X \times_A \mathbb{P}_A^k$ together with the projection $\tilde{X} \to X$ onto the first factor. Moreover this construction is independent of the set $\{g_0, \dots, g_k\}$ of generators.*

**Proposition 1.8** (Blow-up of ideals generated by a regular sequence)**.** [EH00, Proposition IV-25] *Let $X = \operatorname{Spec}(R)$ be an affine scheme and $Z = \operatorname{Spec}(R/I)$ a closed subscheme. If $g_0, \dots, g_k$ is a regular sequence*[1] *of generators of $I$, then we can construct the blow-up $\operatorname{BL}_Z(X)$ as the scheme $\operatorname{Proj}(R[U_0, \dots U_k]/(U_i g_j - U_j g_i | i, j \leq k))$ together with the morphism associated to $R \subseteq R[U_0, \dots U_k]/(U_i g_j - U_j g_i | i, j \leq k)$. Furthermore this construction is independent of the regular sequence.*

*Proof.* For the proof it is enough to give an isomorphism

$$R[U_0, \dots U_k]/(U_i g_j - U_j g_i | i, j \leq k) \to \operatorname{Rees}(I).$$

We will give this isomorphism in the next lemma. ☐

**Lemma 1.9.** [EH00, Proposition IV-25] *Let the ideal $I$ be generated by a regular sequence $g_0, \dots g_k$, then we get an isomorphism:*

$$\begin{aligned} R[U_0, \dots U_k]/(U_i g_j - U_j g_i | i, j \leq k) &\xrightarrow{\cong} R[g_0 t, \dots g_k t] = \bigoplus_{i \in \mathbb{N}} I^i \cdot t^i = \operatorname{Rees}(I) \\ U_j &\longmapsto g_j t \end{aligned}$$

*Proof.* At first we invert $g_0$ and set $U_0' \coloneqq U_0 g_0^{-1} \in R[g_0^{-1}][U_0', U_1, \dots, U_k]$. Hence we get a morphism

$$\begin{aligned} R[g_0^{-1}][U_0', U_1, \dots, U_k] &\longrightarrow R[g_0^{-1}][t] \\ U_0' &\longmapsto t \\ U_i &\longmapsto g_i t \ \forall 0 < i \leq k \end{aligned}$$

with kernel $(U_0' g_i - U_i | i \leq k) = (U_0 g_i - U_i g_0 | i \leq k) = (U_j g_i - U_i g_j | i, j \leq k)$.
It is clear that the sequence $g_0, U_0 g_1 - U_1 g_0, \dots, U_0 g_k - U_k g_0$ is regular, since we have $U_0 g_i - U_i g_0 \equiv U_0 g_i \pmod{g_0}$ and the $g_i$ are a regular sequence by definition. In general this does not imply that permutations of this sequence is regular, but we will end this proof by showing that in our situation the sequence $U_0 g_1 - U_1 g_0, \dots, U_0 g_k - U_k g_0, g_0$ is indeed regular. Especially $g_0$ will

[1] $g_0, \dots, g_k$ is called regular sequence, if $(g_0, \dots, g_k) \neq R$ and $g_i$ is not a zero divisor in $R/(g_0, \dots g_{i-1})$ $\forall i \leq k$.

not be a zero divisor modulo $(U_0g_i - U_ig_0 | i \leq k)$ and hence the morphism above restrict to the isomorphism of the statement.
At first we observe, that the element $U_0g_1 - U_1g_0$ is not a zero divisor, else if $F(U_0, \dots, U_k)$ would be an annihilator, then the leading term in $U_0$ would be an annihilator of $g_0$. The same argument shows that $U_0g_i - U_ig_0$ is a non-zero divisor modulo $(U_0g_j - U_jg_0 | j < i)$.
We are left to show that $g_0$ is not a zero divisor modulo $(U_jg_i - U_ig_j | i, j \leq k)$. But for two non zero divisors $x, y$ in a noetherian ring the sequence $x, y$ is regular if and only if the sequence $y, x$ is regular (cf. [Eis95, Chapter 17.1]) and the claim follows by induction. □

In [Gen00] a slightly different point of view on blow-up was used. As we have seen before, understanding a blow-up explicitly means understanding its charts. But using the Reese algebra we also have a more concise description. In the following we use $\mathbb{G}_m$-quotients to give a description which is both concise and accessible to explicit calculations.

**Proposition 1.10** (Blow-up as a $\mathbb{G}_m$-quotient)**.** *Let $\mathbb{A}^n_R = \operatorname{Spec}(R[\underline{x}])$ be the affine space over $R$ and $\{0\} = \operatorname{Spec}\left(R[\underline{x}]/(x_1, \dots, x_n)_{R[\underline{x}]}\right)$ its origin. Set $\mathbb{A}^{n+1}_R = \operatorname{Spec}(R[\lambda, \underline{X}])$ and define a $\mathbb{G}_m$-operation on $V(X_1, \dots, X_n)^c \subseteq \mathbb{A}^{n+1}_R$ via $\mu.(\lambda, \underline{X}) \coloneqq (\mu^{-1}\lambda, \mu\underline{X})$. Then we can construct the blow-up $\mathrm{BL}_{\{0\}}(\mathbb{A}^n_R)$ as the categorial quotient $V(X_1, \dots, X_n)^c // \mathbb{G}_m$ together with the morphism induced by $x_i \mapsto \lambda X_i$.*

*Proof.* If we look at the morphism $\pi_1 \colon V(X_1, \dots, X_n)^c \to \mathbb{A}^n_R$ in the diagram

$$\begin{tikzcd} V(X_1, \dots, X_n)^c \arrow[r, dashed, "\exists!\varphi"] \arrow[d, "\pi_1"'] & \mathrm{BL}_{\{0\}}(\mathbb{A}^n_R) \arrow[dl, "\pi_2"] \\ \mathbb{A}^n & \end{tikzcd},$$

we see, that $\pi_1^{-1}(\{0\}) = V(\lambda) \cap V(X_1, \dots, X_n)^c$ is a Cartier divisor and from the universal property of the blow-up we get a unique morphism $\varphi \colon V(X_1, \dots, X_n)^c \to \mathrm{BL}_{\{0\}}(\mathbb{A}^n_R)$. Locally the diagram above looks like:

$$\begin{tikzcd} R[\lambda, \underline{X}, X_k^{-1}] & R\left[\underline{x}, \frac{x_1}{x_k}, \dots, \frac{x_n}{x_k}\right] \arrow[l, dashed, "\exists!\varphi"'] \\ R[\underline{x}] \arrow[u, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}.$$

And the morphism $\varphi$ is uniquely determined by $\varphi(x_i) = \lambda X_i$ and $\varphi(\frac{x_i}{x_k}) = \frac{X_i}{X_k}$. The morphism $\varphi$ locally coincide with the inclusion

$$R\left[\underline{x}, \frac{x_1}{x_k}, \dots, \frac{x_n}{x_k}\right] = R\left[\lambda, \underline{X}, \frac{X_1}{X_k}, \dots, \frac{X_n}{X_k}\right] = R[\lambda, \underline{X}, X_k^{-1}]^{\mathbb{G}_m} \subseteq R[\lambda, \underline{X}, X_k^{-1}]$$

and therefore it is the categorical quotient. □

For a polynomial $f(x_1, \dots, x_n)$ in $R[x_1, \dots, x_n]$ we denote by $f^{\mathrm{tot}}$ the total transform of the element $f$ defined as the polynomial $f(\lambda X_0, \dots, \lambda X_n)$ in $R[\lambda, \underline{X}]$. The ideal $\left(f^{\mathrm{tot}} | f \in I\right)_{R[\lambda, \underline{X}]}$ is the total transformed ideal and will be denoted by $I^{\mathrm{tot}}$. Let $\mathfrak{v}(f)$ be the degree of the smallest non vanishing homogeneous part of $f$. The element $f^{\mathrm{s}} \coloneqq \lambda^{-\mathfrak{v}(f)} f^{\mathrm{tot}}$ is called the strict transform of $f$. The strict transformed ideal $I^{\mathrm{s}}$ of $I$ is generated by the elements $\{f^{\mathrm{s}} | f \in I\}$.

**Corollary 1.11.** *For an ideal $I \trianglelefteq R[\underline{x}]$ the total transform $\pi^{-1}(Z)$ of the closed subscheme $Z \coloneqq V(I) \subseteq \mathbb{A}^n$ in $\mathrm{BL}_{\{0\}}(\mathbb{A}^n)$ is given as the $\mathbb{G}_m$-quotient of:*

$$V(I^{\mathrm{tot}}) \cap V(X_1, \dots, X_n)^c.$$

*The strict transform* $\overline{\pi^{-1}(Z) \setminus \pi^{-1}(\{0\})}$ *can be described as the* $\mathbb{G}_m$*-quotient of the subscheme defined by*

$$V(I^{\mathrm{s}}) \cap V(X_1, \ldots, X_n)^c.$$

*Proof.* The morphism $\pi_1$ in the diagram

$$\begin{tikzcd} V(X_1,\ldots,X_n)^c \arrow[r, "\pi_1"] \arrow[d, hook] & \mathbb{A}^n \\ \mathbb{A}^{n+1} \arrow[ur, dashed, "\exists!\pi_u"'] & \end{tikzcd}$$

from the construction of the blow-up in Proposition 1.10 is extending uniquely to the surrounding space $\mathbb{A}^{n+1}$. Therefore the preimage $\pi_1^{-1}(Z)$ is the intersection of the preimage $\pi_u^{-1}(Z) = V(I^{\mathrm{tot}})$ with $V(X_1, \ldots, X_n)^c$.
The strict transform can be written as

$$\overline{V(I^{\mathrm{tot}}) \cap V(\lambda)^c} \cap V(X_1, \ldots, X_n)^c$$

and therefore we have to find the largest ideal in $R[\lambda, \underline{X}]$, that coincide with $I^{\mathrm{tot}}$ after the localization of $\lambda$. This ideal coincides with the set

$$R[\lambda, \underline{x}] \cap I^{\mathrm{tot}} R[\lambda, \underline{X}]_\lambda = \left\{ f \in R[\lambda, \underline{X}] \middle| \exists n \in \mathbb{N} \text{ such that } \lambda^n f \in I^{\mathrm{tot}} \right\} = I^{\mathrm{s}}.$$

$\square$

The ring $R[\underline{x}] = \bigoplus_{i \in \mathbb{N}} (x_1, \ldots, x_n)_R^i$ carries a natural grading. For this grading we can define a notion of initial forms. Let $f$ be an element in $R[\underline{x}]$, then we define $(f)^{\mathrm{in}}$ to be the non vanishing homogeneous part of $f$ of smallest degree. Similarly we define for an ideal $I \trianglelefteq R[\underline{x}]$ the corresponding initial ideal $(I)^{\mathrm{in}} := \left( (f)^{\mathrm{in}} \middle| f \in I \right)$.

**Lemma 1.12.** *Fix $R$ a noetherian domain and an ideal $I \trianglelefteq R[\underline{x}]$ that is generated by $f_1, \ldots, f_m$, such that the initial forms $(f_i)^{\mathrm{in}}$ generate the initial ideal $(I)^{\mathrm{in}}$. Then the strict transforms $f_1^{\mathrm{s}}, \ldots, f_m^{\mathrm{s}}$ generate the ideal $I^s$.*

*Proof.* Since $I^{\mathrm{s}}$ is generated by the elements $f^{\mathrm{s}}$ for $f \in I$, it is enough to show, that these elements lie in the ideal generated by $f_1^{\mathrm{s}}, \ldots, f_m^{\mathrm{s}}$.
Let $f = \sum g_i f_i \in I$ and assume we have $\mathfrak{v}(f) \le \mathfrak{v}(f_i g_i)$ for all $i$, then we get:

$$\begin{aligned} f^{\mathrm{s}} =& \lambda^{-\mathfrak{v}(f)} f^{\mathrm{tot}} = \lambda^{-\mathfrak{v}(f)} \left( \sum f_i g_i \right)^{\mathrm{tot}} \\ =& \sum \lambda^{-\mathfrak{v}(f)} f_i^{\mathrm{tot}} g_i^{\mathrm{tot}} = \sum \lambda^{\mathfrak{v}(f_i g_i) - \mathfrak{v}(f)} f_i^{\mathrm{s}} g_i^{\mathrm{s}}. \end{aligned}$$

It is left to show that every $f \in I$ has a presentation like above. This is equivalent to the following identity: $I^d := I \cap (x_1, \ldots, x_n)_{R[\underline{x}]}^d = J^d := \sum_i (x_1, \ldots, x_n)_{R[\underline{x}]}^{d - \mathfrak{v}(f_i)} f_i$.[2]
For $d = 0$ this is trivial and we always have the inclusion $J^d \subseteq I^d$. Let $d > 0$, if we show that the ideals becomes equal in every localization $R[\underline{x}]_{\mathrm{m}}$ for every maximal ideal $\mathrm{m} \trianglelefteq R[\underline{x}]$ we get the desired inclusion in $R[\underline{x}]$.
If m does not contain the ideal $(x_1, \ldots, x_n)_{R[\underline{x}]}$, then at least one $x_i$ becomes invertible in $R[\underline{x}]_{\mathrm{m}}$ and we have $I^d R[\underline{x}]_{\mathrm{m}} \subseteq I R[\underline{x}]_{\mathrm{m}} = J^d R[\underline{x}]_{\mathrm{m}}$.
Let m be a maximal ideal containing $(x_1, \ldots, x_n)_{R[\underline{x}]}$, $f \in I^d$ and $(f)^{\mathrm{in}} = \sum (f_i)^{\mathrm{in}} g_i$ with $g_i$ homogeneous, then we get $f - \sum f_i g_i \in I^{d+1}$ and $\sum f_i g_i \in J^d$. After a simple induction we get the inclusion $I^d \subseteq J^d + (x_1, \ldots, x_n)_{R[\underline{x}]}^k \subseteq J^d + \mathrm{m}^k$ for all $k \in \mathbb{N}$.
If $\widehat{R[\underline{x}]}$ is the completion with respect to the maximal ideal m, then the above shows the inclusion $I^d \widehat{R[\underline{x}]} \subseteq J^d \widehat{R[\underline{x}]}$. But the ring $\widehat{R[\underline{x}]}$ is faithfully flat over the localization $R[\underline{x}]_{\mathrm{m}}$ and we get the inclusion in the localization at m. $\square$

[2]For $n \le 0$ we also use the notation $(x_1, \ldots, x_n)^n$ for the unit ideal $R[\underline{x}]$.

**Example.** Let $R$ be a noetherian domain and $(f_1, \ldots, f_n)_R = I \trianglelefteq R$ an ideal generated by a regular sequence $f_1, \ldots, f_n$ in $R$. We can identify the scheme $\mathrm{Spec}\,(R)$ with the closed subscheme $V\,(x_i = f_i | i \leq n)$ of the affine space $\mathrm{Spec}\,(R[x_1, \ldots, x_n])$. Under this identification the subscheme $\mathrm{Spec}\,(R/I) \subseteq \mathrm{Spec}\,(R)$ is $V\,(x_i = f_i | i \leq n) \cap V\,(x_i | i \leq n)$.
The blow-up $\mathrm{BL}_I\,(R)$ can now be identified with the strict transform of $V\,(x_i = f_i | i \leq n)$ in $\mathrm{BL}_{\{0\}}\,(R[\underline{x}])$. Since the initial forms $(x_i - f_i)^{\mathrm{in}} = f_i$ clearly generates the initial ideal of the ideal $(x_1 - f_1, \ldots, x_n - f_n)$, we get $\mathrm{BL}_I\,(R) = V\,(\lambda X_i = f_i | i \leq n) \subseteq \mathrm{BL}_{\{0\}}\,(R[\underline{x}])$.

## 2. The local model $\mathcal{M}^{\mathrm{loc}}$

In this chapter we will first provide some background on Alcoves, the extended affine Weyl group $W$ and Coxeter groups in general. In particular we will analyse certain projections of a Weyl group $W$ to simpler Coxeter groups. In the second part we will discuss affine flag varieties and its projections to affine Grassmannians. In the last section we will finally define the local model $\mathcal{M}^{\mathrm{loc}}$ as a functor given by a moduli problem. The definition will be followed by a discussion on the embedding of the special fiber $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}$ into the flag variety. Furthermore we will use the calculations in the first part to make the natural projections $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}} \to \mathrm{Gr}^k(\Lambda)$ explicit.

2.1. **Alcoves and the extended affine Weyl group.** For a more detailed discussion on the Bruhat length in the first part of this section see [IM65]. More details on general Coxeter groups can be found in [BB05]. None of this section is claimed to be original.
Let $W$ be the Weyl group of $\mathrm{GL}_n$. It is isomorphic to the symmetric group of the set of $n$ elements $[n] = \{0,\dots,n-1\}$ and is generated by the set of neighbour transpositions $S = \{s_i\}_{1\le i\le n-1}$. This group is a finite Coxeter group und thus contains a unique maximal element $w_S = (n, n-1,\dots,2,1)$. The extended affine Weyl group is the semi-direct product $\widetilde{W} := W \ltimes \mathbb{Z}^n$. The elements in $\mathbb{Z}^n$ are represented as $t^\chi$ with $\chi \in \mathbb{Z}^n$ and the operation from $W$ is defined as $x.t^\chi := t^{x(\chi)}$. The extended affine Weyl group is in general not a Coxeter group, but it contains the Coxeter group $W^{\mathrm{aff}}$ generated by $W$ and $s_0 := t^{(1,0\dots,0,-1)}(n,2,\dots,n-1,1)$.
The extended affine Weyl group $\widetilde{W}$ is generated by the subgroup $W^{\mathrm{aff}}$ and the element $\tau := t^{(1,0\dots,0)}(n,1,\dots,n-1)$ and therefore every element in $\widetilde{W}$ can be written as $\tau^k w$ with $k \in \mathbb{Z}$ and $w \in W^{\mathrm{aff}}$. Using the Bruhat order $\le_{W^{\mathrm{aff}}}$ and the length function $l_{W^{\mathrm{aff}}}$ on the Coxeter group $W^{\mathrm{aff}}$, we define a partial order on $\widetilde{W}$ by $\tau^{d_1}w_1 \le \tau^{k_2}w_2$ if $k_1 = k_2$ and $w_1 \le_{W^{\mathrm{aff}}} w_2$ and a length function via $l(\tau^k w) := l_{W^{\mathrm{aff}}}(w)$. We also fix a root system $\phi$ for $\mathrm{GL}_n$. We also fix the set of *positive roots* $\phi^+$ and the set of *negative roots* $\phi^-$ such that $\mu = (1^k, 0^{n-k}) \in \mathbb{Z}^n$ is a minuscule dominant coweight, i.e. $\langle \mu, \alpha\rangle \ge 0$ for all positive roots $\alpha$. In the following we will describe the Bruhat length of $\widetilde{W}$ in a different way.
An alcove is an element $x = (x_1,\dots,x_n) \in (\mathbb{Z}^n)^n$ such that there exist a $\sigma \in W$ with $x_{i+1} = x_i + e_{\sigma(i)}$ where $\{e_i\}_{i\in[n]}$ are the standard basis vectors of $\mathbb{Z}^n$.

**Definition 2.1.** For the set of *alcoves* we write Alcov. We say an alcove $(x_i)_{i\in[n]}$ is of size $k \in \mathbb{N}$ if $\sum_{j\in[n]} x_0(j) = k$ and write $\mathrm{Alcov}_n$ for the set of alcoves of size $n$.

**Definition 2.2.** For $k \in \mathbb{Z}$ and $i \in [n]$ define the *affine hyperplanes* $H_{\alpha_i,k} \subseteq \mathbb{R}^n$ to be the set $\{v \in \mathbb{R}^n | \langle v, \alpha_i\rangle = k\}$. We say $H_{\alpha_i,k}$ is a *wall* of the alcove $x \in$ Alcov if there is a $j \in [n]$ and $x_i \in H_{\alpha_i,k}$ for all $i \ne j$. We say that an alcove $x \in$ Alcov is lying on the positive side of a wall $H_{\alpha,k}$ if for all $i \in [n]$ we have $\langle x_i, \alpha\rangle \ge k$ and we write $x \in H^+_{\alpha,k}$.

**Definition 2.3.** Fix two alcoves $x, y \in \mathrm{Alcov}_n$. We say they are *adjacent* and write $x \to y$, if $\{x_i\}_{i\in[n]}$ and $\{y_i\}_{i\in[n]}$ differ by just one element. In this case the set $\{x_i\}_{i\in[n]} \cap \{y_i\}_{i\in[n]}$ is contained in wall $H_{\alpha,k}$ for some root $\alpha$ and $k \in \mathbb{Z}$. In this case we say they share that common wall. A sequence $a_1 \to a_2 \to \dots \to a_n$ of elements $\{a_i\} \subseteq \mathrm{Alcov}_n$, such that $a_i$ and $a_{i+1}$ are adjacent, is called a gallery between the alcoves $a_1$ and $a_n$.

**Definition 2.4.** For two alcoves $x, y$ of size $n$ we define their *distance* as

$$d(x,y) := \min\{n \in N | \exists (x = a_0 \to a_1 \to \dots \to a_n = y)\}.$$

**Lemma 2.5.** [IM65, Section 1.9] *Fix $x, y \in \mathrm{Alcov}_n$ and let $x = a_0 \to a_1 \to \dots \to a_n = y$ be a not necessarily minimal galery between $x$ and $y$. For $\alpha \in \phi^+$ we define the two numbers $d^+_\alpha := \{a_i | \exists k \in \mathbb{Z};\ a_{i+1} \notin H^+_{\alpha,k} \ni a_i\}$ and $d^-_\alpha := \{a_i | \exists k \in \mathbb{Z};\ a_i \notin H^+_{\alpha,k} \ni a_{i+1}\}$. Then we can calculate the distance as*

$$d(x,y) = \sum_{\alpha\in\phi^+} |d^+_\alpha - d^-_\alpha|.$$

**Lemma 2.6.** [IM65, Section 1.9] *For $v \in \mathbb{Z}^n$ and $x \in$ Alcov the difference $d_\alpha^+ - d_\alpha^-$ for the alcoves $x$ and $x + v := (x_1 + v, x_2 + v, \dots, x_n + v)$ can be computed as $\langle \alpha, v\rangle$ and we get the distance $d(x, x+v) = \sum_{\alpha \in \phi^+} |\langle v, \alpha\rangle|$.*

*Proof.* Let $x = a_0 \to a_1 \to \dots \to a_n = x + v$ be a gallery of minimal length and for $\alpha \in \phi^+$ the number of pairs $a_i \to a_{i+1}$ adjacent via a wall $H_{\alpha,k}$ for some $k$ in this gallery can be identified with $|\langle v, \alpha\rangle|$. Therefore we get the identification. □

We define an action of $\widetilde{W}$ on Alcov by $t^\chi.(x_1, \dots, x_n) := (x_1 + \chi, \dots, x_n + \chi)$ and for $y \in W$ set $y.(x_1, \dots, x_n) := (y(x_1), \dots, y(x_n))$. If we denote by $\mathrm{a}_b$ the standard alcove $(0, e_1, \dots, \sum_{i \leq n-1} e_i)$ we get the following Lemma.

**Lemma 2.7.** *The action of $\widetilde{W}$ on* Alcov *defined above is simply transitive and restricts to simply transitive actions of $W^{\mathrm{aff}}$ on* $\mathrm{Alcov}_n$.

**Lemma 2.8.** [IM65, Proposition 1.10.] *For the standard alcove $\mathrm{a}_b$ the set of adjacent alcoves is $\{s_i(\mathrm{a}_b)\}_{0 \leq i \leq n}$ and we get the equality $l(\tau^k x) = d(\tau^k.\mathrm{a}_b, \tau^k x.\mathrm{a}_b) = d(\mathrm{a}_b, x.\mathrm{a}_b)$.*

This identification can now be used to give an alternative definition of the length in $\widetilde{W}$.

**Lemma 2.9.** [IM65, Proposition 1.23] *For an element $xt^\chi \in \widetilde{W}$ we can compute the length as*

$$l(xt^\chi) = \sum_{\alpha \in \phi^+;\ x(\alpha) \in \phi^+} |\langle \chi, \alpha\rangle| + \sum_{\alpha \in \phi^+;\ x(\alpha) \in \phi^-} |\langle \chi, \alpha\rangle + 1|.$$

**Definition 2.10.** [BB05, Section 2.5] For a Coxeter group $(\mathcal{W}, \mathcal{S})$ and a set $S \subseteq \mathcal{S}$ of simple reflection, we denote with $\mathcal{W}_S$ the subgroup of $\mathcal{W}$ generated by the elements in $S$. The orbit $\mathcal{W}_S x$ contains unique element $x^S$ of minimal length. We write $\mathcal{W}^S$ for the set $\{x \in \mathcal{W} | xs > x \forall s \in S\}$ of these elements of minimal length.
Similarly for the orbit $x\mathcal{W}_S$ we denote the unique element of minimal length with ${}^S x$ and denote with ${}^S\mathcal{W} = \{x \in \mathcal{W} | sx > x \forall s \in S\}$ the set of these elements.

For the definition above we have the following Proposition:

**Proposition 2.11.** [BB05, Proposition 2.5.1] *For a Coxeter group $(\mathcal{W}, \mathcal{S})$ as above the Projections from $\mathcal{W}$ to $\mathcal{W}^S$ and ${}^S\mathcal{W}$ are order preserving and for $x \in \mathcal{W}$ we can calculate the length by $l(x) = l(x^S) + l(x_S) = l({}^S x) + l(x_S)$.*

**Definition 2.12.** [HL15, Chapter 2.1] In the following we will denote by $J$ the set of generators $\{s \in \mathcal{S} | s(\mu) = \mu\}$ where $\mu = (1^k, 0^{n-k})$ is the fixed minuscule dominant coweight. The poset $Q_J$ is defined as $\{(x, y) \in W^J \times W | y \leq x\}$ with ordering $(x', y') \leq (x, y) :\Leftrightarrow \exists u \in W_J : xu \leq x', yu \geq y'$. Let us also give a slightly different but isomorphic poset ${}^J Q := \{(x, y) \in W^J \times W | y^J \geq x(y_J)^{-1}\}$ with the partial ordering defined by the bijection

$$\begin{aligned} {}^J Q &\longrightarrow Q_J \\ (x, y) &\longmapsto ((y^J), x(y_J)^{-1}) \end{aligned}$$

of posets.

**Definition 2.13.** [BB05, Definition in equation 2.18] Let $\binom{[n]}{k}$ be the set of subsets $S \subseteq [n]$ of seize $k$. For two sets $S, S' \in \binom{[n]}{k}$ let $\lambda_1, \dots, \lambda_k$ and $\lambda'_1, \dots, \lambda'_k$ be the elements of $S$ respectively $S'$ ordered by size, then we define an order on $\binom{[n]}{k}$ in the following way: $S \leq S'$ if $\lambda_i \leq \lambda'_i$ for all $i \leq r$. Using the same notation for the elements the length of a Set $S \in \binom{[n]}{k}$ is defined as $l(S) := \sum_{i \leq n} \lambda_i - i$.

**Lemma 2.14.** [BB05, prop. 2.4.8] *We have a order preserving bijection*

$$\begin{aligned} W^J &\longrightarrow \binom{[n]}{k} \\ x &\longmapsto S_x := \{x(i) | i \leq r\}. \end{aligned}$$

**Corollary 2.15.** *The projection ${}^JQ \to W^J$ is order preserving. In particular the composition*

$$\begin{aligned} {}^JQ &\longrightarrow \binom{[n]}{k} \\ (x,y) &\longmapsto S_x. \end{aligned}$$

*is order preserving.*

*Proof.* Let $(x,y) \leq_{{}^JQ} (x',y')'$. Then by definition $(y^J, xy_J^{-1}) \leq_{Q_J} (y'^J, x'y'^{-1}_J)$ an hence there exists an element $u \in W_J$ such that and $xy_J^{-1} \leq x'y'^{-1}_J u$. But since $x, x' \in W^J$ we get $x \leq x'$. □

Later we will need to use a bijection

$$\begin{aligned} W^J \times W &\longrightarrow Wt^\mu W \\ (x,y) &\longmapsto yt^\mu x^{-1} \end{aligned}$$

defined in [HL15] to relate ${}^JQ$ to a stratification of the special fibers of local models. It will be help full to compute the length of elements of $Wt^\mu W \subseteq \widetilde{W}$.

**Lemma 2.16.** [HL15, Chapter 2.1] *For an element $yt^\mu x \in \widetilde{W}$ with $x \in {}^JW$ and $y \in W$ we compute the length $l(xt^\mu y)$ to be $l(t^\mu) + l(y) - l(x)$.*

*Proof.* **Step I** Since $\mu$ is dominant by definition we get $\langle \mu, \alpha \rangle \geq 0$ for all $\alpha \in \phi^+$ and therefore we can compute:

$$\begin{aligned} l(yt^\mu) &= \sum_{\alpha \in \phi^+;\ y(\alpha) \in \phi^+} |\langle \mu, \alpha \rangle| + \sum_{\alpha \in \phi^+;\ y(\alpha) \in \phi^-} |\langle \mu, \alpha \rangle + 1| \\ &= \sum_{\alpha \in \phi^+;\ id(\alpha) \in \phi^+} |\langle \mu, \alpha \rangle| + \sharp \{\alpha \in \phi^+ | y(\alpha) \in \phi^-\} = l(t^\mu) + l(y). \end{aligned}$$

**Step II** For $x \in {}^JW$ we get $\langle \mu, \alpha \rangle \geq 1$ for all $\alpha \in \phi^+$ with $x^{-1}(\alpha) \in \phi^-$. To see this let $\alpha \in \phi^+$ be a root with $x^{-1}(\alpha) \in \phi^-$ then write $\alpha = \sum m_i \alpha_i$ as a sum of simple roots. We get $\langle \mu, \alpha_i \rangle = 0$ for $i \neq k$ and $\langle \mu, \alpha_k \rangle = 1$. But since $l(t_\alpha x) < l(x)$ for the reflection $t_\alpha$ corresponding to $\alpha$ (cf. [BB05, Chapter 4.4]) we see that $t_\alpha$ is not in $W_J$ and therefore we have $m_k > 0$.

$$\begin{aligned}
l(yt^\mu x) =& l(yxt^{x^{-1}(\mu)}) = \sum_{\alpha\in\phi^+;\ y(x(\alpha))\in\phi^+} |\langle x^{-1}(\mu),\alpha\rangle| + \sum_{\alpha\in\phi^+;\ y(x(\alpha))\in\phi^-} |\langle x^{-1}(\mu),\alpha\rangle + 1|\\
=& \sum_{\alpha\in\phi^+;\ y(x(\alpha))\in\phi^+} |\langle \mu, x(\alpha)\rangle| + \sum_{\alpha\in\phi^+;\ y(x(\alpha))\in\phi^-} |\langle \mu, x(\alpha)\rangle + 1|\\
=& \sum_{x(\alpha)\in\phi^+;\ y(x(\alpha))\in\phi^+} |\langle \mu, x(\alpha)\rangle| + \sum_{x(\alpha)\in\phi^+;\ y(x(\alpha))\in\phi^-} |\langle \mu, x(\alpha)\rangle + 1|\\
&+\left(\sum_{\alpha\in\phi^+;\ x(\alpha)\in\phi^-;\ y(x(\alpha))\in\phi^+} |\langle \mu, x(\alpha)\rangle| + \sum_{\alpha\in\phi^+;\ x(\alpha)\in\phi^-;\ y(x(\alpha))\in\phi^-} |\langle \mu, x(\alpha)\rangle + 1|\right)\\
&-\left(\sum_{\alpha\in\phi^-;\ x(\alpha)\in\phi^+;\ y(x(\alpha))\in\phi^+} |\langle \mu, x(\alpha)\rangle| + \sum_{\alpha\in\phi^-;\ x(\alpha)\in\phi^+;\ y(x(\alpha))\in\phi^-} |\langle \mu, x(\alpha)\rangle + 1|\right)\\
=& l(yt^\mu) + \left(\sum_{x^{-1}(\alpha)\in\phi^+;\ \alpha\in\phi^-;\ y(\alpha)\in\phi^+} |\langle \mu,\alpha\rangle| + \sum_{x^{-1}(\alpha)\in\phi^+;\ \alpha\in\phi^-;\ y(\alpha)\in\phi^-} |\langle \mu,\alpha\rangle| - 1\right)\\
&-\left(\sum_{x^{-1}(\alpha)\in\phi^-;\ \alpha\in\phi^+;\ y(\alpha)\in\phi^+} |\langle \mu,\alpha\rangle| + \sum_{x^{-1}(\alpha)\in\phi^-;\ \alpha\in\phi^+;\ y(\alpha)\in\phi^-} |\langle \mu,\alpha\rangle| + 1\right)\\
=& l(t^\mu) + l(y) - \sharp\left\{\alpha\in\phi^- | x^{-1}(\alpha)\in\phi^+, y(\alpha)\in\phi^-\right\} - \sharp\left\{\alpha\in\phi^+ | x^{-1}(\alpha)\in\phi^-, y(\alpha)\in\phi^-\right\}\\
=& l(t^\mu) + l(y) - \sharp\left\{\alpha\in\phi^- | x(\alpha)\in\phi^+\right\} = l(t^\mu) + l(y) - l(x)
\end{aligned}$$

□

2.2. **Affine flag varieties and projections.** In this section we will give a brief overview on affine flag varieties. We will define them as fppf quotients and then cite a result identifying them with moduli functors of lattice chains. Furthermore we will describe a stratification into Schubert varieties. A more detailed discussion can be found in [Gör01]. After discussing projections to affine Grassmannians, we will use a result in [Deo77] to describe a Schubert variety by the set of its projections to the affine Grassmannians. Throughout this section we will fix an algebraically closed field $\overline{\kappa}$ of positive characteristic and a natural number $n$.

**Definition 2.17.** Fix a $\overline{\kappa}$-algebra $R$. An $R[[t]]$-submodule $\mathcal{L}$ of $R((t))^n$ is called *lattice* if for some $N\in\mathbb{N}$ we have $t^N R[[t]]^n \subseteq \mathcal{L} \subseteq t^{-N}R[[t]]^n$ and the quotient $t^{-N}R[[t]]^n/\mathcal{L}$ is projective.

**Definition 2.18.** A sequence $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{L}_n := t^{-1}\mathcal{L}_0$ of lattices in $R((t))^n$ such that $\mathcal{L}_{i+1}/\mathcal{L}_i$ is a locally free of rank 1 is called *complete lattice chain.*

**Definition 2.19.** For $I\subseteq[n]$ and a complete lattice chain $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{L}_n := t^{-1}\mathcal{L}_0$, the sequence $(\mathcal{L}_i)_{i\in I}$ is called *$I$-partial lattice chain.*

**Definition 2.20.** A lattice chain $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{L}_n := t^{-1}\mathcal{L}_0$ is called *$k$-special* if we have $\bigwedge^n \mathcal{L}_0 = t^k R[[t]]^n$ as submodules of $\bigwedge^k R((t))^n = R((t))$.

**Definition 2.21.** For a $\overline{\kappa}$-algebra $R$ and $k\in\mathbb{Z}$ we define the *standard lattice chain* in $R((t))^n$ to be the $k$-special lattice chain

$$\ldots \longrightarrow \lambda_0 \longrightarrow \lambda_1 \longrightarrow \ldots \longrightarrow \lambda_n = t^{-1}\lambda_0 \longrightarrow \ldots$$

with $\lambda_i := R[[t]]^{n-k+i} \oplus tR[[t]]^{k-i}$ for $i\in[k+1]$ and $\lambda_i := t^{-1}R[[t]]^{i-k+1} \oplus R[[t]]^{n-i+k-1}$ for $i\in[n]\smallsetminus[k+1]$.

**Definition 2.22.** The *standard Iwahori subgroup $B$* of $\mathrm{SL}_n(\overline{\kappa}((t)))$ is defined as the stabiliser of the $k$-special standard lattice chain

$$\ldots \longrightarrow \lambda_0 \longrightarrow \lambda_1 \longrightarrow \ldots \longrightarrow \lambda_n = t^{-1}\lambda_0 \longrightarrow \ldots$$

in $R((t))^n$ under the action of $\mathrm{SL}_n(\overline{\kappa}((t)))$ on the set of $k$-special lattice chains in $R((t))^n$.

*Remark* 2.23. Note that for the $k$-special standard lattice chain $(\lambda_i)_i$ and $j \in \mathbb{Z}$ $(\lambda'_i)_i$ the shift with $\lambda'_i = \lambda_{i+j}$ is the $(k-j)$-special standard lattice chain (cf. [Gör01, Corollary 3.6]). In particular the stabiliser of both chains agree and hence the definition of the standard Iwahori subgroup is independent of the choice of $k$.

**Definition 2.24.** The *affine flag variety* $\mathcal{F}$ is defined as the fppf quotient $\mathrm{SL}_n(\overline{\kappa}((t)))/B$ where $B$ is the standard Iwahori subgroup of $\mathrm{SL}_n(\overline{\kappa}((t)))$.
For $I \subseteq [n]$ let $P_I \supseteq B$ be the parahoric subgroup of of $\mathrm{SL}_n(\overline{\kappa}((t)))$ stabilising the $I$-partial subchain $(\lambda_i)_{i\in I}$ of the $k$-special standard lattice chain. The fppf quotient $\mathrm{SL}_n(\overline{\kappa}((t)))/P_I$ is called the *partial affine flag variety.* In the special case $I = \{i\}$ we denote the functor with $\mathcal{G}_i$ and call it the *affine Grassmannian variety* .

*Remark* 2.25. All the fppf quotients $\mathrm{SL}_n(\overline{\kappa}((t)))/P_I$ for some $I \subseteq [n]$ carry a natural ind-scheme structure (for references see [Gör01, Section 3.1]).

**Proposition 2.26.** [Gör01, Proposition 3.5] *For $I \subseteq [n]$ and a $\overline{\kappa}$-algebra $R$ we have a functorial bijection*

$$\begin{aligned} \mathcal{F}_I(R) &\xrightarrow{\cong} \{k\text{-special } I\text{-partial lattice chains in } R((t))^n\} \\ \overline{g} &\longmapsto g \cdot (\lambda_i)_{i\in I} \end{aligned} .$$

**Corollary 2.27.** *We can embed $\mathcal{F}$ in $\prod_{i\in[n]} \mathcal{G}_i$ using the projections $\pi_i : \mathcal{F} \to \mathcal{G}_i$ for $i$ in $[n]$.*

**Definition 2.28.** For $w \in \widetilde{W}$ we denote with $X_w^\circ$ the orbit $BwB/B$ of $w$ under the Iwahori action in $\mathcal{F}$. This orbit is called the *Schubert cell* corresponding to $w$ and the Zariski closure $X_w := \overline{X_w^\circ}$ endowed with the reduced scheme structure is called *Schubert variety.*

**Lemma 2.29.** [Gör01, Chapter 3.2] *For $w \in \widetilde{W}$ the Schubert variety $X_w$ is set theoretically the union $\bigcup_{w' \leq w} X_{w'}^\circ$ of Schubert cells and the Schubert cell $X_w^\circ$ is isomorphic to the affine space of dimension* $\dim(X_w^\circ) = l(w)$.

**Proposition 2.30.** [HR08, Proposition 8] *For a subset $I \subseteq [n]$ we have the Bruhat decomposition $\mathcal{F}_I = \bigcup_{w \in \widetilde{W}(I)} X_w^\circ / P_I$ where the group $\widetilde{W}(I)$ is the quotient $\widetilde{W}/(P_I/B \cap \widetilde{W})$ of the extended affine Weyl group.*

**Lemma 2.31.** *For the set of simple reflections $S_i := \{s_0, \dots, \hat{s_i}, \dots, s_{n-1}\}$ in $W^{\mathrm{aff}}$ for some $i \in [n]$ we have $P_i/B \cap \widetilde{W} = W_{S_i}^{\mathrm{aff}}$.*

*Proof.* First we note that $P_i/B \subseteq \mathcal{F}$ is invariant under the action of the Iwahori group $B$. In particular set theoretically $P_i/B$ is the union of Schubert cells. For a $\overline{\kappa}$-algebra $R$ we can identify the $R$-valued points of $P_i/B$ with the orbit of $(\lambda_j)_{j\in[n]}$. But this orbit is nothing but the set of flags in $\lambda_i/t\lambda_i$. In particular $P_i/B$ is a flag variety and hence complete. Since now $P_i/B$ is closed in $\mathcal{F}$, the set $\widetilde{W} \cap P_i/B$ is closed under the Bruhat order, i.e. for $w \in \widetilde{W} \cap P_i/B$ all the simple reflections appearing in a reduced expression of $w$ has to be in $\widetilde{W} \cap P_i/B$. It follows that the group $\widetilde{W} \cap P_i/B$ is contained in $W^{\mathrm{aff}}$, but a simple reflection $s_j$ is stabilising $\lambda_i$ exactly if $j \neq i$ and the claim is proven. ☐

**Theorem 2.32.** [Deo77, Lemma 3.6] *Fix a Coxeter group $(\mathcal{W}, \mathcal{S})$ and a family of subsets $\{S_i\}_i$ of $\mathcal{S}$ with $\bigcap_i S_i = \varnothing$. For $x, y$ in $\mathcal{W}$ we have $x \leq y$ if and only if $x^{S_i} \leq y^{S_i}$ for all $i$ in $[n]$.*

**Lemma 2.33.** *For $w \in \widetilde{W}$ the Schubert variety $X_w$ in the affine flag variety $\mathcal{F}$ is the intersection $\left(\prod_{i\in[n]} \pi_i(X_w)\right) \cap \mathcal{F}$ taken in the product $\prod_{i\in[n]} \mathcal{G}_i$.*

*Proof.* Fix a Schubert variety $X_w$ in $\mathcal{F}$ for some $w \in \widetilde{W}$. This variety is the union $\bigcup_{w' \leq w} X_{w'}^\circ$ of Schubert cells. Since the inverse image $\pi_i^{-1}(\pi_i(X_w))$ of the image of $X_w$ in $\mathcal{G}_i$ is invariant under the Iwahori action for all $i$, it is a union of Schubert cells. In particular this union is precisely the union over all elements $w'$ in $\widetilde{W}$ congruent modulo $W^{\mathrm{aff}}_{S_i}$ to an element $w'' \leq w$. With out loss of generality we can assume that $w$ is in $W^{\mathrm{aff}}$ and hence also $w''$ and $w'$ are in $W^{\mathrm{aff}}$. Now we note that $(\prod \pi_i(X_w)) \cap \mathcal{F}$ is the intersection $\bigcap_i \pi_i^{-1}(\pi_i(X_w))$ and hence by Lemma 2.31 consist of all Schubert cells $X^\circ_{w'}$ with $w'^{S_i} \leq w^{S_i}$ for all $i$ in $[n]$. By applying Theorem 2.32 we have proven the desired result. □

2.3. **The definition of $\mathcal{M}^{\mathrm{loc}}$ and some combinatorics of the special fiber.** First let us fix some notation. Let $\mathcal{O}$ be a complete discrete valuation ring with perfect residue class field $\kappa$. Fix a uniformizer $\pi$ and an algebraic closure $\overline{\kappa}$ of $\kappa$. Denote with K the quotient field of $\mathcal{O}$. Further fix two natural numbers $k < n$. We denote with $\mu$ the minuscule coweight $(1^k, 0^{n-k}) \in \mathbb{Z}^n$ of $\mathrm{GL}_n$.
We start this section by defining the local model. Then we will follow [Gör01] and describe an embedding of the special fiber $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}$ into the affine flag variety.

**Definition 2.34.** For a fixed basis $\{e_i\}_{i\in[n]}$ of $\mathrm{K}^n$ set $\Lambda_i = \langle \pi^{m(i,j)} e_j \rangle_{\mathcal{O}}$ with $m(i,j) = -1$ for $j \in [i]$ and $m(i,j) = 0$ for $j \in [n] \setminus [i]$. The lattice chain

$$\cdots \to \Lambda_0 \to \Lambda_1 \to \cdots \to \Lambda_n = \pi^{-1}\Lambda_0 \to \ldots$$

is called the *standard lattice chain* in $\mathrm{K}^n$ and will be denoted by $\Gamma^{\mathrm{st}}$.

**Definition 2.35.** For an $\mathcal{O}$-scheme $S$ we write $\Lambda_{i,S}$ for $\Lambda_i \otimes_{\mathcal{O}} \mathcal{O}_S$ and define the $S$-points of the functor $\mathcal{M}^{\mathrm{loc}}$ to be the set of diagrams of the form

$$\begin{tikzcd}
\Lambda_{0,S} \arrow[r] & \Lambda_{1,S} \arrow[r] & \ldots \arrow[r] & \Lambda_{n-1,S} \arrow[r, "\pi"] & \Lambda_{0,S} \\
\mathcal{F}_0 \arrow[r] \arrow[u, hook] & \mathcal{F}_1 \arrow[r] \arrow[u, hook] & \ldots \arrow[r] & \mathcal{F}_{n-1} \arrow[r] \arrow[u, hook] & \mathcal{F}_0 \arrow[u, hook]
\end{tikzcd}$$

where the $\mathcal{F}_i's$ are locally free $\mathcal{O}_S$-submodules of $\Lambda_{i,S}$ of rank $k$ that are Zariski-locally direct summands.

*Remark* 2.36. The functor $\mathcal{M}^{\mathrm{loc}}$ is easy seen to be represented by a projective scheme of finite type over $\mathcal{O}$. Further more we have the following theorem.

**Theorem 2.37.** [Gör01, Theorem 4.19] *The scheme $\mathcal{M}^{loc}$ is flat over $\mathcal{O}$.*

For a $\overline{\kappa}$-algebra $R$ we can identify the lattices $\Lambda_{i,R} = R^n$ of the standard lattice chain in $\mathrm{K}^n$ with the quotients $\lambda_i/t\lambda_i$ of the lattices of the 0-special standard lattice chain

$$\ldots \longrightarrow \lambda_0 \longrightarrow \lambda_1 \longrightarrow \ldots \longrightarrow \lambda_n = t^{-1}\lambda_0 \longrightarrow \ldots$$

in $R((t))^n$. With this identification we can lift an element $(\mathcal{F}_i)_i \in \mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}(R)$ to a lattice chain $(\mathcal{L}_i)_i$ by taking the inverse images of the $\mathcal{F}_i$ under the projection $\lambda_i \to \lambda_i/t\lambda_i$. With this construction we get the following Proposition.

**Proposition 2.38.** [Gör01, Chapter 4.2] *The complete lattice chain $(\mathcal{L}_i)_i$ constructed above is $(n-k)$-special. In particular we get a closed immersion*

$$\mathcal{M}^{loc}_{\overline{\kappa}} \longrightarrow \mathcal{F}$$

*in this way.*

In the following we want to understand the implications of the Bruhat decomposition of Proposition 2.30 on the special fiber of the local model $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}$ via the embedding above. Therefore we need the following definition.

**Definition 2.39.** [KR00] We define the set $\mathrm{Adm}(\mu)$ of *admissible alcoves* to be the set of all $w \in \widetilde{W}$ with $w \leq t^{x(\mu)}$ for some $x \in W$.

**Proposition 2.40.** [Gör01, Chapter 4.3] *For the special fiber of the local model we have the decomposition*

$$\mathcal{M}^{loc}_{\overline{\kappa}} = \bigcup_{w \in \mathrm{Adm}(\mu)} X^{\circ}_{w}$$

*induced by the Bruhat decomposition of the affine flag variety $\mathcal{F}$.*

**Proposition 2.41.** [Gör01, Proposition 4.5] *The irreducible components $C$ of $\mathcal{M}^{loc}_{\overline{\kappa}}$ are the Schubert varieties $X_w$ for $w \in \widetilde{W}$ of the form $t^{x(\mu)}$ for some $x \in W$.*

*Remark* 2.42. Using the Proposition above we can easily compute the number of irreducible components in $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}$ to be $\sharp W/W_J = \binom{n}{k}$.

We also have described in Lemma 2.33 the Schubert varieties in $\mathcal{F}$ using their projections to the affine Grassmannians $\mathcal{G}_i$ for $i \in [n]$. For $i \in [n]$ let us denote by abuse of notation the natural projection

$$\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}} \subseteq \prod_{j \in [n]} \mathrm{Gr}^k\left(\Lambda_j\right)_{\mathbb{F}_p} \longrightarrow \mathrm{Gr}^k\left(\Lambda_i\right)_{\mathbb{F}_p}$$

with $\pi_i$. The embedding $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}} \subseteq \mathcal{F}$ of Proposition 2.38 is clearly compatible with the projections, i.e. we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}} \arrow[r] \arrow[d, "\pi_i"'] & \mathcal{F} \arrow[d, "\pi_i"] \\
\mathrm{Gr}^k\left(\Lambda_i\right)_{\mathbb{F}_p} \arrow[r] & \mathcal{G}_i
\end{tikzcd}$$

where the embeddings $\mathrm{Gr}^k\left(\Lambda_i\right) \to \mathcal{G}_i$ for $i \in [n]$ are constructed similarly to the discussion leading to Proposition 2.38. In particular we can apply Lemma 2.33 to irreducible components of $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}$ and immediately get the following lemma.

**Lemma 2.43.** *An irreducible component $C$ of $\mathcal{M}^{loc}_{\overline{\kappa}}$ can be described as the intersection of $\prod \pi_i(C)$ with $\mathcal{M}^{loc}_{\overline{\kappa}}$ taken in $\prod_{i \in [n]} \mathrm{Gr}^k\left(\Lambda_i\right)_{\mathbb{F}_p}$.*

The projections $\pi_i$ above from the special fiber of the local model $\mathcal{M}^{\mathrm{loc}}_{\mathbb{F}_p}$ to the Grassmannians $\mathrm{Gr}^k\left(\Lambda_i\right)_{\mathbb{F}_p}$ will play a central roll in the chapter to follow. Let us now try to describe the projection for $i = 0$ explicitly. For this purpose we cite a result identifying $\mathrm{Adm}(\mu)$ with the set $Q^J$ discussed in Section 2.1.

**Theorem 2.44.** [HL15, Theorem 2.2] *We have an order preserving bijection*

$$\begin{aligned} Q_J &\longrightarrow \mathrm{Adm}(\mu) \\ (x, y) &\longmapsto w_{x,y} := y t^{\mu} x^{-1} \end{aligned}$$

*of posets.*

**Corollary 2.45.** *We have an order preserving bijection*

$$\begin{aligned} {}^{J}Q &\longrightarrow \mathrm{Adm}(\mu) \\ (x, y) &\longmapsto w'_{x,y} := x t^{\mu} y^{-1} \end{aligned}$$

*of posets.*

*Proof.* Since for $(x, y) \in {}^{J}Q$ using the notation of the last theorem, we can identify $w'_{x,y} := x t^{\mu} y^{-1}$ with $x(y_J)^{-1} t^{\mu} (y^J)^{-1} = w_{y^J, x(y_J)^{-1}}$ the claimed bijection is now the composition of ${}^{J}Q \to Q_J$ with $Q_J \to \mathrm{Adm}(\mu)$. □

**Corollary 2.46.** *Fix $x \in W^J$ and denote with $w$ the element of maximal length in $W$. Then in the set $\{w \in \mathrm{Adm}(\mu) | w = w'_{x,y} = xt^\mu y^{-1},$ for $y \in W\}$ the element $xt^\mu({}^J w)$ is unique element of minimal length $l(x)$ and the element $t^{x(\mu)} = xt^\mu x^{-1}$ is the unique element with maximal length $l(t^\mu)$.*

*Proof.* Recall from Lemma 2.16 that the length of an element $xt^\mu y = xy_J t^\mu({}^J y)$ can be computed as $l(t^\mu) - l({}^J y) + l(xy_J) = (t^\mu) - l({}^J y) + l(x) + l(y_J)$. The length is minimal if $y = {}^J y$ and the length $l(y)$ is maximal. We get this minimum for $y = {}^J w$ where $w$ is the maximal element of $W$. Since $t^\mu({}^J w) = \tau^k$ is of length 0, we compute the length $l(xt^\mu({}^J w))$ to be $l(x)$.
From the theorem above and the definition of ${}^J Q$ we get ${}^J y \geq x^{-1} y_J$. Hence the difference $l(t^\mu) - l({}^J y) + l(xy_J)$ is maximal if $y = x^{-1}$ and we compute the length to be $l(t^\mu)$. □

**Corollary 2.47.** *We can index the Bruhat decomposition using ${}^J Q$ instead of $\mathrm{Adm}(\mu)$ and get:*

$$\mathcal{M}^{loc}_{\overline{\kappa}} = \bigcup_{(x,y)\in {}^J Q} X^\circ_{w'_{x,y}}.$$

*Furthermore $\overline{Iw'_{x,y}}$ is the union $\bigcup_{(x',y')\leq_{{}^J Q}(x,y)} Iw'_{x',y'}$ and the Schubert cell $X^\circ_{w'_{x,y}}$ is an affine space of dimension $\dim(X^\circ_{w'_{x,y}}) = (t^\mu) - l({}^J y) + l(x) + l(y_J)$.*

*Remark* 2.48. Recall the classical Bruhat decomposition $\mathrm{Gr}^{n,r}_{\overline{\kappa}} = \bigcup_{S\in\binom{[n]}{k}} X^\circ_S$ of the Grassmannian into into Schubert cells which we denote by abuse of notation with $X^\circ_S$ for $S \in \binom{[n]}{k}$.
The projection

$$\pi_0\colon \bigcup_{(x,y)\in {}^J Q} X^\circ_{w_{x,y}} = \mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}} \subseteq \prod_{i\in[n]} \mathrm{Gr}^k(\Lambda_i)_{\overline{\kappa}} \longrightarrow \mathrm{Gr}^k(\Lambda_0)_{\overline{\kappa}} = \mathrm{Gr}^{n,r}_{\overline{\kappa}} = \bigcup_{S\in\binom{[n]}{k}} X^\circ_S$$

restricts to projections of a Schubert cell $X^\circ_{w_{x,y}}$ of $\mathcal{M}^{\mathrm{loc}}_{\overline{\kappa}}$ to the Schubert cell $X^\circ_{S_x}$ of $\mathrm{Gr}^{n,r}_{\overline{\kappa}}$.
Indeed the action of the Iwahori $B$ restricted to $\mathrm{Gr}^{n,r}_{\overline{\kappa}}$ via the projection to Borel of upper triangular matrices. It is clear that now the projection is $I$-equivariant and Iwahori orbits surject to Borel orbits. And since the elements in $\mathrm{Adm}(\mu)$ are mapped to $\binom{[n]}{r}$ via the maps in Corollary 2.15 the claim follows. For the explicit calculation of the projection we have attached a complete list for the case $n = 4$ and $k = 2$ in Appendix A. And to get an idea of the combinatorics of the Schubert varieties we have attached a list of some examples in Appendix B.

**Lemma 2.49.** *For $x \in W^J$ the set $\pi_0^{-1}(x)$ is the interval $[xt^\mu({}^J w), xt^\mu x^{-1}]$, i.e. for $w \in \widetilde{W}$ we have $\pi_0(w) = x$ if and only if $xt^\mu({}^J w) \leq w \leq xt^\mu x^{-1}$.*

*Proof.* From the bijection in Corollary 2.45 we see that the elements in $\pi_0^{-1}(x)$ are of the form $w'_{x,y}$ for some $y \in W$. Using Corollary 2.46 we already determined the unique elements of maximal and minimal length. Now fix $y \in W$. Then $w'_{x,y} \leq w'_{x,x} = xt^\mu x^{-1}$ is equivalent to $(x,y) \leq_{{}^J Q} (x,x)$. Using the bijection with $Q_J$ this is equivalent to $(y^J, xy_J^{-1}) \preceq_{Q_J} (x^J, xx_J^{-1})$ but this is true since $y^J \geq xy_J^{-1} \geq x = x^J$ and $xy_J^{-1} \leq x = xx_J^{-1}$. Now we check that $w'_{x,y} \geq w'_{x,({}^J w)^{-1}} = xt^{\mu J} w$. Again this is equivalent to $(y^J, xy_J^{-1}) \succeq_{Q_J} (w^J, x(w^J)_J) = (w^J, x)$. But this is true since $y^J \leq w^J$ and $xy_J^{-1} \geq x$.

□

## 3. Mustafin Varieties

In this chapter we will start by defining Mustafin varieties $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma)$ for two natural numbers $n$ and $k$ and a finite set $\Gamma$ of lattice classes in $\mathbb{Q}_p^n$. These schemes first appeared in [Mum72, Chapter 2] $k = 1$ and $n$ arbitrary and were studied later by Mustafin in [Mus78, Chapter 2]. The name Mustafin variety was introduced in [CHSW11, Definition 1.1.] and generalised to $n$ and $k$ arbitrary (and even arbitrary flag types) in [Häb14, Definition 2.1]. Once we have defined Mustafin varieties the flatness of the local model $\mathcal{M}^{\mathrm{loc}}$ immediately identifies the local model with the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ for the standard lattice chain.
In the second part of this chapter we will follow [Fal01] and describe Mustafin varieties $\mathcal{M}_{\mathbb{P}}(\Gamma)$ for a convex set of lattice classes as blow-ups of $\mathbb{P}(\Lambda)$ for any class $[\Lambda]$ in $\Gamma$ and show its semi-stability.

3.1. **The local model as a Mustafin Variety.** The definition of Mustafin varieties requires the rather classical construction of the join of two schemes. The definition can be found for example in [Mum72, Chapter 2] or [Häb11, Definition 2.1].

**Definition 3.1.** For two reduced and separated $\mathbb{Z}_p$-schemes $X_1$ and $X_2$ with identical generic fiber $X_{1,\mathbb{Q}_p} = X_{2,\mathbb{Q}_p}$ the *join* $X_1 \vee X_2$ is defined as the scheme theoretic closure of the generic fiber $X_{1,\mathbb{Q}_p} = X_{2,\mathbb{Q}_p}$ diagonally embedded in $X_1 \times_{\mathbb{Z}_p} X_2$.

For the join we will frequently use the properties described in the following two lemmas. Both lemmas are quite immediate and proofs can be found for example in [Häb11].

**Lemma 3.2.** [Häb11, Lemma 2.2] *Let $X$, $Y$ and $Z$ be reduced and separated $\mathbb{Z}_p$-schemes with identical generic fiber $H$ then we have:*

(i) $(X \vee Y)_{\mathbb{Q}_p} = H$
(ii) $X \vee Y = Y \vee X$
(iii) $(X \vee Y) \vee Z = X \vee (Y \vee Z)$
(iv) $X \vee X = X$
(v) $X \vee Y \vee Z = (X \vee Y) \vee (Y \vee Z)$

**Lemma 3.3.** [Häb11, Lemma 2.3] *Fix two reduced and separated $\mathbb{Z}_p$-schemes $X_1$ and $X_2$ with identical generic fiber. Let $Y$ be a flat $\mathbb{Z}_p$-scheme with two morphism $f_i\colon Y \to X_i$ for $i = 1, 2$. If the two morphism are identical on the generic fiber the natural morphism $Y \to X_1 \times_{\mathbb{Z}_p} X_2$ factors over the closed subscheme $X_1 \vee X_2 \subseteq X_1 \times_{\mathbb{Z}_p} X_2$.*

**Definition 3.4.** Let $\Gamma$ be a finite set of homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$ for a fixed $n \in \mathbb{N}$ and $\Gamma^{\mathrm{rep}}$ be a set of representatives. For a lattice $\Lambda \in \Gamma^{\mathrm{rep}}$ and a fixed $k \in [n]$ the inclusion $\Lambda \subseteq \mathbb{Q}_p^n$ identifies $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{Q}_p^n$ and hence the generic fiber $\mathrm{Gr}^k(\Lambda)$ is naturally isomorphic to $\mathrm{Gr}^k\left(\mathbb{Q}_p^n\right)$. The *Mustafin variety* is now defined as the join $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma) \coloneqq \bigvee_{\Lambda \in \Gamma^{\mathrm{rep}}} \mathrm{Gr}^k(\Lambda)$ over $\mathbb{Z}_p$.

*Remark* 3.5. For two representatives $\Lambda$ and $p^m\Lambda$ of the same homothety class we get a canonical isomorphism $\mathrm{Gr}^k(\Lambda) \cong \mathrm{Gr}^k(p^m\Lambda)$ and hence up to canonical isomorphism the definition is independent of the choice of representatives.

*Remark* 3.6. Using Theorem 2.37 we know that $\mathcal{M}^{\mathrm{loc}}$ is flat and since we have the embedding $\mathcal{M}^{\mathrm{loc}} \subseteq \prod_{[\Lambda] \in \Gamma^{\mathrm{st}}} \mathrm{Gr}^k(\Lambda)$ we can identify it with the closure of its generic fiber. Hence we have an identification with the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$.

**Definition 3.7.** For a finite set $\Gamma$ of $\mathbb{Z}_p$-lattice classes in $\mathbb{Q}_p^n$ and a subset $\Gamma' \subseteq \Gamma$ the projection $\prod_{[\Lambda] \in \Gamma} \mathrm{Gr}^k(\Lambda) \to \prod_{[\Lambda] \in \Gamma'} \mathrm{Gr}^k(\Lambda)$ induces a projection $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma) \to \mathcal{M}_{\mathrm{Gr}^k}(\Gamma')$ which we denote by $\pi_{\Gamma',\Gamma}$. For the case $\Gamma' = \{\Lambda]\}$ we will also write $\pi_\Lambda$ for $\pi_{\Gamma',\Gamma}$ when ever $\Gamma$ is clear from the context.

These projections will play an important roll in the following so let us collect some of their properties.

**Lemma 3.8.** [Häb14, Lemma 3.1] *For finite sets of lattice classes $\Gamma \subseteq \Gamma'$ and an irreducible component $C$ of $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma)_{\mathbb{F}_p}$ there is a unique irreducible component $C'$ of $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma')_{\mathbb{F}_p}$ such that $\pi_{\Gamma,\Gamma'}(C') = C$. Further more the map $\pi_{\Gamma',\Gamma}|_{C'}: C' \to C$ is birational.*

**Definition 3.9.** A finite set $\Gamma$ of $\mathbb{Z}_p$-lattice classes in $\mathbb{Q}_p^n$ is called *convex* if for any two classes $[\Lambda], [\Lambda'] \in \Gamma$ and any representatives $\Lambda$ and $\Lambda'$ the class of the intersection $\Lambda \cap \Lambda'$ is again in $\Gamma$. For an arbitrary finite set of $\mathbb{Z}_p$-lattice classes $\Gamma$ the intersection of all convex sets containing $\Gamma$ is called the *convex closure* and is denoted by $\overline{\Gamma}$.

*Remark* 3.10. This notion of convexity plays an important role for example in [Fal01]. In [JSY07] a reformulation to the notion of *tropical convexity* was given. This reformulation is based on the identification of an apartment of the Bruhat-Tits building with the points $\mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$ of the tropical projective torus $\mathbb{R}^n/\mathbb{R}(1,\dots,1)$ (see for example [CHSW11, Chapter 4]).
In [CHSW11, Chapter 2] also the relation to the more intrinsic notion of *metrically convexity* was discussed. We call $\Gamma$ metrically convex if for $[\Lambda]$ and $[\Lambda']$ in $\Gamma$ all geodesics for the graph metric of the Bruhat-Tits building are contained in $\Gamma$, i.e. any $[\Lambda'']$ with

$$\mathrm{dist}([\Lambda],[\Lambda'']) + \mathrm{dist}([\Lambda''],[\Lambda']) = \mathrm{dist}([\Lambda],[\Lambda'])$$

is contained in $\Gamma$. Since this equality is satisfied for $[\Lambda'']$ with $\Lambda'' = p^n\Lambda \cap p^{n'}\Lambda'$ we see that metrical convexity implies convexity in the sense of Definition 3.9.

**Definition 3.11.** We call two distinct lattices classes $[\Lambda]$ and $[\Lambda']$ neighbours and denote this by $[\Lambda] \sim [\Lambda']$ if $\{[\Lambda],[\Lambda']\}$ is convex. This is equivalent to the existence of two representatives $\Lambda$ and $\Lambda'$ with $p\Lambda \subseteq \Lambda' \subseteq \Lambda$.

*Remark* 3.12. In general it is hard to compute the convex closure for a finite set $\Gamma$. But for two lattices classes $[\Lambda]$ and $[\Lambda']$ the convex closure $\overline{\{[\Lambda],[\Lambda']\}}$ can be described in the following way. Fix a representative $\Lambda$ for $[\Lambda]$ and let $\Lambda'$ be the representative of $[\Lambda']$ minimal with $\Lambda \subseteq \Lambda'$. Then this description gives us a chain ob neighbouring classes

$$[\Lambda] = [\Lambda \cap \Lambda'] \sim [p^1\Lambda \cap \Lambda'] \sim \dots \sim [p^k\Lambda \cap \Lambda'] = [\Lambda']$$

for $k$ minimal with $[p^k\Lambda\cap\Lambda'] = [\Lambda']$. The classes $[p^i\Lambda\cap\Lambda'] = [\Lambda']$ and $[p^k\Lambda\cap\Lambda'] = [\Lambda']$ for $i < j$ in $[k+1]$ are neighbours if and only if $i+1 = j$. We now identify $\overline{\{[\Lambda],[\Lambda']\}}$ with $\{[p^i\Lambda\cap\Lambda'] | i \in \mathbb{Z}\}$.

To indicate the importance of convexity let us cite the following lemma.

**Lemma 3.13.** [CHSW11, Lemma 5.8] *For a finite convex set of lattice classes $\Gamma$ and an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ there exist a unique class $[\Lambda]$ in $\Gamma$ such that $C$ is the unique irreducible component mapping birationally to $\mathbb{P}(\Lambda)_{\mathbb{F}_p}$. In particular the number of irreducible components of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ coincides with the number of lattice classes in $\Gamma$.*

3.2. **Mustafin varieties as blow-ups.** In this section we focus on the much better understood case of Mustafin varieties of projective spaces. We will prove in Proposition 3.25 that for a convex set of lattice classes $\Gamma$ the Mustafin variety $\mathcal{M}_{\mathbb{P}}(\Gamma)$ is semi-stable. This was first proven by Mustafin in [Mus78, Proposition 2.1] for the case that $\Gamma$ forms a simplex and generalised by Faltings in [Fal01, Chapter 5] to the convex case. For the proof Faltings is using the moduli description of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ and then easily reduce to the case of a simplex.
In contrast to this approach we will analyse the description of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ as a sequence of blow-ups (cf. [Fal01, proof of Lemma 5]) and will reprove with Proposition 3.22 that for any $\Lambda$ in $\Gamma$ the Mustafin variety is obtained by a sequence of blow-ups starting with $\mathbb{P}(\Lambda)$ in smooth centers. Using [Gen00, Lemma 3.2.1] saying that blow-ups preserve semi-stability under some hypothesis, we are able to reprove the semi-stability of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ for convex sets of lattice classes in this way. Before we proof the claims above, we will start with some technical preparations.

**Lemma 3.14.** [Mus78, Lemma in Chapter 2] *Let $X$ be a integral noetherian regular scheme with smooth subschemes $Y_1, Y_2 \subset X$ such that $Y_1 \subseteq Y_2$ or $Y_1 \cap Y_2 = \varnothing$, then we can identify the* *blow-ups* $\mathrm{BL}_{Y_2^{\mathrm{s}}}(\mathrm{BL}_{Y_1}(X))$ *and* $\mathrm{BL}_{Y_1^{\mathrm{tot}}}(\mathrm{BL}_{Y_2}(X))$*. Moreover the blow-ups coincide with join* $\mathrm{BL}_{Y_1}(X) \vee \mathrm{BL}_{Y_2}(X)$ *together with the projections* $\pi_1$ *and* $\pi_2$ *to its factors. The situation is summarised in the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{BL}_{Y_2^{\mathrm{s}}}(\mathrm{BL}_{Y_1}(X)) \arrow[r, equal] \arrow[d, "\pi_{Y_2^{\mathrm{s}}}"'] & \mathrm{BL}_{Y_1}(X) \vee \mathrm{BL}_{Y_2}(X) \arrow[r, equal] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \mathrm{BL}_{Y_1^{\mathrm{tot}}}(\mathrm{BL}_{Y_2}(X)) \arrow[d, "\pi_{Y_1^{\mathrm{tot}}}"] \\
\mathrm{BL}_{Y_1}(X) \arrow[dr, "\pi_{Y_1}"'] & & \mathrm{BL}_{Y_2}(X) \arrow[dl, "\pi_{Y_2}"] \\
& X &
\end{tikzcd}$$

*Proof.* Let us start with the first statement in the lemma. The case $Y_1 \cap Y_2 = \varnothing$ is clear since the blow-ups are isomorphisms away from the blow-up centers. So let us concentrate on the other case.

First we look at the situation where $X = \mathrm{Spec}(A[x_0,\dots,x_n])$ is an affine space over a noetherian domain $A$ and $Y_1 = V(x_0,\dots,x_n)$ and $Y_2 = V(x_0,\dots,x_k)$ for some $k \le n$. Then the blow-up $\mathrm{BL}_{Y_1}(X)$ is computed as $V(X_0,\dots,X_n)^c//\mathbb{G}_m$ where the complement $V(X_0,\dots,X_n)^c$ is taken in $\mathrm{Spec}(A[\lambda_1,X_0,\dots,X_n])$ and has the $\mathbb{G}_m$-action defined by $\mu.(\lambda_1,\underline{X}) := (\mu^{-1}\lambda_1,\mu\underline{X})$. The strict transform $Y_2{}^{\mathrm{s}}$ is described by the $\mathbb{G}_m$-quotient of $V(X_0,\dots,X_k)\cap V(X_0,\dots,X_n)^c$. Now we get

$$\begin{aligned}
\mathrm{BL}_{Y_2^{\mathrm{s}}}(\mathrm{BL}_{Y_1}(X)) &= \Big(V(\lambda_2X_0^{(2)},\dots,\lambda_2X_k^{(2)},X_{k+1},\dots X_n)^c \cap V(X_0^{(2)},\dots,X_k^{(2)})^c\Big)//(\mathbb{G}_m\times\mathbb{G}_m)\\
&= \Big(V(\lambda_2,X_{k+1},\dots,X_n)^c \cap V(X_0^{(2)},\dots,X_k^{(2)})^c\Big)//(\mathbb{G}_m\times\mathbb{G}_m)
\end{aligned}$$

with the action of $\mathbb{G}_m\times\mathbb{G}_m$ on $\mathrm{Spec}\Big(A[\lambda_1,\lambda_2,X_0^{(2)},\dots,X_k^{(2)},X_{k+1},\dots,X_n]\Big)$ defined by

$$(\mu_1,\mu_2).(\lambda_1,\lambda_2,\underline{X}) := (\mu_1^{-1}\lambda_1,\mu_2^{-1}\lambda_2,\mu_1\mu_2X_0^{(2)},\dots,\mu_1\mu_2X_k^{(2)},\mu_1X_{k+1},\dots,\mu_1X_n).$$

On the other hand we can compute $\mathrm{BL}_{Y_2}(X)$ as $V(X_0,\dots,X_k)^c//\mathbb{G}_m$ with $\mathbb{G}_m$-action on $V(X_0,\dots,X_k)^c \subseteq \mathrm{Spec}(A[\lambda_2,\underline{X}])$ defined as

$$\mu_2.(\lambda_2,\underline{X}) := (\mu_2^{-1}\lambda_2,\mu_2X_0,\dots,\mu_2X_k,X_{k+1},\dots,X_n).$$

Now the transform $Y_1^{\mathrm{tot}}$ is described by

$$\begin{aligned}
Y_1^{\mathrm{tot}} &= (V(\lambda_2X_0,\dots,\lambda_2X_k,X_{k+1},\dots,X_n)\cap V(X_0,\dots,X_k)^c)//\mathbb{G}_m\\
&= (V(\lambda_2,X_{k+1},\dots,X_n)\cap V(X_0,\dots,X_k)^c)//\mathbb{G}_m
\end{aligned}$$

and we get

$$\mathrm{Bl}_{Y_1^{\mathrm{tot}}}(\mathrm{Bl}_{Y_2}(X)) = \Big(V(X_0,\dots,X_k)^c\cap V(\lambda_2,X_{k+1}^{(2)},\dots,X_n^{(2)})^c\Big)//(\mathbb{G}_m\times\mathbb{G}_m)$$

with $\mathbb{G}_m\times\mathbb{G}_m$ action on $\mathrm{Spec}\Big(A[\lambda_1,\lambda_2,X_0,\dots,X_k,X_{k+1}^{(2)},\dots,X_n^{(2)}]\Big)$ defined by

$$(\mu_1,\mu_2).(\lambda_1,\lambda_2,X_0,\dots,X_k,X_{k+1}^{(2)},\dots,X_n^{(2)}) := (\mu_1^{-1}\lambda_1,\mu_1\mu_2^{-1}\lambda_2,\mu_2X_0,\dots,\mu_2X_k,\mu_1X_{k+1}^{(2)},\dots,\mu_1X_n^{(2)}).$$

Now we can twist the $\mathbb{G}_m\times\mathbb{G}_m$ action as follows

$$\begin{aligned}
\mathbb{G}_m\times\mathbb{G}_m &\longrightarrow \mathbb{G}_m\times\mathbb{G}_m\\
(\mu_1,\mu_2) &\longmapsto (\mu_1,\mu_1\mu_2)
\end{aligned}$$

and we get the same quotient as before.

Now take an affine integral regular noetherian scheme $X = \mathrm{Spec}(R)$. Then $R$ is a noetherian domain and by shrinking further we can find a regular sequence $f_1,\dots,f_n \in R$ with $Y_1 = V(f_1,\dots,f_k)$ for $k \le n$ and $Y_2 = V(f_1,\dots,f_n)$. We use the computations described in Example 1 and identify $X = V(x_i = f_i | i \le n) \subseteq \mathrm{Spec}(R[\underline{x}])$. Then the two blow-ups $\mathrm{Bl}_{Y_1}(X)$ and $\mathrm{Bl}_{Y_2}(X)$ can be identified with the strict transforms $V(\lambda_1x_i = f_i | i \le k) \subseteq \mathrm{Bl}_{(x_i|i\le k)}(R[\underline{x}])$ and

$V(\lambda_2 x_i = f_i | i \leq n) \subseteq \mathrm{Bl}_{(x_i|i\leq n)}(R[\underline{x}])$.[3] Now for the strict transforms inside the blow-up $\mathrm{Bl}_{V(x_i|i\leq k)}\left(\mathrm{Bl}_{\{0\}}(R[\underline{x}])\right)$ we get

$$\mathrm{Bl}_{Y_2^{\mathrm{s}}}(\mathrm{Bl}_{Y_1}(X)) = V(\lambda_1\lambda_2 x_i = f_i | i \leq k) \cap V(\lambda_2 x_i = f_i | k < i \leq n) = \mathrm{Bl}_{Y_1^{\mathrm{tot}}}(\mathrm{Bl}_{Y_2}(X)).$$

And since all the construction are functorial, the statement glue for non affine smooth schemes. For the second statement extend the above diagram as follows

$$\begin{tikzcd}
 & & \mathrm{BL}_{Y_2^{\mathrm{s}}}(\mathrm{BL}_{Y_1}(X)) \arrow[ddll, "\pi_{Y_2^{\mathrm{s}}}"'] \arrow[ddrr, "\pi_{Y_1^{\mathrm{tot}}}"] \arrow[d, leftrightarrow] & & \\
 & & \mathrm{BL}_{Y_1}(X) \bigvee \mathrm{BL}_{Y_2}(X) \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & & \\
\mathrm{BL}_{Y_1}(X) \arrow[drr, "\pi_{Y_1}"'] & & & & \mathrm{BL}_{Y_2}(X) \arrow[dll, "\pi_{Y_2}"] \\
 & & X & &
\end{tikzcd}$$

Now $\pi_2^{-1}(Y_1^{\mathrm{tot}}) = \pi_1^{-1}(\pi_{Y_1}^{-1}(Y_1))$ is a Cartier divisor in $\mathrm{BL}_{Y_1}(X) \bigvee \mathrm{BL}_{Y_2}(X)$ since $\pi_{Y_1}^{-1}(Y_1)$ is one. From the universal property of the blow-up we get a morphism

$$g\colon \mathrm{BL}_{Y_1}(X) \bigvee \mathrm{BL}_{Y_2}(X) \longrightarrow \mathrm{BL}_{Y_1^{\mathrm{tot}}}(\mathrm{BL}_{Y_2}(X)) = \mathrm{BL}_{Y_2^{\mathrm{s}}}(\mathrm{BL}_{Y_1}(X))$$

On the other hand we get a morphism $f$ in the other direction from Lemma 3.3. But over the inverse images of $Y_2$ all morphisms in the above diagram are isomorphism and hence $f$ restricted to $\pi_1^{-1}(\pi_{Y_1}^{-1}(Y_2))$ and $g$ restricted to $\pi_{Y_2^{\mathrm{s}}}^{-1}(\pi_{Y_1}^{-1}(Y_2))$ are inverse to each other. Since $X$ and hence also $\mathrm{BL}_{Y_2^{\mathrm{s}}}(\mathrm{BL}_{Y_1}(X))$ and $\mathrm{BL}_{Y_1}(X) \bigvee \mathrm{BL}_{Y_2}(X)$ are reduced and separated the morphism are inverse to each other. $\square$

Let $\Gamma$ be a finite convex set of $\mathbb{Z}_p$-lattice classes in $\mathbb{Q}_p^n$ and $[\Lambda] \in \Gamma$ such that $\Gamma \smallsetminus \{[\Lambda]\}$ is still convex. The projection $\pi_{\Gamma\smallsetminus\{[\Lambda]\},\Gamma}$ will turn out to be a blow-up. First we define a closed subscheme $Z_{\Gamma,\Lambda} \subseteq \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{\Lambda\})$ that will turn out to be the center of the blow-up.
Fix a representative $\Lambda$ for the homothety class $[\Lambda]$ and for every lattice class $[\Lambda'] \in \Gamma \smallsetminus \{[\Lambda]\}$ take the unique representatives $\Lambda'$ such that $\Lambda' \subseteq \Lambda \nsupseteq p^{-1}\Lambda'$. Define a closed subscheme $Z_{\Lambda'\subseteq\Lambda}$ of $\mathbb{P}(\Lambda')_{\mathbb{F}_p}$ endowed with the reduced scheme structure as the complement of the open subscheme where the the induced birational map $\mathbb{P}(\Lambda') \dashrightarrow \mathbb{P}(\Lambda)$ is defined. This closed subscheme can be identified with $\mathbb{P}(V_{\Lambda'\subseteq\Lambda}) \subseteq \mathbb{P}(\Lambda')_{\mathbb{F}_p}$ for the module $V_{\Lambda'\subseteq\Lambda}$ defined as the kernel of the map $\Lambda'_{\mathbb{F}_p} \to \Lambda_{\mathbb{F}_p}$ induced by the inclusion. We obtain the following commutative diagram:

$$\begin{tikzcd}
\pi_{\Lambda'}^{-1}(Z_{\Lambda'\subseteq\Lambda}) \arrow[r] \arrow[d] & \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}) \arrow[d, "\pi_{\Lambda'}"] \\
Z_{\Lambda'\subseteq\Lambda} \arrow[r] & \mathbb{P}(\Lambda')
\end{tikzcd}$$

Now we take all the inverse images $\pi_{\Lambda'}^{-1}(Z_{\Lambda'\subseteq\Lambda})$ under the natural projections and define

$$Z_{\Gamma,\Lambda} \coloneqq \bigcap_{[\Lambda']\in\Gamma\smallsetminus\{[\Lambda]\}} \pi_{\Lambda'}^{-1}(Z_{\Lambda'\subseteq\Lambda}) \subseteq \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}).$$

[3]Note that by abuse of notation use the same variables $\{x_i\}$ for $X$ and also for the blow-up $\mathrm{BL}_{(x_i|i\leq k)}(X)$.

**Lemma 3.15.** *Let $\Gamma$ be a finite convex set of $\mathbb{Z}_p$-lattice classes in $\mathbb{Q}_p^n$ and $[\Lambda]$ in $\Gamma$ such that $\Gamma \setminus \{[\Lambda]\}$ is convex. Then we can find for every $[\Lambda'] \in \Gamma \setminus \{[\Lambda]\}$ a neighbour $[\Lambda''] \in \Gamma \setminus \{[\Lambda]\}$ of $[\Lambda]$ such that $\pi^{-1}_{\Lambda''}(Z_{\Lambda''\subseteq\Lambda}) \subseteq \pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda})$. Therefore we get*

$$Z_{\Gamma,\Lambda} = \bigcap_{\substack{[\Lambda']\in\Gamma\setminus\{[\Lambda]\} \\ [\Lambda']\sim[\Lambda]}} \pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda}) \subseteq \mathcal{M}_{\mathbb{P}}(\Gamma \setminus \{[\Lambda]\}),$$

*where $[\Lambda'] \sim [\Lambda]$ denotes the neighbouring relation defined above.*

*Proof.* Fix $[\Lambda'] \in \Gamma \setminus \{[\Lambda]\}$. In Remark 3.12 we have explicitly calculated the convex set $\overline{\{[\Lambda'], [\Lambda]\}}$ to be $\{[\Lambda'] = [\Lambda_0], [\Lambda_1], \dots, [\Lambda_k] = [\Lambda]\}$ where in particular $[\Lambda_i]$ is a neighbour of $[\Lambda_{i+1}]$ for $i \le k-1$. Since $\Gamma$ is convex the closure above is contained in $\Gamma$. Now we fix a representative $\Lambda'$ and take the representatives for $[\Lambda_i]$ such that $\Lambda_i \supseteq \Lambda' \nsubseteq p\Lambda_i$ for $0 < i \le k$. We claim that for these representatives we have the inclusions

$$\Lambda' = \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_k.$$

We prove this claim by induction. Since the case $k = 1$ is trivial let us assume we have the inclusions above for $k-1$. Now take the representative for $[\Lambda_k]$ such that $\Lambda_k \supseteq \Lambda_{k-1} \supseteq p\Lambda_k$. We want to see that this representative has the property $\Lambda_k \supseteq \Lambda' \nsubseteq p\Lambda_k$. If we assume that $p\Lambda_k \supseteq \Lambda'$, the sequence

$$[\Lambda'] = [\Lambda_0 \cap p\Lambda_k], [\Lambda_1 \cap p\Lambda_k], \dots, [\Lambda_{k-2} \cap p\Lambda_k], [\Lambda_{k-1} \cap p\Lambda_k] = [\Lambda]$$

is a path of smaller length contradicting the minimality. Hence the claim is proven.
Now the inclusion $\Lambda' \subseteq \Lambda$ factors over the representative $\Lambda_{k-1}$ of a neighbour of $[\Lambda]$ and therefore $V_{\Lambda'\subseteq\Lambda}$ is contained in $V_{\Lambda'\subseteq\Lambda_k}$. The inclusion induce the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{M}_{\mathbb{P}}(\Gamma \setminus \{[\Lambda]\}) \arrow[dl, "\pi_{\Lambda'}"'] \arrow[dr, "\pi_{\Lambda_k}"] & \\
\mathbb{P}(\Lambda') \setminus Z_{\Lambda\subseteq\Lambda'} \arrow[rr] \arrow[dr] & & \mathbb{P}(\Lambda_k) \setminus Z_{\Lambda\subseteq\Lambda_k} \arrow[dl] \\
 & \mathbb{P}(\Lambda) &
\end{tikzcd}$$

and therefore we get $\pi^{-1}_{\Lambda_k}(Z_{\Lambda\subseteq\Lambda_k})^c \subseteq \pi^{-1}_{\Lambda'}(Z_{\Lambda\subseteq\Lambda'})^c$. Taking complements and applying $\pi_{\Lambda_k}$ we get $\pi_{\Lambda_k}(\pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda})) \subseteq Z_{\Lambda_k\subseteq\Lambda}$ and hence the desired result. □

**Lemma 3.16.** *Fix three different lattice classes $[\Lambda]$, $[\Lambda']$ and $[\Lambda'']$, such that any two of them are neighbours, then $Z_{\{[\Lambda],[\Lambda'],[\Lambda'']\},\Lambda}$ is either $\pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda})$ or $\pi^{-1}_{\Lambda''}(Z_{\Lambda''\subseteq\Lambda})$.*

*Proof.* Fix a representative $\Lambda$ for $[\Lambda]$ and take representatives $\Lambda'$ and $\Lambda''$ for the classes $[\Lambda']$ and $[\Lambda'']$ such that $p\Lambda \subseteq \Lambda' \subseteq \Lambda$ and $p\Lambda \subseteq \Lambda'' \subseteq \Lambda$. Assume without loss of generality that $\Lambda' \subseteq \Lambda''$ and hence $V_{\Lambda'\subseteq\Lambda''} \subseteq V_{\Lambda'\subseteq\Lambda}$. The inclusions induces the diagram

$$\begin{tikzcd}
 & \mathcal{M}_{\mathbb{P}}(\Lambda', \Lambda'') \arrow[dl, "\pi_{\Lambda'}"'] \arrow[dr, "\pi_{\Lambda''}"] & \\
\mathbb{P}(\Lambda') \setminus Z_{\Lambda'\subseteq\Lambda} \arrow[rr] \arrow[dr] & & \mathbb{P}(\Lambda'') \setminus Z_{\Lambda''\subseteq\Lambda} \arrow[dl] \\
 & \mathbb{P}(\Lambda) &
\end{tikzcd}$$

and we get $\pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda}) \subseteq \pi^{-1}_{\Lambda''}(Z_{\Lambda''\subseteq\Lambda})$. In particular the intersection

$$Z_{\{[\Lambda],[\Lambda'],[\Lambda'']\},\Lambda} = \pi^{-1}_{\Lambda''}(Z_{\Lambda''\subseteq\Lambda}) \cap \pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda})$$

is simply $\pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda})$. □

**Lemma 3.17.** *Fix a convex set of lattice classes $\Gamma$ and two classes $[\Lambda],[\Lambda'] \in \Gamma$ such that $\Gamma \smallsetminus \{[\Lambda]\}$ and $\Gamma \smallsetminus \{[\Lambda']\}$ are still convex. If the set $\{[\Lambda],[\Lambda']\}$ is convex, then one of the two subvarieties $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ and $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ is contained in the other. In the case that $\{[\Lambda],[\Lambda']\}$ is not convex the two subvarieties $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ and $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ are disjoint.*

*Proof.* First we start with the case where $\{[\Lambda],[\Lambda']\}$ is convex. Fix $[\widetilde{\Lambda}] \in \Gamma \smallsetminus \{[\Lambda],[\Lambda']\}$ and choose representatives such that $\widetilde{\Lambda} \subseteq \Lambda, \Lambda' \nsupseteq p^{-1}\widetilde{\Lambda}$. Since $\{[\Lambda],[\Lambda']\}$ is convex, $\Lambda \cap \Lambda'$ is either $\Lambda$ or $\Lambda'$. Let us assume $\Lambda \cap \Lambda' = \Lambda$, then we get $Z_{\widetilde{\Lambda}\subseteq\Lambda} = Z_{\widetilde{\Lambda}\subseteq\Lambda\cap\Lambda'} \subseteq Z_{\widetilde{\Lambda}\subseteq\Lambda'}$ and since $[\widetilde{\Lambda}]$ was arbitrary we get the inclusion $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} \subseteq Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$.
Now let us assume $\{[\Lambda],[\Lambda']\}$ is not convex then since $\Gamma$ is convex we can choose representatives such that $[\Lambda \cap \Lambda'] \in \Gamma \smallsetminus \{[\Lambda],[\Lambda']\}$. Now we compute

$$Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'} \cap Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} = \bigcap_{[\widetilde{\Lambda}]\in\Gamma\smallsetminus\{[\Lambda],[\Lambda']\}} \pi_{\widetilde{\Lambda}}^{-1}\left(Z_{\widetilde{\Lambda}\subseteq\Lambda} \cap Z_{\widetilde{\Lambda}\subseteq\Lambda'}\right) \subseteq \pi_{\Lambda\cap\Lambda'}^{-1}\left(Z_{\Lambda\cap\Lambda'\subseteq\Lambda} \cap Z_{\Lambda\cap\Lambda'\subseteq\Lambda'}\right),$$

but since we have $Z_{\Lambda\cap\Lambda'\subseteq\Lambda} \cap Z_{\Lambda\cap\Lambda'\subseteq\Lambda'} = \emptyset$ we are done. □

Now we prepared enough to prove the claims of the beginning of this section. We will prove these claims by induction. But first we need to cite the following proposition by Mustafin for the induction base and the lemma below by Faltings to prove the induction step.

**Proposition 3.18.** [Mus78, Proposition 2.1] *For two neighbouring lattice classes $[\Lambda]$ and $[\Lambda']$ we can describe the projection $\pi_\Lambda\colon \mathcal{M}_{\mathbb{P}}(\{[\Lambda],[\Lambda']\}) \to \mathbb{P}(\Lambda)$ by the blow-up in the smooth center $Z_{\{\Lambda,\Lambda'\},\Lambda}$.*

**Lemma 3.19.** [Fal01, proof of Lemma 5] *Fix a finite set $\Gamma$ of classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$ with at least two elements. For every $[\Lambda] \in \Gamma$ there is a lattice class $[\Lambda'] \in \Gamma$ different from $[\Lambda]$ such that $\Gamma \smallsetminus \{[\Lambda']\}$ remains convex. In particular we can find $[\Lambda],[\Lambda'] \in \Gamma$ such that $\Gamma \smallsetminus \{[\Lambda]\}$ and $\Gamma \smallsetminus \{[\Lambda']\}$ and hence also $\Gamma \smallsetminus \{[\Lambda],[\Lambda']\}$ remains convex.*

*Remark* 3.20. For every convex set of lattice classes $\Gamma$ there is a unique minimal subset $\Gamma^{\mathrm{gen}} \subseteq \Gamma$ such that the convex closure $\overline{\Gamma^{\mathrm{gen}}}$ is $\Gamma$, cf. [MS15, Proposition 5.2.17.]. In particular for $[\Lambda] \in \Gamma^{\mathrm{gen}}$ also $\Gamma \smallsetminus \{[\Lambda]\}$ has to be convex and hence $\Gamma^{\mathrm{gen}} = \{[\Lambda] \in \Gamma | \Gamma \smallsetminus \{[\Lambda]\}$ is convex$\}$.

**Lemma 3.21.** *Let $\Gamma$ be a finite convex set of $\mathbb{Z}_p$-lattice classes in $\mathbb{Q}_p^n$ with at least two elements and $[\Lambda]$ in $\Gamma$ such that $\Gamma \smallsetminus \{[\Lambda]\}$ is still convex. Then we can describe the map $\pi_{\Gamma,\Lambda}$ by the blow-up $\mathrm{BL}_{Z_{\Gamma,\Lambda}}(\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}))$ in the smooth center $Z_{\Gamma,\Lambda}$. Furthermore for every $[\Lambda']$ in $\Gamma \smallsetminus \{[\Lambda]\}$ such that $\Gamma \smallsetminus \{[\Lambda']\}$ remains convex and $\Gamma \smallsetminus \{[\Lambda],[\Lambda']\}$ is not empty the center $Z_{\Gamma,\Lambda}$ can be described as*

1. *the total transform $Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ if $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} \subseteq Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$*
2. *the strict transform $Z^{\mathrm{s}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ if $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'} \subseteq Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$*
3. *or $Z^{\mathrm{s}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} = Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ if $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ and $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ are disjoint*

*of the blow-up $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{\Lambda\}) \to \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{\Lambda,\Lambda'\})$.*

*Proof.* We prove the statement by induction on the number of lattices in $\Gamma$. The base case $\Gamma = \{[\Lambda_1],[\Lambda_2]\}$ for the induction was shown in Proposition 3.18.
Now for $\sharp\Gamma \geq 3$ fix two lattices $[\Lambda],[\Lambda'] \in \Gamma$ such that $\Gamma \smallsetminus \{[\Lambda]\}$ and $\Gamma \smallsetminus \{[\Lambda']\}$ are still convex. By hypothesis we know that the maps $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}) \to \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda']\})$ and $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda']\}) \to \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda']\})$ are blow-ups in the center $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ and $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ respectively. To indicate this we denote the map $\pi_{\Gamma\smallsetminus\{[\Lambda]\},\Gamma\smallsetminus\{[\Lambda],[\Lambda']\}}$ by $\pi_{Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}}$ and similarly we write $\pi_{Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}}$ for $\pi_{\Gamma\smallsetminus\{[\Lambda]\},\Gamma\smallsetminus\{[\Lambda],[\Lambda']\}}$.
From Lemma 3.17 we know that in the case where $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ and $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ are not disjoint one has to be contained in the other and we can prove the statement for every case separately. We will first prove Case (1) and then deduce Case (2). The remaining Case (3) is proven analogously. We begin by proving the second part of the lemma hence that $Z_{\Gamma,\Lambda}$ is the total transform

of $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ under the blow-up $\pi_{Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}}$. We compute

$$Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} = \pi^{-1}_{Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}}(Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}) = \bigcap_{\substack{\widetilde{\Lambda}\in\Gamma\smallsetminus\{[\Lambda],[\Lambda']\} \\ \widetilde{\Lambda}\backsim\Lambda}} \pi^{-1}_{\widetilde{\Lambda}}(Z_{\widetilde{\Lambda}\subseteq\Lambda})$$

which shows, that we have to see, that we get the inclusion $Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} \subseteq \pi^{-1}_{\Lambda'}(Z_{\Lambda'\subseteq\Lambda})$ but this follows from Lemma 3.15.
We just have proven that the center $Z_{\Gamma,\Lambda}$ is the total transform $Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$. To finish the proof we need to do some identifications summarised in the following diagram:

$$\begin{tikzcd}
\mathrm{BL}_{Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}}(\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda]\})) \arrow[dr, equal] \arrow[ddd] & & \mathrm{BL}_{Z^{\mathrm{s}}_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}}(\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\})) \arrow[dl, equal] \arrow[ddd] \\
& \mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda]\}) \vee \mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\}) \arrow[d, equal] & \\
& \mathcal{M}_{\mathbb{P}}(\Gamma) \arrow[dl] \arrow[dr] & \\
\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda]\}) \arrow[dr, "\pi_{Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}}"'] & & \mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\}) \arrow[dl, "\pi_{Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}}"] \\
& \mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda],[\Lambda']\}) &
\end{tikzcd}$$

By hypothesis the center $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ and $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ are smooth. Hence $Z_{\Gamma,\Lambda}$ is a blow-up of a smooth scheme over a field in a smooth center and in particular it is smooth.
With Lemma 3.14 we than can identify the two blow-ups $\mathrm{BL}_{Z^{\mathrm{tot}}_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}}(\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda]\}))$ and $\mathrm{BL}_{Z^{\mathrm{s}}_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}}(\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\}))$ with the join $\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda]\}) \vee \mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\})$ which we identify with $\mathcal{M}_{\mathbb{P}}(\Gamma)$ using Lemma 3.2.
Now the second case where $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} \subseteq Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ follows easily from the identification in the digram above by interchanging the rolls of $\Lambda$ and $\Lambda'$. $\square$

**Proposition 3.22.** [Fal01, proof of Lemma 5] *Let $\Gamma$ be a finite convex set of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$ and $\Lambda$ representing a class in $\Gamma$. Then we can describe he Mustafin variety $\mathcal{M}_{\mathbb{P}}(\Gamma)$ by the successive blow-up of $\mathbb{P}(\Lambda)$ in smooth centers.*

*Proof.* Proving the statement by induction on the number of classes in $\Gamma$, we can assume the statement is true for a convex set of lattice classes $\Gamma$ with $i$ elements. Now for a convex set of lattice classes $\Gamma$ with $i+1$ elements and a class $[\Lambda]$ in $\Gamma$ we use Lemma 3.19 to find a lattice class $[\Lambda'] \neq [\Lambda]$ such that $\Gamma\smallsetminus\{[\Lambda']\}$ is still convex. Now we are done since $\mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\})$ is obtained by a sequence of blow-ups of $\mathbb{P}(\Lambda)$ in smooth centers by assumption and the map $\mathcal{M}_{\mathbb{P}}(\Gamma) \to \mathcal{M}_{\mathbb{P}}(\Gamma\smallsetminus\{[\Lambda']\})$ is a blow-up in a smooth center by Lemma 3.21. $\square$

*Remark* 3.23. For two classes $[\Lambda]$ and $[\Lambda']$ in a convex set $\Gamma$, the two irreducible components $C_\Lambda$ and $C_{\Lambda'}$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ can be described as the exceptional divisors of the blow-ups $\pi_{Z_{\Gamma,\Lambda}}$ and $\pi_{Z_{\Gamma,\Lambda'}}$ respectively. Using Lemma 3.17 it is easy to see, that $C_\Lambda$ and $C_{\Lambda'}$ are disjoint if $[\Lambda]$ and $[\Lambda']$ are not neighbours. Also the converse is true and a proof can be found in [CHSW11, Theorem 2.10].

We are now prepared to use the description of Mustafin varieties as a sequence of blow-ups and the following lemma on semi-stability under blow-ups to get a new proof of the semi-stability of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ for $\Gamma$ convex (cf. [Fal01, Chapter 5]).

**Lemma 3.24.** [Gen00, Lemma 3.2.1] *Let $X$ be semi-stable $\mathbb{Z}_p$-scheme and $Y \subseteq X_{\mathbb{F}_p}$ a closed subscheme of the special fiber. Suppose $Y$ is smooth over $\mathbb{F}_p$ and the intersection $Y \cap X_{\mathbb{F}_p}^{\text{sing}}$ is a simple normal crossing divisor on $Y$. Then blow-up $\mathrm{Bl}_Y(X)$ is semi-stable.*

**Proposition 3.25.** *For a finite convex set $\Gamma$ of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$ the Mustafin variety $\mathcal{M}_{\mathbb{P}}(\Gamma)$ is semi-stable.*

*Proof.* We will prove by induction on the number of elements of $\Gamma$ that the center $Z_{\Gamma,\Lambda}$ in $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})$ satisfies the conditions of Lemma 3.24 and hence $\mathcal{M}_{\mathbb{P}}(\Gamma)$ is semi-stable. Since we already know that $Z_{\Gamma,\Lambda}$ is smooth, we are left to show that $Z_{\Gamma,\Lambda} \cap \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})_{\mathbb{F}_p}^{\text{sing}}$ is a simple normal crossing divisor on $Z_{\Gamma,\Lambda}$.
Assume that we can find a class $[\Lambda']$ different from $[\Lambda]$ such that $\Gamma \smallsetminus \{[\Lambda']\}$ is still convex and $[\Lambda']$ is not a neighbour of $[\Lambda]$. Then $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ and $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ are disjoint and hence the blow-up $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}) \to \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda']\})$ restricts to an isomorphism in a neighbourhood of $Z_{\Gamma,\Lambda} \cong Z_{\Gamma\smallsetminus\{[\Lambda']\}}$ and the statement is trivial.
Now take any $[\Lambda']$ different from $[\Lambda]$ such that $\Gamma \smallsetminus \{[\Lambda']\}$ is still convex. Using the previous step we can assume that $[\Lambda]$ and $[\Lambda']$ are neighbours. Then we know by Lemma 3.21 that one of $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ and $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ contains the other. Let us assume that we are in the situation $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'} \subseteq Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$. The other case is proven analogously. In particular by Case (2) of Lemma 3.21 this implies that $Z_{\Gamma,\Lambda}$ is the strict transform of $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$. By induction hypothesis we know that $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})$ is semi-stable and hence étale locally of the form

$$\mathbb{Z}_p[x_0,\dots,x_m]/\left(\prod_{i\le r} x_i - p\right)$$

Again by induction hypothesis we know that both $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda} \cap \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda']\})_{\mathbb{F}_p}^{\text{sing}}$ and $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'} \cap \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda']\})_{\mathbb{F}_p}^{\text{sing}}$ are a simple normal crossing divisor on $Z_{\Gamma\smallsetminus\{[\Lambda']\},\Lambda}$ and $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'}$ respectively. In particular as shown in [Gen00, proof of Lemma 3.2.1] étale locally $Z_{\Gamma\smallsetminus\{[\Lambda]'\},\Lambda}$ is of the form $V(x_r,\dots,x_{m_1})$ and $Z_{\Gamma\smallsetminus\{[\Lambda]\},\Lambda'} = V(x_r,\dots,x_{m_2})$ for some $m_1 < m_2 \le m$. Now the strict transform $Z_{\Gamma,\Lambda}$ is cut out by the equations $V(X_r,\dots,X_{m_1})$ in the blow-up described as the $\mathbb{G}_m$-quotient of

$$V(X_r,\dots X_{m_2})^c \subseteq \operatorname{Spec}\left(\mathbb{Z}_p[\lambda,x_1,\dots,x_{r-1},X_r,\dots X_{m_2},x_{m_2+1},\dots,x_m]/\left(\left(\prod_{i\le r-1} x_i\right)X_r\lambda - p\right)\right)$$

The intersection of $Z_{\Gamma,\Lambda}$ with singular locus is now cut out of $Z_{\Gamma,\Lambda}$ by the product of the regular sequence $x_1,\dots,x_{r-1},\lambda$. This clearly defines a simple normal crossing divisor before taking the $\mathbb{G}_m$-quotient. But $x_1,\dots,x_{r-1},\lambda$ also define regular sequences in the charts of the $\mathbb{G}_m$-quotient and hence $Z_{\Gamma,\Lambda} \cap \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})_{\mathbb{F}_p}^{\text{sing}}$ is a simple normal crossing divisor on $Z_{\Gamma,\Lambda}$.
In summary $Z_{\Gamma,\Lambda}$ again satisfies the condition of Lemma 3.24. □

To end this section let us prove a useful lemma on the behaviour of irreducible components under the blow-ups of Lemma 3.21.

**Lemma 3.26.** *Fix a convex set of lattice classes $\Gamma$ and a lattice $\Lambda$ in $\Gamma$ such that $\Gamma \smallsetminus \{[\Lambda]\}$ is again convex. For the projection $\pi\colon \mathcal{M}_{\mathbb{P}}(\Gamma) \to \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})$ the inverse image $\pi^{-1}(\pi(C))$ of the image of an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ is a union of irreducible components.*

*Proof.* From Lemma 3.13 we get for every class $[\Lambda']$ in $\Gamma$ a unique irreducible component $C_{\Lambda'}$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ surjecting to $\mathbb{P}(\Lambda')_{\mathbb{F}_p}$ under the natural projection. Note that for an irreducible component $C'$ of $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})_{\mathbb{F}_p}$ the strict transform under the blow-up $\pi\colon \mathcal{M}_{\mathbb{P}}(\Gamma) \to \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})$ is again an irreducible component. Hence the irreducible component $C_\Lambda$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)$ coincides with the exceptional divisor of $\pi$.
Let us prove the statement by induction on the number of elements in $\Gamma$. The Fix $[\Lambda']$ in $\Gamma$

different from $[\Lambda]$. If the classes $[\Lambda]$ and $[\Lambda']$ are not neighbours, then $C_\Lambda$ and $C_{\Lambda'}$ are disjoint by Remark 3.23 and hence $\pi(C_{\Lambda'})$ is disjoint to the center of the blow-up $\pi$. In particular $\pi^{-1}(\pi(C_{\Lambda'}))$ is $C_{\Lambda'}$.
Let us assume from now on that $[\Lambda]$ and $[\Lambda']$ are neighbours. Assume there is a class $[\Lambda'']$ in $\Gamma$ not neighbouring $[\Lambda']$ and such that $\Gamma \smallsetminus \{[\Lambda'']\}$ is still convex. We fix notation as in the following diagram

$$\begin{tikzcd} & \mathcal{M}_{\mathbb{P}}(\Gamma) \arrow[dl, "\pi_\Lambda"'] \arrow[dr, "\pi_{\Lambda''}"] & \\ \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}) \arrow[dr, "\pi_{\Lambda,\Lambda''}"'] & & \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda'']\}) \arrow[dl, "\pi_{\Lambda'',\Lambda}"] \\ & \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda'']\}) & \end{tikzcd}$$

The blow-up $\pi_{\Lambda,\Lambda''}$ restricts to an isomorphism on $\pi_\Lambda(C'_\Lambda)$ and similarly $\pi_{\Lambda''}$ restricts to an isomorphism on $C_{\Lambda'}$. Now the image $\pi_{\Lambda''}(C_{\Lambda'})$ is again an irreducible component and by induction hypothesis we know that $\pi^{-1}_{\Lambda'',\Lambda}(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}(C_{\Lambda'}))$ is either $\pi_{\Lambda''}(C_{\Lambda'})$ or $\pi_{\Lambda''}(C_{\Lambda'})\cup\pi_{\Lambda''}(C_\Lambda)$. In particular $\pi_\Lambda^{-1}(\pi_\Lambda(C_{\Lambda'}))$ is either $C_{\Lambda'}$ or $C_{\Lambda'}\cup C_\Lambda$.
Assume there is a class $[\Lambda'']$ in $\Gamma$ not neighbouring $[\Lambda]$ and such that $\Gamma \smallsetminus \{[\Lambda'']\}$ is still convex. Using the previous step we can assume that $[\Lambda'']$ is a neighbour of $[\Lambda']$. Now $\pi_{\Lambda''}$ restricts to an isomorphism on $C_\Lambda$ and similarly $\pi_{\Lambda,\Lambda''}$ restricts to an isomorphism on $\pi_\Lambda(C_\Lambda)$. We now compute

$$\begin{aligned}\pi_\Lambda^{-1}\left(\pi_\Lambda\left(C_{\Lambda'}\right)\right)\cap C_\Lambda &= \pi_\Lambda^{-1}\left(\pi_\Lambda\left(C_{\Lambda'}\cap C_\Lambda\right)\right) = \left(\pi_{\Lambda,\Lambda''}\circ\pi_\Lambda\right)^{-1}\left(\left(\pi_{\Lambda,\Lambda''}\circ\pi_\Lambda\right)\left(C_{\Lambda'}\cap C_\Lambda\right)\right)\\ &= \left(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}\right)^{-1}\left(\left(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}\right)\left(C_{\Lambda'}\cap C_\Lambda\right)\right)\\ &= \pi_{\Lambda''}^{-1}\left(\pi^{-1}_{\Lambda'',\Lambda}(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}(C_{\Lambda'}))\cap\pi_{\Lambda''}(C_\Lambda)\right)\end{aligned}$$

and again by induction hypothesis we know that $\pi^{-1}_{\Lambda'',\Lambda}(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}(C_{\Lambda'}))$ is either $\pi_{\Lambda''}(C_{\Lambda'})$ or $\pi_{\Lambda''}(C_{\Lambda'})\cup\pi_{\Lambda''}(C_\Lambda)$. We conclude that for $\pi_\Lambda^{-1}\left(\pi_\Lambda\left(C_{\Lambda'}\right)\right) = C_{\Lambda'}\cup\left(\pi_\Lambda^{-1}\left(\pi_\Lambda\left(C_{\Lambda'}\right)\right)\cap C_\Lambda\right)$ we get

$$\pi_\Lambda^{-1}\left(\pi_\Lambda\left(C_{\Lambda'}\right)\right) = \begin{cases} C_{\Lambda'} & \text{if } \pi^{-1}_{\Lambda'',\Lambda}(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}(C_{\Lambda'})) = \pi_{\Lambda''}(C_{\Lambda'}) \\ C_{\Lambda'}\cup C_\Lambda & \text{if } \pi^{-1}_{\Lambda'',\Lambda}(\pi_{\Lambda'',\Lambda}\circ\pi_{\Lambda''}(C_{\Lambda'})) = \pi_{\Lambda''}(C_{\Lambda'})\cup\pi_{\Lambda''}(C_\Lambda). \end{cases}$$

Now we assume that there are two distinct classes $[\Lambda'']$ and $[\Lambda''']$ such that $\Gamma \smallsetminus \{[\Lambda'']\}$ and $\Gamma \smallsetminus \{[\Lambda''']\}$ are convex and $[\Lambda'']$ and $[\Lambda''']$ are not neighbours. By reductions above we can further assume that $[\Lambda'']$ and $[\Lambda''']$ are are different from $[\Lambda]$ and $[\Lambda']$. Let us again fix some notation and extend the diagram above

$$\begin{tikzcd} & \mathcal{M}_{\mathbb{P}}(\Gamma) \arrow[dl, "\pi_{\Lambda''}"'] \arrow[d, "\pi_\Lambda"'] \arrow[dr, "\pi_{\Lambda'''}"] & \\ \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda'']\})] \arrow[d, "\pi_{\Lambda'',\Lambda}"'] & \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\}) \arrow[dl, "\pi_{\Lambda,\Lambda''}"'] \arrow[dr, "\pi_{\Lambda,\Lambda'''}"] & \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda''']\}) \arrow[d, "\pi_{\Lambda''',\Lambda}"] \\ \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda'']\}) & & \mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda],[\Lambda''']\}) \end{tikzcd}$$

Then $C_{\Lambda''}$ and $C_{\Lambda'''}$ are disjoint. In particular for every irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ we get

$$C \subseteq \pi^{-1}_{\Lambda''}(\pi_{\Lambda''}(C))\cap\pi^{-1}_{\Lambda'''}(\pi_{\Lambda'''}(C)) \subseteq (C\cup C_{\Lambda''})\cap(C\cup C_{\Lambda'''}) = C$$

and similarly for an irreducible component of $\mathcal{M}_{\mathbb{P}}(\Gamma \smallsetminus \{[\Lambda]\})$ and the projections $\pi_{\Lambda,\Lambda''}$ and $\pi_{\Lambda,\Lambda'''}$. Hence $\pi_\Lambda^{-1}(\pi_\Lambda(C_{\Lambda'}))$ is the intersection of the inverse images of the images of $C_{\Lambda'}$ under the projections $\pi_{\Lambda,\Lambda''}\circ\pi_\Lambda$ and $\pi_{\Lambda,\Lambda'''}\circ\pi_\Lambda$. Now using the induction hypothesis we know that the inverse image of the image of $\pi_{\Lambda''}(C_{\Lambda'})$ under $\pi_{\Lambda'',\Lambda}$ is either $\pi_{\Lambda''}(C_{\Lambda'})$ or $\pi_{\Lambda''}(C_{\Lambda'}) \cup \pi_{\Lambda''}(C_{\Lambda})$ and similarly for $\Lambda'''$. Together we get the inverse image of the image of $C_{\Lambda'}$ under $\pi_{\Lambda}$ is either $C_{\Lambda'}$ or $C_{\Lambda'} \cup C_{\Lambda}$.

For the remaining case we recall from Remark 3.20 that $\Gamma$ is the convex closure of the set of $[\Lambda''] \in \Gamma$ such that $\Gamma \smallsetminus \{[\Lambda'']\}$ is convex. By the previous steps we can assume that the set $\{[\Lambda''] \in \Gamma | \Gamma \smallsetminus \{[\Lambda'']\}$ is convex$\}$ is contained in a simplex. Hence $\Gamma$ is contained in a simplex and we refer to the explicit calculations of the blow-up in [Mus78, proof of Proposition 2.1] . $\square$

# 4. The Plücker embedding for $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})$

In this chapter we are fixing two integers $n \in \mathbb{N}$ and $0 \neq k \in [n]$ and try to get a relation between the irreducible components of the special fibres of the two Mustafin varieties $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma)$ and $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma)$. The image in $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma)_{\mathbb{F}_p}$ of an irreducible component $C^{\mathrm{gr}}$ of the special fiber of the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma)_{\mathbb{F}_p}$ is again irreducible and hence lies in some irreducible component $C^{\mathrm{pr}}$ of $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma)_{\mathbb{F}_p}$. We will discuss that for the standard lattice chain $\Gamma^{\mathrm{st}}$ with $k = 2$ or $n \leq 5$ and conjecturally for all $n$ this component $C^{\mathrm{pr}}$ is unique. On the other hand we show that in these cases every irreducible component of $\mathcal{M}_{\mathbb{P}}(\bigwedge^k \Gamma^{\mathrm{st}})_{\mathbb{F}_p}$ arises in this way and hence we get a bijective correspondence of irreducible components of the two Mustafin varieties.

4.1. **Irreducible components and linear subspaces.** As a first step we will show that there are linear subspaces $\mathbb{P}(V_I)$ indexed by $I \in \binom{[n]}{k}$ such that for every $I$ there is a unique irreducible component surjecting to $\mathbb{P}(V_I)$ under the projection $\overline{\pi^0_{\mathbb{P}}}\colon \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ and every irreducible component is obtained in that way. In particular $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})_{\mathbb{F}_p}$ and $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ have the same number of irreducible components.
For the rest of this chapter we fix the basis $\{e_I\}_{I\in\binom{[n]}{k}}$ of $\bigwedge^k \mathbb{Q}_p^n$ where $e_I = e_{i_0} \wedge \cdots \wedge e_{i_{k-1}}$ for every $I = \{i_0, \ldots, i_{k-1}\} \in \binom{[n]}{k}$ and $\{e_i\}$ is the standard basis $\mathbb{Q}_p^n$. We define a partial order on this basis by setting $\{i_0 < \cdots < i_{k-1}\} \leq \{j_0 < \cdots < j_{k-1}\}$ if $i_t \leq j_t$ for all $t \in [k]$.

**Definition 4.1.** [HP94, Chapter XIV 3] For $I \in \binom{[n]}{k}$ we write $V_I$ for the $\mathbb{F}_p$-submodule of $\bigwedge^k \Lambda_{0,\mathbb{F}_p}$ generated by $\{e_J | J \leq I\}$.

**Lemma 4.2.** [HP94, Chapter XIV 3] *Using the Plücker embedding* $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p} \to \mathbb{P}(\bigwedge^k \Lambda_0)_{\mathbb{F}_p}$ *we identify the Schubert variety $X_I$ in* $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p}$ *for $I \in \binom{[n]}{k}$ with the intersection of $\mathbb{P}(V_I)$ with* $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p}$.

*Remark* 4.3. Let us recall from Lemma 3.13 that for an irreducible component $C$ in $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ there is a unique lattice $\Lambda_C$ in $\overline{\bigwedge^k \Gamma^{\mathrm{st}}}$ such that under the projection

$$\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \subseteq \prod_{\Lambda \in \bigwedge^k \Gamma^{\mathrm{st}}} \mathbb{P}(\Lambda) \longrightarrow \mathbb{P}(\Lambda_C)$$

$C$ is surjecting to $\mathbb{P}(\Lambda_C)_{\mathbb{F}_p}$. And conversely for every lattice $\Lambda$ in $\overline{\Gamma^{\mathrm{st}}}$ there is a unique irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ surjecting to $\mathbb{P}(\Lambda)$.

For $[\Lambda] \in \overline{\bigwedge^k \Gamma^{\mathrm{st}}}$ we now chose the unique maximal representative of $[\Lambda]$ contained in $\bigwedge^k \Lambda_0$. The inclusion $\Lambda \subseteq \bigwedge^k \Lambda_0$ gives us a birational map $\mathbb{P}(\Lambda) \dashrightarrow \mathbb{P}(\bigwedge^k \Lambda_0)$. Note that for an other representative $\Lambda'$ of $[\Lambda]$ we can use the identification $\Lambda' = p^r \Lambda$ for some $r \in \mathbb{Z}$ to precompose the birational map with the induced isomorphism $\mathbb{P}(\Lambda') \cong \mathbb{P}(\Lambda)$ and to get a birational map with the same image. We now can identify the special fibers of the images of these birational maps with the images of irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$.

**Lemma 4.4.** *Using the notation above we have an equality of sets*

$$\left\{ \overline{\pi^0_{\mathbb{P}}}(C) \,\middle|\, C \text{ irr. component in } \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p} \right\} = \left\{ \mathrm{Im}(\mathbb{P}(\Lambda)_{\mathbb{F}_p} \dashrightarrow \mathbb{P}(\bigwedge^k \Lambda_0)_{\mathbb{F}_p}) \,\middle|\, [\Lambda] \in \overline{\bigwedge^k \Gamma^{\mathrm{st}}} \right\}$$

$$\overline{\pi^0_{\mathbb{P}}}(C) = \mathrm{Im}(\mathbb{P}(\Lambda_C) \dashrightarrow \mathbb{P}(\bigwedge^k \Lambda_0))_{\mathbb{F}_p}.$$

*Proof.* Recall from Lemma 3.8 that for a class $[\Lambda]$ in $\overline{\Gamma^{\mathrm{st}}}$ the projection $\mathcal{M}_{\mathbb{P}}\left(\overline{\Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\Lambda)$ restricts to a birational morphism $C_\Lambda \to \mathbb{P}(\Lambda)$. This implies that the two images $\overline{\pi^0_{\mathbb{P}}}(C)$ and $\mathrm{Im}(\mathbb{P}(\Lambda_C) \dashrightarrow \mathbb{P}(\bigwedge^k \Lambda_0))_{\mathbb{F}_p}$ coincides. Using the bijection between irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ and lattices in $\overline{\Gamma^{\mathrm{st}}}$ discussed in the remark above, the result follows. $\square$

*Remark* 4.5. For a natural two $\mathbb{Z}_p$-lattices $\Lambda$, $\Lambda'$ in $\bigwedge^k \mathbb{Q}_p^n$ and $\Lambda$ maximal in the class $[\Lambda]$ with $\Lambda \subseteq \Lambda'$ the induced birational map $\mathbb{P}(\Lambda)_{\mathbb{F}_p} \dashrightarrow \mathbb{P}(\Lambda')_{\mathbb{F}_p}$ is defined away from the linear subspace $\mathbb{P}(\ker(\Lambda_{\mathbb{F}_p} \to \Lambda'_{\mathbb{F}_p})) \subseteq \mathbb{P}(\Lambda)_{\mathbb{F}_p}$. In particular the image $\mathrm{Im}(\mathbb{P}(\Lambda) \dashrightarrow \mathbb{P}(\Lambda))_{\mathbb{F}_p}$ can be computed as $\mathbb{P}(\mathrm{Im}(\Lambda_{\mathbb{F}_p} \to \Lambda'_{\mathbb{F}_p}))$. Therefore we will focus in the following on understanding the submodules $\mathrm{Im}(\Lambda_{\mathbb{F}_p} \to \Lambda'_{\mathbb{F}_p})$ for a representative $\Lambda$ of the class $[\Lambda]$ maximal with $\Lambda \subseteq \Lambda'$ instead of the subvarieties $\mathrm{Im}(\mathbb{P}(\Lambda) \dashrightarrow \mathbb{P}(\Lambda'))_{\mathbb{F}_p}$.

**Lemma 4.6.** *Fix a representative $\Lambda$ of a class in $\overline{\bigwedge^k \Gamma^{st}}$ maximal in its class with $\Lambda \subseteq \bigwedge^k \Lambda_0$. Then the image $\mathrm{Im}(\Lambda_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p})$ is of the form $V_I$ for some $I \in \binom{[n]}{k}$ and every module $V_I$ for $I \in \binom{[n]}{k}$ arises in this way.*

*Proof.* Let us start by determining the images of $\bigwedge^k \Lambda_i$ for $\Lambda_i$ in $\Gamma^{\mathrm{st}}$. For the lattice $p^l \bigwedge^k \Lambda_i$ with $l \in \mathbb{Z}$ we get

$$p^l \bigwedge^k \Lambda_i = \langle p^{m_I^i} e_I \rangle_{I \in \binom{[n]}{k}} \text{ with } m_I^i = l - \sharp(I \cap [i]) \text{ for } I \in \binom{[n]}{k}$$

and hence

$$p^l \bigwedge^k \Lambda_i \cap \bigwedge^k \Lambda_0 = \langle p^{m_I^{i,l}} e_I \rangle_{I \in \binom{[n]}{k}} \text{ with } m_I^{i,l} = \max\{l - \sharp(I \cap [i]), 0\} \text{ for } I \in \binom{[n]}{k}$$

and hence we can determine the image as

$$\mathrm{Im}((p^l \bigwedge^k \Lambda_i \cap \bigwedge^k \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) = \langle e_I | \sharp(I \cap [i]) - l \geq 0 \rangle.$$

Now if $l > \min\{k, i\}$ no $I$ satisfies the condition for $e_J$ to appear and the image is trivial. Hence let us assume that $l \leq \min\{k, i\}$. Defining

$$I_i^l := \begin{cases} \{i - l, \dots, i - 1\} \cup \{n - 1 - k + l, \dots, n - 1\} & \text{if } i + k < n + l \\ \{n - 1 - k, \dots, n - 1\} & \text{otherwise} \end{cases}$$

gives the reformulation of the condition $\sharp(I \cap [i]) \geq l$ to the condition $I \leq I_i^l$. All together we conclude $\mathrm{Im}((p^l \bigwedge^k \Lambda_i \cap \bigwedge^k \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) = V_{I_i^l}$.

Since every lattice $\Lambda$ representing a class in $\overline{\bigwedge^k \Gamma^{st}}$ is the intersection of lattices of the form as above we need to understand the behaviour of the images under intersections. Consider two representatives $\Lambda$, $\Lambda'$ of classes in $\overline{\bigwedge^k \Gamma^{st}}$ with images described as

$$\mathrm{Im}(\Lambda_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) = V_{I(\Lambda)}$$
$$\mathrm{Im}(\Lambda'_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) = V_{I(\Lambda')}$$

for two subsets $I(\Lambda), I(\Lambda') \in \binom{[n]}{k}$. Now we set

$$\min\{I(\Lambda), I(\Lambda')\} := \{\min\{i_0, i'_0\} < \dots < \min\{i_{k-1}, i'_{k-1}\}\}$$

with $I(\Lambda) = \{i_0 < \dots < i_{k-1}\}$ and $I(\Lambda') = \{i'_0 < \dots < i'_{k-1}\}$ and get

$$\mathrm{Im}((\Lambda \cap \Lambda')_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) = V_{I(\Lambda)} \cap V_{I(\Lambda')} = V_{\min\{I(\Lambda), I(\Lambda')\}}.$$

Hence for all lattices $\Lambda$ representing a class in $\overline{\bigwedge^k \Gamma^{st}}$ we can find a set $I(\Lambda) \in \binom{[n]}{k}$ such that $\mathrm{Im}(\Lambda_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p})$ coincides with $V_{I(\Lambda)}$.
The converse will follow directly from the following lemma. $\square$

**Lemma 4.7.** *Every subset $I \in \binom{[n]}{k}$ can be described as the minimum $I = \min\{I_i^l | I \le I_i^l\}$ for the subsets $I_i^l$ constructed above.*

*Proof.* We prove this claim by induction on the number of maximal intervals in $I$. For $I \in \binom{[n]}{k}$ let $I = I_1 \cup \cdots \cup I_m$ be the decomposition on $m$ maximal intervals. Consider the two subsets $J_1 := \{\min I_2 - \sharp I_1 \dots, \min I_2 - 1\} \cup I_2 \cup \cdots \cup I_m$ and $J_2 := I_1 \cup \{n-1-k+\sharp I_1, \dots, n-1\}$. Now we easily see that $\min\{J_1, J_2\} = I$ where $J_1$ has $m-1$ maximal intervals and $J_2 = I_i^l$ for $l = \sharp I_1$ and $i = l + \min I_1$. $\square$

**Lemma 4.8.** *Fix a natural number $N$, the standard lattice $\Lambda_0 = \langle e_i \rangle_{i \in [N]}$ of $\mathbb{Q}_p^N$ and a second lattice $\Lambda = \langle p^{m_i} e_i \rangle_{i \in [N]}$ with $m_i \in \mathbb{Z}$. Take lattices $\{\Lambda_j\}_{j \in J}$ representing the classes in $\overline{\{[\Lambda_0], [\Lambda]\}}$ and are the unique maximal representative in the homothety class contained in $\Lambda_0$. Then the set $\{m_i | i \in [N]\}$ is an interval if and only if for $j \in J$ we have pairwise different images $\mathrm{Im}(\Lambda_{j,\mathbb{F}_p} \to \Lambda_{0,\mathbb{F}_p})$. In this case the class $[\Lambda]$ is fully determined by the set $\left\{\mathrm{Im}(\Lambda_{j,\mathbb{F}_p} \to \Lambda_{0,\mathbb{F}_p})\right\}_{j \in J}$ of images.*

*Proof.* Fix a lattice $\Lambda = \langle p^{m_i} e_i | m_i \in \mathbb{Z} \rangle_{i \in [N]}$ maximal in its homothety class with $\Lambda \subseteq \Lambda_0$. We have $m_i \ge 0$ for all $i \in [N]$ since $\Lambda \subseteq \Lambda_0$ and $\min\{m_i | i \in [N]\} = 0$ since $\Lambda$ is chosen to be maximal. Assuming both the set of representatives of the convex closure of $\{\Lambda_0, \Lambda\}$ are of the form

$$p^{-l}\Lambda \cap \Lambda_0 = \langle p^{m(l,i)} e_i \rangle_{i \in [N]} \text{ with } m(l,i) = \max\{m_i - l, 0\} \text{ for all } i \in [N]$$

for $l \in [\max\{m_i | i \in [n]\}]$. The corresponding images are

$$\mathrm{Im}(p^{-}l\Lambda \cap \Lambda_0 \to \Lambda_0)_{\mathbb{F}_p} = \langle e_i | m(l,i) = 0 \rangle = \langle e_i | m_i \le l \rangle$$

and hence are pairwise different if and only if $\{m_i | i \in [N]\}$ is an interval.
In this case $\Lambda$ is uniquely determined since we get

$$m_i = \min\{l | e_i \in \mathrm{Im}(p^{-}l\Lambda \cap \Lambda_0 \to \Lambda_0)_{\mathbb{F}_p}\}$$

for all $i \in [N]$. $\square$

**Lemma 4.9.** *For $\Lambda = \langle p^{-m_I} e_I \rangle \in \overline{\bigwedge^k \Gamma^{\mathrm{st}}}$ and $I, J \in \binom{[n]}{k}$ such that $J$ is maximal with $J < I$, we get $m_I - m_J \in \{0, 1\}$. In particular the set $\{m_I | I \in \binom{[n]}{k}\}$ is an interval.*

*Proof.* First we recall that a lattice $\bigwedge^k \Lambda_i$ with $\Lambda_i \in \Gamma^{st}$ has the form $\bigwedge^k \Lambda_i = \langle p^{-m_I^i} e_I \rangle_{I \in \binom{[n]}{k}}$ with $m_I^i = \sharp(I \cap [i])$ and for $J \le I$ in $\binom{[n]}{k}$ we get $m_I^i - m_J^i = \sharp(I \cap [i]) - \sharp(J \cap [i])$. Now if $J$ is maximal with $J < I$ the two sets differ by just one element and hence $\sharp(I \cap [i]) - \sharp(J \cap [i])$ takes values in $\{0, 1\}$.
If for two lattices $\Lambda_1, \Lambda_2 \in \overline{\bigwedge^k \Gamma^{st}}$ with $\Lambda_i = \langle p^{m_I(\Lambda_i)} e_I \rangle$ for $i \in \{1, 2\}$ and such that for all $I, J \in \binom{[n]}{k}$ and $J$ maximal with $J < I$ we have $m_I(\Lambda_i) - m_J(\Lambda_i) \in \{0, 1\}$, then

$$\begin{aligned} m(\Lambda_1 \cap \Lambda_2)_I - m(\Lambda_1 \cap \Lambda_2)_J &= \min\{m(\Lambda_i)_I | i = 1, 2\} - \min\{m(\Lambda_i)_J | i = 1, 2\} \\ &= \min\{m(\Lambda_i)_I - m(\Lambda_i)_J | i = 1, 2\} \in \{0, 1\} \end{aligned}$$

and hence the lemma is proven for all $\Lambda \in \overline{\bigwedge^k \Gamma^{\mathrm{st}}}$. $\square$

*Remark* 4.10. Using Lemma 4.9 and Lemma 4.8 every lattice $\Lambda \in \overline{\bigwedge^k \Gamma^{st}}$ is determined by the images $\mathrm{Im}((p^l \Lambda \cap \bigwedge^k \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p})$ for varying $l$.
But in Lemma 4.6 we already computed the images $\mathrm{Im}((p^l \bigwedge^k \Lambda_i \cap \bigwedge^k \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) = V_{I_i^l}$ for $i \in [n]$ and $0 \le l \le k, i$. In particular for a fixed $i \in [n]$ these images for all $0 \le l \le \min\{k, i\}$

are determined by the image for $l = \min\{k, i\}$. In the lemma below we will generalise this to arbitrary lattices in $\overline{\bigwedge^k \Gamma^{st}}$.

**Example 4.11.** *For $n = 5$ and $k = 2$ we will illustrate the last remark on the example $p^2 \bigwedge^2 \Lambda_2$. In the basis $\{e_I | I \in \binom{[5]}{2}\}$ this lattice is spanned by $e_{\{0,1\}}$, $pe_{\{i,j\}}$ for $0 \le i \le 1 < j \le 4$ and $p^2 e_{\{i,j\}}$ for $2 \le i < j \le 4$.*
*The image $\mathrm{Im}(p^2 \bigwedge^2 \Lambda_{2,\mathbb{F}_p} \to \bigwedge^2 \Lambda_{0,\mathbb{F}_p})$ is the submodule spanned by $e_{0,1}$ and we recover $I_2^2 = \{0,1\}$. Now the image $\mathrm{Im}((p \bigwedge^2 \Lambda_2 \cap \bigwedge^2 \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^2 \Lambda_{0,\mathbb{F}_p})$ is generated by the $e_{\{i,j\}}$ with $i \le 1$. But this is equivalent to $I \le \{1,4\} = I_2^1$. Finally $p^2 \bigwedge^2 \Lambda_2$ is contained in $\bigwedge^2 \Lambda_0$ and hence the image $\mathrm{Im}((\bigwedge^2 \Lambda_2 \cap \bigwedge^2 \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^2 \Lambda_{0,\mathbb{F}_p})$ is spanned by all $\{e_I\}_I$ and we recover $I_2^0 = \{3,4\}$.*

**Lemma 4.12.** *Fix two classes $[\Lambda_1] \ne [\Lambda_2]$ in $\overline{\bigwedge^k \Gamma^{st}}$ and representatives $\Lambda_i$ for $i = 1,2$ maximal in their class with $\Lambda_i \subseteq \bigwedge^k \Lambda_0$. Then the images $\mathrm{Im}(\Lambda_{1,\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p})$ and $\mathrm{Im}(\Lambda_{2,\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p})$ are different.*

*Proof.* For a subset $I \in \binom{[n]}{k}$ let us define a shift $I[l]$ for $l \in \mathbb{N}$ as follows. Using Lemma 4.7 every subset $I$ is of the form $I = \min\{I_i^l | I \le I_i^l\}$. Now the shift is $I[1] := \min\{I_i^{l-1} | I \le I_i^l\}$. Note that that for every $i_0 \in [n]$ and $0 < l_0 \le \min\{k, i\}$

$$I_{i_0}^{l_0}[1] = \min\{I_i^{l-1} | I_{i_0}^{l_0} \le I_i^l\} = \min\{I_i^l | I_{i_0}^{l_0-1} \le I_i^l\} = I_{i_0}^{l_0-1}.$$

For $I$ in $\binom{[n]}{k}$ we use that for $i$ and $0 < l \le \min\{i,k\}$ we have

$$I = \{i_0, \dots, i_{n-1}\} \le I_i^l = \{i-l, \dots, i-1\} \cup \{n-1-k+l, \dots, n-1\}$$

if and only if $i_{l-1} \le i-1$. In particular for two subsets $I = \{i_0, \dots, i_{n-1}\}$ and $J = \{j_0, \dots, j_{n-1}\}$ in $\binom{[n]}{k}$ we see that for $\min\{I,J\} = \{\min\{i_0, j_0\}, \dots, \min\{i_{n-1}, j_{n-1}\}\}$ we get

$$\min\{I,J\} \le I_i^l \Leftrightarrow \min\{i_{l-1}, j_{l-1}\} \le i-1 \Leftrightarrow i_{l-1} \le i-1 \text{ or } j_{l-1} \le i-1 \Leftrightarrow I \le I_i^l \text{ or } J \le I_i^l.$$

Hence we get an equality of sets

$$\{I_i^l | \min\{I,J\} \le I_i^l\} = \{I_i^l | I \le I_i^l, \text{ or } J \le I_i^l\}$$

and derive

$$\min\{I,J\}[1] = \min\{I_i^{l-1} | \min\{I,J\} \le I_i^l\} = \min\{I_i^{l-1} | I \le I_i^l, \text{ or } J \le I_i^l\} = \min\{I[1], J[1]\}.$$

Inductively we now define $I[l]$ to be the shift $I[l-1][1]$.
All representatives of a class $[\Lambda]$ in $\overline{\bigwedge^k \Gamma^{st}}$ have the form $\Lambda = \bigcap_{j \in J} p^{l_j} \bigwedge^k \Lambda_j$ for some $J \subseteq [n]$ and $l_j \in \mathbb{Z}$. If we now take the unique representative maximal with $\Lambda \subseteq \bigwedge^k \Lambda_0$ we can can assume that $0 \in J$, $l_0 = 0$ and $l_i > 0$ for $i \in J \setminus \{0\}$. If we note that

$$p^{-1}\Lambda \cap \bigwedge^k \Lambda_0 = \bigcap_{0 \ne j \in J} p^{l_j - 1} \bigwedge^k \Lambda_j \cap \bigwedge^k \Lambda_0$$

and $V_{I_0^0}$ is $\bigwedge^k \Lambda_0$ we now get

$$\begin{aligned} \mathrm{Im}((p^{-1}\Lambda \cap \bigwedge^k \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) &= \bigcap_{0 \ne j \in J} \mathrm{Im}((p^{-1+l_j} \bigwedge^k \Lambda_j \cap \bigwedge^k \Lambda_0)_{\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p}) \\ = \bigcap_{0 \ne j \in J} V_{I_j^{-1+l_j}} &= \bigcap_{0 \ne j \in J} V_{I_j^{-1+l_j+1}[1]} = V_{I(\Lambda)[1]}. \end{aligned}$$

Inductively we can determine the images for all lattices in $\overline{\{\Lambda, \bigwedge^k \Lambda_0\}}$ from the image of $\Lambda$ and hence the class $[\Lambda]$ is fully determined by its image. $\square$

**Example 4.13.** *Let us give an example for the last Lemma. We fix $n = 6$ and $k = 3$ and want to find the unique lattice $\Lambda$ in $\bigwedge^3 \Gamma^{\mathrm{st}}$ with $I(\Lambda) = \{0,2,3\}$. We write $\Lambda = \bigcap_{i\in[6]} p^{n_i} \bigwedge^3 \Lambda_i$. If $\Lambda$ is maximal in its homothety class with $\Lambda \subseteq \bigwedge^3 \Lambda_0$ then it is easy to see that $n_0 = 0$ and $n_i \geq 0$ for $i \neq 0$. Write $\Lambda$ in the basis of $\bigwedge^3 \mathbb{Q}_p^6$ as $\langle p^{m_I} e_I | I \in \binom{[6]}{3}\rangle$. By assumption we have $m_I = \max\{n_i - \sharp I \cap [i] \mid i \in [6]\} = 0$ exactly when $I \leq I(\Lambda)$ hence $m_I = 0$ precisely if $I \leq I_i^{n_i}$ for all $i$. Using the notation from Remark 4.10 we get $I(\Lambda) = \min_i\{I_i^{n_i}\}$. Note that the subsets smaller than $I(\Lambda)$ are $\{0,1,2\}$, $\{0,1,3\}$ and $\{0,2,3\}$.*
*Now the lattice $p^{-1}\Lambda \cap \bigwedge^2 \Lambda_0$ is of the form $\langle p^{m'_I} e_I | I \in \binom{[6]}{3}\rangle$ with $m'_I = \max\{0, m_I - 1\}$. Hence image of $\mathrm{Im}((p^{-1}\Lambda \cap \bigwedge^2 \Lambda_0)_{\mathbb{F}_p})$ is generated by the $e_I$ with $n_i - \sharp I \cap [i] \leq 1$ for all $i$. This is equivalent to $I \leq \min\{I_i^{n_i-1} | i \in [6]\} = I(\Lambda)[1]$. But on the other hand the subsets of the form $I_i^{l-1}$ with $I(\Lambda) \leq I_i^l$ are the following $\{2,3,5\}$, $\{2,4,5\}$ and $\{3,4,5\}$. Hence the shift is simply $I(\Lambda)[1] = \{2,3,5\}$. Shifting again we get $I(\Lambda)[2] = \{3,4,5\}$ and hence $m_I \leq 2$ for all $I$.*
*We can now use the explicit values $m_I$ for $I \in \binom{[6]}{3}$ and calculate the exponents $n_i$ for $i \in [6]$:*

*(i) for $i = 1$ and $I \leq I(\Lambda)$: $0 = m_I \geq n_1 - \sharp I \cap [1] = n_1 - 1$ hence $n_1 = 1$*
*(ii) for $i = 2$ and $I = \{0,2,3\}$: $0 = m_I \geq n_2 - \sharp I \cap [2] = n_2 - 1$ hence $n_2 = 1$*
*(iii) for $i = 3$ and $I = \{0,2,3\}$: $0 = m_I \geq n_3 - \sharp I \cap [3] = n_3 - 2$ hence $n_3 \leq 2$*
*(iv) for $i = 4$ and $I = \{2,3,5\}$: $1 = m_I \geq n_4 - \sharp I \cap [4] = n_4 - 2$ hence $n_4 \leq 3$*
*(v) for $i = 5$ and $I = \{0,2,3\}$: $0 = m_I \geq n_5 - \sharp I \cap [5] = n_3 - 3$ hence $n_3 \leq 3$*
*(vi) for $I = \{2,4,5\}$ and $i \neq 4$: $n_i - \sharp I \cap [i] \leq 1 < m_I$ hence $2 = n_4 - \sharp I \cap [4] = n_4 - 1$ and $n_4 = 3$*

*But now we note that $p \bigwedge^2 \Lambda_1 \subseteq p \bigwedge^2 \Lambda_2 \cap p \bigwedge^2 \Lambda_3 \cap \bigwedge^2 \Lambda_0$ and $p^3 \bigwedge^2 \Lambda_4 \subseteq p^2 \bigwedge^2 \Lambda_3 \cap p^3 \bigwedge^2 \Lambda_5$. Hence we get $\Lambda = p \bigwedge^2 \Lambda_1 \cap p^3 \bigwedge^2 \Lambda_4$.*
*Now we can check $I_{\bigwedge^2 \Lambda_1} = I_2^1 = \{0,4,5\}$ and $I_{p^3 \bigwedge^2 \Lambda_4} = I_4^3 = \{1,2,3\}$ and hence $I(\Lambda)$ is* $\min\{\{0,4,5\},\{1,2,3\}\} = \{0,2,3\}$.

**Definition 4.14.** For $I$ in $\binom{[n]}{k}$ Lemma 4.12 gives us a unique lattice in $\overline{\Gamma^{\mathrm{st}}}$ such that the image $\mathrm{Im}(\Lambda_{I,\mathbb{F}_p} \to \bigwedge^k \Lambda_{0,\mathbb{F}_p})$ is $I$. In the following this lattice will be denoted by $\Lambda_I$. Using Lemma 4.4 we now get a unique irreducible component of $\mathcal{M}_{\mathbb{P}}\left(\overline{\Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ surjecting to $\mathbb{P}(\Lambda_I)$. This component will be denoted b $C_I$.

**Proposition 4.15.** *We get a bijection*

$$\left\{ C \,\middle|\, C \text{ irr. component of } \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p} \right\} \longrightarrow \left\{ \mathbb{P}(V_I) \subseteq \mathbb{P}(\bigwedge^k \Lambda_0)_{\mathbb{F}_p} \,\middle|\, I \in \binom{[n]}{k} \right\}$$
$$C \longmapsto \overline{\pi_{\mathbb{P}}^0}(C).$$

*And moreover for a linear subspace $\mathbb{P}(V_I)$ for $I \in \binom{[n]}{k}$ the inverse image $(\overline{\pi_{\mathbb{P}}^0})^{-1}(\mathbb{P}(V_I))$ is the union $\bigcup_{J \leq I} C_J$ of irreducible components.*

*Proof.* In Lemma 3.13 we showed that the number of irreducible components of $\mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ coincides with the number of elements in $\Gamma$ when ever $\Gamma$ is a convex set of lattice classes. Using Lemma 4.12 we now get $\binom{n}{k}$ as the number of irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\Gamma^{\mathrm{st}}}\right)$.
Using Lemma 4.6 the images $\overline{\pi_{\mathbb{P}}^0}(C)$ for an irreducible component $C$ in $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ are of the form $\mathbb{P}(V_I)$ for some $I \in \binom{[n]}{k}$. Now with Lemma 4.12 the above map is injective and hence bijective by a cardinality argument.
The second part of the statement now follows directly from Lemma 3.26 using that an irreducible component $C_J$ for some $J$ in $\binom{[n]}{k}$ is mapped into $\mathbb{P}(V_I)$ if and only if $\overline{\pi_{\mathbb{P}}^0}(C_J) = \mathbb{P}(V_J)$ is contained in $\mathbb{P}(V_I)$ which is equivalent to $J \leq I$. ☐

4.2. **Irreducible components and the convex closure.** With the last proposition we got a quite precise relation between the irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ and linear subspaces in $\mathbb{P}(\bigwedge^k \Lambda_0)$. Since we have a similar relation for the irreducible components of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ and the Schubert varieties in $\mathrm{Gr}^k(\Lambda_0)$, the number of irreducible components of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ and $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ are the same. To make this relation more precise we need to specify an explicit bijection on the sets of irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ and $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$. This is done in the conjecture below.

**Conjecture 4.1.** *We get a bijection*

$$\left\{C \middle| C \text{ irr. component of } \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}\right\} \longrightarrow \left\{C \middle| C \text{ irr. component of } \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}\right\}$$
$$C \longmapsto \overline{\pi_{\mathbb{P}}}(C)$$

*where* $\overline{\pi_{\mathbb{P}}}\colon \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ *is the natural projection.*

*Remark* 4.16. We already know from Lemma 3.8 that for an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ there exist a unique irreducible component $\overline{C}$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ surjecting to $C$. Hence to prove the conjecture we just need to show that the images $\overline{\pi_{\mathbb{P}}}(C)$ are irreducible components.

As evidence for the conjecture we can calculate the cases for $n \leq 7$ (see Appendix C) and prove the following proposition.

**Proposition 4.17.** *For $k = 2$ Conjecture 4.1 is true.*

The proof of this proposition will occupy us for the rest of this section therefore let us first indicate one important immediate consequence of the Conjecture. Furthermore we will describe a method to approach the conjecture in general before we prove the proposition.

**Theorem 4.18.** *Assume Conjecture 4.1. Then we get a bijection*

$$\left\{C \middle| C \text{ irr. component of } \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}\right\} \longrightarrow \left\{\mathbb{P}(V_I) \subseteq \mathbb{P}(\bigwedge^k \Lambda_0)_{\mathbb{F}_p} \middle| I \in \binom{[n]}{k}\right\}$$
$$C \longmapsto \pi^0_{\mathbb{P}}(C)$$

*where* $\pi^0_{\mathbb{P}}\colon \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ *is the natural projection. And moreover for a linear subspace $\mathbb{P}(V_I)$ for $I \in \binom{[n]}{k}$ the inverse image $(\pi^0_{\mathbb{P}})^{-1}(\mathbb{P}(V_I))$ is the union $\bigcup_{J \leq I} C_J$ of irreducible components.*

*Proof.* Fix $I$ in $\binom{[n]}{k}$ and take the linear subspace $\mathbb{P}(V_I)$ of $\mathbb{P}(\bigwedge^k \Lambda_0)$. By Proposition 4.15 there exist a unique irreducible component $\overline{C}$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ surjecting to $\mathbb{P}(V_I)$. Now by Conjecture 4.1 the image $C \coloneqq \overline{\pi_{\mathbb{P}}}(\overline{C})$ of $\overline{C}$ in $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ is an irreducible component surjecting to $\mathbb{P}(V_I)$.
Conversely for an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ there is by Lemma 3.8 an irreducible component $\overline{C}$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ surjecting to $C$. Hence by Proposition 4.15 the image of $C$ is of the form $\mathbb{P}(V_I)$ for some $I$ in $\binom{[n]}{k}$.
The second part of the statement follows similarly. ∎

*Remark* 4.19. In [AL17] a combinatorial method was described to compute dimensions of certain images of rational maps using a result by [Li18]. To describe this method and the implication we want to use, we need the following setup. Note that we are using the dual notion of projective space compared to the reference.
Fix a finite set of lattice classes $\Gamma$ in $\mathbb{Q}_p^n$, an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ and a class $[\Lambda]$ in $\Gamma$. Take a representative $\Lambda = \langle p^{m_I(\Lambda)} e_I \rangle$ of $[\Lambda]$ and a representative $\Lambda_C = \langle p^{m_I(\Lambda_C)} e_I \rangle$ of the class corresponding to the irreducible component $C$. Choose $\Lambda_C$ to be maximal with $\Lambda_C \subseteq \Lambda$. We define the subset

$$W_\Lambda := \{ i \in [n] | m(\Lambda)_i - m(\Lambda_C)_i < \max_{j \in [n]} \{ m(\Lambda)_j - m(\Lambda_C)_j \} \}$$

of $[n]$ and construct the set

$$M(h, C) := \{ (a_\Lambda)_{[\Lambda] \in \Gamma} \in \mathbb{N}^\Gamma | \sum_\Gamma a_\Lambda = h \text{ and } n - \sum_{\Lambda \in I} a_\Lambda > \sharp \bigcap_{\Lambda \in I} W_\Lambda \text{ for all subsets } \varnothing \neq I \subseteq \Gamma \}.$$

In the following proposition we describe how this combinatorial data encodes information of the image $\overline{\pi_{\mathbb{P}}}(C)$.

**Proposition 4.20.** *([Li18] and [AL17, Theorem 2.18]) For a finite set of lattice classes $\Gamma$ and an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\Gamma}\right)_{\mathbb{F}_p}$ the dimension of $\overline{\pi_{\mathbb{P}}}(C) \subseteq \mathcal{M}_{\mathbb{P}}(\Gamma)_{\mathbb{F}_p}$ is computed by*

$$\dim(\overline{\pi_{\mathbb{P}}}(C)) = \max\{ h | M(h, C) \neq \varnothing \}.$$

*Remark* 4.21. Let us take the time to explain an idea to approach Conjecture 4.1 using the proposition above. Fix an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ and take the corresponding class $[\Lambda_C]$ in $\overline{\bigwedge^k \Gamma^{\mathrm{st}}}$. First we recall that using Lemma 3.8 the image of $C$ in $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ is clearly an irreducible component if $[\Lambda_C]$ is already in $\bigwedge^k \Gamma$. Hence we just have to check the cases where $\Lambda$ is in $\overline{\bigwedge^k \Gamma^{\mathrm{st}}} \smallsetminus \bigwedge^k \Gamma^{\mathrm{st}}$.
Now we can find a subset $\Gamma_C$ of $\bigwedge^k \Gamma^{\mathrm{st}}$ minimal such that $\overline{\Gamma_C}$ contains $[\Lambda_C]$. With out loss of generality we can further assume that the homothety class of $\bigwedge^k \Lambda_0$ is contained in $\Gamma_C$. By Lemma 3.8 and Lemma 3.13 the image of $C$ in $\mathcal{M}_{\mathbb{P}}(\Gamma_C)_{\mathbb{F}_p}$ is an irreducible component. And using Lemma 3.8 again we see that the image of $C$ in $\mathcal{M}_{\mathbb{P}}(\Gamma_C)_{\mathbb{F}_p}$ is an irreducible component if and only if the image in $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ is an irreducible component.
But since the image of $C$ in $\mathcal{M}_{\mathbb{P}}(\Gamma_C)$ is irreducible, it is an irreducible component if its dimension is $\binom{[n]}{k} - 1$. Using Proposition 4.20 this is equivalent to $M(\binom{[n]}{k} - 1, C)$ not being empty for the set $\Gamma_C$ of lattice classes.
In general the sets $\Gamma_C$ can be difficult to determine, but for $k = 2$ we have the following easy description.

**Lemma 4.22.** *For $[\Lambda] \in \overline{\bigwedge^2 \Gamma^{\mathrm{st}}}$ there are classes $[\Lambda_1]$ and $[\Lambda_2]$ in $\bigwedge^2 \Gamma^{\mathrm{st}}$ such that $[\Lambda]$ is in the convex closure $\overline{\{[\Lambda_1], [\Lambda_2]\}}$.*

*Proof.* Take $[\Lambda]$ in $\overline{\bigwedge^2 \Gamma^{\mathrm{st}}}$ arbitrary. Then $\Lambda$ is of the form $\bigcap_{i \in I} p^{n_i} \bigwedge^2 \Lambda_i$ for some $I \subseteq [n]$ and without loss of generality we have $0 \in I$, $n_0 = 0$ and $n_i > 0$ for all $i \in I \smallsetminus \{0\}$. Further assume that $I$ is minimal, i.e. $\Lambda$ is properly contained in $\bigcap_{i \in J} p^{n_i} \bigwedge^2 \Lambda_i$ for all proper subsets $J \subset I$. Now we note that for all $i \in I$ we have $p^{n_i} \bigwedge^2 \Lambda_i \subseteq \bigwedge^2 \Lambda_0$ if $n_i \geq 2$. Using minimality of $I$ we conclude that $n_i = 1$ for all $i \in I \smallsetminus \{0\}$. But since $p \bigwedge^2 \Lambda_i \subseteq p \bigwedge^2 \Lambda_j$ for $i \leq j$ we again see by minimality of $I$ that $\sharp I \leq 2$. □

*Proof of Proposition 4.17* . Fix an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^2 \Gamma^{\mathrm{st}}\right)$. Following the idea described in Remark 4.21 and Lemma 4.22 we just have to prove $M(\binom{n}{2} - 1, C) \neq \varnothing$ for $\Gamma = \left\{ [\bigwedge^2 \Lambda_0], [\bigwedge^2 \Lambda_i] \right\}$ for some $i \in [n]$.

But for $p\bigwedge^2\Lambda_0 \cap \bigwedge^2\Lambda_i = \langle p^{m_I^i} e_I | m_I^i = \max\{\sharp(I\cap[i]),1\}\rangle_{I\in\binom{[n]}{k}}$ hence $W_{\bigwedge^2\Lambda_0} = \{I | m_I^i > 1\}$ and $W_{\bigwedge^2\Lambda_i} = \{I | m_I^i < 1\}$. Now for $a_{\bigwedge^2\Lambda_0} := \sharp W_{\bigwedge^2\Lambda_i}$ and $a_{\bigwedge^2\Lambda_i} := \sharp W_{\bigwedge^2\Lambda_0} + \sharp\{I | m_I^i = 1\} - 1$ we have an element $(a_{\bigwedge^2\Lambda_0}, a_{\bigwedge^2\Lambda_i}) \in M(\binom{n}{2}-1, C)$.

□

4.3. **The Plücker embedding.** To relate the local model as a Mustafin variety to the well understood Mustafin varieties for projective spaces, we want to use the Plücker embedding. For every lattice $\Lambda_i$ in $\Gamma^{\mathrm{st}}$ we consider the Plücker embedding

$$\mathrm{pl}_{\Lambda_i}\colon \mathrm{Gr}^k(\Lambda) \longrightarrow \mathbb{P}(\bigwedge^k \Lambda_i)$$

and the projections $\pi^{\mathrm{i}}_{\mathrm{Gr}}\colon \mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}}) \to \mathrm{Gr}^k(\Lambda_i)$ and $\pi^{\mathrm{i}}_{\mathbb{P}}\colon \mathcal{M}_{\mathbb{P}}(\bigwedge^k\Gamma^{\mathrm{st}}) \to \mathbb{P}(\bigwedge^k\Lambda_i)$.

Together we now get a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}}) \arrow[r, "\mathrm{pl}_{\Gamma^{\mathrm{st}}}"] \arrow[d, hook, "\prod \pi^{\mathrm{i}}_{\mathrm{Gr}}"'] & \mathcal{M}_{\mathbb{P}}(\bigwedge^k\Gamma^{\mathrm{st}}) \arrow[d, hook, "\prod \pi^{\mathrm{i}}_{\mathbb{P}}"] \\
\prod_{[\Lambda_i]\in\Gamma^{\mathrm{st}}} \mathrm{Gr}^k(\Lambda_i) \arrow[r, "\prod \mathrm{pl}_{\Lambda_i}"] & \prod_{[\bigwedge^k\Lambda_i]\in\bigwedge^k\Gamma^{\mathrm{st}}} \mathbb{P}(\bigwedge^k\Lambda_i)
\end{tikzcd}$$

where we get the map $\mathrm{pl}_{\Gamma^{\mathrm{st}}}$ since we have it on the generic fiber and $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})$ is flat.

*Remark* 4.23. In [Häb11, Proposition 4.5] (or [Häb14, discussion after Definition 2.1]) it was claimed, that the diagram above is cartesian, i.e. inside $\prod_{[\Lambda]\in\Gamma^{\mathrm{st}}}\mathbb{P}(\bigwedge^k\Lambda)$ the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})$ is the intersection of $\prod_{[\Lambda]\in\Gamma^{\mathrm{st}}}\mathrm{Gr}^k(\Lambda)$ and $\mathcal{M}_{\mathbb{P}}(\bigwedge^k\Gamma^{\mathrm{st}})$. Let us use the moduli description of both Mustafin varieties to explain why this is not true. An $\mathbb{F}_p$-valued point $(\mathcal{F}_0,\dots,\mathcal{F}_{n-1}) \in \prod_{i\in[n]}\mathrm{Gr}^k(\Lambda_i)(\mathbb{F}_p)$ is in $\mathcal{M}_{\mathbb{P}}(\bigwedge^k\Gamma^{\mathrm{st}})(\mathbb{F}_p)$ if for $i \mod n$ the image of $\bigwedge^k\mathcal{F}_i$ in $\bigwedge^k\Lambda_{i+1,\mathbb{F}_p}$ either vanishes or is $\bigwedge^k\mathcal{F}_{i+1}$. But the vanishing of the image of $\bigwedge^k\mathcal{F}_i$ in $\bigwedge^k\Lambda_{i+1,\mathbb{F}_p}$ is equivalent to $\mathcal{F}_i$ intersecting the kernel of $\Lambda_{i,\mathbb{F}_p}\to\Lambda_{i+1,\mathbb{F}_p}$ non trivially and hence does not imply that the image of $\mathcal{F}_i$ in $\Lambda_{i+1,\mathbb{F}_p}$ is contained in $\mathcal{F}_{i+1}$.
To give an explicit example for $n=4$ and $k=2$ let us identify $\Lambda_{i,\mathbb{F}_p}$ for $i\in\{0,\dots,3\}$ with $\mathbb{F}_p^4$. For $i\in\{0,\dots,3\}$ the maps $\Lambda_{i,\mathbb{F}_p}\to\Lambda_{i+1,\mathbb{F}_p}$ are now given by multiplying the $i$-th basis vector $e_i$ with 0. Consider the tuple $(\mathcal{F}_0,\dots,\mathcal{F}_3)$ in $\prod_{i\in[4]}\mathrm{Gr}^k(\Lambda_i)$ with $\mathcal{F}_i = \langle e_i, e_{i-1}\rangle$ for $i \mod 4$. Then the element $(\bigwedge^2\mathcal{F}_0,\dots,\bigwedge^2\mathcal{F}_3)$ is in $\mathcal{M}_{\mathbb{P}}(\bigwedge^2\Gamma^{\mathrm{st}})$ but $(\mathcal{F}_0,\dots,\mathcal{F}_3)$ is not in $\mathcal{M}_{\mathrm{Gr}^2}(\Gamma^{\mathrm{st}})$.
But assuming Conjecture 4.1 we can still get a relation in the following sense.

**Proposition 4.24.** *Assume Conjecture 4.1. Then the Plücker embedding*

$$\mathrm{pl}_{\Gamma^{\mathrm{st}}}\colon \mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{st}) \longrightarrow \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k\Gamma^{st}\right)$$

*induces a bijection*

$$\left\{C \middle| C \text{ irr. component of } \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}\right\} \longrightarrow \left\{C \middle| C \text{ irr. component of } \mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})_{\mathbb{F}_p}\right\}$$
$$C \longmapsto C\cap\mathcal{M}_{\mathrm{Gr}^k}(\Gamma^{\mathrm{st}})_{\mathbb{F}_p}$$

*between the sets of irreducible components of the special fibres.*

*Proof.* First recall from Lemma 4.2 that for $I\in\binom{[n]}{k}$ the Schubert variety $X_I$ is the intersection of the linear subspace $\mathbb{P}(V_I)$ with the image of $\mathrm{Gr}^k(\Lambda_0)$ under the Plücker embedding. Although this statement was formulated for Schubert varieties in $\mathrm{Gr}^k(\Lambda_0)$ and the linear subspaces in

$\mathbb{P}(\bigwedge^k \Lambda_0)$ it is certainly true for all lattices in $\Gamma^{\mathrm{st}}$ and we will make use of this fact.
Let us consider the following diagram

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \arrow[r, "\mathrm{pl}_{\Gamma^{\mathrm{st}}}"] \arrow[d, "\pi^0_{\mathrm{Gr}}"] & \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{st}\right) \arrow[d, "\pi^0_{\mathbb{P}}"] \\ \mathrm{Gr}^k\left(\Lambda_0\right) \arrow[r, "\mathrm{pl}_{\Lambda_0}"] & \mathbb{P}(\bigwedge^k \Lambda_0). \end{tikzcd}$$

For an irreducible component $C_I^{\mathrm{gr}}$ of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{st}\right)_{\mathbb{F}_p}$ the open part $C_I^{\mathrm{gr},\circ} := C_I^{\mathrm{gr}} \smallsetminus \bigcup\limits_{J<I} C_J^{\mathrm{gr}}$ is the inverse image $(\pi^0_{\mathrm{Gr}})^{-1}(X_I^\circ)$ of the Schubert cell $X_I^\circ = X_I \smallsetminus \bigcup_{J<I} X_J$. Furthermore this is also the inverse image

$$(\pi^0_{\mathrm{Gr}} \circ \mathrm{pl}_{\Lambda_0})^{-1}(\mathbb{P}(V_I)^\circ) = (\mathrm{pl}_{\Gamma^{\mathrm{st}}} \circ \pi^0_{\mathbb{P}})^{-1}(\mathbb{P}(V_I)^\circ)$$

of the open $\mathbb{P}(V_I)^\circ = \mathbb{P}(V_I) \smallsetminus \bigcup_{J<I} \mathbb{P}(V_J)$. Using Theorem 4.18 we now identify $(\pi^0_{\mathbb{P}})^{-1}(\mathbb{P}(V_I)^\circ)$ with the open part $C_I^{\mathrm{pr},\circ} := C_I^{\mathrm{pr}} \smallsetminus \bigcup\limits_{J<I} C_J^{\mathrm{pr}}$. In conclusion we get $C_I^{\mathrm{gr},\circ} = C_I^{\mathrm{pr},\circ} \cap \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ and in particular $C_I^{\mathrm{gr}} \subseteq C_I^{\mathrm{pr}} \cap \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$.
On the other hand first using Lemma 2.43 and then Lemma 4.2 we get

$$C_I^{\mathrm{gr}} = \left(\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathrm{Gr}}\right)^{-1}\left(\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathrm{Gr}}\left(C_I^{\mathrm{gr}}\right)\right) = \left(\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathrm{Gr}}\right)^{-1}\left(\prod_{i\in[n]} \mathrm{Gr}^k\left(\Lambda_i\right)_{\mathbb{F}_p} \cap \pi^{\mathrm{i}}_{\mathbb{P}}\left(C_I^{\mathrm{pr}}\right)\right)$$

and by looking at the injections in the diagram

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \arrow[r, "\mathrm{pl}_{\Gamma^{\mathrm{st}}}"] \arrow[d, "\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathrm{Gr}}"] & \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{st}\right) \arrow[d, "\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathbb{P}}"] \\ \prod_{i\in[n]} \mathrm{Gr}^k\left(\Lambda_i\right) \arrow[r, "\prod_{i\in[n]} \mathrm{pl}_{\Lambda_i}"] & \prod_{i\in[n]} \mathbb{P}(\bigwedge^k \Lambda_i) \end{tikzcd}$$

we compute $C_I^{\mathrm{gr}}$ to be

$$\left(\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathrm{Gr}}\right)^{-1}\left(\prod_{i\in[n]} \mathrm{Gr}^k\left(\Lambda_i\right)_{\mathbb{F}_p} \cap \pi^{\mathrm{i}}_{\mathbb{P}}\left(C_I^{\mathrm{pr}}\right)\right) = \left(\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathbb{P}}\right)^{-1}\left(\prod_{i\in[n]} \pi^{\mathrm{i}}_{\mathbb{P}}\left(C_I^{\mathrm{pr}}\right)\right) \cap \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$$

which clearly contains $C_I^{\mathrm{pr}} \cap \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$. We have shown that we get $C_I^{\mathrm{gr}} = C_I^{\mathrm{pr}} \cap \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ for all $I \in \binom{[n]}{k}$. But since all of the irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ and $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ are of the form $C_I^{\mathrm{pr}}$ and $C_I^{\mathrm{gr}}$ for some $I \in \binom{[n]}{k}$ we have shown the result.

$\square$

## 5. A candidate for a semi-stable resolution of $\mathcal{M}^{\mathrm{loc}}$

With the notation from the last chapter our goal is to find a semi-stable resolution of the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ for the standard lattice chain $\Gamma^{\mathrm{st}}$. In general it is not known to be possible, but in Section 3 we have proven that for $k = 1$ the Mustafin variety $\mathcal{M}_{\mathbb{P}}\left(\overline{\Gamma}\right)$ is indeed semi-stable for every convex set $\overline{\Gamma}$ of lattice classes (cf. [Fal01]). This generalises the classical case $\mathcal{M}_{\mathbb{P}}\left(\Gamma^{\mathrm{st}}\right)$ of Drinfeld. In [Gen00] a semi-stable resolution for $n \leq 6$ was constructed for a symplectic analogue of the problem via a blow-up of the Grassmannian of isotropic submodules in Schubert subvarieties of the special fiber. Adapting this idea as indicated in [Gen00, remark (3) following Theorem 2.4.2] we define a candidate for a semi-stable resolution as follows.

**Definition 5.1.** We set $\mathcal{G}_0 \coloneqq \mathrm{Gr}^k(\Lambda_0)$ and inductively for $1 \leq i < (n-k)k$ define $\mathcal{G}_i$ to be the blow-up of $\mathcal{G}_{i-1}$ in the union of the strict transforms of the Schubert varieties of dimension $i-1$ in the special fiber of $\mathrm{Gr}^k(\Lambda_0)$. The last blow-up $\mathcal{G}_{(n-k)k-1}$ will be denoted by $\mathcal{G}$.

Now two questions arise.

(i) Is the blow-up $\mathcal{G}$ semi-stable?
(ii) Does the map $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ factor over the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}(\Lambda_0)$?

It is well known that the singular locus $X_I^{\mathrm{sing}}$ of a Schubert variety is again a union of Schubert varieties. This gives some hope that the centers in this sequence are in fact smooth. And to analyse the first question Genestier has proven Lemma 3.24 showing that blow-ups preserve semi-stability when the centers lie in the special fiber, are smooth over $\mathbb{F}_p$ and intersect the singular locus nicely.
Unfortunately in general the answer to the second question is no. In Appendix D we explicitly calculate the case $n = 5$ and $k = 2$. The result is semi-stable, but does not factor over the Mustafin variety $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$. In the following we construct a blow-up $\mathcal{M}$ of $\mathrm{Gr}^k(\Lambda_0)$ as the strict transform of the Plücker embedding $\mathrm{Gr}^k(\Lambda_0) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ under the blow-up $\overline{\pi^0_{\mathbb{P}}}$. We also define the strict transform $\mathcal{S}$ of the projection $\pi^0_{\mathrm{Gr}}$ under the blow-up $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$. Then we will show that the blow-up $\mathcal{M} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ and $\mathcal{S}$. In summary we will construct the following commutative diagram:

$$\begin{tikzcd}
\mathcal{G} \arrow[ddrr] & \mathcal{S} \arrow[l] \arrow[dr] & \mathcal{M} \arrow[l, dotted] \arrow[d, dotted] \arrow[r] & \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \arrow[d, "\overline{\pi_{\mathbb{P}}}"] \arrow[dd, bend left, "\overline{\pi^0_{\mathbb{P}}}"] \\
 & & \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \arrow[r] \arrow[d, "\pi^0_{\mathrm{Gr}}"] & \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right) \arrow[d, "\pi^0_{\mathbb{P}}"] \\
 & & \mathrm{Gr}^k(\Lambda_0) \arrow[r] & \mathbb{P}(\bigwedge^k \Lambda_0)
\end{tikzcd}$$

In the second part of this chapter we will proof some relations between the candidates. The easiest is to see that if $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ then $\mathcal{S} \to \mathcal{G}$ is an isomorphism. We will see that $\mathcal{S}$ restricts to the semi-stable resolution in a neighbourhood of the most singular point for $n = 5$ and $k = 2$ constructed in [Gör04]. In particular $\mathcal{S}$ generalises the candidate $\mathcal{G}$ and is stronger in at least one case.
If we further assume Conjecture 4.1 holds for $n$ and $k$ (e.g. $k = 2$ or $n \leq 7$), Theorem 5.16 will show that under some technical conditions, $\mathcal{M}$ and $\mathcal{G}$ coincide whenever $\mathcal{G}$ is semi-stable and the map $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ factors over the local model $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$.

5.1. **Construction of $\mathcal{M}$, $\mathcal{S}$ and related objects.** For the rest of this section we will fix a set $\{\Lambda_0^{\mathrm{pl}}, \dots, \Lambda_{\binom{n}{k}-1}^{\mathrm{pl}}\}$ of representatives of the classes in $\overline{\bigwedge^k \Gamma^{\mathrm{st}}}$ such that $\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_i^{\mathrm{pl}}]\}$ is convex for every $i \in \left[\binom{n}{k}\right]$ and $\Lambda_0^{\mathrm{pl}} = \bigwedge^k \Lambda_0$. Due to Lemma 3.19 it is possible to find such representatives.

**Definition 5.2.** Set $\mathcal{M}_0 := \mathrm{Gr}^k(\Lambda_0)$ and consider $\mathcal{M}_0$ via the Plücker embedding as a closed subscheme of $\mathbb{P}(\bigwedge^k \Lambda_0) = \mathcal{M}_{\mathbb{P}}\left(\{[\bigwedge^k \Lambda_0]\}\right)$. For $i \in \left[\binom{n}{k}\right]$ define $\mathcal{M}_i \subseteq \mathcal{M}_{\mathbb{P}}\left(\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_i^{\mathrm{pl}}]\}\right)$ to be the strict transform $\mathcal{M}_{i-1}^{\mathrm{s}}$ under the blow-up

$$\mathcal{M}_{\mathbb{P}}\left(\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_i^{\mathrm{pl}}]\}\right) \to \mathcal{M}_{\mathbb{P}}\left(\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_{i-1}^{\mathrm{pl}}]\}\right).$$

We denote the last blow-up $\mathcal{M}_{\binom{n}{k}-1}$ by $\mathcal{M}$.

**Lemma 5.3.** *The sequence $\mathcal{M} \to \mathrm{Gr}^k(\Lambda_0)$ of blow-ups defined above factors over the projection $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \to \mathrm{Gr}^k(\Lambda_0)$.*

*Proof.* Since the centers of the blow-ups are all contained in the special fiber the generic fiber of $\mathcal{M}$ is still dense and maps isomorphically to $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{Q}_p}$. Hence $\mathcal{M}$ is the closure in the Mustafin variety $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ of the image under the Plücker embedding

$$\mathrm{Gr}^k(\Lambda_0)_{\mathbb{Q}_p} \to \mathbb{P}(\bigwedge^k \Lambda_0)_{\mathbb{Q}_p} = \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{Q}_p}.$$

Now since $\overline{\pi_{\mathbb{P}}}(\mathcal{M}_{\mathbb{Q}_p})$ is the image of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{Q}_p}$ under the embedding of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ in $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ and $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ is flat by construction, the map $\overline{\pi_{\mathbb{P}}}|_{\mathcal{M}}$ will factor over $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$. □

**Definition 5.4.** Similarly to the construction of the blow-up $\mathcal{G}$ we can set $\mathcal{G}_0^{\mathrm{pl}} := \mathbb{P}(\bigwedge^k \Lambda_0)$ and for $i \in [(n-k)k]$ inductively define $\mathcal{G}_i^{\mathrm{pl}}$ to be the blow-up of $\mathcal{G}_i^{\mathrm{pl}}$ in the union of strict transforms of the linear subspaces corresponding to Schubert varieties of dimension $i-1$.

*Remark* 5.5. Using the relation in Lemma 4.2 between the Schubert varieties in $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p}$ and the linear subspaces of $\mathbb{P}(\bigwedge^k \Lambda_0)$, the blow-up $\mathcal{G}_j \to \mathcal{G}_{j-1}$ agrees with the strict transform of $\mathcal{G}_{j-1} \to \mathcal{G}_{j-1}^{\mathrm{pl}}$ under the blow-up $\mathcal{G}_j^{\mathrm{pl}} \to \mathcal{G}_{j-1}^{\mathrm{pl}}$.

**Lemma 5.6.** *The blow-up $\mathcal{G}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$ can be described as the successive blow-up of the total transforms of the linear subspaces $\mathbb{P}(V_I)$ for $I \in \binom{[n]}{k}$ in any order.*

*Proof.* Fix an integer $i \in [(n-k)k]$. We need to show that the blow-up $\mathcal{G}_{i+1}^{\mathrm{pl}} \to \mathcal{G}_i^{\mathrm{pl}}$ defined in Remark 5.5 can be described as the sequence blow-ups in the total transforms of the linear subspaces corresponding to Schubert varieties of dimension $i$. First we note that the intersection of two distinct linear subspaces corresponding to Schubert varieties of dimension $i$ are linear subspaces corresponding to Schubert varieties of dimension $i-1$ and hence the strict transforms of the linear subspaces corresponding to Schubert varieties of dimension $i$ are disjoint in $\mathcal{G}_i^{\mathrm{pl}}$. Therefore the blow-up $\mathcal{G}_{i+1}^{\mathrm{pl}} \to \mathcal{G}_i^{\mathrm{pl}}$ can be split up in the blow-ups on the individual strict transforms of the linear subspaces.
Now by induction the strict transforms of linear subspaces corresponding to Schubert varieties are smooth since they are blow-ups of smooth schemes over a field in smooth centers. But the strict transforms in $\mathcal{G}_{i-1}^{\mathrm{pl}}$ of a linear subspaces $L_i$ corresponding to a Schubert variety of dimension $i$ and a linear subspaces $L_{i-1}$ corresponding to a Schubert variety of dimension $i-1$ are either disjoint or $L_{i-1}$ is contained in $L_i$. Using Lemma 3.14 we identify the blow-up $\mathrm{Bl}_{L_i^{\mathrm{s}}}\left(\mathrm{Bl}_{L_{i-1}}\left(\mathcal{G}_i^{\mathrm{pl}}\right)\right)$ with $\mathrm{Bl}_{L_{i-1}^{\mathrm{tot}}}\left(\mathrm{Bl}_{L_i}\left(\mathcal{G}_i^{\mathrm{pl}}\right)\right)$. Since for blow-ups in total transforms the order does not matter, these blow-ups also agree with $\mathrm{Bl}_{L_i^{\mathrm{tot}}}\left(\mathrm{Bl}_{L_{i-1}}\left(\mathcal{G}_i^{\mathrm{pl}}\right)\right)$. □

**Corollary 5.7.** *The projection $\mathcal{M} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$.*

*Proof.* First recall from Proposition 4.15 that for $I \in \binom{[n]}{k}$ the inverse of the linear subspace $\mathbb{P}(V_I)$ in $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ is a union of irreducible components. From Proposition 3.25 we know that $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ is semi-stable and hence union of irreducible components of the special fiber are effective Cartier divisors. Using the sequence of blow-ups from Lemma 5.6 the universal property of the blow-ups inductively gives a factorisation of the projection $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ over $\mathcal{G}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$.
Now recall that for the Plücker embedding $\mathrm{Gr}^k(\Lambda_0) \subseteq \mathbb{P}(\bigwedge^k \Lambda_0)$ the scheme $\mathcal{M}$ is defined as the strict transform under the blow-up $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ and $\mathcal{G}$ is identified with the strict transform under the blow-up $\mathcal{G}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$. Hence we get the desired factorisation of $\mathcal{M} \to \mathrm{Gr}^k(\Lambda_0)$ over $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$. □

**Lemma 5.8.** *Fix the following commutative diagram*

$$\begin{tikzcd} Y \arrow[d, "f"'] & B \arrow[l, "g"'] \arrow[dl, "\pi"] \\ X & \end{tikzcd}$$

*of $\mathbb{Z}_p$-schemes. Assume that $f$ induces an isomorphism $f_{\mathbb{Q}_p}$ on the generic fibers and $\pi$ is blow-up with center $Z$ contained in the special fiber $X_{\mathbb{F}_p}$. Then the map $g\colon B \to Y$ agrees with the strict transform $\mathrm{Bl}_{f^{-1}(Z)}(Y) \to Y$.*

*Proof.* Let $Z \subseteq X_{\mathbb{F}_p}$ be the center of $\pi$ and $Z' := f^{-1}(Z)$ be the inverse image in $Y$. By the universal property the strict transform $\mathrm{Bl}_{Z'}(Y)$ admits a unique map to $B$ compatible with the maps to $X$. On the other hand $g^{-1}(Z')$ is the inverse image $\pi^{-1}(Z)$ and hence the universal property of the blow-up $\pi$ admits a unique map $B \to \mathrm{Bl}_{Z'}(Y)$ compatible with the maps to $Y$. Now uniqueness shows that $B$ and $\mathrm{Bl}_{Z'}(Y)$ are isomorphic via the constructed maps. □

*Remark* 5.9. Note that sequences of blow-ups can always be computed as a single blow-up and successively taking the strict transform under a sequence of blow-ups gives the strict transform under the composition. Hence the lemma above can be reformulated for $\pi$ to be a sequence of blow-ups and $g$ to be the sequence of strict transforms.
Now it is easy to see that $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ and hence $\mathcal{M}$ are the successive blow-ups of $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ and $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ in images of unions of irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)$ and $\mathcal{M}$ respectively.

**Definition 5.10.** For the sequence of blow-ups $\mathcal{G}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$ let us denote the sequence of strict transforms of the projection $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ by $\mathcal{S}^{\mathrm{pl}} \to \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$. More precisely we set $\mathcal{S}_0^{\mathrm{pl}} = \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ and inductively construct $\mathcal{S}_{i+1}^{\mathrm{pl}}$ as the blow-up $\mathrm{Bl}_{\mathrm{pr}^{-1}(Z)}\left(\mathcal{S}_i^{\mathrm{pl}}\right)$ of $\mathcal{S}_i^{\mathrm{pl}}$ in the inverse image of the center $Z$ of the blow-up $\mathcal{G}_{i+1}^{\mathrm{pl}} \to \mathcal{G}_i^{\mathrm{pl}}$ under the projection $\mathrm{pr}\colon \mathcal{S}_i^{\mathrm{pl}} \to \mathcal{G}_i^{\mathrm{pl}}$. Similarly we define $\mathcal{S} \to \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ to be the strict transform under the blow-up $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ of the projection $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \to \mathrm{Gr}^k(\Lambda_0)$.

**Lemma 5.11.** *The blow-up $\mathcal{S} \to \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ defined above can be alternatively constructed in the following way:*

*(i) $\mathcal{S}_0$ is set to be $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$*
*(ii) $\mathcal{S}_1$ is the blow-up of $\mathcal{S}_0$ in the irreducible component surjecting to the 0-dimensional Schubert variety in $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p}$*
*(iii) $\mathcal{S}_i$ is the blow-up of $\mathcal{S}_{i-1}$ in the union of all strict transforms of irreducible component surjecting to $i-1$-dimensional Schubert varieties in $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p}$*

*Proof.* Recall that by definition $\mathcal{G}_{i+1}$ is the blow-up of $\mathcal{G}_i$ in union of the strict transform of Schubert varieties of dimension $i$. We will inductively show that $\mathcal{S}_{i+1}$ is the strict transform of the map $\mathrm{pr}_i\colon \mathcal{S}_i \to \mathcal{G}_i$ under the blow-up $\mathcal{G}_{i+1} \to \mathcal{G}_i$.
We have to show that for the strict transform $X_I^{\mathrm{s}}$ of a Schubert variety $X_I$ of dimension $i$ under the blow-up $\mathcal{G}_i \to \mathrm{Gr}^k(\Lambda_0)$ the inverse image $\mathrm{pr}_i^{-1}(X_I^{\mathrm{s}})$ in $\mathcal{S}_i$ is the strict transform $C_I^{\mathrm{s}}$ of the irreducible component $C_I$ of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ under the blow-up $\mathcal{S}_i \to \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$. But $\mathcal{G} \to \mathrm{Bl}_{X_I^{\mathrm{s}}}(\mathcal{G}_i)$ is surjective and the exceptional divisor $E$ in $\mathrm{Bl}_{X_I^{\mathrm{s}}}(\mathcal{G}_i)$ is an irreducible component in the special fiber. Hence there exists an irreducible component in $\mathcal{G}_{\mathbb{F}_p}$ surjecting to $E$. Again since $\mathcal{S} \to \mathcal{G}$ is surjective there exists an irreducible component of $\mathcal{S}_{\mathbb{F}_p}$ surjecting to $E$. Now the image of $E$ in $\mathrm{Bl}_{\mathrm{pr}_i^{-1}(X_I^{\mathrm{s}})}(\mathcal{S}_i)$ is the exceptional divisor and its image in $\mathcal{G}_i$ is an irreducible component.
Using that the irreducible components in $\mathcal{S}_{i,\mathbb{F}_p}$ are precisely the strict transforms of irreducible components of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ and that the irreducible component surjecting to $X_I$ is unique, we have shown the claim. ☐

**Lemma 5.12.** *The blow-up $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ factors over $\mathcal{S}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$.*

*Proof.* Using Lemma 5.6 we can argue in the same way as in the proof of Corollary 5.7. ☐

5.2. **Comparisons of the candidates.** Before we go on and prove some relation between the objects constructed above, let us summarise all of them with the maps between them in the following diagram:

$$\begin{tikzcd}
\mathcal{S} \arrow[d] \arrow[dr] & \mathcal{M} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \arrow[r] \arrow[d] & \mathcal{S}^{\mathrm{pl}} \arrow[dl] \arrow[d] \\
\mathcal{G} \arrow[dr] & \mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right) \arrow[r] \arrow[d] & \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right) \arrow[d] & \mathcal{G}^{\mathrm{pl}} \arrow[dl] \\
 & \mathrm{Gr}^k(\Lambda_0) \arrow[r] & \mathbb{P}(\bigwedge^k \Lambda_0) &
\end{tikzcd}$$

**Lemma 5.13.** *Assume Conjecture 4.1. Then the image in $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ of an irreducible component $C$ of the special fiber $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ is an irreducible component.*

*Proof.* For an irreducible component $C$ of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ the image in $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ is irreducible and hence lies in an irreducible component. But the images in $\mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$ of two different irreducible components of $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ are not contained in the same irreducible component. Hence for an irreducible component of $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ there is at most one irreducible component mapping to it. Now since the map $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p} \to \mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ is surjective the image of $C$ is a full irreducible component. ☐

**Proposition 5.14.** *Assume Conjecture 4.1 and that $\mathcal{S}^{\mathrm{pl}}$ is semi-stable. Then the projection $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathcal{S}^{\mathrm{pl}}$ of Lemma 5.12 is an isomorphism. In this case also the projection $\mathcal{M} \to \mathcal{S}$ is an isomorphism.*

*Proof.* Let us assume that for $i \in \left[\binom{n}{k}\right]$ the map $\mathcal{S}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$ factors over the sequence of blow-ups $\mathcal{M}_{\mathbb{P}}\left(\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_i^{\mathrm{pl}}]\}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$ defined in the beginning of this chapter. Denote the center of the blow-up $\mathcal{M}_{\mathbb{P}}\left(\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_{i+1}^{\mathrm{pl}}]\}\right) \to \mathcal{M}_{\mathbb{P}}\left(\{[\Lambda_0^{\mathrm{pl}}] \dots, [\Lambda_i^{\mathrm{pl}}]\}\right)$ by $Z_{\Lambda_{i+1}^{\mathrm{pl}}}$.

Then the inverse image of $Z_{\Lambda^{\mathrm{pl}}_{i+1}}$ in $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right)_{\mathbb{F}_p}$ is by Lemma 3.26 a union of irreducible components. Hence by Lemma 5.13 also the inverse image of $Z_{\Lambda^{\mathrm{pl}}_{i+1}}$ in $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ is a union of irreducible components. Since $\mathcal{S}^{\mathrm{pl}}$ is semi-stable by assumption, unions of irreducible components of its special fiber are effective Cartier divisors. Therefore the map $\mathcal{S}^{\mathrm{pl}} \to \mathbb{P}(\bigwedge^k \Lambda_0)$ factors over $\mathcal{M}_{\mathbb{P}}\left(\{[\Lambda^{\mathrm{pl}}_0]\dots,[\Lambda^{\mathrm{pl}}_{i+1}]\}\right) \to \mathbb{P}(\bigwedge^k \Lambda_0)$. By induction we now have an inverse for the map $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) \to \mathcal{S}^{\mathrm{pl}}$.
For the second statement we recall that by construction both $\mathcal{M}$ and $\mathcal{S}$ are the strict transforms of the Plücker embedding under the blow-up $\mathcal{M}_{\mathbb{P}}\left(\overline{\bigwedge^k \Gamma^{\mathrm{st}}}\right) = \mathcal{S}^{\mathrm{pl}} \to \mathcal{M}_{\mathbb{P}}\left(\bigwedge^k \Gamma^{\mathrm{st}}\right)$ hence they agree. □

**Proposition 5.15.** *Assume Conjecture 4.1. Then $\mathcal{M} \to \mathcal{S}$ is an isomorphism whenever $\mathcal{S}$ is semi-stable and for every irreducible component $C$ of $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ the intersection $C \cap \mathcal{S}$ is a union of irreducible components.*

*Proof.* The proof argues in the same way as for the lemma above. For convenience let us recall the arguments.
Let us assume that for $i \in \left[\binom{n}{k}\right]$ the map $\mathcal{S} \to \mathrm{Gr}^k(\Lambda_0)$ factors over the blow-up $\mathcal{M}_i \to \mathrm{Gr}^k(\Lambda_0)$. Denote the center of the blow-up $\mathcal{M}_{i+1} \to \mathcal{M}_i$ with $Z_{i+1}$.
Using the notation in the proof of the proposition above we write $Z_{i+1}$ as the intersection $\mathcal{S} \cap Z_{\Lambda^{\mathrm{pl}}_{i+1}}$. Hence again inverse image of $Z_{i+1}$ in $\mathcal{M}_{\mathbb{F}_p}$ is the intersection of a union of irreducible components of $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ with $\mathcal{S}$. Using the hypothesis this is an intersection of irreducible components of $\mathcal{S}_{\mathbb{F}_p}$. Since $\mathcal{S}$ is semi-stable by assumption, unions of irreducible components of its special fiber are effective Cartier divisors. Therefore the map $\mathcal{S} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{M}_{i+1} \to \mathrm{Gr}^k(\Lambda_0)$. By induction we now have an inverse for the map $\mathcal{M} \to \mathcal{S}$. □

**Theorem 5.16.** *Assume Conjecture 4.1 and that the blow-up $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$. Then the map $\mathcal{M} \to \mathcal{G}$ of Corollary 5.7 is an isomorphism in any of the following cases:*

*(i) $\mathcal{S}^{\mathrm{pl}}$ is semi-stable*
*(ii) $\mathcal{S}$ is semi-stable and for every irreducible component $C$ of $\mathcal{S}^{\mathrm{pl}}_{\mathbb{F}_p}$ the intersection $C \cap \mathcal{S}$ is a union of irreducible components*

*Proof.* Since by hypothesis $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ factors over $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$ we the map $\mathcal{S} \to \mathcal{G}$ is an isomorphism by Lemma 5.8. By Proposition 5.15 or Proposition 5.14 we now see that $\mathcal{M} \to \mathcal{G}$ is an isomorphism. □

As mentioned before in the case of $n = 5$, $k = 2$ the candidate $\mathcal{G}$ of Genestier is indeed semi-stable, but the blow up $\mathcal{G} \to \mathrm{Gr}^k(\Lambda_0)$ does not factor over $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$. Hence it does not give a semi-stable resolution. But for this case Görtz constructed in [Gör04] a semi-stable resolution $\widetilde{\mathcal{G}}$ in a neighbourhood of the worst singularity by blowing up irreducible components of $\mathcal{M}_{\mathrm{Gr}^k}\left(\Gamma^{\mathrm{st}}\right)$. We will prove that in this case also the candidate $\mathcal{S}$ is given by this blow ups of irreducible components and hence is indeed a semi-stable resolution.

**Theorem 5.17.** *For $n = 5$ and $k = 2$ the candidate $\mathcal{S}$ restricts to the semi-stable resolution defined in* [Gör04] *in a neighbourhood of the worst singularity of $\mathcal{M}^{loc}$.*

*Proof.* In Lemma 5.11 we have described $\mathcal{S}$ as a sequence of blow-ups of $\mathcal{M}_{\mathrm{Gr}^2}\left(\Gamma^{\mathrm{st}}\right)$ in irreducible components of $\mathcal{M}_{\mathrm{Gr}^2}\left(\Gamma^{\mathrm{st}}\right)_{\mathbb{F}_p}$. Since the resolution $\widetilde{\mathcal{G}}$ is constructed in the same way, but blowing up just the strict transforms of irreducible components not corresponding to a lattice in $\Gamma^{\mathrm{st}}$ we have locally a factorisation of $\mathcal{S} \to \mathcal{M}_{\mathrm{Gr}^2}\left(\Gamma^{\mathrm{st}}\right)$ over $\widetilde{\mathcal{G}}$. Using that $\widetilde{\mathcal{G}}$ is semi-stable we now see that we get an inclusion $\widetilde{\mathcal{G}} \to \mathcal{S}$. □

## Appendix A. Projection of $\mathcal{M}^{\mathrm{loc}}_{\mathbb{F}_p}$ to $\mathrm{Gr}^k(\Lambda_0)_{\mathbb{F}_p}$ for $n = 4$ and $k = 2$

In this Appendix we give a complete list of elements in $\mathrm{Adm}(\mu)$ for $n = 4$ and $k = 2$ together with their projections to $\binom{[4]}{2} = W^J$. For a discussion of this projection see Section 2.1 and Section 2.3. In the third column the corresponding alcove is presented. For simplicity each row just has an entry if it is different from the one in the first row in the same block.

| $w\tau$ | $xt^\mu y_J({}^J y)$ | | | | | length | $W^J$ | $W^J$-length |
|---|---|---|---|---|---|---|---|---|
| $\tau$ | $t^\mu(s_2s_1s_3s_2)$ | (1100) | (1110) | (1111) | (2111) | 0 | Id | 0 |
| $s_0\tau$ | $t^\mu(s_2s_1s_3)$ | | | (2110) | | 1 | Id | 0 |
| $s_1\tau$ | $t^\mu s_1(s_2s_1s_3s_2)$ | | | | (1211) | 1 | Id | 0 |
| $s_3\tau$ | $t^\mu s_3(s_2s_1s_3s_2)$ | | (1101) | | | 1 | Id | 0 |
| $s_1s_3\tau$ | $t^\mu s_1s_3(s_2s_1s_3s_2)$ | | (1101) | | (1211) | 2 | Id | 0 |
| $s_0s_3\tau$ | $t^\mu(s_2s_3)$ | | (2100) | (2110) | | 2 | Id | 0 |
| $s_1s_0\tau$ | $t^\mu s_1(s_2s_3s_1)$ | | | (1210) | (1211) | 2 | Id | 0 |
| $s_3s_0\tau$ | $t^\mu s_3(s_2s_1s_3)$ | | (1101) | (2101) | | 2 | Id | 0 |
| $s_0s_1\tau$ | $t^\mu(s_2s_1)$ | | | (2110) | (2210) | 2 | Id | 0 |
| $s_3s_0s_3\tau$ | $t^\mu s_3(s_2s_1)$ | | (2100) | (2101) | | 3 | Id | 0 |
| $s_0s_1s_3\tau$ | $t^\mu s_2$ | | (2100) | (2110) | (2210) | 3 | Id | 0 |
| $s_0s_1s_0\tau$ | $t^\mu s_1(s_2s_1)$ | | | (1210) | (2210) | 3 | Id | 0 |
| $s_3s_1s_0\tau$ | $t^\mu s_3s_1(s_2s_1s_3)$ | | (1101) | (1201) | (1211) | 3 | Id | 0 |
| $s_0s_1s_3s_0\tau$ | $t^\mu$ | | (2100) | (2200) | (2210) | 4 | Id | 0 |
| $s_2\tau$ | $s_2t^\mu(s_2s_1s_3s_2)$ | (1010) | (1110) | (1111) | (2111) | 1 | $s_2$ | 1 |
| $s_0s_2\tau$ | $s_2t^\mu(s_2s_1s_3)$ | | | (2110) | | 2 | $s_2$ | 1 |
| $s_2s_3\tau$ | $s_2t^\mu s_3(s_2s_1s_3s_2)$ | | (1011) | | | 2 | $s_2$ | 1 |
| $s_2s_1\tau$ | $s_2t^\mu s_1(s_2s_1s_3s_2)$ | | | | (1121) | 2 | $s_2$ | 1 |
| $s_0s_2s_3\tau$ | $s_2t^\mu(s_2s_3)$ | | (2010) | (2110) | | 3 | $s_2$ | 1 |
| $s_2s_1s_3\tau$ | $s_2t^\mu s_1s_3(s_2s_1s_3s_2)$ | | (1011) | | (1121) | 3 | $s_2$ | 1 |
| $s_0s_2s_1\tau$ | $s_2t^\mu(s_2s_1)$ | | | (2110) | (2120) | 3 | $s_2$ | 1 |
| $s_0s_2s_1s_3\tau$ | $s_2t^\mu(s_2)$ | | (2010) | (2110) | (2120) | 4 | $s_2$ | 1 |
| $s_1s_2\tau$ | $s_1s_2t^\mu(s_2s_1s_3s_2)$ | (0110) | (1110) | (1111) | (1211) | 2 | $s_1s_2$ | 2 |
| $s_1s_0s_2\tau$ | $s_1s_2t^\mu(s_2s_1s_3)$ | | | (1210) | | 3 | $s_1s_2$ | 2 |
| $s_1s_2s_1\tau$ | $s_1s_2t^\mu s_1(s_2s_1s_3s_2)$ | | | | (1121) | 3 | $s_1s_2$ | 2 |
| $s_1s_0s_2s_1\tau$ | $s_1s_2t^\mu(s_2s_1)$ | | | (1210) | (1220) | 4 | $s_1s_2$ | 2 |
| $s_3s_2\tau$ | $s_3s_2t^\mu(s_2s_1s_3s_2)$ | (1001) | (1101) | (1111) | (2111) | 2 | $s_3s_2$ | 2 |
| $s_3s_0s_2\tau$ | $s_3s_2t^\mu(s_2s_1s_3)$ | | | (2101) | | 3 | $s_3s_2$ | 2 |
| $s_2s_3s_2\tau$ | $s_3s_2t^\mu s_3(s_2s_1s_3s_2)$ | | (1011) | | | 3 | $s_3s_2$ | 2 |
| $s_3s_0s_2s_3\tau$ | $s_3s_2t^\mu(s_2s_3)$ | | (2001) | (2101) | | 4 | $s_3s_2$ | 2 |
| $s_3s_1s_2\tau$ | $s_3s_1s_2t^\mu(s_2s_1s_3s_2)$ | (0101) | (1101) | (1111) | (1211) | 3 | $s_3s_1s_2$ | 3 |
| $s_3s_1s_2s_0\tau$ | $s_1s_3s_2t^\mu(s_2s_3s_1)$ | | | (1201) | | 4 | $s_1s_3s_2$ | 3 |
| $s_2s_1s_3s_2\tau$ | $s_2s_1s_3s_2t^\mu(s_2s_3s_1s_2)$ | (0011) | (1011) | (1111) | (1121) | 4 | $s_2s_1s_3s_2$ | 4 |

## Appendix B. Examples of the combinatorics of Schubert cells

In this Appendix we want to give some examples for the combinatorics of the Schubert varieties in the Grassmanians $\mathrm{Gr}_{n,r}(\mathbb{F}_p)$. The arrows in the diagrams below indicate an inclusion of the corresponding Schubert varieties.

**Example.** In $\mathrm{Gr}^{4,2}(\mathbb{F}_p)$ the Schubert cells are:

$$\begin{tikzcd}
 & & X_{\{4,1\}} \arrow[dl] & & \\
X_{\{4,3\}} & X_{\{4,2\}} \arrow[l] & & X_{\{3,1\}} \arrow[ul] \arrow[dl] & X_{\{2,1\}} \arrow[l] \\
 & & X_{\{3,2\}} \arrow[ul] & &
\end{tikzcd}$$

**Example.** In $\mathrm{Gr}^{4,23}(\mathbb{F}_p)$ the Schubert cells are:

$$X_{\{4,3,2\}} \longleftarrow X_{\{4,3,1\}} \longleftarrow X_{\{4,2,1\}} \longleftarrow X_{\{3,2,1\}}$$

**Example.** In $\mathrm{Gr}^{5,2}(\mathbb{F}_p)$ the Schubert cells are:

$$\begin{tikzcd}
 & & X_{\{5,2\}} \arrow[dl] & X_{\{5,1\}} \arrow[l] & X_{\{4,1\}} \arrow[l] \arrow[ddl] & & \\
X_{\{5,4\}} & X_{\{5,3\}} \arrow[l] & & & & X_{\{3,1\}} \arrow[ul] \arrow[dl] & X_{\{2,1\}} \arrow[l] \\
 & & X_{\{4,3\}} \arrow[ul] & X_{\{4,2\}} \arrow[l] \arrow[uul] & X_{\{3,2\}} \arrow[l] & &
\end{tikzcd}$$

**Example.** In $\mathrm{Gr}^{5,3}(\mathbb{F}_p)$ the Schubert cells are:

$$\begin{tikzcd}
 & & X_{\{5,4,1\}} \arrow[dl] & X_{\{5,3,1\}} \arrow[l] \arrow[ddl] & X_{\{5,2,1\}} \arrow[l] & & \\
X_{\{5,4,3\}} & X_{\{5,4,2\}} \arrow[l] & & & & X_{\{4,2,1\}} \arrow[ul] \arrow[dl] & X_{\{3,2,1\}} \arrow[l] \\
 & & X_{\{5,3,2\}} \arrow[ul] & X_{\{4,3,2\}} \arrow[l] & X_{\{4,3,1\}} \arrow[l] \arrow[uul] & &
\end{tikzcd}$$

**Example.** In $\mathrm{Gr}^{6,2}(\mathbb{F}_p)$ the Schubert cells are:

$$\begin{tikzcd}
 & & X_{\{6,3\}} \arrow[dl] & X_{\{6,2\}} \arrow[l] & X_{\{6,1\}} \arrow[l] & X_{\{5,1\}} \arrow[l] \arrow[dl] & X_{\{4,1\}} \arrow[l] \arrow[ddl] & & \\
X_{\{6,5\}} & X_{\{6,4\}} \arrow[l] & & & X_{\{5,2\}} \arrow[ul] \arrow[dl] & & & X_{\{3,1\}} \arrow[ul] \arrow[dl] & X_{\{2,1\}} \arrow[l] \\
 & & X_{\{5,4\}} \arrow[ul] & X_{\{5,3\}} \arrow[l] \arrow[uul] & X_{\{4,3\}} \arrow[l] & X_{\{4,2\}} \arrow[l] \arrow[ul] & X_{\{3,2\}} \arrow[l] & &
\end{tikzcd}$$

## Appendix C. Sage calculatons for Conjecture 4.1

In this Appendix we describe a Sage program explicitly proving Conjecture 4.1 for $n \leq 7$ using the discussion made in Remark 4.21.

Defining the functions for creating the graph of lattices in $\overline{\Gamma^{st}}$:

```
def secreps(lat1,lat2):
   global n
   global k
   return [(stats.IntList([lat1[j],lat2[j]]).max()) for j in range(binomial(n,k))]
def sec(v,w):
   global n
   global k
   con = [v,w]
   lb = min(w[i]-v[i] for i in range(binomial(n,k)))
   ub = max(w[i]-v[i] for i in range(binomial(n,k)))
   for j in range(ub-lb):
      if not any(same(secreps(w,[(v[i]+j+lb)for i in range(binomial(n,k))]),t ) for t in con):
         con.append(secreps(w,[(v[i]+j+lb)for i in range(binomial(n,k))]))
   return [lat for lat in con if lat not in [v,w]]
def same(v,w):
   global n
   global k
   minimum =  min(w[j]-v[j] for j in range(binomial(n,k)))
   for i in range(binomial(n,k)) :
     if w[i]-v[i]-minimum >0:
        return False
   return True
def testconvex(lis,G):
   for tpl in Combinations(lis,2).list():
      for intersection in sec(G.get_vertex(tpl[0]),G.get_vertex(tpl[1])):
         if not any(same(intersection,t) for t in [G.get_vertex(v) for v in lis]):
            return False
   return True
```

```
def convex(G):
   while testconvex(G.vertices(),G)==False:
      for tpl in Combinations(G.vertices(),2).list():
         i = 0
         for intsec in sec(G.get_vertex(tpl[0]),G.get_vertex(tpl[1])):
            if not any(same(intsec,t) for t in [G.get_vertex(v) for v in G.vertices()]):
               G.add_vertex('(%s cap %s)_%d' %(tpl[0],tpl[1],i))
               G.set_vertex('(%s cap %s)_%d' %(tpl[0],tpl[1],i), intsec)
               i=i+1

def gr(n,k):
   global G
   G=graphs.EmptyGraph()
   for i in range(n):
      G.add_vertex('L%d' %i)
      lat=[]
      for j in Combinations(range(n),k).list():
         ent = 0
         for b in j:
            if b<i:
               ent = ent + 1
         lat.append(ent)
      G.set_vertex('L%d' %i, lat)
   vert1=G.get_vertices()
   convex(G)
   vert2=[v for v in G.get_vertices() if not v in vert1]
   print('case (n,k)= (%d , %d)' %(n,k) )
   return [vert1,vert2]

def conj(a,b):
   global n
   global k
   n=a
   k=b
   vert=gr(n,k)
   for lambda1 in vert[1]:
```

```
      wl=[]
      for lambda2 in vert[0]:
         wl.append(kern(G.get_vertex(lambda1),G.get_vertex(lambda2)))
      if condition(wl,binomial(n,k),n)==False:
         print(False)
         return wl
   print(True)
   return True
def kern(lambda1,lambda2):
   w=Set([])
   maximaldiff =  max(lambda2[j]-lambda1[j] for j in range(binomial(n,k)))
   for j in range(binomial(n,k)):
      if not lambda2[j]-lambda1[j]==maximaldiff:
         w=Set({j}).union(Set(w))
   return w
```

Defining the function checking the condition for the set of subsets:

```
def condition(sset,d,n):
  bol = False
  B=[]
  for i in range(n):
     B.append(range(d-sset[i].cardinality()))
  for m in cartesian_product(B):
     if iselement(m,sset,d,n):
        return True
  print(False)
  return False
def iselement(li_1,li_2,d,n):
  if not sum(li_1)==d-1:
     return false
  for I in Subsets(range(n)):
    if not I.is_empty():
      Wi=Set(range(d))
      for i in I:
         Wi=Wi.intersection(li_2[i])
```

```
            if d-sum((li_1[i]) for i in I) <= Wi.cardinality():
                return False
        return True
```

Defining the function for creating an output:

```
def out(start,end):
    ta=[]
    for n in range(end-start):
        for k in range((n+start)/2.floor()-2):
            ta.append([n+start,k+3, conj(n+start,k+3)])
    return table(ta, header_row=['n', 'k', 'Conjecture'], frame=True)
```

The result:

| n | k | Conjecture |
|---|---|---|
| 6 | 3 | True |
| 7 | 3 | True |

# Appendix D. Sage calculation of $\mathcal{G}$ for $n = 5$ and $k = 2$

In this Appendix we describe a Sage program computing $\mathcal{G}$ for $n = 5$ and $k = 2$ and showing that we do not have a factorisation over the local model $\mathcal{M}^{\mathrm{loc}}$.

```
global n
n=5
global k
k=2
global Base
Base = QQ
```

Defining the function intersecting latices

```
def secreps(lat1,lat2):
   global n
   global k
   return [(stats.IntList([lat1[j],lat2[j]]).min()) for j in range(binomial(n,k))]
def sec(v,w):
   global n
   global k
   con = [v,w]
   lb = min((w[i]-v[i]) for i in range(binomial(n,k)))
   ub = max((w[i]-v[i]) for i in range(binomial(n,k)))
   for j in range(ub-lb):
      if not any(same(secreps(w,[(v[i]+j+lb)for i in range(binomial(n,k))]),t) for t in con):
         con.append(secreps(w,[(v[i]+j+lb)for i in range(binomial(n,k))]))
   return [lat for lat in con if lat not in [v,w]]
```

Defining the function constructing the convex closure

```
def testconvex(lis):
   for tpl in Combinations(lis,2).list():
      for intersect in sec(G.get_vertex(tpl[0]),G.get_vertex(tpl[1])):
         if not any(same(intersect,t) for t in [G.get_vertex(v) for v in lis]):
            return False
   return True
def convex():
   while testconvex(G.vertices())==False:
      for tpl in Combinations(G.vertices(),2).list():
         i = 0
         for intersct in sec(G.get_vertex(tpl[0]),G.get_vertex(tpl[1])):
```

```
            if not any(same(intersct,t) for t in [G.get_vertex(v) for v in G.vertices()]):
                G.add_vertex('$(%s \cap %s)_%d$' %(tpl[0],tpl[1],i))
                G.set_vertex('$(%s \cap %s)_%d$' %(tpl[0],tpl[1],i), intersct)
                i=i+1
```

Defining the function deciding identification for lattices

```
def same(v,w):
   global n
   global k
   minimum =  min(w[j]-v[j] for j in range(binomial(n,k)))
   for i in range(binomial(n,k)) :
     if w[i]-v[i]-minimum >0:
        return False
   return True
```

Defining the function deciding neighbourhood for lattices

```
def neighbour(v,w):
   if same(v,w)==True:
      return False
   minimum =  min(w[j]-v[j] for j in range(binomial(n,k)))
   for i in range(binomial(n,k)) :
      if w[i]-v[i]-minimum >1:
         return False
   return True
```

Defining the function for the output

```
def out(cent):
   global q
   A=AffineSpace(R)
   ta = [(c[0],compareschub(c)[0],c[2]) for c in cent if c[0] not in [d[0] for d in donecenter]]
   return table(ta , header_row=['adm. center', 'Schub', 'smooth'], frame=True)

def outschub(cent):
   global q
   A=AffineSpace(R)
   if [(c[0],c[2]) for c in cent if c[0] not in [d[0] for d in donecenterschub]]==[]:
      return ['All blow-ups are done']
   ta = [(c[0],c[2]) for c in cent if c[0] not in [d[0] for d in donecenterschub]]
   return table(ta , header_row=['Schub', 'smooth'], frame=True)

def compareschub(cen):
```

```
    for sch in schub:
        if sch[1]==cen[1]+R.ideal(q):
            return sch
    return [False,False,False]
```

Defining the function computing the blow up

```
def blowup():
    global proj
    global admcenter
    global count
    global donecenter
    global donecenterschub
    global var
    global R
    global A
    global q
    global V
    global schub
    blcenter = small(admcenter)
    if blcenter==False:
        return ['There is no admissible center']
    print('The blow-up in %s' %blcenter[0])
    count = count +1
    donecenter.append([blcenter[0],'l%d' %count])
    donecenterschub.append([compareschub(blcenter)[0],'l%d' %count])
    var.append('l%d' %count)
    R = PolynomialRing(Base, var, order='deglex')
    R.inject_variables()
    A=AffineSpace(R)

    V =R.ideal([tot(f,R,blcenter[1],R.gens()[count+variablecount+2]) for f in V.gens()])
    V= R.ideal(R.ideal(blcenter[1]).intersection(R.ideal(V).radical()).groebner_basis())
    q =R.ideal([strict(f,R,blcenter[1],R.gens()[count+variablecount+2]) for f in q.groebner_basis()])
    schub=schubtrans(schub,blcenter[1])
    proj = [ tot(f,R,blcenter[1],R.gens()[count+variablecount+2]) for f in proj]
    admcenter = admissiblecenter(admcenter)
    return out(admcenter)

def blowupschub():
```

```
   global proj
   global admcenter
   global count
   global donecenter
   global donecenterschub
   global var
   global R
   global A
   global q
   global V
   global schub
   blcenter = smallschub(schub)
   if blcenter==False:
      return ['There is no admissible center']
   print('The blow-up in %s' %blcenter[0])
   count = count +1
   donecenter.append([blcenter[0],'l%d' %count])
   donecenterschub.append([blcenter[0],'l%d' %count])
   var.append('l%d' %count)
   R = PolynomialRing(Base, var, order='deglex')
   R.inject_variables()
   A=AffineSpace(R)

   V =R.ideal([tot(f,R,blcenter[1],R.gens()[count+variablecount+2]) for f in V.gens()])
   V= R.ideal(R.ideal(blcenter[1]).intersection(R.ideal(V).radical()).groebner_basis())
   q =R.ideal([strict(f,R,blcenter[1],R.gens()[count+variablecount+2]) for f in q.groebner_basis()])
   schub=schubtrans(schub,blcenter[1])
   proj = [ tot(f,R,blcenter[1],R.gens()[count+variablecount+2]) for f in proj]
   return outschub(schub)

def schubtrans(li,cen):
   global count
   schnew = []
   for sch in li:
      if not sch[1]==cen:
         idl = transf(sch[1],cen,V)
         print('checking %s' %sch[0])
         sm=true
```

```
         if sch[2]== false:
            sm = smooth(idl+R.ideal(q),V)
         print(sm)
         schnew.append([sch[0],idl,sm])
   return schnew
def small(lis):
   global n
   global k
   global count
   for j in range(binomial(n,k)+n+count):
      for c in lis:
         if not c[0] in  [d[0] for d in donecenter]:
            if c[2]==True:
                  if AffineSpace(R).subscheme(R.ideal(c[1])).dimension()==j:
                     return c
   return False
def smallschub(lis):
   global n
   global k
   global count
   for j in range(binomial(n,k)+n+count):
      for c in lis:
         if not c[0] in  [d[0] for d in donecenterschub]:
            if c[2]==True:
                  if AffineSpace(R).subscheme(R.ideal(c[1])).dimension()==j:
                     return c
   return False
def quasicont(Idl1,Idl2,U):
   return R.ideal(Idl2)+R.ideal(q)<=R.ideal(Idl1) +R.ideal(q)
def quasieq(Idl1,Idl2,U):
   if quasicont(Idl1,Idl2,U)== False:
      return False
   if quasicont(Idl2,Idl1,U)== False:
      return False
   return True
def transf(I,cen,V):
   idl = R.ideal(R(1))
```

```
    if quasicont(I,cen,V)==False:
       idl = R.ideal([strict(f,R,cen,R.gens()[count+variablecount+2]) for f in I.groebner_basis()]).radical()
    if quasicont(I,cen,V)==True:
       idl =R.ideal([tot(f,R,cen,R.gens()[count+variablecount+2]) for f in I.gens()]).radical() + R.ideal(q)
    idl =  prod([co.defining_ideal().gens() for co in A.subscheme(idl).irreducible_components() if not V<=co.defining_ideal()]).radical()
    return idl
def strict(f,am,I,l):
  mon = [m*f.monomial_coefficient(m) for m in f.monomials()]
  fs = sum(m*l^ip(m,I) for m in mon)
  while fs/l in am:
    fs = fs/l
  return fs
def tot(f,am,I,l):
  if f.is_constant()==True:
     return f
  mon = [m*f.monomial_coefficient(m) for m in f.monomials()]
  fs = sum(m*l^ip(m,I) for m in mon)
  return fs
def ip(m,I):
  n=1
  while m in I^n:
    n=n+1
  return n -1
```

Defining the function creating admissible center

```
def admissiblecenter(lis):
    global proj
    adm = []
    poly = P
    for j in range(count):
        poly= poly*R.gens()[j+variablecount+3]
    for new in convexext([d[0] for d in donecenter]):
       if new in [ad[0] for ad in lis if ad[2]==True]:
          idl=R.ideal([0])
          gcd = 0
          biratmap = [proj[j]*(poly)^(dif(G.get_vertex(new),G.get_vertex('L0'))[j]) for j in   range(variablecount+1)]
          for va in biratmap:
             gcd = va.gcd(gcd)
```

```
            biratmap = [va/gcd for va in biratmap]
            idl = R.ideal(biratmap).radical()
            idl = prod([com.defining_ideal() for com in A.subscheme(R.ideal(idl)).irreducible_components() if not V<=com.defining_ideal()])
            adm.append([new,idl,True])
        if not new in [ad[0] for ad in lis if ad[2]==True]:
            idl=R.ideal([0])
            gcd = 0
            biratmap = [proj[j]*(poly)^(dif(G.get_vertex(new),G.get_vertex('L0'))[j]) for j in range(variablecount+1)]
            for va in biratmap:
                gcd = va.gcd(gcd)
            biratmap = [va/gcd for va in biratmap]
            idl = R.ideal(biratmap).radical()
            idl = prod([com.defining_ideal() for com in A.subscheme(R.ideal(idl)).irreducible_components() if not V<=com.defining_ideal()])
            print('checking %s for smoothnes' %new)
            sm = smooth(R.ideal(idl)+R.ideal(q),V)
            print(sm)
            adm.append([new,idl,sm])
    return adm
```

```
def prod(li):
    p=R.ideal([1])
    for l in li:
        p=p*l
    return p
```

computing minimal difference

```
def dif(v,w):
    global variablecount
    dif = []
    minimum =  min(v[j]-w[j] for j in range(variablecount+1))
    for j in range(variablecount+1):
        dif.append(v[j]-w[j]-minimum)
    return dif
```

Defining the images

```
def im(v,w):
    global variablecount
    I = R.ideal([0])
    minimum =  min(G.get_vertex(w)[j]-G.get_vertex(v)[j] for j in range(variablecount+1))
    for j in range(variablecount):
```

```
        if G.get_vertex(w)[j+1]-minimum==G.get_vertex(v)[j+1]:
            I = I+R.ideal([var[j+3]])
    return R.ideal(I)
```

Defining the function finding smallest admissible center

```
def convexext(order):
    global n
    global k
    ext = []
    for v in G.vertices():
        set = [w for w in order]
        if not v in set:
            set.append(v)
            if testconvex(set)== True:
                ext.append(v)
    return ext
def find(center):
    global n
    global k
    global count
    for j in range(binomial(n,k)+n+count):
        for c in center:
            if not c[0] in  [d[0] for d in donecenter]:
                if c[2]==True:
                    lis = [d[0] for d in donecenter]
                    lis.append(c[0])
                    if testconvex(lis)==True:
                        if AffineSpace(R).subscheme(R.ideal(c[1])).dimension()==j:
                            return c
    return False
```

Checking for smoothness

```
def smooth(I,U):
    Rsing= R._singular_()
    Ising = singular.ideal(I)
    GI = Ising.groebner()
    codimI = Rsing.nvars() - GI.dim()
    slocsing = (GI.jacob().minor(codimI) + I).radical().groebner()
    sloc = slocsing.sage()
```

```
    if V<= sloc:
      return True
    return False
```

Defining the function deciding the containment of lattice in a set of vertices

```
def cont(v,list):
  for w in list:
    if same(v,Gr.get_vertex(w))==True:
      return True
  return False
```

Creating the graph

```
  G=graphs.EmptyGraph()
  for i in range(n):
    G.add_vertex('L%d' %i)
    lat=[]
    for j in Combinations(range(n),k).list():
      ent = 0
      for b in j:
        if b<i:
          ent = ent + 1
      lat.append(ent)
    G.set_vertex('L%d' %i, lat)

  for v in G.vertices():
    for w in G.vertices():
      if neighbour(G.get_vertex(v),G.get_vertex(w)):
        G.add_edge(v,w)
```

Setting up the starting situation

```
def index(li,i):
  ind=0
  for j in range(len(li)):
    ind=ind+10^j*(i-li[j])
  return ind
```

creating the graph Schubert varieties

```
def schubertintexrelation(i,j):
  bool = True
  for pos in range(len(i)):
```

```
        if j[pos]<i[pos]:
            bool = False
    return bool

    Gschub=graphs.EmptyGraph()
    for i in Combinations(range(n),k).list():
        Gschub.add_vertex('Lschu%d' %-index(i,0))
        lat=[]
        for j in Combinations(range(n),k).list():
            ent = 1
            if schubertintexrelation(j,i)==False:
                ent=0
            lat.append(ent)
        Gschub.set_vertex('Lschu%d' %-index(i,0), lat)
    for v in Gschub.vertices():
        for w in Gschub.vertices():
            if neighbour(Gschub.get_vertex(v),Gschub.get_vertex(w)):
                Gschub.add_edge(v,w)

def schubertvar():
    sch=[]
    vbase = []
    for j in range(variablecount +1):
        vbase.append(0)
    for ver in Gschub.vertices():
        if not Gschub.get_vertex(ver)[variablecount]==1:
            idl=R.ideal([0])
            gcd = 0
            biratmap = [proj[j]*(P)^(dif(Gschub.get_vertex(ver),vbase)[j]) for j in   range(variablecount+1)]
            for va in biratmap:
                gcd = va.gcd(gcd)
            biratmap = [va/gcd for va in biratmap]
            idl = R.ideal(biratmap)+R.ideal(q)
            if not idl==R.ideal([R(1)]):
                sch.append([ver,idl,smooth(idl+R.ideal(q),V)])
    return sch

def mapping():
    global proj
    adm = []
```

```
maps = []
poly = P
for j in range(count):
    poly= poly*R.gens()[j+variablecount+3]
for vert in G.vertices():
   idl=R.ideal([0])
   gcd = 0
   biratmap = [proj[j]*(poly)^(dif(G.get_vertex(vert),G.get_vertex('L0'))[j]) for j in   range(variablecount+1)]
   for va in biratmap:
      gcd = va.gcd(gcd)
   biratmap = [va/gcd for va in biratmap]
   idl = R.ideal(biratmap).radical()+R.ideal(q)
   idl = prod([com.defining_ideal() for com in A.subscheme(R.ideal(idl)).irreducible_components() if not V<=com.defining_ideal()])
   maps.append([vert,idl==R.ideal(R(1))])
return table(maps,header_row=['Projective space', 'mapping'], frame=True)

global n
global k
global Base
global var
global count
global admcenter
global blowvar
global donecenter
global donecenterschub
global proj
global R
global V
global q
global schub

count =0
var = ['P','p','x']
for j in range(k):
   for i in range(n-k):
      var.append('x%d%d' %(n-k-i,k-j))
   for col in Combinations(range(j+1)):
      if len(col)>1 and j in col:
```

```
         for rw in Combinations(range(n-k),len(col)):
            var.append('d%d%d' %(index(col,k),index(rw,n-k)))
variablecount = len(var)-3
blowvar = []

R = PolynomialRing(Base, var, order='deglex')
A = AffineSpace(R)

V = R.ideal([1])
R.inject_variables();

q = R.ideal([P-p])
M = matrix([[R('x%d%d'%(n-k-i,k-j)) for j in range(k)] for i in range(n-k)])
for col in Combinations(range(k)):
   if len(col)>1:
      for row in Combinations(range(n-k),len(col)):
         det = M.matrix_from_rows_and_columns(row, col).determinant()
         q=q+R.ideal([R('d%d%d' %(index(col,k),index(row,n-k)))-det])
donecenter=[['L0',P]]
donecenterschub=[]
proj =[R(1)]
for v in R.gens():
   if not v in [P,p,x]:
      proj.append(v)
admcenter=admissiblecenter([])
schub=schubertvar()
```

We will present the list of Schubert varieties and check their smoothness:

| Schub | smooth |
|---|---|
| Lschu10 | True |
| Lschu20 | True |
| Lschu21 | True |
| Lschu30 | True |
| Lschu31 | False |
| Lschu32 | True |
| Lschu40 | True |
| Lschu41 | False |
| Lschu42 | False |

In the blow-up with center `Lschu10` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu20 | True |
| Lschu21 | True |
| Lschu30 | True |
| Lschu31 | True |
| Lschu32 | True |
| Lschu40 | True |
| Lschu41 | True |
| Lschu42 | False |

In the blow-up with center `Lschu20` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu21 | True |
| Lschu30 | True |
| Lschu31 | True |
| Lschu32 | True |
| Lschu40 | True |
| Lschu41 | True |
| Lschu42 | False |

In the blow-up with center `Lschu21` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu30 | True |
| Lschu31 | True |
| Lschu32 | True |
| Lschu40 | True |
| Lschu41 | True |
| Lschu42 | True |

In the blow-up with center `Lschu30` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu31 | True |
| Lschu32 | True |
| Lschu40 | True |
| Lschu41 | True |
| Lschu42 | True |

In the blow-up with center `Lschu31` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu32 | True |
| Lschu40 | True |
| Lschu41 | True |
| Lschu42 | True |

In the blow-up with center `Lschu40` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu32 | True |
| Lschu41 | True |
| Lschu42 | True |

In the blow-up with center `Lschu32` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu41 | True |
| Lschu42 | True |

In the blow-up with center `Lschu41` we are left with the following transforms of Schubert varieties:

| Schub | smooth |
|---|---|
| Lschu42 | True |

In the blow-up with center `Lschu42` we are left with the following transforms of Schubert varieties:

[`All blow-ups are done`]

We will end by checking the existence of maps into $\mathbb{P}(\Lambda)$ for $[\Lambda] \in \Gamma^{\mathrm{st}}$ compatible with the maps on the generic fibers:

| Projective space | mapping |
|---|---|
| L0 | True |
| L1 | True |
| L2 | False |
| L3 | False |
| L4 | True |

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