On the Bloch-Kato conjecture for Hilbert modular forms

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Abstract

In this thesis we are concerned with arithmetic properties of Hilbert modular forms and Hilbert modular varieties. In the first part we prove inequalities towards instances of the Bloch-Kato conjecture for Hilbert modular forms of parallel weight two, when the order of vanishing of the L-function at the central point is zero or one. The proof relies on explicit reciprocity laws for cohomology classes constructed using congruences of Hilbert modular forms and special points on Shimura curves. The aim of the second part is to study analogues of the plectic conjectures formulated by Nekovář and Scholl. We first investigate a function field counterpart, involving moduli spaces of Shtukas for groups arising via restriction of scalars along a cover of curves; finally we obtain results towards a p-adic version of the conjectures for Hilbert modular surfaces, exploiting Scholze's theory of diamonds and mixed characteristic Shtukas.

Zusammenfassung

In dieser Arbeit befassen wir uns mit arithmetischen Eigenschaften von Hilbertschen Modulformen und Modulvarietäten. Im ersten Teil beweisen wir Ungleichungen in Richtung der Bloch-Kato Vermutung für Hilbertsche Modulformen von parallelem Gewicht Zwei, im Fall, dass die L-Funktion bei dem zentralen Punkt Nullstellenordnung Null oder Eins besitzt. Der Beweis beruht auf expliziten Reziprozitätsgesetzen für Kohomologieklassen, welche mittels Kongruenzen von Hilbertschen Modulformen und speziellen Punkten auf Shimura-Kurven konstruiert werden. Das Ziel des zweiten Teils ist das Studieren von Analoga der plektischen Vermutung von Nekovář und Scholl. Wir untersuchen zunächst das Pendant für Funktionenkörper, was Modulräume von Shtukas für solche Gruppen involviert, die als Skalarrestriktion entlang einer Überdeckung von Kurven auftreten; schließlich erhalten wir Resultate in Richtung einer p-adischen Version der Vermutung für Hilbertsche Flächen, wobei wir Scholzes Theorie von Diamanten und Shtukas gemischter Charakteristik ausnutzen.

Contents

Introduction

0.1. Let F be a number field with ring of integers $\mathcal{O}_F$. It is well known that the group of units $\mathcal{O}_F^\times$ is finitely generated, and its rank (i.e. the rank of of its free part) equals $r_1 + r_2 - 1$, where r_1 (resp. r_2) is the number of real embeddings of F (resp. half the number of complex embeddings of F).

On the other hand, to the number field F one can attach an invariant of a more analytic nature, its *Dedekind zeta function*, defined as

$$\zeta_F(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$$

where the sum ranges over all non zero ideals of $\mathcal{O}_F$ and $N(\mathfrak{a})$ is the cardinality of $\mathcal{O}_F/\mathfrak{a}$. The above series defines a holomorphic function on the half plane $Re\ s > 1$, which admits a meromorphic continuation to the whole complex plane. Examining the behaviour of $\zeta_F(s)$ at $s = 0$ one discovers the following remarkable facts:

(1) the order of vanishing at $s = 0$ of $\zeta_F(s)$ equals:

$$ord_{s=0}\zeta_F(s) = r_1 + r_2 - 1 = rk(\mathcal{O}_F^\times).$$

(2) The leading term in the Taylor expansion of $\zeta_F(s)$ at $s = 0$ is given by:

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_F(s) = -\frac{h_F R_F}{w_F}$$

where h_F is the cardinality of the class group of F, R_F is the regulator of F, and w_F is the number of roots of unity in F.

There are several proofs of these classical results. The original one relies on ideas from geometry of numbers; alternatively, as proposed in Tate's thesis, one can make use of Fourier analysis on adeles to study $\zeta_F(s)$ and deduce the above properties. An analogous statement over function fields can also be approached using the cohomological interpretation of zeta functions of smooth projective curves over finite fields.

There is however a more direct approach available when $F/\mathbf{Q}$ is abelian. Let us take for example $F = \mathbf{Q}(\sqrt{p})$ where $p \equiv 1 \pmod 4$ is a prime. Then F can be embedded in $\mathbf{Q}(\zeta_p)$, which contains *explicit* units of the form $\frac{1-\zeta_p^k}{1-\zeta_p}$, where k is an integer coprime with p. These units, called *cyclotomic units*, can be used to produce an explicit element in $\mathcal{O}_F^\times$ that can be related to $\zeta_F'(0)$. One then has to show that this element generates a finite index subgroup of $\mathcal{O}_F^\times$ and, in order to prove (2), express this index in terms of arithmetic invariants of F (precisely, its class number). These two tasks are of a *purely algebraic* nature, hence it is natural to look for an algebraic approach to them. The theory of *Euler systems* was developed by Kolyvagin and Rubin for this purpose.

0.2. It is expected that the previous story is an instance of a general phenomenon: the zeta function of any smooth projective variety over a global field F should encode deep arithmetic invariants of the variety. There are several conjectures making this precise; the results in this document are motivated by those proposed by Bloch and Kato [**BK90**] (later reformulated by Fontaine and Perrin-Riou [**FPR94**]). They relate the order of vanishing of the L-function of a *motive* M (in an old-fashioned sense) over F at a suitable point to the rank of a *Selmer group* attached to the motive. Furthermore they express the leading term in the Taylor expansion of the L-function of M at the point in terms of arithmetic invariants of M. We refer the reader to the original sources for the precise statement of the conjectures, and only point out that in the case of the motive attached to an elliptic curve those are closely related to the Birch and Swinnerton-Dyer conjecture.

0.3. Geometric proofs of instances of the above conjectures can be given over function fields, relying on the expression of L-functions of varieties over *finite fields* via étale cohomology. This allows to deduce almost for free one inequality in the conjectural equalities; the other inequality is instead much harder, requiring the Riemann hypothesis for the relevant L-function and the Tate conjecture. Several attempts have been made to carry over this cohomological approach to number fields, as well as to generalize Tate's thesis to higher dimensions; we would like to mention here Hesselholt's work [**Hes18**] and Parshin's survey [**Par10**].

However for the time being only the explicit approach described above for the Dedekind zeta function of quadratic number fields has been successfully generalized to a limited number of motives over number fields. Such a generalization relies first of all on instances of the global Langlands correspondence, roughly speaking predicting that every *motivic* L-function should be the L-function of an *automorphic* representation of a suitable reductive algebraic group G. Conversely, to an automorphic representation satisfying a suitable algebraicity assumption one should be able to attach a motive having the same L-function. In favourable situations this motive can be constructed from explicit varieties attached to the algebraic group G, known as *Shimura varieties*. These are locally symmetric spaces with a canonical model over a suitable number field (called the *reflex field*), and are endowed with distinguished classes of *special cycles* defined over explicit number fields. In a similar way as in the case of cyclotomic units, one exploits these cycles to produce a distinguished element in the Selmer group of the motive having a precise connection with its L-function, and then relates the index of this element with arithmetic invariants of the motive.

For $G = GL_{2,\mathbf{Q}}$ the two directions in the Langlands correspondence are given (partially) by the Eichler-Shimura correspondence and the modularity theorem of Taylor-Wiles [**Wil95**]. The relevant Shimura varieties are the familiar modular curves, and the relation between special points on them (CM points) and L-functions is expressed by the celebrated Gross-Zagier formula [**GZ86**]. Joint with Kolyvagin's work [**Kol90**], this allows to prove instances of the BSD conjecture for elliptic curves over $\mathbf{Q}$ in analytic rank 0 and 1.

0.4. In this document we are concerned with the Bloch-Kato conjecture for the base change of the motive attached to a Hilbert newform f of parallel weight 2 to a CM extension K of the base field F, a totally real field. In other words the relevant algebraic group is $GL_{2,F}$, or more generally the group of units of a quaternion algebra B over F. Taking B ramified at every (resp. at all but one) infinite place one finds that the associated locally symmetric space is a finite set (resp. a one dimensional Shimura variety). According to the sign of the functional equation of the L-function of f over K this space contains special points related to the central value (resp. the first derivative) of the L-function.

It is often the case that the Galois representation attached to f can be realized in the Tate module of the Jacobian of a suitable Shimura curve. Precisely, recall that if f is an eigenform with trivial central character then the ring generated by its Hecke eigenvalues is an order in a totally real number field. Its completion at a prime $\mathfrak{p}$ where it is maximal is a DVR denoted by $\mathcal{O}_{\mathfrak{p}}$; we denote by ϖ a uniformizer. Let $T(f)$ be an $\mathcal{O}_{\mathfrak{p}}$-lattice in the self-dual $\mathfrak{p}$-adic Galois representation $V(f)$ attached to f and $A(f) = V(f)/T(f)$. Then in favourable circumstances $T(f)$ is a quotient of the Tate module of the Jacobian of a Shimura curve, and a special point with CM by the ring of integers of K can be used to construct a cohomology class $c_K \in Sel(K, T(f))$. We can now state a special instance of our first main result (conditional to Ihara's lemma for Shimura curves; see Remark 1.1, Chapter II):

0.5. THEOREM. *(cf. Chapter II, Theorem 1.2) Let f be a newform of parallel weight 2, level $\mathfrak{n}$ and with trivial central character. Assume that*

(1) The level $\mathfrak{n}$ of f, the discriminant $disc(K/F)$ and the prime p below $\mathfrak{p}$ are coprime to each other. Moreover $p > 3$ is unramified in F, and $\mathfrak{n}$ is squarefree and all its factors are inert in K.

(2) The image of the residual Galois representation $\bar{\rho}$ attached to f contains $SL_2(\mathbf{F}_p)$.

(3) For every prime $\mathfrak{q}|\mathfrak{n}$ we have $N(\mathfrak{q}) \not\equiv -1 \pmod p$. Moreover if $N(\mathfrak{q}) \equiv 1 \pmod p$ then $\bar{\rho}$ is ramified at $\mathfrak{q}$.

Then the following statements hold true.

Definite case: *If $|\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}\}| \equiv [F : \mathbf{Q}] \pmod 2$ and $L(f_K, 1) \neq 0$ then $Sel(K, A(f))$ is finite. Denoting by $L^{alg}(f_K, 1)$ the algebraic part of the special value of $L(f_K, 1)$, the following inequality holds:*

$$length_{\mathcal{O}_{\mathfrak{p}}} Sel(K, A(f)) \leq ord_{\varpi}(L^{alg}(f_K, 1)).$$

Indefinite case: *If $|\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}\}| \not\equiv [F : \mathbf{Q}] \pmod 2$ and $L'(f_K, 1) \neq 0$ then $Sel(K, A(f))$ has $\mathcal{O}_{\mathfrak{p}}$-corank one. Denoting by $Sel(K, A(f))/div$ the quotient by its divisible part, the following inequality holds:*

$$length_{\mathcal{O}_{\mathfrak{p}}} Sel(K, A(f))/div \leq 2ord_{\varpi}(c_K).$$

We also give a criterion under which we can say that the above inequalities are equalities: what we need to know is a GL_2-version of Ribet's seminal result [**Rib76**] (see remark 3.13, Chapter II).

0.6. The proof of the above theorem relies on a "level raising-length lowering" method based on explicit reciprocity laws first introduced in [**BD05**]. More precisely, one first produces a cohomology class $c(\mathfrak{l}) \in H^1(K, A(f)[\varpi])$ whose localization at *all but one* place $\mathfrak{l}$ lies in the finite part of $H^1(K_{\mathfrak{l}}, A(f)[\varpi])$, and whose localization at $\mathfrak{l}$ is related to $L^{alg}(f_K, 1)$ (via the so-called *first reciprocity law*). If $L^{alg}(f_K, 1)$ is a unit in $\mathcal{O}_{\mathfrak{p}}$ one then shows via global duality that the existence of such a class for sufficiently many $\mathfrak{l}$ forces the vanishing of the Selmer group. If $L^{alg}(f_K, 1)$ is non zero but not a unit, one constructs a level raising g of f at two distinct primes such that $ord_{\varpi} L^{alg}(g_K, 1) < ord_{\varpi} L^{alg}(f_K, 1)$, and proves the desired inequality by induction. This step requires the use of a *second reciprocity law* describing the localization of $c(\mathfrak{l})$ at a suitable prime different from $\mathfrak{l}$ in terms of a level raising of f at two primes. This second reciprocity law is also the key ingredient to prove the result in the indefinite case, essentially allowing to reduce the statement to the definite one. It relies on the description of the supersingular locus in a fiber of good reduction of a Shimura curve in terms of a suitable quaternionic set.

0.7. Of course, several results in the spirit of theorem 0.5 have already been established: the implication "$L(f_K,1) \neq 0 \Rightarrow Sel(K,A(f))$ is finite" was proved in much greater generality in [**Nek12**], again relying on the first reciprocity law. The main point in our theorem is that we are able to prove inequalities towards Bloch-Kato's special value formulas, and to provide a criterion for equality. Such inequalities are usually deduced as a consequence of one divisibility in the relevant Iwasawa main conjecture; conversely, proving (in)equalities in the style of theorem 0.5 for the Selmer group of the twist of $A(f)$ by sufficiently many anticyclotomic Hecke characters allows to deduce (one inequality in) the Iwasawa main conjecture. This is the approach taken in the preprint [**BLV19**], where the authors establish the anticyclotomic Iwasawa main conjecture for elliptic curves over $\mathbf{Q}$ for both ordinary and supersingular primes. Our work was inspired by the observation that the heart of the argument in [**BLV19**] does not rely on Iwasawa theory, and can be adapted to prove directly the inequalities in theorem 0.5. The advantage is that we can treat both ordinary and supersingular primes at the same time, whereas an Iwasawa-theoretic approach requires different arguments in the two cases. In particular, in the supersingular case our result is new (since Iwasawa-theoretic methods are not available in this setting for Hilbert modular forms for the time being). Notice however that we are only able to prove *one inequality* towards the conjectured Bloch-Kato special value formulas, whereas in [**BLV19**] the authors are able to establish the full Iwasawa main conjecture. This is due to the fact that, when K is an imaginary quadratic field, the converse theorem *à la* Ribet which we need to prove that our inequalities are actually equalities can be deduced from the analogous theorem over $\mathbf{Q}$, which in turn is a consequence of the work of Skinner-Urban and Wan ([**SU14**], [**Wan14**]) and Kato ([**Kat04**]). It is an interesting question whether such a converse theorem can also be proved without Iwasawa theory, via a more direct generalization of Ribet's original work (cf. Chapter II, Remark 3.13).

Some ideas which we use in the proof of Theorem 0.5 were already present in more or less disguised form in the literature. Besides the above mentioned preprint [**BLV19**], we wish to mention Zhang's paper [**Zha14**], from which we originally learned the "level raising-length lowering" method (which Zhang borrowed from [**GP12**]). Part of this process was formalized in [**How06**]; finally, instances of our result in rank one can also be deduced from Kolyvagin's work (or its totally real version [**Nek07**]). It is worth remarking that our approach is quite different from Kolyvagin's one: in particular we proceed from rank 0 to rank 1, whereas Kolyvagin does the opposite.

0.8. In order to study higher rank cases of the Bloch-Kato conjecture, it is clear that a special value formula relating higher derivatives of L-functions to suitable special cycles is the crucial missing ingredient. Such a formula has been established over function fields in [**YZ17**], [**YZ18**]. The relevant geometric objects are moduli spaces of Shtukas with *several legs* (as many as the order of the derivative of interest). It is not at all clear how to define their number field analogue for the time being, due to the lack of a meaningful object "$\mathbf{Z} \times \mathbf{Z}$" - interestingly, the lack of a suitable "base field" is the same issue which prevents a cohomological study of L-functions over number fields from working. However it has been observed by Nekovář [**Nek09**] that Shimura varieties attached to groups arising via restriction of scalars seem to possess extra symmetry suggesting that they are "shadows" of such non-existing spaces. At a purely cohomological level, this speculation implies that étale cohomology of these Shimura varieties should be endowed with a canonical action of a *plectic Galois group* which is in general larger than the absolute Galois group of the reflex field. Precise conjectures in this direction are formulated in [**NS16**], where potential arithmetic applications are also discussed. We verify in chapter 3 that an analogous phenomenon happens over function fields: moduli spaces of Shtukas with one leg for groups arising as restriction of scalars are pullbacks of moduli spaces of Shtukas with several legs (see Proposition 3.12, Chapter III).

Finally, we examine the mixed characteristic situation, where products over a "deeper base" can be made sense of in the world of diamonds introduced by Scholze. For example, let $F/\mathbf{Q}$ be a real quadratic extension and H_N the Hilbert modular surface with full level N structure (for an integer $N \geq 3$) parametrizing abelian surfaces with $\mathcal{O}_F$-action. For a prime p of good reduction which splits in F, let H_N^{good} be the good reduction locus inside the analytification of the base change of H_N to $\mathbf{Q}_p$. Then we construct a sheaf H_N^{plec} (which we hope to be a diamond, though we still cannot prove it) fibred over $\mathbf{Q}_p^\diamond \times \mathbf{Q}_p^\diamond$ and fitting into a diagram, cartesian on geometric points

$$\begin{tikzcd} H_N^{good,\diamond} \arrow[r] \arrow[d] & H_N^{plec} \arrow[d] \\ \mathbf{Q}_p^\diamond \arrow[r, "\Delta"] & (\mathbf{Q}_p^\diamond)^2 \end{tikzcd}$$

where Δ is the diagonal map (cf. Theorem 4.3, Chapter III). The existence of such a diagram can be regarded as a geometric, p-adic version of the conjectures in [**NS16**]. It is worth pointing out that in the archimedean situation, examined by Nekovář and Scholl in ongoing work undertaken in [**NS17**], there is no way to make sense of the above diagram; however the authors of *loc. cit.* remarkably manage to prove an archimedean version of their conjectures and to obtain concrete information on special values of higher derivatives of certain L-functions. In our p-adic setting, as we explain in chapter 3, we are interested in investigating whether a form of the second reciprocity law holds for suitable special cycles on H_N^{plec}.

0.9. In order to construct the above diagram, the function field picture suggests that one should first of all try to interpret $H_N^{good,\diamond}$ as a suitable "moduli space of Shtukas". However our object is *semi-global*, and not purely local, hence Scholze's local shtukas are not enough for our purposes. Drawing inspiration from [**CS17**] we describe $H_N^{good,\diamond}$ in terms of (families of) moduli spaces of local Shtukas with $\mathcal{O}_F$-action and Igusa varieties. We then construct H_N^{plec} by gluing a moduli space of *couples of* local Shtukas *with possibly different legs* and a moduli space in positive characteristic which resembles a family of Igusa varieties. In fact we hope that ours is an easy example of a class of "p-adic Shimura varieties". Those should exist in greater generality than their archimedean counterparts, due to the fact that Shtukas with several legs exist in the p-adic world, and may have interesting arithmetic applications.

Notations and conventions

We collect here some notation which is frequently used in the text. In few occasions we will use some of the following symbols with a different meaning from the one indicated below. We always make it explicit in the body of the text when this is the case.

- The cardinality of a set A is denoted by $|A|$.
- The ring of adeles of a global field E is denoted by $\mathbf{A}_E$. We let $\mathbf{A}_f$ denote the ring of finite adeles of $\mathbf{Q}$, while $\mathbf{A}_f^p$ stands for finite adeles outside a prime p.
- If A is an abelian group, we set $\hat{A} = A \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$ (hence $\mathbf{A}_f = \hat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q} = \hat{\mathbf{Q}}$).
- The absolute Galois group of a field L is denoted by Γ_L.
- F denotes most of the time a totally real field of degree r and with ring of integers $\mathcal{O}_F$; $F^+ \subset F$ denotes the subset of totally positive elements and $\mathcal{O}_F^+ = F^+ \cap \mathcal{O}_F$. The set of infinite places of F is denoted by Σ_∞.
- For a place v of F we denote by F_v the completion of F at v. For v finite, we denote by $\mathcal{O}_v$ the ring of integers of F_v, by ϖ_v a uniformizer of O_v and by $N(v)$ the cardinality of the residue field $\mathcal{O}_v/\varpi_v$.
- K is a totally imaginary quadratic extension of F with ring of integers $\mathcal{O}_K$. If $\mathfrak{c} \subset \mathcal{O}_F$ is an ideal, then $\mathcal{O}_\mathfrak{c} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K \subset \mathcal{O}_K$ denotes the $\mathcal{O}_F$-order of conductor $\mathfrak{c}$.
- B is a quaternion algebra over F of discriminant $\mathfrak{D}$; $R \subset B$ is an Eichler order. The symbol R will also often denote a ring; we did our best to avoid that this conflict of notation generates confusion.
- $\mathbf{H}$ is the $\mathbf{R}$-algebra of Hamilton quaternions and $\mathbf{H}^1$ denotes Hamilton quaternions of reduced norm 1.
- $\mathbf{S} = Res_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{m,\mathbf{C}}$ is the Deligne torus.
- $k = \mathbf{F}_q$ is the finite field with q elements, where q is a power of a prime p. If S is a scheme over k, we denote by $F_S : S \to S$ the absolute Frobenius morphism $x \to x^q$.
- C is a smooth, projective, geometrically irreducible curve over k with function field E.
- Perf is the category of perfectoid spaces in characteristic p.

Conventions. Fix a rational prime p, an embedding $\iota_\infty : \bar{\mathbf{Q}} \to \mathbf{C}$, an embedding $\iota_p : \bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_p$ and an isomorphism $\bar{\mathbf{Q}}_p \overset{\sim}{\leftrightarrow} \mathbf{C}$ compatible with the two embeddings. In particular, for any number field $L \subset \bar{\mathbf{Q}}$, the above embeddings determine a distinguished infinite (resp. p-adic) place.

The symbol Fr denotes *geometric* Frobenius, unless stated otherwise. Accordingly, the Artin map of global class field theory is normalised so that uniformizers correspond to geometric Frobenius elements.

Numeration of subsections, lemmas, propositions, theorems, remarks, examples follows the simplest logic: subsequent objects in the same section are marked by increasing numbers. Whenever quoting an item from a different chapter we add a roman number to specify to which chapter we are referring. For example, the last paragraph of this document is III.4.17.

CHAPTER 1

Shimura varieties and Shtukas

1. Outline of the chapter

In this chapter, mainly of an expository nature, we introduce the main objects which we are going to use. Over number fields we will be primarily concerned with quaternionic Shimura varieties and automorphic forms, their L-functions and Galois representations. We also state Zhang's special value formulas for the central value and first derivative of L functions of Hilbert modular forms, which will be crucial for us in chapter 2. In the function field setting we will instead work with the moduli spaces of Shtukas, which exist in bigger generality than their number field counterparts. In particular one can define moduli spaces of Shtukas with several legs, and special cycles in some of these spaces are related with higher derivatives of L-functions, as proved in [**YZ17**], [**YZ18**]. We state the main result in *loc. cit.*, which was the starting point of our investigations in chapter 3. Finally, we recall the definition of local Shtukas in mixed characteristic, introduced in [**SW17**], and explain their relation with p-divisible groups, which will be used in chapter 3.

With the exception of a couple of statements in section 8 (which are mild generalizations of results in [**SW17**]) all the material in this chapter is already known, hence we only provide references or sketches of proofs of few results.

2. Quaternionic Shimura varieties

2.1. Let $F/\mathbf{Q}$ be a totally real field, $r = [F : \mathbf{Q}]$ and B a quaternion algebra over F. Let us denote by $\Sigma_\infty = \{\sigma_1, \dots, \sigma_r\}$ the set of real embeddings of F, and write $\Sigma_\infty = \Sigma_s \sqcup \Sigma_r$, where Σ_s (resp. Σ_r) contains all infinite places of F where B is split (resp. ramified). Let us suppose that $\Sigma_r \neq \Sigma_\infty$ and let $d = |\Sigma_s|$. Let us denote by G the algebraic group $Res_{F/\mathbf{Q}} B^\times$ and by Z its center. Then we have

$$G_{\mathbf{R}} = \prod_{\sigma \in \Sigma_s} GL_{2,\mathbf{R}} \times \prod_{\sigma \in \Sigma_r} \mathbf{H}^\times.$$

Let

$$\begin{aligned} h : \mathbf{S} \longrightarrow & G_{\mathbf{R}} \\ a + ib \mapsto & \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}_{\sigma \in \Sigma_s}, (1)_{\sigma \in \Sigma_r} \right). \end{aligned}$$

Then the $G(\mathbf{R})-$conjugacy class of h, denoted by X, is naturally identified with

$$G(\mathbf{R})/(F \otimes_{\mathbf{Q}} \mathbf{R})^\times U_\infty \simeq (\mathbf{C} \setminus \mathbf{R})^d$$

where $U_\infty = \prod_{\sigma \in \Sigma_s} SO_2(\mathbf{R}) \times \prod_{\sigma \in \Sigma_r} \mathbf{H}^1$ is a maximal compact connected subgroup of $G(\mathbf{R})$. The couple (G, X) is a Shimura datum, with reflex field $E = \mathbf{Q}(\sum_{\sigma \in \Sigma_s} \sigma(a), a \in F) \subset \bar{\mathbf{Q}}$. It follows from the general theory of Shimura varieties that, for every compact open subgroup

$U \subset G(\mathbf{A}_f)$, the space

$$G(\mathbf{Q}) \setminus X \times G(\mathbf{A}_f)/U \tag{2.1.1}$$

which is a finite disjoint union of quotients of $(\mathbf{C} \setminus \mathbf{R})^d$ by arithmetic subgroups of $GL^d_{2,\mathbf{R}}$ hence is a complex analytic space of dimension d, actually has a *canonical* structure of a quasi-projective algebraic variety over the reflex field E, denoted by $Sh(G,X)_U$. The transition maps $Sh(G,X)_{U'}(\mathbf{C}) \mapsto Sh(G,X)_U(\mathbf{C})$ for $U' \subset U$ are defined over E.

2.2. Remark. The varieties $Sh(G,X)_U$ are proper if and only if $B \neq M_{2,F}$.

2.3. Remark. Let us denote a point in $Sh(G,X)_U(\mathbf{C}) = G(\mathbf{Q}) \setminus X \times G(\mathbf{A}_f)/U$ by $[z,a]$. For $g \in G(\mathbf{A}_f)$ the map $[z,a] \mapsto [z,ag]$ induces a morphism $Sh(G,X)_U \to Sh(G,X)_{g^{-1}Ug}$ which is defined over the reflex field. In particular we obtain an action of $G(\mathbf{A}_f)$ of the tower $\{Sh(G,X)_U\}_{U \subset G(\mathbf{A}_f)}$. As a special case, the group $Z(\mathbf{A}_f) = \hat{F}^\times$ acts on each $Sh(G,X)_U$ via morphisms defined over E, given on complex points by $g \cdot [x,a] = [x,ag]$. This action factors through the finite group $\hat{F}^\times / \hat{F}^\times \cap U$, hence the quotient $Sh(G,X)_U/Z(\mathbf{A}_f)$ exists as a quasi-projective scheme. It is the Shimura variety attached to the Shimura datum $(G/Z,X)$ and its complex points are given by $Sh(G/Z,X)_U(\mathbf{C}) = G(\mathbf{Q}) \setminus X \times G(\mathbf{A}_f)/\hat{F}^\times U$.

2.4. Example.
1. If $F = \mathbf{Q}$ and $B = M_{2,\mathbf{Q}}$ then one obtains the classical modular curves.
2. If $\Sigma_s = \Sigma_\infty$, i.e. B is *totally indefinite*, then $E = \mathbf{Q}$. In this case $Sh(G,X)_U$ is a d-dimensional algebraic variety with a canonical model over $\mathbf{Q}$. If $B = M_{2,F}$ then this can be described as a coarse moduli space of d-dimensional abelian schemes with real multiplication by $\mathcal{O}_F$ and U-level structure; the resulting varieties are called *Hilbert modular varieties* (see section 3). For $B \neq M_{2,F}$ totally indefinite one can still construct the canonical model as a coarse moduli space of abelian schemes with $\mathcal{O}_B$-action (see [**LT17**, Section 2.5]).
3. If Σ_s contains only one infinite place then $E = F$ and $Sh(G,X)_U$ is a curve, called *Shimura curve*, with a canonical model over F. In this case $Sh(G,X)_U$ is *not* a moduli space of abelian schemes with extra-structure; however its base-change to a suitable CM-extension F'/F is related to a unitary Shimura variety of PEL-type, hence having such a moduli description (this is explained for general quaternionic Shimura varieties in [**Nek18**, Appendix A.3] and [**TX16a**, Section 3]). By remark 2.2 the Shimura curves $Sh(G,X)_U$ are always compact if $F \neq \mathbf{Q}$. They are furthermore smooth over F for U small enough. The same holds true for the quotient Shimura curves $Sh(G/Z,X)_U$.
4. Compact open subgroups $U \subset G(\mathbf{A}_f)$ can be obtained concretely as follows: let $R \subset B$ be an Eichler order. Then $\hat{R}^\times \subset G(\mathbf{A}_f)$ is a compact open subgroup. We will often work with Shimura varieties arising from such subgroups.

2.5. CM-points. Let K/F be a totally imaginary extension, $T = Res_{K/\mathbf{Q}}\mathbf{G}_m$ and

$$\begin{aligned} h : \mathbf{S} \to & T_{\mathbf{R}} \simeq \mathbf{G}^r_{m,\mathbf{C}} \\ a+bi \mapsto & (a+bi)^r. \end{aligned}$$

The $T(\mathbf{R})$ conjugacy class of h is reduced to a point, and $(T,\{*\})$ is a Shimura datum. For $V \subset T(\mathbf{A}_f) = \hat{K}^\times$ compact open the associated Shimura variety is just the finite set

$$Sh(T,\{*\})_V = K^\times \setminus \hat{K}^\times / V \xrightarrow{Art} Gal(K_V/K)$$

where K_V is the abelian extension of K corresponding via global class field theory to the compact open subgroup V of $\mathbf{A}_f \otimes_{\mathbf{Q}} K$ and Art is the Artin map. For example, for $V = \hat{\mathcal{O}}_K^\times$ we have $K^\times \setminus \hat{K}^\times / V \simeq Pic(\mathcal{O}_K)$ and K_V/K is the Hilbert class field of K (the maximal unramified abelian extension of K). An embedding $\iota : K \to B$ (whenever it exists) induces a morphism of Shimura data $(T, \{*\}) \to (G, X)$ sending $*$ to the unique point $z_0 \in X = (\mathbf{C} \setminus \mathbf{R})^d$ whose coordinates belong to the upper half plane and which is fixed by the action of $K^\times$ induced by $K^\times \hookrightarrow B^\times \to GL_2(\mathbf{R})^d$ and by the natural action of $GL_2(\mathbf{R})^d$ on X. Taking $V = U \cap T(\mathbf{A}_f)$ we obtain an injection

$$\begin{aligned} Sh(T, \{*\})_V = K^\times \setminus \hat{K}^\times / V \hookrightarrow & Sh(G, X)_U(\mathbf{C}) \\ [a] \mapsto & [z_0, a]. \end{aligned}$$

The points in the image of the above map are called CM points (more precisely, points with CM by K). While a priori they are complex points, the key result in the theory of Shimura varieties states that they are actually algebraic points of $Sh(G, X)_U$, defined over the abelian extension K_V of K, and the action of $g \in Gal(K_V/K)$ on $[z_0, a]$ is given by $g \cdot [z_0, a] = [z_0, Art^{-1}(g)a]$. This property actually characterizes uniquely the canonical model.

The previous discussion depends on the choice of an embedding $\iota : K \to B$. Any two such embeddings are conjugate by an element of $B^\times$. Furthermore the $G(\mathbf{A}_f)$-action on the tower $\{Sh(G, X)_U\}_{U \subset G(\mathbf{A}_f)}$ described in remark 2.3 preserves CM-points. As a consequence the set of all points with CM by K on $Sh(G, X)_U$ is given by

$$CM(Sh(G, X)_U, K) = \{[b \cdot z_0, g], b \in B^\times, g \in \hat{B}^\times\} \simeq K^\times \setminus \hat{B}^\times / U. \tag{2.5.1}$$

We have $CM(Sh(G, X)_U, K) \subset Sh(G, X)_U(K^{ab})$ and the Galois action on $CM(Sh(G, X)_U, K)$ is induced by the composition of the inverse of the Artin map $K^\times \setminus \hat{K}^\times \xrightarrow{\sim} Gal(K^{ab}/K)$ and the map $K^\times \setminus \hat{K}^\times \to K^\times \setminus \hat{B}^\times / U$.

2.6. Definition. Let $\mathfrak{c} \subset \mathcal{O}_F$ be an ideal and $P \in CM(Sh(G, X)_U, K)$ a point with CM by K. We say that P is a CM-point *of conductor* $\mathfrak{c}$ if the action of $Gal(K^{ab}/K)$ on P factors through $K^\times \setminus \hat{K}^\times / \hat{\mathcal{O}}_{\mathfrak{c}}^\times \simeq Pic(\mathcal{O}_{\mathfrak{c}})$, and $\mathfrak{c}$ is maximal among the ideals with this property.

2.7. Remark. The above discussion carries over to the Shimura varieties $Sh(G/Z, X)_U$, replacing T by T/Z. In terms of Galois action, this amounts to killing the action of $Pic(\mathcal{O}_F)$. For example, for $V = \hat{\mathcal{O}}_K^\times$, we have

$$Sh(T/Z, X)_V = K^\times \setminus \hat{K}^\times / \hat{F}^\times \hat{\mathcal{O}}_K^\times \simeq (K^\times \setminus \hat{K}^\times / \hat{\mathcal{O}}_K^\times)/(F^\times \setminus \hat{F}^\times / \hat{\mathcal{O}}_F^\times) = Pic(\mathcal{O}_K)/i_* Pic(\mathcal{O}_F);$$

in general, the Galois action on a CM point of conductor $\mathfrak{c}$ factors through $Pic(\mathcal{O}_{\mathfrak{c}})/i_* Pic(\mathcal{O}_F)$, where $i_* : Pic(\mathcal{O}_F) \to Pic(\mathcal{O}_{\mathfrak{c}})$ denotes the map induced by the inclusion $i : \mathcal{O}_F \to \mathcal{O}_{\mathfrak{c}}$.

2.8. Quaternionic sets. Let B/F be a *totally definite* quaternion algebra, i.e. let us suppose, unlike in the previous section, that $\Sigma_r = \Sigma_\infty$. In this setting we do not strictly speaking obtain a Shimura datum. However, for $U \subset G(\mathbf{A}_f)$ compact open, we can still define, in analogy with 2.1.1, the space

$$Sh(G, X)_U = G(\mathbf{Q}) \setminus X \times G(\mathbf{A}_f)/U$$

where $X = G(\mathbf{R})/(F \otimes_{\mathbf{Q}} \mathbf{R})^\times U_\infty$, with $(\mathbf{H}^1)^r = U_\infty \subset G(\mathbf{R})$ being a maximal compact subgroup. Since $\mathbf{H}^\times = \mathbf{H}^1 \mathbf{R}^\times$, we see that X reduces to a point. In other words, the "archimedean part" of the space $Sh(G, X)_U$ is just a point. Consequently $Sh(G, X)_U$ is the finite set

$$Sh(G, X)_U = B^\times \setminus \hat{B}^\times / U.$$

As before the tower $\{Sh(G,X)_U\}_{U\subset G(\mathbf{A}_f)}$ comes equipped with a $G(\mathbf{A}_f)$-action.

2.8.1. *CM-points on quaternionic sets.* As in the previous section, an embedding $K \hookrightarrow B$ induces an injection $K^\times \setminus \hat{K}^\times/V \hookrightarrow B^\times \setminus \hat{B}^\times/U$. where $V = U \cap \hat{K}^\times$. The set of all points with CM by K is defined, in analogy with 2.5.1, as $CM(Sh(G,X)_U, K) = K^\times \setminus \hat{B}^\times/U$ with Galois action induced by the composition of the inverse of the Artin map $K^\times \setminus \hat{K}^\times \xrightarrow{\sim} Gal(K^{ab}/K)$ and the map $K^\times \setminus \hat{K}^\times \to K^\times \setminus \hat{B}^\times/U$. The definition 2.6 of CM point with conductor $\mathfrak{c} \subset \mathcal{O}_F$ then carries over to this context. Finally, replacing G by G/Z we obtain the quotient quaternionic set $B^\times \setminus \hat{B}^\times/\hat{F}^\times U$, and remark 2.7 holds for this object, too.

3. Hilbert modular varieties: moduli interpretation and integral model

3.1. Let $B = M_{2,F}$; the Shimura varieties $Sh(G,X)_U$ are known as *Hilbert modular varieties*, and have a moduli interpretation which allows to construct integral models of $Sh(G,X)_U$ over $\mathbf{Z}_p$. We will need all this in chapter 3, hence we now recall the relevant facts, following [**TX16b**, Section 2]. Let us suppose that p is unramified in F and U is of the form U_pU^p, with $U^p \subset G(\mathbf{A}_f^p)$ and $U_p \subset G(\mathbf{Q}_p)$ hyperspecial, i.e. $U_p \simeq GL_2(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)$.

Fix a set I of prime-to-p fractional ideals in F which are representatives of the narrow class group of F. For each $\mathfrak{c} \in I$ let $\mathfrak{c}^+$ be the cone of totally positive elements in $\mathfrak{c}$. Let us consider the functor $Sch_{\mathbf{Z}_p}^{op} \to Sets$ which associates to a scheme S the set of isomorphism classes of quadruples $(A, \iota, \lambda, \eta)$, where:

(1) (A,ι) is an abelian scheme over S of relative dimension r with real multiplication by $\mathcal{O}_F$, i.e. a morphism $\iota : \mathcal{O}_F \to End(A)$ such that the Rapoport condition is satisfied: $Lie(A)$ is a Zariski-locally free $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathcal{O}_S$-module of rank 1.
(2) λ is a $\mathfrak{c}$-polarisation on A, for some $\mathfrak{c} \in I$, i.e. an isomorphism (given étale locally on S) of $\mathcal{O}_F$-modules preserving positive cones:
$$(\mathfrak{c}, \mathfrak{c}^+) \xrightarrow{\lambda} (Hom_{\mathcal{O}_F}^{Sym}(A, A^\vee), Hom_{\mathcal{O}_F}^{Sym}(A, A^\vee)^+)$$
where $Hom_{\mathcal{O}_F}^{Sym}(A, A^\vee)$ denotes the set of $\mathcal{O}_F$-linear, symmetric homomorphisms from A to its dual and $Hom_{\mathcal{O}_F}^{Sym}(A, A^\vee)^+$ is the cone of polarizations.
(3) η is a U-level structure on (A,ι). For $U = Ker(GL_2(\hat{\mathcal{O}}_F) \to GL_2(\mathcal{O}_F/N))$, where N is an integer coprime with p, an U-level structure is an $\mathcal{O}_F$-linear isomorphism of étale group schemes $(\mathcal{O}_F/N)^2 \xrightarrow{\sim} A[N]$. For the definition of U-level structure for general U see [**TX16b**, pag. 9].

3.2. Remark. In our setting, since p does not divide the discriminant of F, the Rapoport condition is equivalent to the Kottwitz determinant condition and to the Deligne-Pappas condition. Moreover, by [**Rap78**, pag. 258], those are automatic over base schemes of characteristic zero. We will call a couple (A,ι) as above, such that any of these conditions is satisfied, an abelian scheme with real multiplication ($RMAS$ for short). Triples (A,ι,λ) will be called polarized abelian schemes with real multiplication, shortened as $PRMAS$.

3.3. For U^p sufficiently small (which we will always assume in what follows) the above functor is representable by a smooth quasi-projective scheme $\tilde{\mathsf{H}}_U$ over $\mathbf{Z}_p$.

3.4. Remark. The moduli problem in [**TX16b**] is formulated over the category of *locally noetherian* $\mathbf{Z}_{(p)}$-schemes. Such a restriction is however not necessary when working with moduli problems of abelian varieties up to *isomorphism* and not up to *isogeny*. See [**Lan08**, Section

1.4] for most general statement concerning this. It is crucial for us to work without noetherian assumptions as we will later have to evaluate our functors on perfectoid rings, which are highly non-noetherian.

3.5. There is a natural action of $\mathcal{O}_F^{\times,+}$ on $\tilde{\mathsf{H}}_U$ given by $\beta \cdot (A, \iota, \lambda, \eta) = (A, \iota, \iota(\beta)\lambda, \eta)$. Moreover the subgroup $(O_F^\times \cap U)^2$ acts trivially. Indeed if $\beta \in \mathcal{O}_F^\times$ then $\iota(\beta) : A \to A$ is an $\mathcal{O}_F$-linear isomorphism, and

$$\iota(\beta)^\vee \circ \lambda \circ \iota(\beta)(c) = \iota(\beta)^\vee \circ \lambda(c) \circ \iota(\beta) = \lambda(c) \circ \iota(\beta^2) \ \forall c \in \mathfrak{c}.$$

Therefore multiplication by $\iota(\beta)$ induces an isomorphism $(A, \iota, \lambda) \simeq (A, \iota, \iota(\beta)^2\lambda)$ which respects level structures if $\beta \in U$. Hence the action of $\mathcal{O}_F^{\times,+}$ on $\tilde{\mathsf{H}}_U$ factors through an action of the finite group $\Delta_U = \mathcal{O}_F^{\times,+}/(\mathcal{O}_F^\times \cap U)^2$, and the quotient $\mathsf{H}_U = \tilde{\mathsf{H}}_U/\Delta_U$ is a quasi-projective scheme over $\mathbf{Z}_p$ giving an integral model of $Sh(G, X)_U$ over $\mathbf{Z}_p$.

For U satisfying a further assumption [**TX16b**, Hypothesis 2.7] the group Δ_U acts freely on the set of geometric connected components of $\tilde{\mathsf{H}}_U$.

3.6. The quotient H_U is not a fine moduli space any more, but it is a coarse moduli space for the functor $Sch_{\mathbf{Z}_p}^{op} \to Sets$ sending S to the set of isomorphism classes of triples $(A, \iota, \bar{\lambda}, \bar{\eta})$, where:

(1) (A, ι) is an abelian scheme over S of relative dimension r with real multiplication by $\mathcal{O}_F$.
(2) $\bar{\lambda}$ is an $\mathcal{O}_F^{\times,+}$-orbit of $\mathfrak{c}$-polarisations on A, for some $\mathfrak{c} \in I$.
(3) $\bar{\eta}$ is an orbit of U^p-level structures on (A, ι).

3.7. Remark. For B totally indefinite *division* algebra the Shimura varieties $Sh(G, X)_U$ have a similar interpretation as coarse moduli spaces of abelian schemes with quaternionic multiplication (i.e. $\mathcal{O}_B$-action), which allows again to construct an integral model. In this case the orbit of $\mathcal{O}_F$-linear polarisations $\bar{\lambda}$ is unique, by [**Zin82**, Lemma 3.8], hence it can be omitted from the above description of the moduli problem.

3.8. Remark. Since Δ_U acts only on the polarisation, one can show (cf. [**LT17**, Remark 2.9]) that the universal abelian scheme with real multiplication over $\tilde{\mathsf{H}}_U$ descends to an abelian scheme with real multiplication over H_U.

4. Quaternionic automorphic forms

4.1. General definition. Assume in this section that $r = [F : \mathbf{Q}] > 1$. Let $U \subset G(\mathbf{A}_f) = \hat{B}^\times$ be a compact open subgroup. As in the section 2, write $G_{\mathbf{R}} = \prod_{\sigma \in \Sigma_s} GL_{2,\mathbf{R}} \times \prod_{\sigma \in \Sigma_r} \mathbf{H}^\times$ and set $U_\infty = \prod_{\sigma \in \Sigma_s} SO_2(\mathbf{R}) \times \prod_{\sigma \in \Sigma_r} \mathbf{H}^1 \subset G(\mathbf{R})$. We denote an element of U_∞ as $(r(\theta_1), \dots, r(\theta_d), s_{d+1}, \dots, s_r)$ where $r(\theta_i) = \begin{pmatrix} cos(\theta_i) & -sin(\theta_i) \\ sin(\theta_i) & cos(\theta_i) \end{pmatrix} \in SO_2(\mathbf{R})$ and $s_j \in \mathbf{H}^1$.
Let $GL_2(\mathbf{R})^+ \subset GL_2(\mathbf{R})$ be the subgroup of matrices with positive determinant, which acts transitively on the Poincaré upper half plane $\mathcal{H}$ via Möbius transformations. The stabiliser of i is $\mathbf{R}^\times SO_2(\mathbf{R})$, hence this action induces an identification $GL_2(\mathbf{R})^+/\mathbf{R}^\times SO_2(\mathbf{R}) \xrightarrow{\sim} \mathcal{H}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{R})^+$ define the automorphy factor

$$j(\gamma, i) = ci + d$$

and set, for $\underline{\gamma} = (\gamma_1, \dots, \gamma_d) \in (GL_2(\mathbf{R})^+)^d$, $j(\underline{\gamma}, \underline{i}) = \prod_{j=1}^d j(\gamma_j, i)$. Similarly, define $det(\underline{\gamma}) = \prod_j det(\gamma_j)$.

4.2. Definition. The space $M_2^{B^\times}(U)$ of automorphic forms for G with level U and weight two (at every $\sigma \in \Sigma_s$) is the $\mathbf{C}$-vector space of continuous functions

$$f : G(\mathbf{A_Q}) = B^\times(\mathbf{A}_F) \longrightarrow \mathbf{C}$$

satisfying the following properties:

(1) f is left $G(\mathbf{Q})$-invariant: $f(gb) = f(b)\ \forall g \in G(\mathbf{Q}), b \in G(\mathbf{A}_f)$.
(2) f is right U-invariant: $f(bu) = f(b)\ \forall u \in U, b \in G(\mathbf{A_Q})$.
(3) f is $Z(\mathbf{R})$-invariant: $f(bz) = f(b)\ \forall z \in Z(\mathbf{R}), b \in G(\mathbf{A_Q})$.
(4) For every $g \in G(\mathbf{A_Q})$ and $(r(\theta_1), \dots, r(\theta_d), s_{d+1}, \dots, s_r) \in U_\infty$ we have

$$f(g(r(\theta_1), \dots, r(\theta_d), s_{d+1}, \dots, s_r)) = e^{-2i\theta_1} \times \cdots \times e^{-2i\theta_d} f(g)$$

(5) For every $g \in G(\mathbf{A_Q})$ the function

$$\begin{aligned} f_g : \prod_{\sigma \in \Sigma_s} GL_2(\mathbf{R})^+ &\longrightarrow \mathbf{C} \\ \underline{\gamma} &\mapsto det(\underline{\gamma})^{-1} j(\underline{\gamma}, \underline{i})^2 f(g\underline{\gamma}) \end{aligned}$$

which factors through $\prod_{\sigma \in \Sigma_s} GL_2(\mathbf{R})^+ / \mathbf{R}^\times SO_2(\mathbf{R}) = \mathcal{H}^d$ because of (4), induces a holomorphic function $\mathcal{H}^d \to \mathbf{C}$.

We say that f as above is a cusp form if $B \neq M_2(F)$ or if $B = M_2(F)$ and the following condition holds:

$$\int_{F \backslash \mathbf{A}_F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \mathrm{d}x = 0\ \ \forall g \in G(\mathbf{A}_f)$$

and we denote by $S_2^{B^\times}(U)$ the space of cusp forms of weight 2 and level U.

4.3. Remark. (1) In general an additional condition on the growth at infinity must be imposed in (5). However in our situation, since $F \neq \mathbf{Q}$, either $B \neq M_2(F)$ hence this condition is automatically satisfied since the corresponding locally symmetric spaces are compact 2.2, or $B = M_2(F)$ and the locally symmetric space has dimension larger than one hence the Koecher principle assures that the growth condition is satisfied.
(2) The general cuspidality condition for automorphic forms for a reductive group G requires that the integral of f over (the adelic points of) parabolic subgroups of G vanishes. If $B \neq M_2(F)$ then $B^\times$ has no parabolic subgroups, which justifies our imposition that $S_2^{B^\times}(U) = M_2^{B^\times}(U)$ in this case (this is related once again to the compactness phenomenon remarked in 2.2).

4.4. The Hecke algebra. The Hecke algebra $\mathcal{H}(U \backslash G(\mathbf{A}_f)/U)$ of compactly supported, left and right U-invariant functions $\alpha : G(\mathbf{A}_f) \to \mathbf{C}$ with composition given by convolution:

$$\alpha \star \beta(h) = \int_{G(\mathbf{A}_f)} \alpha(g)\beta(g^{-1}h) dg$$

(where the Haar measure $dg = \prod_v d_{g_v}$ is normalized requiring that the measure of each local maximal compact open subgroup equals one) acts on $M_2^{B^\times}(U)$ via the formula

$$\alpha \cdot f(h) = \int_{G(\mathbf{A}_f)} \alpha(g) f(hg) dg.$$

and the action preserves $S_2^{B^\times}(U)$.

4.4.1. *Central character.* In particular, for $z \in Z(\mathbf{A}_f)$ the action of the function $\frac{1}{\mu(zU)}\mathbf{1}_{zU}$, where $\mu(zU)$ is the measure of zU and $\mathbf{1}_{zU}$ its characteristic function, sends f to

$$z \cdot f(h) = \frac{1}{\mu(zU)} \int_{zU} f(hg)dg = f(hz)$$

This induces an action of $Z(\mathbf{A}_f)$ on $M_2^{B^\times}(U)$ and $S_2^{B^\times}(U)$ which factors through the finite quotient $F^\times \backslash \mathbf{A}_F^\times/(F \otimes_{\mathbf{Q}} \mathbf{R})^\times V$, where $V = U \cap \hat{F}^\times$. This determines a decomposition

$$M_2^{B^\times}(U) = \bigoplus_{\varphi} M_2^{B^\times}(U, \varphi)$$
$$S_2^{B^\times}(U) = \bigoplus_{\varphi} S_2^{B^\times}(U, \varphi)$$

where the sum runs over all characters $\varphi : F^\times \backslash \mathbf{A}_F^\times/(F \otimes_{\mathbf{Q}} \mathbf{R})^\times V \longrightarrow \mathbf{C}^\times$. The character φ is called the *central character* of f. We denote the space of automorphic forms with *trivial* central character by $M_2^{B^\times/Z}(U)$ and the subspace of cuspidal automorphic forms by $S_2^{B^\times/Z}(U)$ (indeed, those are automorphic forms for the group $G^\times/Z = Res_{F/\mathbf{Q}}PGL_{2,F}$). By construction these spaces inherit an action of the Hecke algebra, which is trivial on elements coming from $Z(\mathbf{A}_f)$.

4.4.2. *Spherical and physical Hecke algebras.* Let v be a finite place of F where B is unramified and such that $U = U_v U^v$ with $U_v \subset GL_2(F_v)$ hyperspecial. Let $A_v = U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U$, where ϖ_v is a uniformizer at v, and $B_v = U \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U$. The sub-algebra of $\mathcal{H}(U \setminus G(\mathbf{A}_f)/U)$ generated by the characteristic functions of these sets is called the *spherical Hecke algebra*. We denote by $T_v : M_2^{B^\times}(U) \to M_2^{B^\times}(U)$ (resp. $S_v : M_2^{B^\times}(U) \to M_2^{B^\times}(U)$) the Hecke operator corresponding to the function

$$\frac{1}{\mu(A_v)}\mathbf{1}_{A_v} \ \left(\text{resp. } \frac{1}{\mu(B_v)}\mathbf{1}_{B_v}\right);$$

the restriction of the above operators to $S_2^{B^\times}(U)$, as well as to the space of automorphic forms with trivial central character, will be denoted by the same symbol. Notice that the action of every S_v on $M_2^{B^\times/Z}(U)$ is trivial. We will denote by $\mathbf{T}^{B^\times}(U)$ (resp. $\mathbf{T}^{B^\times/Z}(U)$) the subring of $End(S_2^{B^\times}(U))$ (resp. $End(S_2^{B^\times/Z}(U))$) generated by the operators T_v, S_v (the *physical Hecke algebra*).

4.5. $B = M_2(F)$**: Hilbert modular forms (of parallel weight 2).** Let $B = M_2(F)$. In this case we obtain the space $M_2^{GL_{2,F}}(U)$ of *Hilbert modular forms* of parallel weight 2 and level U. Those can be interpreted in more classical terms as holomorphic functions on (several copies of) $\mathcal{H}^d$ satisfying a suitable transformation law, componentwise similar to the classical one for modular forms, with respect to the action of arithmetic subgroups of $GL_2(\mathcal{O}_F)$. One can also define the q-expansion of Hilbert modular forms and check that cusp forms are those with vanishing constant term in the q-expansion. Furthermore one can interpret Hilbert modular forms as sections of suitable vector bundles on $Sh(G, X)_U$ (see [**TX16b**]). We will however not make use of these facts, except for the following definition: we say that a Hilbert cusp form f is *normalized* if the first non zero term in its q-expansion equals 1.

4.5.1. *Eigenforms and newforms.* Let $U = U_1(\mathfrak{n})$, where $\mathfrak{n} \subset \mathcal{O}_F$ is an ideal and

$$U_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) \ : c, d-1 \equiv 0 \pmod{\hat{\mathfrak{n}}} \right\}.$$

Let us denote by $M_2(\mathfrak{n})$ (resp. $S_2(\mathfrak{n})$) the space of Hilbert modular (resp. cusp) forms of level $U_1(\mathfrak{n})$ with trivial central character. In particular those are right-invariant for the action of

$$U_0(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) \ : c \equiv 0 \pmod{\hat{\mathfrak{n}}} \right\}.$$

In addition to the operators T_v acting on $M_2(\mathfrak{n})$ let us denote by $U_v : M_2(\mathfrak{n}) \to M_2(\mathfrak{n})$ the operator induced from $\frac{1}{\mu(A_v)}\mathbf{1}_{A_v}$ where $A_v = U_1(\mathfrak{n}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_1(\mathfrak{n})$ and v divides $\mathfrak{n}$.

The Hecke operators $\{T_v \ v \nmid \mathfrak{n}\}$ form a commuting family of normal operators with respect to the Petersson inner product

$$\langle \cdot, \cdot \rangle_{Pet} : S_2(\mathfrak{n}, \varphi) \times S_2(\mathfrak{n}, \varphi) \longrightarrow \mathbf{C}$$
$$(f, g) \mapsto \int_{G(\mathbf{Q})Z(\mathbf{Q})\backslash G(\mathbf{A_Q})} f(x)\overline{g(x)}|det(x)|^2 dx$$

hence they can be simultaneously diagonalised, but, as in the classical case of modular forms, the corresponding eigenspaces need not be one dimensional. Precisely, let $f \in S_2(\mathfrak{n})$ be an eigenform for every T_v and w a place dividing $\mathfrak{n}$. We say that f is w-old if there exists $f' \in S_2(\mathfrak{n}/w)$ with the same eigenvalues as f for the Hecke operators T_v, $v \not| \mathfrak{n}$. If this is not the case we say that f is w-new, and we say that f is new if it is w-new for every $w|\mathfrak{n}$. We denote by $S_2^{new}(\mathfrak{n}) \subset S_2(\mathfrak{n})$ the subspace generated by newforms. The strong multiplicity one theorem states that $S_2^{new}(\mathfrak{n})$ decomposes as a direct sum of one dimensional subspaces under the action of $\{T_v, \ v \nmid \mathfrak{n}\}$. In particular, every $f \in S_2^{new}(\mathfrak{n})$ which is an eigenvector for all the operators T_v is also an eigenvector of the U_v operators, and the vector space it generates contains a unique *normalized* form. We call such an normalized eigenform a *newform* (of level $\mathfrak{n}$ and parallel weight 2).

4.6. $d = 1$**: differentials on Shimura curves.** Let us now take a quaternion algebra B/F split at *exactly one* infinite place of F. For $U \subset \hat{B}^\times$ compact open we have the space $S_2^{B^\times}(U)$ of (cuspidal) automorphic forms of level U and weight 2 with an action of $\mathcal{H}(U \setminus G(\mathbf{A}_f)/U)$. On the other hand the Hecke algebra gives rise to correspondences on the Shimura curve $Sh(G, X)_U$ as follows: the algebra $\mathcal{H}(U \setminus G(\mathbf{A}_f)/U)$ is generated by characteristic functions of sets of the form UgU, for $g \in G(\mathbf{A}_f)$. To such a g we associate the correspondence

$$\begin{array}{ccc} Sh(G,X)_{U \cap gUg^{-1}} & \xrightarrow{[z,h]\mapsto[z,hg]} & Sh(G,X)_{g^{-1}Ug \cap U} \\ \downarrow & & \downarrow \\ Sh(G,X)_U & & Sh(G,X)_U. \end{array}$$

This induces an action of the Hecke algebra $\mathcal{H}(U \setminus G(\mathbf{A}_f)/U)$ on $H^0(Sh(G, X)_U^{an}, \Omega_\mathbf{C})$. In fact, one can show that there is a canonical identification

$$S_2^{B^\times}(U) \xrightarrow{\simeq} H^0(Sh(G, X)_U^{an}, \Omega_\mathbf{C})$$

which is Hecke equivariant up to a normalization factor [**Nek07**, Section 1.12].

The comparison isomorphism between Betti and étale cohomology and the Hodge decomposition $H^1_{Betti}(Sh(G, X)_U^{an}, \mathbf{C}) = H^0(Sh(G, X)_U^{an}, \Omega_\mathbf{C}) \oplus \overline{H^0(Sh(G, X)_U^{an}, \Omega_\mathbf{C})}$ together with the above identification imply that the action of the abstract Hecke algebra $\mathcal{H}(U \setminus G(\mathbf{A}_f)/U)$ on $H^1_{\acute{e}t}(Sh(G,X)_{U,\bar{F}}, \bar{\mathbf{Q}}_p)$ also induces an action of the physical Hecke algebra $\mathbf{T}^{B^\times}(U)$. Finally, this story is $Z(\mathbf{A}_f)$-equivariant. In particular we have an induced identification

$$S_2^{B^\times/Z}(U) \xrightarrow{\simeq} H^0(Sh(G/Z,X)_U^{an}, \Omega_{\mathbf{C}}).$$

4.7. Notation. In the special case $U = \hat{R}^\times$, where $R \subset B$ is an Eichler order of level $\mathfrak{n} \subset \mathcal{O}_F$, we denote the corresponding space of automorphic forms (with trivial central character) by $S_2^{B^\times}(\mathfrak{n})$. The subspace generated by newforms (defined similarly to the Hilbert modular form case) is denoted by $S_2^{B^\times,new}(\mathfrak{n})$. Beware that we are dropping the center Z from the notation, as we only use this notation for forms with trivial central character in this document.

4.8. $d = 0$: functions on quaternionic sets. Finally let us consider the case of a *totally definite* quaternion algebra B, i.e. take $d = 0$. In this case conditions (3) and (4) in definition 4.2 imply that an automorphic form $f : G(\mathbf{R}) \times G(\mathbf{A}_f) \to \mathbf{C}$ is $G(\mathbf{R})$-invariant, and condition (5) is empty. It follows that f factors through a function

$$\tilde{f} = B^\times \setminus \hat{B}^\times/U \to \mathbf{C}$$

i.e. (weight two) automorphic forms for $B^\times$ with level U are just complex-valued functions on the quaternionic set $Sh(G,X)_U$. Automorphic forms with trivial central character are those functions which factor through $Sh(G/Z,X)_U = B^\times \setminus \hat{B}^\times/\hat{F}^\times U$. An analogue of the theory of old and new forms can be partially developed in this context, as explained for example in [**Nek12**, Section 1].

4.8.1. *Integral automorphic forms.* We will need to deal with congruences between automorphic forms, and for this a notion of automorphic form *modulo* (powers of) p will be useful. This can be developed for Hilbert modular forms either exploiting the q-development or, more geometrically, via their interpretation as sections of automorphic vector bundles, suitably extended to integral models of Hilbert modular varieties (see [**TX16b**, Section 2]). However for our (minimal) needs it will be enough to dispose of this notion in the setting of quaternionic sets, in which case the definition is straightforward:

4.9. Definition. Let B be a totally definite quaternion algebra and A a ring. We define the space of A-valued automorphic forms for $B^\times$ of level 2 as

$$S_2^{B^\times}(U,R) = \{f : \hat{B}^\times \to A \ : f(bgu) = f(g)\, \forall b \in B^\times, g \in \hat{B}^\times, u \in U\} = A[B^\times \backslash \hat{B}^\times/U]$$

and we define $S_2^{B^\times/Z}(U,A)$ by requiring $\hat{F}^\times$-invariance in addition.

4.10. Notation. As before (4.7), for $U = \hat{R}^\times$ where $R \subset B$ is an Eichler order of level $\mathfrak{n} \subset \mathcal{O}_F$, we introduce the notations $S_2^{B^\times}(\mathfrak{n}, A)$, $S_2^{B^\times,new}(\mathfrak{n}, A)$.

4.11. Jacquet-Langlands correspondence. The Jacquet-Langlands correspondence allows to transfer, under suitable conditions, an eigenform for $GL_{2,F}$ to an eigenform for other quaternion algebras, preserving (almost all) Hecke eigenvalues. The correct, and most general, statement of the correspondence is given in the language of automorphic representations, and its proof crucially relies on automorphic techniques (precisely, the simple trace formula). We will however content ourselves of the following more elementary statement in terms of automorphic forms with trivial central character and level coming from Eichler orders. It can be deduced from the general one considering the automorphic representations attached to automorphic forms and studying their local components in terms of the level the automorphic form [**Nek06**, Lemma 12.3.10].

4.12. THEOREM. *(cf. [**DV13**, Theorem 3.9]) Let B/F be a quaternion algebra of discriminant $\mathfrak{D}$ and $\mathfrak{n} \subset \mathcal{O}_F$ an ideal coprime to $\mathfrak{D}$. There is a natural injection*

$$S_2^{B^\times}(\mathfrak{n}) \hookrightarrow S_2(\mathfrak{D}\mathfrak{n})$$

which is Hecke-equivariant for the action of Hecke operators outside $\mathfrak{D}$ on both sides and whose image coincides with the space generated by Hilbert modular forms which are new at all primes dividing $\mathfrak{D}$.

5. L-functions and special value formulas

5.1. We will now recall the special value formulas relating CM points on quaternionic sets and Shimura curves with special values of L-functions of Hilbert modular forms. The results in this section come from [**Zha04**]; a more up-to-date reference on the subject, with more general results, is [**YZZ13**]. A lucid discussion from a point of view similar to ours can be found in [**CV07**].

5.2. Hecke characters. Fix a totally imaginary quadratic extension K/F and a character $\chi : K^\times\backslash\hat{K}^\times \to \mathbf{C}^\times$ of finite order, such that $\chi_{|\hat{F}^\times} = 1$ (i.e. χ is an *anticyclotomic character*). Let $\bar{\chi} : K^\times\backslash\hat{K}^\times \to \mathbf{C}^\times$ be defined as $\bar{\chi}(a) = \chi(c(a))$ where $c \in Gal(K/F)$ is the unique non trivial element. Then $\chi \cdot \bar{\chi} = \chi \circ N_{K/F} = 1$, hence $\bar{\chi}(a) = 1/\chi(a) = \overline{\chi(a)}$, which justifies our notation. Via the inverse of the Artin reciprocity map $Gal(K^{ab}/K) \to K^\times\backslash\hat{K}^\times$ we will see χ as a one dimensional representation of Γ_K. The conductor of χ is the ideal $c(\chi) \subset \mathcal{O}_F$ which is maximal among the ideals $\mathfrak{c}$ such that χ factors through

$$K^\times\backslash\hat{K}^\times/\hat{F}^\times\hat{\mathcal{O}}_{\mathfrak{c}}^\times = Pic(\mathcal{O}_{\mathfrak{c}})/i_*Pic(\mathcal{O}_F) = Gal(H_{\mathfrak{c}}/K)$$

where $H_{\mathfrak{c}}$ is the abelian extension of K corresponding to $Pic(\mathcal{O}_{\mathfrak{c}})/i_*Pic(\mathcal{O}_F)$ via class field theory, $i : F \hookrightarrow K$ being the inclusion.

5.3. L-functions. Let $f \in S_2(\mathfrak{n})$ be a *newform.* For every finite place $v \nmid \mathfrak{n}$ (resp. $v|\mathfrak{n}$) let $\lambda_f(v)$ be the eigenvalue of T_v (resp. U_v) acting on f. The (incomplete) L-function of f is defined as the Euler product

$$L^\infty(f,s) = \prod_{v|\mathfrak{n}}(1-\lambda_f(v)N(v)^{-s})^{-1}\prod_{v\nmid\mathfrak{n}}(1-\lambda_f(v)N(v)^{-s}+N(v)^{1-2s})^{-1}.$$

The above expression converges for $Re\, s > 3/2$ and defines a holomorphic function on this half-plane. The *completed* L- function

$$L(f,s) = \left(\prod_{\sigma\in\Sigma_\infty} 2(2\pi)^{-s}\Gamma(s)\right)\cdot L^\infty(f,s)$$

admits a holomorphic continuation to the whole complex plane, satisfying a functional equation (beware that our normalization is *not* the automorphic one)

$$L(f,s) = \epsilon(f,s)L(f,2-s).$$

In particular the parity of the vanishing order of $L(f,s)$ at $s=1$ depends only on $\epsilon(f) = \epsilon(f,1)$.

One can similarly define the L-function of f *twisted by* χ (in scientific terms, denoting by π the automorphic representation of $GL_{2,F}$ attached to f, it is the Rankin-Selberg L-function associated to π and the automorphic representation of $GL_{2,F}$ attached to χ [**CV07**, Section 1.1]). We denote it by $L(f,\chi,s)$; it satisfies a functional equation of the form $L(f,\chi,s) = \epsilon(f,\chi,s)L(f,\chi,2-s)$.

5.4. Assumption. The results in [**Zha04**] are proved under the following assumptions, which will be in force in this section: the level $\mathfrak{n}$ of f, the conductor $c(\chi)$ of χ and the discriminant of K/F are coprime to each other. Write $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$ where $v|\mathfrak{n}^+$ (resp. $v|\mathfrak{n}^-$) if and only if v splits in K (resp. v is inert in K) and assume that $\mathfrak{n}^-$ is *squarefree.*

5.4.1. *Sign of the functional equation.* Let $S = \{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}^-\} \cup \{v|\infty\}$ and denote by $|S|$ the cardinality of S. Under the above assumption, the value $\epsilon(f,\chi) = \epsilon(f,\chi,1)$ equals (cf. [**CV07**, p. 4]):

$$\epsilon(f,\chi) = (-1)^{|S|}$$

We will distinguish in what follows two cases according to the value of $\epsilon(f,\chi)$.

(1) If $r = [F : \mathbf{Q}] \equiv |\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}^-\}| \pmod 2$ then $\epsilon(f,\chi) = 1$; this is called the *definite case.* In this situation we will be interested in the special value $L(f,\chi,1)$, which by results of Cornut-Vatsal [**CV07**] is non zero for most χ.
(2) If $r \not\equiv |\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}^-\}| \pmod 2$ then $\epsilon(f,\chi) = -1$. This is called the *indefinite case.* In this situation the functional equation forces the vanishing of the central value $L(f,\chi,1)$, and one looks instead at $L'(f,\chi,1)$.

5.5. The definite case. Let B/F be the quaternion algebra ramified at all primes dividing $\mathfrak{n}^-$ as well as at all infinite places. Then f can be transferred, via the Jacquet-Langlands correspondence, to a newform

$$f_B : B^\times\backslash\hat{B}^\times/\hat{F}^\times\hat{R}^\times \to \mathbf{C}$$

where $R \subset B$ is an Eichler order of level $\mathfrak{n}^+$. We normalize f_B requiring its Petersson norm to be 1 (the Petersson inner product being just a finite sum in this case). Fix an R-optimal embedding $\iota : K \hookrightarrow B$ (i.e. such that $\iota^{-1}(R) = \mathcal{O}_K$) and let $P_\chi \in CM(Sh(G/Z,X)_U, K)$ be a CM point of conductor $c(\chi)$. Let

$$a(f,\chi) = \sum_{\sigma\in Gal(H_{c(\chi)}/K)} \bar{\chi}(\sigma) f_B(\sigma(P_\chi)) \in \mathbf{C}. \tag{5.5.1}$$

5.6. Theorem. *([**Zha04**, Theorem 7.1]) The following equality holds:*

$$L(f,\chi,1) = \frac{2^r}{N(c(\chi))\sqrt{N(disc(K/F))}} \cdot \langle f,f\rangle_{Pet} \cdot |a(f,\chi)|^2. \tag{5.6.1}$$

5.7. The indefinite case. Let B/F be the quaternion algebra ramified at all primes dividing $\mathfrak{n}^-$ and at all but one infinite place, and f_B the Jacquet-Langlands transfer of f to $G = Res_{F/\mathbf{Q}}B^\times$. Let $R \subset B$ be an order of conductor $\mathfrak{n}^+$ and fix an R-optimal embedding $K \hookrightarrow R$ as before. Let $P_\chi \in CM(Sh(G/Z,X)_U, K)$ be a CM point of conductor $c(\chi)$ and $Q_\chi = \sum_{\sigma\in Gal(H_{c(\chi)}/K)} \bar{\chi}(\sigma)(\sigma(P_\chi)) \in CH^1(Sh(G/Z,X)_{U,\bar{F}})$. Let $a(f,\chi)$ be the f_B-isotypical part of $Q_\chi - deg(Q_\chi)\xi \in Jac(Sh(G/Z,X)_U)(H_{c(\chi)}) \otimes \mathbf{C}$ where $\xi \in CH^1(Sh(G/Z,X)_U) \otimes \mathbf{Q}$ is the Hodge class [**Zha04**, pag. 202].

5.8. Theorem. *([**Zha04**, Theorem 6.1]) The following equality holds:*

$$L'(f,\chi,1) = \frac{2^{r+1}}{N(c(\chi))\sqrt{N(disc(K/F))}} \cdot \langle f,f\rangle_{Pet} \cdot \langle a(f,\chi), a(f,\chi)\rangle_{NT}$$

where $\langle -, - \rangle_{NT}$ *is the Neron-Tate height.*

5.9. Remark. For χ trivial, in the definite case we can also write

$$a(f, \mathbf{1}) = \sum_{P \in Pic(\mathcal{O}_K)/i_* Pic(\mathcal{O}_F)} f_B(\hat{\iota}(P))$$

where $\hat{\iota} : K^\times \backslash \hat{K}^\times / \hat{F}^\times \hat{\mathcal{O}}_K^\times \to B^\times \backslash \hat{B}^\times / \hat{F}^\times \hat{R}^\times$ is the map induced by $\iota : K \hookrightarrow B$ and $i : F \hookrightarrow K$ is the inclusion. In other words $a(f, \mathbf{1})$ is the average of f_B on the image of $Pic(\mathcal{O}_K)/i_* Pic(\mathcal{O}_F)$ in $Sh(G, X)_U$. Similarly, in the indefinite case $a(f, \chi)$ is the f_B-isotypic part of the image of $Pic(\mathcal{O}_K)/i_* Pic(\mathcal{O}_F)$ in $CH^1(Sh(G/Z, X)_{U, \bar{F}})$.

6. Galois representations attached to Hilbert modular forms

6.1. Let $f \in S_2(\mathfrak{n})$ be a newform and $\mathcal{O}$ the ring generated by the eigenvalues $\lambda_f(v)$ of the Hecke operators acting on f. It is an order in the ring of integers of a number field E which is totally real (since f has trivial central character). One can attach to f a compatible system of Galois representations (coming in most cases from a motive M over F of weight one with coefficients in E by [**BR93**])

$$\rho_{f,\pi} : \Gamma_F \to Aut(V_{f,\pi})$$

where, for each finite place π of E, $V_{f,\pi}$ is a 2-dimensional vector space over the completion E_π of E at π. Recall that at the very end of the introduction we have fixed a prime p and a distinguished embedding $E \to \bar{\mathbf{Q}}_p$. Assume that p does not divide $\mathfrak{n}$. We denote simply by

$$\rho_f : \Gamma_F \to Aut(V_f)$$

the Galois representation corresponding to the place $\mathfrak{p}$ of E induced by this embedding. We denote by $\mathcal{O}_\mathfrak{p}$ the ring of integers of $E_\mathfrak{p}$ and by ϖ a uniformiser of $\mathcal{O}_\mathfrak{p}$. The representation ρ_f enjoys the following properties:

(1) it is unramified outside $\mathfrak{n}p$;
(2) for every finite place v of F not dividing $\mathfrak{n}p$ we have

$$det(1 - Fr_v N(v)^{-s} | V_f) = 1 - \lambda_f(v) N(v)^{-s} + N(v)^{1-2s};$$

(3) for $v \nmid \mathfrak{n}p$ the eigenvalues of Fr_v acting on V_f are v-Weil numbers of weight 1;
(4) it is absolutely irreducible.

By (3) and the Cebotarev density theorem ρ_f is uniquely characterised up to isomorphism by the property (2), which determines the trace of almost all Frobenius elements. Moreover (2) implies that

$$det(V_f) = \wedge^2 V_f \simeq E_\mathfrak{p}(-1)$$

hence $V_f^* = Hom(V_f, E_\pi) \simeq V_f(1)$. Letting $V(f) = V_f(1)$ it follows that $V(f)$ is *self-dual*, i.e. there is a skew-symmetric, non degenerate, Γ_F-equivariant pairing

$$V(f) \times V(f) \to E_\mathfrak{p}(1)$$

yielding an identification $V(f) \simeq Hom_{\Gamma_F}(V(f), E_\mathfrak{p}(1))$.

We choose a Γ_F-stable $\mathcal{O}_\mathfrak{p}$-lattice $T(f) \subset V(f)$ such that the above pairing (possibly scaled by a constant) induces a perfect pairing

$$T(f) \times T(f) \to \mathcal{O}_\mathfrak{p}(1)$$

hence perfect pairings

$$\begin{aligned} T(f)\times A(f) \to & E_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}(1) \\ T_n(f)\times A_n(f) \to & E_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}(1),\ n\geq 0, \end{aligned}$$

where $A(f)=V(f)/T(f)$, $A_n(f)=A(f)[\varpi^n]\simeq T_n(f)=T(f)/\varpi^n$.

6.2. Assumption. Assume that the residual Galois representation $T_1(f)$ is irreducible (hence absolutely irreducible).

Under the above assumption the Galois representations $T(f), T_n(f)$ do not depend on the choice of the lattice $T(f)$.

We will need the following information on the local structure of the Γ_F-module $V(f)$:

6.3. Lemma. *(cf. [**Nek06**, 12.4.4.2, 12.4.5]) If v is a place of F dividing exactly $\mathfrak{n}$ then $V(f)_{|G_{F_v}}$ is of the form*

$$\begin{pmatrix} \mu\chi_{cyc} & * \\ 0 & \mu \end{pmatrix}$$

where χ_{cyc} is the cyclotomic character and μ is a quadratic unramified character.

6.4. Remark. The proof of the existence of the Galois representation ρ_f attached to f is due to the work of many people, and was completed by Taylor [**Tay89**] and Blasius-Rogawski [**BR89**]. The construction is easier, and was known much earlier, if either $[F:\mathbf{Q}]$ is odd or it is even and there exists a finite place v of F such that the component at v of the automorphic representation attached to f is special or supercuspidal (which happens if v divides the conductor of f exactly). Indeed, in this case the Jacquet-Langlands correspondence allows to transfer f to an automorphic form for a quaternion algebra B ramified at all but one infinite place. The Eichler-Shimura relation for Shimura curves implies that the sought for Galois representation can be realised in the étale cohomology of the Shimura curve attached to B. Taylor then reduces the general case to this one exploiting congruences of automorphic forms.

6.5. Hecke twists. Let $\chi: K^\times\backslash\hat{K}^\times\to\mathbf{C}^\times$ be an anticyclotomic character of finite order as in 5.2. Up to replacing $E_{\mathfrak{p}}$ by a finite extension we can, and will, assume that it contains the values of χ. If M is a $\mathcal{O}_{\mathfrak{p}}[\Gamma_F]$-module we will denote by $M(\chi)$ the twist $M_{|\Gamma_K}\otimes_{\mathcal{O}_{\mathfrak{p}}}\mathcal{O}_{\mathfrak{p}}(\chi)$. The above pairings induce perfect pairings

$$\begin{aligned} T(f)(\bar\chi)\times A(f)(\chi) \to & E_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}(1) \\ T_n(f)(\bar\chi)\times A_n(f)(\chi) \to & E_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}(1) \end{aligned}$$

which yield isomorphisms of Γ_K-modules

$$Hom(T_n(f)(\chi), E_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}(1)) = A_n(f)(\bar\chi)\simeq {}^c(A_n(f)(\chi)).$$

In the above equation the notation ${}^c(A_n(f)(\chi))$ stands for the module $A_n(f)(\chi)$ with twisted Γ_K-action given by $(g,m)\mapsto(\tilde{c}^{-1}g\tilde{c})m$ where $\tilde{c}\in\Gamma_F$ is any lift of $1\neq c\in Gal(K/F)$. For $i\geq 0$ there is a canonical isomorphism

$$\begin{aligned} H^i(\Gamma_K, A_n(f)(\chi)) \xrightarrow{\sim} & H^i(\Gamma_K, {}^c(A_n(f)(\chi))) \\ z\mapsto & z'(g_1,\ldots,g_i) = z(\tilde{c}^{-1}g_1\tilde{c},\ldots,\tilde{c}^{-1}g_i\tilde{c}). \end{aligned}$$

7. Shtukas in positive characteristic and higher Gross-Zagier formulas

7.1. The aim of this section is to recall the definition of the moduli spaces of global Shtukas over function fields and state Yun and Zhang's higher Gross-Zagier formula [**YZ17**].

In this section we work over a finite base field $k = \mathbf{F}_q$; fiber products are taken over $Spec\,k$ whenever no base is specified. Let C be a smooth projective geometrically irreducible curve over k, E its function field and $\mathcal{N}$ a closed subscheme of C.

The absolute k-Frobenius on a k-scheme S, raising functions to the q-th power, is denoted by F_S.

The graph of a morphism of schemes $f : S \to C$ is denoted by $\Gamma_f \subset S \times C$. If $\underline{f} = (f_1, \dots, f_n)$ is a n-tuple of morphisms from S to C we denote by $\Gamma_{\underline{f}} \subset S \times C$ the union of the graphs of f_i.

Let $\mathcal{G}$ be a smooth, affine group scheme over C whose generic fibre $G = \mathcal{G} \times_C E$ is reductive.

7.2. Definition. Let S be a $k-$scheme and let $\underline{x} = (x_1, \dots, x_n) : S \to C^n$ be an n-tuple of S-points of C. A $\mathcal{G}$-Shtuka over S with legs at $\underline{x}$ is a $\mathcal{G}$-bundle $\mathcal{F}$ over $C \times S$ together with an isomorphism

$$\phi : \mathcal{F}_{|C\times S\setminus \Gamma_{\underline{x}}} \longrightarrow {}^{\tau}\mathcal{F}_{|C\times S\setminus \Gamma_{\underline{x}}}$$

where ${}^{\tau}\mathcal{F}$ is the pullback of $\mathcal{F}$ via the morphism $Id_C \times F_S : C \times S \to C \times S$.

A Shtuka with (full) $\mathcal{N}$-level structure and legs at $\underline{x} = (x_1, \dots, x_n) : S \to (C \setminus \mathcal{N})^n$ is a $\mathcal{G}$-Shtuka with legs at $\underline{x}$ together with an isomorphism, compatible with ϕ,

$$u : \mathcal{F}_{|\mathcal{N}\times S} \xrightarrow{\sim} (\mathcal{G} \times_C \mathcal{N}) \times S.$$

7.3. Definition. The moduli stack of $\mathcal{G}$-Shtukas with n legs and level $\mathcal{N}$-structure, denoted by $Sht^n(\mathcal{G})_{\mathcal{N}}$ is the stack whose S-points, for any k-scheme S, are given by

$$Sht^n(\mathcal{G})_{\mathcal{N}}(S) = \{\underline{x} : S \to (C \setminus \mathcal{N})^n, \mathcal{F} \text{ a } \mathcal{G}-\text{Shtuka on } S \text{ with legs at } \underline{x} \text{ and } \mathcal{N}\text{-level structure}\}.$$

Remembering only the legs of Shtukas we obtain a morphism

$$Sht^n(\mathcal{G})_{\mathcal{N}} \to (C \setminus \mathcal{N})^n.$$

7.4. The moduli space of Shtukas we just defined was studied, slightly more in general, in [**RH13**], generalising Varshavsky's work [**Var04**] for constant split reductive groups. We gave the definition in the above generality since we will be interested in non-split groups arising via restriction of scalars. In order to state higher Gross-Zagier formulas we will also need the following variation of the above definitions.

7.5. Definition. Let S be a $k-$scheme and let $\underline{x} = (x_1, \dots, x_n) : S \to C^n$ be an n-tuple of S-points of C. An *iterated* $\mathcal{G}$-Shtuka over S with legs at $\underline{x}$ is a $(n+1)-$tuple of $\mathcal{G}$-bundles $(\mathcal{F}_0, \dots, \mathcal{F}_n)$ over $C \times S$ together with isomorphisms

$$\tau_i : \mathcal{F}_{i|C\times S\setminus \Gamma_{x_{i+1}}} \longrightarrow \mathcal{F}_{i+1|C\times S\setminus \Gamma_{x_{i+1}}},\ i = 0, \dots, n-1$$

and an isomorphism

$$\phi : \mathcal{F}_n \longrightarrow {}^{\tau}\mathcal{F}_0.$$

We will denote by $Sht^{\underline{n}}(\mathcal{G})_{\mathcal{N}}$ the stack of iterated $\mathcal{G}$-Shtukas with $\mathcal{N}$-level structure (defined in the obvious way).

Forgetting all the $\mathcal{G}$-bundles except the first yields a natural map

$$Sht^{\underline{n}}(\mathcal{G})_{\mathcal{N}} \to Sht^n(\mathcal{G})_{\mathcal{N}}.$$

7.6. The Hecke stack. The moduli stack of Shtukas has a useful description in terms of the Hecke stack which we now recall. Let us denote by $Bun(\mathcal{G})_{\mathcal{N}}$ the stack classifying $\mathcal{G}$-bundles with $\mathcal{N}$-level structure. There is a map of stacks

$$Sht^n(\mathcal{G})_{\mathcal{N}} \to Bun(\mathcal{G})_{\mathcal{N}}$$

forgetting everything but the $\mathcal{G}$-bundle $\mathcal{F}$. Let us define the *Hecke stack* $Hk^n(\mathcal{G})_{\mathcal{N}}$ as the stack whose S-points are given by

$$\begin{aligned} Hk^n(\mathcal{G})_{\mathcal{N}}(S) = \{&\underline{x} : S \to (C \setminus \mathcal{N})^n,\ \mathcal{F},\ \mathcal{F}' \in Bun(\mathcal{G})_{\mathcal{N}}(S), \\ &\phi : \mathcal{F}_{|C\times S\setminus\Gamma_{\underline{x}}} \xrightarrow{\sim} \mathcal{F}'_{|C\times S\setminus\Gamma_{\underline{x}}} \text{ respecting the level structures}\}. \end{aligned}$$

Remembering only the $\mathcal{G}$-bundles gives a morphism

$$Hk^n(\mathcal{G})_{\mathcal{N}} \to Bun(\mathcal{G})_{\mathcal{N}} \times Bun(\mathcal{G})_{\mathcal{N}}$$

and our previous definition can be reformulated saying that $Sht^n(\mathcal{G})_{\mathcal{N}}$ is the intersection of the Hecke stack and the graph of the Frobenius map $Fr : Bun(\mathcal{G})_{\mathcal{N}} \to Bun(\mathcal{G})_{\mathcal{N}}$, i.e. the following diagram is cartesian

$$\begin{tikzcd} Sht^n(\mathcal{G})_{\mathcal{N}} \arrow[r] \arrow[d] & Hk^n(\mathcal{G})_{\mathcal{N}} \arrow[d] \\ Bun(\mathcal{G})_{\mathcal{N}} \arrow[r, "Id\times Fr"] & Bun(\mathcal{G})_{\mathcal{N}} \times Bun(\mathcal{G})_{\mathcal{N}}. \end{tikzcd}$$

The stack of iterated $\mathcal{G}$-Shtukas has an analogous description involving the iterated Hecke stack $Hk^{\underline{n}}(\mathcal{G})_{\mathcal{N}}$, defined in the obvious way. As above, there is a forgetful morphism

$$Hk^{\underline{n}}(\mathcal{G})_{\mathcal{N}} \to Hk^n(\mathcal{G})_{\mathcal{N}}.$$

7.7. Remark. The stacks of Shtukas introduced above have a number of unpleasant properties which must be taken into account when working with them:

(1) They have infinitely many connected components, since the same is true for $Bun_{\mathcal{G}}$ (for example its connected components for $G = GL_d$ are the substacks parametrising vector bundles of a given degree).
(2) Connected components of are *not* of finite type. Again, the problem comes from $Bun_{\mathcal{G}}$ (for example, the vector bundles $\mathcal{O}(m) \oplus \mathcal{O}(-m)$ on $\mathbf{P}^1_k$ correspond to infinitely many k-points in the connected component parametrising rank two vector bundles, which therefore is not of finite type).
(3) They are *not* proper nor smooth in general.

The first two issues are commonly addressed working with suitable truncations of the space of interest. We will recall how this works in the easiest case of $G = GL_d$ and iterated Shtukas with *minuscule* modifications. The general discussion can be found in [**Var04**], [**Laf12**] and [**RH13**]. Concerning the last point, depending on one's purposes one may either try to compactify the moduli stack (but compactifications will not be smooth in general), or live with the issue and use intersection cohomology in place of usual cohomology.

7.8. Truncation by relative position. From now on in this section we let $\mathcal{G} = GL_{d,C}$, unless stated otherwise. Then a $\mathcal{G}$ bundle over $C \times S$ is a vector bundle of rank d on $C \times S$. Take two rank d vector bundles $\mathcal{E}$, $\mathcal{E}'$ on $C \times S$, an S-point $x : S \to C$ and a morphism

$$\tau : \mathcal{E}_{|C\times S\setminus\Gamma_x} \longrightarrow \mathcal{E}'_{C\times S\setminus\Gamma_x}.$$

We say that τ is a *positive* (resp. *negative*) *minuscule* modification if τ (resp. τ^{-1}) extends to an injection $\mathcal{E} \hookrightarrow \mathcal{E}'$ (resp. $\mathcal{E}' \hookrightarrow \mathcal{E}$) whose cokernel is supported on Γ_x and is free of rank one on its support. If $x : S \to C$ is a closed geometric point of C this means concretely the following: the pullbacks $\mathcal{E}_x$, $\mathcal{E}'_x$ of $\mathcal{E}$ and $\mathcal{E}'$ to the completed local ring $\hat{\mathcal{O}}_x$ of C at x are identified via τ after pulling back to the fraction field $\hat{K}_x$ of $\hat{\mathcal{O}}_x$. Hence they are two $\hat{\mathcal{O}}_x$-lattices in the same $\hat{K}_x$-vector space. Then τ is a positive (resp. negative) minuscule modification if, in a suitable basis of $\mathcal{E}_x$ and $\mathcal{E}'_x$, it can be written as the matrix

$$\begin{pmatrix} \varpi_x & 0 \\ 0 & (1)_{d-1} \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} (1)_{d-1} & 0 \\ 0 & \varpi_x^{-1} \end{pmatrix}$$

where $\varpi_x \in \hat{\mathcal{O}}_x$ is a uniformizer and $(1)_{d-1}$ is the identity matrix of rank $d-1$.

Let $\underline{\mu} = (\mu_1, \dots, \mu_n)$ be a n-tuple of signs, $\mu_i \in \{\pm 1\}$. We denote by $Hk^{\underline{n},\underline{\mu}}(\mathcal{G})_{\mathcal{N}}$ the substack of $Hk^{\underline{n}}(\mathcal{G})_{\mathcal{N}}$ parametrizing objects $(\mathcal{E}_i, \tau_j)$ such that τ_j is a positive (resp. negative) minuscule modification if $\mu_i = 1$ (resp. $\mu_i = -1$). Let $Sht^{\underline{n},\underline{\mu}}(\mathcal{G})_{\mathcal{N}}$ be the preimage of $Hk^{\underline{n},\underline{\mu}}(\mathcal{G})_{\mathcal{N}}$ under the map $Sht^{\underline{n}}(\mathcal{G})_{\mathcal{N}} \to Hk^{\underline{n}}(\mathcal{G})_{\mathcal{N}}$.

7.9. Truncation by Harder-Narasimhan filtration. Recall that the *slope* of a vector bundle $\mathcal{E}$ over C is defined as $\mu(\mathcal{E}) = \frac{deg(\mathcal{E})}{rk(\mathcal{E})}$. One says that $\mathcal{E}$ is *semistable* if $\mu(\mathcal{E}') \le \mu(\mathcal{E})$ for every sub-bundle $\mathcal{E}'$ of $\mathcal{E}$. Harder and Narasimhan proved that every vector bundle $\mathcal{E}$ over $C_{\bar{k}}$ has a unique filtration

$$\mathcal{E}_0 = 0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_l = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and $\mu(\mathcal{E}_i/\mathcal{E}_{i-i}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ for every i, called the *Harder-Narasimhan filtration.* For example if $C = \mathbf{P}^1_{\bar{k}}$ then semistable vector bundles are those of the form $\mathcal{O}(m)^k$, and the Harder-Narasimhan filtration splits.

For any integer $h > 0$ define $Hk^{\underline{n},\le h}(\mathcal{G})_{\mathcal{N}}$ as the substack parametrising chains $(\mathcal{F}_0, \dots, \mathcal{F}_n)$ such that the maximal slope in the Harder-Narasimhan filtration of the pullback of $\mathcal{F}_0$ at any closed geometric point is at most h. One then defines $Sht^{\underline{n},\le h}(\mathcal{G})_{\mathcal{N}}$ as the preimage of $Hk^{\underline{n},\le h}(\mathcal{G})_{\mathcal{N}}$. As one sees examining the case $C = \mathbf{P}^1_k$, this truncation (together with the previous one) has the aim to make connected components of the resulting stack of finite type. It remains to kill infinitely many connected components.

7.10. Quotient by the action of $Z(\mathbf{A}_E)$. Let $Pic(C)_{\mathcal{N}}(k) = E^\times \setminus \mathbf{A}_E^\times / U_{\mathcal{N}}$ be the group of isomorphism classes of line bundles on C with level $\mathcal{N}$-structure. The rule $(\mathcal{L}, \mathcal{E}) \mapsto \mathcal{E} \otimes \mathcal{L}$ for $\mathcal{E} \in Bun(\mathcal{G})(S)$ and $\mathcal{L} \in Pic(C)_{\mathcal{N}}(k)$ induces an action of $Pic(C)_{\mathcal{N}}(k)$ on $Sht^{\underline{n}}(\mathcal{G})_{\mathcal{N}}$ which respects the truncations introduced above. The quotient of $Sht^{\underline{n}}(\mathcal{G})_{\mathcal{N}}$ by this action will be denoted by $Sht^{\underline{n}}(\mathcal{G}/Z)_{\mathcal{N}}$.

7.11. PROPOSITION. *(cf. [**Var04**, Corollary 2.21, Proposition 2.31]; [**YZ17**, Lemma 5.6, Corollary 5.7]) Let n be even and take $\mu = (\mu_1, \dots, \mu_n)$ such that $\sum_i \mu_i = 0$. Then:*

1. *The stacks $Sht^{\underline{n},\underline{\mu}}(\mathcal{G})_{\mathcal{N}}$, $Sht^{\underline{n},\underline{\mu}}(\mathcal{G}/Z)_{\mathcal{N}}$ are Deligne-Mumford stacks.*
2. *The leg morphism $Sht^{\underline{n},\underline{\mu}}(\mathcal{G},\mathcal{N}) \to (C \setminus \mathcal{N})^n$, and the induced morphism from the quotient $Sht^{\underline{n},\underline{\mu}}(\mathcal{G}/Z)_{\mathcal{N}}$, are separated and smooth of relative dimension $n(d-1)$.*

(3) *For* $h > 0$, *the stack* $Sht^{\underline{n},\underline{\mu},\le h}(\mathcal{G}/Z)_{\mathcal{N}}$ *is a quotient of a quasi-projective scheme by a finite group.*
(4) *If* $d = 2$ *then for different choices of* $\underline{\mu}$ *such that* $\sum_i \mu_i = 0$ *the stacks* $Sht^{\underline{n},\underline{\mu}}(\mathcal{G}/Z)$ *are canonically isomorphic.*

7.12. Shtukas with no legs. Let us suppose that $n = 0$. Then for a scheme S/k the objects of $Sht^0(\mathcal{G})_{\mathcal{N}}(S)$ are rank d-vector bundles $\mathcal{E}$ over $C \times S$ together with an isomorphism $\mathcal{E} \xrightarrow{\sim} {}^\tau\mathcal{E}$ (and with level structure).

Letting $p_C : C \times S \to C$ be the projection, such an $\mathcal{E}$ is a twisted form of $p_C^*(\mathcal{E}_{\mathcal{C}})$ for a suitable vector bundle $\mathcal{E}_C \in Bun(\mathcal{G})_{\mathcal{N}}(k)$. This is given by an $Aut(\mathcal{E}_{\mathcal{C}})$-torsor on S, hence we find

$$Sht^0(\mathcal{G})_{\mathcal{N}} = \coprod_{\mathcal{E} \in Bun(\mathcal{G})_{\mathcal{N}}(k)} [k/Aut(\mathcal{E})].$$

7.13. The GL_1-case: class field theory. Let $\mathcal{G} = GL_1$. In this case an object

$$(\mathcal{L}_0, \dots, \mathcal{L}_n; x_1, \dots x_n; \tau_0, \dots \tau_{n-1}; \phi) \in Sht^{\underline{n},\underline{\mu}}(\mathcal{G})_{\mathcal{N}}(S)$$

is uniquely determined by the first line bundle $\mathcal{L}_0$. Indeed, given $\mathcal{L}_i$, one has $\mathcal{L}_{i+1} = \mathcal{L}_i(\Gamma_{x_{i+1}})$ or $\mathcal{L}_{i+1} = \mathcal{L}_i(-\Gamma_{x_{i+1}})$ depending on the sign of μ_{i+1}. In particular one has $\mathcal{L}_n = \mathcal{L}_0(\sum_{j=1}^n \mu_j \Gamma_{x_j})$. Let us denote by $j : (C \setminus \mathcal{N})^n \to Pic^0(C)_{\mathcal{N}}$ be the map sending $\underline{x}$ to $\mathcal{O}(\sum_{j=1}^n \mu_j \Gamma_{x_j})$. Then our discussion implies that the following diagram is cartesian:

$$\begin{array}{ccc} Sht^{\underline{n},\underline{\mu}}(\mathcal{G})_{\mathcal{N}} & \longrightarrow & Pic(C)_{\mathcal{N}} \\ \downarrow & & \downarrow {\scriptstyle Id-Fr} \\ (C \setminus \mathcal{N})^n & \xrightarrow{j} & Pic^0(C)_{\mathcal{N}}. \end{array}$$

The Lang isogeny $Pic(C)_{\mathcal{N}} \xrightarrow{Id-Fr} Pic^0(C)_{\mathcal{N}}$ is an étale Galois cover with Galois group $Pic(C)_{\mathcal{N}}(k)$, hence the same is true for the map $Sht^{\underline{n}}(\mathcal{G})_{\mathcal{N}} \to (C \setminus \mathcal{N})^n$. Restricting to the connected component $Sht^{\underline{n},0}(\mathcal{G})_{\mathcal{N}}$ parametrizing chains of line bundles whose first element has degree 0 one obtains an étale map $Sht^{\underline{n},0}(\mathcal{G})_{\mathcal{N}} \to (C \setminus \mathcal{N})^n$ which induces a surjective continuous morphism $\pi_1((C \setminus \mathcal{N})^n, *) \to Pic^0(C)_{\mathcal{N}}(k)$. Hence in a sense the moduli spaces of Shtukas for $\mathcal{G} = GL_1$ give a geometric realization of class field theory for curves (this is explained much better in the notes [**Laf**]).

7.14. From GL_1 to GL_2: Heegner-Drinfeld cycles. Let us now take $d = 2$, i.e. $\mathcal{G} = GL_{2,C}$, and $\mathcal{N} = \emptyset$. Then for any even integer n and any choice of signs $\underline{\mu}$ such that $\sum_i \mu_i = 0$ we have a Deligne-Mumford stack

$$Sht^{\underline{n},\underline{\mu}}(\mathcal{G}/Z) \xrightarrow{p} C^n$$

where the projection map p is separated and smooth of relative dimension n.

Let $C' \xrightarrow{q} C$ be an double cover of smooth projective geometrically irreducible curves over k and $E \hookrightarrow E'$ the induced extension of function fields. For any k-scheme S, the pushforward of a rank d vector bundle over $C' \times S$ along $q \times Id_S$ is a vector bundle of rank $2d$ on $C \times S$. Moreover, for any S-point $x : S \to C'$, an isomorphism $\mathcal{E}_{|C' \times S \setminus \Gamma_x} \to \mathcal{E}'_{|C' \times S \setminus \Gamma_x}$ induces an isomorphism $q_*\mathcal{E}_{|C \times S \setminus \Gamma_{q(x)}} \to q_*\mathcal{E}'_{|C \times S \setminus \Gamma_{q(x)}}$. We obtain therefore a commutative square

$$\begin{tikzcd} Sht^{\underline{n},\underline{\mu}}(GL_{1,C'}) \arrow[r] \arrow[d] & Sht^{\underline{n},\underline{\mu}}(GL_{2,C}) \arrow[d] \\ (C')^n \arrow[r] & C^n. \end{tikzcd}$$

After quotienting by the action of $Pic(C)(k)$ the above diagram induces a map

$$\theta_{C'} : Sht^{\underline{n},\underline{\mu}}(GL_{1,C'})/Pic(C)(k) \to Sht^{\underline{n},\underline{\mu}}(GL_{2,C}/Z)_{C'}$$

where we denote by $Sht^{\underline{n},\underline{\mu}}(GL_{2,C}/Z)_{C'}$ the base change of $Sht^{\underline{n},\underline{\mu}}(GL_{2,C}/Z)$ to $(C')^n$.

The discussion in section 7.13 implies that $Sht^{\underline{n},\underline{\mu}}(GL_{1,C'})/Pic(C)(k) \to (C')^n$ is a torsor for the finite group $Pic(C')(k)/Pic(C)(k)$. In particular, $Sht^{\underline{n},\underline{\mu}}(GL_{1,C'})/Pic(C)(k)$ is proper smooth of dimension n over k. The image of $Sht^{\underline{n},\underline{\mu}}(GL_{1,C'})/Pic(C)(k)$ via $\theta_{C'}$ defines therefore a proper cycle in middle dimension

$$\theta_{C',*}[Sht^{\underline{n},\underline{\mu}}(GL_{1,C'})/Pic(C)(k)] \in CH^n_c(Sht^{\underline{n},\underline{\mu}}(GL_{2,C}/Z)_{C'}) \otimes \mathbf{Q}$$

which is called a *Heegner-Drinfeld cycle* and will be denoted by $HD_{C'}$.

7.15. Higher Gross-Zagier formulas. We will now state the main result of [**YZ17**], to which we refer the reader for further details. Let us suppose that the map q is étale; let π be an *unramified* cuspidal automorphic representation of $GL_{2,E}$ and $\lambda_\pi : \mathcal{H}(U\backslash GL_2(\mathbf{A}_E)/U) \to E_\pi$ the associated character of the spherical Hecke algebra, with values in the field of coefficients E_π of π and with kernel $\mathbf{m}_\pi$.

The intersection pairing

$$\langle\cdot,\cdot\rangle : CH^n_c(Sht^{\underline{n},\underline{\mu}}(GL_{2,C}/Z)_{C'}) \otimes \mathbf{Q} \times CH^n_c(Sht^{\underline{n},\underline{\mu}}(GL_{2,C}/Z)_{C'}) \otimes \mathbf{Q} \to \mathbf{Q}$$

restricts to a pairing on the sub-$\mathcal{H}(U\backslash GL_2(\mathbf{A}_E)/U)$-module $\tilde{W}$ generated by $HD_{C'}$ and induces a non-degenerate pairing on the quotient W of $\tilde{W}$ by the radical. Letting $W_\pi = W[\mathbf{m}_\pi]$ be the π-isotypic component of W, there is a unique E_π-linear symmetric pairing

$$\langle\cdot,\cdot\rangle_\pi : W_\pi \times W_\pi \to E_\pi$$

such that $Tr_{E_\pi/\mathbf{Q}}(e \cdot \langle w, w'\rangle_\pi) = \langle ew, w'\rangle \ \forall e \in E_\pi, w, w' \in W_\pi$. We will denote by $HD_{C',\pi}$ the π-isotypic component of $HD_{C'}$.

Let $\pi_{E'}$ be the base change of π to E' and $L(\pi_{E'}, s)$ its (complete) L-function. Set $\epsilon(\pi_{E'}, s) = q^{-8(g_C-1)(s-1/2)}$ where g_C is the genus of C. Finally, let $L(\pi, Ad, s)$ be the adjoint L-function of π.

7.16. THEOREM. *(cf. [**YZ17**, Theorem 1.2]) With the notations introduced above, the following equality holds:*

$$\frac{q^{2-2g_C}}{2(log(q))^n L(\pi, Ad, 1)} \frac{d^n}{ds^n} \Big(\epsilon(\pi_{E'}, s)^{-1/2} L(\pi_{E'}, s)\Big)_{|s=1/2} = \langle HD_{C',\pi}, HD_{C',\pi}\rangle_\pi.$$

7.17. Extension to the ramified case. In [**YZ18**] the authors extend the previous result to the case where both the cover $q : C' \to C$ and the automorphic representation π are allowed to be ramified. We will briefly discuss how the relevant moduli spaces of Shtukas look like, in order to underline the similarities with the situation over number fields 5.6.1, 5.7.

Let π be a cuspidal automorphic representation of $GL_{2,E}$ ramified at a finite set Σ of closed points of C, which we assume for simplicity to be all k-points. Suppose that π is isomorphic to an unramified twist of the Steinberg representation at every $x \in \Sigma$. Let us also assume that the ramification locus of the cover q is disjoint from Σ. It follows that we can write $\Sigma = \Sigma_f \coprod \Sigma_\infty$ where every point in Σ_f (resp. Σ_∞) is split (resp. inert) in C'. The sign of the functional equation of $L(\pi_{E'}, s)$ equals $(-1)^{|\Sigma_\infty|}$; this situation is akin to the one discussed in 5.4.1.

Let $n \equiv |\Sigma_\infty| \pmod 2$, $m = n + |\Sigma_\infty|$ and $\underline{\mu}$ a m-tuple of signs which sum to 0. Let us consider the fiber product

$$\begin{array}{ccc} Sht^{\underline{n},\underline{\mu},\Sigma_\infty}(GL_{2,C})_{\Sigma_f} & \longrightarrow & Sht^{\underline{m},\underline{\mu}}(GL_{2,C})_{\Sigma_f} \\ \downarrow & & \downarrow \\ (Spec\, k)^{|\Sigma_\infty|} \times C^n & \longrightarrow & C^m \end{array}$$

where the lower horizontal map sends $(Spec\, k)^{|\Sigma_\infty|}$ to Σ_∞ and $Sht^{\underline{m},\underline{\mu}}(GL_{2,C})_{\Sigma_f}$ is the moduli space of Shtukas with m legs and Iwahori level structure at Σ_f. Let $Sht^{\underline{n},\underline{\mu},s\Sigma_\infty}(GL_{2,C})_{\Sigma_f} \subset Sht^{\underline{n},\underline{\mu},\Sigma_\infty}(GL_{2,C})_{\Sigma_f}$ be the substack parametrizing Shtukas which are *supersingular* at all places in Σ_∞ (meaning that the associated Dieudonné module is non trivial). The Heegner-Drinfeld cycle whose self intersection is related to the n-th derivative of $L(\pi_{E'}, s)$ at the central point is a proper cycle in middle dimension inside the quotient by the action of $Pic(C)(k)$ of the base change of $Sht^{\underline{n},\underline{\mu},s\Sigma_\infty}(GL_{2,C})_{\Sigma_f}$ to $(C')^n$, constructed in a similar way as in the unramified case (more details can be found in [**YZ18**, Section 4]).

7.18. Analogies. In the general situation we just considered, one looks at the moduli space of Shtukas with $|\Sigma_\infty|$ fixed legs at Σ_∞, n varying legs on C and Iwahori level structure at Σ_f, with the additional requirement that Shtukas are "supersingular at infinity". A first crucial lesson one can draw from the results we discussed can be summarized in the following slogan:

n-th derivative of the L function $\leftrightarrow$ Shtukas with n legs.

The moduli space of Shtukas with n legs being fibred over C^n, an analogue over number fields can be made sense of (for the time being) only for $n = 0, 1$. The parallel objects in this case should be Shimura varieties (or discrete Shimura sets), although those exist in a much more restricted generality. We will attempt throughout chapter 3 to better understand this analogy. For the time being let us just compare the special value formulas over (totally real) number fields and function fields.

7.19. Example. Let $n = 0$ (hence $|\Sigma_\infty| \equiv 0 \pmod 2$). The situation is then similar to what we discussed in 5.5: the moduli stack $Sht^{0,s\Sigma_\infty}(GL_{2,C})_{\Sigma_f}$ is zero-dimensional, and the set of isomorphism classes of its $\bar{k}$ points is canonically identified with the double coset

$$B^\times(E)\backslash B^\times(\mathbf{A}_E)/U_{\Sigma_f}$$

where B is the quaternion algebra over E ramified exactly at places in Σ_∞ and $U_{\Sigma_f} \subset B^\times(\mathbf{A}_E)$ is a compact open subgroup which is maximal at all places outside Σ_f and Iwahori at places in Σ_f (see [**YZ18**, 3.2.2]). After taking the quotient by $Pic(C)(k)$ we end up with a set which is analogous to the quaternionic Shimura set defined in 2.8. As discussed in remark 5.9, the quantity $a(f, \mathbf{1})$ appearing in Zhang's special value formula 5.6.1 can be seen as the f-isotypic part of the image of $Pic(\mathcal{O}_K)/Pic(\mathcal{O}_F)$ in the relevant quaternionic set. This mirrors the construction of the Heegner-Drinfeld cycle over function fields. Notice that, if one wants to take the analogy seriously, one should work with the Arakelov class groups $\widehat{Pic}(\mathcal{O}_F)$ and $\widehat{Pic}(\mathcal{O}_K)$ instead, since in the function field situation vector bundles on *proper* curves always show up. However one does not see the difference in this context: since K/F is a CM extension the groups of units $\mathcal{O}_F^\times$ and $\mathcal{O}_K^\times$ have the same rank, hence the archimedean part disappears when taking the quotient.

7.20. Example. In the one leg case, if Σ_∞ has only one element then, after quotienting out by $Pic(C)(k)$, the relevant moduli space of Shtukas is isomorphic to the quotient by $Pic(C)(k)$ of the moduli space of elliptic sheaves [**YZ18**, 3.2.3], which are the same thing as Drinfeld modules if one in addition requires the leg to be disjoint from Σ_∞. In this case we obtain a function field analogue of the classical Gross-Zagier formula.

8. Shtukas in mixed characteristic

In this section, which is mostly a condensed form of some parts of [**SW17**] (to which we refer the reader for further details) we will recall the definition of local Shtukas in mixed characteristic. It is a local version of the definition given in the previous section, using as a base curve $\mathbf{Z}_p$. We also explain the relation between Shtukas and p-divisible groups over perfectoid rings, which will play a crucial role in chapter 3. Let us start by recalling some basic notions.

Recall that a Huber ring R is *Tate* if it contains a topologically nilpotent unit ϖ, called a *pseudo-uniformizer*. It is *uniform* if the subring R^o of power-bounded elements is bounded. A Huber pair is a couple (R, R^+) where R is a Huber ring and $R^+ \subset R^o$ is an open, integrally closed subring. The adic spectrum of (R, R^+), whose points are equivalence classes of continuous valuations on R bounded by 1 on R^+, is denoted by $Spa(R, R^+)$. If (R, R^+) is sheafy then $Spa(R, R^+)$ acquires the structure of an *affinoid adic space*, and general adic spaces are obtained gluing such affinoid pieces. For any Huber ring R the notation $Spa(R)$ stands for $Spa(R, R)$.

8.1. Definition.
(1) A *perfectoid ring* is a complete, Tate, uniform ring R such that there exists a pseudo-uniformizer ϖ such that $\varpi^p | p$ in R^o and the p-power Frobenius map
$$\Phi : R^o/\varpi \to R^o/\varpi^p$$
is an isomorphism
(2) A *perfectoid Huber pair* is a Huber pair (R, R^+) with R perfectoid. A perfectoid space is an adic space covered by affinoids of the form $Spa(R, R^+)$ where (R, R^+) is a perfectoid Huber pair.
(3) An *integral perfectoid ring* is a p-complete $\mathbf{Z}_p$-algebra R such that the p-Frobenius map on R/p is surjective, there is an element $\varpi \in R$ such that $\varpi^p = pu$ for some $u \in R^\times$ and the kernel of the Fontaine map
$$\theta : A_{inf}(R) = W(R^\flat) \to R$$
is principal. In the above the equation the *tilt* $R^\flat$ of R is the characteristic p ring $R^\flat = \varprojlim_{x \to x^p} R/p$.

8.2. Remark. The last definition can be recast more naturally in the language of *prisms*: integral perfectoid rings are the same thing as *perfect prisms*. Since we won't need this we will not say more on this point of view, discussed in [**Bha19**, Lecture 4].

Notice that if (R, R^+) is a perfectoid Huber pair then R^+ is an integral perfectoid ring. Indeed, since R^+ is open and integrally closed the Frobenius map on R^+/p is surjective since it is so on R^o/p. To show the existence of ϖ such that $\varpi^p = pu$ for some $u \in R^{+,\times}$ one first checks that ϖ as in (1) can be chosen so that $\varpi^p | p$ in R^+ and then applies [**BMS18**, Lemma 3.9]. The fact that the kernel of the Fontaine map is principal is proved in [**SW17**, Lemma 6.2.7].

8.3. Diamonds. Let Perf be the category of perfectoid spaces in characteristic p, endowed with the *pro-étale* topology. A *diamond* is a sheaf on Perf which can be written as a quotient of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$ (i. e. such that the two projections $R \to X$ are pro-étale).

For an analytic adic space X over $Spa(\mathbf{Z}_p)$, define a presheaf $X^\diamond$ on Perf sending a perfectoid space $T \in \mathsf{Perf}$ to the set of isomorphism classes of data $(T^\sharp, T^\sharp \to X)$ where $T^\sharp$ is an untilt of T. Notice that if X is perfectoid then $Hom(T^\sharp, X) = Hom(T, X^\flat)$, hence $X^\diamond$ is just the functor of points of the tilt of X.

8.4. Theorem. *[**SW17**, Proposition 10.2.4, Theorem 10.4.2] The association $X \mapsto X^\diamond$ induces a functor*

$$\{\textit{Analytic adic spaces over } Spa(\mathbf{Z}_p)\} \to \{\textit{diamonds}\}$$

which restricts to an fully faithful functor from seminormal rigid analytic spaces over $\mathbf{Q}_p$ to diamonds endowed with a map to $Spa(\mathbf{Q}_p, \mathbf{Z}_p)^\diamond$.

Moreover, the functor $X \to X^\diamond$ induces an equivalence of sites $X_{et} \simeq (X^\diamond)_{et}$.

8.5. Remark. In other words the previous theorem tells us that the functor $X \to X^\diamond$, which generalizes the tilting functor, somehow forgets the structure map to $\mathbf{Q}_p$ but remembers all the topological information.

8.6. Example. We will denote $Spa(\mathbf{Q}_p, \mathbf{Z}_p)^\diamond$ just by $\mathbf{Q}_p^\diamond$. It is the sheaf sending a perfectoid spaces $T \in \mathsf{Perf}$ to the set of isomorphism classes of its characteristic zero untilts. Similarly, if $L/\mathbf{Q}_p$ is a finite extension, then $L^\diamond(T)$ is the set of isomorphism classes of untilts $T^\sharp$ of T together with a structure map $T^\sharp \to Spa(L, \mathcal{O}_L)$.

Taking products, we can construct the diamond $(\mathbf{Q}_p^\diamond)^k$ for any positive integer k. As a functor, it sends $T \in \mathsf{Perf}$ to the set of isomorphism classes of k-tuples of characteristic zero untilts of T.

8.7. We will need a refinement of theorem 8.4 to adic spaces which are not necessarily analytic. One can define in the same way as above a functor

$$\begin{aligned} \{\text{Adic spaces over } \mathbf{Z}_p\} \to & Sh(\mathsf{Perf}) \\ X \mapsto & Spd(X) \end{aligned}$$

but one finds sheaves which are in general not diamonds (for example for $X = \mathbf{F}_p$ one obtains the constant sheaf sending everything to a point, which is not a diamond). However $Spd(X)$ is a sheaf in the v-topology (generated by definition by open covers and *all* surjective maps of affinoids). For example $Spd(\mathbf{Z}_p)$ sends T to the set of isomorphism classes of *all* the untilts of T. Removing the point corresponding to the positive characteristic untilt one finds the diamond $\mathbf{Q}_p^\diamond$. Every diamond is also a v-sheaf [**SW17**, Theorem 17.1.6], hence in particular the functor of points of any $T \in \mathsf{Perf}$ is a v-sheaf, which we abusively denote still by T.

8.8. Proposition. *Let $L/\mathbf{Q}_p$ be a finite extension and $T = Spa(S, S^+)$ be an affinoid perfectoid space in characteristic p. Then $T \times Spd(\mathcal{O}_L)$ coincides with the diamond attached to the analytic adic space $Spa(W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L) \setminus V([\varpi])$ where $[\varpi]$ is the Teichmuller lift of any pseudo-uniformizer in S^+.*

PROOF. This is a direct generalization of [**SW17**, Proposition 11.2.1]. First of all, the space $Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])$ is an increasing union of the rational subsets $Spa(R_n,R_n^+)$ defined for $n\geq 1$ by $|p|\leq |[\varpi]^{\frac{1}{p^n}}|\neq 0$. Hence R_n is the ring obtained by first taking the $[\varpi]$-adic completion of $W(R^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L[p/[\varpi^{1/p^n}]]$ and then inverting $[\varpi]$. After base change to $\mathcal{O}_L[p^{1/p^\infty}]$ and completion R_n becomes a perfectoid ring, which implies that (R_n,R_n^+) is sheafy. It follows that $Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])$ is an analytic adic space. Let us now check that the associated diamond coincides with $T\times Spd(\mathcal{O}_L)$.

Giving an (A,A^+)-point of $T\times Spd(\mathcal{O}_L)$ amounts to giving a map of Huber pairs $(S,S^+)\xrightarrow{f}(A,A^+)$ plus an untilt (B,B^+) of (A,A^+) with a structure map $\mathcal{O}_L\to B^+$. Giving the map f is in turn the same as giving a continuous map $S^+\to A^+$ sending ϖ to a unit in A.

On the other hand, a map $Spa(A,A^+)\to\big(Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])\big)^\diamond$ is the datum of an untilt (B,B^+) of (A,A^+) together with a map $Spa(B,B^+)\to Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])$. This is the same as a continuous map $g: W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L\to B^+$ such that the image of $[\varpi]$ is invertible in B. On the other hand, g corresponds to a continuous map $W(S^+)\to B^+$ plus a structure map $\mathcal{O}_L\to B^+$. Finally, maps from $W(S^+)$ to B^+ are naturally identified with maps from S^+ to A^+, and those which send $[\varpi]$ to a unit in B correspond to those sending ϖ to a unit in A. Hence the proposition is proved. □

8.9. REMARK. Observe that the ring R_n in the proof of the above proposition can be written as $R_n = R_{n,0}\otimes_{\mathbf{Z}_p}\mathcal{O}_L$ where $R_{n,0}$ is the the ring obtained by $[\varpi]-$adically completing $W(R^+)[p/[\varpi^{1/p^n}]]$ and then inverting $[\varpi]$. In other words, the space $Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])$ is covered by the adic spectra of the rings covering $Spa(W(S^+))\setminus V([\varpi])$ tensored by $\mathcal{O}_L$.

8.10. EXAMPLE. Let $\mathbf{C}_p$ be the completion of the algebraic closure of $\mathbf{Q}_p$ and $C=\mathbf{C}_p^\flat$. Let $\mathcal{O}_C$ be the ring of integers of C. Then $Spa(C,\mathcal{O}_C)\times Spd(\mathbf{Z}_p) = Spa(A_{inf}(\mathcal{O}_{\mathbf{C}_p}))\setminus V([\varpi])^\diamond$, where $A_{inf}(\mathcal{O}_{\mathbf{C}_p})$ is the classical period ring of Fontaine. Removing the divisor $p=0$ and dividing by the action of the Frobenius φ on $Spa(C,\mathcal{O}_C)$ one obtains (the diamond associated to) the Fargues-Fontaine curve:

$$X^{FF,\diamond}_{\mathbf{Q}_p,C}=\varphi^{\mathbf{Z}}\backslash Spa(C,\mathcal{O}_C)\times\mathbf{Q}_p^\diamond.$$

Fargues and Fontaine proved that this curve is geometrically simply connected: every étale cover is of the form $\varphi^{\mathbf{Z}}\backslash Spa(C,\mathcal{O}_C)\times L^\diamond$ for a finite extension $L/\mathbf{Q}_p$. It follows that $\pi_1(X^{FF}_{\mathbf{Q}_p,C})\simeq\Gamma_{\mathbf{Q}_p}$, hence $\pi_1(\varphi^{\mathbf{Z}}\backslash\mathbf{Q}_p^\diamond\times\mathbf{Q}_p^\diamond)\simeq(\Gamma_{\mathbf{Q}_p})^2$. This is a version of Drinfeld lemma in the world of diamonds, which extends to a similar statement for a product of an arbitrary number of copies of $\mathbf{Q}_p^\diamond$ [**SW17**, Lecture 16]. Notice in particular that $\mathbf{Q}_p^\diamond\times\mathbf{Q}_p^\diamond$ is a geometric object with fundamental group close to the product $\Gamma_{\mathbf{Q}_p}\times\Gamma_{\mathbf{Q}_p}$. This is crucial for our purposes.

8.11. Vector bundles. A vector bundle on an adic space X is a locally free sheaf of $\mathcal{O}_X$-modules. This notion is well behaved thanks to a result of Kedlaya-Liu [**SW17**, Theorem 5.2.8] stating that, if $X=Spa(A,A^+)$ is affinoid analytic (and sheafy), then the functor from finite projective A-modules of rank d to rank d vector bundles on X is an equivalence of categories. As a consequence we obtain the following (not surprising) lemma:

8.12. LEMMA. *Let $L/\mathbf{Q}_p$ be a finite extension of degree r and $T=Spa(S,S^+)\in\mathsf{Perf}$. Then the pushforward of a rank d vector bundle on $Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])$ is a vector bundle of rank rd on $Spa(W(S^+))\setminus V([\varpi])$.*

PROOF. As remarked in 8.9, $Spa(W(S^+))\setminus V([\varpi])$ (resp. $Spa(W(S^+)\otimes_{\mathbf{Z}_p}\mathcal{O}_L)\setminus V([\varpi])$) is an increasing union of affinoids $Spa(R_{n,0},R^+_{n,0})$ (resp. $Spa(R_{n,0}\otimes_{\mathbf{Z}_p}\mathcal{O}_L,R^+_{n,0}\otimes_{\mathbf{Z}_p}\mathcal{O}_L)$). By the above mentioned result a rank d vector bundle $\mathcal{E}$ on $Spa(W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L) \setminus V([\varpi])$ is a compatible sequence of finite projective modules of rank d on $R_{n,0} \otimes_{\mathbf{Z}_p} \mathcal{O}_L$. Pushing forward amounts to remembering only the $R_{n,0}$-module structure, yielding a compatible sequence of projective modules of rank rd, hence a vector bundle on $Spa(W(S^+)) \setminus V([\varpi])$ of rank rd. $\square$

8.13. NOTATION. With the notations as in the lemma, we will often speak of *vector bundles on* $Spd(\mathbf{Z}_p)\times T$ meaning vector bundles on $Spa(W(S^+))\setminus V([\varpi])$. With this terminology the lemma says that the pushforward of a vector bundle of rank d along the map $Spd(\mathcal{O}_L)\times T \to Spd(\mathbf{Z}_p)\times T$ is a vector bundle of rank rd.

8.14. Let $T = Spa(S, S^+) \in \mathsf{Perf}$. Then maps $x : T \to Spd(\mathbf{Z}_p)$ correspond to untilts $T^\sharp = Spa(S^\sharp, S^{+,\sharp})$ of T. For any such untilt the Fontaine map $\theta : W(S^+) \to S^{+,\sharp}$ induces a map $\Gamma_x : T^\sharp \to Spa(W(S^+)) \setminus V([\varpi])$. This map identifies $T^\sharp$ with a closed Cartier divisor in $Spa(W(S^+)) \setminus V([\varpi])$. This means that locally on $Spa(W(S^+)) \setminus V([\varpi])$ multiplication by a generator of the kernel of θ is an injective map with closed image, which is proved in [**SW17**, Proposition 11.3.1]. We will call Γ_x the *graph* of the map x. If $\underline{x} = (x_1, \ldots, x_n) : T \to (Spd(\mathbf{Z}_p))^n$ we define $\Gamma_{\underline{x}}$ as the union of the graphs of the x_i. We are finally ready to define local Shtukas (for the general linear group).

8.15. DEFINITION. Let $T \in \mathsf{Perf}$ and $\underline{x} : T \to (Spd(\mathbf{Z}_p))^n$. A Shtuka of rank d over T with legs at $\underline{x}$ is a rank d vector bundle $\mathcal{E}$ over $T \times Spd(\mathbf{Z}_p)$ together with an isomorphism

$$\phi : \mathcal{E}_{|T\times Spd(\mathbf{Z}_p)\setminus\Gamma_{\underline{x}}} \longrightarrow {}^\tau\mathcal{E}_{|T\times Spd(\mathbf{Z}_p)\setminus\Gamma_{\underline{x}}}$$

where ${}^\tau\mathcal{E}$ is the pullback of $\mathcal{E}$ via the morphism $Id_{\mathbf{Z}_p} \times \varphi_T : Spd(\mathbf{Z}_p) \times T \to Spd(\mathbf{Z}_p) \times T$. We furthermore require ϕ to be meromorphic along $\Gamma_{\underline{x}}$.

8.16. REMARK. The definition is formally analogous to the one over function fields, except for the additional meromorphicity condition which was not necessary in that setting.

8.17. Shtukas with no legs. Let us start describing the easiest case: Shtukas with no leg, i.e. vector bundles $\mathcal{E}$ over $T \times Spd(\mathbf{Z}_p)$ together with an isomorphism ${}^\tau\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. The restriction of any such $\mathcal{E}$ to $Spa(W(S^+)) \setminus V(p[\varpi])$ is Frobenius-equivariant, hence it descends to a vector bundle $\tilde{\mathcal{E}}$ on the relative Fargues-Fontaine curve $X^{FF}_{\mathbf{Q}_p,T} = \varphi_T^{\mathbf{Z}}\backslash T \times \mathbf{Q}_p^\diamond$. Moreover, the fact that $\mathcal{E}$ lives on $Spa(W(S^+)) \setminus V([\varpi])$ implies that $\tilde{\mathcal{E}}$ is semistable of degree 0, and the associated étale $\mathbf{Q}_p$-local system on $Spa(S, S^+)$ comes from a $\mathbf{Z}_p$-local system. In fact we have the following

8.18. LEMMA. *([**KL15**, Theorem 8.5.3]) The category of rank d Shtukas with no legs over T is equivalent to the category of étale $\mathbf{Z}_p$-local systems of rank d on T.*

The main point in the proof of the above result is to show that $\mathbf{Z}_p$-local systems on T are the same as φ-modules over $W(S^+)$, which is a generalization of Artin-Schreier theory. Those are in turn the same as φ-modules over the integral Robba ring, which coincide with Shtukas with no legs. Once again, the situation is analogous to the one over function fields 7.12 (to compare the two descriptions one has to pass from étale local systems to $GL_d(\mathbf{Z}_p)$-torsors).

8.19. Shtukas with one leg. Let $T = Spa(S, S^+) \in \mathsf{Perf}$ and $\mathcal{E}$ a rank d Shtuka on T with a leg at $x : T \to Spd(\mathbf{Z}_p)$, corresponding to an untilt $T^\sharp$ of T. Let ξ be a generator of the kernel of the corresponding Fontaine map. Then $\mathcal{E}$ is given locally on $Spa(W(S^+)) \setminus V([\varpi])$ by a projective module M together with an isomorphism between $M[\frac{1}{\xi}]$ and its Frobenius twist. This is close to the following familiar object in p-adic Hodge theory:

8.20. Definition. A *Breuil-Kisin-Fargues* module over an integral perfectoid ring R is a finite projective $A_{inf}(R)$-module M together with an isomorphism $\phi_M : (\varphi^* M)[1/\xi] \xrightarrow{\sim} M[1/\xi]$, where $\varphi^* M := M \otimes_{A_{inf}(R),\varphi} A_{inf}(R)$. We say that M is *minuscule* if ϕ_M comes from a map $(\varphi^* M) \to M$ whose cokernel is killed by ξ.

8.21. Let $T = Spa(S, S^+) \in \mathsf{Perf}$ and $T^\sharp = Spa(S^\sharp, S^{+,\sharp})$ an untilt. Any Breuil-Kisin-Fargues module M over $S^{+,\sharp}$ gives rise by restriction to a Shtuka over $T \times Spd(\mathbf{Z}_p)$ with one leg at $T^\sharp$. Together with the content of the next paragraph, this is what allows to make the connection between p-divisible groups over $S^{+,\sharp}$ and Shtukas with one leg. Before explaining this, let us record a different description of Shtukas with a leg in characteristic zero.

8.22. Lemma. *(cf. [**Ked17**, Lemma 4.5.13, Remark 4.5.14]) Let $T \in \mathsf{Perf}$ and $x : T \to \mathbf{Q}_p^\diamond$. Then the following categories are equivalent:*

1. *Shtukas over T with one leg at x.*
2. *Triples $(\mathcal{F}, \mathcal{L}, \beta)$ where $\mathcal{F}$ is a vector bundle on the relative Fargues-Fontaine curve $X^{FF}_{\mathbf{Q}_p,T}$, $\mathcal{L}$ is an étale $\mathbf{Z}_p$-local system on T and β is an isomorphism between $\mathcal{F}$ and the vector bundle attached to $\mathcal{L}$ outside the image of Γ_x.*

The equivalence is constructed as follows: given a Shtuka $\mathcal{E}$, restricting it to a sufficiently small neighbourhood of $V(p)$ in $Spa(W(S^+)) \setminus V([\varpi])$ we obtain a φ^{-1}-equivariant vector bundle (here we use that the leg factors through $\mathbf{Q}_p^\diamond$), which comes from a unique local system $\mathcal{L}$. On the other hand, restricting to a suitable punctured open around $V([\varpi])$ we obtain another φ-equivariant vector bundle, which gives a vector bundle $\mathcal{F}$ on the Fargues-Fontaine curve. The modification β tells how to glue $\mathcal{F}$ and the vector bundle attached to $\mathcal{L}$ to get $\mathcal{E}$.

8.23. Notation. If $\mathcal{E}$ is a Shtuka over T with one leg, we will denote by $\mathcal{E}^\infty$ the vector bundle on $X^{FF}_{\mathbf{Q}_p,T}$ induced by the restriction of $\mathcal{E}$ to an open neighbourhood of $V([\varpi])$ not meeting the leg.

8.24. Shtukas and p-divisible groups. Let us finally come to the relation between Shtukas with one leg and p-divisible groups. This is mostly explained in the Berkeley notes [**SW17**], building on the results in [**SW13**]. Most results have also been proved in a different way in [**Lau18**]. For any ring R which is either integral perfectoid or a quotient of a perfect ring by a finitely generated ideal, let $A_{cris}(R)$ be the p-adic completion of the PD-hull of $A_{inf}(R) \to R$.

Let $BT(R)$ be the category of p-divisible groups over $Spec\, R$ and $BT^0(R)$ the category of p-divisible groups up to isogeny. Evaluating the Dieudonné crystal of a p-divisible group $\mathcal{G}$ on $A_{cris}(R) \to R$ gives a finite projective $A_{cris}(R)$-module $M(\mathcal{G})$.

Dieudonné theory. Let us now suppose that R is of characteristic p. Then $A_{cris}(R)$ has a natural Frobenius map φ, and is the universal p-complete PD-thickening of R. We obtain a functor

$$\mathbb{D} : BT(R) \to D(R)$$

where $D(R)$ is the category whose objects are finite projective $A_{cris}(R)$-modules M together with maps

$$\begin{aligned} F :& M \otimes_{A_{cris}(R),\varphi} A_{cris}(R) \to M \\ V :& M \to M \otimes_{A_{cris}(R),\varphi} A_{cris}(R) \end{aligned}$$

such that $FV = VF = p$. The universal property of $A_{cris}(R)$ implies that this category is equivalent to the category of Dieudonné crystals over $Spec(R)$ (as defined in [**CL17**, Section 2.4]). Set $B^+_{cris}(R) = A_{cris}(R)[\frac{1}{p}]$. Inverting p we obtain a functor from $BT^0(R)$ to the category of finite projective $B^+_{cris}(R)$-modules N together with an isomorphism $N \otimes_{B^+_{cris}(R),\varphi} B^+_{cris}(R) \to N$, which is equivalent to the category $F - isoc(Spec(R))$ of F-isocrystals on $Spec(R)$. Finally, one can check that elements in $B^+_{cris}(R)$ converge in a suitable neighbourhood of $[\varpi]$ in $Spa(W(R^+))$, hence any such module N gives a φ-equivariant vector bundle on this neighbourhood.

8.25. Theorem. *Let $T = Spa(S, S^+) \in \mathsf{Perf}$ and $\varpi \in S^+$ a pseudo-uniformizer. Then the following statements hold true:*

*(1) ([**Lau18**, Corollary 5.15]) The Dieudonné module functor $\mathbb{D} : BT(S^+/\varpi) \to D(S^+/\varpi)$ is fully faithful.*

*(2) ([**Far16**, Proposition 6.1]) The category $F - isoc(Spec(S^+/\varpi))$ is equivalent to the category of φ-equivariant vector bundles on $Spa(W(S^+)) \setminus V(p)$.*

*(3) ([**Far16**, Corollary 6.3]) Composing the embedding in (1) (up to isogeny) , the equivalence in (2) and restriction to $Spa(W(S^+)) \setminus V(p[\varpi])$ induces a fully faithful functor*

$$BT^0(S^+/\varpi) \to Bun_{X^{FF}_{\mathbf{Q}_p,T}}$$

where $Bun_{X^{FF}_{\mathbf{Q}_p,T}}$ denotes the category of vector bundles on the relative Fargues-Fontaine curve $X^{FF}_{\mathbf{Q}_p,T}$.

p-divisible groups and Breuil-Kisin-Fargues modules. Let us now take R integral perfectoid (e.g. the ring of bounded elements in a perfectoid Huber pair). In this case p-divisible groups over $Spec(R)$ are the same as minuscule Breuil-Kisin-Fargues modules over R:

8.26. Theorem. *([**SW17**, Theorem 17.5.2], [**Lau18**, Theorem 1.5]) There is an equivalence of categories, functorial in R, between p-divisible groups over $Spec(R)$ and* minuscule *Breuil-Kisin-Fargues modules over R. Under this equivalence, if M is the module attached to a p-divisible group $\mathcal{G}$, then $M \otimes_{A_{inf}(R)} A_{cris}(R)$ coincides with the evaluation of the Dieudonné crystal of $\mathcal{G}$ at $A_{cris}(R) \to R$.*

After restriction to $Spa(A_{inf}(R)) \setminus V([\varpi])$, compatibility of the above classifications yields:

8.27. Corollary. *Let (R, R^+) be a perfectoid Huber pair. Then there is a functor from p-divisible groups over $Spec(R^+)$ to Shtukas over $T = Spa(R^\flat, R^{+,\flat})$ with one leg at (R, R^+). Moreover, if $\mathcal{E}$ is the Shtuka attached to a p-divisible group $\mathcal{G}$, then $\mathcal{E}^\infty \in Bun_{X^{FF}_{\mathbf{Q}_p,T}}$ is the vector bundle attached to $\mathcal{G} \times_{R^+} R^+/\varpi$ via the functor in 8.25 (notice that $R^+/\varpi = R^{+,\flat}/\varpi^\flat$).*

In general the functors defined above relating the categories of p-divisible groups, minuscule Breuil-Kisin-Fargues modules and (minuscule) Shtukas are *not* equivalences. There is however an important special case in which all these notions coincide, i.e. when $(R, R^+) = (C, \mathcal{O}_C)$ with $C/\mathbf{Q}_p$ a complete algebraically closed nonarchimedean field. Let $X^{FF} = X^{FF}_{\mathbf{Q}_p, Spa(C^\flat, \mathcal{O}_{C^\flat})}$ and $\infty \in X^{FF}$ the point given by the untilt $(C, \mathcal{O}_C)$ of $(C^\flat, \mathcal{O}^\flat_C)$. Let B^+_{dR} be the completed local ring of $Spa(A_{inf}(\mathcal{O}_C)) \setminus V([\varpi])$ at C. This is a complete discrete valuation ring with residue field C, isomorphic to the completion of $A_{inf}(\mathcal{O}_C)[\frac{1}{p}]$ at the kernel (ξ) of the Fontaine map. Let $B_{dR} = B^+_{dR}[\frac{1}{\xi}]$. Then we have the following:

8.28. THEOREM. *(Fargues; cf. [**SW17**, Theorem 14.1]) Let $C/\mathbf{Q}_p$ be a complete algebraically closed non archimedean field with ring of integers $\mathcal{O}_C$. Then the following categories are equivalent:*

1. *Shtukas over $Spa(C^\flat, \mathcal{O}_C^\flat)$ with one leg at $(C, \mathcal{O}_C)$.*
2. *Pairs (T, Ξ) where T is a finite free $\mathbf{Z}_p$-module and $\Xi \subset T \otimes_{\mathbf{Z}_p} B_{dR}$ is a B_{dR}^+-lattice.*
3. *Quadruples $(\mathcal{F}, \mathcal{F}', \beta, T)$ where $\mathcal{F}$ is the trivial vector bundle on X^{FF}, $T \subset H^0(X_{FF}, \mathcal{F})$ a $\mathbf{Z}_p$-lattice, $\mathcal{F}' \in Bun(X^{FF})$ and $\beta : \mathcal{F}_{|X^{FF}\setminus\infty} \xrightarrow{\sim} \mathcal{F}'_{|X^{FF}\setminus\infty}$.*
4. *Breuil-Kisin-Fargues modules over $\mathcal{O}_C$.*

Restricting to minuscule objects the above equivalences induce equivalences among:

1. *Shtukas over $Spa(C^\flat, \mathcal{O}_C^\flat)$ with one leg at $(C, \mathcal{O}_C)$ such that the the isomorphism between ${}^\tau\mathcal{E}$ and $\mathcal{E}$ outside the graph of $(C, \mathcal{O}_C)$ extends to a map ${}^\tau\mathcal{E} \to \mathcal{E}$ whose cokernel is killed by ξ.*
2. *Pairs (T, Ξ) such that $\xi(T \otimes_{\mathbf{Z}_p} B_{dR}^+) \subset \Xi \subset T \otimes_{\mathbf{Z}_p} B_{dR}^+$.*
3. *Pairs (T, W) where T is a finite free $\mathbf{Z}_p$-module and $W \subset T \otimes_{\mathbf{Z}_p} C$ a C-subvectorspace.*
4. *Minuscule Breuil-Kisin-Fargues modules over $\mathcal{O}_C$.*
5. *p-divisible groups over $\mathcal{O}_C$.*

8.29. REMARK. The proof is explained in [**SW17**, Lectures 12-14]; we just make some remarks here. To establish the first set of equivalences the hardest point is to prove an extension result for Shtukas from $Spa(A_{inf}(\mathcal{O}_C) \setminus V([\varpi])$ to the whole $Spa(A_{inf}(\mathcal{O}_C))$, yielding the equivalence between (1) and (4). The equivalence between (1) and (3) is a special case of Lemma 8.22, and (1) $\Leftrightarrow$ (2) is easy (the $\mathbf{Z}_p$-lattice gives the vector bundle near $V(p)$ and the B_{dR}^+-lattice tells us how to modify it at the leg).

The second set of equivalences follows immediately from the first and from Theorem 8.26. The new entry (3) is equivalent to (2) because $B_{dR}^+/(\xi) = C$. It is worth pointing out that in fact Scholze-Weinstein at first proved the equivalence between (3) and (5) in [**SW13**] and deduced the rest of the statement from it.

CHAPTER 2

The Bloch-Kato conjecture in analytic rank 0 and 1

1. Outline of the chapter

The aim of this chapter is to study formulas relating special values of L-functions of Hilbert modular forms with arithmetic invariants attached to them, as predicted by the Bloch-Kato conjecture [**BK90**].

We fix the following notations, and make the following assumptions, throughout the chapter:

- Since we only work with automorphic forms of parallel weight 2, we will drop this index from our notations, for simplicity. For example $S_2(\mathfrak{n})$ will be denoted by $S(\mathfrak{n})$. Recall that automorphic forms in this space have *trivial* central character.
- Fix a newform $f \in S(\mathfrak{n})$, and let $V(f) = V_f(1)$ the $\mathfrak{p}$-adic Galois representation attached to f, containing an $\mathcal{O}_\mathfrak{p}$-lattice $T(f)$ as in section I.6 (in particular, $\mathfrak{p}$ is a place of the field generated by the Hecke eigenvalues of f lying above a rational prime p not dividing $\mathfrak{n}$).
- Fix a totally imaginary quadratic extension K/F and an anticyclotomic character of finite order $\chi : K^\times \backslash \hat{K}^\times \to \mathbf{C}$ of conductor $c(\chi)$, and extend $\mathcal{O}_\mathfrak{p}$ so that it contains the values of χ. Let $K_\chi = \bar{K}^{ker(\chi)}$. We assume that $K_\chi \cap H_F^+ = F$, where H_F^+ is the narrow Hilbert class field of F.
- Assume that $p > 3$, p is unramified in K, and the ideals $\mathfrak{n}, disc(K/F)$ and $cond(\chi)$ are coprime to each other. Write $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$ as in I.5.4 and assume that $\mathfrak{n}^-$ is squarefree. For a prime $\mathfrak{q}|\mathfrak{n}^-$ we will often denote with the same symbol the unique prime of K above it.
- Assume that the image of the residual Galois representation $\bar{\rho} : \Gamma_F \to Aut(T_1(f))$ contains (a subgroup conjugated to) $SL_2(\mathbf{F}_p)$, where $T_1(f) = T(f)/\varpi$.

1.1. Warning: The main results in this chapter rely on Ihara lemma for Shimura curves over totally real fields. This was proved by Cheng, but we were informed that his proof contains a gap. The recent preprint [**Zho19**] contains a proof for quaternionic Shimura surfaces which may adapt to the case of Shimura curves, but this has not been written down. Work in progress by Manning and Shotton aims at giving a proof of the lemma (using a different strategy from the one of the previous authors). In what follows we will assume that the lemma holds true.

With the above warning in mind, the main result of this chapter is the following

1.2. THEOREM. *Suppose that the following assumptions are satisfied:*

(1) The level $\mathfrak{n}$ of f, the discriminant $disc(K/F)$, the conductor of χ and the prime p are coprime to each other. Moreover $p > 3$ is unramified in F and does not divide $[K_\chi : K]$, and $K_\chi \cap H_F^+ = F$.

(2) Writing $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$, $\mathfrak{n}^-$ is squarefree.

(3) The image of the residual Galois representation $\bar{\rho}$ contains $SL_2(\mathbf{F}_p)$.

(4) If $\mathfrak{q}|\mathfrak{n}^-$ *then* $N(\mathfrak{q}) \not\equiv -1 \pmod p$. *Moreover if* $N(\mathfrak{q}) \equiv 1 \pmod p$ *then* $\bar{\rho}$ *is ramified at* $\mathfrak{q}$.

(5) If $\mathfrak{q}|\mathfrak{n}^+$ *then* $A_1(f)^{I_\mathfrak{q}} = 0$.

(6) The level $\mathfrak{n}^+$ *is* minimal *for* f_1 *(see definition 3.10).*

Then the following statements hold true.

Definite case: *If* $|\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}^-\}| \equiv [F : \mathbf{Q}] \pmod 2$ *and* $L(f,\chi,1) \neq 0$ *then* $Sel(K, A(f)(\chi))$ *is finite. Denoting by* $L^{alg}(f,\chi,1)$ *the algebraic part of the special value of* $L(f,\chi,1)$ *(see 3.1), the following inequality holds:*

$$length_{\mathcal{O}_\mathfrak{p}} Sel(K, A(\chi)) \leq ord_\varpi(L^{alg}(f,\chi,1)).$$

Indefinite case: *If* $|\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}^-\}| \not\equiv [F : \mathbf{Q}] \pmod 2$ *and* $L'(f,\chi,1) \neq 0$ *then* $Sel(K, A(f)(\chi))$ *has* $\mathcal{O}_\mathfrak{p}$*-corank one. Denoting by* $Sel(K, A(f)(\chi))/div$ *the quotient by its divisible part and by* $c_\chi \in Sel(K, T(f)(\bar{\chi}))$ *the cohomology class obtained from a CM point of conductor* $c(\chi)$ *(defined in section 6), the following inequality holds:*

$$length_{\mathcal{O}_\mathfrak{p}} Sel(K, A(\chi))/div \leq 2ord_\varpi(c_\chi).$$

Moreover both inequalities above are equalities provided that the following implication holds true: if g *is an admissible automorphic form mod* ϖ *(see definition 3.10) and* $Sel(K, A(g)(\chi)) = 0$ *then* $L^{alg}(g,\chi,1)$ *is a* ϖ*-adic unit.*

1.3. Remark.

(1) The relation between the above (in)equalities and the ones predicted by the Bloch-Kato conjecture is explained in remark 3.4.
(2) See remark 3.12 for a discussion of which assumptions in our theorem may be weakened.
(3) The implication guaranteeing that our inequalities are equalities can be proved to hold as a consequence of one divisibility in the Iwasawa main conjecture, when this is known. In fact a weaker result, in the spirit of Ribet's converse of Herbrand theorem, would suffice. This is discussed in remark 3.13.

2. Selmer groups

2.1. Bloch-Kato Selmer groups. Let $V = V(f), V(f)(\chi)$ or $V(f)(\bar{\chi})$. The Bloch-Kato Selmer group of V is defined as

$$Sel(K,V) = ker\left(H^1(K,V) \to \prod_v \frac{H^1(K_v,V)}{H^1_f(K_v,V)}\right)$$

where, for a finite place v of K,

$$H^1_f(K_v,V) = ker\left(H^1(K_v,V) \to \begin{cases} H^1(I_v,V) \text{ if } v \not| p \\ H^1(K_v, V \otimes B_{cris}) \text{ if } v|p \end{cases}\right).$$

We also define Selmer groups $Sel(K,M)$ for $M = T(f)(\bar{\chi}), T_n(f)(\bar{\chi}), A(f)(\chi), A_n(f)(\chi)$ imposing as local conditions $H^1_f(K_v,M)$ those coming from $H^1_f(K_v,V)$ by propagation. In particular under the local Tate pairing at a place v

$$H^1(K_v, T_n(f)(\bar{\chi})) \times H^1(K_v, A_n(f)(\chi)) \to E_p/\mathcal{O}_\mathfrak{p}$$

the local conditions $H^1_f(K_v, T_n(f)(\bar{\chi}))$ and $H^1_f(K_v, A_n(f)(\chi))$ are annihilators of each other, since the same is true for the Bloch-Kato local condition on V. Let us also remark that, under the identifications $T_n(f)(\bar{\chi}) = {}^c(A_n(f)(\chi))$, inducing an isomorphism $H^1(K, T(g_n)(\bar{\chi})) \simeq H^1(K, A_n(f)(\chi))$, the local Tate pairing becomes a perfect pairing

$$H^1(K_v, A_n(f)(\chi)) \times H^1(K_{\bar{v}}, A_n(f)(\chi)) \to E_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}} \tag{2.1.1}$$

under which $H^1_f(K_v, A_n(f)(\chi))$ and $H^1_f(K_{\bar{v}}, A_n(f)(\chi))$ are annihilators of each other.

2.2. Our aim is to determine the local conditions defining $Sel(K, A_n(f)(\chi))$ more explicitly. Precisely, imposing suitable hypotheses on the Galois representation $T_1(f)$, we wish to describe these local conditions *purely in terms of the Galois representation* $A_n(f)$ and of the level $\mathfrak{n}$.

2.3. LEMMA. *Let v be a finite place of K not dividing p. Then:*

(1) $H^1_f(K_v, V(f)(\chi)) = 0$;
(2) $H^1_f(K_v, A(f)(\chi)) = 0$;
(3) $H^1(K_v, T(f)(\bar{\chi})) = H^1_f(K_v, T(f)(\bar{\chi}))$ *is finite.*

PROOF. This follows from the fact that, if E/F is a finite extension and w is a finite place of E not dividing p, then $H^1(E_w, V(f)) = 0$ (cf. [**Nek12**, Proposition 2.7.8]), which is a consequence of the fact that $V(f)$ has weight -1 and of local duality. This implies the first statement; the second and third statement follow from the fact that the other local conditions are obtained from $V(f)(\chi)$ by propagation. □

2.4. Local conditions at places above p. Fix a place v of K above p. We will describe the Bloch-Kato local condition at v in terms of flat cohomology of p-divisible groups. This material is discussed more in detail in the appendix of [**Nek12**]; we collect here the main results which we will need. Let us set $\mathcal{K} = K_v$ and let $\mathcal{R}$ be the ring of integers of $\mathcal{K}$. If $\mathcal{G}$ is a commutative finite flat group scheme over $\mathcal{R}$ then $\mathcal{G} \times_{\mathcal{R}} \mathcal{K}$ is a finite étale group scheme over $\mathcal{K}$, determined by the $\Gamma_{\mathcal{K}}$-module $\mathcal{G}(\bar{\mathcal{K}})$. We obtain a functor $\mathcal{G} \mapsto \mathcal{G}(\bar{\mathcal{K}})$ from the category of finite flat group schemes over $\mathcal{R}$ to the category of $\Gamma_{\mathcal{K}}-$modules. This functor induces maps

$$H^i_{fl}(\mathcal{R}, \mathcal{G}) \to H^i_{ét}(\mathcal{K}, \mathcal{G}_{\mathcal{K}}) = H^i(\Gamma_{\mathcal{K}}, \mathcal{G}(\bar{\mathcal{K}})).$$

2.5. PROPOSITION. *(Local flat duality, [**Nek12**, A.2.3]) Let $\mathcal{G}^D$ be the Cartier dual of $\mathcal{G}$.*

(1) $H^0_{fl}(\mathcal{R}, \mathcal{G}) \simeq H^0(\mathcal{K}, \mathcal{G}(\bar{\mathcal{K}}))$.
(2) $H^i_{fl}(\mathcal{R}, \mathcal{G}) = 0$ *for* $i > 1$.
(3) Under the Tate pairing

$$H^1(\mathcal{K}, \mathcal{G}_{\mathcal{K}}) \times H^1(\mathcal{K}, \mathcal{G}^D_{\mathcal{K}}) \to H^2(\mathcal{K}, \mathbf{G}_m) \simeq \mathbf{Q}/\mathbf{Z}$$

the orthogonal complement of $H^1_{fl}(\mathcal{R}, \mathcal{G})$ *is* $H^1_{fl}(\mathcal{R}, \mathcal{G}^D)$.

2.6. PROPOSITION. *(cf. [**Nek12**, A.2.6]) Let $\mathcal{G} = \{\mathcal{G}_n\}_{n\geq 1}$ be a p-divisible group over $\mathcal{R}$, with dual $\mathcal{G}^D = \{\mathcal{G}^D_n\}_{n\geq 1}$. Let $T_p(\mathcal{G}) := \varprojlim_n \mathcal{G}_n(\bar{\mathcal{K}})$ be the Tate module of $\mathcal{G}$ and $V_p(\mathcal{G}) = T_p(\mathcal{G}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. The following statements hold true.*

(1) $V_p(\mathcal{G})$ *is a cristalline representation of* $\Gamma_{\mathcal{K}}$.
(2) For $m, n \geq 1$ the multiplication map $j_m : \mathcal{G}_{m+n} \to \mathcal{G}_n$ induces an isomorphism

$$H^1_{fl}(\mathcal{R}, \mathcal{G}_{m+n}) \otimes \mathbf{Z}/p^n\mathbf{Z} \to H^1_{fl}(\mathcal{R}, \mathcal{G}_n).$$

(3) $X(\mathcal{G}) = \varprojlim_n H^1_{fl}(\mathcal{R}, \mathcal{G}_n)$ *is a $\mathbf{Z}_p$-module of finite type and, for $n \geq 1$, we have*

$$X(G)/p^n X(G) \simeq H^1_{fl}(\mathcal{R}, \mathcal{G}_n).$$

(4) *The subgroup* $X(\mathcal{G}) \hookrightarrow \varprojlim_n H^1(\mathcal{K}, \mathcal{G}_n(\bar{\mathcal{K}})) = H^1(\mathcal{K}, T_p(\mathcal{G}))$ *coincides with the Bloch-Kato subspace*

$$H^1_f(\mathcal{K}, T_p(\mathcal{G})) = ker(H^1(\mathcal{K}, T_p(\mathcal{G})) \to H^1(\mathcal{K}, V_p(\mathcal{G}) \otimes B_{cris})).$$

PROOF. The first point is [**Fon82**, Theorem 6.2]. Let us now sketch the argument for (2) − (4). The long exact sequence in cohomology obtained from the short exact sequence $0 \to \mathcal{G}_m \xrightarrow{i_n} \mathcal{G}_{m+n} \xrightarrow{j_m} \mathcal{G}_n \to 0$, together with (2) in the last proposition, yields:

$$H^1_{fl}(\mathcal{R}, \mathcal{G}_m) \xrightarrow{i_n} H^1_{fl}(\mathcal{R}, \mathcal{G}_{m+n}) \xrightarrow{j_m} H^1_{fl}(\mathcal{R}, \mathcal{G}_n) \to 0$$

(we are abusively denoting with the same symbol a map between sheaves and the induced map in cohomology) hence $H^1_{fl}(\mathcal{R}, \mathcal{G}_n) = H^1_{fl}(\mathcal{R}, \mathcal{G}_{m+n})/i_n(H^1_{fl}(\mathcal{R}, \mathcal{G}_m))$. Moreover the composite $\mathcal{G}_{m+n} \xrightarrow{j_n} \mathcal{G}_m \xrightarrow{i_n} \mathcal{G}_{m+n}$ is multiplication by p^n. Since $H^1_{fl}(\mathcal{R}, \mathcal{G}_{m+n}) \xrightarrow{j_n} H^1_{fl}(\mathcal{R}, \mathcal{G}_m)$ is surjective (because of the vanishing of $H^2_{fl}(\mathcal{R}, \mathcal{G}_n)$) we deduce the equality $p^n H^1_{fl}(\mathcal{R}, \mathcal{G}_{m+n}) = Im(i_n : H^1_{fl}(\mathcal{R}, \mathcal{G}_m) \to H^1_{fl}(\mathcal{R}, \mathcal{G}_{m+n}))$. This proves (2), and (3) follows taking the inverse limit over m.

To show (4), one first argues, using (1), that $X(\mathcal{G}) \hookrightarrow H^1_f(\mathcal{K}, T_p(\mathcal{G})) \subset H^1(\mathcal{K}, T_p(\mathcal{G}))$. An analogous inclusion holds with $\mathcal{G}^D$ in place of $\mathcal{G}$. Moreover by (3) of proposition 2.5 we see that $H^1_{fl}(\mathcal{R}, T_p(\mathcal{G}))$ and $H^1_{fl}(\mathcal{R}, T_p(\mathcal{G}^D))$ are annihilators of each other under the local Tate pairing. Since the same is true for the Bloch-Kato local condition, the statement follows. □

2.7. PROPOSITION. (1) *Let v be a place of F above p. Then there exists a p-divisible group $\mathcal{G}/\mathcal{O}_{F_v}$ with an action of $\mathcal{O}_\mathfrak{p}$ such that $T_p(\mathcal{G}) = T(f)_{|G_{F_v}}$.*

(2) *For $\mathcal{G}$ as in* (1) *and $n \geq 1$ we have*

$$H^1_f(K_w, T_n(f)) = H^1_{fl}(\mathcal{O}_w, \mathcal{G}[\varpi^n])$$

where w is a place of K above v.

(3) *Let w be a place of K above p and $\mathcal{H}/\mathcal{O}_w$ a finite flat group scheme such that $T_n(f) \simeq \mathcal{H}(\bar{K}_w)$ as $\mathcal{O}_\mathfrak{p}/\varpi^n[\Gamma_{K_w}]$-modules. Then $H^1_f(K_w, T_n(f)) = H^1_{fl}(\mathcal{O}_w, \mathcal{H})$.*

PROOF. The existence of $\mathcal{G}$ such that $T_p(\mathcal{G}) = T(f)_{|G_{F_v}}$ is proved in [**Tay95**, Theorem 1.6]. Invoking proposition 2.6 we obtain that $H^1_f(K_w, T(f)) = X(\mathcal{G})$. It follows that

$$\begin{aligned} H^1_f(K_w, T_n(f)) =& Im(H^1_f(K_w, T(f)) \to H^1(K_w, T_n(f))) \\ =& Im(X(\mathcal{G}) \to H^1(K_w, \mathcal{G}[\varpi^n](\bar{K}_w))) = H^1_{fl}(\mathcal{O}_w, \mathcal{G}[\varpi^n]) \end{aligned}$$

where the last equality is proved as in (3) in proposition 2.6.

Finally let us prove the third statement. We have $\mathcal{G}[\varpi^n](\bar{K}_w) \simeq T_n(f) \simeq \mathcal{H}(\bar{K}_w)$. Since p is odd and unramified in K, by [**Ray74**, Corollary 3.3.6] the isomorphism $\mathcal{G}[\varpi^n](\bar{K}_w) \simeq \mathcal{H}(\bar{K}_w)$ comes from an isomorphism $\mathcal{G}[\varpi^n] \simeq \mathcal{H}$, under which $H^1_{fl}(\mathcal{O}_w, \mathcal{G}[\varpi^n])$ and $H^1_{fl}(\mathcal{O}_w, \mathcal{H})$ are identified in $H^1(K_w, T_n(f))$. □

2.8. REMARK. This kind of description of the local condition at a finite place obtained in the third point of the above proposition, expressed purely in terms of the place itself and of the G_{K_w}-module $T_n(f)$, is the prototypical kind of statement we are looking for, as it allows to compare local conditions defining Selmer groups of automorphic forms of different level which are congruent modulo ϖ^n.

2.9. Local condition at places outside $\mathfrak{n}p$. If v is a finite place of K not dividing $\mathfrak{n}p$ then $V(f)$ is unramified at v. Since the unramified local condition is stable under propagation on unramified Γ_K-modules it follows that the local condition on $H^1(K_v, M)$ for $M = A_n(f), T_n(f)$ is the unramified one, i.e.

$$H^1_f(K_v, M) = ker\left((H^1(K_v, M) \to H^1(I_v, M)\right).$$

2.10. Local condition at places dividing $\mathfrak{n}^-$. We want to express local conditions at places dividing $\mathfrak{n}^-$ in terms of the Galois module $A_n(f)$. To do this we will need the following

2.11. Assumption. Assume that, if $\mathfrak{q}|\mathfrak{n}^-$ and $N(\mathfrak{q}) \equiv \pm 1 \pmod p$, then $T_1(f)$ is ramified at $\mathfrak{q}$.

2.12. Lemma. *(cf. [**PW11**, Lemma 3.5]) Let $\mathfrak{q}|\mathfrak{n}^-$ and $n \geq 1$. Under the above assumption there exists a unique submodule $A^{\mathfrak{q}}_n(f)$ of $A_n(f)$ free of rank one over $\mathcal{O}_{\mathfrak{p}}/\varpi^n$ on which $\Gamma_{K_{\mathfrak{q}}}$ acts via the cyclotomic character.*

Proof. For $n = 1$ this follows from assumption 2.11. Indeed by lemma I.6.3 the $\Gamma_{K_{\mathfrak{q}}}$-module $A_1(f)$ is of the form

$$\begin{pmatrix} \chi_{cyc} & * \\ 0 & 1 \end{pmatrix}$$

If $A_1(f)^{I_{\mathfrak{q}}}$ is one dimensional, which is always the case when $N(\mathfrak{q}) \equiv \pm 1 \pmod p$, then it is the only subspace on which $\Gamma_{K_{\mathfrak{q}}}$ acts via the cyclotomic character. Otherwise $N(\mathfrak{q}) \not\equiv \pm 1 \pmod p$, so $Fr_{\mathfrak{q}}$ acting on $A_1(f)^{I_{\mathfrak{q}}}$ has distinct eigenvalues, hence the claim follows. The statement for general n can then be established by induction. □

2.13. Proposition. *With the notations of the previous lemma, we have:*

$$H^1_f(K_{\mathfrak{q}}, A_n(f)) = Im(H^1(K_{\mathfrak{q}}, A^{\mathfrak{q}}_n(f)) \to H^1(K_{\mathfrak{q}}, A_n(f))).$$

Proof. Let us denote the $\Gamma_{K_{\mathfrak{q}}}-$module $T(f)$ by T. Then we have an exact sequence

$$0 \to T^+ \to T \to T^- \to 0$$

where $T^+ \simeq \mathcal{O}_{\mathfrak{p}}(1)$ and $T^- \simeq \mathcal{O}_{\mathfrak{p}}$. By lemma 2.3 we have $H^1_f(K_{\mathfrak{q}}, T) = H^1(K_{\mathfrak{q}}, T)$; examining the long exact sequence in cohomology coming from the above short exact sequence one sees that the map $H^1(K_{\mathfrak{q}}, T^+) \to H^1(K_{\mathfrak{q}}, T)$ is surjective, hence $H^1_f(K_{\mathfrak{q}}, T) = Im(H^1(K_{\mathfrak{q}}, T^+) \to H^1(K_{\mathfrak{q}}, T))$.

Now the map

$$H^1(K_{\mathfrak{q}}, T^+) = K_{\mathfrak{q}}^\times \hat{\otimes} \mathcal{O}_{\mathfrak{p}} \to H^1(K_{\mathfrak{q}}, T^+/\varpi^n) = K_{\mathfrak{q}}^\times \otimes \mathcal{O}_{\mathfrak{p}}/\varpi^n$$

is surjective, which implies that $H^1_f(K_{\mathfrak{q}}, T/\varpi^n) = Im(H^1(K_{\mathfrak{q}}, T^+/\varpi^n) \to H^1(K_{\mathfrak{q}}, T/\varpi^n))$. The proposition then follows from the fact that, under Assumption 2.11, one has $T^+/\varpi^n = A^{\mathfrak{q}}_n(f)$. □

2.14. Local condition at places dividing $\mathfrak{n}^+$. In this case we will impose the following rather strong condition on the local Galois representation $A_1(f)$.

2.15. Assumption. Assume that $A_1(f)^{I_{\mathfrak{q}}} = 0$ for every place $\mathfrak{q}|\mathfrak{n}^+$.

Under the above assumption we have in particular $H^0(K_{\mathfrak{q}}, A_n(f)) = 0$, hence the same holds true for $H^2(K_{\mathfrak{q}}, A_n(f))$ by duality and for $H^1(K_{\mathfrak{q}}, A_n(f))$ by the local Euler characteristic formula. Therefore we find

$$0 = H^1(K_{\mathfrak{q}}, A_n(f)) = H^1_f(K_{\mathfrak{q}}, A_n(f))\ \forall\ \mathfrak{q}|\mathfrak{n}^+.$$

2.16. Galois representations and Selmer groups for automorphic forms modulo ϖ^n. In what follows we will work with automorphic forms with values in $\mathcal{O}_{\mathfrak{p}}/\varpi^n$ as defined in I.4.9, and we need to extend to them the discussion made so far. We will however only encounter automorphic forms $g_n \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_{\mathfrak{p}}/\varpi^n)$ (which we will often just call *automorphic forms modulo* ϖ^n) such that:

(1) the automorphic form g_n is *congruent modulo* ϖ^n to the Hilbert newform $f \in S(\mathfrak{n})$ fixed at the beginning of the chapter (this means that f and g_n have the same eigenvalues modulo ϖ^n for almost all the Hecke operators);
(2) the discriminant $\mathfrak{D}$ of B is divisible by $\mathfrak{n}^-$ and by primes $\mathfrak{l}$ inert in K such that the conclusion of Lemma 2.12 holds true for $A_n(f)_{|\Gamma_{K_{\mathfrak{l}}}}$.

In fact, as we shall see later, we will produce automorphic forms modulo ϖ^n via level raising of f modulo ϖ^n at *admissible primes* (see 3.9), which is why the two conditions above will always hold true. It will therefore be enough for us to work with the following somewhat ad hoc definitions.

Let B be a totally definite quaternion algebra of discriminant $\mathfrak{D}$ and $g_n \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_{\mathfrak{p}}/\varpi^n)$ an automorphic form satisfying conditions $(1), (2)$ above. Then we define the Galois representation attached to g_n as $T(g_n) = A(g_n) = T_n(f)$. We define $Sel(K, A(g_n))$ imposing as local condition the unramified one outside $\mathfrak{D}\mathfrak{n}^+p$. At a place $\mathfrak{q}|\mathfrak{D}$ we define $H^1_f(K_{\mathfrak{q}}, A(g_n)) = Im(H^1(K_{\mathfrak{q}}, A^{\mathfrak{q}}(g_n)) \to H^1(K_{\mathfrak{q}}, A(g_n))$ (the notations being as in lemma 2.12). If $\mathfrak{q}|\mathfrak{n}^+$ the local cohomology group vanishes hence there is nothing to be done. Finally, if v divides p then we set $H^1_f(K_v, A(g_n)) = H^1_{fl}(\mathcal{O}_{K_v}, \mathcal{G})$ where $\mathcal{G}/\mathcal{O}_{K_v}$ is a finite flat group scheme such that $A(g_n) = \mathcal{G}(\bar{K}_v)$. Notice that this is well defined because of point (3) of Proposition 2.7.

In other words, in what follows we will work with a *fixed* Galois representation $T_n(f)$, but the local conditions defining the relevant Selmer groups will vary.

2.17. Assumption. Assume that the conductor of χ is coprime with p and that $p \nmid [K_\chi : K]$.

2.18. Hecke twists. Under the above assumption the local assumptions we made of the Γ_K-module $A_1(f)$ still hold for $A_1(f)_{|\Gamma_{K_\chi}}$, so the same description of the local conditions for the Selmer group $Sel(K_\chi, A_n(f))$ holds. We have $Sel(K, A(f)(\chi)) = Sel(K_\chi, A(f))^{(\bar{\chi})}$ and we set $Sel(K, A(g_n)(\chi)) = Sel(K_\chi, A(g_n))^{(\bar{\chi})}$ for an automorphic form g_n modulo ϖ^n as above. Notice that the isomorphism $H^1(K, A(g_n)(\chi)) \simeq H^1(K, A(g_n)(\bar{\chi}))$ from section I.6.5 induces an isomorphism $Sel(K, A(g_n)(\chi)) \simeq Sel(K, A(g_n)(\bar{\chi}))$.

3. Statement of the main theorem, and a first dévissage.

3.1. Until further notice, we are now going to work in the *definite case*, i.e. we suppose that $[F : \mathbf{Q}] \equiv |\{\mathfrak{q}, \mathfrak{q}|\mathfrak{n}^-\}| \pmod 2$. Then the sign of the functional equation of $L(f, \chi, 1)$ is 1, and Zhang's special value formula I.5.6.1 states that

$$L(f, \chi, 1) = \frac{2^r}{N(c(\chi))\sqrt{N(disc(K/F))}} \cdot \langle f, f \rangle_{Pet} \cdot |a(f, \chi)|^2.$$

Recall that $a(f,\chi) = \sum_{\sigma\in Gal(H_{c(\chi)}/K)} \bar{\chi}(\sigma) f_B(\sigma(P_\chi))$ where $f_B \in S^{B^\times}(\mathfrak{n}^+, \mathbf{C})$ is the Jacquet-Langlands transfer of f of Petersson norm 1. Since we need to work with integral automorphic forms we will however need to make a different choice of f_B, which results in a different period appearing in the special value formula in place of $\langle f, f\rangle_{Pet}$. For a discussion of this issue we refer the reader to [**Vat03**, Section 2]. For our purposes, let us recall that we can, and will, choose $f_B \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O})$, where $\mathcal{O}$ is the ring generated by the Hecke eigenvalues of f, and such that the image of f_B in $\mathcal{O}_\mathfrak{p}$ contains a $\mathfrak{p}$-adic unit. For such a choice the above formula translates into

$$C \cdot \frac{L(f,\chi,1)}{\Omega^{Gr}} = |a(f,\chi)|^2$$

where $C = \frac{N(c(\chi))\sqrt{N(disc(K/F))}}{2^r}$ and $\Omega^{Gr} = \frac{\langle f,f\rangle_{Pet}}{\eta_B}$ is the *Gross period*, quotient of $\langle f, f\rangle_{Pet}$ and the *congruence number* η_B.

In particular the value $C \cdot \frac{L(f,\chi,1)}{\Omega^{Gr}}$ is an algebraic number, called the *algebraic part* of the special value $L(f,\chi,1)$ and denoted by $L^{alg}(f,\chi,1)$.

3.2. Notation. For $a \in \mathcal{O}_\mathfrak{p}$ we denote by $ord_\varpi(a)$ its ϖ-adic valuation. More generally, if M is an $\mathcal{O}_\mathfrak{p}$-module of finite type and $m \in M$ we let $ord_\varpi(m) = sup\{n \geq 0, m \in \varpi^n M\}$ (which we declare to be equal to ∞ if $m = 0$). The length of an $\mathcal{O}_\mathfrak{p}$-module M will be denoted by $l_{\mathcal{O}_\mathfrak{p}}(M)$. Hence $ord_\varpi(a) = l_{\mathcal{O}_\mathfrak{p}}(\mathcal{O}_\mathfrak{p}/(a))$ for $a \in \mathcal{O}_\mathfrak{p}$, and the cardinality of a finite $\mathcal{O}_\mathfrak{p}$-module M equals $|\mathcal{O}_\mathfrak{p}/\varpi|^{l_{\mathcal{O}_\mathfrak{p}}(M)}$.

3.3. With the above notation, our aim is to prove the following equality (or at least one inequality):

$$2ord_\varpi(a(f,\chi)) = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f)(\chi)). \tag{3.3.1}$$

3.4. Remark. Bloch and Kato predict that in our situation the following formula holds:

$$\frac{L(f,\chi,1)}{\Omega^{BK}} = |Sel(K, A(f)(\chi))| \prod_{\mathfrak{q}} t_\mathfrak{q}$$

where $t_\mathfrak{q}$ is the $\mathfrak{q}$-th Tamagawa number of $A(f)(\chi)$ and Ω^{BK} is a suitable period. In our sought-for formula 3.3.1 Tamagawa number are missing. The point is that the period Ω^{Gr} in Zhang's special value formula is different from the one showing up in Bloch-Kato's conjecture. To show that our formula 3.3.1 is equivalent to the one predicted by Bloch-Kato one needs to compare the quantities Ω^{Gr} and Ω^{BK}. This is done in [**PW11**, Theorem 6.8] for modular forms over $\mathbf{Q}$ and $\chi = 1$; there the ratio between the two periods is shown to be equal precisely to the product of the missing Tamagawa numbers. We do not know whether the analogous result over totally real fields has been proved, and we did not address this issue.

Let us show first of all that it is enough to prove a mod-ϖ^n version of the equality 3.3.1.

3.5. Lemma. *Let B be the totally definite quaternion algebra over F of discriminant $\mathfrak{n}^-$ and $R \subset B$ an Eichler order of level $\mathfrak{n}^+$. Let n be a positive integer and $f_n : B^\times\backslash\hat{B}^\times/\hat{F}^\times\hat{R}^\times \to \mathcal{O}_\mathfrak{p}/\varpi^n$ be the reduction of the Jacquet-Langlands transfer of f modulo ϖ^n. Let $a(f_n,\chi) \in \mathcal{O}_\mathfrak{p}/\varpi^n$ be the reduction of $a(f,\chi)$. Assume that $L(f,\chi,1) \neq 0$ and the equality*

$$2ord_\varpi(a(f_n,\chi)) = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f_n)(\chi))$$

holds true for n large enough. Then

$$2ord_{\varpi}(a(f,\chi)) = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f)(\chi)).$$

PROOF. If $L(f,\chi,1) \neq 0$ then $a(f,\chi) \neq 0$, hence $a(f,\chi) \not\equiv 0 \pmod{\varpi^n}$ for n large enough. For any such n we have $ord_{\varpi} a(f,\chi) = ord_{\varpi} a(f_n,\chi)$. By hypothesis we have the equality $l_{\mathcal{O}_\mathfrak{p}} Sel(K, A_n(f)(\chi)) = 2ord_{\varpi} a(f_n,\chi)$ for large n. Now $A_n(f)(\chi) = A(f)(\chi)[\varpi^n]$, and by the next control result (Proposition 3.6) we have the equality

$$Sel(K, A(f)(\chi)[\varpi^n]) = Sel(K, A(f)(\chi))[\varpi^n].$$

Hence we obtain, for large n:

$$\begin{aligned} 2ord_{\varpi}(a(f,\chi)) =& 2ord_{\varpi}(a(f_n,\chi)) = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A_n(f)(\chi)) \\ =& l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f)(\chi))[\varpi^n] = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f)(\chi)). \end{aligned}$$

Let us justify the last equality: we have $Sel(K, A(f)(\chi)) = Sel(K, A(f)(\chi))[\varpi^\infty]$, hence we have a chain of inclusions:

$$Sel(K, A(f)(\chi))[\varpi] \subset Sel(K, A(f)(\chi))[\varpi^2] \subset \ldots \subset Sel(K, A(f)(\chi)) = \cup_n Sel(K, A_n(f)(\chi))$$

Since $l_{\mathcal{O}_\mathfrak{p}} Sel(K, A_n(f)(\chi)) = 2ord_{\varpi}(a(f_n,\chi))$ is constant for n large enough, for such an n we have $Sel(K, A(f)(\chi)) = Sel(K, A_n(f)(\chi))$. $\square$

3.6. PROPOSITION. *(cf. [**MR04**, Lemma 3.5.3]) For $n \geq 1$ the natural map*

$$Sel(K, A(f)(\chi)[\varpi^n]) \longrightarrow Sel(K, A(f)(\chi))[\varpi^n]$$

is an isomorphism.

PROOF. To shorten the notation let us denote $A(f)(\chi)$ by M in this proof. Let Σ be the set consisting of all infinite places of K and all places dividing $\mathfrak{n}c(\chi)p$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} Sel(K, M[\varpi^n]) & \hookrightarrow & H^1(K_\Sigma/K, M[\varpi^n]) & \longrightarrow & \oplus_{v\in\Sigma} H^1(K_v, M[\varpi^n])/H^1_f(K_v, M[\varpi^n]) \\ \downarrow & & \downarrow & & \downarrow \\ Sel(K, M)[\varpi^n] & \hookrightarrow & H^1(K_\Sigma/K, M)[\varpi^n] & \longrightarrow & \oplus_{v\in\Sigma} H^1(K_v, M)/H^1_f(K_v, M) \end{array}$$

where K_Σ/K is the maximal extension unramified outside Σ.

Since the Selmer structure on $M[\varpi^n]$ is propagated from the Selmer structure on M, the rightmost vertical map in injective. Therefore by the snake lemma it is enough to show that the central vertical map is an isomorphism. We have an exact sequence:

$$0 \longrightarrow M[\varpi^n] \longrightarrow M \xrightarrow{\cdot\varpi^n} M \longrightarrow 0.$$

Taking long exact sequences in cohomology we find an exact sequence:

$$H^0(K_\Sigma/K, M) \longrightarrow H^1(K_\Sigma/K, M[\varpi^n]) \longrightarrow H^1(K_\Sigma/K, M) \xrightarrow{\varpi^n} H^1(K_\Sigma/K, M).$$

To end the proof it suffices to notice that $H^0(K, M) = 0$ since $M[\varpi] = A(f)(\chi)[\varpi]$ is irreducible. $\square$

3.7. REMARK. If we know the inequality

$$l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f_n)(\chi)) \le 2ord_\varpi(a(f_n,\chi))$$

for n large enough we deduce that $l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f_n)(\chi))$ is bounded, as n goes to infinity. This implies, in a similar way as in lemma 3.5, that the above inequality also holds for $Sel(K, A(f)(\chi))$.

3.8. In view of the above reduction step, it is enough to study the length of the Selmer group attached to automorphic forms modulo ϖ^n.

3.9. DEFINITION. A prime $\mathfrak{l}$ of $\mathcal{O}_F$ is called n-admissible if:

(1) $\mathfrak{l} \not| p\, \mathfrak{n}\, disc(K/F)\, c(\chi)$.
(2) $\mathfrak{l}$ is inert in K.
(3) $p \not| N(\mathfrak{l})^2 - 1$.
(4) $(N(\mathfrak{l})+1)^2 \equiv \lambda_f(\mathfrak{l})^2 \pmod{\varpi^n}$.

where $\lambda_f(\mathfrak{l})$ is the eigenvalue of $T_\mathfrak{l}$ acting on f.

3.10. DEFINITION. (1) An eigenform $g_n \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi^n)$ with trivial central character is called admissible if B is a totally definite quaternion algebra of discriminant $\mathfrak{D}$ divisible by $\mathfrak{n}^-$ and by n-admissible primes, g_n is not zero modulo ϖ and f, g_n have Hecke eigenvalues for the Hecke operators outside $\mathfrak{D}/\mathfrak{n}^-$ which are congruent modulo ϖ^n.

(2) We say that $\mathfrak{n}^+$ is the *minimal level* of f_1 if $\hat{R}^\times \subset \prod_v B^\times(\mathcal{O}_{F,v})$ is the minimal level of $\bar\rho$ in the sense of [**Man19**, pag. 6] (where R is an Eichler order of level $\mathfrak{n}^+$) for every totally definite quaternion algebra B such that there exists an admissible form $g_1 \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi)$.

For example, if $\mathfrak{n}^+ = \mathcal{O}_F$ then it is the minimal level of f_1.

Finally, for an admissible eigenform g_n we define $a(g_n,\chi) \in \mathcal{O}_\mathfrak{p}/\varpi^n$ as in I.5.5.1, and we denote by g_k the reduction of g_n modulo ϖ^k. We will now devote our efforts to the proof of the following result.

3.11. THEOREM. *Suppose that the following assumptions are satisfied:*

(1) The level $\mathfrak{n}$ of f, the discriminant $disc(K/F)$, the conductor of χ and the prime p are coprime to each other. Moreover $p > 3$ is unramified in F and does not divide $[K_\chi : K]$, and $K_\chi \cap H_F^+ = F$.
(2) Writing $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$, $\mathfrak{n}^-$ is squarefree and $\#\{\mathfrak{q} : \mathfrak{q}|\mathfrak{n}^-\} \equiv [F:\mathbf{Q}] \pmod 2$.
(3) The image of the residual Galois representation $\bar\rho$ contains $SL_2(\mathbf{F}_p)$.
(4) If $\mathfrak{q}|\mathfrak{n}^-$ then $N(\mathfrak{q}) \not\equiv -1 \pmod p$. Moreover if $N(\mathfrak{q}) \equiv 1 \pmod p$ then $\bar\rho$ is ramified at $\mathfrak{q}$.
(5) If $\mathfrak{q}|\mathfrak{n}^+$ then $A_1(f)^{I_\mathfrak{q}} = 0$.
(6) The level $\mathfrak{n}^+$ is minimal *for f_1.*

Let $n = 2k$ and let $g_n \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi^n)$ be an admissible eigenform such that $a(g_n,\chi) \not\equiv 0 \pmod{\varpi^k}$. Then the following inequality holds:

$$l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(g_k)(\chi)) \le 2ord_\varpi(a(g_k,\chi)).$$

Moreover the above inequality is an equality provided that the following implication holds true: if g is an admissible automorphic form mod ϖ and $Sel(K, A(g)(\chi)) = 0$ then $a(g,\chi)$ is a ϖ-adic unit.

3.12. Remark. Certain assumptions we make could probably be relaxed. Precisely, (5) could be weakened via a more careful analysis of the local condition for the mod ϖ Selmer group at primes dividing $\mathfrak{n}^+$ (cf. the hypotheses in [**Zha14**]). Condition (4) is slightly stronger than what we needed in our study of the local conditions at primes dividing $\mathfrak{n}^-$; we need this stronger form only to invoke [**Man19**, Theorem 1.1]. Similarly, condition (6) is only needed to apply the main result in *loc. cit.*

3.13. Remark.

(1) As it will become clear later (see remark 5.9), we have to work modulo ϖ^{2k} in order to establish a result modulo ϖ^k because we will need to make use of a certain *freeness* property of the Euler system we construct. This is immaterial as long as we are interested in the special value formula 3.3.1, which concerns modular forms in characteristic zero.

(2) Let us say a word concerning the condition under which we can say that the equality holds true. Suppose that g lifts to characteristic zero, and let $\tilde{g}$ be a lift. In light of the special value formula, what we need to know is the implication

$$Sel(K, A(\tilde{g})(\chi)) = 0 \Longrightarrow L^{alg}(\tilde{g}, \chi, 1) \text{ is a unit.}$$

This is currently deduced from Skinner-Urban's divisibility in the Iwasawa main conjecture [**SU14**] (therefore in our case we could obtain the full equality in the ordinary case, thanks to Wan's work [**Wan15a**], under additional hypotheses).

However the result in *loc. cit.* is stronger than what we need, and our theorem shows that a generalisation to $GL_{2,F}$ of Ribet's converse of Herbrand's theorem [**Rib76**] would suffice to obtain the sought-for equality. It would be interesting to investigate whether one can use the Langlands-Shahidi method from GL_2 to GL_3, joint with Scholze's results on Galois representations attached to (mod p) automorphic forms for GL_3 [**Sch15**], to prove such a generalisation. Let us remark that this approach would also lead some support to Sharifi's conjectures [**FKS14**].

4. Explicit reciprocity laws

4.1. Admissible primes. We will prove theorem 3.11 exploiting a system of cohomology classes in the group $H^1(K, T_k(f)(\bar{\chi}))$ whose localisation at suitable admissible primes falls in a subspace of the local cohomology group which is *not* the finite subspace, and such that the failure of this localisation being equal to zero is measured by (the ϖ-adic valuation of) $a(g_k, \chi)$. Global duality will then yield annihilation results for the Selmer group $Sel(K, A(g_k)(\chi))$ allowing us to prove our result. The construction of such cohomology classes is based on level raising of quaternionic automorphic forms at admissible primes, which we will now discuss. From now on we will use the notations of theorem 3.11 and assume that all its hypotheses hold. In particular we fix $g_n \in S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi^n)$ and let $\mathfrak{D}$ be the discriminant of B.

4.2. Lemma. *Let $\mathfrak{l}$ be an n-admissible prime. Then:*

(1) $T_n(f) \simeq \mathcal{O}_\mathfrak{p}/\varpi^n \oplus \mathcal{O}_\mathfrak{p}/\varpi^n(1)$ as $\Gamma_{K_\mathfrak{l}}$-modules, and this decomposition is unique.

(2) $H^1(K_\mathfrak{l}, T_n(f)(\bar{\chi})) \simeq H^1(K_\mathfrak{l}, \mathcal{O}_\mathfrak{p}/\varpi^n) \oplus H^1(K_\mathfrak{l}, \mathcal{O}_\mathfrak{p}/\varpi^n(1))$ where both direct summands are free $\mathcal{O}_\mathfrak{p}/\varpi^n$-modules of rank one, and the first one is identified with the unramified cohomology group $H^1_{ur}(K_\mathfrak{l}, T_n(f)(\bar{\chi}))$.

4.3. Notation. We denote the summand $H^1(K_\mathfrak{l}, \mathcal{O}_\mathfrak{p}/\varpi^n(1))$ in the above decomposition by $H^1_{tr}(K_\mathfrak{l}, T_n(f)(\bar{\chi}))$, so that $H^1(K_\mathfrak{l}, T_n(f)(\bar{\chi})) = H^1_{ur}(K_\mathfrak{l}, T_n(f)(\bar{\chi})) \oplus H^1_{tr}(K_\mathfrak{l}, T_n(f)(\bar{\chi}))$.

PROOF. The direct sum decomposition in (1) comes from the fact that, by (2) and (4) in definition 3.9, the polynomial $det(1-xFr_{K,\mathfrak{l}}|T_n(f))$ splits as a product $(1-x)(1-N_{F/\mathbf{Q}}(\mathfrak{l})^{-2}x)$. Moreover (3) guarantees that $1 \not\equiv N(\mathfrak{l})^2 \pmod{\varpi^n}$, hence the decomposition is unique. This also implies that $H^1_{ur}(K_\mathfrak{l}, T_n(f)) = T_n(f)/(Fr_\mathfrak{l} - 1)T_n(f) = \mathcal{O}_\mathfrak{p}/\varpi^n$. On the other hand $H^1(K_\mathfrak{l}, T_n(f))/H^1_{ur}(K_\mathfrak{l}, T_n(f)) \simeq H^1(I_\mathfrak{l}, T_n(f))^{G_{K_\mathfrak{l}}} = Hom(I_\mathfrak{l}, T_n(f))^{G_{K_\mathfrak{l}}}$. Any such morphism factors through the tame inertia, and has image contained in $\mathcal{O}_\mathfrak{p}/\varpi^n(1)$ since the Frobenius at $\mathfrak{l}$ acts on the tame inertia as multiplication by $N(\mathfrak{l})^{-2}$. Finally, since $\mathfrak{l}$ is inert in K and does not divide $c(\chi)$, it splits completely in $K_\chi \subset H_{c(\chi)}$, as $Gal(H_{c(\chi)}/K) \simeq K^\times \backslash \hat{K}^\times / \hat{F}^\times \hat{\mathcal{O}}^\times_{c(\chi)}$. Hence $H^1(K_\mathfrak{l}, T_n(f)(\bar{\chi})) = (\oplus_{\mathfrak{l}'|\mathfrak{l}} H^1(K_{\chi,\mathfrak{l}'}, T_n(f)))^\chi$ can be identified with $H^1(K_\mathfrak{l}, T_n(f))$. □

The following lemma tells us that there are enough admissible primes to distinguish whether a given cohomology class is non zero.

4.4. LEMMA. *Let $c \in H^1(K, T(f_1)(\bar{\chi}))$ be a non zero class. Then there are infinitely many n-admissible primes $\mathfrak{l}$ such that $loc_\mathfrak{l}(c) \neq 0$.*

PROOF. Let us denote by $\rho_n : G_F \to Aut_{\mathcal{O}_\mathfrak{p}}(A_n(f))$ the mod ϖ^n representation attached to f and by F_n the extension of F fixed by $ker(\rho_n)$. Then F_n and K are linearly disjoint over F, since their sets of finite ramification places are assumed to be disjoint and $K \cap H_F^+ = F$ by assumption. Hence $K_\chi \cap F_n = F$. Indeed let $E = K_\chi \cap F_n$. Then E/F is an abelian extension of F unramified at all finite places, hence $E \subset H_F^+$. But $K_\chi \cap H_F^+ = F$, hence $E = F$.

Since $Gal(K_\chi/F) = Gal(K_\chi/K) \rtimes Gal(K/F)$ (because $H_{c(\chi)}/F$ is dihedral) we see that, denoting by τ the non trivial element in $Gal(K/F)$ and by M the compositum of F_n and K_χ, there is an inclusion:

$$Gal(M/F) = Gal(K_\chi/F) \times Gal(F_n/F) \subset (Gal(K_\chi/K) \rtimes \{1, \tau\}) \times Aut_{\mathcal{O}_\mathfrak{p}}(A_n(f)).$$

Furthermore our large image assumption and the fact that $p > 3$ imply that $Gal(F_n/F) \subset Aut_{\mathcal{O}_\mathfrak{p}}(A_n(f))$ contains an element T which has eigenvalues δ and λ, with $\delta = \pm 1 \neq \lambda \in (\mathcal{O}_\mathfrak{p}/\varpi^n)^\times$ and the order of λ is coprime with p. With this in hand, the rest of the proof proceeds as in [**LV10**, Proposition 4.5]. □

4.5. The next theorem 4.9 collects the essential ingredients needed to construct the cohomology classes we will use in our Euler system argument. It is a level raising result at an admissible prime $\mathfrak{l} \nmid \mathfrak{D}$, stating that the representation $T_n(f)$ appears in the mod ϖ^n-cohomology of a Shimura curve attached to a quaternion algebra $B_\mathfrak{l}$ of discriminant $\mathfrak{D}\mathfrak{l}$. In the form we will use it, theorem 4.9 rests on a multiplicity one result for automorphic forms modulo ϖ. We will use the main result of [**Man19**], which in our situation states the following

4.6. THEOREM. *[**Man19**, Theorem 1.1] Let $\mathfrak{m} \subset \mathbf{T}^{B^\times} \otimes \mathcal{O}_\mathfrak{p}$ be the maximal ideal in the Hecke algebra which is the kernel of the morphism attached to the reduction of g_n modulo ϖ. Assume that:*

(1) If $\mathfrak{q}|\mathfrak{D}$ then $N(\mathfrak{q}) \not\equiv -1 \pmod p$ and $\bar{\rho}$ is Steinberg at $\mathfrak{q}$. Moreover if $N(\mathfrak{q}) \equiv 1 \pmod p$ then $\bar{\rho}$ is ramified at $\mathfrak{q}$.

(2) The restriction $\bar{\rho}_{|\Gamma_{F(\zeta_p)}}$ is absolutely irreducible.

(3) $\bar{\rho}$ is finite flat at places above p.

(4) The level $\mathfrak{n}^+$ is minimal *for g_1.*

Then the dimension of the $\mathcal{O}_\mathfrak{p}/\varpi$-vector space $S^{B^\times}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi)[\mathfrak{m}]$ is one.

4.7. REMARK. In fact [**Man19**, Theorem 1.1] is more precise, stating that the dimension of $S^B(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi)[\mathfrak{m}]$ equals 2^k, where k is the number of finite places at which B ramifies, $\bar{\rho}$ is unramified and $\bar{\rho}(Fr_v)$ is a scalar. Assumption (1) guarantees that $k = 0$. Notice that the assumptions of the theorem are satisfied in our case. In particular, recall that $\bar{\rho}$ being Steinberg at $\mathfrak{q}$ means that its restriction at $\Gamma_{F_\mathfrak{q}}$ can be written in the form

$$\begin{pmatrix} \mu\chi_{cyc} & * \\ 0 & \mu \end{pmatrix}$$

where χ_{cyc} is the cyclotomic character and μ is an unramified character. This is true if $\mathfrak{q}|\mathfrak{n}^-$ by Lemma I.6.3, and also holds at $\mathfrak{q}|\mathfrak{D}/\mathfrak{n}^-$ by Lemma 4.2. Furthermore the Taylor-Wiles condition (2) follows from our large image assumption (3) in theorem 3.11.

4.8. Let us denote by $X_\mathfrak{l}$ the quotient Shimura curve attached to $B_\mathfrak{l}$ and to an Eichler order of level $\mathfrak{n}^+$. Let $J_\mathfrak{l}$ be its Jacobian, $\phi^\mathfrak{l}$ the group of connected components of its Néron model over $\mathcal{O}_{K_\mathfrak{l}}$ and $\phi^\mathfrak{l}_{\mathcal{O}_\mathfrak{p}} = \phi^\mathfrak{l} \otimes \mathcal{O}_\mathfrak{p}$. Let $\mathbf{T}_\mathfrak{l}$ be the Hecke algebra acting on $S^{B_\mathfrak{l}/Z}(\mathfrak{n}^+)$ (hence on the Tate module $T_p(J_\mathfrak{l})$) generated by the spherical Hecke operators and by the operators $U_\mathfrak{q}$ for $\mathfrak{q}|\mathfrak{D}\mathfrak{n}^+\mathfrak{l}$, where, for $\mathfrak{q}|\mathfrak{D}\mathfrak{l}$, $U_\mathfrak{q}$ is the Hecke operator attached to an element in $B_\mathfrak{l} \otimes_F F_\mathfrak{q}$ of norm $N(\mathfrak{q})$. In what follows we will work most of the time with $\mathbf{T}_\mathfrak{l} \otimes_\mathbf{Z} \mathcal{O}_\mathfrak{p}$ and tacitly extend scalars to the objects on which $\mathbf{T}_\mathfrak{l}$ acts, although we will often suppress this from the notation.

The eigenvalue of the Hecke operator T_v (or U_v if $v|\mathfrak{D}\mathfrak{n}^+$) acting on g_n will be denoted by $\lambda_{g_n}(v)$.

4.9. THEOREM. *(1) There exists a surjective morphism $g^\mathfrak{l}_n : \mathbf{T}_\mathfrak{l} \otimes_\mathbf{Z} \mathcal{O}_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}/\varpi^n$ such that $g^\mathfrak{l}_n(T_v) = \lambda_{g_n}(v)\ \forall v \not| \mathfrak{D}\mathfrak{n}^+\mathfrak{l}$, $g^\mathfrak{l}_n(U_v) = \lambda_{g_n}(v)$ for $v|\mathfrak{D}\mathfrak{n}^+$ and $g^\mathfrak{l}_n(U_\mathfrak{l}) = \epsilon$, where $\epsilon \in \{\pm 1\}$ is such that $N(\mathfrak{l}) + 1 \equiv \epsilon\lambda_{g_n}(\mathfrak{l}) \pmod{\varpi^n}$. Furthermore there is an isomorphism of Γ_F-modules*

$$T_p(J_\mathfrak{l}) \otimes_{\mathbf{Z}_p} \mathcal{O}_\mathfrak{p}/I_\mathfrak{l} \simeq T_n(f).$$

where $I_\mathfrak{l} = ker(g^\mathfrak{l}_n)$.

(2) The module $\phi^\mathfrak{l}_{\mathcal{O}_\mathfrak{p}}/I_\mathfrak{l}$ is a free $\mathcal{O}_\mathfrak{p}/\varpi^n$-module of rank one.

(3) There are commutative diagrams

$$\begin{array}{ccc} J_\mathfrak{l}(K_\mathfrak{l})/I_\mathfrak{l} & \xrightarrow{\kappa} & H^1(K_\mathfrak{l}, T_n(f)) \\ \downarrow & & \downarrow \\ \phi^\mathfrak{l}_{\mathcal{O}_\mathfrak{p}}/I_\mathfrak{l} & \longrightarrow & H^1_{tr}(K_\mathfrak{l}, T_n(f)) \end{array}$$

$$\begin{array}{ccc} Div^{CM}(X_\mathfrak{l})(K_\mathfrak{l})/I_\mathfrak{l} & \xrightarrow{\psi} & J_\mathfrak{l}(K_\mathfrak{l})/I_\mathfrak{l} \\ \downarrow{\scriptstyle sp} & & \downarrow \\ B^\times\backslash\hat{B}^\times/\hat{F}^\times\hat{R}^\times \times \mathbf{Z}/2\mathbf{Z} & \longrightarrow & \phi^\mathfrak{l}_{\mathcal{O}_\mathfrak{p}}/I_\mathfrak{l} \end{array}$$

where the bottom horizontal line in the first diagram is an isomorphism.

4.10. The first diagram is (the totally real version of) [**BD05**, Corollary 5.18.3]; the map κ is the composition of the Kummer map and the isomorphism coming from (1). In the second diagram, the map ψ is the restriction to divisors supported on CM points of the inverse of the natural map $J_\mathfrak{l}(K_\mathfrak{l})/I_\mathfrak{l} \to Div(X_\mathfrak{l})(K_\mathfrak{l})/I_\mathfrak{l}$, which is an isomorphism since the maximal ideal containing $I_\mathfrak{l}$ is not Eisenstein. The map sp is defined as follows: one looks at the dual graph of the special fibre of $X_\mathfrak{l}$ at $\mathfrak{l}$, whose vertices are irreducible components of the fibre, and an edge connects two vertices if the corresponding components intersect. One can show that the set of vertices can be identified with $B^\times\backslash\hat{B}^\times/\hat{R}^\times \times \mathbf{Z}/2\mathbf{Z}$, and under this identification the map sp sends a CM point to the irreducible component where its reduction lands. The bottom map in the second diagram is the map induced by the map $\omega_\mathfrak{l}$ in [**BD05**, Corollary 5.12] and the identification we just mentioned (for a more detailed discussion we refer the reader to [**Lon07**], [**LV10**], [**Nek12**]).

4.11. Remarks on the proof. We will not enter into the details of the proof of the above theorem, which already appeared many times in the literature. It was first proved over totally real fields by Longo [**Lon07**], following the strategy in [**BD05**, Section 5], under the assumption that f is p-isolated, i.e. that the completion of $S^{B^\times/Z}(\mathfrak{n}^+,\mathcal{O}_\mathfrak{p})$ at the maximal ideal which is the kernel of the mod ϖ character of the Hecke algebra attached to f is free of rank 1 over $\mathcal{O}_\mathfrak{p}$ ([**Lon07**, Definition 3.2]). However, as remarked in [**CH15**] (generalised to totally real fields by [**Wan15b**]), one only needs to know that $S^{B^\times/Z}(\mathfrak{n}^+,\mathcal{O}_\mathfrak{p})/I_{g_n} \simeq \mathcal{O}_\mathfrak{p}/\varpi^n$, where I_{g_n} is the kernel of the mod ϖ^n character of the Hecke algebra attached to g_n. So, letting $\mathfrak{m}_{g_n}$ be the maximal ideal in the Hecke algebra containing I_{g_n}, it suffices to show that $S^B(\mathfrak{n}^+,\mathcal{O}_\mathfrak{p}/\varpi^n)/\mathfrak{m}_{g_n}$ is one dimensional. This follows from Theorem 4.6 (and the self-duality of $S^B(\mathfrak{n}^+,\mathcal{O}_\mathfrak{p}/\varpi)$).

Besides the latter multiplicity one result, the proof rests on the study of the bad reduction of the Shimura curve $X_\mathfrak{l}$ and its Jacobian $J_\mathfrak{l}$ at the prime $\mathfrak{l}$, and Cerednik-Drinfeld uniformisation plays a crucial role. In fact a similar result has been proved in a weaker form, but with far less assumptions, in [**Nek12**]. The ideas in *loc. cit.* could certainly help improving this part of our proof, but we should remark that the multiplicity one result 4.6, which is the reason why we make many of our assumptions, will however play a crucial role later 4.17, so it does not seem possible to us to get rid of it.

4.12. Construction of the cohomology class $c_\chi(\mathfrak{l})$. The notations being as in theorem 4.9, let $Q_\chi = \sum_{\sigma\in Gal(H_{c(\chi)}/K)} \bar{\chi}(\sigma)P_\chi^\sigma \in (Pic(X_\mathfrak{l})(K_\chi)\otimes\mathcal{O}_\mathfrak{p})/I_\mathfrak{l}$, where P_χ is a CM point whose conductor equals the conductor of χ. The point Q_χ gives rise to a cohomology class $c_\chi(\mathfrak{l}) \in H^1(K_\chi, T_p(J_\mathfrak{l})\otimes_{\mathbf{Z}_p}\mathcal{O}_\mathfrak{p}/I_\mathfrak{l})^{(\chi)} = H^1(K_\chi, T_n(f))^{(\chi)} = H^1(K, T_n(f)(\bar{\chi}))$.

4.13. Localisation of $c_\chi(\mathfrak{l})$ at $\mathfrak{l}$: the first reciprocity law.

4.14. THEOREM. *(First reciprocity law)*

(1) $loc_v c_\chi(\mathfrak{l}) \in H^1_f(K_v, T(g_n)(\bar{\chi}))$ *if* $v \nmid \mathfrak{l}$.

(2) $loc_\mathfrak{l} c_\chi(\mathfrak{l}) \in H^1_{tr}(K_\mathfrak{l}, T(g_n)(\bar{\chi})) \simeq \mathcal{O}_\mathfrak{p}/\varpi^n$ *and we have an equality, up to ϖ-adic unit:*

$$loc_\mathfrak{l}(c_\chi(\mathfrak{l})) = a(g_n,\chi).$$

PROOF. Let v be a place of K_χ not dividing $\mathfrak{D}\mathfrak{n}^+p\mathfrak{l}$. Then $loc_v(c_\chi(\mathfrak{l})) \in H^1((K_\chi)_v, T_n(f))$ belongs to the unramified part because the Jacobian $J_\mathfrak{l}$ has good reduction at v. At places dividing $\mathfrak{D}$ the class $loc_v c_\chi(\mathfrak{l})$ falls in the finite part of the local cohomology (for the Selmer structure attached to g_n) because $J_\mathfrak{l}$ has purely toric reduction at these places (see [**GP12**, Lemma 8]). For the same reason the localisation at $\mathfrak{l}$ of $c_\chi(\mathfrak{l})$ must fall in the transverse part. At places above $\mathfrak{n}^+$ there is nothing to do since the local cohomology groups vanish.

For a place v above p, since the Jacobian $J_\mathfrak{l}$ has good reduction at v, the image of the Kummer map in $H^1((K_\chi)_v, J_\mathfrak{l}[p^n])$ lies in $H^1_{fl}(\mathcal{O}_{(K_\chi)_v}, \mathcal{J}_\mathfrak{l}[p^n])$, where $\mathcal{J}_\mathfrak{l}$ is the Neron model of $J_\mathfrak{l}$. This can be proved by a direct generalisation of [**LV10**, Proposition 3.2]. Since K_χ is unramified at v the map $J_\mathfrak{l}[p^n] \to T(g_n)$ induced by $T_p(J_\mathfrak{l})\otimes_{\mathbf{Z}_p}\mathcal{O}_\mathfrak{p}/I_\mathfrak{l} \simeq T(g_n)$ comes from a map $\mathcal{J}_\mathfrak{l}[p^n] \to \mathcal{G}$

where $\mathcal{G}$ is a finite flat group scheme with generic fiber $T(g_n)$. The description of the finite condition at places above p in Proposition 2.7 then shows that $loc_v(c) \in H^1_f((K_\chi)_v, T(g_n))$.

The equality $loc_{\mathfrak{l}}(c_\chi(\mathfrak{l})) = a(g_n, \chi)$ (which only makes sense up to unit, but this is all we need) follows from the two diagrams in theorem 4.9 and from the fact that the specialisation map on CM points is Galois-equivariant. □

4.15. Localisation of $c_\chi(\mathfrak{l})$ at $\mathfrak{l}' \neq \mathfrak{l}$: the second reciprocity law.

4.16. THEOREM. *Let $\mathfrak{l}' \neq \mathfrak{l}$ be an n-admissible prime not dividing $\mathfrak{D}$. Then*

$$loc_{\mathfrak{l}'} c_\chi(\mathfrak{l}) \in H^1_{ur}(K_{\mathfrak{l}'}, T(g_n)(\bar{\chi})) \simeq \mathcal{O}_{\mathfrak{p}}/\varpi^n$$

and ve have an equality (up to unit)

$$loc_{\mathfrak{l}'} c_\chi(\mathfrak{l}) = a(h_n, \chi)$$

where $h_n : B'^\times \backslash \hat{B}'^\times / \hat{F}^\times \hat{R}'^\times \to \mathcal{O}_{\mathfrak{p}}/\varpi^n$ is an automorphic form on the quaternion algebra B' of discriminant $\mathfrak{D}\mathfrak{l}\mathfrak{l}'$ and $R' \subset B'$ is an Eichler order of level $\mathfrak{n}^+$. Moreover h_n satisfies $T_v(h_n) = \lambda_{g_n}(v)h_n$ if $v \not| \mathfrak{D}\mathfrak{n}^+\mathfrak{l}\mathfrak{l}'$, $U_v(h_n) = \lambda_{g_n}(v)h_n$ if $v|\mathfrak{D}\mathfrak{n}^+$, $U_{\mathfrak{l}}(h_n) = \epsilon_{\mathfrak{l}} h_n$, $U_{\mathfrak{l}'}(h_n) = \epsilon_{\mathfrak{l}'} h_n$, where $\epsilon_{\mathfrak{l}}$, $\epsilon_{\mathfrak{l}'}$ are defined as in theorem 4.9. Finally, it induces a surjective *map $\mathcal{O}_{\mathfrak{p}}[B'^\times \backslash \hat{B}'^\times / \hat{F}^\times \hat{R}'^\times] \to \mathcal{O}_{\mathfrak{p}}/\varpi^n$ (in particular, h_n is admissible).*

4.17. REMARK. The above theorem implies the following equality (up to unit), which will be used repeatedly later:

$$loc_{\mathfrak{l}'} c_\chi(\mathfrak{l}) = loc_{\mathfrak{l}} c_\chi(\mathfrak{l}'). \tag{4.17.1}$$

Indeed, the left hand side (resp. right hand side) equals, by the reciprocity law, $a(h^1_n, \chi)$ (resp. $a(h^2_n, \chi)$), where h^1_n and h^2_n have the same Hecke eigenvalues and are non zero modulo ϖ. It follows from theorem 4.6 that they differ by a unit.

4.18. In the rest of this section we will outline the proof of the above theorem. It rests on a geometric realization of level raising of automorphic forms using the supersingular stratum in the special fiber of a Shimura curve at a prime of good reduction. The use of *both* reciprocity laws is what will allow us to prove, via an induction process, the inequality in our theorem 3.11. This is the main difference between our work and several others proving vanishing results for Selmer groups only relying on the first reciprocity law [**Lon07**], [**LV10**], [**Nek12**]. Furthermore, the second reciprocity law will allow us to prove results in rank one by essentially reducing them to the rank zero case. As we will explain, the results in the next chapter were motivated by our attempt to understand whether this strategy could be applied to higher rank situations.

Because of its importance for us, we will give an account of the proof of Theorem 4.16, essentially borrowed from [**LT17**, Proposition 4.8] (this proof differs from the original one, given for example in [**Lon12**, Section 7]).

Everything happens locally at $\mathfrak{l}'$, which splits completely in K_χ, hence we can (and will) forget about χ. Since the Jacobian $J_{\mathfrak{l}}$ has good reduction at $\mathfrak{l}'$, the cohomology class arising from a $K_{\mathfrak{l}'}-$point of the Jacobian falls in $H^1_{ur}(K_{\mathfrak{l}'}, T_p(J_{\mathfrak{l}}))$. Taking $g^{\mathfrak{l}}_n$-isotypic components we get a map

$$J_{\mathfrak{l}}(K_{\mathfrak{l}'})/I_{\mathfrak{l}} \longrightarrow H^1_{ur}(K_{\mathfrak{l}'}, T(g_n)) \simeq \mathcal{O}_{\mathfrak{p}}/\varpi^n$$

which fits in a commutative diagram

$$\begin{array}{ccc} J_{\mathfrak{l}}(K_{\mathfrak{l}'})/I_{\mathfrak{l}} & \longrightarrow & H^1_{ur}(K_{\mathfrak{l}'}, T(g_n)) \\ \downarrow{\scriptstyle red} & & \downarrow \\ J_{\mathfrak{l}}(\mathbf{F}_{\mathfrak{l}'^2})/I_{\mathfrak{l}} & \longrightarrow & H^1(\mathbf{F}_{\mathfrak{l}'^2}, T_p(J_{\mathfrak{l}})/I_{\mathfrak{l}}) \end{array}$$

where the vertical arrows are isomorphisms, and *red* denotes reduction modulo $\mathfrak{l}'$.

There is a stratification of the special fibre $X_{\mathfrak{l},\mathbf{F}_{\mathfrak{l}'}}$:

$$X_{\mathfrak{l},\mathbf{F}_{\mathfrak{l}'}} \supset X^{ss}_{\mathfrak{l},\mathbf{F}_{\mathfrak{l}'}}$$

where $X^{ss}_{\mathfrak{l},\mathbf{F}_{\mathfrak{l}'}}$ is the set of supersingular points in the special fiber of $X_{\mathfrak{l}}$. It is a finite set which can be identified with $B'^{\times}\backslash\hat{B}'^{\times}/\hat{F}^{\times}\hat{R}'^{\times}$, and the reduction of CM points on $X_{\mathfrak{l}}$ lands in the supersingular stratum. From the bottom row of the above diagram we get a map

$$\gamma : B'^{\times}\backslash\hat{B}'^{\times}/\hat{F}^{\times}\hat{R}'^{\times} \longrightarrow H^1(\mathbf{F}_{\mathfrak{l}'^2}, T_p(J_{\mathfrak{l}})/I_{\mathfrak{l}}) \simeq \mathcal{O}_{\mathfrak{p}}/\varpi^n.$$

By commutativity of the above diagram we have $loc_{\mathfrak{l}'}(c_{\chi}(\mathfrak{l})) = \gamma(red(Q_{\chi}))$. To complete the proof one has to show that γ satisfies the conclusion of the theorem. It is clear that it is an eigenform for the Hecke operators outside $\mathfrak{l}'$, and the $U_{\mathfrak{l}'}$-eigenvalue is determined in [**Lon12**, Proposition 7.21]. Up to this point however γ may very well be the zero map, and the main point consists in showing that this does not happen. Notice that in order for the map not to be zero one needs first of all that $H^1(\mathbf{F}_{\mathfrak{l}'^2}, T(g_n)) \neq 0$, i.e. that $H^0(\mathbf{F}_{\mathfrak{l}'^2}, T(g_n)) \neq 0$. This is guaranteed by condition (4) in the definition of admissible primes (primes satisfying it are usually called *level raising primes*).

We have to show that γ induces a *surjective* map

$$\phi : \mathcal{O}_{\mathfrak{p}}[B'^{\times}\backslash\hat{B}'^{\times}/\hat{F}^{\times}\hat{R}'^{\times}] \longrightarrow \mathcal{O}_{\mathfrak{p}}/\varpi^n. \tag{4.18.1}$$

To establish the surjectivity of 4.18.1 it is enough to show that the map

$$\phi_{\mathfrak{m}} : \mathcal{O}_{\mathfrak{p}}/\varpi[B'^{\times}\backslash\hat{B}'^{\times}/\hat{F}^{\times}\hat{R}'^{\times}]_{\mathfrak{m}} \longrightarrow H^1(\mathbf{F}_{\mathfrak{l}'^2}, H^1((X_{\mathfrak{l}})_{\bar{\mathbf{F}}_{\mathfrak{l}}}, \mathcal{O}_{\mathfrak{p}}/\varpi(1))_{\mathfrak{m}})$$

is surjective, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{T}_{\mathfrak{l}}$ containing $I_{\mathfrak{l}}$.

Let us fix the following notation: we let $k_{\varpi} = \mathcal{O}_{\mathfrak{p}}/\varpi$, $X = X_{\mathfrak{l},\mathbf{F}_{\mathfrak{l}'}}$, $X^{ss} \subset X$ is the supersingular locus and $X^{ord} = X \setminus X^{ss}$. Let $X(\mathfrak{l}')/\mathbf{F}_{\mathfrak{l}'}$ be the (quotient) Shimura curve attached to the quaternion algebra $B_{\mathfrak{l}}$, but with Iwahori level structure at $\mathfrak{l}'$ (and same level structure as X elsewhere).

Consider the exact sequence:

$$H^0(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_{\varpi}) \to H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_{\varpi}) \to H^1_c(X^{ord}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_{\varpi}) \to H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_{\varpi}) \to 0.$$

Localising at $\mathfrak{m}$ and taking Galois cohomology we obtain a map

$$\phi^*_{\mathfrak{m}} : H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_{\varpi})_{\mathfrak{m}}^{G_{\mathbf{F}_{\mathfrak{l}'^2}}} \to H^1(\mathbf{F}_{\mathfrak{l}'^2}, H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_{\varpi})_{\mathfrak{m}}).$$

As our notation suggests, the latter map can be identified with the dual of the map $\phi_{\mathfrak{m}}$, hence we have to show that $\phi^*_{\mathfrak{m}}$ is injective. Here the auxiliary curve $X(\mathfrak{l}')$ comes into play. There are two degeneracy maps

$$\pi_1, \pi_2 : X(\mathfrak{l}') \to X$$

and the special fiber $X(\mathfrak{l}')_{\mathbf{F}_{\mathfrak{l}'}}$ consists of two copies of $X_{\mathbf{F}_{\mathfrak{l}'}}$ meeting at supersingular points. Let $i_1 : X_{\mathbf{F}_{\mathfrak{l}'}} \to X(\mathfrak{l}')_{\mathbf{F}_{\mathfrak{l}'}}$ (resp. $i_2 : X_{\mathbf{F}_{\mathfrak{l}'}} \to X(\mathfrak{l}')_{\mathbf{F}_{\mathfrak{l}'}}$) be the copy such that $\pi_1 \circ i_1$ is the identity (resp. $\pi_2 \circ i_2$ is the identity). Let $\delta : X_{\mathbf{F}_{\mathfrak{l}'}} \coprod X_{\mathbf{F}_{\mathfrak{l}'}} \to X(\mathfrak{l}')_{\mathbf{F}_{\mathfrak{l}'}}$ be the normalisation map and

$i : X^{ss}_{\mathbf{F}_{\mathfrak{l}'}} \to X(\mathfrak{l}')_{\mathbf{F}_{\mathfrak{l}'}}$ the inclusion of the singular locus. Then we have an exact sequence of sheaves on $X(\mathfrak{l}')_{\mathbf{F}_{\mathfrak{l}'}}$:

$$0 \to k_\varpi \to \delta_* k_\varpi \to i_* k_\varpi \to 0$$

where the third arrow sends a germ (a, b) over a supersingular point to $a - b$. This induces an exact sequence
(4.18.2)

$$0 = H^0(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)^2_{\mathfrak{m}} \to H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}} \to H^1(X(\mathfrak{l}')_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}} \xrightarrow{(i_1^*, i_2^*)} H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)^2_{\mathfrak{m}} \to 0.$$

On the other hand, the degeneracy maps π_1, π_2 induce a map

$$\pi_1^* \oplus \pi_2^* : H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)^2_{\mathfrak{m}} \to H^1(X(\mathfrak{l}')_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}$$

and the composite

$$\theta : H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)^2_{\mathfrak{m}} \xrightarrow{\pi_1^* \oplus \pi_2^*} H^1(X(\mathfrak{l}')_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}} \xrightarrow{(i_1^*, i_2^*)} H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)^2_{\mathfrak{m}}$$

is given by the matrix $\begin{pmatrix} 1 & Fr_{\mathfrak{l}'} \\ S_{\mathfrak{l}}^{-1} Fr_{\mathfrak{l}'} & 1 \end{pmatrix}$. Since $S_{\mathfrak{l}}$ acts trivially we see that $ker(\theta)$ coincides with the image of the injection

$$H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}^{Fr^2_{\mathfrak{l}'}=1} \xrightarrow{(-Fr_{\mathfrak{l}'}, Id)} H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)^{\oplus 2}_{\mathfrak{m}}.$$

Ihara's lemma states that the map $\pi_1^* + \pi_2^*$ is injective, which implies that we obtain an injection

$$\phi^* : ker(\theta) \simeq H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}^{Fr^2_{\mathfrak{l}'}=1} \to ker(i_1^*, i_2^*) \simeq H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}.$$

where the last isomorphism follows from 4.18.2.

Finally the action of $Fr_{\mathfrak{l}'^2}$ on $H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}$ is trivial (because supersingular points are all defined over $\mathbf{F}_{\mathfrak{l}'^2}$) hence there is a canonical isomorphism

$$H^1(\mathbf{F}_{\mathfrak{l}'^2}, H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}) \simeq H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}.$$

To conclude the proof, one checks (cf. [**LT17**, pag. 35]) that under the above isomorphism the map

$$\phi^* : H^1(X_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}}^{G_{\mathbf{F}_{\mathfrak{l}'^2}}} \to H^1(\mathbf{F}_{\mathfrak{l}'^2}, H^0(X^{ss}_{\bar{\mathbf{F}}_{\mathfrak{l}'}}, k_\varpi)_{\mathfrak{m}})$$

is identified with the map $\phi^*_{\mathfrak{m}}$, which is therefore injective.

4.19. Remark.
(1) In the proof of the second reciprocity law we made crucial use of Ihara's lemma for Shimura curves, which we are assuming to hold for general totally real fields.
(2) Let us point out that in order to construct the level raising in theorem 4.16 and prove the second reciprocity law one only needs the base change to $K_{\mathfrak{l}'}$ of the Shimura curve $X_{\mathfrak{l}}$ and its integral model over the ring of integers of $K_{\mathfrak{l}'}$. In other words, everything in the theorem happens purely at a *semiglobal* level. Notice that the action of the *full* Hecke algebra is however crucial, so purely *local* objects are not enough.
(3) The key fact behind the theorem is that one sees, inside the special fiber at $\mathfrak{l}'$ of the Shimura curve $X_{\mathfrak{l}}$, a quaternionic set attached to a quaternion algebra ramified at finite places where $B_{\mathfrak{l}}$ is *and* at the additional prime $\mathfrak{l}'$.

(4) If one knew already that $loc_{\mathfrak{l}'}(c_\chi(\mathfrak{l})) \not\equiv 0 \pmod{\varpi}$ then surjectivity of 4.18.1 would be automatic. Because of the special value formula 5.7, this is the case for a suitable choice of the prime $\mathfrak{l}'$ in the *indefinite* setting.

5. The Euler system argument.

5.1. We will now run the Euler system argument which proves theorem 3.11. The main idea in the proof is to raise the level of g_n at *two* well-chosen admissible primes $\mathfrak{l}_1, \mathfrak{l}_2$, and construct an admissible automorphic form $h_n \in S^{B'^\times/Z}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi^n)$, where B' has discriminant $\mathfrak{D}\mathfrak{l}_1\mathfrak{l}_2$, such that $ord_\varpi(a(h_n,\chi)) < ord_\varpi(a(g_n,\chi))$ and we have

$$ord_\varpi(a(g_k,\chi)) - ord_\varpi(a(h_k,\chi)) = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(g_k)(\chi)) - l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(h_k)(\chi)).$$

One is thus reduced to prove the (in)equality in the case when $a(g_n,\chi)$ is a unit, which follows from the first reciprocity law. This level raising-length lowering method is a refinement of ideas already used, for other purposes, by Wei Zhang in [**Zha14**]. We will also make use in the first steps of our argument of few lemmas essentially borrowed from [**How06**].

5.2. We will always work in what follows with admissible primes not dividing the discriminant $\mathfrak{D}$ of the quaternion algebra where g_n lives. Let $\mathfrak{l}$ be such an n-admissible prime and let M denote either $A(g_n)(\chi)$ or $T(g_n)(\bar\chi)$. We denote by $Sel_\mathfrak{l}(K,M)$ (resp. $Sel^\mathfrak{l}(K,M)$, $Sel_{(\mathfrak{l})}(K,M)$) the Selmer group defined by the same local conditions as those for M at all places except at $\mathfrak{l}$, where the local condition is the zero subspace (resp. $H^1(K_\mathfrak{l},M)$, resp. $H^1_{tr}(K_\mathfrak{l},M)$). The self-duality of $T(g_n)$ and the self-orthogonality of the local conditions defining $Sel(K,M)$ and $Sel_{(\mathfrak{l})}(K,M)$ have the following consequences on the structure of the Selmer group.

5.3. Theorem. *There exists $e \in \{0,1\}$ and an $\mathcal{O}_\mathfrak{p}/\varpi^n$-module N such that*

$$Sel(K,M) = (\mathcal{O}_\mathfrak{p}/\varpi^n)^e \oplus N \oplus N.$$

The same holds for $Sel_{(\mathfrak{l})}(K,M)$.

5.4. Proposition. *Let $C \subset Sel(K,M)$ be a submodule isomorphic to $\mathcal{O}_\mathfrak{p}/\varpi^n$. Then there exist infinitely many n-admissible primes $\mathfrak{l}$ such that $loc_\mathfrak{l} : Sel(K,M) \to H^1_{ur}(K_\mathfrak{l},M)$ is an isomorphism when restricted to C.*

Proof. Let c be a generator of C. Then $\varpi^{n-1}c \in Sel(K,M)[\varpi] = Sel(K,M[\varpi])$ is non zero, hence by lemma 4.4 there are infinitely many n-admissible primes $\mathfrak{l}$ such that $loc_\mathfrak{l}(\varpi^{n-1}c) \neq 0$. For such a $\mathfrak{l}$ the localisation map $loc_\mathfrak{l} : C \to H^1_{ur}(K_\mathfrak{l},M) \simeq \mathcal{O}_\mathfrak{p}/\varpi^n$ is injective, hence an isomorphism. □

5.5. First of all, let us show that the first reciprocity law and the assumption that $a(g_n,\chi)$ does not vanish yield a weak annihilation result for the Selmer group.

5.6. Proposition. *The $\mathcal{O}_\mathfrak{p}/\varpi^n$-module $Sel(K,A(g_n)(\chi))$ is killed by ϖ^{n-1}.*

Proof. Suppose by contradiction that there exists $c \in Sel(K,A(g_n)(\chi))$ which generates a submodule $C \simeq \mathcal{O}_\mathfrak{p}/\varpi^n$. By Proposition 5.4 we can choose $\mathfrak{l}$ admissible such that $loc_\mathfrak{l} : C \to H^1_{ur}(K_\mathfrak{l},A(g_n)(\chi)) \simeq \mathcal{O}_\mathfrak{p}/\varpi^n$ is an isomorphism. In particular $loc_\mathfrak{l} : Sel(K,A(g_n)(\chi)) \to H^1_{ur}(K_\mathfrak{l},A(g_n)(\chi))$ is surjective. We have two exact sequences:

$$0 \to Sel_\mathfrak{l}(K,A(g_n)(\chi)) \to Sel(K,A(g_n)(\chi)) \xrightarrow{loc_\mathfrak{l}} H^1_{ur}(K_\mathfrak{l},A(g_n)(\chi))$$

$$0 \to Sel(K,T(g_n)(\bar\chi)) \to Sel^\mathfrak{l}(K,T(g_n)(\bar\chi)) \xrightarrow{loc_\mathfrak{l}} H^1_{tr}(K_\mathfrak{l},T(g_n)(\bar\chi)).$$

By global duality the images of the two localisations maps are annihilators of each other. Since $loc_{\mathfrak{l}} : Sel(K, A(g_n)(\chi)) \to H^1_{ur}(K_{\mathfrak{l}}, A(g_n)(\chi))$ is surjective and the pairing

$$H^1_{ur}(K_{\mathfrak{l}}, A(g_n)(\chi)) \times H^1_{tr}(K_{\mathfrak{l}}, T(g_n)(\bar{\chi})) \to \mathcal{O}_{\mathfrak{p}}/\varpi^n$$

is perfect we deduce that

$$loc_{\mathfrak{l}} : Sel^{\mathfrak{l}}(K, T(g_n)(\bar{\chi})) \to H^1_{tr}(K_{\mathfrak{l}}, T(g_n)(\bar{\chi}))$$

is the zero map. In particular $loc_{\mathfrak{l}}(c_{\chi}(\mathfrak{l})) = 0$. But by the first reciprocity law $loc_{\mathfrak{l}}(c_{\chi}(\mathfrak{l})) = a(g_n, \chi)$ and $a(g_n, \chi)$ is non zero by hypothesis, which gives a contradiction. □

5.7. Corollary. *(cf. [**How06**, Corollary 2.2.10, Remark 2.2.11])*

(1) *There exists an $\mathcal{O}_{\mathfrak{p}}/\varpi^n$-module N such that*

$$Sel(K, A(g_n)(\chi)) = N \oplus N.$$

(2) *There exists an $\mathcal{O}_{\mathfrak{p}}/\varpi^n$-module N' such that*

$$Sel_{(\mathfrak{l})}(K, T(g_n)(\bar{\chi})) = \mathcal{O}_{\mathfrak{p}}/\varpi^n \oplus N' \oplus N'.$$

Proof. The first point follows immediately from the previous proposition and the structure theorem 5.3. In order to prove (2), it is enough to show that $dim_{\mathcal{O}_{\mathfrak{p}}/\varpi} Sel(K, A(g_1)(\chi))$ and $dim_{\mathcal{O}_{\mathfrak{p}}/\varpi} Sel_{(\mathfrak{l})}(K, T(g_1)(\bar{\chi}))$ do not have the same parity. To prove this we argue as follows: recall that $A(g_1)(\chi) = {}^c(T(g_1)(\bar{\chi}))$, inducing an isomorphism $H^1(K, A(g_1)(\chi)) = H^1(K, T(g_1)(\bar{\chi}))$ under which $Sel(K, A(g_1)(\chi))$ is identified with $Sel(K, T(g_1)(\bar{\chi}))$. Hence we obtain two exact sequences

$$0 \to Sel_{\mathfrak{l}}(K, T(g_1)(\bar{\chi})) \to Sel(K, T(g_1)(\bar{\chi})) \xrightarrow{loc_{\mathfrak{l}}} H^1_{ur}(K_{\mathfrak{l}}, T(g_1)(\bar{\chi}))$$
$$0 \to Sel(K, T(g_1)(\bar{\chi})) \to Sel^{\mathfrak{l}}(K, T(g_1)(\bar{\chi})) \xrightarrow{loc_{\mathfrak{l}}} H^1_{tr}(K_{\mathfrak{l}}, T(g_1)(\bar{\chi})).$$

If the upper localisation map is non zero then the bottom one is zero, hence we obtain

$$Sel(K, T(g_1)(\bar{\chi})) = Sel^{\mathfrak{l}}(K, T(g_1)(\bar{\chi}))$$

therefore

$$Sel_{(\mathfrak{l})}(K, T(g_1)(\bar{\chi})) = Sel_{\mathfrak{l}}(K, T(g_1)(\bar{\chi})).$$

Hence

$$dim\ Sel(K, T(g_1)(\bar{\chi})) - dim\ Sel_{(\mathfrak{l})}(K, T(g_1)(\bar{\chi})) =$$
$$dim\ Sel(K, T(g_1)(\bar{\chi})) - dim\ Sel_{\mathfrak{l}}(K, T(g_1)(\bar{\chi})) = 1.$$

If the upper localisation map is zero then the bottom one is non zero and one argues similarly. □

5.8. Proposition. *(cf. [**How06**, Lemma 3.3.6]) There exists a free $\mathcal{O}_{\mathfrak{p}}/\varpi^k$-submodule of rank one of $Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$ which contains (the reduction modulo ϖ^k of) $c_{\chi}(\mathfrak{l})$.*

Proof. By the structure theorem we can write

$$Sel_{(\mathfrak{l})}(K, T(g_n)(\bar{\chi})) = \mathcal{O}_{\mathfrak{p}}/\varpi^n \oplus N \oplus N$$
$$Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi})) = \mathcal{O}_{\mathfrak{p}}/\varpi^k \oplus M \oplus M.$$

We know that $c_\chi(\mathfrak{l})$ is non zero, since this is true for its localisation at $\mathfrak{l}$. We claim that this implies that $\varpi^{k-1}M = 0$. If this is not the case, then $Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$ contains a free $\mathcal{O}_\mathfrak{p}/\varpi^k$-submodule of rank 2, hence, for any admissible prime $\mathfrak{l}' \neq \mathfrak{l}$, the kernel $Sel_{(\mathfrak{l}\mathfrak{l}')}(K, T(g_k)(\bar{\chi}))$ of the localisation map

$$loc_{\mathfrak{l}'} : Sel^{\mathfrak{l}'}_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi})) \to H^1_{ur}(K_{\mathfrak{l}'}, T(g_k)(\bar{\chi}))$$

contains a free $\mathcal{O}_\mathfrak{p}/\varpi^k$-submodule. Hence, writing $Sel_{(\mathfrak{l}\mathfrak{l}')}(K, T(g_k)(\bar{\chi})) = P \oplus P$, we have $\varpi^{k-1}P \neq 0$. With the same argument as in the proof of proposition 5.6 we deduce that $a(h_k, \chi) = 0$, where h_k is a level raising of g_k at $\mathfrak{l}\mathfrak{l}'$. The second reciprocity law yields $loc_{\mathfrak{l}'}c_\chi(\mathfrak{l}) = a(h_k, \chi) = 0$; since this is true for every admissible prime $\mathfrak{l}'$ we get $c_\chi(\mathfrak{l}) = 0$, contradiction.

Hence $\varpi^{k-1}M = 0$. We have a commutative diagram

$$\begin{array}{ccc} Sel_{(\mathfrak{l})}(K, T(g_n)(\bar{\chi})) & \xrightarrow{\varpi^k} & Sel_{(\mathfrak{l})}(K, T(g_n)(\bar{\chi}))[\varpi^k] \\ \downarrow & \nearrow & \\ Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi})) & & \end{array}$$

where the diagonal arrow is an isomorphism. Since $\varpi^{k-1}M = 0$ we deduce that the $\mathcal{O}_\mathfrak{p}/\varpi^k$-module $\varpi^{k-1}Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$ is cyclic, hence the same holds for $\varpi^{k-1}Sel_{(\mathfrak{l})}(K, T(g_n)(\bar{\chi}))[\varpi^k]$. Therefore $\varpi^{k-1}N = 0$, i.e. N is killed by the horizontal map, which implies that the image of the vertical arrow is free of rank one; since it contains $c_\chi(\mathfrak{l})$, the proof is complete. □

5.9. Remark. The above property, which will play an important role in the proof of the sought-for annihilation results for the Selmer group, explains why we need to work with the reduction modulo ϖ^k of automorphic forms modulo ϖ^{2k}. Let us say (following Howard, whose proof of a very similar result we also closely followed) that our Euler system is free if it enjoys the property in the above proposition. Then the Euler system modulo ϖ^{2k} may not be free, but its reduction modulo ϖ^k is.

5.10. Let us set $t_\chi(g_k) = ord_\varpi(a(g_k, \chi))$ and $t_\chi(g_k, \mathfrak{l}) = ord_\varpi(c_\chi(\mathfrak{l}))$. We remark that $ord_\varpi(c_\chi(\mathfrak{l}))$ can be calculated in any submodule $C \simeq \mathcal{O}_\mathfrak{p}/\varpi^k$ containing $c_\chi(\mathfrak{l})$, whose existence is guaranteed by the previous proposition. Indeed, let $c_\chi(\mathfrak{l}) = \varpi^a u$, where $u \in C$ is a unit. Then clearly a is smaller than the order of $c_\chi(\mathfrak{l})$ in $Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$. We claim that equality holds. Indeed, suppose that there exists $v \in Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$ and $b > a$ such that $\varpi^b v = c_\chi(\mathfrak{l})$. Then we have $\varpi^b v = \varpi^a u$, hence

$$\varpi^{k-1}u = \varpi^{a+k-a-1}u = \varpi^{b+k-a-1}v$$

The left hand side is non zero, but the right hand side is zero, since $b+k-a-1 \geq 1+k-1 = k$; this proves our claim.

5.11. We have the following chain of inequalities:

$$t_\chi(g_k, \mathfrak{l}) = ord_\varpi(c_\chi(\mathfrak{l})) \leq ord_\varpi(loc_\mathfrak{l} c_\chi(\mathfrak{l})) = t_\chi(g_k) < k$$

where the last equality follows from the first reciprocity law, and the last inequality holds because of our assumption that $a(g_k, \chi) \not\equiv 0 \pmod{\varpi^k}$. Hence there exists a class $\kappa_\chi(\mathfrak{l}) \in Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$ such that $c_\chi(\mathfrak{l}) = \varpi^{t_\chi(g_k, \mathfrak{l})}\kappa_\chi(\mathfrak{l})$. Our previous discussion implies that the class $\kappa_\chi(\mathfrak{l})$ can (and will) be taken to be in a submodule C as above. It enjoys the following properties:

(1) $\kappa_\chi(\mathfrak{l}) \in Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$.
(2) $ord_\varpi \kappa_\chi(\mathfrak{l}) = 0$.
(3) $ord_\varpi(loc_\mathfrak{l}(\kappa_\chi(\mathfrak{l}))) = t_\chi(g_k) - t_\chi(g_k, \mathfrak{l})$.

5.12. Lemma. *Suppose that $Sel(K, A(g_k)(\chi)) \neq 0$. Then there exist infinitely many admissible primes $\mathfrak{l}$ such that $t_\chi(g_k, \mathfrak{l}) < t_\chi(g_k)$.*

Proof. Let $c \in Sel(K, A(g_k)(\chi))$ be a non zero class, and $\mathfrak{l}$ an admissible prime such that $loc_\mathfrak{l}(c) \neq 0$. By global duality and the fact that the local conditions defining $Sel_{(\mathfrak{l})}(K, T(g_k)(\bar{\chi}))$ and $Sel(K, A(g_k)(\chi))$ are everywhere orthogonal except at $\mathfrak{l}$ we have:

$$0 = \sum_v \langle loc_v(c), loc_v(\kappa_\chi(\mathfrak{l}))\rangle = \langle loc_\mathfrak{l}(c), loc_\mathfrak{l}(\kappa_\chi(\mathfrak{l}))\rangle.$$

Since the pairing between $H^1_{ur}(K_\mathfrak{l}, A(g_k)(\chi))$ and $H^1_{tr}(K_\mathfrak{l}, T(g_k)(\bar{\chi}))$ is perfect and $loc_\mathfrak{l}(c) \neq 0$ we deduce that $loc_\mathfrak{l}(\kappa_\chi(\mathfrak{l}))$ cannot be a unit. By property (3) above, this proves the lemma. □

5.13. Corollary. *If $a(g_k, \chi)$ is a unit then $Sel(K, A(g_k)(\chi)) = 0$.*

Proof. This follows immediately from the previous lemma. We remark that this can also be deduced from proposition 5.6, replacing $A(g_n)$ with $A(g_1)$ and using the hypothesis that $a(g_k, \chi)$ is not congruent to 0 modulo ϖ. These two proofs essentially rely on the same argument. □

5.14. Recall that we want to prove theorem 3.11. We will prove it by induction on $t_\chi(g_k) = ord_\varpi(a(g_k, \chi))$, which is finite by assumption. The above corollary deals with the base case $t_\chi(g_k) = 0$; to treat the general case we will make use of the following

5.15. Lemma. *Suppose that $Sel(K, A(g_k)(\chi))$ is non zero. Then there exist two n-admissible primes $\mathfrak{l}_1 \neq \mathfrak{l}_2$ and an admissible automorphic form $h_n \in S^{B^\times_{\mathfrak{l}_1\mathfrak{l}_2}/Z}(\mathfrak{n}^+, \mathcal{O}_\mathfrak{p}/\varpi^n)$, where $B_{\mathfrak{l}_1\mathfrak{l}_2}$ is the definite quaternion algebra of discriminant $\mathfrak{D}\mathfrak{l}_1\mathfrak{l}_2$, such that:*

(1) $t_\chi(g_k, \mathfrak{l}_1) = t_\chi(g_k, \mathfrak{l}_2) < t_\chi(g_k)$.
(2) $t_\chi(h_k) = t_\chi(g_k, \mathfrak{l}_i)$, $i = 1, 2$.
(3) $ord_\varpi loc_{\mathfrak{l}_1}(k_\chi(\mathfrak{l}_2)) = ord_\varpi loc_{\mathfrak{l}_2}(k_\chi(\mathfrak{l}_1)) = 0$.
(4) $Sel(K, A(h_k)(\chi)) = Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(g_k)(\chi))$.

Proof. Take $\mathfrak{l}_1$ admissible such that $t_\chi(g_k, \mathfrak{l}_1) = min\{t_\chi(g_k, \mathfrak{l}), \mathfrak{l}\ n-\text{admissible prime}\}$. Lemma 5.12 and the assumption that the Selmer group is non trivial imply that $t_\chi(g_k, \mathfrak{l}_1) < t_\chi(g_k)$. We know that $ord_\varpi(\kappa_\chi(\mathfrak{l}_1)) = 0$ and that $\kappa_\chi(\mathfrak{l}_1) \in C \subset Sel_{(\mathfrak{l}_1)}(K, T(g_k)(\bar{\chi}))$, where $C \simeq \mathcal{O}_\mathfrak{p}/\varpi^k$. Hence $0 \neq \varpi^{k-1}\kappa_\chi(\mathfrak{l}_1) \in Sel_{(\mathfrak{l}_1)}(K, T(g_1)(\bar{\chi}))$. We can therefore choose an admissible prime $\mathfrak{l}_2$ distinct from $\mathfrak{l}_1$ such that $loc_{\mathfrak{l}_2}(\varpi^{k-1}\kappa_\chi(\mathfrak{l}_1)) \neq 0$, i.e.

$$ord_\varpi(loc_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_1))) = 0.$$

We now have the following chain of equalities:

$$\begin{aligned} t_\chi(g_k, \mathfrak{l}_1) + ord_\varpi(loc_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_1))) =& ord_\varpi(loc_{\mathfrak{l}_2}(c_\chi(\mathfrak{l}_1))) \\ =& t_\chi(h_k) = ord_\varpi(loc_{\mathfrak{l}_1}(c_\chi(\mathfrak{l}_2))) \\ =& t_\chi(g_k, \mathfrak{l}_2) + ord_\varpi(loc_{\mathfrak{l}_1}(\kappa_\chi(\mathfrak{l}_2))) \end{aligned}$$

where the second and third equalities follow from the second reciprocity law in the form given in remark 4.17.

Now

$$t_\chi(g_k, \mathfrak{l}_1) \leq t_\chi(g_k, \mathfrak{l}_2)$$

by minimality of $t_\chi(g_k, \mathfrak{l}_1)$, and $ord_\varpi(loc_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_1))) = 0$. Comparing the first, third and last member in the chain of equalities above we deduce that

$$\begin{aligned} t_\chi(g_k, \mathfrak{l}_1) = t_\chi(g_k, \mathfrak{l}_2) =& t_\chi(h_k), \\ ord_\varpi(loc_{\mathfrak{l}_1}(\kappa_\chi(\mathfrak{l}_2))) =& 0. \end{aligned}$$

Hence claims (1), (2) and (3) are proved.
It remains to show (4). We have two exact sequences:

$$Sel(K, T(h_k)(\bar{\chi})) \hookrightarrow Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K, T(h_k)(\bar{\chi})) \xrightarrow{v_{\mathfrak{l}_1} \oplus v_{\mathfrak{l}_2}} H^1_{ur}(K_{\mathfrak{l}_1}, T(h_k)(\bar{\chi})) \oplus H^1_{ur}(K_{\mathfrak{l}_2}, T(h_k)(\bar{\chi}))$$

$$Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(h_k)(\chi)) \hookrightarrow Sel(K, A(h_k)(\chi)) \xrightarrow{\delta_{\mathfrak{l}_1} \oplus \delta_{\mathfrak{l}_2}} H^1_{tr}(K_{\mathfrak{l}_1}, A(h_k)(\chi)) \oplus H^1_{tr}(K_{\mathfrak{l}_2}, A(h_k)(\chi))$$

where $v_{\mathfrak{l}_i}$ (resp. $\delta_{\mathfrak{l}_i}$) denotes the composition of the localisation map and the projection onto the unramified (resp. transverse) part.

By Poitou-Tate global duality the images of $v_{\mathfrak{l}_1} \oplus v_{\mathfrak{l}_2}$ and $\delta_{\mathfrak{l}_1} \oplus \delta_{\mathfrak{l}_2}$ are orthogonal complements with respect to the local Tate pairing. Now, the classes $\kappa_\chi(\mathfrak{l}_1)$ and $\kappa_\chi(\mathfrak{l}_2)$ belong to $Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K, T(h_k)(\bar{\chi}))$, and because of (3) and the fact that the localisation at $\mathfrak{l}_i$ of $\kappa_\chi(\mathfrak{l}_i)$ falls in the transverse part we have, up to unit:

$$\begin{aligned} v_{\mathfrak{l}_1} \oplus v_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_1)) =& (0, 1) \\ v_{\mathfrak{l}_1} \oplus v_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_2)) =& (1, 0). \end{aligned}$$

This implies that the map

$$v_{\mathfrak{l}_1} \oplus v_{\mathfrak{l}_2} : Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K, T(h_k)(\bar{\chi})) \to H^1_{ur}(K_{\mathfrak{l}_1}, T(h_k)(\bar{\chi})) \oplus H^1_{ur}(K_{\mathfrak{l}_2}, T(h_k)(\bar{\chi}))$$

is surjective. Since the pairing between $H^1_{ur}(K_{\mathfrak{l}_i}, T(h_k)(\bar{\chi}))$ and $H^1_{tr}(K_{\mathfrak{l}_i}, A(h_k)(\chi))$ is perfect for $i = 1, 2$ we deduce that

$$\delta_{\mathfrak{l}_1} \oplus \delta_{\mathfrak{l}_2} : Sel(K, A(h_k)(\chi)) \longrightarrow H^1_{tr}(K_{\mathfrak{l}_1}, A(h_k)(\chi)) \oplus H^1_{tr}(K_{\mathfrak{l}_2}, A(h_k)(\chi))$$

is the zero map, therefore we have an isomorphism:

$$Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(h_k)(\chi)) \simeq Sel(K, A(h_k)(\chi)).$$

Since the local conditions defining the Selmer groups $Sel(K, A(g_k)(\chi))$ and $Sel(K, A(h_k)(\chi))$ coincide outside $\{\mathfrak{l}_1, \mathfrak{l}_2\}$, we have $Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(h_k)(\chi)) = Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(g_k)(\chi))$, which finally yields:

$$Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(g_k)(\chi)) \simeq Sel(K, A(h_k)(\chi)).$$

□

5.16. Let us now prove the inequality

$$l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(g_k)(\chi)) \leq 2t_\chi(g_k)$$

by induction on $t_\chi(g_k)$. If $t_\chi(g_k) = 0$ then the inequality follows from corollary 5.13, hence let us suppose that $t_\chi(g_k) > 0$. If $Sel(K, A(g_k)(\chi))$ is trivial then there is nothing to prove. If $Sel(K, A(g_k)(\chi))$ is non trivial, choose two n-admissible primes $\mathfrak{l}_1$, $\mathfrak{l}_2$ as in lemma 5.15.

We have two exact sequences:

$$Sel(K, T(g_k)(\bar{\chi})) \hookrightarrow Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K, T(g_k)(\bar{\chi})) \xrightarrow{\delta_{\mathfrak{l}_1} \oplus \delta_{\mathfrak{l}_2}} H^1_{tr}(K_{\mathfrak{l}_1}, T(g_k)(\bar{\chi})) \oplus H^1_{tr}(K_{\mathfrak{l}_2}, T(g_k)(\bar{\chi}))$$

$$Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K, A(g_k)(\chi)) \hookrightarrow Sel(K, A(g_k)(\chi)) \xrightarrow{v_{\mathfrak{l}_1} \oplus v_{\mathfrak{l}_2}} H^1_{ur}(K_{\mathfrak{l}_1}, A(g_k)(\chi)) \oplus H^1_{ur}(K_{\mathfrak{l}_2}, A(g_k)(\chi)).$$

Let us identify, for $i=1,2$, $H^1_{tr}(K_{\mathfrak{l}_i},T(g_k)(\bar\chi))$ with $H^1_{ur}(K_{\mathfrak{l}_i},A(g_k)(\chi))^\vee$ via the local Tate pairing at $\mathfrak{l}_i$. Taking the dual of the lower exact sequence above and using Poitou-Tate global duality we find an exact sequence:

$$Sel(K,T(g_k)(\bar\chi))\hookrightarrow Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))\xrightarrow{\delta_{\mathfrak{l}_1}\oplus\delta_{\mathfrak{l}_2}} H^1_{tr}(K_{\mathfrak{l}_1},T(g_k)(\bar\chi))\oplus H^1_{tr}(K_{\mathfrak{l}_2},T(g_k)(\bar\chi))$$
$$\xrightarrow{v^\vee_{\mathfrak{l}_1}\oplus v^\vee_{\mathfrak{l}_2}} Sel(K,A(g_k)(\chi))^\vee\longrightarrow Sel_{\mathfrak{l}_1\mathfrak{l}_2}(K,A(g_k)(\chi))^\vee\longrightarrow 0.$$

Using (4) of lemma 5.15 we deduce that:

$$\begin{aligned} &l_{\mathcal{O}_\mathfrak{p}}(Sel(K,A(g_k)(\chi)))-l_{\mathcal{O}_\mathfrak{p}}(Sel(K,A(h_k)(\chi)))=\\ &l_{\mathcal{O}_\mathfrak{p}}Sel(K,T(g_k)(\bar\chi))-l_{\mathcal{O}_\mathfrak{p}}Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))+2k. \end{aligned}\tag{5.16.1}$$

Let us now compute $l_{\mathcal{O}_\mathfrak{p}}Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))-l_{\mathcal{O}_\mathfrak{p}}Sel(K,T(g_k)(\bar\chi))$. Choose an element $\zeta_\chi(\mathfrak{l}_1)\in Sel^{\mathfrak{l}_1}(K,T(g_k)(\bar\chi))$ such that $\delta_{\mathfrak{l}_1}(\zeta_\chi(\mathfrak{l}_1))$ generates the image of the map

$$Sel^{\mathfrak{l}_1}(K,T(g_k)(\bar\chi))\xrightarrow{\delta_{\mathfrak{l}_1}} H^1_{tr}(K_{\mathfrak{l}_1},T(g_k)(\bar\chi))\simeq\mathcal{O}_\mathfrak{p}/\varpi^k.$$

We find an exact sequence:

$$0\longrightarrow Sel(K,T(g_k)(\bar\chi))\longrightarrow Sel^{\mathfrak{l}_1}(K,T(g_k)(\bar\chi))\xrightarrow{\delta_{\mathfrak{l}_1}}\delta_{\mathfrak{l}_1}(\zeta_\chi(\mathfrak{l}_1))\mathcal{O}_\mathfrak{p}/\varpi^k\longrightarrow 0.\tag{5.16.2}$$

The cohomology class $\kappa_\chi(\mathfrak{l}_1)$ belongs to $Sel^{\mathfrak{l}_1}(K,T(g_k)(\bar\chi))$; hence, possibly after multiplying it by a unit of $\mathcal{O}_\mathfrak{p}/\varpi^k$, there exists an integer $m_1\geq 0$ such that

$$\delta_{\mathfrak{l}_1}(\varpi^{m_1}\zeta_\chi(\mathfrak{l}_1)-\kappa_\chi(\mathfrak{l}_1))=0.$$

This implies:

$$m_1+ord_\varpi(\delta_{\mathfrak{l}_1}(\zeta_\chi(\mathfrak{l}_1)))=ord_\varpi(\delta_{\mathfrak{l}_1}(\kappa_\chi(\mathfrak{l}_1)))=t_\chi(g_k)-t_\chi(g_k,\mathfrak{l}_1)=t_\chi(g_k)-t_\chi(h_k)$$

where the third equality follows from (2) of lemma 5.15. Using this and the exact sequence 5.16.2 we obtain:

$$\begin{aligned} l_{\mathcal{O}_\mathfrak{p}}Sel^{\mathfrak{l}_1}(K,T(g_k)(\bar\chi))-l_{\mathcal{O}_\mathfrak{p}}Sel(K,T(g_k)(\bar\chi)) =&k-ord_\varpi(\delta_{\mathfrak{l}_1}(\zeta_\chi(\mathfrak{l}_1)))\\ =&k+m_1-t_\chi(g_k)+t_\chi(h_k). \end{aligned}$$

Similarly, take $\zeta_\chi(\mathfrak{l}_2)\in Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))$ such that we have an exact sequence:

$$0\longrightarrow Sel(K,T(g_k)(\bar\chi))^{\mathfrak{l}_1}\longrightarrow Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))\xrightarrow{\delta_{\mathfrak{l}_2}}\delta_{\mathfrak{l}_2}(\zeta_\chi(\mathfrak{l}_2))\mathcal{O}_\mathfrak{p}/\varpi^k\longrightarrow 0.$$

Then there exists $m_2\geq 0$ such that $\delta_{\mathfrak{l}_2}(\varpi^{m_2}\zeta_\chi(\mathfrak{l}_2)-\kappa_\chi(\mathfrak{l}_2))=0$, hence we find:

$$\begin{aligned} l_{\mathcal{O}_\mathfrak{p}}Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))-l_{\mathcal{O}_\mathfrak{p}}Sel^{\mathfrak{l}_1}(K,T(g_k)(\bar\chi)) =&k-ord_\varpi(\delta_{\mathfrak{l}_2}(\zeta_\chi(\mathfrak{l}_2)))\\ =&k+m_2-t_\chi(g_k)+t_\chi(h_k). \end{aligned}$$

Therefore we obtain:

$$l_{\mathcal{O}_\mathfrak{p}}Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K,T(g_k)(\bar\chi))-l_{\mathcal{O}_\mathfrak{p}}Sel(K,T(g_k)(\bar\chi))=2k+m_1+m_2-2t_\chi(g_k)+2t_\chi(h_k).$$

This, together with equation 5.16.1, yields:

$$l_{\mathcal{O}_\mathfrak{p}}(Sel(K,A(g_k)(\chi)))-l_{\mathcal{O}_\mathfrak{p}}(Sel(K,A(h_k)(\chi)))=-m_1-m_2+2t_\chi(g_k)-2t_\chi(h_k)$$

which finally implies:

$$l_{\mathcal{O}_\mathfrak{p}}(Sel(K,A(g_k)(\chi)))-2t_\chi(g_k)=l_{\mathcal{O}_\mathfrak{p}}(Sel(K,A(h_k)(\chi)))-2t_\chi(h_k)-m_1-m_2.\tag{5.16.3}$$

Since $t_\chi(h_k) < t_\chi(g_k)$, by induction we have $l_{\mathcal{O}_\mathfrak{p}}(Sel(K, A(h_k)(\chi))) - 2t_\chi(h_k) \leq 0$, hence by 5.16.3 we also have $l_{\mathcal{O}_\mathfrak{p}}(Sel(K, A(g_k)(\chi))) - 2t_\chi(g_k) \leq 0$.

5.17. We have completed the proof of the inequality in the statement of theorem 3.11. It remains to prove that the equality also holds, under the additional hypothesis that the implication

$$Sel(K, A(g)(\chi)) = 0 \Longrightarrow t_\chi(g) = 0$$

holds true for every admissible automorphic form g modulo ϖ.
As before, the proof is by induction on $t_\chi(g_k)$, and the case $t_\chi(g_k) = 0$ is covered by lemma 5.13. So let us suppose that $t_\chi(g_k) > 0$. Then, *since we are assuming that*

$$Sel(K, A(g_1)(\chi)) = 0 \Longrightarrow t_\chi(g_1) = 0 \tag{5.17.1}$$

we deduce that $Sel(K, A(g_k)(\chi))$ *cannot be trivial*, hence we can invoke lemma 5.15. Let us stress, before continuing the proof, that it is at this point that the proof of the equality differs substantially from the proof of the inequality we gave above, and the assumption 5.17.1 is crucially needed.

Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be two admissible primes as in lemma 5.15, and let h_n be the automorphic form given by the lemma.
We proved above the following equality (5.16.3):

$$l_{\mathcal{O}_\mathfrak{p}}(Sel(K, A(g_k)(\chi))) - 2t_\chi(g_k) = l_{\mathcal{O}_\mathfrak{p}}(Sel(K, A(h_k)(\chi))) - 2t_\chi(h_k) - m_1 - m_2$$

Moreover we know that $t_\chi(h_k) < t_\chi(g_k)$. Therefore by induction we have

$$l_{\mathcal{O}_\mathfrak{p}}(Sel(K, A(h_k)(\chi))) = 2t_\chi(h_k).$$

In order to complete the proof it is therefore enough to show that $m_1 = m_2 = 0$.

5.18. Let us first show that $m_1 = 0$. Recall that m_1 was chosen in such a way that the equality $\delta_{\mathfrak{l}_1}(\varpi^{m_1}\zeta_\chi(\mathfrak{l}_1) - \kappa_\chi(\mathfrak{l}_1)) = 0$ is satisfied. In other words, the class $\varpi^{m_1}\zeta_\chi(\mathfrak{l}_1) - \kappa_\chi(\mathfrak{l}_1)$, which a priori lives in $Sel^{\mathfrak{l}_1}(K, T(g_k)(\bar{\chi}))$, actually belongs to $Sel(K, T(g_k)(\bar{\chi}))$. Lemma 5.6, which holds true modulo ϖ^k since we have $a(g_k, \chi) \not\equiv 0 \pmod{\varpi^k}$ by hypothesis, yields the equality:

$$\varpi^{k-1}\kappa_\chi(\mathfrak{l}_1) = \varpi^{m_1+k-1}\zeta_\chi(\mathfrak{l}_1)$$

hence:

$$\varpi^{k-1}loc_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_1)) = \varpi^{m_1+k-1}loc_{\mathfrak{l}_2}(\zeta_\chi(\mathfrak{l}_1)) \in H^1_{ur}(K, T(g_k)(\bar{\chi})) \simeq \mathcal{O}_\mathfrak{p}/\varpi^k.$$

By lemma 5.15 we have $ord_\varpi(loc_{\mathfrak{l}_2}(\kappa_\chi(\mathfrak{l}_1))) = 0$, hence the left hand side of the above equality is non zero. Therefore the right hand side must also be non trivial, yielding $m_1 + k - 1 < k$. Hence $m_1 = 0$.

5.19. Let us finally show that $m_2 = 0$. Since we already know that $m_1 = 0$ we have $\delta_{\mathfrak{l}_1}(\zeta_\chi(\mathfrak{l}_1)) = \delta_{\mathfrak{l}_1}(\kappa_\chi(\mathfrak{l}_1))$. By definition of $\zeta_\chi(\mathfrak{l}_1)$, this implies that $\delta_{\mathfrak{l}_1}(\kappa_\chi(\mathfrak{l}_1))$ generates the image of the map $Sel^{\mathfrak{l}_1}(K, T(g_k)(\bar{\chi})) \xrightarrow{\delta_{\mathfrak{l}_1}} H^1_{tr}(K_{\mathfrak{l}_1}, T(g_k)(\bar{\chi}))$.
Now recall that m_2 was chosen so that $\delta_{\mathfrak{l}_2}(\varpi^{m_2}\zeta_\chi(\mathfrak{l}_2) - \kappa_\chi(\mathfrak{l}_2)) = 0$, which implies that

$$\varpi^{m_2}\zeta_\chi(\mathfrak{l}_2) - \kappa_\chi(\mathfrak{l}_2) \in Sel^{\mathfrak{l}_1}(K, T(g_k)(\bar{\chi})) \subset Sel^{\mathfrak{l}_1\mathfrak{l}_2}(K, T(g_k)(\bar{\chi})).$$

Therefore there exists $m_3 \geq 0$ such that:

$$\delta_{\mathfrak{l}_1}(\varpi^{m_2}\zeta_\chi(\mathfrak{l}_2) - \kappa_\chi(\mathfrak{l}_2) - \varpi^{m_3}\kappa_\chi(\mathfrak{l}_1)) = 0.$$

In other words, we have $\varpi^{m_2}\zeta_\chi(\mathfrak{l}_2) - \kappa_\chi(\mathfrak{l}_2) - \varpi^{m_3}\kappa_\chi(\mathfrak{l}_1) \in Sel(K, T(g_k)(\bar\chi))$. Invoking lemma 5.6 again we obtain

$$\varpi^{m_2+k-1}\zeta_\chi(\mathfrak{l}_2) - \varpi^{k-1}\kappa_\chi(\mathfrak{l}_2) = \varpi^{m_3+k-1}\kappa_\chi(\mathfrak{l}_1)$$

hence:

$$loc_{\mathfrak{l}_1}(\varpi^{m_2+k-1}\zeta_\chi(\mathfrak{l}_2)) - loc_{\mathfrak{l}_1}(\varpi^{k-1}\kappa_\chi(\mathfrak{l}_2)) = loc_{\mathfrak{l}_1}(\varpi^{m_3+k-1}\kappa_\chi(\mathfrak{l}_1)).$$

Suppose by contradiction that $m_2 > 0$. Then the first term in the above equation dies, and we get:

$$-loc_{\mathfrak{l}_1}(\varpi^{k-1}\kappa_\chi(\mathfrak{l}_2)) = loc_{\mathfrak{l}_1}(\varpi^{m_3+k-1}\kappa_\chi(\mathfrak{l}_1)).$$

Notice that

$$loc_{\mathfrak{l}_1}(\varpi^{k-1}\kappa_\chi(\mathfrak{l}_2)) \in H^1_{ur}(K_{\mathfrak{l}_1}, T(g_k)(\bar\chi))$$
$$loc_{\mathfrak{l}_1}(\varpi^{m_3+k-1}\kappa_\chi(\mathfrak{l}_1)) \in H^1_{tr}(K_{\mathfrak{l}_1}, T(g_k)(\bar\chi)).$$

Hence both terms must be zero. On the other hand, since $ord_\varpi(loc_{\mathfrak{l}_1}(\kappa_\chi(\mathfrak{l}_2))) = 0$ the left hand side of the above equality is non trivial. This gives a contradiction, and completes the proof of theorem 3.11.

6. The indefinite case

6.1. We now switch to the indefinite setting, i.e. we assume that

$$[F : \mathbf{Q}] \not\equiv \#\{\mathfrak{q} : \mathfrak{q} | \mathfrak{n}^-\} \pmod 2.$$

We keep all the other assumptions unchanged. Recall from section I.5.7 that in the indefinite setting the functional equation forces the vanishing of the central value $L(f, \chi, 1)$, and one has the special value formula

$$L'(f, \chi, 1) = \frac{2^{r+1}}{N(c(\chi))\sqrt{N(disc(K/F))}} \cdot \langle f, f\rangle_{Pet} \cdot \langle a(f,\chi), a(f,\chi)\rangle_{NT}$$

Recall that $a(f, \chi)$ is the f_B- isotypic component of the point

$$Q_\chi = \sum_{\sigma\in Gal(H_{c(\chi)}/K)} \bar\chi(\sigma)(\sigma(P_\chi)) \in CH^1(Sh(G/Z, X)_{\hat{R}^\times}).$$

where $G = Res_{F/\mathbf{Q}}B^\times$ is the group of invertible elements in the quaternion algebra B ramified at all but one infinite place and at places dividing $\mathfrak{n}^-$, $R \subset B$ is an Eichler order of level $\mathfrak{n}^+$ and f_B is a Jacquet-Langlands transfer of the Hilbert modular form f to G. Observe that under our big image assumption on the residual Galois representation $\bar\rho$ the maximal ideal of the Hecke algebra containing the kernel I_{f_B} of the $\mathcal{O}_\mathfrak{p}$-valued character attached to f_B is not Eisenstein, hence there is an isomorphism $CH^1(Sh(G/Z, X)_{\hat{R}^\times})/I_{f_B} \simeq CH^1_0(Sh(G/Z, X)_{\hat{R}^\times})/I_{f_B}$. So after taking f_B-isotypic components we can see Q_χ as an element of $Jac(Sh(G/Z, X)_{\hat{R}^\times})/I_{f_B}(K_\chi)$, with no need to use the Hodge class (a similar trick was used in [**Zha14**]).

The point Q_χ gives rise to a cohomology class $c_\chi \in Sel(K, T(f)(\bar\chi))$. Our aim in this section is to prove that, if $L'(f, \chi, 1) \neq 0$, then $Sel(K, A(f)(\chi))$ has $\mathcal{O}_\mathfrak{p}$-corank one and we have the inequality (which is an equality under the same additional assumption as in theorem 3.11)

$$l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f)(\chi))/div \leq 2ord_\varpi(c_\chi), \tag{6.1.1}$$

where we denote by $Sel(K, A(f)(\chi))/div$ the quotient of $Sel(K, A(f)(\chi))$ by its maximal divisible submodule.

We are going to prove the following result

6.2. THEOREM. *Let $n = 2k$, and suppose that $c_\chi \not\equiv 0 \pmod{\varpi^k}$. Then the following inequality holds:*

$$l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f_k)(\chi)) \leq k + 2ord_\varpi(c_\chi).$$

Moreover the above inequality is an equality provided that the following implication holds true: if g is an admissible automorphic form mod ϖ and $Sel(K, A(g)(\chi)) = 0$ then $a(g, \chi)$ is a ϖ-adic unit.

6.3. Theorem 6.2 implies equation 6.1.1. Indeed, assume that the theorem holds and write $Sel(K, A(f)(\chi)) = (E_\mathfrak{p}/\mathcal{O}_\mathfrak{p})^r \oplus M$ with M finite. Then $l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f_k)(\chi)) = kr + l_{\mathcal{O}_\mathfrak{p}} M[\varpi^k]$, hence $r = 1$ and for k large enough we have

$$l_{\mathcal{O}_\mathfrak{p}}(M) = l_{\mathcal{O}_\mathfrak{p}}(M[\varpi^k]) = l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f_k)(\chi)) - k \leq 2ord_\varpi(c_\chi),$$

hence $l_{\mathcal{O}_\mathfrak{p}} Sel(K, A(f)(\chi))/div \leq 2ord_\varpi(c_\chi)$, as we had to show.

In order to prove theorem 6.2 we will make use of the second reciprocity law 4.16, describing the localisation of the class c_χ at an admissible prime $\mathfrak{l}$ in terms of the algebraic part of the special value of the L-function of a level raising of f at $\mathfrak{l}$. Choosing $\mathfrak{l}$ appropriately we will reduce the statement to the definite case, where we can invoke theorem 3.11. We find it an important and very interesting fact that the second reciprocity law allows us to prove special value formulas in rank one by reducing them to the rank zero case. In the next chapter we will investigate to what extent this phenomenon may carry over to higher rank situations.

6.4. Let us prove theorem 6.2. Let $t_\chi(f_n) = ord_\varpi(c_\chi)$. The reduction modulo ϖ^k of c_χ is contained in a free $\mathcal{O}_\mathfrak{p}/\varpi^k$-module C of rank one. This is proved in the same way as in proposition 5.8, once one we know that

$$Sel(K, T(f_n)(\bar{\chi})) = \mathcal{O}_\mathfrak{p}/\varpi^n \oplus N \oplus N.$$

To show this, choose $\mathfrak{l}$ admissible such that $loc_\mathfrak{l}(c_\chi) \neq 0$. By the second reciprocity law this implies that $a(g_n, \chi) \neq 0$, where g_n is a level raising of f at $\mathfrak{l}$ modulo ϖ^n. Hence $Sel(K, T(g_n)(\bar{\chi})) \simeq M \oplus M$. By corollary 5.7 we conclude.

There exists a class $k_\chi \in C \subset Sel(K, T(f_k)(\bar{\chi}))$ such that $\varpi^{t_\chi(f_n)} k_\chi = c_\chi$. Hence we can choose an admissible prime $\mathfrak{l}$ such that $ord_\varpi(loc_\mathfrak{l}(k_\chi)) = 0$. Using the second reciprocity law we find:

$$t_\chi(f_n) = ord_\varpi(loc_\mathfrak{l}(c_\chi)) = ord_\varpi(a(h_n, \chi)) = ord_\varpi(a(h_n, \bar{\chi})) = ord_\varpi(loc_\mathfrak{l}(c_{\bar{\chi}})), \tag{6.4.1}$$

where h_n is a level raising of f at $\mathfrak{l}$ modulo ϖ^n. Since $t_\chi(f_n) = t_{\bar{\chi}}(f_n)$, the above equation implies that the class $k_{\bar{\chi}} \in Sel(K, T(f_k)(\chi))$ satisfies $ord_\varpi(loc_\mathfrak{l}(k_{\bar{\chi}})) = 0$.

6.5. To prove theorem 6.2 we shall first compare the Selmer groups $Sel(K, A(f_k)(\chi))$ and $Sel(K, A(h_k)(\chi))$.

We have a square of Selmer groups:

$$\begin{tikzcd}
& Sel^{\mathfrak{l}}(K, A(f_k)(\chi)) = Sel^{\mathfrak{l}}(K, A(h_k)(\chi)) & \\
Sel(K, A(f_k)(\chi)) \arrow[ur, "c"] & & Sel(K, A(h_k)(\chi)) \arrow[ul, "d"'] \\
& Sel_{\mathfrak{l}}(K, A(f_k)(\chi)) = Sel_{\mathfrak{l}}(K, A(h_k)(\chi)) \arrow[ul, "a"] \arrow[ur, "b"'] &
\end{tikzcd}$$

Global duality yields an exact sequence:

$$0 \longrightarrow Sel(K,T(h_k)(\bar{\chi})) \longrightarrow Sel^{\mathfrak{l}}(K,T(h_k)(\bar{\chi})) \xrightarrow{v_{\mathfrak{l}}} H^1_{ur}(K_{\mathfrak{l}},T(h_k)(\bar{\chi})) \tag{6.5.1}$$
$$\xrightarrow{\delta_{\mathfrak{l}}^{\vee}} Sel(K,A(h_k)(\chi))^{\vee} \longrightarrow Sel_{\mathfrak{l}}(K,A(h_k)(\chi))^{\vee} \longrightarrow 0.$$

Since $k_\chi \in Sel^{\mathfrak{l}}(K,T(h_k)(\bar{\chi}))$ satisfies $ord_\chi(loc_{\mathfrak{l}}(k_\chi)) = 0$ the map $v_{\mathfrak{l}}$ is surjective, therefore $\delta_{\mathfrak{l}}^{\vee}$ is the zero map, which yields:

$$Sel(K,A(h_k)(\chi)) \simeq Sel_{\mathfrak{l}}(K,A(h_k)(\chi)).$$

In other words, the map b in the diagram above is an isomorphism. This implies that

$$l_{\mathcal{O}_{\mathfrak{p}}} Sel^{\mathfrak{l}}(K,A(h_k)(\chi)) - l_{\mathcal{O}_{\mathfrak{p}}} Sel_{\mathfrak{l}}(K,A(h_k)(\chi)) = \tag{6.5.2}$$
$$l_{\mathcal{O}_{\mathfrak{p}}} Sel^{\mathfrak{l}}(K,A(h_k)(\chi)) - l_{\mathcal{O}_{\mathfrak{p}}} Sel(K,A(h_k)(\chi)) \leq k.$$

Since the class $k_{\bar{\chi}} \in Sel(K,T(f_k)(\chi)) \simeq Sel(K,A(f_k)(\chi))$ satisfies $ord_\varpi(loc_{\mathfrak{l}}(k_{\bar{\chi}})) = 0$, we find an exact sequence

$$0 \longrightarrow Sel_{\mathfrak{l}}(K,A(f_k)(\chi)) \longrightarrow Sel(K,A(f_k)(\chi)) \xrightarrow{v_{\mathfrak{l}}} H^1_{ur}(K_{\mathfrak{l}},A(f_k)(\chi)) \longrightarrow 0$$

which yields

$$l_{\mathcal{O}_{\mathfrak{p}}} Sel(K,A(f_k)(\chi)) - l_{\mathcal{O}_{\mathfrak{p}}} Sel_{\mathfrak{l}}(K,A(f_k)(\chi)) = k.$$

Because of 6.5.2 we see that the map c is an isomorphism. Collecting everything we get

$$l_{\mathcal{O}_{\mathfrak{p}}} Sel(K,A(h_k)(\chi)) = l_{\mathcal{O}_{\mathfrak{p}}} Sel_{\mathfrak{l}}(K,A(f_k)(\chi)) = l_{\mathcal{O}_{\mathfrak{p}}} Sel(K,A(f_k)(\chi)) - k.$$

The automorphic form h_n now lives over a totally definite quaternion algebra, and the conclusion of theorem 3.11 holds for it, hence

$$l_{\mathcal{O}_{\mathfrak{p}}} Sel(K,A(h_k)(\chi)) \leq 2t_\chi(h_k). \tag{6.5.3}$$

On the other hand

$$t_\chi(h_k) = ord_\varpi(loc_{\mathfrak{l}}(c_\chi)) = t_\chi(f_k)$$

where the first equality follows from the second reciprocity law, and the second from the fact that $loc_{\mathfrak{l}}(k_\chi)$ is a unit. Therefore we have:

$$l_{\mathcal{O}_{\mathfrak{p}}} Sel(K,A(f_k)(\chi)) \leq 2t_\chi(f_k) + k$$

and equality holds whenever it does in equation 6.5.3. Hence the proof is complete.

CHAPTER 3

The plectic conjecture in positive and mixed characteristic

1. Outline of the chapter

The aim of this chapter is to prove partial results towards analogues in positive and mixed characteristic of the plectic conjectures of Nekovář and Scholl [**NS16**]. Our research in this direction was motivated by the desire to understand the relation between these conjectures and the results in [**YZ17**], [**YZ18**], and to figure out whether an analogue of the second reciprocity law discussed in chapter 2 could hold in higher rank situations. We will at first illustrate the plectic conjecture in the special case of Hilbert modular surfaces, emphasizing the role it could play in extending the results of the previous chapter. We then investigate a function field analogue, and point out the relation between "plectic objects" in this setting and moduli spaces of Shtukas with several legs. Finally, we report the result we were able to obtain so far towards a mixed characteristic version of the conjecture (Theorem 4.3).

2. The plectic conjecture for Hilbert modular surfaces

2.1. In this section we discuss a special case of the general plectic conjectures formulated by Nekovář and Scholl in [**NS16**], restricting ourselves to the minimal non trivial case of Hilbert modular surfaces. We fix therefore a *real quadratic* number field F. We will underline in our presentation the role that these conjectures should play in the construction of special cycles (or at least cohomology classes) related to higher rank cases of the Bloch-Kato conjecture. Similar ideas are discussed in Nekovář's notes [**Nek16**].

Our starting point is the phenomenon observed in section 6 in the previous chapter, where we explained how the second reciprocity law II.4.16, together with special value formulas for the L-function of Hilbert newforms, essentially allows to deduce cases of the Bloch-Kato conjecture in rank 1 from the knowledge of the rank 0 case. We wondered long ago whether this phenomenon may carry over to higher rank situations, and this motivated our work on the plectic conjectures.

2.2. The geometric objects of interest in the previous chapter were quaternionic sets or Shimura curves defined over F. The former, zero dimensional, contain points related to the central value $L(f_K, 1)$ where K/F is a CM extension and $f \in S_2(\mathfrak{n})$ is a newform. Special points on Shimura curves are instead related with $L'(f_K, 1)$. A very naive guess may suggest that CM points on Hilbert modular surfaces, which are two dimensional, could be related to the value $L^{(2)}(f_K, 1)$. The main aim of this chapter is to explain why in fact this guess may not be entirely wrong.

2.3. The superspecial locus and level raising. Recall that the second reciprocity law relied crucially on the fact that the supersingular locus $X_{\mathfrak{p}}^{ss}$ in the special fiber at a prime $\mathfrak{p}$ of good reduction of a Shimura curve of level $\mathfrak{n}^+$ attached to a quaternion algebra B can be identified with a quaternionic set attached to a totally definite quaternion algebra ramified at all finite places where B is *and at the additional prime* $\mathfrak{p}$. This allows to give a geometric realization of a level raising of f at $\mathfrak{p}$, whose L-function has central value "congruent" to $L'(f_K, 1)$.

Let us now take (the integral model over $\mathbf{Z}_p$ of) a Hilbert modular surface $\mathsf{H}_\mathfrak{n}$, where $\mathfrak{n} \subset \mathcal{O}_F$ and $\mathsf{H}_\mathfrak{n}$ stands for the choice of level $U_0(\mathfrak{n})$. Assume that p does not divide $\mathfrak{n}$ and splits in F. Then there is a stratification in the special fiber at p of $\mathsf{H}_\mathfrak{n}$, defined by Goren-Oort, whose strata have been described explicitly in [**TX16a**]. In particular its smallest stratum, called the *superspecial locus* of $\mathsf{H}_\mathfrak{n}$, is a finite set which can be identified with the quaternionic Shimura set:

$$\mathsf{H}_\mathfrak{n}^{ss} = B^\times \backslash \hat{B}^\times / \hat{U}_0(\mathfrak{n})$$

where B is the totally definite quaternion algebra ramified at *both* places above p. This description is analogous to the one of $X_\mathfrak{p}^{ss}$ recalled above, except that now we see a quaternion algebra whose invariants at two finite places differ from those of the original one ($M_2(\mathcal{O}_F)$). This suggests that one may be able to perform a level raising at *two* primes, hence passing from rank 2 phenomena to rank 0 ones.

2.4. Cohomology of Hilbert modular surfaces. Let us try and use CM points on $\mathsf{H}_\mathfrak{n}$ to construct interesting cohomology classes in the Selmer group attached to a Hilbert newform. For this purpose we need at first a description of the étale cohomology of Hilbert modular surfaces, which is given by the following

2.5. Theorem. *Let $f \in S_2(\mathfrak{n})$ be a newform. Then:*

(1) $H^i_{et}(\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}, \bar{\mathbf{Q}}_l)[f] = 0 \; \forall\, i \neq 2$.
(2) $H^2_{et}(\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}, \bar{\mathbf{Q}}_l)[f] = Ind^{\otimes}_{F/\mathbf{Q}} V_f$

where V_f is the Galois representation attached to f, and $Ind^{\otimes}_{F/\mathbf{Q}}$ denotes tensor induction from Γ_F to $\Gamma_\mathbf{Q}$.

Proof. The first statement is easy; the second one was proved up to semisimplification in [**BL84**]. Semisimplicity of the relevant part of the cohomology of $\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}$ was established in [**Nek18**]. □

2.6. Let K/F be a CM extension and $P \in \mathsf{H}_\mathfrak{n}(K)$ (for example, P could be the trace of a point with CM by $\mathcal{O}_K$). After making P nullhomologous (we will completely neglect this issue here) and applying the Abel-Jacobi map we obtain a cohomology class

$$AJ(P) \in H^1(K, H^3_{et}(\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}, \bar{\mathbf{Q}}_l(2)))$$

whose f-isotypic part will however necessarily vanish because of Theorem 2.5. Going one step further in the filtration induced by the Hochschild-Serre spectral sequence giving rise to AJ we obtain an element

$$AJ^2(P) \in H^2(K, H^2_{et}(\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}, \bar{\mathbf{Q}}_l(2))).$$

This time the étale cohomology group is potentially interesting, but $H^2(K, -)$ is not. Hence in either case nothing useful could be done with P.

2.7. The Plectic Conjecture: statement and sci-fi geometric explanation. To understand what is going on, let us look back at Theorem 2.5. Recall that tensor induction is defined as follows: we have a canonical embedding $\Gamma_\mathbf{Q} \hookrightarrow Aut(F \otimes_\mathbf{Q} \bar{\mathbf{Q}}/F)$ and a *non-canonical* isomorphism $Aut(F \otimes_\mathbf{Q} \bar{\mathbf{Q}}/F) \simeq \Gamma_F^2 \rtimes S_2$ (depending on the choice of a lift to $\Gamma_\mathbf{Q}$ of the non trivial element in $Gal(F/\mathbf{Q})$). The latter group, which we will denote by Γ_F^{plec}, acts on $V_f \otimes V_f$, and the restriction of this action to $\Gamma_{\mathbf{Q}}$ endows $V_f \otimes V_f$ with a well defined $\Gamma_{\mathbf{Q}}$-action, giving by definition the module $Ind^{\otimes}_{F/\mathbf{Q}} V_f$. This description suggests that the cohomology of the Hilbert modular surface appears to have more symmetry than that of a random algebraic variety over $\mathbf{Q}$. Let us suppose that one can endow this cohomology with a *canonical* action of Γ_F^{plec}, and that we can replace Γ_K by Γ_K^{plec} in the above computation. We would then find a class

(2.7.1)
$$AJ^{2,pl}(P) \in H^2(\Gamma_K^{pl}, H^2_{et}(\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}, \bar{\mathbf{Q}}_l(2))[f]) = H^2(\Gamma_K^2 \rtimes S_2, V_f(1) \otimes V_f(1)) = \wedge^2 H^1(\Gamma_K, V_f(1)).$$

This would be consistent with our hope that the point P could be related to rank 2 phenomena! This motivates the following

2.8. CONJECTURE. *(cf. [**NS16**, Conjecture 6.1, Remark 6.2]) There is a canonical lift of* $\mathbf{R}\Gamma_{et}(\mathsf{H}_{\mathfrak{n},\bar{\mathbf{Q}}}, \bar{\mathbf{Q}}_l) \in D^+(\bar{\mathbf{Q}}_l[\Gamma_{\mathbf{Q}}])$ *to an object in* $D^+(\bar{\mathbf{Q}}_l[\Gamma_F^{pl}])$.

Assuming the above conjecture (and that the resulting *plectic cohomology* behaves well), the formal computation in 2.7.1 would become meaningful. However working with points on $\mathsf{H}_{\mathfrak{n}}$ will still not produce anything interesting, and one needs a stronger *geometric version* of the conjecture, roughly along the following lines: there should be an object "$F \times F/S_2$" having "fundamental group" Γ_F^{pl}, a fiber map "$\iota : Spec\,\mathbf{Q} \to F \times F/S_2$" arising from the double cover $Spec\,F \to Spec\,\mathbf{Q}$ and an object $\mathsf{H}_{\mathfrak{n}}^{pl}$ fitting in a cartesian diagram

$$\begin{tikzcd} \mathsf{H}_{\mathfrak{n}} \arrow[r, dashed] \arrow[d, "\pi"] & \mathsf{H}_{\mathfrak{n}}^{pl} \arrow[d, dashed, "\pi_{plec}"] \\ Spec(\mathbf{Q}) \arrow[r, dashed, "\iota"] & F \times F/S_2. \end{tikzcd}$$

The desired lift predicted by conjecture 2.8 would then be the cohomology of $\mathsf{H}_{\mathfrak{n}}^{pl}$. Moreover special cycles on this plectic object may potentially give rise to interesting cohomology classes.

In what follows we will describe our attempt to study analogues of this sci-fi picture in contexts when it can be made meaningful, i.e. over function fields (where products over the base field behave as we need) and in mixed characteristic (where Scholze's theory of diamonds makes it possible to construct the sought-for objects).

3. The plectic conjecture over function fields

3.1. Fix a finite field $k = \mathbf{F}_q$ and a degree r cover $q : C' \to C$ of smooth projective geometrically irreducible curves over k. Let $E \hookrightarrow E'$ be the corresponding extension of function fields.

Fix an integer $d \geq 1$ and set $\mathcal{H} = GL_{d,C'}$ and $\mathcal{G} = Res_{C'/C}\mathcal{H} = q_*\mathcal{H}$. Let $\mathcal{N} \subset C$ be a finite set of closed points.

We are interested in the moduli stack $Sht^1(\mathcal{G})_{\mathcal{N}}$ of $\mathcal{G}$-Shtukas with one leg over C and $\mathcal{N}$-level structure, and in the cohomology of (suitable truncations of) its generic fibre

$$\begin{tikzcd} Sht^1_E(\mathcal{G})_{\mathcal{N}} \arrow[r] \arrow[d] & Sht^1(\mathcal{G})_{\mathcal{N}} \arrow[d] \\ E \arrow[r] & C \setminus \mathcal{N}. \end{tikzcd}$$

By definition 7.3 $Sht^1_{G,D}$ is the stack whose S-points are:

$$Sht^1(\mathcal{G})_{\mathcal{N}}(S) = \{x : S \to C \setminus \mathcal{N},\ \mathcal{F} \in Bun(\mathcal{G})_{\mathcal{N}}(S), \phi : \mathcal{F}_{|C \times S \setminus \Gamma_x} \xrightarrow{\sim} {}^{\tau}\mathcal{F}_{|C \times S \setminus \Gamma_x}\}.$$

As explained in section 2.7, an analogue of the plectic conjecture in this situation should predict that this moduli space is related to a suitable moduli space of Shtukas with r *legs* over C'. Indeed, we will now show that for any k-scheme S, a $\mathcal{G}$-shtuka on $C \times S$ with one leg at x is the same thing as a $\mathcal{H}$-shtuka on $C' \times S$ with "r unordered legs" at the fibre of x. Let us make this precise.

3.2. The Hilbert stack and the fiber map. To the curve C' and the integer r we can associate the Hilbert stack $\mathcal{H}^r_{C'}$ whose S-points, for any k-scheme S, are given by

$$\mathcal{H}^r_{C'}(S) = \{T \to C' \times S : T \to S \text{ is finite flat of rank } r\}$$

and the substack $\mathcal{E}t^r_{C'}$ parametrising families which are étale of rank r.

Taking the fiber of $q : C' \to C$ defines a morphism

$$\begin{aligned} \iota : C &\longrightarrow \mathcal{H}^r_{C'} \\ (S \to C) &\mapsto S \times_C C' \end{aligned}$$

Moreover, if $U \subset C$ is the open subscheme such that q is unramified on C and $V = q^{-1}(U)$ then the restriction of the fibre map to $U \to \mathcal{H}^r_V$ lands inside $\mathcal{E}t^r_V$.

Let $Sym^r(V)$ be the r-th symmetric power $[V^r/S_r]$. We have the following result, proven in greater generality in [**Ryd11**, Theorem 5.1]:

3.3. Lemma. *There is a canonical isomorphism*

$$\mathcal{E}t^r_V \xrightarrow{\sim} Sym^r(V).$$

Proof. We sketch the proof for completeness. See [**Ryd11**, Theorem 5.1] for more details. Given $T \in \mathcal{E}t^r_V(S)$ let $Sec^r_{T/S}$ be the complement of the diagonals in $(T/S)^r$. It is an S_r-torsor over S and it comes equipped with a natural equivariant map $Sec^r_{T/S} \to V^r$, which yields an S-point of $Sym^r(V)$.

Conversely, given a S_r-torsor W/S together with an S_r-equivariant map $W \to V^r$, let $Z = W/S_{r-1}$, the action of S_{r-1} being on the first $r-1$ components. Then Z/S is étale of rank r, and the map $W \to V^r$ induces a map $Z \to V$, projecting to the last component. This gives an S-point of $\mathcal{E}t^r_V$. One checks that these two constructions define inverse isomorphisms of stacks. □

3.4. $\mathcal{G}$-bundles and Weil restriction. Let $\mathcal{N}' = q^{-1}\mathcal{N}$. We will now compare $Bun(\mathcal{G})_{\mathcal{N}}$ and $Bun(\mathcal{H})_{\mathcal{N}'}$. A similar, and more general, result has been proved by Damiolini [**Dam17**, Appendix A], generalising [**BS15**]. We will closely follow their proof, the only difference being that we do not require $C' \xrightarrow{q} C$ to be Galois, and we take care of level structures. A similar result is also proved (in much more detail) in [**Bre19**, Lemma 3.3].

3.5. Lemma. *(cf. [**BS15**, Lemma 4.1.3], [**Bre19**, Lemma 3.2]) Let S be a k-scheme and $\mathcal{E}$ an $\mathcal{H}$-bundle over $C' \times S$. Then $q_*\mathcal{E}$ is a $q_*\mathcal{H}$-bundle over $C \times S$.*

Proof. By definition, $\mathcal{E}$ is a smooth $C' \times S$-scheme with an action of $\mathcal{H}$ such that the natural map

$$\begin{aligned} \mathcal{H} \times_{C' \times S} \mathcal{E} &\longrightarrow \mathcal{E} \times_{C' \times S} \mathcal{E} \\ (g, e) &\mapsto (e, ge) \end{aligned} \tag{3.5.1}$$

is an isomorphism. One shows that the sheaf $\pi_*\mathcal{E}$ is representable by a smooth $C \times S$-scheme. Moreover, since Weil restriction is functorial and commutes with products, $q_*\mathcal{H}$ acts on $q_*\mathcal{E}$, and pushing forward the isomorphism 3.5.1 we get an isomorphism

$$q_*\mathcal{H} \times_{C\times S} q_*\mathcal{E} \longrightarrow q_*\mathcal{E} \times_{C\times S} q_*\mathcal{E}.$$

By [**BLR90**, Section 2.2, Proposition 14] $q_*\mathcal{E}$ has étale locally a section, hence it is a $q_*\mathcal{H}$-bundle. □

3.6. Using the above lemma we obtain a map:

$$q_* : Bun(\mathcal{H}) \longrightarrow Bun(\mathcal{G})$$

which induces a map

$$q_* : Bun(\mathcal{H})_{\mathcal{N}'} \longrightarrow Bun(\mathcal{G})_{\mathcal{N}}$$

since the pushforward of the trivial $\mathcal{H}$-bundle on $\mathcal{N}' \times S$ is the trivial $\mathcal{G}$-bundle on $\mathcal{N} \times S$.

Let us now construct an inverse. There is a natural map $\iota : q^*\mathcal{G} = q^*q_*\mathcal{H} \longrightarrow \mathcal{H}$, through which $q^*\mathcal{G}$ acts on $\mathcal{H}$ on the left. Now let $\mathcal{F}$ be a $\mathcal{G}$-bundle on $C\times S$, so that $q^*\mathcal{F}$ is a $q^*\mathcal{G}$-bundle. We can therefore define the $\mathcal{H}$-bundle:

$$q^\star\mathcal{F} = q^*\mathcal{F} \times^{q^*\mathcal{G}} \mathcal{H}$$

where $q^*\mathcal{F} \times^{q^*G} \mathcal{H}$ denotes the quotient of $q^*\mathcal{F} \times_{C'\times S} \mathcal{H}$ by the action of $q^*\mathcal{G}$ induced by its action on $q^*\mathcal{F}$ and its left action on $\mathcal{H}$ given above. Then the right $\mathcal{H}$-action on $\mathcal{H}$ makes $q^\star\mathcal{F}$ an $\mathcal{H}$-bundle.

The above construction yields a map $Bun(\mathcal{G})_{\mathcal{N}} \xrightarrow{q^\star} Bun(\mathcal{H})_{\mathcal{N}'}$.

3.7. Proposition. *The functors*

$$Bun(\mathcal{H})_{\mathcal{N}'} \xrightarrow{q_*} Bun(\mathcal{G})_{\mathcal{N}}$$

$$Bun(\mathcal{H})_{\mathcal{N}'} \xleftarrow{q^\star} Bun(\mathcal{G})_{\mathcal{N}}$$

are inverse isomorphisms of stacks.

Proof. (cf. [**BS15**, Theorem 4.1.5], [**Bre19**, Lemma 3.3]) Let $\mathcal{E}$ be a $\mathcal{H}$-bundle on $C' \times S$, $\mathcal{F} = q_*\mathcal{E}$. By adjunction we have a natural map $q^*\mathcal{F} \longrightarrow \mathcal{E}$, hence a map $q^*\mathcal{F} \times_{C'\times S} \mathcal{H} \to \mathcal{E} \times_{C'\times S} \mathcal{H} \to \mathcal{E}$ inducing a morphism of $\mathcal{H}$-bundles:

$$q^*\mathcal{F} \times^{\pi^*\mathcal{G}} \mathcal{H} \longrightarrow \mathcal{E}$$

which is an isomorphism. This can be checked locally, and since q is finite it is enough to check this on pullbacks of étale neighbourhoods of points in $C \times S$ where $\mathcal{F}$ is trivial. Hence we may assume that both $\mathcal{E}$ and $\mathcal{F}$ are trivial, in which case the statement is clear.

Conversely, let $\mathcal{F}$ be a $\mathcal{G}$-bundle. Pushing forward the natural map

$$q^*\mathcal{F} \to q^*\mathcal{F} \times^{\pi^*\mathcal{G}} \mathcal{H}$$

and composing with the adjunction map $\mathcal{F} \to q_*q^*\mathcal{F}$ we get a natural map

$$\mathcal{F} \to q_*q^\star\mathcal{F}$$

To check that this is an isomorphism we can work étale locally on $C\times S$, hence suppose that $\mathcal{F}$ is trivial, in which case the claim holds.

By the above discussion the functors $q_*, q^\star$ induce an isomorphism $Bun(\mathcal{G}) \simeq Bun(\mathcal{H})$. Since they clearly respect level structures, they induce isomorphisms $Bun(\mathcal{G})_{\mathcal{N}} \simeq Bun(\mathcal{H})_{\mathcal{N}'}$. □

3.8. EXAMPLE. Let us make the affine situation more explicit, i.e. let $Spec(A) = O \subset C$ be an open affine subscheme and $Spec(B) = O' = q^{-1}(O)$. In this case an $\mathcal{H}$-bundle on V is just a projective B-module M of finite rank. The pushforward q_*M is the module M itself, seen as a projective A-module (of rank dr). Hence we see that torsors for the restriction of scalars of GL_d from B to A are projective A-modules of rank dr having a B-module structure.

3.9. Let $U \subset C$ be the complement of the ramification locus of q and of $\mathcal{N}$, and let $V = q^{-1}(U)$. If

$$\begin{array}{ccc} y & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbf{F}_q \end{array}$$

is an S-point of $\mathcal{E}t^r_V$ we define Γ_y as the image of y in $S \times V$. It is a closed subscheme of $S \times V$ since $y \to S \times V$ is the composite of the graph $y \to y \times V$, which is a closed embedding, and of the finite map $y \times V \to S \times V$.

3.10. DEFINITION. The moduli space of $\mathcal{H}$-shtukas with r unordered legs on V and level structure $\mathcal{N}'$ is the stack $Sht^{(r)}(\mathcal{H})_{\mathcal{N}'}$ whose S-points, for any k-scheme S, are given by

$$Sht^{(r)}(\mathcal{H})_{\mathcal{N}'}(S) = \{y \in \mathcal{E}t^r_V(S),\ \mathcal{E} \in Bun(\mathcal{H})_{\mathcal{N}'}(S),\ \phi : \mathcal{E}_{|C' \times S \setminus \Gamma_y} \xrightarrow{\sim} {}^\tau\mathcal{E}_{|C' \times S \setminus \Gamma_y}\}.$$

3.11. There is a forgetful map $Sht^{(r)}(\mathcal{H})_{\mathcal{N}'}(S) \to \mathcal{E}t^r_V$. Moreover, as usual $Sht^{(r)}(\mathcal{H})_{\mathcal{N}'}$ fits into the following cartesian diagram

$$\begin{array}{ccc} Sht^{(r)}(\mathcal{H})_{\mathcal{N}'} & \longrightarrow & Hk^{(r)}(\mathcal{H})_{\mathcal{N}'} \\ \downarrow & & \downarrow \\ Bun(\mathcal{H})_{\mathcal{N}'} & \xrightarrow{Id \times Fr} & Bun(\mathcal{H})_{\mathcal{N}'} \times Bun(\mathcal{H})_{\mathcal{N}'} \end{array}$$

where the Hecke stack $Hk^{(r)}(\mathcal{H})_{\mathcal{N}'}$ is defined in the usual way.

3.12. PROPOSITION. *Let $\iota : U \to \mathcal{E}t^r_V \simeq Sym^r(V)$ be the fiber map. The functor $q^\star$ induces cartesian diagrams:*

$$\begin{array}{ccc} Hk^1(\mathcal{G})_{\mathcal{N}} & \longrightarrow & Hk^{(r)}(\mathcal{H})_{\mathcal{N}'} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\iota} & Sym^r(V) \end{array}$$

$$\begin{array}{ccc} Sht^1(\mathcal{G})_{\mathcal{N}} & \longrightarrow & Sht^{(r)}(\mathcal{H})_{\mathcal{N}'} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\iota} & Sym^r(V). \end{array}$$

PROOF. Let S be a k-scheme, $\mathcal{F}, \mathcal{F}' \in Bun(\mathcal{G})_{\mathcal{N}}(S)$ and $\phi : \mathcal{F}_{|C\times S\setminus\Gamma_x} \to \mathcal{F}'_{|C\times S\setminus\Gamma_x}$ an isomorphism. It induces an isomorphism $q^\star\phi : q^\star\mathcal{F}_{|C'\times S\setminus\Gamma_{\iota(x)}} \to q^\star\mathcal{F}'_{|C'\times S\setminus\Gamma_{\iota(x)}}$, which yields an S-point of $Hk^{(r)}(\mathcal{H})_{\mathcal{N}'}$ (with legs in the image of the fibre map). Conversely, given $\mathcal{H}$-bundles $\mathcal{E}$, $\mathcal{E}'$ on $C' \times S$ together with an isomorphism $\phi : \mathcal{E}_{|C'\times S\setminus\Gamma_{\iota(x)}} \to \mathcal{E}'_{|C'\times S\setminus\Gamma_{\iota(x)}}$ for some $x : S \to U$, pushing forward we obtain $\mathcal{G}$-bundles on $C \times S$ together with an isomorphism $q_*\phi : q_*\mathcal{E}_{|C\times S\setminus\Gamma_x} \to q_*\mathcal{E}'_{|C\times S\setminus\Gamma_x}$. The equivalence in proposition 3.7 implies that the first diagram is cartesian.

The corresponding result for Shtukas follows, since:

$$\begin{aligned} Sht^1(\mathcal{G})_{\mathcal{N}} =& Hk^1(\mathcal{G})_{\mathcal{N}} \times_{Bun(\mathcal{G})_{\mathcal{N}}\times Bun(\mathcal{G})_{\mathcal{N}}} Bun(\mathcal{G})_{\mathcal{N}} \\ =& \left(Hk^{(r)}(\mathcal{H})_{\mathcal{N}'} \times_{Sym^r(V)} U\right) \times_{Bun(\mathcal{H})_{\mathcal{N}'}\times Bun(\mathcal{H})_{\mathcal{N}'}} Bun(\mathcal{H})_{\mathcal{N}'} \\ =& Sht^{(r)}(\mathcal{H})_{\mathcal{N}'} \times_{Sym^r(V)} U. \end{aligned}$$

□

3.13. EXAMPLE. Let us check the affine situation, with notations as in example 3.8. A $\mathcal{G}$-bundle on $Spec\ A$ is a projective B-module M whose A-module structure is only remembered. Let $x \in Spec\ A(k)$ and assume for simplicity that it corresponds to a principal maximal ideal (ξ). Then an isomorphism

$$\phi : M\left[\frac{1}{\xi}\right] \xrightarrow{\sim} M\left[\frac{1}{\xi}\right]$$

gives rise to a Shtuka with a leg at x. Seeing M as an $\mathcal{H}$-bundle (which amounts to remembering the B-module structure), the same ϕ makes M an $\mathcal{H}$-Shtuka with legs at the primes of B lying above x.

3.14. It follows from the above proposition that there is a cartesian diagram

$$\begin{array}{ccc} Sht^1_E(\mathcal{G})_{\mathcal{N}} & \longrightarrow & Sht^{(r)}_{E'}(\mathcal{H})_{\mathcal{N}'} \\ \downarrow & & \downarrow \\ E & \xrightarrow{\iota} & Sym^r(E') \end{array}$$

where $Sht^{(r)}_{E'}(\mathcal{H})_{\mathcal{N}'}$ is by definition the fibre of $Sht^{(r)}(\mathcal{H})_{\mathcal{N}'}$ over $Sym^r(E')$. This is (almost) the sought-for analog of the diagram in 2.7. Indeed, since the map $E'^r \longrightarrow Sym^r(E')$ is étale we have an exact sequence (cf. [**BH15**, Lemma 4.2])

$$1 \longrightarrow \pi_1(E'^r, x) \longrightarrow \pi_1(Sym^r(E'), y) \longrightarrow S_r \longrightarrow 1$$

where x is a geometric point of E'^r and y its image in $Sym^r(E')$. The isotropy group of y is S_r, and yields a canonical splitting $S_r = \pi_1([y/S^r], y) \longrightarrow \pi_1(Sym^r(E'), y)$ of the above short exact sequence. We deduce that $\pi_1(Sym^r(E'), y) \simeq S_r \ltimes \pi_1(E'^r)$. Let us point out that it is crucial at this point to be working with stacks: the fundamental group of the symmetric power (as a scheme) of a curve is isomorphic to the abelianized fundamental group of the curve, which would be useless for our purposes.

3.15. However, let us remark that the existence of the above diagram does not directly imply that the cohomology of $Sht^1_E(\mathcal{G})_{\mathcal{N}}$ has an action of $S_r \ltimes \pi_1(E')^r$ because of the following two issues:

(1) The stack $Sht^{(r)}_{E'}(\mathcal{H})_{\mathcal{N}'}$ is not of finite type over $Sym^r(E')$, hence we cannot directly apply any base change theorem.
(2) The fundamental group of E'^r is *not* isomorphic to $\Gamma^r_{E'}$.

The first issue can be addressed by taking suitable truncations of the relevant moduli space, as discussed in I.7.8, I.7.9; the second one is related to Drinfeld's lemma, measuring the difference between $\pi_1((E')^r)$ and $\pi_1(E')^r$ in terms of partial Frobenius morphisms. Those are however not defined on $Sym^r(E')$.

3.16. Relation with the work [YZ17]. The most important lesson we learn from this function field picture is the fact that

Shtukas with one leg for the restriction of scalars of the group $\mathcal{H}$ are the same as Shtukas with several (unordered) legs for the group $\mathcal{H}$.

Together with I.7.18, this suggests that moduli spaces of Shtukas with one leg for a restriction of scalars are related to the spaces containing special cycles connected with higher derivatives of L-functions.

For example taking $d = 2$ (i.e. $\mathcal{H} = GL_{2,C'}$), r even and $\mathcal{N} = \emptyset$ we see that $Sht_E(\mathcal{G})$ is related to the stack used in [**YZ17**] to formulate higher Gross-Zagier formulas (Theorem I.7.16). Indeed in this case the moduli space $Sht^{(r)}_{E'}(\mathcal{H})$ receives a map from $Sht^{\underline{r}}_{(E')^r}(\mathcal{H})$. After quotienting by the action of the center, this contains (a base change of) the space used in section I.7.14 to construct the cycles appearing in theorem 7.16.

4. The mixed characteristic situation

4.1. Let us now come back to the number field setting: let $F/\mathbf{Q}$ be a totally real field. We will assume that F is *quadratic* throughout this section to simplify the notation, although most of our arguments extend to a more general setting. The map $Spec(\mathcal{O}_F) \to Spec(\mathbf{Z})$ is analogous to the cover $C' \xrightarrow{q} C$ in the previous section (with $C = \mathbf{P}^1_k$ and $deg(q) = 2$). Let $H = GL_{2,F}$ and $G = Res_{F/\mathbf{Q}}H$. The Shimura varieties $Sh(G,X)_U$ introduced in I.3, or better their integral models over $Spec\ \mathbf{Z}$, should be the analogue of suitable moduli spaces of $\mathcal{G}$-Shtukas with one leg, where $\mathcal{G} = Res_{C'/C}GL_{2,C'}$. In this section we are interested in a p-adic analogue of the plectic conjecture, i.e. in (the generic fiber of) the Hilbert modular variety H_U over $\mathbf{Z}_p$ defined in section I.3 and its cover $\tilde{\mathsf{H}}_U$. Recall that H_U is a coarse moduli space, quotient of the fine moduli space $\tilde{\mathsf{H}}_U$ by the action of a finite group. The latter parametrizes data $(A, \iota, \lambda, \eta)$ where (A, ι, λ) is a polarized abelian surface with real multiplication by $\mathcal{O}_F$ (PRMAS) and η is a U-level structure. We fix for simplicity U to be the full level N-structure, with $N \geq 3$ and $(p, N) = 1$. Hence η is the datum of an $\mathcal{O}_F$-linear isomorphism of étale group schemes $(\mathcal{O}_F/N)^2 \xrightarrow{\sim} A[N]$. We will denote the generic fiber of H_U (resp. $\tilde{\mathsf{H}}_U$) by X_U (resp. $\tilde{X}_U$).

The function field counterpart of H_U is the moduli space of $\mathcal{G}$-Shtukas over $\mathbf{P}^1_k$ with a leg factoring through a fixed completed local ring $\hat{\mathcal{O}}_{\mathbf{P}^1_k,x}$ where x is a closed point of $\mathbf{P}^1_k$. Notice that this moduli space is still parametrizing *global* Shtukas; it is *only the leg* which is restricted at a given place. Hence our objects of interest are *not* purely local, but are instances of *semi-global* objects.

4.2. Let us take for example a closed point $x \in \mathbf{P}^1_k$ which splits in C'. Denoting by y_1, y_2 the two points above it, we have $Spec(\hat{\mathcal{O}}_{\mathbf{P}^1_k,x}) \times_{\mathbf{P}^1_k} C' = Spec(\hat{\mathcal{O}}_{C',y_1} \times \hat{\mathcal{O}}_{C',y_2})$. Both local rings on the right hand side are isomorphic to $\mathcal{O} = \hat{\mathcal{O}}_{\mathbf{P}^1_k,x}$, and the fiber map, when restricted to points of $\mathbf{P}^1_k$ factoring through $Spec(\hat{\mathcal{O}}_{\mathbf{P}^1_k,x})$, sends S to

$$S \times_{\mathbf{P}^1_k} C = S \times_{Spec(\hat{\mathcal{O}}_{\mathbf{P}^1_k,x})} Spec(\hat{\mathcal{O}}_{\mathbf{P}^1_k,x}) \times_{\mathbf{P}^1_k} C' = S \times_{Spec(\mathcal{O})} Spec(\mathcal{O} \times \mathcal{O}).$$

In other words we can see the fiber map as the diagonal map $Spec(\mathcal{O}) \to Spec(\mathcal{O} \otimes \mathcal{O})$.

Let now p be a prime which splits in F. The above discussion suggests that there should be a diamond X_U^{plec} and a cartesian diagram

$$\begin{tikzcd} X_U^{\diamond} \arrow[r] \arrow[d, "p"] & X_U^{plec} \arrow[d, "q"] \\ \mathbf{Q}_p^{\diamond} \arrow[r, "\Delta"] & (\mathbf{Q}_p^{\diamond})^2 \end{tikzcd}$$

where Δ is the diagonal map and $X_U^{\diamond}$ is the diamond attached to the p-adic analytification of X_U. Moreover the map q should be at the very least surjective (without such a requirement the existence of the above diagram would be trivial).

We are at the moment only able to prove the following weaker result:

4.3. THEOREM. *Suppose that* p *splits in* F. *Let* $\tilde{\mathcal{X}}_U^{\diamond}$ *be the diamond attached to the generic fiber (as an adic space) of the completion of* $\tilde{\mathsf{H}}_U$ *along its special fiber. Then there exists an étale sheaf* $\tilde{\mathcal{X}}_U^{plec}$ *on* Perf *fitting into a diagram*

$$\begin{tikzcd} \tilde{\mathcal{X}}_U^{\diamond} \arrow[r] \arrow[d, "p"] & \tilde{\mathcal{X}}_U^{plec} \arrow[d, "q"] \\ \mathbf{Q}_p^{\diamond} \arrow[r, "\Delta"] & (\mathbf{Q}_p^{\diamond})^2. \end{tikzcd}$$

which is cartesian at the level of geometric points and such that q *is surjective on geometric points.*

4.4. REMARK. The diamond $\tilde{\mathcal{X}}_U^{\diamond}$ embeds canonically into $\tilde{X}_U^{\diamond}$, but it is smaller, as $\tilde{X}_U$ is not proper. Precisely, $\tilde{\mathcal{X}}_U^{\diamond}$ is identified with the locus of good reduction of (the analytification of) the universal abelian scheme over $\tilde{X}_U$. The reason why we (have to) use $\tilde{\mathcal{X}}_U^{\diamond}$ instead of $\tilde{X}_U^{\diamond}$ is that, in order to construct the upper horizontal map in the above diagram, we need to know the functor of points of the space at the top left corner. This is known, to the best of our knowledge, for the good reduction locus, but not for the full variety. There could be two ways to fix this issue: either try to work with a compactification of $\tilde{X}_U$, or avoid the problem proving the same theorem for the Shimura variety attached to a totally indefinite division algebra. Such a Shimura variety is compact, hence the generic fiber of the completion of an integral model coincides with the analytification of the full space.

Secondly, we still have to prove that the object $\tilde{\mathcal{X}}_U^{plec}$ we constructed is not just an étale sheaf but a diamond, or at least a v-sheaf. One can then try to apply [**SW17**, Corollary 17.4.10] to deduce that the diagram in the statement of the theorem is cartesian from the fact that this holds true after taking geometric points, or we could try to adapt the proof of [**SW17**, Theorem 25.1.2]. Lastly, the action of the finite group Δ_U has to be taken into account.

4.5. The function field situation suggests that to construct $\tilde{\mathcal{X}}_U^{plec}$ one should interpret $\tilde{\mathcal{X}}_U$ as a moduli space of *global G-Shtukas over* $\mathbf{Z}$ with a leg factoring through $\mathbf{Q}_p$, and then show that those come from GL_2-Shtukas over $\mathcal{O}_F$ with two legs. It is however not clear how to make this precise, and we will instead proceed as follows:

(1) Describe objects parametrized by $\tilde{\mathcal{X}}_U$ as triples $(\mathcal{E}_p, \mathcal{E}^p, \psi)$, where $\mathcal{E}_p$ is a local Shtuka (coming from a Breuil-Kisin-Fargues module), $\mathcal{E}^p$ is an object recording information *outside* p and ψ is a gluing datum between $\mathcal{E}_p$ and $\mathcal{E}^p$.
(2) Observe that the $\mathcal{O}_F$-action forces $\mathcal{E}_p$ to split as a sum of two Shtukas $\mathcal{E}_p^1, \mathcal{E}_p^2$ with same leg.
(3) Define $\tilde{\mathcal{X}}_U^{pl}$ as the moduli space of quadruples $(\mathcal{E}_p^1, \mathcal{E}_p^2, \mathcal{E}^p, \psi)$ where $\mathcal{E}_p^1, \mathcal{E}_p^2$ are local Shtukas with possibly different legs.

The object $\mathcal{E}^p$ should ideally be a *Shtuka over* $\mathbf{Z}[\frac{1}{p}]$ *with no legs.* For the time being, we will instead simply take it to be an abelian variety in positive characteristic. We will therefore obtain a description of $\tilde{\mathcal{X}}_U$ as a space gluing a moduli space of local Shtukas with a family of *Igusa varieties.* The cohomological version of this fact for perfectoid (compact unitary) Shimura varieties is the content of [**CS17**], from which we drew inspiration for our construction.

In fact, point (1) above is not specific to Hilbert modular varieties, and should work for arbitrary Shimura varieties allowing to develop a p-adic theory of Shimura varieties (= semi-global Shtukas) and extend it to the case of several legs. We hope that this can be of independent interest, hence we will at first explain the argument in the easier case of modular curves.

4.6. Modular curves. Let $N > 3$ be an integer and $Y_1(N)$ the moduli space of elliptic curves with a marked point of order N. This is a smooth, quasi-projective curve over $\mathbf{Z}[\frac{1}{N}]$. Let p be a prime not dividing N and $\mathcal{Y}_1(N)$ the adic generic fiber of the completion of $Y_1(N)$ along the special fiber at p. It is the good reduction locus inside the adic space attached to $Y_1(N)/\mathbf{Q}_p$. The functor of points of the diamond attached to $\mathcal{Y}_1(N)$ has the following description:

4.7. LEMMA. *The diamond $\mathcal{Y}_1(N)^\diamond$ is the sheafification (in the analytic topology) of the functor sending a perfectoid affinoid $Spa(S, S^+) \in \mathsf{Perf}$ to the set of isomorphism classes of data $((R, R^+), (E, P))$ where (R, R^+) is an untilt of (S, S^+) in characteristic zero and (E, P) is an elliptic curve over R^+ with a marked point of order N.*

PROOF. Let F be the functor of points of $Y_1(N)$ and $Nilp_{\mathbf{Z}_p}$ the category of rings in which p is nilpotent. Then the completion of $Y_1(N)$ along its special fiber represents the restriction of F to $Nilp_{\mathbf{Z}_p}$. By [**SW13**, Proposition 2.2.2] the functor of points of the generic fiber of the completion of $Y_1(N)$ at p, restricted to perfectoid pairs, is the sheafification of the functor F' sending (R, R^+) to $\varprojlim_n F(R^+/\varpi^n)$ (notice that according to *loc. cit.* one should take $\varinjlim_{R_0 \subset R^+} \varprojlim_n F(R_0/\varpi^n)$ instead, where the direct limit runs over all open bounded subrings $R_0 \subset R^+$, but R^+ is always open bounded if (R, R^+) is a perfectoid Huber pair). To complete the proof of the lemma it remains to show that $\varprojlim_n F(R^+/\varpi^n) = F(R^+)$, which is a kind of algebraization result for formal elliptic curves. Notice however that R^+ is in general not noetherian, hence Grothendieck's algebraization theorem cannot be applied. However in this case it suffices to argue that, since $R^+ = \varprojlim_n R^+/\varpi^n$ and $Y_1(N)$ is affine (say $Y_1(N) = Spec(A)$), we have

$$\begin{aligned} F(R^+) =& Hom(Spec(R), Y_1(N)) = Hom(A, \varprojlim_n R^+/\varpi^n) = \varprojlim_n Hom(A, R/\varpi^n) \\ =& \varprojlim_n Hom(Spec(R/\varpi^n), Y_1(N)) = \varprojlim_n F(R^+/\varpi^n). \end{aligned}$$

□

4.8. PROPOSITION. *Let (R, R^+) be a perfectoid Huber pair in characteristic 0. Then there are canonical bijections, functorial in (R, R^+), between:*

(1) The set of isomorphism classes of elliptic curves over R^+ with a point of order N;

(2) The set of isomorphism classes of triples $(\mathcal{G}, (\bar{E}, P), \psi)$ where $\mathcal{G}$ is a p-divisible group over R^+, $(\bar{E}, P)$ is an elliptic curve with level structure over R^+/ϖ and $\psi : \bar{E}[p^\infty] \simeq \mathcal{G} \times_{R^+} (R^+/\varpi)$ is an isomorphism.

(3) The set of isomorphism classes of triples $(M, (\bar{E}, P), \rho)$ where M is a minuscule Breuil-Kisin-Fargues module over R^+ of rank 2, $(\bar{E}, P)$ is an elliptic curve with level structure over $R^{+,\flat}/\varpi^\flat$ up to p-isogeny (respecting the level structure) and $\rho : \mathbb{D}(\bar{E}[p^\infty])[\frac{1}{p}] \simeq M \otimes_{A_{inf}(R^+)} B^+_{cris}(R^+/\varpi)$ is an isomorphism of F-isocrystals.

Furthermore the datum of $(\bar{E}, P)$ over $R^{+,\flat}/\varpi^\flat$ in (3) does not depend on the choice of the pseudouniformizer $\varpi^\flat$.

PROOF. With the same argument as in the proof of the previous lemma one shows that giving (E, P) over R^+ is the same as giving a compatible sequence of couples (E_n, P_n) over R^+/ϖ^n. We claim that this is the same as giving a triple $(\mathcal{G}, (\bar{E}, P), \psi)$ where $(\bar{E}, P)$ is an elliptic curve over R^+/ϖ with level structure, $\mathcal{G}$ is a p-divisible group over R^+ and $\psi : \mathcal{G} \times_{R^+} R^+/\varpi \to \bar{E}[p^\infty]$ is an isomorphism. Indeed, given (E_1, P_1), to obtain E_n it suffices to dispose of a deformation $\mathcal{G}_n$ of the p-divisible group of E_1, by Serre-Tate. The level structure P_1 automatically lifts because N is coprime to p hence $E_1[N]$ is an étale group scheme. Finally, by [**Mes72**, Lemma 4.16], p-divisible groups over R^+ are the same as compatible collections of p-divisible groups over R^+/ϖ^n, hence the data in (1) and (2) are equivalent.

To show that (2) and (3) are equivalent we first observe (following [**CS17**, Lemma 4.3.4]) that given $\bar{E}$ over R^+/ϖ, a p-divisible group $\mathcal{H}$ over R^+/ϖ and an quasi-isogeny of p-divisible groups $\bar{E}[p^\infty] \xrightarrow{\rho} \mathcal{H}$, there is a unique elliptic curve $\bar{E}'$ (with level structure) in the p-isogeny class of E such that ρ induces an isomorphism $\bar{E}'[p^\infty] \to \mathcal{H}$. Therefore giving $(\mathcal{G}, (\bar{E}, P), \psi)$ as in (2) is the same as giving $(\mathcal{G}, (\bar{E}, P), \rho)$ where $(\bar{E}, P)$ is regarded up to p-isogeny and ρ is a quasi-isogeny. The equivalence between (2) and (3) follows from this and the classification theorems for p-divisible groups I.8.25, I.8.26.

It remains to show that the datum of $(\bar{E}, P)$ over $R^+/\varpi \simeq R^{+,\flat}/\varpi^\flat$ is independent of the choice of pseudouniformizer. This can be checked directly as follows: Frobenius gives an isomorphism $\Phi : R^{+,\flat}/\varpi^\flat \to R^{+,\flat}/(\varpi^\flat)^p$. Pullback along Φ gives an elliptic curve $\tilde{E}$ over $R^{+,\flat}/(\varpi^\flat)^p$ whose reduction modulo $\varpi^\flat$ is the Frobenius-pullback of $\bar{E}$. Since the composition of Verschiebung and Frobenius is multiplication by p, the reduction of $\tilde{E}$ modulo $\varpi^\flat$ is in the same p-isogeny class as $\bar{E}$, hence the p-isogeny class of $\bar{E}$ lifts modulo $(\varpi^\flat)^p$. As $R^{+,\flat}$ has characteristic p, we can repeat the process along the chain of isomorphisms $R^{+,\flat}/(\varpi^\flat)^{p^k} \simeq R^{+,\flat}/(\varpi^\flat)^{p^{k+1}}$, $k \geq 1$, obtaining a lift of $\bar{E}$ up to p-isogeny modulo arbitrary high powers of $\varpi^\flat$ (such a lift is necessarily unique because of rigidity of abelian schemes up to p-isogeny). If $\varpi' \in R^{+,\flat}$ is another pseudouniformizer then $\varpi' | (\varpi^\flat)^n$ for some n, hence we conclude. □

If $(R, R^+) = (C, \mathcal{O}_C)$ with $C/\mathbf{Q}_p$ algebraically closed non archimedean field we can invoke theorem I.8.28 and relate elliptic curves with local Shtukas. We will call a Shtuka $\mathcal{E}$ satisfying condition (1) in the second part of theorem I.8.28 a *minuscule Shtuka*. Recall that $\mathcal{E}^\infty$ denotes the vector bundle on the Fargues-Fontaine curve induced by the restriction of $\mathcal{E}$ to a suitable neighbourhood of $V([\varpi])$. Moreover, by theorem I.8.25 one can attach to a p-divisible group $\mathcal{G}$ over $\mathcal{O}_C/\varpi$ a vector bundle on $X^{FF}_{(C,\mathcal{O}_C),\mathbf{Q}_p}$ which we will denote by $\mathcal{E}(\mathcal{G})$

4.9. Corollary. *Let $C/\mathbf{Q}_p$ be an algebraically closed non archimedean field with ring of integers $\mathcal{O}_C$. Then there is a canonical bijection between the set of isomorphism classes of elliptic curves with a point of order N over $\mathcal{O}_C$ and the set of isomorphism classes of triples $(\mathcal{E}, (\bar{E}, P), \rho)$ where $\mathcal{E}$ is a minuscule Shtuka with a leg at C, $(\bar{E}, P)$ is an elliptic curve with level structure over $\mathcal{O}^\flat_C/\varpi^\flat$ up to p-isogeny and $\rho : \mathcal{E}(\bar{E}[p^\infty]) \xrightarrow{\sim} \mathcal{E}^\infty$ is an isomorphism.*

4.10. Motivated by the above results, we define the following objects. Let us denote by Perf^{aff} the category of affinoid perfectoid spaces in characteristic p. Let Bun_2 be the stack of rank 2 vector bundles on the Fargues-Fontaine curve.

Define

$$Ig(N) : \mathsf{Perf}^{aff} \to \text{ Groupoids}$$

as follows: for $T = Spa(S, S^+) \in \mathsf{Perf}^{aff}$ let $Ig(N)(T)$ be the groupoid of elliptic curves $\bar{E}$ with a point of order N on S^+/ϖ up to p-isogeny (as explained in Proposition 4.8, the choice of ϖ is immaterial). Taking the vector bundle attached to $\bar{E}[p^\infty]$ gives a map

$$\mathrm{p} : Ig(N) \to Bun_2.$$

Let

$$Sht_2 : \mathsf{Perf}^{aff} \to \text{ Groupoids}$$

be the stack of minuscule Shtukas of rank 2 with a leg in characteristic zero. Its $(S, S^+)-$points are triples $((R, R^+), \mathcal{E}, \phi)$ where (R, R^+) is a characteristic 0 untilt of (S, S^+), $\mathcal{E}$ is a vector bundle of rank 2 on $Spa(S, S^+) \times Spd(\mathbf{Z}_p)$ and ϕ is an isomorphism between $\mathcal{E}$ and its Frobenius twist outside (the graph of) (R, R^+) (such that the minuscule condition is satisfied). Sending $\mathcal{E}$ to $\mathcal{E}^\infty$ yields a map

$$\mathrm{q} : Sht_2 \to Bun_2.$$

Let $\tilde{\mathcal{Y}}_1(N)$ be the fiber product

$$\begin{tikzcd} \tilde{\mathcal{Y}}_1(N) \arrow[r] \arrow[d] & Sht_2 \arrow[d, "\mathrm{q}"] \\ Ig(N) \arrow[r, "\mathrm{p}"] & Bun_2; \end{tikzcd}$$

then our previous discussion implies that $\tilde{\mathcal{Y}}_1(N)$ enjoys the following properties:

(1) There is a canonical map (of sheaves) from $\mathcal{Y}_1(N)^\diamond$ to the sheafification (with respect to the étale topology) of the presheaf attached to $\tilde{\mathcal{Y}}_1(N)$, which is a bijection on geometric points.
(2) Let $\mathcal{G}$ be the p-divisible group of an elliptic curve over $\bar{\mathbf{F}}_p$, inducing a map $x : Spd(\bar{\mathbf{F}}_p) \to Bun_2$. Then the fiber of p at x parametrizes elliptic curves with level structure up to p-isogeny together with a quasi-isogeny between their p-divisible group and $\mathcal{G}$. This is an Igusa variety as defined in [**CS17**]. Hence $Ig(N)$ can be thought of as a family of Igusa varieties.

(3) The fiber of q at x parametrizes minuscule Shtukas $\mathcal{E}$ together with an isomorphism $\mathcal{E}^\infty \xrightarrow{\sim} \mathcal{E}(\mathcal{G})$. This is the moduli space of local Shtukas defined by Scholze [**SW17**, Lecture 23] (in particular it is a diamond).

4.11. In fact, we would like to construct an object related to $\tilde{\mathfrak{Y}}_1(N)$ which is a diamond isomorphic to $\mathcal{Y}_1(N)^\diamond$. A problem is that $Ig(N)$ as we defined it parametrizes elliptic curves, which do not satisfy any reasonable descent in the adic world. It would be much better to find a replacement of $Ig(N)$ by a moduli space of objects which naturally satisfy v-descent, much in the same spirit as Rapoport-Zink spaces are shown to be isomorphic to suitable moduli spaces of Shtukas [**SW17**]. To achieve this the first step would be to answer the following

4.12. Question: let C be an algebraically closed perfectoid field of *positive* characteristic. Can one describe the category of elliptic curves over $\mathcal{O}_C/\varpi$ up to p-isogeny in terms of objects satisfying v-descent?

4.13. Proof of theorem 4.3. We will now explain how to construct the object $\tilde{\mathcal{X}}_U^{plec}$ in theorem 4.3. Recall that we are assuming p split in F and F real quadratic, hence we have $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$. The main point is to give a description of the functor of points of $\tilde{\mathcal{X}}_U^\diamond$ analogous to the one we found in Proposition 4.8. First of all, arguing as in the proof of Lemma 4.7, invoking at the end Bhatt's Theorem 4.15 reported below instead of the fact that the relevant moduli space is affine, we see that, denoting by $\tilde{\mathcal{X}}_U$ the generic fiber of the completion of $\tilde{\mathsf{H}}_U$ along the special fiber at p, its functor of points on affinoid perfectoid Huber pairs is the sheafification of the functor sending (R, R^+) to the set of isomorphism classes of data $(A, \iota, \lambda, \eta)$ where A is an abelian surface over R^+ with $\mathcal{O}_F$-action given by ι, λ is a $\mathfrak{c}$-polarization for some fractional ideal $\mathfrak{c}$ and η is full level N-structure. The analogue of proposition 4.8 in this context is then given by the following

4.14. Proposition. *Let (R, R^+) be a perfectoid Huber pair in characteristic 0. Then there are canonical bijections, functorial in (R, R^+), between:*

(1) The set of isomorphism classes of data $(A, \iota, \lambda, \eta)$ of abelian surfaces over R^+ with $\mathcal{O}_F$-action, a $\mathfrak{c}$-polarization and U-level structure.
(2) The set of isomorphism classes of data $(\mathcal{G}, (\bar{A}, \iota, \lambda, \eta), \psi)$ where $\mathcal{G}$ is a p-divisible group over R^+ with $\mathcal{O}_F$-action, $(\bar{A}, \iota, \lambda, \eta)$ is a PRMAS with level structure over R^+/ϖ and $\psi : \bar{A}[p^\infty] \simeq \mathcal{G} \times_{R^+} (R^+/\varpi)$ is an $\mathcal{O}_F$-equivariant isomorphism.
(3) The set of isomorphism classes of data $(M_1, M_2, (\bar{A}, \iota, \lambda, \eta), \rho)$ where M_1, M_2 are two minuscule Breuil-Kisin-Fargues modules over R^+ of rank 2, $(\bar{A}, \iota, \lambda, \eta)$ is a PRMAS with level structure over $R^{+,\flat}/\varpi^\flat$ and $\rho = \rho_1 \oplus \rho_2$ is an isomorphism

$$\mathbb{D}(\bar{A}[\mathfrak{p}_1^\infty])[\frac{1}{p}] \oplus \mathbb{D}(\bar{A}[\mathfrak{p}_2^\infty])[\frac{1}{p}] \simeq M_1 \otimes_{A_{inf}(R^+)} B^+_{cris}(R^+/\varpi) \oplus M_2 \otimes_{A_{inf}(R^+)} B^+_{cris}(R^+/\varpi);$$

abelian surfaces over $R^{+,\flat}/\varpi^\flat$ are taken up to p-isogeny (hence are independent of the choice of $\varpi^\flat$).

Proof. The proof is a variation of the argument used to prove Proposition 4.8, with the novelty that in presence of $\mathcal{O}_F$-action linear algebra objects split. Using Bhatt's theorem 4.15 one sees that giving $(A, \iota, \lambda, \eta)$ over R^+ is the same as giving a compatible sequence of objects over R^+/ϖ^n. This in turn is the same as having $(\bar{A}, \iota, \lambda, \eta)$ over R^+/ϖ together with a p-divisible group $\mathcal{G}$ over R^+ with $\mathcal{O}_F$-action deforming $(\bar{A}, \iota)$. Indeed by Serre-Tate this datum will give a lift of $(\bar{A}, \iota)$ to R^+/ϖ^n for arbitrary n. The level structure will also lift because it is entirely outside p by assumption. The Rapoport (=Kottwitz) determinant condition lifts in view of [**Roz**, Proposition 2.12] (see also [**Rap78**, Remarque 1.2 (ii)]). Finally, by [**vdG88**, Chapter 10, Corollary 1.8] the functor sending an abelian variety with real multiplication A to $(Hom^{sym}_{\mathcal{O}_F}(A, A^\vee), Hom^{sym}_{\mathcal{O}_F}(A, A^\vee)^+)$ is formally étale, hence λ lifts uniquely, too. This proves the equivalence of (1) and (2).

To pass from (2) to (3) one first passes from isomorphisms to p-isogenies as in the proof of proposition 4.8. Then, by Theorem I.8.26 the datum of $\mathcal{G}$ is the same as the datum of a minuscule Breuil-Kisin-Fargues module M over R^+ with $\mathcal{O}_F$-action. Hence M is a module over

$$\mathcal{O}_F \otimes_{\mathbf{Z}} A_{inf}(R^+) = (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} A_{inf}(R^+) = A_{inf}(R^+) \times A_{inf}(R^+).$$

One checks that this yields a decomposition of M as a direct sum of two Breuil-Kisin-Fargues modules M_1, M_2, and ψ splits accordingly. Conversely, given M_1, M_2, the sum $M_1 \oplus M_2$ has a natural $\mathcal{O}_F$ action induced by the embedding $\mathcal{O}_F \hookrightarrow \mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq \mathbf{Z}_p \times \mathbf{Z}_p$ and by the natural $\mathbf{Z}_p$-action on each M_i. The fact that the isomorphism ψ in (2) is $\mathcal{O}_F$-equivariant reflects into the splitting of the isomorphism ρ in (3). $\square$

4.15. Theorem. *(Bhatt) [**Bha16**, Theorem 4.1, Remark 4.6] Let A be a ring and $I \subset A$ an ideal such that A is I-adically complete. Then for any scheme X we have $X(A) = lim_n X(A/I^n)$.*

4.16. Passing from Breuil-Kisin-Fargues modules to Shtukas, we see that an abelian surface over R^+ *with $\mathcal{O}_F$-action* gives rise to a *couple* of minuscule Shtukas over $Spa(R^\flat, R^{+,\flat})$ with *same* leg at $Spa(R, R^+)$. It is now clear that in order to "spread" $\tilde{\mathcal{X}}_U^\diamond$ to $\mathbf{Q}_p^\diamond \times \mathbf{Q}_p^\diamond$ it suffices to replace this datum with a couple of Shtukas with *arbitrary* legs. We define $\tilde{\mathcal{X}}_U^{plec}$ as the étale sheafification of the functor on affinoid perfectoid Huber pairs in positive characteristic given by

$$\begin{aligned}\bar{\mathcal{X}}_U^{plec}(S, S^+) = \Big\{ & (R_1, R_1^+), (R_2, R_2^+) \text{ char. } 0 \text{ untilts of } (S, S^+), \\ & (A, \iota, \lambda, \eta)\ PRMAS \text{ over } S^+/\varpi \text{ up to p-isogeny with } U\text{-level structure}, \\ & \mathcal{E}_1, \mathcal{E}_2 \text{ minuscule Shtukas over } (S, S^+) \text{ with legs at } (R_1, R_1^+), (R_2, R_2^+), \\ & \rho_i : \mathcal{E}_i^\infty \xrightarrow{\sim} \mathcal{E}(A[\mathfrak{p}_i^\infty]),\ i = 1, 2 \Big\}.\end{aligned}$$

Then above discussion implies the existence of the sought-for diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}}_U^\diamond & \longrightarrow & \tilde{\mathcal{X}}_U^{plec} \\ \big\downarrow{\scriptstyle \mathrm{p}} & & \big\downarrow{\scriptstyle \mathrm{q}} \\ \mathbf{Q}_p^\diamond & \xrightarrow{\Delta} & (\mathbf{Q}_p^\diamond)^2. \end{array}$$

The fact that the diagram is cartesian on geometric points follows from Theorem I.8.28 and from the fact that the étale sheafification does not affect geometric points; surjectivity of q on geometric points is clear.

4.17. The ramified case. Let us finally sketch the Shtuka part of the story in the ramified case. Let$F/\mathbf{Q}$ be real quadratic and p *ramified* in F. Let L be the completion of F at the prime above p and $\mathcal{O}_L$ the ring of integers of L. In this case it is not true any more that $\mathcal{O}_F$-action forces Breuil-Kisin-Fargues modules (and Shtukas) to split. However we saw in lemma I.8.12 that for $T = Spa(S, S^+) \in \mathsf{Perf}^{aff}$ the pushforward of a vector bundle of rank r over $T \times Spd(\mathcal{O}_L)$ to $T \times Spd(\mathbf{Z}_p)$ is a vector bundle of rank $2r$. Conversely, using the notation in the proof of lemma I.8.12, if $\mathcal{E}$ is a vector bundle over $T \times Spd(\mathbf{Z}_p)$ of rank $2r$ with $\mathcal{O}_F$-action, its restriction to $Spa(R_{n,0}, R_{n,0}^+)$ is a projective module over $R_{n,0}$ of rank $2r$ with $\mathcal{O}_F$-action, i.e. a projective module of rank r over $\mathcal{O}_L \otimes_{\mathbf{Z}_p} R_{n,0}$. Hence we obtain a vector bundle of rank r over $T \times Spd(\mathcal{O}_L)$.

Furthermore if $\xi \in W(S^+)$ is the primitive element of degree one, corresponding to a characteristic zero untilt R^+ of S^+, then its image in $W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L$ is a primitive element of degree 2. Indeed, let π be a uniformizer of $\mathcal{O}_L$. Recall that an element $\sum_{n \geq 0} [a_n]\pi^n$, $a_n \in S^+$ is called primitive of degree k if a_0 is a topologically nilpotent unit, $a_1, \ldots, a_{k-1}$ are topologically nilpotent and $a_k \in S^{+\times}$. Now since p is ramified we have $p = \pi^2$, hence a primitive degree one element $[a_0] + [a_1]p \in W(S^+)$, with $a_1 \in S^{+,\times}$ becomes the degree two element $[a_0] + [a_1]\pi^2 \in W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L$.

By [**Far16**, Section 1.4.3] points of $[(L_p^\diamond)^2/S_2]$ can be described as

$$[(L^\diamond)^2/S_2](S, S^+) = \{\text{primitive degree 2 elements in } W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L\}/\mathcal{R}$$

where $\mathcal{R}$ is the equivalence relation induced from $(W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L)^\times$-action. We therefore obtain a fiber map

$$\iota : \mathbf{Q}_p^\diamond \to [(L_p^\diamond)^2/S_2].$$

4.18. Definition. Let $\eta \in W(S^+) \otimes_{\mathbf{Z}_p} \mathcal{O}_L$ be a degree 2 primitive element. A Shtuka of rank n over $Spa\mathcal{O}_{F_\mathfrak{p}} \times Spa(R, R^+)$ with a leg at η is a rank n vector bundle $\mathcal{E}$ over $Spd\mathcal{O}_L \times Spa(S, S^+)$ together with an isomorphism

$$\varphi_{\mathcal{E}} : \varphi^*(\mathcal{E})_{|Spa\mathcal{O}_L \times Spa(S,S^+) \setminus V(\eta)} \to \mathcal{E}_{|Spa\mathcal{O}_L \times Spa(S,S^+) \setminus V(\eta)}.$$

The above discussion shows that rank $2n$ Shtukas over $\mathbf{Z}_p$ with one leg and with $\mathcal{O}_F-$action are the same as rank n Shtukas over $\mathcal{O}_L$ with a leg at the image of the fiber map ι. This suggests that the relevant plectic object in this case will be fibred over $[(L_p^\diamond)^2/S_2]$.

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