Stochastic Methods in Risk Management

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Wirtschaftswissenschaften

(Dr. rer. pol.)

durch die Fakultät für Wirtschaftswissenschaften
der Universität Duisburg-Essen, Campus Essen

vorgelegt von

Jinsong Zheng
geboren in Zhejiang, China

Essen, 2018

DOI: 10.17185/duepublico/70203
URN: urn:nbn:de:hbz:464-20190624-093546-8

Tag der mündlichen Prüfung: 12. Juni 2019
Erstgutachter: Prof. Dr. Rüdiger Kiesel
Zweitgutachter: Prof. Dr. Antje Mahayni

Zusammenfassung

Stochastische Methoden sind in der Finanzbranche weit verbreitet und werden z.B. in der stochastischen Modellierung und Simulation, der risikoneutralen Bewertung, der Derivatebewertung und vielen weiteren Anwendungen eingesetzt. Unter den Rahmenbedingungen von Solvency II müssen Versicherungsunternehmen adäquat kapitalisiert sein um jene aus Solvency II erwachsenen Kapitalanforderungen, zum Schutze der Aktionäre und Versicherungsnehmer, zu erfüllen. Daher müssen zwei wesentliche Größen betrachtet werden; das vorhandene Risikokapital (bzw. die Basiseigenmittel) und das benötigte Risikokapital. Im Allgemeinen werden diese Größen anhand der Mittelwerte von stochastischen Simulationen berechnet und folglich wird ein Economic Scenario Generator (ESG) verwendet, um die potentielle Entwicklung von Risikofaktoren der Ökonomie und des Finanzmarktes im Zeitverlauf zu simulieren. Für die Berechnung des vorhandenen Risikokapitals (definiert als die Differenz zwischen dem Marktwert der Vermögenswerte abzüglich der Verbindlichkeiten) wird ein stochastische Cash-Flow Projektionsmodell verwendet, um eine marktkonsistente Bewertung der Vermögenswerte und der Verbindlichkeiten, unter Verwendung von risikoneutralen Szenarien, durchzuführen. Die Berechnung des benötigten Risikokapitals erfolgt anhand der Wahrscheinlichkeitsverteilung des vorhandenen Risikokapitals über einen einjährigen Zeithorizont mithilfe eines Risikomaßes. Beispielsweise wird die Solvency Capital Requirement (SCR) anhand des Value-at-Risk zum Konfidenzniveau 99,5% gemessen.

Zunächst haben wir einen Überblick über die bestehende Literatur gegeben. Hierbei haben wir festgestellt, dass die allermeisten Autoren sich bei ihrer Betrachtung hinsichtlich der Verwendung stochastischer Methoden im Rahmen von Solvency II auf eine der drei Modellkomponenten des interne Partialmodells, also dem Inputmodell, dem Bewertungsmodell und dem Risikomodell, konzentrieren. In dieser Arbeit wollten wir ein internes Partialmodell mit allen Komponenten aufbauen und Schritt für Schritt zeigen, wie wir mit stochastischen Methoden eine marktkonsistente Bewertung vornehmen und das benötigte Risikokapital berechnen können.

Für das Inputmodel haben anstatt eines akademisch bevorzugten einfachen ESG Modells, mit einem Ein-Faktor Zinsmodells und einem durch eine geometrische Brownsche Bewegung getriebenen Aktienmodell, ein komplexeres Modell entwickelt, welches in der Praxis besser geeignet ist. Für die Modellierung des Zinssatzes haben wir das erweiterte Drei-Faktoren-Modell von Cox-Ingersoll-Ross, das die drei Hauptkomponenten der Zinsstrukturkurve erfassen kann, verwendet. Wir haben den Preis von Nullkupon-Optionen mithilfe der Fourier-Transformation der charakteristischen Funktion einer Linearkombination von Zustandsvariablen sowie einer anschließender Bewertung von Swaptions anhand einer stochastischen Durationsapproximation hergeleitet. Für die Modellierung von Aktien haben wir ein stochastisches Volatilitätsmodell (Heston-Modell) zusammen

mit der oben beschriebenen stochastischen Zinsmodellierung verwendet. In ähnlicher Weise haben wir die geschlossene Form der diskontierten charakteristischen Funktion des logarithmierten Aktienpreises ermittelt, indem wir ein System von gewöhnlichen Differentialgleichungen lösen, welches von einer affinen partiellen Differentialgleichung stammt. Anschließend haben wir den Preis von europäischen Optionen ebenfalls anhand Fouriertechniken hergeleitet. Zusätzlich haben wir die Methode der Erzeugung ökonomischer Szenarien mithilfe einer Monte Carlo Simulation, unter Verwendung eines Euler Diskretisierungsschemas und Varianzreduktionstechniken formuliert.

Für das Bewertungsmodell haben wir ein stochastisches Cashflow-Projektionsmodell entwickelt, um die Entwicklung der Bilanz sowie des aus Kuponanleihen und Aktien bestehenden Vermögensportfolios und des aus überschussberechtigten Lebensversicherungsverträgen bestehenden Passivportfolios zu erfassen. Anschließend haben wir eine marktkonsistente Bewertung der Vermögenswerte und Verbindlichkeiten vorgenommen, die auf den vom stochastischen Modell projizierten Cashflows und den Input risikoneutraler ökonomischen Szenarien basiert. Darüber hinaus haben wir die Managementregeln modelliert. Beispielsweise haben wir eine konstante Asset-Allocation-Strategie entwickelt, um das Asset-Portfolio wieder ins Gleichgewicht zu bringen. Wir haben den nicht realisierten Gewinn und Verlust durch Modellierung des Buchwerts und des Marktwerts von Vermögenswerten berücksichtigt. Darüber hinaus haben wir den MUST-Fall für die Überschussbeteiligung zwischen Anteilseigner und Versicherungsnehmer modelliert.

Für das Risikokapitalmodell haben wir zunächst die verschachtelte stochastische Simulation („Nested Stochastic Simulation“) implementiert, um das benötigten Risikokapital zu bestimmen. Da für die verschachtelte Simulation eine hohe Rechenzeit erforderlich ist, haben wir auch die Proxy-Methoden „Least Squared Monte Carlo“ (LSMC), „Replicating Portfolio“ und „Curve Fitting“ untersucht. Insbesondere haben wir eine allgemeine Strategie entwickelt, um ein gutes replizierendes Portfolio zusammenzustellen. Hierbei haben wir zuerst den Aufbau eines Asset-Pools beschrieben. Danach haben wir die Konstruktion von Sensitivitätssätzen durch Rekalibrierungs- oder Neugewichtungstechniken veranschaulicht. Als nächstes haben wir ein Kalibrierungsverfahren vorgeschlagen, bei dem sowohl die Optimierungsmethode der kleinsten Quadrate als auch die Auswahl von Teilmengen anhand bestimmter Kriterien verwendet werden, um das optimale Replikationsportfolio auszuwählen und das benötigte Risikokapital zu berechnen.

Schließlich haben wir anhand einer empirischen Anwendung den gesamten Prozess von der Kalibrierung der ESG-Modelle anhand von realen Marktdaten, der Erzeugung und Validierung der ökonomischen Szenarien, der marktkonsistenten Bewertung sowie der Bestimmung des SCR durch die verschachtelte Simulation und des replizierenden Portfolios, illustriert.

Abstract

Stochastic methods, such as stochastic modeling and simulation, risk neutral valuation, derivative pricing, etc., are widely used in the finance industry. Under Solvency II framework, in order to protect the benefit of shareholder and policyholder, the insurance company should be adequately capitalized to fulfill the capital requirement for solvency. Therefore, two main quantities are taken into account, i.e. the available capital (or basic own funds) and the required capital. In general, these two quantities are calculated by means of stochastic simulation and hence an Economic Scenario Generator (ESG) is used to simulate the potential evolution of risk factors of the economies and financial markets over time. For the calculation of available capital (defined as the difference between the market value of assets and liabilities), the stochastic cash flow projection model is used to perform the market consistent valuation of assets and liabilities given the risk neutral scenarios. For the calculation of required capital, the probability distribution of available capital over a one-year time horizon and a risk measure based on such distribution is taken into account. For instance, the Solvency Capital Requirement (SCR) is measured by the Value-at-Risk at confidence level of 99.5%.

We began by reviewing the existing literature and found that most authors used stochastic methods in risk management under Solvency II framework on one of the three components of the partial internal model, i.e. the input model, the valuation model or the risk capital model. In this thesis, we aimed to build a partial internal model including all components and show how we can use stochastic methods to do market consistent valuation and calculate the required capital.

For the input model, instead of using academic preferred simple ESG models, e.g. one factor short rate interest rate model along with geometric Brownian motion equity model, we developed advanced models that are more suitable in practice. For the modeling of interest rate, we used the extended three-factor Cox-Ingersoll-Ross model, which is able to capture the three main principle components of yield curve. We derived the pricing of zero coupon options by Fourier transformation of the characteristic function of the linear combination of state variables and subsequently the pricing of swaption using stochastic duration approximation. For the modeling of equity, we used the stochastic volatility model (Heston model) along with above-mentioned stochastic interest rate. Similarly, we first showed the closed-form of discounted characteristic function of log equity price by solving a system of Ordinary Differential Equations (ODEs) resulting from an affine Partial Differential Equation (PDE). We then derived the price of European options by Fourier techniques as well. In addition, we formulated the method of generating economic scenarios by using Monte Carlo simulation with Euler discretization scheme and variance reduction technique of antithetic variates.

For the valuation model, we built a stochastic cash flow projection model to capture

the development of balance sheet as well as the asset portfolio consisting of coupon bonds and stocks and the liability portfolio consisting of German traditional participating life insurance contracts. We then derived market consistent valuation of assets and liabilities based on the cash flows projected by the stochastic model along with the input of risk neutral economic scenarios. Furthermore, we modeled the management rules. For instance, we developed a constant asset allocation strategy to rebalance the asset portfolio. We considered the unrealized gain and loss by modeling the book value and market value of assets. Additionally, we modeled the MUST-case for the investment surplus distribution between shareholders and policyholders.

For the risk capital model, we first implemented the nested stochastic simulation to determine the required risk capital. Since nested simulation requires high computational time, we also investigated the proxy methods of least squared Monte Carlo, replicating portfolio and curve fitting. In particular, we developed a general strategy to construct a good replicating portfolio. First, we described the construction of asset pool. Second, we illustrated the construction of sensitivity sets through recalibration or reweighting techniques. Third, we proposed a calibration procedure, by using the least square optimization and subset selection with certain criteria, to select the optimal replicating portfolio and calculate the required capital.

Finally, we performed an empirical application to illustrate the full process, including the calibration of ESG models to real market data, economic scenario generation and validation, market consistent valuation and determination of SCR by nested simulation and replicating portfolio.

Acknowledgements

I would like to take the opportunity here to express my thanks to all those people who helped me to complete my PhD thesis.

First I am deeply indebted to Prof. Dr. Gerhald Stahl, CRO and head of Group Risk Management of Talanx AG, for offering me the great opportunity to do the PhD project while working full-time at Talanx AG. It would be impossible for me to finish this thesis without his support, continuous encouragement and useful guidance from start to end.

I would like to express my sincere gratitude to my supervisor, Prof. Dr. Rüdiger Kiesel, who supported me with his encouragement, patience and kindness and made this work possible. His guidance and expert advice have been invaluable throughout all stages of writing this dissertation.

I would also like to thank Prof. Dr. Antje Mahayni for being my co-examiner.

This thesis has been written during my work at Talanx AG. I would like to thank Talanx AG for the financial support. Furthermore, I would like to extend my thanks to all my colleagues in the department of Group Risk Management at Talanx AG for all sorts of help. Special mention goes to Mr. Lehner and Mr. Bamal.

Finally, I also thank my family and friends for their support, understanding and constant encouragement throughout these years.

Contents

List of Figures

List of Tables

1. Introduction

1.1. Motivation

Risk management plays an increasingly important role for companies and financial institutions, as they need to deal with an ever increasing dimension of risks necessitated by the complicated developments of economy. In insurance industry, the regulators of European Union had been developing the new risk based supervisory regime Solvency II since last decades and came into force on 1 January 2016. It reflects new risk management practices and requires more elaborate risk management systems for insurers. Furthermore, the rating agencies have paid extensive attention to risks and risk management in the insurance sector, e.g. Standard & Poors has added a formal evaluation of insurer enterprise risk management (ERM) as one of the new category for the overall rating decision since 2005. The importance of adequate and holistic risk management system was also highlighted by the financial crisis in 2007-08 as one of the main contributing factor of which was the poor risk management practices at banks. Therefore, the insurance companies need to establish a holistic enterprise risk management system to better manage the risks and comply the risk-based regulatory and rating agency requirements according to new risk management standards such as the international standard of risk management, ISO 31000. There are various risks to which insurance sector is exposed, such as market risk, underwriting risk, credit risk etc. Our motivation is to see how we can quantify such risks by means of stochastic methods for the purpose of risk management.

For quantifying these risks, an internal model is usually used to model the risks and afterwards a risk measure is required to measure the risk. The internal model is based on application of stochastic methods, such as stochastic modeling and simulation, risk neutral valuation, derivative pricing etc. In the mean while, there is a variety of risk measures with different confidence levels that could be chosen to measure the risk. Under Solvency II framework, the Solvency Capital Requirement (SCR), which corresponds to the Value-at-Risk at confidence level of 99.5% of available capital over a one-year time horizon, is chosen to measure the total risk. It means that the quantities of available capital and SCR should be quantified.

In general these two quantities are calculated by means of stochastic simulation. An Economic Scenario Generator (ESG) is used to simulate the potential evolution of risk factors of the economies and financial markets over time. For the stochastic modeling and calibration of ESG, stochastic techniques such as the risk neutral valuation and derivative pricing need to be adapted. There are two types of ESG scenarios that are used, the market consistent risk neutral scenarios and the real world scenarios.

For the calculation of available capital (defined as the difference between the market value of assets and liabilities), the stochastic cash flow projection model is used to perform the market consistent valuation of assets and liabilities with usage of risk neutral scenarios.

For the calculation of required capital, the probability distribution of available capital over a one-year time horizon and a risk measure based on such distribution should be taken into account. In principle, the so called nested stochastic simulation should be applied to get the probability distribution, however, it results quite high computational time and is not quite practical to use this approach. Therefore, the proxy methods such as replicating portfolio are taken into consideration.

In order to demonstrate the usage of stochastic methods in risk management described above, we build a partial internal model to illustrate the calculation of available capital and SCR. We develop an ESG consisting of the interest rate model and equity model. We then generate the economic scenarios after calibrating the ESG models to the market data. Furthermore, we develop a stochastic cash flow projection model to capture the evolution of balance sheet and cash flows of assets and liabilities, where the asset portfolio consists of coupon bonds and stocks while the liability portfolio consists of traditional life insurance products with profit sharing and interest rate guarantee. Besides the nested stochastic simulation, we also develop the proxy method of replicating portfolio that is widely used in insurance industry to determine the SCR and compare it to the nested simulation to check the estimation quality.

1.2. Literature Review and Contribution

In the literature, there are many applications of stochastic methods in risk management in insurance. In the beginning of literature review, we need to figure out the following questions:

- What is risk?
- How to manage risk or what is the process of risk management for an insurance company?
  - What kinds of requirements should be considered?
  - What kinds of risk management standards could be followed?
- In which part of risk management process we need to apply the stochastic methods?

Risk can be defined in a variety of ways. The definitions are mainly based on probability or uncertainty. Probability based definitions of risk (see e.g. Kaplan and Garrick (1981), Aven (2010), Hansson (2012)) could be formalized by a triplet (A, C, P) (see Aven (2011)), where A is the events, C and P are the corresponding consequences and probabilities of A. Probability is a tool for expressing the uncertainty. However, it lacks the informative description of the uncertainties related to the event. Therefore, the definitions of risk with uncertainties beyond the probabilities should be considered.

Similarly they could be described with another triplet (A, C, U), where U is the uncertainty, examples of these definitions could be seen e.g. ISO (2009a,b), Holton (2004), Aven and Renn (2009).

The insurance company should have an enterprise wide risk management process according to the required risk management standards as well as compliance with the requirements from stakeholders such as regulatory and rating agencies. Therefore, we need to review the regulatory and rating agency requirements as well as the enterprise risk management process.

For the regulatory requirements, we need to consider the Solvency II framework (see Directive 2009/138/EC (2009), Directive 2014/51/EU (2014)) in European Union, which could be dividend into three pillars, i.e. quantitative requirements, qualitative requirements and disclosure requirements. Pillar I contains harmonised rules of valuation of assets and liabilities including technical provisions, own funds, solvency capital requirement, minimum capital requirement and investment in securitisation positions (see Delegated Regulation (2015, Chapter II-VIII) or Directive 2009/138/EC (2009, Chapter VI)). Pillar II contains the rules relating to supervisory review system, system of governance, risk management system and Own Risk and Solvency Assessment (ORSA) (see Directive 2009/138/EC (2009, Chapters III-IV), Delegated Regulation (2015, Chapters IX-XI)). Pillar III contains the rules relating to public disclosure, regular supervisory reporting, transparency (see Delegated Regulation (2015, Chapters XII-XIV)).

Furthermore, the rating agencies, such as Standard & Poors, Moody's and A. M. Best, assess the ratings of the enterprise risk management (see e.g. Standard & Poors (2005b), Standard & Poors (2013), Moody's Research Methodology (2004), Harris (2009)), as part of the overall rating of an insurance company. Consequently, the insurance company should also fulfill the requirements or criteria from rating agencies in order to get a strong rating, especially when the insurance company has been listed on the stock exchanges.

The well-established risk management processes in the literature are IRM standard (AIRMIC, ALARM, IRM (2002)), COSO ERM framework (Committee of Sponsoring Organizations of the Treadway Commission (2004)), ISO 31000 (ISO (2009b)), ERM framework proposed by ERM Committee of the American Academy of Actuaries (2013). AIRMIC, Alarm, IRM (2010) describe a structured approach to ERM by considering the requirements of ISO 31000. Following the international standard ISO 31000 (ISO (2009a), ISO (2009b)) and AIRMIC, Alarm, IRM (2010), we review the risk management process combined with the Solvency II framework and focus on the risk assessment consisting of risk identification, risk analysis and risk evaluation, which is highly related to Pillar I, e.g the market consistent valuation, available capital and required capital. The risk analysis tools could be established by using internal model (see Directive 2009/138/EC (2009)) for the quantification of risk. Aven (2011, Section 8.2) presents a model based framework for risk assessment. Therefore, the stochastic methods are mainly applied in the step of (quantitative) risk assessment.

Up to now, we have answered the above listed questions. In the next step, we need to consider how we could construct a partial internal model in the process of risk assessment. We then need to figure out further questions:

- How to mimic an insurance company using a stochastic model?
- How to do market consistent valuation?
- How to determine the SCR?

In order to mimic an insurance company, the Asset-Liability framework, including the balance sheet, asset and liability models as well as the related management rules should be built.

First of all, a simplified balance sheet should be constructed to reflect the most important items of the real balance sheet of the insurance company. For instance, Bauer et al. (2006), Kling et al. (2007) and Bauer et al. (2009) give simplified balance sheet consisting of asset and liability sides, where asset side is the market value of asset portfolio and liability side consists of two parts, the book value of policyholder's account and reserve account (a hybrid determined as the difference between a market value and book value). Similar balance sheet could be seen in e.g. Grosen and Jørgensen (2000), Reuß et al. (2013), Burkhart et al. (2014) for life insurance companies. A more general balance sheet is proposed by Gerstner et al. (2008), they separate the policyholder's account into actuarial reserve and allocated bonus. Furthermore, they separate the reserve account to company account called equity and a buffer account called free reserve for the future bonus payment to achieve more stable return of the policyholders.

Secondly, the asset model is used for modeling the development of asset portfolio for the asset side. In practice, the asset portfolio consists of the various financial assets, such as the treasury bonds, corporate bonds, stocks, real estate etc. Since the asset allocation depends on the evolution of financial market, the management rules for determining the proportion of financial asset classes are usually considered. Therefore, asset allocation strategy should be defined to reflect the management rules in the asset model. The constant strategic asset allocation, i.e. keeping constant proportion of market value of bonds and stocks, is widely adopted, e.g. Kling et al. (2007), Bauer et al. (2009), Gerstner et al. (2008), Reuß et al. (2013), Burkhart et al. (2014). Moreover, the book value might not be equal to the market value due to local GAAP accounting rules, which leads to unrealized gain and losses (UGL). In practice, the company may realize some of the gains to get higher returns and release the losses in the equity investments. Therefore, the corresponding management rules should be incorporated as well, see e.g. Reuß et al. (2013).

Thirdly, the liability model is used for modeling the development of insurer's liabilities for the liability side. In practice, the liability portfolio consists of different insurance products, such as endowment policies, life annuities, unit-link products etc. For the sake of simplicity, Kling et al. (2007), Bauer et al. (2006) and Bauer et al. (2009) use the participating single-premium term-fix insurance (ignoring any charges and mortality rates), which is an image of the life insurance company's general financial situation, and hence the evolution of corresponding liability portfolio could be served as the development of the insurer's liabilities. Gerstner et al. (2008), Seemann (2009) use liability portfolios including participating endowment assurance with and without surrender options by considering mortality rates. Reuß et al. (2013) and Burkhart et al. (2014) use traditional participating life insurance contracts (endowment assurance) by considering the charges and mortality rates in their liability portfolio.

Finally, the management rules for the surplus participation should be considered, since the profit or bonus should be shared between the policyholders and shareholders for the traditional participating life insurances. There are several sources of surplus, namely the investment surplus, risk surplus, cost surplus and other surplus as described in German Minimum Surplus Participation (Mindestzuführungsverordnung - MindZV). Most literatures such as Bauer et al. (2006), Grosen and Jørgensen (2000), Gerstner et al. (2008), Kling et al. (2007) focus on investment surplus. Burkhart et al. (2014) considers the cost surplus as well by introducing the cost model. For the investment surplus distribution mechanism, a point to point guarantee framework is used by Briys and de Varenne (1997), i.e. a fixed guaranteed interest as well as bonus determined a certain fraction of financial gains are that received by the policyholders. The cliquet-style guarantee is considered in Grosen and Jørgensen (2000) and Gerstner et al. (2008) using the average interest principle, Bauer et al. (2006) and Kling et al. (2007) for MUST-case and IS-case etc. The MUST-case considers only obligatory payments to the policyholders as required in the German market. The IS-case reflects closely the behavior of typical life German insurance companies over the last few years.

Given the asset-liability framework, we could then proceed to do the market consistent valuation of assets and liabilities through a stochastic cash flow projection model, i.e. the stochastic modeling and simulation of the development of balance sheet and future cash flows generated from the Asset-Liability framework. There are a number of papers that relate to the development of such models in the recent years, such as Briys and de Varenne (1997), Grosen and Jørgensen (2000), Bacinello (2001), Grosen and Jørgensen (2002), Bacinello (2003), Tanskanen and Lukkarinen (2003), Bauer et al. (2006), Kling et al. (2007), Gerstner et al. (2008), Graf et al. (2011), Bauer et al. (2009), Reuß et al. (2013), Burkhart et al. (2014).

The inputs of the stochastic cash flow projection model are the economic scenarios generated by an ESG. The economic scenarios include the financial market risk factors such as the risk free yield curve, option implied volatilities of interest rates, equity returns and dividends, credit spreads, transition probabilities among credit ratings, property returns, inflation rates etc. Therefore, the interest rate model, equity model, credit model, property model and inflation model are usually required. For instance, these models are all included in the market leading ESGs providers such as Barrie & Hibbert ESG and Conning ESG. In academic, most of the literature focuses on the interest rate and equity models. The most widely used combinations are the classic short rate models (Vasicek, Cox-Ingersoll-Ross, Hull-White) for the stochastic interest rate along with geometric Brownian motion for the stock or asset portfolio, see e.g. Bauer et al. (2010), Burkhart et al. (2014), Reuß et al. (2013), Jørgensen (2001), Briys and de Varenne (1997), Gerstner et al. (2008), De Felice and Moriconi (2005), Jørgensen (2001), Rühlicke (2013), DAV (2015) and de Boer (2009) etc. Rühlicke (2013) further considers the Cox-Ingersoll-Ross model for interest rate and Heston model for the equity. In practice, the ESG providers use either the short rate models or market model for the interest rate, e.g. extended three factor Cox-Ingersoll-Ross model (CIR3++) in Conning's ESG (Conning

(2012)), extended two factor Black-Karasinski model and Libor Market Model in Barrie & Hibbert's ESG (see Morrison (2007), Barrie & Hibbert (2010)). Both Conning's ESG and the Barrie & Hibbert's ESG use the stochastic volatility jump diffusion (SVJD) model for equity modeling (see Conning (2012), Lawson (2011)). The German Actuarial Society DAV (2015) constructs a benchmark ESG with Hull-white model for interest rate and geometric Brownian motion model for the stock and property.

After the market consistent valuation, we need to further determine the capital requirements, which could be measured by a risk measure. Therefore, we first review the axiomatic approach to risk measures used for the determination of capital requirements. The literature review of risk measures could be seen e.g. Szegö (2002) from a probabilistic perspective, Albrecht (2004) from an actuarial perspective and Föllmer and Weber (2015) from an perspective of capital requirement. The textbook of Föllmer and Schied (2011) offers mathematical insights into risk measures. Heyde et al. (2007) make a very important contribution to the concept of risk measures by introducing natural risk statistics. This concept bridges the gap between risk measures and statistics by defining natural risk statistics on a sample space in contrast to a probability space.

In general, the so called nested stochastic simulation should be applied (see Gordy and Juneja (2010), Broadie et al. (2011), Bauer et al. (2010)), i.e. it requires Monte-Carlo simulation based market consistent valuation for each real world path in one year horizon. Since the SCR corresponds to the 99.5%-quantile of random loss, the number of simulation should be large to reduce the estimation error of the quantile. It then results quite high computational time and is not quite practical to use this approach by obtaining the results in required time lines. Consequently, a number of proxy methods have been developed to make the calculation more practical. For instance, the methods of replicating portfolio, curve fitting and least square Monte-Carlo simulation etc are applied in the insurance industry.

All the proxy approaches are based on finding a linear combination of basis functions to approximate the present value of future profits. The valuation function could be approximated by a function of risk factors, and hence a curve could be used to fit the valuation function. One example of curve fitting is the Swiss Solvency Test (SST) standard formula that applies Delta-(Gamma) curve fitting (see FINMA (2012)). The least square Monte Carlo approach was first introduced by Longstaff and Schwartz (2001), who use least squares regression on a countable set of basis functions to approximate the conditional expectation. Bauer et al. (2009) apply the idea and propose a faster approach for the calculation of required risk capital under Solvency II. The replicating portfolio consists of a set of financial assets that could be used as a computationally efficient proxy to evaluate the PVFP under real world in one year horizon. An introduction to this approach is given by Oechslin et al. (2007) and Boekel et al. (2009). Seemann (2009), Erixon and Tubis (2008) and Kalberer (2007) use replicating portfolio for valuation and hedging of life insurance products. Burmeister and Mausser (2009) and Burmeister et al. (2010) apply the trading restrictions as further constrains to get a relative smaller effective replicating portfolio.

After answering the further questions, we summarized that a partial internal model could be constructed with three main components, namely the input model, the valuation model and the risk model. The structure of such partial internal model is illustrated in Figure 1.1, which gives the interrelationship among the main components as well as the main sub-components.

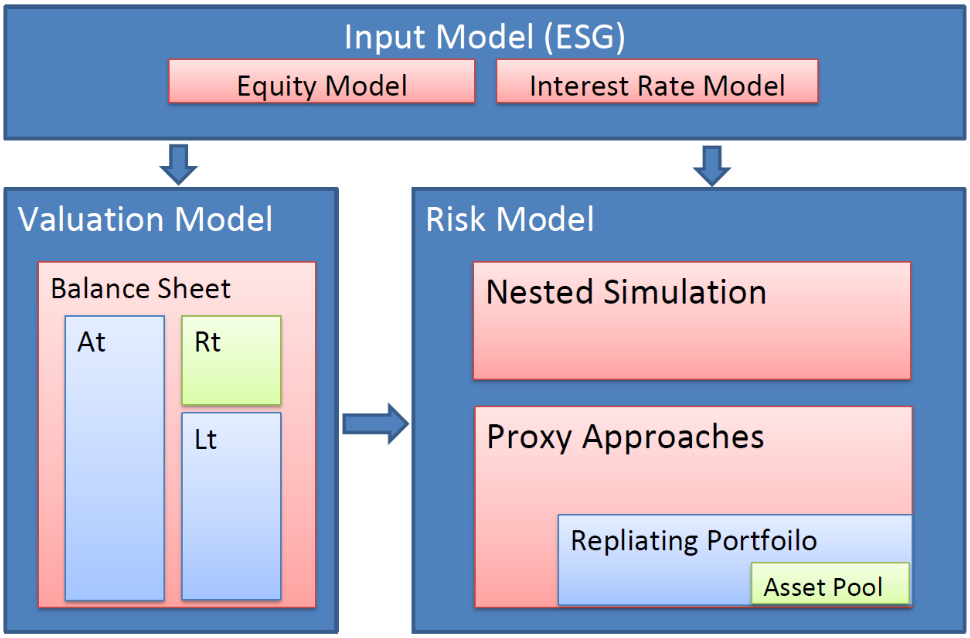

Figure 1.1.: Structure of a partial internal model.

After reviewing the literature, we could figure out that most authors use stochastic methods on specific component of the partial internal model. In this thesis, we aim to build a partial internal model including all components and show step by step how we can use stochastic methods to do market consistent valuation and calculate the required capital by the valuation model and risk model using the economic scenarios generated by the input model.

First of all, instead of using academic preferred simple financial ESG models, e.g. one factor short rate model along with geometric Brownian motion equity model, we introduce the more advanced models that are used more widely in practice. We implement the extended multi-factor Cox-Ingersoll-Ross model for modeling the interest rates as Conning (2012). Additionally, we also implement the stochastic volatility Heston model joint with the above stochastic interest rate. Afterwards, we calibrate the models to real market data for both the risk neutral and real world measures, which is more meaningful especially under the current low interest rate environment.

Secondly, we integrate the valuation model introduced in Burkhart et al. (2014) and Reuß et al. (2013) into the framework of Bauer et al. (2009) for doing not only the market consistent valuation but also the determination of required capital.

Thirdly, we develop a general strategy and describe in details how to select an optimal replicating portfolio to calculate the required capital, including the construction of financial asset pool, subset selection techniques etc.

Finally, we illustrate the full process from the ESG model calibration to market data, scenario generation, market consistent valuation and determination of SCR by nested simulation and proxy methods.

1.3. Structure

Chapter 2 starts with the definition of risk. Afterwards we review briefly the road from Solvency I to Solvency II, and give a short description to the three-pillars of Solvency II, namely the quantitative requirements, qualitative requirements and disclosure requirements. Besides the regulatory requirements, we further address the rating agency requirements for the overall rating decision. We proceed to the construction of the Enterprise Risk Management system while considering these requirements. Consequently, the concepts of risk management, risk management process based on international standard ISO 31000 are discussed.

Chapter 3 reviews the axiomatic approach to risk measures used for the determination of capital requirements.

Chapter 4 presents the Economic Scenario Generator. We first describe the interest rate model, including the bond and swaption pricing, model calibration and estimation under risk neutral and real world. We then further provide the mathematical description of equity model, including the European option pricing and model estimation.

Chapter 5 focuses on the market consistent valuation of assets and liabilities. We first set up the mathematical framework for the risk neutral valuation. Under the framework, we present a stochastic cash flow projection model, i.e. the stochastic modeling and simulation of the development of balance sheet and future cash flows generated from the Asset-Liability framework. Consequently, the balance sheet for modeling the most important balance sheet items, asset model and liability model for modeling the asset portfolio and liability portfolio, are then given. In addition, the management rules of e.g. the asset allocation strategies, unrealized gains and losses, surplus distribution, are described. In the end of Chapter 5 we then derive the computation of Market Consistent Embedded Value.

Chapter 6 focuses on the risk modeling for the SCR calculation. We start the method of nested stochastic simulation, which requires Monte-Carlo simulation based market consistent valuation for each real world path in one year horizon. Due to the high computational time of nested simulation in practice, we further present a number of proxy methods, such as replicating portfolio, curve fitting and least square Monte-Carlo simulation etc. In particular, Chapter 7 gives more details about replicating portfolio by presenting a general strategy and describing step by step for constructing a 'good' replicating portfolio.

Chapter 8 illustrates the application. We first describe the required market data, to which we calibrate the ESG models. After calibration, we generate the scenarios and proper validations are performed to check the quality of scenarios. Then we do the market consistent valuation based on the stochastic cash flow projection model. In addition to the nested simulation for the calculation of SCR, we conduct the proxy method of replicating portfolio and check the approximation error of SCR by comparing to the result of nested simulation.

Chapter 9 gives the conclusions and suggestions for further work.

2. Insurance Enterprise Risk Management

2.1. Risk

Risk can be defined in a variety of ways. Hansson (2012) describes quantitative senses of risk defined as the probability or statistical expectation value of an unwanted event which may or may not occur. Aven (2011) gives a more broad review of the common definitions of risk. The definitions of risk, besides based on expected values or probabilities, could be based on uncertainty. In general, the risk defined as expected value is misleading as it losses the information about the events with low probabilities and high consequences. Take the nature catastrophe as an example, it happens in low probability but could lead to extreme consequences. Special attention should be paid for it even though the expected value might be small. That means both severity and frequency of the nature catastrophe should be taken into account.

In line with this idea, the definitions of risk based on probability are suggested. Aven (2011) uses a triplet (A, C, P) to define the risk, where A is the events, C and P are the corresponding consequences and probabilities of A. The P could be referred to frequentist probability or subjective (or knowledge-based) probability. The frequentist probability is interpreted as relative frequency, which is the relative fraction of times of occurrence of event on an infinite number of repetitions of the statistical experiment. One example for the risk definition based on frequentist probability is Kaplan and Garrick (1981). On the other hand, the subjective probability is referred to Bayesian probability by specifying the prior probability based on a state of background knowledge. See for example Aven (2010).

Probability is a tool for quantifying the uncertainty. However, this value of probability may loss the information for describing the uncertainty. Therefore, the uncertainties beyond the probabilities should be considered as mentioned by Aven (2011). The International Organization for Standardization (ISO) (ISO (2009a,b)) defines the risk as "risk is the effect of uncertainty on objective". ISO (2009b) gives the note that "risk is often expressed in terms of a combination of the consequences of an event (including changes in circumstances) and the associated likelihood of occurrence". Holton (2004) refers the risk is "exposure to a proposition of which one is uncertain". One more example about risk definition through uncertainty is given by Aven and Renn (2009) "Risk is uncertainty about and severity of the consequences (or outcomes) of an activity with respect to something that humans value". Similar to the definition based on probability (A, C, P), all these definitions based on uncertainty could be described with another triplet (A, C, U), where U is the uncertainty.

After understanding the concept of risk, we start to discuss how to manage the risks in insurance, such as underwriting risk, market risk, credit risk and operational risk etc. In order to manage the risk better, the enterprise wide risk management is usually taken into account. The enterprise risk management (ERM) should be risk based and comply the regulatory and rating agency requirements. In the following sections, we will first describe the regulatory and rating agency requirements and the enterprise risk management based on these requirements and risk management standards by ISO.

2.2. Risk-based Regulatory and Rating Agency Requirements

The regulatory requirements play an important role for the risk management for an insurance company. The insurance supervision in European Union started firstly in 1970s with first non-life insurance Directive (Directive 73/239/EEC) and first life assurance Directive (Directive 79/267/EEC). With amendments in the 1990s (Directives 88/357/EEC, 90/619/EEC, 92/49/EEC and 92/96/EEC) and mostly recently (life assurance Directive 2002/83/EC and non-life insurance Directive 2002/13/EC), the supervisory regime Solvency I was completed in 2002. The Solvency I mainly focused on the underwriting risk and had structure weakness. According to the examination of Müller Report (see Müller (1997)), it does not account for all risks.

In order to remedy the weakness of Solvency I, the Solvency II project was initiated. It has been divided into two phases (see European Commission (2002), European Commission (2003)).

In the first phase, the general design of a future solvency system was determined by conducting several studies on a number of areas. There were two important general studies in the first phase, namely the KPMG report (see KPMG (2002)) and Sharma Report (see Conference of the Insurance Supervisory Services of the Member States of the European Union (2002)). In the KPMG report, it concludes with recommendations for a three-pillar approach, which is similar to Basel II (The New Basel Capital Accord, see Basel Committee on Banking Supervision (2001)) that is adapted by the banking sector. The Sharma Report concludes that the capital requirement and solvency levels should be more risk-sensitive, a range of early-warning tools is needed to detect the potential threats to the solvency, internal factors such as quality management, corporate governance practices and risk management systems should be paid more attention.

In the second phase, details of the system such as specific rules and guidelines were further developed. The new rules have been created by following the Lamfalusssy process (see The Committee of Wise Men's (2001), Commission of the European Communities (2004)). It is a four-level legislative process, each of which focuses on a certain stage of the legislative process. At Level 1, the European Parliament and Council of the European Union adopt a piece of legislation, containing framework principles, with implementing powers being delegated to the second level (see The Committee of Wise Men's

(2001)). For Solvency II project, Solvency II Directive (Directive 2009/138/EC (2009)) was adopted. It was further amended by Omnibus II Directive (Directive 2014/51/EU (2014)). At Level 2, The European Commission enacted the delegated act with implementing details of framework Directives and Regulations. Between Level 2 and Level 3, there is Level 2.5, which binds the technical standards developed by the European Insurance and Occupational Pensions Authority (EIOPA) (before called as the Committee of European Insurance and Occupational Pension Supervisors (CEIOPS)). At Level 3, the EIOPA develops supervisory guidelines and recommendations. At Level 4, the European Commission monitors unit implementation of the EU regulations in close co-operation with Member States and EOIPA.

On the 1st of January 2016, the new supervisory regime Solvency II framework came into force. As proposed by KPMG Report, Solvency II (European Commission (2002, p. 28)) is dividend into three pillars as Basel II[1]. As the supplement of Directive 2009/138/EC, Delegated Regulation (Delegated Regulation (2015)) specifies more detailed implementing rules for the three pillars, based on which an overview of the three pillars of Solvency II could be given in Figure 2.1.

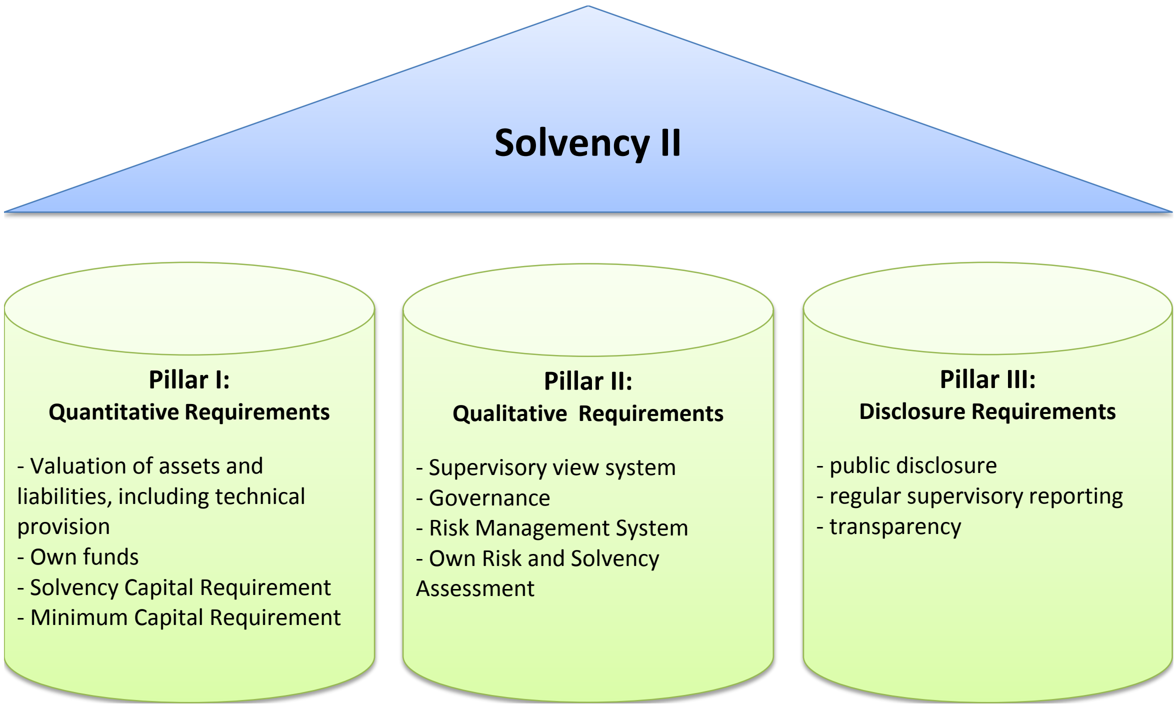

Figure 2.1.: The three-pillars approach of Solvency II.

Pillar I of Solvency II sets out the quantitative requirements. It contains harmonised

[1]Note that compared to Basel II, the Solvency II focuses more on a holistic risk management approach rather than on management of single risks independently (see Eling et al. (2007)). More detailed comparison of Basel II/III and Solvency II could be seen in Gatzert and Wesker (2012).

rules of valuation of assets and liabilities including technical provisions, own funds, solvency capital requirement, minimum capital requirement and investment in secularization positions (see Delegated Regulation (2015, Chapter II-VIII) or Directive 2009/138/EC (2009, Chapter VI)).

The starting point in Solvency II is the economic balance sheet, where the assets and liabilities should be valued according to market consistent principles. The market consistent valuation allows for the comparison and disclosure of the balance sheet in a harmonized manner. For instance, the Market Consistent Embedded Value (MCEV) should be calculated and disclosed by the life insurances. The technical provisions, which correspond to the expected amount that another undertaking would require to take over and fulfill the underlying obligations (see Directive 2009/138/EC (2009, p. 6)), should also be calculated consistently as the valuation of assets and other liabilities. In order to do the right valuation of liabilities, a risk free yield curve is used to price embedded options and guarantees. Therefore, the methodologies for the determination of relevant risk free interest rate term structure need to be specified. According to Delegated Regulation (2015, Chapter VIII, Section 4) (see also the technical specification by EIOPA (2014)), the basic risk free interest rate term structure should be constructed based on swap rates (from deep, liquid and transparent financial market) or rates of government bonds with adjustment of credit spreads, extrapolation to the ultimate forward rate, as well as considering the long term guarantee measure[2], e.g. volatility adjustment or matching adjustment.

In order to absorb the unexpected financial losses and to cover risks inherent to the insurance business, the own funds corresponding to the amount of sufficient financial resources should be hold by the insurance undertakings. The own funds consist of basic own funds (the excess of assets over liabilities valued as described above, plus the subordinated liabilities, see Directive 2009/138/EC (2009, Article 88)) and ancillary own funds (items other than basic own funds that can be called up to absorb losses, see Directive 2009/138/EC (2009, Article 89)). Own funds shall be further classified into three tiers according to some characteristics and features such as permanent availability, subordination, sufficient duration etc (Directive 2009/138/EC (2009, Article 93-97)).

For the capital requirements in Solvency II, the Minimum Capital Requirement (MCR) and Solvency Capital Requirement (SCR) shall be covered by the eligible amount of own funds. The Solvency Capital Requirement, which corresponds to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period (see Directive 2009/138/EC (2009, Article 101)), could be calculated by the standard formula or an internal model. The calculation of Minimum Capital Requirement (MCR) could be seen in Directive 2009/138/EC (2009, Article 129), it should be calibrated to 85% Value-at-Risk of basic own funds in one year horizon. More detailed implementing mathematical formula of MCR is given in Delegated Regulation (2015, Chapter VI, Article 248).

Pillar II of Solvency II sets out the qualitative requirements. It contains the rules

[2]The long-term guarantee measures for insurance products with long-term guarantees were first discussed in EIOPA (2013) and adopted in Omnibus II Directive to reduce so called 'artificial volatility'.

relating to supervisory review system, system of governance, risk management system and Own Risk and Solvency Assessment (ORSA) (see Directive 2009/138/EC (2009, Chapters III-IV), Delegated Regulation (2015, Chapters IX-XI)).

The supervisory review system (see Directive 2009/138/EC (2009, Article 36)) is essential for Solvency II since the complexity of insurance business and risk management techniques in the future will be such that no formulas or models could capture the situation fully (see Linder and Ronkainen (2004, p. 471)). It also specifies the aspects related to the calculation of capital add-ons that may be imposed for supervisory purposes. Furthermore, an effective system of governance (see Directive 2009/138/EC (2009, Chapter II, Section 2)) should be in place by undertakings to provide for sound and prudent management of the business. It mainly contains the risk management system, internal control, internal audit and actuarial function. As part of risk management system, the undertakings should conduct its Own Risk and Solvency Assessment (ORSA) after the risks are identified and quantified in Pillar I under risk management system.

Pillar III of Solvency II sets out disclosure requirements. It contains the rules relating to public disclosure, regular supervisory reporting, transparency (see Delegated Regulation (2015, Chapters XII-XIV)).

The extensive attention has been paid to risks and risk management in insurance by the rating agencies for their rating analysis. Since 2005 along with the existing categories of competitive position, management and corporate strategy, operating performance, capitalization, liquidity, investments, and financial flexibility, Standard & Poors (2005a) has added a formal evaluation of insurer enterprise risk management (ERM) as one new category for the rating process. The overall rating decision is then based on the combination of the quality in each of these categories.

For the purpose of evaluating risk management, Standard & Poors (see Standard & Poors (2005b), Standard & Poors (2013)) look at the companies' ERM in five main components, i.e. *the risk management culture, risk control, emerging risk management, risk models and strategic management*. Standard & Poors (2013) describes how each of these components is assessed by some criteria and then combined to derive the insurer's ERM score.

Similarly other rating agencies assess the ratings with the enterprise risk management, see e.g. Moody's Research Methodology (2004), Harris (2009) for the rating agency Moody's and A. M. Best (2013) for the rating agency A. M. Best.

Therefore, the insurance company should also fulfill the requirements or criteria related the five components in order to get strong rating, especially when the insurance company has been listed on the stock exchanges.

After the discussion of the regulatory and rating agency requirements, in the next section we start to construct an enterprise risk management system while fulfilling these requirements.

2.3. Enterprise Risk Management

2.3.1. Risk Management and Enterprise Risk Management

Given the definition of risk in Section 2.1, we proceed do discuss the risk management. The concept of risk management in current usage began in the early 1950s as mentioned in Vaughan and Vaughan (2008). One of the earliest references in literature was Gallagher (1956), in which the author proposed the idea that someone within the organization should be responsible for managing the organization's pure risks. There are various ways to define risk management.

ISO (2009a) defines risk management as coordinated activities to direct and control an organization with regard to risk. Rejda (2008) defines risk management is a process that identifies loss exposures faced by an organization and selects the most appropriate techniques for treating such exposures. Vaughan and Vaughan (2008) defines risk management is a scientific approach to dealing with risks by anticipating possible losses and designing and implementing procedures that minimize the occurrence of loss or the financial impact of the losses that do occur.

Traditional risk management is focused on the specific individual risks. More recently, the concept of enterprise risk management has been drawn great attention. Compared to traditional risk management, ERM is a more holistic approach to integrate the management of all types of risks. Casualty Actuarial Society Enterprise Risk Management Committee (2003) defines the ERM as "the discipline by which an enterprise in any industry assesses, controls, exploits, finances, and monitors risks from all sources for the purpose of increasing the enterprise's short- and long-term value to its stakeholders". Furthermore, Committee of Sponsoring Organizations of the Treadway Commission (2004) (COSO) defines ERM as "a process, effected by an entity's board of directors, management and other personnel, applied in strategy setting and across the enterprise, designed to identify potential events that may affect the entity, and manage risk to be within its risk appetite, to provide reasonable assurance regarding the achievement of entity objectives." Hopkin (2010) illustrates more definitions of ERM and summarize the definition of ERM requires three components: 1) a ERM process; 2) identification of the outputs of the process; and 3) the intended impact of those outputs for e.g. risk-based decision making.

2.3.2. Risk Management Process

There are many ways to describe the ERM process. The well-established risk management processes are IRM standard (AIRMIC, ALARM, IRM (2002)) , COSO ERM framework (Committee of Sponsoring Organizations of the Treadway Commission (2004)), ISO 31000 (ISO (2009b)), ERM framework proposed by ERM Committee of the American Academy of Actuaries (2013). AIRMIC, Alarm, IRM (2010) describe a structured approach to ERM by considering the requirements of ISO 31000. Here we follow the international standard ISO 31000 (ISO (2009a), ISO (2009b)) and AIRMIC, Alarm, IRM (2010).

The graphical illustration could be seen in Figure 2.2. In the following we will describe in greater detail for each step of the risk management.

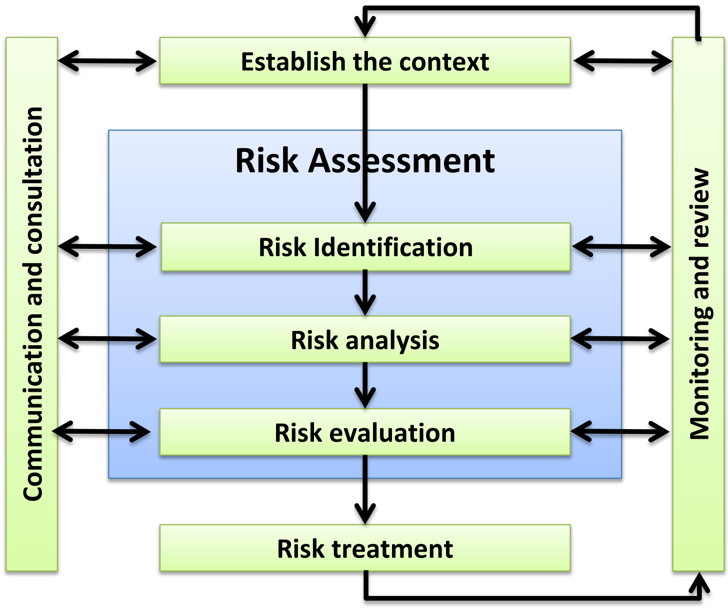

Figure 2.2.: The risk management process. (See ISO (2009b))

Communication and consultation

The *communication and consultation* are continual and iterative processes that organization conducts to provide, share or obtain information, and to engage in dialog with stakeholders regarding the management of risk (see ISO (2009b, Section 3.2.1)).

It shows up during all stages of the risk management process and should be planned in early stage. The communication and consultation should be effective in order to ensure the smooth implementation of risk management process and the basis and reasons for the decision making are understood by the stakeholders.

The communication and consultation with stakeholders are important, as stakeholders may have different perceptions or views on a risk that may have a significant impact of decisions making.

Establish the context

By *establishing the context*, it includes the internal and external, risk management contexts as well as the setting of risk criteria:

- *Establishing the external context*: It focuses on the objectives and concerns of external stakeholders (e.g. regulators, rating agencies, external audits etc). The external context can include cultural, political, legal, regulatory, financial and economic environments etc.

- *Establishing the internal context*: It focuses on the objectives of the organization. The internal context can include the organization's culture and structure, governance, policies, strategies, internal stakeholders etc.

- *Establishing the context of the risk management process*: It focuses on how to establish the risk management processes. The context of the risk management process can include the definition of objectives, strategies, scope and parameters of the risk management activities, the definition of risk assessment methodologies etc.

- *Defining risk criteria*: It focuses on defining the risk criteria for the risk assessment by considering organization's objectives, regulatory, rating agencies and other requirements. The criteria could include how to define likelihood, measure consequences, acceptable level of risks etc.

Risk Assessment

The *risk assessment* consists of risk identification, risk analysis and risk evaluation.

In order to manage risk effectively, all sources of risk to which the organization is exposed should be identified. The *risk identification* is the process of finding, recognizing and describing risks (ISO (2009b, Section 3.5.1)), the tools and techniques that are suited to its objectives should be applied by the organization. For an insurance organization, the sources of risk could be aggregated into six main risk categories (see Delegated Regulation (2015)):

- non-life underwriting risk: non-life premium and reserve risk, catastrophe risk and lapse risk;

- life underwriting risk: longevity, mortality, disability, life expense, revision, lapse, life catastrophe;

- health underwriting risk;

- Market risk: interest rate risk, equity risk, property risk, spread risk, concentration risk and currency risk;

- Counterparty default risk;

- Operational risk.

Furthermore, the sources of emerging risk should also be anticipated, since they may develop to large risk in the future quickly.

After the risk identification, the next step is *risk analysis*, which involves the understanding and definition of risk. It is the process to comprehend the nature of risk and determine the level of risk. It provides the input for risk evaluation, risk treatment as well as the decision making.

The risk analysis tools could be established by using several methods for the quantification of risk. The most two common methods are standard formula and internal model (see Directive 2009/138/EC (2009)). Furthermore, the stress tests, reverse stress tests and hybrid methods with combination of prior methods are also mentioned in ERM Committee of the American Academy of Actuaries (2013).

For the method of standard formula, the Solvency Capital Requirement is calculated as the sum of Basic Solvency Capital Requirement, capital requirement of operational risk, the adjustment for loss-absorbing capacity of technical provisions and deferred taxes (see Directive 2009/138/EC (2009, Article 103)). The Basic Solvency Capital Requirement comprise individual risk modules, which are aggregated by so called "square-root formula" using linear correlation matrix (see Directive 2009/138/EC (2009, Annex IV). The risk modules are the main risk categories as mentioned in the previous paragraphs, each of them shall be calibrated as 99.5% Value-at-Risk in one year horizon. More details about the standard formula could be seen in Directive 2009/138/EC (2009, Article 103-111) and Delegated Regulation (2015, Chapter V)). The square-root formula is correct if the risk categories are normally distributed (see Pfeifer and Strassburger (2008)), however, this is not the case in real world. For instance, Sandström (2007) shows that outcome could be problematic if the marginal distribution are skewed. Furthermore, the overall SCR would be underestimated if the dependency structures are based on heavy tailed copula.

Therefore, in order to better measure the required capital, the usage of more sophisticated partial and full internal models are preferred. However, the calculation of Solvency Capital Requirement using the internal model must be approved by supervisory authorities (see Directive 2009/138/EC (2009, Article 112)). For the approval of internal model, there are many requirements, i.e. use test, statistical quality standards, calibration standards, profit and loss attribution, validation standards, and documentation standards should be fulfilled (see Directive 2009/138/EC (2009, Articles 120-127)). Besides the calculation of capital requirement for Solvency II by the regulation, the rating agency measures of required capital (e.g. 99.97% confidence interval for AA rating) or measures defined by organization internally in line with risk strategy could be calculated based on the internal model.

In addition, the capital adequacy that is assessed by the solvency ratio (i.e. the ratio of available capital to the required capital) could be derived as well. These two quantities are usually calculated by means of stochastic simulation. Therefore, an Economic Scenario Generator (ESG) will be used to simulate the potential evolution of risk factors of the economies and financial markets over time. Two types of ESG scenarios are used, namely the market consistent risk neutral scenarios and real world scenarios.

For the calculation of available capital (defined as the difference between the market value of assets and liabilities), the stochastic cash flow projection model with usage of risk neutral scenarios is used for the market consistent valuation of assets and liabilities.

For the calculation of required capital, the probability distribution of available capital in one year horizon and a risk measure based on such distribution should be taken into account. There are various of risk measures with different confidence level of risk could be chosen to measure the risk. For instance, the Solvency Capital Requirement is measured by the Value-at-Risk at confidence level of 99.5%.

After the risk analysis, the outcomes of which should then be used for the process of *risk evaluation*. It examines the level of risk by the consequences and likelihood and assists the decision making by considering the organizations' risk attitude and risk criteria. It also includes the *risk appetite*, *risk tolerance*, *risk acceptance* etc (see ISO (2009b)).

Risk Treatment

Risk treatment is the process to modifying risk (see ISO (2009b, Section 3.8.1)). Risk treatments are sometimes referred to risk mitigation when dealing with negative consequences. It includes the *risk control*, *risk mitigation*, *risk avoidance*, *risk sharing* (risk transfer) and *risk financing* etc.

Monitoring and review

Monitoring is the activity of continual checking, supervising, critically observing or determining the status in order to identify change from the performance level required or expected (see ISO (2009b, Section 3.8.2.1)). *Review* is the activity of determining the suitability, adequacy and effectiveness of the subject matter to achieve established objectives (see ISO (2009b, Section 3.8.2.2)).

The monitoring and review should take place in all stages of the risk management process. It involves the *risk reporting* for providing information and recording with respect to the current stage of risk, the creation of *risk profile* for the description of any sorts of risks, the *risk management audit* for determining the adequacy and effective of risk management framework by the evidence obtained from systematic, independent and documented process.

3. Risk Measures

In this chapter, we review the axiomatic approach to risk measures used for the determination of capital requirements. The literature review of risk measures could be seen e.g. Szegö (2002) from a probabilistic perspective, Albrecht (2004) from an actuarial perspective and Föllmer and Weber (2015) from a perspective of capital requirement. The textbook of Föllmer and Schied (2011) offers mathematical insights into risk measures.

3.1. Risk Measures on a Probability Space

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra on Ω, $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$.

Let X be a random variable defined as a function $X : \Omega \to \mathbb{R}$ with $X^{-1} : \mathcal{B} \to \mathcal{F}$ where $\mathcal{B}$ is the Borel σ-algebra on Ω. Let $F_X(x) := \mathbb{P}(X \leq x)$ be the *distribution function* of X, for a non-continuous distribution function F_X the α quantile may not be uniquely defined. Therefore we introduce the lower and upper α quantile for univariate random variable X by:

$$q^l_\alpha(X) = \inf\{x : \mathbb{P}(X \leq x) \geq \alpha\} \quad x \in \mathbb{R} \tag{3.1}$$

$$q^u_\alpha(X) = \inf\{x : \mathbb{P}(X \leq x) > \alpha\} \quad x \in \mathbb{R} \tag{3.2}$$

Evidently, $q^l_\alpha(X) \leq q^u_\alpha(X)$, and the equality $q^l_\alpha(X) = q^u_\alpha(X)$ holds if and only if $\mathbb{P}(X \leq x) = \alpha$ for at most one x (see Acerbi and Tasche (2002)). Therefore, if F_X is strictly increasing and continuous, the lower quantile and upper quantile are same and the quantile can expressed by the inverse function $F_X^{-1}(\cdot)$ of F,

$$q_\alpha(X) = F_X^{-1}(\alpha) \tag{3.3}$$

In the following, we will use $F(\cdot)$ as the short notation of $F_X(\cdot)$ and $F^{-1}(\cdot)$ as the short notation of $F_X^{-1}(\cdot)$.

Definition 3.1.1 (Risk Measure)**.** Let $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all bounded measurable random variables on $(\Omega, \mathcal{F})$, a risk measure is a mapping from $\mathcal{X}$ into $\mathbb{R}$, i.e.

$$\rho : \mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \tag{3.4}$$

The risk measures could be classified into different classes. The main classes are *monetary risk measure, convex risk measure, coherent risk measure, distortion risk measure, spectral risk measure* etc. Each of them fulfills certain properties of risk measure. Before we discuss the different classes of risk measures mentioned in the literature, we first give a variety of properties of risk measure for the purpose of classification.

Definition 3.1.2 (Properties of risk measure)**.** For a risk measure $\rho : \mathcal{X} \to \mathbb{R}$, it has the properties of: [1]

(A.1) *translation invariance* if for $m \in \mathbb{R}$ and $X \in \mathcal{X}$, we yield:

$$\rho(X + m) = \rho(X) + m. \tag{3.5}$$

(A.2) *monotonicity* if for $X \leq Y$ and $X, Y \in \mathcal{X}$, we yield:

$$\rho(X) \leq \rho(Y). \tag{3.6}$$

(A.3) *convexity* ρ if for all $0 \leq \lambda \leq 1$ and $X, Y \in \mathcal{X}$, we yield:

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y). \tag{3.7}$$

(A.4) *positive homogeneity* if for all $\lambda \geq 0$ and $X \in \mathcal{X}$, we yield:

$$\rho(\lambda X) = \lambda\rho(X). \tag{3.8}$$

(A.5) *subadditivity* if for any $X, Y \in \mathcal{X}$, we yield:

$$\rho(X + Y) \leq \rho(X) + \rho(Y). \tag{3.9}$$

(A.6) *law invariance* if for all X and Y with the same distribution under probability measure $\mathbb{P}$, we yield:

$$\rho(X) = \rho(Y) \tag{3.10}$$

(A.7) *comonotonic additivity* if X and Y are comonotonic, we yield

$$\rho(X + Y) = \rho(X) + \rho(Y) \tag{3.11}$$

where random variables are comonotonic if and only if

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \text{ for all } \quad (\omega_1, \omega_2) \in \Omega \times \Omega \tag{3.12}$$

(A.8) *continuity* if for $d \in \mathbb{R}$ it satisfies

$$\lim_{d \to 0} \rho((X - d)^+) = \rho(X^+), \lim_{d \to \infty} \rho(\min(X, d)) = \rho(X), \lim_{d \to -\infty} \rho(\max(X, d)) = \rho(X) \tag{3.13}$$

where $(X - d)^+ = \max(X - d, 0)$.

[1] Note that the notations are different compared to Artzner et al. (1999). Artzner et al. (1999) define the risk measure on portfolios by profit and loss value functions. Here we define the risk measure on loss functions.

(A.9) *scale normalization* if $\rho(1) = 1$, in which 1 represents the degenerate random variable which equals 1 with probability 1.

Translation invariance (A.1) is motivated by the interpretation of $\rho(X)$ as a capital requirement, a amount of sure loss m is added, the capital requirement will be increased by m.

Monotonicity (A.2) is the minimum requirement for a reasonable risk measure and it states that the risk measure is increasing for higher risk (loss).

The convexity property (A.3) translates the *diversification* principle. Markowitz (1952) and later the CAPM model introduce the concepts of diversification. The diversifiable or unsystematic risk can be diversified by a variety of securities in the portfolio, while the non-diversifiable or systematic risk can not be diversified. Therefore, diversification can reduce the risk or keep the same risk. In other words, *diversification should not increase risk*. Suppose one investor has two possible investment strategies X and Y. If one *diversifies*, i.e. puts the proportion of λ into X and $1-\lambda$ into Y, it will lead to the outcome of $\lambda X + (1-\lambda)Y$. The risk of $\lambda X + (1-\lambda)Y$ should be smaller than maximum risk of X or Y. Using a risk measure ρ, the mathematical translation is that:

$$\rho(\lambda X + (1-\lambda)Y) \leq \max(\rho(X), \rho(Y)) \text{ for } 0 \leq \lambda \leq 1 \tag{3.14}$$

(3.14) is the weaker version (3.7) due to $\lambda\rho(X) + (1-\lambda)\rho(Y) \leq \max(\rho(X), \rho(Y))$ for $0 \leq \lambda \leq 1$. Therefore, the convexity property gives the idea of the *diversification* principle. Note that under the property of translation invariance (A.1) and monotonicity (A.2), (3.14) is equivalent to (3.7) (see Föllmer and Schied (2011)).

Positive homogeneity (A.4) states that the risk of a financial position is *proportional* to the size of the position. Combined with the property of translation invariance (A.1), one can standardize the model by centering (subtracting the mean) and scaling (dividing by the standard deviation). The advantage of standardization is that the standardized data are easier to visualize, compare and do further statistical analysis. Note that when the risk grows in a non-linear way, one may not insist on positive homogeneity.

Subadditivity (A.5) means that "a merger does not create extra risk" (See Artzner et al. (1999)). It shows that the aggregate risk is bounded by the sum of the individual risks. Under the condition of positive homogeneity (A.4), the property of subadditivity (A.5) is equivalent to convexity (A.3) (see Föllmer and Schied (2011)). Therefore, subadditivity (A.5) also means the diversification when the ρ is positive homogeneous (A.4).

Law invariance (A.6) means that the risk of a position is only determined by the distribution. The risk measure does no depend on the dependence structure of the risks.

Comonotonic additivity states the idea that the risk should be simply added up when there is no way for X to work as *a hedge for* Y[2]. If two random variables are comonotonic, then $X(\omega)$ and $Y(\omega)$ move in the same direction as the state ω changes, that means they can not hedge against each other, leading to the additivity of risks.

The continuity (A.8) is important. Nonfulfilment of the continuity property implies that even small change in the data can lead to big difference of risk measures.

[2] X and Y may hedge, if they are negative correlated.

For more interpretation of the properties, we refer to Föllmer and Schied (2011).

Given the properties of risk measure, we start to describe different classes of risk measures. Fist of all, we start the monetary risk measure first initiated by Artzner et al. (1999).

Definition 3.1.3 (Monetary Risk Measure)**.** A risk measure is called a *monetary risk measure* if it satisfies the properties of translation invariance (A.1) and monotonicity (A.2).

The monetary risk measure could be viewed as minimum capital, i.e. the amount needs to be raised in order to make the financial position acceptable from the point of view of regulators (see Föllmer and Weber (2015)). One example of monetary risk measure is the popular Value-at-Risk.

As monetary risk measure only fulfills to two properties, more potentially desirable properties need to be considered. We start with the coherent risk measure by considering additional properties.

Definition 3.1.4 (Coherent Risk Measure)**.** (Artzner et al. (1999)) A risk measure is called *a coherent risk measure* if it satisfies the properties of translation invariance (A.1), monotonicity (A.2), positive homogeneity (A.4) and subadditivity (A.5).

Artzner et al. (1999) show that a risk measure is coherent if and only if there exists a family Q of probability measures on the finite set of all possible states at a future date, such that $\rho(X) = \sup_{Q\in\mathbb{Q}}\{\mathbb{E}^Q[X]\}$. Delbaen (2000) extends the results from finite dimensional space to the space of bounded measure functions $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 3.1.1 (Delbaen (2000))**.** *A risk measure is coherent if and only if there exists a norm closed and convex set of probability measure $\mathbb{Q}$, all absolutely continuous with respect to $\mathbb{P}$, such that*

$$\rho(X) = \sup_{Q\in\mathbb{Q}}\{\mathbb{E}^Q[X]\}, X \in \mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}).$$

In some cases the risk might increase in nonlinear way with the size of position. Therefore it contradicts the property of positive homogeneity (A.4), Föllmer and Schied (2002) replace the properties of positive homogeneity (A.4) and subadditivity (A.5) to convexity (A.3) and propose the convex risk measure.

Definition 3.1.5 (Convex Risk Measure)**.** (Föllmer and Schied (2002)) A risk measure is called a *convex risk measure* if it satisfies the properties of translation invariance (A.1), monotonicity(A.2) and convexity (A.3).

The convex risk measure has a dual representation for scenario analysis, which is formulated in the following proposition.

Proposition 3.1.2. *(Föllmer and Schied (2011)) A convex risk measure can be represented as*

$$\rho(X) = \sup_{Q\in\mathcal{M}}\{\mathbb{E}^Q[X] - \alpha(Q)\} \qquad \forall X \in \mathcal{X} \tag{3.15}$$

where $\mathcal{M}$ is the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$ and $\alpha : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ is the penalty function such that $\inf_{Q \in \mathcal{M}} \alpha(Q) \in \mathbb{R}$.

The elements of $\mathcal{M}$ can be interpreted as possible scenarios or probabilistic models (different probability measures), $\rho(X)$ is then computed as the worst case expectation taken over all these scenarios $Q \in \mathcal{M}$ and penalized by $\alpha(Q)$. If one sets the penalty function $\alpha(Q) = 0$ for $Q \in \mathbb{Q}$ where $\mathbb{Q}$ is a subset of of $\mathcal{M}$ defined in Proposition 3.1.1 and $\alpha(Q) = +\infty$ for others, then the risk measure possess the property of positive homogeneity. Therefore, coherent risk measure is a special case of convex risk measure, i.e. a convex risk measure turns to be coherent risk measure if the property of positive homogeneity (A.4) is satisfied.

If the risk measure needs to be law invariant and comonotonic additive, the distortion risk measure introduced by Wang et al. (1997) comes into play.

Definition 3.1.6 (Distortion Risk Measure)**.** (Wang et al. (1997)) A risk measure is called *distortion risk measure* if it satisfies the properties of monotonicity (A.2), law invariance (A.6), comonotonic additivity (A.7), continuity (A.8) and scale normalization (A.9).

Proposition 3.1.3. *A distortion risk measure ρ can be represented as a Choquet integral representation*

$$\rho_g(X) = \int_{-\infty}^{0} [g(S_X(x)) - 1]dx + \int_{0}^{\infty} g(S_X(x))dx$$

where $S_X(x) = \mathbb{P}(X > x)$ and $g : [0,1] \longrightarrow [0,1]$ is a nondecreasing function with $g(0) = 0$ and $g(1) = 1$.

Proposition 3.1.4. *A distortion risk measure ρ is subadditive (A.5) if and only if the distortion function $g(x)$ is concave.*

Proof. The proof is given in Wirch and Hardy (1999). $\square$

Therefore, whenever the distortion function of a distortion risk measure is concave, the risk measure is also a coherent risk measure.

Proposition 3.1.5. *A distortion risk measure ρ can also be represented as*

$$\rho_g(X) = \mathbb{E}[Xg'(S_X(X))] = \int_{0}^{1} g'(1-x)F^{-1}(x)dx$$

where g' is the first derivative of the distortion function and $S_X(x) = \mathbb{P}(X > x)$.

Proof. According to the Proposition 3.1.3, the distortion can be represented as follows:

$$
\begin{aligned}
\rho_g(X) &= \int_{-\infty}^{0} [g(S_X(s)) - 1]ds + \int_{0}^{\infty} g(S_X(s))ds \\
&= \int_{-\infty}^{0} [g(1 - F(s)) - 1]ds + \int_{0}^{\infty} g(1 - F(s))ds \\
&= \int_{-\infty}^{0} [g(1 - F(s)) - g(1 - F(-\infty))]ds - \int_{0}^{\infty} [g(1 - F(\infty)) - g(1 - F(s))]ds \\
&= \int_{-\infty}^{0} ds \int_{-\infty}^{s} dg(1 - F(t)) - \int_{0}^{\infty} ds \int_{s}^{\infty} dg(1 - F(t)) \\
&= \int_{-\infty}^{0} dg(1 - F(t)) \int_{t}^{0} ds - \int_{0}^{\infty} dg(1 - F(t)) \int_{0}^{t} ds \qquad (3.16) \\
&= -\int_{-\infty}^{0} tdg(1 - F(t)) - \int_{0}^{\infty} tdg(1 - F(t)) \\
&= -\int_{-\infty}^{\infty} tdg(1 - F(t)) \\
&= \int_{-\infty}^{\infty} tg'(1 - F(t))F'(t)dt \\
&= \mathbb{E}[Xg'(1 - F(X)] = \mathbb{E}[Xg'(S_X(X))] \qquad (3.17)
\end{aligned}
$$

where (3.16) we change the order of integration. Then by setting $F(t) = x$, we have[3]

$$\rho_g(X) = \mathbb{E}[Xg'(S_X(X))] = -\int_{-\infty}^{\infty} tdg(1 - F(t)) = \int_{0}^{1} g'(1 - x)F^{-1}(x)dx.$$

□

By considering the $g'(1 - x)$ as a weight function, Proposition 3.1.5 shows that the distortion risk measure can be calculated as a weighted sum of quantiles.

Definition 3.1.7 (Spectral Risk Measure)**.** (Acerbi (2002)) Let F_X^{-1} be the inverse function of F_X [4], a risk measure is called a *spectral risk measure* if

$$\rho_\phi(X) = \int_{0}^{1} \phi(x)F_X^{-1}(x)dx \qquad (3.18)$$

where ϕ is an admissible risk spectrum i.e. a non-negative, non-decreasing and normalized with $||\phi|| = \int_0^1 \phi(x)dx = 1$. Note that ϕ can be interpreted as a weight function reflecting an investor's (subjective) risk aversion.

[3] When $F(x)$ is not continuous, we can prove it by separating the integral into finite or infinite intervals according to the discontinuous points.

[4] Here F_X^{-1} is defined as $F_X^{-1} := \inf\{x : \mathbb{P}(X \leq x) \geq \alpha\}$, it can also be any other sensible definition for the inverse of F_X, see Acerbi (2002).

Proposition 3.1.6. *(Kusuoka (2001)) A spectral risk measure coincides with a coherent, law invariance (A.6) and comonotonic additivity (A.7) risk measure.*

Proposition 3.1.6 implies that the spectral risk measure is a subclass of coherent risk measure. Furthermore, the spectral risk measure is highly related to the distortion risk measure by setting the distortion function $g(x)$ as a function of risk spectrum $\phi(x)$. The relationship could be seen in Proposition 3.1.7.

Proposition 3.1.7. *Let ϕ be a piecewise continuous admissible spectrum. Then $\rho_g = \rho_\phi$ is a coherent distortion risk measure with concave distortion function satisfying $g'(1-u) = \phi(u)$. (See Gzyl and Mayoral (2006)).*

Proof. It follows immediately from the Proposition 3.1.5 and $\int_0^1 g'(1-x)dx = 1$. □

In the following, we give some examples of risk measures which are popular in financial industry and see which properties they can fulfill and which kinds of risk measure they are

Example 3.1.1 (Examples of risk measures)**.**

- **Value at Risk.** The Value at Risk is given by

$$\mathrm{VaR}_\alpha(X) = q_\alpha(X) \tag{3.19}$$

  for $0 < \alpha < 1$. VaR satisfies all the properties except the convexity (A.3) and subadditivity (A.5). Therefore, it is a monetary risk measure and distortion risk measure, but not coherent or convex or spectral risk measure (see also Denuit et al. (2005)).

- **Standard Deviation based risk measure.**

$$\mathbb{S}td_c(X) := c \cdot \mathbb{S}td(X) + \mathbb{E}(X) \text{ for } \mathbb{S}td(X) > 0$$

  $\mathbb{S}td_c(X)$ satisfies (A.1), subadditivity (A.5), positive homogeneity (A.4), convexity (A.3), law invariance (A.6), continuity (A.8) and scale normalization (A.9). However, it is not monotonous (see the counterexample in Kalkbrener (2005)[5]) and is not comonotonic additive.

- **Tail VaR.** Tail-VaR [6] is defined as

$$\mathrm{TVaR}_\alpha(X) = (1-\alpha)^{-1} \int_\alpha^1 \mathrm{VaR}_\alpha(X)dx = (1-\alpha)^{-1} \int_\alpha^1 q_\alpha(X)dx \tag{3.20}$$

  TVaR satisfies all the properties except the continuity w.r.t weak topology. It is in the class of convex, coherent, spectral and distortion risk measure.

[5] Actually, for $X \le Y$, if $\mathbb{S}td(X) > \mathbb{S}td(Y) + (\mathbb{E}(Y) - \mathbb{E}(X))/c$, then $\mathbb{S}td_c(X) > \mathbb{S}td_c(Y)$, i.e. when $\mathbb{S}td(X)$ is much larger $\mathbb{S}td(Y)$, the $\mathbb{S}td_c$ may not fulfill the monotonic property.

[6] It is also called as expected shortfall (see Acerbi and Tasche (2002)), however, it is different with tail conditional expectation $\mathrm{TCE}_\alpha(X) = \mathbb{E}(X \mid X > VaR_\alpha)$, which is not coherent in general. TCE and TVaR coincide for all α only if F_X is continuous.

- **Distortion risk measure**. According to the Proposition 3.1.5, both the VaR and TVaR are all distortion risk measures by setting different distortion functions.
  (i) The distortion function of VaR is

$$g(x) = \begin{cases} 0 & \text{if } x < 1-\alpha \\ 1 & \text{if } x \geq 1-\alpha \end{cases}$$

  The first derivative at $1-x$ of $g(x)$ then can be calculated as $g'(1-x) = \mathbb{1}_{\{x=\alpha\}}$.

  (ii) The distortion function of TVaR is

$$g(x) = \begin{cases} \frac{x}{1-\alpha} & \text{if } x < 1-\alpha \\ 1 & \text{if } x \geq 1-\alpha \end{cases}$$

  The first derivative at $1-x$ of $g(x)$ for TVaR then can be calculated as

$$g'(1-x) = \begin{cases} \frac{1}{1-\alpha} & \text{if } x > \alpha \\ 0 & \text{if } x \leq \alpha \end{cases}$$

  (iii) Wang's Distortion function (see Wang (2002)). Let α be a pre-selected security level. let $\lambda = \Phi^{-1}(\alpha)$, the distortion function

$$g_\lambda(x) = \Phi(\Phi^{-1}(x) + \lambda) \tag{3.21}$$

  where Φ is the standard normal distribution. When $\alpha > 0.5$, the distortion function is non-decreasing and concave. According to the Proposition 3.1.4 the corresponding risk measure $WT_\lambda(X)$ is also a coherent risk measure and also a spectral risk measure. The first derivative at x of $g_\lambda(x)$ then can be calculated as

$$\begin{aligned} g'_\lambda(x) &= \phi(\Phi^{-1}(x) + \lambda)\frac{\partial(\Phi^{-1}(x) + \lambda)}{\partial x} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\Phi^{-1}(x) + \lambda)^2}{2}\right] \frac{\partial \Phi^{-1}(x)}{\partial x} \\ &= \exp\left[-\frac{\lambda^2}{2} - \lambda\Phi^{-1}(x)\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\Phi^{-1}(x))^2}{2}\right] \frac{\partial \Phi^{-1}(x)}{\partial x} \\ &= \exp\left[-\frac{\lambda^2}{2} - \lambda\Phi^{-1}(x)\right] \frac{\partial \Phi(\Phi^{-1}(x))}{\partial x} \\ &= \exp\left[-\frac{\lambda^2}{2} - \lambda\Phi^{-1}(x)\right] \end{aligned}$$

  where ϕ the density function of standard normal distribution. Therefore $g'_\lambda(1-x) = \exp[-\lambda^2/2 - \lambda\Phi^{-1}(1-x)]$. The motivation of $WT_\lambda(X)$ is that it contains all the information of the data, while VaR is determined only by a quantile and TVaR only reflects losses exceeding the quantile and consequently losses the information below the quantile. Furthermore, the TVaR is calculated by setting the same weights to the tail and it may be not suitable for low frequency but high severity losses.

(iv) The beta family of distortion risk measures, proposed by Wirch and Hardy (1999), utilizes the incomplete beta functions:

$$g(x) = I_x(a,b) = \int_0^x \frac{1}{\beta(a,b)} t^{a-1}(1-t)^{b-1}dt \tag{3.22}$$

where $\beta(a,b)$ is the beta function with $a, b > 0$. The Harrel Davis (HD) estimator of VaR (Harrell and Davis (1982)) is a special case of it by setting $a = (n+1)p+1$ and $b = (n+1)(1-p)+1$, where n is the number of empirical data.

(v) The Proportional Harzard (PH) transform is a special case of the beta-distortion by setting $a = 1/\gamma$ and $b = 1$. The corresponding risk measure is

$$g(x) = u^{\frac{1}{\gamma}} \tag{3.23}$$

The distortion function is concave when $\gamma > 1$.

(vi) The exponential distortion (ED) function:

$$g(x) = \frac{1 - \mathrm{e}^{-hx}}{1 - \mathrm{e}^{-h}} \tag{3.24}$$

The distortion function is concave when $h > 0$. The first derivative at $1 - x$ is

$$g'(1-x) = \frac{h\mathrm{e}^{-h(1-x)}}{1 - \mathrm{e}^{-h}} \tag{3.25}$$

The comparison of different kinds of distortion functions with various of parameter sets is shown in Figure 3.1. The distortion risk measure may not fulfill the property of subadditivity, for instance when choosing $\gamma < 1$ for PH transform, the corresponding distortion function would be convex, and the distortion risk measure based on PH transform is not subadditive. However, we are more interested in the property of subadditivity. Therefore, we consider the criteria for the choice of a distortion function is checking if it is concave, continuous and differentiable. The distortion function of VaR is discontinuous at point $1-\alpha$ and is not concave, which determines that VaR does not fulfill property of subaddtivity by Proposition 3.1.4. The distortion function of TVaR is not differentiable at $1-\alpha$, it losses information by setting weights only to the tail, see also the weight function $g'(1-x)$ plot in Figure 3.2. Compared with the shorts of distortion functions for VaR and TVaR, the distortion functions of Wang transform, PH transform and exponential distortion are all continuous and differentiable. Furthermore, by making them fulfill the concavity, we should choose $\lambda > 0$ and $\alpha > 0.5$ for Wang's transform, $\gamma > 1$ for PH transform, and $h > 0$ for exponential distortion function.

Figure 3.2 illustrates the selected distortion function with specified parameters and their corresponding weight functions. Higher value of tangency (first derivative)

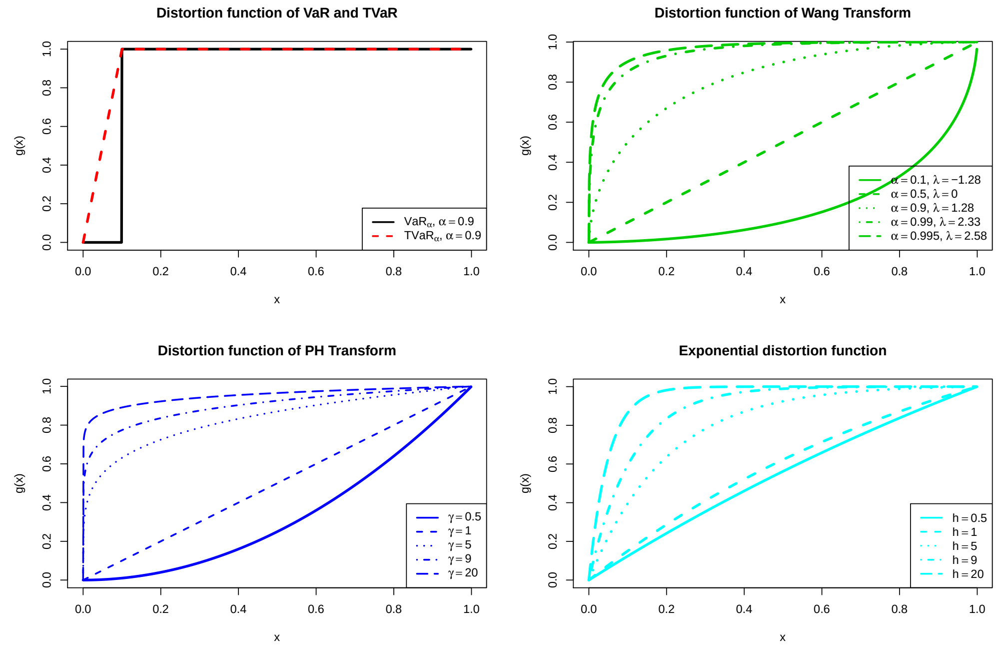

Figure 3.1.: Comparison of different kinds of distortion functions with various of parameter sets.

near 0 in distortion function will lead to higher weights in the tail. By choosing different kinds of distortion function and corresponding parameter sets, we can obtain a variety of weight functions to take into account the severities of extreme values.

3.2. Risk Measures on the Sample Space

Heyde et al. (2007) make a very important contribution to the concept of risk measures by introducing natural risk statistics. This concept bridges the gap between risk measures and statistics by defining natural risk statistics on a sample space in contrast to a probability space as in (3.4).

Let $\mathbb{R}^n$ denote the sample space associated to $(\Omega, \mathcal{F}, \mathbb{P})$, and $\tilde{x} = \{x_1, \ldots, x_n\} \in \mathbb{R}^n$ be a collection of observations on the random variable X, where $x_i = X(w_i)$.

Definition 3.2.1 (Risk Statistic)**.** (Heyde et al. (2007))A risk statistic $\hat{\rho}$ is a mapping from sample space $\mathbb{R}^n$ into $\mathbb{R}$, i.e.

$$\hat{\rho} : \mathbb{R}^n \longrightarrow \mathbb{R}. \tag{3.26}$$

Similar to risk measure, we define the associated properties of risk statistics.

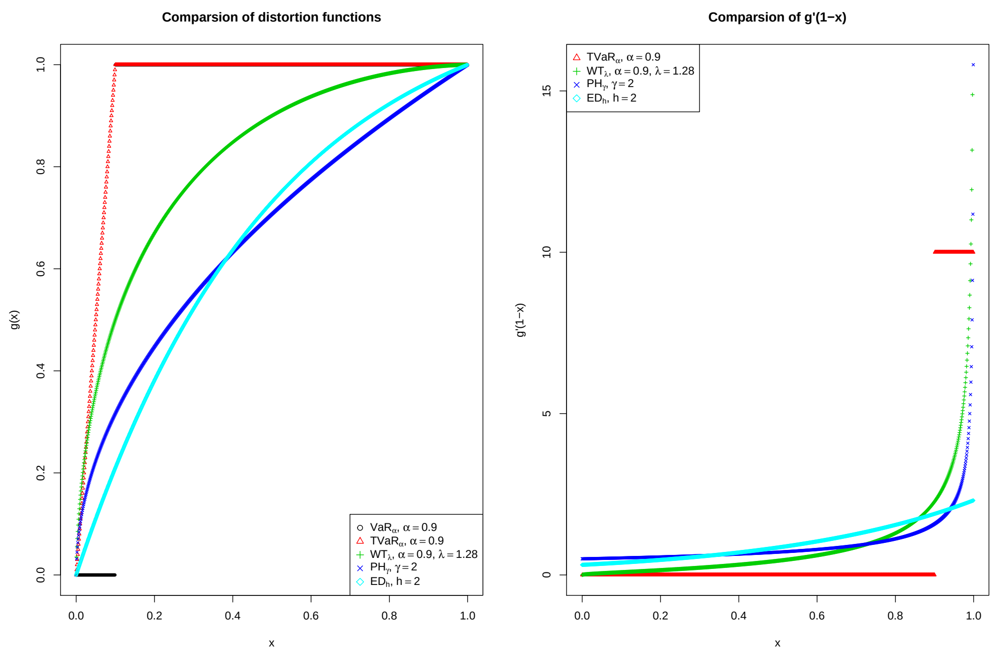

Figure 3.2.: Comparison of different kinds of distortion functions and its corresponding weights function $g'(1-x)$.

Definition 3.2.2 (Properties of Risk Statistic)**.** For a risk statistic $\hat{\rho} : \mathbb{R}^n \to \mathbb{R}$, it have the the properties of:

(B.1) *translation invariance* if for $m \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^n$, we yield:

$$\hat{\rho}(\tilde{x} + m\mathbf{1}) = \hat{\rho}(\tilde{x}) + m \qquad \forall \tilde{x} \in \mathbb{R}^n \tag{3.27}$$

where $\mathbf{1} = (1, 1, \ldots, 1)'$.

(B.2) *monotonicity* if for $\tilde{x} \leq \tilde{y}$, where $\tilde{x} \leq \tilde{y}$ means $x_i \leq y_i$ for all $i = 1, \ldots, n$, we yield:

$$\hat{\rho}(\tilde{x}) \leq \hat{\rho}(\tilde{y}). \tag{3.28}$$

(B.3) *convexity* if for all $0 \leq \lambda \leq 1$ and $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we yield:

$$\rho(\lambda\tilde{x} + (1-\lambda)\tilde{y}) \leq \lambda\hat{\rho}(\tilde{x}) + (1-\lambda)\hat{\rho}(\tilde{y}). \tag{3.29}$$

(B.4) *positive homogeneity* if for all $\lambda \geq 0$ and $\tilde{x} \in \mathbb{R}^n$, we yield:

$$\hat{\rho}(\lambda\tilde{x}) = \lambda\hat{\rho}(\tilde{x}). \tag{3.30}$$

(B.5) *subadditivity* if for any $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we yield:

$$\hat{\rho}(\tilde{x} + \tilde{y}) \leq \hat{\rho}(\tilde{x}) + \hat{\rho}(\tilde{y}). \tag{3.31}$$

(B.6) *Permutation invariance* if for any permutation $(i_1, \ldots, i_n)$, we yield:

$$\hat{\rho}((x_1, \ldots, x_n)) = \hat{\rho}((x_{i_1}, \ldots, x_{i_n})) \tag{3.32}$$

(B.7) *comonotonic additivity* if $\tilde{x}$ and $\tilde{y}$ are comonotonic, we yield

$$\hat{\rho}(\tilde{x} + \tilde{y}) = \hat{\rho}(\tilde{x}) + \hat{\rho}(\tilde{y}) \tag{3.33}$$

where $\tilde{x}$ and $\tilde{y}$ are comonotonic if and only if $(x_i - x_j)(y_i - y_j) \geq 0$, for any $i \neq j$.

(B.8) *continuity* if for any $\tilde{x} \in \mathbb{R}^n$, $\epsilon > 0$ and $\tilde{y}$ satisfying $|y_i - x_i| < \epsilon$ for $i = 1, \ldots, n$, we yield:

$$|\hat{\rho}(\tilde{x}) - \hat{\rho}(\tilde{y})| < \epsilon \tag{3.34}$$

(B.9) *scale normalization* if $\rho(\mathbf{1}) = 1$, where $\mathbf{1} = (1, 1, \ldots, 1)'$.

(B.10) *comonotonic subadditivity* if $\tilde{x}$ and $\tilde{y}$ are comonotonic, we yield:

$$\hat{\rho}(\tilde{x} + \tilde{y}) \leq \hat{\rho}(\tilde{x}) + \hat{\rho}(\tilde{y}) \tag{3.35}$$

(B.11) *weak continuity* if $d_k(F_{\tilde{x}}, F_{\tilde{y}}) \to 0$, then $\tilde{y}$ converges to $\tilde{x}$, where d_k denotes the weak-metric. [7]

(B.12) *continuity w.r.t wasserstein metric* if $d_w(F_{\tilde{x}}, F_{\tilde{y}}) \to 0$, then $\tilde{y}$ converges to $\tilde{x}$, where d_w denotes the wasserstein metric.

Properties (B.1)-(B.9) can be considered as the counterpart of the properties (A.1)-(A.9) in terms of data. Note that the combination of Property (B.1), (B.2) and (B.4) imply the Property of (B.8).

Similar to the law invariance Property (A.6), the permutation invariance Property (B.6) means that if $\tilde{x}$ and $\tilde{y}$ have the same empirical distribution, then the risk statistic should be the same for the given $\tilde{x}$ and $\tilde{y}$.

The property of (B.8) can also be extend to the continuity w.r.t to a probability metric, i.e. (B.11) and (B.12). The comonotonic subadditivity Property (B.10) relaxes the comonotonic additivity Property (B.7). Property (B.10) is consistent with the prospect theory of risk in psychology when the preference is only specified by the comonotonic random variables (see Heyde et al. (2007)).

Definition 3.2.3 (Natural Risk Statistic)**.** (Heyde et al. (2007))A risk statistic is called a natural risk statistic, if it fulfills the properties of translation invariance (B.1), monotonicity (B.2), positive homogeneity (B.4), permutation invariance (B.6), comonotonic subadditivity (B.10).

[7] Such as Kolmogorov metric.

By means of natural risk statistics, Heyde et al. (2007) bridge the gap to statistics and all the statistical concepts can be applied. The following representation theorem shows that the risk statistics can be represented by L-statistics.

Theorem 3.2.1. *Let $x_{(1)}, \dots, x_{(n)}$ be the order statistics for the observation $\tilde{x}$. If $\hat{\rho}$ is a natural risk statistic, then there exists a set of weights $\mathcal{W} = \{\tilde{w} = (w_1, \dots, w_n)\} \subset \mathbb{R}^n$ with each $\tilde{w} \in \mathcal{W}$ satisfying $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0$ for $i = 1, \dots, n$ such that*

$$\hat{\rho}(\tilde{x}) = \sup_{\tilde{w} \in \mathcal{W}} \left\{ \sum_{i=1}^{n} w_i x_{(i)} \right\}, \forall \tilde{x} \in \mathbb{R}^n \tag{3.36}$$

Proof. For the proof and more details, see Heyde et al. (2007). □

3.3. Robustness of Risk Measures

As described in Chapter 1, certain tests and standards should be fulfilled for the approval of internal model (see Directive 2009/138/EC (2009, Articles 120-127)). CEIOPS (2009) gives further details on the requirements for the approval of internal model, in which the requirement of robustness of a model is explicitly mentioned. The objective of internal model is the determination of capital requirement, which is measured by a risk measure with certain confidence level. Therefore, in this section we discuss the robustness of risk measure.

In order to discuss the concept of robustness, we first start with the definition of statistical functional.

Definition 3.3.1 (Statistical Functional)**.** (see Fernholz (1983, p. 5)) Let $x_1, \dots, x_n$ be a sample from a population with distribution function F. Let $T_n = T_n(x_1, \dots, x_n)$ be a statistic. If T_n can be written as a functional T of the empirical distribution F_n, say $T_n = T(F_n)$, where T does not depend on n, then T will be called as a statistical functional. More precisely, T is a functional defined on $\mathbb{F}$,

$$T : \mathbb{F} \longrightarrow \mathbb{R} \tag{3.37}$$

where $\mathbb{F}$ denotes the set of all distributions.

For a distribution based or law invariant risk measure ρ, it could be written as a statistical functional (see Föllmer and Knispel (2013)), i.e. $\rho(X) = T_\rho(F)$. Furthermore, given the empirical distribution F_n based on sample data from historical data or generated by Monte Carlo simulation, $\rho(X)$ could be estimated by $T_\rho(F_n)$. For instance, the distortion risk measure in Example 3.1.1 could be estimated by

$$\rho_g(X) = T(F_n) = \int_0^1 g'(1-x) F_n^{-1}(x) dx \tag{3.38}$$

$g'(1-x) = \mathbb{1}_{\{x=\alpha\}}$ for VaR and $g'(1-x) = \frac{1}{1-\alpha}$ if $x > \alpha$ and 0 if $x \leq \alpha$ for TVaR. If F_n in (3.38) is replaced by F, we obtain the associated risk measures.

The robustness of risk measure then can be related to the continuity of statistical functional. Here, the robustness means that if the risk measure is insensitive to the small variations of distribution function. Actually according to Definition 3.3.1 if F_n is in a neighborhood of F, then the statistical functional $T(F)$ can be approximated with $T(F_n)$. It leads to consider of F as a "point" in $\mathbb{F}$. We therefore now investigate the continuity for statistical functionals defined on $\mathbb{F}$.

Definition 3.3.2 (Continuity of statistical functional)**.** (see Kiesel et al. (2016)) Let $\bar{\mathbb{F}} \subset \mathbb{F}$ be a convex class of distribution functions on $\mathbb{R}$ containing all degenerate distributions. A functional T defined on $\mathbb{F}$ is continuous at $F \in \bar{\mathbb{F}}$ if

$$T(G) - T(F) = o(1) \text{ as } d(G,F) \to 0, \quad F, G \in \bar{\mathbb{F}}$$

where $d(G,F)$ denotes the distance of two distribution G and F.

Examples for d may be the Kolmogorov distance, Wasserstein distance, L_2-distance, on $\mathbb{F}$. The VaR is weak continuous w.r.t to weak topology (see Huber (1981, Theorem 3.1)). However, the TVaR is not weak continuous as shown in Kiesel et al. (2016, Example 2).

Kiesel et al. (2016) then propose to use a right metric with respect to which the statistical functionals associated with important risk measures are continuous. Therefore, the Wasserstein metric (the definition could be seen in Bickel and Freedman (1981), Mallows (1972), and convenient representation on the real line Salvemini (1943), Dall' Aglio (1956)) is taken into account. Wasserstein metric has nice properties such as scaling property, convexity, sub-additivity etc and is a one-ideal metric (see Bickel and Freedman (1981)). The VaR and TVaR are continuous w.r.t W_1-Wasserstein metric, see Krätschmera and Zähle (2011). Furthermore, they show that the L-statistical functional are continuous w.r.t Wasserstein metric, which indicates that the trimmed mean, mean and TVaR are also continuous w.r.t to Wasserstein metric.

The weak continuity of statistical functional is related to qualitative robustness for statistics could be represented by T_n as described in Huber (1981). In the following we discuss further the quantitative robustness. It concerns how greatly a small deviation in the distribution F changes the statistical functional T_n. We start with the von Mises expansion. von Mises (1947) uses a Taylor expansion to approximate the statistical functionals. Let $F, G \in \bar{\mathbb{F}}$, the Taylor expansion in the first order is given by

$$T(G) = T(F) + T'_F(G - F) + Rem(G - F), \tag{3.39}$$

where $Rem(G-F)$ is the remainder term and T'_F is the *Gateaux derivative* or *von Mises derivative* is defined as

$$T'_F(G - F) = \frac{d}{dt} T(F + t(G - F))|_{t=0}, \tag{3.40}$$

if there exists a measurable function $\phi_F(x)$ independent of G such that

$$T'_F(G - F) = \int \phi_F(x) d(G - F)(x). \tag{3.41}$$

The function $\phi_F(x)$ is uniquely defined up to an additive constant, and should be normalized by making $\int \phi_F(x)dF(x) = 0$. It is called *influence function* or *influence curve* by Hampel (1974) and is defined as

$$\phi_F(x) = \frac{d}{dt}T(F + t(\delta_x - F))\,|_{t=0}\ , \tag{3.42}$$

where δ_x is the function of the point mass one at x. The influence curve is useful in assessing the robustness of an estimator, since it measures the effect on T_n by adding one more observation with value x to a very large sample (see Hampel (1974)).

Huber (1981, p. 56) gives the influence function of α-quantile, i.e. $T(F) = F^{-1}(\alpha)$. If F has nonzero finite derivative f at $F^{-1}(\alpha)$, i.e. all the quantile values are uniquely determined, then

$$\phi_F(x, VaR_\alpha) = \begin{cases} \frac{\alpha-1}{f(F^{-1}(\alpha))} & \text{for } x < F^{-1}(\alpha) \\ \frac{\alpha}{f(F^{-1}(\alpha))} & \text{for } x > F^{-1}(\alpha) \end{cases}. \tag{3.43}$$

Therefore $\phi_F(x, VaR_\alpha)$ is bounded. In contrast, influence function of Tail-VaR is unbounded with value

$$\phi_F(x, TVaR_\alpha) = \begin{cases} F^{-1}(\alpha) - TVaR_\alpha(F) & \text{for } x < F^{-1}(\alpha) \\ \frac{x}{1-\alpha} - TVaR_\alpha(F) - \frac{\alpha}{1-\alpha}F^{-1}(\alpha) & \text{for } x > F^{-1}(\alpha) \end{cases}, \tag{3.44}$$

see e.g. Heyde et al. (2007, Section 8.3), Cont et al. (2010).

The von Mises expansion (3.39) could be used to analyze the asymptotic behavior. For $G = F_n$, the expansion could be written as:

$$\begin{aligned} \sqrt{n}(T(F_n) - T(F)) &= \sqrt{n}\left(\frac{1}{n}\int \phi_F(x)d(F_n - F) + Rem(F_n - F)\right) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \phi_F(x_i) + \sqrt{n}Rem(F_n - F). \end{aligned} \tag{3.45}$$

If

$$0 < \mathbb{E}(\phi_F(X))^2 = A(F,T) < \infty \tag{3.46}$$

and

$$\sqrt{n}Rem(F_n - F) \to 0, \quad \text{in probability} \tag{3.47}$$

then the central limit theorem and Slutsky's lemma imply that

$$\sqrt{n}(T(F_n) - T(F)) \to N(0, A(F,T)) \quad \text{in distribution,}$$

as $n \to \infty$ (see Fernholz (1983)). To satisfy the condition (3.47), the Fréchet derivative and the Hadamard (or compact) derivative (see more details in Fernholz (1983)) need to be considered. Note that the differentiability of statistical functional is also helpful for using bootstrap.

Subsequently one can discuss the quantitative robustness of T according to the behavior of its asymptotic bias and variance in some neighborhood P_ϵ (e.g. Lévy neighborhood) of the model distribution F_0. For instance, Huber (1981) gives the maximum bias

$$b_1(\epsilon) = \sup_{F \in P_\epsilon} |T(F) - T(F_0)|,$$

and maximum variance

$$v_1(\epsilon) = \sup_{F \in P_\epsilon} A(F, T).$$

4. Economic Scenario Generator

In the context of Solvency II, an Economic Scenario Generator (ESG) is used to generate economic scenarios for calculation of Market Consistent Embedded Value (MCEV) and determination of Solvency Capital Requirement (SCR). There are two types of ESG scenarios are used, the market consistent risk neutral scenarios and real world scenarios. For the calculation of MCEV, the risk neutral scenarios are used for the market consistent valuation of assets and liabilities. For the determination of SCR, the real-world scenarios including all relevant risk factors are used to calculate the distribution of shareholders' net asset in one year horizon.

The economic scenarios include the financial market risk factors such as the risk free yield curve, option implied volatilities of interest rates, equity returns and dividends, credit spreads, transition probabilities among credit ratings, property returns, inflation rates etc. Therefore, the interest rate model, equity model, credit model, property model and inflation model are usually required.

There are two types of interest rate models that are used widely in the industry, i.e. short rate model and market model. The short rate model is based on modeling the instantaneous spot rate (short rate) through a one or multivariate dimensional diffusion process. It is convenient since all the fundamental quantities such as rates and bonds could be defined as a function of short rate process. Especially for the family of Affine Term Structure Models (ATSMs), which are widely used due to their analytical tractable formula for calculating the bond prices (see Duffie and Kan (1996)). The classical one factor interest rate models such as the Vasicek Model (See Vasicek (1977)), the Cox-Ingersoll-Ross (CIR) model (See Cox et al. (1985)) are all affine models. The one factor model could be extended by adding a deterministic shift on the short rate in order to fit perfectly the initial yield curve, for instance the Hull White extended Vasicek model (see Hull and White (1990)) and the extended CIR model (CIR++) (see Brigo and Mercurio (2006, p. 102, Seciton 3.9)). In addition, multi-factor models are used to capture the main factors of yield curve (e.g. level, slope and curvature), such as multi-factor Cox-Ingersoll-Ross model (see Chen and Scott (1993)). The market model is modeling directly on the forwrd-LIBOR rates or forward swap rates via multi-dimensional diffusion processes. The main advantage is that such models, i.e. lognormal forward-LIBOR model and lognormal forward-swap model, are compatible with the well-established market formulas for basic derivatives, i.e. caps and swaptions. More descriptions of short rate models and market model we refer the book Brigo and Mercurio (2006).

In practice, the market leading ESG providers use either the short rate models or market model. For instance, the extended two factor Black-Karasinski model (Morrison (2007)) and Libor Market Model (LMM) are implemented in the Barrie & Hibbert ESG.

[1] In contrast, a special case of multi-factor affine term structure model, i.e. the extended three factor Cox-Ingersoll-Ross model (CIR3++), is implemented in GEMS, which is a ESG provided by Conning. [2]

There are many popular models for modeling the equity price. The Black-Scholes model is the most classical model, which has the closed form of option price. Nevertheless, it has the main drawback that the stock return is normally distributed and the volatility is constant, which is not consistent with the empirical studies of skewed stock returns as well as time-varying volatilities. In order to overcome the drawback, popular stochastic volatility models such as Heston model (see Heston (1993)) as well as stochastic jump diffusion model (see Bates (1996), Pan (2002), Bates (2006)) considering jumps in the stock returns are taken into account. In practice, both GEMS by Conning and the Barrie & Hibbert ESG use the stochastic volatility jump diffusion (SVJD) model for equity modeling (see Conning (2012), Lawson (2011)).

The credit risk spreads could be modeled by the so called Jarrow-Lando-Turnbull (JLT) model (see Jarrow et al. (1997)). The transition matrices are time dependent under risk neutral measure for JLT model, it leads to time dependent credit spreads. Therefore, there is a lack of stochastic spreads (transition probabilities) for the JLT model. One extension could be modeling the risk premia by the Cox-Ingersoll-Ross process (see Arvanitis et al. (1998) and Dubrana (2011)). In practice, GEMS by Conning uses so called "Corporate Yield Model" based on Feldhütter and Lando (2008) and the Barrie & Hibbert ESG uses the extension of JLT model by allowing the risk premia to be stochastic (see Morrison (2003)).

In the following we build a simple ESG by choosing the extended multi-factor Cox-Ingersoll-Ross model for modeling the interest rates and Heston model for modeling the equity index and ignoring the other risk factors. The calibration and simulation of both models under risk neutral measure and real world measure are shown in the next sections.

4.1. Interest rate model

In this section, we describe the extended multi-factor Cox-Ingersoll-Ross model for the modeling of interest rates. The dynamics of short rate is driven by N dimensional vector of state-variable $X_1(t), \dots, X_N(t)$. A constant shift on the short rate δ_0 is added to allow negative interest rates. Furthermore, a deterministic shift term $\delta(t)$ could be further added in order to fit the initial yield curve at the current time exactly if necessary. In all, the short rate dynamics under real world measure is

$$\begin{aligned} r(t) &= \delta(t) + \delta_0 + \sum_{i=1}^{N} X_i(t), \\ dX_i(t) &= \kappa_i'(\theta_i' - X_i(t))dt + \sigma_i \sqrt{X_i(t)} dW_i^{\mathbb{P}}(t), \end{aligned} \tag{4.1}$$

[1] www.barrhibb.com
[2] www.conning.com

where $W_1^{\mathbb{P}}(t), \ldots, W_N^{\mathbb{P}}(t)$ are independent Wiener processes. According to the Girsanov Theorem, we could change the short rate dynamics from real world measure to risk neutral measure by applying

$$dW_i^{\mathbb{Q}}(t) = dW_i^{\mathbb{P}}(t) + \frac{\lambda_i^0 + \lambda_i^1 X_i(t)}{\sigma_i \sqrt{X_i(t)}} dt. \tag{4.2}$$

Plugging (4.2) into (4.1), we then get:

$$\begin{aligned} r(t) &= \delta(t) + \delta_0 + \sum_{i=1}^{N} X_i(t), \\ dX_i(t) &= \kappa_i(\theta_i - X_i(t))dt + \sigma_i \sqrt{X_i(t)} dW_i^{\mathbb{Q}}(t), \end{aligned} \tag{4.3}$$

where the Wiener processes $W_1^{\mathbb{Q}}(t), \ldots, W_N^{\mathbb{Q}}(t)$ are independent and

$$\kappa_i' = \kappa_i - \lambda_i^1, \quad \theta_i' = \frac{\kappa_i \theta_i + \lambda_i^0}{\kappa_i - \lambda_i^1} \quad \text{for } i = 1, \ldots, N,$$

where λ_i^0, λ_i^1 are the parameters for market price of risk. Note that there are two parameters for the market price of risk to make the speed and mean reversion parameters κ_i and θ_i to be different between risk neutral and real world measure. Hence it is more flexible to fit better under two different measures.

4.1.1. Pricing zero coupon bonds and swaptions

The zero coupon bond is the most basic interest rate instrument. We start to price the zero coupon bond. Since the model is in the family of affine model, the zero coupon bond could then be priced as (see Duffie and Kan (1996), Dai and Singleton (2000)),

$$P(t,T) = e^{C(t,T) - \sum_{i=1}^{N} B^{X_i}(t,T) X_i(t)}. \tag{4.4}$$

Given the money market account $M_t = \exp\{\int_0^t r(s)ds\}$, then $\frac{P(t,T)}{M_t}$ is a martingale under risk neutral measure $\mathbb{Q}$ with the numeraire M_t. Therefore, applying the Ito's formula to $d\frac{P(t,T)}{M_t}$, then the drift term of $d\frac{P(t,T)}{M_t}$ should be zero according to the martingale representation theorem (see Bingham and Kiesel (2004)). Then we will have the ordinary differential equations (ODEs):

$$\begin{aligned} \frac{\partial B^{X_i}(t,T)}{\partial t} &= \kappa_i B^{X_i}(t,T) + \frac{1}{2} B^{X_i}(t,T)^2 \sigma_i^2 - 1 \qquad && B^{X_i}(T,T) = 0, \\ \frac{\partial C(t,T)}{\partial t} &= \sum_{i=1}^{N} \theta_i \kappa_i B^{X_i}(t,T) + \delta(t) + \delta_0 && C(T,T) = 0. \end{aligned}$$

By solving the ODEs we could have:

$$B^{X_i}(t,T) = \frac{2(e^{h_i(T-t)} - 1)}{2h_i + (\kappa_i + h_i)(e^{h_i(T-t)} - 1)}, \tag{4.5}$$

$$C(t,T) = -\int_t^T \delta(s) + \delta_0 \; ds + \sum_{i=1}^{N} C^{X_i}(t,T), \tag{4.6}$$

where

$$
\begin{aligned}
h_i &= \sqrt{\kappa_i^2 + 2\sigma_i^2}, \\
C^{X_i}(t,T) &= \frac{2\kappa_i\theta_i}{\sigma_i^2} \log\left[\frac{2h_i e^{(\kappa_i+h_i)(T-t)/2}}{2h_i + (\kappa_i+h_i)(e^{h_i(T-t)}-1)}\right].
\end{aligned}
$$

As we described before, the shift term $\delta(t)$ is used to fit exactly the zero-coupon curve observed in the market. The determination of shift we follow the CIR++ model introduced in Brigo and Mercurio (2006, p.102). Let

$$
\tilde{\delta}(t_0, t_0+t) = f^M(t_0, t_0+t) - \sum_{i=1}^{N} f^{X_i}(t_0, t_0+t) - \delta_0,
$$

be the difference between market instantaneous forward rate and the sum of factor instantaneous forward rate f^{X_i} for state variable X_i at time t_0 for time to maturity t. Under the current time $t_0 = 0$, then the $\delta(t)$ is defined as

$$
\delta(t) = \tilde{\delta}(0,t) = f^M(0,t) - \sum_{i=1}^{N} f^{X_i}(0,t) - \delta_0,
$$

where

$$
f^{X_i}(0,t) = -\frac{\partial \ln P^{X_i}(0,t)}{\partial t} = \frac{2\kappa_i\theta_i(e^{th_i}-1)}{2h_i + (\kappa_i+h_i)(e^{th_i-1})} + X_i(0)\frac{4h_i^2 e^{th_i}}{[2h_i + (\kappa_i+h_i)(e^{th_i}-1)]^2}.
$$

Therefore, we have $e^{-\int_0^T \delta(s)+\delta_0 \ ds} = P^M(0,T)/\prod_{i=1}^N P^{X_i}(0,T)$, where $P^M(0,T)$ is the market zero coupon bond price with maturity T and $P^{X_i}(t,T)$ is the zero coupon bond price of single factor $X_i(t)$, i.e.

$$
P^{X_i}(t,T) = e^{C^{X_i}(t,T) - B^{X_i}(t,T)X_i(t)}.
$$

We now could get the explicit bond-price for the extended multi-factor CIR,

$$
P(t,T) = \xi^X(t,T)\prod_{i=1}^{N} P^{X_i}(t,T),
$$

where $\xi^X(t,T)$ is the extended term on the bond price

$$
\xi^X(t,T) = e^{-\int_t^T \delta(s)+\delta_0 \ ds} = \frac{P^M(0,T)\prod_{i=1}^N P^{X_i}(0,t)}{P^M(0,t)\prod_{i=1}^N P^{X_i}(0,T)}.
$$

Given the bond price, we now start to consider the European option on zero-coupon bond. For an European call option on zero coupon bond with option expiration T and the bond maturity $S > T$ with payoff $H_T = (P(T,S) - K)^+$, the price is given by:

$$\begin{aligned} ZBC(t,T,S,K) &= \mathbb{E}^{\mathbb{Q}}[M_t \frac{H_T}{M_T} | \mathcal{F}_t] \\ &= P(t,T)\mathbb{E}^{\mathbb{Q}^T}[H_T|\mathcal{F}_t] \\ &= P(t,T)\mathbb{E}^{\mathbb{Q}^T}[(P(T,S) - K)^+|\mathcal{F}_t] \\ &= P(t,T)\mathbb{E}^{\mathbb{Q}^T}[(P(T,S)\mathbf{1}_A|\mathcal{F}_t] - KP(t,T)\mathbb{Q}^T(A) \\ &= P(t,S)\mathbb{E}^{\mathbb{Q}^S}[\mathbf{1}_A|\mathcal{F}_t] - KP(t,T)\mathbb{Q}^T(A) \\ &= P(t,S)\mathbb{Q}^S(A) - KP(t,T)\mathbb{Q}^T(A), \end{aligned}$$

where $\mathbb{Q}^T$ is the T-forward measure and A is the exercise region, which satisfies

$$\left\{ (X_1, \ldots, X_N) \in A \middle| \sum_{i=1}^{N} B^{X_i}(T,S)X_i(T) \le Z \right\} \tag{4.7}$$

with $Z = -\ln K + C(T,S)$. As we know the $\sum_{i=1}^N B^{X_i}(t,T)X_i(T)$ is a linear function of the N-state variables, more precisely a linear combination of non-central random χ^2 variables. The linear combination of non-central random variables is not non-central distributed any more, and therefore it is not straightforward to calculate the probabilities $\mathbb{Q}^S(A)$ and $\mathbb{Q}^T(A)$. Longstaff and Schwartz (1992) suggest that bivariate numerical integrations should be used to calculate these probabilities. Chen and Scott (1992) develop a method reduce the bivariate integration to one-dimensional integral.

Here we refer to the method proposed by Chen and Scott (1995), who use Fourier transformation and characteristic functions to reduce the multi-dimensional integration to univariate integration. Given a probability distribution function $G(y) = P(Y \le y)$, the corresponding characteristic function could be given as

$$\Psi(u) = \int_{-\infty}^{\infty} e^{iuy} dG(y) = \int_{-\infty}^{\infty} e^{iuy} g(y) dy, \tag{4.8}$$

where $g(y)$ is the probability density function of $G(y)$. In the multi-factor CIR models, each state variable has a non-central χ^2 distribution, which has a probability density function and a closed form for the characteristic function. The characteristic function of linear combination of independent state variables could be calculated as the product of characteristic functions for single state variable, i.e.

$$\begin{aligned} \Psi(u) = &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{iu\sum_{i=1}^N B^{X_i}(T,S)X_i(T)} g_1(B^{X_1}(T,S)X_1(T)) \ldots g_N(B^{X_N}(T,S)X_N(T)) \\ &dB^{X_1}(T,S)X_1(T) \ldots dB^{X_N}(T,S)X_N(T) \\ = &\Psi_1(B^{X_1}(T,S)u) \ldots \Psi_N(B^{X_N}(T,S)u). \end{aligned}$$

Suppose that the probability density function exists, one can use an inverse Fourier transform of the characteristic function to recover the probability density function,

$$g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(u) e^{-iuy} du.$$

Furthermore, if the random variables are non-negative, as it is in the CIR model, the probability function can be calculated by a version of the Fourier inversion formula as stated in Chen and Scott (1995),

$$P(Y \leq y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin uy}{u}\Psi(u)du. \tag{4.9}$$

In the CIR model, for the single process under T-forward measure $\mathbb{Q}^T$, i.e. using T-bond $P(t,T)$ as the numeriare, the density function of the $X_i(t)$ conditional on $X_i(s)$, $s \leq t \leq T$, is given by (see Brigo and Mercurio (2006, p.67))

$$\begin{aligned}
p^T_{X_i(t)|X_i(s)}(y_i) &= p_{\chi^2(\nu_i,\lambda_i(t,s))/q_i(t,s)}(y_i) = q_i(t,s)p_{\chi^2(\nu_i,\lambda_i(t,s))}(q_i(t,s)y_i)\\
q^T_i(t,s) &= 2[\psi_i + \phi_i(t-s) + B^{X_i}(t,T)],\\
\lambda^T_i(t,s) &= \frac{4\phi_i(t-s)^2 X_i(s)e^{h_i(t-s)}}{q_i(t,s)},
\end{aligned}$$

where

$$\begin{aligned}
\nu_i &= \frac{4\kappa_i\theta_i}{\sigma_i^2},\\
\phi_i(t-s) &= \frac{2h_i}{\sigma_i^2(e^{h_i(t-s)}-1)},\\
\psi_i &= \frac{\kappa_i + h_i}{\sigma_i^2}.
\end{aligned}$$

Hence the density function of the $X_i(T)$ conditional on $X_i(t)$ is:

$$\begin{aligned}
p^T_{X_i(T)|X_i(t)}(y_i) &= p_{\chi^2(\nu_i,\lambda_i(t,s))/q_i(T,t)}(y_i) = q_i(T,t)p_{\chi^2(\nu_i,\lambda_i(T,t))}(q_i(T,t)y_i),\\
q^T_i(T,t) &= 2[\psi_i + \phi_i(T-t) + B^{X_i}(T,T)] = 2[\psi_i + \phi_i(T-t)],\\
\lambda^T_i(T,t) &= \frac{4\phi_i(T-t)^2 X_i(t)e^{h_i(T-t)}}{q^T_i(T,t)}.
\end{aligned}$$

Under measure $\mathbb{Q}^S$, the density function of $X_i(t)$ conditional on $X_i(s)$, $s \leq t \leq S$, is given by

$$\begin{aligned}
p^S_{X_i(t)|X_i(s)}(y_i) &= p_{\chi^2(\nu_i,\lambda_i(t,s))/q_i(t,s)}(y_i) = q_i(t,s)p_{\chi^2(\nu_i,\lambda_i(t,s))}(q_i(t,s)y_i),\\
q^S_i(t,s) &= 2[\psi_i + \phi_i(t-s) + B^{X_i}(t,S)],\\
\lambda^S_i(t,s) &= \frac{4\phi_i(t-s)^2 X_i(s)e^{h_i(t-s)}}{q^S_i(t,s)}.
\end{aligned}$$

Hence the distribution of the $X_i(T)$ conditional on $X_i(t)$ is:

$$\begin{aligned}
p^S_{X_i(T)|X_i(t)}(y_i) &= p_{\chi^2(\nu_i,\lambda_i(T,T))/q_i(T,t)}(y_i) = q_i(T,t)p_{\chi^2(\nu_i,\lambda_i(T,t))}(q_i(T,t)y_i)\\
q^S_i(T,t) &= 2[\psi_i + \phi_i(T-t) + B^{X_i}(T,S)],\\
\lambda^S_i(T,t) &= \frac{4\phi_i(T-t)^2 X_i(t)e^{h_i(T-t)}}{q^S_i(T,t)}.
\end{aligned}$$

Given the characteristic function for the non-central χ^2 (see Johnson and Kotz (1970)):

$$F(u,\nu,\lambda) = \mathbb{E}[e^{iux}] = (1-2iu)^{-\frac{1}{2}\nu} e^{\frac{iu\lambda}{1-2iu}}, \tag{4.10}$$

where ν is the degrees of freedom parameters, and λ is the non-central parameter, we could get the characteristic function of each state variable $X_i(T)$ conditional on $X_i(t)$ as

$$\Psi_i^T(u) = \mathbb{E}_t^{\mathbb{Q}^T}[e^{iuX_i(t)}] = F(u/q_i^T(T,t), \nu_i, \lambda_i^T(T,t))$$

and hence the characteristic function of the linear combination of $\sum_{i=1}^M B^{X_i}(T,S)X_i(T)$ can be given by:

$$\begin{aligned}
\Psi^T(u) &= \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{iu\sum_{i=1}^N B^{X_i}(T,S)X_i(T)}\right] \\
&= \prod_{i=1}^N \Psi_i^T(B^{X_i}(T,S)u) \\
&= \prod_{i=1}^N F\left(\frac{B^{X_i}(T,S)u}{q_i^T(T,t)}, \nu_i, \lambda_i^T(T,t)\right).
\end{aligned}$$

Therefore, as the case of multi-factor CIR model, we get (see Chen and Scott (1995)):

$$\begin{aligned}
ZBC(t,T,S,K) &= P(t,S)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin uZ}{u}\Psi^S(u)du\right) - KP(t,T)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin uZ}{u}\Psi^T(u)du\right) \\
&= P(t,S)\left(\frac{2}{\pi}\int_0^{\infty}\frac{\sin uZ}{u}Re(\Psi^S(u))du\right) \\
&\quad - KP(t,T)\left(\frac{2}{\pi}\int_0^{\infty}\frac{\sin uZ}{u}Re(\Psi^T(u))du\right),
\end{aligned} \tag{4.11}$$

where Z is defined in (4.7), i.e. $Z = -\ln K + C(T,S)$ where $C(T,S)$ is calculated by (4.6) and

$$\Psi^S(u) = \prod_{i=1}^N F\left(\frac{B^{X_i}(T,S)u}{q_i^S(T,t)}, \nu_i, \lambda_i^S(T,t)\right), \tag{4.12}$$

$$\Psi^T(u) = \prod_{i=1}^N F\left(\frac{B^{X_i}(T,S)u}{q_i^T(T,t)}, \nu_i, \lambda_i^T(T,t)\right). \tag{4.13}$$

Another way to calculate the bond option price is to get the characteristic function of $\ln P(T,S)$ proposed in Carr and Madan (1999) or Lee (2004). As mentioned above we have known the conditional characteristic function of $\sum_{i=1}^N B^{X_i}(T,S)X_i(T)$, therefore

$$g(u,t,T,S) = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{iu\ln P(T,S)}\right] = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{iuC(T,S)-iu\sum_{i=1}^N B^{X_i}(T,S)X_i(T)}\right] = e^{iuC(T,S)}\Psi^T(-u).$$

Furthermore, let

$$c_t(k) = \exp\{ak\}ZBC(t;K,S,T)$$

be the dampened call price, where k is the log strike $k = \ln K$ and $a > 0$ is the dampening coefficient. Then we have the Fourier transform of $c_t(k)$,

$$\begin{aligned}
\zeta(u,t,T,S) &= \int_{-\infty}^{\infty} e^{iuk} c_t(k)dk \\
&= \int_{-\infty}^{\infty} e^{iuk} e^{ak} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} (e^{\ln P(T,S)} - e^k)^+ dk \right] \\
&= P(t,T)\mathbb{E}_t^{\mathbb{Q}^T} \left[\int_{-\infty}^{\infty} e^{iuk} e^{ak} (e^{\ln P(T,S)} - e^k)^+ dk \right] \\
&= P(t,T)\mathbb{E}_t^{\mathbb{Q}^T} \left[\int_{-\infty}^{\infty} e^{iuk} e^{ak} (e^{\ln P(T,S)} - e^k)^+ dk \right] \\
&= P(t,T)\mathbb{E}_t^{\mathbb{Q}^T} \left[\int_{-\infty}^{\ln P(T,S)} e^{iuk} e^{ak} (e^{\ln P(T,S)} - e^k) dk \right] \\
&= P(t,T)\mathbb{E}_t^{\mathbb{Q}^T} \left[\left(\frac{e^{(iu+a+1)\ln P(T,S)}}{iu+a} - \frac{e^{(iu+a+1)\ln P(T,S)}}{iu+a+1} \right) \right] \\
&= P(t,T)\mathbb{E}_t^{\mathbb{Q}^T} \left[\frac{e^{(iu+a+1)\ln P(T,S)}}{(iu+a)(iu+a+1)} \right] \\
&= P(t,T) \frac{g(-i(iu+a+1),t,T,S)}{(iu+a)(iu+a+1)}.
\end{aligned}$$

The price of call option can now be calculated by a single integration by using inverse transform,

$$\begin{aligned}
ZBC(t;T,S,K) &= \frac{\exp\{-a \ln K\}}{2\pi} \int_{-\infty}^{\infty} \exp\{-iu \ln K\} \zeta(u,t,T,S) du \\
&= \frac{\exp\{-a \ln K\}}{\pi} \int_{0}^{\infty} Re[\exp\{-iu \ln K\} \zeta(u,t,T,S)] du. \qquad (4.14)
\end{aligned}$$

Note that the pricing of European call option on zero coupon bond based on (4.14) is faster than (4.11) proposed by Chen and Scott (1995) since only one integration is required. However, it requires the dampening coefficient a and one needs to determine an appropriate choice of a.

Finally, the European put option on zero coupon bonds could be calculated according to the Put-Call-Parity, i.e.

$$ZBP(t;T,S,K) = ZBC(t;T,S,K) - (P(t,S) - KP(t,T)).$$

For the extended multi-factor Cox-Ingersoll-Ross model, there is no analytical formula for the pricing of European swpation. There are several approximation methods for the pricing of swaption. Munk (1999) introduces the method called stochastic duration approximation, which approximate the swaption as a zero-bond option with maturity equal to stochastic duration of the zero bond. Singleton and Umantsev (2002) propose a method which is based on approximation on the excise region using straight-line segments. Then the probability of the exercise can be easily calculated according Fourier transformation we already had described above. Schrager and Pelsser (2006) also propose an approximation method which is based on the derivation of approximate swap rate of dynamics. All the numerical comparison about the quality of the approximations according to Monte Carlo simulations can be seen in Zheng (2009).

Here we describe the stochastic duration approximation by Munk (1999). Consider dynamics of the prices of the zero coupon bond maturing at time T for the affine term structure model under measure $\mathbb{Q}$,

$$dP(t,T) = P(t,T)\left[\mu(t,T) + \sum_{j=1}^{N} v_j(t,T)dW_j^{\mathbb{Q}}(t)\right]$$

where $W_1, \ldots, W_N$ are independent standard Brownian motions, with drift term $\mu(t,T)$, and the diffusion terms $v_j(t,T)$. Again we denote $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, ..., T_\beta\}$ as the set of payment times, and the year fraction $\tau = \{\tau_{\alpha+1}, \ldots, \tau_\beta\}$, which τ_i the difference between T_{i-1} to T_i, $i = \alpha+1, \ldots, \beta$. Furthermore, we set $c_i = K\tau_i$ for $i = \alpha+1, \ldots, \beta-1$ and $c_\beta := 1 + K\tau_i$. Apply the Ito's formula,

$$\begin{aligned}
d\sum_{i=\alpha+1}^{\beta} c_i P(t,T_i) &= \sum_{i=\alpha+1}^{\beta} c_i P(t,T_i)\mu(t,T_i)dt + \sum_{i=\alpha+1}^{\beta} c_i P(t,T_i) \sum_{j=1}^{N} v_j(t,T_i)dW_j^{\mathbb{Q}}(t) \\
&= \sum_{i=\alpha+1}^{\beta} c_i P(t,T_i)\left[\sum_{i=\alpha+1}^{\beta} w(t,T_i)\mu(t,T_i)dt + \sum_{j=1}^{N}\left(\sum_{i=\alpha+1}^{\beta} w(t,T_i)v_j(t,T_i)\right)dW_j^{\mathbb{Q}}(t)\right]
\end{aligned}$$

where Munk (1999) introduces the weights

$$w(t,T_i) = \frac{c_i P(t,T_i)}{\sum_{i=\alpha+1}^{\beta} c_i P(t,T_i)}$$

that are non-negative and sum to one. Then he defines the stochastic duration $D(t)$ of the coupon bond as the time to maturity of the zero-coupon bond having the same relative volatility as the coupon bond, i.e. the same instantaneous variance of relative price changes. More formally, $D(t)$ is given by

$$\sum_{j=1}^{N} v_j(t, t+D(t))^2 = \sum_{j=1}^{N}\left(\sum_{i=\alpha+1}^{\beta} w(t,T_i)v_j(t,T_i)\right)^2. \tag{4.15}$$

For the general properties of stochastic duration and the approximation errors we refer to Munk (1999). Here we only give the formula approximate the price of swaptions using the stochastic duration.

Let $ZBC(t; T_\alpha, T, K)$ be the time t price of a European call option maturing at time T_α, written on a zero-coupon bond paying one dollar at time T (bond maturity T), and having an exercise price K. Let $CBC(t; T_\alpha, K)$ be the time t price of a European call option maturing at time T_α with coupon payments c_i at future times $T_{\alpha+1}, \ldots, T_\beta$, written on a coupon bond and having an exercise price K. Similarly, denote by CBP

the European put option on coupon bond. Then we have the following approximation (see Munk (1999)):

$$CBC(t, T_\alpha, K) \approx CBC^{app}(t, T_\alpha, K) \equiv \eta ZBC(t, T_\alpha, t + D(t), K/\eta), \tag{4.16}$$

where $D(t)$ is the stochastic duration defined in (4.15), and

$$\eta = \sum_{i=\alpha+1}^{\beta} \frac{c_i P(t, T_i)}{P(t, t + D(t))}.$$

Furthermore, the swaption can be viewed as an option on a coupon bond. i.e.

$$\begin{aligned}
PS(t, \mathcal{T}, K) &= \mathbb{E}_t^{\mathbb{Q}} \left(D(t, T_\alpha) \left[P(T_\alpha, T_\alpha) - P(T_\alpha, T_\beta) - \sum_{i=\alpha+1}^{\beta} \tau_i K P(T_\alpha, T_i) \right]^+ \right) \\
&= P(t, T_\alpha) \mathbb{E}_t^{\mathbb{Q}^T} \left(\left[1 - P(T_\alpha, T_\beta) - \sum_{i=\alpha+1}^{\beta} \tau_i K P(T_\alpha, T_i) \right]^+ \right) \\
&= P(t, T_\alpha) \mathbb{E}_t^{\mathbb{Q}^T} \left(\left[1 - \sum_{i=\alpha+1}^{\beta} c_i P(T_\alpha, T_i) \right]^+ \right) \\
&= CBP(t, T_\alpha, 1) \\
&\approx \eta ZBP(t, T_\alpha, t + D(t), 1/\eta).
\end{aligned} \tag{4.17}$$

For the extended multi-factor CIR model, we could have

$$dP(t, T) = P(t, T) \left[\mu(t, T) - \sum_{j=1}^{N} \sigma_j \sqrt{X_j(t)} B^{X_j}(t, T) dW_j^{\mathbb{Q}}(t) \right]$$

by applying Ito's formula with $v_j(t, T) = -\sigma_j \sqrt{X_j(t)} B^{X_j}(t, T)$. Therefore, we can get the $D(t)$ under enxtended multi-factor model according the equation of (4.15).

4.1.2. Model calibration

To estimate the multi-factor CIR model, we follow the state-space approach described in Chen and Scott (1993) and Geyer and Pichler (1999).

Let R^M denote the historical continuous compounded spot rates bootstrapped from the swap rates. The difference between the market and model spot rates is captured in the measurement error. The *measurement equation* for the state-space model for the multi-factor CIR model could be written as:

$$R^M(t, T) = \delta_0 - \frac{\sum_{i=1}^N C^{X_i}(t, T)}{T - t} + \frac{\sum_{i=1}^N B^{X_i}(t, T) X_i(t)}{T - t} + u_T \varepsilon_T(t). \tag{4.18}$$

Note that the shift term $\delta(t)$ is not taken into account in the Kalman filter estimation based on historical data, since $\delta(t)$ only refers to the current time in order to fit the initial yield curve. u_T is a constant and $\varepsilon_T(t)$ are independent standard normal variables for all T and t.

For the construction of transition equation, some approximation should be made to use standard linear Kalman filter, since the transition density for a single process is the non-central χ^2 density (see Cox et al. (1985)). Following Chen and Scott (1993) and Geyer and Pichler (1999), a normal transition density is used as a good approximation of the exact transition density by matching the first two moments. We now need to calculate the first two moments, i.e. the conditional expectation and variance for the exact transition density. By applying the Ito's formula to $d(e^{\kappa_i' t}X_i(t))$, we have

$$d(e^{\kappa_i' t}X_i(t)) = e^{\kappa_i' t}dX_i(t) + d(e^{\kappa_i' t})X_i(t) = e^{\kappa_i' t}\kappa_i'\theta_i' dt + e^{\kappa_i' t}\sigma_i\sqrt{X_i(t)}dW^{\mathbb{P}}(t), \tag{4.19}$$

and integrate on both side,

$$e^{\kappa_i' t}X_i(t) = e^{\kappa_i' s}X_i(s) + \kappa_i'\theta_i'\int_s^t e^{\kappa_i' u}du + \sigma_i\int_s^t e^{\kappa_i' u}\sqrt{X_i(u)}dW^{\mathbb{P}}(u). \tag{4.20}$$

Therefore, the expectation of $X_i(t)$ conditional on $X_i(s)$ is:

$$\mathbb{E}[X_i(t)|X_i(s)] = \theta_i'\left(1 - e^{-\kappa_i'(t-s)}\right) + e^{-\kappa_i'(t-s)}X_i(s) \tag{4.21}$$

Furthermore, by applying the Ito's isometry [3] and then the corresponding conditional variance could be calculated

$$\mathbb{V}ar[X_i(t)|X_i(s)] = \frac{\theta_i'{\sigma_i}^2}{2\kappa_i'}\left(1 - e^{-\kappa_i'(t-s)}\right)^2 + \frac{{\sigma_i}^2}{\kappa_i'}\left(e^{-\kappa_i'\Delta t} - e^{-2\kappa_i'(t-s)}\right)X_i(s). \tag{4.22}$$

Given the first two moments of exact transition density, i.e. non-central χ^2 density, we use a normal density by matching these first two moments approximate the exact density. Under small discrete time interval, the *transition equation* for the state-space model is

$$X_i(t) = \theta_i'\left(1 - e^{-\kappa_i'\Delta t}\right) + e^{-\kappa_i'\Delta t}X_i(t - \Delta t) + H_i(t)\eta_i(t), \tag{4.23}$$

where $\eta_i(t)$ are independent standard normal variables for all i, t and independent of $\varepsilon_T(t)$. $H_i(t)^2$ is the conditional variance of the state variable given as:

$$\begin{aligned} H_i(t)^2 &= \mathbb{V}ar[X_i(t)|X_i(t-\Delta t)] \\ &= \frac{\theta_i'{\sigma_i}^2}{2\kappa_i'}\left(1 - e^{-\kappa_i'\Delta t}\right)^2 + \frac{{\sigma_i}^2}{\kappa_i'}\left(e^{-\kappa_i'\Delta t} - e^{-2\kappa_i'\Delta t}\right)X_i(t - \Delta t). \end{aligned} \tag{4.24}$$

[3] $\mathbb{E}[(\int_s^t e^{\kappa_i' u}\sqrt{X_i(u)}dW^{\mathbb{P}}(u))^2|X_i(s)] = \mathbb{E}[(\int_s^t e^{2\kappa_i' u}X_i(u)du)|X_i(s)] = \int_s^t e^{2\kappa_i' u}\mathbb{E}[X_i(u)|X_i(s)]du.$

The initial updating moments of state variables at time 0 are the unconditional moments of state variables, i.e.

$$\mathbb{E}[X_i(0)] = \lim_{t\to\infty} \mathbb{E}[X(t)] = \theta_i', \tag{4.25}$$

$$\mathbb{V}ar[X_i(0)] = \lim_{t\to\infty} \mathbb{V}ar[X(t)] = \frac{\theta_i'{\sigma_i}^2}{2\kappa_i'}, \tag{4.26}$$

which do not depend on the initial value of X_i due to the stationary property of X_i.

Under the assumption of normality of transition density and given initial values, we could then follow the main steps for performing the standard linear Kalman filter algorithm in Appendix C. Furthermore, since there is an extra non-negative restriction on state variable $X_i(t) \geq 0$, we modify the Kalman filter by assuming that there is no negative updating of state variable, i.e. $a_{t|t} = \max(a_{t|t}, 0)$.

Consequently, an approximate Kalman filter is used to estimate the unobservable state variables and the model parameters are estimated by applying a quasi maximum likelihood estimator in a state space model.

Besides fitting to the historical spot rates, the model should also calibrate to the market ATM swaption prices with different option expiries and swap tenor at cut-off date. The market swaption prices are calculated by the Black-like formula given the swaption implied volatilities by (A.5). The model swaption prices could be calculated by the stochastic duration approximation given in (4.17) for the multi-factor extended CIR model.

In the end, we construct the object function of the optimization problem as minimizing the negative log-likelihood of quasi maximum likelihood through Kalman filter and a penalty on the the mean squared error of model and market ATM swaption prices with different swap tenors and option expires.

$$\min_{\Theta} \left\{ -\ln L(\Theta) + w \left(\frac{1}{n} \sum_{i=1}^{n} (P_i^{mod}(\Theta) - P_i^{mkt})^2 \right) \right\} \tag{4.27}$$

4.2. Equity model

In this section, we describe the Heston model introduced by Heston (1993) for modeling the equity index. We start the dynamics of equity index under risk neutral measure:

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= (r(t) - q)dt + \sqrt{v(t)} \left(\rho dW_v^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dW_s^{\mathbb{Q}}(t) \right), \quad S(0) > 0, \\
dv(t) &= \kappa_v(\theta_v - v(t))dt + \sigma_v \sqrt{v(t)} dW_v^{\mathbb{Q}}(t), \quad v(0) > 0,
\end{aligned}$$

where $dW_s^{\mathbb{Q}}(t) dW_v^{\mathbb{Q}}(t) = 0$. The $v(t)$ represents the variance rather than the standard deviation, which follows a mean reversion square root process, i.e. the CIR process (see Cox et al. (1985)). The constraint $2\kappa_v\theta_v \geq \sigma_v^2$ is usually considered to make sure that the zero can not be reached. Furthermore, the dynamics of short rate $r(t)$ is given in (4.3) with $dW_s^{\mathbb{Q}}(t) dW_i^{\mathbb{Q}}(t) = 0$ and $dW_v^{\mathbb{Q}}(t) dW_i^{\mathbb{Q}}(t) = 0$ for $i = 1, \ldots, N$.

Now we consider the dynamic of the logarithm of $S(t)$, i.e. $s(t) = \ln S(t)$,

$$\begin{aligned}
ds(t) =& d\ln S(t) = \frac{1}{S(t)} dS(t) + \frac{1}{2}\left(-\frac{1}{S(t)^2} d\langle S(t)\rangle\right) \\
=& (r(t) - q)dt + \sqrt{v(t)}\left(\rho dW_v^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dW_s^{\mathbb{Q}}(t)\right) + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)\left(v(t)S(t)^2\right) dt \\
=& \left(r(t) - q - \frac{1}{2}v(t)\right) dt + \sqrt{v(t)}\left(\rho dW_v^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dW_s^{\mathbb{Q}}(t)\right).
\end{aligned}$$

Therefore, the model is affine in terms of $s(t)$ and $v(t)$ and we rewrite the dynamics under risk neutral measure as:

$$ds(t) = (r(t) - q - \frac{1}{2}v(t))dt + \sqrt{v(t)}\left(\rho dW_v^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dW_s^{\mathbb{Q}}(t)\right), \tag{4.28}$$

$$dv(t) = \kappa_v(\theta_v - v(t))dt + \sigma_v\sqrt{v(t)}dW_v^{\mathbb{Q}}(t), \quad v(0) > 0. \tag{4.29}$$

It is important that the model is written to be affine, since we could get closed pricing form of European options that will be discussed in the next section.

Now we need to get the dynamics under real world measure. According to the Girsanov Theorem, the change of measure could be done by applying

$$dW_s^{\mathbb{Q}}(t) = dW_s^{\mathbb{P}}(t) + \frac{\lambda_s^0 + \lambda_s^1 v(t)}{\sqrt{1-\rho^2}\sqrt{v(t)}} dt, \tag{4.30}$$

$$dW_v^{\mathbb{Q}}(t) = dW_v^{\mathbb{P}}(t) + \frac{\lambda_v^0 + \lambda_v^1 v(t)}{\sigma_v\sqrt{v(t)}} dt. \tag{4.31}$$

Therefore, the dynamics under real world measure could be derived by plugging (4.30) and (4.31) into (4.28) and (4.29),

$$\begin{aligned}
ds(t) &= (r(t) - q + a + bv(t))dt + \sqrt{v(t)}(\rho dW_v^{\mathbb{P}}(t) + \sqrt{1-\rho^2} dW_s^{\mathbb{P}}(t)), \\
dv(t) &= \kappa_v'(\theta_v' - v(t))dt + \sigma_v\sqrt{v(t)}dW_v^{\mathbb{P}}(t), \quad v(0) > 0,
\end{aligned}$$

with $dW_v^{\mathbb{P}}(t)dW_s^{\mathbb{P}}(t) = 0$ and where

$$\begin{aligned}
a &= \frac{\rho}{\sigma_v}\lambda_v^0 + \lambda_s^0, \\
b &= \frac{\rho}{\sigma_v}\lambda_v^1 + \lambda_s^1 - \frac{1}{2}, \\
\kappa_v' &= \kappa_v - \lambda_v^1, \\
\theta_v' &= \frac{\kappa_v\theta_v + \lambda_v^0}{\kappa_v - \lambda_v^1}.
\end{aligned}$$

4.2.1. Pricing European options

Given the state vector $\boldsymbol{X}(t) = [s(t), v(t), X_1(t), \ldots, X_N(t)]$, in order to price the European option by Heston model with stochastic interest rate, the discounted characteristic function of log price $s(t) = \ln S(t)$ is needed, which is defined as follows

$$\phi_{HCIR}(u; \boldsymbol{X}(t), t, T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)ds + ius(T)\right)\bigg|\mathcal{F}_t\right). \tag{4.32}$$

According to the law of iterated expectations, $\frac{\phi_{HCIR}(t)}{M_t}$ is a martingale, for $t < \ell$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\frac{\phi_{HCIR}(u,\ell,T)}{M_\ell}\bigg|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_\ell^T r(s)ds + ius(T)\right)\Big|\mathcal{F}_\ell\right)}{M_\ell}\Bigg|\mathcal{F}_t\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\exp\left(-\int_\ell^T r(s)ds + ius(T)\right)}{M_\ell}\bigg|\mathcal{F}_t\right] = \frac{\phi_{HCIR}(u,t,T)}{M_t}\end{aligned}$$

then $d\frac{\phi_{HCIR}(u,t,T)}{M_t}$ should has no drift and we have the following partial differential equation (PDE),

$$\begin{aligned}0 =& \frac{\partial\phi}{\partial t} + (r - q - \frac{1}{2}v)\frac{\partial\phi}{\partial s} + \kappa_v(\theta_v - v)\frac{\partial\phi}{\partial v} + \sum_{j=1}^{N}\kappa_j(\theta_j - X_j)\frac{\partial\phi}{\partial X_j} + \frac{1}{2}v\frac{\partial^2\phi}{\partial s^2}\\ &+ \frac{1}{2}\sigma_v^2 v\frac{\partial^2\phi}{\partial v^2} + \frac{1}{2}\sum_{j=1}^{N}\sigma_j^2 X_j\frac{\partial^2\phi}{\partial X_j^2} + \rho\sigma_v v\frac{\partial^2\phi}{\partial s\partial v} - r\phi,\end{aligned} \tag{4.33}$$

subject to terminal condition $\phi_{HCIR}(u; \boldsymbol{X}(t), T, T) = \exp(ius(T))$.

Since the PDE is affine, according to Duffie et al. (2000), the discounted characteristic function has the following closed form:

$$\phi_{HCIR}(u; \boldsymbol{X}(t), \tau) = \exp\left(\bar{A}(u,\tau) + \bar{B}(u,\tau)s(t) + \sum_{j=1}^{N}\bar{C}_j(u,\tau)X_j(t) + \bar{D}(u,\tau)v(t)\right), \tag{4.34}$$

for $\tau = T - t$. We then get the following Riccati ordinary differential equations (ODEs) by substituting ϕ into the PDE (4.33):

$$\begin{aligned}\frac{\partial\bar{B}(\tau)}{\partial\tau} &= 0, \quad \bar{B}(u,0) = iu,\\ \frac{\partial\bar{C}_j(\tau)}{\partial\tau} &= \bar{B}(\tau) - \kappa_j\bar{C}_j(\tau) + \frac{1}{2}\sigma_j^2\bar{C}_j^2(\tau) - 1, \quad \bar{C}_j(u,0) = 0,\\ \frac{\partial\bar{D}(\tau)}{\partial\tau} &= \frac{1}{2}B(\tau)(\bar{B}(\tau) - 1) + (\rho\sigma_v\bar{B}(\tau) - \kappa_v)\bar{D}(\tau) + \frac{1}{2}\sigma_v^2\bar{D}^2(\tau), \quad \bar{D}(u,0) = 0,\\ \frac{\partial\bar{A}(\tau)}{\partial\tau} &= \kappa_v\theta_v\bar{D}(\tau) + \sum_{i=1}^{N}\kappa_i\theta_i\bar{C}_i(\tau) + (\delta(T-\tau) + \delta_0)(\bar{B}(\tau) - 1) - q\bar{B}(\tau), \quad \bar{A}(u,0) = 0.\end{aligned}$$

We then follow the techniques of proof for Lemma 5.1 of Grzelak and Oosterlee (2011) to solve these ODEs and get:

$$
\begin{aligned}
\bar{B}(u,\tau) &= iu,\\
\bar{C}_j(u,\tau) &= \frac{1-e^{-D_j\tau}}{\sigma_j^2(1-G_je^{-D_j\tau})}(\kappa_j - D_j), \quad j=1,\dots,N,\\
\bar{D}(u,\tau) &= \frac{1-e^{-D_v\tau}}{\sigma_v^2(1-G_ve^{-D_v\tau})}(\kappa_v - \sigma_v\rho iu - D_v),\\
\bar{A}(u,\tau) &= \frac{\kappa_v\theta_v}{\sigma_v^2}\left[\tau(\kappa_v - \sigma_v\rho iu - D_v) - 2\ln\left(\frac{1-G_ve^{-D_v\tau}}{1-G_v}\right)\right]\\
&\quad + \sum_{j=1}^{N}\frac{\kappa_j\theta_j}{\sigma_j^2}\left[\tau(\kappa_j - D_j) - 2\ln\left(\frac{1-G_je^{-D_j\tau}}{1-G_j}\right)\right] + (iu-1)\int_t^T \delta(s)+\delta_0\,ds - iuq\tau,\\
D_j &= \sqrt{\kappa_j^2 + 2\sigma_j^2(1-iu)}, \quad j=1,\dots,N,\\
D_v &= \sqrt{(\sigma_v\rho iu - \kappa_v)^2 - (iu-1)iu\sigma_v^2},\\
G_j &= \frac{\kappa - D_j}{\kappa + D_j}, \quad j=1,\dots,N,\\
G_v &= \frac{\kappa_v - \sigma_v\rho iu - D_v}{\kappa_v - \sigma_v\rho iu + D_v}.
\end{aligned}
$$

Note that the value of $\bar{D}(u,\tau)$ is consistent with the corresponding part of Heston (1993) by setting $\widetilde{G}_v = 1/G_v$ and it leads to the formula of $\bar{D}(u,\tau)$ depends on $e^{D_v\tau}$. The reason for preferring of $e^{-D_v\tau}$ here is that it is more numerical stable (see Albrecher et al. (2006)).

Let $C(t;T,K) = \mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r(t)dt)(S_T-K)^+]$ be the price of European call option with maturity T and strike K at time t. Furthermore, following Carr and Madan (1999), we define the dampened call price $c_t(k) = \exp\{ak\}C(t;e^k,T)$, where $k=\ln K$, and with dampening coefficient $a>0$ for the dampened call transform. According to Lee (2004, Theorem 4.2 and 4.3), we have:

$$\zeta_c(u;t,T) = \int_{-\infty}^{\infty} e^{iuk}c_t(k)dk = \frac{\phi(u-(a+1)i, \boldsymbol{X}(t), T-t)}{a^2+a-u^2+i(2a+1)u}.$$

The price of call option can now be calculated by a single integration,

$$C(t;K,T) = \frac{\exp\{-a\ln K\}}{\pi}\int_0^{\infty} Re[\exp\{-iu\ln K\}\zeta_c(u;t,T)]du. \tag{4.35}$$

Similarly, with the corresponding damped put price $p_t(k) = \exp\{-ak\}P(t;e^k,T)$, we have

$$\zeta_p(u;t,T) = \int_{-\infty}^{\infty} e^{iuk}p_t(k)dk = \frac{\phi(u-(-a+1)i, \boldsymbol{X}(t), T-t)}{a^2-a-u^2+i(-2a+1)u}.$$

The price of put option can now be calculated by a single integration,

$$P(t; K, T) = \frac{\exp\{a \ln K\}}{\pi} \int_0^\infty Re[\exp\{-iu \ln K\}\zeta_p(u; t, T)]du. \tag{4.36}$$

4.2.2. Model calibration

The calibration of equity model is separated into two steps. First of all, the model is calibrated under risk neutral measure to the market observed European put and call options prices with different strikes and maturities. The calibration is done by minimizing the mean squared error of model and market European option prices, where the model European Call and Put option prices are calculated by (4.35) and (4.36) for the Heston model.

In the next step, the parameters for market price of risk are estimated by maximum likelihood estimation in closed-form proposed by Aït-Sahalia and Kimmel (2007).

4.3. Monte Carlo simulation

The MCEV can be considered as a path-dependent complexity structured product based on all kinds of the risk factors. It does not have analytical formula and usually should be priced with Monte-Carlo techniques. Therefore, once the models in ESG are calibrated under risk neutral measure, one needs to do the Monte-Carlo simulation to generate all kinds of risk factors. In order to be path identical, the same random seed should be used for the random number generation for all models with same random number generation method. Due to the complexity of MCEV, the limited number of scenarios (e.g. 5000) are used the MCEV calculation and hence the variance reduction techniques such as antithetic variates might be applied during the simulation process.

There are many ways for the discretization of stochastic differential equations (SDE) for Monte Carlo simulation. Since the conditional distribution of square root process follows non-central χ^2 distribution, it is possible to do exact simulation of the CIR process, which could be seen in Glasserman (2004, Section 3.4). In addition, the exact simulation of Heston stochastic volatility model called Broadie-Kaya scheme could be seen in Broadie and Kaya (2006). However, the Broadie-Kaya scheme has some practical drawbacks for simulation under risk neutral measure, such as complex and more time-consuming. The usage of acceptance-rejection sampling might "scramble" random paths when parameters are perturbed as mentioned in Andersen (2008) and might lead some problems if one needs to combine the variance reduction techniques. The simplest way is to use the Euler discretization scheme to get around these drawbacks. However, for the simulation of square root process with Euler discretization, some modification (see e.g. Lord et al. (2010)) should be done to prevent the value to be negative and make the computation of square root possible. Furthermore, Andersen (2008) proposes some schemes for efficient simulation of Heston model.

In this thesis, we use the simple Euler discretization scheme with partial truncation for the Monte Carlo simulation under risk neutral measure. Furthermore, the antithetic variates (see Glasserman (2004, Section 4.2)) technique is used to reduce the variance, which is easy to implement under Euler discretization scheme.

The Euler scheme for the square root diffusion processes in interest rate model is:

$$X_i(t+\Delta t) - X_i(t) = \kappa_i(\theta_i - X_i(t))\Delta t + \sigma_i\sqrt{X_i(t)^+}\sqrt{\Delta t}Z_i, \tag{4.37}$$

where Z_i, $i = 1, \ldots, N$ are independent standard Gaussian random variables. Here we take the positive part of $X_i(t)$, $X_i(t)^+ = \max(0, X_i(t))$ inside the square root, since the value of $X_i(t)$ produced by Euler discretization might be negative (see partial truncation in Lord et al. (2010)). Compared to the original path, the antithetic path could be generated by pairing the standard Gaussian random variable Z_i with $\widetilde{Z}_i = -Z_i$,

$$\widetilde{X}_i(t+\Delta t) - \widetilde{X}_i(t) = \kappa_i(\theta_i - \widetilde{X}_i(t))\Delta t + \sigma_i\sqrt{\widetilde{X}_i(t)^+}\sqrt{\Delta t}\widetilde{Z}_i, \tag{4.38}$$

which represents the reflection of original path that may result lower variance.

Similarly, the Euler scheme for the equity model is:

$$s(t+\Delta t) - s(t) = (r(t) - q - \frac{1}{2}v(t))\Delta t + \sqrt{v(t)^+}\sqrt{\Delta t}\left(\rho Z_v + \sqrt{1-\rho^2}Z_s\right) \tag{4.39}$$

$$v(t+\Delta t) - v(t) = \kappa_v(\theta_v - v(t))\Delta t + \sigma_v\sqrt{v(t)^+}\sqrt{\Delta t}Z_v, \tag{4.40}$$

where $v(t)^+ = \max(0, v(t))$ and Z_s and Z_v are independent standard Gaussian random variables, which are independent with Z_i as well. For the antithetic path,

$$\widetilde{s}(t+\Delta t) - \widetilde{s}(t) = (r(t) - q - \frac{1}{2}\widetilde{v}(t))\Delta t + \sqrt{\widetilde{v}(t)^+}\sqrt{\Delta t}\left(\rho \widetilde{Z}_v + \sqrt{1-\rho^2}\widetilde{Z}_s\right) \tag{4.41}$$

$$\widetilde{v}(t+\Delta t) - \widetilde{v}(t) = \kappa_v(\theta_v - \widetilde{v}(t))\Delta t + \sigma_v\sqrt{\widetilde{v}(t)^+}\sqrt{\Delta t}\widetilde{Z}_v, \tag{4.42}$$

where $\widetilde{Z}_v = -Z_v$ and $\widetilde{Z}_s = -Z_s$.

For the real world simulation, the exact simulation scheme is used for the Monte Carlo simulation under real world in order to avoid generating negative values of the square root process, especially when the nested simulation is performed.

5. Market Consistent Valuation

Under Solvency II framework, in order to protect the benefit of shareholder and policyholder, the insurance company should be adequately capitalized to fulfill the capital requirement for solvency. Therefore, two main components should be taken into account, the available capital and the solvency capital requirement.

The available capital is the amount of financial resources available to absorb the potential or unexpected financial losses. Under solvency II, it is called as "own fund" (see Directive 2009/138/EC (2009, Article 87-99)). The own funds consist of basic own funds (the excess of assets over liabilities, plus the subordinated liabilities, see Directive 2009/138/EC (2009, Article 88)) and ancillary own funds (items other than basic own funds that can be called up to absorb losses, see Directive 2009/138/EC (2009, Article 89)). In this thesis, we ignore the subordinated liabilities and ancillary own funds. Therefore, the basic own funds is the same as own funds.

The available capital is defined as the difference between the market value of assets and liabilities. Therefore, the available capital is assumed to be equal to the basic own funds (BOF). In general, the market consistent valuation of assets is straight forward since the market values (prices) of financial instruments in the investment portfolio could either be observable in capital market (mark-to-market) or replicable by a combination of market observable financial instruments (combination of market-to-market and mark-to-model). However, this is not the case for the market consistent valuation of liabilities since the life insurance contracts in the liability portfolio usually contain the embedded options and guarantees. Therefore, a stochastic model (mark-to-model) is generally used to perform the market consistent valuation of the life insurance company.

In order to ensure that the market consistent values of life companies could be comparable, the so called Market Consistent Embedded Value (MCEV) from the shareholders' perspective is developed. CFO Forum (2009) gives the definition of MCEV that it is a measure of the consolidated value of the shareholders' interests in the covered business. This is very similar to available capital, in this thesis we follow the same assumption mentioned in paper Bauer et al. (2009) that these two quantities are identical.

Based on the stochastic model, the valuation of shareholder's interest is in line with the valuation of traded financial instruments, by calculating the sum of expected discounted future shareholder's cash flows using the risk neutral valuation. The stochastic model and the risk neutral valuation are discussed in greater detail in the following sections.

5.1. Risk neutral valuation

In order to do the risk neutral valuation, there should exist a risk neutral (equivalent martingale) measure, under which every asset earns the same expected return as the risk free rate regardless of risk preference and could be valued by taking the present value of its expected payoff.

The existence of equivalent martingale measure should fulfill some requirements. For the discrete time, the no arbitrage condition should be fulfilled. For the continuous model, the condition "No Free Lunch With Vanishing Risk" (NFLVR) (see Delbaen and Schachermayer (1994)) should be satisfied followed by the "Fundamental Theorem of Asset Pricing" (see Bingham and Kiesel (2004), Theorem 6.1.2), i.e. *for a financial market model with bounded prices, there exists an equivalent martingale measure if and only if the condition NFLVR holds.*

Furthermore, in order to ensure that all cash flows could be evaluated, the market should be complete as well, i.e. any contingent claim is attainable.[1] The condition of completeness of the market could follow Theorem 6.1.5 of Bingham and Kiesel (2004), i.e. *if the strong equivalent martingale measure is the unique martingale measure, then the financial market is complete.*

Under these conditions, we describe the mathematical framework in the following section for the risk neutral valuation.

5.1.1. Mathematical Framework

The setup of mathematical framework follows Bergmann et al. (2009) and Bauer et al. (2009). It is assumed that investors can trade continuously in a frictionless arbitrage free market for the risk neutral valuation. Furthermore, let T be the largest maturity of life insurance contracts in the liability portfolio and let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]})$ be a complete filtered probability space. The Ω is the sample space of all possible outcomes and $\mathbb{P}$ is the real world measure. The σ-algebra $\mathcal{F}_t$ represents all information about the financial market up to time t and the $\mathbb{F}$ represents the information flow evolving with time. We assume further that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]})$ fulfills the 'usual condition', i.e. $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous (see Bingham and Kiesel (2004), p. 153).

The uncertainty of the insurance company's future profits is mainly influenced by the uncertainty of the financial risk factors in the capital market, e.g. interest rates, equity returns etc. Therefore, the so called *state process*, which is a d-dimensional sufficiently regular Markov process $Y = (Y_t)_{t\in[0,T]} = (Y_{t,1}, \ldots, Y_{t,d})_{t\in[0,T]}$, is introduced to capture the uncertainty of the financial market. Consequently, all risk factors and risky assets in the market can be expressed in terms of Y. In particular, it is assumed that there exists numéraire process $(M_t)_{t\in[0,T]}$ with $M_t = \int_0^t r_s\, ds$, where $r_s = r(Y_s)$ is the instantaneous risk-free interest rate at time t.

[1] A contingent claim is called attainable if there exists at least one admissible trading strategy that could be replicated. See Bingham and Kiesel (2004), Definition 6.1.8.

Furthermore, a stochastic cash flow projection model defined in Section 5.2 is given to generate the future profits f_t at time t depending on the development of financial market up to time t for $t = 0, \ldots, T$. Since shareholders' cash flows are assumed to be paid out in discrete times $t \in \{0, 1, \ldots, T\}$ as described in Section 5.2, we model the future profits as a sequence of random variables $X = (X_1, \ldots, X_T)$ where $X_t = f_t(Y_s, 0 \leq s \leq t)$, i.e. X_t is $\mathcal{F}_t$-measurable, $t = 1, \ldots, T$. Note that the cash flow process X_t is not Markov, as X_t depends on Y_s, $0 \leq s \leq t$ not only just on Y_t.

Finally, we assume that there is a risk neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ so that the present value of future profits can be evaluated as expected sum of discounted cash flows w.r.t to numéraire process $(M_t)_{t\in[0,T]}$.

5.1.2. Valuation at $t = 0$

Based on the mathematical framework, we can calculate market consistent value of the cash flows at $t = 0$ denoted by V_0 under risk neutral measure $\mathbb{Q}$ at time $t = 0$ according the risk neutral valuation formula (see Bingham and Kiesel (2004), Theorem 6.1.14):

$$V_0 := E^{\mathbb{Q}} \underbrace{\left[\sum_{t=1}^{T} \exp\left(- \int_0^t r_u \, du \right) X_t \right]}_{=:PV_0}, \tag{5.1}$$

where PV_0 is the sum of discounted cash flows. We see that V_0 is expressed as the expectation of PV_0.

There are two ways to estimate V_0, i.e. Monte Carlo simulation and Certainty Equivalent approach. In general, V_0 can not be computed analytically due to complex insurance liabilities with embedded options and guarantees. Therefore, it is common to compute V_0 by means of Monte Carlo simulation. However, if there are no embedded options and guarantees, the cash flows X_t depend linearly on the development of assets such as traditional non participating life insurance, the Monte Carlo simulation is then not necessary and estimation could be replaced by Certainty Equivalent approach using a deterministic Certainty Equivalent scenario to estimate the V_0. Note that we apply the same management rules for these two approaches.

First of all, let us give the definition of Certainty Equivalent scenario (see Oechslin et al. (2007)). It assumes that the (total) returns of assets are the same as the risk free forward rates implied by the reference risk free rate at $t = 0$ and the cash flows are discounted by the same risk free reference rate. Let X_t^{CE} be the shareholder's future cash flows in the Certainty Equivalent scenario. In consequence, V_0 is given by

$$\widehat{V_0}(CE) := \sum_{t=1}^{T} \frac{X_s^{CE}}{(1 + rr(0, t))^t}, \tag{5.2}$$

where $rr(0, t)$ is the risk free annually compounded spot rate at time 0 with maturity t.

Furthermore, if there are embedded options and guarantees in the insurance liabilities, the cash flows depend non-linearly or asymmetrically on the development of assets such as traditional life insurance with profit sharing, and then the Monte Carlo simulation is needed.

Let K_0 be simulated sample paths under risk neutral measure $\mathbb{Q}$, then we get the estimation of V_0 through Monte Carlo simulation by averaging the sum of discounted cash flows over all K_0 sample paths, i.e.

$$\widehat{V}_0(K_0) := \frac{1}{K_0} \sum_{k=1}^{K_0} \underbrace{\sum_{t=1}^{T} \exp\left(- \int_0^t r_u^{(k)}\, du \right) X_t^{(k)}}_{:= PV_0^{(k)}} \tag{5.3}$$

where $r_u^{(k)}$ is the realization of instantaneous risk-free interest rate at time u for sample path k and $PV_0^{(k)}$ is the sum of discounted cash flows in the k-th sample path. The estimator $\widehat{V}_0(K_0)$ is unbiased, i.e.

$$\mathbb{E}^{\mathbb{Q}}[\widehat{V}_0(K_0)] = V_0. \tag{5.4}$$

Furthermore, according to the "Strong Law of Large Numbers" (SLLN), the estimator $\widehat{V}_0(K_0)$ is also consistent since it convergences with probability 1 (almost surely) to V_0, i.e.

$$\mathbb{P}(\lim_{K_0 \to \infty} \widehat{V}_0(K_0) = V_0) = 1. \tag{5.5}$$

As described above, the difference of Monte Carlo simulation and certainty equivalent approach rises from the embedded options and guarantees. Given the present value of future cash flows calculated both by certainty scenario and Monte Carlo simulation, the resulting difference based on these two approaches leads to the time value of financial options and guarantees.

5.2. Stochastic cash flow projection model

In order to perform the market consistent valuation of assets and liabilities, especially when there are embedded options and guarantees in the insurance liabilities, a stochastic cash flow projection model, i.e. the stochastic modeling and simulation of the development of balance sheet and future cash flows generated from the Asset-Liability framework, should be taken into account. There are a number of papers relate to the development of such models in the recent years, such as Briys and de Varenne (1997), Grosen and Jørgensen (2000), Bacinello (2001), Grosen and Jørgensen (2002), Bacinello (2003), Tanskanen and Lukkarinen (2003), Bauer et al. (2006), Kling et al. (2007), Gerstner et al. (2008), Graf et al. (2011), Bauer et al. (2009), Reuß et al. (2013), Burkhart et al. (2014).

In the next subsections, we start with the balance sheet for modeling the most important balance sheet items. Afterwards, the asset model and liability model are given for the modeling of the asset portfolio in the asset side and liability portfolio in the liability side. In the meanwhile, the management rules of e.g. the asset allocation strategies, unrealized gains and losses, surplus distribution, are described.

5.2.1. Balance sheet

The starting point of the stochastic model is the balance sheet. Therefore, a simplified balance sheet should be constructed to reflect the most important items of the real balance sheet of the insurance company. Following Bauer et al. (2006), Kling et al. (2007) and Bauer et al. (2009), we give the insurer's simplified balance sheet at time t in Table 5.1. The asset side of the the balance sheet is the market value of asset portfolio

Assets	Liabilities
${}^{MV}A_t$	L_t
	R_t

Table 5.1.: The simplified balance sheet at time t.

${}^{MV}A_t$. The liability side of the balance sheet consists of two parts. The first part L_t is the book value of policyholder's account value or policy reserve (see Grosen and Jørgensen (2000)) at time t consisting of actuarial reserve and bonus reserve. The second part R_t is the reserve account, which is a hybrid determined as the difference between a market value and book value, i.e.

$$R_t = {}^{MV}A_t - L_t. \tag{5.6}$$

The difference between the book value of assets and liabilities is the shareholder's equity, i.e. $E_t = {}^{BV}A_t - L_t$.

Similar balance sheet could be seen in e.g. Grosen and Jørgensen (2000), Reuß et al. (2013), Burkhart et al. (2014) for life insurance companies. A more general balance sheet is proposed by Gerstner et al. (2008), they separate the policyholder's account into actuarial reserve and allocated bonus (the part of surpluses that have been credited the policyholder's account through surplus distribution). Furthermore, they separate the reserve account to company account called equity and a buffer account called free reserve for the future bonus payment to achieve more stable return of the policyholders.

5.2.2. Asset model

In the asset side, the asset model is used for modeling the development of asset portfolio. In practice, the asset portfolio consists of the various financial assets, such as the treasury bonds, corporate bonds, stocks, real estate etc. Here we assume that the life insurance company only invests the money in coupon bonds and stocks, i.e. the asset portfolio consists of coupon bonds and stocks. Since the asset allocation depends on the evolution of financial market, the management rules for determining the proportion of the different financial asset classes are usually considered. Therefore, asset allocation strategy should be defined to reflect the management rules in the asset model. The constant strategic asset allocation, i.e. keeping constant proportion of market value of bonds and stocks, is widely adopted, e.g. Kling et al. (2007), Bauer et al. (2009), Gerstner et al. (2008), Reuß et al. (2013), Burkhart et al. (2014) with slightly difference on the allocation among coupon or zero coupon bonds. Furthermore, the book value might not be equal to the market value due to local GAAP accounting rules, which leads to unrealized gain and losses (UGL). In practice, the company may realize some of the gains to get higher returns and release the losses in the equity investments. Therefore, the corresponding management rules should be incorporated as well. Here we follow the same management rules as in Reuß et al. (2013).

In the projection of stochastic model, incoming and outgoing cash flows occur in each year. The incoming cash flows include the premium payments at the start of year, the coupon payments and repayments of nominal for the coupon bonds at maturity at the end of year, the dividends of stocks at the end of year, and the capital contribution from shareholders. In contrast, the outgoing cash flows include the (dividends) payments of shareholder's profit and the benefit payments to the policyholders at the end of year. At the end of year, with the fulfillment of paying cash flows to shareholders and policyholders the asset portfolio is rebalanced based on market value with a constant strategic asset allocation such that:

- The proportion of market value of stocks in the asset portfolio is p^{SAA}.
- The proportion of market value of coupon bonds in the asset portfolio is $1 - p^{SAA}$. If additional coupon bonds need to be bought, the corresponding amount is withdrawn from bank account and invested in coupon bonds yield at par with term T_B. When the longest remaining term of insurance contracts is less than T_B, then the coupon bonds with term the same as longest remaining term of insurance contracts are invested. If bonds need to be sold, they are sold proportionally to the market values of the different bonds in the existing portfolio.

The minimum participation rate is based on the earnings on book values according to the German regulation. The earnings on book values are usually not the same as the earnings on market values due to local GAAP accounting rules, since the assets such as coupon bonds and stocks have differences between market value and book value and hence results in unrealized gains and losses (UGL). The book value of coupon bond is always assumed to be the nominal amount and the book value of single stock is the market value of stock when the stock was bought and entered into the existing portfolio. The management rule for UGL is assumed that a ratio p^{UGL} of the UGL of stocks is realized. More precisely, p^{UGL} of the UGL of stocks is realized annually if the UGL is positive, i.e. there exists unrealized gains. However, if the UGL of stocks is negative, i.e. there exists unrealized losses, $p^{UGL} = 100\%$ of the UGL is realized annually according to the legal framework that unrealized losses on stocks are not possible.

Let ${}^{MV}A_t^-$ and ${}^{MV}A_t^+$ be the market values of asset portfolio before and after the in/out cash flows ${}_{sh}X_t$ (dividends or capital contributions) payments of shareholder's profit and the benefit payments ${}_{ph}X_t$ to the policyholders. Furthermore, let ${}^{BV}A_t^-$ and ${}^{BV}A_t^+$ be the corresponding book values of the asset portfolio before and after the in/out cash flows ${}_{sh}X_t$ and benefit payments ${}_{ph}X_t$. That is:

$$ {}^{MV}A_t^+ = {}^{MV}A_t^- - {}_{sh}X_t - {}_{ph}X_t, \tag{5.7} $$
$$ {}^{BV}A_t^+ = {}^{BV}A_t^- - {}_{sh}X_t - {}_{ph}X_t. \tag{5.8} $$

The market value of asset portfolio ${}^{MV}A_t^-$ at the end of year t consists of market value of assets (coupon bonds and stocks) and the cashes including the dividends of stocks received at the end of year t and the nominal repayments of coupon bonds at maturity at the end of year t, as well as the cash flows of premium received at the beginning of year CF_{t-1}^P that invested in the bank account earning at one year risk free annually compounded spot rate $rr(t-1,t)$. Therefore, we have:

$$ \begin{aligned} {}^{MV}A_t^- = {} & N_{t-1}^S \cdot {}^{MV}S_t + \sum_{i=t+1}^{\widetilde{T}_{t-1}} N_{t-1,i}^{CB} \cdot {}^{MV}CB_t(i) \\ & + N_{t-1}^S Div(t-1,t) + N_{t-1,t}^{CB} \cdot {}^{MV}CB_t(t) + CF_{t-1}^P(1+rr(t-1,t)) \end{aligned} $$

where N_{t-1}^S is the number of stocks at time $t-1$ and $N_{t-1,i}^{CB}$ is the number of coupon bonds at time $t-1$ with maturity i determined by the constant strategic asset allocation at the end of year $t-1$. Furthermore, $\widetilde{T}_{t-1} = \min\{T_B + t - 1, T\}$ is the maximum maturity at time $t-1$. ${}^{MV}S_t$ is the market value of stock at time t and ${}^{MV}CB_t(i)$ is the market value of coupon bond at time t with maturity i. Note that ${}^{MV}CB_t(t) = 1$ since the coupon bond is expired at maturity time t, i.e. nominal repayment of coupon bond at maturity at the end of year t.

We also consider the unrealized gain and loss for the stock. Let ${}^{UGL}S_t = {}^{MV}S_t - {}^{BV}S_t$ be the unrealized gain or loss for the stock at time t. We assume that p^{UGL} of the ${}^{UGL}S_t$ is realized if ${}^{UGL}S_t$ is positive and 100% is realized if ${}^{UGL}S_t$ is negative. The realization is done by selling the corresponding amount of stocks and then receives cash flows of ${}^{UGL}CF_t = p^{UGL} \cdot {}^{UGL}S_t$.

Actually, the market value after payment of cash flows and re-balancing at the beginning of year t (end of year $t-1$) is given by:

$$ {}^{MV}A_{t-1}^+ = N_{t-1}^S \cdot {}^{MV}S_{t-1} + \sum_{i=t}^{\widetilde{T}_{t-1}} N_{t-1,i}^{CB} \cdot {}^{MV}CB_{t-1}(i). \tag{5.9} $$

Let ${}^{MV}I_t$ and ${}^{BV}I_t$ be the investment earnings based on market value and book value between year $t-1$ and t, i.e.

$$ {}^{MV}I_t = {}^{MV}A_t^- - {}^{MV}A_{t-1}^+, \tag{5.10} $$
$$ {}^{BV}I_t = {}^{BV}A_t^- - {}^{BV}A_{t-1}^+. \tag{5.11} $$

It could be further decomposed into the earnings on asset backing shareholder's equity and assets backing liabilities, i.e. ${}^{BV}I_t = {}^{BV}I_t^{AbE} + {}^{BV}I_t^{AbL}$. Furthermore, the corresponding return on the book value is could be written as:

$$ {}^{BV}r_t = \frac{{}^{BV}I_t}{{}^{BV}A_{t-1}^- + CF_{t-1}^P}. \tag{5.12} $$

5.2.3. Liability model

In the liability side, the liability model is used for modeling the development of insurer's liabilities. In practice, the liability portfolio consists of different insurance products, such as endowment policies, life annuities, unit-link products etc. For the sake of simplicity, Kling et al. (2007), Bauer et al. (2006) and Bauer et al. (2009) use the participating single-premium term-fix insurance (ignoring any charges and mortality rates), which is an image of the life insurance company's general financial situation, and hence the evolution of corresponding liability portfolio could be served as the development of the insurer's liabilities. Gerstner et al. (2008), Seemann (2009) use liability portfolios including participating endowment assurance with and without surrender options by considering mortality rates. Reuß et al. (2013) and Burkhart et al. (2014) use traditional participating life insurance contracts (endowment assurance) by considering the charges and mortality rates in their liability portfolio.

In order to better reflect the life insurance company's liabilities but still keep simplicity, we use traditional participating life insurance contracts (endowment assurance) with a cliquet style guarantee ignoring any costs and the surrender options for the construction of liability portfolio. The contract pays a guaranteed benefit G (sum assured) to a policyholder now aged x at the end of the year of death, if death occurs during the next n years, or after n years if the life is still alive. The technical interest rate for pricing is set to be the same as the guaranteed interest rate g (See Section 2.1 of Eling and Holder (2012)). Furthermore, we do not consider any charges such as initial acquisition charge and administration charge etc.

According to the actuarial principle of equivalence (see e.g. Bowers et al. (1997)), the premium payable annually in advance throughout the duration of the contract is given by:

$$P_{x,\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} G \tag{5.13}$$

where $A_{x:\overline{n}|}$ is the expected present value of payment of the contract with sum assured of 1, which is payable not on death but at the end of the year of death. $\ddot{a}_{x:\overline{n}|}$ is the expected present value of premiums paying annual amount of 1 at the start of each year. They are calculated as follows:

$$A_{x:\overline{n}|} = \left(\frac{1}{1+g}\right)^n {}_np_x + \sum_{k=0}^{n-1} \left(\frac{1}{1+g}\right)^{k+1} {}_kp_x \; q_{x+k} \tag{5.14}$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(\frac{1}{1+g}\right)^k {}_kp_x, \tag{5.15}$$

where q_x is the mortality rate in one-year for age x and ${}_kp_x$ is the survival probability in k years for age x.

During the lifetime of the contract, the insurer should set aside the provision or actuarial reserve to meet the future payment of guaranteed benefit. Note that liability portfolio consists of contracts issued to policyholders with different age and duration at different inception dates. Let AR_t^{x,n,t_0} be the actuarial reserve at time t of a contract with n year duration issued to a policyholder aged x incepted at the beginning of year $t_0 + 1$, it could be calculated recursively as:

$$AR_t^{x,n,t_0} = \frac{(AR_{t-1}^{x,n,t_0} + P_{x:\overline{n}|}) \cdot (1+g) - Gq_{x+t-t_0-1}}{p_{x+t-t_0-1}} \tag{5.16}$$

for $t = t_0 + 1, \dots, t_0 + n$ and $AR_{t_0}^{x,n,t_0} = 0$.

For the traditional German participating life insurance contracts, besides the minimum interest rate of g should be guaranteed on the actuarial reserves, German regulation requires that a minimum participation rate δ of the earnings on book values should be credited to the policyholder's account. The part of surplus Sp_t^{x,n,t_0} to policyholder due to minimum participation, denoted by PS_t^{x,n,t_0}, as well as the guaranteed interest rate g are credited to a bonus reserve account BR_t^{x,n,t_0}, i.e.

$$BR_t^{x,n,t_0} = BR_{t-1}^{x,n,t_0}(1+g) + PS_t^{x,n,t_0}, \tag{5.17}$$

for $t = t_0 + 1, \dots, t_0 + n$ and $BR_{t_0}^{x,n,t_0} = 0$.

The policyholder's account value is the sum of actuarial reserve and bonus reserve, i.e. $AV_t^{x,n,t_0} = AR_t^{x,n,t_0} + BR_t^{x,n,t_0}$. In the event of a claim, that is if the policyholder dies during the lifetime of the contract or he/she is still alive at maturity of the contract, the benefit ${}_{ph}X_t^{x,n,t_0}$ consists of bonus reserve and the guaranteed benefit G should be paid out to the policyholder, i.e. ${}_{ph}X_t^{x,n,t_0} = G + BR_t^{x,n,t_0}$. The sum of benefit payments to the policyholders are then defined as ${}_{ph}X_t$.

Let l_t^{x,n,t_0} be the number of policyholders at time t aged x at inception date t_0 and duration n. The number of policyholders at end of year t depends on the number of policyholder at the beginning of year t and the mortality rate q_{x+t} during the year.

$$l_t^{x,n,t_0} = l_{t-1}^{x,n,t_0}(1 - q_{x+t-t_0-1}) \tag{5.18}$$

for $t = t_0 + 1, \dots, t_0 + n$ and $l_{t0+n}^{x,n,t_0} = 0$ at the maturity date. Note that for the simplification, we assume that actual (best estimate) second-order mortality rate is the same as the first-order mortality rates used for the premium calculations, i.e. the mortality rates are based on DAV 2008 T (German standard mortality table).

The total population of the portfolio at time t is

$$l(t) = \sum_{t_0=t-n+1}^{t} \sum_x \sum_n l_t^{x,n,t_0} \tag{5.19}$$

5.2.4. Surplus distribution

For the traditional participating life insurances, the profit or bonus will be shared between the policyholders and shareholders. Therefore, the management rules for the surplus participation should be considered. There are several sources of surplus, namely the investment surplus, risk surplus, cost surplus and other surplus as described in

MindZV[2]. Most literatures such as Bauer et al. (2006), Grosen and Jørgensen (2000), Gerstner et al. (2008), Kling et al. (2007) focus on investment surplus. Burkhart et al. (2014) consider the cost surplus as well by introducing the cost model.

For the investment surplus distribution mechanism, a point to point guarantee framework is used by Briys and de Varenne (1997), i.e. a fixed guaranteed interest as well as bonus determined a certain fraction of financial gains are received by the policyholders. The cliquet-style guarantee is considered in Grosen and Jørgensen (2000), Bauer et al. (2006), Gerstner et al. (2008), Bauer et al. (2006), Kling et al. (2007) etc. Grosen and Jørgensen (2000) consider the cliquet-style guarantees and use the average interest principle. Their mechanism is that a fixed faction of the reserve quote R_t/L_t over the target buffer ratio is credited as bonus rate to the policyholder's account, only if such rate exceed the the guaranteed rate. Gerstner et al. (2008) follow this mechanism of Grosen and Jørgensen (2000). Bauer et al. (2006) describe two cases, namely the MUST-case and IS-case. The MUST-case considers only obligatory payments to the policyholders as required in the German market. The IS-case reflects closely the behavior of typical life German insurance companies over the last few years. In order to keep surplus stable, especially in years with adverse market conditions, the insurance companies accumulate hidden reserves by crediting a target rate of interest to policyholder each year. Therefore, the IS-case sets a target ratio of interest that is credited to policyholder if the reserve quote R_t/L_t is within a given range. The surplus will be reduced or increased whenever the reserve quote is out of range (see more details in Bauer et al. (2006), Kling et al. (2007)).

In order to keep simplicity, we follow the MUST-case for the surplus distribution and only consider the investment surplus, i.e. the difference between actual investment earnings on book value of assets backing liabilities and the amount I_t^g credited to the policyholder account due to profit sharing and guaranteed interest rate

$$Sp_t = {}^{BV}I_t^{AbL} - I_t^g = \sum_{t_0=t-n+1}^{t} \sum_x \sum_n \left[\left(AR_{t-1}^{x,n,t_0} + BR_{t-1}^{x,n,t_0} + P_{x:\overline{n}|} \right) l_{t-1}^{x,n,t_0} \right] \cdot ({}^{BV}r_t - g). \tag{5.20}$$

According to the German regulatory, a minimum surplus participation rate δ (based on MindZV) of the earnings on book values should be credited the policyholders' account.

$$PS_t = \max(\delta \cdot {}^{BV}I_t^{AbL} - I_t^g, 0)$$

The remaining part of surplus ${}_{sh}X_t$ goes to shareholders, which represents the in/out cash flow payment to shareholders. ${}_{sh}X_t$ could be positive and negative referring to dividends and capital contributions respectively. If it is negative, we assume that the insurance company does not exercise its limited liability (put) option (see Gatzert and Schmeiser (2008)).

[2]Mindestzuführungsverordnung (MindZV) - Verordnung über die Mindestbeitragsrückerstattung in der Lebensversicherung. `http://www.bafin.de/SharedDocs/Aufsichtsrecht/DE/Verordnung/MindZV_080404_va.html`

The dividends or profits on assets backing liabilities are

$$ {}^{L}X_t = \begin{cases} Sp_t - PS_t & \text{if } Sp_t > PS_t \\ 0 & \text{if } Sp_t \leq PS_t \end{cases}, \tag{5.21} $$

and the dividends on assets backing required capital or shareholder's equity are defined as:

$$ {}^{RC}X_t = \begin{cases} {}^{BV}I_t^{AbE} & \text{if } Sp_t > PS_t \\ {}^{BV}I_t^{AbE} - (PS_t - Sp_t) & \text{if } Sp_t \leq PS_t \leq {}^{BV}I_t - I_t^g \\ 0 & \text{if } {}^{BV}I_t - I_t^g \leq PS_t \end{cases}, \tag{5.22} $$

If capital contribution is required then the value is defined as:

$$ c_t = \max\{L_t - {}^{MV}A_t^-, 0\}, \tag{5.23} $$

where

$$ L_t = AV_t = \sum_{t_0=t-n+1}^{t} \sum_x \sum_n AV_t^{x,n,t_0}. \tag{5.24} $$

Therefore ${}_{sh}X_t$ could be given by

$$ {}_{sh}X_t = {}^{L}X_t + {}^{RC}X_t - c_t. \tag{5.25} $$

The investment earnings need to be distributed to all policyholders. In order to simplify the distribution, we assume that the earnings are distributed such that all policyholders receive the same total yield on their account.

Therefore, the surplus credited a single policyholder is calculated as

$$ PS_t^{x,n,t_0} = (AR_{t-1}^{x,n,t_0} + BR_{t-1}^{x,n,t_0} + P_{x:\overline{n}|}) \cdot \max(\delta \cdot {}^{BV}r_t - g, 0). $$

5.3. Market consistent embedded value

According to the Market-Consistent Embedded Value Principles (See CFO Forum (2009)), the Market Consistent Embedded Value (MCEV) could be defined as follows:

$$ MCEV := FS + RC + {}^{RC}PVFP^{CE} - TVFOG - CoC - CoNHR \tag{5.26} $$

where:

- FS: Free surplus allocated to the covered business (Principle 4)
- RC: Required capital (Principle 5)
- ${}^{RC}PVFP^{CE}$: Present value of future profits of post taxation shareholder cash flows from the in-force covered business and the assets backing the associated liabilities (Principle 6)

- $TVFOG$: Time value of financial options and guarantees (Principle 7)
- CoC: Frictional cost of required capital (Principle 8)
- $CoRNHR$: Cost of residual non hedgeable risks (Principle 9)

Following Talanx AG (2016), the CoC is defined as the difference between the amount of required capital and the present value of future releases of the required capital in the certainty equivalent scenario, allowing for future after-tax investment income on the associated assets. If the tax rate is assumed to be 0, then CoC could be defined as:

$$\begin{aligned} CoC :&= RC_0 - {}^{RC}PVFP^{CE} \\ &= RC_0 - \sum_{t=1}^{T} \frac{{}^{RC}X_t^{CE} - c_t^{CE}}{(1+rr(0,t))^t} - \frac{RC_T^{CE}}{(1+rr(0,T))^T}, \end{aligned} \tag{5.27}$$

where ${}^{RC}X_t^{CE}$ is the profit on the assets backing the required capital and c_t^{CE} is the capital contribution based on certainty equivalent scenario. RC_T^{CE} is the residual required capital at end.

We assume that at time $t = 0$, the market value is equal to book value and hence there is no UGL at time $t = 0$. The free surplus is always added to be required capital and assumed to be 0. Furthermore, we ignore the CoRNHR and then we have:

$$\begin{aligned} MCEV &= NAV + VIF \\ &= RC + PVFP - TVFOG - CoC \\ &= RC_0 + {}^{L}PVFP^{CE} + \big(\big({}^{L}PVFP^{Stoch} + {}^{RC}PVFP^{Stoch}\big) \\ &\quad - \big({}^{L}PVFP^{CE} + {}^{RC}PVFP^{CE}\big)\big) + {}^{RC}PVFP^{CE} - RC_0 \\ &= {}^{L}PVFP^{Stoch} + {}^{RC}PVFP^{Stoch} \end{aligned} \tag{5.28}$$

where

$${}^{L}PVFP^{Stoch} = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} \frac{{}^{L}X_t}{B_t}\right), \tag{5.29}$$

$${}^{RC}PVFP^{Stoch} = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} \frac{{}^{RC}X_t - c_t}{B_t} + \frac{RC_T}{B_T}\right), \tag{5.30}$$

If we combine these two together, we can write the MCEV as:

$$BOF_0 = AC_0 = MCEV_0 = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} \frac{X_t}{B_t}\right), \tag{5.31}$$

where

$$X_t = \begin{cases} {}_{sh}X_t & \text{if } t \in 1,\dots,T-1 \\ {}_{sh}X_t + RC_T & \text{if } t = T \end{cases}.$$

Furthermore, the company's future obligations in the policyholder's account are given by:

$$ {}^{MV}L_0 = \mathbb{E}^{\mathbb{Q}} \left(\sum_{t=1}^{T} \left(-\frac{CF^P_{t-1}}{B_{t-1}} + \frac{{}_{ph}X_t}{B_t} \right) \right). \tag{5.32} $$

If all cash flows are properly captured by the cash flow projection model, the following relationship holds:

$$ AC_0 + {}^{MV}L_0 = {}^{MV}A_0. \tag{5.33} $$

The leakage test should be performed to check if (5.33) is satisfied when the Monte Carlo simulation is used to estimate the value of AC_0 and ${}^{MV}L_0$.

6. Risk Modeling for SCR Calculation

For the Solvency Capital Requirement (SCR), the distribution of available capital at $t = 1$ is taken into account. Directive 2009/138/EC (2009, Article 101) describes the calculation of SCR: "*the SCR shall correspond to the Value-at-Risk of the basic own funds of an insurance and reinsurance undertaking subject to a confidence level of 99.5% over a one-year period*".

As mentioned in Chapter 5, the basic own funds is assumed to be the same as available capital. Let AC_1 be the value of available capital at $t = 1$ and

$$L := BOF_0 - \frac{1}{1 + rr(0,1)} BOF_1 = AC_0 - \frac{1}{1 + rr(0,1)} AC_1 \tag{6.1}$$

be the one-year loss function at $t = 1$. The SCR is calculated as:

$$\begin{aligned} SCR :&= \mathrm{VaR}_\alpha(L) \\ &= \inf\{x \in \mathbb{R} : \mathbb{P}[L > x] \geq 1 - \alpha\} = \inf\{x \in \mathbb{R} : \mathbb{P}[L \leq x] \geq \alpha\}, \end{aligned} \tag{6.2}$$

where α represents the confidence level and is set to be equal to 99.5%. Therefore, the SCR is just the α quantile of L.

The SCR reflects that the probability of loss L exceeds the SCR is less or equal to $1 - \alpha$. Furthermore, the solvency ratio defined as the available capital over required capital (i.e. AC_0/SCR) is used to compare the solvency among insurance companies. The solvency ratio should be larger than 100%. If not, the supervisor will require actions and the rating of the company will be affected.

In the previous chapter, we know that the AC_0 or MCEV is actually the present value of future profits (PVFP), i.e. sum of present value of future profits for required capital and asset backing liabilities calculated stochastically, under certain conditions[1], which needs to be determined by stochastic models through Monte-Carlo simulation. Therefore in the following we focus on the calculation of PVFP. Furthermore, in order to estimate the SCR, the so called nested stochastic simulation should be applied, i.e. it requires Monte-Carlo simulation based market consistent valuation for each real world path at $t = 1$. Since the SCR is the 99.5%-quantile of random loss, the number of simulation should be large to reduce the estimation error of the quantile. It then results quite high computational time and is not quite practical to use this approach by obtaining the results in required time lines. Consequently, a number of proxy methods have been developed to make the calculation more practical. For instance, the methods of replicating portfolio, curve fitting and least square Monte-Carlo simulation etc are applied in the insurance industry.

[1] More generally, the MCEV would be the PVFP calculated stochastically plus some deterministic parts.

6.1. Available capital at $t = 1$

In addition to the calculation of AC_0, we also need the distribution of AC_1 under real world measure for the determination of Solvency Capital Requirement. Therefore, we evaluate the PVFP at $t = 1$ denoted by V_1, conditional on one year evolution of the financial market under real world measure, i.e.

$$V_1 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=2}^{T} \exp\left(-\int_1^t r_u\, du\right) X_t \middle| Y_s,\, 0 \le s \le 1\right] + X_1 \tag{6.3}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\, du\right) X_t \middle| Y_s,\, 0 \le s \le 1\right]. \tag{6.4}$$

Here we assume that the profit of the first year (denoted by X_1) has not been paid to shareholders yet and is included in the PVFP at $t = 1$. As mentioned before, the PVFP is the same as available capital, therefore we have $AC_1 = V_1$.

The corresponding $\mathbb{P}$-distribution of V_1 could then be given by the cumulative distribution function as:

$$F_{V_1}(x) := \mathbb{P}(V_1 \le x). \tag{6.5}$$

Furthermore, in order to determine the SCR, the α-quantile should be taken into consideration for the calculation of VaR:

$$q_\alpha(V_1) = \inf\{x : \mathbb{P}(V_1 \le x) \ge \alpha\}. \tag{6.6}$$

Note that the α is set to be 0.5% for the computation of $VaR_{99.5\%}$ defined in (6.2).

In practice, the calculation of AC_1 does not depend on the whole continuous history of financial market up to $t = 1$, all necessary information can be contained in the state of financial market at certain discrete times. Bauer et al. (2009) introduce so-called *Markov State Variables*, and assume that (Y_1, D_1), where $D_1 \in \mathbb{R}^m$, contains all the information we need from $\mathcal{F}_1$. Then V_1 can be written as:

$$\begin{aligned} V_1 &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\, du\right) X_t \middle| Y_s,\, 0 \le s \le 1\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\, du\right) X_t}_{=:PV_1} \middle| (Y_1, D_1)\right] \end{aligned} \tag{6.7}$$

V_1 is calculated under risk neutral measure $\mathbb{Q}$ at time $t = 1$, which is dependent on (Y_1, D_1). After one year evolution of risk factors under real world measure, we change back to the risk neutral measure $\mathbb{Q}$ as $t = 0$. Here we assume that there is only one risk neutral measure $\mathbb{Q}$ both at time $t = 1$ and $t = 0$ if we do not change the market price of risk parameters. [2]

[2]Note that in practice, different realizations of (Y_1, D_1), e.g. $(Y_1^{(i)}, D_1^{(i)})$ would have different risk

6.2. Nested simulation

In Section 5.1.2, we have given the valuation formula at $t = 0$. Since calculation of V_0 could be performed by means of Monte Carlo simulation, the determination of distribution of V_1 could also be performed by Monte Carlo simulation, i.e. we first generate N outer real world scenarios up to $t = 1$ and then for each real world scenario we generate K_1 inner risk neutral scenarios to calculate the V_1 conditional on the financial market up to $t = 1$. This kinds of simulation is called full nested simulation and is quite computationally expensive if one chooses efficient large number of N and K_1.

Gordy and Juneja (2010) show how to choose optimal relative smaller number of inner simulation K_1 can still yield accurate estimates such as VaR and analyze how to allocate the computational budget for the inner and outer simulations by minimizing the mean square error of resultant estimator. Furthermore, they introduce a jackknife technique to reduce the bias in the estimator. Instead of using a constant number of inner samples, i.e. allocating the computational burden uniformly across all scenarios, Broadie et al. (2011) show an algorithm to allocate sequentially the computational effort in the inner simulation based on marginal changes in the risk estimator. For instance, larger numbers of inner simulations are employed for the corresponding real world outer paths that might have a greater expected marginal change to the risk measure. Bauer et al. (2010) apply screening procedure, which screens out those scenarios that are unlikely to be in the tail of distribution and generate more inner simulations per real world outer scenario for the survived scenarios during the screening process, to get more reliable (smaller length of confidence interval) and efficient estimation of SCR based on nested simulation. Instead of generating new inner risk neutral scenarios for each real world outer scenarios, Bergmann et al. (2009) use so called basis scenarios (i.e. the set of risk neutral scenarios at $t = 0$ used to calculate V_0) to approximate the real world scenarios and hence avoid the change of measure from real world to risk neutral measure.

In the following we describe in greater detail how the nested simulation procedure is performed. First of all, we simulate N outer scenarios under the real world measure $\mathbb{P}$ for the evolution of financial market up to $t = 1$. Then the PVFP at $t = 1$ conditional on the financial market development for scenario i can be written as

$$\begin{aligned} V_1^{(i)} :&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \,\middle|\, Y_s = Y_s^{(i)},\, 0 \le s \le 1\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \,\middle|\, (Y_1, D_1) = (Y_1^{(i)}, D_1^{(i)})}_{=:PV_1^{(i)}}\right]. \end{aligned} \tag{6.8}$$

neutral (no arbitrage pricing) measure $\mathbb{Q}_1^{(i)}$, since one might adjust the start yield curve and long term option volatility at $t = 1$ considering the extrapolation to ultimate forward rate and ultimate forward volatility as $t = 0$ for each realization and require different re-calibration.

Then we can estimate the distribution function F of V_1 by the empirical distribution function

$$\hat{F}(x; N) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{V_1^{(i)} \leq x\}}. \tag{6.9}$$

By the Glivenko-Cantelli theorem we have that

$$\|\hat{F}(x, N) - F\|_\infty = \sup_{x \in \mathbb{R}} |\hat{F}(x, N) - F(x)| \to 0 \qquad \mathcal{P} \text{ a.s. as } N \to \infty. \tag{6.10}$$

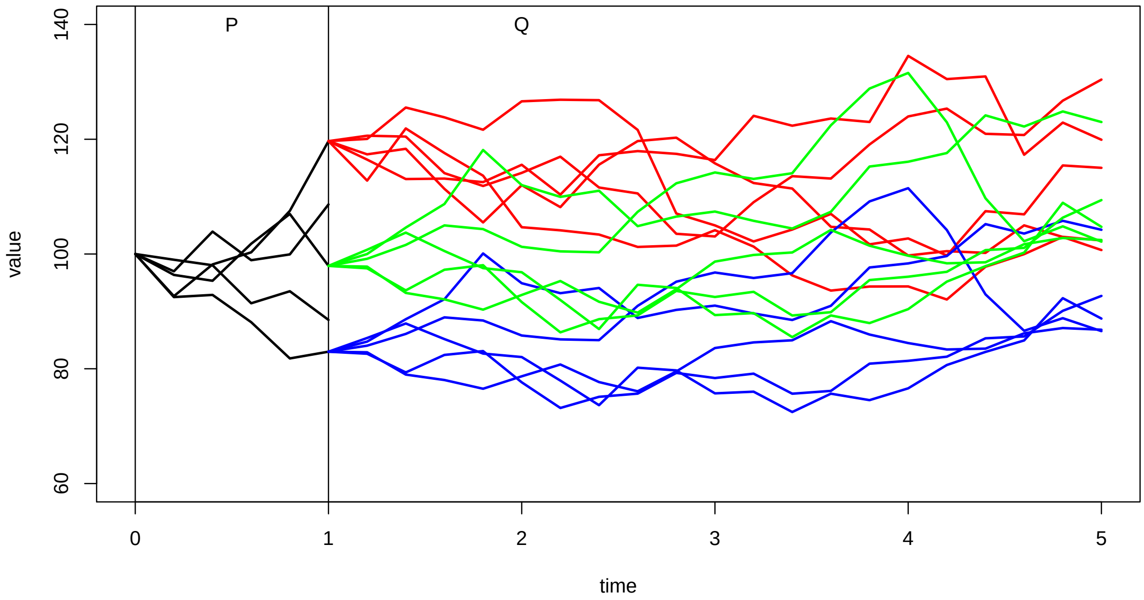

Figure 6.1.: Nested simulations

The same as the calculation of V_0 described in 5.1.1, $V_1^{(i)}$ should also be computed numerically by means of Monte Carlo simulation. Therefore, in the second step we perform inner simulation for each real world outer scenario, i.e. we simulate $K_1^{(i)}$ risk neutral inner scenarios per real world outer scenario i. Figure 6.1 shows the idea of nested simulation. Then $V_1^{(i)}$ could be evaluated by taking the average of the sum of discounted future profits generated by the inner scenarios $K_1^{(i)}$, i.e.

$$\widehat{V}_1^{(i)}(K_1^{(i)}) := \frac{1}{K_1^{(i)}} \sum_{k=1}^{K_1^{(i)}} \underbrace{\sum_{t=1}^{T} \exp\left(-\int_1^t r_u^{(i,k)}\, du\right) X_t^{(i,k)}}_{=: PV_1^{(i,k)}}, \quad i = 1, \ldots, N. \tag{6.11}$$

Now we use the empirical distribution function G to estimate the distribution function F of V_1 . It is constructed by putting a mass $1/N$ for the each realization of $\widehat{V}_1^{(i)}(K_1^{(i)})$ estimated by inner simulation per real world scenario, therefore it can be represented as:

$$\hat{G}(x; K_1^{(1)}, \ldots, K_1^{(N)}, N) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\widehat{V}_1^{(i)}(K_1^{(i)}) \leq x\}}. \tag{6.12}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function.

Let us denote $\tilde{u} = (u_1, \ldots, u_N) = (\widehat{V}_1^{(i)}(K_1^{(i)}), \ldots, \widehat{V}_1^{(N)}(K_1^{(N)}))$ by the realizations of random variable V_1. Let $u_{(1)}, \ldots, u_{(N)}$ be the corresponding order statistics with $u_{(1)} \leq \ldots \leq u_{(N)}$. Then the quantile could be estimated as:

$$\hat{q}_\alpha(\tilde{u}) = u_{(\lceil N\alpha \rceil)}, \tag{6.13}$$

where $\lceil x \rceil$ is the smallest integer not less than x, i.e. $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$.

Since our interest is the efficient estimation of SCR, we now start to analyze the errors when estimating the quantiles of PVFP at $t = 1$ approximated by nested simulation. There are two sources of error accrued during the nested simulation. Firstly, only N real world outer scenarios are used to estimate the distribution. Secondly, only $K_1^{(i)}$ risk neutral inner scenarios are used to estimate the PVFP at $t = 1$ conditional on outer scenario i.

To analyze the influence of the errors on the estimation of SCR, the mean square error of the α quantile of PVFP at $t = 1$ are taken into account.

We now decompose the mean square error of α-quantile into variance and squared bias:

$$\mathbb{E}\left[(\hat{q}_\alpha(\tilde{u}) - q_\alpha(V_1))^2\right] = \mathbb{V}ar[\hat{q}_\alpha(\tilde{u})] + \left(\mathbb{E}\left[\hat{q}_\alpha(\tilde{u}) - q_\alpha(V_1)\right]\right)^2. \tag{6.14}$$

In order to get the approximation of bias and variance, the random error as well as the joint distribution to V_1 need to be clarified. We then follow the analysis of Gordy and Juneja (2010). First of all, the number of inner scenarios are set to be constant for simplicity, i.e. $K_1^{(i)} = K_1$.[3] As we describe before, the $V_1(Y_1, D_1)$ needs to be estimated via inner simulations, in which each inner scenario k gives an unbiased estimate of it. Then the $V_1(Y_1, D_1)$ could be approximated by taking the average over all inner scenarios:

$$\tilde{V}_1(Y_1, D_1) = \frac{1}{K_1} \sum_{k=1}^{K_1} \sum_{t=1}^{T} \exp\left(-\int_1^t r_u^{(k)}\, du\right) X_t^{(k)} \Big| (Y_1, D_1). \tag{6.15}$$

It is associated with a zero mean random error $\epsilon_{K_1}(Y_1, D_1) \equiv \tilde{V}_1(Y_1, D_1) - V_1(Y_1, D_1)$. By the law of large numbers,

$$\tilde{V}_1(Y_1, D_1) \to V_1(Y_1, D_1) \text{ and } \epsilon_{K_1}(Y_1, D_1) \to 0 \text{ as } K_1 \to \infty. \tag{6.16}$$

[3]Broadie et al. (2011) call this kind of nested simulation as uniform sampling and they propose so called sequential sampling nested simulation by choosing non constant number of inner scenarios sequential under certain criteria.

Define $\tilde{\epsilon}(K_1) := \epsilon(K_1)\sqrt{K_1}$ and let $g_N(\cdot,\cdot)$ be the joint probability density function of V_1 and $\tilde{\epsilon}(K_1)$. According to the Proposition 2 of Gordy and Juneja (2010), the bias and variance could be approximated as:

$$\mathbb{E}\left[\hat{q}_\alpha(\tilde{u})\right] - q_\alpha(V_1) = \frac{\theta(\alpha)}{K_1 f(q_\alpha(V_1))} + o_{K_1}(1/K_1) + O_N(1/N) + o_{K_1}(1)O_N(1/N), \quad (6.17)$$

$$\mathbb{V}ar\left[\hat{q}_\alpha(\tilde{u})\right] = \frac{\alpha(1-\alpha)}{(N+2)f^2(q_\alpha(V_1))} + O_N(1/N^2) + o_{K_1}(1)O_N(1/N), \quad (6.18)$$

where f is the density function of V_1 and

$$\begin{aligned}\theta(\alpha) &= -\frac{1}{2}\int_{-\infty}^{\infty} z^2 \frac{\partial}{\partial u} g_N(u,z)dz\bigg|_{u=q_\alpha(V_1)} \\ &= -\frac{1}{2}\frac{d}{du}\big[f(u)\mathbb{E}\left[\mathbb{V}ar(\tilde{\epsilon}(K)|(Y_1,D_1))|V_1=u\right]\big]\bigg|_{u=q_\alpha(V_1)}. \end{aligned} \quad (6.19)$$

Equation (6.17) shows that the approximation of bias is based on the number of inner simulation K_1 as well as the number of outer simulation N, since the bias turns to zero if $K_1 \to \infty$ and $N \to \infty$. However, the approximation of variance in (6.18) is mainly based on the number of outer simulation N, since the variance turns to zero if $N \to \infty$.

Gordy and Juneja (2010) then determine the optimal K_1 and N for the computational budget allocation by minimizing the mean square error. Bauer et al. (2009) apply the same methodology for the estimation of SCR by considering the number of risk neutral simulation at $t = 0$ for the calculation of MCEV. The result is that one can use larger number of outer scenarios and relative smaller inner scenarios to perform the nested simulation.

Finally we summarize the procedure of nested simulation as follows:

1. Generate N outer scenarios under real world measure up to time $t = 1$.
2. For each outer real world scenario i:
   - Generate $K^{(i)}$ inner scenarios under risk neutral measure.
   - For each inner risk neutral scenario k, compute the sum of discounted future profits $PV_1^{(i,k)}$.
   - Evaluate the PVFP at $t = 1$ conditional on outer scenario i, $\widehat{V}_1^{(i)}(K^{(i)})$, by taking the average of $PV_1^{(i,k)}$ for $k = 1,\dots,K^{(i)}$ over all inner risk neutral scenarios.

6.3. Proxy approaches

In Section 6.2 we have discussed the nested simulation. Although one can choose optimal number of outer and inner scenarios by minimizing the mean square of error for the resultant risk measure, it still requires quite high computational effort and is not suitable to apply by achieving the results in required deadline. In this section, we discuss so called proxy approaches to make the calculation faster and hence more practical. There are several proxy approaches applied in the insurance industry, i.e.

- curve fitting
- least-square Monte-Carlo
- replicating portfolio

All the proxy approaches are based on finding a linear combination of basis functions to approximate the PVFP. Let $e_k(Y_1, D_1)$ be the k-th basis function, then the finite linear combination of basis functions VA_1 is used to approximate value of V_1, i.e.

$$V_1 \approx VA_1 = \sum_{k=1}^{M} \beta_k e_k(Y_1, D_1) \tag{6.20}$$

where M is the number of basis functions. The basis functions for replicating portfolio are financial assets. The basis functions for curve fitting and least square Monte-Carlo are the risk factors.

In the following, we describe in more detail of these proxy methods.

6.3.1. Curve fitting

The insurance company uses stochastic cash flow projection model (e.g. Prophet ALS) to calculate the PVFP. The corresponding inputs are risk neutral scenarios consisting of all kinds of risk factors. Therefore, we can approximate the valuation function by a fitted function of risk factors. Let $RF_t(Y_t, D_t) \in \mathbb{R}^n$ be all the input risk factors conditional on (Y_t, D_t) for the calculation of PVFP. Since the calculation of PVFP in the stochastic cash flow projection model only depends on the risk factors, we have

$$V_0 \approx VA_0^{CF} = \sum_{k=1}^{M} \beta_k e_k(RF_0(Y_0, D_0)) = \sum_{k=1}^{M} \beta_k e_k(RF_0) \tag{6.21}$$

$$V_1 \approx VA_1^{CF} = \sum_{k=1}^{M} \beta_k e_k(RF_1(Y_1, D_1)) = \sum_{k=1}^{M} \beta_k e_k(RF_1), \tag{6.22}$$

under the assumption that the stochastic model does not change from $t = 0$ to $t = 1$.

We now describe how to run the curve fitting step by step. First of all, we generate N^{CF} (small, e.g. 50) outer scenarios under the real world measure $\mathbb{P}$ form $t = 0$ to $t = 1$. Subsequently, for each outer scenario i, we change from the real world measure $\mathbb{P}$ to risk neutral measure $\mathbb{Q}$, and generate K_1 inner scenario from $t = 1$ to $t = T$. Afterwards, we compute the realized sum of discounted future profits $\widehat{V}_1^{(i)}(K_1)$. Then we determine the coefficients $\beta = (\beta_1, \ldots, \beta_M)$ by running the curve fitting:

$$\hat{\beta} = \operatorname{argmin}_{\beta \in \mathbb{R}^M} \left\{ \sum_{i=1}^{N^{CF}} \left(\widehat{V}_1^{(i)}(K_1) - \sum_{k=1}^{M} \beta_k e_k(RF_1^{(i)}) \right)^2 \right\} \tag{6.23}$$

Finally, the estimated coefficients $\hat{\beta}$ are used to approximate V_1:

$$V_1 \approx VA_1^{CF}(RF_1) \approx \widehat{VA}_1^{CF}(RF_1) = \sum_{k=1}^{M} \hat{\beta} e_k(RF_1). \tag{6.24}$$

In practice, one could use predefined stressed scenario sets at $t = 0$ by changing one risk factor (e.g yield curve, equity volatility, swaption volatility) instead of generating new real world scenarios to represent the different combination of risk factors. Therefore, we could generate $N^{CF,Sensi}$ risk neutral stressed scenario sets at $t = 0$. Subsequently, for each stressed scenario set i, we generate K_0 inner scenario from $t = 0$ to $t = T$. Afterwards, we compute the realized sum of discounted future profits $\widehat{V}_0^{(i)}(K_0)$. Then we determine the coefficients $\beta = (\beta_1, \ldots, \beta_M)$ by running the curve fitting:

$$\hat{\beta} = \operatorname{argmin}_{\beta \in \mathbb{R}^M} \left\{ \sum_{i=1}^{N^{CF,Sensi}} \left(\widehat{V}_0^{(i)}(K_0) - \sum_{k=1}^{M} \beta_k e_k(RF_0^{(i)}) \right)^2 \right\} \tag{6.25}$$

Finally, we use the estimated coefficients $\hat{\beta}$ to get the approximation of V_0 and V_1:

$$V_0 \approx VA_0^{CF}(RF_0) \approx \widehat{VA}_0^{CF}(RF_0) = \sum_{k=1}^{M} \hat{\beta} e_k(RF_0) \tag{6.26}$$

$$V_1 \approx VA_1^{CF}(RF_1) \approx \widehat{VA}_1^{CF}(RF_1) = \sum_{k=1}^{M} \hat{\beta} e_k(RF_1). \tag{6.27}$$

Note that there is usually no linear relationship between risk factors and the PVFP. It is not clear to which risk factors to be used and which function to be used for the interpolation. Furthermore, it often requires cross terms among risk factors due to the complexity of the stochastic liability model. In practice first and second of cross terms are considered only.

One example of curve fitting is the Swiss Solvency Test (SST) standard formula applies Delta-(Gamma) curve fitting (see FINMA (2012)). It approximates the V_1 by the second order Taylor expansion as follows:

$$\begin{aligned} V_1 = V(RF_1) \approx & V(RF_0) + \sum_{i=1}^{d} \frac{\partial V(RF_0)}{\partial RF^i}(RF_1^i - RF_0^i) \\ & + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 V(RF_0)}{\partial RF^i RF^j}(RF_1^i - RF_0^i)(RF_1^j - RF_0^j). \end{aligned} \tag{6.28}$$

The coefficients are the first and second derivatives, which represent the sensitivities of V with respect to the risk factors. These could be estimated through sensitivity analysis, i.e. using the change in V with respect to changes in the risk factors. Therefore, numbers of stressed (sensitivity) scenario sets should be constructed. Technical description of the Delta-Gamma approximation is referred to Cardi and Rusnak (2007) and the formal procedure is referred to FINMA (2012).

In the end we also summarize the procedure of curve fitting approach as follows:

1. Generate N outer scenario sets (risk neutral scenario sets at time $t = 1$ based on the recalibration of realized real world path up to time $t = 1$, or stressed scenario sets)
2. For each outer scenario i:
   - Generate K inner scenarios.
   - Evaluate the PVFP by taking the average of the sum of discounted cash flows over all inner scenarios.
3. Determine the coefficients $\beta = (\beta_1, \ldots, \beta_M)$ by running the curve fitting.
4. Calculate the approximated value of V_1.

6.3.2. Least Square Monte Carlo

The least square Monte Carlo approach was first introduced by Longstaff and Schwartz (2001), who use least squares regression on a countable set of basis functions to approximate the conditional expectation. Bauer et al. (2009) apply the idea and propose a faster approach for the calculation of required risk capital under Solvency II.

Assume that the conditional expectation V_1 is an element of the Hilbert space

$$L^2(\Omega, \sigma(Y_1, D_1), \mathbb{P}),$$

then it has a countable orthonormal basis and the conditional expectation could be represented as a linear function of the elements of the basis (see Longstaff and Schwartz (2001) and Bauer et al. (2009)). Therefore, one can replace the conditional expectation in (6.7) for calculation of V_1 by a finite combination of basis functions as follows:

$$V_1 \approx VA_1^{LSMC}(Y_1, D_1) = \sum_{k=1}^{M} \beta_k e_k(Y_1, D_1) \tag{6.29}$$

where the sequence of basis functions $(e_k(Y_1, D_1))$ is assumed to linearly independent (orthogonal) and complete on the Hilbert space.

We now describe how to run the least square Monte Carlo step by step. First of all, we generate N outer scenarios under the real world measure $\mathbb{P}$ form $t = 0$ to $t = 1$. Subsequently, for each outer scenario i, we change back from the real world measure $\mathbb{P}$ to risk neutral measure $\mathbb{Q}$, and then extend the path by generating an inner scenario from $t = 1$ to $t = T$. Afterwards, we compute the realized sum of discounted future profits $PV_1^{(i)}$ of this path. Then we determine the coefficients $\beta = (\beta_1, \ldots, \beta_M)$ by running the least square regression:

$$\hat{\beta}^{(N)} = \operatorname{argmin}_{\beta \in \mathbb{R}^M} \left\{ \sum_{i=1}^{N} \left(PV_1^{(i)} - \sum_{k=1}^{M} \beta_k e_k(Y_1^{(i)}, D_1^{(i)}) \right) \right\} \tag{6.30}$$

Finally, we use the estimated coefficients $\hat{\beta}^{(N)}$ to get the approximation of V_1:

$$V_1 \approx VA_1^{LSMC}(Y_1, D_1) \approx VA_1^{LSMC,(N)}(Y_1, D_1) = \sum_{k=1}^{M} \hat{\beta}_k^{(N)} e_k(Y_1, D_1) \tag{6.31}$$

In addition, we have the following convergence for the approximation calculated by the least square Monte Carlo:

$$\begin{aligned} VA_1^{LSMC} &\to V_1 \text{ as } M \to \infty, \text{ and} \\ VA_1^{LSMC,(N)} &\to VA_1^{LSMC} \text{ as } N \to \infty. \end{aligned}$$

For the proof we refer to Proposition 3.1 of Bauer and Ha (2013).

In the end we also summarize the procedure of least square Monte Carlo approach as follows:

1. Generate N outer scenarios under real world measure up to time $t = 1$.
2. For each outer real world scenario i:
   - Change back from the real world measure $\mathbb{P}$ to risk neutral measure $\mathbb{Q}$.
   - Extend the path by generating an inner scenario from $t = 1$ to $t = T$ under risk neutral measure.
   - Compute the sum of discounted future profits $PV_1^{(i,k)}$.
3. Determine the coefficients $\beta = (\beta_1, \ldots, \beta_M)$ by running the least square regression.
4. Calculate the approximated value of V_1 by $VA_1^{LSMC,(N)}(Y_1^{(i)}, D_1^{(i)})$.

6.3.3. Replicating portfolio

The replicating portfolio consists of a set of financial assets could be used as a computationally efficient proxy to evaluate the PVFP under real world at $t = 1$. It is motivated by no arbitrage pricing that if one portfolio of assets (e.g. bond and stock) has identical cash flows with a given asset (e.g option) in all states, then we say this given asset could be replicated by the portfolio of assets. This indicates a replicating portfolio could be used to approximate the original liability of a life insurance company that sells life insurance contracts containing embedded options and guarantees, if this portfolio could replicate the liability cash flows in all states.

An introduction of the approach is given by Oechslin et al. (2007). Oechslin et al. (2007) illustrate how to use replicating portfolio to the market consistent valuation and management of option and guarantees embedded in the life insurance contracts. Given a pool of candidate assets that could be evaluated easily, they find a linear combination of these assets in order to replicate the liability cash flows. The weights for the linear combination of assets could be determined by minimizing a metric, which measures the distance between the cash flows of replicating portfolio and the original liability. Furthermore, some constraints could also be incorporated in the optimization of the weights to force market-consistent and the certainty-equivalent values of the original cash flows of liability could be matched exactly by the replicating portfolio. Oechslin et al. (2007) finally give a case study based on a real traditional life insurance portfolio consisting of products such as single and regular premium endowments, deferred annuities and term insurances, which are mostly profit sharing with policyholders.

Boekel et al. (2009) give an introduction of replicating portfolio as well. They illustrate how to derive the replicating portfolio, including the determination of the replication methods, selection of economic scenarios, definition of asset pool and practical constrains as well as the fitting methodology and criteria.

Seemann (2009) follows the framework proposed by Oechslin et al. (2007) for the constructing of replicating portfolio. The replicating portfolio approach is applied to German products such as single premium endowment with profit-sharing and interest rate guarantee as well as combining the surrender and mortality risk. In addition, Erixon and Tubis (2008) and Kalberer (2007) use replicating portfolio approach for the hedging and valuation of unit-link products with investment guarantee.

Burmeister and Mausser (2009) and Burmeister et al. (2010) focus on constructing a replicating portfolio including a relative smaller number of assets. An effective replicating portfolio should not only match closely to the original liability under various market conditions but also be quickly priced and easily interpret the relation to the liability. Clearly, small portfolio is easier to meet the above criteria and is more robust to the unanticipated or stressed market conditions. Burmeister and Mausser (2009) and Burmeister et al. (2010) apply the trading restrictions as further constrains to get a relative smaller effective replicating portfolio.

Let $C = \{C_1, \ldots, C_K\}$ be the asset pool of financial assets, where C_k be the k-th candidate asset in the asset pool. Let CF_{t,C_k} be the cash flow generated by the candidate C_k at time t.

Let $G \subseteq C$ be a subset of C, which represents the set of assets in a selected replicating portfolio. Let $w^G = \{w_k\}$ for $C_k \in G$ be the weights of the candidate assets in the selected replicating portfolio, where w_k is the weight for the candidate C_k in G.

Let $Z^G = (Z_1^G, \ldots, Z_T^G)$ be the cash flows of the replicating portfolio, where Z_t^G is the cash flow of the replicating portfolio at time t. It is calculated as the weighted sum of cash flows of candidate assets in the replicating portfolio, i.e. $Z_t^G = \sum_{k \in G} w_k CF_{t,C_k}$. We consider to replicate the cash flows of future profits by the cash flows generated by a portfolio of financial assets, i.e. we replicate $X = (X_1, \dots, X_T)$ by Z^G:

$$\begin{aligned} V_1 &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \middle| (Y_1, D_1)\right] \\ &\approx \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) Z_t^G \middle| (Y_1, D_1)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) \left(\sum_{k\in G} w_k CF_{t,C_k}\right) \middle| (Y_1, D_1)\right] \\ &= \sum_{k\in G} w_k \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_1^t r_u\,du\right) CF_{t,C_k} \middle| (Y_1, D_1)\right]}_{:=V_1^{C_k}(Y_1,D_1)}. \end{aligned} \tag{6.32}$$

Therefore, the PVFP at $t = 1$ could be approximated by the replicating portfolio G:

$$V_1 \approx VA_1^{RP} = \sum_{k\in G} w_k V_1^{C_k}(Y_1, D_1). \tag{6.33}$$

Analog to $t = 1$, we could replicate the V_0 by the replicating portfolio G:

$$V_0 \approx VA_0^{RP} = \sum_{k\in G} w_k V_0^{C_k}(Y_0, D_0). \tag{6.34}$$

The objective is to find out an optimal replicating portfolio G^* and the corresponding optimal weights, which can match the cash flows X as well as possible, i.e.

$$\min_{w^G,\ G} d(X, Z^G) \tag{6.35}$$

where d is the norm that measures the distance between X and Z^G. For instance, one can choose d as L_2-norm.

According to no arbitrage pricing, if a portfolio of assets has the same cash-flows as another asset or liability, then their price has to be the same for there to be no arbitrage. If not, there exists arbitrage opportunity. Therefore, if the cash flows of replicating portfolio Z^G are identical with the cash flows of future profits X, then VA_1^{RP} equals to V_1. The more detailed theoretical foundation of replicating portfolio could be seen in Natolski and Werner (2017).

7. Replicating portfolio

7.1. General strategy

We have introduced the replicating portfolio in Section 6.3.3 as the approximation of PVFP for the calculation of SCR. In this chapter, we develop a general strategy and describe step by step how to construct a 'good' replicating portfolio.

So what is a 'good' replicating portfolio? Are there any criteria for judgment of the quality of replicating portfolio? Of course the answer is yes. As we described before the motivation of replicating portfolio is used to replicate the cash flows of future profits. Hence, the first criterion is the quality of cash flow matching. If the cash flow could be replicated very well, then we say the replicating portfolio is a computationally efficient proxy to the PVFP. Therefore, the second criterion is the (in sample) calibration error for determining the replicating portfolio. After the calibration of replicating portfolio, it is further applied to approximate the PVFP under real world at $t = 1$ for the estimation of SCR. Therefore, the next criterion is that the estimation error of SCR. Furthermore, with sufficient large pool of candidate assets, the likelihood of getting an effective replicating portfolio would be higher. However one should not select all these assets into the replicating portfolio, which is not robust and usually causes the problem of over fitting and large long short positions or offsetting effects. Consequently the further two criteria are the robustness and offsetting effects. We now summarize all the criteria as follows:

(i) the quality of cash flow matching

(ii) the calibration error

(iii) the estimation error of SCR

(iv) the robustness and over fitting

(v) long short positions and offsetting effects

For criterion (i), in order to match the cash flows of future profits, we create a sufficient large pool of candidate assets containing different kinds of assets [1] to increase the likelihood of cash flow matching. However, in practice it is quite challenging for replicating the cash flows in each year for each risk neutral scenario since the cash flows

[1] Typically the candidate assets are functions in one risk factor only. However, the cash-flows which need to be approximated are typically functions of whole portfolios of risk factors. For instance, the embedded option and guarantees are based on whole portfolio. Therefore, the cross term of financial assets could be introduced.

might be complicated due to options and guarantees, dynamic surrender values and costs etc. Actually we are more interested in the approximation quality of PVFP, and hence it is reasonable to aggregate the cash flows over all years, i.e. match the sum of discounted cash flows.

The criterion (ii) is highly related to the criterion (i). If the present value of cash flows could be matched very well, then the calibration error would be small and hence the fitting quality is good. In addition, the fitting quality should also be validated by the out of sample test. Note that besides the basis set, more stressed sensitivity risk neutral scenario sets might be taken into account as calibration scenarios. Furthermore, we apply the subset selection procedure to select the most significant candidate assets into the replicating portfolio.

The calibration scenarios consists of basis scenario set (or more sensitivity scenario sets as well) might still not be able to cover the evolution of possible financial market states in future. In such case, though the criterion (ii) could be fulfilled with small calibration errors, it does not mean that it leads to small estimation error of SCR. Therefore, in order to get smaller estimation error of SCR as for criterion (iii), we need to first analyze the real world scenarios of risk factors (or historical time series of risk factors) such as yield curve, swaption etc. Afterwards, we construct (artificial) risk neutral sensitivity scenario sets to cover the possible financial market states and combine these scenario sets as the calibration scenarios. Note that the construction of artificial sensitivity risk neutral scenario sets usually requires new calibration in practice or applies weighted Monte-Carlo techniques on the risk neutral basis set (see Avellaneda et al. (2001))

For criterion (iv), in order to avoid over fitting to get more robust results, we limit the number of candidate assets in the replicating portfolio. For criterion (v), high collinearity of candidate assets in asset pool could lead to strong offset effect if one longs and shorts these assets simultaneously. Therefore, one should avoid large long short positions, especially on assets with high multi-collinearity. For the reduction of large long-short positions, one could apply shrinkage method e.g. LASSO by setting constraints on the weights or simply define selection criterion when performing the subset selection.

With the consideration of the above criteria, we propose the following general strategy to get a ‘good’ replicating portfolio and afterwards the estimation of SCR:

1. Construct a pool of standard financial assets.
2. Construct the calibration scenarios.
3. Perform the calibration procedure:
   - Define the cost function of the optimization problem for determining the weights of replicating portfolio.
   - Optimization procedure: (weighted) least square optimization, mixed integer quadratic programming, subset selection techniques, shrinkage method etc.
   - Out of sample test.
4. Select the replicating portfolio with given criteria.

5. Evaluate the PVFP at $t = 1$ conditional on each real world scenario.
6. Estimate the SCR.

7.2. Construction of the pool of financial assets

In this section, we construct the asset pool with different kinds of assets, which might be selected into the replicating portfolio. The purpose of replicating portfolio is that using a linear combination of candidate assets to match the liability cash flows. That means the cash flows of liability could be decomposed into simple cash flows generated by the standard financial assets. For instance, the cash flows of immediate annuity or endowment products could be represented as zero coupon bonds, the minimum interest rate guarantee products as interest rate option, unit-link products linked to equity portfolio as the equity index and options. Furthermore, the candidate assets in the asset pool should be priced in closed form in order to evaluate the replicating portfolio very fast.

Therefore, the pool of financial assets should ideally include all types of assets (traded and synthetic assets) that could replicate the cash flows of future profits. Since these cash flows might depend on the risk factors such as interest rates, equity etc, the assets based on these risk factors are taken into account and listed as below:

- interest rate related assets
  - risk free zero coupon bonds with different maturities
  - total return indices of zero coupon bond
  - total return indices of constant maturity zero coupon bond
  - Interest rate (receiver) swaps with different strikes
  - (Receiver) swaption with different option expiries and swap tenors
- equity related assets
  - total return indices of equity at different years
  - European (put) options with different option maturities and strikes

Note that we choose receiver swaption and European put option, since the life insurance product with guaranteed interest rate is more likely to be put style option from shareholder's point of view. The mathematical definition of these financial assets and the corresponding cash flow payments could be seen in the Appendix A.

7.3. Construction of calibration scenarios

7.3.1. Construction of sensitivity scenario sets through recalibration

In order to do the sensitivity analysis of MCEV or the estimation of SCR, the sensitivity scenario sets would be used. The sensitivity scenario sets could be e.g.:

- interest rate +50bps
- interest rate -50bps
- swaption volatility +25%
- equity volatility +25%

Given the stressed market input data, the recalibration is usually required. In practice, it costs some time for the recalibration of all ESG models and generates the new scenario set by fulfilling all the criteria such as martingale tests.

Alternatively, one could construct sensitivity scenario sets by changing specific calibrated parameters in order to avoid recalibration.

7.3.2. Construction of artificial scenario set through reweighting

Avellaneda et al. (2001) use the weighted Monte Carlo to correct the price mis-specifications in the simulation by assigning "probability weights" to the simulated paths. Hörig and Wechsung (2014) follow this re-weighting technique for targeting any arbitrary volatility assumptions. Here we apply this idea to construct a new sensitivity scenario set by re-weighting the risk neutral scenarios of basis or sensitivity scenario set.

Let $\mathbb{Q}$ be the risk neutral measure and H be the attainable contingent claim with maturity time T, then the fair price of H could be given by the risk neutral valuation formula:

$$\pi_H(0) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_u\,du\right)H\right] = \mathbb{E}^{\mathbb{Q}}\left[PV_H\right], \tag{7.1}$$

where the PV_H is the present value of H. Then $\pi_H(0)$ could be approximated by Monte Carlo simulation as

$$\pi_H(0) \approx \frac{1}{K}\sum_{i=1}^{K}\exp\left(-\int_0^T r_u^{(i)}\,du\right)H^{(i)} = \frac{1}{K}\sum_{i=1}^{K} PV_H^{(i)}. \tag{7.2}$$

Let $\widetilde{\mathbb{Q}}$ be a new risk neutral measure which is absolutely continuous with respect to $\mathbb{Q}$. According to the Radon-Nikodým Theorem (see Williams (1991, p. 145)) there exists an (nonnegative) $\mathcal{F}$-measurable function (random variable) g such that

$$\widetilde{\mathbb{Q}}(A) = \int_A d\widetilde{\mathbb{Q}} = \int_A g d\mathbb{Q} \quad \forall A \in \mathcal{F} \tag{7.3}$$

where $g = \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$ is called the Radon-Nikodým derivative and

$$\mathbb{E}^{\mathbb{Q}}[g] = \int g d\mathbb{Q} = \int \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} d\mathbb{Q} = \int d\widetilde{\mathbb{Q}} = 1. \tag{7.4}$$

The fair price of H under risk neutral measure $\widetilde{\mathbb{Q}}$ is:

$$\widetilde{\pi}_H(0) = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[PV_H\right] = \int PV_H\,d\widetilde{\mathbb{Q}} = \int PV_H \cdot g\,d\mathbb{Q} = \mathbb{E}^{\mathbb{Q}}\left[PV_H \cdot g\right]. \tag{7.5}$$

Consequently, $\widetilde{\pi}_H(0)$ could be approximated as

$$\widetilde{\pi}_H(0) \approx \frac{1}{K}\sum_{i=1}^{K} PV_H^{(i)} g^{(i)}, \tag{7.6}$$

where $\frac{1}{K}\sum_{i=1}^{K} g^{(i)} = 1$ since $\mathbb{E}^{\mathbb{Q}}[g] = 1$. Therefore, we could see that the fair price of H under risk neutral measure $\widetilde{\mathbb{Q}}$ could be estimated by averaging the present values over all risk neutral scenarios under $\mathbb{Q}$ multiplied by the respective weights.

Let $C_1, \ldots, C_N$ be the target (stressed market) prices of financial assets for the sensitivity scenario set representing the stressed financial market state. In general, given the target prices, the re-calibration process should be performed and a new risk neutral scenario set should be generated through Monte-Carlo simulation under the calibrated risk neutral measure $\widetilde{\mathbb{Q}}$. However, as we described before, we can apply the weighted Monte Carlo techniques to avoid the re-calibration and re-generation of scenarios.

Let $PV_C^{(i,j)}$ be the present value of cash flows for the risk neutral scenario i of the j-th financial asset. We need to determine the weights $(g^{(1)}, \ldots, g^{(K)})$ such that the expected value of the present values should coincide (either exactly or within tolerance) target price of j-th financial asset, i.e. the price relationship as follows holds:

$$C_j = \frac{1}{K}\sum_{i=1}^{K} g^{(i)} PV_C^{(i,j)} = \sum_{i=1}^{K} p_i PV_C^{(i,j)} \text{ for } j = 1, \ldots, N, \tag{7.7}$$

where p_i is the normalized version of $g^{(i)}$, i.e. $p_i = g^{(i)}/K$.

In general, there are many solutions for the linear equations (7.7) since usually the number of simulation K is larger than the number of target financial assets N. We could follow the criterion proposed by Avellaneda et al. (2001) for finding the weights by minimizing the Kullback-Leibler relative entropy between the posterior measure with non-uniformly sampling and the prior measure with uniformly sampling. The prior corresponds to the information available for the risk neutral measure $\mathbb{Q}$ while the posterior reconciles the prior information with the target prices. It means that besides fulfilling price relationship (7.7), we try to keep the weights for non-uniformly sampling as close as possible to the uniformly sampling. Therefore under linear constraints (7.7), the weights are determined by minimizing the relative entropy:

$$D(p|q) = \sum_{i=1}^{k} p_i \log(\frac{p_i}{q_i}) = \log K + \sum_{i=1}^{K} p_i \log p_i \tag{7.8}$$

where $q_i = 1/K$ for uniformly sampling. Avellaneda et al. (2001) reformulate this optimization problem by a min-max program as follows by introducing the Lagrange multipliers:

$$\min_{\lambda} \max_{p} \left\{ -\log K - \sum_{i=1}^{K} p_i \log p_i + \sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{K} p_i PV_C^{(i,j)} - C_j \right) \right\} \tag{7.9}$$

The maximum of p for each λ is realized by Boltzmann-Gibbs form:

$$p_i = \frac{1}{\zeta(\lambda)} \exp\left(\sum_{j=1}^{N} PV_C^{(i,j)} \lambda_j\right) \tag{7.10}$$

where $\zeta(\lambda)$ is the normalized factor defined as $\zeta(\lambda) = \sum_{i=1}^{K} \exp\left(\sum_{j=1}^{N} PV_C^{(i,j)} \lambda_j\right)$. Then the optimization turns to minimize:

$$\min_{\lambda} \left\{ \log(\zeta(\lambda)) - \sum_{j=1}^{N} \lambda_j C_j \right\}. \tag{7.11}$$

Equation (7.11) is a nonlinear optimization program which could be solved by L-BFGS, simplex method etc. Note that it is local optimization and might convergence to a local minimum.

In addition, it is possible that there are no solution for the linear equations (7.7) if $\mathbb{Q}$ does not contain enough information. Consequently, instead of matching the target prices exactly by solving linear equations (7.7), we could determine the weights by minimizing the least square errors:

$$\begin{aligned}
&\min_{g} \left\{ \left(\frac{1}{K} \sum_{i=1}^{K} g^{(i)} PV_C^{i,j} - C_j \right)^2 \right\} \\
&s.t. \\
&\quad \frac{1}{K} \sum_{i=1}^{K} g^{(i)} = 1 \\
&\quad g^{(i)} \geq 0 \text{ for } i = 1, \ldots, K.
\end{aligned} \tag{7.12}$$

It is an optimization problem that could be solved by quadratic programming.

Finally we summarize the process how to construct the artificial sensitivity set by re-weighting the given risk neutral scenario set.

1. Set the target (stressed) market input data: initial yield curve, European option prices or implied volatility, swaption implied volatilities.

2. Map these market input data to corresponding prices of financial assets
   - initial yield curve $\mapsto$ risk free zero coupon bonds
   - European option implied volatilities $\mapsto$ European option prices
   - swaption implied volatilities $\mapsto$ Swaption prices

3. Compute the present value $PV_C^{(i,j)}$ for each risk neutral scenario under $\mathbb{Q}$ of each financial asset.

4. Determine the weights for the risk neutral scenarios.

7.4. Calibration procedure

7.4.1. Definition of criteria and objective functions

In Section 6.3.3 we introduced the replicating portfolio and Section 7.1 pointed out the criteria for constructing a 'good' replicating portfolio. In this section, we now define the criteria precisely in mathematically and then combine them into the objective function.

For criterion (i) (the quality of cash flow matching), we could use a metric $d(X, Z^G)$ to measure the distance between the two random variables of cash flows, $X = (X_1, \ldots, X_T)$ and $Z^G = (Z_1^G, \ldots, Z_T^G)$. If we only consider the cash flow matching, then objective function could be given as:

$$\min_{w^G,\ G} d(X, Z^G) \tag{7.13}$$

Therefore, the objective is only to find out an optimal replicating portfolio G^* and the corresponding optimal weights, which can match the cash flows X as well as possible.

In order to consider the criterion (iii) (the estimation of SCR), a replicating portfolio that could replicate the cash flows under different market condition is preferred. Therefore, not only the basis scenario set but also the further stressed sensitivity scenario sets or artificial scenario sets should be considered as the calibration scenarios.

Let $\mathbb{Q}_j$ be a risk neutral measure derived by the calibration to different market data (market consistent or stressed market data). For instance, $\mathbb{Q}_{0,Basis}$ is the risk neutral measure derived by the calibration to market consistent data at $t = 0$. $\mathbb{Q}_{0,Stress}$ is a risk neutral measure derived by the calibration to the stressed marked data at $t = 0$, e.g. 50 basis points decrease of reference yield curve, 25% increase of swpation volatility etc. $\mathbb{Q}_{1,(i)}$ is a risk neutral measure derived by the calibration to market data at $t = 1$ conditional on the financial market development for real world path i up to $t = 1$.

Let $X^{\mathbb{Q}_j}$ and $Z^{\mathbb{Q}_j,G}$ be the corresponding random variables under the probability measure $\mathbb{Q}_j$. Then the objective function turns to be:

$$\min_{w^G,\ G} \sum_{j=1}^{J} q^{\mathbb{Q}_j} d(X^{\mathbb{Q}_j}, Z^{\mathbb{Q}_j,G}), \tag{7.14}$$

where J is the number of scenario sets and $q_{\mathbb{Q}_j}$ is the weight for different probability measures. Equation (7.14) gives the objective function for cash flow replication under different market conditions without any further constrains.

Now we denote the realization of a random variable by the corresponding lower case letter. Note that a random variable is always denoted by a upper case letter. For instance, x_t^i is the realization of X_t.

Let $x_t^{i,\mathbb{Q}_j}$ be the i-th realization of $X_t^{\mathbb{Q}_j}$ simulated under the probability measure $\mathbb{Q}_j$, $i = 1, \ldots, K^{\mathbb{Q}_j}$. $K^{\mathbb{Q}_j}$ is the number of realization for the Monte-Carlo simulation. If we choose d as the L_2 norm, then the optimization can be explicitly written as:

$$\min_{w^G,\ G} \sum_{j=1}^{J} \sum_{i=1}^{K^{\mathbb{Q}_j}} \sum_{t=1}^{T} q^{i,\mathbb{Q}_j,t} \left(x_t^{i,\mathbb{Q}_j} - z_t^{i,\mathbb{Q}_j,G} \right)^2, \tag{7.15}$$

where $q^{i,\mathbb{Q}_j,t}$ is the weights at time t for scenario i under probability measure $\mathbb{Q}_j$. Therefore (7.15) gives the weighted least square optimization of cash flow replication for different types of risk neutral scenarios representing different market conditions without any constrains.

In practice, the good matching of cash flow at each time period could not be achieved easily due to complexity of liability cash flows and the MCEV calculation mainly focus on the present value of cash flows. Furthermore, the criterion (ii) (calibration error) is also measured by the difference of PVFP and the value of replicating portfolio, $d(V_0, VA_0^{RP})$. The value of PVFP and replicating portfolio are both calculated by averaging the present values.

Therefore one can determine the replicating portfolio with objective function on matching the present value instead of matching cash flows at each period. Then the dimensions of the optimization problem could be reduced to $J \times K^{\mathbb{Q}_j}$. Note that besides using the present value, i.e. the sum of discounted cash flows, one could also use the accumulated terminal cash flows, see Oechslin et al. (2007) and Boekel et al. (2009).

Let X_{PV} and Z^G_{PV} be the present value (sum of the discounted cash flows) of future profits and the replicating portfolio C^G, i.e.

$$Z^G_{PV} = \sum_{t=0}^{T} \exp\left(-\int_0^t r_u\, du\right) Z^G_t, \tag{7.16}$$

$$X_{PV} = \sum_{t=0}^{T} \exp\left(-\int_0^t r_u\, du\right) X_t. \tag{7.17}$$

Let $x^{i,\mathbb{Q}_j}_{PV}$ be the i-th realization of X_{PV} simulated under the risk neutral measure $\mathbb{Q}_j$, $i = 1, \ldots, K^{\mathbb{Q}_j}$. Let $z^{i,\mathbb{Q}_j,G}_{PV}$ be the i-th realization of Z^G_{PV} simulated under the risk neutral measure $\mathbb{Q}_j$, $i = 1, \ldots, K^{\mathbb{Q}_j}$.

The optimization problem turns to be:

$$\min_{w^G,\ G} \sum_{j=1}^{J} \sum_{i=1}^{K^{\mathbb{Q}_j}} q^{i,\mathbb{Q}_j} \left(x^{i,\mathbb{Q}_j}_{PV} - z^{i,\mathbb{Q}_j,G}_{PV}\right)^2, \tag{7.18}$$

where $q^{i,\mathbb{Q}_j}$ be the weights for the i-th realization under risk neutral measure $\mathbb{Q}_j$.

The optimization problem could be separated in two steps, i.e. select an appropriate set of candidate assets from the asset pool and then do the linear regression to determine the weights. It is identical to the variable or subset selection problem in linear regression. Therefore, the subset selection techniques could be applied to solve (7.18). See Miller (2002).

Furthermore, in the linear regression, with the larger number of assets selected into the replicating portfolio, the overall fit will be increased. The optimization tends to produce a replicating portfolio with large number of assets, which leads to over fitting and offset effects of long short positions.

In order to control the over fitting and the long-short positions for criterion (iv) and (v), one solution is that setting cardinality constraints, i.e. specify the maximum number of assets N^G could be selected into the replicating portfolio G. Then we still need to find out the optimal G with smallest the sum of squared error. Therefore, we reformulate the optimization problem as:

$$
\begin{aligned}
&\min_{w^G,\ G} \sum_{j=1}^{J} \sum_{i=1}^{K^{\mathbb{Q}_j}} q^{i,\mathbb{Q}_j} \left(x_{PV}^{i,\mathbb{Q}_j} - z_{PV}^{i,\mathbb{Q}_j,G} \right)^2 \\
&s.t. \\
&\quad y_n \in \{0,1\} \quad n = 1, \ldots, N^{Pool} \\
&\quad \sum_{n=1}^{N^{Pool}} y_n \leq N^G,
\end{aligned} \tag{7.19}
$$

where N^{Pool} is the number of assets in the asset pool. y_i is the binary variable, $y_n = 1$ means that candidate asset C_n is selected and $y_n = 0$ means that C_n is not selected. Therefore, the optimization problem is reformulated to be a Mixed Integer Quadratic Programming (MIQP) (7.19). It could be solved by branch-and bound algorithm, see e.g. Bonami and Lejeune (2009), who use such approach to solve MIQP in the portfolio optimization problems.

The other solution is that using shrinkage method or regularization by setting upper limit on the total size of position when performing least square optimization for the replicating portfolio. Burmeister and Mausser (2009) add the trading restrictions, i.e. the linear constraints on the weights of replicating portfolios, to obtain small replicating portfolios as well as avoiding the offsetting effects of long short positions. Burmeister et al. (2010) extend the methodology by using instrument-dependent trading costs, which is more effective when constructing replicating portfolios.

$$
\begin{aligned}
&\min_{w} \sum_{j=1}^{J} \sum_{i=1}^{K^{\mathbb{Q}_j}} q^{i,\mathbb{Q}_j} \left(x_{PV}^{i,\mathbb{Q}_j} - z_{PV}^{i,\mathbb{Q}_j,G} \right)^2 \\
&s.t. \\
&\quad \sum_{k \in G} a_k |w_k| \leq b,
\end{aligned} \tag{7.20}
$$

where $a_k > 0$ represents a trading cost specific to candidate asset C_k and

$$
z_{PV}^{i,\mathbb{Q}_j,G} = \sum_{t=1}^{T} \exp\left(- \int_0^t r_u \, du \right) \left(\sum_{k \in G} w_k CF_{t,k}^{i,\mathbb{Q}_j} \right).
$$

For instance, a_k could account both correlation between PVFP and asset C_k and the ratio of standard deviations of the asset C_k and PVFP, i.e. asset with higher correlation and lower standard deviation will be less penalized.

Actually, the optimization problem (7.20) could be mapped into the shrinkage method called LASSO (Least Absolute Shrinkage and Selection Operator) introduced by Tibshirani (1996), who uses a penalty of L_1 norm of coefficients of predict variables for ordinary least squares regression. There are many methods for solving the LASSO and hence we could apply them for solving optimization problem (7.20).

Note that further constraints could be taken into consideration. For instance, Oechslin et al. (2007) mention that it makes sense that the market consistent and the certainty equivalent values of the replicating portfolio match the respective original value. So one can set some constrains for this specific purpose. Furthermore, it is also possible to add constraints to consider that the average cash flows at each time point of replicating portfolio could match the respective original PVFP cash flows.

7.4.2. Selection of candidate assets for replicating portfolio

In general all candidate assets should not be chosen into the replicating portfolio, since it leads to over fitting and larger variance in the prediction. Furthermore, a replicating portfolio should be meaningful and have clear interpretation, which is easier to be achieved with small subset of candidate assets. Therefore we only choose a subset of candidates while the coefficients of the selected candidates are estimated by least squares regression.

The selection of replicating portfolio is now turned into the subset selection with linear regression. More description of subset selection and shrinkage method could be seen in Appendix B.

8. Application

Given the mathematical descriptions in previous chapters, we illustrate the application of a constructed partial internal model for the market consistent valuation and the determination of capital requirements.

First of all, we calibrate the interest rate model and equity model to the market data. Given the calibrated parameters, we then generate the risk neutral and real world scenarios. The martingale tests are performed to validate the quality of risk neutral scenarios. The simulated and historical distributions of total returns of zero coupon bonds and equity are also compared in order to check if the real world scenarios could mimic the historical behavior, especially if the worst economic movements are captured. Finally, we perform the market consistent valuation of the life insurance portfolio and determine the risk capital based on nested simulation. Since the nested simulation requires high computational efforts in practice, the proxy method of replicating portfolio is taken into account. Given a pool of candidate assets, we illustrate the procedure how to select the optimal replicating portfolio with subset variable selection strategy and use such replicating portfolio to determine the required risk capital.

8.1. Model calibration

The ESG models are calibrated to market data. In this paper, we use the market data for the economy of EUR on the cut-off date of 12/31/2014 as an illustration.

8.1.1. Calibration of interest rate model

Before the calibration of interest rate model, we start to see the historical behavior of interest rates. Figure 8.1 shows the historical continuous compounded spot rates (bootstrapped from the swap rates quoted form Bloomberg) with different maturities.

The market data are only available from 1999 for the economy of EUR. We could see in Figure 8.1 that the interest rates are much lower after year 2010, i.e. we enter into the lower interest rate environment. The interest rates are lowest on the cut-off date during the observed period of historical data. They have all decreasing trends, especially the long term rates for the available historical data. According to the principle component analysis of the historical spot rates, there are three main components, 92.87% for the level, 6.43% for the slope, and 0.49% for the curvature, see Figure 8.2. These three factors are necessary to characterize the most of the variation of the bond returns as founded by Litterman and Scheinkman (1991).

The three factor CIR model could capture these three main principle components (see

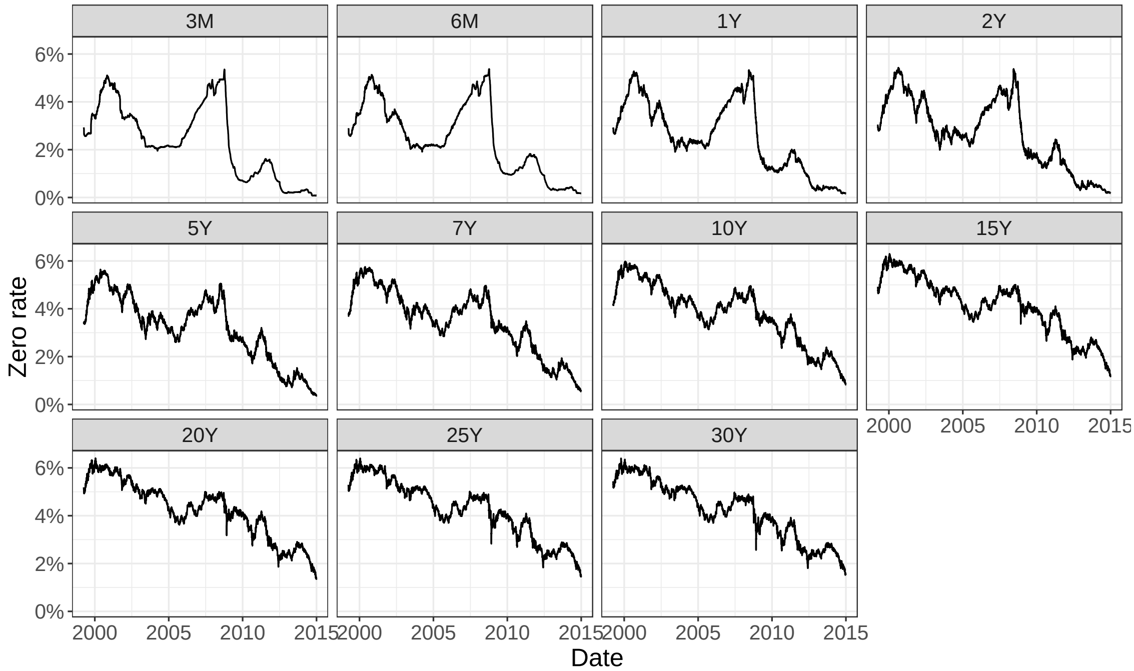

Figure 8.1.: The continuous compounded spot rates bootstrapped from historical daily data of swap rates up to 12/31/2014.

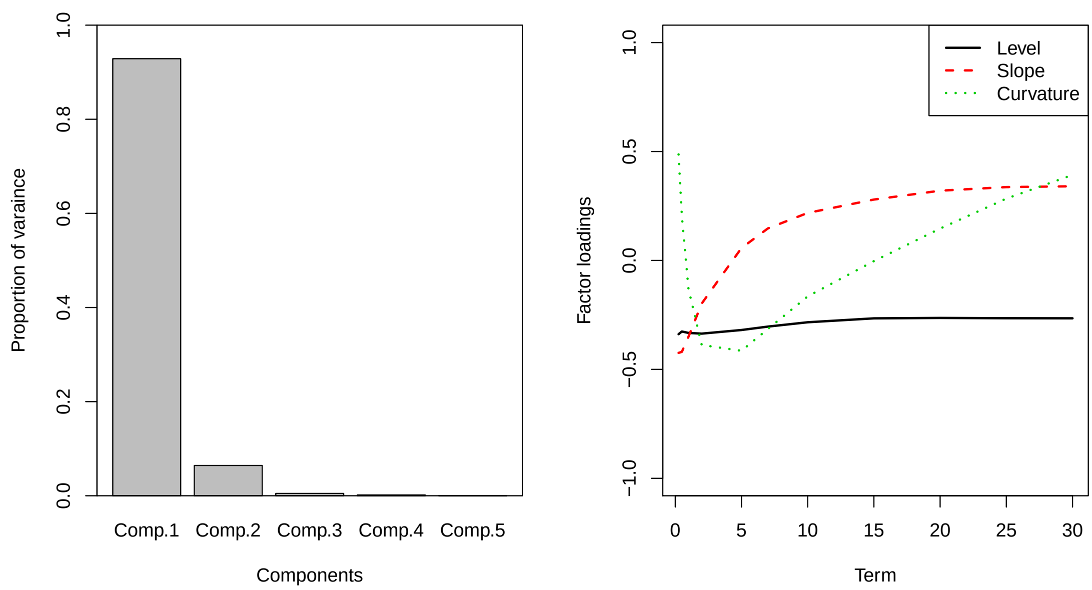

Figure 8.2.: PCA of the historical continuous compounded spot rates.

Chen and Scott (1993)). Furthermore, by using a constant shift $\delta_0 < 0$ on the short rate, it allows interest rates to be negative, which is necessary for the current low interest rate environment. Furthermore, it is plausible to assume the interest rates are mean reverting according to the economic theory. There are also statistical evidence that the short term rates revert to the long-term average yields with a strong tendency (see e.g. van den End (2013)). Though the available historical data of swap rates do not show strong evidence of mean reversion, we think it is due to short period of data and the mean reversion property for the three factor CIR model is still meaningful. Regarding to the choosing of time horizon for the real world estimation, we prefer to use longest historical data to better capture the volatility of interest rates.

For the risk neutral calibration of interest rate model on the cutoff date 12/31/2014, the following market data are used:

1. Swap rates from the deep, transparent, liquid market on the cutoff date.

2. ATM swpation implied volatilities on the cutoff date.

The swap rates quoted from Bloomberg (see Table 8.1) are used to derive the risk free yield curve on the cutoff date. In order to be most consistent with market data, we use the swap rates up to 30 years without considering volatility adjustment. Furthermore, we assume that the swap rate is risk free and there is no credit risk adjustment. For the interpolation among the missing buckets and extrapolation after 30 years, the Smith Wilson methodology is applied (see Smith and Wilson (2001)). The extrapolation parameter is optimized such that at year 60, the difference to the ultimate forward rate (4.2%) is smaller than one basis point. The risk free yield curve is then illustrated in Figure 8.3.

For capturing the interest rate volatilities, we also calibrate to the instruments of ATM swaption implied volatilities (see Table 8.2).

As we described before, the asset portfolio consists of coupon bonds with maturity up to 10 years, we do not consider the swaption with swap tenor larger than 10 years. Furthermore, the implied volatilities with larger tenor and larger option expires are more important, we therefore minimize the mean square error of market and model swaption prices that implicitly put more weights on swaption with large swap tenor and option expires.

Generally, the parameters of the interest rate model (i.e. extended three factor CIR model) could be estimated by two steps. First of all, the parameters of κ_i, θ_i, σ_i and $X_i(0)$ could be estimated by minimizing the mean squared errors between the market swaption prices and the model swaption prices. Secondly, the parameters related to the market price of risk λ_i^0 and λ_i^1 are then estimated by the method of quasi maximum likelihood using Kalman filter. However, there are many parameters for the three factor model. It is easily to convergence a local minimum which has good fit to the market swaption prices but with very poor fit to the historical data through maximum likelihood. Therefore, here we estimate the parameters based on maximizing the log-likelihood through Kalman filter as well as minimizing the sum of squared errors for the selected (Receiver-) swaption

Table 8.1.: The market data of swap rates from Bloomberg. The values are in percent.

Bloomberg Ticker	Maturity	Swap Rate
EUR003M CMPL Index	0.2500	0.0780
EUR006M CMPL Index	0.5000	0.1710
EUSA1 CMPL Index	1.0000	0.1615
EUSA2 CMPL Index	2.0000	0.1750
EUSA3 CMPL Index	3.0000	0.2211
EUSA4 CMPL Index	4.0000	0.2844
EUSA5 CMPL Index	5.0000	0.3600
EUSA7 CMPL Index	7.0000	0.5280
EUSA10 CMPL Index	10.0000	0.8120
EUSA12 CMPL Index	12.0000	0.9765
EUSA15 CMPL Index	15.0000	1.1480
EUSA20 CMPL Index	20.0000	1.3210
EUSA25 CMPL Index	25.0000	1.4120
EUSA30 CMPL Index	30.0000	1.4610

Table 8.2.: The market data of ATM swaption volatilities (Bloomberg Tickers with prefix EUSV) on 12/31/2014. The values are in percent.

	swap tenor								
option expiry	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y
1Y	345.20	93.62	84.30	80.32	74.17	66.57	57.85	46.81	43.47
2Y	192.79	88.13	77.75	72.70	68.64	60.14	53.42	44.09	41.22
3Y	136.00	82.66	73.39	67.23	62.03	54.65	49.57	41.97	39.45
4Y	108.29	74.79	67.51	61.07	56.20	50.18	46.33	39.88	37.76
5Y	86.63	65.84	59.27	54.31	50.66	46.41	43.57	37.93	36.03
7Y	58.30	50.22	47.20	44.55	42.79	39.90	39.86	35.67	33.83
10Y	42.34	41.66	41.15	40.13	39.28	36.06	38.36	34.65	32.48
15Y	37.15	37.36	37.57	37.25	36.84	34.41	37.06	32.99	30.52
20Y	34.19	36.65	36.72	36.39	35.94	33.32	35.57	31.43	28.37
25Y	35.07	34.85	34.69	34.17	33.60	31.20	33.25	29.03	26.65
30Y	33.42	34.41	34.01	33.49	32.79	29.87	31.25	28.06	25.99

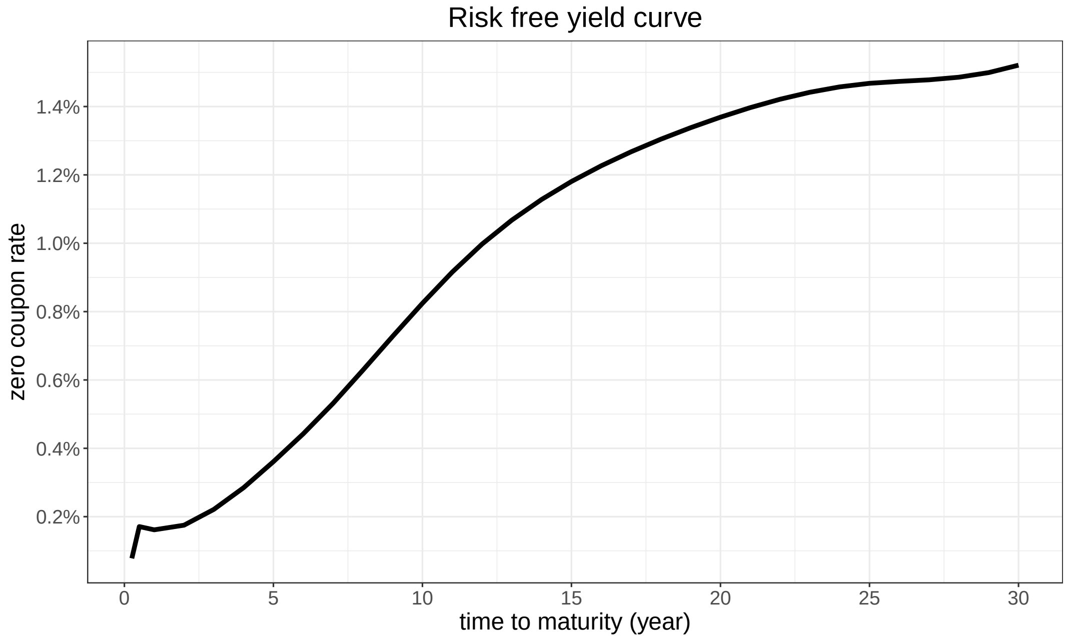

Figure 8.3.: The risk free zero coupon curve on the cut-off date of 13/31/2014.

prices. In summary, the following data are used for the calibration of the extended three factor CIR model:

- The market ATM Swaptions with option expires of 2Y, 5Y, 10Y, 15Y, 20Y and swap tenors of 2Y, 5Y, 10Y conditioned on the final date (sum of option expiry and swap ternor) between 5Y and 30Y, are chosen to be calibrated.
- The monthly historical data of continuous compounded spot rates (bootstrapped from the swap rates) with time to maturities of 3M, 6M, 1Y, 2Y, 5Y, 7Y, 10Y, 15Y, 20Y, 25Y, 30Y.

The λ_i^0 is set to be 0. [1] The target of long term mean reversion is assumed to be 4.2%. The shift term $\delta(t)$ of extended three factor CIR model is determined such that the model initial yield curve fits the market initial risk free yield curve on the cutoff date exactly.

The calibrated results are illustrated in Figure 8.4 and Figure 8.5. Figure 8.4 gives the comparison of historical observed spot rate and the forecasted spot rate through Kalman filter. We could see that forecasted spot rates are quite comparable to the historical observed spot rates for almost all terms, with only small deviation in terms of 1 and 2 years. Additionally, Figure 8.5 shows the comparison of the model calculated ATM swaption implied volatilities and prices based on the calibrated parameters and the market observed values with selected option expiries and swap tenors . We see that the

[1]If we set these values to be non zero, the fitting results are only improved slightly. Therefore, we set them to be zero to keep simplicity.

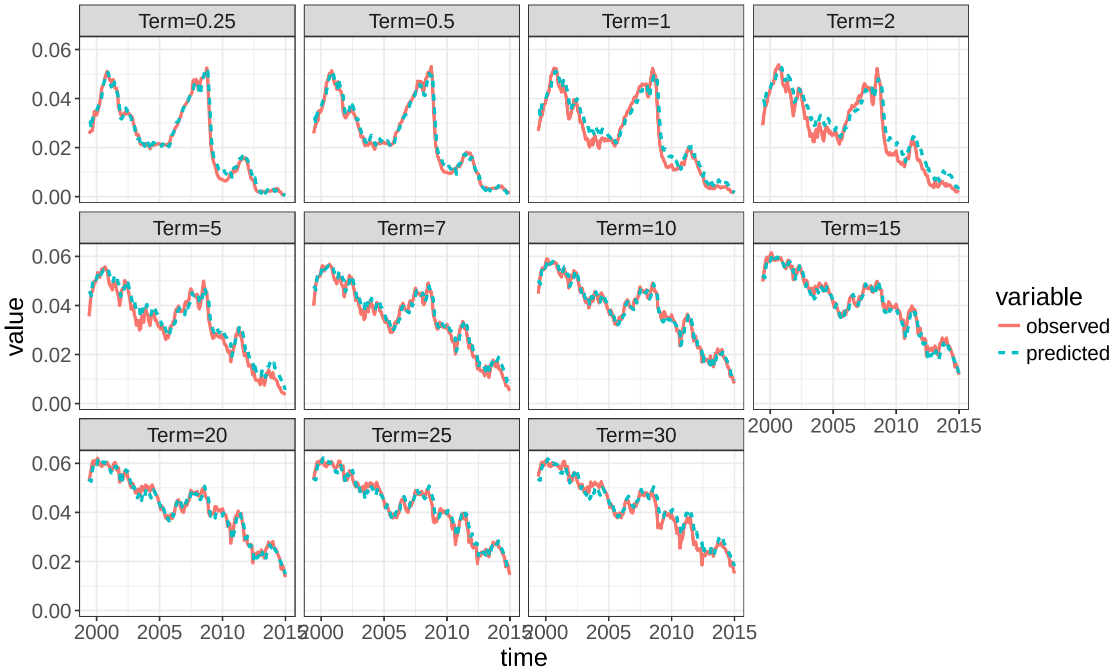

Figure 8.4.: The comparison of the observed spot rates and predicted spot rates by the Kalman Filter for the selected terms (time to maturities).

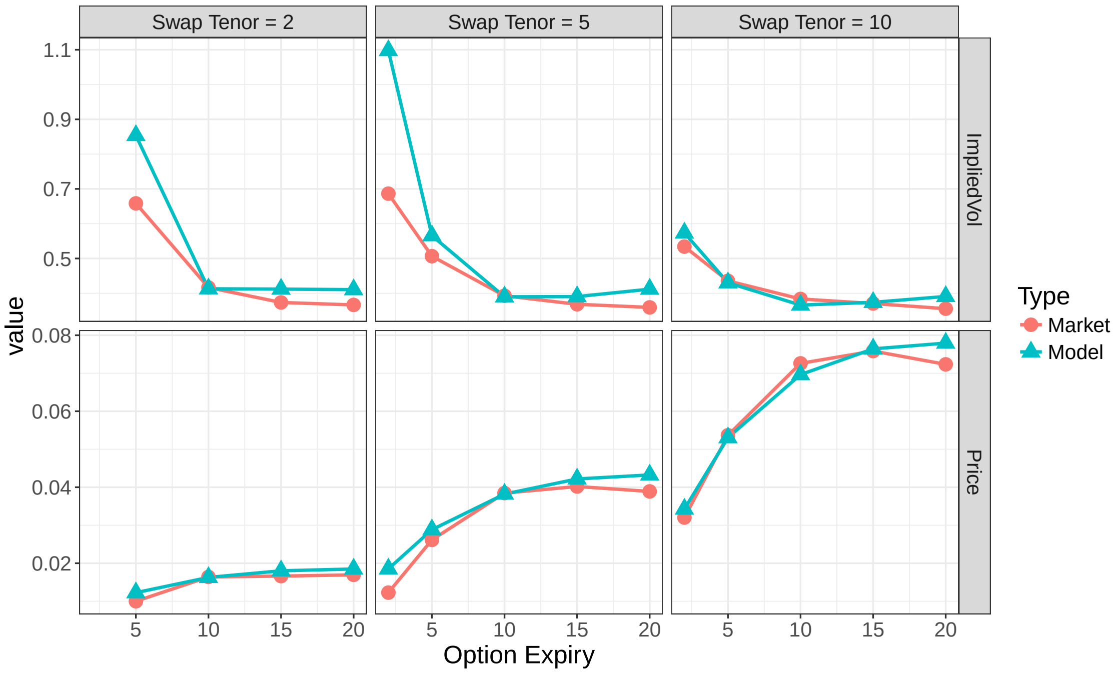

Figure 8.5.: Comparison of the model calculated and market observed swaption implied volatilities and prices for different swap tenors.

ATM swaption volatilities fit well for large option expires along with large swap tenors, while swaption implied volatilities could not be fitted very well for small option expiries and swap tenors, especially the implied volatilities with 2Y (5Y) option expiry and 5Y (2Y) swap tenor. It is acceptable, since we put more weights on long term volatilities.

Table 8.3 and Table 8.4 give the estimated parameters. We compute the standard errors of parameters by estimating the covariance matrix (or inverse of the Fisher information matrix) followed by Chen and Scott (1993, p. 9, eq.(4)) based on the log-likelihood function. The estimated standard errors of parameters for $\kappa_j, \theta_j, \sigma_j$ for $j = 1, 2, 3$ and the parameters of u_T are all small, while the standard errors are relative large for the market price of risk parameters λ_j^1. The large uncertainty for the λ_j estimates is probably not surprising, given that the short period of available historical data and they are typically poorly estimated even with larger sample.

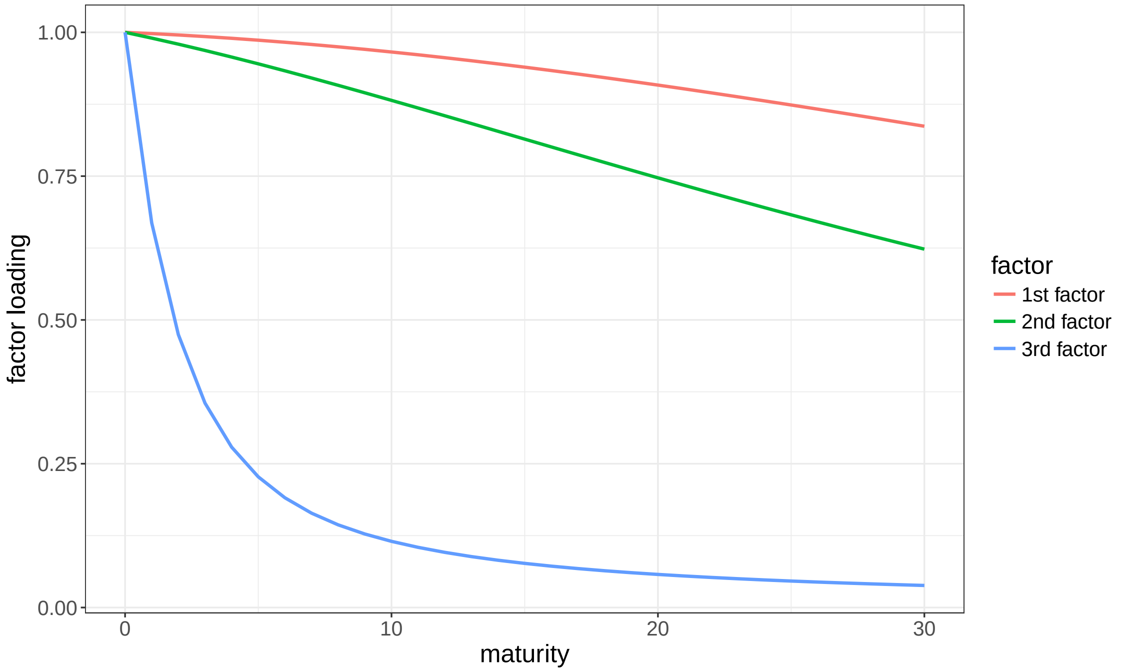

Figure 8.6.: The factor loadings of state variables.

The estimates of the risk-premia parameters λ_j^1 have the expected negative sign as in Chen and Scott (1993) and Geyer and Pichler (1999), Bolder (2001). The factor loadings of the three factors are illustrated in Figure 8.6. For the first factor, it has weak mean reversion ($\kappa_1' = 0.0085$) and low volatility ($\sigma_1 = 0.030$) (see Table 8.3). In agreement with Chen and Scott (1993) and Litterman and Scheinkman (1991) we conclude that the first factor may represent a 'general level' of interest rates which is closely related to the yield with the longest maturity.

The second factor has stronger mean reversion ($\kappa_2' = 0.281$) and a rather large volatility ($\sigma_2 = 0.045$) compared to first factor. It relates to the 'slope' of the term structure. The third factor has strongest mean reversion ($\kappa_2' = 0.866$) with largest volatility ($\sigma_3 =$

Table 8.3.: The estimated parameters of the extended three factor CIR model. The estimated standard errors are in parentheses. $\delta_0 = -0.07$.

	factor 1	factor 2	factor 3
κ	0.004	0.01949	0.86616
	(0)	(7e-05)	(0.00538)
θ	0.18673	0.06504	0.0236
	(8e-05)	(0.00033)	(0.00014)
σ	0.03029	0.04505	0.06931
	(0.00031)	(3e-04)	(0.01838)
λ^1	-0.00447	-0.26155	-0.18049
	(0.00606)	(0.01478)	(0.08269)

Table 8.4.: The additional parameters of the extended three factor CIR model

	factor 1	factor 2	factor 3
X_0	0.0458	0.0001	0.0253
λ^0	0.0000	0.0000	0.0000
κ'	0.0085	0.2810	1.0467
θ'	0.0881	0.0045	0.0195

Table 8.5.: The estimated parameters of error term u_T in 4.18.

terms	u	(S.E.)
0.25	0.0025	(8e-04)
0.5	0.0015	(4e-04)
1	0.0017	(0)
2	0.0025	(1e-04)
5	0.0022	(2e-04)
7	0.0014	(1e-04)
10	8e-04	(1e-04)
15	0.0017	(1e-04)
20	0.0021	(5e-04)
25	0.002	(3e-04)
30	0.0026	(1e-04)

0.069). The value of factor loading decreases sharply for the short term maturities, which indicates the third factor is mainly relevant for the short maturities.

8.1.2. Calibration of equity model

For the calibration of equity Heston model, we follow the description in Section 4.2.2. The model is first calibrated to the market European option prices under the risk neutral measure, by minimizing the mean squared error between the model and market European option prices (see Table 8.6).

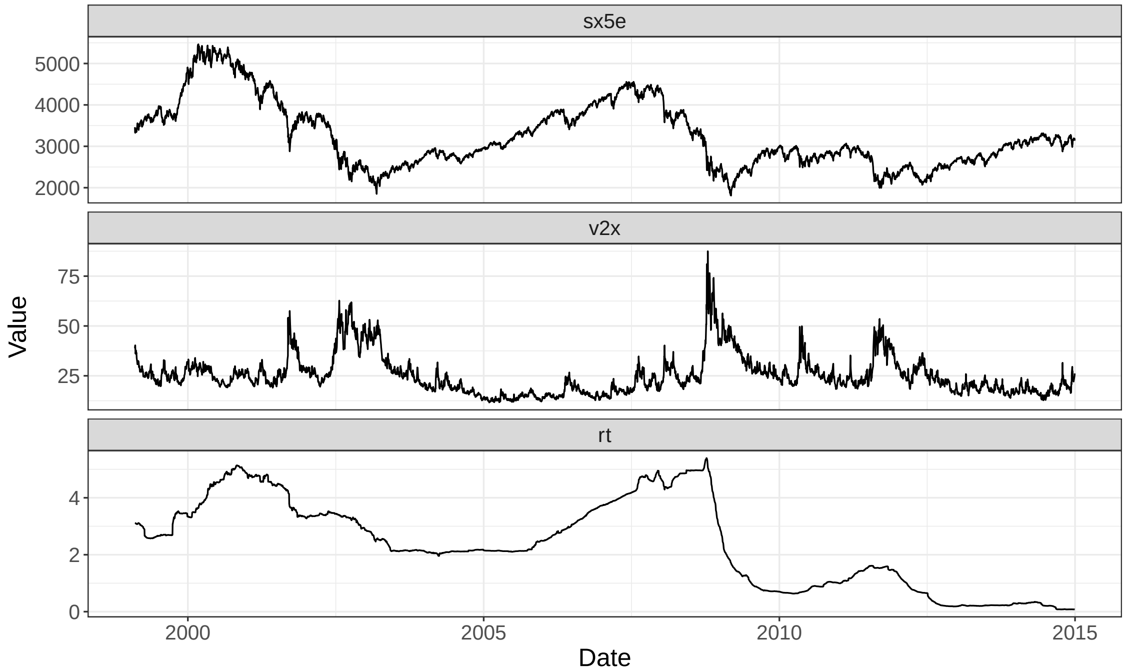

Figure 8.7.: The available historical data of EuroStoxx (Bloomberg Ticker: SX5E Index), the corresponding volatility index (Bloomberg Ticker: V2X index) and 1-week Euribor (in percent) as instantaneous short rate (Bloomberg Ticker: EUR001W Index) up to 12/31/2014.

The calibration results of the Heston model are illustrated in Figure 8.8, which compares the model calculated European option prices to the market observed European option prices for different strikes and maturities (1Y to 5Y). The root mean squared relative error between model and market European option prices is 0.4%.

In the next step, the parameters of market price of risk are estimated by the maximum likelihood method under the real world measure. Two of these parameters λ_v^0 and λ_s^0 are set to be zero for keeping simplicity and without losing the fitting quality.

Let $y(t) = \log S(t) - \int_0^t r(s)ds$ be the excess log-return over the risk free rate, then we

Table 8.6.: The market prices of European options for EuroStoxx (Bloomberg Ticker: SX5E Index) on 12/31/2014 with index value of 3146.43.

		maturity				
Type	strike	1	2	3	4	5
Call	2500	629.0	618.4	617.1	622.5	626.8
Call	2600	549.0	549.1	556.4	567.5	576.4
Call	2700	473.0	484.2	500.0	516.3	529.3
Call	2800	401.5	423.8	447.8	468.7	485.4
Call	2900	335.3	368.4	399.6	424.5	444.8
Call	3000	274.9	317.9	355.1	383.6	407.0
Call	3100	221.2	272.2	314.9	346.2	372.2
Call	3200	174.4	231.7	278.1	311.9	340.1
Call	3300	134.6	195.7	244.9	280.6	310.5
Call	3400	101.7	164.1	215.2	252.2	283.2
Call	3500	75.1	136.9	188.6	226.3	258.3
Call	3600	54.1	113.4	164.7	202.9	235.5
Call	3700	38.2	93.5	143.7	181.7	214.7
Call	3800	26.4	76.6	124.8	162.6	195.5
Put	2500	84.6	161.1	232.1	303.7	364.1
Put	2600	104.6	191.8	271.5	348.5	413.2
Put	2700	128.6	226.7	314.8	397.0	465.6
Put	2800	157.1	266.4	362.5	449.0	521.2
Put	2900	190.8	310.8	414.0	504.5	580.1
Put	3000	230.5	360.2	469.7	563.4	641.7
Put	3100	276.7	414.5	529.1	625.8	706.3
Put	3200	330.0	473.9	592.4	691.2	773.7
Put	3300	390.2	537.9	659.2	759.6	843.5
Put	3400	457.2	606.5	729.3	830.7	915.9
Put	3500	530.7	679.2	802.5	904.7	990.2
Put	3600	609.6	755.5	878.5	981.0	1066.9
Put	3700	693.6	835.5	957.4	1059.5	1145.5
Put	3800	781.8	918.6	1038.7	1140.2	1225.8

have

$$dy(t) = (-q + bv(t))dt + \sqrt{v(t)}(\rho dW_v^{\mathbb{P}}(t) + \sqrt{1-\rho^2}dW_s^{\mathbb{P}}(t)),$$
$$dv(t) = \kappa_v'(\theta_v' - v(t))dt + \sigma_v\sqrt{v(t)}dW_v^{\mathbb{P}}(t), \quad v(0) > 0,$$

with $dW_v^{\mathbb{P}}(t)dW_s^{\mathbb{P}}(t) = 0$.

The $y(t)$ could be calculated using the historical data of equity index (EuroStoxx) and instantaneous short rate (1-Week Euribor). For the instantaneous volatility state variable $v(t)$, we use the so-called an unadjusted Black-Scholes proxy (see Aït-Sahalia and Kimmel (2007)), i.e. using the implied volatility of a short-maturity at-the-money option instead of the true instantaneous volatility state variable. Since the V2X index measures the 30-day implied volatility of the EuroStoxx, we then set $v(t) = \mathrm{V2X}^2(t)$. The corresponding historical data could be seen in Figure 8.7. We could see that V2X index is mean reversion and non-negative, it indicates that it is suitable to use the Cox-Ingersoll-Ross process to the model the volatility state variable $v(t)$.

We then can follow the maximum likelihood estimation procedure proposed by Aït-Sahalia and Kimmel (2007) with state variables $(y(t), v(t))$ to estimate the risk premia parameters λ_v^1, λ_s^1. All the calibrated parameters are shown in Table 8.7 and 8.8.

Table 8.7.: The calibrated parameters of the Heston model.

κ_v	θ_v	σ_v	v_0	q	ρ
5.4144	0.0461	0.7054	0.0494	0.0307	-0.6411

Table 8.8.: The calibrated parameters for market prices of risks of Heston model as well as the implied other parameters.

λ_v^0	λ_v^1	λ_s^0	λ_s^1	a	b	κ_v'	θ_v'
0.0000	1.9437	0.0000	2.2779	0.0000	0.0111	3.4707	0.0719

8.2. Scenario generation

8.2.1. Risk neutral scenarios

Given the calibrated parameters, the risk neutral scenarios could be generated for the market consistent valuation through Monte Carlo simulation. Here we use the standard Euler scheme to generate the scenarios. The time step is chosen to be 0.004, i.e. 250 time steps per year. Furthermore, the Antithetic Variates (see e.g. Glasserman (2004)) is used for the variance reduction. We choose the number of risk neutral simulation is 10000 (i.e. 5000 pairs).

Before the usage of risk neutral scenarios for market consistent valuation, we need to do the validation of the risk neutral scenarios. First of all, the martingale test should

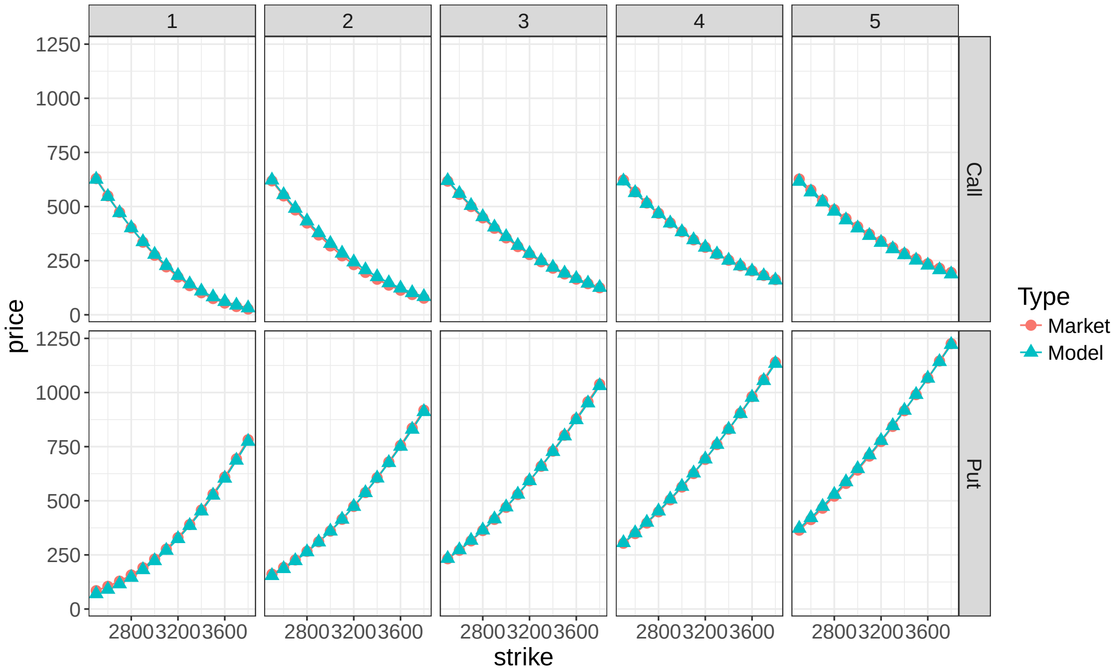

Figure 8.8.: Comparison of the model calculated and market observed European option prices for different strikes and maturities (1Y-5Y).

be performed for the scenarios in order to check the risk neutrality by validating if the discounted assets are martingale.

The deflated total return indices should be 1 for different years under risk neutral measure. Therefore if the martingale property is fulfilled, then the true value of 1 should be in the confidence interval (95%) of the deflated TRI based on Monte Carlo simulation. The assets of total return indices such as the total return indices (TRI) of constant maturity zero coupon bond (CM ZCB) with maturities 1Y, 5Y and 10Y, as well as the total return index of equity, are taken into account for the martingale test. Figure 8.9 illustrates the results of martingale test of the 10000 risk neutral antithetic scenarios. The results show that the scenarios fulfill the martingale test property.

Secondly, we check the pricing quality of swaptions through Monte Carlo simulation. We compare the model and Monte Carlo based ATM receiver swaption prices with different option expiries and swap tenors. The comparison results could be seen in Table 8.9. The relative errors of MC prices with respect to model prices are all less than 1.9%. The root mean squared relative error between Monte Carlo and model receiver swaption prices is 0.9%. All the Monte Carlo based swaption prices are in the 95% confidence interval.

Lastly, we check further the pricing quality of European option prices through Monte Carlo simulation. We compare the model and Monte Carlo based European put prices with different option expiries and swap tenors. The comparison results could be seen in Table 8.10. The relative errors of MC prices with respect to model prices are all

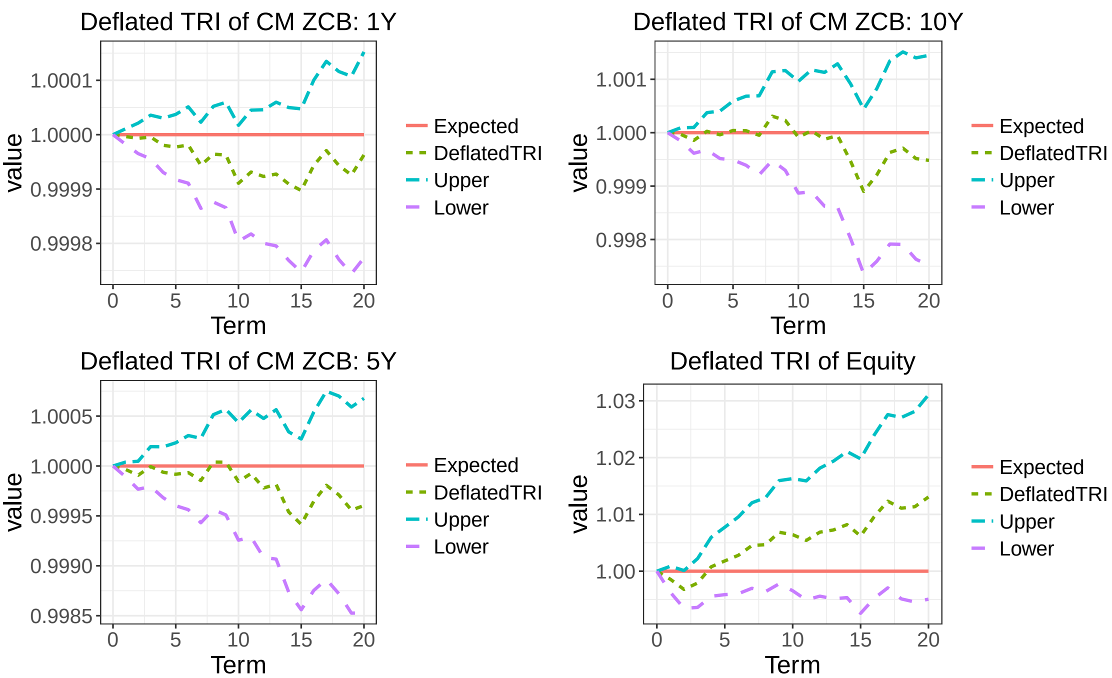

Figure 8.9.: Martingale test of the risk neutral antithetic scenarios.

less than 3.3%. The root mean squared relative error between Monte Carlo and model put European option prices is 1.17%. All the Monte Carlo based swaption prices are in the 95% confidence interval. Note that the relative error could be reduced with larger number of simulation and smaller time step.

Based on the martingale tests and Monte Carlo pricing quality of options, we conclude that the risk neutral scenarios are acceptable for the market consistent valuation.

Table 8.9.: Comparison of model and Monte Carlo based ATM receiver swaption prices for different swap tenors and option expiries.

Option Expiry	Swap Tenor	Model Price	MC Price	Rel. Error	95% CI
5	2	0.0123	0.0124	1.1%	(0.0121, 0.0126)
10	2	0.0163	0.0163	0.4%	(0.016, 0.0167)
15	2	0.0180	0.0179	-0.5%	(0.0175, 0.0184)
20	2	0.0185	0.0185	0%	(0.018, 0.019)
2	5	0.0185	0.0187	0.9%	(0.0183, 0.019)
5	5	0.0288	0.0293	1.6%	(0.0287, 0.0299)
10	5	0.0382	0.0384	0.4%	(0.0375, 0.0392)
15	5	0.0422	0.0419	-0.6%	(0.0409, 0.043)
20	5	0.0432	0.0432	0%	(0.042, 0.0445)
2	10	0.0343	0.0346	0.9%	(0.0338, 0.0353)
5	10	0.0530	0.0540	1.9%	(0.0529, 0.0552)
10	10	0.0696	0.0699	0.3%	(0.0682, 0.0716)
15	10	0.0764	0.0759	-0.7%	(0.0738, 0.078)
20	10	0.0779	0.0779	0%	(0.0754, 0.0804)

Table 8.10.: Comparison of the model and Monte Carlo based equity option put prices for different strikes and maturities.

Strike	Maturity	Model Price	MC Price	Rel. Error	95% CI
2500	1	70	72	3.3%	(68, 76)
2800	1	146	148	1.5%	(143, 153)
3100	1	270	272	0.6%	(266, 278)
3500	1	525	529	0.7%	(522, 535)
3800	1	774	778	0.6%	(772, 785)
2500	2	154	159	3%	(153, 164)
2800	2	264	268	1.7%	(261, 275)
3100	2	414	419	1.2%	(411, 427)
3500	2	676	682	0.9%	(674, 690)
3800	2	911	918	0.7%	(909, 927)
2500	3	233	237	1.5%	(230, 244)
2800	3	364	368	1%	(360, 376)
3100	3	530	533	0.7%	(525, 542)
3500	3	799	801	0.3%	(792, 811)
3800	3	1031	1033	0.2%	(1023, 1043)
2500	4	306	306	-0.1%	(299, 313)
2800	4	452	452	-0.1%	(443, 460)
3100	4	628	626	-0.2%	(617, 636)
3500	4	903	899	-0.4%	(889, 910)
3800	4	1134	1130	-0.3%	(1119, 1141)
2500	5	372	375	0.7%	(367, 383)
2800	5	529	531	0.5%	(522, 541)
3100	5	712	713	0.2%	(703, 723)
3500	5	990	990	0%	(979, 1001)
3800	5	1221	1219	-0.2%	(1208, 1231)

8.2.2. Real world scenarios

Given the calibrated parameters, the real world scenarios in one year horizon could also be generated for the determination of risk capital. The summary statistics of the simulated spot rates in one year horizon for different terms could be seen in Table 8.11. The values are compared to the spot rates on cutoff date $t = 0$. We have lower average spot rates in short terms (smaller than 5 years) compared to $t = 0$ value, which indicates that the expected interest rate would be lower or even to be negative in one year horizon. [2] These statistics are also shown in graphically in Figure 8.10.

Table 8.11.: Summary statistics of simulated spot rates (in percent) at $t = 1$ for different terms.

	0.25	0.5	1	2	5	7	10	15	20	25	30
t=0	0.08	0.17	0.16	0.18	0.36	0.53	0.82	1.18	1.37	1.47	1.52
Mean	-0.04	0.04	0.07	0.15	0.42	0.61	0.92	1.28	1.46	1.56	1.61
Std	0.91	0.87	0.81	0.74	0.67	0.66	0.65	0.63	0.61	0.59	0.56
Min	-3.19	-2.97	-2.73	-2.35	-1.69	-1.39	-1.05	-0.69	-0.46	-0.31	-0.19
0.5%	-2.14	-1.99	-1.84	-1.59	-1.15	-0.95	-0.61	-0.20	0.05	0.20	0.30
1%	-1.98	-1.81	-1.66	-1.44	-1.04	-0.80	-0.47	-0.06	0.17	0.32	0.42
5%	-1.46	-1.32	-1.20	-1.01	-0.63	-0.42	-0.10	0.29	0.51	0.64	0.73
25%	-0.66	-0.55	-0.48	-0.36	-0.05	0.15	0.46	0.84	1.04	1.15	1.21
50%	-0.09	-0.00	0.04	0.12	0.39	0.58	0.89	1.25	1.44	1.53	1.58
75%	0.54	0.60	0.59	0.63	0.86	1.04	1.34	1.69	1.86	1.94	1.98
95%	1.52	1.53	1.44	1.40	1.57	1.74	2.02	2.35	2.50	2.56	2.56
99%	2.29	2.24	2.09	1.99	2.12	2.27	2.54	2.84	2.97	3.00	2.99
99.5%	2.59	2.55	2.40	2.20	2.29	2.46	2.73	3.02	3.15	3.18	3.17
Max	3.69	3.53	3.25	3.04	3.04	3.20	3.47	3.75	3.85	3.86	3.81

The proportional of variances for the first three components based on the principle component analysis of the simulated spot rates are 92.76%, 7.23% and 0.01%, which are comparable to the values based on historical spot rates, i.e. 92.87%, 6.43%, 0.49%.

Furthermore, we compute the one year absolute changes of the spot rates with different maturities of the historical data, i.e. the difference of spot rates at time t and $t-1$, for t before than 12/31/2014. And we calculate the absolute changes of the simulated spot rates in one year horizon and compare them to the historical changes in order to check if the simulated spot rates could cover the one year historical changes. Figure 8.12 shows the comparison results. We could see that simulated absolute changes cover the range of historical changes for maturities larger than 5Y. For the short term maturities, the simulated 1Y changes do not cover the historical 1Y changes, which is still reasonable due to the interest rate is already quite low on the cutoff date. From the historical data, the large absolute decrease relates to high interest rate at one year before, for instance, the

[2] Actually we have observed the negative interest rates in year 2015 under the current low interest rate environment.

largest absolute decrease 4.6% of the 3M spot rates happened between dates 10/07/2008 (5.34%) and 10/06/2009 (0.74%). Therefore, instead of absolute changes in one year, we compare the one year returns of constant maturity zero coupon bond, especially for the short term spot rates, the results are illustrated in Figure 8.13. We could then see that the simulated one year returns could cover all the lower tails of the historical 1 year returns.

For the Heston model, we compare the simulated and historical state variables $(S(t), v(t))$ as displayed in Figure 8.11. $S(t)$ is the equity price index sx5e and $v(t)$ is the squared value of the volatility index v2x. The range of historical data is from 07/01/1999 to 12/31/2014. Both of the distributions are comparable, the simulated data could cover the range of historical values. The range of simulated one year excess total return is (-81%, 106%) and the range of historical one year excess total return is (-50%, 62%).

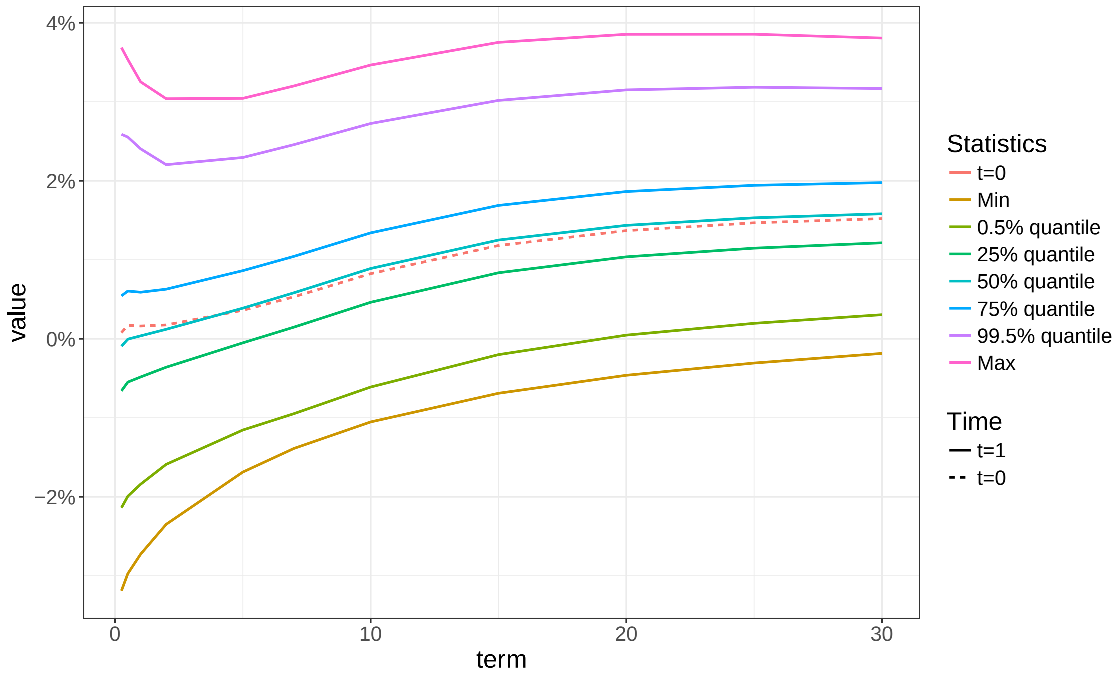

Figure 8.10.: The simulated spot rates at $t = 1$.

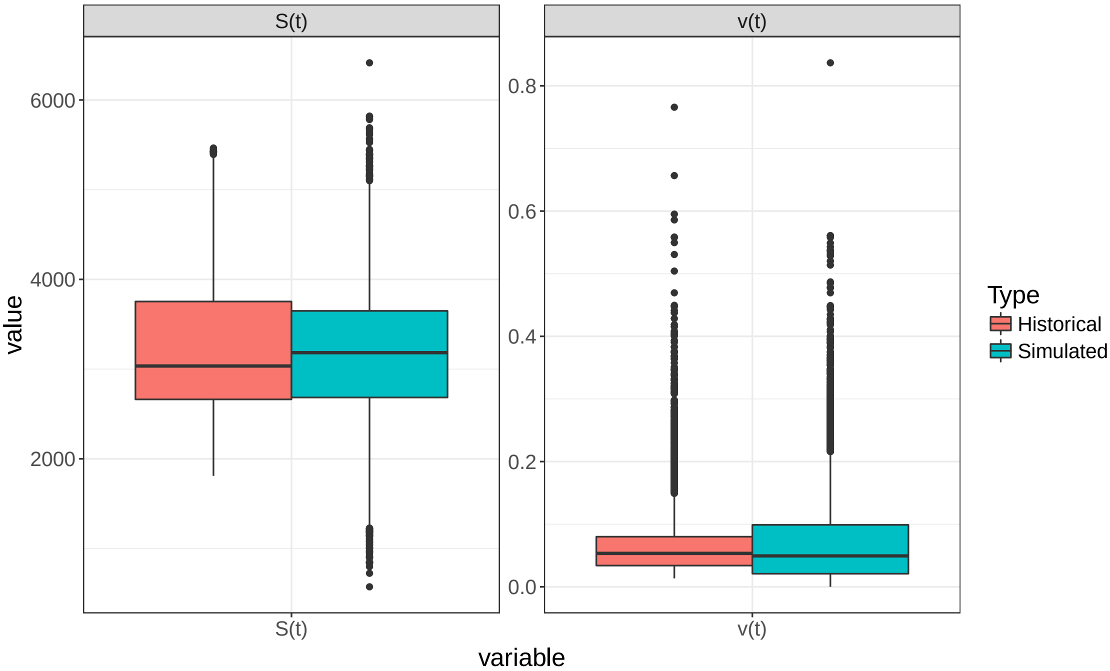

Figure 8.11.: Boxplots of simulated and historical state variable $(S(t), v(t))$.

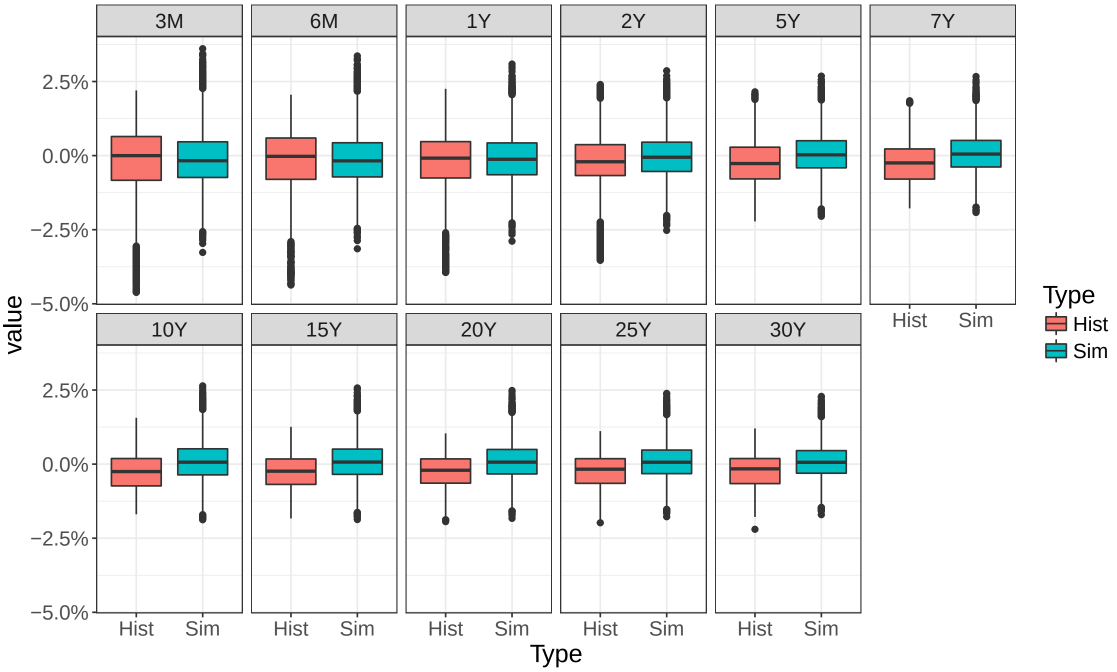

Figure 8.12.: Comparison of historical and simulated absolute change of spot rates in 1 year.

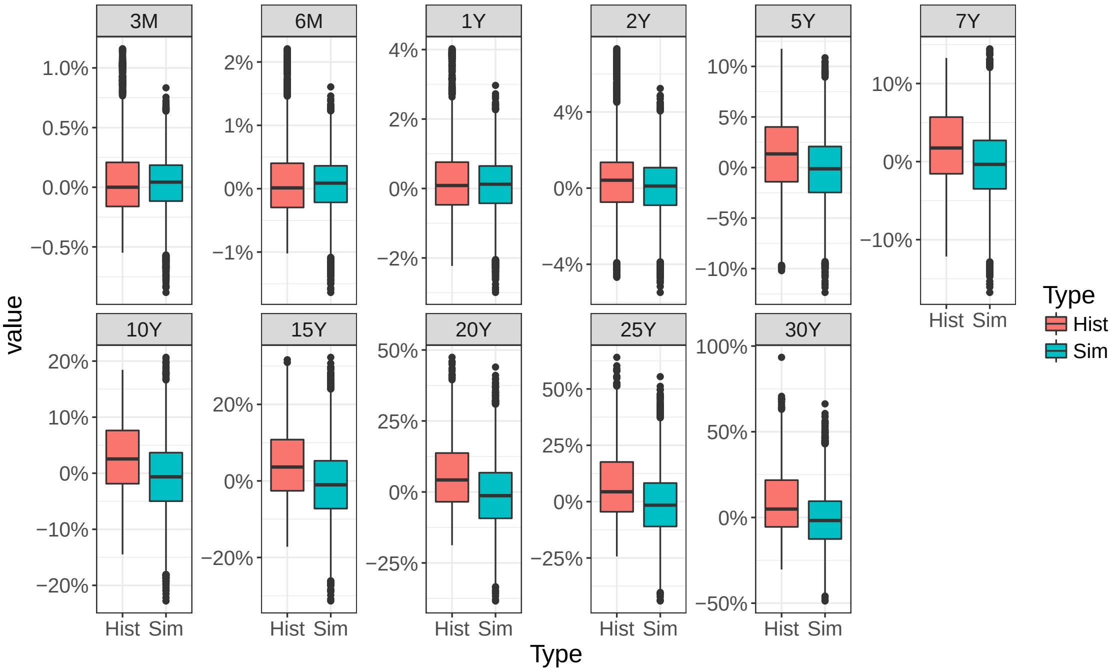

Figure 8.13.: Comparison of historical and simulated one year returns of constant maturity zero coupon bond.

8.3. Market consistent valuation

Given the risk neutral scenarios generated by the interest rate and equity model, we start to do the market consistent valuation of the stochastic cash flow projection model in Section 5.2 for a life insurance company.

The liability portfolio consists of identical traditional participating life insurance contracts with a cliquet style guarantee ignoring any costs and the surrender options. For simplicity, we assume that all policyholders are 50 years old at inception date of the contract. For each contract, the guaranteed rate is $g = 1.75\%$ and the guaranteed benefit $G = 20$ TEUR (Thousand Euros).

The liability portfolio is built up at the valuation date $t = 0$, such that there are 1000 new policies for each duration $n = 1, \ldots, 10$ entering into the portfolio. Hence, we have the portfolio at the beginning of projections with remaining time to maturities 1 to 10 years.

The asset portfolio consists of stocks and coupon bonds. We set the proportion of market value of stocks $p^{SAA} = 5\%$ and bond maturity $T_B = 10$ years for the constant asset allocation strategy. The coupon bond portfolio at $t = 0$ is constructed with coupon bonds at par with time to maturities 1 to 10 years proportionally on market value. Furthermore, we assume that $p^{UGL} = 20\%$ is realized if the unrealized gain and loss for the stock ${}^{UGL}S_t$ is positive and 100% is realized if ${}^{UGL}S_t$ is negative.

Based on the setting as described before, the market consistent valuation could be performed for the life insurance company. In order to check all the cash flows are properly captured the cash flow projection model, the leakage test is performed to check if the relationship (5.33) is satisfied. Figure 8.14 shows the results for the leakage test, we see that mean value of $AC_0 + {}^{MV}L_0 - {}^{MV}A_0$ convergences to 0 and is in the 95% confidence interval as well. Therefore, the market consistent valuation is fine according to the leakage test.

We now calculate the AC_0 or MCEV through Monte Carlo simulation, the value is 3072.57 TEUR with standard error of 14.18 and therefore 95% confidence interval (3044.78, 3100.36) TEUR. The standard error of AC_0 is relative small compared to the value of AC_0. Figure 8.15 displays the density plot of present value of shareholder's future profits, which indicates that the corresponding distribution is left heavy tailed.

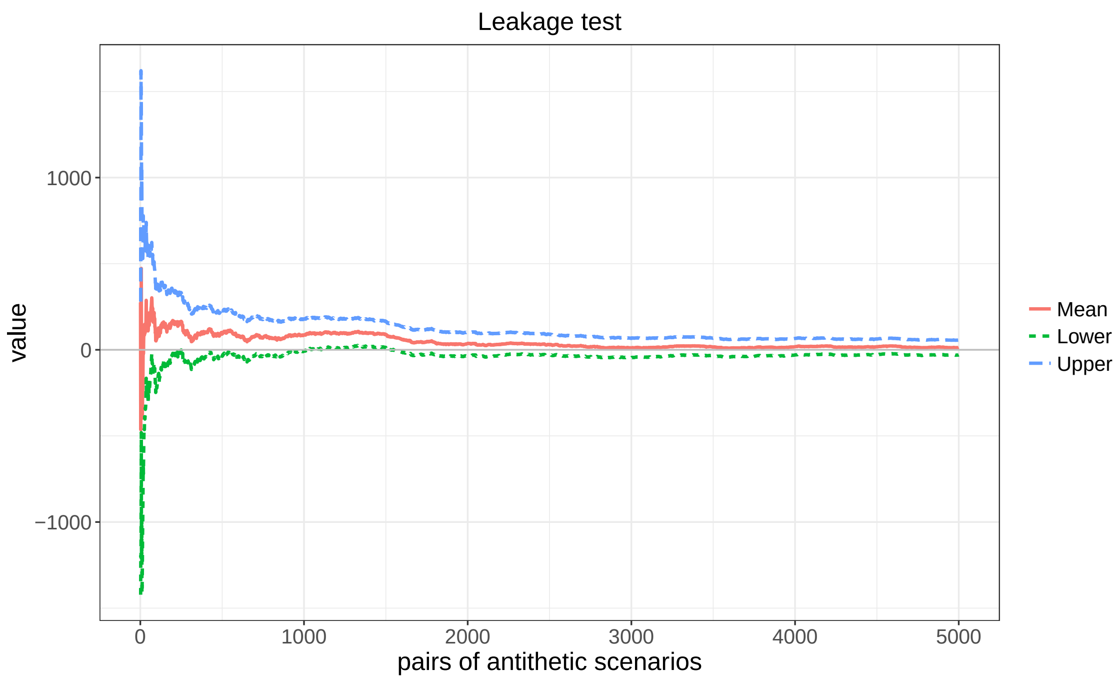

Figure 8.14.: The leakage test of the valuation of the asset-liability-model based on 10000 (5000 pairs) paths of antithetic scenarios.

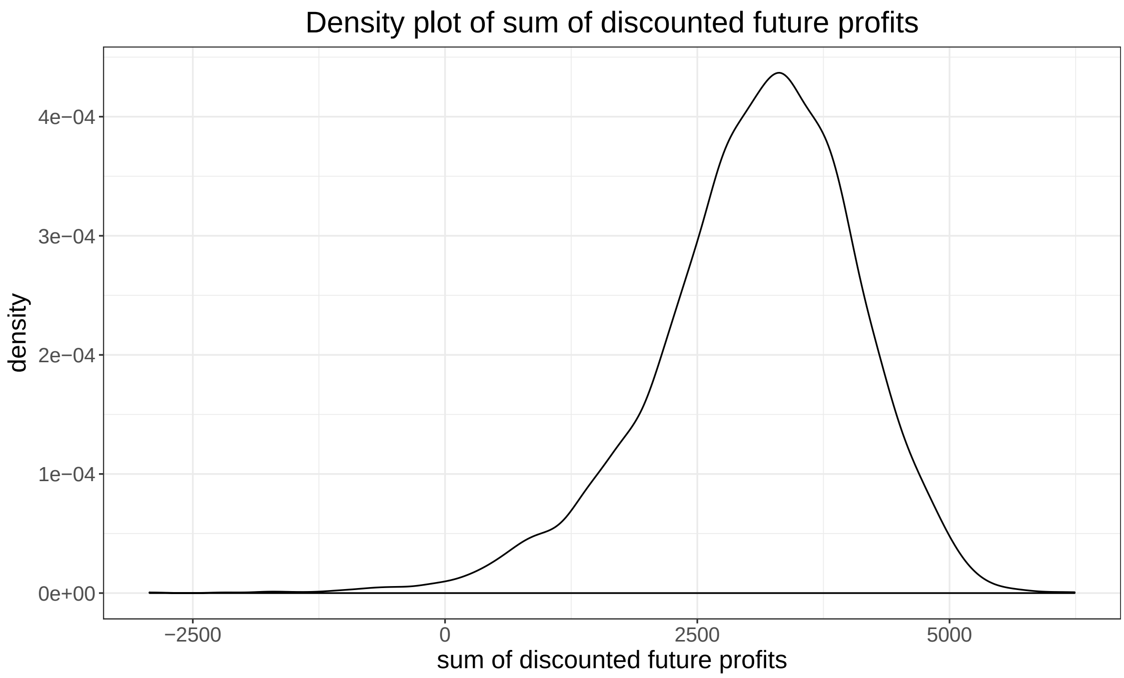

Figure 8.15.: The density plot of the present value of shareholder's future profits PV_0.

8.4. Risk modeling

In this section, we perform nested simulation and the proxy method of replicating portfolio to estimate the SCR.

8.4.1. Nested simulation

For the nested simulation, the number of outer and inner simulation needs to be predefined. Here the number of outer simulation is chosen to be 10000 and the number of inner simulation is chosen to be 5000 with 250 time steps per year, which is the same as risk neutral simulation at $t = 0$.

The estimated 0.5% quantile of AC_1 is 1436.9. As we know that the AC_0 is the MCEV at $t = 0$ with value 3072.57 TEUR and one-year risk free rate $rr(0, 1) = 0.1615\%$. Therefore, the estimated SCR based on nested simulation is 1638 TEUR.

8.4.2. Replicating portfolio

For the replicating portfolio, we follow the general strategy described in Chapter 7.

We first construct a pool of financial assets. Under the cut-off date of 12/31/2014, the following assets are selected into the asset pool:

- Receiver swaptions with option expiries 1Y to 10Y, swap tenor 1Y to 10Y and strikes of 0.5%, 1.0%, 2.0%, 3.0%, 4.0%.
- European put options with time to maturities $1, \ldots, 10$ years and moneyness of 0.5, 0.75, 1.0, 1.25, 1.5.
- Total return indices of equity at year end of $2015, \ldots, 2024$ years.
- Total return indices of constant maturity zero coupon bond with time to maturity $1, \ldots, 10$ at year end of $2015, \ldots, 2024$.

Second, we construct different risk neutral scenarios sets for the calibration of replicating portfolio.

Given the calibrated parameters in Section 8.1, we could generate a set of risk neutral scenarios by means of Monte Carlo simulation. We call such risk neutral scenario set as a basis set. Besides the basis set, further sensitivity sets are usually required to better calibrate the replicating portfolio. Here we simply modify some of the calibrated parameters and then do Monte Carlo simulation to construct further sensitivity sets of risk neutral scenarios without doing recalibration. By modifying corresponding parameters, following sensitivity scenarios are taken into account for calibration of replicating portfolio:

1. The interest rate shift-up sensitivity set (i.e. the parameter of $X_1(0)$ is added by 50 bps).

2. The interest rate shift-down sensitivity set (i.e. the parameter of $X_1(0)$ is subtracted by 50 bps).
3. The interest rate volatility sensitivity set (i.e. the parameters of σ_j for $j = 1, 2, 3$ are increased by 25%).
4. The equity volatility sensitivity set (i.e. the mean reversion parameter θ_v is increased by 25%).

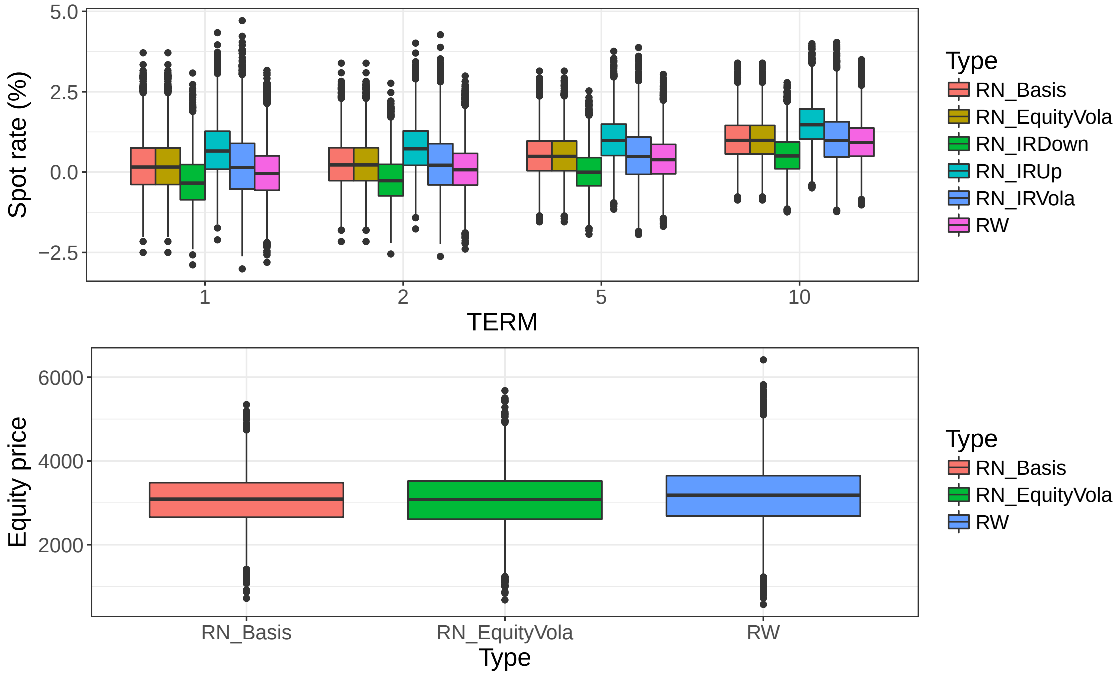

Figure 8.16.: Comparison of spot rates and equity price in one year horizon under risk neutral and real world measure.

We then construct the risk neutral calibration scenarios for replicating portfolio by combining the basis set and further sensitivity sets.

If the risk factors of real world scenarios could be covered sufficiently by the risk factors of risk neutral calibration scenarios, then it is more possible to get a replicating portfolio with smaller error estimation of SCR.

We therefore compare the risk factors of risk neutral calibration scenarios to the real world scenarios. Figure 8.16 illustrates the comparison of box-plots of main risk factors, i.e. the spot rates with different terms and the equity prices, at $t = 1$ under risk neutral and real world measure.

The real world scenarios of spot rates at $t = 1$ could be fully covered by the risk neutral scenarios of spot rates. The coverage of equity prices is considered to be fine as well. There are two reasons. First, the uncovered range of equity prices focuses mainly on the right tail that does not affect the estimation of SCR. Second, the equity risk is a small proportion since only 5% of the portfolio is invested in equity.

Before the subset selection, the linear dependency should be checked by calculating the rank of matrix with all candidate assets in the asset pool. If the rank is smaller than the number of candidate assets, then there exists linear dependency or perfect multicollinearity among the candidate assets. Therefore, the pre-selection of candidate assets should be performed.

Third, we perform the calibration procedure. We determine the replicating portfolio by matching the present values of replicating portfolio and PVFP, i.e. using cost function of sum squared error of the present values as given in (7.18). The subset selection techniques could be applied to perform the optimization procedure.

Now we start to do the subset selection with method of backward selection. Note that one could also use method of sequential replacement, which convergences faster but with more computation time. Compared to backward selection, the forward selection turns to select some candidate assets that their estimated weights are not quite significant in the final selected portfolio, since once the asset is selected in then it could not be drawn out any more.

For each risk neutral scenario set, it consists of 5000 pairs of antithetic scenarios. In each pair of antithetic scenarios, the two scenarios are driven by the same random seed, therefore only one of them should be taken into account for the calibration scenarios. These selected 5000 scenarios for each set are further split into two parts: the first 4000 (80% of the selected antithetic scenarios) into the calibration set and the others into the validation set.

Since the asset prices with different types are in different orders of magnitude. For instance, the order to magnitude (measured on a 10-base logarithmic scale) of European option prices is usually 2 to 3, while order to magnitude of swaption prices is usually -2 to -3. It leads to different order of magnitude of the estimated weights. Let N^C be the number of asset types of all assets in the asset pool C and C^j be the set of assets with type j. For a given replicating portfolio G with weight w_k for candidate asset $C_k \in G$, the Long-Short-Position (LSP) is then defined as

$$LSP = \frac{1}{N^C} \sum_{j=1}^{N^C} \frac{|\sum_{C_k \in C^j} w_k|}{\sum_{C_k \in C^j} |w_k|},$$

in order to consider magnitudes of weights with different asset types, as well as the offsetting effect in the same type.

Figure 8.17 illustrates the selection criterion (see definitions in B.2), i.e. the R-squared (R^2), adjusted R-squared, Mallows'C_p, BIC, LSP and Out-of-Sample (OOS) R-squared, for determining the number of candidate assets into the replicating portfolio. We see that the R-squared, adjusted R-squared are monotonic increasing and the Mallows'C_p is monotonic decreasing along with the number of assets. It is not sufficient to choose such single criterion to determine the number of candidate assets. We therefore construct a total index as the combination of In-Sample R-squared, Out-of-Sample R-squared and

LSP, i.e.

$$TotalIndex = R^2 + R^2_{OOS} + c \cdot LSP \tag{8.1}$$

where c is the penalty coefficient for LSP. We assume that the penalty is applied only if fitting quality is acceptable, i.e. R-squared value larger than 0.9. Mathematically, c is assumed to be 0.1 if R-squared is larger than 0.9 and 0 for the others.

Figure 8.18 illustrates the total index and 0.5%-Quantile of AC_1 approximated by the selected replicating portfolios with different number of candidate assets. The dashed horizontal line is the 0.5%-Quantile of AC_1 based on nested simulation. The vertical line represents the number of candidate assets with maximum value of total index.

By maximizing the total index, we choose 42 candidate assets for replicating portfolio. Figure 8.19 gives the comparison of QQ-plot, density plot and boxplot of the value of selected replicating portfolio and the sum of discounted shareholder's future profits for calibration scenarios with different scenario types. The results show that the fitting quality is quite good for the in sample calibration. Figure 8.20 shows the shares of asset type of the selected replicating portfolio based on calibration scenarios with different scenario types. The proportion related to equity assets is around 5.5% (calculating the proportion based on taking absolute values of shares), which is comparable with the shares of equity in the asset portfolio. Furthermore, we see that the PVFP is most sensitive to the shift of interest rates.

8.4.3. Comparison of SCRs

Now we proceed to compare the resulting AC_1 based on the nested simulation and replicating portfolio. They are high correlated with correlation coefficient of 96.5%. The corresponding density plots, boxplots as well as the QQ-plot are illustrated in Figure 8.21. We see that the distributions of AC_1 based on replicating portfolio and nested simulation are quite similar, especially in the lower tail for determining 0.5% quantile.

In addition, we calculate the 0.5% quantile of AC_1, consequently the SCR and solvency ratio by the methods of nested simulation and replicating portfolio. As shown in Table 8.12, the SCR calculated by the value of replicating portfolio is 1655 TEUR. The SCR calculated by nested simulation is 1638 TEUR. The percentage difference is 1.04%.

Table 8.12.: Comparison of the SCRs determined by nested simulation and replicating portfolio.

	SCR	Solvency Ratio
RP	1655	1.86
NS	1638	1.88

Therefore, we conclude that the proxy method of replicating portfolio for calculating the SCR is quite well compared to the method of nested simulation.

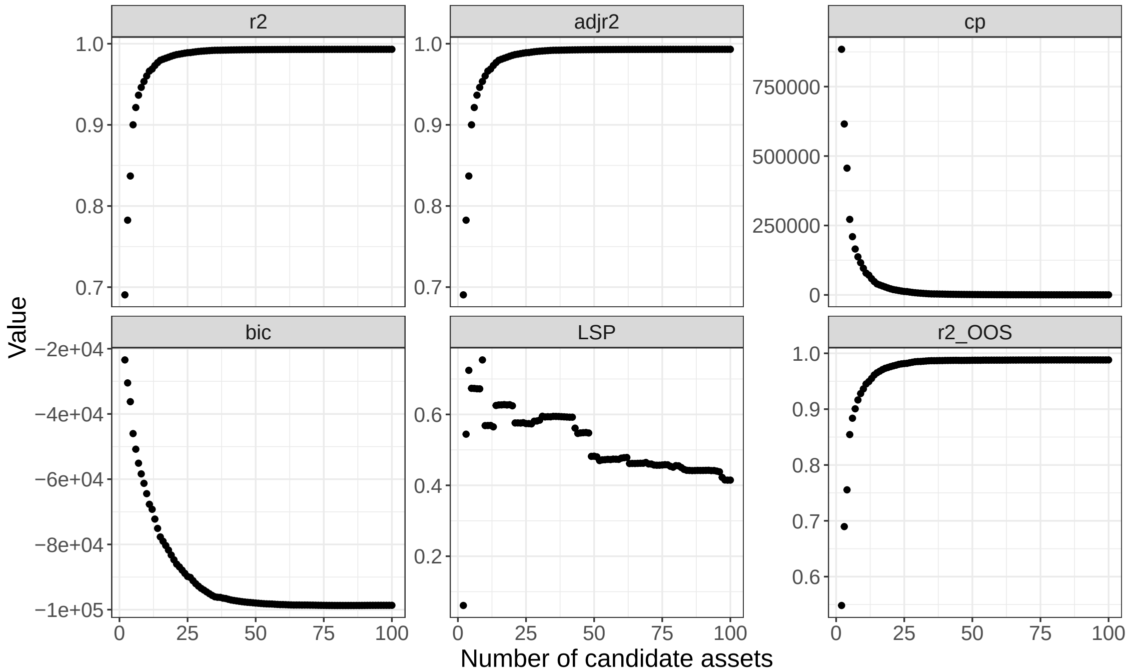

Figure 8.17.: Comparison of the different criterion for choosing the number of candidate assets.

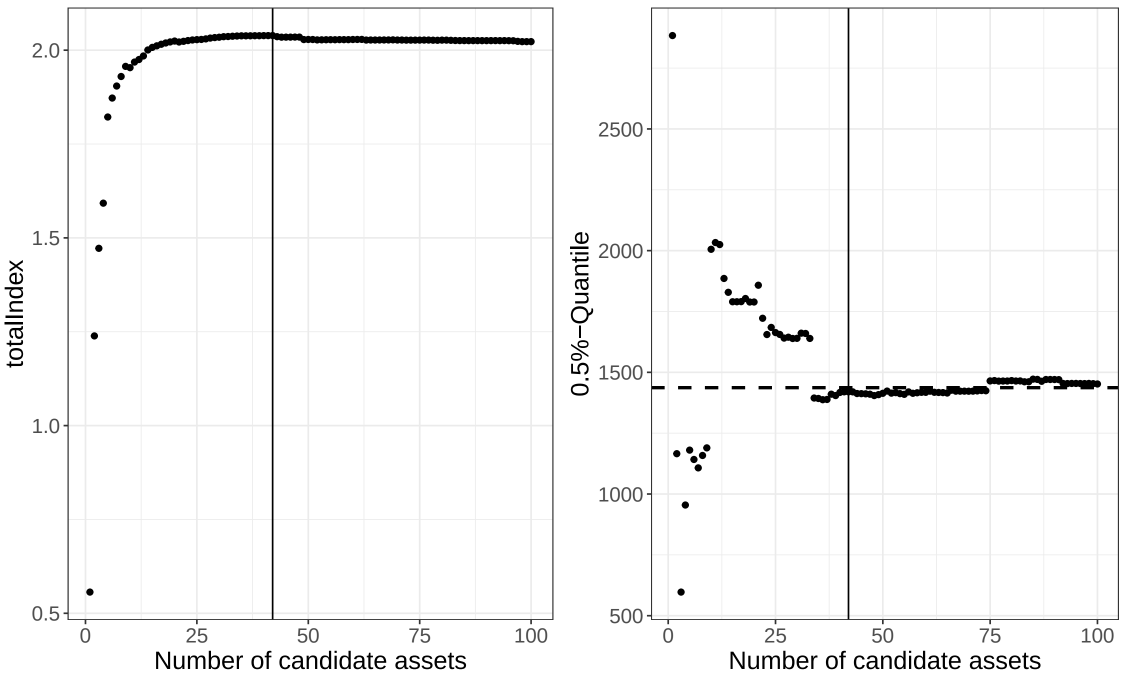

Figure 8.18.: Total index and 0.5%-Quantile for value of the selected replicating portfolios at $t = 1$ with different number of candidate assets.

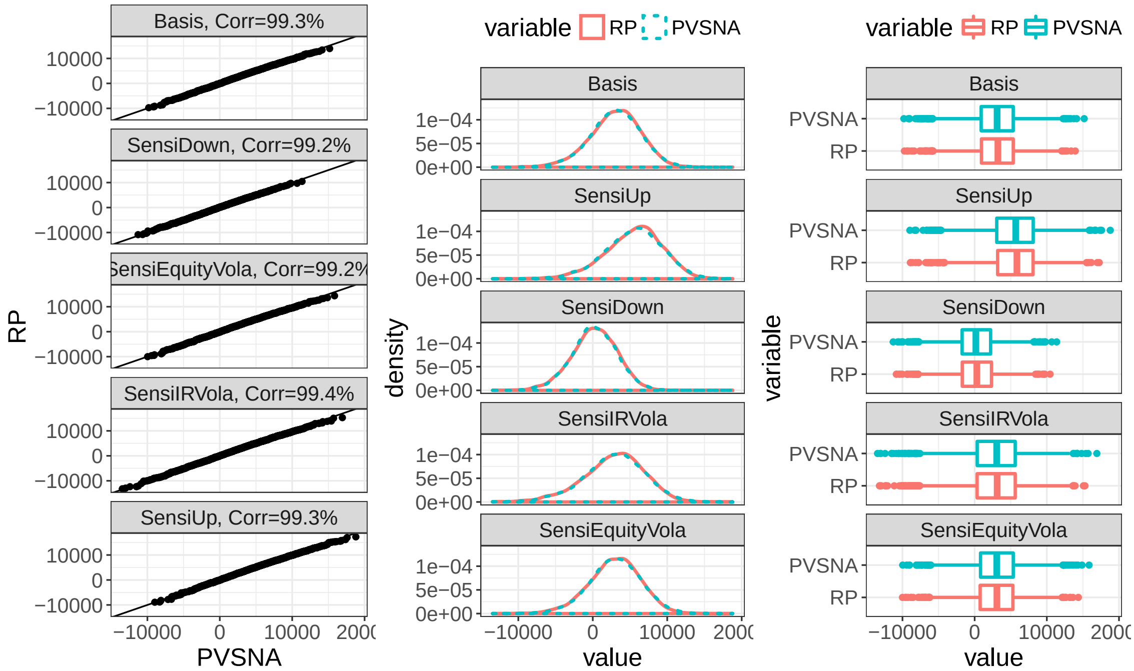

Figure 8.19.: The comparison of the sum of discounted shareholder's future profits and value of selected replicating portfolio for the calibration scenarios.

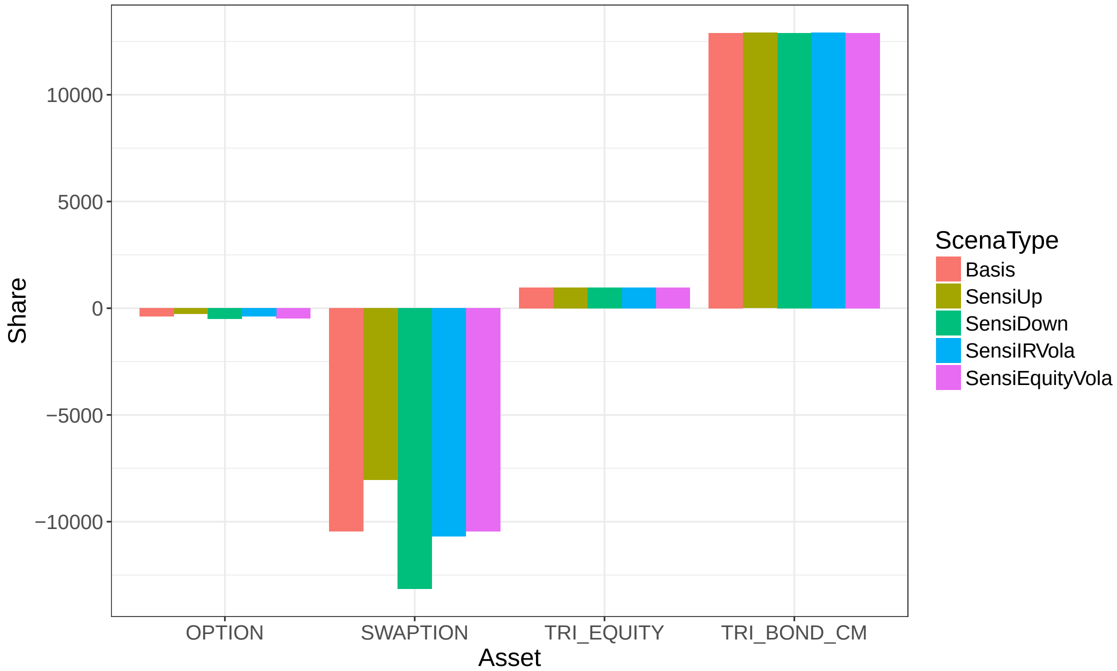

Figure 8.20.: The asset shares of the selected replicating portfolio based on the calibration scenarios.

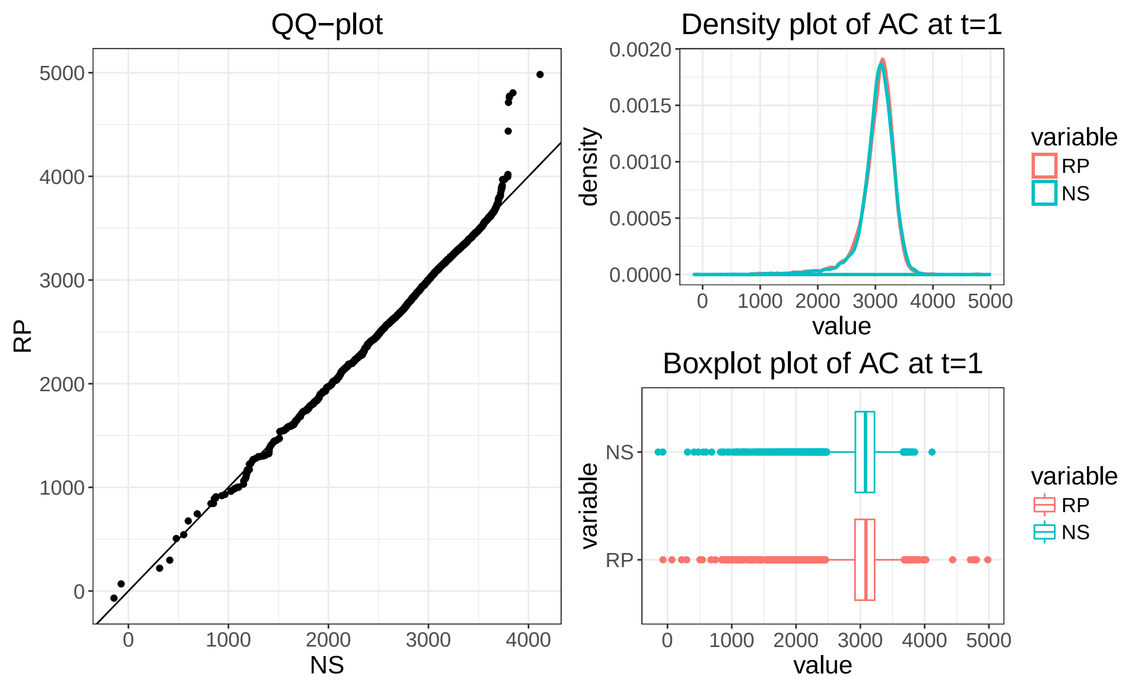

Figure 8.21.: Comparison of AC_1 based on nested simulated and replicating portfolio.

9. Conclusions

In this thesis, we have illustrated the application of stochastic methods in risk management under solvency II framework.

We started with the concept of risk and then proceeded to discuss to the risk management in the insurance sector under the enterprise wide risk management system, which should consider the risk based regulatory and rating agency requirements, as well as the risk management standards by ISO. We figured out that the stochastic methods are mainly applied in the risk assessment in the enterprise risk management process for the quantification of risk. We then focused on giving a detailed description of quantifying the risk of an insurance company, i.e. doing market consistent valuation and determining the Solvency Capital Requirement under Solvency II framework, through a partial internal model by means of stochastic methods. The partial internal model separates into three components: input model, valuation model and risk capital model.

For the input model, we developed a simple ESG consisting of interest rate and equity models to generate the economic scenarios. For the modeling of interest rate, we used the extended three-factor Cox-Ingersoll-Ross model, which is able to capture the three main principle components of yield curve. Since it is an affine model, we first derived the pricing formula of zero coupon bonds in a closed form. We then derived the pricing of zero coupon options by Fourier transformation of the characteristic function of the linear combination of state variables and subsequently the pricing of swaption using stochastic duration approximation. For the modeling of equity, we used the Heston model along with stochastic interest rate from above model to better capture the equity volatility. Similarly, we first showed the closed-form of discounted characteristic function of log equity price by solving a system of ODEs resulting from an affine PDE. Afterwards, we derived the price of European options by Fourier techniques as well. Finally, we described the Euler discretization scheme and variance reduction technique of antithetic variates for economic scenario generation .

For the valuation model, we built a stochastic cash flow projection model to capture the development of balance sheet as well as the asset portfolio consisting of coupon bonds and stocks and the liability portfolio consisting of participating life insurance contracts. We then did market consistent valuation of assets and liabilities based on the cash flows projected by the stochastic model along with the input of risk neutral economic scenarios. Furthermore, we modeled the management rules. For instance, we developed a constant asset allocation strategy to rebalance the asset portfolio. We considered the unrealized gain and loss by modeling the book value and market value of assets. Additionally, we modeled the MUST-case for surplus distribution for the profit sharing between shareholder and policyholders, etc.

For the risk capital model, we first implemented the nested stochastic simulation to determine the required risk capital. Since nested simulation requires high computational time, we also investigated the proxy methods of least squared Monte Carlo, replicating portfolio and curve fitting. In particular, we developed a general strategy to construct a good replicating portfolio. First, we described the construction of asset pool. Second, we illustrated the construction of sensitivity sets through recalibration or reweighting techniques. Third, we proposed a calibration procedure, by using the least square optimization and subset selection with certain criteria, to select the optimal replicating portfolio and calculate the required capital.

Finally, we performed an empirical application to illustrate how to conduct market consistent valuation and required risk capital calculation through the partial internal model. We first calibrated the ESG models to the real market data and then generated the economic scenarios through Monte Carlo simulation. We did appropriate validations by doing the martingale tests for risk neutrality, the comparison of model and market option prices, the comparison of simulated and historical distributions of risk factors etc. Afterwards, we performed the market consistent valuation with leakage-test for checking the correctness of all cash flows projection. We calculated the available capital as the average of sum of discounted shareholder's cash flows generated by the stochastic model. We further estimated the required capital by the nested simulation as well as the replicating portfolio proxy method by following the general strategy. We then illustrated that the replicating portfolio method produces a good approximation of SCR by comparing it to nested simulation method.

There are many further studies to extend the current research. For the input model, as the credit spread risk is quite important in practice, one could further investigate a stochastic credit spread model and integrate into the ESG. For valuation model, one could consider more life products such as life annuities, costs and lapse as well as more management rules for surplus distribution. One could then perform sensitivity analyses for some parameters such as the guaranteed interest rates. For the risk model, one could implement all proxy methods and compare the results. Regarding to the replicating portfolio, instead of subset selection, one could also investigate the shrinkage method of LASSO as well as other machine learning techniques. Moreover, given the framework of such partial internal model, one could do future researches, such as developing new insurance products under low interest rate environment, constructing a benchmark portfolio for validation purpose, constructing hedging strategy based on the replicating portfolio to reduce the required capital, doing capital allocation etc.

In summary, we comprehensively showed the usage of stochastic methods in risk management.

A. Definition of the financial assets in the asset pool

A.1. Interest rate related financial assets

In this section, we give the basic definitions of interest rate related financial assets which follows the textbook of Brigo and Mercurio (2006).

Risk free zero coupon bond

Let $P(t,T)$ be the value of time t of a risk free zero coupon bond with maturity T. The cash payment is only occurred at the maturity time T with value of 1 and without any periodic coupon payments.

Total return index of risk free zero coupon bond

For given maturity T, the total return index of risk free zero coupon bond at tme $t \leq T$, is defined as

$$TRI(t;T) = \prod_{i=1}^{t} \frac{P(i,T)}{P(i-1,T)}. \tag{A.1}$$

Total return index of constant (time to) maturity risk free zero coupon bond

For given time to maturity $\widetilde{T}$, the total return index of constant (time to) maturity risk free zero coupon bond at tme $t \leq T$ is defined as

$$TRI_{CM}(t;\widetilde{T}) = \prod_{i=1}^{t} \frac{P(i,i+\widetilde{T})}{P(i-1,i-1+\widetilde{T})}. \tag{A.2}$$

Interest rate swaps

The Interest-Rate Swap (IRS) is a contract that exchanges payments between fixed and floating legs, starting from a future time instant. Let $\mathcal{T} := \{T_\alpha, \ldots, T_\beta\}$ be the payment dates and $\tau := \{\tau_{\alpha+1}, \ldots, \tau_\beta\}$ be the time fractions between two payment dates with $\tau_i = T_i - T_{i-1}$. Let $L(T_{i-1}, T_i)$ be the interest rate resetting at instant time T_i. The payment of fixed leg is the amount of $\tau_i K$, whereas the payment of the floating leg is $\tau_i L(T_{i-1}, T_i)$. The Payer (Forward-start) Interest-Rate Swap (PFS) is defined as paying the fixed legs and receiving the floating legs, whereas the Receiver (Forward-start) Interest-Rate Swap (RFS) is defined as receiving the fixed legs and paying the floating legs. Therefore, for a PFS, the cash flow at time T_i is $(L(T_{i-1}, T_i) - K)$. Then the discounted payoff at time $t < T_\alpha$ of a PFS can be written as

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K)$$

and find the value of PFS as:

$$\begin{aligned} PFS(t, \mathcal{T}, \tau, K) &= \mathbb{E}_t\left(\sum_{i=\alpha+1}^{\beta} D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K)\right) \\ &= \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)\mathbb{E}_t^i(F(T_{i-1}, T_{i-1}, T_i) - K) \\ &= \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(F(t; T_{i-1}, T_i) - K) \\ &= P(t, T_\alpha) - P(t, T_\beta) - \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) \end{aligned} \tag{A.3}$$

where $\mathbb{E}_t^i$ means the conditional expectation under forward measure $\mathbb{Q}^{T_i}$ with numeraire $P(t, T_i)$. $F(t; T_{i-1}, T_i)$ is the simply-compounded forward interest rate with $F(t; T_{i-1}, T_i) = \frac{1}{\tau_i}(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1)$.

Swaption

The swaption or swap option is the option on the IRS. A European payer swaption is an option grants the buyer the right, but not the obligation, to enter a payer swaption. The market quotes of the swaptions are usually including two parts: the option maturity (expiry) and the length of the swaption (tenor of swaption). For the option maturity is the first reset date or settlement date of the underlying IRS, can be denoted by T_α, whereas, the tenor of swaption can be denoted by $T_\beta - T_\alpha$.

According to (A.3), The discounted payoff of the payer swaption at its first reset date T_α can be calculated as :

$$\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F(T_\alpha, T_{i-1}, T_i) - K) = \mathcal{A}_{\alpha,\beta}(T_\alpha)[S_{\alpha,\beta}(T_\alpha) - K]^+, \tag{A.4}$$

where $\mathcal{A}_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$. Therefore, for the payer swaption, the cash flow will only generated at the option expiry time T_α with value of $\mathcal{A}_{\alpha,\beta}(T_\alpha)[S_{\alpha,\beta}(T_\alpha) - K]^+$.

The option will be exercised only if this value is positive. So then discount the payoff to the current time t is:

$$D(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F(T_\alpha, T_{i-1}, T_i) - K)\right)^+ = D(t, T_\alpha)\mathcal{A}_{\alpha,\beta}(T_\alpha)[S_{\alpha,\beta}(T_\alpha) - K]^+.$$

In practice, we value swaptions with a Black-like formula, by assuming the swap rate is lognormal distribution.[1] The formulation of the price of PFS (at time zero) is given by:

$$PS^{Black}(0, \mathcal{T}, \tau, K, \sigma_{\alpha,\beta}) = Bl(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, 1) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i), \tag{A.5}$$

where

$$\begin{aligned} Bl(K, F, v, \omega) &= F\omega\Phi(\omega d_1(K, F, v)) - K\omega\Phi(\omega d_2(K, F, v)) \\ d_1(K, F, v) &= \frac{\log(F/K) + (v^2/2)T}{v\sqrt{T}} \\ d_2(K, F, v) &= \frac{\log(F/K) - (v^2/2)T}{v\sqrt{T}} \end{aligned}$$

with Φ denoting the standard Gaussian cumulative distribution function. $\sigma_{\alpha,\beta}$ is now a volatility parameter quoted in the market. A similar formula is used for a receiver swaption, which gives the holder the right to enter at time T_α a receiver IRS, with payment dates in $\mathcal{T}$. The formula is:

$$RS^{Black}(0, \mathcal{T}, \tau, K, \sigma_{\alpha,\beta}) = Bl(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, -1) \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i).$$

A payer swaption is said to be at-the-money (ATM) if and only if

$$K = K_{ATM} := S_{\alpha,\beta}(0) = \frac{P(0, T_\alpha) - P(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)},$$

while in-the-money (ITM) for $K < K_{ATM}$ and out-of-the-money (OTM) for $K > K_{ATM}$, with the converse holding for a receiver swaption.

A.2. Equity related financial assets

Total return index of equity and European option on equity price

Let $S(t)$ denote the spot price of equity stock at t and q is the dividend yield. A total return index $T(t)$ for equity can be calculated by

$$T(t) = T(t-1)\frac{S(t)}{S(t-1)}(1+q), \quad \text{for} \quad t > 0 \quad \text{and} \quad T(0) = 1. \tag{A.6}$$

[1] Under current negative interest rate environment, new convention such as the shifted lognormal or normal distribution assumption on forward swap might be used which leads to normal implied volatility or shifted Black implied volatility, see e.g. d-fine (2012).

The payoff of a European Call Option on the price of equity with maturity T and strike K is given by

$$H(T) = [S(T) - K]^+. \tag{A.7}$$

Therefore, for the European Call option, the cash flow will only be generated only at the maturity time T with value of $[S(T) - K]^+$.

Under the risk-neutral valuation principle its price at time t is given by

$$C(t; T, K) = \mathbb{E}_t^{\mathbb{Q}} \left[(S(T) - K)^+ D(t, T) \right]$$

Since the equity index is dividend paying, the Black formula based on forward price is usually preferred to price the option, i.e. (see Joshi (2003, section 6.15))

$$C(t; T, K) = P(t, T) Bl(K, F_T(t), \sigma(T, K), 1). \tag{A.8}$$

The forward price of equity is defined as $F_T(t) = S(t) \exp((R(t, T) - q)(T - t))$ (see Rebonato (2004, section 3.2)). $F_T(T)$ equals to $S(T)$ at time T. Finally, the implied volatility $\sigma(T, K)$ could be derived by Equation (A.8).

B. Linear model selection and regularization

B.1. Linear regression models and least-squares computations

Let $X_1, \ldots, X_k$ be the explanatory variables, where X_1 could be identically equal to one in some practical cases and Y be the dependent variable, the linear regression model is used to model linear relationship between explanatory variables and dependent variables, i.e.

$$Y = \sum_{j=1}^{k} \beta_j X_j + \epsilon \tag{B.1}$$

where ϵ is the *error or disturbance* term to capture all other factors that influence the dependent variable other than the explanatory variables.

Let $\boldsymbol{y} = (y_1, \ldots, y_n)'$ be the n observations of the dependent variable and $\boldsymbol{x_j} = (x_{1j}, \ldots, x_{nj})'$ be the n observations of the j-th explanatory variable X_j. Note that a column vector of values will be always denoted by a lowercase letter with boldface. A matrix will always be denoted by a boldface upper letter. For instance, let $\boldsymbol{X} = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_k})$ be the $n \times k$ matrix where each column is an $n \times 1$ vector. Then we have:

$$y_i = \sum_{j=1}^{k} \beta_j x_{ij} + \epsilon_i \tag{B.2}$$

and the corresponding matrix form could be written as:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon} \tag{B.3}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)'$ and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)'$.

The objective is to use observations in a sample to estimate the unknown parameters or coefficients $\boldsymbol{\beta}$ in the model, and then later to predict the value of Y. Before the estimation of the parameters, we need to distinguish between population quantities such as $\boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ and the sample estimates of them, $\boldsymbol{b}$ and $\boldsymbol{e}$ (see Greene (2012, p.26, Chapter 3)). Given the estimate of $\boldsymbol{\beta}$, i.e. $\boldsymbol{b}$, we so called *residual* $\boldsymbol{e}$ is calculated as $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{Xb}$. Therefore, we have

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon} = \boldsymbol{Xb} + \boldsymbol{e}. \tag{B.4}$$

The least squares estimation of the coefficients $\boldsymbol{\beta}$ is performed by minimizing the residual sum of squares:

$$RSS(\boldsymbol{\beta}) = \sum_{i=1}^{n}(y_i - \sum_{j=1}^{k}\beta_j x_{ij})^2 = (\boldsymbol{y} - \boldsymbol{X\beta})'(\boldsymbol{y} - \boldsymbol{X\beta}). \tag{B.5}$$

Differentiating the (B.5) with respect to $\boldsymbol{\beta}$ and setting the first derivatives to zero, under the assumption that there is no linear dependent or perfect multicollinearity among the explanatory variables (i.e. $\boldsymbol{X}$ has full column rank, $\boldsymbol{X'X}$ is positive definite and hence invertible) we get the unique solution for the estimates of the coefficients as follows:

$$\boldsymbol{b} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'y}. \tag{B.6}$$

Another way to do the estimation of coefficients could be done by the triangular factorization based methods. The matrix $\boldsymbol{X}$ could be factored as

$$\boldsymbol{X} = \boldsymbol{QR} \tag{B.7}$$

where $\boldsymbol{Q}$ is the $n \times k$ orthogonal matrix, i.e. $\boldsymbol{Q'Q} = \boldsymbol{I}$ where $\boldsymbol{I}$ is $k \times k$ identity matrix and $\boldsymbol{R}$ is the $k \times k$ upper triangular matrix.

According to the QR decomposition, each column of $\boldsymbol{X}$ could be represented by a linear combination of orthogonal vectors $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_k$, i.e.

$$\begin{aligned}
\boldsymbol{x}_1 &= r_{11}\boldsymbol{Q}_1 \\
\boldsymbol{x}_2 &= r_{12}\boldsymbol{Q}_1 + r_{22}\boldsymbol{Q}_2 \\
\boldsymbol{x}_3 &= r_{13}\boldsymbol{Q}_1 + r_{23}\boldsymbol{Q}_2 + r_{33}\boldsymbol{Q}_3, etc.
\end{aligned}$$

where r_{ij} is the element of matrix $\boldsymbol{R} = \{r_{ij}\}$. We could have

$$\boldsymbol{y} = \gamma_1\boldsymbol{Q}_1 + \ldots + \gamma_k\boldsymbol{Q}_k + \boldsymbol{e} = \boldsymbol{Q\gamma} + \boldsymbol{e} \tag{B.8}$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)'$ are the least-squares estimates by running regression of Y upon on the $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_k$. And the values are:

$$\boldsymbol{\gamma} = (\boldsymbol{Q'Q})^{-1}\boldsymbol{Q'y} = \boldsymbol{Q'y}. \tag{B.9}$$

As we know

$$\boldsymbol{y} = \boldsymbol{Xb} + \boldsymbol{e} = \boldsymbol{Q\gamma} + \boldsymbol{e}, \tag{B.10}$$

then by substituting $\boldsymbol{\gamma} = \boldsymbol{Q'y}$ and $\boldsymbol{X} = \boldsymbol{QR}$, we have

$$\boldsymbol{b} = \boldsymbol{R}^{-1}\boldsymbol{Q'y}. \tag{B.11}$$

The orthogonal reduction method based on QR decomposition is quite meaningful in the context of subset selection. First of all, it could speed up the subset selection. According to (B.7), we have $\boldsymbol{X'X} = \boldsymbol{R'R}$, i.e. $\boldsymbol{R}$ is the Cholesky factor of $\boldsymbol{X'X}$. This means that if we run the regression of Y against a subset of $\boldsymbol{X}$ such as the first p columns of $\boldsymbol{X}$, we can just use the same calculations of the regression of Y against $\boldsymbol{X}$ by skipping the last $(k-p)$ rows and columns. If the subset variables are not in the first p columns, one could use planar rotation algorithm to change the order of variables and then omitting the later variables.

The sum of squares of Y is

$$\boldsymbol{y}'\boldsymbol{y} = \boldsymbol{\gamma}_1^2 + \boldsymbol{\gamma}_1^2 + \ldots + \boldsymbol{\gamma}_k^2 + \boldsymbol{e}'\boldsymbol{e}. \tag{B.12}$$

The residual sum of squares after regressing Y against $X_1, X_2, \ldots, X_p$ is

$$\boldsymbol{\gamma}_{p+1}^2 + \ldots + \boldsymbol{\gamma}_k^2 + \boldsymbol{e}'\boldsymbol{e}. \tag{B.13}$$

In the subset selection, the square of the i-th projection, is the reduction in the residual sum of squares when the variable in position i is added to the linear model containing the first $(i-1)$ variables.

Furthermore, it has the advantage of accuracy. The estimation of coefficients based on QR decomposition given in (B.11) is more accuracy than the methods based on the $\boldsymbol{X'X}$ and its inverse given in (B.6). (see Miller (2002, p.31)). The accuracy is quite important for the subset selection. There are reasons as given by Miller (2002). Firstly, there is a choice of selection among high co-linearity variables during the subset selection procedure. Secondly, the searching procedures of best subsets require a very substantial number of arithmetic operations, the rounding errors should be accumulated as slowly as possible.

B.2. Subset selection

There are a number of different approaches to choose the optimal subset. It is feasible to find the best subset through exhaustive evaluation of all subsets by examining the residual sum of squares. The number of possible subsets of one or more variables out of n is $2^n - 1$ and efficient algorithm such as *branch-and-bound* techniques (see Furnival and Wilson (1974)) is used to perform the exhaustive search. However, it is still quite time consuming and therefore not quite feasible in practice if the number of possible subsets are quite large. In order to overcome the computational disadvantage, some greedy algorithms such as *forward selection*, *Efroymson's stepwise regression*, *backward elimination*, *sequential replacement algorithms* and so on are used to find the best-fitting subset instead of exhaustive search.

The forward selection starts with one variable and adds sequentially more variables that could improve most the fit (i.e. reduce the residual sum of squares) until some stopping criteria is satisfied.

Efroymson's stepwise regression proposed by Efroymson (1960) is an extension on forward selection. After the addition of one new variable into selected set, a test is performed to check if any variable of the previously selected variables could be deleted with relative low increase of the residual sum of squares.

The backward elimination starts with all variables and deletes sequentially one variable that has the least impact on the fit (i.e. yields smallest residual sum of squares after deletion) until some stopping criteria is satisfied.

The sequential replacement algorithm tries to replace any of selected variables with another gives smaller residual sum of squares provided that two or more variables have been selected. Sequential replacement requires more computation time but improves the chances of finding the best-fitting subset than the than forward selection or the Efroymson algorithm.

More descriptions of these algorithm could be seen in Miller (2002).

There are many criteria could be used to determine which subset should be chosen and stop the subset selection.

First of all, we look at the coefficient of determination R^2, which is a measure of how well the regression line fitted to the data. It is defined as:

$$R_p^2 = 1 - \frac{RSS_p}{RSS_1}, \tag{B.14}$$

where RSS_p is the residual sum of squares of model with p explanatory variables and RSS_1 is the total sum of squares for the linear regression with intercept. It is not a good criterion, since R^2 is always increasing when one more variable is added into selected subset. For this reason, the adjusted R^2 that penalizes the number of parameters in the model is preferred and given as:

$$\bar{R}_p^2 = 1 - (1 - R_p^2)\frac{n-1}{n-p} \tag{B.15}$$

Note that for linear regression with no intercept, the R^2 is redefined as $R_p^2 = 1 - \frac{RSS_p}{RSS_0}$ where RSS_0 is the total sum of squares for the regression with no intercept and $\bar{R}_p^2 = 1 - (1 - R_p^2)\frac{n-1}{n-p}$.

Another criterion is the Mallows' C_p (see Mallows (1973)) defined by

$$C_p = \frac{RSS_p}{\sigma^2} - (n - 2p), \tag{B.16}$$

where σ^2 is replaced by the unbiased estimate of residual variance $\hat{\sigma}^2 = \frac{RSS_k}{n-k}$ for the full model under consideration, which includes all k explanatory variables.

For a model that fits the data adequately $E(C_p)$ is approximately p and therefore, C_p itself should be approximately equals to p for an adequate model. The criterion clearly can be used to compare subsets of the same size, but it can also be used more generally by looking for those models for which $C_p \approx p$. The quantity estimated by C_p is the normalized mean squared prediction error (MSPE) with p predictors.

An alternative family of criteria is based on likelihood. Assume that the disturbances ϵ_i are i.i.d normal distributed, the observations y_i are also normal distributed conditioned on given data $(x_{i1}, \ldots, x_{ik})$ and the corresponding density for model M_p with p predictors is:

$$f(y_i; \sum_{j\in M_p} x_{ij}\beta_j, \sigma_p^2, M_p) = \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left\{ -\frac{\left(y_i - \sum_{j\in M_p} x_{ij}\beta_j\right)^2}{2\sigma_p^2} \right\}. \tag{B.17}$$

The likelihood based on the n i.i.d observations could be given as:

$$l(\boldsymbol{\beta}, \sigma_p^2; \boldsymbol{X}, \boldsymbol{y}, M_p) = \prod_{i=1}^{n} f(y_i; \sum_{j\in M_p} x_{ij}\beta_j, \sigma_p^2). \tag{B.18}$$

The log-likelihood by taking the natural log of likelihood is

$$L_p := \ln l(\boldsymbol{\beta}, \sigma^2; \boldsymbol{X}, \boldsymbol{y}, M_p) = -\frac{n}{2}\ln(2\pi\sigma_p^2) - RSS_p/(2\sigma_p^2). \tag{B.19}$$

The maximum likelihood estimate of σ_p^2 is $\hat{\sigma}_p^2 = RSS_p/n$, and then the maximum value of log likelihood is

$$L_p = -\frac{n}{2}\ln(2\pi\hat{\sigma}_j^2) - \frac{n}{2} = \text{Const} - \frac{n}{2}\ln(RSS_p). \tag{B.20}$$

The Akaike information criterion (AIC) is

$$AIC_p = -2L_p + 2p. \tag{B.21}$$

The Schwarz criterion, often known as the Bayesian Information Criterion (BIC) is

$$BIC_p = -2L_p + p\ln(n). \tag{B.22}$$

B.3. Shrinkage method

The LASSO (see Tibshirani (1996)) is a shrinkage method defined by:

$$\begin{aligned} &\min_{\beta} \sum_{i=1}^{N} (y_i - \sum_{j=1}^{p} x_{ij}\beta_j)^2 \\ &s.t. \sum_{j=1}^{p} |\beta_j| \leq b. \end{aligned} \tag{B.23}$$

The closely related optimization problem is:

$$\min_{\beta} \quad \frac{1}{2}\sum_{i=1}^{N}(y_i - \sum_{j=1}^{p} x_{ij}\beta_j)^2 + \lambda\sum_{j=1}^{p}|\beta_j| \tag{B.24}$$

Problems (B.23) and (B.24) are equivalent, since for a given λ, $0 \leq \lambda < \infty$, there exists a $b \geq 0$ such that the two problems share the same solution, and vice versa. (see Osborne et al, Chen et al. (1998)). Note that with sufficient small b, some of the coefficients turns to be exactly zero due to the nature of the constraint. Therefore, the LASSO treats as a kind of continuous subset selection.

C. Linear Kalman filter

The Kalman filter developed by Kalman (1960) is an approach to linear filtering and prediction, which could be applied to various models, in particular the state space model that including unobservable variables. In the following we summarize the main steps of linear Kalman filter algorithm (see more details in Chapter 2 of Tanizaki (1996)).

Let m be the dimension of the state variables, d be the dimension of the observations, and n the number of observations. In general, *a state space model* with unobservable variable can be represented as following two equations:

$$\text{Measurement equation} \qquad y_t = c_t + Z_t\alpha_t + G_t\epsilon_t, \tag{C.1}$$

$$\text{Transition equation} \qquad \alpha_t = d_t + T_t\alpha_{t-1} + H_t\eta_t, \tag{C.2}$$

where y_t is the observed data and α_t is the unobservable state variable. $\epsilon_t \sim N(0, I_d)$ and $\eta_t \sim N(0, I_m)$ are unit matrices. We assume that both of ϵ_t and η_t are serially uncorrelated. Furthermore, ϵ_t and η_s are uncorrelated for all time periods t, s and uncorrelated with the initial state variable. The dimensions of the parameters are:

$$y_t \in \mathbb{R}^d,\ c_t \in \mathbb{R}^d,\ Z_t \in \mathbb{R}^{d\times m}, G_t \in \mathbb{R}^{d\times d},\ \epsilon_t \in \mathbb{R}^d$$

$$\alpha_t \in \mathbb{R}^m,\ d_t \in \mathbb{R}^m,\ T_t \in \mathbb{R}^{m\times m},\ H_t \in \mathbb{R}^{m\times m}, \eta_t \in \mathbb{R}^m.$$

Let Y_s denote the information of the observation up to time s, i.e., $Y_s := \{y_1, y_2, \cdots, y_s\}$. Let $a_{t|s} := \mathbb{E}[\alpha_t|Y_s]$ and $\Sigma_{t|s} := \mathbb{C}ov[\alpha_t|Y_s]$ be the conditional expectation and covariance of α_t given Y_s. Similarly, let $y_{t|s} := \mathbb{E}[y_t|Y_s]$ and $F_{t|s} := \mathbb{C}ov[y_t|Y_s]$ be the conditional expectation and variance of y_t given Y_s.

The algorithm of Kalman filter is given by following equations.

- **Step 1: Initialization.** The first step is to give the initial values of state variables. Here we use the unconditional mean and variance for $a_{0|0}$ and $\Sigma_{0|0}$, i.e.

$$a_{0|0} = \lim_{t\to\infty} \mathbb{E}[\alpha_t] \tag{C.3}$$

$$\Sigma_{0|0} = \lim_{t\to\infty} \mathbb{V}ar[\alpha_t], \tag{C.4}$$

- **Step 2: Prediction.** In this step, it gives the prediction of α_t and y_t and associated variance conditional on the information up to time $t-1$.

$$a_{t|t-1} = T_t a_{t-1|t-1} + d_t, \tag{C.5}$$

$$\Sigma_{t|t-1} = T_t\Sigma_{t-1|t-1}T_t' + H_tH_t', \tag{C.6}$$

$$y_{t|t-1} = Z_t a_{t|t-1} + c_t, \tag{C.7}$$

$$F_{t|t-1} = Z_t\Sigma_{t|t-1}Z_t' + G_tG_t'. \tag{C.8}$$

- **Step 2: Kalman gain.** The Kalman gain K_t is chosen such that the $a_{t|t}$ has minimum variance.
$$K_t = \Sigma_{t|t-1} Z_t' F_{t|t-1}^{-1}. \tag{C.9}$$

- **Step 3: Updating.** In this step, it updates the state variable and associated variance by combining the new observation obtained at time t (i.e. y_t) .
$$\Sigma_{t|t} = \Sigma_{t|t-1} - K_t F_{t|t-1} K_t', \tag{C.10}$$
$$a_{t|t} = a_{t|t-1} + K_t(y_t - y_{t|t-1}). \tag{C.11}$$

- **Step 4: Maximum likelihood function.** $a_{t|t}$ and $\Sigma_{t|t}$ for $t = 1, \ldots, T$ are then calculated recursively by the previous three steps, once $a_{0|0}$, $\Sigma_{0|0}$, H_t and G_t are given. Under the assumption of normality of error terms ϵ_t and η_t, we have $y_t|_{Y_{t-1}} \sim N(y_{t|t-1}, F_{t|t-1})$ and the maximum likelihood function could be constructed as
$$\begin{aligned} & \mathbb{P}[y_T, y_{T-1}, \cdots, y_1] \\ = & \prod_{t=1}^{T} \mathbb{P}[y_t|Y_{t-1}] \\ = & \prod_{t=1}^{T} (2\pi)^{-\frac{m}{2}} |F_{t|t-1}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y_t - y_{t|t-1})' F_{t|t-1}^{-1} (y_t - y_{t|t-1})\right). \end{aligned} \tag{C.12}$$

Finally the parameters could be estimated by maximizing the (log-) likelihood function.
$$\ln L(\Theta) = -\frac{mT}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln|F_{t|t-1}| - \frac{1}{2}\sum_{t=1}^{T}(y_t - y_{t|t-1})' F_{t|t-1}^{-1}(y_t - y_{t|t-1}) \tag{C.13}$$

Bibliography

A. M. Best (2013). Risk management and the rating process for insurance companies. Available at: `http://www3.ambest.com/ambv/ratingmethodology/OpenPDF.aspx?rc=197707` [Accessed Oct 10, 2016].

Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking and Finance* **26**: 1505–1518.

Acerbi, C. and D. Tasche (2002). On the coherence of expected shortfall. *Journal of Banking & Finance* **26**(7): 1487–1503.

AIRMIC, ALARM, IRM (2002). A risk management standard. Available at: `https://www.theirm.org/media/886059/ARMS_2002_IRM.pdf` [Accessed Oct 11, 2016].

AIRMIC, Alarm, IRM (2010). A structured approach to Enterprise Risk Management (ERM) and the requirements of ISO 31000. Available at: `https://www.theirm.org/media/886062/ISO3100_doc.pdf` [Accessed Oct 11, 2016].

Aït-Sahalia, Y. and R. Kimmel (2007). Maximum likelihood estimation of stochastic volatility models. *Journal of Econometrics* **83**: 413–452.

Albrecher, H., P. Mayer, W. Schoutens, and J. Tistaert (2006). The little Heston trap.

Albrecht, P. (2004). Risk measures. *Encyclopedia of Actuarial Science* .

Andersen, L. (2008). Efficient simulation of the Heston stochastic volatility model. *The Journal of Computational Finance* **11**(3): 1–42.

Artzner, P., F. Delbaen, J. Eber, and D. Heath (1999). Coherent measures of risk. *Mathematical Finance* **9**(3): 203–228.

Arvanitis, A., J. Gregory, and J.-P. Laurent (1998). Building models for credit spreads. Available at: `http://www.ressources-actuarielles.net/EXT/ISFA/1226.nsf/0/7c1d935203184237c1257a4f006b127a/$FILE/building_models_for_credit_spreads.pdf`. [Accessed Oct 10, 2016].

Avellaneda, M., R. Buff, C. Friedman, N. Grandchamp, L. Kruk, and J. Newman (2001). Weighted Monte Carlo: a new technique for calibrating asset-pricing models. *International Journal of Theoretical and Applied Finance* **4**(1): 91–119.

Aven, T. (2010). Some reflections on uncertainty analysis and management. *Rel. Eng. & Sys. Safety* **95**(3): 195–201.

Aven, T. (2011). *Quantitative Risk Assessment*. Cambridge University Press.

Aven, T. and O. Renn (2009). On risk defined as an event where the outcome is uncertain. *Journal of Risk Research* **12**: 1–11.

Bacinello, A. R. (2001). Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. *Astin Bulletin* **31**(2): 275–298.

Bacinello, A. R. (2003). Fair valuation of a guaranteed life insurance participating contract embedding a surrender option. *Journal of risk and insurance* **70**(3): 461–487.

Barrie & Hibbert (2010). LMMPlus – a new addition to Barrie & Hibbert's ESG 7 armoury. Available at: `http://s3.amazonaws.com/zanran_storage/www.barrhibb.com/ContentPages/2465694088.pdf` [Accessed Nov 09, 2016].

Basel Committee on Banking Supervision (2001). The New Basel Capital Accord.

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. *The Review of Financial Studies* **9**(1): 66–107.

Bates, D. S. (2006). Maximum likelihood estimation of latent affine processes. *The Review of Financial Studies* **19**(3): 909–965.

Bauer, D., D. Bergmann, and A. Reuss (2009). Solvency II and Nested Simulations - a Least-Squares Monte Carlo Approach. Preprint Series, Ulm University. Available at: `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi2/dokumente/preprint-server/2009/0905_200905_solvency_preprint-server.pdf` [Accessed Oct 11, 2016].

Bauer, D., D. Bergmann, and A. Reuss (2010). On the calculation of the Solvency Capital Requirement based on nested simulations. Available at: `https://www.ifa-ulm.de/fileadmin/user_upload/download/forschung/2012_ifa_Bauer-etal_On-the-Calculation-of-the-S%olvency-Capital-Requirement-based-on-Nested-Simulations.pdf` [Accessed Aug 6, 2018].

Bauer, D. and H. Ha (2013). A least-squares Monte Carlo approach to the calculation of capital requirements. Technical report, Georgia State University.

Bauer, D., R. Kiesel, A. Kling, and J. Ruß(2006). Risk neutral valuation of with-profits life insurance. *Insurance: Mathematics and Economics* **39**: 171–183.

Bergmann, D., A. Reuss, A. Siebert, G. Stahl, and H.-J. Zwiesler (2009). Computational aspects of nested Monte Carlo simulations for risk management purposes. Working Paper.

Bickel, P. and D. Freedman (1981). Some asymptotic theory for the bootstrap. *Annals of Statistics* **9**: 1196–1217.

Bingham, N. and R. Kiesel (2004). *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivative*. Springer, 2nd edition.

Boekel, P., L. van Delft, T. Hoshine, R. Ino, C. Reynolds, and H. Verheugen (2009). Replicating Portfolios. An Introduction: Analysis and Illustrations. Available at: `https://web.actuaries.ie/sites/default/files/erm-resources/replicating_portfolios_rr.pdf` [Accessed Aug 6, 2018].

de Boer, P. (2009). Risk neutral versus real world valuation. Available at: `http://dare.uva.nl/cgi/arno/show.cgi?fid=164512` [Accessed Nov 09, 2016].

Bolder, D. J. (2001). Affine term-structure models: Theory and implementation. Working paper. Bank of Canada. Available at: `https://www.bankofcanada.ca/wp-content/uploads/2010/02/wp01-15a.pdf` [Accessed Aug 6, 2018].

Bonami, P. and M. Lejeune (2009). An exact solution approach for integer constrained portfolio optimization problems under stochastic constraints. *Operations Research* **57**: 650–670.

Bowers, N. L., H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt (1997). *Actuarial Mathematics*. The Society of Actuaies.

Brigo, D. and F. Mercurio (2006). *Interest Rate Models- Theory and Practice*. Springer, 2nd edition.

Briys, E. and F. de Varenne (1997). On the risk of insurance liabilities: Debunking some common pitfalls. *Journal of Risk and Insurance* : 673–694.

Broadie, M., Y. Du, and C. C. Moallemi (2011). Efficient risk estimation via nested sequential simulation. *Management Science* **57**(6): 1172–1194.

Broadie, M. and O. Kaya (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations Research* **54**(2): 217–231.

Burkhart, T., A. Reuß, and H.-J. Zwiesler (2014). Participating life insurance contracts under Solvency II: Inheritance effects and allowance for a going concern reserve. Preprint Series. Available at: `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi2/dokumente/preprint-server/2014/1405-paper_gcr.pdf` [Accessed Aug 6, 2018].

Burmeister, C. and H. Mausser (2009). Using trading restrictions in replicating portfolios. *Life & Pension* : 36–40.

Burmeister, C., H. Mausser, and O. Romanko (2010). Using trading costs to construct better replicating portfolio. Technical report, Algorithmics Softward LLC.

Cardi, G. G. and R. Rusnak (2007). When the SST standard model underestimates market risk. Available at: https://www.finma.ch/FinmaArchiv/bpv/download/f/DeltaGammaCardiRusnak.pdf [Accessed Aug 6, 2018].

Carr, P. and D. B. Madan (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance* **3**: 463–520.

Casualty Actuarial Society Enterprise Risk Management Committee (2003). Overview of Enterprise Risk Management. Available at: https://www.casact.org/area/erm/overview.pdf [Accessed Oct 11, 2016].

CEIOPS (2009). CEIOPS' Advice for Level 2 Implementing Measures on Solvency II: Articles 120 to 126 Tests and Standards for Internal Model Approval. CEIOPS-DOC 48/09.

CFO Forum (2009). Market Consistent Embedded Value Principles. Technical report. Available at: http://www.cfoforum.eu/downloads/MCEV_Principles_and_Guidance_October_2009.pdf [Accessed Oct 11, 2016].

Chen, P. and L. Scott (1992). Pricing interest rate options in a two factor Cox-Ingersoll-Ross model of the term structure. *Review of Financial Studies* **5**: 613–636.

Chen, R.-R. and L. Scott (1993). Multi-factor Cox-Ingersoll-Ross models of the term structure: Estimates and tests from a Kalman filter model. Working paper.

Chen, R.-R. and L. Scott (1995). Interest rate options in multifactor Cox-Ingersoll-Ross models of the term structure. *The Journal of Derivatives* : 53–72.

Chen, S., D. L. Donoho, and M. A. Saunders (1998). Atomic decomposition by basis pursuit. *SIAM Journal on Scientific Computing* **20**(1): 33–61.

Commission of the European Communities (2004). The application of the Lamfalussy process to EU securities markets legislation.

Committee of Sponsoring Organizations of the Treadway Commission (2004). Enterprise risk management – integrated framework. Available at: www.coso.org/documents/coso_erm_executivesummary.pdf [Accessed Oct 11, 2016].

Conference of the Insurance Supervisory Services of the Member States of the European Union (2002). Prudential supervision of insurance undertakings. Report prepared under the chairmanship of Paul Sharma, Head of the Prudential Risks Department of the UK's Financial Services Authority.

Conning (2012). GEMS model guide. Technical report, Conning, INC.

Cont, R., R. Deguest, and G. Scandolo (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quantitative Finance* **10**(6): 593 – 606.

Cox, J., J. Ingersoll, and S. Ross (1985). A theory of the term structure of interest rates. *Econometrica* **53**: 385–407.

d-fine (2012). New volatility conventions for in negative interest rate environment. Available at: `http://www.d-fine.com/fileadmin/d-fine/hochgeladen/Fachartikel/WhitePaper_Vols_NegIR_V1_1_en.pdf` [Accessed September 11, 2016].

Dai, Q. and K. J. Singleton (2000). Specification analysis of affine term stucture models. *The Journal of Finance* **55**(5): 1943–1978.

Dall' Aglio, G. (1956). Sugli estremi dei momenti delle funzioni di ripartizione doppia. *Ann. Scuola Normale Superiore Di Pisa,* **3**(1): 33–74.

DAV (2015). Zwischenbericht zur Kalibrierung und Validierung spezieller ESG unter Solvency II. Available at: `https://aktuar.de/unsere-themen/fachgrundsaetze-oeffentlich/2015-11-09_DAV-Ergebnisbericht_Kalibrierung%20und%20Validierung%20spezieller%20ESG_Update.pdf` [Accessed Nov 09, 2016].

De Felice, M. and F. Moriconi (2005). Market based tools for managing the life insurance company. *Astin Bulletin* **35**(01): 79–111.

Delbaen, F. (2000). Coherent risk measures on general probability spaces. Available at: `https://people.math.ethz.ch/~delbaen/ftp/preprints/RiskMeasuresGeneralSpaces.pdf` [Accessed Aug 6, 2018].

Delbaen, F. and W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* **300**: 463–520.

Delegated Regulation (2015). Commission Delegated Regulation (EU) 2015/35 Directive of 10 October 2014 supplementing Directive 2009/138/EC of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II). Official Journal of the European Union.

Denuit, M., J. Dhaene, M. Goovaerts, and R. Kaas (2005). *Actuarial Theory for Dependent Risks-Measures, Orders and Models.* John Wiley & Sons Ltd.

Directive 2009/138/EC (2009). Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II). Official Journal of the European Union.

Directive 2014/51/EU (2014). Directive 2014/51/EU of the European Parliament and of the Council of 16 April 2014 amending Directives 2003/71/EC and 2009/138/EC and Regulations (EC) No 1060/2009, (EU) No 1094/2010 and (EU) No 1095/2010 in respect of the powers of the European Supervisory Authority (European Insurance and Occupational Pensions Authority) and the European Supervisory Authority (European Securities and Markets Authority). Official Journal of the European Union.

Dubrana, L. (2011). A stochastic model for credit spreads under a risk-neutral framework through the use of an extended version of the Jarrow, Lando and Turnbull model. Available at SSRN: `https://ssrn.com/abstract=1964459` [Accessed Aug 6, 2018].

Duffie, D. and R. Kan (1996). A yield factor model of interest rates. *Mathematical Finance* **6**(4): 379–406.

Duffie, D., J. Pan, and K. Singleton (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68**(6): 1343–1376.

Efroymson, M. A. (1960). *Multiple Regression Analysis.* Mathematical Methods for Digital Computers (Ralston, A. and Wilf, H.S., ed.), John Wiley, New York.

EIOPA (2013). Technical specification on the Long Term Guarantee Assessment.

EIOPA (2014). Technical specification for the preparatory phase.

Eling, M. and S. Holder (2012). Maximum technical interest rates in life insurance: An international overview. Technical report, University of St. Gallen. Working Papers On Risk Management and Insuranc No. 121. Available at: `http://www.ivw.unisg.ch/~/media/internet/content/dateien/instituteundcenters/ivw/wps/wp121.pdf`. [Accessed August 11, 2014].

Eling, M., H. Schmeiser, and J. T. Schmit (2007). The Solvency II process: Overview and critical analysis. *Risk management and insurance review* **10**(1): 69–85.

van den End, J. W. (2013). Statistical evidence on the mean reversion of interest rates. *Journal of Investment Strategies* **2**(3): 91–122.

Erixon, E. and J. Tubis (2008). Replicating portfolios for variable annuity hedging. *Emphasis* **4**.

ERM Committee of the American Academy of Actuaries (2013). Insurance enterprise risk management practices. Available at: `www.actuary.org/files/ERM_practice_note_030713_exposure.pdf` [Accessed Oct 11, 2016].

European Commission (2002). Considerations on the design of a future prudential supervisory system. MARKT/2535/02.

European Commission (2003). Design of a Future Prudential Supervisory System in the EU – Recommendations by the Commission Services. MARKT/2509/03.

Feldhütter, P. and D. Lando (2008). Decomposing swap spreads. *Journal of Financial Economics* **88**: 375–405.

Fernholz, L. T. (1983). *von Mises Calculus for Statistical Functionals.* Lecture notes in statistics v.19. Springer-Verlag New York Inc.

FINMA (2012). *Wegleitung zum SST-Marktrisiko-Standardmodell.*

Föllmer, H. and T. Knispel (2013). Convex risk measures: Basic facts, law-invariance and beyond, asymptotics for large portfolios. *Handbook of the Fundamentals of Financial Decision Making, Part II* : 507–554.

Föllmer, H. and A. Schied (2002). Convex measures of risk and trading constraints. *Finance and stochastics* **6**(4): 429–447.

Föllmer, H. and A. Schied (2011). *Stochastic Finance: An Introduction in Discrete Time*. De Gruyter, 3rd edition.

Föllmer, H. and S. Weber (2015). The axiomatic approach to risk measures for capital determination. *Annual Review of Financial Economics* **7**: 301–337.

Furnival, G. and R. Wilson (1974). Regression by leaps and bounds. *Technometrics* **16**: 499–511.

Gallagher, R. B. (1956). Risk management: A new phase of cost control. *Harvard Business Review* **34**(5): 75–86.

Gatzert, N. and H. Schmeiser (2008). Combining fair pricing and capital requirements for non-life insurance companies. *Journal of Banking & Finance* **32**(12): 2589 – 2596.

Gatzert, N. and H. Wesker (2012). A comparative assessment of Basel II/III and Solvency II. *The Geneva Papers on Risk and Insurance Issues and Practice* **37**: 539–570.

Gerstner, T., M. Griebel, and M. Holtz (2008). A general asset-liability management model for the efficient simulation of portfolios of life insurance policies. *Insurance: Mathematics and Economics* **42**(2): 704–716.

Geyer, A. and S. Pichler (1999). A state-space approach to estimate and test multifactor Cox-Ingersoll-Ross models of the term structure. *Journal of Financial Research* **22**(1): 107–130.

Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering*. Springer, New York.

Gordy, M. B. and S. Juneja (2010). Nested simulation in portfolio risk measurement. *Management Science* **56**(10): 1833–1848.

Graf, S., A. Kling, and J. Ruß (2011). Risk analysis and valuation of life insurance contracts: Combining actuarial and financial approaches. *Insurance: Mathematics and Economics* **49**(1): 115–125.

Greene, W. H. (2012). *Econometric Analysis*. Prentice Hall, 7th edition.

Grosen, A. and P. L. Jørgensen (2000). Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics* **26**(1): 37–57.

Grosen, A. and P. L. Jørgensen (2002). Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *Journal of risk and insurance* **69**(1): 63–91.

Grzelak, L. A. and C. W. Oosterlee (2011). On the Heston model with stochastic interest rate. *SIAM J. Financial Math.* **2**: 255–286.

Gzyl, H. and S. Mayoral (2006). On a relationship between distorted and spectral risk measures. Available at `http://mpra.ub.uni-muenchen.de/916/1/MPRA_paper_916.pdf` [Accessed Aug 6, 2018].

Hampel, F. R. (1974). The influence curve and its role in robust estimation. *Journal of the American Statistical Association* **69**(346): 383–393.

Hansson, S. O. (2012). Risk. In *The Stanford Encyclopedia of Philosophy*, editor E. N. Zalta. Winter 2012 edition. Available at: `http://plato.stanford.edu/archives/win2012/entries/risk/` [Accessed Oct 10, 2016].

Harrell, F. E. and C. Davis (1982). A new distribution-free quantile estimator. *Biometrika* **69**(3): 635–640.

Harris, S. (2009). A rating agency's view of risk management. Available at: `https://www.actuaries.org.uk/documents/rating-agencys-view-risk-management-handout` [Accessed Oct 10, 2016].

Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bonds and currency options. *Review of Financial Studies* **6**: 327–343.

Heyde, C., S. Kou, and X. Peng (2007). What is a good external risk measure: Bridging the gaps between robustness, subadditivity, and insurance risk measures. Technical report, Columbia University. Available at: `http://www.ressources-actuarielles.net/EXT/ISFA/1226.nsf/0/eaa017fa2b5f7e37c12577ae0025963e/$FILE/Heyde_Kou_07_07.pdf` [Accessed Aug 6, 2018].

Holton, G. A. (2004). Defining risk. *Financial Analysts Journal* **60**(6): 19–25.

Hopkin, P. (2010). *Fundamentals of Risk Management: Understaning, evaluating and implementing effective risk management.* The Institute of Risk management.

Hörig, M. and F. Wechsung (2014). Weighted Monte-Carlo for fast re-calibration of pricing scenarios: Applications in insurance risk management. *Der Aktuar* : 21–25.

Huber, P. J. (1981). *Robust Statistics.* Wiely.

Hull, J. and A. White (1990). Pricing interest rate derivative securities. *The Review of Financial Studies* **3**: 573–592.

ISO (2009a). *Risk Management-Principles and guidelines.* ISO 31000:2009.

ISO (2009b). *Risk Management-Vocabulary.* ISO Guide 73:2009.

Jarrow, R., D. Lando, and S. Trunbull (1997). A Markov model for the term structure of credit risk spreads. *The Review of Financial Studies* **10**(2): 481–523.

Johnson, N. and S. Kotz (1970). *Distribution in Statistics: Continuous Univariate Distribution-2.* Boston: Houghton Mifflin Company.

Jørgensen, P. L. (2001). Life insurance contracts with embedded options: Valuation, risk management, and regulation. *Journal of Risk Finance* **3**(1): 19–30.

Joshi, M. (2003). *The Concepts and Practice of Mathematical Finance.* Mathematics, Finance and Risk. Cambridge University Press.

Kalberer, T. (2007). Guaranteed links to the life market. *Life & Pension* : 39–44.

Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance* **15**(3): 425–437.

Kalman, R. E. (1960). A new approach to linear filtering and prediction problems. *Journal of Basic Engineering* **82**: 35–45.

Kaplan, S. and B. Garrick (1981). On the quantitative definition of risk. *Risk Analysis* **1**: 11–27.

Kiesel, R., R. Rühlicke, G. Stahl, and J. Zheng (2016). The Wasserstein metric and robustness in risk management. *Risks* **4**(3): 32.

Kling, A., A. Richter, and J. Ruß(2007). The interaction of guarantees, surplus distribution, and asset allocation in with profit life insurance policies. *Insurance: Mathematics and Economics* **40**(1): 164–178.

KPMG (2002). Study into the methodologies to assess the overall financial position of an insurance undertaking from the perspective of prudential supervision. Available at: `http://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=5DB5C86CDA72CD1F998686D3ED3BFE6C?doi=10.1.1.122.3292&rep=rep1&type=pdf` [Accessed Aug 6, 2018].

Krätschmera, V. and H. Zähle (2011). Sensitivity of risk measures with respect to the normal approximation of total claim distributions. *Insurance: Mathematics and Economics* **49**(3): 335–344.

Kusuoka, S. (2001). On law-invariant coherent risk measures. *Advances in Mathematical Economics* **3**: 83–95.

Lawson, G. (2011). Stochastic volatility jump diffusion with time dependent parameters calibration, dynamics & implementation. Technical report, Barrie & Hibbert.

Lee, R. W. (2004). Option pricing by transform methods: Extensions, unification and error control. *Journal of Computational Finance* **7**(3): 51–86.

Linder, U. and V. Ronkainen (2004). Solvency II – towards a new insurance supervisory system in the EU. *Scand. Actuarial J.* **6**: 462–474.

Litterman, R. and J. Scheinkman (1991). Common factors affecting bond returns. *The Journal of Fixed Income* **June**: 54–61.

Longstaff, F. and E. Schwartz (2001). Valuing American options by simulation: A simple least-squares approach. *The Review of Financial Studies* **14**: 113–147.

Longstaff, F. A. and E. S. Schwartz (1992). Interest rate volatility and the term structure: A two factor general equilibrium model. *Journal of Finance* **47**(4): 1259–1282.

Lord, R., R. Koekkoek, and D. van Dijk (2010). Comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance* **10**(2): 177–194.

Mallows, C. (1973). Some commmnets on C_p. *Technometrics* **15**: 661–675.

Mallows, C. L. (1972). A note on asymptotic joint normality. *The Annals of Mathematical Statistics* **43**(2): 508–515.

Markowitz, H. M. (1952). Portfolio selection. *The Journal of Finance* **7**(1): 77–91.

Miller, A. (2002). *Subset Selection in Regression*. Chapman & Hall/CRC, 2nd edition.

von Mises, R. (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Statist.* **18**: 309–348.

Moody's Research Methodology (2004). Risk management assessments. Available at: `https://www.moodys.com/sites/products/AboutMoodysRatingsAttachments/2002900000432768.pdf` [Accessed Oct 10, 2016].

Morrison, S. (2003). A stochastic model for credit spreads. Technical report, Barrie & Hibbert.

Morrison, S. (2007). Implementation of the extended 2-factor Black-Karasinski model. Technical report, Barrie & Hibbert.

Müller, H. (1997). Solvency of insurance undertaking. Technical report, Conference of the Insurance Supervisory Services of the Member States of the European Union.

Munk, C. (1999). Stochastic duration and fast coupon bond option pricing in multifactor models. *Review of Derivatives Research* **3**: 157–181.

Natolski, J. and R. Werner (2017). Mathematical foundation of the replicating portfolio approach. *Scandinavian Actuarial Journal* **0**(0): 1–24.

Oechslin, J., O. Aubry, M. Aellig, A. Käppeli, D. Brönniman, A. Tandonnet, and G. Valois (2007). Replicating embedded options. *Life & Pension* : 47–52.

Pan, J. (2002). The jump-risk premia implicit in options: evidence from an integrated time-series study. *Journal of Financial Economics* **63**(1): 3–50.

Pfeifer, D. and D. Strassburger (2008). Solvency II: stability problems with the SCR aggregation formula. *Scandinavian Actuarial Journal* **61-77**(1).

Rebonato, R. (2004). *Volatility and Correlation.* John Wiley & Sons, Ltd, 2nd edition.

Rejda, G. E. (2008). *Principles of risk management and insurance.* Pearson Education, 10th edition.

Reuß A., J. Ruß, and J. Wieland (2013). Participating life insurance contracts under risk based solvency frameworks: How to increase capital efficiency by product design. Available at: `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.140/Forschung/reuss_russ_wieland_2013.pdf` [Accessed Aug 6, 2018].

Rühlicke, R. (2013). *Robust Risk Management in the Context of Solvency II Regulations.* Ph.D. thesis, University of Duisburg-Essen.

Salvemini, T. (1943). Sul calcolo degli indici di concordanza tra due caratteri quantitativi. Technical report, Atti della VI Riunione della Soc. Ital. di Statistica, Roma.

Sandström, A. (2007). Solvency II: Calibration for skewness. *Scandinavian Actuarial Journal* **126-134**(2).

Schrager, D. F. and A. Pelsser (2006). Pricing swaptions and coupon bond options in affine term structure models. *Mathematical Finance* **16**(4): 673–694.

Seemann, A. (2009). Replizierende Portfolios in der Lebensversicherung. Preprint Series, Ulm University. Available at: `https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi2/dokumente/preprint-server/2009/0902_replicatingportfolios.pdf` [Accessed Aug 6, 2018].

Singleton, K. J. and L. Umantsev (2002). Pricing coupon-bond options and swaptions in affine term structure models. *Mathematical Finance* **12**(4): 427–446.

Smith, A. and T. Wilson (2001). Fitting yield curves with long-term constraints. Research notes, Bacon and Woodrow.

Standard & Poors (2005a). Evaluating the Enterprise Risk Management practices of insurance companies.

Standard & Poors (2005b). Refining the focus of insurer Enterprise Risk Management criteria.

Standard & Poors (2013). Enterprise Risk Management. Available at: `http://www.maalot.co.il/publications/MT20151123154908.pdf` [Accessed Aug 6, 2018].

Szegö, G. (2002). Measures of risk. *Journal of Banking & Finance* **26**(7): 1253–1272.

Talanx AG (2016). MCEV – Market Consistent Embedded Value Report 2015.

Tanizaki, H. (1996). *Non linear filters, estimation and applications.* Springer-Verlag, Berlin, 2nd edition.

Tanskanen, A. J. and J. Lukkarinen (2003). Fair valuation of path-dependent participating life insurance contracts. *Insurance: Mathematics and Economics* **33**(3): 595–609.

The Committee of Wise Men's (2001). The regulation of European securities markets. Final report.

Tibshirani, R. (1996). Regression shrinkage and selection via the Lasso. *Journal of the Royal Statistical Society Series B* **58**: 267–288.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics* **5**: 177–188.

Vaughan, E. J. and T. Vaughan (2008). *Fundamentals of Risk and Insurance.* John Wiley & Sons, Inc., 10th edition.

Wang, S. (2002). A set of new methods and tools for enterprise risk capital management and portfolio optimization. Technical report, SCOR Reinsurance Co.

Wang, S. S., V. R. Young, and H. H. Panjer (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics* **21**.

Williams, D. (1991). *Probability With Martingales.* Cambridge University Press.

Wirch, J. and M. Hardy (1999). A synthesis of risk measures for capital adequacy. *Insurance: Mathematics and Economics* **25**: 337–347.

Zheng, J. (2009). *Comparison of Model Risk in Calibrating Affine Models and Libor Market Model.* Master's thesis, Ulm University.

Ehrenwörtliche Erklärung

Ich versichere an Eides statt durch meine Unterschrift, dass ich die vorstehende Arbeit selbständig und ohne fremde Hilfe angefertigt und alle Stellen, die ich wörtlich oder annähernd wörtlich aus Veröffentlichungen entnommen habe, als solche kenntlich gemacht habe, mich auch keiner anderen als der angegebenen Literatur oder sonstiger Hilfsmittel bedient habe. Die Arbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Jinsong Zheng
Essen, 03.12.2018