On the Automorphy of Potentially Semi-stable Deformation Rings

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Abstract

Using p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and an ordinary $R = \mathbb{T}$ theorem, we prove that the support of patched modules for quaternionic forms meet every irreducible component of the potentially semi-stable deformation ring. This gives a new proof of the Breuil-Mézard conjecture for 2-dimensional representations of the absolute Galois group of $\mathbb{Q}_p$, which is new in the case $p = 2$ or $p = 3$ and $\bar{r}$ a twist of an extension of the trivial character by the mod p cyclotomic character. As a consequence, a local restriction in the proof of Fontaine-Mazur conjecture in [48, 57] is removed.

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Chapter 1

Introduction

Let p be a prime number, and let f be a normalized cuspidal eigenform of weight $k \geq 2$ and level $N \geq 1$. There exists a Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathscr{O})$$

attached to f by Deligne, where $\mathscr{O}$ is a ring of integers of a finite extension E of $\mathbb{Q}_p$. Due to many people, the representation is known to be irreducible, odd (i.e. $\det \rho_f(c) = -1$ with c the complex conjugation), unramified at primes $l \nmid pN$, and de Rham (in the sense of Fontaine) at p with Hodge-Tate weights $(0, k-1)$.

In [32] Fontaine and Mazur made a conjecture which asserts the converse:

Conjecture (Fontaine-Mazur)**.** *Let*

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathscr{O})$$

be a continuous, irreducible representation such that

- *ρ is odd;*
- *ρ is unramified outside all but finitely many places;*
- *the restriction of ρ at the decomposition group at p is de Rham with distinct Hodge-Tate weights.*

Then (up to a twist) $\rho \cong \rho_f$ for some cuspidal eigenform f.

We will say that ρ is modular if it is isomorphic to a twist of ρ_f by a character. Similarly, we will say that $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$ is modular if $\overline{\rho} \cong \overline{\rho}_f$ up to a twist, where k is the residue field of $\mathscr{O}$ and $\overline{\rho}$ is obtained by reducing the matrix entries of ρ_f modulo the maximal ideal of $\mathscr{O}$. This conjecture has been proved in several cases under different assumptions. We will only focus on those related to the groundbreaking work of Kisin in [48].

Theorem (Kisin, Paškūnas, Hu-Tan). *Let ρ be as in the conjecture. Let $\overline{\rho}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$ be the reduction of ρ modulo the maximal ideal of $\mathscr{O}$. Assume furthermore that*

- *$\overline{\rho}$ is modular;*
- *$\overline{\rho}$ has non-solvable image if $p=2$; $\overline{\rho}|_{\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})}$ is absolutely irreducible if $p>2$;*
- *if $p=2$ or 3, then $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \not\sim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ for any character $\chi:\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to k^\times$, where ω denotes the mod p cyclotomic character (which is the trivial character when $p=2$).*

Then ρ is modular.

A result of such form is known as a modularity lifting theorem, which says that if $\overline{\rho}$ is modular, then any lift ρ of $\overline{\rho}$ satisfying necessary local conditions is also modular. We note that since we work over $\mathbb{Q}$, the condition on the modularity of $\overline{\rho}$ follows from a deep theorem of Khare-Wintenberger [44] and Kisin [49]. Establishing a modularity lifting theorem comes down to proving that certain surjection $\tilde{R}_\infty \twoheadrightarrow \mathbb{T}_\infty$ of a patched global deformation ring $\tilde{R}_\infty$ onto a patched Hecke algebra $\mathbb{T}_\infty$ is an isomorphism after inverting p, both of which acts on a patched module $\tilde{M}_\infty$ coming from applying Taylor-Wiles-Kisin method to algebraic modular forms on a definite quaternion algebra.

A key ingredient in Kisin's approach to the Fontaine-Mazur conjecture is a purely local statement, known as the Breuil-Mézard conjecture [13], which predicts that μ_{Gal}, the Hilbert-Samuel multiplicity of certain quotients of the framed deformation ring of $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ parametrizing deformations subjected to p-adic Hodge theoretical conditions modulo the maximal ideal of $\mathscr{O}$, is equal to μ_{Aut}, an invariant which can be computed from the representation theory of $\mathrm{GL}_2(\mathbb{Z}_p)$ over k. A refined version of this conjecture replacing multiplicities with cycles is formulated by Emerton and Gee in [30].

In his work, Kisin establishes a connection between $\tilde{R}_\infty[1/p] \cong \mathbb{T}_\infty[1/p]$ and the Breuil-Mézard conjecture (when $p>2$). He shows that $\tilde{R}_\infty \twoheadrightarrow \mathbb{T}_\infty$ implies $\mu_{\mathrm{Gal}} \geq \mu_{\mathrm{Aut}}$, with equality if and only if $\tilde{R}_\infty[1/p] \cong \mathbb{T}_\infty[1/p]$. It follows that in each case where one can prove the reverse inequality, one would simultaneously obtain both the Breuil-Mézard conjecture and a modularity lifting theorem. A similar argument when $p=2$ was carried out in [57] using the results of Khare-Wintenberger [44].

The key ingredient to prove the reverse inequality $\mu_{\mathrm{Gal}} \leq \mu_{\mathrm{Aut}}$ is the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ due to Berger, Breuil, Colmez, Emerton, Kisin and

Paškūnas. The correspondence is given by Colmez's Montreal functor in [22], which is an exact, covariant functor $\check{\mathbf{V}}$ sending certain $\mathrm{GL}_2(\mathbb{Q}_p)$-representations on $\mathscr{O}$-modules to finite $\mathscr{O}$-modules with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Moreover, via reduction modulo p it is compatible with Breuil's (semi-simple) mod p Langlands correspondence in [10].

By using p-adic local Langlands correspondence, [48] deduces the inequality $\mu_{\mathrm{Aut}} \geq \mu_{\mathrm{Gal}}$ (and thus the Breuil-Mézard conjecture) in the cases that p is odd and $\bar{r}$ $(:= \overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})$ is not (a twist of) an extension of $\mathbf{1}$ by ω. Later on, a purely local proof of the Breuil-Mézard conjecture for all continuous representations $\bar{r}$, which has only scalar endomorphisms and is not (a twist of) an extension of $\mathbf{1}$ by ω if $p = 2,3$, is given in [56, 57] using the results in [55]. The cases that $\bar{r}$ is a direct sum of two distinct characters whose ratio is not ω when $p = 2,3$ are proved in [42, 58] by a similar local method. The combined work of Kisin, Hu-Tan and Paškūnas handle the Breuil-Mézard conjecture in all cases except when $p = 2$ or $p = 3$ and $\bar{r} \sim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$. As a consequence, the theorem is proved.

1.1 Summary of results

In this thesis, we give a new proof of this theorem which works for all p without any restriction on $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$. Here is our result:

Theorem A. *Let ρ be as in the conjecture. Assume furthermore that*

- *$\overline{\rho}$ is modular;*
- *$\overline{\rho}$ has non-solvable image if $p = 2$; $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_p))}$ is absolutely irreducible if $p > 2$.*

Then ρ is modular.

Instead of proving $\mu_{\mathrm{Aut}} \geq \mu_{\mathrm{Gal}}$ (or the Breuil-Mézard conjecture), we prove $\tilde{R}_\infty[1/p] \cong \mathbb{T}_\infty[1/p]$ directly. As a result, we can prove the Breuil-Mézard conjecture for 2-dimensional Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ completely. We explain our method in more detail below.

Let $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the absolutely Galois group of the field of p-adic number $\mathbb{Q}_p$ and let $\bar{r}: G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a continuous representation. We denote the fixed determinant universal framed deformation ring of $\bar{r}$ by $R_{\bar{r}}^{\square}$. It can be shown that $\bar{r}$ is isomorphic to the restriction to a decomposition group at p of a mod p Galois representation $\overline{\rho}$ associated to an algebraic modular form on some definite quaternion algebra. By applying the Taylor-Wiles-Kisin patching method in [18] to algebraic modular forms on a definite quaternion algebra, we construct an R_∞-module M_∞ equipped with a commuting action of $\mathrm{GL}_2(\mathbb{Q}_p)$, where R_∞

is a complete local noetherian $R_{\bar{r}}^{\square}$-algebra with residue field k. For simplicity, one may think of R_∞ as $R_{\bar{r}}^{\square}[\![x_1,\cdots,x_m]\!]$. In particular, there is no local deformation condition at place p.

If $y \in \text{m-Spec}\, R_\infty[1/p]$, then

$$\Pi_y := \mathrm{Hom}_{\mathscr{O}}^{\mathrm{cont}}(M_\infty \otimes_{R_\infty,y} E_y, E)$$

is an admissible unitary E-Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, where m-Spec$(R_\infty[1/p])$ is the set of maximal ideals of $R_\infty[1/p]$ and E_y is the residue field at y. Since Π_y lies in the range of p-adic local Langlands, we may apply the Colmez's functor $\check{\mathbf{V}}$ to Π_y and obtain a R_∞-module $\check{\mathbf{V}}(\Pi_y)$ equipped with an action of $G_{\mathbb{Q}_p}$. On the other hand, the composition $x : R_{\bar{r}}^{\square} \to R_\infty \xrightarrow{y} E_y$ defines a continuous Galois representation $r_x : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E_y)$. It is expected that the Banach space representation Π_y depends only on x and that it should be related to r_x by the p-adic local Langlands correspondence (see theorem C below). This has been proved in [19] under the assumptions that $p > 2$, $\bar{r}$ has only scalar endomorphisms and is not a twist of an extension of $\mathbf{1}$ by ω.

Our patched module M_∞ is related to Kisin's $\tilde{M}_\infty$ as follows. The patching in Kisin's paper is always with fixed Hodge-Tate weights and a fixed inertial type. This information can be encoded by an irreducible locally algebraic representation σ of $\mathrm{GL}_2(\mathbb{Z}_p)$ over E. Let $R_p^{\square}(\sigma)$ be a quotient of $R_p^{\square}$ parameterizing the lifts of $\bar{\rho}$ of type σ. We define $R_\infty(\sigma) = R_\infty \otimes_{R_p^{\square}} R_p^{\square}(\sigma)$ (which is Kisin's patehced global deformation ring $\tilde{R}_\infty$ before) and $M_\infty(\sigma^\circ) = M_\infty \otimes_{\mathscr{O}[\![\mathrm{GL}_2(\mathbb{Z}_p)]\!]} \sigma^\circ$ with σ° a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathscr{O}$-lattice of σ. Then $M_\infty(\sigma^\circ)$ is a finitely generated R_∞-module with the action of R_∞ factoring through $R_\infty(\sigma)$. Moreover, an argument using Auslander-Buchsbaum theorem shows that the support of $M_\infty(\sigma^\circ)$ is equal to a union of irreducible components of $R_\infty(\sigma)$. It can be shown that Kisin's patched module $\tilde{M}_\infty$ is isomorphic to $M_\infty(\sigma^\circ)$. The main theorem in this thesis is the following:

Theorem B. *Every irreducible component of $\tilde{R}_\infty$ is contained in the support of $\tilde{M}_\infty$.*

By the local-global compatibility for the patched module M_∞, this amounts to showing that if r_x is de Rham with distinct Hodge-Tate weights, then (a subspace of) locally algebraic vectors in Π_y can be related to $\mathrm{WD}(r_x)$ via the classical local Langlands correspondence, where $\mathrm{WD}(r_x)$ is the Weil-Deligne representation associated to r_x defined by Fontaine.

One of the ingredients to show this is a result in [31], which implies that the action of R_∞ on M_∞ is faithful. Note that this does not imply that $\Pi_y \neq 0$ since M_∞ is not finitely generated over R_∞. We overcome this problem by applying Colmez's functor $\check{\mathbf{V}}$ to M_∞ and showing that $\check{\mathbf{V}}(M_\infty)$ is a finitely generated R_∞-module. This finiteness result is a key idea in my thesis. Since $\check{\mathbf{V}}(M_\infty)$ is a finitely generated R_∞-module, the specialization of $\check{\mathbf{V}}(M_\infty)$ at any $y \in$ m-Spec $R_\infty[1/p]$ is non-zero by Nakayama's lemma, which in turn implies that Π_y is nonzero. Combining these and results from p-adic local Langlands, we prove the following:

Theorem C. *If r_x is absolutely irreducible, then $\check{\mathbf{V}}(\Pi_y) \cong r_x$.*

This shows that Kisin's patched module $\tilde{M}_\infty$ is supported at every generic point whose associated local Galois representation at place p is absolutely irreducible. So we only have to handle the reducible (thus ordinary) locus, which can be shown to be modular by using an ordinary modularity lifting theorem (c.f. [36, 2, 64, 63]). This finishes the proof of theorem B and gives a new proof of the Breuil-Mézard conjecture by the formalism in [48, 34, 30, 56], which is new in the cases that $\bar{r}$ is a twist of the $\mathbf{1}$ by ω and $p = 2,3$. As a consequence, we prove new cases of Fontaine-Mazur conjecture without the local restriction at p.

Let me note that the proof of Breuil-Mézard given in [56, 42] uses results of [55], which are not available in the exceptional cases when $p = 2$ or $p = 3$, in an essential way. So our argument differs substantially from [56, 42]. While our thesis is written, we were notified that Lue Pan also obtains a similar finiteness result independently in his thesis [53]. However, we would like to emphasize that our results are most interesting in the cases not covered by [55], whose results Pan uses in an essential way.

The thesis is organized as follows. We first recall some background knowledge and properties in chapter 2, 3 and 4 on representation theory, automorphic forms and Galois deformation theory respectively. In chapter 5, we introduce completed homology and construct the patched module. We relate our patched module to the Breuil-Mézard conjecture in chapter 6 and to the p-adic Langlands correspondence in chapter 7 using a faithfulness result in [31]. In chapter 8, we construct some partially ordinary Galois representations by an ordinary $R = \mathbb{T}$ theorem. In the last chapter, we put all these results together and prove our main theorem, and use it to give a new proof of the Breuil-Mézard conjecture and the Fontaine-Mazur conjecture.

1.2 Notations

We fix a prime p, and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Throughout the thesis, we work with a finite extension $E/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, which will be our coefficient field. We write $\mathcal{O}$ for the ring of integers in E, ϖ for a uniformizer, $k = \mathcal{O}/\varpi$ for the residue field. Furthermore, we will assume that E and k is sufficient large in the sense that if we are working with representations of the absolute Galois group of a p-adic field L, then the image of all embeddings $L \to \overline{\mathbb{Q}}_p$ are contained in E.

If F is a field with a fixed algebraic closure $\overline{F}$, then we write $G_F = \mathrm{Gal}(\overline{F}/F)$ for its absolutely Galois group. We write $\varepsilon : G_F \to \mathbb{Z}_p^\times$ for the p-adic cyclotomic character, and ω

for the mod p cyclotomic character. If F is a finite extension of $\mathbb{Q}_p$, we write I_F for the inertia subgroup of G_F, ϖ_F for a uniformizer of the ring of integers $\mathscr{O}_F$ of F and $k_F = \mathscr{O}_F/\varpi_F$ its residual field.

If F is a number field and v is a place of F, we let F_v be the completion of F at v and $\mathbb{A}_F$ its ring of adeles. If S is a finite set of places of F, we let $\mathbb{A}_F^S$ denote the resticted tensor product $\prod'_{v\notin S} F_v$. In particular, $\mathbb{A}_F^\infty$ denotes the ring of finite adeles. For each finite place v of F, we will denote by q_v the order of residue field at v, and by $\varpi_v \in F_v$ a uniformizer and Frob_v an arithmetic Frobenius element of G_{F_v}.

We let

$$\mathrm{Art}_F = \prod_v \mathrm{Art}_{F_v} : \mathbb{A}_F^\times/\overline{F^\times (F_\infty^\times)^\circ} \xrightarrow{\sim} G_F^{ab}$$

be the global Artin map, where the local Artin map $\mathrm{Art}_{F_v} : F_v^\times \to W_{F_v}^{ab}$ is the isomorphism provided by local class field theory, which sends our fixed uniformizer to a geometric Frobenius element.

We will consider locally algebraic character $\psi : \mathbb{A}_F^\times/\overline{F^\times (F_\infty^\times)^\circ} \to \mathscr{O}^\times$ in the sense that there exists an open compact subgroup U of $(\mathbb{A}_F^\infty)^\times$ such that $\psi(u) = \prod_{v|p} \mathbf{N}_v(u_v)^{t_v}$ for $u \in U$, where u_v is the projection of u to the place v, $\mathbf{N}_v$ the local norm, and t_v an integer. When $\overline{F^\times(F_\infty^\times)}$ lies in the kernel of ψ, we consider ψ as a character $\psi : (\mathbb{A}_F^\infty)^\times/F^\times \to \mathscr{O}^\times$, whose corresponding Galois character is totally even.

If L is a p-adic field and W is a de Rham representation of G_L over E. For $\kappa : L \to E$, we will write $\mathrm{HT}_\kappa(W)$ for the multiset of Hodge-Tate weights of W with respect to κ, which contains i with multiplicity $\dim_E(W \otimes_{\kappa,L} \hat{\overline{L}}(i))^{G_L}$. Thus for example $\mathrm{HT}_\kappa(\varepsilon) = \{-1\}$. We say that W has regular Hodge-Tate weights if for each κ, the elements of $\mathrm{HT}_\kappa(W)$ are pairwise distinct.

Let $\mathbb{Z}_+^2$ denote the set of tuples (λ_1, λ_2) of integers with $\lambda_1 \geq \lambda_2$. If W be a 2-dimensional de Rham representation with regular Hodge-Tate weights, there is a $\lambda = (\lambda_\kappa) \in (\mathbb{Z}_+^2)^{\mathrm{Hom}_{\mathbb{Q}_p}(L,E)}$ such that for each $\kappa : L \to E$,

$$\mathrm{HT}_\kappa(W) = \{\lambda_{\kappa,2}, \lambda_{\kappa,1} + 1\},$$

and we say that W is regular of weight λ.

For any $\lambda \in (\mathbb{Z}_+^2)$, we write $\Xi_\lambda = \mathrm{Sym}^{\lambda_1-\lambda_2} \otimes \det^{\lambda_2}$ for the algebraic $\mathscr{O}_L$-representation of GL_n with highest weight λ and M_λ for the $\mathscr{O}$-representation of $\mathrm{GL}_2(\mathscr{O}_L)$ obtained by evaluating Ξ_λ on $\mathscr{O}_L$. For any $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}(F,E)}$, we write M_λ for the $\mathscr{O}$-representation of $\mathrm{GL}_2(\mathscr{O}_L)$ defined by

$$M_\lambda := \otimes_{\kappa:F\hookrightarrow E} M_{\lambda_\kappa} \otimes_{\mathscr{O}_L,\kappa} \mathscr{O}.$$

An inertial type is a representation $\tau : I_L \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with open kernel which extends to the Weil group W_L. We say a de Rham representation $\rho : G_F \to \mathrm{GL}_2(E)$ has inertial type τ if the restriction to I_L of the Weil-Deligne representation $\mathrm{WD}(\rho)$ associated to ρ is equivalent to τ. Given an inertia type τ, by a result of Henniart in the appendix of [13], there is a (unique if $p > 2$) finite dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\tau)$ $\big($resp. $\sigma^{cr}(\tau)\big)$ of $\mathrm{GL}_2(\mathcal{O}_L)$, such that for any infinite dimensional smooth absolutely irreducible representation π of G and the associated Weil-Deligne representation $\mathrm{LL}(\pi)$ attached to π via the classical local Langlands correspondence, we have $\mathrm{Hom}_K(\sigma(\tau),\pi) \neq 0$ (resp. $\mathrm{Hom}_K(\sigma^{cr}(\tau),\pi) \neq 0$) if and only if $\mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ (resp. $\mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ and the monodromy operator N is trivial). Enlarging E if needed, we may assume $\sigma(\tau)$ is defined over E.

If L be a finite extension of $\mathbb{Q}_p$, we let rec for the local Langlands correspondence for $\mathrm{GL}_2(L)$, as defined in [16, 38]. By definition, it is a bijection between the set of isomorphism classes of irreducible admissible representation of $\mathrm{GL}_2(L)$ over $\mathbb{C}$, and the set of Frobenius semi-simple Weil-Deligne representation of W_L over $\mathbb{C}$. Fix once and for all an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. We define the local Langlands correspondence rec_p over $\overline{\mathbb{Q}}_p$ by $\iota \circ \mathrm{rec}_p = \mathrm{rec} \circ \iota$, which depends only on $\iota^{-1}(\sqrt{p})$. If we set $r_p(\pi) := \mathrm{rec}_p(\pi \otimes |\det|^{-1/2})$, then r_p is independent of the choice of ι. Furthermore, if V is a Frobenius semi-simple Weil-Deligne representation Weil-Deligne representation of W_L over E, then $r_p^{-1}(V)$ is also defined over E.

If $r : G_L \to \mathrm{GL}_2(E)$ is de Rham of regular weight λ, then we write $\pi_{\mathrm{alg}}(r) = M_\lambda \otimes_{\mathcal{O}} E$, $\pi_{\mathrm{sm}}(r) = r_p^{-1}(\mathrm{WD}(r_x)^{F-ss})$ and $\pi_{\mathrm{l.alg}}(r) = \pi_{\mathrm{alg}}(r) \otimes \pi_{\mathrm{sm}}(r)$, all of which are E-representations of $\mathrm{GL}_2(E)$.

Recall that a linearly topological $\mathcal{O}$-module is a topological $\mathcal{O}$-module which has a fundamental system of open neighborhoods of the identity which are $\mathcal{O}$-submodules. If A is a linear topological $\mathcal{O}$-module, we write $A^\vee$ for its Pontryagin dual $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, E/\mathcal{O})$, where $E/\mathcal{O}$ has the discrete topology, and we give $A^\vee$ the compact open topology. We write A^d for the Schikhof dual $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, \mathcal{O})$, which induces an anti-equivalence of categories between the category of compact, $\mathcal{O}$-torsion free linear-topological $\mathcal{O}$-modules A and the category of ϖ-adically complete separated $\mathcal{O}$-torsion free $\mathcal{O}$-modules. A quasi-inverse is given by $B \mapsto B^d := \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O})$, where the target is given the weak topology of pointwise convergence. Note that if A is an $\mathcal{O}$-torsion free profinite linearly topological $\mathcal{O}$-module, then A^d is the unit ball in the E-Banach space $\mathrm{Hom}_{\mathcal{O}}(A, E)$.

Let $(A, \mathfrak{m})$ be a complete local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, we will denote CNL_A the category of complete local A-algebra with residue field k.

Chapter 2

Preliminaries in representation theory

2.1 Generalities

Let G be a p-adic analytic group, K be a compact open subgroup of G, and Z be the center of G.

Let $(A, \mathfrak{m}_A) \in \mathrm{CNL}_O$. We denote by $\mathrm{Mod}_G(A)$ the category of $A[G]$-modules and by $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ the full subcategory with objects V such that

$$V = \cup_{H,n} V^H[\mathfrak{m}^n],$$

where the union is taken over all open subgroups of G and integers $n \geq 1$ and $V[\mathfrak{m}^n]$ denotes elements of V killed by all elements of $\mathfrak{m}^n$. Let $\mathrm{Mod}_G^{\mathrm{l.fin}}(A)$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ with objects smooth G-representation which are locally of finite length, this means for every $v \in V$, the smallest $A[G]$-submodule of V containing v is of finite length.

An object V of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is called admissible if $V^H[\mathfrak{m}^i]$ if a finitely generated A-module for every open subgroup H of G and every $i \geq 1$; V is called locally admissible if for every $v \in V$ the smallest $A[G]$-submodule of V containing v is admissible. Let $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of locally admissible representations.

For a continuous character $\zeta : Z \to A^\times$, adding the subscript ζ in any of the above categories indicates the corresponding full subcategory of G-representations with central character ζ. These categories are abelian and are closed under direct sums, direct limits and subquotients. It follows from [27, Theorem 2.3.8] that if $G = \mathrm{GL}_2(\mathbb{Q}_p)$ or G is a torus then $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.fin}}(A) = \mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$.

Let H be a compact open subgroup of G and $A[\![H]\!]$ the complete group algebra of H. Let $\mathrm{Mod}_G^{\mathrm{pro}}(A)$ be the category of profinite linearly topological $A[\![H]\!]$-modules with an action of $A[G]$ such that the two actions are the same when restricted to $A[H]$ with morphisms

G-equivariant continuous homomorphisms of topological $A[\![H]\!]$-modules. The definition does not depend on H since any two compact open subgroups of G are commensurable. By [27, Lemma 2.2.7], this category is anti-equivalent to $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ under the Pontryagin dual $V \mapsto V^\vee := \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})$ with the former being equipped with the discrete topology and the latter with the compact-open topology. We denote $\mathfrak{C}_{G,\zeta}(A)$ the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro}}(A)$ anti-equivalent to $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.fin}}(A)$.

An E-Banach space representation Π of G is an E-Banach space Π together with a G-action by continuous linear automorphisms such that the inducing map $G \times \Pi \to \Pi$ is continuous. A Banach space representation Π is called unitary if there is a G-invariant norm defining the topology on Π, which is equivalent to the existence of an open bounded G-invariant $\mathscr{O}$-lattice Θ in Π. An unitary E-Banach space representation is admissible if $\Theta \otimes_{\mathscr{O}} k$ is an admissible smooth representation of G, which is independent of the choice of Θ. We denote $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)$ the category of admissible unitary E-Banach space representations on which Z acts by ζ.

2.2 Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and B be the upper triangular Borel. We recall the classification of the absolutely irreducible smooth k-representations of G with a central character.

Let σ be an irreducible smooth representation of K. It is of the form $\mathrm{Sym}^r k^2 \otimes \det^a$ for uniquely determined integers $0 \le r \le p-1$ and $0 \le a \le p-2$. There exists an isomorphism of algebras

$$\mathrm{End}_G(\text{c-Ind}_K^G \sigma) \cong k[T, S^{\pm 1}]$$

for certain Hecke operators $T, S \in \mathrm{End}_G(\text{c-Ind}\,\sigma)$. It follows from [6, Theorem 33] and [10, Theorem 1.6] that the absolutely irreducible smooth k-representations of G with a central character fall into four disjoint classes:

- characters $\eta \circ \det$;
- special series $\mathrm{Sp} \otimes \eta \circ \det$;
- principal series $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$, with $\chi_1 \neq \chi_2$;
- supersingular $\text{c-Ind}_K^G(\sigma)/(T, S-\lambda)$, with $\lambda \in k^\times$,

where the Steinberg representation Sp is defined by the exact sequence

$$0 \to \mathbf{1} \to \mathrm{Ind}_B^G \mathbf{1} \to \mathrm{Sp} \to 0.$$

2.2.1 Blocks

Let $\mathrm{Irr}_{G,\zeta}$ be the set of equivalent classes of smooth irreducible k-representations of G with central character ζ. We write $\pi \leftrightarrow \pi'$ if $\pi \cong \pi'$ or $\mathrm{Ext}^1_{G,\zeta}(\pi,\pi') \neq 0$ or $\mathrm{Ext}^1_{G,\zeta}(\pi',\pi) \neq 0$. We write $\pi \sim \pi'$ if there exists $\pi_1,\cdots,\pi_n \in \mathrm{Irr}_{G,\zeta}$ such that $\pi \cong \pi_1$, $\pi' \cong \pi_n$ and $\pi_i \leftrightarrow \pi_{i+1}$ for $1 \leq i \leq n-1$. The relation $\sim$ is an equivalence relation on $\mathrm{Irr}_{G,\zeta}$. A block is an equivalence class of $\sim$. The classification of blocks can be found in [59, Corollary 1.2]. Moreover, by [55, Proposition 5.34], the category $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathscr{O})$ decomposes into a direct product of subcategories

$$\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathscr{O}) \cong \prod_{\mathfrak{B}} \mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathscr{O})^{\mathfrak{B}}$$

where the product is taken over all the blocks $\mathfrak{B}$ and the objects of $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathscr{O})^{\mathfrak{B}}$ are representations with all the irreducible subquotients lying in $\mathfrak{B}$. Dually we obtain

$$\mathfrak{C}_{G,\zeta}(\mathscr{O}) \cong \prod_{\mathfrak{B}} \mathfrak{C}_{G,\zeta}(\mathscr{O})^{\mathfrak{B}}$$

with $\mathfrak{C}_{G,\zeta}(\mathscr{O})^{\mathfrak{B}}$ are representations with all the dual of irreducible subquotients lying in $\mathfrak{B}$.

2.2.2 Capture

Let $\zeta : Z(K) \to \mathscr{O}^\times$ be a continuous character. Let $\{V_i\}_{i\in I}$ be a family of continuous representations of K on finite-dimensional E-vector spaces, and let $M \in \mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathscr{O})$. We say that $\{V_i\}_{i\in I}$ captures M if the smallest quotient $M \twoheadrightarrow Q$ such that $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M,V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(Q,V_i^*)$ for all $i \in I$ is equal to M, where V_i^* is the E-linear dual of V_i. The following proposition is due to [57, Proposition 2.7].

Proposition 2.2.1. *Let $K = \mathrm{GL}_2(\mathbb{Z}_p)$, and let $\zeta : Z(K) \to \mathscr{O}^\times$ be a continuous character. There is a smooth irreducible representation τ of K which is a type for a Bernstein component containing a principal series representation, but not containing a special series representation, such that*

$$\{\tau \otimes \mathrm{Sym}^a E^2 \otimes \eta' \circ \det\}_{a\in\mathbb{N},\eta'}$$

captures every projective object in $\mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathscr{O})$. Here, for each $a \in \mathbb{N}$, η' runs over all continuous character $\eta' : \mathbb{Z}_p^\times \to E^\times$ such that $\tau \otimes \mathrm{Sym}^a E^2 \otimes \eta' \circ \det$ has central character ζ.

Let M,K and $\{V_i\}_{i\in I}$ be as above. Let R be a complete local noetherian $\mathscr{O}$-algebra with residue field k. We assume that M is a compact $R[\![K]\!]$-module, such that the action of R is faithful. For each $i\in I$, let $\mathfrak{a}_i$ be the R-annihilator of $\mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K]\!]}(M,V_i^*)$.

Let m-Spec $R[1/p]$ be the set of maximal ideal of $R[1/p]$. If $x\in$ m-Spec $R[1/p]$ then its residue field E_x is a finite extension of E, and we denote by $\mathscr{O}_{E_x}$ the ring of integers of E_x. Then $M\otimes_R \mathscr{O}_{E_x}$ is a compact $\mathscr{O}[\![K]\!]$-module and we define

$$\Pi_x = \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}}(M\otimes_R \mathscr{O}_{E_x}, E).$$

Then Π_x equipped with the supremum norm is a unitary E-Banach space representation of K.

Proposition 2.2.2. *Assume that M is a finitely generated $R[\![K]\!]$-module, that $\{V_i\}_{i\in I}$ captures M, and that $\sqrt{\mathfrak{a}_i}=\mathfrak{a}_i$ for all $i\in I$. Then the set*

$$\Sigma := \{x\in \text{m-Spec}\, R[1/p] \mid \mathrm{Hom}_K(V_i,\Pi_x)\neq 0 \textit{ for some } i\in I\}$$

is dense in Spec R.

Proof. The proof of [31, Proposition 2.11] works verbatim to our setting. □

2.2.3 Colmez's Montreal functor

Let $\mathrm{Mod}^{\mathrm{fin}}_{G,Z}(\mathscr{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_{G}(\mathscr{O})$ consisting of representations of finite length with a central character. Let $\mathrm{Mod}^{\mathrm{fin}}_{G_{\mathbb{Q}_p}}(\mathscr{O})$ be the category of continuous $G_{\mathbb{Q}_p}$-representations on $\mathscr{O}$-modules of finite length with the discrete topology.

In [22], Colmez has defined an exact and covariant functor $\mathbf{V}:\mathrm{Mod}^{\mathrm{fin}}_{G,Z}(\mathscr{O})\to \mathrm{Mod}^{\mathrm{fin}}_{G_{\mathbb{Q}_p}}(\mathscr{O})$. If $\psi:\mathbb{Q}_p^\times\to\mathscr{O}^\times$ is a continuous character, then we may also consider it as a continuous character $\psi: G_{\mathbb{Q}_p}\to\mathscr{O}^\times$ via class field theory and for all $\pi\in\mathrm{Mod}^{\mathrm{fin}}_{G,Z}(\mathscr{O})$ we have $\mathbf{V}(\pi\otimes\psi\circ\det)\cong\mathbf{V}(\pi)\otimes\psi$. Moreover we have:

$$\begin{aligned}&\mathbf{V}(\mathbf{1})=0,\quad \mathbf{V}(\mathrm{Sp})=\omega,\quad \mathbf{V}(\mathrm{Ind}_B^G\chi_1\otimes\chi_2\omega^{-1})=\chi_2,\\ &\mathbf{V}(\text{c-Ind}_K^G(\mathrm{Sym}^r k^2)/(T,S))\cong \mathrm{ind}\,\omega_2^{r+1},\end{aligned}$$

where ω is the mod p cyclotomic character, $\omega_2: I_{\mathbb{Q}_p}\to k^\times$ is Serre's fundamental character of level 2, and $\mathrm{ind}\,\omega_2^{r+1}$ is the unique irreducible representation of $G_{\mathbb{Q}_p}$ of determinant ω^r and such that $\mathrm{ind}\,\omega_2^{r+1}|_{I_{\mathbb{Q}_p}}\cong\omega_2^{r+1}\oplus\omega_2^{p(r+1)}$ for $0\le r\le p-1$.

Let $\mathrm{Rep}_{G_{\mathbb{Q}_p}}(\mathscr{O})$ be the category of continuous $G_{\mathbb{Q}_p}$-representations on compact $\mathscr{O}$-modules. Following [56, Section 3], we define an exact covariant functor $\check{\mathbf{V}}:\mathfrak{C}_{G,\zeta}(\mathscr{O})\to$

$\mathrm{Rep}_{G_{\mathbb{Q}_p}}(\mathscr{O})$ as follows: Let M be in $\mathfrak{C}_{G,\zeta}(\mathscr{O})$, if it is of finite length, we define $\check{\mathbf{V}}(M) := \mathbf{V}(M^\vee)^\vee(\varepsilon\zeta)$ where $\vee$ denotes the Pontryagin dual. For general $M \in \mathfrak{C}_{G,\zeta}(\mathscr{O})$, write $M \cong \varprojlim M_i$, with M_i of finite length in $\mathfrak{C}_{G,\zeta}(\mathscr{O})$ and define $\check{\mathbf{V}}(M) := \varprojlim \check{\mathbf{V}}(M_i)$. With this normalization of $\check{\mathbf{V}}$ we have

$$\check{\mathbf{V}}(\mathbf{1}) = 0, \quad \check{\mathbf{V}}((\mathrm{Sp}\otimes\eta\circ\det)^\vee) = \eta, \quad \check{\mathbf{V}}((\mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1})^\vee) = \chi_1, \quad \check{\mathbf{V}}(\pi^\vee) = \mathbf{V}(\pi),$$

where π is a supersingular representation.

Let $\Pi \in \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)$, we define $\check{\mathbf{V}}(\Pi) = \check{\mathbf{V}}(\Theta^d) \otimes_{\mathscr{O}} E$ with Θ any open bounded G-invariant $\mathscr{O}$-lattice in Π, so that $\check{\mathbf{V}}$ is exact and contravariant on $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(E)$ and $\check{\mathbf{V}}(\Pi)$ does not depend on the choice of Θ.

2.2.4 Extension Computations when $p = 3$

In this subsection we assume $p = 3$ and $\zeta = 1$.

For $\pi', \pi \in \mathrm{Mod}_{G,\zeta}^{\mathrm{l.fin}}(k)$, we write $\mathrm{Ext}^1_{G/Z}(\pi', \pi)$ for the extension group of π' by π in $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.fin}}(k)$. By [54, Thereom 11.4], we have the following table for $\dim_k \mathrm{Ext}^1_{G/Z}(\pi', \pi)$:

$\pi'\backslash\pi$	$\mathbf{1}$	Sp	$\omega\circ\det$	$\mathrm{Sp}\otimes\omega\circ\det$
$\mathbf{1}$	0	2	0	0
Sp	1	0	1	0
$\omega\circ\det$	0	0	0	2
$\mathrm{Sp}\otimes\omega\circ\det$	1	0	1	0

For $\tau \in \mathrm{Hom}(\mathbb{Q}_p, k)$ ($\cong \mathrm{Ext}^1_{G/Z}(\mathbf{1}, \mathrm{Sp})$), we denote E_τ the extension of $\mathbf{1}$ by Sp defined in [22, Proposition VII.4].

Proposition 2.2.3. *For $\pi' \in \{\mathrm{Sp}, \mathrm{Sp}\otimes\omega\circ\det\}$, $\mathrm{Ext}^1_{G/Z}(\pi', E_\tau)$ is 1-dimensional over k. Moreover, the unique non-split extension D_τ of π' by E_τ gives rise to the class of* $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1}, \check{\mathbf{V}}(\pi'^\vee)) \cong \mathrm{Hom}(\mathbb{Q}_p, k)$ *given by τ under the Colmez's functor* $\check{\mathbf{V}}$.

Proof. If $\pi' = \mathrm{Sp}$, then every non-zero element of $\mathrm{Ext}^1_{G/Z}(\pi', E_\tau)$ would give an element of $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1}, \mathrm{Sp})$ ($\mathrm{Ext}^1_{G/Z}(\mathrm{Sp}, \mathrm{Sp}) = 0$ and the only non-split extension of Sp by $\mathbf{1}$ is given by $\mathrm{Ind}_B^G \mathbf{1}$). Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G 1, \mathrm{Ind}_B^G 1) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1}, \mathrm{Sp}) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{G/Z}(\mathbf{1}, \mathrm{Sp}) \\
\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1}, \mathbf{1}) \arrow[r, equal] & \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1}, \mathbf{1}), &
\end{tikzcd} \tag{2.1}$$

where the horizontal maps come from the functoriality of $\mathrm{Ext}^1_{G/Z}$ and the vertical maps are induced by the Colmez's funtor $\check{\mathbf{V}}$. Applying $\mathrm{Hom}_{G/Z}(-,\mathrm{Sp})$ to the short exact sequence

$$0 \to \mathbf{1} \to \mathrm{Ind}_B^G \mathbf{1} \to \mathrm{Sp} \to 0,$$

we obtain the following exact sequence

$$0 \to \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathrm{Sp}).$$

Note that the second map is an injection ($\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) = 0$). Since both $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp})$ and $\mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathrm{Sp})$ are 2-dimensional ([28, Proposition 4.3.12 (1)] and the table above), the first assertion follows.

Appying $\mathrm{Hom}_{G/Z}(\mathrm{Ind}_B^G 1,-)$ to the short exact sequence

$$0 \to \mathbf{1} \to \mathrm{Ind}_B^G \mathbf{1} \to \mathrm{Sp} \to 0,$$

we have the following exact sequence

$$0 \to \mathrm{Hom}_G(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathbf{1}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Ind}_B^G \mathbf{1}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp}).$$

Since both $\mathrm{Hom}_G(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp})$ and $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathbf{1})$ are 1-dimensional (c.f. [28, Proposition 4.3.13 (2)]), we see that the second map is an injection. It is indeed an isomorphism since both the source and target are isomorphic to $\mathrm{Hom}(\mathbb{Q}_p,k)$ ([22, Proposition VII.4.13 (i)(ii)]). On the other hand, the first vertical map in (2.1) is an isomorphism by [22, Proposition VII.4.13 (iii)], hence the second vertical map in (2.1) is an isomorphism also and the second assertion follows.

If $\pi' = \mathrm{Sp}\otimes\omega\circ\det$, then we consider following diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{G/Z}(\mathrm{Sp}\otimes\omega\circ\det,E_\tau) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\,\omega\otimes\omega,E_\tau) \arrow[d] \\
\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\omega) \arrow[r, equal] & \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\omega),
\end{tikzcd}$$

where the horizontal map comes from the functoriality of $\mathrm{Ext}^1_{G/Z}$ and the vertical map is induced by the Colmez's funtor $\check{\mathbf{V}}$. Using the long exact sequence coming from applying $\mathrm{Hom}_{G/Z}(-,E_\tau)$ to the short exact sequence

$$0 \to \omega\circ\det \to \mathrm{Ind}_B^G\,\omega\otimes\omega \to \mathrm{Sp}\otimes\omega\circ\det \to 0,$$

we have the long exact sequence

$$\mathrm{Hom}_{G/Z}(\omega\circ\det,E_\tau)\to\mathrm{Ext}^1_{G/Z}(\mathrm{Sp}\otimes\omega\circ\det,E_\tau)\to\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\,\omega\otimes\omega,E_\tau)\to\mathrm{Ext}^1_{G/Z}(\omega\circ\det,E_\tau).$$

Since both $\mathrm{Hom}_{G/Z}(\omega\circ\det,E_\tau)$ and $\mathrm{Ext}^1_{G/Z}(\omega\circ\det,E_\tau)$ are zero, we see that the second map is an isomorphism. This proves the first assertion since $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\,\omega\otimes\omega,E_\tau)$ is 1-dimensional [22, Proposition VII.4.25], and the second assertion follows from [22, Proposition VII.4.24]. □

2.2.5 Extension Computations when $p=2$

In this subsection we assume $p=2$ and $\zeta=1$.

Lemma 2.2.4. *We have* $\dim_k\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathbf{1})=1$*. In particular, the unique non-split extension of* Sp *by* $\mathbf{1}$ *is* $\mathrm{Ind}_B^G\,\mathbf{1}$..

Proof. We first note that $\dim_k\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\,\mathbf{1},\mathbf{1})=1$ [28, Theorem 4.3.13 (2)]. Applying $\mathrm{Hom}_G(-,\mathbf{1})$ to the short exact sequence

$$0\to\mathbf{1}\to\mathrm{Ind}_B^G\,\mathbf{1}\to\mathrm{Sp}\to 0,$$

we obtain the following long exact sequence

$$0\to\mathrm{Hom}_G(\mathbf{1},\mathbf{1})\to\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathbf{1})\to\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\,\mathbf{1},\mathbf{1})\xrightarrow{f}\mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathbf{1}).$$

It follows that $\dim_k\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathbf{1})=2$ if the map f is the zero map and $\dim_k\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathbf{1})=1$ if f is nonzero.

On the other hand, consider the spectral sequence coming from the pro-p Iwahori invariant functor $\mathscr{I}$

$$0\to\mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\,\mathbf{1}),\mathscr{I}(\mathbf{1}))\to\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\,\mathbf{1},\mathbf{1})\to\mathrm{Hom}_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\,\mathbf{1}),\mathbb{R}^1\mathscr{I}(\mathbf{1})),$$

where $\mathscr{H}$ is the pro-p Iwahori Hecke algebra (same as the Iwahori Hecke algebra since Iwahori subgroups are pro-p when $p=2$) and $\mathscr{I}$ is the pro-p Iwahori invariant functor. We claim that $\mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\,\mathbf{1}),\mathscr{I}(\mathbf{1}))$ is nonzero.

If the claim holds, by applying $\mathrm{Hom}_{\mathscr{H}}(-,\mathscr{I}(\mathbf{1}))$ to the short exact sequence

$$0\to\mathscr{I}(\mathbf{1})\to\mathscr{I}(\mathrm{Ind}_B^G\,\mathbf{1})\to\mathscr{I}(\mathrm{Sp})\to 0,$$

we have the following exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{H}}(\mathscr{I}(\mathbf{1}),\mathscr{I}(\mathbf{1})) \to \mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Sp}),\mathscr{I}(\mathbf{1})) \to \mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\mathbf{1}),\mathscr{I}(\mathbf{1})) \to \mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathbf{1}),\mathscr{I}(\mathbf{1})).$$

Since $\dim \mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Sp}),\mathscr{I}(\mathbf{1})) = 1$ by [54, Lemma 11.3], we see that the last map is an injection. It follows that any nonzero element in $\mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\mathbf{1}),\mathscr{I}(\mathbf{1}))$ would give an element of $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\mathbf{1},\mathbf{1})$ whose image under f is nonzero by the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\mathbf{1}),\mathscr{I}(\mathbf{1})) \arrow[r] \arrow[d, hook] & \mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathbf{1}),\mathscr{I}(\mathbf{1})) \arrow[d, hook] \\
\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\mathbf{1},\mathbf{1}) \arrow[r, "f"] & \mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathbf{1})
\end{tikzcd}$$

coming from the injections $\mathbf{1} \to \mathrm{Ind}_B^G\mathbf{1}$, $\mathscr{I}(\mathbf{1}) \to \mathscr{I}(\mathrm{Ind}_B^G\mathbf{1})$ and the pro-p Iwahori invariant functor. This proves the lemma.

To prove the claim, we construct a non-trivial extension of $\mathscr{I}(\mathrm{Ind}_B^G\mathbf{1})$ by $\mathscr{I}(\mathbf{1})$ explicitly. Note that $\mathscr{H}$ is the k-algebra with two generators T,S satisfying two relations $T^2 = 1$ and $(S+1)S = 0$ (c.f. [74, Section 1.1]), $\mathscr{I}(\mathbf{1})$ is the simple (right) $\mathscr{H}$-module given by $vT = v$; $vS = 0$, $\mathscr{I}(\mathrm{Sp})$ is the simple $\mathscr{H}$-module given by $vT = v$; $vS = v$ and $\mathscr{I}(\mathrm{Ind}_B^G\mathbf{1})$ is the $\mathscr{H}$-module given by $v_1T = v_1$; $v_2T = v_2$; $v_1S = 0$; $v_2S = v_1 + v_2$. Since the unique non-split extension of $\mathscr{I}(\mathbf{1})$ by itself is given by $v_1T = v_1$; $v_2T = v_1 + v_2$; $v_1S = 0$; $v_2S = 0$ (note that $2 = 0$ in k), it follows that

$$\begin{aligned} v_1T = v_1 \quad v_2T = v_1 + v_2 \quad v_3T = v_3; \\ v_1S = 0 \quad v_2S = 0 \quad v_3S = v_2 + v_3 \end{aligned}$$

gives a desired non-trivial element in $\mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\mathrm{Ind}_B^G\mathbf{1}),\mathscr{I}(\mathbf{1}))$. □

By [28, Proposition 4.3.21, Proposition 4.3.22], [22, Proposition VII.4.18] and the above lemma, we have the following table for $\dim_k \mathrm{Ext}^1_{G/Z}(\pi',\pi)$:

$\pi'\backslash\pi$	$\mathbf{1}$	Sp
$\mathbf{1}$	3	3
Sp	1	3

Lemma 2.2.5. *(i)* $\dim_k \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\mathbf{1},\mathrm{Sp}) = 3$.

(ii) The natural map $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\mathbf{1},\mathrm{Sp})$ *is a bijection.*

Proof. The first assertion is due to [28, Theorem 4.3.12 (2)]. To show the second assertion, consider the exact sequence

$$0 \to \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathrm{Sp}).$$

coming from applying $\mathrm{Hom}_G(-,\mathrm{Sp})$ to the short exact sequence $0 \to \mathbf{1} \to \mathrm{Ind}_B^G \mathbf{1} \to \mathrm{Sp} \to 0$. Since both $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp})$ and $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp})$ are 3-dimensional by the table above and (1), we see that the first map is a bijection (and the second map is identical zero). □

Proposition 2.2.6. *Let N be a smooth G-representation of finite length over k whose Jordan-Holder factors are all* $\mathbf{1}$ *and let E be an extension of N by* Sp *with socle* Sp*. Then the natural map* $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},E)$ *is a bijection.*

Proof. We first note that when $N = \mathbf{1}$, $E \cong E_\tau$ for some $\tau \in \mathrm{Hom}(\mathbb{Q}_p^\times,k)$ constructed in [22, Proposition VII.4] and the natural map $\mathrm{Ext}^1_{G/Z}(\mathbf{1},E_\tau) \to \mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathbf{1})$ is identical zero [22, Proposition VII.5.4]. It follows that N can't contain any non-split extension of $\mathbf{1}$ by itself. Thus we may reduce to the case $N = \mathbf{1}$ and the assertion is equivalent to showing that the second map in the following exact sequence

$$0 \to \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},E) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathbf{1})$$

is zero. Suppose this is not the case, then there is an extension E' of Sp by E with socle Sp such that $E'/\mathrm{Sp} \cong \mathrm{Ind}_B^G \mathbf{1}$ ($\mathrm{Ind}_B^G \mathbf{1}$ is the unique non-split extension of Sp by $\mathbf{1}$ by lemma 2.2.4). It follows that E' would give an element of $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp})$ whose image in $\mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathrm{Sp})$ is nonzero. On the other hand, lemma 2.2.5 (2) says that the map $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathbf{1},\mathrm{Sp})$ is identical zero, which gives a contradiction. This finishes the proof. □

Proposition 2.2.7. *The map* $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\mathbf{1})$ *induced by Colmez's functor* $\check{\mathbf{V}}$ *is a bijection.*

Proof. For $\tau \in \mathrm{Hom}(\mathbb{Q}_p^\times,k)$, we denote Y_τ the extension of $\mathbf{1}$ by itself given by τ (as B-representation). Note that $\mathrm{Ind}_B^G Y_\tau$ is an extension of $\mathrm{Ind}_B^G \mathbf{1}$ by itself. It is shown in [22, Proposition VII.4.12] that the map $\mathrm{Hom}(\mathbb{Q}_p^\times,k) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\mathbf{1})$ induced by $\tau \mapsto \check{\mathbf{V}}(\mathrm{Ind}_B^G Y_\tau)$ is a bijection. By lemma 2.2.5 (2), the map $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp})$ is a bijection. Composing its inverse with the map $\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Ind}_B^G \mathbf{1}) \to \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G \mathbf{1},\mathrm{Sp})$, we obtain the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Q}_p^\times,k) \arrow[r] & \mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\mathbf{1},\mathrm{Ind}_B^G\mathbf{1}) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\mathbf{1}) \arrow[d, equal] \\
 & \mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp}) \arrow[r] & \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\mathbf{1}).
\end{tikzcd}$$

Note that the right square is commutative since there is no extension of $\mathrm{Ind}_B^G\mathbf{1}$ by Sp with socle Sp (by the bijectivity of $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp})\to\mathrm{Ext}^1_{G/Z}(\mathrm{Ind}_B^G\mathbf{1},\mathrm{Sp})$). It follows that the lower map $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp})\to\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbf{1},\mathbf{1})$ is a bijection since both $\mathrm{Hom}(\mathbb{Q}_p^\times,k)$ and $\mathrm{Ext}^1_{G/Z}(\mathrm{Sp},\mathrm{Sp})$ are 3-dimensional over k. This proves the lemma. $\square$

2.2.6 Some miscellaneous results

Lemma 2.2.8. *Let $\pi,\pi'\in\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathscr{O})^{\mathfrak{B}}$ be of finite length such that $\pi^{\mathrm{SL}_2(\mathbb{Q}_p)}=0$ and π' irreducible non-character, then $\check{\mathbf{V}}$ induces:*

$$\begin{aligned}
\mathrm{Hom}_G(\pi',\pi) &\cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}\big(\check{\mathbf{V}}(\pi^\vee),\check{\mathbf{V}}(\pi'^\vee)\big),\\
\mathrm{Ext}^1_{G,\zeta}(\pi',\pi) &\hookrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}\big(\check{\mathbf{V}}(\pi^\vee),\check{\mathbf{V}}(\pi'^\vee)\big).
\end{aligned}$$

Proof. In [54, Lemma A1], Paškūnas proved this lemma for supersingular blocks. For other blocks, the argument in his proof reduces the assertion to the case that $\check{\mathbf{V}}(\pi^\vee)$ is irreducible. We include the argument there for the sake of completeness. We argue by induction on the length l of $\check{\mathbf{V}}(\pi^\vee)$. Assume that $l>1$, then there exists an exact sequence

$$0\to\pi_1\to\pi\to\pi_2\to0 \tag{2.2}$$

with $\pi_1^{\mathrm{SL}_2(\mathbb{Q}_p)}=\pi_2^{\mathrm{SL}_2(\mathbb{Q}_p)}=0$. Since $\check{\mathbf{V}}$ is exact, it induces an exact sequence

$$0\to\check{\mathbf{V}}(\pi_2^\vee)\to\check{\mathbf{V}}(\pi^\vee)\to\check{\mathbf{V}}(\pi_1^\vee)\to0. \tag{2.3}$$

Applying $\mathrm{Hom}_G(\mathrm{Sp},-)$ to (2.2) and $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(-,\check{\mathbf{V}}(\mathrm{Sp}^\vee))$ to (2.3), we obtain two long exact sequences, and a map between then induced by $\check{\mathbf{V}}$. With the obvious notation, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r] \arrow[d, equal] & B^0 \arrow[r] \arrow[d] & C^0 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, hook] & B^1 \arrow[r] \arrow[d] & C^1 \arrow[d, hook] \\
0 \arrow[r] & \mathscr{A}^0 \arrow[r] & \mathscr{B}^0 \arrow[r] & \mathscr{C}^0 \arrow[r] & \mathscr{A}^1 \arrow[r] & \mathscr{B}^1 \arrow[r] & \mathscr{C}^1.
\end{tikzcd}$$

By induction hypothesis, the first and third vertical maps are bijections, and the fourth and sixth vertical maps are injections. This implies that the second vertical map is an bijection and the fifth vertical map is an injection.

The case $l = 1$ is dealt in [22, Theorem VII.4.7, Proposition VII.4.10, Proposition VII.4.13, Proposition VII.4.24] except the following two cases:

(i) $p = 3$ and $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}, \omega \circ \det, \mathrm{Sp} \otimes \omega \circ \det\} \otimes \delta \circ \det$;

(ii) $p = 2$ and $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}\} \otimes \delta \circ \det$,

where $\delta : \mathbb{Q}_p^\times \to k^\times$ is a smooth character and $\omega : \mathbb{Q}_p^\times \to k^\times$ is the character $\omega(x) = x|x| \bmod \varpi$. Without loss of generality we may assume that $\delta = 1$.

In case (i), by symmetry of the table in subsection 2.2.4, we may assume the socle of π is Sp. We claim that $l = 1$ implies that π is an extension of (at most two) copies of $\mathbf{1}$ by Sp. If the claim holds, then the assertion follows from proposition 2.2.3. To show the claim, we note that the only absolutely irreducible representation which allows a non-trivial extension by Sp is $\mathbf{1}$ (see the table in subsection 2.2.4) and there is no non-trivial extension of $\mathbf{1}$ (resp. $\omega \circ \det$) by $\mathbf{1}$. This proves the claim.

In case (ii), since $l = 1$ implies that π is of the form E_N for some N in proposition 2.2.6, the assertion follows from proposition 2.2.6 and proposition 2.2.7. □

Proposition 2.2.9. *If $\pi \in \mathrm{Mod}_{G,\zeta}^{\mathrm{l.fin}}(k)$ is admissible, then $\check{\mathbf{V}}(\pi^\vee)$ is finitely generated as $k[\![G_{\mathbb{Q}_p}]\!]$-module.*

Proof. Without loss of generality, we can assume $\pi \in \mathrm{Mod}_G^{\mathrm{l.fin}}(\mathscr{O})^{\mathfrak{B}}$, hence has finitely many irreducible subquotients $\pi_1, \dots, \pi_n$ up to isomorphism. Since $\rho_i := \check{\mathbf{V}}(\pi_i^\vee)$ is a finite dimensional $G_{\mathbb{Q}_p}$-representation over k, $\ker \rho_i$ has finite index in $G_{\mathbb{Q}_p}$. It follows that $\mathfrak{K} := \bigcap_i \ker \rho_i$ is of finite index in $G_{\mathbb{Q}_p}$ and $\mathfrak{H} := G_{\mathbb{Q}_p}/\mathfrak{K}$ is a finite group. Denote $\mathfrak{P}$ by the maximal pro-p quotient of $\mathfrak{K}$ and $\mathfrak{G}$ by the quotient of $G_{\mathbb{Q}_p}$ defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K} \arrow[r] \arrow[d, two heads] & G_{\mathbb{Q}_p} \arrow[r] \arrow[d, two heads] & \mathfrak{H} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{P} \arrow[r] & \mathfrak{G} \arrow[r] & \mathfrak{H} \arrow[r] & 0
\end{tikzcd}$$

We claim that the action of $G_{\mathbb{Q}_p}$ on $\check{\mathbf{V}}(\pi^\vee)$ factors through $\mathfrak{G}$. Since π can be written as inductive limit of finite length smooth representations, it suffices to prove the claim for π of finite length by the definition of $\check{\mathbf{V}}$. Set $\rho = \check{\mathbf{V}}(\pi^\vee)$ a finite dimensional $G_{\mathbb{Q}_p}$-representation over k. Since $\mathfrak{K}$ acts trivially on each irreducible pieces, $\rho(\mathfrak{K})$ has to be contained in the upper triangular unipotent matrices, hence it is a p-group. The claim follows because the action of $\mathfrak{K}$ factors through $\mathfrak{P}$.

We have the following equivalent conditions:

$$
\begin{aligned}
&\check{\mathbf{V}}(\pi^\vee) \text{ is a finitely generated } k[\![G_{\mathbb{Q}_p}]\!]\text{-module}\\
\Longleftrightarrow\ &\check{\mathbf{V}}(\pi^\vee) \text{ is a finitely generated } k[\![\mathfrak{G}]\!]\text{-module (thus a finitely generated } k[\![\mathfrak{P}]\!]\text{-module)}\\
\Longleftrightarrow\ &\check{\mathbf{V}}(\pi^\vee)_{\mathfrak{P}} \text{ is a finitely generated } k[\mathfrak{H}]\text{-module (thus a finite dimensional } k\text{-vector space)}\\
\Longleftrightarrow\ &\text{the cosocle of } \check{\mathbf{V}}(\pi^\vee) \text{ is of finite length}
\end{aligned}
$$

where cosocle is defined to be the maximal semi-simple quotient. The first equivalence is due to the claim and the second equivalence follows from Nakayama lemma for compact modules (cf. [14, Corollary 1.5]). The last equivalence is because the cosocle of $\check{\mathbf{V}}(\pi^\vee)$ in the category of compact $k[\![G_{\mathbb{Q}_p}]\!]$-modules coincides with the cosocle of $\check{\mathbf{V}}(\pi^\vee)_{\mathfrak{P}}$ in the category of $k[\mathfrak{H}]$-modules since $\mathfrak{P}$ is a pro-p normal subgroup of $\mathfrak{G}$.

Consider the quotient $\pi/\pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ of π. It is admissible since its Pontryagin dual $(\pi/\pi^{\mathrm{SL}_2(\mathbb{Q}_p)})^\vee$ is a finitely generated over $k[\![K]\!]$ (it is a submodule of $\pi^\vee$, which is finitely generated over $k[\![K]\!]$ since it is admissible). Applying the $\mathrm{SL}_2(\mathbb{Q}_p)$-invariant to the exact sequence

$$0 \to \pi^{\mathrm{SL}_2(\mathbb{Q}_p)} \to \pi \to \pi/\pi^{\mathrm{SL}_2(\mathbb{Q}_p)} \to 0,$$

we obtain

$$(\pi/\pi^{\mathrm{SL}_2(\mathbb{Q}_p)})^{\mathrm{SL}_2(\mathbb{Q}_p)} \hookrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Q}_p), \pi^{\mathrm{SL}_2(\mathbb{Q}_p)}) \cong \mathrm{Hom}(\mathrm{SL}_2(\mathbb{Q}_p), \pi^{\mathrm{SL}_2(\mathbb{Q}_p)}),$$

which is 0 since the derived subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is $\mathrm{SL}_2(\mathbb{Q}_p)$. Since $\check{\mathbf{V}}(\pi^\vee) \cong \check{\mathbf{V}}((\pi/\pi^{\mathrm{SL}_2(\mathbb{Q}_p)})^\vee)$ and $\check{\mathbf{V}}((\pi/\pi^{\mathrm{SL}_2(\mathbb{Q}_p)})^\vee)$ is of finite length by lemma 2.2.8, we prove the proposition. $\square$

Chapter 3

Automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$

In this section we define the class of automorphic representations whose associated Galois representations we wish to study. Throughout this section, we let F be a totally real field and fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$.

If $\lambda = (\lambda_\kappa)_{\kappa : F \to \mathbb{C}} \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(F,\mathbb{C})}$, let Ξ_λ denote the irreducible algebraic representation of $(\mathrm{GL}_2)^{\mathrm{Hom}(F,\mathbb{C})}$ which is the tensor product over $\kappa \in \mathrm{Hom}(F,\mathbb{C})$ of irreducible representations of GL_2 with highest weight λ_κ. We say that $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(F,\mathbb{C})}$ is an algebraic weight if it satisfies the parity condition, i.e. $\lambda_{\kappa,1} + \lambda_{\kappa,2}$ is independent of κ.

Definition 3.0.1. We say that a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is regular algebraic if the infinitesimal character of π_∞ has the same infinitesimal character as $\Xi_\lambda^\vee$ for an algebraic weight λ.

Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight λ. For any place $v|p$ of F and any integer $a \geq 1$, let $\mathrm{Iw}_v(a,a)$ denote the subgroup of $\mathrm{GL}_2(\mathscr{O}_{F_v})$ of matrices that reduce to an upper triangular matrix modulo ϖ_v^a. We define the Hecke operator

$$\mathbf{U}_{\varpi_v} = \left[\mathrm{Iw}_v(a,a) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_v(a,a) \right]$$

and the modified Hecke operator

$$\mathbf{U}_{\lambda,\varpi_v} = \left(\prod_{\kappa : F_v \hookrightarrow \overline{\mathbb{Q}}_p} \kappa(\varpi_v)^{-\lambda_{\iota\kappa,2}} \right) \mathbf{U}_{\varpi_v}.$$

Definition 3.0.2. Let v be a place of F above p. We say that π is ι-ordinary at v, if there is an integer $a \geq 1$ and a nonzero vector in $(\iota^{-1}\pi_v)^{\mathrm{Iw}_v(a,a)}$ that is an eigenvector for $\mathbf{U}_{\lambda,\varpi_v}$ with an eigenvalue which is a p-adic unit. This definition does not depend on the choice of ϖ_v.

The following theorem is due to the work of many people. We refer the reader to [20] and [69] for the existance of Galois representations, to [20] for part (2) when $v \nmid p$, to [62] for part (1) and part (2) when $v|p$, and to [39, 76] for part (3).

Theorem 3.0.3. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight λ. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$. Then there exists a continuous semi-simple representation*

$$\rho_{\pi,\iota} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

satisfying the following conditions:

1. *For each place $v|p$ of F, $\rho_{\pi,\iota}|_{G_{F_v}}$ is de Rham, and for each embedding $\kappa : F \to \overline{\mathbb{Q}}_p$, we have*
$$\mathrm{HT}_\kappa(\rho_{\pi,\iota}|_{G_{F_v}}) = \{\lambda_{\iota\kappa,2}, \lambda_{\iota\kappa,1}+1\}.$$
2. *For each finite place v of F, we have $\mathrm{WD}(\rho_{\pi,\iota}|_{G_{F_v}})^{F-ss} \cong r_p(\iota^{-1}\pi_v)$.*
3. *If π is ι-ordinary at $v|p$, then there is an isomorphism*
$$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix},$$
where for $i = 1,2$, $\psi_{v,i} : G_{F_v} \to \overline{\mathbb{Q}}_p^\times$ is a continuous character satisfying
$$\psi_{v,i}(\mathrm{Art}_{F_v}(\sigma)) = \prod_{\kappa : F_v \hookrightarrow \overline{\mathbb{Q}}_p} \kappa(\sigma)^{-(\lambda_{\iota\kappa,3-i}+i-1)}$$
for all σ in some open subgroup of $\mathscr{O}_{F_v}^\times$.

These conditions characterize $\rho_{\pi,\iota}$ uniquely up to isomorphism.

Definition 3.0.4. We call a Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ automorphic of weight $\iota^*\lambda = (\lambda_{\iota^{-1}\kappa,1}, \lambda_{\iota^{-1}\kappa,2}) \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ if there exists a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\lambda := (\lambda_{\kappa,1}, \lambda_{\kappa,2}) \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(F,\mathbb{C})}$ such that $\rho \cong \rho_{\pi,\iota}$. Moreover, if π is ι-ordinary at a place $v|p$ then we say ρ is ι-ordinary at v.

Lemma 3.0.5. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight λ. If π is ι-ordinary at a place $v|p$, then there exist smooth characters $\chi_{v,1}, \chi_{v,2} : F_v^\times \to \overline{\mathbb{Q}}_p^\times$ satisfying the following conditions:*

1. *π_v is a subquotient of the (unnormalized) induction $\mathrm{Ind}_B^{\mathrm{GL}_2(F_v)}(\iota\chi_{v,1}|\cdot|^{\frac{1}{2}} \otimes \iota\chi_{v,2}|\cdot|^{-\frac{1}{2}})$, where B is the subgroup of upper triangular matrices in $\mathrm{GL}_2(F_v)$ and $|\cdot|$ is the absolutely valued defined by $|p| = p^{-1}$.*

2. *For a uniformizer ϖ_v of F_v and $i = 1,2$, we have*

$$\mathrm{val}_p(\chi_{v,i}(\varpi_v)) = \frac{1}{e_v} \sum_{\kappa: F_v \hookrightarrow \mathbb{C}} (\lambda_{\iota^{-1}\kappa, n-i+1} - \frac{3}{2} + i),$$

where e_v is the absolutely ramification index of F_v. In particular, we have $\mathrm{val}_p(\chi_{v,1}(\varpi_v)) < \mathrm{val}_p(\chi_{v,2}(\varpi_v))$*, where* val_p *is the valuation given by* $|\cdot|$ *(i.e.* $\mathrm{val}_p(p) = 1$*).*

Proof. See [40, Corollary 2.2] for part (1). Part (2) is a consequence of the local-global compatibility in [39, 76] (see [72, Theorem 2.4] for a proof). □

Lemma 3.0.6. *Let F'/F be a solvable totally real extension, and let $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ be an isomorphism.*

1. *Let π be a regular algebraic cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$*, and suppose that $\rho_{\pi,\iota}|_{G_{F'}}$ is irreducible. Then there exists a regular algebraic cuspidal automorphic representation $\pi_{F'}$ of* $\mathrm{GL}_2(\mathbb{A}_{F'})$ *such that $\rho_{\pi_{F'},\iota} \cong \rho_{\pi,\iota}|_{G_{F'}}$. Moreover, π is ι-ordinary at a place v if and only if $\pi_{F'}$ is ι-ordinary at all places $w|v$ of F'.*

2. *Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation. Suppose that $\rho|_{G_{F'}}$ is irreducible and that there exists a regular algebraic cuspidal automorphic representation π' of* $\mathrm{GL}_2(\mathbb{A}_{F'})$ *such that $\rho|_{G_{F'}} \cong \rho_{\pi',\iota}$. Then there exists a regular algebraic cuspidal automorphic representation π of* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that $\rho \cong \rho_{\pi,\iota}$.*

Proof. Both statement can be deduced from [51]. See [36, Lemma 5.1.6] for the ordinariness. □

Chapter 4

Galois deformation theory

4.1 Global deformation problems

Let F be a number field and p be a prime. We fix a continuous absolutely irreducible $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ and a continuous character $\psi : G_F \to \mathscr{O}^\times$ such that $\psi\varepsilon$ lifts $\det\overline{\rho}$. We fix a finite set S of places of F containing those above p, ∞ and the places at which $\overline{\rho}$ and ψ are ramified. For each $v \in S$, we fix a ring $\Lambda_v \in \mathrm{CNL}_\mathscr{O}$ and define $\Lambda_S = \hat{\otimes}_{v\in S,\mathscr{O}}\Lambda_v \in \mathrm{CNL}_\mathscr{O}$.

For each $v \in S$, we denote $\overline{\rho}|_{G_{F_v}}$ by $\overline{\rho}_v$ and write $\mathscr{D}_v^\square : \mathrm{CNL}_{\Lambda_v} \to \mathrm{Sets}$ (resp. $\mathscr{D}_v^{\square,\psi} : \mathrm{CNL}_{\Lambda_v} \to \mathrm{Sets}$) for the functor associates $R \in \mathrm{CNL}_{\Lambda_v}$ the set of all continuous homomorphisms $r : G_{F_v} \to \mathrm{GL}_2(R)$ such that $r \bmod \mathfrak{m}_R = \overline{\rho}_v$ (resp. and $\det r$ agrees with the composition $G_{F_v} \to \mathscr{O}^\times \to R^\times$ given by $\psi\varepsilon|_{G_{F_v}}$), which is represented by an object $R_v^\square \in \mathrm{CNL}_{\Lambda_v}$ (resp. $R_v^{\square,\psi} \in \mathrm{CNL}_{\Lambda_v}$).

Definition 4.1.1. Let $v \in S$, a local deformation problem for $\overline{\rho}_v$ is a subfunctor $\mathscr{D}_v \subset \mathscr{D}_v^\square$ satisfying the following conditions:

- $\mathscr{D}_v$ is represented by a quotient R_v of $R_v^\square$.
- For all $R \in \mathrm{CNL}_{\Lambda_v}$, $a \in \ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$ and $r \in \mathscr{D}_v(R)$, we have $ara^{-1} \in \mathscr{D}_v(R)$.

We will write $\rho_v^\square : G_{F_v} \to \mathrm{GL}_2(R_v^\square)$ for the universal lifting of $\overline{\rho}_v$.

Definition 4.1.2. A global deformation problem is a tuple

$$\mathscr{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathscr{D}_v\}_{v\in S})$$

where

- The object $\overline{\rho}$, S and $\{\Lambda_v\}_{v\in S}$ are defined at the beginning of this section.
- For each $v \in S$, $\mathscr{D}_v$ is a local deformation problem for $\overline{\rho}_v$.

Definition 4.1.3. Let $\mathscr{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathscr{D}_v\}_{v\in S})$ be a global deformation problem. Let $R \in \mathrm{CNL}_{\Lambda_S}$, and let $\rho : G_F \to \mathrm{GL}_2(R)$ be a lifting of $\overline{\rho}$. We say that ρ is of type $\mathscr{S}$ if it satisfies the following conditions:

1. ρ is unramified outside S.
2. For each $v \in S$, $\rho_v := \rho|_{G_{F_v}}$ is in $\mathscr{D}_v(R)$, where R has a natural Λ_v-algebra structure via the homomorphism $\Lambda_v \to \Lambda_S$.

We say that two liftings $\rho_1, \rho_2 : G_F \to \mathrm{GL}_2(R)$ are strictly equivalent if there exists $a \in \ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$ such that $\rho_2 = a\rho_1 a^{-1}$. It's easy to see that strictly equivalence preserves the property of being type $\mathscr{S}$.

We write $\mathscr{D}^{\square}_{\mathscr{S}}$ for the functor $\mathrm{CNL}_{\Lambda_S} \to \mathrm{Sets}$ which associates to $R \in \mathrm{CNL}_{\Lambda_S}$ the set of liftings $\rho : G_F \to \mathrm{GL}_2(R)$ which are of type $\mathscr{S}$, and write $\mathscr{D}_{\mathscr{S}}$ for the functor $\mathrm{CNL}_{\Lambda_S} \to \mathrm{Sets}$ which associates to $R \in \mathrm{CNL}_{\Lambda_S}$ the set of strictly equivalence classes of liftings of type $\mathscr{S}$.

Definition 4.1.4. If $T \subset S$ and $R \in \mathrm{CNL}_{\Lambda_S}$, then a T-framed lifting of $\overline{\rho}$ to R is a tuple $(\rho, \{\alpha_v\}_{v\in T})$, where ρ is a lifting of $\overline{\rho}$, and for each $v \in T$, α_v is an element of $\ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$. Two T-framed liftings $(\rho, \{\alpha_v\}_{v\in T})$ and $(\rho', \{\alpha'_v\}_{v\in T})$ are strictly equivalent if there is an element $a \in \ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$ such that $\rho' = a\rho a^{-1}$ and $\alpha'_v = a\alpha_v$ for each $v \in T$.

We write $\mathscr{D}^{T}_{\mathscr{S}}$ for the functor $\mathrm{CNL}_{\Lambda_S} \to \mathrm{Sets}$ which associates to $R \in \mathrm{CNL}_{\Lambda_S}$ the set of strictly equivalence classes of T-framed liftings $(\rho, \{\alpha_v\}_{v\in T})$ to R such that ρ is of type $\mathscr{S}$. Similarly, we may consider liftings of type $\mathscr{S}$ with determinant $\psi\varepsilon$, which we denote the corresponding functor by $\mathscr{D}^{\psi}_{\mathscr{S}}$, $\mathscr{D}^{\square,\psi}_{\mathscr{S}}$ and $\mathscr{D}^{T,\psi}_{\mathscr{S}}$.

Theorem 4.1.5. *Let $\mathscr{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathscr{D}_v\}_{v\in S})$ be a global deformation problem. Then the functor $\mathscr{D}_{\mathscr{S}}$, $\mathscr{D}^{\square}_{\mathscr{S}}$, $\mathscr{D}^{T}_{\mathscr{S}}$, $\mathscr{D}^{\psi}_{\mathscr{S}}$, $\mathscr{D}^{\square,\psi}_{S}$ and $\mathscr{D}^{T,\psi}_{S}$ are represented by objects $R_{\mathscr{S}}$, $R^{\square}_{\mathscr{S}}$, $R^{T}_{\mathscr{S}}$, $R^{\psi}_{\mathscr{S}}$, $R^{\square,\psi}_{\mathscr{S}}$ and $R^{T,\psi}_{\mathscr{S}}$, respectively, of* CNL_{Λ_S}.

Proof. For $\mathscr{D}_{\mathscr{S}}$, this is due to [37, Theorem 9.1]. The representability of the functors $\mathscr{D}^{\square}_{\mathscr{S}}$, $\mathscr{D}^{T}_{\mathscr{S}}$, $\mathscr{D}^{\psi}_{\mathscr{S}}$, $\mathscr{D}^{\square,\psi}_{S}$ and $\mathscr{D}^{T,\psi}_{S}$ can be deduced easily from this. ◻

Lemma 4.1.6. *Let $\mathscr{S}$ be a global deformation problem, and let $\rho_{\mathscr{S}} : G_F \to \mathrm{GL}_2(R_{\mathscr{S}})$ be the universal solution of deformations of type $\mathscr{S}$. Choose $v_0 \in T$, and let $\mathscr{T} = \mathscr{O}[\![X_{v,i,j}]\!]_{v\in T, 1\le i,j\le 2}/(X_{v_0,1,1})$. There is a canonical isomorphism $R^{T}_{\mathscr{S}} \cong R_{\mathscr{S}} \hat{\otimes}_{\mathscr{O}} \mathscr{T}$.*

Proof. Note that the centralizer in $id_2 + M_2(\mathfrak{m}_{R_{\mathscr{S}}})$ of ρ_S is the scalar matrices, Thus the T-framed lifting over $R_{\mathscr{S}}\hat{\otimes}_{\mathscr{O}}\mathscr{T}$ given by the tuple $(\rho_S, \{id_2 + (X_{v,i,j})\}_{v\in T})$ is a universal framed deformation of $\bar{r}$ over $R_{\mathscr{S}}\hat{\otimes}_{\mathscr{O}}\mathscr{T}$. This shows that the induced map $R^T_{\mathscr{S}} \to R_{\mathscr{S}}\hat{\otimes}_{\mathscr{O}}\mathscr{T}$ is an isomorphism. □

Let $\mathscr{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathscr{D}_v\}_{v\in S})$ be a global deformation problem and denote $R_v \in \mathrm{CNL}_{\Lambda_v}$ the representing object of $\mathscr{D}_v$ for each $v \in S$. We write $A^T_{\mathscr{S}} = \hat{\otimes}_{v\in T,\mathscr{O}} R_v$ for the complete tensor product of R_v over $\mathscr{O}$ for each $v \in T$, which has a canonical $\Lambda_T := \hat{\otimes}_{v\in T,\mathscr{O}}\Lambda_v$ algebra structure. The natural transformation $(\rho, \{\alpha_v\}_{v\in T}) \mapsto (\alpha_v^{-1}\rho|_{G_{F_v}}\alpha_v)_{v\in T}$ induces a canonical homomorphism of Λ_T-algebras $A^T_{\mathscr{S}} \to R^T_{\mathscr{S}}$. Moreover, lemma 4.1.6 allows us to consider $R_{\mathscr{S}}$ as an $A^T_{\mathscr{S}}$ algebra via the map $A^T_{\mathscr{S}} \to R^T_{\mathscr{S}} \twoheadrightarrow R_{\mathscr{S}}$.

Proposition 4.1.7. *Suppose that*

- *T contains all the places above p;*
- *either $S\backslash T$ contains a finite place or $H^0(G_{F,S}, (\mathrm{ad}^0\overline{\rho})^*(1)) = 0$.*
- *$\mathscr{D}_v \subset \mathscr{D}_v^{\square,\psi}$ for $v \in T$ and $\mathscr{D}_v = \mathscr{D}_v^{\square,\psi}$ for $v \in S - T$.*

Let $r = \sum_{v|\infty, v\notin T} \dim_k H^0(G_{F_v}, (\mathrm{ad}^0\overline{\rho})^*(1))$. Then for some $s \geq 0$, there is an isomorphism of $A^T_{\mathscr{S}}$-algebras

$$R^{T,\psi}_{\mathscr{S}} \cong A^T_{\mathscr{S}}[\![x_1, \cdots, x_{s+|T|-1}]\!]/(f_1, \cdots, f_{r+s}).$$

Proof. This follows from the proof of [46, Proposition 4.1.5]. □

Remark 4.1.8. Note that $H^0(G_{F,S}, (\mathrm{ad}^0\overline{\rho})^*(1)) = 0$ if $p > 2$ and $\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible with ζ_p a primitive p-th roots of unity (c.f. [41, Lemma 8.1]).

Proposition 4.1.9. *Let $\mathscr{S}$ be a global deformation problem as before and F' be a finite Galois extension of F. Suppose that*

- $\mathrm{End}_{G_{F'}}(\overline{\rho}) = k$.
- *$\mathscr{S}' = (\overline{\rho}|_{G_{F'}}, S', \{\Lambda_w\}_{v\in S'}, \{\mathscr{D}_w\}_{w\in S'})$ is a deformation problem where*
  - *S' is the set of places of F' above S;*
  - *T' is the set of places of F' above T;*
  - *for each $w|v$, $\Lambda_w = \Lambda_v$ and $\mathscr{D}_w$ is a local deformation problem equipped with a natural map $R_w \to R_v$ induced by restricting deformations of $\overline{\rho}_v$ to $G_{F'_w}$.*

Then the natural map $R^{T',\psi}_{\mathscr{S}'} \to R^{T,\psi}_{\mathscr{S}}$ induced by restricting deformations of $\overline{\rho}$ to $G_{F'}$, make $R^{T,\psi}_{\mathscr{S}}$ into a finitely generated $R^{T',\psi}_{\mathscr{S}'}$-module.

Proof. Let $\mathfrak{m}'$ be the maximal ideal of $R^{T',\psi}_{\mathscr{S}'}$. It follows from [43, Lemma 3.6] and Nakayama's lemma that it is enough to show the image of $G_{F,S} \to \mathrm{GL}_2(R^{T,\psi}_{\mathscr{S}}) \to \mathrm{GL}_2(R^{T,\psi}_{\mathscr{S}}/\mathfrak{m}' R^{T,\psi}_{\mathscr{S}})$ is finite. Since $G_{F',S'}$ is of finite index in $G_{F,S}$ and it gets mapped to the finite subgroup $\overline{\rho}(G_{F',S'})$, we are done. $\square$

4.2 Local deformation problems

In this section, we define some local deformation problems we will use later.

The following lemma is due to [5, Lemma 3.2].

Lemma 4.2.1. *Let $R_v \in \mathrm{CNL}_{\Lambda_v}$ be a quotient of $R_v^\square$ satisfying the following conditions:*

- *The ring R_v is reduced, and not isomorphic to k.*
- *Let $r : G_{F_v} \to \mathrm{GL}_2(R_v)$ denote the specialization of the universal lifting, and let $a \in \ker(\mathrm{GL}_2(R_v) \to \mathrm{GL}_2(k))$. Then the homomorphism $R_v^\square \to R_v$ associated to the representation ara^{-1} by universality factors through the canonical projection $R_v^\square \to R_v$.*

Then the subfunctor of $\mathscr{D}_v^\square$ defined by R_v is a local deformation problem.

4.2.1 Ordinary deformations

In this section we define ordinary deformations following [2, Section 1.4].

Suppose that $v|p$ and that E contains the image of all embeddings $F_v \hookrightarrow \overline{\mathbb{Q}}_p$. We will assume throughout this subsection that there is some line $\overline{L}$ in $\overline{\rho}_v$ that is stable by the action of G_{F_v}. Let $\overline{\eta}$ denote the character of G_{F_v} giving the action on $\overline{L}$. Note that the choice of $\overline{\eta}$ is unique unless $\overline{\rho}_v$ is the direct sum of two distinct characters. In this case we simply make a choice of one of these characters.

We write $\mathscr{O}_{F_v}^\times(p)$ for the maximal pro-p quotient of $\mathscr{O}_{F_v}^\times$. Set $\Lambda_v = \mathscr{O}[\![\mathscr{O}_{F_v}^\times(p)]\!]$ and write $\psi^{univ} : G_{F_v} \to \Lambda_v^\times$ for the universal character lifting $\overline{\psi}$. Note that Art_{F_v} restricts to an isomorphism $\mathscr{O}_{F_v}^\times \cong I_{F_v}^{ab}$, where $I_{F_v}^{ab}$ is the abelianization of I_{F_v}.

Let $\mathbb{P}^1$ be the projective line over $\mathscr{O}$. We denote $\mathscr{L}_\Delta$ the subfunctor of $\mathbb{P}^1 \times_{\mathscr{O}} \operatorname{Spec} R_v^{\square,\psi}$, whose A-points for any $\mathscr{O}$-algebra A consist of an $\mathscr{O}$-algebra homomorphism $R_v^{\square,\psi} \to A$ and a line $L \in \mathbb{P}^1(A)$ such that the filtration is preserved by the action of G_{F_v} on A^2 induced from $\rho_v^\square$ and such that the action of G_{F_v} on L is given by pushing forward ψ^{univ}. This subfunctor is represented by a closed subscheme (c.f. [2, Lemma 1.4.2]), which we denote by $\mathscr{L}_\Delta$ also.

We define R_v^{Δ} to be the maximal reduced, $\mathscr{O}$-torsion free quotient of the image of the map $R_v^{\square,\psi} \to H^0(\mathscr{L}_{\Delta}, \mathscr{O}_{\mathscr{L}_{\Delta}})$.

Proposition 4.2.2. *The ring R_v^{Δ} defines a local deformation problem. Moreover,*

1. *An $\mathscr{O}$-algebra homomorphism $x : R_v^{\square,\psi} \to \overline{\mathbb{Q}}_p$ factors through R_v^{Δ} if and only if the corresponding Galois representation is $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$-conjugate to a representation*
$$\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$
where $\psi_1|_{G_{F_v}} = x \circ \psi^{univ}$.

2. *assume the image of $\bar{\rho}|_{G_{F_v}}$ is either trivial or has order p, and that if $p = 2$, then either F_v contains a primitive fourth roots of unity or $[F_v : \mathbb{Q}_2] \geq 3$. Then for each minimal prime $Q_v \subset \Lambda_v$, R_v^{Δ}/Q_v is an integral domain of relative dimension $3 + 2[F_v : \mathbb{Q}_p]$ over $\mathscr{O}$, and its generic point is of characteristic 0.*

Proof. The first assertion follows from [2, Proposition 1.4.4] and the second assertion is due to [2, Proposition 1.4.12]. $\square$

We define $\mathscr{D}_v^{\Delta}$ to be the local deformation problem represented by R_v^{Δ}.

4.2.2 Potentially semi-stable deformations

Suppose that $v|p$ and that E contains the image of all embeddings $F_v \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\Lambda_v = \mathscr{O}$.

Proposition 4.2.3. *For each $\lambda_v \in (\mathbb{Z}_+^2)^{\mathrm{Hom}(F_v,E)}$ and inertial type $\tau_v : I_v \to \mathrm{GL}_2(E)$, there is a unique (possibly trivial) quotient $R_v^{\lambda_v,\tau_v}$ (resp. $R_v^{\lambda_v,\tau_v,cr}$) of the universal lifting ring $R_v^{\square,\psi}$ with the following properties:*

1. *$R_v^{\lambda_v,\tau_v}$ (resp. $R_v^{\lambda_v,\tau_v,cr}$) is reduced and p-torsion free, and all the irreducible components of $R_v^{\lambda_v,\tau_v}[1/p]$ (resp. $R_v^{\lambda_v,\tau_v,cr}[1/p]$) is formally smooth and of relative dimension $3+[F_v : \mathbb{Q}_p]$ over $\mathscr{O}$.*

2. *If E'/E is a finite extension, then an $\mathscr{O}$-algebra homomorphism $R_v^{\square,\psi} \to E'$ factors through $R_v^{\lambda_v,\tau_v}$ (resp. $R_v^{\lambda_v,\tau_v,cr}$) if and only if the corresponding Galois representation $G_{F_v} \to \mathrm{GL}_2(E')$ is potentially semi-stable (resp. potentially crystalline) of weight λ and inertial type τ.*

3. *$R_v^{\lambda_v,\tau_v}/\varpi$ (resp. $R_v^{\lambda_v,\tau_v,cr}/\varpi$) is equidimensional.*

Proof. This is due to [47] (see also [1, Corollary 1.3.5]). □

In the case that $R_v^{\lambda_v,\tau_v} \neq 0$ (resp. $R_v^{\lambda_v,\tau_v,cr} \neq 0$), we define $\mathscr{D}_v^{\lambda_v,\tau_v,ss}$ (resp. $\mathscr{D}_v^{\lambda_v,\tau_v,cr}$) to be the local deformation problem represented by $R_v^{\lambda_v,\tau_v}$ (resp. $R_v^{\lambda_v,\tau_v,cr}$).

4.2.3 Fixed weight potentially semi-stable deformations

For $\lambda_v \in (\mathbb{Z}_+^2)^{\mathrm{Hom}(F_v,E)}$, we define characters $\psi_i^{\lambda_v} : I_{F_v} \to \mathscr{O}^\times$ for $i = 1,2$ by

$$\psi_i^{\lambda_v} : \sigma \mapsto \varepsilon(\sigma)^{-(i-1)} \prod_{\kappa_v : F_v \hookrightarrow E} \kappa_v(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-\lambda_{\kappa_v,3-i}}.$$

Definition 4.2.4. Let $\lambda_v \in (\mathbb{Z}_+^2)^{\mathrm{Hom}(F_v,E)}$ and $\rho_v : G_{F_v} \to \mathrm{GL}_2(\mathscr{O})$ be a continuous representation. We say ρ is ordinary of weight λ_v if there is an isomorphism

$$\rho_v \sim \begin{pmatrix} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix},$$

where for $i = 1,2$, $\psi_{v,i} : G_{F_v} \to \mathscr{O}^\times$ is a continuous character agrees with $\psi_i^{\lambda_v}$ on an open subgroup of I_{F_v}.

Proposition 4.2.5. *For each λ_v, τ_v there is a unique (possibly trivial) reduced and p-torsion free quotient $R_v^{\Delta,\lambda_v,\tau_v}$ of R_v^{Δ} satisfying the following properties:*

1. *If E'/E is a finite extension, then the $\mathscr{O}$-algebra homomorphism $R_v^{\square,\psi} \to E'$ factors through $R_v^{\Delta,\lambda_v,\tau_v}$ if and only if the corresponding Galois representation $G_v \to \mathrm{GL}_2(E')$ is ordinary and potentially semi-stable of Hodge type λ and inertial type η.*

2. $\mathrm{Spec}\, R_v^{\Delta,\lambda_v,\tau_v}$ *is a union of irreducible components of* $\mathrm{Spec}\, R_v^{\lambda_v,\tau_v}$.

Proof. This follows from [36, Lemma 3.3.3]. □

Lemma 4.2.6. *If $R_v^{\Delta,\lambda_v,\tau_v}$ is non-zero, then $\tau = \alpha_1 \oplus \alpha_2$ is a sum of smooth characters of I_v. Moreover, the natural surjection $R_v^{\Delta} \twoheadrightarrow R_v^{\Delta,\lambda_v,\tau_v}$ factors through $R_v^{\Delta} \otimes_{\mathscr{O}[\![\mathscr{O}_{F_v}^\times(p)]\!],\eta} \mathscr{O}$, where $\eta : \mathscr{O}[\![\mathscr{O}_{F_v}^\times(p)]\!] \to \mathscr{O}$ is given by $u \mapsto \alpha_1(\mathrm{Art}_{F_v}(u)) \prod_{\kappa_v : F_v \hookrightarrow E} \kappa_v(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-\lambda_{\kappa_v,2}}$ for $u \in \mathscr{O}_{F_v}^\times(p)$.*

Proof. The first assertion is due to [36, Lemma 3.3.2]. For the second assertion, consider the following diagram

$$\begin{array}{ccccc} \mathscr{L}_{\lambda_v,\tau_v} & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{L}_\Delta \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R_v^{\lambda_v,\tau_v} & \hookrightarrow & \operatorname{Spec} R_v^{\square,\psi} & \hookrightarrow & \operatorname{Spec} \tilde{R}_v^{\square,\psi}, \end{array}$$

where $\tilde{R}_v^{\square,\psi} = R_v^{\square,\psi}\hat{\otimes}_{\mathscr{O}}\mathscr{O}[\![\mathscr{O}_{F_v}^\times(p)]\!]$, $R_v^{\square,\psi} \hookrightarrow \tilde{R}_v^{\square,\psi}$ is induced by the surjection $\tilde{R}_v^{\square,\psi} \twoheadrightarrow R_v^{\square,\psi}$ given by η, $\mathscr{L}_{\lambda_v,\tau_v}$ is the closed subscheme of $\mathbb{P}^1 \times_{\mathscr{O}} R_v^{\lambda_v,\tau_v}$, whose R-valued points, R an $R_v^{\lambda_v,\tau_v}$-algebra, consist of a R-line $L \subset R^2$ on which I_{F_v} acts via the character η composed with Art_{F_v}, and $\mathscr{L}$ is the closed subscheme of $\mathbb{P}^1_{R_v^{\square,\psi}}$ defined by the same way using $R_v^{\square,\psi}$ instead of $R_v^{\lambda_v,\tau_v}$.

It's easy to see that the left square (induced by the quotient $R_v^{\square,\psi} \twoheadrightarrow R_v^{\lambda_v,\tau_v}$) is cartesian and the right square is commutative. This proves the proposition since R_v^Δ is the scheme theoretical image of $\mathscr{L}$ in $\tilde{R}_v^{\square,\psi}$ and $R_v^{\Delta,\lambda_v,\tau_v}$ is the scheme theoretical image of $\mathscr{L}_{\lambda_v,\tau_v}$ in $\operatorname{Spec} R_v^{\lambda_v,\tau_v}$ (c.f. [36, Section 3.3]). □

4.2.4 Irreducible components of potentially semi-stable deformations

Suppose that $\mathscr{C}_v$ is an irreducible component of $\operatorname{Spec} R_v^{\lambda_v,\tau_v}[1/p]$. Then we write $R_v^{\mathscr{C}_v}$ for the maximal reduced, p-torsion free quotient of $R_v^{\lambda_v,\tau_v}$ such that $\operatorname{Spec} R_v^{\mathscr{C}_v}[1/p]$ is supported on the component $\mathscr{C}_v$.

Lemma 4.2.7. *Say that a lifting $\rho : G_{F_v} \to \mathrm{GL}_2(R)$ is of type $\mathscr{D}_v^{\mathscr{C}_v}$ if the induced map $R_v^{\square,\psi} \to R$ factors through $R_v^{\mathscr{C}_v}$. Then $\mathscr{D}_v^{\mathscr{C}_v}$ is a local deformation problem.*

Proof. It follows from [4, Lemma 1.2.2] and lemma 4.2.1. □

4.2.5 Odd deformations

Assume that $F_v = \mathbb{R}$ and $\overline{\rho}|_{G_{F_v}}$ is odd, i.e. $\det\overline{\rho}(c) = -1$ for c the complex conjugation. Let $\Lambda_v = \mathscr{O}$.

Proposition 4.2.8. *There is a reduced and p-torsion free quotient R_v^{-1} of $R_v^{\square,\psi}$ such that if E'/E is a finite extension, a $\mathscr{O}$-homomorphism $R_v^{\square,\psi} \to E'$ factors through R_v^{-1} if and only if the corresponding Galois representation is odd. Moreover,*

- *R_v^{-1} is a complete intersection domain of relative dimension 2 over $\mathscr{O}$ and is formally smooth if $p > 2$.*

- $R_v^{-1}[1/p]$ *is formally smooth over* E.
- $R_v^{-1} \otimes_{\mathcal{O}} k$ *is a domain.*

Proof. See [44, Proposition 3.3]. □

We write $\mathscr{D}_v^{-1}$ for the local deformation problem defined by R_v^{-1}.

4.2.6 Irreducible components of unrestricted deformations

Let $v \nmid p$ and $\Lambda_v = \mathcal{O}$.

Lemma 4.2.9. *Let* $x, y : R_v^{\square,\psi} \to \overline{\mathbb{Q}}_p$ *with* $\rho_x, \rho_y : G_{F_v} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *be the associated framed deformations.*

1. *If* x *and* y *lie on the same irreducible component of* $\operatorname{Spec} R_v^{\square,\psi} \otimes \overline{\mathbb{Q}}_p$, *then*
$$(\rho_x)|_{I_{F_v}}^{ss} \cong (\rho_y)|_{I_{F_v}}^{ss}.$$
2. *Suppose that moreover neither* x *nor* y *lie on any other irreducible component of* $\operatorname{Spec} R_v^{\square,\psi} \otimes \overline{\mathbb{Q}}_p$. *Then*
$$(\rho_x)|_{I_{F_v}} \cong (\rho_y)|_{I_{F_v}}.$$

Proof. See [4, Lemma 1.3.4]. □

Suppose that $\mathscr{C}_v$ is an irreducible component of $\operatorname{Spec} R_v^{\square,\psi}[1/p]$. Then we write $R_v^{\mathscr{C}_v}$ for the maximal reduced, p-torsion free quotient of $R_v^{\square,\psi}$ such that $R_v^{\mathscr{C}_v}$ is supported on the component $\mathscr{C}_v$, which defines a local deformation problem $\mathscr{D}_v^{\mathscr{C}_v}$ by lemma 4.2.1. Moreover, it follows from lemma 4.2.9 that all points of $\operatorname{Spec} R_v^{\mathscr{C}_v}[1/p]$ are of same inertial type if E is large enough.

4.2.7 Unramified deformations

Let $v \nmid p$ and $\Lambda_v = \mathcal{O}$.

Proposition 4.2.10. *Suppose* $\overline{\rho}|_{G_{F_v}}$ *is unramified and* ψ *is unramified at* v. *There there is a reduecd,* $\mathcal{O}$*-torsion free quotient* R_v^{ur} *of* $R_v^{\square,\psi}$ *corresponding to unramified deformations. Moreover,* R_v^{ur} *is formally smooth over* $\mathcal{O}$ *of relative dimension 3.*

Proof. This is due to [49, prop 2.5.3]. □

We denote $\mathscr{D}_v^{ur}$ the local deformation problem defined by R_v^{ur}.

4.2.8 Special deformations

Let $v \nmid p$ and $\Lambda_v = \mathscr{O}$.

Proposition 4.2.11. *There is a reduced, $\mathscr{O}$-torsion free quotient R_v^{St} of $R_v^{\square,\psi}$ satisfying the following properties:*

1. *If E'/E is a finite extension then an $\mathscr{O}$-algebra homomorphism $R_v^{\square,\psi} \to E'$ factors through R_v^{St} if and only if the corresponding Galois representation is an extension of γ_v by $\gamma_v(1)$, where $\gamma_v : G_{F_v} \to \mathscr{O}^\times$ is an unramified character such that $\gamma_v^2 = \psi|_{G_{F_v}}$.*

2. *R_v^{St} is a domain of relative dimension 3 over $\mathscr{O}$ and $R_v^{St}[1/p]$ is regular.*

Proof. This follows from [50, Proposition 2.6.6] and [44, Theorem 3.1]. $\square$

We denote $\mathscr{D}_v^{St}$ the local deformation problem defined by R_v^{St}.

4.2.9 Taylor-Wiles deformations

Suppose that $q_v \equiv 1 \bmod p$, that $\overline{\rho}|_{G_{F_v}}$ is unramified, and that $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues $\alpha_{v,1}, \alpha_{v,2} \in k$. Let $\Delta_v = k(v)^\times(p)$ and $\Lambda_v = \mathscr{O}[\Delta_v^{\oplus 2}]$.

Proposition 4.2.12. *$R_v^\square$ is a formally smooth Λ_v-algebra. Moreover, $\rho_v^\square \cong \chi_{v,1} \oplus \chi_{v,2}$ with $\chi_{v,i}$ a character satisfying $\chi_{v,i}(\mathrm{Frob}_v) \equiv \alpha_{v,i}$ mod $\mathfrak{m}_{R_v^\square}$ and $\chi_{v,i}|_{I_{F_v}}$ agrees, after the composition composition with the Artin map, with the character $k(v)^\times \to \Delta_v^{\oplus 2} \to \Lambda_v^\times$ defined by mapping $k(v)^\times$ to its image in the i-th component of Δ_v.*

Proof. This follows from the proof of [25, Lemma 2.44] (see [66, Proposition 5.3] for an explicit computation of $R_v^\square$). $\square$

In this case, we write $\mathscr{D}_v^{\mathrm{TW}}$ for $\mathscr{D}_v^\square$.

4.3 Irreducible component of p-adic framed deformation rings of $G_{\mathbb{Q}_p}$

Let $\overline{r} : G_L \to \mathrm{GL}_2(k)$ and $\psi : G_L \to \mathscr{O}^\times$ be a lifting of $\det \overline{r}$. We write $R_{\overline{r}}$ (resp. $R_{\overline{r}}^\psi$) for the universal lifting ring of $\overline{r}$ (resp. universal lifting ring of $\overline{r}$ with determinant ψ). The goal of this section is to prove the following theorem.

Theorem 4.3.1. $\mathrm{Spec}\, R_{\overline{r}}^\psi[1/p]$ *is irreducible if one of the following conditions holds:*

- *$p > 2$.*
- *$p = 2$ and $L = \mathbb{Q}_p$.*

Remark 4.3.2. The ring $R_{\bar{r}}^{\square}$ has been calculated when $p > 2$ in [8], $p = 2$ and $L = \mathbb{Q}_p$ in [21, 24, 3].

When $p > 2$, this is due to [8, Theorem 1.5].

When $p = 2$, this is proved in [21, Proposition 4.1] when $\bar{r}$ absolutely irreducible or reducible indecomposable with non-scalar semi-simplification. Assume that $\bar{r}$ is split reducible with non-scalar semi-simplification (i.e. $\bar{r} \cong \left(\begin{smallmatrix} \bar{\chi}_1 & 0 \\ 0 & \bar{\chi}_2 \end{smallmatrix}\right)$ with $\bar{\chi}_1\bar{\chi}_2^{-1} \neq \mathbf{1}$). It is proved in [58, Proposition 5.2] that $R^{\mathrm{ver}} \cong R^{\mathrm{ps}}[\![x,y]\!]/(xy-c)$, where R^{ver} is the versal deformation ring of $\bar{r}$, R^{ps} is the pseudo deformation ring of (the pseudo-character associated to) $\bar{r}$, and $c \in R^{\mathrm{ps}}$ is the element generating the reducibility ideal. Since R^{ps} is isomorphic to the universal deformation ring of $\bar{r}' = \left(\begin{smallmatrix} \bar{\chi}_1 & * \\ 0 & \bar{\chi}_2 \end{smallmatrix}\right)$ with $* \neq 0$ by [58, Proposition 3.6] and $xy - c$ is irreducible in $R^{\mathrm{ps}}[\![x,y]\!]$, wee see that $\operatorname{Spec} R^{\mathrm{ver}}$ is irreducible. This implies $\operatorname{Spec} R_{\bar{r}}$ is irreducible since $R_{\bar{r}}$ is formally smooth over R^{ver} [44, Proposition 2.1].

Thus the only remaining case is that $\bar{r}$ has scalar semi-simplification. We prove this case following the strategy of [24] and [3], which shows that $R_{\bar{r}}$ has 2 irreducible components.

If $A, B \in \mathrm{GL}_2(R)$, then we let $[A,B] = ABA^{-1}B^{-1}$. If $A \in M_2(R)$, we write R/A for the quotient of R by the ideal generated by the entries of A.

Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \; Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \; Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}.$$

We consider the matrix entries of X, Y, and Z as indeterminates and let

$$\mathscr{O}[\![X,Y,Z]\!] := \mathscr{O}[\![x_{11}, x_{12}, x_{21}, x_{22}, \cdots, z_{11}, \cdots, z_{22}]\!].$$

Given $\overline{\lambda}, \overline{\mu}, \overline{\kappa} \in k$, choose lifts λ, μ, κ of $\overline{\lambda}, \overline{\mu}, \overline{\kappa}$ to $\mathscr{O}$. We define the matrices $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathrm{GL}_2(R)$ by

$$\tilde{X} = \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} := \begin{pmatrix} 1 + x_{11} & \lambda + x_{12} \\ x_{21} & 1 + x_{22} \end{pmatrix},$$

$$\tilde{Y} = \begin{pmatrix} \tilde{y}_{11} & \tilde{y}_{12} \\ \tilde{y}_{21} & \tilde{y}_{22} \end{pmatrix} := \begin{pmatrix} 1 + y_{11} & \mu + y_{12} \\ y_{21} & 1 + y_{22} \end{pmatrix},$$

$$\tilde{Z} = \begin{pmatrix} \tilde{z}_{11} & \tilde{z}_{12} \\ \tilde{z}_{21} & \tilde{z}_{22} \end{pmatrix} := \begin{pmatrix} 1 + z_{11} & \kappa + z_{12} \\ z_{21} & 1 + z_{22} \end{pmatrix}.$$

Let $S = S_{\lambda,\mu,\kappa,\alpha,\beta,\gamma} := \mathscr{O}[\![X,Y,Z]\!]/(\tilde{X}^2\tilde{Y}^4[\tilde{Y},\tilde{Z}]-1, \det\tilde{X}-\alpha, \det\tilde{Y}-\beta, \det\tilde{Z}-\gamma)$, which is independent of the choice of lifts λ,μ,κ. Since $\det(\tilde{X}^2\tilde{Y}^4) = 1$, S is nonzero if and only if $\alpha^2\beta^4 = 1$.

Lemma 4.3.3. *Given $\overline{r} = \left(\begin{smallmatrix}\mathbf{1} & * \\ 0 & \mathbf{1}\end{smallmatrix}\right)$ be a continuous representation of $G_{\mathbb{Q}_2}$ and $\psi : G_{\mathbb{Q}_2} \to \mathscr{O}^\times$ be a continuous character lifting $\mathbf{1}$, we have an isomorphism of $\mathscr{O}$-algebras $S \cong R^{\psi}_{\overline{r}}$ (for some $\lambda,\mu,\kappa,\alpha,\beta,\gamma$).*

Proof. Let $G_{\mathbb{Q}_2}(2)$ be the maximal pro-2 quotient of $G_{\mathbb{Q}_2}$, which is topologically generated by three generators x,y,z with the relation $x^2y^4[y,z] = 1$ (cf. [65]). Define $\overline{\lambda},\overline{\mu},\overline{\kappa} \in k$ by

$$\overline{\rho}(x) = \begin{pmatrix}1 & \overline{\lambda} \\ 0 & 1\end{pmatrix},\ \overline{\rho}(y) = \begin{pmatrix}1 & \overline{\mu} \\ 0 & 1\end{pmatrix},\ \overline{\rho}(z) = \begin{pmatrix}1 & \overline{\kappa} \\ 0 & 1\end{pmatrix},$$

and $\alpha,\beta,\gamma \in \mathscr{O}^\times$ by $\psi(x) = \alpha, \psi(y) = \beta, \psi(z) = \gamma$. Let S be the ring defined above with respect to $\overline{\lambda},\overline{\mu},\overline{\kappa},\alpha,\beta,\gamma$. Since $\tilde{X},\tilde{Y},\tilde{Z}$ are congruent to the identity matrix modulo $\mathfrak{m}_S$, sending $x \mapsto \tilde{X}, y \mapsto \tilde{Y}, z \mapsto \tilde{Z}$ induces a continuous representation $r_S : G_{\mathbb{Q}_2}(2) \to \mathrm{GL}_2(S)$. This gives a homomorphism of $\mathscr{O}$-algebras $\phi : R^{\psi}_{\overline{r}} \to S$.

Let $(A,\mathfrak{m}_A)$ be a local Artinian $\mathscr{O}$-algebra with residue field k. A lifting r of $\overline{r}$ to A with determinant ψ has image contained in $\left(\begin{smallmatrix}1+\mathfrak{m}_A & A \\ \mathfrak{m}_A & 1+\mathfrak{m}_A\end{smallmatrix}\right)$, which is a 2-group. Every such r factors through the maximal pro-2 quotient $G_{\mathbb{Q}_2}(2)$ of $G_{\mathbb{Q}_2}$. Thus mapping $r : G_{\mathbb{Q}_2}(2) \to \mathrm{GL}_2(A)$ to $\left(r(x) - \left(\begin{smallmatrix}1 & \lambda_A \\ 0 & 1\end{smallmatrix}\right), r(y) - \left(\begin{smallmatrix}1 & \mu_A \\ 0 & 1\end{smallmatrix}\right), r(z) - \left(\begin{smallmatrix}1 & \kappa_A \\ 0 & 1\end{smallmatrix}\right)\right)$, with λ_A,μ_A,κ_A lifts of $\overline{\lambda},\overline{\mu},\overline{\kappa}$, induces a bijection between the set of such r and the set of triples $(X_A,Y_A,Z_A) \in M_2(\mathfrak{m}_A)^3$ satisfying $\tilde{X}_A^2\tilde{Y}_A^4[\tilde{Y}_A,\tilde{Z}_A]$, $\det\tilde{X}_A = \alpha$, $\det\tilde{Y}_A = \beta$, $\det\tilde{Z}_A = \gamma$, where $\tilde{X}_A = \left(\begin{smallmatrix}1 & \lambda_A \\ 0 & 1\end{smallmatrix}\right) + X_A, \tilde{Y}_A = \left(\begin{smallmatrix}1 & \mu_A \\ 0 & 1\end{smallmatrix}\right) + Y_A, \tilde{Z}_A = \left(\begin{smallmatrix}1 & \kappa_A \\ 0 & 1\end{smallmatrix}\right) + Z_A$. These are in bijection with $\mathrm{Hom}_{\mathscr{O}}(S,A)$. Thus $\phi : R^{\psi}_{\overline{r}} \to S$ is an isomorphism. $\square$

Denote

$$S' = \mathscr{O}[\![X,Y,Z]\!]/(\tilde{X}^2\tilde{Y}^4[\tilde{Y},\tilde{Z}]-1, \det\tilde{Y}-\beta, \det\tilde{Z}-\gamma),$$
$$S^- = \mathscr{O}[\![X,Y,Z]\!]/(\tilde{X}^2\tilde{Y}^4[\tilde{Y},\tilde{Z}]-1, \det\tilde{X}+\alpha, \det\tilde{Y}-\beta, \det\tilde{Z}-\gamma).$$

Lemma 4.3.4. $S'[1/p] = S[1/p] \times S^-[1/p]$.

Proof. Note that the image of $\det\tilde{X} - \alpha$ and $\det\tilde{X} + \alpha$ in S' are zero divisors since

$$0 = \det\tilde{X}^2\det\tilde{Y}^4 - 1 = (\det\tilde{X}\det\tilde{Y}^2 + 1)(\det\tilde{X}\det\tilde{Y}^2 - 1) = \beta^4(\det\tilde{X} + \alpha)(\det\tilde{X} - \alpha)$$

in S'. Denote a the image of $\det\tilde{X}/\alpha$ in S', we see that $(1+a)(1-a) = 0$. It follows that $(1+a)/2$ and $(1-a)/2$ gives the desired idempotents for such a decomposition. $\square$

Lemma 4.3.5. *S' is a complete intersection of dimension 7. The elements*

$$\varpi, x_{21} - \operatorname{tr}\tilde{X}, x_{12}, y_{21}, z_{21}, z_{12}$$

form a regular sequence on S'. In particular, S' is flat over $\mathscr{O}$.

Proof. Since S' has dimension at least 7 (it is a formal power series in 12 variables quotient by 6 relations), it is enough to prove that the ring $A := (S'/(\varpi, x_{21} - \operatorname{tr}(\tilde{X}), x_{12}, y_{21}, z_{21}, z_{12}))^{\mathrm{red}}$ is at most 1-dimensional as S' has dimension at least 7.

Note that the image of $\tilde{Y}, \tilde{Z}$ in A are upper triangular matrices, the relation $\tilde{X}^2\tilde{Y}^4[\tilde{Y}, \tilde{Z}] = 1$ gives that $x_{21}^2 = x_{21}\operatorname{tr}\tilde{X} = 0$ in A, which implies $x_{21} = \operatorname{tr}\tilde{X} = 0$. We see that $x_{11} = x_{22}$ in A and the relation $\tilde{X}^2\tilde{Y}^4[\tilde{Y}, \tilde{Z}] = 1$ simplifies to $\tilde{Y}^4[\tilde{Y}, \tilde{Z}] = 1$. By considering its diagonal entries, we have $(1+y_{11})^4 = (1+y_{22})^4 = 1$ in A, which implies $y_{11} = y_{22} = 0$. Thus the relation $\tilde{Y}^4[\tilde{Y}, \tilde{Z}] = 1$ simplifies to $[\tilde{Y}, \tilde{Z}] = 1$, which is equivalent to $(z_{22} - z_{11})(y_{12} + \mu) = 0$ in A.

To sum up, we have

$$\begin{aligned} A &\cong \frac{k[\![X,Y,Z]\!]}{\left(x_{21}, x_{11} - x_{22}, \det\tilde{X} - 1, x_{12}, y_{21}, y_{11} - y_{22}, \det\tilde{Y} - 1, z_{21}, z_{12}, \det\tilde{Z} - 1, (z_{22} - z_{11})(y_{12} + \overline{\mu})\right)} \\ &\cong \frac{k[\![y_{12}, z_{11}, z_{22}]\!]}{\left((z_{22} - z_{11})(y_{12} + \overline{\mu}), z_{11} + z_{22} + z_{11}z_{22}\right)}, \end{aligned}$$

which is of dimension 1. This proves the lemma. □

Proposition 4.3.6. *The singular locus of* $\operatorname{Spec} S'[\frac{1}{2}]$ *has dimension* ≤ 4.

Proof. Since $S'[\frac{1}{2}]$ is excellent, the singular locus is closed in $S'[\frac{1}{2}]$. Since $S'[\frac{1}{2}]$ is Jacobson, this implies that the singular locus is also Jacobson.

It follows from [24, Lemma 4.1] and lemma 4.3.4 that all singular closed points of $S'[\frac{1}{2}]$ is contained in $V(I)$, where I is the ideal of S' generated by the elements

$$\operatorname{tr}(\rho^{\square}(g))^2 - (\varepsilon(g)+1)^2\varepsilon(g)^{-1}\psi^{\pm}(g), \tag{4.1}$$

as g varies over $G_{\mathbb{Q}_2}$, where $\psi^+ = \psi$ and ψ^- is the determinant defined by S^-. Hence, the singular locus is also contained in $V(I)$ and it is enough to prove that $\dim S'/I \leq 5$ as $\dim(S'/I)[\frac{1}{2}] \leq \dim S'/I - 1$.

Let $J := \sqrt{(\varpi, I)}$ and let $\tilde{\rho} : G_{\mathbb{Q}_2} \to \mathrm{GL}_2(S'/J)$ be the representation obtained by reducing the entries of $\rho^{\square}$ modulo J. It is enough to bound $\dim S'/J$ by 4. Since $\varepsilon \equiv 1 \pmod{\varpi}$, we deduce from 4.1 that $\operatorname{tr}(\rho^{\square}(g))^2 \equiv 0 \pmod{(\varpi, I)}$, and hence $\operatorname{tr}(\tilde{\rho}(g)) = 0$ for all $g \in G_{\mathbb{Q}_2}$.

Hence, the surjection $S' \twoheadrightarrow S'/J$ factors through

$$B := \frac{k[\![X,Y,Z]\!]}{(\det\tilde{X}-1, \det\tilde{Y}-1. \det\tilde{Z}-1, \operatorname{tr}\tilde{X}, \operatorname{tr}\tilde{Y}, \operatorname{tr}\tilde{Z}, \operatorname{tr}\tilde{X}\tilde{Y}, \operatorname{tr}\tilde{Y}\tilde{Z}, \operatorname{tr}\tilde{X}\tilde{Z})} \twoheadrightarrow S'/J \tag{4.2}$$

Note that $\operatorname{tr}\tilde{Y} = \operatorname{tr}\tilde{Z} = 0$ then $\operatorname{tr}\tilde{Y}\tilde{Z} = \tilde{y}_{12}z_{21} + y_{21}\tilde{z}_{12}$, as we are in characteristic 2. It follows from [15, Proposition 1.1] (see [66, Proposition 2.7] also) that

$$\frac{k[\tilde{x}_{12}, x_{21}, \tilde{y}_{12}, y_{21}, \tilde{z}_{12}, z_{21}]}{(\tilde{x}_{12}y_{21} + y_{21}\tilde{x}_{12}, \tilde{y}_{12}z_{21} + y_{21}\tilde{z}_{12}, \tilde{x}_{12}z_{21} + z_{21}\tilde{x}_{12})}$$

is four dimnesional, hence

$$A := \frac{k[\![x_{12}, x_{21}, y_{12}, y_{21}, \tilde{z}_{12}, z_{21}]\!]}{(\tilde{x}_{12}y_{21} + y_{21}\tilde{x}_{12}, \tilde{y}_{12}z_{21} + y_{21}\tilde{z}_{12}, \tilde{x}_{12}z_{21} + z_{21}\tilde{x}_{12})}$$

is four dimnesional also (see [3, Proposition 2.13]). The relations $\det\tilde{X} = \det\tilde{Y} = \det\tilde{Z} = 1$ and $\operatorname{tr}\tilde{X} = \operatorname{tr}\tilde{Y} = \operatorname{tr}\tilde{Z} = 0$ gives that B is finite over A and hence $\dim B \leq 4$. Thus $\dim S'/J \leq 4$. $\square$

Proposition 4.3.7. *The rings $S'[\frac{1}{2}]$ is normal.*

Proof. Since the ring is Cohen-Macaulay by lemma 4.3.5, it satisfies Serre's condition S_2. By Serre's criterion of normality, it suffices to prove that $S'[\frac{1}{2}]$ is regular in codimension 1, which follows from proposition 4.3.6. $\square$

Proposition 4.3.8. *Let A be a local Noetherian ring and let $x_1, \cdots, x_k$ be a system of parameters of A with $k \geq 2$. If A is equidimensional, then each irreducible component of* $\operatorname{Spec} A[1/x_k]$ *meets the closed subset* $\operatorname{Spec}(A/(x_1, \cdots, x_{k-1})[1/x_k])$.

Proof. See [57, Lemma 3.9]. $\square$

Corollary 4.3.9. *Each irreducible component of* $\operatorname{Spec} S'[\frac{1}{2}]$ *meets the vanishing locus of*

$$\operatorname{tr}\tilde{X}, x_{21}, y_{21}, z_{21}.$$

Proof. Following from lemma 4.3.5 and proposition 4.3.8, each irreducible component of $\operatorname{Spec} S'[\frac{1}{2}]$ meets the closed subset $\operatorname{Spec} S'[\frac{1}{2}]/(x_{21} - \operatorname{tr}\tilde{X}, x_{12}, y_{21}, z_{21}, z_{12})$. This proves the theorem since $\operatorname{Spec}\left(S'[\frac{1}{2}]/(x_{21} - \operatorname{tr}\tilde{X}, x_{12}, y_{21}, z_{21}, z_{12})\right)^{red}$ is a closed subset of $\operatorname{Spec} S'[\frac{1}{2}]/(\operatorname{tr}\tilde{X}, x_{21}, y_{21}, z_{21})$ by the proof of lemma 4.3.5. $\square$

Corollary 4.3.10. *Let K be a sufficiently large finite extension of $\mathbb{Q}_2$. Then each irreducible component of* $\operatorname{Spec} S[\frac{1}{2}]$ *contains K-valued point of the form*

$$(\tilde{X},\tilde{Y},\tilde{Z}) = (\begin{pmatrix} \xi & \lambda \\ 0 & -\xi \end{pmatrix}, \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \begin{pmatrix} * & * \\ 0 & * \end{pmatrix})$$

with $\xi^2 = -\alpha$.

Proof. The corollary follows immediately from lemma 4.3.4 and corollary 4.3.9. □

Proposition 4.3.11. *The ring $S[\frac{1}{2}]/(x_{21}, y_{21}, z_{21})$ has 2 irreducible components given by $\tilde{x}_{11}\tilde{y}_{11}^2 = \varepsilon$ for $\varepsilon = \{\pm 1\}$.*

Proof. We have the following relations in $\mathscr{O}[\![X,Y,Z]\!]/(x_{21}, y_{21}, z_{21})$

$$\tilde{X}^2\tilde{Y}^4 = \begin{pmatrix} \tilde{x}_{11}^2\tilde{y}_{11}^4 & \tilde{x}_{11}^2\tilde{y}_{12}(\tilde{y}_{11}+\tilde{y}_{22})(\tilde{y}_{11}^2+\tilde{y}_{22}^2)+\tilde{x}_{12}(\tilde{x}_{11}+\tilde{x}_{22})\tilde{y}_{22}^4 \\ 0 & \tilde{x}_{22}^2\tilde{y}_{22}^4 \end{pmatrix}, \tag{4.3}$$

$$[\tilde{Y},\tilde{Z}] = \begin{pmatrix} 1 & ((\tilde{y}_{11}-\tilde{y}_{22})\tilde{z}_{12}+\tilde{y}_{12}(\tilde{z}_{22}-\tilde{z}_{11}))\tilde{y}_{22}^{-1}\tilde{z}_{22}^{-1} \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

Thus the relation $\tilde{X}^2\tilde{Y}^4[\tilde{Y},\tilde{Z}] = 1$ is equivalent to

$$\tilde{x}_{11}^2\tilde{y}_{11}^4 = \tilde{x}_{22}^2\tilde{y}_{22}^4 = 1,$$
$$\tilde{x}_{11}^2\tilde{y}_{12}(\tilde{y}_{11}+\tilde{y}_{22})(\tilde{y}_{11}^2+\tilde{y}_{22}^2)+\tilde{x}_{12}(\tilde{x}_{11}+\tilde{x}_{22})\tilde{y}_{22}^4+((\tilde{y}_{11}-\tilde{y}_{22})\tilde{z}_{12}+\tilde{y}_{12}(\tilde{z}_{22}-\tilde{z}_{11}))\tilde{y}_{22}^{-1}\tilde{z}_{22}^{-1} = 0.$$

Since $\det\tilde{X} = \alpha$, $\det\tilde{Y} = \beta$ and $\alpha^2\beta^4 = 1$ in S, the first equation implies that on an irreducible component of $S[\frac{1}{2}]/(x_{21}, y_{21}, z_{21})$ we have $\tilde{x}_{11}\tilde{y}_{11}^2 = \varepsilon$ with $\varepsilon \in \{\pm 1\}$. It is enough to show that $A := S[\frac{1}{2}]/(x_{21}, y_{21}, z_{21}, \tilde{x}_{11}\tilde{y}_{11}^2 - \varepsilon)$ is an integral domain.

Since $\tilde{x}_{11} = \varepsilon\tilde{y}_{11}^{-2}$, $\tilde{x}_{22} = \alpha\tilde{x}_{11}^{-1}$, $\tilde{y}_{22} = \beta\tilde{y}_{11}^{-1}$ and $\tilde{z}_{22} = \gamma\tilde{z}_{11}^{-1}$ in A, the second equation is equivalent to

$$\begin{aligned}\tilde{y}_{11}^{-4}\tilde{y}_{12}(\tilde{y}_{11}+\beta\tilde{y}_{11}^{-1})(\tilde{y}_{11}^2+\beta^2\tilde{y}_{11}^{-2})+\tilde{x}_{12}(\varepsilon\tilde{y}_{11}^{-2}+\alpha\varepsilon\tilde{y}_{11}^2)\beta^4\tilde{y}_{11}^{-4}+ \\ ((\tilde{y}_{11}-\beta\tilde{y}_{11}^{-1})\tilde{z}_{12}+\tilde{y}_{12}(\gamma\tilde{z}_{11}^{-1}-\tilde{z}_{11}))\beta^{-1}\tilde{y}_{11}\gamma^{-1}\tilde{z}_{11} = 0.\end{aligned}$$

Multiply the equation by $\tilde{y}_{11}^7$, we have

$$(\tilde{y}_{11}^2+\beta)(\tilde{y}_{11}^4+\beta^2)\tilde{y}_{12}+\varepsilon\beta^4\tilde{y}_{11}(\alpha\tilde{y}_{11}^4+1)\tilde{x}_{12}+\beta^{-1}\gamma^{-1}\tilde{y}_{11}^7(\tilde{y}_{11}^2-\beta)\tilde{z}_{11}\tilde{z}_{12}+\beta^{-1}\gamma^{-1}\tilde{y}_{11}^8(\gamma-\tilde{z}_{11}^2)\tilde{y}_{12} = 0.$$

Thus it suffices to show that

$$f := (\tilde{y}_{11}^2+\beta)(\tilde{y}_{11}^4+\beta^2)\tilde{y}_{12}+\varepsilon\beta^4\tilde{y}_{11}(\alpha\tilde{y}_{11}^4+1)\tilde{x}_{12}+\beta^{-1}\gamma^{-1}\tilde{y}_{11}^7(\tilde{y}_{11}^2-\beta)\tilde{z}_{11}\tilde{z}_{12}+\beta^{-1}\gamma^{-1}\tilde{y}_{11}^8(\gamma-\tilde{z}_{11}^2)\tilde{y}_{12}$$

is irreducible in $\mathscr{O}[\![x_{12},y_{11},y_{12},z_{11},z_{12}]\!][\frac{1}{2}]$. This is equivalent to the irreducibility of f in $\mathscr{O}[\![x_{12},y_{11},y_{12},z_{11},z_{12}]\!]$ because the coefficient $\beta^{-1}\gamma^{-1}$ of $z_{11}z_{12}y_{11}^9$ is not divisible by ϖ and hence $\varpi \nmid f$.

We may write f as $az_{12}+b$, where

$$a = \beta^{-1}\gamma^{-1}\tilde{y}_{11}^7(\tilde{y}_{11}^2-\beta)\tilde{z}_{11}$$
$$b = (\tilde{y}_{11}^2+\beta)(\tilde{y}_{11}^4+\beta^2)\tilde{y}_{12}+\varepsilon\beta^4\tilde{y}_{11}(\alpha\tilde{y}_{11}^4+1)\tilde{x}_{12}+\beta^{-1}\gamma^{-1}\tilde{y}_{11}^7(\tilde{y}_{11}^2-\beta)\tilde{z}_{11}\kappa+\beta^{-1}\gamma^{-1}\tilde{y}_{11}^8(\gamma-\tilde{z}_{11}^2)\tilde{y}_{12}$$

are elements in $B := \mathscr{O}[\![x_{12},y_{11},y_{12},z_{11}]\!]$. Note that b lies in the maximal ideal of B since the factors $(\tilde{y}_{11}^2+\beta)$, $(\tilde{y}_{11}^4+\beta^2)$, $(\alpha\tilde{y}_{11}^4+1)$ and $(\gamma-\tilde{z}_{11}^2)$ do. We see that f is not an unit and thus to prove f is irreducible, it is enough to show that $(a,b)=1$.

Because a is the unit $\beta^{-1}\gamma^{-1}\tilde{y}_{11}^7$ times a distinguished polynomial $\tilde{y}_{11}^2-\beta = y_{11}^2+2y_{11}+(1-\beta)$, $(a,b)=1$ if b is non-zero when evaluating at $\tilde{y}_{11}=t$ for both square root t of β, which can be seen easily since the coefficient of $z_{11}^2y_{12}$ $(=\beta^{-1}\gamma^{-1}\tilde{y}_{11}^8)$ is equal to $\gamma^{-1}\beta^3 \neq 0$ after evaluating. Hence the theorem is proved. □

Lemma 4.3.12. *Let K be a sufficiently large finite extension of $\mathbb{Q}_2$ and let $\mathscr{C}$ be an irreducible component of* $\operatorname{Spec} S[\frac{1}{2}]$. *Assume $\mathscr{C}$ contains a K-valued point of the form*

$$(\tilde{X}_\varepsilon,\tilde{Y}_\varepsilon,\tilde{Z}_\varepsilon) = (\begin{pmatrix}\xi & \lambda\\ 0 & -\xi\end{pmatrix},\begin{pmatrix}* & *\\ 0 & *\end{pmatrix},\begin{pmatrix}* & *\\ 0 & *\end{pmatrix})$$

with $\xi^2=-\alpha$ and $\tilde{x}_{11}\tilde{y}_{11}^2=\varepsilon\in\{\pm1\}$. Then $\mathscr{C}$ also contains the point

$$(\tilde{X}_{-\varepsilon},\tilde{Y}_{-\varepsilon},\tilde{Z}_{-\varepsilon}) = (\begin{pmatrix}-\xi & \lambda\\ 0 & \xi\end{pmatrix},\tilde{Y}_\varepsilon,\tilde{Z}_\varepsilon).$$

Proof. By [24, Lemma 6.5], it sufficient to show that the points $(\tilde{X}_\varepsilon,\tilde{Y}_\varepsilon,\tilde{Z}_\varepsilon)$ with $\varepsilon\in\{\pm1\}$ are arc connected in $X(K)$. That is, there exists matries $(\tilde{X}(t),\tilde{Y}(t),\tilde{Z}(t))$ in $M_2(\mathscr{O}_K[\![t]\!][\frac{1}{2}])$ such that the following hold:

1. the entries $\tilde{X}(t)-(\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix})$, $\tilde{Y}(t)-(\begin{smallmatrix}1 & \mu\\ 0 & 1\end{smallmatrix})$, and $\tilde{Z}(t)-(\begin{smallmatrix}1 & \kappa\\ 0 & 1\end{smallmatrix})$ are topologically nilpotent in $\mathscr{O}_K[\![t]\!][\frac{1}{2}]$.
2. $\tilde{X}(t)^2\tilde{Y}(t)^4[\tilde{Y}(t),\tilde{Z}(t)]=1$, $\det\tilde{X}(t)=\alpha$, $\det\tilde{Y}(t)=\beta$, and $\det\tilde{Z}(t)=\gamma$.

3. $(\tilde{X}_1,\tilde{Y}_1,\tilde{Z}_1)=(\tilde{X}(0),\tilde{Y}(0),\tilde{Z}(0))$, $(\tilde{X}_{-1},\tilde{Y}_{-1},\tilde{Z}_{-1})=(\tilde{X}(1),\tilde{Y}(1),\tilde{Z}(1))$.

Note that $\operatorname{tr}\tilde{X}_\varepsilon=0$ in both cases and that $\tilde{X}^2=-\det\tilde{X}$ if $\operatorname{tr}\tilde{X}=0$. Write $\tilde{X}_1=\left(\begin{smallmatrix}1+a & b\\ 0 & 1+d\end{smallmatrix}\right)$, we have $\tilde{X}_{-1}=\left(\begin{smallmatrix}1+d & b\\ 0 & 1+a\end{smallmatrix}\right)$. The matrices

$$\tilde{X}(t)=\begin{pmatrix}1+a+(d-a)t & b\\ \frac{(d-a)^2t(1-t)}{b} & 1+d+(a-d)t\end{pmatrix},\ \tilde{Y}(t)=\tilde{Y}_\varepsilon,\ \tilde{Z}(t)=\tilde{Z}_\varepsilon$$

if $\overline{\lambda}\neq 0$ (thus $b\neq 0$) and matrices

$$\tilde{X}(t)=\begin{pmatrix}1+a+(d-a)t^2 & (d-a)t(1-t)\\ (d-a)t(1+t) & 1+d+(a-d)t^2\end{pmatrix},\ \tilde{Y}(t)=\tilde{Y}_\varepsilon,\ \tilde{Z}(t)=\tilde{Z}_\varepsilon$$

if $\overline{\lambda}=0$ (take $\lambda=b=0$) give the desired arcs. □

Theorem 4.3.13. *$S[\frac{1}{2}]$ is an integral domain.*

Proof. It follows from proposition 4.3.7 that $S[\frac{1}{2}]$ is a product of integral domains. Thus, it is enough to prove that $X:=\operatorname{Spec}S[\frac{1}{2}]$ is irreducible. Suppose this is not the case, by corollary 4.3.10 we may find two points of the form

$$(\tilde{X}_{\varepsilon_i},\tilde{Y}_{\varepsilon_i},\tilde{Z}_{\varepsilon_i})=(\begin{pmatrix}\xi & \lambda\\ 0 & -\xi\end{pmatrix},\begin{pmatrix}* & *\\ 0 & *\end{pmatrix},\begin{pmatrix}* & *\\ 0 & *\end{pmatrix})$$

with $\xi^2=-\alpha$, $\tilde{x}_{11}\tilde{y}_{11}^2=\varepsilon_i\in\{\pm1\}$ and $i\in\{1,2\}$, lying on two different irreducible components $\mathscr{C}_1,\mathscr{C}_2$ of X. If $\varepsilon_1=\varepsilon_2$, then by proposition 4.3.11 the two points $(\tilde{X}_{\varepsilon_1},\tilde{Y}_{\varepsilon_1},\tilde{Z}_{\varepsilon_1})$ and $(\tilde{X}_{\varepsilon_2},\tilde{Y}_{\varepsilon_2},\tilde{Z}_{\varepsilon_2})$ lies on the same irreducible component of $S[\frac{1}{2}]/(x_{21},y_{21},z_{21})$, thus on the same irreducible component of X which is impossible by our assumption. If $\varepsilon_1\neq\varepsilon_2$, the same argument applying to the points $(\tilde{X}_{-\varepsilon_1},\tilde{Y}_{-\varepsilon_1},\tilde{Z}_{-\varepsilon_1})$ and $(\tilde{X}_{\varepsilon_2},\tilde{Y}_{\varepsilon_2},\tilde{Z}_{\varepsilon_2})$ with

$$(\tilde{X}_{-\varepsilon_1},\tilde{Y}_{-\varepsilon_1},\tilde{Z}_{-\varepsilon_1})=(\begin{pmatrix}-\xi & \lambda\\ 0 & \xi\end{pmatrix},\tilde{Y}_{\varepsilon_1},\tilde{Z}_{\varepsilon_1})$$

gives a contradiction again since $(\tilde{X}_{-\varepsilon_1},\tilde{Y}_{-\varepsilon_1},\tilde{Z}_{-\varepsilon_1})$ lies on $\mathscr{C}_1$ by lemma 4.3.12. This proves the theorem. □

As a consequence, theorem 4.3.1 is proved.

Chapter 5

The patching argument

5.1 Quaternionic forms and completed cohomology

Let F be a totally real field and let D be a quaternion algebra with center F, which is ramified at all infinite places and at a set of finite places Σ, which does not contain any primes dividing p. We will write $\Sigma_p = \Sigma \cup \{v|p\}$. We fix a maximal order $\mathscr{O}_D$ of D, and for each finite places $v \notin \Sigma$ an isomorphism $(\mathscr{O}_D)_v \cong M_2(\mathscr{O}_{F_v})$. For each finite place v of F, we will denote by $\mathbf{N}(v)$ the order of the residue field at v, and by $\varpi_v \in F_v$ a uniformizer.

Denote by $\mathbb{A}_F^\infty \subset \mathbb{A}_F$ the finite adeles and adeles respectively. Let $U = \prod_v U_v$ be a compact open subgroup contained in $\prod_v (\mathscr{O}_D)_v^\times$. We may write

$$(D \otimes_F \mathbb{A}_F^\infty)^\times = \bigsqcup_{i \in I} D^\times t_i U (\mathbb{A}_F^\infty)^\times$$

for some $t_i \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ and a finite index I. We say U is sufficiently small if it satisfies the following condition:

$$(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times = 1 \quad \text{for all } t \in (D \otimes_F \mathbb{A}_F^f)^\times. \tag{5.1}$$

For example, U is sufficiently small if for some place v of F, at which D splits and not dividing $2M$ with M being the integer defined in [57, Lemma 3.1], U_v is the pro-v Iwahori subgroup (i.e. the subgroup whose reduction modulo ϖ_v are the upper triangular unipotent matrices). We will assume this is the case from now on and denote the place by v_1.

Write $U = U^p U_p$, where $U_p = \prod_{v|p} U_v$ and $U^p = \prod_{v \nmid p} U_v$. If A is a topological $\mathscr{O}$-algebra, we let $S(U^p, A)$ be the space of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / U^p \to A.$$

The group $G_p = (D \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \prod_{v|p} \mathrm{GL}_2(F_v)$ acts continuously on $S(U^p, A)$. It follows from (5.1) that there is an isomorphism of A-modules

$$S(U^p, A) \xrightarrow{\sim} \bigoplus_{i \in I} C(F^\times \backslash K_p (\mathbb{A}_F^\infty)^\times, A) \tag{5.2}$$

$$f \mapsto (u \mapsto f(t_i u))_{i \in I}, \tag{5.3}$$

where C denotes the space of continuous functions. Let $\psi : (\mathbb{A}_F^\infty)^\times / F^\times \to \mathscr{O}^\times$ be a continuous character such that ψ is trivial on $(\mathbb{A}_F^\infty)^\times \cap U^p$. We may view ψ as an A-valued character via $\mathscr{O}^\times \to A^\times$. Denote $S_\psi(U^p, A)$ be the A-submodule of $S(U, A)$ consisting of functions such that $f(gz) = \psi(z) f(g)$ for all $z \in (\mathbb{A}_F^\infty)^\times$. The isomorphism (5.2) induces an isomorphism of U_p-representations:

$$S_\psi(U^p, A) \xrightarrow{\sim} \bigoplus_{i \in I} C_\psi(K_p (\mathbb{A}_F^\infty)^\times, A), \tag{5.4}$$

where C_ψ denotes the continuous functions on which the center acts by the character ψ. One may think of $S_\psi(U^p, A)$ as the space of algebraic automorphic forms on $D^\times$ with tame level U^p and no restrictions on the level or weight at spaces dividing p.

Let σ be a continuous representation of U_p on a free $\mathscr{O}$-module of finite rank, such that $(\mathbb{A}_F^\infty)^\times \cap U_p$ acts on σ by the restriction of ψ to this group. We let

$$S_{\psi,\sigma}(U, A) := \mathrm{Hom}_{U_p}(\sigma, S_\psi(U^p, A)).$$

We will omit σ as an index if it is the trivial representation. If the topology on A is discrete (e.g. $A = E/\mathscr{O}$ or $A = \mathscr{O}/\varpi^s$), then we have

$$S_\psi(U^p, A) \cong \varinjlim_{U_p} S_\psi(U^p U_p, A),$$

where U_p runs through compact open subgroup of K_p. The action of G_p on $S_\psi(U^p, A)$ by right translations is continuous. The module $S_\psi(U^p, A)$ is naturally equipped with an A-linear action of $G_p := (D \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \prod_{v|p} \mathrm{GL}_2(F_v)$, which extends the K_p-action. To be precisely, for $g \in G_p$, right multiplication by g induces an map

$$\cdot g : S_\psi(U^p U_p, A) \to S_\psi(U^p U_p^g, A)$$

for each U_p, where $U_p^g = g^{-1}U_p^g g$. As U_p runs through the cofinal subset of open subgroups of K_p with $U_p^g \subset K_p$, the subgroups U_p^g also runs through a cofinal subset of open subgroups of K_p, so we may identify $\varinjlim_{U_p} S_\psi(U^pU_p^g, A)$ with $S_\psi(U^p, A)$.

Denote $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_v F_v$ and $\mathscr{O}_{F_p} = \mathscr{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_v \mathscr{O}_{F_v}$. Let $\zeta : F_p^\times \to \mathscr{O}^\times$ be the character obtained restricting ψ to $F_p^\times$.

Lemma 5.1.1. *The representation* $S_\psi(U^p, E/\mathscr{O})$ *lie in* $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(\mathscr{O})$. *Moreover,* $S_\psi(U^p, E/\mathscr{O})$ *is admissible and injective in* $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(\mathscr{O})$.

Proof. This follows from (5.4). □

Let S_p be the set of places of F above p, S_∞ be the set of places of F above ∞, and let S be a union of the places containing Σ_p, S_∞, and all the places v of F such that $U_v \neq (\mathscr{O}_D)_v^\times$. Write $W = S - (\Sigma_p \cup S_\infty)$. We will assume that for $v \in W$, $U_v \subset \mathrm{GL}_2(\mathscr{O}_{F_v})$ is contained in the Iwahori subgroup and contains the pro-v Iwahori subgroup.

We denote $\mathbb{T}^S = \mathscr{O}[T_v, S_v, \mathbf{U}_{\varpi_w}]_{v \notin S, w \in W}$ be the commutative $\mathscr{O}$-polynomial algebra in the indicated formal variables. If A is a topological $\mathscr{O}$-algebra then $S_\psi(U^p, A)$ and $S_{\psi,\sigma}(U^p, A)$ become $\mathbb{T}^S$-modules with S_v acting via the double coset operator $[U_v\left(\begin{smallmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{smallmatrix}\right)U_v]$, T_v acting via $[U_v\left(\begin{smallmatrix}\varpi_v & 0\\ 0 & 1\end{smallmatrix}\right)U_v]$, and $\mathbf{U}_{\varpi_w}$ acting via $[U_w\left(\begin{smallmatrix}\varpi_w & 0\\ 0 & 1\end{smallmatrix}\right)U_w]$. Note that the operators T_v and S_v do not depend on the choice of ϖ_v but $\mathbf{U}_{\varpi_w}$ does.

5.2 Completed homology and big Hecke algebras

Let $S = S_p \cup S_\infty \cup \Sigma \cup \{v_1\}$, where S_p be the set of places of F above p and S_∞ be the set of places of F above ∞. We define an open compact subgroup $U^p = \prod_{v \nmid p} U_v$ of $G(\mathbb{A}_F^{\infty,p})$ as follows:

- $U_v = G(\mathscr{O}_{F_v})$ if $v \notin S$ or $v \in \Sigma$.
- U_{v_1} is the pro-v_1 Iwahori subgroup.

Due to the choice of v_1, U^pU_p is sufficiently small for any open compact subgroup U_p of $G(F_p)$. It follows that the functor $V \mapsto S_\psi(U^pU_p, V)$ is exact by (5.4).

Definition 5.2.1. We define the completed homology groups $M_\psi(U^p)$ by

$$M_\psi(U^p) := \varprojlim_{U_p} S_\psi(U^pU_p, \mathscr{O})^d$$

equipped with an $\mathscr{O}$-linear action of G_p extending the K_p-action coming from the $\mathscr{O}[\![K_p]\!]$-module structure.

Following from the definition, there is a natural G_p-equivariant homeomorphism

$$M_\psi(U^p) \cong S_\psi(U^p, E/\mathscr{O})^\vee.$$

Corollary 5.2.2. *The representation $M_\psi(U^p)$ is a projective object in* $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(\mathscr{O})$.

Proof. Note that we have natural G_p-equivariant homeomorphism

$$M_\psi(U^p) \cong S_\psi(U^p, E/\mathscr{O})^\vee$$

by definition. Thus the corollary follows from lemma 5.1.1. □

For $U = U^p U_p$, we write $S_\psi(U,s)$ for $S_\psi(U, \mathscr{O}/\varpi^s)$. Define $\mathbb{T}^S_\psi(U,s)$ to be the image of the abstract Hecke algebra $\mathbb{T}^S$ in $\mathrm{End}_{\mathscr{O}/\varpi^s[K_p/U_p]}(S_\psi(U,s))$.

Definition 5.2.3. We define the big Hecke algebra $\mathbb{T}^S_\psi(U^p)$ by

$$\mathbb{T}^S_\psi(U^p) = \varprojlim_{U_p,s} \mathbb{T}^S_\psi(U^p U_p, s)$$

where the limit is over compact open normal subgroups U_p of K_p and $s \in \mathbb{Z}_{\geq 1}$, and the surjective transition maps come from

$$\mathrm{End}_{\mathscr{O}/\varpi^{s'}[K_p/U'_p]}(S_\psi(U'_p U^p, s')) \to \mathrm{End}_{\mathscr{O}/\varpi^s[K_p/U_p]}(\mathscr{O}/\varpi^s[K_p/U_p] \otimes_{\mathscr{O}/\varpi^{s'}[K_p/U'_p]} S_\psi(U'_p U^p, s'))$$

for $s' \geq s$ and $U'_p \subset U_p$ and the natural identification

$$\mathscr{O}/\varpi^s[K_p/U_p] \otimes_{\mathscr{O}/\varpi^{s'}[K_p/U'_p]} S_\psi(U'_p U^p, s') \cong S_\psi(U_p U^p, s).$$

We equip $\mathbb{T}^S_\psi(U^p)$ with the inverse limit topology. It follows from the definition that the action of $\mathbb{T}^S_\psi(U^\mathfrak{p})$ on $M_\psi(U^p)$ is faithful and commutes with the action of G_p.

Lemma 5.2.4. *$\mathbb{T}^S_\psi(U^p)$ is a profinite $\mathscr{O}$-algebra with finitely many maximal ideals. Denote its finitely many maximal ideals by $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ and let $J = \cap_i \mathfrak{m}_i$ denote the Jacobson radical. Then $\mathbb{T}^S_\psi(U^p)$ is J-adically complete and separated, and we have*

$$\mathbb{T}^S_\psi(U^p) = \mathbb{T}^S_\psi(U^p)_{\mathfrak{m}_1} \times \cdots \times \mathbb{T}^S_\psi(U^p)_{\mathfrak{m}_r}.$$

For each i, $\mathbb{T}^S_\psi(U^p)/\mathfrak{m}_i$ is a finite extension of k.

Proof. This is indeed [35, Lemma 2.1.14]. It suffices to prove when $U'_p \subset U_p$ are open normal pro-p subgroups such that $\psi|_{U'_p \cap \mathscr{O}_{F,p}^\times}$ is trivial modulo $\varpi^{s'}$, the map

$$\mathbb{T}_\psi^S(U^p U'_p, s') \to \mathbb{T}_\psi^S(U^p U_p, 1)$$

induces a bijection of maximal ideals.

Let $\mathfrak{m}$ be a maximal ideal of the artinian ring $\mathbb{T}_\psi^S(U^p U'_p, s')$. Since $\mathbb{T}_\psi^S(U^p U'_p, s')$ acts faithfully on $S_\psi(U^p U'_p, s')$, we know that

$$S_\psi(U^p U'_p, s')[\mathfrak{m}] \neq 0.$$

The p-group U_p/U'_p acts naturally on this k-vector space, hence has a non-zero fixed vector, which belong to $S_\psi(U^p U_p, 1)$. Thus $S_\psi(U^p U_p, 1)[\mathfrak{m}] \neq 0$ and $\mathfrak{m}$ is also a maximal ideal of $T_\psi^S(U^p U_p, 1)$. □

Let $\mathfrak{m} \subset \mathbb{T}_\psi^S(U^p)$ be a maximal ideal with residue field k. There exists a continuous semi-simple representation $\overline{\rho}_\mathfrak{m} : G_{F,S} \to \mathrm{GL}_2(k)$ such that for any finite place $v \notin S$ of F, $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - T_v X + q_v S_v \in k[X]$. If $\overline{\rho}_\mathfrak{m}$ is absolutely reducible, we say that the maximal ideal $\mathfrak{m}$ is Eisenstein; otherwise, we say that $\mathfrak{m}$ is non-Eisenstein.

We define a global deformation problem

$$\mathscr{S} = (\overline{\rho}_\mathfrak{m}, F, S, \{\mathscr{O}\}_{v\in S}, \{\mathscr{D}_v^{\square,\psi}\}_{v\in S_p} \cup \{\mathscr{D}_v^{-1}\}_{v\in S_\infty} \cup \{\mathscr{D}_v^{St}\}_{v\in\Sigma} \cup \{\mathscr{D}_{v_1}^{\square,\psi}\}).$$

Proposition 5.2.5. *Suppose that $\mathfrak{m}$ is non-Eisenstein. Then there exists a lifting of $\overline{\rho}_\mathfrak{m}$ to a continuous homomorphism*

$$\rho_\mathfrak{m} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}_\psi^S(U^p)_\mathfrak{m})$$

such that for any finite place $v \notin S$ of F, $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - T_v X + q_v S_v \in \mathbb{T}_\psi^S(U^p)_\mathfrak{m}[X]$. Moreover, $\rho_\mathfrak{m}$ is of type $\mathscr{S}$ and has determinant $\psi\varepsilon$.

Proof. By the proof of lemma 5.2.4, the surjective map $\mathbb{T}_\psi^S(U^p) \twoheadrightarrow \mathbb{T}_\psi^S(U^p U_p, s)$ induces bijection of maximal ideals for U_p small enough. By taking projective limit, it suffices to show that there exist continuous homomorphism $\overline{\rho}_{\mathfrak{m},U_p,s} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}_\psi^S(U^p U_p, s)/\mathfrak{m})$ and $\rho_{\mathfrak{m},U_p,s} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}_\psi^S(U^p U_p, s)_\mathfrak{m})$ satisfying the same conditions as in the statement, which follows from the well-known assertion for $S_\psi(U^p U_p, \mathscr{O})$. □

5.3 Globalization

Keeping the setting of section 5.2. We fix a continuous representation

$$\overline{\rho} : G_{F,S} \to \mathrm{GL}_2(k)$$

which comes from a non-Eisenstein maximal ideal of $\mathbb{T}^S_\psi(U^p)$ (i.e. $\overline{\rho} \cong \overline{\rho}_\mathfrak{m}$). Assume $\overline{\rho}$ satisfying the following properties:

(i) If $p > 2$, then $\overline{\rho}|_{G(\zeta_p)}$ is absolutely irreducible. If $p = 5$ and $\mathrm{Proj}(\overline{\rho}) \cong A_5$, then $[F(\zeta_p) : F] = 4$.

(ii) If $p = 2$, then $\overline{\rho}$ has non-solvable image.

(iii) $\overline{\rho}$ is unramified at all finite places $v \nmid p$;

(iv) $\overline{\rho}(\mathrm{Frob}_{v_1})$ has distinct eigenvalues.

In application to the modularity lifting theorem, assumption (iii) is satisfied after a solvable base change. The following lemma will allow us to reduce to situations where (iv) holds.

Lemma 5.3.1. *Suppose $\overline{\rho}$ is odd if $p > 2$ and has non-solvable image if $p = 2$. Then there exists a place v_1 of F not dividing $2Mp$ such that the eigenvalues of $\overline{\rho}(\mathrm{Frob}_{v_1})$ are distinct.*

Proof. If $p > 2$ then the image of complex conjugations have determinant -1 and order 2, thus have distinct eigenvalues ± 1. By Chebotarev density theorem, there are infinite many places v of F (not dividing $2Mp$) with the same image as a complex conjugation. This gives a desired place v_1.

If $p = 2$, then the projective image of $\overline{\rho}$ is conjugate to $\mathrm{PGL}_2(\mathbb{F}_{2^r})$ for some $r > 1$ by Dickson's theorem, which contains elements with distinct Frobenius eigenvalues $\big($e.g. $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ $\big)$. Thus by Chebotarev density theorem, there are infinite many places v of F with distinct Frobenius eigenvalues. This proves the lemma. □

Definition 5.3.2. Let L be a finite extension of $\mathbb{Q}_p$. Given a continuous representation $\overline{r} : G_L \to \mathrm{GL}_2(k)$, we will say that $\overline{r}$ has a suitable globalization if there is a totally real field F and a continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ satisfying the properties $(i)-(iv)$ above and moreover,

- $\overline{\rho}|_{G_{F_v}} \cong \overline{r}$ for each $v|p$ (hence $F_v \cong L$);
- $[F : \mathbb{Q}]$ is even;

- there exists a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $(0,0)^{\mathrm{Hom}(F,\mathbb{C})}$ and level prime to p satisfying $\overline{\rho}_{\pi,\iota} \cong \overline{\rho}$.

Given a suitable globalization of $\bar{r}$, we set $S = S_p \cup S_\infty \cup \{v_1\}$, $\Sigma = \emptyset$, D the quaternion algebra with center F which is ramified exactly at S_∞, and U^p as in section 5.2. Let $\psi : G_{F,S} \to \mathscr{O}^\times$ be the totally even finite order character such that $\det \rho_{\pi,\iota} = \psi\varepsilon$ and view ψ as a character of $(\mathbb{A}_F^\infty)^\times / F^\times \to \mathscr{O}^\times$ via global class field theory. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_\psi^S(U^p)$ corresponding to $\overline{\rho}$ and γ be the character given by π. Together with the last property, we are in the same situation as section 5.2.

Lemma 5.3.3. *Given $\bar{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$, a suitable globalization exists.*

Proof. This is essentially [30, Proposition 3.2.1] when $p > 2$. By [17, Proposition 3.2], we may find F and $\overline{\rho}$ satisfying all but the last two conditions. By [67, Proposition 8.2.1], there is a finite Galois extension F'/F in which all places above p split completely such that $\overline{\rho}|_{G_{F'}}$ is modular. Note that Snowden proves the result by a theorem of Moret-Bailly under the assumption p is odd. This assumption can be removed using the proof [44, Theorem 6.1], which shows the existance of points with values in local fields for some Hilbert-Blumenthal Abelian varieties when $p = 2$.

If $[F' : \mathbb{Q}]$ is odd, we make a further quadratic extension F'' linearly disjoint form $\overline{F}^{\ker \overline{\rho}}$ over F, and in which all primes above p splits completely. The result follows by replacing F with F''. □

The following lemma says we may change the weight of a globalization $\overline{\rho}$ when p splits completely in F.

Lemma 5.3.4. *Assume that p splits completely in F and that $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ is automorphic. Then $\overline{\rho}$ is automorphic of small weight, i.e. there is a regular algebraic cuspidal automorphic representation π of weight $\lambda = (0, k_v - 2)_v$ with $2 \le k_v \le p$ for each $v|p$ such that $\overline{\rho} \cong \overline{\rho}_{\pi,\iota}$. Moreover,*

1. *at each $v|p$, $\rho_{\pi,\iota}|_{G_{F_v}}$ is crystalline if $p > 2$ and is semi-stable if $p = 2$;*
2. *π is ι-ordinary at those $v|p$ such that $\overline{\rho}|_{G_{F_v}}$ is reducible.*

Proof. When p is odd, the small weight assertion and assertion (1) are a consequence of [33, Theorem 4.6.9], which says that if $\overline{\rho}$ is automorphic, then it is automorphic of weight given by a crystalline lift of a Serre weight (see section 6.2.2). When p is even, this is due to [57, Lemma 3.29], which shows that if $\overline{\rho}$ is automorphic, then it is automorphic of weight $(0,0)^{\mathrm{Hom}(F,\mathbb{C})}$ and semi-stable at each $v|p$.

The second assertion follows from [44, Lemma 3.5], which proves that for a continuous representation $r : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$,

- if r is crystalline of weight $(0,k-2)$ with $2 \le k \le p$, then it is ordinary if and only if residually it is ordinary.
- if r is semi-stable non-crystalline of weight $(0,0)$, then it is ordinary.

This finishes the proof. $\square$

5.4 Auxiliary primes

Let Q be a set of places disjoint from S, such that for each $v \in Q$, $q_v \equiv 1 \bmod p$ and $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues. For each $v \in Q$, we fix a choice of eigenvalue α_v. We refer to the tuple $(Q, \{\alpha_v\}_{v \in Q})$ as a Taylor-Wiles datum, and define the augmented deformation problem

$$\begin{aligned}\mathscr{S}_Q = (\overline{\rho}, S \cup Q, \{\mathscr{O}\}_{v \in S} \cup \{\mathscr{O}[\Delta_v]\}_{v \in Q}, \{\mathscr{D}_v^{\square,\psi}\}_{v \in S_p} \cup \{\mathscr{D}_v^{-1}\}_{v \in S_\infty} \cup \{\mathscr{D}_v^{St}\}_{v \in \Sigma} \cup \{\mathscr{D}_{v_1}^{\square,\psi}\} \\ \cup \{\mathscr{D}_v^{\mathrm{TW}}\}_{v \in Q}).\end{aligned}$$

Let $\Delta_Q = \prod_{v \in Q} \Delta_v = \prod_{v \in Q} k(v)^\times(p)$. Then $R_{\mathscr{S}_Q}$ is naturally a $\mathscr{O}[\Delta_Q]$-algebra, If $\mathfrak{a}_Q \subset \mathscr{O}[\Delta_Q]$ is the augmentation ideal, then there is a canonical isomorphism $R_{\mathscr{S}_Q}/\mathfrak{a}_Q R_{\mathscr{S}_Q} \cong R_{\mathscr{S}}$ (resp. $R^T_{\mathscr{S}_Q}/\mathfrak{a}_Q R^T_{\mathscr{S}_Q} \cong R^T_{\mathscr{S}}$).

Lemma 5.4.1. *Let $T = S$. For every $N \gg 0$, there exists a Taylor-Wiles datum $(Q_N, \{\alpha_v\}_{v \in Q_N})$ satisfying the following conditions:*

1. *$\#Q_N := q = \dim_k H^1(G_{F,S}, \mathrm{ad}^0 \overline{\rho}(1))$ if $p > 2$ (resp. $q = \dim_k H^1(G_{F,S}, \mathrm{ad}\, \overline{\rho}) - 2$ if $p = 2$).*
2. *For each $v \in Q_N$, $q_v \equiv 1$ (mod p^N).*
3. *The ring $R^{S,\psi}_{\mathscr{S}_{Q_N}}$ is topologically generated by $q + |S| - 1$ elements over $A^S_{\mathscr{S}}$ if $p \geq 2$ (resp. $R^S_{\mathscr{S}_{Q_N}}$ is topologically generated by $2q+1$ elements over $A^S_{\mathscr{S}}$ if $p = 2$).*
4. *If $p = 2$, let G_{Q_N} be the Galois group of the maximal abelian 2-extension of F which is unramified outside Q_N and is split at primes in S. Then we have $G_{Q_N}/2^N G_{Q_N} \cong (\mathbb{Z}/2^N\mathbb{Z})^t$ with $t := 2 - |S| + q$.*

Proof. See [44, Lemma 5.3, Lemma 5.5, Lemma 5.10]. $\square$

5.4.1 Action of Θ_Q

If $p = 2$ and Q is a finite set of finite primes of F disjoint from S, we denote by Θ_Q the Galois group of the maximal abelian 2-extension of F which is unramified outside Q and in which every prime in S splits completely. Let Θ_Q^* be the formal group scheme defined over $\mathscr{O}$ whose A-valued points is given by the group $\mathrm{Hom}(\Theta_Q, A)$ of continuous characters on Θ_Q that reduce to the trivial character modulo $\mathfrak{m}_A$.

It follows that $\mathrm{Spf} R_{\mathscr{S}_Q}$ (resp. $\mathrm{Spf} R^T_{\mathscr{S}_Q}$) has a natural action by Θ_Q^* given by $\chi_A \times V_A \mapsto V_A \otimes \chi_A$ on A-valued points, which is free if $\overline{\rho}$ has non-solvable image [44, Lemma 5.1]. Moreover, there is a Θ_Q^*-equivariant map

$$\delta_Q : \mathrm{Spf} R^T_{\mathscr{S}_Q} \to \Theta_Q^*; \qquad V_A \mapsto \det V_A \cdot (\psi\varepsilon)^{-1}$$

where Θ_Q^* acts on itself via the square of the identity map, and $\mathrm{Spf} R^{T,\psi}_{\mathscr{S}_Q} = \delta_Q^{-1}(1)$.

5.5 Auxiliary levels

A choice of Taylor-Wiles datum $(Q, \{\alpha_v\}_{v\in Q})$ having been fixed, we have defined an auxiliary deformation problem $\mathscr{S}_Q$.

Let U^p be the open compact subgroup of $G(\mathbb{A}_F^{\infty,p})$ in section 5.2. We define compact open subgroups $U_0^p(Q) = \prod_{v\nmid p} U_0(Q)_v$ and $U_1^p(Q) = \prod_{v\nmid p} U_1(Q)_v$ of $U^p = \prod_{v\nmid p} U_v$ by:

- if $v \notin Q$, then $U_0(Q)_v = U_1(Q)_v = U_v$.
- if $v \in Q$, then $U_0(Q)_v$ is the Iwahori subgroup of $\mathrm{GL}_2(\mathscr{O}_{F_v})$ and $U_1(Q)_v$ is the set $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(Q)_v$ such that ad^{-1} maps to 1 in Δ_v.

In particular, $U_1(Q)_v$ contains the pro-v Iwahori subgroup of $U_0(Q)_v$, so we may identify $\prod_{v\in Q} U_0(Q)_v / U_1(Q)_v$ with Δ_Q.

Let $\mathfrak{m}_Q$ denote the ideal of $\mathbb{T}^{S\cup Q}$ generated by $\mathfrak{m} \cap \mathbb{T}^{S\cup Q}$ and the elements $\mathbf{U}_{\varpi_v} - \tilde{\alpha}_v$ for $v \in Q$, where $\tilde{\alpha}_v$ is an arbitrary lift of α_v. We denote by $\mathbb{T}_\psi^{S\cup Q}(U_i^p(Q)U_p, s)$ the image of $\mathbb{T}^{S\cup Q}$ in $\mathrm{End}_{\mathscr{O}/\varpi^s}(S_\psi(U_i^p(Q)U_p, s))$. Exactly as [48, Section 2.1], we have the following:

1. The maximal ideal $\mathfrak{m}_Q$ induces proper, maximal ideals in $\mathbb{T}_\psi^{S\cup Q}(U_i^p(Q)U_p, s)$. Moreover, the map

$$S_\psi(U^pU_p, s)_{\mathfrak{m}} \to S_\psi(U_1^p(Q)U_p, s)_{\mathfrak{m}_Q}$$

is an isomorphism.

2. $S_\psi(U_1^p(Q)U_p, s)_{\mathfrak{m}_Q}$ is a finite projective $\mathscr{O}/\varpi^s[\Delta_Q]$-module with

$$S_\psi(U_1^p(Q)U_p, s)_{\mathfrak{m}_Q}^{\Delta_Q} \xrightarrow{\sim} S_\psi((U_0^p(Q)U_p, s)_{\mathfrak{m}_Q}.$$

3. There is a deformation

$$\rho_{\mathfrak{m},Q,s} : G_F \to \mathrm{GL}_2(\mathbb{T}_\psi^{S\cup Q}(U_1^p(Q)U_p, s))$$

of $\overline{\rho}$ which is of type $\mathscr{S}_Q$ and has determinant $\psi\varepsilon$. In particular, $S_\psi(U_1^p(Q)U_p, s)_{\mathfrak{m}_Q}$ is a finite $R^\psi_{\mathscr{S}_Q}$-module.

The following proposition is an immediate consequence of (3).

Proposition 5.5.1. *Let $(Q, \{\alpha_v\}_{v\in Q})$ be a Taylor-Wiles datum. Then there exists a lifting of $\overline{\rho}_\mathfrak{m}$ to a continuous morphism*

$$\rho_{\mathfrak{m},Q} : G_{F,S\cup Q} \to \mathrm{GL}_2(\mathbb{T}_\psi^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_{Q,1}})$$

satisfying the following conditions:

- *for each place $v \notin S\cup Q$ of F, $\rho_{\mathfrak{m},Q}(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - T_vX + q_vS_v \in \mathbb{T}_\psi^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_{Q,1}}[X]$;*
- *for each place $v \in Q$, $\rho_{\mathfrak{m},Q}|_{G_{F_v}} \sim \left(\begin{smallmatrix}\chi_v & * \\ 0 & *\end{smallmatrix}\right)$ such that $\chi_v(\varpi) = \mathbf{U}_{\varpi_v}$.*

In particular, $\rho_{\mathfrak{m},Q}$ is of type $\mathscr{S}_Q$ and has determinant $\psi\varepsilon$.

It follows that we have an $\mathscr{O}[\Delta_Q]$-algebra surjection

$$R^\psi_{\mathscr{S}_Q} \twoheadrightarrow \mathbb{T}_\psi^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_Q} \tag{5.5}$$

such that for $v \notin S$ the trace of Frob_v on the universal deformation of type $\mathscr{S}_Q$ maps to T_v and $\chi_v(\varpi_v)$ maps to $\mathbf{U}_{\varpi_v}$ for $v \in Q$.

5.5.1 Action of Θ_Q

If $p = 2$, let $\chi \in \Theta_Q^*(\mathscr{O})[2]$ be a character of G_Q of order 2. As χ is split at infinite places, we can regard χ also as a character $(\mathbb{A}_F^\infty)^\times$. Given $f \in S_\psi(U_1^p(Q)U_p, \mathscr{O})$, we define

$$f_\chi(g) := f(g)\chi(\det(g)),$$

which also lies in $S_\psi(U_1^p(Q)U_p,\mathscr{O})$. This induces an action of $\Theta_Q^*(\mathscr{O})[2]$ on $S_\psi(U_pU_1^p(Q),s)$ for each $s\in\mathbb{N}$. By Proposition 7.6 of [44], we may also define an action χ on $\mathbb{T}_\psi^{S\cup Q}(U_1^p(Q))$ and $\mathscr{O}[\Delta_N]$ by sending T_v to $\chi(\pi_v)T_v$, S_v to $\chi(\pi_v)S_v$ and $\langle h\rangle$ to $\chi(h)\langle h\rangle$, which is compatible with the action of χ on $S_\psi(U_pU_1^p(Q),s)$. Moreover, the action of χ on $\mathbb{T}_\psi^{S\cup Q}(U_1^p(Q))$ preserves its maximal ideal $\mathfrak{m}_Q$ and the homomorphism $R_{\mathscr{S}_Q}^\psi\to\mathbb{T}_\psi^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_Q}$ is $\Theta_Q^*(\mathscr{O})[2]$-equivariant.

5.6 Patching

We write G_p for $\prod_{v|p}\mathrm{GL}_2(F_v)$, K_p for $\prod_{v|p}\mathrm{GL}_2(\mathscr{O}_{F_v})$ and $Z_p\cong\prod_{v|p}F_v^\times$ for the center of G_p.

We let $(Q_N,\{\alpha_v\}_{v\in Q_N})$ be a choice of Taylor-Wiles datum for each $N\gg 0$ and $T=S$ be the subset as in lemma 5.4.1. Choose $v_0\in S$, and let $\mathscr{T}=\mathscr{O}[\![X_{v,i,j}]\!]_{v\in T,1\le i,j\le 2}/(X_{v_0,1,1})$. By lemma 4.1.6, there is a canonical isomorphism $R_{\mathscr{S}}^T\cong R_{\mathscr{S}}\hat{\otimes}_{\mathscr{O}}\mathscr{T}$ (resp. $R_{\mathscr{S}}^{T,\psi}\cong R_{\mathscr{S}}^\psi\hat{\otimes}_{\mathscr{O}}\mathscr{T}$). Let $\Delta_\infty=\mathbb{Z}_p^q$, which is endowed with a natural surjection $\Delta_\infty\twoheadrightarrow\Delta_N$ for each N. This induces a surjection $\mathscr{O}_\infty:=\mathscr{T}[\![\Delta_\infty]\!]\to\mathscr{O}_N:=\mathscr{T}[\![\Delta_N]\!]$ of $\mathscr{T}$-algebras. Denote the kernel of the homomorphism $\mathscr{O}_\infty\to\mathscr{O}$ which sends Δ_∞ to 1 and all $4|T|-1$ variables of $\mathscr{T}$ to 0 by $\mathfrak{a}$.

We write R^{loc} for $A_{\mathscr{S}}^T$ and denote $g=q+|T|-1$.

- If $p>2$, we define $R_\infty:=R^{loc}[\![X_1,\dots,X_g]\!]$ and fix a surjective R^{loc}-algebra map $R_\infty\twoheadrightarrow R_{\mathscr{S}_{Q_N}}^{T,\psi}$ for each N.
- If $p=2$, we fix a surjection $\mathbb{Z}_2^t\to\Theta_{Q_N}$ for each N. This induces an embedding of formal group scheme $\iota:\Theta_{Q_N}^*\hookrightarrow(\hat{\mathbb{G}}_m)^t$, where $\hat{\mathbb{G}}_m$ denotes the completion of the $\mathscr{O}$-group scheme $\mathbb{G}_m$ along the identity section. We define
  - $R'_\infty=R^{loc}[\![X_1,\dots,X_{g+t}]\!]$. Then $\mathrm{Spf}R'_\infty$ is equipped with a free action of $(\hat{\mathbb{G}}_m)^t$, and a $(\hat{\mathbb{G}}_m)^t$-equivariant morphism $\delta:\mathrm{Spf}R'_\infty\to(\hat{\mathbb{G}}_m)^t$, where $(\hat{\mathbb{G}}_m)^t$ acts on itself by the square of the identity map.
  - R_∞ by $\mathrm{Spf}R_\infty=\delta^{-1}(1)$ and R_∞^{inv} by $\mathrm{Spf}R_\infty^{\mathrm{inv}}:=\mathrm{Spf}R'_\infty/(\hat{\mathbb{G}}_m)^t$ (cf. [44, Proposition 2.5]). By [44, Lemma 9.4], $\mathrm{Spf}R'_\infty$ is a $(\hat{\mathbb{G}}_m)^t$-torsor over $\mathrm{Spf}R_\infty^{\mathrm{inv}}$.

  We fix a $\Theta_{Q_N}^*$-equivariant surjective R^{loc}-algebra homomorphism $R'_\infty\twoheadrightarrow R_{\mathscr{S}_{Q_N}}^T$ for each N, which induces a $\Theta_{Q_N}^*[2]$-equivariant surjective R^{loc}-algebra homomorphism $R_\infty\twoheadrightarrow R_{\mathscr{S}_{Q_N}}^{T,\psi}$.

Definition 5.6.1. Let U_p be a compact open subgroup of K_p and let J be an open ideal in $\mathscr{O}_\infty$. Let I_J be the subset of $N\in\mathbb{N}$ such that J contains the kernel of $\mathscr{O}_\infty\to\mathscr{O}_N$. For $N\in I_J$, define

$$M(U_p,J,N):=\mathscr{O}_\infty/J\otimes_{\mathscr{O}_N}S_\psi(U_1^p(Q_N)U_p,\mathscr{O})_{\mathfrak{m}_{Q_N}}^d.$$

From the definition, it follows that $M(U_p,J,N)$ satisfying the following properties:

- We have a map
$$R^{T,\psi}_{\mathscr{S}_{Q_N}} \to \mathscr{T}\hat{\otimes}_{\mathscr{O}}\mathbb{T}^S_{\psi}(U_1^p(Q_N))_{\mathfrak{m}_{Q_N}}, \tag{5.6}$$
and a map
$$\mathscr{T}\hat{\otimes}_{\mathscr{O}}\mathbb{T}^S_{\psi}(U_1^p(Q_N))_{\mathfrak{m}_{Q_N}} \to \mathrm{End}_{\mathscr{O}_\infty/J}(M(U_p,J,N)). \tag{5.7}$$
In particular, for all J and $N \in I_J$ we have a ring homomorphism
$$R_\infty \to \mathrm{End}_{\mathscr{O}_\infty/J}\left(M(U_p,J,N)\right)$$
which factors through our chosen quotient map $R_\infty \to R^{T,\psi}_{\mathscr{S}_{Q_N}}$ and the maps (5.6), (5.7). Moreover, it is $\Theta^*_{Q_N}[2]$-equivariant if $p=2$.
- If U_p' is an open normal subgroup of U_p, then $M(U_p',J,N)$ is projective in the category of $\mathscr{O}_\infty/J[U_p/U_p']$-module with central character $\psi^{-1}|_{\mathscr{O}^\times_{F_p}}$.
- Suppose that $\mathfrak{a} \subset J$. Then $M(U_p,J,N) = S_\psi(U^pU_p,s(J))^\vee_{\mathfrak{m}}$, where $\mathscr{O}_\infty/J \cong \mathscr{O}/\varpi^{s(J)}$.

Definition 5.6.2. For $d \geq 1$, J an open ideal in $\mathscr{O}_\infty$ and $N \in I_J$, we define
$$R(d,J,N) := \mathscr{O}_\infty/J \otimes_{\mathscr{O}_N} (R^{T,\psi}_{\mathscr{S}_{Q_N}}/\mathfrak{m}^d_{R^{T,\psi}_{\mathscr{S}_{Q_N}}}).$$

We have the following properties:

- Each ring $R(d,J,N)$ is a finite commutative local $\mathscr{O}_\infty/J$-algebra, equipped with a surjective $\mathscr{O}$-algebra homomorphism
$$R_\infty \twoheadrightarrow R(d,J,N).$$
- For d sufficiently large, the map $R_\infty \to \mathrm{End}_{\mathscr{O}_\infty/J}(M(U_p,J,N))$ factors through $R(d,J,N)$.
- We have an isomorphism
$$R(d,J,N)/\mathfrak{a}R(d,J,N) \cong R^{\psi}_{\mathscr{S}}/(\mathfrak{m}^d_{R^{\psi}_{\mathscr{S}}}, \varpi^{s(\mathfrak{a}+J)}).$$
- For all open ideals $J' \subset J$ and open normal subgroups $U_p' \subset U_p$, we have a surjective map
$$M(U_p',J',N) \to M(U_p,J,N)$$

inducing an isomorphism

$$\mathscr{O}_\infty/J \otimes_{\mathscr{O}_\infty/J'[U_p/U_p']} M(U_p', J', N) \to M(U_p, J, N).$$

- Let K_1 be the pro-p Sylow subgroup of K_p and let U_p be an open normal subgroup of K_1. Then $\{M(U_p, J, N)\}_{N \in I_J}$ is a set of projective objects in the category of $\mathscr{O}_\infty/J[K_1/U_p]$-modules with central character $\psi^{-1}|_{\mathscr{O}_{F_p}^\times}$.

We fix a non-principal ultrafilter $\mathfrak{F}$ on the set $\mathbb{N}$.

Definition 5.6.3. Let $x \in \operatorname{Spec}\big((\mathscr{O}_\infty/J)_{I_J}\big)$ corresponding to $\mathfrak{F}$. We define

$$M(U_p, J, \infty) := (\mathscr{O}_\infty/J)_{I_J,x} \otimes_{(\mathscr{O}_\infty/J)_{I_J}} \left(\prod_{N \in I_J} M(U_p, J, N) \right),$$
$$R(d, J, \infty) := (\mathscr{O}_\infty/J)_{I_J,x} \otimes_{(\mathscr{O}_\infty/J)_{I_J}} \left(\prod_{N \in I_J} R(d, J, N) \right).$$

We have the following

- If U_p' is an open normal subgroup of U_p, then $M(U_p', J, \infty)$ is is projective in the category of $\mathscr{O}_\infty/J[U_p/U_p']$-module with central character given by ψ^{-1}.
- If $\mathfrak{a} \subset J$, there is a natural isomorphism

$$M(U_p, J, \infty)/\mathfrak{a}M(U_p, J, \infty) \cong S_\psi(U^p U_p, s(J))_{\mathfrak{m}}^\vee. \tag{5.8}$$

- For d sufficiently large, the map

$$R_\infty \to \operatorname{End}_{\mathscr{O}_\infty/J}(M(U_p, J, \infty)) \tag{5.9}$$

factors through $R(d, J, \infty)$ and the map

$$R(d, J, \infty) \to \operatorname{End}_{\mathscr{O}_\infty/J}(M(U_p, J, \infty)) \tag{5.10}$$

is an $\mathscr{O}_\infty$-algebra homomorphism. Moreover, if $p = 2$ both (5.9) and (5.10) are $\Theta_{Q_N}^*[2]$-equivariant.

- We have an isomorphism

$$R(d, J, \infty)/\mathfrak{a} \cong R_{\mathscr{S}}/(\mathfrak{m}_{R_{\mathscr{S}}}^d, \varpi^{s(\mathfrak{a}+J)}). \tag{5.11}$$

- For all open ideals $J' \subset J$ and open normal subgroups $U'_p \subset U_p$, the natural map

$$M(U'_p, J', \infty) \to M(U_p, J, \infty)$$

is surjective, and induce an isomorphism

$$\mathscr{O}_\infty/J \otimes_{\mathscr{O}_\infty/J'[U_p/U'_p]} M(U'_p, J', \infty) \to M(U_p, J, \infty). \tag{5.12}$$

- Let U_p be an open normal subgroup of K_1. Then $M(U_p, J, \infty)$ is projective in the category of $\mathscr{O}_\infty/J[K_1/U_p]$-module with central character $\psi^{-1}|_{\mathscr{O}_{F_p}^\times}$. If U_p is moreover normal in K_p, then $M(U_p, J, \infty)$ is a projective in the category of $\mathscr{O}_\infty/J[K_p/U_p]$-module with central character $\psi^{-1}|_{\mathscr{O}_{F,p}^\times}$.

Definition 5.6.4. We define an $\mathscr{O}_\infty[\![K]\!]$-module

$$M_\infty := \varprojlim_{J, U_p} M(U_p, J, \infty).$$

We claim the following hold.

- M_∞ is endowed with an action of R_∞ via the map $\alpha : R_\infty \to \varprojlim_{J,d} R(d, J, \infty)$. Since α contains the image of $\mathscr{O}_\infty$, $\alpha(R_\infty)$ is naturally an $\mathscr{O}_\infty$-algebra. Since $\mathscr{O}_\infty$ is formally smooth, we can choose a lift of the map $\mathscr{O}_\infty \to \alpha(R_\infty)$ to a map $\mathscr{O}_\infty \to R_\infty$. We make such a choice, and regard R_∞ as an $\mathscr{O}_\infty$-algebra and α as an $\mathscr{O}_\infty$-algebra homomorphism.

- The module M_∞ is naturally equipped with an $\mathscr{O}_\infty$-linear action of G_p, which extends the K_p-action coming from the $\mathscr{O}_\infty[\![K_p]\!]$ structure. To be precisely, for $g \in G_p$, right multiplication by g induces an map

$$\cdot g : M(U_p, J, N) \to M(g^{-1}U_p g, J, N)$$

for each U_p, J, N. Suppose that $g^{-1}U_p g \subset K_p$, our construction gives a map

$$\cdot g : M(U_p, J, \infty) \to M(g^{-1}U_p g, J, \infty).$$

As U_p runs through the cofinal subset of open subgroups of K_p with $g^{-1}U_p g \subset K_p$, the subgroups $g^{-1}U_p g$ also runs through a cofinal subset of open subgroups of K_p, so we may identify $\varprojlim_{J,U_p} M(g^{-1}U_p g, J, \infty)$ with M_∞. Taking the inverse limit over J and U_∞ gives the action of g on M_∞.

Proposition 5.6.5. *1. For all open ideals J and open compact subgroups U_p of K, we have a sujective map*

$$M_\infty \to M(U_p, J, \infty)$$

inducing isomorphism

$$\mathscr{O}_\infty/J \otimes_{\mathscr{O}_\infty/J[U_p]} M_\infty \to M(U_p, J, \infty).$$

2. There is a ring homomorphism $R_\infty \to \mathrm{End}_{\mathscr{O}_\infty[\![K]\!]}(M_\infty)$ which factors as the composite of $\mathscr{O}_\infty$-homomorphisms $R_\infty \to \varprojlim_{J,d} R(d,J,\infty)$ and $\varprojlim_{J,d} R(d,J,\infty) \to \mathrm{End}_{\mathscr{O}_\infty[\![K_p]\!]}(M_\infty)$ given by the homomorphisms above. Moreover, it is $\Theta^*_{Q_N}[2]$-equivariant if $p=2$.

3. M_∞ is finitely generated over $\mathscr{O}_\infty[\![K_p]\!]$ and projective in the category $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(\mathscr{O}_\infty)$, with $\zeta = \psi|_{\mathscr{O}^\times_{F_p}}$. In particular, it is finitely generated over $R_\infty[\![K_p]\!]$ and projective in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(\mathscr{O})$.

Proof. The first assertion follows from the isomorphism (5.12) and the second assertion can be deduced easily by the definition of M_∞.

It is proved in [18, Proposition 2.10] (see [35, Proposition 3.4.16 (1)] also) that M_∞ is finitely generated over $\mathscr{O}_\infty[\![K_p]\!]$ and projective in the category $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(\mathscr{O}_\infty)$. We claim that for any compact module M over a complete local ϖ-torsion free $\mathscr{O}$-algebra R, we have the following equivalent conditions:

$$\begin{aligned}
&M \text{ is projective in } \mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R)\\
\Longleftrightarrow &M \text{ is } \varpi\text{-torsion free and } M/\varpi M \text{ is projective in } \mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R/\varpi)\\
\Longleftrightarrow &M \text{ is } \varpi\text{-torsion free and } M/\varpi M \text{ is projective in } \mathrm{Mod}^{\mathrm{pro}}_{I_p,\zeta}(R/\varpi)\\
\Longleftrightarrow &M \text{ is } \varpi\text{-torsion free, and } M/\varpi M \cong \prod_{i\in J} R/\varpi[\![I_p/I_p\cap Z_p]\!]
\end{aligned}$$

where I_p is the pro-p Iwahori subgroup of G_p and J is an index set. Given the claim, we see that $M_\infty/\varpi M_\infty \cong \prod_J \mathscr{O}_\infty/\varpi[\![I_p/I_p\cap Z_p]\!]$. Since $\mathscr{O}_\infty/\ \varpi \cong k[\![x_1,\dots,x_q]\!] \cong \prod_{J'} k$ for some index set J', we have $M_\infty/\varpi M_\infty \cong \prod_J \prod_{J'} k[\![I_p/I_p\cap Z_p]\!]$ and thus M_∞ is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(\mathscr{O})$ by the claim.

To show the first equivalence, we first assume that M is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R)$. Note that the map $K'_p \to (K'_p/K'_p\cap Z_p)\times\Gamma_p,\ g\mapsto (g(K'_p\cap Z_p), (\det g)^{-1})$, where $K'_p = \{g = sz \mid s=(s_v)\in\prod_{v|p}\mathrm{SL}_2(\mathscr{O}_{F_v}),\ s_v \equiv \left(\begin{smallmatrix}1&*\\ *&1\end{smallmatrix}\right) \bmod \varpi_v^2,\ z\in\prod_{v|p}(1+\varpi_v^2\mathscr{O}_v)\}$ and $\Gamma_p = (K'_p\cap Z_p)^2$, is an isomorphism of groups. It follows that $R[\![K'_p]\!] \cong R[\![(K'_p/K'_p\cap Z_p)]\!]\hat{\otimes}_R R[\![\Gamma_p]\!]$. Viewing M as compact $R[\![K'_p]\!]$-module, we see that it is a quotient of $\prod_j R[\![K'_p]\!]$ and thus a quotient of $\prod_j R[\![K'_p]\!]/(z-\zeta^{-1}(z))_{z\in K'_p\cap Z_p} \cong \prod_j R[\![(K'_p/K'_p\cap Z_p)]\!]$. Since M is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R)$, it is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K'_p,\zeta}(R)$ and hence a direct summand of $\prod_j R[\![(K'_p/K'_p\cap Z_p)]\!]$. This shows that M is ϖ-torsion free. Note that for every N in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R/\varpi)$ we have $\mathrm{Hom}(M,N)\cong \mathrm{Hom}(M/\varpi M,N)$ thus $M/\varpi M$ is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R/\varpi)$. On the other hand, suppose that M is ϖ-torsion free and $M/\varpi M$ is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R/\varpi)$. Let P be the projective envelope of $M/\varpi M$ in $\mathrm{Mod}^{\mathrm{pro}}_{K_p,\zeta}(R)$. It follows that there is a morphism $P\to M$ lifting $P\twoheadrightarrow M/\varpi M$. This morphism is surjective by Nakayama's lemma for compact modules ($P/\varpi P\cong M/\varpi M$). Denote K to be the kernel of this morphism, we have $K/\varpi K=0$ because $P/\varpi P\cong M/\varpi M$ and $0\to K/\varpi K\to P/\varpi P\to M/\varpi M$ is exact (5-lemma). This implies $K=0$ (by the Nakayama's lemma for compact modules) and thus $M\cong P$. The second equivalence is because I_p is the pro-p Sylow subgroup of K_p. Since ζ mod ϖ is trivial on $I_p/I_p\cap Z_p$, $M/\varpi M$ is a compact module over $R/\varpi[\![I_p/I_p\cap Z_p]\!]$ and the third equivalence follows from the fact that a compact $R/\varpi[\![I_p/I_p\cap Z_p]\!]$-module is projective if and only if it is pro-free (because $R/\varpi[\![I_p/I_p\cap Z_p]\!]$ is local, projectivity coincides with freeness). This proves the proposition. □

Proposition 5.6.6. *Let $\mathfrak{a}=\ker(\mathscr{O}_\infty\to\mathscr{O})$, we have a natural (G-equivariant) isomorphism*

$$M_\infty/\mathfrak{a}M_\infty\cong M_\psi(U^p)_{\mathfrak{m}}.$$

There is a surjective map $R_\infty/\mathfrak{a}R_\infty\to R^\psi_{\mathscr{S}}\to\mathbb{T}^S_\psi(U^p)_{\mathfrak{m}}$ and the above isomorphism intertwines the action of R_∞ on the left hand side with the action of $\mathbb{T}^S_\psi(U^p)_{\mathfrak{m}}$ on the right hand side.

Proof. Note that we have a isomorphism (5.8). To prove the first part, it suffices to show that we have an isomorphism

$$M_\infty/\alpha M_\infty\cong \varprojlim_{J,U_p} M(U_p,J,\infty)/\alpha M(U_p,J,\infty),$$

which follows from [35, Lemma A.33] (see also [18, Corollary 2.11]). The second part is an immediate consequence of isomorphism (5.11). □

Chapter 6

Patching and Breuil-Mézard conjecture

6.1 Background on multiplicities and cycles

6.1.1 Hilbert-Samuel multiplicities

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ of dimension d, and M be a finite A-module. There is a polynomial $P_M^A(X)$ of degree at most d (the Hilbert-Samuel polynomial), uniquely determined by the property that, for $n \gg 0$, the value $P_M^A(n)$ is equal to the length of $M/\mathfrak{m}^{n+1}M$ as A-module. The Hilbert-Samuel multiplicity $e(M,A)$ is defined to be $d!$ times the coefficient of X^d in $P_M^A(X)$. We write $e(A)$ for $e(A,A)$.

6.1.2 Cycles

Let $\mathscr{X}$ be a Noetherian scheme, $\mathscr{M}$ be a coherent sheaf and write $\mathscr{Z}$ the scheme-theoretic support of $\mathscr{M}$. For any point $x \in \mathscr{X}$, we define $e(\mathscr{M},x)$ to be the Hilbert-Samuel multiplicity $e(\mathscr{M}_x, \mathscr{O}_{\mathscr{Z},x})$. If $\mathscr{Z}$ is a closed subscheme of $\mathscr{X}$, we write $e(\mathscr{Z},x) := e(\mathscr{O}_{\mathscr{Z}},x)$ for all $x \in \mathscr{X}$.

We say a point $x \in \mathscr{X}$ is of dimension d, write $\dim(x) = d$, if its closure $\overline{\{x\}}$ is of dimension d. A d-dimensional cycle on $\mathscr{X}$ is a formal finite linear combination of points of $\mathscr{X}$ of dimension d.

If $d \geq 0$ is a non-zero integer, and $\mathscr{M}$ be a coherent sheaf on $\mathscr{X}$ whose support is of dimension $\leq d$, then we define the d-dimensional cycle $Z_d(\mathscr{M})$ associate to $\mathscr{M}$ by $Z_d(\mathscr{M}) := \sum_{\dim(x)=d} e(\mathscr{M},x)x$, where the sum ranges over all points $\mathscr{X}$ of dimension d. If the support of $\mathscr{M}$ is finite dimensional of dimension d, then we write $Z(\mathscr{M}) = Z_d(\mathscr{M})$.

Lemma 6.1.1. *If $\mathscr{M}$ is a coherent sheaf on $\mathscr{X}$ with support of dimension d, and if $Z(\mathscr{M})$ is the cycle associated to $\mathscr{M}$, then for any point $y \in X$, we have the formula*

$$e(\mathscr{M},y) = e(Z(\mathscr{M}),y)$$

Proposition 6.1.2. *Let $\mathscr{X}$ be a Noetherian scheme of finite dimension d, and let $f \in \mathscr{O}_{\mathscr{X}}(\mathscr{X})$ be regular (i.e. non-zero divisor in each stalk of $\mathscr{O}_{\mathscr{X}}$). If $\mathscr{M}$ is an f-torsion free coherent sheaf on $\mathscr{X}$, and if $Z_d(\mathscr{M}) = \sum_{\dim(x)=d} n_x x$, where x runs over the d-dimensional points of $\mathscr{X}$, then the support has dimension $\leq d-1$, and*

$$Z_{d-1}(\mathscr{M}/f\mathscr{M}) = \sum_{\dim(x)=d} n_x Z_{d-1}(\overline{\{x\}} \cap V(f)).$$

Proof. See [30, Proposition 2.2.13]. □

If A and B are complete Noetherian local k-algebra, and if $\mathfrak{p}$ and $\mathfrak{q}$ are primes of A and B respectively, such that $A/\mathfrak{p}$ is of dimension d and $B/\mathfrak{q}$ is of dimension e, then $A\hat{\otimes}_k B$ is of dimension $d+e$. Hence $\operatorname{Spec} A/\mathfrak{p}\hat{\otimes}_k B/\mathfrak{q}$ is a closed subscheme of $\operatorname{Spec} A\hat{\otimes}_k B$ of dimension $d+e$, and we write

$$Z(\operatorname{Spec} A/\mathfrak{p}) \times_k Z(\operatorname{Spec} B/\mathfrak{q}) := Z(\operatorname{Spec} A/\mathfrak{p}\hat{\otimes}_k B/\mathfrak{q}),$$

and then extend this by linearity to a bilinear product from d-dimensional cycles on $\operatorname{Spec} A$ and e-dimensional cycles on $\operatorname{Spec} B$ to $d+e$ dimensional cycles on $\operatorname{Spec} A\hat{\otimes}_k B$.

If M and N are finitely generated A- and B-module respectively, giving rise to coherent sheaves $\mathscr{M}$ and $\mathscr{N}$ on $\operatorname{Spec} A$ and $\operatorname{Spec} B$ respectively, then the complete tensor product $M\hat{\otimes}_k N$ gives rise to a coherent sheaf on $\operatorname{Spec} A\hat{\otimes}_k B$, which we denote by $\mathscr{M}\hat{\boxtimes}_k\mathscr{N}$. If the supports of $\mathscr{M}$ and $\mathscr{N}$ are of dimension d and e respectively, then the support of $\mathscr{M}\hat{\boxtimes}_k\mathscr{N}$ is of dimension $d+e$, and $Z_{d+e}(\mathscr{M}\hat{\boxtimes}_k\mathscr{N}) = Z_d(\mathscr{M}) \times_k Z_e(\mathscr{N})$.

6.2 Local results

In this section, we let L be a finite extension of $\mathbb{Q}_p$ and $\bar{r} : G_L \to \mathrm{GL}_2(k)$ be a continuous representation.

6.2.1 Locally algebraic type

Fix a Hodge type λ, and inertia type τ, and a continuous character $\psi : G_L \to \mathscr{O}^\times$ such that $\psi|_{I_L} = (\prod_\kappa \kappa^{\lambda_{\kappa,1}+\lambda_{\kappa,2}} \circ \mathrm{Art}_L^{-1}) \cdot \det \tau$. We define $\sigma(\lambda,\tau) = \sigma(\lambda) \otimes_E \sigma(\tau)$, where $\sigma(\lambda) = \sigma(\lambda) = M_\lambda \otimes_{\mathscr{O}} E$ and $\sigma(\tau)$ be the smooth type corresponding to τ. Since $\sigma(\lambda,\tau)$ is a finite dimensional E-vector space and K is compact and the action of K on $\sigma(\lambda,\tau)$ is continuous, there is a K-stable $\mathscr{O}$-lattice $\sigma^\circ(\lambda,\tau)$ in $\sigma(\lambda,\tau)$. Then $\sigma^\circ(\lambda,\tau)/(\varpi)$ is a smooth finite length k-representation of K, we will denote by $\overline{\sigma(\lambda,\tau)}$ its semi-simplification. One may show that $\overline{\sigma(\lambda,\tau)}$ does not depends on the choice of a lattice. The same assertion holds for $\sigma^{cr}(\lambda,\tau) = \sigma(\lambda) \otimes \sigma^{cr}(\tau)$.

A locally algebraic type σ is an absolutely irreducible representation of $\mathrm{GL}_2(L)$ of the form $\sigma(\lambda,\tau)$ or $\sigma^{cr}(\lambda,\tau)$ for some inertial type τ and Hodge type λ. We say that a continuous representation $r : G_L \to \mathrm{GL}_2(E)$ has type $\sigma = \sigma(\lambda,\tau)$ (resp. $\sigma^{cr}(\lambda,\tau)$) if it is potentially semi-stable (resp. potentially crystalline) of inertial type τ and Hodge type λ. Denote $R_{\bar{r}}^\psi(\sigma)$ the local universal lifting ring of type σ and determinant $\psi\varepsilon$ for $\bar{r}$.

If x is a point of $\operatorname{Spec} R_{\bar{r}}^\psi(\sigma)[1/p]$ with residue field E_x, we denote by $r_x : G_{F_\mathfrak{p}} \to \mathrm{GL}_2(E_x)$ the lifting of $\bar{r}$ given by x. We define the locally algebraic G-representation $\pi_{\mathrm{l.alg}}(r_x) = \pi_{\mathrm{sm}}(r_x) \otimes_{E_x} \pi_{\mathrm{alg}}(r_x)$. Note that $\mathscr{H}(\sigma) := \mathrm{End}_G(\text{c-Ind}_K^G(\sigma))$ acts via a character on the one-dimensional space $\mathrm{Hom}_{\mathrm{GL}_2(\mathscr{O}_L)}(\sigma, \pi_{\mathrm{l.alg}}(r_x))$.

Theorem 6.2.1. *There is an E-algebra homomorphism*

$$\phi : \mathscr{H}(\sigma) \to R_{\bar{r}}^\psi(\sigma)[1/p]$$

which interpolates the local Langlands correspondence. More precisely, for any closed point x of $\operatorname{Spec} R_{\bar{r}}^\psi(\sigma)[1/p]$, *the $\mathscr{H}(\sigma)$-action on* $\mathrm{Hom}_{\mathrm{GL}_2(\mathscr{O}_{F_\mathfrak{p}})}(\sigma, \pi_{\mathrm{l.alg},x})$ *factors as ϕ composed with the evaluation map $R_{\bar{r}}^\psi(\sigma)[1/p] \to E_x$.*

Proof. This follows from [18, Theorem 4.1] for $\sigma = \sigma^{cr}(\lambda,\tau)$ and [60, Theorem 3.4] for $\sigma = \sigma(\lambda,\tau)$. □

6.2.2 Serre weights

By a Serre weight we mean an absolutely irreducible representation of $\mathrm{GL}_2(k_L)$ on an k-vector space, up to isomorphism. Let $(\mathbb{Z}_+^2)_{\mathrm{sw}}^{\mathrm{Hom}(k_L,k)} \subset (\mathbb{Z}_+^2)^{\mathrm{Hom}(k_L,k)}$ be the subset consisting of elements a such that

- for each $\varsigma \in \mathrm{Hom}(k_L,k)$, we have $p-1 \geq a_{\varsigma,1} - a_{\varsigma,2}$.
- for each ς we have $0 \leq a_{\varsigma,2} \leq p-1$, and not all $a_{\varsigma,2} = p-1$.

Then

$$\overline{\sigma}_a := \otimes_\varsigma \det^{a_{\varsigma,2}} \otimes \mathrm{Sym}^{a_{\varsigma,1}-a_{\varsigma,2}} k_L^2 \otimes_{k_L,\varsigma} k,$$

where ς runs over the embedding $k_L \to k$, is a Serre weight, and every Serre weight is of this form. We say that $a, a' \in (\mathbb{Z}_+^2)_{\mathrm{sw}}^{\mathrm{Hom}(k_L,k)}$ are equivalent if $\overline{\sigma}_a \cong \overline{\sigma}_{a'}$, which happens if and only if $a_{\varsigma,1} = a'_{\varsigma,1}$ and $a_{\varsigma,2} = a'_{\varsigma,2}$ for all ς.

6.2.3 The Breuil-Mézard conjecture

We now state the Breuil-Mézard conjecture [13].

Conjecture 6.2.2 (Breuil-Mézard)**.** *There exist non-negative integers μ_a for each Serre weight a of $\mathrm{GL}_2(k)$ such that for each locally algebraic type σ, we have*

$$e(R_{\bar{r}}^{\psi}(\sigma)/\varpi) = \sum_a m_a(\sigma)\mu_a(\bar{r})$$

where a runs over all Serre weights, and $m_a(\sigma)$ is the multiplicity of $\overline{\sigma}_a$ as a Jordan-Holder factor of $\overline{\sigma}$.

There is also a geometric version of the Breuil-Mézard conjecture due to [30].

Conjecture 6.2.3. *For each Serre weight a of $\mathrm{GL}_2(k)$, there exists a $(3+[L:\mathbb{Q}_p])$-dimensional cycle $\mathscr{C}_a(\bar{r})$ of $R_{\bar{r}}^{\psi}$, independent of λ and τ, such that for each λ, τ, we have equalities of cycles:*

$$Z(R_{\bar{r}}^{\psi}(\sigma)/\varpi) = \sum_a m_a(\sigma)\mathscr{C}_a(\bar{r})$$

where a runs over all Serre weights and $m_a(\sigma)$ is as in the previous conjecture.

Remark 6.2.4. Given two characters ψ, ψ' lifting $\det \bar{r}$, we have $R_{\bar{r}}^{\psi}/\varpi \cong R_{\bar{r}}^{\psi'}/\varpi$ by the universality. Thus $R_{\bar{r}}^{\psi}(\sigma)/\varpi \cong R_{\bar{r}}^{\psi'}(\sigma)/\varpi$ if both characters are compatible with σ (thus $\psi = \psi'\mu$ with μ an unramified charater). This implies that the two conjectures above are independent of the choice of ψ.

6.3 Local-global compatibility

We now return to the global setting in section 5.6.

6.3.1 Actions of Hecke algebras

Let σ be a representation of K_p over E. Fix a K_p-stable $\mathscr{O}$-lattice σ° in σ. Let $\mathscr{H}(\sigma) = \mathrm{End}_G(\text{c-Ind}_{K_p}^{G_p}\sigma)$ and $\mathscr{H}(\sigma^\circ) := \mathrm{End}_G(\text{c-Ind}_{K_p}^{G_p}\sigma^\circ)$, which is an $\mathscr{O}$-subalgebra of $\mathscr{H}(\sigma)$.

Since M_∞ is a pseudocompact $\mathscr{O}_\infty[\![K_p]\!]$-module equipped with a compatible action of G_p, the $\mathscr{O}_\infty$-module $M_\infty(\sigma^\circ) := \sigma^\circ \otimes_{\mathscr{O}[\![K_p]\!]} M_\infty$ has a natural action of $\mathscr{H}(\sigma^\circ)$ commuting with the action of R_∞ via isomorphisms

$$(\sigma^\circ \otimes_{\mathscr{O}[\![K_p]\!]} M_\infty)^d \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K_p]\!]}(\sigma^\circ, M_\infty^d) \cong \mathrm{Hom}_{G_p}(\text{c-Ind}_{K_p}^{G_p}(\sigma^\circ), (M_\infty)^d),$$

where the first isomorphism is induced by Schkhof duality and the second isomorphism is given by Frobenius reciprocity. In particular, $M_\infty(\sigma^\circ)$ is a $\mathscr{O}$-torsion free, profinite, linearly topological $\mathscr{O}$-module.

6.3.2 Local-global compatibility

We say a representation σ of K_p is a locally algebraic type if $\sigma = \otimes_{v|p}\sigma_v$, where $\sigma_v = \sigma(\lambda_v, \tau_v)$ or $\sigma^{cr}(\lambda_v, \tau_v)$ is a locally algebraic type of $\mathrm{GL}_2(F_v)$ for each $v|p$. We denote $R_p^{loc} = \hat{\otimes}_{v|p} R_v^{\square,\psi}$ and $R_p^{loc}(\sigma) = \hat{\otimes}_{v|p} R_v^{\square,\psi}(\sigma_v)$. Define $R^{loc}(\sigma) = R^{loc} \otimes_{R_p^{loc}} R_p^{loc}(\sigma)$ and $R_\infty(\sigma) = R_\infty \otimes_{R_p^{loc}} R_p^{loc}(\sigma)$ (similarly, $R'_\infty(\sigma) = R'_\infty \otimes_{R_p^{loc}} R_p^{loc}(\sigma)$ and $R_\infty^{\mathrm{inv}}(\sigma) = R_\infty^{\mathrm{inv}} \otimes_{R_p^{loc}} R_p^{loc}(\sigma)$ if $p = 2$).

Lemma 6.3.1. *Assume $p = 2$.*

1. *There are $a_1, \cdots, a_t \in \mathfrak{m}_\infty$ such that*

$$R_\infty(\sigma) = \frac{R_\infty^{inv}(\sigma)[\![z_1]\!]}{((1+z_1)^2 - (1+a_1))} \otimes_{R_\infty^{inv}(\sigma)} \cdots \otimes_{R_\infty^{inv}(\sigma)} \frac{R_\infty^{inv}(\sigma)[\![z_t]\!]}{((1+z_t)^2 - (1+a_1))}.$$

   In particular, $R_\infty(\sigma)$ is a free $R_\infty^{inv}(\sigma)$-module of rank 2^t.

2. *Let $\mathfrak{p} \in \mathrm{Spec}\, R_\infty^{inv}(\sigma)$. The group $(\hat{\mathbb{G}}_m[2])^t(\mathscr{O})$ acts transitively on the set of prime ideals of $R_\infty(\sigma)$ lying above $\mathfrak{p}$.*

Proof. See [57, Lemma 3.3] for the first part and [57, Lemma 3.4] for the second part. □

Proposition 6.3.2. 1. *The action of R_∞ on $M_\infty(\sigma^\circ)$ factors through $R_\infty(\sigma)$.*

2. *The action of $\mathscr{H}(\sigma)$ on $M_\infty(\sigma^\circ)[1/p]$ coincides with the composition*

$$\mathscr{H}(\sigma) \xrightarrow{\prod_{v|p}\phi_v} R_p^{loc}(\sigma)[1/p] \to R_\infty(\sigma)[1/p],$$

where ϕ_v is the map defined in theorem 6.2.1.

3. *The module $M_\infty(\sigma^\circ)$ is finitely generated over $R_\infty(\sigma)$ and Cohen-Macaulay, and moreover $M_\infty(\sigma^\circ)[1/p]$ is locally free of rank 1 over the regular locus of its support in $R_\infty(\sigma)[1/p]$. The topology on $M_\infty(\sigma^\circ)$ coincides with its $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ denote the maximal ideal of $R_\infty(\sigma)$.*

Proof. The first part is an immediate consequence of local-global compatibility at $v|p$. The second part follows from the first part and theorem 6.2.1.

The third part is a consequence of numerical coincidence (cf. [48, Lemma 2.2.11]). We give the proof for the sake of completeness. We first note that $M_\infty(\sigma^\circ)$ is a finite free over $\mathscr{O}_\infty$. Since the $\mathscr{O}_\infty$-action on $M_\infty(\sigma^\circ)$ factors through the action of R_∞, it also factors through $R_\infty(\sigma)$ by (1). This shows that $M_\infty(\sigma^\circ)$ is finitely generated over $R_\infty(\sigma)$.

Since $M_\infty(\sigma^\circ)$ is free over $\mathscr{O}_\infty$, the topology on $M_\infty(\sigma^\circ)$ coincides with its $\mathfrak{n}$-adic topology, where $\mathfrak{n}$ is the maximal ideal of $\mathscr{O}_\infty$. Furthermore, since $R_\infty(\sigma)$ maps into $\mathrm{End}_{\mathscr{O}_\infty}(M_\infty(\sigma^\circ))$, we see that its image if finite as an $\mathscr{O}_\infty$-algebra. In particular, the $\mathfrak{n}$-adic topology on $M_\infty(\sigma^\circ)$ coincide with the $\mathfrak{m}$-adic topology, and the depth of $M_\infty(\sigma^\circ)$ as $R_\infty(\sigma)$-module is at least the krull dimension of $\mathscr{O}_\infty$. Since $R_\infty(\sigma)$ is equidimensional with Krull dimension

$$1+g+\sum_{v\in S_p}(3+[F_v:\mathbb{Q}_p])+\sum_{v\in S_\infty}2+\sum_{v\in\Sigma}3=q+4|T|,$$

which is equal to the Krull dimension of $\mathscr{O}_\infty$, it follows that it is Cohen-Macaulay over $R_\infty(\sigma)$. Moreover, the support of $M_\infty(\sigma^\circ)$ is a union of irreducible components of $R_\infty(\sigma)$ by [70, Lemma 2.3].

Let $\mathfrak{m}$ be a smooth point in the support of $M_\infty(\sigma^\circ)[1/p]$. By the same argument, we have $\operatorname{depth} R_\infty(\sigma)[1/p]_{\mathfrak{m}} = \operatorname{depth} M_\infty(\sigma^\circ)[1/p]_{\mathfrak{m}}$. Then by the Auslander-Buchsbaum formula, $M_\infty(\sigma^\circ)[1/p]_{\mathfrak{m}}$ is free over $R_\infty(\sigma)[1/p]_{\mathfrak{m}}$. It follows that $M_\infty(\sigma^\circ)[1/p]$ is locally free over the regular locus of $\operatorname{Supp} M_\infty(\sigma^\circ)[1/p]$. That it is actually generically of rank 1 can be checked at finite level, which follows from [57, Lemma 3.10] and the fact we include $U_{\varpi_{v_1}}$ in our Hecke algebra $\mathbb{T}^S_\psi(U^p)$ (note that the Hecke algebra in [57] does not contain $U_{\varpi_{v_1}}$, thus their patched module is generically free of rank 1 instead of 2). □

Definition 6.3.3. It follows from proposition 6.3.2 (3) that the support of $M_\infty(\sigma^\circ)[1/p]$ in $\operatorname{Spec} R_\infty(\sigma)[1/p]$ is a union of irreducible components, which we call the set of automorphic components of $\operatorname{Spec} R_\infty(\sigma)[1/p]$.

6.4 Breuil-Mézard via patching

Define $R^{T,\psi}_{\mathscr{S}}(\sigma) = R^{T,\psi}_{\mathscr{S}} \otimes_{R^{loc}_p} R^{loc}_p(\sigma)$ and $R^{\psi}_{\mathscr{S}}(\sigma) = R^{\psi}_{\mathscr{S}} \otimes_{R^{loc}_p} R^{loc}_p(\sigma)$.

Proposition 6.4.1. *For some $s \geq 0$, there is an isomorphism of $R^{loc}(\sigma)$-algebras*

$$R^{S,\psi}_{\mathscr{S}}(\sigma) \cong R^{loc}(\sigma)[\![x_1, \cdots, x_{s+|S|-1}]\!]/(f_1, \cdots, f_s).$$

In particular, $\dim R^{S,\psi}_{\mathscr{S}}(\sigma) \geq 4|S|$ *and* $\dim R^{\psi}_{\mathscr{S}}(\sigma) \geq 1$.

Proof. This follows from lemma 4.1.6 and proposition 4.1.7. □

We define a global Serre weight to be an absolutely irreducible mod p representations of $K_p = \prod_{v \in S_p} \mathrm{GL}_2(\mathscr{O}_{F_v})$. It is of the form

$$\overline{\sigma}_a = \otimes \overline{\sigma}_{a_v}$$

with $a_v \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k_v,k)}_{\mathrm{sw}}$ for each $v \in S_p$, k_v the residue field of v, and K_p acting on $\overline{\sigma}_a$ by reduction modulo p.

We define

$$M^a_\infty = M_\infty \otimes_{\mathscr{O}[\![K_p]\!]} \overline{\sigma}_a \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}[\![K_p]\!]}(M_\infty, \overline{\sigma}^\vee_a)^\vee,$$

which is an R_∞/ϖ-module. We set

$$\mu'_a(\overline{\rho}) := \begin{cases} e(M^a_\infty, R_\infty/\varpi) & \text{if } p > 2, \\ \frac{1}{2^t} e(M^a_\infty, R^{\mathrm{inv}}_\infty/\varpi) & \text{if } p = 2 \end{cases}$$

and cycles $Z'_a(\overline{\rho})$ of R_∞/ϖ (resp. $R^{\mathrm{inv}}_\infty/\varpi$ if $p = 2$) by

$$Z'_a(\overline{\rho}) := \begin{cases} Z(M^a_\infty) & \text{if } p > 2, \\ \frac{1}{2^t} Z(M^a_\infty) & \text{if } p = 2. \end{cases}$$

For each $v|p$, we write

$$\overline{\sigma^\circ_v} \xrightarrow{\sim} \oplus_{a_v} \overline{\sigma}^{m_{a_v}}_{a_v},$$

so that

$$\overline{\sigma^\circ} \xrightarrow{\sim} \oplus_a \overline{\sigma}^{m_a}_a,$$

where $m_a = \prod_v m_{a_v}$.

Due to [48, Lemma 2.2.11], [34, Lemma 4.3.9], [30, Lemma 5.5.1] and [57, Proposition 3.17], we have the following equivalent conditions.

Lemma 6.4.2. *For any locally algebraic type σ, the following conditions are equivalent.*

1. *The support of $M(\sigma^\circ)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ meets every irreducible component of $\operatorname{Spec} R^{loc}(\sigma)[1/p]$.*

2. *$M_\infty(\sigma^\circ)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a faithful $R_\infty(\sigma)[1/p]$-module which is locally free of rank 1 over the smooth locus of its support.*

3. *$R^{\psi}_{\mathscr{S}}(\sigma)$ is a finite $\mathscr{O}$-algebra and $M(\sigma)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a faithful $R^{\psi}_{\mathscr{S}}(\sigma)[1/p]$-module.*

4. *$e(R_\infty(\sigma)/\varpi)=\sum_a m_a e(M^a_\infty,R_\infty/\varpi)=\sum_a \mu'_a(\overline{\rho})$ if $p>2$.*
   (resp. $e(R^{inv}_\infty(\sigma)/\varpi)=\frac{1}{2^t}\sum_a m_a e(M^a_\infty,R^{inv}_\infty/\varpi)=\sum_a \mu'_a(\overline{\rho})$ if $p=2$)

5. *$Z(R_\infty(\sigma)/\varpi)=\sum_a m_a Z(M^a_\infty)=\sum_a Z'_a(\overline{\rho})$ if $p>2$.*
   (resp. $Z(R^{inv}_\infty(\sigma)/\varpi)=\frac{1}{2^t}\sum_a m_a Z(M^a_\infty)=\sum_a Z'_a(\overline{\rho})$ if $p=2$)

Proof. We only prove for $p>2$, the proof for $p=2$ is similar using lemma 6.3.1. By our assumption, $R_\infty(\sigma)[1/p]$ is equidimensional with dimension equal to $\mathscr{O}_\infty[1/p]$. Furthermore, the morphism $\operatorname{Spec} R_\infty(\sigma)[1/p]\to R_{\mathfrak{p}}(\sigma)[1/p]$, corresponding to the $R^{loc}_p(\sigma)[1/p]$-algebra structure on $R_\infty(\sigma)[1/p]$, induces a bijection on irreducible components; we denote the bijection by $Z\mapsto Z'$.

If $Z\subset \operatorname{Spec} R_\infty(\sigma)[1/p]$ is an irreducible component in the support of $M_\infty(\sigma^\circ)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, then Z is finite over $\operatorname{Spec}\mathscr{O}_\infty[1/p]$ and of the same dimension. Hence the map $Z\to \operatorname{Spec}\mathscr{O}_\infty[1/p]$ is surjective. In particular, the fiber over Z over the closed point $\mathfrak{a}$ of $\operatorname{Spec}\mathscr{O}_\infty[1/p]$ is non-zero, and hence gives a point in the support of $M_\infty(\sigma^\circ)/\mathfrak{a}M_\infty(\sigma^\circ):=M(\sigma^\circ)$ lying in the component Z' of $\operatorname{Spec} R^{loc}_p(\sigma)[1/p]$. This shows that (1) and (2) are equivalent.

If $M_\infty(\sigma^\circ)$ is a faithful $R_\infty(\sigma)$-module, then $R_\infty(\sigma)$, being a subring of the ring of $\mathscr{O}_\infty$-endomorphism of the finite $\mathscr{O}_\infty$-module M_∞, is finite over $\mathscr{O}_\infty$. Hence $R^{S,\psi}_{\mathscr{S}}(\sigma)$, which is a quotient of $R_\infty(\sigma)/\mathfrak{a}$, is a finite $\mathscr{O}$-module. This shows that (2) implies (3).

Given (3), it follows from proposition 6.4.1 that $f_1,\cdots,f_s,\varpi$ is a part of system of parameters of $R^{loc}(\sigma)[\![x_1,\cdots,x_{s+|S|-1}]\!]$, and 4.3.8 implies that every irreducible component of that ring contains a closed point of $R^{\psi}_{\mathscr{S}}(\sigma)[1/p]$. We deduce that every irreducible components of $R^{loc}(\sigma)[1/p]$ contains a closed point of $R^{\psi}_{\mathscr{S}}(\sigma)[1/p]$. Thus (3) implies (1).

By the projectivity of M_∞, we have

$$e(R_\infty(\sigma)/\varpi)\geq e(M_\infty(\sigma^\circ)/\varpi M_\infty(\sigma^\circ),R_\infty(\sigma)/\varpi)=\sum_a m_a e(M^a_\infty,R_\infty(\sigma)/\varpi),$$

with equality if and only if $M_\infty(\sigma^\circ)$ is a faithful $R_\infty(\sigma)$-module. Thus (2) and (4) are equivalent.

By lemma 6.1.1, we see that (5) implies (4). Now assume that (2) holds. Then $M_\infty(\sigma^\circ)$ is ϖ-torsion free and generically free of rank 1 over each component of $\operatorname{Spec} R_\infty(\sigma)$, so by proposition 6.1.2, we have

$$Z(R_\infty(\sigma)/\varpi) = Z(M_\infty(\sigma^\circ)/\varpi M_\infty(\sigma^\circ)) = \sum_a m_a Z(M_\infty^a).$$

Thus (2) implies (5). □

For each Serre weight $a_v \in (\mathbb{Z}_+^2)_{\mathrm{sw}}^{\mathrm{Hom}(k_v,k)}$, we choose a fixed lift λ_{a_v}, i.e. for each $\varsigma \in \operatorname{Hom}(k_v,k)$ fix an element $\kappa \in \operatorname{Hom}(F_v,E)$ lifting ς and take $\lambda_{a_v,\kappa} = a_\varsigma$, and for all other $\kappa' \in \operatorname{Hom}_{\mathbb{Q}_p}(F_v,E)$ lifting ς we take $\lambda_{a_v,\kappa'} = 0$. By definition, we have $M_{\lambda_{a_v}} \otimes_{\mathscr{O}} k \cong \overline{\sigma}_{a_v}$.

Define

$$\mu_{a_v}(\overline{\rho}_v) = e(R_v^{\lambda_{a_v},\mathbf{1},cr}/\varpi) \in \mathbb{Z}_{\geq 0}$$

and

$$\mathscr{C}_{a_v}(\overline{\rho}_v) = Z(R_v^{\lambda_{b_v},\mathbf{1},cr}/\varpi)$$

a $3+[F_v:\mathbb{Q}_p]$-dimensional cycle of $\operatorname{Spec} R_v^{\square,\psi}$. We obtain the following analogue of [30, Theorem 5.5.2].

Theorem 6.4.3. *Suppose the equivalence conditions of lemma 6.4.2 hold for $\sigma = \otimes_{v|p}\sigma^{cr}(\lambda_{a_v},\mathbf{1})$ with a_v some Serre weights of $\mathrm{GL}_2(F_v)$. Then if $\sigma = \otimes_{v|p}\sigma_v$ is a locally algebraic type with $\sigma_v = \sigma^*(\lambda_v,\tau_v)$ and $* \in \{\emptyset, cr\}$, and if we write*

$$\overline{\sigma^\circ} \xrightarrow{\sim} \oplus_a \overline{\sigma}_a^{m_a},$$

then the following conditions are equivalent.

1. *The equivalent conditions of lemma 6.4.2 hold for σ.*
2. *$e(R_v^{\lambda_v,\tau_v,*}/\varpi) = \sum_{a_v} m_{a_v}\mu_{a_v}(\overline{\rho}_v)$ for each $v|p$.*
3. *$Z(R_v^{\lambda_v,\tau_v,*}/\varpi) = \sum_{a_v} m_{a_v}\mathscr{C}_{a_v}(\overline{\rho}_v)$ for each $v|p$.*

Proof. We only prove for $p>2$, the case $p=2$ can be deduced similarly. Consider conditions (4) and (5) of Lemma 6.4.2 in the case that each $\lambda_v = \lambda_{a_v,\mathbf{1}}$ for some Serre weights a_v and each $\tau_v = \mathbf{1}$. Note that in this case we have $\overline{\sigma^\circ(\lambda,\mathbf{1})} \xrightarrow{\sim} \overline{\sigma}_a$.

For $v \in S - S_p$, we write R_v for the corresponding local deformation ring of the global deformation problem $\mathscr{S}$. Our assumption that the equivalent conditions of 6.4.2 hold implies that

$$\begin{aligned}\mu_a'(\overline{\rho}) &= e(R_\infty(\sigma)/\varpi)\\ &= \prod_{v|p} e(R_v^{\lambda_{a_v},\mathbf{1},cr}/\varpi) \prod_{v\in S\setminus S_p} e(R_v/\varpi)\\ &= \prod_{v|p} \mu_{a_v}(\overline{\rho}_v) \prod_{v\in S\setminus S_p} e(R_v/\varpi).\end{aligned}$$

Similarly,

$$\begin{aligned}Z_a'(\overline{\rho}) =& Z(R_\infty(\sigma)/\varpi)\\ =& \prod_{v|p} Z(R_v^{\lambda_v,\mathbf{1},cr}/\varpi) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!])\\ =& \prod_{v|p} \mathscr{C}_{a_v}(\overline{\rho}_v) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!]).\end{aligned}$$

This shows that for each global Serre weight a, we have

$$\mu_a'(\overline{\rho}) = \prod_{v|p} \mu_{a_v}(\overline{\rho}_v) \prod_{v\in S\setminus S_p} e(R_v/\varpi)$$

and

$$Z_a'(\overline{\rho}) = \prod_{v|p} \mathscr{C}_{a_v}(\overline{\rho}_v) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!]).$$

We now consider the general case $\sigma = \otimes_{v|p}\sigma_v$, where $\sigma_v = \sigma^*(\lambda_v,\tau_v)$ with $* \in \{\emptyset, cr\}$. By definition, we have

$$Z(R_\infty(\sigma)/\varpi) = \prod_{v|p} Z(R_v^{\lambda_v,\tau_v,*}/\varpi) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!]).$$

Using the relation above, we have

$$\begin{aligned}\sum_a m_a Z_a'(\overline{\rho}) =& \sum_a m_a \left(\prod_{v|p} \mathscr{C}_{a_v}(\overline{\rho}_v) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!]) \right)\\ =& \prod_{v|p} \left(\sum_{a_v} m_{a_v} \mathscr{C}_{a_v}(\overline{\rho}_v) \right) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!]).\end{aligned}$$

Now, condition (5) of 6.4.2 says that

$$Z(R_\infty(\sigma)/\varpi) = \sum_a m_a(\overline{\rho}) Z_a'(\overline{\rho}),$$

which is equivalent to

$$\begin{aligned}&\prod_{v|p} Z(R_v^{\lambda_v,\tau_v,*}/\varpi) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!])\\ =&\prod_{v|p}\left(\sum_{a_v} m_{a_v}\mathscr{C}_{a_v}(\overline{\rho}_v)\right) \times \prod_{v\in S\setminus S_p} Z(R_v/\varpi) \times Z(k[\![x_1,\cdots,x_g]\!]).\end{aligned}$$

This shows that

$$Z(R_v^{\lambda_v,\tau_v,*}) = \sum_{a_v} m_{a_v}\mathscr{C}_{a_v}(\overline{\rho}_v)$$

for each $v|p$, which gives the equivalence of (1) and (3). The equivalence of (1) and (2) can be deduced similarly. □

Remark 6.4.4. The assumption of theorem 6.4.3 may be viewed as a strong form of the weight part of Serre's conjecture, saying that each set of components of the local crystalline deformation rings in low weight is realized by a global automorphic Galois representation. Under this assumption, 6.4.3 says that the Breuil-Mézard conjecture are equivalent to certain $R = \mathbb{T}$ theorems.

6.5 The support at v_1

Let σ be a locally algebraic type for G_p. Suppose that $M_\infty(\sigma^\circ) \neq 0$.

Proposition 6.5.1. *The support of $M_\infty(\sigma^\circ)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ meets every irreducible component of* $\operatorname{Spec} R_{v_1}^{\square,\psi}[1/p]$.

Proof. By assumption and proposition 6.3.2 (3), $M_\infty(\sigma^\circ)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is supported at an irreducible component $\mathscr{C}$ of $\operatorname{Spec} R_\infty(\sigma)[1/p]$. We write $\mathscr{C}_v$ for the corresponding irreducible component at $v \in S$. Let $\tilde{\mathscr{C}}_{v_1}$ be an irreducible component of $\operatorname{Spec} R_{v_1}^{\square,\psi}[1/p]$. It suffices to show that $M_\infty(\sigma^\circ)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is supported at the irreducible component $\tilde{\mathscr{C}}$ defined by $\{\mathscr{C}_v\}_{v\in S-\{v_1\}}$ and $\tilde{\mathscr{C}}_{v_1}$.

Choose a finite solvable totally real extension F' of F such that

- For each place w of F' above $v \in S_p$, $F'_w \cong F_v$;
- For each place w of F' above v_1, the map $R_w^{\square,\psi} \to R_{v_1}^{\square,\psi}$ induced by restriction to $G_{F'_w}$ factors through R_w^{ur}.

Fix a place w_1 of F' above v_1. Let $S' = S'_p \cup S'_\infty \cup \Sigma' \cup \{w_1\}$, where S'_p is the set of places of F' dividing p, S'_∞ is the set of places of F above ∞, and Σ' is the set of places of F' lying above Σ. Consider the following global deformation problems

$$\begin{aligned}\mathscr{R} =&(\overline{\rho}, S, \{\mathscr{O}\}_{v\in S}, \{\mathscr{D}_v^{\mathscr{C}_v}\}_{v\in S_p} \cup \{\mathscr{D}_v^{-1}\}_{v\in S_\infty} \cup \{\mathscr{D}_v^{St}\}_{v\in\Sigma} \cup \{\mathscr{D}_{v_1}^{\tilde{\mathscr{C}}_{v_1}}\}),\\ \mathscr{R}' =&(\overline{\rho}|_{G_{F'}}, S', \{\mathscr{O}\}_{w\in S'}, \{\mathscr{D}_w^{\mathscr{C}_w}\}_{w\in S'_p} \cup \{\mathscr{D}_w^{-1}\}_{w\in S'_\infty} \cup \{\mathscr{D}_w^{St}\}_{w\in\Sigma'} \cup \{\mathscr{D}_{w_1}^{ur}\}).\end{aligned}$$

We claim that $R^{\psi}_{\mathscr{R}'}$ is a finite $\mathscr{O}$-algebra. Given this, since the morphism $R^{\psi}_{\mathscr{R}'} \to R^{\psi}_{\mathscr{R}}$ is finite by proposition 4.1.9, $R^{\psi}_{\mathscr{R}}$ is a finite $\mathscr{O}$-module. On the other hand, $R^{\psi}_{\mathscr{R}}$ has a $\overline{\mathbb{Q}}_p$-point since it has Krull dimension at least 1 by proposition 6.4.1. This gives a lifting ρ of $\overline{\rho}$ of type $\mathscr{R}$. Note that ρ is automorphic because $\rho|_{G_{F'}}$ lies in the automorphic component defined by $\mathscr{C}$ restricted to F' and F'/F is solvable (lemma 3.0.6). It follows that ρ gives a point on $\tilde{\mathscr{C}}$ and the theorem is proved.

To prove the claim, we denote M'_∞ the patched module constructed in the same way as M_∞ replacing F with F' and S with S', which is endowed with an $\mathscr{O}'_\infty$-linear action R'_∞. Write $\mathfrak{a}'$ for the ideal of $\mathscr{O}'_\infty$ defined by its formal variables, $\mathscr{S}'$ for corresponding global deformation problem (as in section 5.2) and σ' for the locally algebraic type defined by σ restricting to F'. It follows that $M'_\infty(\sigma'^{,\circ}) \otimes_{A^{S'}_{\mathscr{S}'}} A^{S'}_{\mathscr{R}'}$ is a faithful $R'_\infty(\sigma') \otimes_{A^{S'}_{\mathscr{S}'}} A^{S'}_{\mathscr{R}'}$-module by proposition 6.3.2 (3) and the irreducibility of $R'_\infty(\sigma') \otimes_{A^{S'}_{\mathscr{S}'}} A^{S'}_{\mathscr{R}'}$. Thus $R^{\psi}_{\mathscr{R}'} \cong (R'_\infty(\sigma') \otimes_{A^{S'}_{\mathscr{S}'}} A^{S'}_{\mathscr{R}'})/\mathfrak{a}'(R'_\infty(\sigma') \otimes_{A^{S'}_{\mathscr{S}'}} A^{S'}_{\mathscr{R}'})$ is a finite $\mathscr{O}$-algebra by the proof of lemma 6.4.2. $\square$

Chapter 7

Patching and p-adic local Langlands correspondence

Throughout this section, we will use freely the notations in chapter 5 and chapter 6. We fix a place $\mathfrak{p}$ of F lying above p and denote $L = F_{\mathfrak{p}}$. Let $G = \mathrm{GL}_2(L)$, $K = \mathrm{GL}_2(\mathscr{O}_L)$, T be the subgroup of diagonal matrices in G, and T_0 be the subgroup of diagonal matrices in K.

7.1 Patching and Banach space representations

For each place $v \neq \mathfrak{p}$ above p, we fix a locally algebraic type σ_v and an irreducible component $\mathscr{C}_v$ of the corresponding deformation ring $R_v^{\lambda_v,\tau_v,*}$. Write $\sigma^{\mathfrak{p}} = \otimes_{v\in S_p-\{\mathfrak{p}\}}\sigma_v$, which is a representation of $K^{\mathfrak{p}} = \prod_{v\in S_p-\{\mathfrak{p}\}} \mathrm{GL}_2(\mathscr{O}_{F_v})$.

We denote $R^{\mathrm{loc},\mathfrak{p}} = \hat{\otimes}_{\mathscr{O},v\in S_p\cup\{v_1\}-\{\mathfrak{p}\}} R_v^{\square,\psi}$, $R^{\mathrm{loc},\mathfrak{p}}(\sigma^{\mathfrak{p}}) = \hat{\otimes}_{\mathscr{O},v\in S_p-\{\mathfrak{p}\}} R_v^{\lambda_v,\tau_v,*} \hat{\otimes} R_{v_1}^{\square,\psi}$. Define

$$\tilde{M}_\infty = M_\infty \otimes_{\mathscr{O}[\![K^{\mathfrak{p}}]\!]} (\sigma^{\mathfrak{p}})^\circ.$$

It follows that $\tilde{M}_\infty$ is an $\mathscr{O}_\infty[\![K]\!]$-module endowed with an $\mathscr{O}_\infty$-linear action of $\tilde{R}_\infty = R_\infty \otimes_{R^{\mathrm{loc},\mathfrak{p}}} R^{\mathrm{loc},\mathfrak{p}}(\sigma^{\mathfrak{p}})$. Assume that $\tilde{N}_\infty$ is non-zero.

Remark 7.1.1. The assumption is satisfied when $\overline{\rho}$ admits an automorphic lift ρ such that its associated local Galois representation $\rho|_{G_{F_v}}$ lies on $\mathscr{C}_v$ for each $v \in S_p - \{\mathfrak{p}\}$ and that $\rho|_{G_{F_{v_1}}}$ is unramified.

The following proposition is a direct consequence of proposition 5.6.5 (3).

Proposition 7.1.2. *$\tilde{M}_\infty$ is finitely generated over $\mathscr{O}_\infty[\![K]\!]$ and projective in the category $\mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathscr{O}_\infty)$, with $\zeta = \psi|_{\mathscr{O}^\times_{F_\mathfrak{p}}}$. In particular, it is finitely generated over $\tilde{R}_\infty[\![K]\!]$ and projective in $\mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathscr{O})$.*

Remark 7.1.3. $\tilde{M}_\infty$ is the same as the patched module considered in [18].

Let us denote by $\Pi_\infty := \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}}(\tilde{M}_\infty, \mathscr{O})$. If $y \in \text{m-Spec}\,\tilde{R}_\infty[1/p]$, then

$$\Pi_y := \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{O}}(\tilde{M}_\infty \otimes_{\tilde{R}_\infty, y} E_y, E) = \Pi_\infty[\mathfrak{m}_y]$$

is an admissible unitary E-Banach space representation of $\mathrm{GL}_2(L)$ (by [18, Proposition 2.13]). The composition $R^{\square,\psi}_{\mathfrak{p}} \to R_\infty \xrightarrow{y} E_y$ defines an E_y-valued point $x \in \mathrm{Spec}\,R^{\square,\psi}_{\mathfrak{p}}[1/p]$ and thus a continuous representation $r_x : G_L \to \mathrm{GL}_2(E_y)$.

Proposition 7.1.4. *Let $y \in \text{m-Spec}\,\tilde{R}_\infty[1/p]$ be a closed E-valued point whose the associated local Galois representation r_x is potentially semi-stable of type $\sigma_\mathfrak{p}$. Assume that y lies on an automorphic component of $R_\infty(\sigma)$ with $\sigma = \sigma_\mathfrak{p} \otimes \sigma^\mathfrak{p}$ and $\pi_{\mathrm{sm}}(r_x)$ is generic. Then*

$$\Pi_y^{\mathrm{l.alg}} \cong \pi_{\mathrm{l.alg}}(r_x).$$

Proof. The proof of [18, Theorem 4.35] (r_x potentially crystalline) and [60, Theorem 7.7] (r_x potentially semi-stable) works verbatim to our setting. □

7.2 Patched eigenvarieties

We write R_1 for the universal deformation ring of the trivial character $1 : G_L \to k^\times$ and 1^{univ} for the universal character. Via the natural map $\mathscr{O}[Z] \to R_1[Z]$, the maximal ideal of $R_1[Z]$ generated by ϖ and $z - 1^{univ} \circ \mathrm{Art}_L(z)$ gives a maximal ideal of $\mathscr{O}[Z]$. If we denote by Λ_Z the completion of the group algebra $\mathscr{O}[Z]$ at this maximal ideal, then the character $1^{univ} \circ \mathrm{Art}_L$ induces an isomorphism $\Lambda_Z \xrightarrow{\sim} R_1$.

Proposition 7.2.1. *The morphism $\mathrm{Spec}(R^{\square,\psi}_{\mathfrak{p}} \hat{\otimes}_{\mathscr{O}} R_1)[1/p] \to \mathrm{Spec}\,R^{\square}_{\mathfrak{p}}[1/p]$ induced by $(r,\chi) \mapsto r \otimes \chi$ is finite étale.*

Proof. To prove the proposition, it suffices to show that the map $R^{\square}_{\mathfrak{p}}[1/p] \to (R^{\square,\psi}_{\mathfrak{p}} \hat{\otimes}_{\mathscr{O}} R_1)[1/p]$ is finite étale after localizing at each $y \in \text{m-Spec}(R^{\square,\psi}_{\mathfrak{p}} \hat{\otimes}_{\mathscr{O}} R_1)[1/p]$ (c.f. [68, tag 02GU]). Suppose that $x \in \text{m-Spec}\,R^{\square}_{\mathfrak{p}}[1/p]$ is a point coming from $y \in \text{m-Spec}(R^{\square,\psi}_{\mathfrak{p}} \hat{\otimes}_{\mathscr{O}} R_1)[1/p]$. We denote r_x the framed deformation of $\bar{r}$ given by x, r_y the framed deformation given by y and χ_y the lifting of trivial character defined by y. Thus we have $r_x \cong r_y \otimes \chi_y$.

Note that $(R_{\mathfrak{p}}^{\square}[1/p])_{\mathfrak{m}_x}$ is a complete E_x-algebra representing the functor sending local artinian E_x-algebra A to the set of liftings r of r_x to A and $(R_{\mathfrak{p}}^{\square,\psi}\hat{\otimes}_{\mathscr{O}}R_1[1/p])_{m_y}$ is the complete E_y ($\cong E_x$)-algebra representing the functor sending local artinian E_y-algebra A to the set of pairs (r',χ'), where r' is a lifting of r_y with determinant $\psi\varepsilon$ and χ' is a character lifting χ_y by For any lifting r of r_x to A, since the character $\chi' := \psi^{-1}\det r$ is a lifting of $\psi^{-1}\det r_x = \psi^{-1}(\det r_y)\chi_y^2 = \chi_y^2$, it admits a square roots χ which lifts χ_y by Hensel's lemma. Hence the map $r \mapsto (r\otimes\chi^{-1},\chi)$ gives an inverse of the map $(r,\chi)\mapsto r\otimes\chi$. This shows that $(R_{\mathfrak{p}}^{\square}[1/p])_{\mathfrak{m}_x} \to (R_{\mathfrak{p}}^{\square,\psi}\hat{\otimes}_{\mathscr{O}}R_1[1/p])_{m_y}$ is an isomorphism and the proposition is proved. $\square$

Remark 7.2.2. Assume that $E=\mathbb{Q}_p$ if $p=2$. Since that the morphism $\operatorname{Spec} R_{\mathfrak{p}}^{\square} \to \operatorname{Spec} R_1$ defined by $r\mapsto \psi(\det r)^{-1}$ induces a bijection on irreducible components (see remark 4.3.2), $\operatorname{Spec}(R_{\mathfrak{p}}^{\square,\psi}\hat{\otimes}_{\mathscr{O}}R_1)[1/p] \to \operatorname{Spec} R_{\mathfrak{p}}^{\square}[1/p]$ is a double covering onto the irreducible component given by ψ.

We define the patched eigenvarieties following [12, Section 3] and [31, Section 6]. Denote $\tilde{R}'_\infty = R_{\mathfrak{p}}^{\square}\hat{\otimes}_{\mathscr{O}}\tilde{R}^{\mathrm{loc},\mathfrak{p}}[\![X_1,\dots,X_g]\!]$. We let $\mathfrak{X}_\infty := \mathrm{Spf}(\tilde{R}'_\infty)^{\mathrm{rig}}$, $\mathfrak{X}_{\mathfrak{p}} = \mathrm{Spf}(R_{\mathfrak{p}}^{\square})^{\mathrm{rig}}$ and $\mathfrak{X}^{\mathfrak{p}} = \mathrm{Spf}(\tilde{R}^{\mathrm{loc},\mathfrak{p}})^{\mathrm{rig}}$ so that

$$\mathfrak{X}_\infty = \mathfrak{X}_{\mathfrak{p}} \times \mathfrak{X}^{\mathfrak{p}} \times \mathbb{U}^g,$$

where $\mathbb{U} := \mathrm{Spf}(\mathscr{O}_L[\![x]\!])^{\mathrm{rig}}$ is the open unit disk over L.

We define $\tilde{M}'_\infty = \tilde{M}_\infty\hat{\otimes}_{\mathscr{O}}1^{univ}$ and $\tilde{\Pi}_\infty = \mathrm{Hom}(\tilde{M}'_\infty, E)$, which is equipped with an $\tilde{R}'_\infty$-action (via $R_{\mathfrak{p}}^{\square} \to R_{\mathfrak{p}}^{\square,\psi}\hat{\otimes}_{\mathscr{O}}R_1 \to \tilde{R}_\infty\hat{\otimes}_{\mathscr{O}}R_1$) and a commuting action of $\mathrm{GL}_2(L)$ coming from $\tilde{M}_\infty$ twisting by the character

$$\mathrm{GL}_2(L) \xrightarrow{\det} L^\times \to \Lambda_Z \xrightarrow{\sim} R_1.$$

Let $\hat{T}$ be the rigid analytic space over E parametrizing continuous character of T and $\hat{T}^0$ be the rigid analytic space over E parametrizing continuous character of T_0. Define the patched eigenvariety X_∞^{tri} as the support of coherent $\mathscr{O}_{\mathfrak{X}_\infty\times\hat{T}}$-module

$$J_{B_p}(\tilde{\Pi}_\infty^{\tilde{R}'_\infty-\mathrm{an}})'$$

on $\mathfrak{X}_\infty\times\hat{T}$, where J_B is Emerton's Jacquet functor with respect to B defined in [26], $\tilde{\Pi}_\infty^{\tilde{R}'_\infty-\mathrm{an}}$ is the subspace of $\tilde{R}'_\infty$-analytic vectors defined in [12, Definition 3.2], and $'$ is the strong dual. This is a reduced closed analytic subset of $\mathfrak{X}_\infty\times\hat{T}$ [12, Corollary 3.20] whose points are

$$\{x=(y,\delta)\in\mathfrak{X}_\infty\times\hat{T} \mid \mathrm{Hom}_T\big(\delta, J_B\big(\tilde{\Pi}_\infty^{\tilde{R}'_\infty-\mathrm{an}}[\mathfrak{p}_y]\otimes_{E_y}E_x\big)\big)\neq 0\}$$

with $\mathfrak{p}_y \subset \tilde{R}'_\infty$ the prime ideal corresponding to the point $y \in \mathfrak{X}_\infty$ and E_y the residue field of $\mathfrak{p}_y$.

Let $\mathscr{W}_\infty = \mathrm{Spf}(\mathscr{O}_\infty)^{\mathrm{rig}} \times \hat{T}^0$ be the weight space of the patched eigenvariety. We define the weight map $\omega_X : X_\infty^{\mathrm{tri}} \to \mathscr{W}_\infty$ by the composite of the inclusion $X_\infty^{\mathrm{tri}} \to \mathfrak{X}_\infty \times \hat{T}$ with the map from $\mathfrak{X}_\infty \times \hat{T}$ to $\mathrm{Spf}(\mathscr{O}_\infty)^{\mathrm{rig}} \times \hat{T}^0$ induced by the $\mathscr{O}_\infty$-structure of $\tilde{R}_\infty$ and by the restriction $\hat{T} \to \hat{T}^0$.

Proposition 7.2.3. *The rigid analytic space X_∞^{tri} is equidimensional of dimension $q-1+4|T|+2[L:\mathbb{Q}_p]$ and has no embedded component.*

Proof. The proof of [12, Proposition 3.11], which shows that the weight map ω_X is locally finite, works verbatim to our setting. Thus the dimension of X_∞^{tri} is equal to the dimension of $\mathscr{W}_\infty$, which is given by

$$\begin{aligned}\dim \mathscr{W}_\infty &= \dim \mathrm{Spf}(\mathscr{O}_\infty)^{\mathrm{rig}} + \dim \hat{T}^0 \\ &= q + 4|T| - 1 + 2[L:\mathbb{Q}_p].\end{aligned}$$

□

Let ι be an automorphism of $\hat{T}$ given by

$$\iota(\delta_{v,1}, \delta_{v,2}) = (\mathrm{unr}(q)\delta_{v,1}, \mathrm{unr}(q^{-1})\delta_{v,2} \prod_{\kappa_v \in \mathrm{Hom}(L,E)} \kappa_v^{-1}),$$

which induces an isomorphism of rigid spaces

$$\begin{aligned}\mathfrak{X}_\infty \times \hat{T} &\xrightarrow{\sim} \mathfrak{X}_\infty \times \hat{T} \\ (x,\delta) &\mapsto (x, \iota^{-1}(\delta)),\end{aligned}$$

and thus a morphism of reduced rigid spaces over E:

$$X_\infty^{\mathrm{tri}} \to X_{\mathfrak{p}}^{\mathrm{tri}} \times \mathfrak{X}^{\mathfrak{p}} \times \mathbb{U}^g,$$

where $X_{\mathfrak{p}}^{\mathrm{tri}}$ is the space of trianguline deformation of $\overline{\rho}|_{G_{F_{\mathfrak{p}}}}$ [12, Definition 2.4].

Theorem 7.2.4. *This morphism induces an isomoprhism from X_∞^{tri} to a union of irreducible components of $X_{\mathfrak{p}}^{\mathrm{tri}} \times \mathfrak{X}^p \times \mathbb{U}^g$.*

Proof. This can be proved in the same way as in [12, Theorem 3.21]. □

7.2.1 The support of $\tilde{M}_\infty$

Proposition 7.2.5. *The support of $\tilde{M}_\infty$ in* $\operatorname{Spec}\tilde{R}_\infty$ *is equal to a union of irreducible components in* $\operatorname{Spec}\tilde{R}_\infty$.

Proof. By the same proof of [31, Theorem 6.3], it can be shown that the support of $\tilde{M}'_\infty$ as $\tilde{R}'_\infty$-module is a union of irreducible components of $\operatorname{Spec}\tilde{R}'_\infty$. Thus by proposition 7.2.1, the support of $\tilde{M}'_\infty$ as $\tilde{R}_\infty\hat{\otimes}_{\mathscr{O}}R_1$-module is a union of irreducible components of $\operatorname{Spec}\tilde{R}_\infty$ times $\operatorname{Spec}R_1$ and the theorem follows. □

Corollary 7.2.6. *The Zariski closure in* $\operatorname{Spec}\tilde{R}_\infty$ *of the set of points with type given by the family of locally algebraic types in proposition 2.2.1 lying in the support of $\tilde{M}_\infty$ is equal to a union of irreducible components of* $\operatorname{Spec}\tilde{R}_\infty$.

Proof. Since $\tilde{M}_\infty$ is projective in $\operatorname{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathscr{O})$ by proposition 7.1.2, it is captured by the family of locally algebraic types in proposition 2.2.1. Applying proposition 7.1.2 to $M = \tilde{M}_\infty$ and $R = \tilde{R}_\infty/\operatorname{Ann}_{\tilde{R}_\infty}(M_\infty)$, we see that the set of points with types defined by this family are dense in $\tilde{R}_\infty/\operatorname{Ann}_{\tilde{R}_\infty}(M_\infty)$, which is equal to a union of irreducible components in $\operatorname{Spec}\tilde{R}_\infty$ by proposition 7.2.5. This proves the corollary. □

7.3 Relations with Colmez's functor

From now on, we assume $L = F_{\mathfrak{p}} = \mathbb{Q}_p$.

Lemma 7.3.1. *$\tilde{M}_\infty$ lies in $\mathfrak{C}_{G,\psi}(\mathscr{O})$.*

Proof. This follows immediately from proposition 7.1.2. □

As a result, we may apply Colmez's functor $\check{\mathbf{V}}$ to $\tilde{M}_\infty$ and obtain an $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]$-module $\check{\mathbf{V}}(\tilde{M}_\infty)$.

Proposition 7.3.2. *$\check{\mathbf{V}}(\tilde{M}_\infty)$ is finitely generated over $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]$.*

Proof. Using Nakayama lemma for compact modules, it is enough to show that $\check{\mathbf{V}}(\tilde{M}_\infty)\otimes_{\tilde{R}_\infty} k \cong \check{\mathbf{V}}(\tilde{M}_\infty\otimes_{\tilde{R}_\infty} k)$ (cf. [55, Lemma 5.50]) is a finitely generated $k[\![G_{\mathbb{Q}_p}]\!]$-module. Note that $\tilde{M}_\infty\otimes_{\tilde{R}_\infty} k$ is a finitely generated $k[\![K]\!]$-module by lemma 7.3.1, so its Pontryagin dual is an admissible K-representation with a smooth G-action, and thus an admissible G-representation. By lemma 2.2.9, the proposition follows. □

Let σ be a locally algebraic type for G. We define $\tilde{R}_\infty(\sigma) = \tilde{R}_\infty\otimes_{R_{\mathfrak{p}}^{\square,\psi}} R_{\mathfrak{p}}^{\square,\psi}(\sigma)$ (resp. $\tilde{R}'_\infty(\sigma) = \tilde{R}'_\infty\otimes_{R_{\mathfrak{p}}^{\square,\psi}} R_{\mathfrak{p}}^{\square,\psi}(\sigma)$) and $\tilde{M}_\infty(\sigma^\circ) = \tilde{M}_\infty\otimes_{\mathscr{O}[\![K]\!]}\sigma^\circ$ (resp. $\tilde{M}'_\infty(\sigma^\circ) = \tilde{M}'_\infty\otimes_{\mathscr{O}[\![K]\!]}\sigma^\circ$), which satisfies a similar local-global compatibility as in section 6.3.

Theorem 7.3.3. *The action of $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]$ on $\check{\mathbf{V}}(\tilde{M}_\infty)$ factors through $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]/J$, where J is a closed two-sided ideal generated by $g^2 - \operatorname{tr}\big(r_\infty(g)\big)g + \det\big(r_\infty(g)\big)$ for all $g \in G_{\mathbb{Q}_p}$, where $r_\infty : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\tilde{R}_\infty)$ is the Galois representation lifting $\bar{r}$ induced by the natural map $R_{\mathfrak{p}}^{\square,\psi} \to \tilde{R}_\infty$.*

Proof. Take $\{\sigma_i\}_{i\in I}$ to be the family of K-representations in proposition 2.2.1. Note that it captures every projective object of $\mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathscr{O})$. We have the following commuting diagram:

$$\begin{tikzcd}
\bigoplus_{i\in I} \mathrm{Hom}_K(\sigma_i, \Pi_\infty)\otimes\sigma_i \arrow[r] & \Pi_\infty \\
\bigoplus_{y\in \text{m-Spec}\,\tilde{R}_\infty[1/p]} \bigoplus_{i\in I} \mathrm{Hom}_K(\sigma_i, \Pi_\infty[\mathfrak{m}_y])\otimes\sigma_i \arrow[u, hook] \arrow[r, "(\star)"] & \bigoplus_{y\in \text{m-Spec}\,\tilde{R}_\infty[1/p]} \Pi_\infty[\mathfrak{m}_y] \arrow[u, hook, "(*)"']
\end{tikzcd}$$

Since $\mathscr{H}(\sigma_i)$ acts on $\mathrm{Hom}_K(\sigma_i,\Pi_\infty)$ via $\mathscr{H}(\sigma_i) \to R_{\mathfrak{p}}^{\square,\psi}(\sigma_i)[1/p] \to \tilde{R}_\infty(\sigma_i)[1/p]$ by proposition 6.3.2 (2), it acts on $\mathrm{Hom}_K(\sigma_i,\Pi_\infty[\mathfrak{m}_y])$ via $\mathscr{H}(\sigma_i) \to R_{\mathfrak{p}}(\sigma_i)[1/p] \to \tilde{R}_\infty(\sigma_i)[1/p] \xrightarrow{y} E_y$. Thus by applying the Frobenius reciprocity to $(\star)$, we obtain a map

$$\mathrm{Hom}_G(\text{c-Ind}_K^G\,\sigma_i \otimes_{\mathscr{H}(\sigma_i),y} E_y, \Pi_\infty[\mathfrak{m}_y]) \otimes \big(\text{c-Ind}_K^G\,\sigma_i \otimes_{\mathscr{H}(\sigma_i),y} E_y\big) \to \Pi_\infty[\mathfrak{m}_y].$$

Since $\Pi_\infty[\mathfrak{m}_y] = \mathrm{Hom}_G^{\mathrm{cont}}\big(\tilde{M}_\infty \otimes_{\tilde{R}_\infty,y} E_y, E\big)$, the image of this map is nonzero if and only if y lies in the support of $\tilde{M}_\infty(\sigma)$, which implies x is potentially crystalline of type σ_i (recall $x \in \text{m-Spec}\,R_{\mathfrak{p}}^{\square,\psi}[1/p]$ is the point induced by y). We can also deduce from proposition 6.3.2 (3) that the dimension of $\mathrm{Hom}_G(\text{c-Ind}_K^G\,\sigma_i \otimes_{\mathscr{H}(\sigma_i),y} E_y, \Pi_\infty[\mathfrak{m}_y])$ over E_y is 1 if y lies in the support of $\tilde{M}_\infty(\sigma_i)[1/p]$ and 0 otherwise. In case it is nonzero, we have $\text{c-Ind}_K^G\,\sigma_i \otimes_{\mathscr{H}(\sigma_i),y} E_y \cong \pi_{\mathrm{l.alg}}(r_x) \hookrightarrow \Pi[\mathfrak{m}_y]$ since $\pi_{\mathrm{l.alg}}(r_x)$ is irreducible. Note that $\pi_{\mathrm{l.alg}}(r_x)$ admits a universal completion $\widehat{\pi_{\mathrm{l.alg}}(r_x)}$, which is topologically irreducible by [7, Corollary 5.3.2, Corollary 5.3.4] and [11, Proposition 2.2.1]. Composing with $(*)$, we get an injection $\widehat{\pi_{\mathrm{l.alg}}(r_x)} \hookrightarrow \Pi_\infty[\mathfrak{m}_y] \hookrightarrow \Pi_\infty$. Let $\widehat{\pi_{\mathrm{l.alg}}(r_x)}^\circ := \widehat{\pi_{\mathrm{l.alg}}(r_x)} \cap (\tilde{M}_\infty)^d$ be a G-invariant $\mathscr{O}$-lattice of $\widehat{\pi_{\mathrm{l.alg}}(r_x)}$. We define N to be the kernel of the $\tilde{R}_\infty$-algebra homomorphism

$$\tilde{M}_\infty \to \prod_{i\in I}\prod_{y} \big(\widehat{\pi_{\mathrm{l.alg}}(r_x)}^\circ\big)^d, \tag{7.1}$$

where $y \in \text{m-Spec}\,\tilde{R}_\infty(\sigma_i)[1/p]$ lies in the support of $\tilde{M}_\infty(\sigma_i^\circ)[1/p]$, and define M by the exact sequence

$$0 \to N \to \tilde{M}_\infty \to M \to 0 \tag{7.2}$$

in $\mathfrak{C}_{G,\psi}(\mathscr{O})$ equipped with a compatible action of $\tilde{R}_\infty$.

Tensoring (7.2) with σ_i over $\mathscr{O}[\![K]\!]$, we obtain a surjection $\tilde{M}_\infty(\sigma_i^\circ) \twoheadrightarrow M(\sigma_i^\circ)$. By the definition of M, we see that $\widehat{\pi_{\mathrm{l.alg}}(r_x)} \hookrightarrow M^d$ if y lies in the support of $\tilde{M}_\infty(\sigma_i^\circ)[1/p]$ for some $i \in I$, thus $M(\sigma_i)[1/p]$ is supported at each point of $\tilde{R}_\infty(\sigma_i^\circ)[1/p]$ at which $\tilde{M}_\infty(\sigma_i^\circ)[1/p]$ is supported. Since $\tilde{M}_\infty(\sigma_i^\circ)[1/p]$ is locally free of rank one over its support and $R_\infty(\sigma_i)$ is p-torsion free, we deduce that $M(\sigma_i^\circ) \cong \tilde{M}_\infty(\sigma_i^\circ)$ for all i, which implies that $M = \tilde{M}_\infty$ by capture and thus $N = 0$. Applying $\check{\mathbf{V}}$ to (7.1), we get an injection

$$\check{\mathbf{V}}(\tilde{M}_\infty) \cong \check{\mathbf{V}}(M) \hookrightarrow \prod_{i\in I}\prod_{y} \check{\mathbf{V}}\left(\left(\widehat{\pi_{\mathrm{l.alg}}(r_x)}^{\circ}\right)^d\right). \tag{7.3}$$

We claim that the action of $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]$ on $\check{\mathbf{V}}\left(\left(\widehat{\pi_{\mathrm{l.alg}}(r_x)}^{\circ}\right)^d\right)$ factors through $\mathscr{O}_{E_y}[\![G_{\mathbb{Q}_p}]\!]/J_x$, where J_x is the closed two-sided ideal generated by $g^2 - \operatorname{tr}\big(r_x(g)\big)g + \det\big(r_x(g)\big)$ for all $g \in G_{\mathbb{Q}_p}$. Given the claim, we see that $g^2 - \operatorname{tr}(r_\infty(g))g + \det(r_\infty(g))$ acts by 0 on the right hand side of (7.3), and thus on $\check{\mathbf{V}}(\tilde{M}_\infty)$. This proves the proposition.

To prove the claim, we note that $\pi_{\mathrm{l.alg}}(r_x)$ is the locally algebraic vectors of the unitary Banach representation $B(r_x)$ constructed in [7, Section 5] and [11, Section 2.3]. Hence we have $r_x \cong \check{\mathbf{V}}\big(B(r_x)\big) \twoheadrightarrow \check{\mathbf{V}}\big(\widehat{\pi_{\mathrm{l.alg}}(r_x)}\big)$, and the claim follows. □

Corollary 7.3.4. $\check{\mathbf{V}}(\tilde{M}_\infty)$ *is finitely generated over* $\tilde{R}_\infty$.

Proof. By proposition 7.3.2 and theorem 7.3.3, $\check{\mathbf{V}}(\tilde{M}_\infty)$ is a finitely generated $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]/J$-module, so it suffices to show that $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]/J$ is finitely generated over $\tilde{R}_\infty$. We note that $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!]/J$ is a Cayley-Hamilton algebra with residual pseudorepresentation associated to $\bar{r}$ in the sense of [75], hence it is finitely generated over $\tilde{R}_\infty$ by [75, Proposition 3.6]. □

Proposition 7.3.5. $\tilde{R}_\infty[1/p]$ *acts on* $\check{\mathbf{V}}(\tilde{M}_\infty)[1/p]$ *faithfully.*

Proof. By corollary 7.2.6, the set of points in $y \in \text{m-Spec}\,\tilde{R}_\infty[1/p]$ with types defined by proposition 2.2.1 are dense in the support of $\tilde{M}_\infty$, which is equal to $\operatorname{Spec}\tilde{R}_\infty$ by theorem 4.3.1 and proposition 6.5.1. On the other hand, it can be shown that $\check{\mathbf{V}}(\Pi_y) \neq 0$ for such y by proposition 7.1.4 ($\Pi_y^{\mathrm{l.alg}} \cong \pi_{\mathrm{l.alg}}$), [7, Theorem 4.3.1] and [11, Proposition 2.2.1] ($\check{\mathbf{V}}(\widehat{\pi_{\mathrm{l.alg}}}) \neq 0$). Since $\check{\mathbf{V}}(\tilde{M}_\infty)$ is finitely generated over $\tilde{R}_\infty$ by corollary 7.3.4, the support of $\check{\mathbf{V}}(\tilde{M}_\infty)[1/p]$ as an $\tilde{R}_\infty[1/p]$-module is equal to the zero set of $\operatorname{Ann}(\check{\mathbf{V}}(\tilde{M}_\infty)[1/p])$. This proves the proposition. □

Proposition 7.3.6. *For all* $y \in \operatorname{Spec}\tilde{R}_\infty[1/p]$, *we have* $\Pi_y \neq 0$.

Proof. It is sufficient to show that $\check{\mathbf{V}}(\Pi_y) \neq 0$. Since $\Pi_y = \operatorname{Hom}^{\mathrm{cont}}_{\mathscr{O}}(\tilde{M}_\infty \otimes_{\tilde{R}_\infty,y} \mathscr{O}_y, E)$, $\check{\mathbf{V}}(\Pi_y) \neq 0$ if and only if $\check{\mathbf{V}}(\tilde{M}_\infty \otimes_{\tilde{R}_\infty,y} \mathscr{O}_y) \cong \check{\mathbf{V}}(\tilde{M}_\infty) \otimes_{\tilde{R}_\infty,y} \mathscr{O}_y \neq 0$. By corollary 7.3.4, $\check{\mathbf{V}}(\tilde{M}_\infty)$ is finitely generated, so Nakayama's lemma says that $\check{\mathbf{V}}(\tilde{M}_\infty) \otimes_{\tilde{R}_\infty,y} \mathscr{O}_y \neq 0$ if and only if $\check{\mathbf{V}}(\tilde{M}_\infty)_y \neq 0$. This proves the proposition. The proposition follows from proposition 7.3.5, which shows that the support of $\check{\mathbf{V}}(\tilde{M}_\infty)$ is all of $\operatorname{Spec}\tilde{R}_\infty[1/p]$. $\square$

Theorem 7.3.7. *For all $y \in$ m-Spec$\tilde{R}_\infty[1/p]$ such that $\Pi_y \neq 0$ and the associate Galois representation r_x is absolutely irreducible, we have $\check{\mathbf{V}}(\Pi_y) \cong r_x$. In particular, $\tilde{M}_\infty(\sigma^\circ)[1/p]$ is supported on every non-ordinary (at $\mathfrak{p}$) component of* $\operatorname{Spec}\tilde{R}_\infty(\sigma)$ *for each locally algebraic type σ for G.*

Proof. Let $x \in$ m-Spec$R_{\mathfrak{p}}^{\square,\psi}[1/p]$ be the image of $y \in$ m-Spec$\tilde{R}_\infty[1/p]$ and let $r_x : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E_y)$ be the corresponding Galois representation. If r_x is absolutely irreducible, then the action of $\tilde{R}_\infty[\![G_{\mathbb{Q}_p}]\!] \otimes_{\tilde{R}_\infty,y} E_y$ on $\check{\mathbf{V}}(\Pi_y)$ factors through $g^2 - \operatorname{tr}\big(r_x(g)\big)g + \det\big(r_x(g)\big)$ by theorem 7.3.3, which implies $\check{\mathbf{V}}(\Pi_y) \cong (r_x)^{\oplus n}$ (see [9, Theorem 1]) for some integer n. As $\Pi_y \neq 0$ by assumption, we get $n \geq 1$.

Since Π_y is an admissible Banach space representation, it contains an irreducible subrepresentation Π, which can be assumed to be absolutely irreducible after extending scalars. If $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$, then $\check{\mathbf{V}}(\Pi)$ is nonzero and $\dim_E \check{\mathbf{V}}(\Pi) \leq 2$ by [23, Corollary 1.7]. Since $(r_x)^{\oplus n} \cong \check{\mathbf{V}}(\Pi_y) \twoheadrightarrow \check{\mathbf{V}}(\Pi)$, it follows that $\check{\mathbf{V}}(\Pi) \cong r_x$. Moreover, by [22, Theorem VI.6.50] and [29, Theorem 3.3.22], the locally algebraic vectors inside Π is nonzero if and only if r_x is de Rham with distinct Hodge-Tate weights, in which case $\Pi^{\mathrm{l.alg}} \cong \pi_{\mathrm{l.alg}}(r_x)$. Suppose that r_x is potentially semi-stable of type σ with $\pi_{sm}(r_x)$ generic. Then we obtain that $n = 1$ and y lies in the support of $\tilde{M}_\infty(\sigma^\circ)$ since

$$1 = \dim_E \operatorname{Hom}_K(\sigma, \Pi) \leq \operatorname{Hom}_K(\sigma, \Pi_y) \leq 1,$$

where the last inequality is by proposition 6.3.2 (3).

If $\Pi^{\mathrm{SL}_2(\mathbb{Q}_p)} \neq 0$, then $\Pi \cong \Psi \circ \det$, where $\Psi : \mathrm{GL}_2(\mathbb{Q}_p) \to E^\times$ is a continuous character. Since the central character ψ of $\tilde{M}_\infty$ is locally algebraic, it follows that $\Pi = \det^a \otimes \eta \circ \det$, where a is an integer and η is a smooth character of $\mathbb{Q}_p^\times$. Thus we have

$$0 \neq \operatorname{Hom}_K(\det^a \otimes \eta \circ \det, \Pi^{\mathrm{SL}_2(\mathbb{Q}_p)}) \hookrightarrow \operatorname{Hom}_K(\det^a \otimes \eta \circ \det, \Pi_y),$$

which implies that y lies in the support of $\tilde{M}_\infty(\det^a \otimes \eta \circ \det)$ (i.e. r_x is crystalline with Hodge-Tate weights $(a, a+1)$ up to a twist) and $n = 1$ by the same reason above.

These proves the first assertion since such points (potentially semi-stable of some type σ with $\pi_{sm}(r_x)$ generic) are dense in the support of $\tilde{M}_\infty$ and the second assertion follows immediately from proposition 7.3.5. $\square$

Corollary 7.3.8. *For all $y \in$ m-Spec$\tilde{R}_\infty[1/p]$ such that $\Pi_y \neq 0$ and the associate Galois representation r_x is absolutely irreducible, we have $\check{\mathbf{V}}(M_\infty)_{\mathfrak{m}_y} \cong (r_{\mathfrak{p}}^{\square})_{\mathfrak{m}_x} \otimes_{(R_{\mathfrak{p}}^{\square,\psi})_{\mathfrak{m}_x}} (\tilde{R}_\infty)_{\mathfrak{m}_y}$.*

Proof. Since $\check{\mathbf{V}}(\Pi_y) \cong r_x$ by theorem 7.3.7, it follows that

$$\check{\mathbf{V}}(\tilde{M}_\infty)_{\mathfrak{m}_y} \cong (r_{\mathfrak{p}}^{\square})_{\mathfrak{m}_x} \otimes_{(\tilde{A}_\infty)_{\mathfrak{m}_y}} U_y$$

for some $(\tilde{R}_\infty)_{\mathfrak{m}_y}$-module U_y by the proof of [73, Theorem 3.1]. Note that U_y is free of rank 1 over $(\tilde{R}_\infty)_{\mathfrak{m}_y}$ since it can be generated by 1 element over $(\tilde{R}_\infty)_{\mathfrak{m}_y}$ (by Nakayama's lemma) and $(\tilde{R}_\infty)_{\mathfrak{m}_y}$ acts on $\check{\mathbf{V}}(\tilde{M}_\infty)_{\mathfrak{m}_y}$ faithfully (by proposition 7.3.5). This proves the corollary. □

Remark 7.3.9. To remove the patching variables, i.e. $\check{\mathbf{V}}(\tilde{M}_\infty) \cong r_{\mathfrak{p}}^{\square} \otimes_{R_{\mathfrak{p}}^{\square,\psi}} \tilde{R}_\infty$, the flatness of $\check{\mathbf{V}}(\tilde{M}_\infty)[1/p]$ is not enough. It is essential to show that $\check{\mathbf{V}}(\tilde{M}_\infty)$ is flat over $\tilde{R}_\infty$. If $\bar{r}$ is absolutely irreducible, this can be proved by the same argument using [73, Theorem 3.1].

Chapter 8

Patching argument: ordinary case

The goal of this chapter is to construct automorphic points on some partial ordinary irreducible components of $R_\infty(\sigma)$. We will follow the strategy in [2, 72, 64, 63] and use freely the notations in section 5.1.

If v is a finite place of F above p and $c \geq b \geq 0$ are integers, then we define an open compact subgroup $\mathrm{Iw}_v(b,c)$ of $\mathrm{GL}_2(\mathscr{O}_{F_v})$ by the formula

$$\mathrm{Iw}_v(b,c) = \left\{ \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \bmod \varpi_v^c \mid t_1 \equiv t_2 \equiv 1 \bmod \varpi_v^b \right\}.$$

Thus $\mathrm{Iw}_v(0,1)$ is the Iwahori subgroup of $\mathrm{GL}_2(\mathscr{O}_{F_v})$ and $\mathrm{Iw}_v(1,1)$ is the pro-v Iwahoric subgroup.

Let $U_v = \mathrm{Iw}_v(b,c)$ for some integers $c \geq b \geq 1$. We define the operator $\mathbf{U}_{\varpi_v}$ by the double coset operator $\mathbf{U}_{\varpi_v} = [U_v(\begin{smallmatrix} \varpi_w & 0 \\ 0 & 1 \end{smallmatrix})U_v]$, and the diamond operator $\langle \alpha \rangle = [U_v(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix})U_v]$ for $\alpha \in \mathscr{O}_{F_v}^\times$.

Lemma 8.0.1. *Let v be a fixed place of F above p. If $U' \subset U$ are open compact subgroups of $G(\mathbb{A}_F^\infty)$ such that $U'_w = U_w$ if $w \neq v$, and $U'_v = \mathrm{Iw}_v(b',c') \subset U_v = \mathrm{Iw}_v(b,c)$ for some $b' \geq b \geq 1$, $c' \geq c$. Then for any topological $\mathscr{O}$-algebra A, the operators $\mathbf{U}_{\varpi_v}$ and $\langle \alpha \rangle$ for $\alpha \in \mathscr{O}_{F_v}^\times$ commutes with each other and with the natural map*

$$S_\psi(U,A) \to S_\psi(U',A).$$

Proof. See [40, Section 1]. $\square$

8.1 Partial Hida families

Let $S \supset S_p \cup S_\infty \cup \Sigma \cup \{v_1\}$ be a set defined as section 5.1. Let $P \subset S_p$ be a subset. For each $v \in S_p - P$, we fix a locally algebraic type σ_v compatible with ψ. Define the open compact subgroup $U^P = \prod_v U_v$ of $(D \otimes_F \mathbb{A}_F^{\infty,P})^\times$ by

- $U_v = (\mathscr{O}_D)_v^\times$ if $v \notin S$ or $v \in \Sigma \cup (S_p - P)$.
- U_{v_1} is the pro-v_1 Iwahori subgroup.

If $c \geq b \geq 1$ are two integers, then we set $U(b,c) = U^P \times \prod_{v \in P} \mathrm{Iw}_v(b,c)$. Let $\sigma^P(b,c) = \otimes_{v \in S_p - P} \sigma_v \bigotimes \otimes_{v \in P} 1$ be a continuous representation of $\prod_{v \in S_p - P} U_v \times \prod_{v \in P} \mathrm{Iw}_v(b,c)$. We will write $S_{\sigma^P,\psi}(U(b,c),\mathscr{O})$ for $S_{\sigma^P(b,c),\psi}(U(b,c),\mathscr{O})$.

We define $\mathscr{O}_P^\times(b,c) = \ker(\prod_{v \in P}(\mathscr{O}_{F_v}/\varpi_v^c)^\times \to \prod_{v \in P}(\mathscr{O}_{F_v}/\varpi_v^b)^\times)$. The group $U(1,c)$ acts on $S_{\sigma^P,\psi}(U(b,c),\mathscr{O})$, which is uniquely determined by the diamond operator action of $\mathscr{O}_P^\times(b,c)$ via the embedding

$$\mathscr{O}_P^\times(b,c) \to U(1,c)/U(b,c) \quad y \mapsto \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod U(b,c).$$

We define $\Lambda_P(b,c) = \mathscr{O}[\mathscr{O}_P^\times(b,c)]$ and $\Lambda_P^b = \varprojlim_c \Lambda_P(b,c)$. If $b=1$, we write Λ_P for Λ_P^1.

We write $\mathbb{T}_{S,P}^{\mathrm{ord}}$ for the polynomial algebra over $\Lambda_P[\Delta_{v_1}]$ in the indeterminates T_v, S_v for $v \notin S$ and the indeterminates $\mathbf{U}_{\varpi_v}$ for $v \in P \cup \{v_1\}$. Define a $\mathbb{T}_{S,P}^{\mathrm{ord}}$-module structure on $S_{\sigma^P,\psi}(U(b,c),\mathscr{O})$ by letting $\Lambda_P[\Delta_{v_1}]$ act via diamond operators and $T_v, S_v, \mathbf{U}_{\varpi_v}$ act as usual. Since for $v \in P$ the operators U_{ϖ_v} and $\langle \alpha \rangle$ commutes with all inclusions $S_{\sigma^P,\psi}(U(b,c),\mathscr{O}) \to S_{\sigma^P,\psi}(U(b',c'),\mathscr{O})$ for every $b' \geq b \geq 1$, $c' \geq c$, these maps become maps of $\mathbb{T}_{S,P}^{\mathrm{ord}}$-modules.

Lemma 8.1.1. *Let R be a commutative Noetherian ring with residue field k.*

1. *Let M be a finite R-module, and let $\mathbf{U} \in \mathrm{End}_R(M)$. There is a unique $\mathbf{U}$-invariant decomposition $M = M_{\mathrm{ord}} \oplus M_{\text{non-ord}}$ with the following property: $\mathbf{U}$ is invertible on M_{ord}, and topologically nilpotent (for the $\mathfrak{m}_R$-adic topology) on $M_{\text{non-ord}}$. Moreover, the limit $e = \lim_{n\to\infty} \mathbf{U}^{n!}$ exists in $\mathrm{End}_R(M)$, is an idempotent, and we have $M_{\mathrm{ord}} = eM$, $M_{\text{non-ord}} = (1-e)M$.*

2. *Let M, N be finite R-modules, let $\mathbf{U}_1 \in \mathrm{End}_R(M)$, $\mathbf{U}_2 \in \mathrm{End}_R(N)$, and let $f \in \mathrm{Hom}_R(M,N)$ intertwine $\mathbf{U}_1$ and $\mathbf{U}_2$. Then we have $fM_{\mathrm{ord}} \subset N_{\mathrm{ord}}$, $fM_{\text{non-ord}} \subset N_{\text{non-ord}}$, the decomposition being taken with respect to $\mathbf{U}_1$ and $\mathbf{U}_2$.*

Proof. This is well-known (c.f. [45, Lemma 2.10]). We sketch the proof.

Suppose R is artinian. Let $\overline{P}(X) \in k[X]$ be the characteristic polynomial of $\mathbf{U}$ on $M/\mathfrak{m}_R$. Let $P(X) \in R[X]$ be a monic lift of $\overline{P}$ such that $P(\mathbf{U}) = 0$ on M. There is a unique factorization

$\overline{P} = \overline{AB}$ in $k[X]$, where the constant term of $\overline{A}$ is a unit and $\overline{B} = X^m$ for some $m \geq 0$. By Hensel's lemma, this factorization lifts uniquely to a factorization $P = AB$ in $R[X]$, with A, B monic. We define $M_{\text{ord}} = B(\mathbf{U})M$ and $M_{\text{non-ord}} = A(\mathbf{U})M$. It is easily to see that it satisfies the properties in (1), (2).

The general case follows by the fact that the $\mathfrak{m}_R$-adic topology on $\mathrm{End}_R(M)$ coincides with the topology induced by kernels of the maps $\mathrm{End}_R(M) \to \mathrm{End}_R(M/(\mathfrak{m}_R^n))$, $n \geq 1$. □

By the lemma, we may take $\mathbf{U} = \mathbf{U}_P := \prod_{v \in P} \mathbf{U}_{\varpi_v}$ and $e = \lim_{n \to \infty} (\mathbf{U}_P)^{n!}$. Define the ordinary subspace of $S_{\sigma^P, \psi}(U(b,c), \mathscr{O})$ (resp. $S_{\sigma^P, \psi}(U(b,c), s)$) by

$$S_{\psi}^{\text{ord}}(U(b,c), \mathscr{O}) = eS_{\sigma^P, \psi}(U(b,c), \mathscr{O}) \quad (\text{resp. } S_{\psi}^{\text{ord}}(U(b,c), s) = eS_{\sigma^P, \psi}(U(b,c), s)).$$

Lemma 8.1.2. *For all $c \geq b \geq 1$, the natural map*

$$S_{\psi}^{\text{ord}}(U(b,b), \mathscr{O}) \to S_{\psi}^{\text{ord}}(U(b,c), \mathscr{O})$$

is an isomorphism.

Proof. See [2, Lemma 2.3.2] and [36, Lemma 2.5.2]. □

We now define the partial Hida family. By Lemma 8.0.1, for $c' \geq c$ the natural maps

$$S_{\psi}(U(c,c), \mathscr{O}) \to S_{\psi}(U(c',c'), \mathscr{O})$$

commute with the action of the Hecke operator $\mathbf{U}_P$ and $\langle \alpha \rangle$, $\alpha \in \mathscr{O}_P^{\times}(p)$.

Definition 8.1.3. We define

$$M_{\psi}^{\text{ord}}(U^P) = \varprojlim_{c} S_{\psi}^{\text{ord}}(U(c,c), \mathscr{O})^d,$$

which is naturally a Λ_P-module.

Proposition 8.1.4. 1. *For every $s, c \geq 1$, there is an isomorphism*

$$M_{\psi}^{\text{ord}}(U^P) \otimes_{\Lambda_P} \Lambda_P(1,c)/(\varpi^s) \xrightarrow{\sim} S_{\psi}^{\text{ord}}(U(c,c), s)^{\vee}.$$

2. *For every $c \geq 1$, the Λ_P^c-module $M_{\psi}^{\text{ord}}(U^P)$ is finite free of rank equal to the $\mathscr{O}$-rank of $S_{\psi}^{\text{ord}}(U(c,c), \mathscr{O})$.*

Proof. See [2, Proposition 2.3.3]. □

The algebra $\mathbb{T}^{\mathrm{ord}}_{S,P}$ acts naturally on $S^{\mathrm{ord}}_{\psi}(U(c,c),s)$. We write $\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U(c,c),\mathscr{O})$ for its image in $\mathrm{End}_{\Lambda_P}(S^{\mathrm{ord}}_{\psi}(U(c,c),\mathscr{O}))$.

Definition 8.1.5. We define

$$\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P) := \varprojlim_{c} \mathbb{T}^{S,\mathrm{ord}}_{\psi}(U(c,c),\mathscr{O})$$

endowed with inverse limit topology. It follows immediately from the definition that $\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)$ acts on $M^{\mathrm{ord}}_{\psi}(U^P)$ faithfully.

Lemma 8.1.6. *$\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)$ is a finite Λ_P-algebra with finitely many maximal ideals. Denote its finitely many maximal ideals by $\mathfrak{m}_1,\cdots,\mathfrak{m}_r$ and let $J=\cap_i \mathfrak{m}_i$ denote the Jacobson radical. Then $\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)$ is J-adically complete and separated, and we have*

$$\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P) = \mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)_{\mathfrak{m}_1} \times \cdots \times \mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)_{\mathfrak{m}_r}.$$

For each i, $\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)/\mathfrak{m}_i$ is a finite extension of k.

Proof. The proof is same as lemma 5.2.4. □

Let $\mathfrak{m} \subset \mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)$ be a maximal ideal with residue field k. There exists a continuous semi-simple representation $\overline{\rho}^{\mathrm{ord}}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_2(k)$ such that $\overline{\rho}^{\mathrm{ord}}_{\mathfrak{m}}$ is totally odd, and for any finite place $v \notin S$ of F, $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - T_v X + q_v S_v \in (\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)/\mathfrak{m})[X]$. If $\overline{\rho}^{\mathrm{ord}}_{\mathfrak{m}}$ is absolutely reducible, we say that the maximal ideal $\mathfrak{m}$ is Eisenstein; otherwise, we say that $\mathfrak{m}$ is non-Eisenstein.

Suppose that $\mathfrak{m}$ is non-Eisenstein. For each $v \in S_p - P$, let λ_v and τ_v be the Hodge type and inerital type given by σ_v. We define a global deformation problem

$$\begin{aligned}\mathscr{S}^P = (&\overline{\rho}^{\mathrm{ord}}_{\mathfrak{m}}, F, S, \{\mathscr{O}[\![\mathscr{O}_v^{\times}(p)]\!]\}_{v\in P} \cup \{\mathscr{O}\}_{v \in S-P}, \{\mathscr{D}_v^{\Delta}\}_{v\in P} \cup \{\mathscr{D}_v^{\lambda_v,\tau_v,ss}\}_{v\in S_p - P} \cup \{\mathscr{D}_v^{-1}\}_{v\in S_\infty} \\ &\cup \{\mathscr{D}_v^{St}\}_{v\in\Sigma} \cup \{\mathscr{D}_{v_1}^{\square,\psi}\}),\end{aligned}$$

where $\mathscr{D}_v^{\Delta}$ is the ordinary deformation problem defined with respect to the character $\overline{\eta}_v$ given by $\overline{\eta}_v(\varpi_v) = U_{\varpi_v} \bmod \mathfrak{m}$ and $\overline{\eta}_v(\alpha) = \langle\alpha\rangle \bmod \mathfrak{m}$ for all $\alpha \in \mathscr{O}_{F_v}^{\times}$.

Proposition 8.1.7. *Suppose that $\mathfrak{m}$ is non-Eisenstein. Then there exists a lifting of $\overline{\rho}^{\mathrm{ord}}_{\mathfrak{m}}$ to a continuous homomorphism*

$$\rho^{\mathrm{ord}}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}^{S,\mathrm{ord}}_{\psi}(U^P)_{\mathfrak{m}})$$

such that

- *for each place $v \notin S$ of F, $\overline{\rho}_{\mathfrak{m}}^{\mathrm{ord}}(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - T_v X + q_v S_v \in \mathbb{T}_{\psi}^{S,\mathrm{ord}}(U^P)_{\mathfrak{m}}[X]$;*
- *for each place $v \in P$, $\overline{\rho}_{\mathfrak{m}}^{\mathrm{ord}}|_{G_{F_v}} \sim \left(\begin{smallmatrix} \chi_v & * \\ 0 & * \end{smallmatrix}\right)$ such that $\chi_v(\varpi) = \mathbf{U}_{\varpi_v}$ and $\chi_v \circ \mathrm{Art}_{F_v}(t) = \langle t \rangle$ for $t \in \mathscr{O}_{F_v}^{\times}$.*

Moreover, $\rho_{\mathfrak{m}}^{\mathrm{ord}}$ is of type $\mathscr{S}^P$ and has determinant $\psi\varepsilon$.

Proof. This is an analog of [2, Proposition 2.4.4]. We sketch the proof. Take $c \geq 1$ such that $\psi|_{U(c,c)\cap((\mathbb{A}_F^\infty)^\times)}$ is trivial. Then $\mathfrak{m}$ is the pullback of a non-Eisenstein maximal ideal of $\mathbb{T}_{\psi}^{S,\mathrm{ord}}(U(c,c),\mathscr{O})$. Let

$$x : \mathbb{T}_{\psi}^{S,\mathrm{ord}}(U(c,c),\mathscr{O})_{\mathfrak{m}} \to \overline{\mathbb{Q}}_p$$

be an $\mathscr{O}$-algebra morphism corresponding to automorphic representation π, and let $\rho_{\pi,\iota}$ denote the corresponding Galois representation. Since $\mathrm{tr}\,\rho_{\pi,\iota}(\mathrm{Frob}_v) = x(T_v)$ for every $v \notin S$, the injection

$$\mathbb{T}_{\psi}^{S,\mathrm{ord}}(U(c,c),\mathscr{O})_{\mathfrak{m}} \to \prod_{\pi} \overline{\mathbb{Q}}_p$$

induces a pseudo-representation $r_c : G_{F,S} \to \mathbb{T}_{\psi}^{S,\mathrm{ord}}(U(c,c),\mathscr{O})_{\mathfrak{m}}$ with $r_c(\mathrm{Frob}_v) = T_v$ for every $v \notin S$, such that

$$\begin{tikzcd} G_{F,S} \arrow[r, "r_{c'}"] \arrow[dr, "r_c"'] & \mathbb{T}_{\psi}^{S,\mathrm{ord}}(U(c,c),\mathscr{O})_{\mathfrak{m}} \arrow[d] \\ & \mathbb{T}_{\psi}^{S,\mathrm{ord}}(U(c',c'),\mathscr{O})_{\mathfrak{m}}. \end{tikzcd}$$

commutes for every $c' \geq c$. We then get a pseudo-representation

$$r = \varprojlim_{c' \geq c} r_{c'} \to \mathbb{T}_{\psi}^{S,\mathrm{ord}}(U^P)_{\mathfrak{m}},$$

such that $r_c(\mathrm{Frob}_v) = T_v$ for every $v \notin S$. Since r modulo $\mathfrak{m}$ is the trace of the absolutely irreducible Galois representation $\overline{\rho}_{\mathfrak{m}}$, a theorem of [52, 61] implies that r is the trace of a Galois representation

$$\rho_{\mathfrak{m}}^{\mathrm{ord}} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}_{\psi}^{S,\mathrm{ord}}(U^P)_{\mathfrak{m}}).$$

By local-global compatibility, it can be seen that $\rho_{\mathfrak{m}}^{\mathrm{ord}}$ is of type $\tilde{\mathscr{S}}^P$. $\square$

8.2 Ordinary patching

Fix a non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{\psi}^{S,\mathrm{ord}}(U^P)$. Let $T = S$ and $(Q_N, \{\alpha_v\}_{v\in Q_N})$ be a Taylor-Wiles datum as in lemma 5.4.1. Define $S_N = \mathscr{O}_N \hat{\otimes}_{\mathscr{O}} \Lambda_P$, $S_\infty = \mathscr{O}_\infty \hat{\otimes}_{\mathscr{O}} \Lambda_P$ and

- if $p > 2$, we define $R_\infty^\Delta := A^S_{\mathscr{S}^P}[\![x_1, \cdots, x_g]\!]$ and fix a surjective $A^S_{\mathscr{S}^P}$-algebra map $R_\infty^\Delta \twoheadrightarrow R^{S,\psi}_{\mathscr{S}^P_{Q_N}}$ for each N.
- if $p = 2$, We define
  - $R_\infty^{\Delta,\prime} := A^S_{\mathscr{S}^P}[\![x_1, \cdots, x_{g+t}]\!]$. Then $\mathrm{Spf} R_\infty^{\Delta,\prime}$ is equipped with a free action of $(\hat{\mathbb{G}}_m)^t$, and a $(\hat{\mathbb{G}}_m)^t$-equivariant morphism $\delta^\Delta : \mathrm{Spf} R_\infty^{\Delta,\prime} \to (\hat{\mathbb{G}}_m)^t$, where $(\hat{\mathbb{G}}_m)^t$ acts on itself by the square of the identity map.
  - R_∞^Δ by $\mathrm{Spf} R_\infty^\Delta = (\delta^\Delta)^{-1}(1)$ and $R_\infty^{\Delta,\mathrm{inv}}$ by $\mathrm{Spf} R_\infty^{\Delta,\mathrm{inv}} := \mathrm{Spf} R_\infty^{\Delta,\prime}/(\hat{\mathbb{G}}_m)^t$.

  We fix a $\Theta^*_{Q_N}$-equivariant surjective $A^S_{\mathscr{S}^P}$-algebra homomoprhism $R_\infty^{\Delta,\prime} \twoheadrightarrow R^S_{\mathscr{S}^P_{Q_N}}$ for each N, which induces a $\Theta^*_{Q_N}[2]$-equivariant surjective $A^S_{\mathscr{S}^P}$-algebra map $R_\infty \twoheadrightarrow R^{S,\psi}_{\mathscr{S}^P_{Q_N}}$.

Let $c \in \mathbb{N}$ and let J be an open ideal in S_∞. Let I_J be the subset of N such that J contains the kernel of $S_\infty \to S_N$. For $N \in I_J$, define

$$M_\psi^{\mathrm{ord}}(c, J, N) := S_\infty/J \otimes_{S_N} S_\psi^{\mathrm{ord}}(U_1(Q_N)(c,c), \mathscr{O})^d_{\mathfrak{m}_{Q_N,1}}.$$

Applying Taylor-Wiles method to $M_\psi^{\mathrm{ord}}(c, J, N)$ by the same way as in section 5.6 (with some choice of ultrafilter $\mathfrak{F}$), we obtain an S_∞-module M_∞^Δ, which is finite free over S_∞ and endowed with a S_∞-linear action of R_∞^Δ. Moreover, we have $M_\infty^\Delta/\mathfrak{a}M_\infty^\Delta \cong M_\psi^{\mathrm{ord}}(U^P)$ with $\mathfrak{a} = \ker(\mathscr{O}_\infty \to \mathscr{O})$.

For each $v \in S_p - P$, we fix an irreducible component $\mathscr{C}_v$ of $R_v^{\lambda_v,\tau_v}$. Define $\mathscr{R}^P$ to be the global deformation problem

$$\mathscr{R}^P = (\overline{\rho}_{\mathfrak{m}}^{\mathrm{ord}}, F, S, \{\mathscr{O}[\![\mathscr{O}_v^\times(p)]\!]\}_{v\in P} \cup \{\mathscr{O}\}_{v\in S-P}, \{\mathscr{D}_v^\Delta\}_{v\in P} \cup \{\mathscr{D}_v^{\mathscr{C}_v}\}_{v\in S_p-P} \cup \{\mathscr{D}_v^{-1}\}_{v\in S_\infty}$$
$$\cup \{\mathscr{D}_v^{St}\}_{v\in\Sigma} \cup \{\mathscr{D}_{v_1}^{ur}\}).$$

Thus there is a natural surjection $R^\psi_{\mathscr{S}^P} \twoheadrightarrow R^\psi_{\mathscr{R}^P}$.

We will write $N_\infty^\Delta = M_\infty^\Delta \otimes_{A^S_{\mathscr{S}^P}} A^S_{\mathscr{R}^P}$, which is an S_∞-module endowed with an S_∞-linear action of $A_\infty^\Delta = R_\infty^\Delta \otimes_{A^S_{\mathscr{S}^P}} A^S_{\mathscr{R}^P}$. Note that A_∞^Δ is a quotient of R_∞^Δ corresponding the irreducible components which is given by the irreducible component $\mathscr{C}_v$ if $v \in S_p - P$ and unramified component at $v = v_1$. Moreover, we have $A_\infty^\Delta/\mathfrak{a} \cong R^\psi_{\mathscr{R}^P}$ and $N_\infty^\Delta/\mathfrak{a}N_\infty^\Delta \cong M_\psi^{\mathrm{ord}}(U^P) \otimes_{R^\psi_{\mathscr{S}^P}} R^\psi_{\mathscr{R}^P}$. We assume N_∞^Δ is nonzero, which is the case if $\overline{\rho}_{\mathfrak{m}}^{\mathrm{ord}}$ admits an automorphic lift of type $\mathscr{R}^P$.

The following proposition is an analog of [36, Theorem 4.3.1] and [64, Theorem 3].

Proposition 8.2.1. *Assume that for each $v \in P$, the image of $\overline{\rho}_{\mathfrak{m}}^{\mathrm{ord}}|_{G_{F_v}}$ is either trivial or has order p, and that if $p = 2$, then either F_v contains a primitive fourth roots of unity or $[F_v : \mathbb{Q}_2] \geq 3$. We have* $\mathrm{Supp}_{A_\infty^\Delta} N_\infty^\Delta = A_\infty^\Delta$.

Proof. Let Q be a minimal prime ideal of Λ_P. Then M_∞^Δ/Q is a finite free S_∞/Q-module. It follows that the depth of M_∞^Δ/Q as an R_∞^Δ-module is at least $\dim S_\infty/Q$. Thus every minimal prime of $(A_\infty^\Delta/Q)/\operatorname{Ann}(N_\infty^\Delta/QN_\infty^\Delta)$ has dimension at least $\dim S_\infty/Q$. On the other hand, by proposition 4.2.2(3), A_∞^Δ/Q is irreducible of dimension

$$\begin{aligned}&g+1+\sum_{v\in P}(3+2[F_v:\mathbb{Q}_p])+\sum_{v\in S_p-P}(3+[F_v:\mathbb{Q}_p])+\sum_{v\in S_\infty}2+\sum_{v\in\Sigma}3\\ =&q+4|T|+\sum_{v\in P}[F_v:\mathbb{Q}_p]\end{aligned}$$

which is equal to $\dim S_\infty/Q$. Thus N_∞^Δ/Q is supported at all of $\operatorname{Spec} A_\infty^\Delta/Q$ and the proposition follows. □

Corollary 8.2.2. *Under the assumption of proposition 8.2.1, the homomorphism $R^\psi_{\mathscr{S}^P} \twoheadrightarrow \mathbb{T}_\psi^{S,\mathrm{ord}}(U^P)_\mathfrak{m}$ induces an isomorphism*

$$(R^\psi_{\mathscr{R}^P})^{\mathrm{red}} \cong (\mathbb{T}_\psi^{S,\mathrm{ord}}(U^P)_\mathfrak{m}\otimes_{R^\psi_{\mathscr{S}^P}} R^\psi_{\mathscr{R}^P})^{\mathrm{red}}.$$

Proof. Reducing modulo $\mathfrak{a}$, we see that $N_\infty^\Delta/\mathfrak{a}N_\infty^\Delta \cong M_\psi^{\mathrm{ord}}(U^P)\otimes_{R^\psi_{\mathscr{S}^P}} R^\psi_{\mathscr{R}^P}$ is a nearly faithful $A_\infty^\Delta/\mathfrak{a}$-module. Note that the action of $R_\infty^\Delta/\mathfrak{a}$ on $M_\psi^{\mathrm{ord}}(U^P)$ factors through the homomorphism $R_\infty^\Delta/\mathfrak{a}\cong R^\psi_{\mathscr{S}^P} \twoheadrightarrow \mathbb{T}_\psi^{S,\mathrm{ord}}(U^P)_\mathfrak{m}$. It follows that the induced map

$$R^\psi_{\mathscr{R}^P} \twoheadrightarrow \mathbb{T}_\psi^{S,\mathrm{ord}}(U^P)_\mathfrak{m}\otimes_{R^\psi_{\mathscr{S}^P}} R^\psi_{\mathscr{R}^P}$$

becomes an isomorphism after passing to the reduced quotient, which proves the corollary. □

Corollary 8.2.3. *Under the assumption of proposition 8.2.1, $R^\psi_{\mathscr{R}^P}$ is a finite Λ_P-module.*

Proof. The proof of [71, Corollary 8.7] works verbatim to our setting. We include the proof for the sake of completeness. Corollary 8.2.2 shows that $R^\psi_{\mathscr{S}^P}/J$ is a quotient of the finite Λ_P-module $\mathbb{T}_\psi^{S,\mathrm{ord}}(U^p)_{\mathfrak{m}^{\mathrm{ord}}}$, for some nilpotent ideal J of $R^\psi_{\mathscr{S}^P}$. This implies that $R^\psi_{\mathscr{S}^P}/\mathfrak{m}'$ is a finite k-algebra, where $\mathfrak{m}'$ is the maximal ideal of Λ_P. Thus the corollary follows from Nakayama's lemma.

Since N_∞^Δ is a faithful A_∞^Δ-module, A_∞^Δ, being a subring of the ring of S_∞-endomorphisms of the finite S_∞-module N_∞^Δ, is finite over S_∞. Thus $R^\psi_{\mathscr{R}^P}\cong A_\infty^\Delta/\mathfrak{a}$ is a finite $\mathscr{O}$-module. □

8.3 Constructing Galois representations

Theorem 8.3.1. *Let F be a totally real field and let*

$$\overline{\rho} : G_F \to \mathrm{GL}_2(k)$$

be a continuous representation unramified outside p. Suppose that $\bar{\rho}|_{F(\zeta_p)}$ is absolutely irreducible if $p > 2$ and that $[F(\zeta_p) : F] = 4$ if $p = 5$ and $\mathrm{Proj}(\overline{\rho}) \cong A_5$; $\bar{\rho}$ has non-solvable image if $p = 2$.

Let Σ be a finite subset of places of F not containing those above p and let $\Sigma_p = \Sigma \cup \{v|p\}$. Given a subset P of $\{v|p\}$ such that $\overline{\rho}|_{G_{F_v}}$ is reducible, and an ordinary lift ρ_v of $\overline{\rho}|_{G_{F_v}}$ for each $v \in P$.

Assume that there is a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that

- $\overline{\rho}_{\pi,\iota} \cong \overline{\rho}$;
- $\det \rho_{\pi,\iota}|_{G_{F_v}} = \det \rho_v$ *for each* $v \in P$;
- π_v *is unramified outside* Σ_p *and is special at* Σ;
- π *is* ι*-ordinary at* $v \in P$.

Then there is an automorphic lift $\rho : G_F \to \mathrm{GL}_2(\mathscr{O})$ of $\overline{\rho}$ such that

- ρ *is unramified outside* Σ_p *and* $\rho(I_v)$ *is unipotent non-trivial at* $v \in \Sigma$;
- *if* $v \in S_p - P$, *then* $\rho|_{G_{F_v}}$ *and* $\rho_{\pi,\iota}|_{G_{F_v}}$ *lies on the same irreducible component of the potentially semi-stable deformation ring given by* $\rho_{\pi,\iota}|_{G_{F_v}}$;
- *if* $v \in P$, *then* $\rho|_{G_{F_v}}$ *and* ρ_v *lies on the same irreducible component of the potentially semi-stable deformation ring (corresponding to* ρ_v*).*

Proof. This theorem is a variant of [71, Theorem 10.2]. Let $\psi = \varepsilon^{-1} \det \rho_{\pi,\iota}$. Choose a finite solvable totally real extension F' of F such that

- $[F' : \mathbb{Q}]$ is even;
- F' is linearly disjoint form $\overline{F}^{\ker \overline{\rho}}(\zeta_p)$;
- $\rho_{\pi,\iota}|_{G_{F'}}$ is ramified at an even number of places outside p;

- for every place w of F' lying above P, the image of $\overline{\rho}|_{G_{F'_w}}$ is either trivial or has order p, and that if $p=2$, then either F'_w contains a primitive fourth roots of unity or $[F'_w:\mathbb{Q}_p]\geq 3$.

Let D be the quaternion algebra with center F' ramified exactly at all infinite places and all w lying above Σ. Choose w_1 to be a place not in Σ such that $v_1\nmid 2Mp$ and Frob_{v_1} has distinct eigenvalues. Fix a place v_1 of F dividing w_1. Let $S=S_p\cup S_\infty\cup\Sigma\cup\{v_1\}$ and $S'=S'_p\cup S'_\infty\cup\Sigma'\cup\{w_1\}$, where S_p (resp. S'_p) is the set of places of F (resp. F') dividing p, S_∞ (resp. S'_∞) is the set of places of F (resp. F') above ∞, and Σ' is the set of places of F' lying above Σ. Denote P' the set of places of F' lying above P and $U^{P'}=\prod_{w\in P'}U_w$ the open compact subgroup of $G(\mathbb{A}^\infty_{F'})$ defined by $U_w=\mathscr{O}_D^\times$ if $w\notin P'\cup\{w_1\}$ and U_{w_1} is the pro-w_1 Iwahori subgroup. Let σ_v be the locally algebraic type given by $\rho_{\pi,\iota}$ if $v\in S'_p-P'$ and let $\mathfrak{m}$ be the maximal ideal in $\mathbb{T}_\psi^{S',\mathrm{ord}}$ defined by $\pi|_{F'}$ and ϖ. Thus we are in the setting of previous sections.

Let λ_v and τ_v be the type given by ρ_v if $v\in P$ (resp. $\rho_{\pi,\iota}$ if $v\in S_p-P$) and let $\mathscr{C}_v$ be an irreducible component of the potentially semi-stable deformation ring containing ρ_v if $v\in P$ (resp. $\rho_{\pi,\iota}$ if $v\in S_p-P$). Define $\lambda_w,\tau_w,\mathscr{C}_w$ similarly for $w\in S'_p$. Let $T=S-\{v_1\}$ and $T'=S'-\{w_1\}$. Let γ be the character given by $\rho_{\pi,\iota}|_{G_{F_{v_1}}}$. Consider the following global deformation problems

$$\begin{aligned}
\mathscr{R}=&(\overline{\rho},S,\{\mathscr{O}\}_{v\in S},\{\mathscr{D}_v^{\mathscr{C}_v}\}_{v\in S_p}\cup\{\mathscr{D}_v^{-1}\}_{v\in S_\infty}\cup\{\mathscr{D}_v^{St}\}_{v\in\Sigma}\cup\{\mathscr{D}_{v_1}^{ur}\}),\\
\mathscr{R}'=&(\overline{\rho}|_{G_{F'}},S',\{\mathscr{O}\}_{w\in S'},\{\mathscr{D}_w^{\mathscr{C}_w}\}_{w\in S'_p}\cup\{\mathscr{D}_w^{-1}\}_{w\in S'_\infty}\cup\{\mathscr{D}_w^{St}\}_{w\in\Sigma'}\cup\{\mathscr{D}_{w_1}^{ur}\}),\\
\mathscr{R}^{P,\prime}=&(\overline{\rho}|_{G_{F'}},S',\{\mathscr{O}[\![\mathscr{O}_{F'_w}^\times(p)]\!]\}_{w\in P}\cup\{\mathscr{O}\}_{w\in S'-P'},\{\mathscr{D}_w^{\Delta}\}_{w\in P}\cup\{\mathscr{D}_w^{\mathscr{C}_w}\}_{w\in S'_p-T'}\cup\{\mathscr{D}_w^{-1}\}_{w\in S'_\infty}\\
&\cup\{\mathscr{D}_w^{St}\}_{w\in\Sigma'}\cup\{\mathscr{D}_{w_1}^{ur}\}).
\end{aligned}$$

Then by corollary 8.2.3, $R^\psi_{\mathscr{R}^{P,\prime}}$ is a finite $\Lambda_{P'}$-module. Note that $R^\psi_{\mathscr{R}^{P,\prime}}$ is a quotient of $R^\psi_{\mathscr{S}^{P,\prime}}\otimes_{\Lambda_P}\mathscr{O}$ by lemma 4.2.6, thus a finite $\mathscr{O}$-module. Since the morphism $R^\psi_{\mathscr{R}'}\to R^\psi_{\mathscr{R}}$ is finite by proposition 4.1.9, $R^\psi_{\mathscr{R}}$ is a finite $\mathscr{O}$-module.

On the other hand, $R^\psi_{\mathscr{R}}$ has a $\overline{\mathbb{Q}}_p$-point since it has Krull dimension at least 1 by proposition 6.4.1. This gives the desired lifting ρ of $\overline{\rho}$. It remains to show that ρ is automorphic, which follows from the automorphy of $\rho|_{G_{F'}}$ and solvable base change (lemma 3.0.6). $\square$

Chapter 9

Main results

Theorem 9.0.1. *Suppose that p splits completely in F (i.e. $F_v \cong \mathbb{Q}_p$ for $v|p$). For each locally algebraic type σ, the support of $M_\infty(\sigma^\circ) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ meets every irreducible component of $R_\infty(\sigma)[1/p]$.*

Proof. Given an arbitrary irreducible component $\mathscr{C}$ of $\operatorname{Spec} R_\infty(\sigma)[1/p]$, we want to show that there is a point y lying on $\mathscr{C}$ such that $M_\infty(\sigma^\circ) \otimes_{R_\infty(\sigma),y} E_y \neq 0$.

For each $v|p$, let $\mathscr{C}_v$ be the irreducible component of $\operatorname{Spec} R_v^{\lambda_v,\tau_v}[1/p]$ given by $\mathscr{C}$ and let $\mathscr{C}'_v$ be the irreducible component of $\operatorname{Spec} R_v^{\lambda'_v,\tau'_v}[1/p]$ given by an automorphic lift of $\overline{\rho}$ (which exists by assumption and can be chosen to be ordinary of weight $(0,0)^{\operatorname{Hom}(F,\overline{\mathbb{Q}}_p)}$ if $\overline{\rho}_v$ is reducible).

Fix a place $\mathfrak{p}|p$ of F. We claim that the support of $M_\infty(\sigma^\circ) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ meets the irreducible component of $R_\infty(\sigma)[1/p]$ defined by $\mathscr{C}_\mathfrak{p}$ and $\mathscr{C}'_v$ for $v \in S_p - \{\mathfrak{p}\}$. In the case $\mathscr{C}_\mathfrak{p}$ is ordinary, the assertion follows from theorem 8.3.1 and proposition 6.5.1, otherwise the assertion is due to theorem 7.3.7. Repeating the argument for each place $v|p$, we obtain a point lying on $\mathscr{C}$. This proves the theorem. $\square$

Due to the equivalent conditions in theorem 6.4.3 and lemma 5.3.3, we obtain the following:

Corollary 9.0.2. *Conjecture 6.2.2 and Conjecture 6.2.3 holds for each continuous representation $\overline{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$*

This gives a new proof of Breuil-Mézard conjecture, which is new in the case $p = 2$ or $p = 3$ and $\overline{r} \sim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ with ω the mod p cyclotomic character and χ an arbitrary character.

Another application of theorem 9.0.1 is an improvement of the theorem in [48, 42] (for $p > 2$) and [57] (for $p = 2$) below, which is new in the case $p = 2$ or $p = 3$ and $\overline{\rho}|_{G_{F_v}} \sim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ for some $v|p$.

Theorem 9.0.3. *Let F be a totally real field in which p splits completely. Let $\rho : G_F \to \mathrm{GL}_2(\mathscr{O})$ be a continuous representation. Suppose that*

1. *ρ is ramified at only finitely many places;*
2. *$\bar{\rho}$ is modular;*
3. *$\bar{\rho}$ is totally odd;*
4. *$\bar{\rho}|_{F(\zeta_p)}$ is absolutely irreducible if $p > 2$; $\bar{\rho}$ has non-solvable image if $p = 2$;*
5. *For every $v|p$, $\rho|_{F_v}$ is potentially semi-stable with distinct Hodge-Tate weights.*

Then (up to twist) ρ comes from a Hilbert modular form.

Proof. Let $\psi = \varepsilon^{-1}\det\rho$. By solvable base change, it is enough to prove the assertion for the restriction of ρ to $G_{F'}$, where F' is a totally real solvable extension of F. Moreover, we can choose F' satisfying

- $[F' : \mathbb{Q}]$ is even;
- F' is linearly disjoint form $\overline{F}^{\ker\overline{\rho}}(\zeta_p)$ and splits completely at p;
- $\overline{\rho}|_{G_{F'}}$ is unramified outside p;
- If $\rho|_{G_{F'}}$ is ramified at $v \neq p$, then the image of inertia is unipotent;
- $\rho|_{G_{F'}}$ is ramified at an even number of places outside p.

Let Σ be the set of places outside p such that $\overline{\rho}|_{G_{F'}}$ is ramified. If $v \in \Sigma$, then

$$\rho|_{G_{F'}} \cong \begin{pmatrix} \gamma_v(1) & * \\ 0 & \gamma_v \end{pmatrix}$$

with γ_v an unramified character such that $\gamma_v^2 = \psi|_{G_{F'_v}}$. After replacing F' by a further solvable extension, we may assume that there is a Hilbert modular form f over F' such that $\det\rho_f = \varepsilon\psi|_{G_{F'}}$, $\overline{\rho}_f \cong \overline{\rho}|_{G_{F'}}$, ρ_f is ramified at $v \nmid p$ iff $v \in \Sigma$ and $\rho_f|_{G_{F_v}} \cong \rho|_{G_{F_v}}$ for $v \in \Sigma$ (cf. the proof of [49, Theorem 3.3.5]).

Let D be the quaternion algebra with center F' ramified exactly at all infinite places and all $v \in \Sigma$. Choose a place v_1 of F' not in Σ such that $v_1 \nmid 2Mp$, and $\overline{\rho}(\mathrm{Frob}_{v_1})$ has distinct eigenvalues. Let S be the union of infinite places, places above p, Σ and v_1. Let $U^p = \prod_{v\nmid p} = U_v$ be an open subgroup of $G(\mathbb{A}_{F'}^{\infty,p})$ such that $U_v = G(\mathscr{O}_{F'_v})$ if $v \neq v_1$ and U_{v_1}

is the pro-v_1 Iwahori subgroup. Let $\mathfrak{m}$ be the maximal ideal in the Hecke algebra $\mathbb{T}^S_\psi(U^p)$ defined by $\overline{\rho}|_{G_{F'}}$. Thus we are in the setting of section 5.3.

By theorem 9.0.1 and lemma 6.4.2 (3) with σ the locally algebraic type associated to $\rho|_{G_{F'}}$, we see that $\rho|_{G_{F'}}$ is automorphic and thus proves the theorem. $\square$

References

[1] Allen, P. B. (2014a). Deformations of Hilbert modular Galois representations and adjoint selmer groups. https://faculty.math.illinois.edu/~pballen/Smooth.pdf.

[2] Allen, P. B. (2014b). Modularity of nearly ordinary 2-adic residually dihedral Galois representations. *Compos. Math.*, 150(8):1235–1346.

[3] Babnik, M. (2015). Irreduzible Komponenten von 2-adischen Deformationsräumen. *arXiv e-prints*, page arXiv:1512.09277.

[4] Barnet-Lamb, T., Gee, T., Geraghty, D., and Taylor, R. (2014). Potential automorphy and change of weight. *Ann. of Math. (2)*, 179(2):501–609.

[5] Barnet-Lamb, T., Geraghty, D., Harris, M., and Taylor, R. (2011). A family of Calabi-Yau varieties and potential automorphy II. *Publ. Res. Inst. Math. Sci.*, 47(1):29–98.

[6] Barthel, L. and Livné, R. (1994). Irreducible modular representations of GL_2 of a local field. *Duke Math. J.*, 75(2):261–292.

[7] Berger, L. and Breuil, C. (2010). Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbf{Q}_p)$. *Astérisque*, (330):155–211.

[8] Böckle, G. and Juschka, A.-K. (2015). Irreducibility of versal deformation rings in the (p,p)-case for 2-dimensional representations. *J. Algebra*, 444:81–123.

[9] Boston, N., Lenstra, Jr., H. W., and Ribet, K. A. (1991). Quotients of group rings arising from two-dimensional representations. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(4):323–328.

[10] Breuil, C. (2003). Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. I. *Compositio Math.*, 138(2):165–188.

[11] Breuil, C. and Emerton, M. (2010). Représentations p-adiques ordinaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ et compatibilité local-global. *Astérisque*, (331):255–315.

[12] Breuil, C., Hellmann, E., and Schraen, B. (2017). Une interprétation modulaire de la variété trianguline. *Math. Ann.*, 367(3-4):1587–1645.

[13] Breuil, C. and Mézard, A. (2002). Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbf{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ en $l = p$. *Duke Math. J.*, 115(2):205–310. With an appendix by Guy Henniart.

[14] Brumer, A. (1966). Pseudocompact algebras, profinite groups and class formations. *J. Algebra*, 4:442–470.

[15] Bruns, W. and Vetter, U. (1988). *Determinantal rings*, volume 1327 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

[16] Bushnell, C. J. and Henniart, G. (2006). *The local Langlands conjecture for* $\mathrm{GL}(2)$, volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.

[17] Calegari, F. (2012). Even Galois representations and the Fontaine–Mazur conjecture. II. *J. Amer. Math. Soc.*, 25(2):533–554.

[18] Caraiani, A., Emerton, M., Gee, T., Geraghty, D., Paškūnas, V., and Shin, S. W. (2016). Patching and the p-adic local Langlands correspondence. *Camb. J. Math.*, 4(2):197–287.

[19] Caraiani, A., Emerton, M., Gee, T., Geraghty, D., Paškūnas, V., and Shin, S. W. (2018). Patching and the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$. *Compos. Math.*, 154(3):503–548.

[20] Carayol, H. (1986). Sur les représentations l-adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)*, 19(3):409–468.

[21] Chenevier, G. (2009). Sur la variété des caractéres p-adique du groupe de Galois absolu de $\mathbf{Q}_p$. http://gaetan.chenevier.perso.math.cnrs.fr/articles/lieugalois.pdf.

[22] Colmez, P. (2010). Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ,Γ)-modules. *Astérisque*, (330):281–509.

[23] Colmez, P., Dospinescu, G., and Paškūnas, V. (2014). The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. *Camb. J. Math.*, 2(1):1–47.

[24] Colmez, P., Dospinescu, G., and Paškūnas, V. (2015). Irreducible components of deformation spaces: wild 2-adic exercises. *Int. Math. Res. Not. IMRN*, (14):5333–5356.

[25] Darmon, H., Diamond, F., and Taylor, R. (1994). Fermat's last theorem. In *Current developments in mathematics, 1995 (Cambridge, MA)*, pages 1–154. Int. Press, Cambridge, MA.

[26] Emerton, M. (2006). Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties. *Ann. Sci. École Norm. Sup. (4)*, 39(5):775–839.

[27] Emerton, M. (2010a). Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties. *Astérisque*, (331):355–402.

[28] Emerton, M. (2010b). Ordinary parts of admissible representations of p-adic reductive groups II. Derived functors. *Astérisque*, (331):403–459.

[29] Emerton, M. (2011). Local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$. http://www.math.uchicago.edu/~emerton/pdffiles/lg.pdf.

[30] Emerton, M. and Gee, T. (2014). A geometric perspective on the Breuil-Mézard conjecture. *J. Inst. Math. Jussieu*, 13(1):183–223.

[31] Emerton, M. and Paskunas, V. (2018). On the density of supercuspidal points of fixed weight in local deformation rings and global Hecke algebras. *arXiv e-prints*, page arXiv:1809.06598.

[32] Fontaine, J.-M. and Mazur, B. (1995). Geometric Galois representations. In *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, Ser. Number Theory, I, pages 41–78. Int. Press, Cambridge, MA.

[33] Gee, T. (2011). Automorphic lifts of prescribed types. *Math. Ann.*, 350(1):107–144.

[34] Gee, T. and Kisin, M. (2014). The Breuil-Mézard conjecture for potentially Barsotti-Tate representations. *Forum Math. Pi*, 2:e1, 56.

[35] Gee, T. and Newton, J. (2016). Patching and the completed homology of locally symmetric spaces. *arXiv e-prints*, page arXiv:1609.06965.

[36] Geraghty, D. J. (2010). *Modularity lifting theorems for ordinary Galois representations*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Harvard University.

[37] Gouvêa, F. Q. (2001). Deformations of Galois representations. In *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 233–406. Amer. Math. Soc., Providence, RI. Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton.

[38] Harris, M. and Taylor, R. (2001). *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ. With an appendix by Vladimir G. Berkovich.

[39] Hida, H. (1989a). Nearly ordinary Hecke algebras and Galois representations of several variables. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 115–134. Johns Hopkins Univ. Press, Baltimore, MD.

[40] Hida, H. (1989b). On nearly ordinary Hecke algebras for $\mathrm{GL}(2)$ over totally real fields. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 139–169. Academic Press, Boston, MA.

[41] Hu, Y. and Paškūnas, V. (2018). On crystabelline deformation rings of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (with an appendix by Jack Shotton). *Mathematische Annalen*.

[42] Hu, Y. and Tan, F. (2015). The Breuil-Mézard conjecture for non-scalar split residual representations. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(6):1383–1421.

[43] Khare, C. and Wintenberger, J.-P. (2009a). On Serre's conjecture for 2-dimensional mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Ann. of Math. (2)*, 169(1):229–253.

[44] Khare, C. and Wintenberger, J.-P. (2009b). Serre's modularity conjecture. II. *Invent. Math.*, 178(3):505–586.

[45] Khare, C. B. and Thorne, J. A. (2017). Potential automorphy and the Leopoldt conjecture. *Amer. J. Math.*, 139(5):1205–1273.

[46] Kisin, M. (2007). Modularity of 2-dimensional Galois representations. In *Current developments in mathematics, 2005*, pages 191–230. Int. Press, Somerville, MA.

[47] Kisin, M. (2008). Potentially semi-stable deformation rings. *J. Amer. Math. Soc.*, 21(2):513–546.

[48] Kisin, M. (2009a). The Fontaine-Mazur conjecture for GL_2. *J. Amer. Math. Soc.*, 22(3):641–690.

[49] Kisin, M. (2009b). Modularity of 2-adic Barsotti-Tate representations. *Invent. Math.*, 178(3):587–634.

[50] Kisin, M. (2009c). Moduli of finite flat group schemes, and modularity. *Ann. of Math. (2)*, 170(3):1085–1180.

[51] Langlands, R. P. (1980). *Base change for* $\mathrm{GL}(2)$, volume 96 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo.

[52] Nyssen, L. (1996). Pseudo-représentations. *Math. Ann.*, 306(2):257–283.

[53] Pan, L. (2018). *The Fontaine-Mazur Conjecture in the Residually Reducible Case*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Princeton University.

[54] Paškūnas, V. (2010). Extensions for supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. *Astérisque*, (331):317–353.

[55] Paškūnas, V. (2013). The image of Colmez's Montreal functor. *Publ. Math. Inst. Hautes Études Sci.*, 118:1–191.

[56] Paškūnas, V. (2015). On the Breuil-Mézard conjecture. *Duke Math. J.*, 164(2):297–359.

[57] Paškūnas, V. (2016). On 2-dimensional 2-adic Galois representations of local and global fields. *Algebra Number Theory*, 10(6):1301–1358.

[58] Paškūnas, V. (2017). On 2-adic deformations. *Math. Z.*, 286(3-4):801–819.

[59] Paškūnas, V. (2014). Blocks for mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In *Automorphic forms and Galois representations. Vol. 2*, volume 415 of *London Math. Soc. Lecture Note Ser.*, pages 231–247. Cambridge Univ. Press, Cambridge.

[60] Pyvovarov, A. (2018). On the Breuil-Schneider conjecture: Generic case. *arXiv e-prints*, page arXiv:1803.01610.

[61] Rouquier, R. (1996). Caractérisation des caractères et pseudo-caractères. *J. Algebra*, 180(2):571–586.

[62] Saito, T. (2009). Hilbert modular forms and p-adic Hodge theory. *Compos. Math.*, 145(5):1081–1113.

[63] Sasaki, S. (2017). Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representations, and weight one forms II. http://www.cantabgold.net/users/s.sasaki.03/hmv1-5-9.pdf.

[64] Sasaki, S. (2019). Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representations, and weight one forms. *Invent. Math.*, 215(1):171–264.

[65] Serre, J.-P. (1995). Structure de certains pro-p-groupes (d'après Demuškin). In *Séminaire Bourbaki, Vol. 8*, pages Exp. No. 252, 145–155. Soc. Math. France, Paris.

[66] Shotton, J. (2016). Local deformation rings for GL_2 and a Breuil-Mézard conjecture when $\ell \neq p$. *Algebra Number Theory*, 10(7):1437–1475.

[67] Snowden, A. (2009). On two dimensional weight two odd representations of totally real fields. *arXiv e-prints*, page arXiv:0905.4266.

[68] Stacks Project Authors, T. (2018). *Stacks Project*. https://stacks.math.columbia.edu.

[69] Taylor, R. (1989). On Galois representations associated to Hilbert modular forms. *Invent. Math.*, 98(2):265–280.

[70] Taylor, R. (2008). Automorphy for some l-adic lifts of automorphic mod l Galois representations. II. *Publ. Math. Inst. Hautes Études Sci.*, (108):183–239.

[71] Thorne, J. (2012). On the automorphy of l-adic Galois representations with small residual image. *J. Inst. Math. Jussieu*, 11(4):855–920. With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne.

[72] Thorne, J. A. (2015). Automorphy lifting for residually reducible l-adic Galois representations. *J. Amer. Math. Soc.*, 28(3):785–870.

[73] Thorne, J. A. (2016). Automorphy of some residually dihedral Galois representations. *Math. Ann.*, 364(1-2):589–648.

[74] Vignéras, M.-F. (2004). Representations modulo p of the p-adic group $\mathrm{GL}(2,F)$. *Compos. Math.*, 140(2):333–358.

[75] Wang-Erickson, C. (2018). Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings. *Math. Ann.*, 371(3-4):1615–1681.

[76] Wiles, A. (1988). On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573.