Motivic stable homotopy groups via framed correspondences

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Abstract

Let k be a perfect field. Using the technique of framed correspondences, we obtain an expression for mapping spaces from suspension spectra to Thom spectra in the motivic stable homotopy ∞-category $\mathcal{SH}(k)$. This result allows us to express some stable homotopy groups of Thom spectra in terms of geometric generators and relations, and we apply this approach to study the unit map of the algebraic special linear cobordism spectrum MSL. We introduce SL-oriented framed correspondences and identify non-positive $\mathbb{G}_m$-homotopy groups of MSL with stabilizations of free abelian groups generated by these correspondences, modulo $\mathbb{A}^1$-homotopy. When k has characteristic 0, we show that the unit map $\mathbb{1}_k \to \mathrm{MSL}$ induces an isomorphism of homotopy modules, by a direct comparison of these abelian groups. As a straightforward corollary, we deduce over a base field of characteristic 0 the known fact that the Chow-Witt groups and the MW-motivic cohomology groups are SL-oriented cohomology theories.

Zusammenfassung

Sei k ein perfekter Körper. Durch Verwendung gerahmter Korrespondenzen erhalten wir eine Beschreibung der Abbildungsräume von Einhängungsspektren nach Thom-Spektren in der motivischen stabilen Homotopie-∞-Kategorie $\mathcal{SH}(k)$. Dadurch wird es möglich, diverse stabile Homotopiegruppen von Thom-Spektren durch geometrische Erzeuger und Relationen auszudrücken, und wir nutzen dies um die Eins-Abbildung des algebraischen speziell-linearen Kobordismusspektrums MSL zu untersuchen. Wir führen SL-orientierte gerahmte Korrespondenzen ein und identifizieren nicht-positive $\mathbb{G}_m$-Homotopiegruppen von MSL mit Stabilisierungen der von solchen Korrespondenzen erzeugten freien abelschen Gruppen modulo $\mathbb{A}^1$-Homotopie. Für k von Charakteristik 0 zeigen wir durch direkten Vergleich dieser abelscher Gruppen, dass die Eins-Abbildung $\mathbb{1}_k \to \mathrm{MSL}$ einen Isomorphismus von Homotopiemoduln induziert. Als unmittelbare Konsequenz erhalten wir daraus die bekannte Aussage, dass über Körpern der Charakteristik 0 die Chow-Witt-Gruppen sowie die MW-motivischen Kohomologiegruppen SL-orientierte Kohomologietheorien sind.

To my grandmother
who has been speaking poetry to me ever since I was born.

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Introduction

One of the main problems of classical algebraic topology is the computation of (stable) homotopy groups of spheres. A breakthrough in this problem was the Pontryagin-Thom theorem, which identifies n-th stable homotopy group with the group of n-dimensional smooth compact *framed* manifolds (i.e. equipped with a trivialization of the stable normal bundle), modulo the bordism equivalence relation.

The problem of computing stable homotopy groups becomes even more tricky in motivic homotopy theory, where topological spaces are replaced with smooth schemes over some base field k. Recall that the motivic stable homotopy ∞-category $\mathcal{SH}(k)$ is constructed as the $\mathbb{P}^1$-stabilization of the ∞-category of $\mathbb{A}^1$-invariant pointed Nisnevich sheaves of spaces on Sm_k [Voe98, Definition 5.7]. Despite the fairly straightforward construction of the ∞-category, its mapping spaces, in general, are not at all explicit. In particular, mapping spaces between suspension spectra of smooth k-schemes are very mysterious objects.

In his unpublished notes [Voe01], Voevodsky suggested an approach to this problem, in the flavour of the Pontryagin-Thom isomorphism. In particular, he introduced the notion of a *framed correspondence* between smooth k-schemes X and Y. Such a correspondence is a bunch of geometric data, which in the simplest case $X = Y = \operatorname{Spec} k$ gives a geometric model of topological 0-dimensional framed manifolds (i.e. framed points). In more detail, a framed correspondence c of level $n \geqslant 0$ is given by a closed subscheme $Z \subset \mathbb{A}^n_X$ (called *support* of c), finite over X; an étale neighborhood U of Z in $\mathbb{A}^n_X$; a morphism $\phi\colon U \to \mathbb{A}^n$, cutting out Z as the preimage of 0 (called *framing* of Z); and a morphism $g\colon U \to Y$. As Voevodsky observed, the set of framed correspondences $\mathrm{Fr}_n(X,Y)$ is in bijection with the set of morphisms between pointed Nisnevich sheaves $(\mathbb{P}^1,\infty)^{\wedge n} \wedge X_+$ and $\mathrm{L}_{\mathrm{Nis}}(T^{\wedge n} \wedge Y_+)$, where $T = \mathbb{A}^1/\mathbb{A}^1 - 0$ (motivically, $T \simeq \mathbb{P}^1$). This bijection provides an explicit map from the set $\mathrm{Fr}_n(X,Y)$ to the mapping space between corresponding suspension spectra in $\mathcal{SH}(k)$. After stabilizing with respect to level n, one gets a map of presheaves

$$(\star) \qquad \mathrm{Fr}(-,Y) \to \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma^\infty_T Y_+).$$

Here the left-hand side is only a presheaf of sets, whereas the right-hand side is an $\mathbb{A}^1$-invariant Nisnevich sheaf of grouplike $\mathcal{E}_\infty$-spaces.

In a series of papers, Ananyevskiy, Druzhinin, Garkusha, Neshitov and Panin developed a theory of framed motives, discovering the features of framed correspondences and of presheaves with additional „wrong way“ maps along them, so called *framed transfers* [GP18a], [AGP18], [GP18b], [GNP18], [DP18]. As a result of their work, they provide an explicit model for the mapping space between suspension spectra when the base field k is perfect. It turns out, in this case it is enough to supply the left-hand side of $(\star)$ with the mentioned good properties of the right-hand side in a minimal fashion, to get an equivalence of presheaves of spaces. The precise statement is formulated in a more general case below.

We extend this result by replacing the suspension spectrum in the target with a Thom space of a virtual vector bundle of rank 0. Let E be a vector bundle over Y of rank r, its Thom space is given by the quotient $\mathrm{Th}_Y(E) = E/E - Y$. We introduce E-framed correspondences $\mathrm{Fr}_{E,n}(X,Y)$ with a framing in the vector bundle $\mathbb{A}^n \times E$ over Y (see Definition 2.1.1) and prove the following result.[1]

Theorem 1 (see Theorem 2.2.2)**.** *Let k be a perfect field, Y a smooth k-scheme, E a vector bundle over Y of rank r. Then there is a canonical equivalence of presheaves of spaces on* Sm_k*:*

$$\mathrm{L}_{\mathrm{Nis}}(\mathrm{L}_{\mathbb{A}^1}\mathrm{Fr}_E(-,Y))^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma^{-r}_T\Sigma^\infty_T \mathrm{Th}_Y(E)).$$

[1]This statement, as well as the definition of an E-framed correspondence, was suggested by Marc Hoyois and Adeel Khan, and they will later be part of [EHK+19].

Here $\mathrm{L}_{\mathbb{A}^1}$ is the (naive) $\mathbb{A}^1$-localization functor, and gp denotes group completion with respect to an $\mathcal{E}_\infty$-structure, which is given, roughly speaking, by taking disjoint union of supports of correspondences (see Proposition 1.2.8 for details).

Let us note that in the case $E = Y$, this is the result of Ananyevskiy-Druzhinin-Garkusha-Neshitov-Panin, described above. To prove the general statement, we first show that the map under consideration is a motivic equivalence, by reducing Nisnevich-locally to the case of a trivial bundle. Afterwards we compute motivic localization of the left-hand side by means of the following fundamental property: „good“ $\mathbb{A}^1$-invariant presheaves with framed transfers become strictly $\mathbb{A}^1$-invariant after Nisnevich localization [GP18b, Theorems 1.1 and 2.1].

In classical homotopy theory, an important example of a map out of the stable homotopy groups of spheres is induced by the unit map of the cobordism spectrum: $\mathbb{S} \to \mathrm{MU}$. This map has two features: 1) it induces isomorphism on $\pi_0^s = \mathbb{Z}$; 2) it detects nilpotence (giving rise to the field of chromatic homotopy theory). In motivic homotopy theory, there is not much known about nilpotence phenomena, so we will concentrate on the first property of the unit map. In motivic settings, the abelian group π_0 of a spectrum is replaced by a richer invariant. For a motivic spectrum $\mathcal{E} \in \mathcal{SH}(k)$ one considers a sequence of Nisnevich sheaves of abelian groups $\{\underline{\pi}_0(\mathcal{E})_l\}_{l\in\mathbb{Z}}$, called a *homotopy module*. One may ask an analogous question: does the unit map of a cobordism spectrum induce an isomorphism of homotopy modules?

The first guess would be to consider the unit map of the algebraic cobordism spectrum MGL, which is the motivic analogue of MU, constructed by Voevodsky [Voe98]. As it turns out, the induced map on homotopy modules kills η, the motivic Hopf element. More precisely, Hoyois has shown that the unit map factors through the map $\mathbb{1}_k/\eta \to \mathrm{MGL}$, which induces isomorphism of homotopy modules [Hoy15, Theorem 3.8]. One could ask if there is another algebraic cobordism spectrum „closer“ to the motivic sphere spectrum $\mathbb{1}_k$. Indeed, for the *algebraic special linear cobordism spectrum* MSL (a motivic analogue of MSU, constructed by Panin and Walter [PW10]) the unit map induces an isomorphism of homotopy modules. This can be shown directly by studying the geometry of oriented grassmanians in the similar fashion to Hoyois' proof (in fact, over arbitrary base schemes), as stated in [BH18, Example 16.34]. But instead we apply the technique of framed correspondences, in order to analyze the homotopy module of MSL in terms of explicit generators.

One can apply π_0 to the equivalence in Theorem 1 to express in concrete terms $\pi_0(\mathcal{E})_l(k) = [\mathbb{1}_k, \Sigma^l_{\mathbb{G}_m}\mathcal{E}]_{\mathrm{SH}(k)}$ for Thom spectra $\mathcal{E}$ and $l \geqslant 0$. For example, it follows that

$$\pi_0(\mathbb{1}_k)_l(k) \simeq \mathrm{Coker}(\mathbb{Z}\mathrm{F}(\mathbb{A}^1_k, \mathbb{G}_m^{\wedge l}) \xrightarrow{i_1^* - i_0^*} \mathbb{Z}\mathrm{F}(\operatorname{Spec} k, \mathbb{G}_m^{\wedge l})),$$

where $\mathbb{Z}\mathrm{F}(X,Y)$ is the stabilized free abelian group on framed correspondences from X to Y, modulo equivalences $c \sqcup d \sim c + d$. One can think of the right-hand side as $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l}))$, i.e. the zeroth homology of the framed version of Suslin complex. In fact, in the case $\operatorname{char} k = 0$, Neshitov has computed $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l}))$ as the Milnor-Witt K-theory $\mathrm{K}^{MW}_l(k)$ [Nes18, Theorem 9.7], recovering in that case the famous computation of the homotopy module of the motivic sphere spectrum by Morel [Mor12, Theorem 6.40].

To express in a similar form $\pi_0(\mathrm{MSL})_l(k)$ we have to introduce SL-*oriented framed correspondences*. Such a correspondence of level n is the same set of data as the usual framed correspondence, except that here a framing is a map $\phi\colon U \to \widetilde{\mathcal{T}}_n$, where $\widetilde{\mathcal{T}}_n \to \widetilde{\mathrm{Gr}}_n$ is the tautological bundle over the oriented grassmanian $\widetilde{\mathrm{Gr}}_n = \widetilde{\mathrm{Gr}}(n, \infty)$. The support is cut out as the preimage of the zero section of $\widetilde{\mathcal{T}}_n$. There is a natural map $\varepsilon_n\colon \mathrm{Fr}_n(X,Y) \to \mathrm{Fr}^{\mathrm{SL}}_n(X,Y)$, given by embedding $\mathbb{A}^n \hookrightarrow \widetilde{\mathcal{T}}_n$ as the fiber over the distinguished point of $\widetilde{\mathrm{Gr}}_n$. It induces a functor $\mathcal{E}\colon \mathrm{Fr}_*(k) \to \mathrm{Fr}^{\mathrm{SL}}_*(k)$ between categories, where objects are smooth k-schemes and morphisms are given by (SL-oriented) framed correspondences. We now reduce the question about the comparison of the homotopic invariants $\underline{\pi}_0(-)_l$ to a concrete question about abelian groups, given in terms of generators and relations of geometric nature.

Proposition 1 (see Corollary 5.3.4)**.** *Let k be a perfect field, and let $l \geqslant 0$. Then the unit map $e_*\colon \pi_0(\mathbb{1}_k)_l(k) \to \pi_0(\mathrm{MSL})_l(k)$ is canonically identified with the induced map*

$$\varepsilon_*\colon H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l})) \to H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l})).$$

Finally, we prove the following comparison result, which was originally suggested by Ivan Panin.

Theorem 2 (see Theorem 9.2.6)**.** *Assume that* $\operatorname{char} k = 0$. *Then the induced map*

$$\varepsilon_*\colon H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \to H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *}))$$

is an isomorphism of graded rings.

The surjectivity of ε_* is proven by providing explicit $\mathbb{A}^1$-homotopies between framed correspondences, which allow us to deform the image of an SL-oriented framing so that the image is contained in the fiber over the distinguished point of $\widetilde{\mathrm{Gr}}_n$. Proving injectivity of ε_* is more involved. To do that, we employ the category $\widetilde{\mathrm{Cor}}_k$ of Milnor-Witt correspondences of Calmès-Fasel [CF17a, Definition 4.15]. This category has smooth k-schemes as objects, and a morphism from X to Y is, roughly speaking, given by a closed subscheme $Z \subset X \times Y$, finite and surjective over components of X, with an „unramified quadratic form“ on Z.

There is a functor $\alpha\colon \mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$, defined in [DF17, Proposition 2.1.12]. We show that Neshitov's isomorphism

$$H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \xrightarrow{\sim} \mathrm{K}^{MW}_*(k)$$

factors via α through the isomorphism $H_0(\widetilde{\mathrm{Cor}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \xrightarrow{\sim} \mathrm{K}^{MW}_*(k)$, constructed in [CF17b, Theorem 2.9]. The functor α can be understood as follows: given a framed correspondence in $\mathrm{Fr}_n(X,Y)$, one considers the oriented Thom class of the trivial bundle of rank n over $\operatorname{Spec} k$ (which is an element of $H^n_0(\mathbb{A}^n_k, \mathbf{K}^{MW}_n)$), takes its pullback along the framing, and then applies pushforward to $X \times Y$. Such functor is naturally extended to the category $\mathrm{Fr}^{\mathrm{SL}}_*(k)$, by applying the same procedure to the oriented Thom class of a tautological bundle over the oriented grassmanian. Altogether, this allows us to define a left inverse map for ε_*.

From Theorem 2 we obtain the following straightforward corollaries.

Corollary 1 (see Proposition 10.1.2)**.** *Assume that* $\operatorname{char} k = 0$. *Then the unit map $e\colon \mathbb{1}_k \to \mathrm{MSL}$ induces an isomorphism of the corresponding homotopy modules:*

$$e_*\colon \underline{\pi}_0(\mathbb{1}_k)_* \xrightarrow{\sim} \underline{\pi}_0(\mathrm{MSL})_*.$$

The spectrum MSL represents a cohomology theory with a *special linear orientation*, and as such has a universal property [PW10, Theorem 5.9]. In particular, a map of commutative monoids $\mathrm{MSL} \to A$ in the homotopy category $\mathrm{SH}(k)$ induces a special linear orientation of the cohomology theory $A^{*,*}$. Thus Corollary 1 immediately implies the following well-known fact.

Corollary 2 (see Corollaries 10.2.3 and 10.2.6)**.** *Assume that* $\operatorname{char} k = 0$. *Then the Chow-Witt groups $H^*(-, \mathbf{K}^{MW}_*)$ and the Milnor-Witt cohomology $H^{*,*}_{MW}(-, \mathbb{Z})$ as ring cohomology theories acquire unique special linear orientations.*

Further generality

There is a recent work by Druzhinin and Kylling, currently in the status of a draft, which extends Neshitov's result (*loc. cit.*) to perfect fields k of char $k \neq 2$ [DK18, Sections 4, 5]. This result would imply that Theorem 2 also holds for such fields. Corollaries 1 and 2 would hold over perfect fields of char $k > 2$ as well, after inverting the characteristic of k.

Outline

This thesis consists of three chapters and one appendix.

In the first chapter, we compute mapping spaces from suspension spectra to Thom spectra in the ∞-category $\mathcal{SH}(k)$. To do so, we recall the main facts about framed correspondences in Section 1, introduce E-framed correspondences in Section 2 and prove Theorem 1 in Section 3.

In the second chapter, we apply Theorem 1 to express homotopy groups of the spectrum MSL in terms of framed correspondences. We obtain an expression in terms of E-framed correspondences in Section 4. Then in Section 5 we introduce SL-oriented framed correspondences and prove Proposition 1. In Section 6 we recall Neshitov's theorem, to state the comparison result of Theorem 2.

In the third chapter, we prove Theorem 2 which implies that the unit map of the spectrum MSL induces an isomorphism of homotopy modules. In more detail, we prove surjectivity of ε_* in Section 7 and injectivity in Section 9, after recalling necessary facts about Milnor-Witt correspondences in Section 8. Finally, we deduce applications in Section 10.

Appendix A collects necessary facts about $\mathcal{E}_\infty$-monoids and group completion in ∞-categorical settings.

Contents

Notation

Notation 1. We will use the language of ∞-categories, following [Lur17b].

- Spc denotes the ∞-category of spaces [Lur17b, Definition 1.2.16.1].
- $\mathrm{Maps}_{\mathcal{C}}(x, y)$ denotes the space of morphisms from x to y in an ∞-category $\mathcal{C}$; $[x, y]_{h\mathcal{C}} = \pi_0 \mathrm{Maps}_{\mathcal{C}}(x, y)$ denotes the set of morphisms in the homotopy category $h\mathcal{C}$ [Lur17b, Definition 1.1.5.14].
- $\mathrm{PSh}(\mathcal{C})$ denotes the ∞-category of presheaves of spaces on an ∞-category $\mathcal{C}$ [Lur17b, Definition 5.1.0.1].

  We consider 1-categories as ∞-categories by applying the nerve functor, which we omit from notation.

Notation 2. The following ∞-categorical constructions of motivic homotopy categories are given, for example, in [BH18, Section 2.2].

- $\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k) \subset \mathrm{PSh}(\mathrm{Sm}_k)$ denotes the full subcategory of Nisnevich sheaves, and $\mathrm{L}_{\mathrm{Nis}} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is the corresponding localization functor (i.e. Nisnevich sheafification).
- $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subset \mathrm{PSh}(\mathrm{Sm}_k)$ is the full subcategory of $\mathbb{A}^1$-invariant presheaves, and $\mathrm{L}_{\mathbb{A}^1} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k)$ is the corresponding localization functor, so called (naive) $\mathbb{A}^1$-*localization*. It can be modelled by the (homotopy) colimit:

$$(\mathrm{L}_{\mathbb{A}^1} P)(X) = \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} P(X \times \Delta^n_k),$$

  where $\Delta^n_k = \mathrm{Spec}\, k[t_0, \ldots, t_n]/(\sum_{i=0}^n t_i - 1)$ is the algebraic n-simplex.
- $\mathrm{PSh}_{\mathrm{Nis}, \mathbb{A}^1}(\mathrm{Sm}_k) = \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \cap \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, and

$$\mathrm{L}_{\mathrm{mot}} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{PSh}_{\mathrm{Nis}, \mathbb{A}^1}(\mathrm{Sm}_k)$$

  is the corresponding localization functor, called *motivic localization*.
- We call a morphism $f \in \mathrm{PSh}(\mathrm{Sm}_k)$ a *Nisnevich equivalence*, an $\mathbb{A}^1$-*equivalence*, or a *motivic equivalence* if $\mathrm{L}_{\mathrm{Nis}}(f)$, $\mathrm{L}_{\mathbb{A}^1}(f)$, or $\mathrm{L}_{\mathrm{mot}}(f)$ is an equivalence.
- $\mathcal{H}(k) = \mathrm{PSh}_{\mathrm{Nis}, \mathbb{A}^1}(\mathrm{Sm}_k)$ is the *unstable homotopy ∞-category* of a field k.
- $\mathcal{SH}(k)$ is the *stable homotopy ∞-category* of k, obtained by stabilization of the pointed unstable homotopy category $\mathcal{H}_\bullet(k)$ with respect to the pointed scheme $(\mathbb{P}^1, \infty)$ [BH18, Section 4.1]. We denote by $\mathrm{SH}(k)$ its homotopy category.

Notation 3. We will also use the following abbreviations.

- k is a perfect field.
- Sm_k is the category of smooth separated schemes of finite type over k.
- $\Delta^\bullet_k$ is the standard cosimplicial object $n \mapsto \Delta^n_k$.
- $\mathbb{A}^1 = \mathbb{A}^1_k$ and $\mathbb{P}^1 = \mathbb{P}^1_k$ when the field k is fixed.
- $\mathbb{G}_m = (\mathbb{A}^1 - 0, 1)$, $\mathbb{P}^1 = (\mathbb{P}^1, \infty)$ are pointed k-schemes.
- $z \colon X \hookrightarrow E$ denotes the zero section of a vector bundle $E \to X$.
- $\mathrm{Th}_X(E) = E/E - z(X)$ is the Thom space of a vector bundle E over a smooth scheme X. In particular, $T = \mathbb{A}^1/\mathbb{A}^1 - 0$.

- $\mathbb{1} = \Sigma^\infty_{\mathbb{P}^1} S^0_k$ is the unit of $\mathcal{SH}(k)$.
- $\underline{\pi}_n(F)_m$ is the Nisnevich sheafification of the presheaf on Sm_k:

$$\pi_n(F)_m \colon U \mapsto [\Sigma^n_{S^1} \Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^m_{\mathbb{G}_m} F]_{\mathrm{SH}(k)}$$

  for $F \in \mathcal{SH}(k)$. Its value is naturally extended to essentially smooth k-schemes.

- $\pi_n(F)_m(L) = \pi_n(F)_m(\operatorname{Spec} L)$ for $F \in \mathcal{SH}(k)$, L/k a finitely generated field extension.

Chapter 1

Homotopy groups of Thom spectra

In this chapter, we compute mapping spaces from suspension spectra to Thom spectra in the ∞-category $\mathcal{SH}(k)$. We start with recalling our main tool, the notion of a framed correspondence, and its features.

1 Framed correspondences

In this section we recollect the main facts about framed correspondences. Our main references are [GP18a] and [EHK$^+$18b].

1.1 Main definitions

First we recall the definition of a framed correspondence, which was first introduced by Voevodsky in [Voe01, Section 2].

Definition 1.1.1. Let X, $Y \in \mathrm{Sm}_k$. A *framed correspondence* $c = (U, \phi, g)$ of level n from X to Y consists of the following data:

- a closed subscheme $Z \subset \mathbb{A}^n_X$, finite over X;
- an étale neighborhood $p\colon U \to \mathbb{A}^n_X$ of Z;
- a morphism $\phi\colon U \to \mathbb{A}^n$ such that $Z = \phi^{-1}(0)$ as closed subscheme in U;
- a morphism $g\colon U \to Y$.

The closed subscheme Z is called the *support* of c and denoted $\mathrm{supp}(c)$. The morphism ϕ is called a *framing* of Z.

Two framed correspondences (U, ϕ, g) and (U', ϕ', g') of level n are said to be *equivalent*, if their supports coincide and there exists an open neighborhood of Z in $U \times_{\mathbb{A}^n_X} U'$ where $g \circ \mathrm{pr}_U = g' \circ \mathrm{pr}_{U'}$ and $\phi \circ \mathrm{pr}_U = \phi' \circ \mathrm{pr}_{U'}$.

The set of framed correspondences of level n from X to Y modulo the equivalence relation is denoted $\mathrm{Fr}_n(X, Y)$.

Remark 1.1.2. In [EHK$^+$18b, Definition 2.1.2], this type of framed correspondences is called *equationally framed*, emphasizing the fact that a framing is given by explicit equations, cutting out the support in its étale neighborhood. Since we will only deal with such kind of correspondences, we drop “equational” from notation.

1.1.3. There is a well-defined *composition* of framed correspondences:

$$\begin{gathered}\circ\colon \mathrm{Fr}_n(X, Y) \times \mathrm{Fr}_m(Y, V) \longrightarrow \mathrm{Fr}_{n+m}(X, V)\\ ((U, \phi, g), (U', \phi', g')) \mapsto (U \times_Y U', (\phi \circ \mathrm{pr}_U, \phi' \circ \mathrm{pr}_{U'}), g' \circ \mathrm{pr}_{U'}).\end{gathered}$$

1.1.4. There is also a well-defined *external product*:

$$\begin{gathered}\boxtimes\colon \mathrm{Fr}_n(X,Y)\times \mathrm{Fr}_m(X',Y') \longrightarrow \mathrm{Fr}_{n+m}(X\times X', Y\times Y')\\ ((U,\phi,g),(U',\phi',g'))\mapsto (U\times U', (\phi\circ \mathrm{pr}_U, \phi'\circ \mathrm{pr}_{U'}), g\times g').\end{gathered}$$

For each level $n\geqslant 0$ there is a unique correspondence with empty support, denoted $0_n\in \mathrm{Fr}_n(X,Y)$.

Definition 1.1.5. The *category of framed correspondences* $\mathrm{Fr}_*(k)$ has smooth k-schemes as objects, and morphisms are given by

$$\mathrm{Fr}_*(X,Y)=\bigvee_{i=0}^{\infty}\mathrm{Fr}_i(X,Y),$$

where each set $\mathrm{Fr}_i(X,Y)$ is pointed by the correspondence $0_i\in \mathrm{Fr}_i(X,Y)$.

Remark 1.1.6. There is a canonical functor:

$$\gamma\colon\ \mathrm{Sm}_k\to \mathrm{Fr}_*(k),$$

which sends $f\colon X\to Y$ to a framed correspondence $(X,\mathrm{const},f)\in \mathrm{Fr}_0(X,Y)$.

By abuse of notation, we will consider morphisms of k-schemes as framed correspondences of level 0.

1.1.7. The definition of framed correspondences is motivated by the following computation of Voevodsky.

Lemma 1.1.8 (Voevodsky)**.** *Let X and Y be smooth k-schemes. Then for every $n\geqslant 0$ there is a natural bijection of pointed sets:*

$$\Theta_n\colon\ \mathrm{Fr}_n(X,Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k)_\bullet}((\mathbb{P}^1,\infty)^{\wedge n}\wedge X_+, \mathrm{L}_{\mathrm{Nis}}(T^{\wedge n}\wedge Y_+)).$$

Proof. The proof is given in [EHK+18b, Appendix A]: this is Corollary A.1.7 combined with Lemma A.1.2(iv). The Lemma can be applied because a smooth k-scheme is normal hence geometrically unibranch [Stacks, Tag 0BQ1]. Here we recall how the map Θ_n is constructed (see also [Ana18, Appendix A]).

Let $c=(U,\phi,g)\in \mathrm{Fr}_n(X,Y)$ be a correspondence with a support Z. Consider the cartesian square:

$$\begin{tikzcd}
U - Z \arrow[r] \arrow[d] & U \arrow[d, "p"] \arrow[ddr, "\overline{(\phi,g)}"] & \\
(\mathbb{P}^1)^{\times n}\times X - Z \arrow[r] \arrow[drr, "\mathrm{const}"'] & (\mathbb{P}^1)^{\times n}\times X \arrow[dr, dotted] & \\
 & & \mathbb{A}^n\times Y/(\mathbb{A}^n-0)\times Y
\end{tikzcd}$$

Here horizontal maps are open embeddings; vertical maps are induced by

$$p\colon U\to \mathbb{A}^n\times X\hookrightarrow (\mathbb{P}^1)^{\times n}\times X,$$

defined via the canonical embeddings at the complement of infinity; $\overline{(\phi,g)}$ is the morphism (ϕ,g) followed by the quotient map; and the constant map sends $(\mathbb{P}^1)^{\times n}\times X-Z$ to the distinguished point. The square is a Nisnevich cover, so we get a morphism of Nisnevich sheaves:

$$(\mathbb{P}^1)^{\times n}\times X\longrightarrow \mathrm{L}_{\mathrm{Nis}}(\mathbb{A}^n\times Y/(\mathbb{A}^n-0)\times Y)=\mathrm{L}_{\mathrm{Nis}}(T^{\wedge n}\wedge Y_+).$$

Since Z is a closed subscheme of $\mathbb{A}^n_X\subset (\mathbb{P}^1)^{\times n}\times X$, we get an induced map of pointed Nisnevich sheaves:

$$\Theta_n(c)\colon (\mathbb{P}^1)^{\wedge n}\wedge X_+\longrightarrow \mathrm{L}_{\mathrm{Nis}}(T^{\wedge n}\wedge Y_+).$$

$\square$

Remark 1.1.9. In these settings the composition of framed correspondences is understood as follows. For morphisms $f\colon (\mathbb{P}^1)^{\wedge n} \wedge X_+ \to T^{\wedge n} \wedge Y_+$ and $g\colon (\mathbb{P}^1)^{\wedge m} \wedge Y_+ \to T^{\wedge m} \wedge V_+$ their composition is given by:

$$g \circ f\colon (\mathbb{P}^1)^{\wedge m} \wedge (\mathbb{P}^1)^{\wedge n} \wedge X_+ \xrightarrow{\mathrm{id} \wedge f} (\mathbb{P}^1)^{\wedge m} \wedge T^{\wedge n} \wedge Y_+ \simeq$$
$$T^{\wedge n} \wedge (\mathbb{P}^1)^{\wedge m} \wedge Y_+ \xrightarrow{\mathrm{id} \wedge g} T^{\wedge n} \wedge T^{\wedge m} \wedge V_+ \simeq T^{\wedge m} \wedge T^{\wedge n} \wedge V_+.$$

1.1.10. Lemma 1.1.8 is the starting point in the computation of mapping spaces in $\mathcal{SH}(k)$, as we will see later. To get to the stable homotopy category, we first need to stabilize maps of sheaves with respect to suspension.

Definition 1.1.11. For $X \in \mathrm{Sm}_k$ define the *suspension morphism*:

$$\sigma_X = (\mathbb{A}^1 \times X, \mathrm{pr}_{\mathbb{A}^1}, \mathrm{pr}_X) \in \mathrm{Fr}_1(X, X).$$

Composition with σ_Y gives a map:

$$\mathrm{Fr}_n(X, Y) \to \mathrm{Fr}_{n+1}(X, Y)\colon (U, \phi, g) \mapsto (U \times \mathbb{A}^1, (\phi, \mathrm{id}_{\mathbb{A}^1}), g \circ \mathrm{pr}_U).$$

Denote the set:

$$\mathrm{Fr}(X, Y) = \mathrm{colim}(\mathrm{Fr}_0(X, Y) \xrightarrow{\sigma_Y} \mathrm{Fr}_1(X, Y) \to \dots).$$

Remark 1.1.12. In the settings of Lemma 1.1.8 the suspension morphism $\sigma_{\mathrm{Spec}\, k}$ corresponds to the canonical motivic equivalence of pointed Nisnevich sheaves $(\mathbb{P}^1, \infty) \xrightarrow{\sim} \mathbb{P}^1/\mathbb{P}^1 - 0 \simeq T$. In particular, for smooth k-schemes X and Y we get an induced map

$$\Theta = \mathrm{colim}_n\, \Theta_n\colon\ \mathrm{Fr}(X, Y) \to \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^\infty_T Y_+), \tag{1.1.13}$$

functorial in X and Y.

1.1.14. One can also consider the linear version of framed correspondences.

Definition 1.1.15. The set of *linear framed correspondences* from X to Y of level n is defined as:

$$\mathbb{Z}\mathrm{F}_n(X, Y) = \mathbb{Z} \cdot \mathrm{Fr}_n(X, Y)/(c \sqcup d - c - d),$$

where $c \sqcup d$ is given by disjoint union of the data of correspondences c and d, whose supports are disjoint as subschemes of $\mathbb{A}^n_X$. Note that $\mathbb{Z}\mathrm{F}_n(X, Y)$ is isomorphic to the free abelian group on framed correspondences with connected support.

The composition law of framed correspondences induces the composition:

$$\mathbb{Z}\mathrm{F}_n(X, Y) \times \mathbb{Z}\mathrm{F}_m(Y, V) \longrightarrow \mathbb{Z}\mathrm{F}_{n+m}(X, V).$$

Stabilization with respect to suspension is denoted as:

$$\mathbb{Z}\mathrm{F}(X, Y) = \mathrm{colim}(\mathbb{Z}\mathrm{F}_0(X, Y) \xrightarrow{\sigma_Y} \mathbb{Z}\mathrm{F}_1(X, Y) \to \dots).$$

1.2 Garkusha-Panin theorems

To formulate the analogue of Voevodsky's theorem on sheafification of presheaves with transfers [MVW06, Theorem 24.1], one needs the following definitions.

Definition 1.2.1. A *framed presheaf* with values in an ∞-category $\mathcal{C}$ is a functor

$$F\colon\ \mathrm{Fr}_*(k)^{\mathrm{op}} \to \mathcal{C}.$$

Definition 1.2.2. A framed presheaf F is called *stable* if for any $X \in \mathrm{Sm}_k$ after passing to the homotopy category holds the following:

$$[\sigma_X^*] = \mathrm{id}_{F(X)} \in [F(X), F(X)]_{h\mathcal{C}}.$$

In particular, σ_X^* induces identity on homotopy groups of $F(X)$.

Definition 1.2.3. A framed presheaf F is called $\mathbb{A}^1$*-invariant* if for any $X \in \mathrm{Sm}_k$ the projection morphism $\mathrm{pr}_X \colon \mathbb{A}^1 \times X \to X$ induces an equivalence:

$$\mathrm{pr}_X^* \colon F(X) \xrightarrow{\sim} F(\mathbb{A}^1 \times X).$$

Definition 1.2.4. Assume that $\mathcal{C}$ admits finite products. A framed presheaf with values in $\mathcal{C}$ is called *radditive* if $F(\emptyset) = *$, and for any pair $X, Y \in \mathrm{Sm}_k$ the inclusions $X \hookrightarrow X \sqcup Y$, $Y \hookrightarrow X \sqcup Y$ induce an equivalence:

$$F(X \sqcup Y) \simeq F(X) \times F(Y).$$

1.2.5. The following result is proven for infinite perfect fields k in [GP18b, Theorems 1.1 and 2.1], complemented in characteristic 2 by [DP18]. The result is extended to finite fields in [DK18, Section 3], or alternatively can be extended using the methods from [EHK$^+$18b, Appendix B].

Theorem 1.2.6 (Garkusha-Panin)**.** *Let k be a perfect field. Let F be an $\mathbb{A}^1$-invariant stable radditive framed presheaf of abelian groups. Then the Nisnevich sheaf* $\mathrm{L}_{\mathrm{Nis}}(F \circ \gamma^{\mathrm{op}})$ *on* Sm_k *is strictly $\mathbb{A}^1$-invariant.*

1.2.7. The necessary definitions for the following theorem are recalled in Appendix A. Consider the presheaf of sets $\mathrm{Fr}(-, Y)$ for $Y \in \mathrm{Sm}_k$. It is a Fin_*-object via the map $\langle n \rangle_+ \mapsto \mathrm{Fr}(-, Y^{\sqcup n})$. By the additivity theorem [GP18a, Theorem 6.4] it becomes an $\mathcal{E}_\infty$-monoid in $\mathrm{PSh}(\mathrm{Sm}_k)$ after $\mathbb{A}^1$-localization (for proof see [EHK$^+$18b, Proposition 2.2.11]):

Theorem 1.2.8 (Garkusha-Panin)**.** *Let $Y_1, \ldots, Y_n \in \mathrm{Sm}_k$. Then the canonical map (see Definition A.0.0.2)*

$$\mathrm{Fr}(-, Y_1 \sqcup \cdots \sqcup Y_n) \to \mathrm{Fr}(-, Y_1) \times \cdots \times \mathrm{Fr}(-, Y_n),$$

is an $\mathbb{A}^1$-equivalence.

1.2.9. Having this $\mathcal{E}_\infty$-structure, we can state the main computation of the theory of framed motives:

Theorem 1.2.10 (Garkusha-Panin)**.** *Let k be a perfect field, and let Y be a smooth k-scheme. Then the map Θ, defined in (1.1.13), induces an equivalence of presheaves of spaces on* Sm_k*:*

$$\Theta \colon \mathrm{L}_{\mathrm{Nis}}(\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}(-, Y))^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma^\infty_T Y_+),$$

where gp *denotes group completion with respect to the $\mathcal{E}_\infty$-structure from Theorem 1.2.8.*

Remark 1.2.11. As explained in Lemma A.1.0.8, it does not matter in which order to apply Nisnevich localization and group completion on the left-hand side.

Proof. An equivalence of presheaves of spaces is explained in [EHK$^+$18b, Corollary 3.5.15]. The fact that the particular map Θ induces an equivalence, is the content of [EHK$^+$18a, Corollary 3.3.8]. □

2 Generalized framed correspondences

To obtain a similar computation as in Theorem 1.2.10 for a Thom spectrum instead of the suspension spectrum $\Sigma_T^\infty Y_+$, we have to consider a generalized version of framed correspondences, associated with a vector bundle.

2.1 Main definition and functoriality

Definition 2.1.1. Let X, Y be smooth k-schemes and E a vector bundle over Y of rank r. An *E-framed correspondence* $c = (U, \phi, g)$ of level n from X to Y consists of the following data:

- a closed subscheme $Z \subset \mathbb{A}^{n+r}_X$, finite over X;
- an étale neighborhood $p\colon U \to \mathbb{A}^{n+r}_X$ of Z;
- a morphism $(\phi, g)\colon U \to \mathbb{A}^n \times E$ such that Z as a closed subscheme of U is the preimage of the zero section $z(0 \times Y) \subset \mathbb{A}^n \times E$.

We say that E-framed correspondences (U, ϕ, g) and (U', ϕ', g') are equivalent if $Z = Z'$ and (ϕ, g) coincides with (ϕ', g') in an étale neighborhood of Z refining both U and U'. We denote the set of E-correspondences modulo this equivalence relation as $\mathrm{Fr}_{E,n}(X, Y)$.

Note that for E a trivial bundle over Y of rank r we get the usual definition of a framed correspondence from X to Y of level $n + r$.

2.1.2. One can compose E-framed correspondences with usual framed correspondences the following way:

$$\begin{gathered}\mathrm{Fr}_n(X, V) \times \mathrm{Fr}_{E,m}(V, Y) \longrightarrow \mathrm{Fr}_{E,n+m}(X, Y)\\ \big((U, \phi, g), (W, \psi, h)\big) \mapsto (U \times_V W, \phi \times \psi, h \circ \mathrm{pr}_W).\end{gathered}$$

In the opposite order one can compose with endomorphisms:

$$\begin{gathered}\mathrm{Fr}_{E,n}(X, Y) \times \mathrm{Fr}_m(Y, Y) \longrightarrow \mathrm{Fr}_{E,n+m}(X, Y)\\ \big((W, \psi, h), (U, \phi, g)\big) \mapsto (W \times_Y U, \psi \times \phi, g \circ \mathrm{pr}_U),\end{gathered}$$

where $W \to Y$ is defined as $W \xrightarrow{h} E \to Y$.

Definition 2.1.3. As before, we define stabliziation with respect to the suspension σ_Y:

$$\mathrm{Fr}_E(X, Y) = \mathrm{colim}(\mathrm{Fr}_{E,0}(X, Y) \xrightarrow{\sigma_Y} \mathrm{Fr}_{E,1}(X, Y) \to \dots).$$

Lemma 2.1.4. *The presheaf* $\mathrm{Fr}_E(-, Y)$ *is a framed presheaf: it defines a functor*

$$\mathrm{Fr}_E(-, Y)\colon \mathrm{Fr}_*(k)^{\mathrm{op}} \longrightarrow \mathrm{Set}\,.$$

Proof. Let $\alpha = (U, \phi, g) \in \mathrm{Fr}_n(X, X')$ be a framed correspondence, we want to define induced map:

$$\alpha^*\colon \mathrm{Fr}_E(X', Y) \longrightarrow \mathrm{Fr}_E(X, Y).$$

Take $a = (W, \psi, h) \in \mathrm{Fr}_{E,m}(X', Y)$. We define:

$$\alpha^*(a) = (U \times_{X'} W, \phi \times \psi, h \circ \mathrm{pr}_W) \in \mathrm{Fr}_{E,n+m}(X, Y)$$

as precomposition with α. Since $\alpha^*(\sigma_Y \circ a) = \sigma_Y \circ \alpha^*(a)$, we deduce that α^* factors through stabilization with respect to the suspension morphism σ_Y. The map defined obviously respects composition in $\mathrm{Fr}_*(k)^{\mathrm{op}}$.

∎

Construction 2.1.5. There is also functoriality with respect to the vector bundles. Assume $f\colon E \to E'$ is a map of rank r vector bundles over smooth k-schemes Y, Y' respectively such that for the zero sections the identity $z(Y) = E \times_{E'} z(Y')$ holds. Then f induces a map:

$$f_{*,n}\colon \mathrm{Fr}_{E,n}(-,Y) \longrightarrow \mathrm{Fr}_{E',n}(-,Y').$$

After stabilization one gets a map $f_*\colon \mathrm{Fr}_E(-,Y) \to \mathrm{Fr}_{E'}(-,Y')$.

We will need an extension of this functoriality to the category Sm_{k+}. Let $f\colon E \dashrightarrow E'$ be a partially-defined map with a clopen domain, that is, $f\colon B \to E'$ where $E = B \sqcup B^c$. Assume that restriction to the zero section gives a map $f\big|_{z(Y)}\colon A \to Y'$ where $z(Y) = A \sqcup A^c$, and that $A = B \times_{E'} z(Y')$. Then f induces a map:

$$\begin{gathered} f_{*,n}(X)\colon \mathrm{Fr}_{E,n}(X,Y) \longrightarrow \mathrm{Fr}_{E',n}(X,Y') \\ (U,\phi,g) \mapsto \big(g^{-1}(B), \phi\big|_{g^{-1}(B)}, f \circ g\big|_{g^{-1}(B)}\big), \end{gathered}$$

functorial in $X \in \mathrm{Sm}_k$, which gives $f_*\colon \mathrm{Fr}_E(-,Y) \to \mathrm{Fr}_{E'}(-,Y')$ after stabilization.

2.1.6. Our motivation to study E-framed correspondences comes from the following generalization of Voevodsky's lemma.

Lemma 2.1.7. *Let X, Y be smooth k-schemes, E a vector bundle over Y of rank r. Then there is a natural bijection:*

$$\Theta_{E,n}\colon \mathrm{Fr}_{E,n}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k)_\bullet}\big((\mathbb{P}^1,\infty)^{\wedge r+n} \wedge X_+, \mathrm{L}_{\mathrm{Nis}}(T^n \wedge \mathrm{Th}_Y(E))\big).$$

Proof. This is a particular case of [EHK+18b, Corollary A.1.5]. □

In view of Lemma 2.1.7 we get an induced map

$$\Theta_E\colon \mathrm{Fr}_E(X,Y) \longrightarrow \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^r_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^\infty_T \mathrm{Th}_Y(E)), \tag{2.1.8}$$

functorial in X.

2.2 Additivity theorem and mapping spaces in $\mathcal{SH}(k)$

We define a structure of a Fin_*-object on the presheaf $\mathrm{Fr}_E(-,Y)$. By Remark A.0.0.3, we can view the category Fin_* as the category with objects $\langle n\rangle = \{1,\dots,n\}$ for $n \geqslant 0$ and partially-defined maps. The functor

$$F\colon \mathrm{Fin}_* \to \mathrm{PSh}(\mathrm{Sm}_k); \quad \langle n\rangle \mapsto \mathrm{Fr}_{E^{\sqcup n}}(-,Y^{\sqcup n})$$

is constructed as follows. Let $a\colon \langle n\rangle \dashrightarrow \langle m\rangle$ be a partially-defined map. The map a induces a partially-defined map $\hat{a}\colon E^{\sqcup n} \dashrightarrow E^{\sqcup m}$ with a clopen domain, satisfying the requirements of Construction 2.1.5. We set $F(a) = \hat{a}_*$.

As for usual framed correspondences, „additivity“ (Theorem 1.2.8) holds for E-framed correspondences.

Proposition 2.2.1. *Let $Y_1,\dots,Y_m$ be smooth k-schemes, and let $E_1,\dots,E_m$ be vector bundles of rank r over $Y_1,\dots,Y_m$ respectively. Then the canonical map (see Definition A.0.0.2)*

$$\alpha\colon \mathrm{Fr}_{E_1\sqcup\cdots\sqcup E_m}(-,Y_1 \sqcup\cdots\sqcup Y_m) \to \mathrm{Fr}_{E_1}(-,Y_1) \times\cdots\times \mathrm{Fr}_{E_m}(-,Y_m)$$

is an $\mathbb{A}^1$-equivalence. In particular, for every $Y \in \mathrm{Sm}_k$ and a vector bundle E over Y the presheaf of spaces $\mathrm{L}_{\mathbb{A}^1}\mathrm{Fr}_E(-,Y)$ is an $\mathcal{E}_\infty$-monoid in $\mathrm{PSh}(\mathrm{Sm}_k)$.

Proof. The proof is the same as in [EHK+18b, Proposition 2.2.11], here we repeat the main steps.

One can assume $m = 2$. Let X be a smooth k-scheme. The map $\alpha(X)$ is the colimit of the maps

$$\alpha_n(X)\colon\ \mathrm{Fr}_{E_1 \sqcup E_2, n}(X, Y_1 \sqcup Y_2) \to \mathrm{Fr}_{E_1,n}(X, Y_1) \times \mathrm{Fr}_{E_2,n}(X, Y_2),$$

where $\alpha_n(X)_i$ is obtained by applying Construction 2.1.5 to partially-defined maps $\mathrm{id}_{E_i}\colon E_1 \sqcup E_2 \dashrightarrow E_i$. We set

$$\begin{gathered}\beta_n(X)\colon\ \mathrm{Fr}_{E_1,n}(X, Y_1) \times \mathrm{Fr}_{E_2,n}(X, Y_2) \longrightarrow \mathrm{Fr}_{E_1 \sqcup E_2, n+1}(X, Y_1 \sqcup Y_2)\\ (U, \phi, g) \times (W, \psi, h) \mapsto (\mathbb{A}^1_U \sqcup \mathbb{A}^1_W, \phi \times t_1 \sqcup \psi \times (t_2 - 1), g \circ \mathrm{pr}_U \sqcup h \circ \mathrm{pr}_V),\end{gathered}$$

where t_1 and t_2 are the coordinate functions on each copy of $\mathbb{A}^1$. Both maps $\alpha_n(X)$ and $\beta_n(X)$ are functorial in $X \in \mathrm{Sm}_k$. Consider the diagram:

$$\begin{tikzcd}
\mathrm{Fr}_{E_1 \sqcup E_2, n}(-, Y_1 \sqcup Y_2) \arrow[r, "\sigma_{Y_1 \sqcup Y_2}"] \arrow[d, "\alpha_n"] & \mathrm{Fr}_{E_1 \sqcup E_2, n+1}(-, Y_1 \sqcup Y_2) \arrow[d, "\alpha_{n+1}"] \\
\mathrm{Fr}_{E_1,n}(-, Y_1) \times \mathrm{Fr}_{E_2,n}(-, Y_2) \arrow[r, "\sigma_{Y_1} \times \sigma_{Y_2}"] \arrow[ur, "\beta_n"] & \mathrm{Fr}_{E_1,n+1}(-, Y_1) \times \mathrm{Fr}_{E_2,n+1}(-, Y_2)
\end{tikzcd}$$

As explained in [EHK+18b, Proposition 2.2.11], after applying $\mathrm{L}_{\mathbb{A}^1}$ this diagram can be made commutative for even n. More precisely, there are constructed homotopies $\sigma_{Y_1 \sqcup Y_2} \rightsquigarrow \beta_n \circ \alpha_n$ and $\sigma_{Y_1} \times \sigma_{Y_2} \rightsquigarrow \alpha_{n+1} \circ \beta_n$ such that outer square remains commutative, when those homotopies are applied simultaneously. This implies that $\mathrm{colim}_i\, \mathrm{L}_{\mathbb{A}^1} \beta_{2i}$ is inverse to $\mathrm{L}_{\mathbb{A}^1} \alpha$. □

Theorem 1.2.10 is then generalized as follows.

Theorem 2.2.2. *Let k be a perfect field, Y a smooth k-scheme, E a vector bundle over Y of rank r. Then the map Θ_E, constructed in (2.1.8), induces an equivalence of presheaves of spaces on Sm_k:*

$$\Theta_E\colon\ \mathrm{L}_{\mathrm{Nis}}(\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-, Y))^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma_T^{-r} \Sigma_T^\infty \mathrm{Th}_Y(E)),$$

where gp *denotes group completion with respect to the $\mathcal{E}_\infty$-structure from Proposition 2.2.1.*

Theorem 2.2.2 will be proved in the next section.

3 Homotopy groups of Thom spectra

Our proof of Theorem 2.2.2 will go as follows: first we will show that the map Θ_E is a motivic equivalence, and then compute the motivic localization of the presheaf $(\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-, Y))^{\mathrm{gp}}$. To begin with, we will need the following facts about E-framed correspondences.

3.1 Lemmas about framed presheaves

Lemma 3.1.1. *The presheaves $\mathrm{Fr}_{E,n}(-, Y)$ and $\mathrm{Fr}_E(-, Y)$ are Nisnevich sheaves. In particular, they are radditive, i.e. take finite coproducts to products.*

Proof. Denote $V = \mathbb{A}^n \times E$, $U = (\mathbb{A}^n - 0) \times (E - z(Y))$, then $U \subset V$ is an open subscheme. By [EHK+18b, Proposition A.1.4], $\mathrm{Fr}_{E,n}(X, Y)$ is the fiber of the pointed map:

$$\mathrm{L}_{\mathrm{Nis}}(V/U)((\mathbb{P}^1)^{n+r} \times X) \longrightarrow \prod_{i=1}^{n+r} \mathrm{L}_{\mathrm{Nis}}(V/U)((\mathbb{P}^1)^{n+r-1} \times X),$$

induced by $n+r$ different embeddings at infinity $(\mathbb{P}^1)^{n+r-1} \hookrightarrow (\mathbb{P}^1)^{n+r}$. Hence the claim follows for the presheaf $\mathrm{Fr}_{E,n}(-,Y)$. We use the fact that Nisnevich localization commutes with filtered colimits to obtain the claim for the presheaf $\mathrm{Fr}_E(-,Y)$:

$$\mathrm{L}_{\mathrm{Nis}} \mathrm{Fr}_E(-,Y) = \mathrm{L}_{\mathrm{Nis}} \operatorname{colim}_n \mathrm{Fr}_{E,n}(-,Y) \simeq \operatorname{colim}_n \mathrm{L}_{\mathrm{Nis}} \mathrm{Fr}_{E,n}(-,Y) \simeq \operatorname{colim}_n \mathrm{Fr}_{E,n}(-,Y) = \mathrm{Fr}_E(-,Y).$$

□

Definition 3.1.2. Let $\mathcal{E}_\bullet \to \mathcal{Y}_\bullet$ be a morphism of simplicial schemes such that for any m the induced map $\mathcal{E}_m \to \mathcal{Y}_m$ is a vector bundle, and for every face or degeneracy map $\mathcal{E}_m \to \mathcal{E}_l$ the identity $z(Y_m) = \mathcal{E}_m \times_{\mathcal{E}_l} z(Y_l)$ holds. We define simplicial presheaves $\mathrm{Fr}_{\mathcal{E}_\bullet,n}(-,\mathcal{Y}_\bullet)$ by setting:

$$\left(\mathrm{Fr}_{\mathcal{E}_\bullet,n}(-,\mathcal{Y}_\bullet)\right)_m = \mathrm{Fr}_{\mathcal{E}_m,n}(-,\mathcal{Y}_m),$$

with face and degeneracy maps induced by those of $\mathcal{E}_\bullet$. The simplicial presheaf $\mathrm{Fr}_{\mathcal{E}_\bullet}(-,\mathcal{Y}_\bullet)$ is defined in the same way.

The following lemma is analogous to [Voe01, Theorem 4.4].

Lemma 3.1.3. *Let Y be a smooth k-scheme, E a vector bundle over Y, and $\{p\colon U \to Y\}$ a Nisnevich cover of Y, given by a single map. Denote by $\mathcal{Y}_\bullet$ the corresponding Čech nerve of Y, i.e. $\mathcal{Y}_n = U \times_Y \cdots \times_Y U$. Let $\mathcal{E}_\bullet$ be the Čech nerve of E associated with the Nisnevich cover $p^*E \to E$. Then the induced maps*

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Fr}_{\mathcal{E}_n,m}(-,\mathcal{Y}_n) \to \mathrm{Fr}_{E,m}(-,Y);$$
$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Fr}_{\mathcal{E}_n}(-,\mathcal{Y}_n) \to \mathrm{Fr}_{E}(-,Y)$$

are Nisnevich equivalences of presheaves of spaces.

Proof. We need to show that for a henselian local k-scheme X the map

$$p_*\colon \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Fr}_{\mathcal{E}_n,m}(X,\mathcal{Y}_n) \to \mathrm{Fr}_{E,m}(X,Y)$$

is an equivalence of spaces. Since face and degeneracy maps of $\mathrm{Fr}_{\mathcal{E}_\bullet,m}(X,\mathcal{Y}_\bullet)$ preserve the support Z and the framing $\phi\colon V \to \mathbb{A}^m$ (for an étale neighborhood V of Z in $\mathbb{A}^{m+r}_X$), there is an equivalence:

$$\mathrm{Fr}_{\mathcal{E}_\bullet,m}(X,\mathcal{Y}_\bullet) \simeq \bigsqcup_{(Z,\phi)} \mathrm{Fr}^{Z,\phi}_{\mathcal{E}_\bullet,m}(X,\mathcal{Y}_\bullet).$$

Here coproduct is taken over all supports Z and their framings ϕ, and $\mathrm{Fr}^{Z,\phi}_{\mathcal{E}_n,m}(X,\mathcal{Y}_n)$ consists out of correspondences of the form $(W,\phi|_W,g)$, where W is an étale neighborhood of Z, refining V. Moreover, the map p_* is the coproduct of maps with fixed support Z and framing $\phi\colon V \to \mathbb{A}^m$:

$$p^{Z,\phi}_*\colon \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Fr}^{Z,\phi}_{\mathcal{E}_n,m}(X,\mathcal{Y}_n) \to \mathrm{Fr}^{Z,\phi}_{E,m}(X,Y).$$

Note that $p^{Z,\phi}_*$ is the Čech construction applied to the map:

$$p^{Z,\phi}\colon \mathrm{Fr}^{Z,\phi}_{p^*E,m}(X,U) \to \mathrm{Fr}^{Z,\phi}_{E,m}(X,Y).$$

It is enough then to check that $p^{Z,\phi}$ is surjective. Consider an element $(Z,\phi,g) \in \mathrm{Fr}^{Z,\phi}_{E,m}(X,Y)$. We claim that the induced map on henselization $\bar{g}\colon (\mathbb{A}^{m+r}_X)^h_Z \to E$ lifts to p^*E. Indeed, since X is henselian local and Z is finite over X, Z is a finite disjoint union of henselian local schemes. Hence the Nisnevich cover $\bar{g}^*(p^*E) \to (\mathbb{A}^{m+r}_X)^h_Z$ has a section over Z, so this cover has an everywhere defined section by [Stacks, Tag 09XD], which provides a lift we wanted. Thus there exists an étale neighborhood V', refining V such that the framing map $(\phi, g)\colon V' \to \mathbb{A}^m \times E$ lifts to $\mathbb{A}^m \times p^*E$, and Z is the preimage of the zero section $0 \times z(U)$. The surjectivity follows.

Since Nisnevich localization commutes with colimits, the result holds for the stabilized presheaf $\mathrm{Fr}_E(-, Y)$ as well. □

Lemma 3.1.4. *The framed presheaf* $\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-, Y)$ *is stable.*

Proof. Let X be a smooth k-scheme. Consider the induced morphisms:

$$\sigma_X^*\colon \mathrm{Fr}_{E,n}(X, Y) \to \mathrm{Fr}_{E,n+1}(X, Y) \quad (U, \phi, g) \mapsto (\mathbb{A}^1 \times U, t \times \phi, g);$$
$$\sigma_{Y,*}\colon \mathrm{Fr}_{E,n}(X, Y) \to \mathrm{Fr}_{E,n+1}(X, Y) \quad (U, \phi, g) \mapsto (U \times \mathbb{A}^1, \phi \times t, g),$$

where t is the coordinate function on $\mathbb{A}^1$. Since $\sigma_{Y,*}$ induces identity on $\mathrm{Fr}_E(X, Y)$, it is enough to provide an $\mathbb{A}^1$-homotopy between the actions of σ_X^* and $\sigma_{Y,*}$ on $\mathrm{Fr}_E(X, Y)$.

Let n be even. Then σ_X^* and $\sigma_{Y,*}$ differ by a cyclic permutation $\Sigma_{n+1} \in \mathrm{SL}_{n+1}(\mathbb{Z})$, acting on $\mathbb{A}^{n+1}$. Since Σ_{n+1} is a product of elementary matrices, there is an $\mathbb{A}^1$-homotopy $A_{n+1}\colon \mathbb{A}^1 \to \mathrm{SL}_{n+1}$ such that $A_{n+1}(0) = \Sigma_{n+1}$ and $A_{n+1}(1) = E_{n+1}$ is the identity matrix. It allows us to define a morphism of presheaves:

$$h_n\colon \mathrm{Fr}_{E,n}(-, Y) \to \mathrm{Fr}_{E,n+1}(-, Y)^{\mathbb{A}^1} \quad (U, \phi, g) \mapsto (s \mapsto (U \times \mathbb{A}^1, A_{n+1}(s) \circ (\phi \times t), g)),$$

where $A_{n+1}(s)$ is the corresponding automorphism of $\mathbb{A}^{n+1}$. It induces a homotopy h_n:

$$\sigma_-^* \rightsquigarrow \sigma_{Y,*}\colon \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,n}(-, Y) \to \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,n+1}(-, Y).$$

Let $i_{n+1}\colon \mathrm{SL}_{n+1} \hookrightarrow \mathrm{SL}_{n+3}$ be the canonical embedding. Define a map $B_{n+1}\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathrm{SL}_{n+3}$ by

$$B_{n+1}(s, v) = (i_{n+1} \circ A_{n+1}(v - vs + s)) \cdot A_{n+3}(sv + 1 - s).$$

Then $B_{n+1}(0, v) = i_{n+1} \circ A_{n+1}(v)$ and $B_{n+1}(1, v) = A_{n+3}(v)$. We let

$$H_n\colon \mathrm{Fr}_{E,n}(-, Y) \to \mathrm{Fr}_{E,n+3}(-, Y)^{\mathbb{A}^1 \times \mathbb{A}^1}$$

be the map of presheaves, sending $(U, \phi, g) \in \mathrm{Fr}_{E,n}(X, Y)$ to

$$(s, v) \mapsto (U \times \mathbb{A}^3, B_{n+1}(s, v) \circ (\phi \times t_1 \times t_2 \times t_3), g) \in \mathrm{Fr}_{E,n+3}(X, Y).$$

This way we get a 2-cell in the Kan complex $\mathrm{Maps}(\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,n}(-, Y), \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,n+3}(-, Y))$ for every even n:

$$H_n\colon \sigma_{Y,*}^2 \circ h_n \rightsquigarrow h_{n+2} \circ \sigma_{Y,*}^2.$$

Let $L \subset N(\mathbb{N})$ be the simplicial subset, consisting of all vertices together with the edges that join consecutive integers. We define a map of simplicial sets $F \in \mathrm{Hom}_{\mathrm{sSet}}(L, \mathrm{PSh}(\mathrm{Sm}_k))^{\Delta^2/\Delta^1}$ by the following diagram:

$$\begin{tikzcd}
\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,0}(-, Y) \arrow[r, "\sigma_{Y,*}^2"] \arrow[d, shift right, "\sigma_-^*"'] \arrow[d, shift left, "\sigma_{Y,*}"] & \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,2}(-, Y) \arrow[r, "\sigma_{Y,*}^2"] \arrow[d, shift right, "\sigma_-^*"'] \arrow[d, shift left, "\sigma_{Y,*}"] & \dots \\
\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,1}(-, Y) \arrow[r, "\sigma_{Y,*}^2"] & \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,3}(-, Y) \arrow[r, "\sigma_{Y,*}^2"] & \dots,
\end{tikzcd}$$

where necessary homotopies are given by h_n and H_n for even n. By the proof of [Lur17b, Proposition 4.4.2.6], the inclusion $L \subset N(\mathbb{N})$ is a categorical equivalence. Hence F extends to a morphism $F \in \mathrm{Fun}(\mathbb{N}, \mathrm{PSh}(\mathrm{Sm}_k))^{\Delta^2/\Delta^1}$. Consider

$$\mathrm{colim}\colon \mathrm{Fun}(\mathbb{N}, \mathrm{PSh}(\mathrm{Sm}_k))^{\Delta^2/\Delta^1} \to (\mathrm{PSh}(\mathrm{Sm}_k))^{\Delta^2/\Delta^1},$$

and let $\operatorname{colim}(F) = \tilde{F}$. By construction,

$$\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,2i}(-,Y) = \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_{E,2i+1}(-,Y) = \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-,Y).$$

Hence $\tilde{F}$, when evaluated on X, provides a homotopy

$$\sigma_X^* \rightsquigarrow \sigma_{Y,*} \colon \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-,Y)(X) \to \mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-,Y)(X).$$

□

3.2 Motivic equivalence

After all the necessary preparation, we first prove that the map Θ_E in Theorem 2.2.2 is a motivic equivalence. The following proposition was originally formulated by Marc Hoyois, inspired by a conjecture of Adeel Khan[1].

Proposition 3.2.1. *Let Y be a smooth k-scheme, E a vector bundle over Y of rank r. Then the induced map of presheaves of spaces on Sm_k*

$$\widehat{\Theta}_E \colon (\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-,Y))^{\mathrm{gp}} \longrightarrow \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma_T^{-r}\Sigma_T^\infty \mathrm{Th}_Y(E))$$

is a motivic equivalence.

Proof. First note that the canonical map

$$\mathrm{L}_{\mathrm{mot}}(\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-,Y))^{\mathrm{gp}} \to (\mathrm{L}_{\mathrm{mot}} \mathrm{Fr}_E(-,Y))^{\mathrm{gp}}$$

is an equivalence, since group completion commutes with motivic localization by Lemma A.1.0.8.

Assume that E is a trivial vector bundle. Then $\mathrm{Fr}_{E,n}(X,Y) = \mathrm{Fr}_{n+r}(X,Y)$, and after stabilization we get an isomorphism $\mathrm{Fr}_E(X,Y) \simeq \mathrm{Fr}(X,Y)$. On the other hand, $\Sigma_T^{-r}\Sigma_T^\infty \mathrm{Th}_Y(E) = \Sigma_T^{-r}\Sigma_T^\infty(T^r \wedge Y_+) = \Sigma_T^\infty Y_+$, and in this case the map $\widehat{\Theta}_E$ becomes a Nisnevich equivalence by Theorem 1.2.10.

Now let $Y = \cup_{i=1}^m U_i$ be an open covering of Y, trivializing E, and denote $p\colon U = \sqcup_{i=1}^m U_i \to Y$. Let $\mathcal{Y}_\bullet \to Y$ be the corresponding Čech nerve of Y, and similarly let $\mathcal{E}_\bullet$ be the Čech nerve of E, corresponding to the cover $p^*E \to E$.

Since $\mathcal{E}_n \to \mathcal{Y}_n$ is a trivial vector bundle for every $n \geqslant 0$, we get the following equivalence of presheaves of spaces, induced by equivalences $\mathrm{L}_{\mathrm{mot}}\,\Theta_{\mathcal{E}_n}$:

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}}(\mathrm{L}_{\mathrm{mot}} \mathrm{Fr}_{\mathcal{E}_n}(-,\mathcal{Y}_n))^{\mathrm{gp}} \xrightarrow{\sim} \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma_T^{-r}\Sigma_T^\infty \mathrm{Th}_{\mathcal{Y}_n}(\mathcal{E}_n)).$$

Since motivic localization and group completion are left adjoint functors, they commute with colimits. By [EHK+18b, Corollary 3.5.14], the functor $\Omega^\infty_{\mathbb{P}^1}\colon \mathcal{SH}^{\mathrm{veff}}(k) \to \mathcal{H}(k)$ commutes with sifted colimits, where $\mathcal{SH}^{\mathrm{veff}}(k) = \mathcal{SH}^{\mathrm{eff}}_{\geqslant 0}(k) \subset \mathcal{SH}(k)$ is the full subcategory of very effective spectra. The spectra $\Sigma_T^{-r}\Sigma_T^\infty \mathrm{Th}_{\mathcal{Y}_n}(\mathcal{E}_n)$ are very effective, so we obtain the following commutative diagram of presheaves of spaces:

$$\begin{tikzcd}
(\mathrm{L}_{\mathrm{mot}} \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Fr}_{\mathcal{E}_n}(-,\mathcal{Y}_n))^{\mathrm{gp}} \arrow[r, "\mathrm{L}_{\mathrm{mot}}\Theta_\bullet", "\sim"'] \arrow[d, "p_*"] & \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \Sigma_T^{-r}\Sigma_T^\infty \mathrm{Th}_{\mathcal{Y}_n}(\mathcal{E}_n)) \arrow[d, "p_*"] \\
(\mathrm{L}_{\mathrm{mot}} \mathrm{Fr}_E(-,Y))^{\mathrm{gp}} \arrow[r, "\mathrm{L}_{\mathrm{mot}}\widehat{\Theta}_E"] & \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \Sigma_T^{-r}\Sigma_T^\infty \mathrm{Th}_Y(E)).
\end{tikzcd}$$

By Lemma 3.1.3 the map $p_*\colon \operatorname{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Fr}_{\mathcal{E}_n}(-,\mathcal{Y}_n) \to \mathrm{Fr}_E(-,Y)$ is a Nisnevich equivalence, hence the left vertical map is an equivalence.

[1] private communication

The cofiber sequences in $\mathrm{PSh}(\mathrm{Sm}_k)$

$$E - z(Y) \to E \to \mathrm{Th}_Y(E); \quad \mathcal{E}_\bullet - z(\mathcal{Y}_\bullet) \to \mathcal{E}_\bullet \to \mathrm{Th}_{\mathcal{Y}_\bullet}(\mathcal{E}_\bullet)$$

allow us to deduce that $p_*\colon \operatorname{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{Th}_{\mathcal{Y}_n}(\mathcal{E}_n) \xrightarrow{\sim} \mathrm{Th}_Y(E)$ is a Nisnevich equivalence, and so induces an equivalence in $\mathcal{SH}(k)$. Hence the right vertical map is also an equivalence, and the claim follows.

□

3.3 Computation of the motivic localization

Theorem 2.2.2 is then deduced from the following proposition.

Proposition 3.3.1. *Let Y be a smooth k-scheme, E a vector bundle over Y. Then the canonical map:*

$$\mathrm{L}_{\mathrm{Nis}}(\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y))^{\mathrm{gp}} \longrightarrow (\mathrm{L}_{\mathrm{mot}}\,\mathrm{Fr}_E(-,Y))^{\mathrm{gp}}$$

is an equivalence of presheaves of spaces on Sm_k.

Proof. We need to show that $\mathrm{L}_{\mathrm{Nis}}(\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y))^{\mathrm{gp}}$ is $\mathbb{A}^1$-invariant. We denote the presheaf of spaces $(\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y))^{\mathrm{gp}}$ as $\mathcal{F}$. It is enough to show that all homotopy groups $\pi_n(\mathrm{L}_{\mathrm{Nis}}\,\mathcal{F})$ are homotopy invariant presheaves. The sheaf $\mathrm{L}_{\mathrm{Nis}}\,\mathcal{F}$ is equivalent to the limit of its Postnikov tower in $\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ (this follows from [Lur18, Corollary 3.7.7.3]), so we can consider the descent spectral sequence:

$$E_2^{p,q} = H^p_{\mathrm{Nis}}(-, \pi_q^{\mathrm{Nis}}\mathcal{F}) \Rightarrow \pi_{q-p}(\mathrm{L}_{\mathrm{Nis}}\,\mathcal{F}).$$

Since the spectral sequence converges [Bro73, Theorem 8], it is enough to show that the sheaves $\pi_q^{\mathrm{Nis}}\mathcal{F}$ are strictly homotopy invariant for any $q \geqslant 0$. We want to deduce this by means of Theorem 1.2.6, and so we have to check that $\pi_q\mathcal{F}$ are $\mathbb{A}^1$-invariant stable radditive presheaves of abelian groups on $\mathrm{Fr}_*(k)$.

The homotopy groups $\pi_q\mathcal{F}$ are presheaves of abelian groups, because $\mathcal{F}$ is a presheaf of grouplike $\mathcal{E}_\infty$-spaces by construction. It suffices to check that $\mathcal{F}$ is an $\mathbb{A}^1$-invariant stable radditive presheaf on $\mathrm{Fr}_*(k)$.

$\mathcal{F}$ is a presheaf on $\mathrm{Fr}_*(k)$: by Lemma 2.1.4, $\mathrm{Fr}_E(-,Y)$ is a framed presheaf, and $\mathrm{Fr}_E(-\times\Delta^n,Y)$ becomes a framed presheaf by applying external product (1.1.4) on $\mathrm{Fr}_*(k)$. One can consider $\operatorname{colim}_{n\in\Delta^{\mathrm{op}}}\mathrm{Fr}_E(-\times\Delta^n,Y)$ in the ∞-category of presheaves with framed transfers $\mathrm{PSh}(\mathrm{Fr}_*(k))$. The forgetful functor

$$(\gamma^{op})^*\colon\ \mathrm{PSh}(\mathrm{Fr}_*(k)) \longrightarrow \mathrm{PSh}(\mathrm{Sm}_k)$$

preserves colimits, because colimits in categories of presheaves are computed objectwise, hence $\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y)$ is also a framed presheaf. By Remark A.1.0.3, group completion in the ∞-category $\mathrm{Mon}_{\mathcal{E}_\infty}(\mathrm{PSh}(\mathrm{Sm}_k))$ is computed objectwise, and we deduce that $\mathcal{F}$ is a framed presheaf by functoriality of group completion.

$\mathcal{F}$ is $\mathbb{A}^1$-invariant by Lemma A.1.0.8, because:

$$\mathrm{L}_{\mathbb{A}^1}\,\mathcal{F} = \mathrm{L}_{\mathbb{A}^1}(\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y))^{\mathrm{gp}} \simeq (\mathrm{L}_{\mathbb{A}^1}\,\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y))^{\mathrm{gp}} = \mathcal{F}.$$

$\mathcal{F}$ is a stable presheaf, because $\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y)$ is stable by Lemma 3.1.4, and group completion is objectwise.

$\mathcal{F}$ is radditive, because $\mathrm{Fr}_E(-,Y)$ is radditive by Lemma 3.1.1, and group completion commutes with finite products by Theorem A.1.0.6. □

Chapter 2

MSL via framed correspondences

In this chapter, we express homotopy groups of the algebraic special linear cobordism spectrum MSL via different versions of framed correspondences.

4 Homotopy groups of MSL

In this section we apply Theorem 2.2.2 to study the homotopy groups of the spectrum MSL. First we briefly recall notations for cobordism T-spectra MGL and MSL. Both constructions are given in [PW10, §4].

4.1 Construction of the spectrum MGL

Let $\Gamma_n = \mathcal{O}_k^{\oplus n}$ be the trivial bundle. For $p \geqslant 1$ consider $\mathrm{Gr}(n, np) = \mathrm{Gr}(n, \Gamma_n^{\oplus p})$ and the tautological bundle $\mathcal{T}(n, np) \to \mathrm{Gr}(n, np)$. The inclusions

$$(\mathrm{id}, 0)\colon \Gamma_n^{\oplus p} \to \Gamma_n^{\oplus p} \oplus \Gamma_n = \Gamma_n^{\oplus p+1}$$

induce closed embeddings $\mathrm{Gr}(n, np) \hookrightarrow \mathrm{Gr}(n, np + p)$ and monomorphisms:

$$\mathrm{Th}_{\mathrm{Gr}(n,np)}(\mathcal{T}(n, np)) \to \mathrm{Th}_{\mathrm{Gr}(n,np+p)}(\mathcal{T}(n, np + p)).$$

We denote:

$$\mathrm{Gr}_n = \mathrm{colim}_p\, \mathrm{Gr}(n, np); \quad \mathcal{T}_n = \mathrm{colim}_p\, \mathcal{T}(n, np).$$

By definition,

$$\mathrm{MGL}_n = \mathrm{Th}_{\mathrm{Gr}_n}(\mathcal{T}_n) = \mathrm{colim}_p\, \mathrm{Th}_{\mathrm{Gr}(n,np)}(\mathcal{T}(n, np)).$$

4.1.1. Concatenation of bases $\Gamma_n \oplus \Gamma_m = \Gamma_{n+m}$ induces morphisms:

$$j_{n,m,p}\colon \mathrm{Gr}(n, np) \times \mathrm{Gr}(m, mp) \to \mathrm{Gr}(n + m, np + mp),$$

which after taking colimit give $j_{n,m}\colon \mathrm{Gr}_n \times \mathrm{Gr}_m \to \mathrm{Gr}_{n+m}$. There are canonical isomorphisms:

$$\mathcal{T}_n \times \mathcal{T}_m \xrightarrow{\sim} j^*_{n,m}\mathcal{T}_{n+m}.$$

Inclusions $(\mathrm{id}, 0, \ldots, 0)\colon \Gamma_n \to \Gamma_n^{\oplus p}$ induce the maps $\mathrm{Gr}(n, n) \hookrightarrow \mathrm{Gr}(n, np)$. They make each $\mathrm{Gr}(n, np)$ a pointed space, and then Gr_n by taking colimit.

The isomorphisms $\mathcal{T}_{n+1}\big|_{\mathrm{Gr}_n} \simeq \mathcal{O}_{\mathrm{Gr}_n} \oplus \mathcal{T}_n$ induce structure maps of the T-spectrum:

$$T \wedge \mathrm{Th}_{\mathrm{Gr}_n}(\mathcal{T}_n) \to \mathrm{Th}_{\mathrm{Gr}_n}(\mathcal{O}_{\mathrm{Gr}_n}) \wedge \mathrm{Th}_{\mathrm{Gr}_n}(\mathcal{T}_n) \simeq \mathrm{Th}_{\mathrm{Gr}_n}(\mathcal{O}_{\mathrm{Gr}_n} \oplus \mathcal{T}_n) \to \mathrm{Th}_{\mathrm{Gr}_{n+1}}(\mathcal{T}_{n+1}).$$

4.2 Construction of the spectrum MSL

4.2.1. For $n \geqslant 1$ consider the line bundle $\det(\mathcal{T}(n, np)) \to \mathrm{Gr}(n, np)$. Removing the zero section gives:

$$\widetilde{\mathrm{Gr}}(n, np) = \det(\mathcal{T}(n, np)) - z(\mathrm{Gr}(n, np)) \in \mathrm{Sm}_k\,.$$

The projection $\pi_{n,np}\colon \widetilde{\mathrm{Gr}}(n, np) \to \mathrm{Gr}(n, np)$ is a principal $\mathbb{G}_m$-bundle. We define:

$$\widetilde{\mathcal{T}}(n, np) = \pi^*_{n,np}(\mathcal{T}(n, np)).$$

The inclusion $\widetilde{\mathrm{Gr}}(n, np) \subset \det(\mathcal{T}(n, np))$ gives a nowhere vanishing section of the line bundle $\det \widetilde{\mathcal{T}}(n, np)$, so defines a trivialization

$$\lambda_{n,np}\colon \mathcal{O}_{\widetilde{\mathrm{Gr}}(n,np)} \xrightarrow{\sim} \det(\widetilde{\mathcal{T}}(n, np)). \tag{4.2.2}$$

Denote:

$$\widetilde{\mathrm{Gr}}_n = \mathrm{colim}_p\, \widetilde{\mathrm{Gr}}(n, np); \quad \widetilde{\mathcal{T}}_n = \mathrm{colim}_p\, \widetilde{\mathcal{T}}(n, np).$$

One defines $\mathrm{MSL}_0 = \mathrm{Spec}\, k_+$ and

$$\mathrm{MSL}_n = \mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n) = \mathrm{colim}_p\, \mathrm{Th}_{\widetilde{\mathrm{Gr}}(n,np)}(\widetilde{\mathcal{T}}(n, np)) \text{ for } n \geqslant 1.$$

4.2.3. There are also canonical morphisms $\widetilde{j}_{n,m}\colon \widetilde{\mathrm{Gr}}_n \times \widetilde{\mathrm{Gr}}_m \to \widetilde{\mathrm{Gr}}_{n+m}$, and isomorphisms:

$$\widetilde{\mathcal{T}}_n \times \widetilde{\mathcal{T}}_m \xrightarrow{\sim} \widetilde{j}^*_{n,m}\widetilde{\mathcal{T}}_{n+m}. \tag{4.2.4}$$

The distinguished point $\mathrm{Gr}(n, n) \to \mathrm{Gr}(n, np)$ induces the map:

$$\mathbb{G}_m \simeq \Lambda^n \mathcal{O}^n_{\mathrm{Gr}(n,n)} - 0 \longrightarrow \Lambda^n \mathcal{T}(n, np) - z(\mathrm{Gr}(n, np)) = \widetilde{\mathrm{Gr}}(n, np).$$

Denote the standard basis of Γ_n as $e_1, \ldots, e_n$. We make each $\widetilde{\mathrm{Gr}}(n, np)$ a pointed space by considering $e_1 \wedge \cdots \wedge e_n \to \widetilde{\mathrm{Gr}}(n, np)$, where $e_1 \wedge \cdots \wedge e_n$ corresponds to $1 \in \mathbb{G}_m$, and then $\widetilde{\mathrm{Gr}}_n$ by taking colimit. Structure morphisms of MSL are defined similarly to those of MGL, each induced by the restriction of $\widetilde{\mathcal{T}}_{n+1}$ to $\widetilde{\mathrm{Gr}}_n$.

Remark 4.2.5. There is an alternative way to define these types of grassmanians. Let $U(n, N)$ be the algebraic subgroup of rank n matrices in $M_{n\times N}$. Then we can express grassmanians as quotient sheaves: $\mathrm{Gr}(n, N) \simeq \mathrm{GL}_n \backslash U(n, N)$ and $\widetilde{\mathrm{Gr}}(n, N) \simeq \mathrm{SL}_n \backslash U(n, N)$.

4.3 Maps into MSL

4.3.1. To investigate homotopy groups of such Thom spectra, we need to generalize results of Theorem 2.2.2 to smooth ind-schemes.

For every smooth k-scheme X we have a morphism by Lemma 2.1.7:

$$\mathrm{Fr}_{\widetilde{\mathcal{T}}(n,np)}(X, \widetilde{\mathrm{Gr}}(n, np)) \xrightarrow{\Theta_{\widetilde{\mathcal{T}}(n,np)}} \mathrm{Maps}_{\mathcal{SH}(k)}\left(\Sigma^n_T\Sigma^\infty_T X_+, \Sigma^\infty_T\, \mathrm{Th}_{\widetilde{\mathrm{Gr}}(n,np)}(\widetilde{\mathcal{T}}(n, np))\right).$$

By applying Construction 2.1.5, we define:

$$\mathrm{Fr}_{\widetilde{\mathcal{T}}_n,m}(X, \widetilde{\mathrm{Gr}}_n) = \mathrm{colim}_p\, \mathrm{Fr}_{\widetilde{\mathcal{T}}(n,np),m}(X, \widetilde{\mathrm{Gr}}(n, np)),$$

and similarly for stabilized correspondences $\mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(X, \widetilde{\mathrm{Gr}}_n)$. Since $\Sigma^n_T\Sigma^\infty_T X_+$ is a compact object in $\mathcal{SH}(k)$, mapping spaces out of it commute with filtered colimits. Thus, one obtains

$$\Theta_{\widetilde{\mathcal{T}}_n}(X)\colon\ \mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(X, \widetilde{\mathrm{Gr}}_n) \longrightarrow \mathrm{Maps}_{\mathcal{SH}(k)}\left(\Sigma^n_T\Sigma^\infty_T X_+, \Sigma^\infty_T\, \mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n)\right)$$

after taking colimits. Since $\mathbb{A}^1$-localization, Nisnevich localization and group completion are left adjoint functors, so commute with colimits in $\mathrm{PSh}(\mathrm{Sm}_k)$, we obtain from Theorem 2.2.2 an induced isomorphism of presheaves of spaces:

$$\Theta_{\widetilde{\mathcal{T}}_n}\colon\ \mathrm{L_{Nis}}(\mathrm{L}_{\mathbb{A}^1}\, \mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(-, \widetilde{\mathrm{Gr}}_n))^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{SH}(k)}\left(\Sigma^\infty_T(-)_+, \Sigma^{-n}_T\Sigma^\infty_T\, \mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n)\right).$$

4.3.2. Recall that $\mathrm{MSL} \simeq \operatorname{colim}_n \Sigma_T^{-n}\Sigma_T^\infty \mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n)$. As a result of the discussion in 4.3.1, we obtain the formula for mapping spaces into the spectrum MSL.

Corollary 4.3.3. *The colimit of maps* $\Theta_{\widetilde{\mathcal{T}}_n}$ *induces an equivalence of presheaves of spaces on* Sm_k*:*

$$\Theta_{\widetilde{\mathcal{T}}}\colon \mathrm{L}_{\mathrm{Nis}}(\mathrm{L}_{\mathbb{A}^1} \operatorname{colim}_n \mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(-, \widetilde{\mathrm{Gr}}_n))^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{SH}(k)}\big(\Sigma_T^\infty(-)_+, \mathrm{MSL}\big).$$

In particular,

$$\pi_0(\mathrm{MSL})_0(k) \simeq \pi_0\big(\mathrm{L}_{\mathbb{A}^1} \operatorname{colim}_n \mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(-, \widetilde{\mathrm{Gr}}_n)(\operatorname{Spec} k)\big)^{\mathrm{gp}}.$$

4.3.4. The homotopy groups $\pi_m(\mathrm{MGL})_{-m}(k)$ have a beautiful computation in terms of the Lazard ring, at least after inverting the exponential characteristic (see [Hoy15, Proposition 8.2]). Meanwhile, the problem of computing $\pi_m(\mathrm{MSL})_{-m}(k)$ remains open, although these groups are already known for the η-completion of MSL (see [LYZ18, Theorem B]). Corollary 4.3.3 suggests a description of these groups: for $l, m \geqslant 0$

$$\pi_m(\mathrm{MSL})_{-m-l}(k) \simeq \pi_0\big(\mathrm{L}_{\mathrm{Nis}}\, \mathrm{L}_{\mathbb{A}^1} \operatorname{colim}_n \mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(-, \widetilde{\mathrm{Gr}}_n)((\mathbb{P}^1)^{\wedge m} \wedge \mathbb{G}_m^{\wedge l})\big)^{\mathrm{gp}}.$$

Here the right-hand side is defined as follows. The ∞-category of presheaves of pointed spaces is symmetric monoidal, with the product induced by the $\wedge$-product of pointed spaces. Hence $(\mathbb{P}^1)^{\wedge m} \wedge \mathbb{G}_m^{\wedge l}$ defines a presheaf of (pointed) spaces. For presheaves of spaces F and P one sets $F(P) = \mathrm{Maps}_{\mathrm{PSh}(\mathrm{Sm}_k)}(P, F) \in \mathrm{Spc}$.

Unfortunately, for positive m we don't know how to compute π_0 on the right-hand side because of the Nisnevich localization applied to the presheaf. But we will investigate the case $m = 0$, when we can actually perform the computation.

4.4 Zeroth homotopy group of MSL

4.4.1. As stated in [GP18a, Corollary 11.3], one can express $\pi_0(\mathbb{1})_0(k)$ in terms of linear framed correspondences:

$$\pi_0(\mathbb{1})_0(k) \simeq \operatorname{Coker}\big(\mathbb{Z}\mathrm{F}(\mathbb{A}^1, \operatorname{Spec} k) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}\mathrm{F}(\operatorname{Spec} k, \operatorname{Spec} k)\big). \tag{4.4.2}$$

In this subsection we explain a similar result for the zeroth homotopy group of MSL. To do that, we extend the definition of linear framed correspondences.

Definition 4.4.3. The abelian group of *linear E-framed correspondences* from X to Y of level n is defined as:

$$\mathbb{Z}\mathrm{F}_{E,n}(X, Y) = \mathbb{Z} \cdot \mathrm{Fr}_{E,n}(X, Y)/(c \sqcup d - c - d).$$

The pairing

$$\mathbb{Z}\mathrm{F}_{E,n}(X, Y) \times \mathbb{Z}\mathrm{F}_m(Y, Y) \longrightarrow \mathbb{Z}\mathrm{F}_{E,n+m}(X, Y)$$

allows one to define stabilization with respect to suspension:

$$\mathbb{Z}\mathrm{F}_E(X, Y) = \operatorname{colim}(\mathbb{Z}\mathrm{F}_{E,0}(X, Y) \xrightarrow{\sigma_Y} \mathbb{Z}\mathrm{F}_{E,1}(X, Y) \to \dots).$$

Lemma 4.4.4. *Let X, Y be smooth k-schemes, and E a vector bundle over Y of rank r. Then the following abelian groups are isomorphic:*

$$\pi_0(\mathrm{L}_{\mathbb{A}^1} \mathrm{Fr}_E(-, Y)(X))^{\mathrm{gp}} \simeq \operatorname{Coker}\big(\mathbb{Z}\mathrm{F}_E(\mathbb{A}^1 \times X, Y) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}\mathrm{F}_E(X, Y)\big).$$

Proof. By definition,

$$\pi_0(\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y)(X)) \simeq \operatorname{coeq}\left(\mathrm{Fr}_E(\mathbb{A}^1_X,Y) \rightrightarrows \mathrm{Fr}_E(X,Y)\right)$$

is the coequalizer along embeddings $i_0, i_1\colon X \hookrightarrow \mathbb{A}^1_X$. The monoid operation is induced by the following map (see Definition A.0.0.2 and the proof of Proposition 2.2.1):

$$\mathrm{Fr}_{E,n}(X,Y)\times \mathrm{Fr}_{E,n}(X,Y) \xrightarrow{\beta_n(X)} \mathrm{Fr}_{E\sqcup E,n+1}(X,Y\sqcup Y) \xrightarrow{(\mathrm{id}\sqcup\mathrm{id})_*} \mathrm{Fr}_{E,n+1}(X,Y).$$

Since taking free abelian group on a set is a left adjoin functor, it preserves colimits. Hence the group completion is computed as follows:

$$\pi_0(\mathrm{L}_{\mathbb{A}^1}\,\mathrm{Fr}_E(-,Y)(X))^{\mathrm{gp}} \simeq \operatorname{Coker}\left(\mathbb{Z}\cdot\mathrm{Fr}_E(\mathbb{A}^1\times X,Y) \xrightarrow{i_0^*-i_1^*} \mathbb{Z}\cdot\mathrm{Fr}_E(X,Y)\right)/\sim_s,$$

where equivalence relation is given by equivalences for each $c_1, c_2 \in \mathrm{Fr}_{E,n}(X,Y)$:

$$[(U_1,\phi_1,g_1)]+_s[(U_2,\phi_2,g_2)] \sim_s [(U_1\times\mathbb{A}^1\sqcup U_2\times\mathbb{A}^1, \phi_1\times t_1\sqcup\phi_2\times(t_2-1), g_1\sqcup g_2\circ\mathrm{pr}_{U_1\sqcup U_2})].$$

Here $[-]$ denotes equivalence classes in the cokernel, and the right-hand side is the equivalence class of a correspondence in $\mathrm{Fr}_{E,n+1}(X,Y)$.

On the other hand, $\mathbb{Z}\mathrm{F}_E(X,Y)$ is constructed as the quotient of the free abelian group $\mathbb{Z}\cdot\mathrm{Fr}_E(X,Y)$, with equivalence relation given by the following equivalences for $c_1, c_2 \in \mathrm{Fr}_{E,n}(X,Y)$ with disjoint supports Z_1 and Z_2 in $\mathbb{A}^{n+r}_X$:

$$(U_1,\phi_1,g_1)+(U_2,\phi_2,g_2) \sim (U_1\times\mathbb{A}^1\sqcup U_2\times\mathbb{A}^1, \phi_1\times t_1\sqcup\phi_2\times t_2, g_1\sqcup g_2\circ\mathrm{pr}_{U_1\sqcup U_2}).$$

Here the right-hand side belongs to $\mathrm{Fr}_{E,n+1}(X,Y)$, because we postcomposed the sum c_1+c_2 with the suspension σ_Y.

As we can see, this equivalence relation is a priori different, but it becomes the same as $+_s$ after applying $\mathbb{A}^1$-homotopy. Indeed, let $c_1, c_2 \in \mathrm{Fr}_{E,n}(X,Y)$ have supports Z_1 and Z_2 that are not disjoint. Then we can make them disjoint by suspending and applying a homotopy:

$$H = (U_2\times\mathbb{A}^1\times\mathbb{A}^1, \phi_2\times(t-s), g_2\circ\mathrm{pr}_{U_2}) \in \mathrm{Fr}_{E,n+1}(\mathbb{A}^1\times X,Y),$$

where s denotes the homotopy coordinate. This way we get: $i_0^*(H) = \sigma_Y\circ c_2$, and $\mathrm{supp}(i_1^*(H)) = Z_2\times 1$ is disjoint with $Z_1\times 0 = \mathrm{supp}(\sigma_Y\circ c_1)$ in $\mathbb{A}^{n+r+1}_X$.

The sums $+$ and $+_s$ become equivalent via an $\mathbb{A}^1$-homotopy in $\mathrm{Fr}_{E,n+1}(\mathbb{A}^1\times X,Y)$ of the same form. The claim follows. □

4.4.5. Applying Lemma 4.4.4 to $X = \operatorname{Spec} k$, $Y_p = \widetilde{\mathrm{Gr}}(n,np)$, and $E_p = \widetilde{\mathcal{T}}(n,np)$ and taking colimit over p, gives the same result for $Y = \widetilde{\mathrm{Gr}}_n$, $E = \widetilde{\mathcal{T}}_n$, since all the functors involved commute with colimits. We get the expression of $\pi_0(\mathrm{MSL})_0(k)$ via linear framed correspondences.

Corollary 4.4.6. *There is an isomorphism of abelian groups:*

$$\pi_0(\mathrm{MSL})_0(k) \simeq \operatorname{colim}_n \operatorname{Coker}\left(\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n}(\mathbb{A}^1,\widetilde{\mathrm{Gr}}_n) \xrightarrow{i_0^*-i_1^*} \mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n}(\operatorname{Spec} k,\widetilde{\mathrm{Gr}}_n)\right).$$

5 SL-oriented framed correspondences

For future comparison with $\pi_0(\mathbb{1})_0(k)$, we now rewrite Corollary 4.4.6 in more convenient terms. To do that, we introduce a notion of SL-oriented framed correspondences.

5.1 Main definitions

Definition 5.1.1. Let X, Y be smooth k-schemes. An SL-*oriented framed correspondence* $c = (U, \phi, g)$ of level n from X to Y consists of the following data:

- a closed subscheme Z in $\mathbb{A}^n_X$, finite over X;
- an étale neighborhood $p\colon U \to \mathbb{A}^n_X$ of Z;
- a morphism $\phi\colon U \to \widetilde{\mathcal{T}}_n$ such that Z as a closed subscheme of U is the preimage of the zero section $z(\widetilde{\mathrm{Gr}}_n) \subset \widetilde{\mathcal{T}}_n$;
- a morphism $g\colon U \to Y$.

Here by a morphism $\phi\colon U \to \widetilde{\mathcal{T}}_n$ we mean a map $U \to \mathrm{colim}_p\, \widetilde{\mathcal{T}}(n, np)$, represented by a morphism $\phi\colon U \to \widetilde{\mathcal{T}}(n, np)$ for some p. We say that two SL-oriented framed correspondences are equivalent if $Z = Z'$ and (ϕ, g) coincides with (ϕ', g') in an étale neighborhood of Z refining both U and U'. We denote the set of SL-oriented framed correspondences modulo this equivalence relation as $\mathrm{Fr}^{\mathrm{SL}}_n(X, Y)$.

Remark 5.1.2. By construction, $\mathrm{Fr}^{\mathrm{SL}}_n(X, Y) = \mathrm{Fr}_{\widetilde{\mathcal{T}}_n \times Y, 0}(X, \widetilde{\mathrm{Gr}}_n \times Y)$.

5.1.3. As for framed correspondences, there is a composition law:

$$\begin{gathered}\circ\colon\ \mathrm{Fr}^{\mathrm{SL}}_n(X, Y) \times \mathrm{Fr}^{\mathrm{SL}}_m(Y, V) \longrightarrow \mathrm{Fr}^{\mathrm{SL}}_{n+m}(X, V)\\ ((U, \phi, g), (U', \phi', g')) \mapsto (U \times_Y U', s_{n,m} \circ (\phi \circ \mathrm{pr}_U, \phi' \circ \mathrm{pr}_{U'}), g' \circ \mathrm{pr}_{U'}),\end{gathered}$$

where $s_{n,m}\colon \widetilde{\mathcal{T}}_n \times \widetilde{\mathcal{T}}_m \simeq j^*_{n,m} \widetilde{\mathcal{T}}_{n+m} \to \widetilde{\mathcal{T}}_{n+m}$ is the composition of the isomorphism (4.2.4) and the projection.

5.1.4. There is as well an external product:

$$\begin{gathered}\boxtimes\colon\ \mathrm{Fr}^{\mathrm{SL}}_n(X, Y) \times \mathrm{Fr}^{\mathrm{SL}}_m(X', Y') \longrightarrow \mathrm{Fr}^{\mathrm{SL}}_{n+m}(X \times X', Y \times Y')\\ ((U, \phi, g), (U', \phi', g')) \mapsto (U \times U', s_{n,m} \circ (\phi \circ \mathrm{pr}_U, \phi' \circ \mathrm{pr}_{U'}), g \times g').\end{gathered}$$

Definition 5.1.5. The *category of* SL-*oriented framed correspondences* $\mathrm{Fr}^{\mathrm{SL}}_*(k)$ has smooth k-schemes as objects, and morphisms are given by:

$$\mathrm{Fr}^{\mathrm{SL}}_*(X, Y) = \bigvee_{i=0}^{\infty} \mathrm{Fr}^{\mathrm{SL}}_i(X, Y),$$

where $\mathrm{Fr}^{\mathrm{SL}}_i(X, Y)$ is pointed by the correspondence with empty support $0_i \in \mathrm{Fr}^{\mathrm{SL}}_i(X, Y)$.

5.1.6. The fiber of $\widetilde{\mathcal{T}}_n$ over the distinguished point of $\widetilde{\mathrm{Gr}}_n$ is $\mathbb{A}^n$. We get the embedding $\mathbb{A}^n \hookrightarrow \widetilde{\mathcal{T}}_n$, which after restriction to zero section is $0 \hookrightarrow \widetilde{\mathrm{Gr}}_n$. For each $X, Y \in \mathrm{Sm}_k$ the embedding induces a natural map between correspondences:

$$\mathrm{Fr}_n(X, Y) \hookrightarrow \mathrm{Fr}^{\mathrm{SL}}_n(X, Y), \tag{5.1.7}$$

which respects the composition and provides a functor

$$\mathcal{E}\colon\ \mathrm{Fr}_*(k) \longrightarrow \mathrm{Fr}^{\mathrm{SL}}_*(k).$$

5.1.8. The following generalization of Voevodsky's lemma holds:

Lemma 5.1.9. *Let X, Y be smooth k-schemes. Then there is a natural bijection:*

$$\Theta_n^{\mathrm{SL}}\colon \mathrm{Fr}_n^{\mathrm{SL}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_k)_\bullet}((\mathbb{P}^1,\infty)^{\wedge n}\wedge X_+, \mathrm{L}_{\mathrm{Nis}}(\mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n)\wedge Y_+)).$$

Proof. Morphisms into the Nisnevich sheafification of $\mathrm{Th}_{\widetilde{\mathrm{Gr}}(n,np)}(\widetilde{\mathcal{T}}(n,np))\wedge Y_+$ are computed as $\mathrm{Fr}_{\widetilde{\mathcal{T}}(n,np)\times Y,0}(X, \widetilde{\mathrm{Gr}}(n,np)\times Y)$ by [EHK+18b, Corollary A.1.5 and Remark A.1.6], and then one passes to colimit along p. □

In view of Lemma 5.1.9 we get an induced map

$$\Theta^{\mathrm{SL}}\colon \mathrm{Fr}^{\mathrm{SL}}(X,Y) \longrightarrow \mathrm{Maps}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}X_+, \mathrm{MSL}\wedge\Sigma^\infty_{\mathbb{P}^1}Y_+),$$

functorial in X.

Definition 5.1.10. We define *linear* SL*-oriented framed correspondences* as:

$$\mathbb{Z}\mathrm{F}_n^{\mathrm{SL}}(X,Y) = \mathbb{Z}\cdot \mathrm{Fr}_n^{\mathrm{SL}}(X,Y)/(c\sqcup d - c - d).$$

The morphism (5.1.7) descends to the map:

$$\varepsilon_n\colon \mathbb{Z}\mathrm{F}_n(X,Y)\hookrightarrow \mathbb{Z}\mathrm{F}_n^{\mathrm{SL}}(X,Y).$$

In particular, we can define an abelian group:

$$\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(X,Y) = \mathrm{colim}(\mathbb{Z}\mathrm{F}_0^{\mathrm{SL}}(X,Y)\xrightarrow{\sigma_Y}\mathbb{Z}\mathrm{F}_1^{\mathrm{SL}}(X,Y)\to\dots),$$

and the induced homomorphism of abelian groups:

$$\varepsilon\colon \mathbb{Z}\mathrm{F}(X,Y)\to\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(X,Y). \tag{5.1.11}$$

5.2 Comparison with generalized framed correspondences

Lemma 5.2.1. *Let V be a smooth k-scheme. Then the following presheaves of abelian groups on Sm_k are isomorphic:*

$$\theta\colon \mathbb{Z}\mathrm{F}^{\mathrm{SL}}(-,V)\xrightarrow{\sim}\mathrm{colim}_n\,\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n\times V}(-,\widetilde{\mathrm{Gr}}_n\times V).$$

Proof. To simplify notations, we assume that $V=\mathrm{Spec}\,k$, since the same argument applies for arbitrary $V\in\mathrm{Sm}_k$. For a smooth k-scheme X set:

$$\theta_n(X)\colon \mathbb{Z}\mathrm{F}_n^{\mathrm{SL}}(X,\mathrm{Spec}\,k)\to\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n,0}(X,\widetilde{\mathrm{Gr}}_n)$$

be the identity map (see Remark 5.1.2). Let

$$\chi_n(X)\colon \mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n,r}(X,\widetilde{\mathrm{Gr}}_n)\to\mathbb{Z}\mathrm{F}_{n+r}^{\mathrm{SL}}(X,\mathrm{Spec}\,k)$$

be the map induced by the canonical embedding $\mathbb{A}^r\times\widetilde{\mathcal{T}}_n\hookrightarrow\widetilde{\mathcal{T}}_{n+r}$, that restricts to $0\times\widetilde{\mathrm{Gr}}_n\hookrightarrow\widetilde{\mathrm{Gr}}_{n+r}$. Clearly, $\chi_n\circ\theta_n=\mathrm{id}$ (in this case $r=0$). For the other composition, consider $\alpha\in\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n,r}(X,\widetilde{\mathrm{Gr}}_n)$. Then:

$$\sigma^r_{\widetilde{\mathrm{Gr}}_{n+r}}\big(\theta_{n+r}(\chi_n(\alpha))\big)=\delta^r(\alpha),$$

where δ denotes the suspension $\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_*}(X,\widetilde{\mathrm{Gr}}_*)\to\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_{*+1}}(X,\widetilde{\mathrm{Gr}}_{*+1})$. So, the correspondences α and $\theta_{n+r}(\chi_n(\alpha))$ become equivalent after taking colimits with respect to $\sigma_{\widetilde{\mathrm{Gr}}_*}$ and δ. Both maps $\theta_n(X)$ and $\chi_n(X)$ respect suspensions, and so stabilize to inverse maps $\theta(X)$ and $\chi(X)$, functorial in X. □

Since cokernels commute with colimits of morphisms, we deduce from Corollary 4.4.6 the following result.

Corollary 5.2.2. *There is an isomorphism of abelian groups:*

$$\pi_0(\mathrm{MSL})_0(k) \simeq \mathrm{Coker}\left(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\mathbb{A}^1, \mathrm{Spec}\, k) \xrightarrow{i_0^*-i_1^*} \mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\mathrm{Spec}\, k, \mathrm{Spec}\, k)\right).$$

Remark 5.2.3. We can rephraze Corollary 5.2.2:

$$\pi_0(\mathrm{MSL})_0(k) \simeq H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathrm{Spec}\, k)).$$

5.2.4. We now compare this expression with the formula (4.4.2) for $\pi_0(\mathbb{1})_0(k)$. Recall that the unit map $e\colon \mathbb{1} \to \mathrm{MSL}$ is induced by the embeddings of distinguished points $\mathrm{Spec}\, k \hookrightarrow \widetilde{\mathrm{Gr}}_n$, giving $e_n\colon T^n \hookrightarrow \mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n)$. We have a commutative diagram of presheaves of spaces on Sm_k:

$$\begin{tikzcd}
\mathrm{L_{Nis}}(\mathrm{L}_{\mathbb{A}^1}\mathrm{Fr}(-, \mathrm{Spec}\, k))^{\mathrm{gp}} \arrow[r, "\Theta", "\sim"'] \arrow[d, "(\varepsilon_n)_*"'] & \mathrm{Maps}_{\mathcal{SH}(k)}\left(\Sigma_T^\infty(-)_+, \mathbb{1}\right) \arrow[d, "(e_n)_*"] \\
\mathrm{L_{Nis}}(\mathrm{L}_{\mathbb{A}^1}\mathrm{Fr}_{\widetilde{\mathcal{T}}_n}(-, \widetilde{\mathrm{Gr}}_n))^{\mathrm{gp}} \arrow[r, "\Theta_{\widetilde{\mathcal{T}}_n}", "\sim"'] & \mathrm{Maps}_{\mathcal{SH}(k)}\left(\Sigma_T^\infty(-)_+, \Sigma_T^{-n}\mathrm{Th}_{\widetilde{\mathrm{Gr}}_n}(\widetilde{\mathcal{T}}_n)\right).
\end{tikzcd}$$

Here the left vertical morphism is induced by stabilization of the maps:

$$\varepsilon_{n,r}\colon \mathrm{Fr}_{n+r}(-, \mathrm{Spec}\, k) \to \mathrm{Fr}_{\widetilde{\mathcal{T}}_{n,r}}(-, \widetilde{\mathrm{Gr}}_n),$$

given by embeddings $\mathbb{A}^n \hookrightarrow \widetilde{\mathcal{T}}_n$ over the distinguished point of $\widetilde{\mathrm{Gr}}_n$.

5.2.5. After taking colimit and applying Corollary 5.2.2, we get the following observation for zeroth homotopy groups.

Lemma 5.2.6. *The following diagram commutes:*

$$\begin{tikzcd}
H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathrm{Spec}\, k)) \arrow[r, "\Theta_*", "\sim"'] \arrow[d, "\varepsilon_*"'] & {[\mathbb{1}, \mathbb{1}]_{\mathrm{SH}(k)}} \arrow[d, "e_*"] \\
H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathrm{Spec}\, k)) \arrow[r, "(\Theta^{\mathrm{SL}})_*", "\sim"'] & {[\mathbb{1}, \mathrm{MSL}]_{\mathrm{SH}(k)}},
\end{tikzcd}$$

where ε_* is induced by the map ε, defined in (5.1.11).

5.3 $\mathbb{G}_m$-homotopy groups of MSL

5.3.1. In a similar form as Corollary 5.2.2, we can express $\pi_0(\mathrm{MSL})_l(k)$ for $l \geqslant 0$.

Let V be a smooth k-scheme and let $p\colon \widetilde{\mathrm{Gr}}(n, np) \times V \to V$ be the projection to V. By applying the reasoning of 4.3.1 to the vector bundles $p^*\widetilde{\mathcal{T}}(n, np) \to \widetilde{\mathrm{Gr}}(n, np) \times V$, we obtain the following isomorphism of presheaves of spaces on Sm_k, generalizing Corollary 4.3.3:

$$\mathrm{colim}_n\, \mathrm{L_{Nis}}(\mathrm{L}_{\mathbb{A}^1}\mathrm{Fr}_{\widetilde{\mathcal{T}}_n \times V}(-, \widetilde{\mathrm{Gr}}_n \times V))^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{SH}(k)}\left(\Sigma_T^\infty(-)_+, \Sigma_T^\infty V_+ \wedge \mathrm{MSL}\right).$$

Applying Lemma 4.4.4 to $X = \mathrm{Spec}\, k$, $Y_p = \widetilde{\mathrm{Gr}}(n, np) \times V$, $E_p = \widetilde{\mathcal{T}}(n, np) \times V$, and taking colimits expresses the abelian group $[\mathbb{1}, \Sigma_T^\infty V_+ \wedge \mathrm{MSL}]_{\mathrm{SH}(k)}$ as:

$$\mathrm{colim}_n\, \mathrm{Coker}\left(\mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n \times V}(\mathbb{A}^1, \widetilde{\mathrm{Gr}}_n \times V) \xrightarrow{i_0^*-i_1^*} \mathbb{Z}\mathrm{F}_{\widetilde{\mathcal{T}}_n \times V}(\mathrm{Spec}\, k, \widetilde{\mathrm{Gr}}_n \times V)\right).$$

By Lemma 5.2.1 we get:

$$[\mathbb{1}, \Sigma_T^\infty V_+ \wedge \mathrm{MSL}]_{\mathrm{SH}(k)} \simeq H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, V)).$$

In particular, for $l \geqslant 0$ we deduce the isomorphism:

$$[\mathbb{1}, \Sigma_T^\infty(\mathbb{G}_m^l)_+ \wedge \mathrm{MSL}]_{\mathrm{SH}(k)} \simeq H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^l)). \tag{5.3.2}$$

Proposition 5.3.3.

$$\pi_0(\mathrm{MSL})_l(k) = [\mathbb{1}, \Sigma_T^\infty \mathbb{G}_m^{\wedge l} \wedge \mathrm{MSL}]_{\mathrm{SH}(k)} \simeq H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l})),$$

where the right-hand side denotes the zeroth homology of the simplicial abelian group

$$\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l}) = \mathrm{Coker}\,\Big(\bigoplus_{i=1}^{l} \mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{l-1}) \xrightarrow{\oplus_{i=1}^l (j_i)_*} \mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{l})\Big),$$

with the maps induced by embeddings $j_i\colon \mathbb{G}_m^{l-1} \hookrightarrow \mathbb{G}_m^l$, inserting 1 at i-th place.

Proof. Consider the cofiber sequence:

$$\bigvee_{i=1}^{l} \Sigma_T^\infty(\mathbb{G}_m^{l-1})_+ \xrightarrow{\vee_{i=1}^l (j_i)_*} \Sigma_T^\infty(\mathbb{G}_m^{l})_+ \longrightarrow \Sigma_T^\infty \mathbb{G}_m^{\wedge l}.$$

This cofiber sequence splits in $\mathrm{SH}(k)$. From isomorphism (5.3.2) we deduce that:

$$[\mathbb{1}, \Sigma_T^\infty \mathbb{G}_m^{\wedge l} \wedge \mathrm{MSL}]_{\mathrm{SH}(k)} \simeq \mathrm{Coker}\,\big(\oplus_{i=1}^l H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{l-1})) \to H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{l}))\big).$$

Since $\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l})$ is a direct summand of the simplicial group $\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{l})$, the claim follows. □

Finally, in the same way as in Lemma 5.2.6 we obtain:

Corollary 5.3.4. *The unit map $e\colon \mathbb{1} \to \mathrm{MSL}$ induces the following commutative diagram of abelian groups for $l \geqslant 0$:*

$$\begin{tikzcd}
H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l})) \arrow[r, "\Theta_*", "\sim"'] \arrow[d, "\varepsilon_*"'] & \pi_0(\mathbb{1})_l(k) \arrow[d, "e_*"] \\
H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l})) \arrow[r, "(\Theta^{\mathrm{SL}})_*", "\sim"'] & \pi_0(\mathrm{MSL})_l(k)
\end{tikzcd}$$

Here horizontal maps are induced by corresponding versions of Voevodsky's lemma (Lemma 1.1.8 and Lemma 5.1.9), and the left vertical map is induced by the homomorphism $\varepsilon\colon \mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l}) \to \mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l})$, defined in (5.1.11).

Proof. The upper isomorphism was stated in [GP18a, Corollary 11.3] and follows from Theorem 1.2.10 by the same reasoning as in Proposition 5.3.3. □

6 Framed correspondences and Milnor-Witt K-theory

After the isomorphism

$$\pi_0(\mathbb{1})_l(k) \simeq H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l}))$$

was obtained in [GP18a, Corollary 11.3] for infinite perfect fields of characteristic $\neq 2$, Neshitov computed the right-hand side for fields of characteristic 0 (see [Nes18, Theorem 9.7]). We recall this result in more detail.

6.1 Neshitov's computation

To study the graded abelian group $H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge *}))$, one first defines a ring structure. As shown in [Nes18, Section 3], the external product structure on $\mathrm{Fr}_*(k)$, defined in (1.1.4), descends to a product:

$$H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet \times X, Y)) \times H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet \times X', Y')) \to H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet \times X \times X', Y \times Y'))$$

for any X, Y, X', $Y' \in \mathrm{Sm}_k$. Taking $X = X' = \mathrm{Spec}\, k$, $Y = \mathbb{G}_m^n, Y' = \mathbb{G}_m^m$, we get a multiplicative structure on the graded abelian group $H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^*))$, which descends to a multiplication on $H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge *}))$, by the same argument as in the proof of Proposition 5.3.3. The main result of [Nes18] is stated as follows.

Theorem 6.1.1 (Neshitov). *Let k be a field of characteristic 0. Then the following graded rings are isomorphic:*

$$H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge *})) \simeq \mathrm{K}_{\geqslant 0}^{MW}(k),$$

where $\mathrm{K}_{\geqslant 0}^{MW}(k)$ means the non-negative part of the Milnor-Witt K-theory of the field k.

6.1.2. By the same argument as in [Nes18, Section 3], the external product (5.1.4) on $\mathrm{Fr}_*^{\mathrm{SL}}(k)$ induces multiplication on $H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge *}))$. The homomorphism

$$\varepsilon_*\colon H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge *})) \longrightarrow H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge *}))$$

becomes then a graded ring homomorphism. We will call ε_* a *unit map*, inspired by Corollary 5.3.4.

The goal of the following sections is to show that the unit map ε_* is an isomorphism, when k is of characteristic 0.

Chapter 3

The unit map of MSL

In this chapter we prove that the unit map ε_*, introduced in 6.1.2, is an isomorphism when k is of characteristic 0, and obtain applications of this result.

7 Surjectivity of the unit map ε_*

Notation 7.0.1. In this subsection we will use the following abbreviations.

- L/k is a finite field extension.
- $s \in X(L)$ and the corresponding L-rational point of $X_L = X \times \operatorname{Spec} L$ are denoted the same way, for $X \in \mathrm{Sm}_k$.
- $c \sim c'$ denotes equality of classes of SL-oriented linear framed correspondences c and c' in $H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, Y))$.

We will prove the surjectivity of the unit map ε_* by means of geometric „moving“ techniques. To do that, we start with preliminary lemmas.

7.1 Auxiliary lemmas

7.1.1. Fix a scheme $Y \in \mathrm{Sm}_k$. In this section we show that for any $c \in \mathbb{Z}\mathrm{F}_n^{\mathrm{SL}}(\operatorname{Spec} k, Y)$ there is $c' \in \mathbb{Z}\mathrm{F}_n(\operatorname{Spec} k, Y)$ such that $c \sim \varepsilon(c')$ in $H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta_k^\bullet, Y))$. This result for $Y = \mathbb{G}_m^l$ for all $l \geqslant 0$ implies the surjectivity of the unit map ε_*, because:

$$\varepsilon_*\big(H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{\wedge l}))\big) = \operatorname{Coker}\big(\oplus_{i=1}^l \varepsilon_*(H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^{l-1}))) \to \varepsilon_*(H_0(\mathbb{Z}\mathrm{F}(\Delta_k^\bullet, \mathbb{G}_m^l)))\big).$$

Since we are working with linear correspondences, we can assume that c is represented by an SL-oriented framed correspondence with a connected support. That is, $\operatorname{supp}(c)_{\mathrm{red}} = \operatorname{Spec} L$, where L is some finite extension of k. We can also assume that c is of level $n > 0$, due to stabilization. We will use the following preliminary lemmas, analogous to [Nes18, Section 2].

Lemma 7.1.2. *Let $c = (U, \phi, g)$ be a correspondence in $Fr_n^{\mathrm{SL}}(\operatorname{Spec} k, Y)$ with support Z such that $Z_{\mathrm{red}} = \operatorname{Spec} L$. Then one can refine U to U', an étale neighborhood of Z such that there is a projection $U' \to \operatorname{Spec} L$.*

Proof. It is enough to show that there is a projection from the henselization $(\mathbb{A}_k^n)_Z^h$, so we can assume $Z = \operatorname{Spec} L$. Since L/k is a separable field extension, the projection $\mathbb{A}_L^n \to \mathbb{A}_k^n$ is an étale neighborhood of Z, so we can consider the composition of projections: $(\mathbb{A}_k^n)_Z^h \to \mathbb{A}_L^n \to \operatorname{Spec} L$. □

Lemma 7.1.3. *Let $c = (U, \phi, g)$ be a correspondence in $Fr_n^{\mathrm{SL}}(\operatorname{Spec} k, Y)$. Assume there is a map $h\colon U \to \operatorname{Spec} L$. Let $A \in \mathrm{SL}(L) = \operatorname{colim}_i \mathrm{SL}_i(L)$ and assume there is given an action of $\mathrm{SL}(L)$ on $\widetilde{\mathcal{T}}_{n,L}$, that induces an endomorphism of the zero section. Denote by $A \cdot \phi$ the composition $U \xrightarrow{\phi \times h} \widetilde{\mathcal{T}}_{n,L} \xrightarrow{A} \widetilde{\mathcal{T}}_{n,L} \xrightarrow{pr} \widetilde{\mathcal{T}}_n$. Then $c \sim c' = (U, A \cdot \phi, g)$.*

Proof. The group $\mathrm{SL}(L)$ is generated by elementary matrices. Thus there is a homotopy $H(t)\colon \mathbb{A}^1 \to \mathrm{SL}$ such that $H(1) = A$ and $H(0) = E$ is the identity matrix. The data $d = (U \times \mathbb{A}^1, H(t) \cdot \phi, g \circ \mathrm{pr}_U)$ define a correspondence in $Fr_n^{\mathrm{SL}}(\mathbb{A}^1, Y)$, because its support $Z \times \mathbb{A}^1$ is finite over $\mathbb{A}^1$. Since $i_0^*(d) = c$ and $i_1^*(d) = (U, A \cdot \phi, g)$, the lemma follows. □

7.2 Proof of the surjectivity

Proposition 7.2.1. *For $n > 0$ let $c = (U, \phi, g) \in \mathrm{Fr}_n^{\mathrm{SL}}(\operatorname{Spec} k, Y)$ be a correspondence with support Z such that $Z_{\mathrm{red}} = \operatorname{Spec} L$. Then there is $c' \in \mathrm{Fr}_n(\operatorname{Spec} k, Y)$ such that $c \sim \varepsilon(c')$.*

Proof. We consider Gr_n and $\widetilde{\mathrm{Gr}}_n$ as embedded in $\mathcal{T}_n$ and $\widetilde{\mathcal{T}}_n$ via the respective zero sections. Denote $p \in \mathrm{Gr}_n$ the distinguished point. Then the distinguished point $q \in \widetilde{\mathrm{Gr}}_n$ is $1 \in \mathbb{G}_m$ in the fiber of $\pi_n\colon \widetilde{\mathrm{Gr}}_n \to \mathrm{Gr}_n$ over p.

The correspondence c has $\phi(U) \subset \widetilde{\mathcal{T}}_n$ and $\phi(Z) = r$, where r is some L-point of $\widetilde{\mathrm{Gr}}_n$. We need to „move“ r to the point $q \in \widetilde{\mathrm{Gr}}_n(k)$ and to „stretch“ $\phi(U)$, so that $\phi(U)$ would be embedded into the fiber of $\widetilde{\mathcal{T}}_n$ over q.

Step 1. First we „move“ r to some point $\hat{r} \in \widetilde{\mathrm{Gr}}_n(L)$ such that $\pi_n(\hat{r}) = p$. Denote $s = \pi_n(r) \in \mathrm{Gr}_n(L)$. The group $\mathrm{SL}_N(L)$ acts transitively on $\mathrm{Gr}(n, N)_L$, so after taking colimit the group $\mathrm{SL}(L)$ acts transitively on $\mathrm{Gr}_{n,L}$. Thus we can choose a matrix $A \in \mathrm{SL}(L)$ such that $A \cdot s = p$ in $\mathrm{Gr}_{n,L}$.

Action of $\mathrm{SL}(L)$ on $\mathrm{Gr}_{n,L}$ lifts to an action on $\mathcal{T}_{n,L}$ and hence on $\widetilde{\mathrm{Gr}}_{n,L}$. Since $\mathcal{T}_{n,L} \to \mathrm{Gr}_{n,L}$ is a colimit of $\mathrm{SL}(L)$-equivariant vector bundles, the action of $\mathrm{SL}(L)$ extends to $\widetilde{\mathcal{T}}_{n,L}$. Hence the matrix A gives an automorphism $\widetilde{\mathcal{T}}_{n,L} \xrightarrow{A} \widetilde{\mathcal{T}}_{n,L}$ where $A \cdot r = \hat{r} \in \widetilde{\mathrm{Gr}}_{n,L}$ and $\pi_{n,L}(\hat{r}) = p$. Note that A induces an automorphism of the zero section of $\widetilde{\mathcal{T}}_{n,L}$. By Lemma 7.1.3 there is an equivalence of correspondences $c \sim c_1 = (U, \phi_1, g)$, where

$$\phi_1(Z) = A \cdot \phi(Z) = \hat{r} \in \widetilde{\mathrm{Gr}}_n(L)$$

for $Z = \mathrm{supp}(c) = \mathrm{supp}(c_1)$.

Step 2. Now we „stretch“ $\phi_1(U)$, so that it would be embedded in the fiber of $\widetilde{\mathcal{T}}_n$ over $\hat{r}$. By definition, ϕ has image in $\widetilde{\mathcal{T}}(n, N)$ for some $N \geqslant n$. The point p has an affine open neighborhood $W \simeq \mathbb{A}^m \subset \mathrm{Gr}(n, N)$ for $m = n(N - n)$, over which $\mathcal{T}(n, N)$ is canonically trivialized, hence so is $\widetilde{\mathrm{Gr}}(n, N)$ [GW10, Corollary 8.15].

We have:

$$\mathcal{T}(n, N) \times_{\mathrm{Gr}(n,N)} W \simeq \mathbb{A}^n \times \mathbb{A}^m;$$
$$\widetilde{\mathrm{Gr}}(n, N) \times_{\mathrm{Gr}(n,N)} W \simeq (\mathbb{A}^1 - 0) \times \mathbb{A}^m;$$
$$\widetilde{\mathcal{T}}(n, N) \times_{\widetilde{\mathrm{Gr}}(n,N)} ((\mathbb{A}^1 - 0) \times \mathbb{A}^m) \simeq \mathbb{A}^n \times (\mathbb{A}^1 - 0) \times \mathbb{A}^m = V.$$

We replace U with its open subscheme $U_1 = \phi_1^{-1}(V) \subset U$ in the correspondence c_1. By Lemma 7.1.2 we can assume that there is a morphism:

$$h\colon U_1 \to \operatorname{Spec} k(Z) \to \operatorname{Spec} L\,.$$

Let p have coordinates $(0, \ldots, 0) \in \mathbb{A}^m$. Denote:

$$\phi_1 = (\rho, \psi, \chi)\colon U_1 \to \mathbb{A}^n \times (\mathbb{A}^1 - 0) \times \mathbb{A}^m.$$

Consider the homotopy: $d = (U_1 \times \mathbb{A}^1, \Phi, g \circ \mathrm{pr}_{U_1}) \in Fr_n^{\mathrm{SL}}(\mathbb{A}^1, Y)$, defined by:

$$\Phi\colon U_1 \times \mathbb{A}^1 \xrightarrow{((\rho,\psi)\circ \mathrm{pr}_{U_1}),(\xi_i)_{i=1}^m} \mathbb{A}^n \times (\mathbb{A}^1 - 0) \times \mathbb{A}^m,$$

where $\xi_i(u,t) = (1-t)\cdot(\chi_i(u))$. Since $\mathrm{supp}(d) = Z \times \mathbb{A}^1$, the correspondence d realizes a homotopy between $i_0^*(d) = (U_1, \phi_1, g)$ and $i_1^*(d) = (U_1, (\rho, \psi, p), g) = c_2$, where p denotes the constant map.

Recall that $\phi_1(Z) = \hat{r}$ where $\hat{r}$ corresponds to $(l,p) \in (\mathbb{A}^1_L - 0) \times \mathbb{A}^m$ for some $l \in L^\times$. Consider the map:

$$\Psi\colon U_1 \times \mathbb{A}^1 \xrightarrow{(1-t)\cdot(\psi,h)(u)+t\cdot l} \mathbb{A}^1_L \xrightarrow{pr} \mathbb{A}^1.$$

Denote $U_2 = \Psi^{-1}(\mathbb{A}^1 - 0) \subset U_1 \times \mathbb{A}^1$, it is an étale neighborhood of $Z \times \mathbb{A}^1$. Consider the homotopy:

$$d' = (U_2, (\rho \circ \mathrm{pr}', \Psi, p), g \circ \mathrm{pr}') \in Fr_n^{\mathrm{SL}}(\mathbb{A}^1, Y),$$

where pr' denotes the projection $U_2 \hookrightarrow U_1 \times \mathbb{A}^1 \to U_1$. We have $\mathrm{supp}(d') = Z \times \mathbb{A}^1$, $i_0^*(d') = c_2$, $i_1^*(d') = (U_3, (\rho, \hat{r}), g) = c_3$. Altogether, we get that $c_1 \sim c_3 = (U_3, \phi_3, g)$ where $\phi_3(U_3) \subset \mathbb{A}^n \times \hat{r}$, which is the fiber of $\widetilde{\mathcal{T}}_n$ over the point $\hat{r} \in (\mathbb{A}^1 - 0) \times \mathbb{A}^m \subset \widetilde{\mathrm{Gr}}_n$.

Step 3. Finally, we „move“ the fiber of $\widetilde{\mathcal{T}}_n$ over $\hat{r}$ to the fiber over $q = (1,p) \in (\mathbb{A}^1 - 0) \times \mathbb{A}^m \subset \widetilde{\mathrm{Gr}}_n$. Both $\hat{r}$ and q are in the fiber of π_n over p. Consider the embedding $p = \mathrm{Gr}(n,n) \subset \mathrm{Gr}(n,n+1) \simeq \mathbb{P}^n$, and note that:

$$\widetilde{\mathrm{Gr}}(n,n+1) \simeq \mathcal{O}_{\mathbb{P}^n}(-1) - z(\mathbb{P}^n) \simeq \mathbb{A}^{n+1} - 0.$$

The smooth scheme $\mathbb{A}^{n+1} - 0$ is $\mathbb{A}^1$-*chain connected* for $n > 0$ (see [AM11, Definition 2.2.2]). That means, there is a finite sequence of $\mathbb{A}^1_L$-paths $\gamma_0, \ldots, \gamma_\ell$ in $\widetilde{\mathrm{Gr}}(n, n+1)$ such that:

$$\gamma_0(0) = \hat{r}; \quad \gamma_\ell(1) = q; \quad \gamma_i(0) = \gamma_{i-1}(1) \text{ for } 1 \leqslant i \leqslant \ell.$$

Each $\mathbb{A}^1_L$-path γ_i will provide a homotopy that „moves“ the fiber of $\widetilde{\mathcal{T}}_n$ over $\gamma_i(0)$ to the fiber over $\gamma_i(1)$.

Let us fix $0 \leqslant i \leqslant \ell$ and consider $\gamma_i\colon \mathbb{A}^1_L \to \widetilde{\mathrm{Gr}}(n,n+1)$. Every vector bundle has a trivialization over affine space, so

$$\widetilde{\mathcal{T}}(n,n+1) \times_{\widetilde{\mathrm{Gr}}(n,n+1)} \mathbb{A}^1_L \simeq \mathbb{A}^n \times \mathbb{A}^1_L.$$

This way we get a homotopy: $\Gamma_i\colon \mathbb{A}^n \times \mathbb{A}^1_L \to \widetilde{\mathcal{T}}(n,n+1)$ where $\Gamma_i(\mathbb{A}^n, t)$ is the fiber of $\widetilde{\mathcal{T}}(n,n+1)$ over $\gamma_i(t) \in \widetilde{\mathrm{Gr}}(n,n+1)(L)$.

Denote $\phi^0 = \mathrm{pr}_{\mathbb{A}^n} \circ \phi_3$, $\bar{U} = U_3$, $c^0 = c_3$. Define $\bar{h}\colon U_3 \hookrightarrow U_2 \xrightarrow{\mathrm{pr}'} U_1 \xrightarrow{h} \mathrm{Spec}\, L$. Consider the correspondence:

$$d_i = (\bar{U} \times \mathbb{A}^1, \Phi_i, g \circ \mathrm{pr}_{\bar{U}}) \in Fr_n^{\mathrm{SL}}(\mathbb{A}^1, Y),$$

given by

$$\Phi_i\colon \bar{U} \times \mathbb{A}^1 \xrightarrow{(\phi^i \circ \mathrm{pr}_1)\times(\mathrm{id}\circ \mathrm{pr}_2)\times(\bar{h}\circ \mathrm{pr}_1)} \mathbb{A}^n \times \mathbb{A}^1 \times \mathrm{Spec}\, L \xrightarrow{\Gamma_i} \widetilde{\mathcal{T}}(n,n+1).$$

We have $\mathrm{supp}(d_i) = Z \times \mathbb{A}^1$, $i_0^*(d_i) = c^i$, and we define by induction:

$$c^{i+1} = i_1^*(d_i) = (\bar{U}, \phi^{i+1}, g).$$

The last correspondence $c^{\ell+1}$ has the properties we wanted: $\phi^{\ell+1}$ maps the support of $c^{\ell+1}$ to q, and $\phi^{\ell+1}(\bar{U})$ is embedded in the fiber of $\widetilde{\mathcal{T}}_n$ over q. Hence c^{r+1} is in the image of the homomorphism ε, and the proposition follows.

$\square$

8 Recollection on Milnor-Witt correspondences

To prove the injectivity of the unit map ε_*, we will provide a left inverse map. To do that, we use the theory of Chow-Witt groups, developed in [Fas07] and [Fas08], and the theory of finite Milnor-Witt correspondences, developed in [CF17a]. Here we recall the main definitions and the construction of the functor $\mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$, given in [DF17, Proposition 2.1.12].

In this section we assume that k is a perfect field, $\operatorname{char} k \neq 2$.

8.1 Chow-Witt groups

Notation 8.1.1. We will use the following conventions.

- $d_X = \dim X$ for an equidimensional scheme $X \in \mathrm{Sm}_k$.
- $\Omega_{X/k}$ is the sheaf of differentials of $X \in \mathrm{Sm}_k$, defined on connected components. We will omit the base field k from notation.
- $\omega_{X/k} = \det \Omega_{X/k}$ is the canonical sheaf of X.
- $\omega_f = \omega_{X/k} \otimes f^*\omega_{Y/k}^{\vee}$ for a morphism of smooth k-schemes $f\colon X \to Y$.
- $\omega_{X\times Y/X} = \omega_f$ for a projection $f\colon X \times Y \to X$.
- $\Lambda_x = \Lambda^n(\mathfrak{m}_x/\mathfrak{m}_x^2)$ for $x \in X^{(n)}$, where $X^{(n)}$ is the set of points of codimension n.

Notation 8.1.2. We will use the Milnor-Witt K-theory and its associated unramified sheaves, as defined in [Mor12, Chapter 2]. Here we recall the notation.

- K_n^M and K_n^{MW} are the n-th Milnor and Milnor-Witt K-theory groups respectively, defined for all fields, $n \in \mathbb{Z}$.
- $\mathbf{K}_n^{MW}$ is the unramified Nisnevich sheaf of Milnor-Witt K-theory on Sm_k.
- GW is the presheaf of Grothendieck-Witt groups on Sm_k (see [Kne77]), its associated Nisnevich sheaf is $\mathbf{K}_0^{MW}$.

The following definition is given in [CF17a, Definition 3.1].

Definition 8.1.3. Let $X \in \mathrm{Sm}_k$, $\mathcal{L}$ — line bundle over X, $Z \subset X$ closed subscheme, $n \in \mathbb{N}$. The *n-th Chow-Witt group* (twisted by $\mathcal{L}$, supported on Z) is defined as:

$$\widetilde{\mathrm{CH}}^n_Z(X, \mathcal{L}) = H^n_Z(X, \mathbf{K}_n^{MW}(\mathcal{L})),$$

where $\mathbf{K}_n^{MW}(\mathcal{L})$ is the Nisnevich sheaf $\mathbf{K}_n^{MW}$, twisted by $\mathcal{L}$ (see [CF17a, Section 1.2] for its construction).

8.1.4. In case of a perfect base field k, Chow-Witt groups can be computed using the Rost-Schmid complex $C^*_{\mathrm{RS}}(X, \mathbf{K}_n^{MW}(\mathcal{L}))$, constructed in [Mor12, Chapter 5]. For a smooth k-scheme X the Rost-Schmid complex provides a flasque resolution of the sheaf $\mathbf{K}_n^{MW}(\mathcal{L})$, restricted to the small Nisnevich site of X:

$$0 \to \mathbf{K}_n^{MW}(\mathcal{L})_X \to \bigoplus_{x\in X^{(0)}} (i_x)_*\mathrm{K}_n^{MW}(k(x), \Lambda_x^{\vee}\otimes\mathcal{L}) \to \bigoplus_{x\in X^{(1)}} (i_x)_*\mathrm{K}_{n-1}^{MW}(k(x), \Lambda_x^{\vee}\otimes\mathcal{L}) \to \dots$$

In a similar way, for a closed subscheme $Z \subset X$ with complement $j\colon U \hookrightarrow X$ one can compute $H^n_Z(X, \mathbf{K}_n^{MW}(\mathcal{L}))$ as cohomology of the Rost-Schmid complex with support, defined as:

$$C^*_{\mathrm{RS},Z}(X, \mathbf{K}_n^{MW}(\mathcal{L})) = \operatorname{Ker}\left(C^*_{\mathrm{RS}}(X, \mathbf{K}_n^{MW}(\mathcal{L})) \xrightarrow{j^*} C^*_{\mathrm{RS}}(U, \mathbf{K}_n^{MW}(\mathcal{L}))\right).$$

It immediately implies that the cohomology only depends on the support as a closed subset:

$$H^n_Z(X, \mathbf{K}^{MW}_n(\mathcal{L})) = H^n_{Z_{\mathrm{red}}}(X, \mathbf{K}^{MW}_n(\mathcal{L})).$$

By construction, d-th term of the Rost-Schmid complex with support is expressed as:

$$C^d_{\mathrm{RS},Z}(X, \mathbf{K}^{MW}_n(\mathcal{L})) = \bigoplus_{x \in X^{(d)} \cap Z} (i_x)_* \mathrm{K}^{MW}_{n-d}(k(x), \Lambda^\vee_x \otimes \mathcal{L}).$$

Let $i\colon Z \hookrightarrow X$ be a closed embedding of codimension c of smooth k-schemes. Then comparison of the corresponding Rost-Schmid complexes gives the *purity isomorphism*

$$\widetilde{\mathrm{CH}}^n_Z(X, \mathcal{L}) \simeq \widetilde{\mathrm{CH}}^{n-c}(Z, i^*\mathcal{L} \otimes \det N_i), \tag{8.1.5}$$

where N_i is the normal bundle of the embedding.

Remark 8.1.6. There is an isomorphism [Mor12, p. 118]:

$$\mathrm{K}^{MW}_i(k(x), \Lambda_x) \simeq \mathrm{K}^{MW}_i(k(x), \Lambda^\vee_x).$$

For $i = 0$ it can be defined as $\langle a \rangle \otimes v \mapsto \langle a^{-1} \rangle \otimes v^\vee = \langle a \rangle \otimes v^\vee$, since $\langle a^2 \rangle = 1$ in $\mathrm{GW}(k(x))$.

Remark 8.1.7. Originally, Chow-Witt groups were defined as:

$$\widetilde{\mathrm{CH}}^n(X, \mathcal{L}) = H^n(C(X, \mathbf{G}^n, \mathcal{L})),$$

where the Gersten-type complex $C(X, \mathbf{G}^n, \mathcal{L})$ is the fiber product of the complexes $C(X, \mathbf{K}^M_n)$ and $C(X, \mathbf{I}^n, \mathcal{L})$ over $C(X, \mathbf{I}^n/\mathbf{I}^{n+1})$ (see [Fas07, Definition 3.21]). These two definitions coincide [Mor12, Theorem 5.47].

8.2 Functoriality of Chow-Witt groups

Let X, Y be smooth k-schemes, $\mathcal{L}$ a line bundle over Y, $Z \subset Y$ and $T \subset X$ closed subsets, $n \in \mathbb{Z}$.

8.2.1. A morphism $f\colon X \to Y$ induces sheaf-theoretic pullback map [AF16, Construction 2.2.4]:

$$f^*\colon \widetilde{\mathrm{CH}}^n_Z(Y, \mathcal{L}) \to \widetilde{\mathrm{CH}}^n_{f^{-1}(Z)}(X, f^*\mathcal{L}).$$

Recall that, for flat morphisms, a pullback map on the corresponding Rost-Schmid complexes was constructed in [Fas08, Corollaire 10.4.3].

Proposition 8.2.2. *When $f\colon X \to Y$ is flat, f^* is induced by flat pullback on Rost-Schmid complexes.*

Proof. The statement for cohomology groups without supports follows from [AF16, Theorem 2.3.4] and [Fas07, Proposition 7.4]. Next, note that the sheaf-theoretic pullback, defined in [AF16, Construction 2.2.4], is immediately generalized to Chow-Witt groups with support, since the construction only uses the induced map $f^*\mathbf{K}^{MW}_n(\mathcal{L}) \to \mathbf{K}^{MW}_n(f^*(\mathcal{L}))$.

The fact that flat pullback on Rost-Schmid complexes induces the sheaf-theoretic pullback on Chow-Witt groups means that the following diagram of complexes of sheaves of abelian groups on Y_{Nis} is commutative:

$$\begin{tikzcd}
\mathbf{K}^{MW}_n(\mathcal{L})_Y \arrow[r] \arrow[d, "f^*"] & C^*_{\mathrm{RS}}(Y, \mathbf{K}^{MW}_n(\mathcal{L})) \arrow[d, "f^*"] \\
f_*\mathbf{K}^{MW}_n(f^*(\mathcal{L}))_X \arrow[r] & f_*C^*_{\mathrm{RS}}(X, \mathbf{K}^{MW}_n(f^*(\mathcal{L}))).
\end{tikzcd}$$

By construction, flat pullback on Rost-Schmid complexes is compatible with restrictions along open immersions. Hence flat pullback induces the correct map on homotopy fibers, that is,

$$f^*\colon C^*_{\mathrm{RS},Z}(Y,\mathbf{K}_n^{MW}(\mathcal{L})) \to f_* C^*_{\mathrm{RS},f^{-1}(Z)}(X,\mathbf{K}_n^{MW}(f^*(\mathcal{L})))$$

induces the sheaf-theoretic pullback on Chow-Witt groups with support. □

8.2.3. If X and Y are equidimensional, a morphism $f\colon X\to Y$ such that $f\big|_T$ is proper, induces the pushforward map [Fas08, Corollaire 10.4.5]:

$$f_*\colon \widetilde{\mathrm{CH}}^n_T(X,\omega_X\otimes f^*\mathcal{L}) \to \widetilde{\mathrm{CH}}^{n+d_Y-d_X}_{f(T)}(Y,\omega_Y\otimes\mathcal{L}).$$

8.2.4. As for Chow groups, there is an external product with support:

$$\times\colon \widetilde{\mathrm{CH}}^n_Z(X,\mathcal{L})\times\widetilde{\mathrm{CH}}^m_W(Y,\mathcal{N}) \to \widetilde{\mathrm{CH}}^{n+m}_{Z\times W}(X\times Y,\mathrm{pr}_1^*(\mathcal{L})\otimes\mathrm{pr}_2^*(\mathcal{N})).$$

The pullback along diagonal morphism $\Delta\colon X\to X\times X$ induces intersection product:

$$\widetilde{\mathrm{CH}}^n_Z(X,\mathcal{L})\times\widetilde{\mathrm{CH}}^m_W(X,\mathcal{N}) \to \widetilde{\mathrm{CH}}^{n+m}_{Z\cap W}(X,\mathcal{L}\otimes\mathcal{N}),$$

denoted by $\cdot = \Delta^*\circ\times$. It makes $\widetilde{\mathrm{CH}}^*_Z(X)$ an associative ring, with the unit given by the pullback of $\langle 1\rangle\in\mathrm{GW}(k)=\mathrm{K}_0^{MW}(k)$ to X (see [Fas07, Section 6]).

8.3 Milnor-Witt correspondences

The folowing definition was introduced in [CF17a, Section 4.1].

Definition 8.3.1. For X,Y smooth k-schemes, Y equidimensional, the abelian group of *finite MW-correspondences* from X to Y is

$$\widetilde{\mathrm{Cor}}_k(X,Y) = \varinjlim_{T\in\mathcal{A}(X,Y)} \widetilde{\mathrm{CH}}^{d_Y}_T(X\times Y,\omega_{X\times Y/X}),$$

where $\mathcal{A}(X,Y)$ is the set of *admissible subsets* of $X\times Y$: closed subsets that are finite and surjective over corresponding irreducible components of X, when endowed with the reduced scheme structure. For $Y=\sqcup_j Y_j$ with equidimensional components Y_j one sets $\widetilde{\mathrm{Cor}}_k(X,Y)=\sqcap_j\widetilde{\mathrm{Cor}}_k(X,Y_j)$.

8.3.2. There is a *category of finite MW-correspondences* $\widetilde{\mathrm{Cor}}_k$, whose objects are smooth k-schemes and morphisms from X to Y are given by $\widetilde{\mathrm{Cor}}_k(X,Y)$ [CF17a, Definition 4.15]. We will write $\widetilde{\mathrm{Cor}}(X,Y)$ for $\widetilde{\mathrm{Cor}}_k(X,Y)$.

Cartesian product of smooth schemes induces a tensor product $\otimes$ on the category $\widetilde{\mathrm{Cor}}_k$, which makes it a symmetric monoidal category [CF17a, Lemma 4.21].

Example 8.3.3. For a smooth k-scheme X of dimension d we have:

$$\widetilde{\mathrm{Cor}}(\mathrm{Spec}\,k,X) = \bigoplus_{x\in X^{(d)}}\widetilde{\mathrm{CH}}^d_{\{x\}}(X,\omega_X) = \bigoplus_{x\in X^{(d)}}\mathrm{GW}(k(x),\omega_{k(x)}).$$

Construction 8.3.4. The functor $\widetilde{\gamma}\colon \mathrm{Sm}_k\to\widetilde{\mathrm{Cor}}_k$ is defined as follows [CF17a, Section 4.3]. On objects one has $\widetilde{\gamma}(X)=X$. Let $f\colon X\to Y$ be a morphism of smooth schemes where Y is equidimensional, and denote $\Gamma_f\subset X\times Y$ the graph of f. We have a pushforward map:

$$(\mathrm{id},f)_*\colon \widetilde{\mathrm{CH}}^0(X,\omega_X)\to\widetilde{\mathrm{CH}}^{d_Y}_{\Gamma_f}(X\times Y,\omega_{X\times Y}).$$

This in turn induces a map:

$$\mathbf{K}_0^{MW}(X)=\widetilde{\mathrm{CH}}^0(X)\to\widetilde{\mathrm{CH}}^{d_Y}_{\Gamma_f}(X\times Y,\omega_{X\times Y/X}).$$

The image of the quadratic form $\langle 1\rangle\in\mathbf{K}_0^{MW}(X)$ is then the finite MW-correspondence $\widetilde{\gamma}(f)\in\widetilde{\mathrm{Cor}}(X,Y)$. For $Y=\sqcup_j Y_j$ with equidimensional components Y_j one has $f=\sqcup_j f_j$, and defines $\widetilde{\gamma}(f)=\sqcap_j\widetilde{\gamma}(f_j)\in\widetilde{\mathrm{Cor}}(X,Y)$.

8.3.5. For smooth schemes the MW-motivic cohomology $H^{p,q}_{MW}(-,\mathbb{Z})$ is defined in [CF17a, Section 6]. The following analogue of the Nesterenko-Suslin-Totaro theorem [MVW06, Theorem 5.1] is proven in [CF17b, Theorem 2.9].

Theorem (Calmès, Fasel)**.** *Let k be a perfect field,* $\operatorname{char} k \neq 2$*, L/k a finitely generated field extension. Then there is a ring isomorphism, natural in L:*

$$\Phi_L\colon \bigoplus_{n\in\mathbb{Z}} \mathrm{K}^{MW}_n(L) \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} H^{n,n}_{MW}(\operatorname{Spec} L,\mathbb{Z}).$$

Remark 8.3.6. Note that for $n \geqslant 0$ the following holds:

$$\begin{aligned} H^{n,n}_{MW}(\operatorname{Spec} L,\mathbb{Z}) &= H_0(\widetilde{\mathrm{Cor}}(\Delta^{\bullet}_L,\mathbb{G}_m^{\wedge n})) \\ &= \operatorname{Coker}(\widetilde{\mathrm{Cor}}(\mathbb{A}^1_L,\mathbb{G}_m^{\wedge n}) \xrightarrow{i_0^*-i_1^*} \widetilde{\mathrm{Cor}}(\operatorname{Spec} L,\mathbb{G}_m^{\wedge n})), \end{aligned}$$

and the multiplication on $H_0(\widetilde{\mathrm{Cor}}(\Delta^{\bullet}_L,\mathbb{G}_m^{\wedge *}))$ is defined by means of the exterior product of Chow-Witt groups, in the same way as in 6.1.

8.4 From framed to Milnor-Witt correspondences

Here we recall the construction of the functor

$$\alpha\colon\ \mathrm{Fr}_*(k) \longrightarrow \widetilde{\mathrm{Cor}}_k,$$

given in [DF17, Proposition 2.1.12]. On objects one has $\alpha(X) = X$. On correspondences of level 0 one defines α as the extension of the functor $\widetilde{\gamma}$ from Construction 8.3.4, by mapping correspondences with empty support to 0. Letting $c = (U,\phi,g) \in \mathrm{Fr}_*(X,Y)$ be a framed correspondence of level $n \geqslant 1$ with support Z, we describe how to associate to it $\alpha(c) \in \widetilde{\mathrm{Cor}}(X,Y)$. We assume that Y is equidimensional, the functor is then extended to general case in the same way as in Construction 8.3.4.

8.4.1. Denote $\phi = (\phi_1,\dots,\phi_n)$, where $\phi_i \in \mathcal{O}(U)$, and let $|\phi_i|$ be the vanishing locus of ϕ_i, then $Z = |\phi_1| \cap \dots \cap |\phi_n|$ as a set. Each $\phi_i \in k(U)^\times$ defines an element of $\mathrm{K}^{MW}_1(k(U))$. For each i the residue map

$$\partial\colon \mathrm{K}^{MW}_1(k(U)) \longrightarrow \bigoplus_{x\in U^{(1)}} \mathrm{K}^{MW}_0(k(x),\Lambda^\vee_x)$$

provides an element $\partial(\phi_i)$ supported on $|\phi_i|$, so by localization sequence for cohomology with support, $\partial(\phi_i)$ defines a cycle $Z(\phi_i) \in H^1_{|\phi_i|}(U,\mathrm{K}^{MW}_1)$. Using the intersection product with support, we get an element:

$$Z(\phi) = Z(\phi_1)\cdot\ldots\cdot Z(\phi_n) \in H^n_Z(U,\mathrm{K}^{MW}_n).$$

As part of the data of c, there is an étale map $p\colon U \to \mathbb{A}^n_X$. It induces an isomorphism $p^*\omega_{\mathbb{A}^n_X} \simeq \omega_U$. Denote the projection by $q\colon \mathbb{A}^n_X \to X$. On $\mathbb{A}^n_X = \operatorname{Spec}_X \mathcal{O}_X[t_1,\dots,t_n]$, the sheaf $\omega_{\mathbb{A}^n_X} \otimes q^*\omega^\vee_X$ has the canonical generator $dt_1\wedge\ldots\wedge dt_n$, giving the canonical isomorphism

$$\mathcal{O}_{\mathbb{A}^n_X} \simeq \omega_{\mathbb{A}^n_X} \otimes q^*\omega^\vee_X.$$

We get the canonical isomorphism:

$$\mathcal{O}_U \simeq p^*(\mathcal{O}_{\mathbb{A}^n_X}) \simeq p^*(\omega_{\mathbb{A}^n_X}\otimes q^*\omega^\vee_X) \simeq \omega_U \otimes (qp)^*\omega^\vee_X.$$

Thus we can consider $Z(\phi)$ as an element of $\widetilde{\mathrm{CH}}^n_Z(U,\,\omega_U\otimes(qp)^*\omega^\vee_X)$.

The map $(qp,g)\colon U \to X\times Y$ sends Z to a closed subscheme T, which is finite and surjective over X by [MVW06, Lemma 1.4]. Since Z is finite over X, the restriction $(qp,g)\big|_Z$ is a finite morphism. We have then the pushforward morphism:

$$(qp,g)_*\colon\ \widetilde{\mathrm{CH}}^n_Z(U,\,\omega_U\otimes(qp)^*\omega^\vee_X) \longrightarrow \widetilde{\mathrm{CH}}^{d_Y}_T(X\times Y,\,\omega_{X\times Y/X}).$$

The image $(qp,g)_*(Z(\phi))$ is the finite MW-correspondence $\alpha(c) \in \widetilde{\mathrm{Cor}}(X,Y)$.

8.4.2. The functor α is naturally extended to linear framed correspondences. By [DF17, Example 2.1.11], for a suspension morphism σ_Y one has:

$$\alpha(\sigma_Y) = Id \in \widetilde{\mathrm{Cor}}(Y, Y).$$

Altogether, for any $X, Y \in \mathrm{Sm}_k$ we obtain a homomorphism of abelian groups:

$$\alpha\colon\ \mathbb{Z}\mathrm{F}(X, Y) \longrightarrow \widetilde{\mathrm{Cor}}(X, Y),$$

inducing a homomorphism of simplicial abelian groups

$$\alpha_l\colon\ \mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l}) \to \widetilde{\mathrm{Cor}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l}).$$

For each $l \geqslant 0$ the homomorphism α_l factors through the zeroth homology and induces:

$$\alpha_*\colon H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l})) \to H_0(\widetilde{\mathrm{Cor}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge l})) = H^{l,l}_{MW}(\mathrm{Spec}\, k, \mathbb{Z}). \tag{8.4.3}$$

Lemma 8.4.4. *The map (8.4.3) induces a ring homomorphism:*

$$\alpha_*\colon H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \longrightarrow H_0(\widetilde{\mathrm{Cor}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})).$$

Proof. We have to check that for correspondences $c = (U, \phi, g) \in \mathrm{Fr}_n(X, Y)$ and $d = (V, \psi, h) \in \mathrm{Fr}_m(X_1, Y_1)$ of levels $n, m \geqslant 1$ with non-empty supports Z and Z' holds the following:

$$\alpha(c \times d) = \alpha(c) \otimes \alpha(d).$$

First we show that the construction of $Z(\phi)$ respects the exterior product. Since the construction is multiplicative, we can assume that $n = m = 1$. The correspondence $c \times d$ has the framing $\chi = (\phi \circ \mathrm{pr}_U, \psi \circ \mathrm{pr}_V)\colon U \times V \longrightarrow \mathbb{A}^2$, and is supported on $Z \times Z'$. Then in $\widetilde{\mathrm{CH}}^2_{Z\times Z'}(U \times V)$ we have:

$$\begin{aligned} Z(\chi) = Z(\phi \circ \mathrm{pr}_U) \cdot Z(\psi \circ \mathrm{pr}_V) = [\partial[\phi \circ \mathrm{pr}_U]] \cdot [\partial[\psi \circ \mathrm{pr}_V]] = \\ [\mathrm{pr}_U^* \,\partial[\phi]] \cdot [\mathrm{pr}_V^* \,\partial[\psi]] = [\partial[\phi] \times 1_V] \cdot [1_U \times \partial[\psi]] = [\partial[\phi] \times \partial[\psi]] = Z(\phi) \times Z(\psi). \end{aligned}$$

The proper pushforward of Chow-Witt groups commutes with exterior product, hence the claim follows.

□

9 Injectivity of the unit map ε_*

In this section we assume that $\mathrm{char}\, k = 0$.

9.0.1. To construct a left inverse map for ε_*, we will consider the following diagram:

$$\begin{tikzcd} & H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \arrow[r, "\varepsilon_*"] \arrow[d, "\alpha_*"] & H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \arrow[dl, dashed, "\alpha_*^{\mathrm{SL}}"] \\ \bigoplus_{l\geqslant 0} \mathrm{K}^{MW}_l(k) \arrow[ur, "\Psi", "\sim"'] \arrow[r, "\Phi", "\sim"'] & \bigoplus_{l\geqslant 0} H^{l,l}_{MW}(\mathrm{Spec}\, k, \mathbb{Z}) & \end{tikzcd} \tag{9.0.2}$$

Here the isomorphisms Ψ and Φ are the ones constructed in [Nes18, Section 8.3] and [CF17b, Theorem 1.8] respectively. We recall how they are constructed on generators $\langle a\rangle \in \mathrm{GW}(k)$ and $[a] \in \mathrm{K}^{MW}_1(k)$ for $a \in k^\times$.

Denote $\mathbb{A}^1 = \mathrm{Spec}\, k[x]$ and $\mathbb{G}_m = \mathrm{Spec}\, k[x, x^{-1}]$. Then the image $\Psi(\langle a\rangle)$ is the class of the correspondence $(\mathbb{A}^1, ax, \mathrm{pr}_k) \in \mathrm{Fr}_1(\mathrm{Spec}\, k, \mathrm{Spec}\, k)$ in $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathrm{Spec}\, k))$, and $\Psi([a])$ is the class of the correspondence $(\mathbb{G}_m, x - a, \mathrm{id}) \in \mathrm{Fr}_1(\mathrm{Spec}\, k, \mathbb{G}_m)$ in $H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge 1}))$. Meanwhile $\Phi(\langle a\rangle) = \langle a\rangle \in \mathrm{GW}(k) = H^{0,0}_{MW}(\mathrm{Spec}\, k, \mathbb{Z})$ and $\Phi([a])$ is the class of $\widetilde{\gamma}(\mathrm{Spec}\, k \xrightarrow{a} \mathbb{G}_m) \in \widetilde{\mathrm{Cor}}(\mathrm{Spec}\, k, \mathbb{G}_m)$ in $H^{1,1}_{MW}(\mathrm{Spec}\, k, \mathbb{Z})$.

For our purposes, it is enough to:

- check that $\Psi \circ \Phi^{-1} \circ \alpha_* = \mathrm{id}$;
- extend α_* to SL-oriented framed correspondences, so that $\alpha_*^{\mathrm{SL}} \circ \varepsilon_* = \alpha_*$.

Then $\Psi \circ \Phi^{-1} \circ \alpha_*^{\mathrm{SL}}$ is a left inverse for ε_*, hence ε_* is injective.

Lemma 9.0.3. *With the notations of the diagram (9.0.2) one has:*

$$\Psi \circ \Phi^{-1} \circ \alpha_* = \mathrm{id}\,.$$

Proof. Equivalently, we need to show that $\Phi = \alpha_* \circ \Psi$, since both Ψ and Φ are isomorphisms. Since all these maps are ring homomorphisms, we only need to check that the equation holds for the generators of $\mathrm{K}^{MW}_{\geqslant 0}(k)$ as a $\mathbb{Z}$-algebra. That is, we need to check it for $\langle a\rangle \in \mathrm{GW}(k)$ and $[a] \in \mathrm{K}^{MW}_1(k)$, where $a \in k^\times$ (see [Nes18, Section 8.3]).

1) For $\langle a\rangle \in \mathrm{GW}(k)$ we have to compute $(\alpha_* \circ \Psi)\langle a\rangle = [\alpha(\mathbb{A}^1, ax, \mathrm{pr}_k)]$. Under the residue map

$$\partial\colon \mathrm{K}^{MW}_1(k(x)) \to \bigoplus_{t\in\mathbb{A}^{1(1)}} \mathrm{K}^{MW}_0(k(t), (\mathfrak{m}_t/\mathfrak{m}_t^2)^\vee)$$

we have the following image (see [Mor12, Remark 3.21]):

$$\partial[ax] = 1 \otimes \overline{ax}^\vee = \langle a\rangle \otimes \overline{x}^\vee \in \mathrm{K}^{MW}_0(k, (\mathfrak{m}_0/\mathfrak{m}_0^2)^\vee).$$

After choosing the canonical orientation of $\mathbb{A}^1$, $\langle a\rangle \otimes \overline{x}^\vee$ corresponds to the class of $\langle a\rangle \in \widetilde{\mathrm{CH}}^1_0(\mathbb{A}^1, \omega_{\mathbb{A}^1})$. The pushfoward of $\langle a\rangle$ under

$$(\mathrm{pr}_k)_*\colon\ \widetilde{\mathrm{CH}}^1_0(\mathbb{A}^1, \omega_{\mathbb{A}^1}) \to \widetilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k)$$

is the class of $\langle a\rangle$, hence $\alpha(\mathbb{A}^1, ax, \mathrm{pr}_k) = \langle a\rangle \in \mathrm{GW}(k)$, coinciding with $\Phi(\langle a\rangle)$.

2) For $[a] \in \mathrm{K}^{MW}_1(k)$ we have to compute $(\alpha_* \circ \Psi)[a] = [\alpha(\mathbb{G}_m, x-a, \mathrm{id})]$. The residue map:

$$\partial\colon \mathrm{K}^{MW}_1(k(x)) \to \bigoplus_{t\in\mathbb{G}_m^{(1)}} \mathrm{K}^{MW}_0(k(t), (\mathfrak{m}_t/\mathfrak{m}_t^2)^\vee)$$

gives

$$\partial[x-a] = 1 \otimes \overline{x-a}^\vee \in \mathrm{K}^{MW}_0(k, (\mathfrak{m}_a/\mathfrak{m}_a^2)^\vee),$$

where a is considered as a k-point of $\mathbb{G}_m$. By construction of the functor α, one applies then the isomorphism $\widetilde{\mathrm{CH}}^1_a(\mathbb{G}_m) \simeq \widetilde{\mathrm{CH}}^1_a(\mathbb{G}_m, \omega_{\mathbb{G}_m})$, induced by the trivialization $\omega_{\mathbb{G}_m} \simeq \langle dx\rangle$. This way, the class of $\partial[x-a]$ is given by

$$1 \otimes d(x-a)^\vee \otimes dx = \langle 1\rangle \in \widetilde{\mathrm{CH}}^1_a(\mathbb{G}_m, \omega_{\mathbb{G}_m}),$$

since $dx = d(x-a)$. The pushforward of $\langle 1\rangle \in \widetilde{\mathrm{CH}}^1_a(\mathbb{G}_m, \omega_{\mathbb{G}_m})$ under $(\mathrm{pr}_k, \mathrm{id})_*$ is the same, hence

$$\alpha(\mathbb{G}_m, x-a, \mathrm{id}) = \langle 1\rangle \in \widetilde{\mathrm{CH}}^1_a(\mathbb{G}_m, \omega_{\mathbb{G}_m}) \subset \widetilde{\mathrm{Cor}}(\operatorname{Spec} k, \mathbb{G}_m).$$

On the other hand, $\Phi([a])$ is the class of $\widetilde{\gamma}(\operatorname{Spec} k \xrightarrow{a} \mathbb{G}_m) \in \widetilde{\mathrm{Cor}}(\operatorname{Spec} k, \mathbb{G}_m)$ in $H^{1,1}_{MW}(\operatorname{Spec} k, \mathbb{Z})$. As follows from Construction 8.3.4,

$$\widetilde{\gamma}(\operatorname{Spec} k \xrightarrow{a} \mathbb{G}_m) = \langle 1\rangle \in \widetilde{\mathrm{CH}}^1_a(\mathbb{G}_m, \omega_{\mathbb{G}_m}).$$

□

9.1 Oriented Thom class

To extend the functor α to SL-oriented framed correspondences, we will use the construction of the oriented Thom class of a vector bundle (see [Lev18, Definition 3.4]).

9.1.1. Let $\xi\colon E \to X$ be a vector bundle of rank r over a smooth k-scheme X. The purity isomorphism (8.1.5) for the zero section $X \hookrightarrow E$ and the twist by the line bundle $\xi^* \det E^\vee$ on E gives the canonical isomorphism:

$$\widetilde{\mathrm{CH}}^0(X) \simeq \widetilde{\mathrm{CH}}^r_X(E,\, \xi^* \det E^\vee).$$

Definition 9.1.2. The *oriented Thom class* of $\xi\colon E \to X$ is defined as the unique class

$$t_\xi \in \widetilde{\mathrm{CH}}^r_X(E,\, \xi^* \det E^\vee),$$

that corresponds under the purity isomorphism to the class $\langle 1 \rangle \in \widetilde{\mathrm{CH}}^0(X)$.

9.1.3. As explained in [Lev18, Remark 3.6], t_ξ can be alternatively understood as follows. A vector bundle $\xi\colon E \to X$ induces the tautological section $t\colon \mathcal{O}_E \to \xi^* E$, so we get the Koszul complex:

$$\mathrm{Kos}(E) = \Lambda^r \xi^* E^\vee \to \ldots \to \xi^* E^\vee \xrightarrow{t^\vee} \mathcal{O}_E,$$

which is a locally free resolution of $\mathcal{O}_{z(X)}$. There is a symmetric isomorphism:

$$\lambda_E\colon \mathrm{Kos}(E) \longrightarrow \mathcal{H}om_{\mathcal{O}_E}(\mathrm{Kos}(E), \xi^* \det E^\vee)[r],$$

given in degree i by the canonical isomorphism:

$$\Lambda^i \xi^* E^\vee \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_E}(\Lambda^{r-i} \xi^* E^\vee, \xi^* \det E^\vee).$$

Consider the triangulated category $D^{perf}_{z(X)}(E)$ of perfect complexes on E supported in the zero section $z(X)$, with duality:

$$D_E = \mathcal{H}om_{\mathcal{O}_E}(-, \xi^* \det E^\vee)[r]; \quad can\colon Id \xrightarrow{\sim} D_E^2.$$

Then $(\mathrm{Kos}(E), \lambda_E)$ is an element of the Grothendieck-Witt group, defined in [Wal03, Section 2]:

$$\begin{gathered}\mathrm{GW}_0(D^{perf}_{z(X)}(E), D_E, can) \xrightarrow{\sim} \mathrm{GW}(X) \\ (\mathrm{Kos}(E), \lambda_E) \mapsto \langle 1 \rangle.\end{gathered}$$

Since $\widetilde{\mathrm{CH}}^0$ is the Nisnevich sheafification of the presheaf GW on Sm_k, the element $(\mathrm{Kos}(E), \lambda_E)$ provides a global section of $\widetilde{\mathrm{CH}}^0$ on X. Hence $(\mathrm{Kos}(E), \lambda_E)$ corresponds to an element in $\widetilde{\mathrm{CH}}^r_X(E,\, \xi^* \det E^\vee)$, and this element is the oriented Thom class t_ξ (see also [FS09, Section 2.4]).

9.2 Construction of α_*^{SL}

Recall that in the construction of the functor $\alpha\colon \mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$ for a framed correspondence $c = (U, \phi, g) \in Fr_n(X, Y)$ with the support Z we defined a cohomology class $Z(\phi) \in \widetilde{\mathrm{CH}}^n_Z(U)$. Here we interpret $Z(\phi)$ in terms of oriented Thom classes.

Proposition 9.2.1. *Let $X, Y \in \mathrm{Sm}_k$ and let $c = (U, \phi, g) \in Fr_n(X, Y)$ be a framed correspondence with non-empty support Z, $n \geqslant 1$. Assume that ϕ is a flat morphism. For the trivial vector bundle $\mathbb{A}^n \to \operatorname{Spec} k$ denote its oriented Thom class by $t_n \in \widetilde{\mathrm{CH}}^n_0(\mathbb{A}^n)$. Then the pullback map $\phi^*\colon \widetilde{\mathrm{CH}}^n_0(\mathbb{A}^n) \to \widetilde{\mathrm{CH}}^n_Z(U)$ gives:*

$$Z(\phi) = \phi^*(t_n).$$

Proof. It is enough to show that $Z(\phi_i) = \phi_i^*(t_1)$, because $Z(\phi) = Z(\phi_1) \cdot \ldots \cdot Z(\phi_n)$ by construction, the oriented Thom class is multiplicative with respect to direct sum of vector bundles by [Lev18, Proposition 3.7(2)], and pullback respects product of Chow-Witt groups by [Fas07, Propositions 7.2 and 7.4].

Assume that $U = \mathbb{A}^1$ and $\phi = \mathrm{id}\colon U \to \mathbb{A}^1$. Then $Z(\mathrm{id})$ is the class of $\langle 1\rangle \in \mathrm{GW}(k)$ in $\widetilde{\mathrm{CH}}^1_0(\mathbb{A}^1)$. The oriented Thom class of $\mathbb{A}^1 \to \operatorname{Spec} k$ is also the class of $\langle 1\rangle \in \widetilde{\mathrm{CH}}^1_0(\mathbb{A}^1)$, and id^* is the identity homomorphism, so $Z(\mathrm{id})$ and $\mathrm{id}^*(t_1)$ coincide.

Now let $\phi\colon U \to \mathbb{A}^n$ be an arbitrary flat morphism, then $\phi_i\colon U \to \mathbb{A}^1$ is flat as a composition of ϕ with a projection. We need to show that $Z(\phi_i) = \phi_i^* Z(\mathrm{id})$. By Proposition 8.2.2, for flat ϕ_i the pullback of Chow-Witt groups ϕ_i^* can be computed using Rost-Schmid complexes as flasque resolutions. Since their differentials are given by residue maps ∂, we obtain:

$$\phi_i^* Z(\mathrm{id}) = \phi_i^*[\partial[\mathrm{id}]] = [\partial(\phi_i^*[\mathrm{id}])],$$

where $[\mathrm{id}] \in \mathrm{K}_1^{MW}(k(\mathbb{A}^1))$ is the image of the generator $[\mathrm{id}] \in \mathrm{K}_1^M(k(\mathbb{A}^1))$.

The pullback map ϕ_i^* on the Rost-Schmid complex for Milnor K-theory is defined as follows (see [Fas08, p. 11]). Denote $u \in U$ and $\eta \in \mathbb{A}^1$ the generic points, let U_η^i be the fiber of ϕ_i over η, and $\psi_i\colon k(\eta) \to k(u)$ the induced map of function fields. By definition, the pullback homomorphism is:

$$\phi_i^* = l_i \cdot \psi_{i,*}\colon \mathrm{K}_*^M(k(\eta)) \to \mathrm{K}_*^M(k(u)),$$

where $l_i = l(\mathcal{O}_{U_\eta^i,u})$ is the multiplicity of U_η^i at u, and $\psi_{i,*}$ is the natural map, induced by ψ_i. In our case, $l_i = 1$, because:

$$U_\eta^i = \varprojlim_{\text{closed } x\in\mathbb{A}^1} U - \phi_i^{-1}(x) = \varprojlim V,$$

where $V \subset U$ are open subschemes, and $\mathcal{O}_{U_\eta^i,u} = \varprojlim k(V) = k(U)$. Thereby,

$$[\partial(\phi_i^*[\mathrm{id}])] = [\partial(l_i \cdot [\phi_i])] = [\partial[\phi_i]] = Z(\phi_i),$$

and the proposition follows. □

9.2.2. We reduce to the case of a flat framing ϕ by means of the following lemma.

Lemma 9.2.3. *Let $c = (U, \phi, g) \in \mathrm{Fr}_n(X, Y)$ be a framed correspondence with non-empty support Z. Then there is an étale neighborhood $U' \subset U$ of Z such that $\phi\big|_{U'}$ is flat.*

Proof. Observe that ϕ is flat at points of Z. Indeed, for $z \in Z$ we have the induced local homomorphism $\phi_z^\sharp\colon \mathcal{O}_{\mathbb{A}^n,0} \to \mathcal{O}_{U,z}$. The sequence $(\phi_1, \ldots, \phi_n)$ is regular in $\mathcal{O}_U$, so

$$\dim \mathcal{O}_{U,z} - \dim \mathcal{O}_{\mathbb{A}^n,0} = \dim \mathcal{O}_{U,z}/\mathfrak{m}\mathcal{O}_{U,z},$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,0}$. Since $\mathcal{O}_{\mathbb{A}^n,0}$ is regular and $\mathcal{O}_{U,z}$ is a Cohen-Macaulay ring, by the miracle flatness [Stacks, Tag 00R4] we get that $\phi_z^\sharp$ is a flat homomorphism.

The property of a morphism being flat is an open condition on the source [Stacks, Tag 0250], so we can choose an open neighborhood $U' \subset U$ of Z, where ϕ is flat. □

9.2.4. Using that, we define for $X, Y \in \mathrm{Sm}_k$ the map

$$\alpha^{\mathrm{SL}}\colon \mathrm{Fr}_*^{\mathrm{SL}}(X, Y) \to \widetilde{\mathrm{Cor}}(X, Y)$$

as follows. For correspondences of level 0 we set $\alpha^{\mathrm{SL}} = \alpha$. Let $c = (U, \phi, g) \in Fr_n^{\mathrm{SL}}(X, Y)$ have the framing represented by a morphism $\phi \colon U \to \widetilde{\mathcal{T}}(n, N)$, and a non-empty support Z. Denote by $\xi_N \colon \widetilde{\mathcal{T}}(n, N) \to \widetilde{\mathrm{Gr}}(n, N)$ the projection and recall that there is a trivialization, defined in (4.2.2):

$$\lambda_{n,N} \colon \mathcal{O}_{\widetilde{\mathrm{Gr}}(n,N)} \xrightarrow{\sim} \det \widetilde{\mathcal{T}}(n, N).$$

This trivialization induces a trivialization of the line bundle $\xi_N^* \det \widetilde{\mathcal{T}}(n, N)^\vee \to \widetilde{\mathcal{T}}(n, N)$. Hence the oriented Thom class of ξ_N is an element of the Chow-Witt group with trivial twist:

$$t_{\xi_N} \in \widetilde{\mathrm{CH}}^n_{\widetilde{\mathrm{Gr}}(n,N)}(\widetilde{\mathcal{T}}(n, N)).$$

As stated in 8.2.1, there is an induced pullback map

$$\phi^* \colon \widetilde{\mathrm{CH}}^n_{\widetilde{\mathrm{Gr}}(n,N)}(\widetilde{\mathcal{T}}(n, N)) \longrightarrow \widetilde{\mathrm{CH}}^n_Z(U),$$

and we define

$$Z(\phi) = \phi^*(t_{\xi_N}) \in \widetilde{\mathrm{CH}}^n_Z(U).$$

The cohomology class $Z(\phi)$ does not depend on the choice of N, because a composition with the canonical embedding $i_{N,M} \colon \widetilde{\mathcal{T}}(n, N) \hookrightarrow \widetilde{\mathcal{T}}(n, N + M)$ induces the equality:

$$(i_{N,M})^*(t_{\xi_{N+M}}) = t_{\xi_N}$$

by [Lev18, Proposition 3.7(1)]. Applying ϕ^* gives us:

$$(i_{N,M} \circ \phi)^*(t_{\xi_{N+M}}) = \phi^*((i_{N,M})^*(t_{\xi_{N+M}})) = \phi^*(t_{\xi_N}),$$

where first equality follows from [AF16, Theorem 2.1.3].

Finally, we set

$$\alpha^{\mathrm{SL}}(c) = (qp, g)_*(Z(\phi)) \in \widetilde{\mathrm{Cor}}(X, Y),$$

where $p \colon U \to \mathbb{A}^n_X$ is the étale neighborhood of Z and $q \colon \mathbb{A}^n_X \to X$ is the projection.

9.2.5. The map α^{SL} factors through stabilization, and we obtain the induced homomorphism on the zeroth homology:

$$\alpha^{\mathrm{SL}}_* \colon H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \longrightarrow H_0(\widetilde{\mathrm{Cor}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})).$$

In case $c = \varepsilon_n(c')$ was in the image of the embedding $\varepsilon_n \colon Fr_n(X, Y) \hookrightarrow Fr_n^{\mathrm{SL}}(X, Y)$, we have:

$$\alpha^{\mathrm{SL}}(c) = \alpha^{\mathrm{SL}}(\varepsilon_n(c')) = \alpha(c'),$$

because we can assume that its framing ϕ is flat by Lemma 9.2.3, and then apply Proposition 9.2.1. Hence we deduce that $\alpha_* = \alpha^{\mathrm{SL}}_* \circ \varepsilon_*$, and the diagram (9.0.2) commutes. Together with Proposition 7.2.1 we have proved the following theorem.

Theorem 9.2.6. *Let k be a field of characteristic 0. Then the map ε_* is a graded ring isomorphism:*

$$\varepsilon_* \colon H_0(\mathbb{Z}\mathrm{F}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})) \xrightarrow{\sim} H_0(\mathbb{Z}\mathrm{F}^{\mathrm{SL}}(\Delta^\bullet_k, \mathbb{G}_m^{\wedge *})).$$

10 Applications

In this section we assume that $\operatorname{char} k = 0$. We show that our work yields the computation of the homotopy module $\underline{\pi}_0(\mathrm{MSL})_*$ and provides special linear orientations for Chow-Witt groups and MW-motivic cohomology.

10.1 Homotopy module of MSL

10.1.1. Recall that Voevodsky has defined homotopy t-structure on $\mathrm{SH}(k)$, whose heart $\mathrm{SH}^{\heartsuit}(k)$ is equivalent to the category of homotopy modules $\Pi_*(k)$ (see [Mor03, Section 5.2]). A *homotopy module* is a sequence of strictly $\mathbb{A}^1$-invariant Nisnevich sheaves of abelian groups $\{\mathcal{E}_i\}_{i\in\mathbb{Z}}$ with isomorphisms $\mathcal{E}_i \xrightarrow{\sim} (\mathcal{E}_{i+1})_{-1}$, and a morphism of homotopy modules is a sequence of maps of sheaves, compatible with the isomorphisms. Here $\mathcal{E}_{-1}$ denotes the *contraction* of $\mathcal{E}$:

$$\mathcal{E}_{-1}(X) = \mathrm{Coker}(\mathcal{E}(X) \xrightarrow{j_*} \mathcal{E}(X \times \mathbb{G}_m)),$$

where j is the embedding at $1 \in \mathbb{G}_m$. The functor

$$\mathrm{F}\colon \mathrm{SH}(k) \longrightarrow \Pi_*(k); \quad E \mapsto \underline{\pi}_0(E)_*$$

induces an equivalence after restriction to $\mathrm{SH}^{\heartsuit}(k)$. Its quasi-inverse functor is denoted by $\mathrm{H}\colon \Pi_*(k) \to \mathrm{SH}^{\heartsuit}(k)$.

Proposition 10.1.2. *Let k be a field of characteristic 0. Then the unit map $e\colon \mathbb{1} \to$ MSL induces an isomorphism of the corresponding homotopy modules:*

$$e_*\colon \underline{\pi}_0(\mathbb{1})_* \xrightarrow{\sim} \underline{\pi}_0(\mathrm{MSL})_*.$$

Proof. As follows from Corollary 5.3.4 together with Theorem 9.2.6, for any finitely field extension L/k and $l \geqslant 0$ the unit map induces isomorphism:

$$e_l(L)\colon \underline{\pi}_0(\mathbb{1}_k)_l(L) \simeq [\mathbb{1}_L, \Sigma^l_{\mathbb{G}_m}\mathbb{1}_L]_{\mathrm{SH}(L)} \xrightarrow{\sim} [\mathbb{1}_L, \Sigma^l_{\mathbb{G}_m}\mathrm{MSL}_L]_{\mathrm{SH}(L)} \simeq \underline{\pi}_0(\mathrm{MSL}_k)_l(L).$$

The first and last isomorphisms follow from the fact that (suspended) spectra $\mathbb{1}$ and MSL are *absolute* in the sense of [Dég18, Definition 1.2.1]. Since $p\colon \mathrm{Spec}\, L \to \mathrm{Spec}\, k$ is an essentially smooth k-scheme, one can express it as a cofiltered limit of smooth k-schemes $p_\alpha\colon X_\alpha \to \mathrm{Spec}\, k$. Then for any absolute spectrum E one has:

$$\begin{aligned}[\mathbb{1}_L, E_L]_{\mathrm{SH}(L)} &\simeq [\mathbb{1}_L, p^*(E_k)]_{\mathrm{SH}(L)} \simeq \operatorname*{colim}_\alpha [\mathbb{1}_{X_\alpha}, p^*_\alpha(E_k)]_{\mathrm{SH}(X_\alpha)} \simeq \\ &\operatorname*{colim}_\alpha [p_{\alpha,\sharp}(\mathbb{1}_{X_\alpha}), E_k]_{\mathrm{SH}(k)} = \operatorname*{colim}_\alpha \pi_0(E_k)_0(X_\alpha) \simeq \pi_0(E_k)_0(L) = \underline{\pi}_0(E_k)_0(L).\end{aligned}$$

Here the second isomorphism is the content of [Hoy15, Lemma A.7(1)], and the rest follows from definitions.

Since $\mathrm{SH}^{\heartsuit}(k)$ is an abelian category, the maps e_l of strictly $\mathbb{A}^1$-invariant Nisnevich sheaves have kernels and cokernels which are also strictly $\mathbb{A}^1$-invariant sheaves, hence unramified [Mor12, Example 2.3]. In case $l \geqslant 0$, we have shown that $\mathrm{Ker}\, e_l(L) = \mathrm{Coker}\, e_l(L) = 0$ for all finitely generated field extensions L/k, which implies that $\mathrm{Ker}\, e_l$ and $\mathrm{Coker}\, e_l$ are zero sheaves. Hence e_* is an isomorphism of the sheaves $\underline{\pi}_0(-)_l$, for $l \geqslant 0$.

Finally, by definition of the morphism of homotopy modules, e_* is compatible with contraction isomorphisms, so the fact that e_l are isomorphisms for all $l \geqslant 0$ implies that e_* is an isomorphism on each level $l \in \mathbb{Z}$. □

10.2 Special linear orientations

10.2.1. Panin and Walter investigated the notion of *special linear orientation* of a bigraded ring cohomology theory on the category Sm_k: it is an extra structure that encodes the data of compatible Thom isomorphisms for vector bundles with trivialized determinants over smooth schemes (see [PW10, Definition 5.1]). A homomorphism of commutative monoids $\mathrm{MSL} \to A$ in $\mathrm{SH}(k)$ induces a special linear orientation of the cohomology theory $A^{*,*}$ [PW10, Theorem 5.5]. We call such homomorphism an SL-*orientation* of the ring spectrum A.

10.2.2. Proposition 10.1.2 gives rise to SL-orientations.

Corollary 10.2.3. *The bigraded ring cohomology theory $H^*(-, \mathbf{K}_*^{MW})$ carries a unique special linear orientation. In this sense, Chow-Witt groups are uniquely specially linearly oriented.*

Proof. The cohomology theory $H^*(-, \mathbf{K}_*^{MW})$ is represented in $\mathrm{SH}(k)$ by the spectrum $\mathrm{H}\underline{\pi}_0(\mathbb{1})_* \in \mathrm{SH}(k)^\heartsuit$. The sequence of ring homomorphisms

$$\mathrm{MSL} \xrightarrow{\pi_0} \mathrm{H}\underline{\pi}_0(\mathrm{MSL})_* \xrightarrow{e_*^{-1}} \mathrm{H}\underline{\pi}_0(\mathbb{1})_*$$

provides the unique SL-orientation of the spectrum $\mathrm{H}\underline{\pi}_0(\mathbb{1})_*$, and hence of the bigraded cohomology theory $H^*(-, \mathbf{K}_*^{MW})$. $\square$

Remark 10.2.4. The system of compatible Thom isomorphisms for $H^*(-, \mathbf{K}_*^{MW})$ is constructed in [AH11, Theorem 4.2.7].

10.2.5. In a similar manner, we obtain an SL-orientation of the spectrum $\mathrm{H}\widetilde{\mathbb{Z}}$ that represents MW-motivic cohomology (see [BF18]).

Corollary 10.2.6. *The spectrum* $\mathrm{H}\widetilde{\mathbb{Z}}$ *is uniquely* SL*-oriented.*

Proof. By [BF18, Theorem 5.2], $\mathrm{H}\widetilde{\mathbb{Z}} \simeq \tau_{\leqslant 0}^{\mathrm{eff}}(\mathbb{1})$, where the right-hand side denotes the image of $\mathbb{1}$ in $\mathrm{SH}(k)^{\mathrm{eff}}_{\leqslant 0}$. Snice MSL is an effective spectrum, the unit map of MSL induces a morphism:

$$e_* \colon \tau_{\leqslant 0}^{\mathrm{eff}}(\mathbb{1}) \longrightarrow \tau_{\leqslant 0}^{\mathrm{eff}}(\mathrm{MSL}),$$

which, as we claim, is an equivalence in $\mathrm{SH}(k)$. Indeed, by [Bac17, Proposition 4.(1)] it is enough to check that e_* induces an isomorphism of $\underline{\pi}_*(-)_0$. But both $\mathbb{1}$ and MSL belong to $\mathrm{SH}(k)_{\geqslant 0}$, so the only of these sheaves of homotopy groups that survive after applying the functor $\tau_{\leqslant 0}^{\mathrm{eff}}$ are $\underline{\pi}_0(-)_0$. By Proposition 10.1.2, e_* induces an isomorphism of $\underline{\pi}_0(-)_0$.

Hence the unique SL-orientation of $\mathrm{H}\widetilde{\mathbb{Z}}$ is given by the following sequence of ring homomorphisms:

$$\mathrm{MSL} \xrightarrow{\tau_{\leqslant 0}^{\mathrm{eff}}} \tau_{\leqslant 0}^{\mathrm{eff}}(\mathrm{MSL}) \xrightarrow{e_*^{-1}} \tau_{\leqslant 0}^{\mathrm{eff}}(\mathbb{1}) \simeq \mathrm{H}\widetilde{\mathbb{Z}}\,.$$

$\square$

Appendix A

$\mathcal{E}_\infty$-monoids

In this Appendix we recall the notion of an $\mathcal{E}_\infty$-monoid in infinity-categorical settings, and discuss group completion of $\mathcal{E}_\infty$-monoids and its properties. Our references are [GGN15, Section 1] and [Lur17a, Section 5.2.6].

A.0.0.1. Denote $\langle n\rangle_+ = \{1,\dots,n\} \sqcup *$ for $n \geqslant 0$. Consider the category of finite pointed sets Fin_*, whose objects are given by $\langle n\rangle_+$ for $n \geqslant 0$ and morphisms are the pointed maps of sets. For $0 \leqslant i \leqslant n$ let $\rho^i\colon \langle n\rangle_+ \to \langle 1\rangle_+$ be the collapse map: $(\rho^i)^{-1}(\{1\}) = \{i\}$.

In this Appendix we assume that $\mathcal{C}$ is an ∞-category with finite products.

Definition A.0.0.2. A Fin_*-*object* in $\mathcal{C}$ is a functor $M\colon \mathrm{Fin}_* \to \mathcal{C}$. A Fin_*-object M is called an $\mathcal{E}_\infty$-*monoid* if for every $n \geqslant 0$ the collapse maps $\{\rho^i\}_{i=1}^n$ induce an equivalence:

$$M(\langle n\rangle_+) \xrightarrow{\sim} \prod_{i=1}^{n} M(\langle 1\rangle_+).$$

They form an ∞-*category of* $\mathcal{E}_\infty$-*monoids* in $\mathcal{C}$, denoted $\mathrm{Mon}_{\mathcal{E}_\infty}(\mathcal{C})$. An $\mathcal{E}_\infty$-*space* is an $\mathcal{E}_\infty$-monoid in Spc.

We abuse notation and call a Fin_*-object (or an $\mathcal{E}_\infty$-monoid) the underlying object $M(\langle 1\rangle_+)$, which we also denote as M. A structure of an $\mathcal{E}_\infty$-monoid provides M with a multiplication map $m\colon M \times M \to M$, determined up to a contractible space of choices, which is coherently associative and commutative. In more detail, a multiplication map is given by

$$M(\langle 1\rangle_+) \times M(\langle 1\rangle_+) \simeq M(\langle 2\rangle_+) \xrightarrow{m} M(\langle 1\rangle_+),$$

induced by $m\colon \langle 2\rangle_+ \to \langle 1\rangle_+$ with $m(\{1,2\}) = \{1\}$. If M is an $\mathcal{E}_\infty$-space, $\pi_0(M)$ is a commutative monoid in the classical sense, with the addition law given by m.

Remark A.0.0.3. Equivalently, one can consider instead of Fin_* an isomorphic category, whose objects are given by $\langle n\rangle$ for $n \geqslant 0$ and morphisms are partially-defined maps of sets $\langle n\rangle \dashrightarrow \langle m\rangle$.

A.1 Group completion

A.1.0.1. The following definition was introduced in [Lur17a, Definition 5.2.6.2].

Definition A.1.0.2. An $\mathcal{E}_\infty$-*monoid* M in $\mathcal{C}$ is called *grouplike* if the shear maps

$$(\mathrm{pr}_1, m)\colon M \times M \to M \times M; \quad (m, \mathrm{pr}_2)\colon M \times M \to M \times M$$

are equivalences.

Remark A.1.0.3. By [Lur17a, Example 5.2.6.4], an $\mathcal{E}_\infty$-space E is grouplike if and only if $\pi_0(E)$ is a group. Furthermore, an $\mathcal{E}_\infty$-monoid $\mathcal{E} \in \mathrm{PSh}(\mathcal{C})$ is grouplike if and only if $\pi_0(\mathcal{E})$ is a presheaf of groups (since shear maps are componentwise).

A.1.0.4. Now let $\mathcal{C}$ be a presentable ∞-category. Consider the full subcategory $\mathrm{Grp}_{\mathcal{E}_\infty}(\mathcal{C})$ of $\mathrm{Mon}_{\mathcal{E}_\infty}(\mathcal{C})$, consisting of grouplike $\mathcal{E}_\infty$-monoids. By [GGN15, Corollary 4.4], the forgetful map is a right adjoint functor. Its left adjoint functor (localization with respect to shear maps) is called *group completion*:

$$(-)^{\mathrm{gp}}\colon \mathrm{Mon}_{\mathcal{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathcal{E}_\infty}(\mathcal{C}).$$

By construction, for an $\mathcal{E}_\infty$-space E the group $\pi_0(E^{\mathrm{gp}})$ is the classical group completion of the monoid $\pi_0(E)$. By Remark A.1.0.3, group completion on a presheaf category is objectwise.

A.1.0.5. We recall the following statement as part of [GGN15, Theorem 5.1].

Theorem A.1.0.6. *Let $\mathcal{C}^{\otimes}$ be a closed symmetric monoidal structure on a presentable ∞-category $\mathcal{C}$. Then ∞-categories $\mathrm{Mon}_{\mathcal{E}_\infty}(\mathcal{C})$ and $\mathrm{Grp}_{\mathcal{E}_\infty}(\mathcal{C})$ admit canoincal closed symmetric monoidal structures, and group completion uniquely extends to a symmetric monoidal functor.*

A.1.0.7. In our work we repeatedly use the following fact [Hoy17, Lemma 5.5].

Lemma A.1.0.8. *Let $\mathrm{L}\colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{PSh}(\mathrm{Sm}_k)$ be a functor that preserves colimits and finite products. Then for every $\mathcal{E}_\infty$-monoid M in $\mathrm{PSh}(\mathrm{Sm}_k)$ the canonical map $\mathrm{L}(M)^{\mathrm{gp}} \to \mathrm{L}(M^{\mathrm{gp}})$ is an equivalence. In particular, $\mathrm{L}_{\mathrm{Nis}}$, $\mathrm{L}_{\mathbb{A}^1}$ and $\mathrm{L}_{\mathrm{mot}}$ commute with group completion.*

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