An explicit geometric Langlands correspondence for the projective line minus four points

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Abstract

This thesis concerns the tamely ramified geometric Langlands correspondence in rank 2 on $\mathbf{P}^1_{\mathbf{F}_q}$ minus four distinct points. For each irreducible pure rank 2 local system E with unipotent monodromy on $\mathbf{P}^1_{\mathbf{F}_q} \setminus D$, where $D = \{\infty, 0, 1, t\} \subset \mathbf{P}^1_{\mathbf{F}_q}(\mathbf{F}_q)$ is a set of four distinct points, we construct the associated Hecke eigensheaf on the moduli space of parabolic rank 2 vector bundles on $\mathbf{P}^1$. The proof of this seems to be new and relies on the observations that the support of the cusp forms can be obtained from a single parabolic vector bundle through Hecke transforms, and on a symmetry that results from this. In addition, we construct a basis of the q-dimensional space of cusp forms in a single degree and provide an explicit formula for the action of the Hecke operator on the cusp forms in terms of that basis.

Zusammenfassung

Das Langlandsprogramm ist ein vielfältiges Netz von Sätzen und Vermutungen, das viele verschiedene Bereiche der Mathematik verbindet. Diese Doktorarbeit ist einem bestimmten Spezialfall dieses Programms gewidmet, einem Fall, der eines der ersten, nicht-trivialen Beispiele der verzweigten geometrischen Langlandskorrespondenz, an die man denkt, ist: die Kurve ist die projektive Gerade $\mathbf{P}^1$ über einem endlichen Körper $\mathbf{F}_q$, die reduktive Gruppe ist die allgemeine lineare Gruppe vom Rang 2 und die Verzweigung ist eine zahme Verzweigung an vier verschiedenen Punkten auf der Gerade. Auf der Galois-Seite dieser Korrespondenz stehen irreduzible lokale Systeme vom Rang 2 auf der viermal punktierten projektiven Gerade mit unipotenter Monodromie. Für jedes Objekt an dieser Seite besteht genau eine Hecke-Eigengarbe auf der anderen Seite, der automorphen Seite, mit der Eigenschaft, dass das lokale System die Eigenwerte der Eigengarbe unter den Hecke-Operatoren gibt. In dieser Arbeit beschreiben wir diese Korrespondenz explizit und geben wir außerdem eine völlig explizite Formel für die Operation der Hecke-Operatoren auf den Spitzenformen.

Die ursprüngliche Motivation war eine Formel, die Kontsevich für diese Operation der Hecke-Operatoren gegeben hat. Die Formel zeigte nämlich eine merkwürdige Symmetrie zwischen auf den ersten Blick beziehungslosen Objekten: einerseits den parabolischen Vektorbündeln, die den Träger der Spitzenform ausmachen, und andererseits dem Ort des Hecke-Operators, der ein Punkt auf der projektiven Geraden ist. Außerdem wurden die Symbole in dieser Formel leider nicht präzise definiert, es fehlte ein Beweis und die Korrekturterme schienen nicht ganz korrekt zu sein. Das Ziel war also eine korrekte Formel zu finden und zu beweisen, und die Symmetrie zu erklären. Die Formel geben wir in Theorem 1.2 an. Der Hauptgrund der Symmetrie ist die Tatsache, dass die parabolischen Bündel im Träger der Spitzenformen alle Hecke-Transformen eines speziellen Bündels sind. Dies liefert eine natürliche Identifikation der projektiven Geraden mit einem Teil des Modulraumes parabolischer Bündel. Diese Erklärung war ein Ziel an sich, aber hat nicht nur die Berechnung der Formel erleichtert, sondern auch zum zweiten Hauptergebnis dieser Arbeit geführt: einem neuen Beweis der Langlands-Korrespondenz im oben angegebenen Fall (Theorem 1.1). Dieser Satz hat den Vorteil, dass er nicht nur ein Existenzsatz ist, sondern die Korrespondenz explizit darstellt.

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Contents

1. Introduction

1.1. Setting and main results. This thesis is about the tamely ramified geometric Langlands correspondence for GL_2 on $\mathbf{P}^1_{\mathbf{F}_q}$, where q is a prime power, with tame ramification at four distinct points $D = \{\infty, 0, 1, t\} \subset \mathbf{P}^1(\mathbf{F}_q)$. We describe in a completely explicit way (1) the action of the Hecke operators on a basis of the cusp forms, which consists of q elements (Theorem 1.2); and (2) the correspondence that assigns to a pure irreducible rank 2 local system E on $\mathbf{P}^1 \setminus D$ with unipotent monodromy its Hecke eigensheaf (Theorem 1.1).

The calculation of the matrix coefficients for the Hecke operators was the original motivation for the work in this thesis. Kontsevich [**Kon09**, Section 0.1] provides a formula, but this formula lacks a proof or explanation and it is not entirely clear what the terms mean. Moreover, Mellit, Golyshev and van Straten noticed that the published formulas of Kontsevich contain misprints, but they were able to able to guess a correction term that made the Hecke operators commute. They used this for their computer computations of Hecke eigensheaves. Lastly, the formula also exhibits interesting symmetries that warrant an explanation. For example, the formula is symmetric in the support of the cusp form, which is a set of rank 2 vector bundles on $\mathbf{P}^1$ with a parabolic structure at D, and the locus of the Hecke operator, which is a point in $\mathbf{P}^1$ — two seemingly unrelated objects. The original aim of this thesis was to prove and provide the correct formulas and to explain this symmetry. A better understanding of this symmetry, which we will explain shortly, also led to a new way to prove the Hecke property for the would-be Hecke eigensheaf corresponding to the local system E (Theorem 1.1). (A proof of the Langlands correspondence for rank 2 local systems with unipotent monodromy appears in [**Dri87**].)

The crucial ingredient in both theorems is the following. The cusp forms, which live on the moduli space of rank 2 parabolic vector bundles, are in fact supported on a smaller open substack that we call the relevant locus. The essential point is that the degree 2 part of the relevant locus contains a canonical parabolic vector bundle, $\tilde{\mathcal{E}}$, and the degree 1 part of the relevant locus can be obtained by taking the Hecke transforms of this bundle $\tilde{\mathcal{E}}$. In the geometric Langlands setting, these Hecke transforms are defined as length 1 lower modifications of $\tilde{\mathcal{E}}$, and the terms Hecke transform and length 1 lower modification can be treated as synonyms in this introduction. The Hecke operators sum over all Hecke transforms.

Taking the Hecke transforms of $\tilde{\mathcal{E}}$ at all points in $\mathbf{P}^1$ provides an isomorphism from a space that is almost $\mathbf{P}^1$ to the relevant locus in degree 1, in the following way. This space that is almost $\mathbf{P}^1$ is in fact isomorphic to $\mathbf{B}\mathbb{G}_\mathrm{m} \times \tilde{\mathbf{P}}^1$, where $\tilde{\mathbf{P}}^1$ is a $\mathbf{P}^1$ with tripled points at each point in D, and one point from each triple has $\mathbb{G}_\mathrm{m}$-automorphisms. (For a precise statement, $\tilde{\mathbf{P}}^1$ is the moduli space $\overline{\mathbf{Coh}_0}^{1,1}$ defined in Section 2.4.1; the isomorphism itself is defined in Section 7.) The modifications of $\tilde{\mathcal{E}}$ at a point $x \in \mathbf{P}^1 \setminus D$ are classified by a $\mathbf{P}^1$. The automorphism group of $\tilde{\mathcal{E}}$ has an additional $\mathbb{G}_\mathrm{m}$ that acts on this $\mathbf{P}^1$ and the modification corresponding to the open orbit is a degree 1 parabolic bundle in the relevant locus. By sending $x \in \mathbf{P}^1 \setminus D$ to this Hecke transform of $\tilde{\mathcal{E}}$ at x and extending this mapping in a similar way to $D \subset \mathbf{P}^1$, we obtain an isomorphism from $\mathbf{B}\mathbb{G}_\mathrm{m} \times \tilde{\mathbf{P}}^1$ to the relevant locus in degree 1.

This observation, together with the simple fact that modifications at different points commute, provides us with a symmetry that is essential in both of the main results mentioned above. For simplicity, assume $x, y \in \mathbf{P}^1 \setminus D$; the situation with x or y in D is essentially the same. We denote by $\mathcal{E}_x^\bullet$ and $\mathcal{E}_y^\bullet$ the parabolic bundles of degree 1 in the relevant locus that correspond to x and y under the above correspondence, respectively; i.e., they are the Hecke transforms of $\tilde{\mathcal{E}}$ at x and y corresponding to the open orbits. Roughly speaking, applying a Hecke operator $\mathbb{H}_x$ at x to a cusp form f and evaluating it in the bundle $\mathcal{E}_y^\bullet$, we get

$$(\mathbb{H}_x f)(\mathcal{E}_y^\bullet) = \sum_{\mathcal{F}^\bullet \overset{x}{\hookrightarrow} \mathcal{E}_y^\bullet (\overset{y}{\hookrightarrow} \tilde{\mathcal{E}})} f(\mathcal{F}^\bullet)$$

where the sum is over length one lower modifications $\mathcal{F}^\bullet$ of $\mathcal{E}_y^\bullet$ at x. Thus, these bundles $\mathcal{F}^\bullet$ are obtained from $\tilde{\mathcal{E}}$ by first modifying at y at the open orbit and then taking all modifications at x. If $x \neq y$, we can show that if we change the order — i.e., we modify $\tilde{\mathcal{E}}$ at x with respect to the open orbit and then sum over all modifications at y — we get exactly the same set of bundles $\mathcal{F}^\bullet$. Therefore, we have

$$(\mathbb{H}_x f)(\mathcal{E}_y^\bullet) = (\mathbb{H}_y f)(\mathcal{E}_x^\bullet). \tag{1.1.1}$$

This idea explains why the formula for the Hecke operators (Theorem 1.2) displays a symmetry in the locus of the Hecke operator, a point in $\mathbf{P}^1$, and the support of the cusp forms, which are points in the relevant locus — two very different objects at first sight, but related via Hecke transforms of the specially chosen bundle $\tilde{\mathcal{E}}$.

In addition to explaining the symmetry in the formula, we use this symmetry to prove that the local system E is its own eigensheaf. In our second main theorem (Theorem 1.1), we prove this Hecke property for perverse sheaves, but Equation (1.1.1) already proves it on the level of trace-of-Frobenius functions: if f in is an eigenform for all Hecke operators $\mathbb{H}_x$, then the eigenvalue of $\mathbb{H}_x$ is equal to $f(x)$.

Before stating the main results, Theorem 1.1 and Theorem 1.2, we introduce the basic objects. In this ramified geometric Langlands setting for GL_2, the objects on the automorphic side of the correspondence are constructible derived sheaves on the moduli space $\mathrm{Bun}_{2,4}$ of rank 2 vector bundles on $\mathbf{P}^1$ with parabolic structure at D. We will simply refer to these bundles with parabolic structure as parabolic (vector) bundles and usually denote them by $\mathcal{E}^\bullet, \mathcal{F}^\bullet$; their underlying vector bundles are then denoted by $\mathcal{E}, \mathcal{F}$. This moduli space $\mathrm{Bun}_{2,4}$ is a disjoint union

$$\mathrm{Bun}_{2,4} = \bigsqcup_{d \in \mathbf{Z}} \mathrm{Bun}_{2,4}^d,$$

where the moduli space $\mathrm{Bun}_{2,4}^d$ classifies parabolic bundles of degree d. Though these are infinitely many components, all odd degrees and all even degrees can be identified: tensoring by $-\otimes\mathcal{O}(1)$ provides canonical isomorphisms for all $d\in\mathbf{Z}$

$$(-\otimes\mathcal{O}(1))\colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d+2}.$$

Moreover, every $x\in D$ defines an isomorphism

$$T_x\colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d-1}$$

that shifts the parabolic structure at x, but these isomorphisms depend on the choice of $x\in D$.

The Hecke operators are defined in terms of a correspondence that involves the Hecke stack of length 1

$$\mathcal{H} := \left\langle \mathcal{F}^\bullet\subset\mathcal{E}^\bullet :\ \begin{array}{c}\mathcal{F}^\bullet,\mathcal{E}^\bullet\in\mathrm{Bun}_{2,4}\\ \mathcal{E}^\bullet/\mathcal{F}^\bullet \text{ is a torsion sheaf of length 1}\\ \text{in every parabolic degree}\end{array}\right\rangle.$$

We denote the moduli stack of parabolic torsion sheaves on $\mathbf{P}^1$ of length 1, such as the quotient $\mathcal{E}^\bullet/\mathcal{F}^\bullet$, by $\mathbf{Coh}_0^{1,1}$. This stack is naturally isomorphic to $\overline{\mathbf{Coh}}_0^{1,1}\times\mathbf{B}\mathbb{G}_{\mathrm{m}}$, where $\overline{\mathbf{Coh}}_0^{1,1}$ is the rigidification of $\mathbf{Coh}_0^{1,1}$ with respect to the diagonal automorphisms. (This rigidified stack $\overline{\mathbf{Coh}}_0^{1,1}$ can also be explicitly described as the stack classifying parabolic torsion sheaves of length 1 in every parabolic degree, equipped with a non-zero global section of the underlying coherent sheaf.) The support map

$$\mathrm{Supp}\colon \overline{\mathbf{Coh}}_0^{1,1}\to\mathbf{P}^1$$

that maps a torsion sheaf to its support, is in fact the universal map of $\overline{\mathbf{Coh}}_0^{1,1}$ to its coarse moduli space. Because parabolic torsion sheaves supported outside of $D\subset\mathbf{P}^1$ have trivial parabolic structure, this map is an isomorphism over $\mathbf{P}^1\setminus D$. In particular, we have a natural inclusion

$$j\colon \mathbf{P}^1\setminus D\hookrightarrow\overline{\mathbf{Coh}}_0^{1,1}.$$

The inverse image of a point $x\in D$ is isomorphic to $\{(x,y)\in\mathbf{A}^2 : xy=0\}/\mathbb{G}_{\mathrm{m}}$, where $\mathbb{G}_{\mathrm{m}}$ acts anti-diagonally, which consists of three points: two without automorphism group and one with automorphism group $\mathbb{G}_{\mathrm{m}}$.

The Hecke correspondence is then the diagram

$$\mathrm{Bun}_{2,4}\times\overline{\mathbf{Coh}}_0^{1,1} \xleftarrow{\ p\ } \mathcal{H} \xrightarrow{\ q\ } \mathrm{Bun}_{2,4} \tag{1.1.2}$$

where p and q are defined by

$$\begin{array}{lll} p\colon \mathcal{H}\to\mathrm{Bun}_{2,4}\times\overline{\mathbf{Coh}}_0^{1,1}, & (\mathcal{F}^\bullet\hookrightarrow\mathcal{E}^\bullet)\mapsto(\mathcal{E}^\bullet,[\mathcal{E}^\bullet/\mathcal{F}^\bullet]) & \text{and}\\ q\colon \mathcal{H}\to\mathrm{Bun}_{2,4}, & (\mathcal{F}^\bullet\hookrightarrow\mathcal{E}^\bullet)\mapsto\mathcal{F}^\bullet & \end{array}$$

and the global Hecke operator is defined as the derived functor between bounded derived categories of constructible sheaves

$$\mathbb{H} := \mathbf{R}p_!q^*[2]\colon D^{\mathrm{b}}(\mathrm{Bun}_{2,4},\mathbf{Q}_\ell)\to D^{\mathrm{b}}(\mathrm{Bun}_{2,4}\times\overline{\mathbf{Coh}}_0^{1,1},\mathbf{Q}_\ell).$$

The local Hecke operator $\mathbb{H}_{\mathcal{T}^\bullet}$ for some $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(\mathbf{F}_q)$ is defined via the global Hecke operator by restricting to $\mathrm{Bun}_{2,4} \times \operatorname{Spec} \mathbf{F}_q$:

$$\mathbb{H}_{\mathcal{T}^\bullet} := (\mathrm{Bun}_{2,4} \times \operatorname{Spec} \mathbf{F}_q \xrightarrow{(\mathrm{id},[\mathcal{T}^\bullet])} \mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1})^* \circ \mathbb{H}.$$

We can also restrict the Hecke correspondence (1.1.2) to a degree $d \in \mathbf{Z}$ to obtain

$$\mathrm{Bun}_{2,4}^d \times \overline{\mathbf{Coh}}_0^{1,1} \xleftarrow{p} \mathcal{H}^d \xrightarrow{q} \mathrm{Bun}_{2,4}^{d-1}; \tag{1.1.3}$$

we will often write $\mathcal{H}$ when it is clear that $\mathcal{H}^d$ is in fact meant.

A Hecke eigensheaf for a pure irreducible rank 2 local system E on $\mathbf{P}^1 \setminus D$ is defined to be a constructible derived sheaf Aut_E on $\mathrm{Bun}_{2,4}$ such that there exists an isomorphism of sheaves on $\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$

$$\mathbb{H} \, \mathrm{Aut}_E \cong \mathrm{Aut}_E \boxtimes j_{!*}E.$$

By Aut_E^d we denote the restriction of Aut_E to $\mathrm{Bun}_{2,4}^d$. Because for all $x \in D$ the isomorphism

$$T_x \colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d-1}$$

is related to the local Hecke operator with respect to a certain torsion sheaf $k_x^{(1,0)} \in \mathbf{Coh}_0^{1,1}(\mathbf{F}_q)$ supported at x, we have for all $d \in \mathbf{Z}$

$$\mathrm{Aut}_E^{d-1} \cong (T_x^{d-1})^* \, \mathrm{Aut}_E^d \otimes (j_{!*}E)|_{k_x^{(1,0)}}^{\otimes -d+1}; \tag{1.1.4}$$

therefore, once we have constructed the correct constructible sheaf Aut_E^1 in degree 1, the Aut_E^d with $d \in \mathbf{Z} \setminus \{1\}$ follow.

We can now state our first main result, Theorem 13.1 in the main text. Note that the sheaf $j_{!*}E$ on $\overline{\mathbf{Coh}}_0^{1,1}$ can be pulled back to a sheaf on $\mathbf{Coh}_0^{1,1}$, which we also denote by $j_{!*}E$.

THEOREM 1.1. *There exists a canonical open embedding*

$$j^{\mathrm{rel}} \colon \mathbf{Coh}_0^{1,1} \hookrightarrow \mathrm{Bun}_{2,4}^1$$

such that for any pure irreducible rank 2 local system E on $\mathbf{P}^1 \setminus D$ with unipotent monodromy, the sheaf Aut_E on $\mathrm{Bun}_{2,4}$ defined by

$$\mathrm{Aut}_E|_{\mathrm{Bun}_{2,4}^1} = j_!^{\mathrm{rel}} j_{!*}E$$

and extended to all of $\mathrm{Bun}_{2,4}$ by (1.1.4), is a Hecke eigensheaf for E, i.e.,

$$\mathbb{H} \, \mathrm{Aut}_E \cong \mathrm{Aut}_E \boxtimes j_{!*}E.$$

In particular, all cusp forms are supported on the image of the open embedding j^{rel}, which has finitely many $\mathbf{F}_q$-points. We use this and an explicit characterization of the cusp conditions to conclude that the space of cusp forms on $\mathrm{Bun}_{2,4}^d$ is q-dimensional for every d and we calculate in Section 11.2 the matrix coefficients of the local Hecke operators with respect to a specific choice of basis forms. These formulas have the following form. For $x \in D$, $M_x \colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ is the unique Möbius transformation that sends $D \subset \mathbf{P}^1$ to D and ∞ to x (Section 8.3). The sets $\{F_z\}_{z \in \mathbf{F}_q}$ and $\{F_z^0\}_{z \in \mathbf{F}_q}$ are specific bases of the cusp forms on $\mathrm{Bun}^1_{2,4}(\mathbf{F}_q)$ and $\mathrm{Bun}^0_{2,4}(\mathbf{F}_q)$, respectively (Definition 11.1 and Definition 11.3).

Theorem 1.2. *Let $z \in \mathbf{F}_q$ and let $x \in \mathbf{P}^1$. Let $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ be a parabolic torsion sheaf supported at x with automorphism group $\mathbb{G}_\mathrm{m}$ (automatic if $x \notin D$) and let $\mathbb{H}_x$ be the Hecke operator with respect to $\mathcal{T}^\bullet$. First suppose $x \neq \infty$. Then*

$$\mathbb{H}_x F_z^0 = \sum_{y \in \mathbf{F}_q} \alpha_{z,y}^x F_y$$

where for all $y \in \mathbf{F}_q \setminus \{x\}$,

$$\begin{aligned}\alpha_{z,y}^x = \#&\left\{ r \in \mathbf{F}_q^* : z = \frac{(yr-x)((y-1)(y-t)r-(x-1)(x-t))}{-(x-y)^2 r} \right\} \\ &- \begin{cases} 0 & \text{if } x \in D \text{ and } y \in D \\ 1 & \text{if } x \in D \text{ or } y \in D\text{, but not both} \\ 2 & \text{otherwise} \end{cases} \\ &- \begin{cases} q & \text{if } z \in D \text{ and } y = M_z(x) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and

$$\alpha_{z,x}^x = \#\left\{ r \in \mathbf{F}_q : z = -(yr-1)((y-1)(y-t)r-(2y-(1+t))) \right\} - q + 1.$$

If $x = \infty$, then the same holds with

$$\alpha_{z,y}^\infty = \begin{cases} -1 & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}.$$

Remark 1.3. In the above theorem, the case $x = \infty$ can be obtained from the formula for $\alpha_{z,y}^x$ with $x \neq y$ and $x \neq \infty$, by replacing r with rx^2 and taking the limit as x goes to infinity. See the proof of Lemma 11.8.

1.2. Overview of the content. In Chapter 1 (Sections 2 to 5) , we provide definitions and basic properties of the objects involved. Most, if not all, of the content of this chapter can also be found elsewhere. Though we mostly restrict ourselves to the situation of interest to us now, with $\mathbf{P}^1_{\mathbf{F}_q}$ as the curve and $D = \{\infty, 0, 1, t\} \subset \mathbf{P}^1(\mathbf{F}_q)$ as the points with parabolic structure, most results can directly be extended to more general situations. In Section 2, we define parabolic structures on coherent sheaves. We first give a definition that applies to all coherent sheaves and give some additional results that apply to either parabolic vector bundles and parabolic torsion sheaves. In particular, we describe in Section 2.4.2 the moduli space of parabolic torsion sheaves of length 1. In the last two subsections, we explain how to define modifications of parabolic coherent sheaves.

Section 3 lists some basic properties of the moduli space of rank 2 vector bundles on $\mathbf{P}^1$ with parabolic structure at D. The Hecke operators are defined in Section 4 in terms of a correspondence involving the Hecke stack

(Definition 4.1), which classifies length 1 lower modifications of parabolic vector bundles. The last section in this chapter, Section 5, defines the cusp conditions and relates them to the cusp conditions as they are known classically.

With Chapter 2 (Sections 6 to 11) , we start our determination of the action of the Hecke operator on the cusp forms. In Section 6, we give a complete characterization of the cusp forms (Theorem 6.2). A part of this characterization is that the cusp forms in degree d are supported on an open substack $\mathrm{Bun}_{2,4}^{d,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^{d}$, which we call the relevant locus. It has only finitely many $\mathbf{F}_q$-points.

A more explicit description of the relevant locus follows in Section 7, where we use the results of Section 6 to construct a canonical isomorphism

$$\alpha\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}};$$

here $\mathbf{Coh}_0^{1,1}$ is almost $\mathbf{P}^1$, but more precisely, it is the moduli space of parabolic torsion sheaves of length 1 defined in Section 2.4.2. The map $j^{\mathrm{rel}}\colon \mathbf{Coh}_0^{1,1} \hookrightarrow \mathrm{Bun}_{2,4}^{1}$ mentioned in Theorem 1.1 is the composition of α with the inclusion $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \hookrightarrow \mathrm{Bun}_{2,4}^{1}$.

The cusp conditions that we have established in Section 6, suffice to specify a basis of the cusp forms. To determine the action of the Hecke operators, however, we need to calculate the length 1 lower modifications of the parabolic bundles in the relevant locus. This is the aim of the rather long and technical section Section 9. This boils down to calculating the isomorphism classes of consecutive length 1 lower modifications of $\tilde{\mathcal{E}}$. This can in principle be calculated in a very mechanical manner; the proofs and calculations provided are hopefully more elucidating.

These calculations are simplified by the results from Section 8, where we study some symmetries of the relevant locus. In Section 11, we define a basis of the cusp forms and provide an explicit description of the action of the Hecke operators using that basis.

The next and last chapter, Chapter 3 (Sections 12 to 13), is concerned with the construction of a Hecke eigensheaf Aut_E associated to a pure irreducible rank 2 local system E on $\mathbf{P}^1 \setminus D$ with unipotent monodromy. The first section, Section 12, gives the construction and some basic properties, while the second section, Section 13, proves it is in fact a Hecke eigensheaf for E.

CHAPTER 1

General

2. Parabolic structures on coherent sheaves

In this section, we define the notion of a parabolic structure on coherent sheaf on a curve. We start with a definition using repeating chains in Section 2.1. This definition has the advantage that it is applicable to all coherent sheaves (other definitions are not suitable for torsion sheaves, for example) and that it allows a succinct description of the maps between sheaves with parabolic structures. In the next section, Section 2.2, we give an equivalent definition in terms of flags in the fibers that is applicable to vector bundles. This definition also shows why parabolic structures appear in the geometric formulation of the Langlands program (Remark 2.8).

After these general definitions, the rest of this section will focus on explicit descriptions of the parabolic sheaves of interest to us: rank 2 parabolic vector bundles with full flags and parabolic torsion sheaves that have degree 1 in every parabolic degree. In Section 2.3, we study extensions of a parabolic line bundle by a parabolic line bundles, which will play a role in a later section on cusp conditions (Section 5). In Section 2.4, we study the moduli space $\mathbf{Coh}_0^{1,1}$ of parabolic torsion sheaves of degree 1. These torsion sheaves are then used in the following sections to define and analyze modifications of parabolic vector bundles (Section 2.5 for the general case and Section 2.6 for rank 2 vector bundles with full flags).

This section is entirely self-contained. The definitions and some of the ideas are based on [**Hei04**, paragraph 3.1].

2.1. Definitions using repeating chains. Let X be a smooth projective curve over $k = \mathbf{F}_q$ (q a prime power) and $S \subset X(\mathbf{F}_q)$ a finite set of points.

Definition 2.1 (Parabolic structure on coherent sheaves)**.** Let $n \in \mathbf{Z}$. An n-*step parabolic structure* on a coherent sheaf $\mathcal{F}$ on X at the points S is the data of sheaves $\mathcal{F}^{(i,p)}$ for every $i \in \mathbf{Z}$ and $p \in S$ and maps $\phi_{(i,p)} : \mathcal{F}^{(i,p)} \to \mathcal{F}^{(i+1,p)}$ such that for all $p \in S$ and $i \in \mathbf{Z}$,

- $\mathcal{F}^{(0,p)} = \mathcal{F}$;
- $\mathcal{F}^{(i+n,p)} = \mathcal{F}^{(i,p)}(p)$;
- the composition

$$\mathcal{F}^{(i,p)} \xrightarrow{\phi_{(i,p)}} \mathcal{F}^{(i+1,p)} \xrightarrow{\phi_{(i+,p)}} \dots \xrightarrow{\phi_{(i+n-1,p)}} \mathcal{F}^{(i+n,p)} = \mathcal{F}^{(i,p)}(p) \tag{2.1.1}$$

coincides with the natural map $\mathcal{F} \otimes (\mathcal{O} \hookrightarrow \mathcal{O}(p))$; and finally

- the chain repeats itself after n steps, but with a twist: $\phi_{(i+n,p)} = \phi_{(i,p)} \otimes \mathrm{id}_{\mathcal{O}(p)}$.

We often denote the sheaf with parabolic structure by $\mathcal{F}^\bullet$, but will also simply denote it by $\mathcal{F} = \mathcal{F}^{(0,p)}$, and we call a sheaf equipped with a parabolic structure a *parabolic sheaf.* We refer to the sheaf $\mathcal{F} = \mathcal{F}^{(0,p)}$ as *the underlying sheaf.* We call each (i,p) a *parabolic degree.*

A map between two sheaves $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ with parabolic structure at S is by definition collection of maps of sheaves

$$f_{(i,p)} \colon \mathcal{F}^{(i,p)} \to \mathcal{G}^{(i,p)} \qquad \text{for all } i \in \mathbf{Z},\, p \in S$$

such that for all $p \in S$, $f_{(\bullet,p)}$ defines a map of chains $\mathcal{F}^{(\bullet,p)} \to \mathcal{G}^{(\bullet,p)}$, and such that for all (i,p), the map $f_{(i+n,p)}$ is identified with $f_{(i,p)} \otimes \mathrm{id}_{\mathcal{O}(p)}$.

Remark 2.2.

- The maps $\phi_{(i,p)}$ are isomorphisms outside of p. In particular, torsion sheaves supported outside of S have trivial parabolic structure, i.e., the maps $\phi_{(i,p)}$ are isomorphisms.
- A parabolic structure on a coherent sheaf is a local structure: the parabolic structure at a point p is determined by the restriction of (2.1.1) to the formal disc $D_p \subset X$ around p.
- Because of the last condition, an n-step parabolic structure $\mathcal{F}^\bullet$ on $\mathcal{F}$ at a point p is completely determined by n consecutive maps $\phi_{i,p}, \phi_{i+1,p}, \ldots, \phi_{i+n-1,p}$.
- The maps $\phi_{(i,p)}$ are injective when restricted to the torsion free parts, since their composition is required to be the map induced by $\mathcal{O} \hookrightarrow \mathcal{O}(p)$. If $\mathcal{T}$ is a torsion sheaf with support at p, the map $\mathcal{T} \otimes (\mathcal{O} \hookrightarrow \mathcal{O}(p))$ is not injective.

Example 2.3 (Parabolic structures on line bundles)**.** Parabolic structures on line bundles are particularly easy and will be used throughout this thesis. Let $\mathcal{L}^\bullet$ be a line bundle with an n-step parabolic structure at p. Because all the maps $\phi_{(i,p)} \colon \mathcal{L}^{(i,p)} \to \mathcal{L}^{(i+1,p)}$ defining the parabolic structure are inclusions that are isomorphisms outside of p, we find that $\phi_{(i,p)}$ is either the identity or the natural map $\mathcal{L}^{(i,p)} \to \mathcal{L}^{(i,p)}(p)$. Therefore, the parabolic structure on $\mathcal{L}$ at p is always given by the repetition of $n-1$ consecutive identity maps and one map induced by $\mathcal{O} \to \mathcal{O}(p)$: up to shifting the parabolic degree by some $0 \leq i < n$, every parabolic structure on $\mathcal{L}$ at p looks like

$$\ldots \subset \mathcal{L}(-2p) \subset \underbrace{\mathcal{L}(-p) \subset \ldots \subset \mathcal{L}(-p)}_{n \text{ inclusions}} \subset \mathcal{L} \subset \ldots$$

Hence, all the information in this structure is given by the unique integer $0 \leq i < n$ for which $\phi_{(i,p)}$ is not the identity, but the map induced by $\mathcal{O} \hookrightarrow \mathcal{O}(p)$.

2.2. Parabolic vector bundles. If $\mathcal{F}$ is a rank n vector bundle on X, then a parabolic structure on $\mathcal{F}$ at X can equivalently be described as flags at all points in S. This description is useful in calculations and allows us to express parabolic vector bundles on X in terms of adèles.

Definition 2.4. Let $\mathcal{F}$ be a vector bundle on X and let $p \in X(\mathbf{F}_q)$ a point. A *flag* of $\mathcal{F}$ at p of length n is a filtration of the fiber of $\mathcal{F}$ at p as $\mathbf{F}_q$-vector space of the form $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n = \mathcal{F}|_p$.

Note that we do not require the inclusions to be proper.

Remark 2.5. An n-step parabolic structure on vector bundle $\mathcal{F}$ is equivalent to the datum of a flag of length n at each $p \in S$. Indeed, given a parabolic structure $(\mathcal{F}^{(i,p)})_{i\in\mathbf{Z},p\in S}$, we can define a flag $0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n := \mathcal{F}|_p$ at any $p \in S$ by setting

$$V_i := \operatorname{im}(\phi_{(i-n,p)}|_p \colon \mathcal{F}^{(i-n),p}|_p \to \mathcal{F}|_p).$$

Conversely, giving such a flag, we can define for each $1 \leq i \leq n$

$$\mathcal{F}^{(i-n,p)} := \ker(\mathcal{F} \to \mathcal{F}|_p/V_i).$$

The sheaves in the other parabolic degrees are then defined as suitable twists (by $\mathcal{O}(p)$) of one of the above sheaves and the maps $\phi_{(i,p)}$ are the natural inclusions. To see that these two operations are inverse to each other, we only need to recall our previous remarks that the $\phi_{(i,p)}$ are isomorphisms outside of p and on torsion-free sheaves, they are inclusions.

A map of rank n parabolic vector bundles can then equivalently be described as a map on the underlying vector bundles that preserves the flags at all points in S, i.e., if the parabolic structure at $p \in S$ is given by the linear subspaces (V_i) on the source and by (W_i) on the target, the image of V_i under the induced map on the fiber at p should be contained in W_i.

For $0 \leq i \leq n$, the degree of $\mathcal{F}^{(-n+i,p)}$ is equal to $\deg \mathcal{F} - \operatorname{rk} \mathcal{F} + \dim_{\mathbf{F}_q} V_i$. Indeed, the cokernel of the inclusion $\mathcal{F}^{(-n+i,p)} \hookrightarrow \mathcal{F}^{(0,p)} = \mathcal{F}$ is the torsion sheaf $\mathcal{F}|_p/V_i$, which has degree $\dim_{\mathbf{F}_q} \mathcal{F}|_p/V_i = \operatorname{rk} \mathcal{F} - \dim_{\mathbf{F}_q} V_i$, so the claim follows from the additivity of the degree in exact sequences.

Example 2.6 (Rank n vector bundles with full flags)**.** Let $\mathcal{E}^\bullet$ be a rank n parabolic vector bundle with an n-step parabolic structure. We will often impose the condition

$$\deg \mathcal{E}^{(i,p)} = \deg \mathcal{E} + i \qquad \text{for all } i \in \mathbf{Z},\, p \in S.$$

(Note that because of the requirement $\mathcal{E}^{(i+n,p)} = \mathcal{E}(p)$ in the definition of parabolic structures, this condition can only hold if the rank of the parabolic bundle agrees with the length of the parabolic structure.) The corresponding condition on the the associated flag $0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathcal{E}|_p$ at p is $\dim V_i = i$; in other words, we require the flag to be a full flag. We therefore refer to parabolic vector bundles satisfying this condition as parabolic bundles with full flags. If $n = 2$, as it will be in most of this thesis, then this is simply the datum of a line (1-dimensional linear subspace) $\ell_p \subset \mathcal{E}|_p$.

Notation 2.7. Let $\mathcal{L}$ be a parabolic line bundle on $\mathbf{P}^1$ with 2-step parabolic structure at S. We write

$$I(\mathcal{L}) := \{p \in S : \mathcal{L}^{(-1,p)} = \mathcal{L}\}$$

and remark that this set determines the parabolic structure on $\mathcal{L}$ by Example 2.3.

Let $\mathcal{M}$ be a line bundle on $\mathbf{P}^1$ and let $I \subset S$. We denote by

$$(\mathcal{M}, I)$$

the parabolic line bundle with 2-step parabolic structure at S given by

$$I(\mathcal{M}) = I.$$

Let $\mathcal{E}$ be a rank 2 vector bundle on $\mathbf{P}^1$ and for every $x \in S$, let $\ell_x \subset \mathcal{E}|_x$ be a line in the fiber of $\mathcal{E}$ at x. We denote by

$$(\mathcal{E}, (\ell_x)_{x \in S})$$

the parabolic vector bundle with 2-step parabolic structure at S corresponding to the flags $(\ell_x)_{x\in S}$ (see Example 2.6).

Remark 2.8. For $p \in S$, let Iw_p denote the Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_p)$, which is defined as the inverse image of $B(\mathbf{F}_q)$ ($B \subset \mathrm{GL}_n$ the standard Borel) under the map $\mathrm{GL}_n(\mathcal{O}_p) \to \mathrm{GL}_n(\mathbf{F}_q)$. (Recall that p is an $\mathbf{F}_q$-point). Let $\mathrm{Bun}_{n,S}$ denote the moduli stack of rank n vector bundles with full flags at S and let $K = K(X)$ denote the function field of X, and $\mathbf{A}$ the adèles. Then there is a natural bijection

$$\mathrm{Bun}_{n,S}(\mathbf{F}_q) = \mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbf{A}) / \prod_{x \in X\setminus S} \mathrm{GL}_n(\mathcal{O}_x) \times \prod_{p\in S} \mathrm{Iw}_p$$

which is essentially due to the uniformization theorem and the fact that the Borel subgroup $B \subset \mathrm{GL}_{n,\mathbf{F}_q}$ is the stabilizer of the standard flag in $\mathbf{F}_q^{\oplus n}$. For $n = 2$ and $D = \{\infty, 0, 1, t\}\} \subset \mathbf{P}^1$ we explain this in more detail in Section 5.3. The bijection can be made into an equivalence of groupoids.

2.3. Extensions of parabolic structures. A sequence of maps of parabolic sheaves is exact when the associated maps of the chains that define the parabolic structures (i.e., the $\mathcal{F}^{(\bullet,p)}$ for all $p \in S$), form an exact sequence of chains. We can then define an extension of parabolic sheaves as an exact sequence that is a short exact sequence in every parabolic degree. In this section, we translate this condition to a condition on the flags that define the parabolic structure as explained in Section 2.2, but only for the particular case we are interested in.

Lemma 2.9. *Let $\mathcal{E}^\bullet = (\mathcal{E}, (\ell_p)_{p\in D})$ be a rank 2 parabolic vector bundle. Suppose we have an exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$$

where $\mathcal{L}$ and $\mathcal{M}$ are line bundles. Then the parabolic structures

$$\mathcal{L}^\bullet := (\mathcal{L}, I), \qquad \mathcal{M}^\bullet := (\mathcal{M}, S \setminus I)$$

with

$$I := \{p \in S : \ell_p = \mathcal{L}|_p\}$$

are the unique parabolic structures on $\mathcal{L}$ and $\mathcal{M}$ that induce a short exact sequence

$$0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet \to \mathcal{M}^\bullet \to 0$$

of parabolic vector bundles.

Note that it is essential for the uniqueness that we require $\mathcal{M}^\bullet$ to be a line bundle in every parabolic degree, as the proof shows. The induced exact sequence in parabolic degree (i,p) for given parabolic structures $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ on $\mathcal{L}$ and $\mathcal{M}$ is, if it exists, the unique short exact sequence $0 \to \mathcal{L}^{(i,p)} \to \mathcal{E}^{(i,p)} \to \mathcal{M}^{(i,p)} \to 0$ that restricts to $0 \to \mathcal{L}|_{X\setminus\{p\}} \to \mathcal{E}|_{X\setminus\{p\}} \to \mathcal{M}|_{X\setminus\{p\}} \to 0$.

Proof. We prove the uniqueness of the parabolic structures, leaving the verification that the parabolic structures from the lemma do indeed induce a short exact sequence to the reader.

Suppose that $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ are parabolic structures on $\mathcal{L}$ and $\mathcal{M}$ such that all $\mathcal{L}^{(i,p)}$ and $\mathcal{M}^{(i,p)}$ are line bundles and that induce a short exact sequence $0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet \to \mathcal{M}^\bullet \to 0$. For every parabolic degree (i,p), we then have $\deg \mathcal{E}^{(i,p)} = \deg \mathcal{L}^{(i,p)} + \deg \mathcal{M}^{(i,p)}$, so that $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ are indeed of the form $\mathcal{L}^\bullet := (\mathcal{L}, I), \mathcal{M}^\bullet := (\mathcal{M}, S \setminus I)$ for some $I \subset D$. To show the uniqueness, it only remains to show that I is as given in the lemma.

Let $p \in S$. If $\ell_p = \mathcal{L}|_p$, then since $\mathcal{L}^{(-1,p)} \hookrightarrow \mathcal{E}^{(-1,p)}$ has a line bundle as its cokernel (the line bundle $\mathcal{M}^{(-1,p)}$), this inclusion is saturated: we thus find $\mathcal{L}^{(-1,p)}|_p = \mathcal{L}|_p$ and $\mathcal{L}^{(-1,p)} = \mathcal{L}$. If $\ell_p \neq \mathcal{L}|_p$, then $\mathcal{L}(-p)$ is the largest subbundle of $\mathcal{L}$ that maps into $\mathcal{E}^{(-1,p)}$, so we conclude $\mathcal{L}^{(-1,p)} = \mathcal{L}(-p)$. This completes the proof. □

2.4. Parabolic torsion sheaves. In this section, we study parabolic torsion sheaves on $\mathbf{P}^1$ that have degree 1 in every parabolic degree. In the next section (Section 2.5), these are used to define modification of parabolic sheaves. In addition, we study the geometry of their moduli stack, $\mathbf{Coh}_0^{1,1}$, which looks like $\tilde{\mathbf{P}}^1 \times \mathbf{B}\mathbb{G}_\mathrm{m}$, where $\tilde{\mathbf{P}}^1$ is an algebraic stack with a map $\tilde{\mathbf{P}}^1 \to \mathbf{P}^1$ that is an isomorphism over $\mathbf{P}^1 \setminus D$ and that has fibers over $x \in D$ isomorphic to $[\{xy = 0\}/\mathbb{G}_\mathrm{m}] \subset [\mathbf{A}^2/\mathbb{G}_\mathrm{m}]$, where $\mathbb{G}_\mathrm{m}$ acts anti-diagonally (see (2.4.4)).

2.4.1. *Isomorphism classes of 2-step parabolic torsion sheaves of degree 1.* We now restrict ourselves to torsion sheaves with 2-step parabolic structure at $p \in \mathbf{P}^1(\mathbf{F}_q)$ that have degree 1 in every parabolic degree. Let $\mathcal{T}$ be such a parabolic torsion sheaf. Then $\mathcal{T}^{(i,p)}$ is isomorphic to k_p, the parabolic torsion sheaf $\mathcal{O}/\mathcal{O}(-p)$ of degree 1 supported at p, for all $i \in \mathbf{Z}$. The maps $\phi_{(i,p)}$ are either zero or an isomorphism (in this case, multiplication by a scalar). Hence, these sheaves are divided into three isomorphism classes, representatives of which we give the following notation:

$$\begin{aligned} k_p^0 &:= (\ldots \xrightarrow{0} k_p \xrightarrow{0} k_p \xrightarrow{0} k_p \xrightarrow{0} \ldots) \\ k_p^{(1,0)} &:= (\ldots \xrightarrow{0} k_p \xrightarrow{1} k_p \xrightarrow{0} k_p \xrightarrow{1} \ldots) \\ k_p^{(0,1)} &:= (\ldots \xrightarrow{1} k_p \xrightarrow{0} k_p \xrightarrow{1} k_p \xrightarrow{0} \ldots) \end{aligned}$$

where the last denoted k_p is placed in degree $(0, p)$.

The automorphism group of k_p^0 is $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$: we can independently scale each of the 2 parabolic degrees $(0, p)$ and $(-1, p)$. (The scaling in all other parabolic degrees are induced by these two.) The automorphism groups of $k_p^{(1,0)}$ and $k_p^{(0,1)}$ are isomorphic $\mathbb{G}_{\mathrm{m}}$: we have to scale each parabolic degree by the same factor, since there is an identity map between the two.

2.4.2. *Moduli spaces of degree 1 torsion sheaves.*

Definition 2.10. We define $\mathbf{Coh}_0^{1,1}$ as the stack classifying torsion sheaves $\mathcal{T}^\bullet$ on $\mathbf{P}^1$ with a parabolic structure at $D = \{\infty, 0, 1, t\}$ such that the degree of $\mathcal{T}^{(i,p)}$ is 1 in every parabolic degree (i, p), $i \in \mathbf{Z}$, $p \in D$. More formally, for T a scheme over $\mathbf{F}_q$, the T-points of $\mathbf{Coh}_0^{1,1}$ form a category

$$\mathbf{Coh}_0^{1,1}(T) := \left\langle \mathcal{T}^\bullet : \begin{array}{c} \mathcal{T}^\bullet \text{ a parabolic sheaf on } \mathbf{P}^1 \times T \\ \text{flat over } T \text{ and for all } t \in |T|, \\ \mathcal{T}^\bullet|_{\mathbf{P}^1 \times \{t\}} \text{ is a parabolic torsion sheaf} \\ \text{of degree 1 in every parabolic degree} \end{array} \right\rangle$$

We define $\overline{\mathbf{Coh}}_0^{1,1}$ as the rigidification of $\mathbf{Coh}_0^{1,1}$ with respect to the central automorphisms of the parabolic torsion sheaves, i.e., the automorphisms that are multiplication by a fixed scalar $\lambda \in k^*$ in every parabolic degree. This has rigidification has a general definition given in theorem 5.1.5 of [**ACV03**], but in our case, we can use the following explicit description: $\overline{\mathbf{Coh}}_0^{1,1}$ is isomorphic to the stack classifying pairs $(\mathcal{T}^\bullet, s)$ with $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ and $s \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{T}) \setminus \{0\}$.

Lastly,

- we define $\mathring{\mathbf{Coh}}_0^{1,1} \subset \mathbf{Coh}_0^{1,1}$ ($\overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}} \subset \overline{\mathbf{Coh}}_0^{1,1}$) as the substack of (rigidified) torsion sheaves with support outside of D; and
- $\mathbf{Coh}_0^1$, $\overline{\mathbf{Coh}}_0^1$, $\mathring{\mathbf{Coh}}_0^1$, $\mathring{\overline{\mathbf{Coh}}}_0^1$ are defined analogously, but classify torsion sheaves without parabolic structure.

Remark 2.11 (Maps between these moduli spaces)**.** The map

$$R\colon \mathbf{Coh}_0^{1,1} \to \overline{\mathbf{Coh}}_0^{1,1}, \qquad \mathcal{T}^\bullet \mapsto (\mathcal{T}^\bullet \otimes (\mathcal{T}^\bullet)^\vee, 1) \tag{2.4.1}$$

has a section

$$\overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{Coh}_0^{1,1}, \qquad (\mathcal{T}^\bullet, s^\bullet) \mapsto \mathcal{T}^\bullet \tag{2.4.2}$$

which exhibits $\mathbf{Coh}_0^{1,1}$ as a trivial $\mathbf{B}\mathbb{G}_{\mathrm{m}}$-gerbe over $\overline{\mathbf{Coh}}_0^{1,1}$.

We have a natural isomorphism

$$\mathbf{P}^1 \to \overline{\mathbf{Coh}}_0^1, \qquad p \mapsto (\mathcal{O}/\mathcal{O}(-p), 1).$$

In addition, we have 2^4 sections

$$\overline{\mathbf{Coh}}_0^1 \to \overline{\mathbf{Coh}}_0^{1,1}$$

of the forgetful map $\overline{\mathbf{Coh}}_0^{1,1} \to \overline{\mathbf{Coh}}_0^1$: for each $x \in D$, we can send the skyscraper sheaf k_x at x to either $k_x^{(1,0)}$ or $k_x^{(0,1)}$, but not to k_x^0. (This follows from Lemma 2.12.) All these maps restrict to the same isomorphism

$$\overline{\mathring{\mathbf{Coh}}}_0^1 \xrightarrow{\sim} \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}$$

where $\overline{\mathring{\mathbf{Coh}}}_0^1 \subset \overline{\mathbf{Coh}}_0^1$ denotes the open substack of sheaves supported outside of D. We thus get isomorphisms

$$\mathbf{P}^1 \setminus D \xrightarrow{\sim} \overline{\mathring{\mathbf{Coh}}}_0^1 \xrightarrow{\sim} \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}$$

with inverse given by the restriction of the map

$$\mathrm{Supp}\colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1, \qquad (\mathcal{T}^\bullet, s^\bullet) \mapsto \mathrm{Supp}\,\mathcal{T}^0. \tag{2.4.3}$$

The map $\mathrm{Supp}\colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$ is the map from $\overline{\mathbf{Coh}}_0^{1,1}$ to its coarse moduli space. The fact that this is the map to the coarse moduli space follows from the following local description of $\mathbf{Coh}_0^{1,1}$.

Locally over $\mathbf{P}^1$, around each $p \in D$, the stack $\mathbf{Coh}_0^{1,1}$ looks like the stack $\mathbf{Coh}^1_{\mathbf{A}^1,0}$ that classifies torsion sheaves on $\mathbf{A}^1$ with parabolic structure on $0 \in \mathbf{A}^1(\mathbf{F}_q)$ that have degree 1 in every parabolic degree. The following lemma tells us we have an isomorphism

$$\mathbf{Coh}^1_{\mathbf{A}^1,0} \cong [\mathbf{A}^2/\mathbb{G}_{\mathrm{m}}] \times \mathbf{B}\mathbb{G}_{\mathrm{m}} \tag{2.4.4}$$

where $\mathbb{G}_{\mathrm{m}}$ acts by $\lambda(x,y) = (\lambda x, \lambda^{-1} y)$.

Lemma 2.12. *The map*

$$\mathbf{A}^2 \to \mathbf{Coh}^1_{\mathbf{A}^1,0}, \qquad (\mu,\nu) \mapsto (\dots \xrightarrow{\mu} k \xrightarrow{\nu} k \xrightarrow{\mu} k \xrightarrow{\nu} k \dots)$$

is a $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$-torsor, with action on $\mathbf{A}^2$ given by $(r,s)(\mu,\nu) = (r\mu, s^{-1}\nu)$. In particular, $\mathbf{Coh}^1_{\mathbf{A}^1,0}$ and therefore $\mathbf{Coh}_0^{1,1}$ are smooth.

PROOF. This is a special case of [**Hei04**, lemma 3.6]. For the reader's benefit, we repeat the argument.

Every degree 1 torsion sheaf on $\mathbf{A}^1 = \operatorname{Spec} k[t]$ can equivalently be given by its $k[t]$-module of global sections. Because we assumed that the degree is one, this module of global sections is isomorphic to k as a k-vector space, so it is given by the action of $t \in k[t]$, which is multiplication by some $\lambda \in k$. This $\lambda \in \mathbf{A}^1$ is exactly the point where the torsion sheaf is supported. A 2-step parabolic structure at 0 on such a torsion sheaf is of the form

$$\dots \xrightarrow{\mu} k \xrightarrow{\nu} k \xrightarrow{\mu} k \xrightarrow{\nu} k \dots$$

for some $\mu, \nu \in k$ with $\mu\nu = \lambda$. (The condition $\mu\nu = \lambda$ comes from the requirement that the composition of two morphisms in the parabolic structure is induced by the map $\mathcal{O} \hookrightarrow \mathcal{O}([0])$.) $\square$

2.5. Modifications of parabolic sheaves. In this section, we introduce the concept of a lower modification of a (parabolic) coherent sheaf. These are essential in defining Hecke operators.

Definition 2.13 (Lower and upper modifications of (parabolic) sheaves)**.** Let $\mathcal{F}^\bullet$ be a parabolic coherent sheaf and let $\mathcal{T}^\bullet$ be a parabolic torsion sheaf supported at a point $p \in X(\mathbf{F}_q)$. A *lower modification* of a $\mathcal{F}^\bullet$ at $p \in X(\mathbf{F}_q)$ with respect to $\mathcal{T}^\bullet$ is a parabolic subbundle $\mathcal{G}^\bullet \subset \mathcal{F}^\bullet$ such that $\mathcal{F}^\bullet/\mathcal{G}^\bullet$ is isomorphic to $\mathcal{T}^\bullet$. The length of $\mathcal{T}^\bullet$ is referred to as the *length of the modification.* If $\mathcal{G}^\bullet \subset \mathcal{F}^\bullet$ is a lower modification with respect to $\mathcal{T}^\bullet$, then we call $\mathcal{F}^\bullet$ an *upper modification* of $\mathcal{G}^\bullet$ with respect to $\mathcal{T}^\bullet$.

These definitions also apply to sheaves without parabolic structures.

Example 2.14. A length 1 lower modification of a parabolic line bundle $(\mathcal{L}, I)$ at x is isomorphic to

$$T_x(\mathcal{L}, I) := \begin{cases} (\mathcal{L}(-x), I) & \text{if } x \notin S \\ (\mathcal{L}, I \setminus \{x\}) & \text{if } x \in I \\ (\mathcal{L}(-x), I \cup \{x\}) & \text{if } x \in S \setminus I \end{cases}.$$

Example 2.15. Let $\mathcal{E}^\bullet$ be a parabolic vector bundle. The vector bundle $\mathcal{E}^{(i,p)}$, considered as a subbundle of $\mathcal{E}^{(i+1,p)}$ via the map $\phi_{(i,p)} \colon \mathcal{E}^{(i,p)} \to \mathcal{E}^{(i+1,p)}$, is a lower modification of $\mathcal{E}^{(i+1,p)}$ at p. This example does not work in general when $\mathcal{E}^\bullet$ has torsion, since in that case, $\phi_{(i,p)}$ is not necessarily an inclusion.

Remark 2.16. A lower modification of $\mathcal{F}^\bullet$ at a point outside of D is the same as a lower modification $\mathcal{E} \hookrightarrow \mathcal{F}$ of the underlying sheaf $\mathcal{F}$.

Example 2.17 (Elementary Hecke operators)**.** Let $\mathcal{F}^\bullet$ be a parabolic sheaf and let $p \in D$. We get a new parabolic sheaf, denoted by $T_p\mathcal{F}^\bullet$, by shifting the parabolic structure at p: we set

$$(T_p\mathcal{F}^\bullet)^{(i,p)} := \mathcal{F}^{(i+1,p)}$$

and the parabolic structure of $T_p\mathcal{F}^\bullet$ at the other points in D is induced by $T_p\mathcal{F} \subset \mathcal{F}$. Suppose now that $\mathcal{F}^\bullet$ is a parabolic vector bundle with full flags, as in Example 2.6. Then this gives an exact sequence

$$0 \to T_p\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to k_p^0 \to 0$$

where $k_p^0 = (\ldots \xrightarrow{0} k_p \xrightarrow{0} k_p \xrightarrow{0} k_p \xrightarrow{0} \ldots)$ (as defined in Section 2.4.1). Up to unique isomorphism, this is the unique lower modification of $\mathcal{F}^\bullet$ with respect to k_x^0: given a surjective map of chains $\mathcal{F}^{(\bullet,p)} \twoheadrightarrow (k_p^0)^{(\bullet,p)}$, we conclude from the commutative squares in the chain map that the composition $\mathcal{F}^{(i,p)} \hookrightarrow \mathcal{F}^{(i+1,p)} \twoheadrightarrow k_p$ is zero, so that $T_p\mathcal{F}^\bullet$ is indeed contained in the kernel; and by counting the degrees, we find that $T_p\mathcal{F}^\bullet$ is all of the kernel.

Example 2.18. A morphism

$$(\mathcal{L}, I) \to (\mathcal{E}, (\ell_p)_{p\in S})$$

from a parabolic line bundle $(\mathcal{L}, I)$ to a parabolic vector bundle $\mathcal{E}^\bullet$ is the same as a morphism of vector bundles

$$\mathcal{L} \to T_I\mathcal{E}.$$

2.6. Modifications of rank 2 parabolic vector bundles. Here we describe the length 1 lower modifications of rank 2 parabolic vector bundles with full flags in terms of flags: see Notation 2.19 for modifications at points away from S and see Notation 2.23 for modifications at points in S. In addition, we similarly describe length 1 lower modifications of length 1 lower modifications (Lemma 2.27) and remark that the moduli stack of modifications of a fixed rank 2 parabolic bundle $\mathcal{E}^\bullet$ is isomorphic to $\mathrm{Bl}_D(\mathcal{E}^\bullet)$ (Proposition 2.28).

We start with length 1 lower modifications of rank 2 vector bundles without parabolic structure.

Notation 2.19. Let $\mathcal{E}$ be a rank 2 vector bundle on $\mathbf{P}^1$ and let $x \in \mathbf{P}^1(\mathbf{F}_q)$. For any $\ell \subset \mathcal{E}|_x$, we write

$$T_x^\ell\mathcal{E} := \ker(\mathcal{E} \twoheadrightarrow \mathcal{E}|_x/\ell).$$

This is length 1 lower modification of $\mathcal{E}$ at x with respect to the skyscraper sheaf k_x with support at x.

Remark 2.20. Every length one lower modification of a rank 2 vector bundle $\mathcal{E}$ is of the form $T_x^\ell\mathcal{E}$ for some $x \in \mathbf{P}^1$ and $\ell \subset \mathcal{E}|_x$. Indeed, every degree 1 torsion sheaf on $\mathbf{P}^1$ is isomorphic to a skyscraper sheaf $k_x = \mathcal{O}/\mathcal{O}(-x)$ supported at some point $x \in \mathbf{P}^1$ and if the lower modification is the kernel of the map $\phi\colon \mathcal{E} \twoheadrightarrow \mathcal{T}$, we can take $\ell := \ker(\phi|_x\colon \mathcal{E}|_x \to \mathcal{T}|_x)$.

Example 2.21. If $\mathcal{E}^{(\bullet,x)}$ is a parabolic structure on $\mathcal{E}$ at x and $\ell_x \subset \mathcal{E}|_x$ is the associated flag (i.e., the image of $\mathcal{E}^{(-1,x)}|_x \to \mathcal{E}|_x$; see Section 2.2), then $\mathcal{E}^{(-1,x)} = T_x^{\ell_x}\mathcal{E}$.

We now turn to modifications of rank 2 parabolic vector bundles.

Remark 2.22. Modifications of a parabolic bundle outside of S are simply modifications of the underlying vector bundle. Indeed, let $(\mathcal{E}, (\ell_p)_{p\in S})$ be a rank 2 parabolic vector bundle with full flags at S and let $(\mathcal{E}', (\ell'_p)_{p\in S}) \subset (\mathcal{E}, (\ell_p)_{p\in S})$ be a length 1 lower modification at some point $x \in \mathbf{P}^1 \setminus S$. By Remark 2.20, $\mathcal{E}'$ is isomorphic vector bundle $T_x^\ell \mathcal{E}$ for some $\ell \subset \mathcal{E}|_x$, and in fact, there is an isomorphism of sub parabolic vector bundles of $(\mathcal{E}, (\ell_p)_{p\in S})$

$$(\mathcal{E}', (\ell'_p)_{p\in S}) \xrightarrow{\sim} (T_x^\ell \mathcal{E}, (\ell_p)_{p\in S}).$$

Modifications at a point $p_0 \in S$ are more complicated. The underlying vector bundle of such a modification is still isomorphic to $T_{p_0}^\ell \mathcal{E}$ for some $\ell \subset \mathcal{E}|_{p_0}$ and the parabolic structures at $S \setminus \{p_0\}$ come from $\mathcal{E}^\bullet$, but the parabolic structure at p_0 depends on the isomorphism class of the cokernel of the modification, as we will now explain.

In Notation 2.23, we introduce for every $x \in S$ three operators

$$T_x, \quad T_x^\ell \quad \text{and} \quad {}^{\ell'}T_x$$

that send a parabolic vector bundle $\mathcal{E}^\bullet$ to a length 1 lower modification

$$T_x\mathcal{E}^\bullet,\ T_x^\ell\mathcal{E}^\bullet,\ {}^{\ell'}T_x\mathcal{E}^\bullet \subset \mathcal{E}^\bullet$$

of $\mathcal{E}^\bullet$ at x. The three operators correspond to the three isomorphism classes of the cokernels, which are degree 1 torsion sheaves supported at x, isomorphic to either k_x^0, $k_x^{(1,0)}$ or $k_x^{(0,1)}$ (see Section 2.4.1). In the process, we classify all length 1 lower modifications of $(\mathcal{E}, (\ell_p)_{p\in S})$ at x.

Notation 2.23. Let $\mathcal{E}^\bullet = (\mathcal{E}, (\ell_p)_{p\in S})$ be a rank 2 parabolic vector bundle with parabolic structure at S and let $x \in S$.

(1) the length 1 lower modifications of $(\mathcal{E}, (\ell_p)_{p\in S})$ with respect to k_x^0 are the modifications of the form

$$0 \to T_x\mathcal{E}^\bullet \to (\mathcal{E}, (\ell_p)_{p\in S}) \to k_x^0 \to 0$$

where $T_x\mathcal{E}^\bullet$ denotes the rank 2 parabolic vector bundle

$$T_x\mathcal{E}^\bullet := (T_x^{\ell_x}\mathcal{E}, (\ell_p)_{p\in S\setminus\{x\}}, \ell'_x),$$

with parabolic structure at x given by the flag

$$\ell'_x := \ker(T_x^{\ell_x}\mathcal{E}|_x \to \mathcal{E}|_x).$$

The operators T_x are the elementary Hecke operators we introduced in Example 2.17.

(2) the length 1 lower modifications of $(\mathcal{E}, (\ell_p)_{p\in S})$ with respect to $k_x^{(0,1)}$ are the modifications of the form

$$0 \to T_x^{\ell}\mathcal{E}^\bullet \to (\mathcal{E}, (\ell_p)_{p\in S}) \to k_x^{(0,1)} \to 0$$

where

$$\ell \subset \mathcal{E}|_x$$

is any line different from ℓ_x, and $T_x^{\ell}\mathcal{E}^\bullet$ denotes the rank 2 parabolic bundle

$$T_x^{\ell}\mathcal{E}^\bullet := (T_x^{\ell}\mathcal{E}, (\ell_p)_{p\in S\setminus\{x\}}, \ell'_x),$$

with parabolic structure at x given by the flag

$$\ell'_x := \ker(T_x^{\ell}\mathcal{E}|_x \to \mathcal{E}|_x).$$

(3) the length 1 lower modifications of $(\mathcal{E}, (\ell_p)_{p\in S})$ with respect to $k_x^{(1,0)}$ are the modifications of the form

$$0 \to {}^{\ell'}T_x\mathcal{E}^\bullet \to (\mathcal{E}, (\ell_p)_{p\in S}) \to k_x^{(0,1)} \to 0$$

where

$$\ell' \subset (T_x^{\ell_x}\mathcal{E})|_x = \mathcal{E}^{(-1,x)}|_x$$

is any line different from $\ell'_x := \ker(T_x^{\ell_x}\mathcal{E}|_x \to \mathcal{E}|_x)$, and ${}^{\ell'}T_x\mathcal{E}^\bullet$ denotes the rank 2 parabolic bundle

$${}^{\ell'}T_x\mathcal{E}^\bullet := (T_x^{\ell}\mathcal{E}, (\ell_p)_{p\in S\setminus\{x\}}, \ell')$$

with parabolic structure at x given by ℓ'.

Remark 2.24. Because modifications change a bundle only locally, modifications at different points commute: if x, y are two different point in $\mathbf{P}^1(\mathbf{F}_q)$ and $\ell_x \subset \mathcal{E}|_x$ and $\ell_y \subset \mathcal{E}|_y$ are lines, then $T_x^{\ell_x}(T_y^{\ell_y}\mathcal{E}) = T_y^{\ell_y}(T_x^{\ell_x}\mathcal{E})$, and similarly for the other operators listed above.

Notation 2.25. For $J \subset S$, we denote by T_J the composition of the operators T_x with $x \in J$:

$$T_J := \prod_{x\in J} T_x.$$

The following lemma lists some straightforward but useful identities for these modifications.

Lemma 2.26. *Let $\mathcal{E}^\bullet$ be a rank 2 parabolic bundle on $\mathbf{P}^1$, let $x \in \mathbf{P}^1(\mathbf{F}_q) \setminus S$ and let $\ell \subset \mathcal{E}|_x$. Define*

$$\ell' := \ker((T_x\mathcal{E})|_x \to \mathcal{E}|_x).$$

Then

$$T_x^{\ell'}T_x^{\ell}\mathcal{E}^\bullet = \mathcal{E}^\bullet(-x).$$

PROOF. By our choice of ℓ', we have $T_x^{\ell'} T_x^{\ell} \mathcal{E} \subset \ker(\mathcal{E} \to \mathcal{E}|_x) = \mathcal{E}(-x)$. By counting the degrees, we can conclude that this is an equality. $\square$

Lemma 2.27. *Let $\mathcal{E}^\bullet = (\mathcal{E}, (\ell_p)_{p\in S})$ be a rank 2 parabolic vector bundle and let $x \in S$. The operators T_x, T_x^{ℓ} and ${}^{\ell'}T_x$ defined in Notation 2.23 satisfy the following identities, where ℓ and ℓ' are lines in $\mathcal{E}|_x$ or in $(T_x\mathcal{E})|_x$ such that the notation makes sense:*

$$T_x \circ T_x = (- \otimes \mathcal{O}(-x)) \tag{2.6.1}$$

$$T_x^{\ell'} \circ {}^{\ell}T_x = T_x^{\ell'} \circ T_x = T_x \circ {}^{\ell'}T_x \tag{2.6.2}$$

$$\begin{aligned} {}^{\ell'}T_x \circ T_x^{\ell} &= {}^{\ell'}T_x \circ T_x = T_x \circ T_x^{\ell' \otimes \mathcal{O}(x)} \\ &= \textit{(replace flag at } x \textit{ by } \ell') \circ (- \otimes \mathcal{O}(-x)) \end{aligned} \tag{2.6.3}$$

PROOF. This follows directly from the descriptions given in Notation 2.23 and Lemma 2.26. $\square$

We now study the geometric structure of the modifications. Let $\mathcal{E}^\bullet$ be a rank 2 parabolic bundle. We denote by

$$\mathrm{Modif}(\mathcal{E}^\bullet) = \langle \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet \rangle$$

the stack that classifies surjections from $\mathcal{E}^\bullet$ to a parabolic torsion sheaf $\mathcal{T}^\bullet$ that is degree 1 in every degree. The morphisms in this stack are commutative diagrams

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r, two heads] \arrow[dr, two heads] & \mathcal{T}^\bullet \arrow[d, "\wr"] \\ & \mathcal{T}'^\bullet \end{tikzcd}$$

The main result is the following proposition. Note that the parabolic structure $\mathcal{E}^\bullet$ on $\mathcal{E}$ induces an inclusion

$$D \subset \mathbf{P}(\mathcal{E}), \qquad x \mapsto (\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}^{(-1,x)}).$$

Proposition 2.28. *The map*

$$\mathrm{Modif}(\mathcal{E}^\bullet) \to \mathbf{P}(\mathcal{E}), \qquad (\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\mathcal{E} \twoheadrightarrow \mathcal{T})$$

induces by the universal properties of blow-ups an isomorphism

$$\mathrm{Modif}(\mathcal{E}^\bullet) \xrightarrow{\sim} \mathrm{Bl}_D(\mathbf{P}(\mathcal{E})).$$

Since we will not need this result, we will not prove but it, but we note that it can be proven using local coordinates around the points in D.

3. The moduli space $\mathrm{Bun}_{2,4}$

In this section, we list some properties of $\mathrm{Bun}_{2,4}$, without proofs. Let $t \in \mathbf{P}^1(\mathbf{F}_q) \setminus \{\infty, 0, 1\}$. We write

$$D := \{\infty, 0, 1, t\} \subset \mathbf{P}^1(\mathbf{F}_q).$$

One of the main geometric objects in this thesis is the stack that classifies rank 2 vector bundles on $\mathbf{P}^1 = \mathbf{P}^1_{\mathbf{F}_q}$ with a 2-step parabolic structure of full flags at S (cf. example 2.6), i.e., satisfying $\deg \mathcal{E}^{(i,p)} = \deg \mathcal{E}^{(0,p)} + i$ for all $p \in S$ and $i \in \mathbf{Z}$. This is a smooth, locally of finite type algebraic stack that we denote by $\mathrm{Bun}_{2,4}$. It decomposes according to the degree:

$$\mathrm{Bun}_{2,4} = \coprod_{d \in \mathbf{Z}} \mathrm{Bun}^d_{2,4}$$

where $\mathrm{Bun}^d_{2,4}$ is the substack classifying parabolic bundles that have an underlying vector bundle of degree d. These $\mathrm{Bun}^d_{2,4}$ are connected.

Tensoring with $\mathcal{O}(1)$ gives us an isomorphism

$$- \otimes \mathcal{O}(1) \colon \mathrm{Bun}^d_{2,4} \xrightarrow{\sim} \mathrm{Bun}^{d-2}_{2,4}.$$

For every $x \in D$, the elementary Hecke operator T_x (defined in Example 2.17) gives an isomorphism

$$T_x \colon \mathrm{Bun}^d_{2,4} \xrightarrow{\sim} \mathrm{Bun}^{d-1}_{2,4} \tag{3.0.1}$$

but this isomorphism depends on the choice of x. (We will describe the automorphism $T_x T_y^{-1}$ for $x, y \in D$ in Section 8.4.) Applying T_x twice, however, is $T_x \circ T_x = (- \otimes \mathcal{O}(x))$, so that on points of stacks, this double application does not depend on x.

Let $d \in \mathbf{Z}$. The stack $\mathrm{Bun}^d_{2,4}$ has an open substack

$$\mathrm{Bun}^{\mathcal{O}(\lfloor d/2 \rfloor) \oplus \mathcal{O}(\lceil d/2 \rceil)}_{2,4} \subset \mathrm{Bun}^d_{2,4}$$

that consists of those parabolic bundle $\mathcal{E}^\bullet$ of degree d such that the underlying bundle $\mathcal{E}$ is isomorphic to $\mathcal{O}(\lfloor d/2 \rfloor) \oplus \mathcal{O}(\lceil d/2 \rceil)$. To see that this substack is indeed open, one can for example use that these are the bundles with an automorphism group of minimal dimension, or alternatively, one can prove it for $d = -1, -2$ with the upper-semicontinuity of $\dim \mathrm{H}^0(\mathbf{P}^1, -)$ and then use the isomorphisms 3.0.1.

4. Geometric Hecke operators

In this section, we define a geometric analogue of the classical Hecke operators. The (geometric) global Hecke operator sends a perverse sheaf on $\mathrm{Bun}_{2,4}$ to a constructible sheaf on $\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$ via a correspondence $\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} \leftarrow \mathcal{H} \rightarrow \mathrm{Bun}_{2,4}$, where $\mathcal{H}$ is the Hecke stack of length 1. We define this stack and the global Hecke operator (Definition 4.2) in Section 4.1. The next section then defines the local Hecke operators by restricting the above correspondence. The last sections study some additional properties of the Hecke stack and the Hecke operators.

4.1. Definition of the global Hecke stack.

Definition 4.1. The *Hecke stack of length 1* is the stack

$$\mathcal{H} := \left\langle 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{T} \to 0 \text{ exact} : \ \mathcal{E}', \mathcal{E} \in \mathrm{Bun}_{2,4}, \quad \mathcal{T} \in \mathbf{Coh}_0^{1,1} \ \right\rangle .$$

The objects are families of such exact sequences and the isomorphisms are isomorphisms of exact sequences. It classifies modifications (see Section 2.5).

We have maps p, q as in the diagram

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_{2,4} \times \mathbf{Coh}_0^{1,1} & & \mathrm{Bun}_{2,4} \end{tikzcd} \tag{4.1.1}$$

defined by

$$\begin{aligned} p(0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{T} \to 0) &= (\mathcal{E}, \mathcal{T}) \\ q(0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{T} \to 0) &= \mathcal{E}' \end{aligned} . \tag{4.1.2}$$

For every degree $d \in \mathbf{Z}$, this restricts to maps

$$\begin{tikzcd} & \mathcal{H}^d \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_{2,4}^d \times \mathbf{Coh}_0^{1,1} & & \mathrm{Bun}_{2,4}^{d-1} \end{tikzcd} \tag{4.1.3}$$

where $\mathcal{H}^d = p^{-1}(\mathrm{Bun}_{2,4}^d \times \mathbf{Coh}_0^{1,1})$.

We can compose the map p with the map $\mathrm{id} \times R \colon \mathrm{Bun}_{2,4} \times \mathbf{Coh}_0^{1,1} \to \mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$, where $R \colon \mathbf{Coh}_0^{1,1} \to \overline{\mathbf{Coh}}_0^{1,1}$ denotes the rigidification with respect to the central automorphisms (see Equation (2.4.1)), which gives us the diagram

$$\begin{tikzcd} & \mathcal{H}^d \arrow[dl, "\bar{p}"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_{2,4}^d \times \overline{\mathbf{Coh}}_0^{1,1} & & \mathrm{Bun}_{2,4}^{d-1} \end{tikzcd} . \tag{4.1.4}$$

Definition 4.2 (The global Hecke operator)**.** The *global Hecke operator* is the map

$$\mathbb{H} \colon D^{\mathrm{b}}(\mathrm{Bun}_{2,4}, \mathbf{Q}_\ell) \to D^{\mathrm{b}}(\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}, \mathbf{Q}_\ell), \qquad F \mapsto \bar{p}_! q^* F$$

on the bounded derived categories of ℓ-adic sheaves on $\mathrm{Bun}_{2,4}$ and $\mathrm{Bun}_{2,4} \times \mathbf{Coh}_0^{1,1}$.

Definition 4.3 (Hecke eigensheaf)**.** Let E be an irreducible local system on $\mathbf{P}^1 \setminus D$ and let F be a perverse sheaf on $\mathrm{Bun}_{2,4}$. We say that F is a *Hecke eigensheaf* for E if there exists an isomorphism in $D^{\mathrm{b}}(\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}, \mathbf{Q}_\ell)$

$$\mathbb{H}F \cong F \boxtimes j_{!*}E$$

where $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1}$ is the inclusion defined in Section 2.4.2.

4.2. Local Hecke operators. Let S be an $\mathbf{F}_q$-scheme and $\mathcal{T}^\bullet \in \overline{\mathbf{Coh}}_0^{1,1}(S)$. We can restrict the global Hecke diagram (Equation (4.1.1)) along the map $S \xrightarrow{\mathcal{T}^\bullet} \overline{\mathbf{Coh}}_0^{1,1}$ to get a "local Hecke diagram"

$$\begin{tikzcd}
 & \mathcal{H}_{\mathcal{T}^\bullet} \arrow[dl, "p_{\mathcal{T}^\bullet}"'] \arrow[d] \arrow[ddr, bend left, "q_{\mathcal{T}^\bullet}"] & \\
\mathrm{Bun}_{2,4} \times S \arrow[r, phantom, "\square"] \arrow[d, "\mathrm{id}\times\mathcal{T}^\bullet"] & \mathcal{H} \arrow[dl, "p"] \arrow[dr, "q"] & \\
\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} & & \mathrm{Bun}_{2,4}
\end{tikzcd}$$

where $(\mathcal{H}_{\mathcal{T}^\bullet}, p_{\mathcal{T}^\bullet})$ is the pullback of $p\colon \mathcal{H} \to \mathbf{P}^1 \times \overline{\mathbf{Coh}}_0^{1,1}$ along $\mathrm{id} \times \mathcal{T}^\bullet$ and $q_{\mathcal{T}^\bullet}$ is the composition $\mathcal{H}_{\mathcal{T}^\bullet} \to \mathcal{H} \xrightarrow{q} \mathbf{P}^1$. This allows us to define a local Hecke operator.

Definition 4.4. Let the notation be as above. The *local Hecke operator* at $\mathcal{T}^\bullet \in \overline{\mathbf{Coh}}_0^{1,1}(S)$ is the map

$$\mathbb{H}_{\mathcal{T}^\bullet}\colon D^{\mathrm{b}}(\mathrm{Bun}_{2,4}, \mathbf{Q}_\ell) \to D^{\mathrm{b}}(\mathrm{Bun}_{2,4} \times S, \mathbf{Q}_\ell), \qquad F \mapsto p_{\mathcal{T}^\bullet,!}q_{\mathcal{T}^\bullet}^* F$$

on the bounded derived categories of ℓ-adic sheaves on $\mathrm{Bun}_{2,4}$ and $\mathrm{Bun}_{2,4} \times S$.

Notation 4.5. Let $k \supset \mathbf{F}_q$ be a finite field extension and let $x \in \mathbf{P}^1(k) \setminus D$. We write

$$\mathbb{H}_x := \mathbb{H}_{k_x}$$

where $k_x \in \overline{\mathbf{Coh}}_0^{1,1}(k)$ is the skyscraper sheaf supported at x, necessarily with trivial parabolic structure. For $x \in D \subset \mathbf{P}^1(\mathbf{F}_q)$, we write

$$\mathbb{H}_x := \mathbb{H}_{k_x^0}, \qquad \mathbb{H}_x^{\mathrm{r}} := \mathbb{H}_{k_x^{(0,1)}}, \quad \text{and} \quad \mathbb{H}_x^{\mathrm{l}} := \mathbb{H}_{k_x^{(1,0)}}$$

where the $k_x^\bullet$ represent the three isomorphism classes of degree 1 parabolic torsion sheaves supported on x (see Section 2.4.1).

The superscript r and l in the notation stand for right and left: $\mathbb{H}_x^{\mathrm{r}}$ sums over modifications of the form T_x^ℓ, whereas $\mathbb{H}_x^{\mathrm{l}}$ sums over modifications of the form ${}^{\ell'}T_x$.

Remark 4.6 (Local Hecke operators on $\mathbf{F}_q$-points)**.** Let F be a perverse sheaf on $\mathrm{Bun}_{2,4}$ and let $x \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$. By taking traces of Frobenii (see e.g. [**Lau87**, (1.1)]), a perverse sheaf F defines a function $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$. The function $\mathbb{H}_x f$ defined by $\mathbb{H}_x F$ is given by

$$(\mathbb{H}_x f)(\mathcal{E}^\bullet) = \sum_{(\mathcal{F}^\bullet \subset \mathcal{E}^\bullet) \in p^{-1}(\mathcal{E}^\bullet, k_x)} f(\mathcal{F}^\bullet).$$

(See [**Lau87**, (1.1)] for a summary of the properties of the associated trace-of-Frobenius functions. Here we use that under the function-sheaf dictionary, the proper pushforward on sheaves corresponds to the operation $g \mapsto (x \mapsto \sum_{y \in g^{-1}(x)} g(y))$ on functions ([**Lau87**, (1.1.1.3)]), and pullback on sheaves corresponds to pullback of functions.)

Recall that for $x \in \mathbf{P}^1 \setminus D$, the isomorphism classes of length 1 lower modifications of $\mathcal{E}^\bullet$ at x are classified by $\mathbf{P}^1(\mathcal{E})$: we have a bijection

$$\begin{aligned}\{\mathcal{F}^\bullet \subset \mathcal{E}^\bullet \quad \text{length 1 modification at } x\}_{/\cong} &\to \mathbf{P}^1(\mathcal{E}|_x),\\ (\mathcal{F}^\bullet \subset \mathcal{E}^\bullet) &\mapsto \operatorname{im}(\mathcal{F}|_x \to \mathcal{E}|_x)\end{aligned}$$

with inverse denoted by

$$\ell \mapsto T_x^\ell \mathcal{E}^\bullet$$

(Remark 2.20). This allows us to rewrite this sum more explicitly as

$$(\mathbb{H}_x f)(\mathcal{E}) = \sum_{\ell \subset \mathcal{E}|_x} f(T_x^\ell \mathcal{E}).$$

For $x \in S$, the three Hecke operators $\mathbb{H}_x$, $\mathbb{H}_x^{\mathrm{l}}$ and $\mathbb{H}_x^{\mathrm{r}}$ correspond to the three different types of modifications. If we denote by $\ell_x \subset \mathcal{E}|_x$ the flag that defines the parabolic structure of $\mathcal{E}^\bullet$ at x and by $\ell'_x \subset (T_x\mathcal{E})|_x$ the flag $\ker((T_x\mathcal{E})|_x \to \mathcal{E}|_x)$ that defines the parabolic structure of $T_x\mathcal{E}^\bullet$, then we have

$$(\mathbb{H}_x f)(\mathcal{E}) = (q-1) f(T_x\mathcal{E}) \qquad (\mathbb{H}_x^{\mathrm{l}} f)(\mathcal{E}) = \sum_{\substack{\ell \subset (T_x\mathcal{E})|_x,\\ \ell \neq \ell'_x}} f({}^\ell T_x \mathcal{E})$$

$$(\mathbb{H}_x^{\mathrm{r}} f)(\mathcal{E}) = \sum_{\substack{\ell \subset \mathcal{E}|_x,\\ \ell \neq \ell_x}} f(T_x^\ell \mathcal{E})$$

Remark 4.7 (Hecke eigenfunctions)**.** Suppose that F is a Hecke eigensheaf for the local system E on $\mathbf{P}^1 \setminus D$, i.e.,

$$\mathbf{R}p_! q^* F \xrightarrow{\sim} F \boxtimes j_{!*} E \tag{4.2.1}$$

where $j \colon \mathbf{P}^1 \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1}$ is the inclusion. Then the trace-of-Frobenius function $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ associated to f is an eigenfunction for the local Hecke operators: for every $\mathcal{T}^\bullet \colon \operatorname{Spec} k \to \overline{\mathbf{Coh}}_0^{1,1}$, restricting (4.2.1) to $\mathbf{P}^1 \times \{\mathcal{T}^\bullet\}$ and taking the associated trace-of-Frobenius functions shows

$$\mathbb{H}_{\mathcal{T}^\bullet} f = \lambda^E_{\mathcal{T}^\bullet} \cdot f$$

where $\lambda^E_{\mathcal{T}^\bullet}$ is the trace of the Frobenius acting on $(j_{!*}E)|_{\mathcal{T}^\bullet}$.

4.3. Dual Hecke stack. Let $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}$ be a parabolic vector bundle. We denote by $\overset{\vee}{\mathcal{E}}{}^\bullet$ the parabolic vector bundle that is $(\mathcal{E}^{(-i,x)})^\vee$ in parabolic degree (i,x). In this section, we study some properties of the maps

$$\mathrm{dual}\colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{-d}, \qquad \mathcal{E}^\bullet \mapsto \overset{\vee}{\mathcal{E}}{}^\bullet$$

and

$$\mathrm{dual}\colon \mathcal{H} \xrightarrow{\sim} \mathcal{H}, \qquad (\mathcal{F}^\bullet \subset \mathcal{E}^\bullet) \mapsto (\overset{\vee}{\mathcal{E}}{}^\bullet \subset \overset{\vee}{\mathcal{F}}{}^\bullet).$$

First, we define a map dual: $\mathbf{Coh}_0^{1,1} \to \mathbf{Coh}_0^{1,1}$, as follows. Let $\mathcal{T}$ be a length 1 torsion sheaf on $\mathbf{P}^1$, without parabolic structure. We define

$$\operatorname{dual}(\mathcal{T}) := \mathscr{H}om(\mathcal{T}, \mathcal{O}_{\operatorname{Supp}\mathcal{T}}) \otimes \mathcal{O}(\operatorname{Supp}\mathcal{T})$$

where $\mathcal{O}_{\operatorname{Supp}\mathcal{T}}$ denotes the pushforward of $\mathcal{O}_{\operatorname{Supp}\mathcal{T}}$ along $\operatorname{Supp}\mathcal{T} \hookrightarrow \mathbf{P}^1$. Given a parabolic length 1 torsion sheaf $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$, we define $\operatorname{dual}(\mathcal{T}^\bullet)$ by applying the above construction in every parabolic degree: $\operatorname{dual}(\mathcal{T}^\bullet)$ is by definition the parabolic length 1 torsion sheaf that is $\operatorname{dual}(\mathcal{T}^{(-i,x)})$ in parabolic degree (i,x). This defines the isomorphism

$$\operatorname{dual}\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathbf{Coh}_0^{1,1}.$$

Lemma 4.8. *Let $\mathcal{T}$ be a torsion sheaf on $\mathbf{P}^1$ supported on $x \in |\mathbf{P}^1|$. Then we have a canonical isomorphism*

$$\operatorname{dual}(\mathcal{T}) \cong \mathscr{E}xt^1(\mathcal{T}, \mathcal{O}_{\mathbf{P}^1}).$$

Proof. This is [**DOPW00**, Lemma A.2]. For the reader's benefit, we quickly give the essential argument. Denote by $i\colon \{x\} \to \mathbf{P}^1$ the inclusion. We calculate this by applying the functor $\mathscr{H}om(\mathcal{T}, -)$ to the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(x) \to i_*\mathcal{O}_{\{x\}} \otimes \mathcal{O}_{\mathbf{P}^1}(x) \to 0.$$

The associated long exact sequences gives us

$$\ldots \to 0 \to \mathscr{H}om(\mathcal{T}, i_*\mathcal{O}_{\{x\}} \otimes \mathcal{O}_{\mathbf{P}^1}(x)) = \operatorname{dual}(\mathcal{T}) \to \mathscr{E}xt^1(\mathcal{T}, \mathcal{O}) \to \mathscr{E}xt^1(\mathcal{T}, \mathcal{O}(x)) \to \ldots$$

Since the last map, $\mathscr{E}xt^1(\mathcal{T}, \mathcal{O}) \to \mathscr{E}xt^1(\mathcal{T}, \mathcal{O}(x))$ is zero, we get the desired isomorphism. $\square$

Since the isomorphism in Lemma 4.8 is canonical, we can apply it in every parabolic degree to get a canonical isomorphism

$$\operatorname{dual}(\mathcal{T}^\bullet) \cong \mathscr{E}xt^1(\mathcal{T}^\bullet, \mathcal{O}_{\mathbf{P}^1})$$

where $\mathscr{E}xt^1(\mathcal{T}^\bullet, \mathcal{O}_{\mathbf{P}^1})$ is to be interpreted as the parabolic sheaf that is $\mathscr{E}xt^1(\mathcal{T}^{(-i,x)}, \mathcal{O}_{\mathbf{P}^1})$ in parabolic degree (i,x).

We introduce a notation for the following three maps from the Hecke stack:

$$\begin{aligned}
\operatorname{big}\colon \mathcal{H} &\to \operatorname{Bun}_{2,4}, & (\mathcal{F}^\bullet \subset \mathcal{E}^\bullet) &\mapsto \mathcal{E}^\bullet \\
\operatorname{small}\colon \mathcal{H} &\to \operatorname{Bun}_{2,4}, & (\mathcal{F}^\bullet \subset \mathcal{E}^\bullet) &\mapsto \mathcal{F}^\bullet \\
\operatorname{quot}\colon \mathcal{H} &\to \mathbf{Coh}_0^{1,1}, & (\mathcal{F}^\bullet \subset \mathcal{E}^\bullet) &\mapsto \mathcal{E}^\bullet/\mathcal{F}^\bullet.
\end{aligned}$$

Lemma 4.9. *The following is a commutative diagram of stacks:*

$$\begin{tikzcd}
\operatorname{Bun}_{2,4}^{d-1} \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[d, "\operatorname{dual}"', "\wr"] & \mathcal{H}^d \arrow[l, "\operatorname{small}\times\operatorname{quot}"'] \arrow[r, "\operatorname{big}"] \arrow[d, "\operatorname{dual}"', "\wr"] & \operatorname{Bun}_{2,4}^d \arrow[d, "\operatorname{dual}"', "\wr"] \\
\operatorname{Bun}_{2,4}^{-d+1} \times \overline{\mathbf{Coh}}_0^{1,1} & \mathcal{H}^{-d+1} \arrow[l, "\operatorname{big}\times\operatorname{quot}"] \arrow[r, "\operatorname{small}"'] & \operatorname{Bun}_{2,4}^{-d}
\end{tikzcd}$$

Proof. The only difficulty lies in providing for every $\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet$ a canonical isomorphism $\operatorname{dual}(\mathcal{E}^\bullet/\mathcal{F}^\bullet) \cong \check{\mathcal{F}}^\bullet/\check{\mathcal{E}}^\bullet$ that makes the left square commute. Indeed, by writing out the definitions

$$\begin{tikzcd}
(\mathcal{F}^\bullet, \mathcal{T}^\bullet) \arrow[d, mapsto] & & (\mathcal{F}^\bullet \subset \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \arrow[ll, mapsto] \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{E}^\bullet) \arrow[d, mapsto] \\
\left(\check{\mathcal{F}}^\bullet, \operatorname{dual}(\mathcal{T}^\bullet)\right) \arrow[r, phantom, "\overset{?}{\cong}"] & (\check{\mathcal{F}}^\bullet, \check{\mathcal{F}}^\bullet/\check{\mathcal{E}}^\bullet) & \left(\check{\mathcal{E}}^\bullet \subset \check{\mathcal{F}}^\bullet\right) \arrow[l, mapsto] \arrow[r, mapsto] & \left(\check{\mathcal{E}}^\bullet\right)
\end{tikzcd}$$

we see that there is nothing else to prove. We establish the existence of such a canonical isomorphism for the underlying vector bundles, which then implies that we have compatible isomorphisms in all other parabolic degrees.

Applying the functor $\mathscr{H}\!om(\,-\,, \mathcal{O})$ to the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{T} \to 0$$

gives us

$$0 \to \check{\mathcal{E}} \to \check{\mathcal{F}} \to \mathscr{E}\!xt^1(\mathcal{T}, \mathcal{O}) \to 0.$$

Then Lemma 4.8 gives the desired isomorphism. $\square$

Let $\mathcal{F} \subset \mathcal{E}$ be a modification of a rank 2 vector bundle $\mathcal{E}$ at x. Then the natural map $\mathcal{E} \hookrightarrow \mathcal{E}(x)$ factors through $\mathcal{F}(x)$, because $\mathcal{F}(x) \subset \mathcal{E}(x)$ is by definition the kernel of $\mathcal{E}(x) \to \mathcal{T}(x)$ and $\mathcal{E} \to \mathcal{E}(x) \to \mathcal{T}(x)$ is zero. This gives us a sequence of natural inclusions

$$\mathcal{E} \hookrightarrow \mathcal{F}(x) \hookrightarrow \mathcal{E}(x). \tag{4.3.1}$$

This can now be extended into a chain

$$\ldots \hookrightarrow \mathcal{E}(-x) \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{F}(x) \hookrightarrow \mathcal{E}(x) \hookrightarrow \ldots \tag{4.3.2}$$

This establishes a bijective correspondence between length 1 modifications of a vector bundle $\mathcal{E}$ at a point $x \in |\mathbf{P}^1|$ and parabolic structures $\mathcal{E}^\bullet$ on $\mathcal{E}$ at x with the property that

$$\deg \mathcal{E}^{(i,x)} = \deg \mathcal{E} + i \qquad \text{for all } i \in \mathbf{Z}.$$

Restricting the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{T} \to 0$$

to the closed subscheme $\operatorname{Supp} \mathcal{T} = \{x\}$ and dualizing gives us a left-exact sequence

$$0 \to (\mathcal{T}|_x)^\vee \to (\mathcal{E}^\vee)|_x \to (\mathcal{F}^\vee)|_x \tag{4.3.3}$$

because dualizing and restricting to $\{x\}$ commute for vector bundles. We denote the image of $(\mathcal{T}|_x)^\vee$ in $(\mathcal{E}^\vee)|_x$ by $\ell^\vee$. Taking the modification with respect to this line, we get

$$\check{\mathcal{E}}(-x) \hookrightarrow T_x^{\ell^\vee} \check{\mathcal{E}} \hookrightarrow \check{\mathcal{E}}. \tag{4.3.4}$$

Lemma 4.10. *Let $\mathcal{F} \subset \mathcal{E} \twoheadrightarrow \mathcal{T}$ be a modification of $\mathcal{E}$ at $x = \operatorname{Supp}\mathcal{T}$ and let $\ell^\vee$ be the image of $(\mathcal{T}|_x)^\vee \to (\check{\mathcal{E}})|_x$. The sheaves $\mathcal{F}(x)$ and $\left(T_x^{\ell^\vee}\check{\mathcal{E}}\right)^\vee$ are equal as subsheaves of $\mathcal{E}(x)$ and the diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[d, equal] & \mathcal{F}(x) \arrow[r, hook] \arrow[d, equal] & \mathcal{E}(x) \arrow[d, equal] \\
\left(\check{\mathcal{E}}\right)^\vee \arrow[r, hook] & \left(T_x^{\ell^\vee}\check{\mathcal{E}}\right)^\vee \arrow[r, hook] & \left(\check{\mathcal{E}}(-x)\right)^\vee
\end{tikzcd}$$

where the top row is (4.3.1) *and the bottom row is the dual of* (4.3.4)*. If we write $\mathcal{F} = T_x^\ell\mathcal{E}$, this statement can be summarized with the equation*

$$(T_x^{\ell^\vee}\check{\mathcal{E}})^\vee = T_x^\ell\mathcal{E}(x).$$

Proof. We have a natural sequence of inclusions

$$\check{\mathcal{F}}(-x) \hookrightarrow \check{\mathcal{E}} \hookrightarrow \check{\mathcal{F}}$$

and $\check{\mathcal{F}}(-x) \hookrightarrow \check{\mathcal{E}}$ is a length 1 modification of $\mathcal{E}$. We recall that by definition of $\ell^\vee$ (see (4.3.3)) and T_x

$$T_x^{\ell^\vee}\check{\mathcal{E}} = \ker(\check{\mathcal{E}} \to \check{\mathcal{F}}|_x).$$

Since $\check{\mathcal{F}}(-x)$ is contained in this kernel and has the same degree, we conclude $\check{\mathcal{F}}(-x) = T_x^{\ell^\vee}\check{\mathcal{E}}$. □

Let $\mathcal{E}^\bullet, \mathcal{F}^\bullet$ be rank 2 parabolic vector bundles with full flags and let $\mathcal{F}^\bullet \subset \mathcal{E}^\bullet$ be a modification. Then the above constructions can be applied in every parabolic degree and because all the maps in the constructions are natural inclusions, the results are compatible with the parabolic structure. We therefore get a chain of inclusions of parabolic bundles

$$\ldots \hookrightarrow \mathcal{E}^\bullet(-x) \hookrightarrow \mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet \hookrightarrow \mathcal{F}^\bullet(x) \hookrightarrow \mathcal{E}^\bullet(x) \hookrightarrow \ldots \tag{4.3.5}$$

and the lemma also generalizes.

Lemma 4.11. *Let $\mathcal{F}^\bullet \subset \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet$ be a length 1 lower modification of a rank 2 parabolic vector bundle $\mathcal{E}^\bullet$ with full flags with respect to $\mathcal{T}^\bullet$. Then $\mathcal{F}^\bullet(x)$ and*

$$\mathcal{G}^\bullet := \left(\ker\left(\check{\mathcal{E}}^\bullet \to \check{\mathcal{F}}^\bullet|_{\operatorname{Supp}\mathcal{T}}\right)\right)^\vee$$

are equal as subbundles of $\mathcal{E}^\bullet(x)$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r, hook] \arrow[d, equal] & \mathcal{F}^\bullet(x) \arrow[r, hook] \arrow[d, equal] & \mathcal{E}^\bullet(x) \arrow[d, equal] \\
\left(\check{\mathcal{E}}^\bullet\right)^\vee \arrow[r, hook] & \check{\mathcal{G}}^\bullet \arrow[r, hook] & \left(\check{\mathcal{E}}^\bullet(-x)\right)^\vee
\end{tikzcd}$$

PROOF. Apply Lemma 4.10 in every parabolic degree. □

4.4. Properties of the Hecke stack and its maps. Recall that in Section 4.1, we defined maps $p\colon \mathcal{H}^d \to \mathrm{Bun}^d_{2,4} \times \mathbf{Coh}_0^{1,1}$, and $\bar{p}\colon \mathcal{H}^d \to \mathrm{Bun}^d_{2,4} \times \mathbf{Coh}_0^{1,1}$, $q\colon \mathcal{H}^d \to \mathrm{Bun}^{d-1}_{2,4}$. In the notation of the previous section, we have $p = (\text{big} \times \text{quot})$ and $q = \text{small}$.

Lemma 4.12. *The maps p, q and $\bar{p}$ are all smooth. The map q is proper. The map $\bar{p}$ is proper over* $\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}$.

PROOF. Instead of proving these statements for p, q and $\bar{p}$, we can also prove them for their dual maps (Lemma 4.9). The smoothness of the maps dual to p, q and the properness of the map dual to q is (part of) the content of [**Hei04**, remark 6.3]. The smoothness of $\bar{p}$ follows from the fact that p and the map $\mathbf{Coh}_0^{1,1} \to \overline{\mathbf{Coh}}_0^{1,1}$ are smooth.

It only remains to prove that $\bar{p}$ is proper. The fiber $\bar{p}^{-1}(\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}})$ is isomorphic to $\mathbf{P}(\mathcal{E}^{\mathrm{univ}})|_{\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}}$, where $\mathcal{E}^{\mathrm{univ}}$ is the parabolic degree 0 part of the universal bundle on $\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}$: the map

$$\bar{p}^{-1}(\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}) \to \mathbf{P}(\mathcal{E}^{\mathrm{univ}}), \qquad (\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\mathcal{E} \twoheadrightarrow \mathcal{T})$$

is an isomorphism, because modifications of a point outside of D are simply modifications of the underlying vector bundle (Remark 2.22). This isomorphism is compatible with the natural maps to $\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}$, i.e., we have a commutative diagram

$$\begin{tikzcd}
\bar{p}^{-1}(\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}) \arrow[r, "\cong"] \arrow[dr, "\bar{p}"'] & \mathbf{P}(\mathcal{E}^{\mathrm{univ}}) \arrow[d] \\
& \mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}
\end{tikzcd}$$

that proves that $\bar{p}$ is indeed proper over $\mathrm{Bun}^d_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1,\mathrm{ur}}$. □

5. Cusp conditions for parabolic vector bundles

Let as before $k = \mathbf{F}_q$ a finite field. We denote by $K = k(\mathbf{P}^1)$ the function field of $\mathbf{P}^1_{\mathbf{F}_q}$. For $x \in |\mathbf{P}^1|$ a closed point, we denote by K_x the completion of K at x, by $\mathcal{O}_x \subset K_x$ the valuation ring and by k_x the residue field. Lastly, by $\mathbf{A} = \mathbf{A}_K := \prod'(K_x, \mathcal{O}_x)$ we denote the adèles of K.

Classically, an automorphic form f on $\mathrm{GL}_2(\mathbf{A})$ is cuspidal if it satisfies

$$\int_{U(K)\backslash U(\mathbf{A})} f(ux)\mathrm{d}u = 0 \qquad (5.0.1)$$

for all $x \in \mathrm{GL}_2(\mathbf{A})$, where U is the unipotent radical of the Borel subgroup $B \subset \mathrm{GL}_2(\mathbf{A})$. By taking the trace of Frobenius (see e.g. [**Lau87**, (1.1)]), a perverse sheaf F on $\mathrm{Bun}_{2,4}$ defines a function

$$f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) = \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0 \to \mathbf{Q}_\ell.$$

The cusp condition on f then has a geometrical analogue on the perverse sheaf F in terms of a correspondence

$$\begin{tikzcd} & \mathcal{S}_I \arrow[dl, "p_I"'] \arrow[dr, "q_I"] & \\ \mathrm{Bun}_{2,4} & & \mathrm{Pic}_I \times \mathrm{Pic}_{S\backslash I} \end{tikzcd}$$

We first explain this geometric cusp condition in Section 5.1 and then in Section 5.2, we explain in detail how it relates to the classical cusp condition.

5.1. Definition of the cusp condition. For $I \subset D$, we define the stack

$$\mathrm{Pic}_I := \left\langle \mathcal{L}^\bullet = (\mathcal{L}, I) : \ \mathcal{L} \text{ a line bundle on } \mathbf{P}^1 \ \right\rangle.$$

This stack is a union of connected components of the stack $\mathrm{Bun}_{1,4}$, classifying line bundles with parabolic structure at D, since the condition $\mathcal{L}^\bullet = (\mathcal{L}, I)$ is a condition on the degrees of the line bundles in the different parabolic degrees. The map $\mathrm{Pic}_I \to \mathrm{Pic}$, $(\mathcal{L}, I) \mapsto \mathcal{L}$ is an isomorphism of stacks.

We denote by $\mathcal{S}_I$ the stack

$$\mathcal{S}_I := \left\langle 0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet \to \mathcal{M}^\bullet \to 0 : \ \mathcal{L}^\bullet \in \mathrm{Pic}_I, \ \mathcal{M}^\bullet \in \mathrm{Pic}_{D\backslash I} \ \right\rangle$$

that classifies certain exact sequences of parabolic sheaves. This stack comes equipped with natural maps

(5.1.1)
$$\begin{tikzcd} & \mathcal{S}_I \arrow[dl, "p_I"'] \arrow[dr, "q_I"] & \\ \mathrm{Bun}_{2,4} & & \mathrm{Pic}_I \times \mathrm{Pic}_{S\backslash I} \end{tikzcd}$$

defined by

$$\begin{aligned} p_I &\colon (0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet \to \mathcal{M}^\bullet \to 0) \mapsto \mathcal{E}^\bullet, \\ q_I &\colon (0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet \to \mathcal{M}^\bullet \to 0) \mapsto (\mathcal{L}^\bullet, \mathcal{M}^\bullet). \end{aligned}$$

Definition 5.1. Let F be a perverse sheaf on $\mathrm{Bun}_{2,4}$. We say that F is *cuspidal* if it satisfies the *cusp condition*: for all $I \subset S$,

$$\mathbf{R}q_{I,!}p_I^* F = 0. \tag{5.1.2}$$

Proposition 5.2. *Hecke operators preserve cusp forms.*

This can be proven using [**FGV02**, Lemma 9.8] and the methods used there. We use this result in Section 6, but it could in principle be avoided by slightly adapting and extending the proofs in that section.

5.2. Relation to classical cusp conditions. The cusp condition on a perverse sheaf F on $\mathrm{Bun}_{2,4}$ (Definition 5.1) defines a condition on the function $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ that is obtained from F by taking traces of Frobenii. In this section, we show that this condition is the same as the classical cusp condition.

Let F be a perverse sheaf on $\mathrm{Bun}_{2,4}$ and let $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ be the corresponding function of traces of Frobenii. We denote by $p_I\colon \mathcal{S}_I \to \mathrm{Bun}_{2,4}$ and $q_I\colon \mathcal{S}_I \to \mathrm{Pic}_I \times \mathrm{Pic}_{D\setminus I}$ the maps defining the cusp condition from Equation (5.1.1). We denote the trace of Frobenius function associated to $\mathbf{R}q_{I,!}p_I^*F$ by $q_{I,!}p_I^*f$. This function satisfies for every

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet)\colon \operatorname{Spec}\mathbf{F}_q \to \mathrm{Pic}_I \times \mathrm{Pic}_{D\setminus I}$$

the equality

$$(q_{I,!}p_I^*f)(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \sum_{\substack{(\mathcal{L}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{M}^\bullet)\\ \in \mathrm{Ext}^1(\mathcal{M}^\bullet, \mathcal{L}^\bullet)(\mathbf{F}_q)}} f(\mathcal{E}^\bullet) \tag{5.2.1}$$

([**Lau87**, (1.1)]) and the cusp condition $\mathbf{R}q_{I,!}p_I^*F = 0$ then translates into the condition

$$\sum_{\substack{(\mathcal{L}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{M}^\bullet)\\ \in \mathrm{Ext}^1(\mathcal{M}^\bullet, \mathcal{L}^\bullet)(\mathbf{F}_q)}} f(\mathcal{E}^\bullet) = 0. \tag{5.2.2}$$

We will now show that this condition, when expressed in terms of adèles instead of parabolic vector bundles, corresponds to classical cusp condition.

To do so, we first describe the maps p_I, q_I for all $I \subset D$ on $\mathbf{F}_q$-points in terms of adèles. We treat the maps p_I, q_I for all $I \subset D$ at the same time, i.e., we describe the diagram

(5.2.3)

$$\begin{tikzcd} & \bigsqcup_{I\subset D} \mathcal{S}_I(\mathbf{F}_q) \arrow[dl, "p=\bigsqcup_{I\subset D} p_I"'] \arrow[dr, "q=\bigsqcup_{I\subset D} q_I"] & \\ \mathrm{Bun}_{2,4}(\mathbf{F}_q) & & \bigsqcup_{I\subset D} \mathrm{Pic}_I(\mathbf{F}_q) \times (\mathrm{Pic}_{S\setminus I})(\mathbf{F}_q) \end{tikzcd}$$

in terms of adèles.

First, we need to describe the $\mathbf{F}_q$-points of the stacks in terms of adèles. It is well-known how to do this; we explain this in Section 5.3.

First some notation. We denote by $B \subset \mathrm{GL}_2$ the standard Borel subgroup of upper-triangular matrices, by $U \subset B$ the unipotent subgroup of strictly upper triangular matrices and by $T \subset \mathrm{GL}_2$ the maximal torus of diagonal matrices, which is isomorphic to $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$. For $x \in |\mathbf{P}^1|$, the Iwahori group Iw_x at x is defined as

$$\mathrm{Iw}_x := \ker\left(\mathrm{GL}_2(\mathcal{O}_x) \to \mathrm{GL}_2(k(x))/B(k(x))\right).$$

Lastly, we define

$$\Gamma_0 := \prod_{x\in|\mathbf{P}^1|} \mathrm{GL}_2(\hat{\mathcal{O}}_x) \times \prod_{x\in D} \mathrm{Iw}_x$$

We get the following bijections:

$$\mathrm{Bun}_{2,4}(\mathbf{F}_q) \xrightarrow{1-1} \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$$

(see Section 5.3.2); the disjoint union $\bigsqcup_I \mathcal{S}_I$ classifies short exact sequences $\mathcal{L}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{M}^\bullet$, but this is equivalent to classifying parabolic sub line bundles $\mathcal{L}^\bullet \subset \mathcal{E}^\bullet$ that are saturated in all parabolic degrees, so we get a bijection

$$\left(\bigsqcup_I \mathcal{S}_I\right)(\mathbf{F}_q) \xrightarrow{1-1} B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$$

(see Section 5.3.4); and lastly, because the forgetful map $\mathrm{Pic}_I \to \mathrm{Pic}, \mathcal{L}^\bullet \mapsto \mathcal{L}$ is an isomorphism, we get a bijection

$$\mathrm{Pic}_I \xrightarrow{1-1} \mathbb{G}_\mathrm{m}(K)\backslash\mathbb{G}_\mathrm{m}(\mathbf{A})/\prod_{x\in|\mathbf{P}^1|}\mathbb{G}_\mathrm{m}(\hat{\mathcal{O}}_x)$$

and therefore also a bijection

$$\bigsqcup_{I\subset S}\mathrm{Pic}(\mathbf{F}_q)\times\mathrm{Pic}(\mathbf{F}_q)$$

$$\xrightarrow{1-1} T(K)\backslash\left(\prod_{x\notin S}T(K_x)/T(\mathcal{O}_x)\times\prod_{x\in S}\left(T(K_x)/T(\mathcal{O}_x)\sqcup T(K_x)/T(\mathcal{O}_x)\right)\right)$$

where the component index by I on the left corresponds to the component that for all $x\in I$ selects the first $T(K_x)/T(\mathcal{O}_x)$, and for all $x\in D\setminus I$ selects the second $T(K_x)/T(\mathcal{O}_x)$ in $T(K_x)/T(\mathcal{O}_x)\sqcup T(K_x)/T(\mathcal{O}_x)$ on the right.

After these identifications, the map

$$p\colon B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0 \to \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$$

is the natural quotient map (see Section 5.3.3) . The map

$$q\colon B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0 \to \left(\mathbb{G}_\mathrm{m}(K)\backslash\mathbb{G}_\mathrm{m}(\mathbf{A})/\prod_{x\in|\mathbf{P}^1|}\mathbb{G}_\mathrm{m}(\hat{\mathcal{O}}_x)\right)^{\times 2}$$

is more difficult to explain, as it requires rewriting the double quotient $B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$. We will now do this and will then conclude that q is essentially modding out the unipotent radical.

Recall that the adèles are defined in terms of a restricted product, so that we have

$$\mathrm{GL}_2(\mathbf{A})/\Gamma_0 = \prod_{x\notin D}{}' \mathrm{GL}_2(K_x)/\mathrm{GL}_2(\mathcal{O}_x)\times\prod_{x\in D}\mathrm{GL}_2(K_x)/\mathrm{Iw}_x\,. \tag{5.2.4}$$

We will now rewrite each of these factor, using the Iwasawa decomposition, Bruhat decomposition and some elementary identities for right cosets.

First we rewrite the factors at $x\in\mathbf{P}^1\setminus D$. The Iwasawa decomposition tells us

$$\mathrm{GL}_2(K_x) = B(K_x)\,\mathrm{GL}_2(\mathcal{O}_x).$$

A standard identity on right cosets then tells us

$$B(K_x)\,\mathrm{GL}_2(\mathcal{O}_x)/\,\mathrm{GL}_2(\mathcal{O}_x) = B(K_x)/(B(K_x)\cap \mathrm{GL}_2(\mathcal{O}_x)) = B(K_x)/B(\mathcal{O}_x).$$

Combining these two equations, we get

$$\mathrm{GL}_2(K_x)/\,\mathrm{GL}_2(\mathcal{O}_x) = B(K_x)/B(\mathcal{O}_x). \tag{5.2.5}$$

Let

$$w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F}_q).$$

For $x \in D$, we use, in addition to Iwasawa decomposition $\mathrm{GL}_2(K_x) = B(K_x)\,\mathrm{GL}_2(\mathcal{O}_x)$, the fact that we get a decomposition

$$\mathrm{GL}_2(\mathcal{O}_x) = \mathrm{Iw}_x \sqcup \mathrm{Iw}_x\, w\, \mathrm{Iw}_x$$

by taking the inverse image along $\mathrm{GL}_2(\mathcal{O}_x) \to \mathrm{GL}_2(k_x)$, of the Bruhat decomposition over k_x

$$B(k_x) = B(k_x) \sqcup B(k_x)wB(k_x).$$

This allows us to write

$$\begin{aligned} \mathrm{GL}_2(K_x)/\,\mathrm{Iw}_x &= B(K_x)\,\mathrm{GL}_2(\mathcal{O}_x)/\,\mathrm{Iw}_x \\ &= (B(K_x)/\,\mathrm{Iw}_x) \sqcup (B(K_x)w\,\mathrm{Iw}_x\,/\,\mathrm{Iw}_x). \end{aligned} \tag{5.2.6}$$

We continue by noting that the Iwahori subgroup Iw_x can be decomposed as $B(\mathcal{O}_x)U^-((\pi_x))$, where $\pi_x \in \mathcal{O}_x$ is a uniformizer and U^- is the opposite of the unipotent radical of B. Since $w^{-1}U^-((\pi_x))w \subset B(\mathcal{O}_x) \subset \mathrm{Iw}_x$, we find

$$\mathrm{Iw}_x\, w\, \mathrm{Iw}_x = B(\mathcal{O}_x)w\,\mathrm{Iw}_x\,.$$

Therefore, the same standard identity on right cosets implies

$$B(K_x)/\,\mathrm{Iw}_x = B(K_x)/(B(K_x)\cap \mathrm{Iw}_x) = B(K_x)/B(\mathcal{O}_x)$$

and

$$\begin{aligned} B(K_x)w\,\mathrm{Iw}_x\,/\,\mathrm{Iw}_x &\overset{1-1}{=} B(K_x)/(B(K_x)\cap w\,\mathrm{Iw}_x\,w^{-1}) \\ &= B(K_x)/(T(\mathcal{O}_x)U_1) \end{aligned}$$

where

$$U_1 := \left\{ \begin{pmatrix} 1 & \pi_x a \\ 0 & 1 \end{pmatrix} : a \in \mathcal{O}_x \right\} \subset B(\mathcal{O}_x).$$

Combining these two equalities with (5.2.6) gives us

$$\mathrm{GL}_x(K_x)/\,\mathrm{Iw}_x = B(K_x)/B(\mathcal{O}_x) \sqcup B(K_x)/(T(\mathcal{O}_x)U_1). \tag{5.2.7}$$

An element $(g_x)_x \in B(K)\backslash\,\mathrm{GL}_2(\mathbf{A})/\Gamma_0$ such that for some $x_0 \in D$, g_{x_0} s in the first component $B(K_{x_0})/B(\mathcal{O}_{x_0})$ in the decomposition (5.2.7), corresponds to a $\mathcal{L}^\bullet \subset \mathcal{E}^\bullet = (\mathcal{E}, (\ell_x)_x)$ with $\ell_x = \mathcal{L}|_x$; if g_{x_0} is in the other component, i.e., in $B(K_x)/(T(\mathcal{O}_x)U_1)$, then $\ell_x \neq \mathcal{L}|_x$. This follows directly from the construction of the flag ℓ_{x_0} (5.3.3) and of $\mathcal{L}^\bullet$ (5.3.4).

Substituting these identities (5.2.7) and (5.2.7) for the right cosets at all $x \in |\mathbf{P}^1|$ into the double coset $B(K)\backslash \mathrm{GL}_2(K_x)/\Gamma_0$, we have proven (5.2.8)

$$B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$$
$$= B(K)\backslash \left(\prod_{x \notin S} B(K_x)/B(\mathcal{O}_x) \times \prod_{x\in S} (B(K_x)/B(\mathcal{O}_x) \sqcup B(K_x)/T(\mathcal{O}_x)U_1) \right)$$

After this identification the map q becomes

$$q\colon B(K)\backslash \left(\prod_{x \notin S} B(K_x)/B(\mathcal{O}_x) \times \prod_{x\in S} (B(K_x)/B(\mathcal{O}_x) \sqcup B(K_x)/T(\mathcal{O}_x)U_1) \right)$$
$$\twoheadrightarrow T(K)\backslash \left(\prod_{x \notin S} T(K_x)/T(\mathcal{O}_x) \times \prod_{x\in S} (T(K_x)/T(\mathcal{O}_x) \sqcup T(K_x)/T(\mathcal{O}_x)) \right)$$

which is the natural map induced by $B \twoheadrightarrow B/U \cong T$

The $\mathbf{F}_q$-points of $\mathcal{S}_I$ correspond to the component

$$B(K)\backslash \left(\prod_{x \notin S} B(K_x)/B(\mathcal{O}_x) \times \prod_{x\in I} B(K_x)/B(\mathcal{O}_x) \times \prod_{x\in S\backslash I} B(K_x)/T(\mathcal{O}_x)U_1 \right).$$

(See the remark just after (5.2.7).)

We thus conclude that the geometric cusp condition on $\mathbf{F}_q$-points (5.2.2)

$$\sum_{\substack{(\mathcal{L}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{M}^\bullet) \\ \in \mathrm{Ext}^1(\mathcal{M}^\bullet, \mathcal{L}^\bullet)(\mathbf{F}_q)}} f(\mathcal{E}^\bullet) = 0$$

has the following form on adèles. Let $g = (g_x)_{x\in|\mathbf{P}^1|} \in \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$. Lift it along $p = \bigsqcup_{I\subset D} p_I$ to $g' = (g_x)_{x\in|\mathbf{P}^1|} \in B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$. The fiber $q^{-1}q(g')$ is

$$F_{g'} := B(K)\backslash B(K)U(\mathbf{A})g'\Gamma_0/\Gamma_0 = U(K)\backslash U(\mathbf{A})/(g'\Gamma_0 g'^{-1} \cap U(\mathbf{A}))$$

where we apply similar identities on cosets as before. The cusp condition (5.2.2) for $q(g')$ becomes

$$\sum_{h\in F_{g'})} f(p(h)) = 0.$$

If we instead of taking right cosets with respect to Γ_0, integrate over Γ_0, this becomes

$$\int_{u\in U(K)\backslash U(\mathbf{A})} f(ug) = 0$$

which is the classical cusp condition.

Every condition of the form (5.2.2) arises in this way from some $g \in \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$: any $(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \in \mathrm{Pic}_I(\mathbf{F}_q) \times \mathrm{Pic}_{D\backslash I}(\mathbf{F}_q)$ is the image of $\mathcal{L}^\bullet \subset \mathcal{L}^\bullet \oplus \mathcal{M}^\bullet$, which is a lift of $\mathcal{L}^\bullet \oplus \mathcal{M}^\bullet \in \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0$.

5.3. Appendix: Adèlic description of vector bundles with parabolic structure. In this section, we give an adèlic description of several stacks of vector bundles with parabolic structure. For simplicity, we do this for rank 2 vector bundles on $\mathbf{P}^1$ with parabolic structure on $D = \{\infty, 0, 1, t\}$, but everything can easily be generalized to vector bundles of any rank on a smooth projective curve X and any simple divisor $D \subset X$. A reference for this is [**HN75**, §2.3],

5.3.1. *Vector bundles.* For every adèle $g = (g_x)_x \in \mathrm{GL}_2(\mathbf{A})$, we can define a vector bundle $\mathcal{E}_g$ by

$$\mathcal{E}_g(U) := \left\{ s \in K \oplus K : \; s \in g_x(\hat{\mathcal{O}}_x)^{\oplus 2} \subset K_x^{\oplus 2} \; \forall x \in |U| \; \right\}.$$

This is a right $\mathcal{O}_{\mathbf{P}^1}$-module, but this does not matter much since $\mathcal{O}_{\mathbf{P}^1}$ is commutative. This assignment $g \mapsto \mathcal{E}_g$ induces a bijection

$$\begin{aligned} \mathrm{GL}_2(K) \backslash \mathrm{GL}_2(\mathbf{A}) / \prod_{x \in |X|} \mathrm{GL}_2(\hat{\mathcal{O}}_x) &\to \mathrm{Bun}_2(\mathbf{F}_q), \\ (g_x)_x &\mapsto \mathcal{E}_{(g_x)_x}. \end{aligned} \tag{5.3.1}$$

We take for granted that this is indeed a bijection and prove two small variations, where we replace the right hand side with the set of bundles equipped with a parabolic structure at D and a sub line bundle.

5.3.2. *Vector bundles with parabolic structure.* We define

$$\Gamma_0 := \prod_{x \in |\mathbf{P}^1|} \mathrm{GL}_2(\hat{\mathcal{O}}_x) \times \prod_{x \in D} \mathrm{Iw}_x \,.$$

We claim there is a bijection

$$\begin{aligned} \mathrm{GL}_2(K) \backslash \mathrm{GL}_2(\mathbf{A}) / \Gamma_0 &\to \mathrm{Bun}_{2,D}(\mathbf{F}_q), \\ (g_x)_x &\mapsto \left(\mathcal{E}_{(g_x)_x}, (\ell_x)_{x \in D}\right) \end{aligned} \tag{5.3.2}$$

where on the right, the flag at $y \in D$ is defined as follows. Let $\pi \in \hat{\mathcal{O}}_y$ be a uniformizer and denote by $i_1 \colon \hat{\mathcal{O}}_y \hookrightarrow (\hat{\mathcal{O}}_y)^{\oplus 2}$ the inclusion into the first summand. Then

$$\ell_y := g_y \left(\hat{\mathcal{O}}_y / \pi_y \hat{\mathcal{O}}_y\right) \overset{g(i_1)}{\subset} g_y \left(\left(\hat{\mathcal{O}}_y / \pi_y \hat{\mathcal{O}}_y\right)^{\oplus 2}\right) \subset \mathcal{E}_{(g_x)_x}|_y. \tag{5.3.3}$$

This map is well-defined, because Iw_x is by definition the stabilizer of $\left(\hat{\mathcal{O}}_y / \pi_y \hat{\mathcal{O}}_y \overset{i_1}{\subset} (\hat{\mathcal{O}}_y / \pi_y \hat{\mathcal{O}}_y\right)^{\oplus 2}$.

This bijection fits into the commutative diagram

$$\begin{tikzcd} \mathrm{GL}_2(K) \backslash \mathrm{GL}_2(\mathbf{A}) / \Gamma_0 \arrow[r] \arrow[d] & \mathrm{Bun}_{2,D}(\mathbf{F}_q) \arrow[d] \\ \mathrm{GL}_2(K) \backslash \mathrm{GL}_2(\mathbf{A}) / \prod_{x \in |\mathbf{P}^1|} \mathrm{GL}_2(\hat{\mathcal{O}}_x) \arrow[r] & \mathrm{Bun}_2(\mathbf{F}_q) \end{tikzcd}$$

where the left vertical map is the quotient map and the right vertical map is the forgetful map. We can prove the bijectivity of the top horizontal map fiberwise. To do this, assume $D = \{y\}$ for simplicity.

Let $(g_x)_x$ be an element from the bottom left and let $\mathcal{E}$ be its image on the bottom right. The fiber on the left over $(g_x)_x$ is the image of the injective map

$$\mathrm{GL}_2(\hat{\mathcal{O}}_y)/\mathrm{Iw}_y \overset{1-1}{\to} B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0, \qquad h \mapsto ((g_x)_{x\neq y}, g_y h)$$

and the fiber on the right over $\mathcal{E}$ is

$$\mathbf{P}^1(\mathcal{E}|_y) = \left\{\text{lines in } g_y\left(\hat{\mathcal{O}}_y/\pi_y\hat{\mathcal{O}}_y\right)\right\}.$$

The map on the fibers is

$$\mathrm{GL}_2(\hat{\mathcal{O}}_y)/\mathrm{Iw}_y \to \left\{\text{lines in } g_y\left(\hat{\mathcal{O}}_y/\pi_y\hat{\mathcal{O}}_y\right)\right\}, \qquad h \mapsto g_y h(\hat{\mathcal{O}}_y/\pi_y\hat{\mathcal{O}}_y)$$

which is a bijection, because $\mathrm{GL}_2(\hat{\mathcal{O}}_y)$ acts transitively on the set of lines in $\hat{\mathcal{O}}_y/\pi_y\hat{\mathcal{O}}_y$ with stabilizer Iw_y.

5.3.3. *Vector bundles with a sub line bundle.* Let $\mathcal{K}_1, \mathcal{K}_2$ denote two copies of the locally constant sheaf of rational functions. For any adèle $(g_x)_x \in \mathrm{GL}_2(\mathbf{A})$, the vector bundle $\mathcal{E}_{(g_x)_x}$ comes equipped with an embedding into $\mathcal{K}_1 \oplus \mathcal{K}_2$ by construction. We claim there is a bijection

(5.3.4)

$$B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\prod_{x\in|\mathbf{P}^1|} \mathrm{GL}_2(\hat{\mathcal{O}}_x) \to \{\mathcal{L}\subset\mathcal{E} \text{ saturated} : \mathcal{L}\in\mathrm{Pic}, \mathcal{E}\in\mathrm{Bun}_2\}_{/\cong},$$

$$(g_x)_x \mapsto (\mathcal{L} := \mathcal{E}_{(g_x)_x}\cap\mathcal{K}_1 \subset \mathcal{E}_{(g_x)_x}).$$

The subsheaf $\mathcal{E}_{(g_x)_x}\cap\mathcal{K}_1$ is a line bundle, because it is torsion free and rank one, and it is saturated because the definition is such that for any $x\in\mathbf{P}^1$, if $s_x\in\mathcal{L}_x$ is such that $\pi_x^{-1}s_x\in\mathcal{E}|_x$, then also $\pi_x^{-1}s_x\in\mathcal{L}_x$. The map is well-defined: for $b\in B(K)$ and $g=(g_x)_x\in\mathrm{GL}_2(\mathbf{A})$, the map b induces an isomorphism $\mathcal{K}_1\oplus\mathcal{K}_2 \xrightarrow{\sim} \mathcal{K}_1\oplus\mathcal{K}_2$ that preserves $\mathcal{K}_1$, so that we get a commutative diagram

$$\begin{tikzcd} \mathcal{E}_g\cap\mathcal{K}_1 \arrow[r, phantom, "\subset"] \arrow[d, "\wr"] & \mathcal{E}_g \arrow[r, phantom, "\subset"] \arrow[d, "\wr"] & \mathcal{K}_1\oplus\mathcal{K}_2 \arrow[d, "b"', "\wr"] \\ \mathcal{E}_{bg}\cap\mathcal{K}_1 \arrow[r, phantom, "\subset"] & \mathcal{E}_{bg} \arrow[r, phantom, "\subset"] & \mathcal{K}_1\oplus\mathcal{K}_2 \end{tikzcd}$$

which shows that bg and g have the same image.

The map fits into a commutative diagram

$$\begin{tikzcd} B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\prod_{x\in|X|}\mathrm{GL}_2(\hat{\mathcal{O}}_x) \arrow[r] \arrow[d] & \{\mathcal{L}\subset\mathcal{E}\text{ saturated}\}_{/\cong} \arrow[d] \\ \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A})/\prod_{x\in|X|}\mathrm{GL}_2(\hat{\mathcal{O}}_x) \arrow[r] & \mathrm{Bun}_2(\mathbf{F}_q) \end{tikzcd}$$

We can again prove the bijectivity fiberwise. Let $g=(g_x)_x\in\mathrm{GL}_2(\mathbf{A})$ represent an double coset on the bottom left and let $\mathcal{E}$ be its image on the bottom right. The fiber over g is the image of the injective map

$$B(K)\backslash \mathrm{GL}_2(K) \to B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\prod_{x\in|X|}\mathrm{GL}_2(\hat{\mathcal{O}}_x), \qquad h\mapsto (hg_x)_x$$

and the map on the fibers is

$$\begin{aligned}
&\mathrm{GL}_2(K)/B(K) \to \{\mathcal{L} \subset \mathcal{E} \text{ saturated}\}_{/\cong},\\
&\qquad h \mapsto (h(\mathcal{E}_g) \cap \mathcal{K}_1 \subset h(\mathcal{E}_g)) \cong (\mathcal{E}_g \cap h^{-1}\mathcal{K}_1 \subset \mathcal{E}_g).
\end{aligned}$$

Since $B(K)$ is the stabilizer of $\mathcal{K}_1$ inside $\mathcal{K}_1 \oplus \mathcal{K}_2$, this is an isomorphism if and only if every saturated sub line bundle of $\mathcal{E}_g$ is of the form $\mathcal{E}_g \cap h^{-1}\mathcal{K}_1$ for some $h \in \mathrm{GL}_2(K)$. But that is easy to see: every $\mathcal{L} \subset \mathcal{E}_g$ generates some $\mathcal{K} \subset \mathcal{K}_1 \oplus \mathcal{K}_2$ and we can take $h \in \mathrm{GL}_2(K)$ such that $h^{-1}(\mathcal{K}_1) = \mathcal{K}$. The saturatedness then implies $\mathcal{L} = \mathcal{E}_g \cap \mathcal{K}$.

5.3.4. *Vectors bundles with parabolic structure and a sub line bundle.* Combining both the parabolic structure and the sub line bundle, we get a bijection

(5.3.5)

$$\begin{aligned}
&B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0 \to \{(\mathcal{E}^\bullet, \mathcal{L} \subset \mathcal{E} \text{ saturated}) : \mathcal{L} \in \mathrm{Pic}, \mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}\}_{/\cong},\\
&\qquad (g_x)_x \mapsto (\mathcal{E}^\bullet := (\mathcal{E}_{(g_x)_x}, (\ell_x)_{x\in D}),\ \mathcal{L} := \mathcal{E}_{(g_x)_x} \cap \mathcal{K}_1 \subset \mathcal{E}_{(g_x)_x}).
\end{aligned}$$

Let $(\mathcal{E}^\bullet = (\mathcal{E}, (\ell_x)), \mathcal{L} \subset \mathcal{E})$ be in the image of this map and define

$$I_{\mathcal{L}\subset\mathcal{E}} := \{x \in D : \mathcal{L}|_x = \ell_x\} \subset D.$$

Then $\mathcal{L}^\bullet = (\mathcal{L}, I)$ is the unique parabolic structure $\mathcal{L}^\bullet$ on $\mathcal{L}$ such that for all parabolic degrees (i,p), the sheaf $\mathcal{L}^{(i,p)}$ is a line bundle and $\mathcal{L}^{(i,p)} \hookrightarrow \mathcal{E}^{(i,p)}$ is saturated (Lemma 2.9). We therefore get a bijection

(5.3.6)

$$\begin{aligned}
&B(K)\backslash \mathrm{GL}_2(\mathbf{A})/\Gamma_0 \to \left\{ (\mathcal{E}^\bullet, \mathcal{L}^\bullet, \mathcal{L}^\bullet \subset \mathcal{E}^\bullet) : \begin{array}{c} \mathcal{L}^\bullet \in \mathrm{Bun}_{1,4},\ \mathcal{E}^\bullet \in \mathrm{Bun}_{2,4} \\ \mathcal{L}^\bullet \subset \mathcal{E}^\bullet \text{ saturated} \\ \text{in all parabolic degrees} \end{array} \right\}_{/\cong},\\
&\qquad g = (g_x)_x \mapsto (\mathcal{E}^\bullet_g := (\mathcal{E}_{(g_x)_x}, (\ell_x)_{x\in D}), \quad \mathcal{L}^\bullet_g = (\mathcal{E}_{(g_x)_x} \cap \mathcal{K}_1, I_{\mathcal{L}_g\subset\mathcal{E}_g}) \subset \mathcal{E}_{(g_x)_x}).
\end{aligned}$$

CHAPTER 2

Cusp forms and lower modifications

6. Cusp conditions and the relevant locus

In this section, we explicitly calculate what the cusp conditions for a function $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ (as defined in Section 5) are. First note that it suffices to give the condition in only one degree: for any $x \in S$, the elementary Hecke operator T_x (see Example 2.17 for the definition) provides an isomorphism

$$T_x\colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d-1}$$

that is compatible with the cusp conditions (Proposition 5.2), in the sense that if f is a cusp form, then so is $f \circ T_x\colon \mathrm{Bun}_{2,4} \to \mathrm{Bun}_{2,4}$. Therefore, a function $f\colon \mathrm{Bun}_{2,4} \to \mathbf{Q}_\ell$ is a cusp form if and only if

$$(f \circ T_x^i)|_{\mathrm{Bun}_{2,4}^1}\colon \mathrm{Bun}_{2,4}^1 \to \mathbf{Q}_\ell$$

satisfies the cusp conditions on $\mathrm{Bun}_{2,4}^1$ for all $i \in \mathbf{Z}$.

The main result is Theorem 6.4, which gives sufficient and necessary conditions for a function to be cuspidal.

One of the conditions is that the function vanishes outside of the image of a certain canonical embedding

$$j\colon \mathbf{Coh}_0^{1,1} \hookrightarrow \mathrm{Bun}_{2,4}^1\,.$$

(Theorem 6.2 and Theorem 6.3). The image of this embedding is the following open substack.

Definition 6.1. We define the *relevant locus* of $\mathrm{Bun}_{2,4}^1$ as the open substack

$$\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^1$$

that classifies parabolic bundles $\mathcal{E}^\bullet = (\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x\in D})$ satisfying

(1) $\ell_x = \mathcal{O}(1)|_x$ for at most 1 $x \in D$; and
(2) the flags do not come from a global section $\sigma\colon \mathcal{O} \to \mathcal{O}(1)$, i.e., there is no $\sigma\colon \mathcal{O} \to \mathcal{O}(1)$ such that for all $x \in D$, $\ell_x = (\sigma|_x : 1)$.

The relevant locus $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ of $\mathrm{Bun}_{2,4}^d$ is defined as the image of $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ under the isomorphism

$$(T_\infty)^{1-d}\colon \mathrm{Bun}_{2,4}^1 \xrightarrow{\sim} \mathrm{Bun}_{2,4}^d$$

where T_∞ is the elementary Hecke operator at ∞ (see Example 2.17).

THEOREM 6.2. *Let $d \in \mathbf{Z}$, $n \in \mathbf{Z}_{\geq 1}$ and let $f\colon \mathrm{Bun}^d_{2,4}(\mathbf{F}_{q^n}) \to \mathbf{Q}_\ell$ be a function that satisfies the cusp conditions. Then f vanishes on $\mathrm{Bun}^d_{2,4}(\mathbf{F}_{q^n}) \setminus \mathrm{Bun}^{d,\mathrm{r}}_{2,4}(\mathbf{F}_{q^n})$.*

The above theorem determines the open substack $\mathrm{Bun}^{d,\mathrm{r}}_{2,4} \subset \mathrm{Bun}^d_{2,4}$ uniquely. We will also see that $\mathrm{Bun}^{d,\mathrm{r}}_{2,4} = T_x^{1-d}\,\mathrm{Bun}^{1,\mathrm{r}}_{2,4}$ for any $x \in D$. so in the definition of the relevant locus in degree $d \neq 1$, we could have taken T_x for $x \in D$ instead of T_∞ We prove this theorem in Section 6.1.

In Section 6.2, we find a canonical point $\tilde{\mathcal{E}} \in \mathrm{Bun}^{2,\mathrm{r}}_{2,4}$. We then prove that all points in $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)$ are length 1 lower modifications of $\tilde{\mathcal{E}}$ (Proposition 6.19). This gives us a canonical bijection $\mathbf{Coh}^{1,1}_0(\mathbf{F}_q) \to \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)$, which is the first step towards proving the following theorem.

THEOREM 6.3. *There is a canonical isomorphism of stacks*

$$\alpha\colon \mathbf{Coh}^{1,1}_0 \xrightarrow{\sim} \mathrm{Bun}^{1,\mathrm{r}}_{2,4}.$$

We finish the proof of this theorem in Section 7.

Finally, in the last subsections, we prove the following complete characterization of cusp forms.

THEOREM 6.4. *A function $f\colon \mathrm{Bun}^1_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ satisfies the cusp conditions if and only if (1) it vanishes outside of $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)$; and (2)*

$$\sum_{\mathcal{E}^\bullet \in \mathcal{P}} \frac{f(\mathcal{E}^\bullet)}{\#\,\mathrm{Aut}(\mathcal{E}^\bullet)} = 0 \tag{6.0.1}$$

for each $\mathcal{P}$ equal to one of the following sets:

(2.1) for each $y \in D$, the set

$$\mathcal{P}^{(1,0)}_y := \mathrm{im}\left(\{k^0_y, k^{(1,0)}_y\} \xrightarrow{\alpha} \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)\right);$$

(2.2) for each $y \in D$, the set

$$\mathcal{P}^{(0,1)}_y := \mathrm{im}\left(\{k^0_y, k^{(0,1)}_y\} \xrightarrow{\alpha} \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)\right);$$

and

(2.3) for each section $\sigma\colon \mathbf{P}^1 \to \mathbf{Coh}^{1,1}_0$ of the support map $\mathrm{Supp}\colon \mathbf{Coh}^{1,1}_0 \to \mathbf{P}^1$, the set

$$\mathcal{P}_\sigma := \mathrm{im}\left(\mathbf{P}^1(\mathbf{F}_q) \xrightarrow{\sigma} \mathbf{Coh}^{1,1}_0(\mathbf{F}_q) \xrightarrow{\alpha} \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)\right).$$

Remark 6.5. Let $f\colon \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ be a function that satisfies (6.0.1) for all $\mathcal{P}$ of the form $\mathcal{P}^\bullet_x$ with $x \in D$ (i.e., one of the $\mathcal{P}$ defined in parts (2.1) and (2.2)). Let $\sigma, \sigma'\colon \mathbf{P}^1 \to \mathbf{Coh}^{1,1}_0$ be two sections of $\mathrm{Supp}\colon \mathbf{Coh}^{1,1}_0 \to \mathbf{P}^1$. Then f satisfies (6.0.1) for $\mathcal{P} = \mathcal{P}_\sigma$ if and only it satisfies (6.0.1) for $\mathcal{P} = \mathcal{P}_{\sigma'}$.

Remark 6.6. The sets $\mathcal{P}$ in the theorem have the following more explicit description:

(6.0.2)
$$\mathcal{P}_y^{(1,0)} = \{(\mathcal{O}(1)\oplus\mathcal{O},(\ell_x)) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q) : \ell_x = \mathcal{O}|_x \ \forall x \in D\setminus\{y\}\}_{/\cong}$$
(6.0.3)
$$\mathcal{P}_y^{(0,1)} = \{(\mathcal{O}(1)\oplus\mathcal{O},(\ell_x)) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q) : \ell_y = \mathcal{O}(1)|_y\}_{/\cong}$$
(6.0.4)
$$\mathcal{P}_\sigma = \{(\mathcal{O}(1)\oplus\mathcal{O},(\ell_x)) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q) : \ell_x \neq \mathcal{O}(1)|_x \ \forall x \in D\}_{/\cong}$$

where $\sigma\colon \mathbf{P}^1 \to \mathbf{Coh}_0^{1,1}$ is the section of $\mathrm{Supp}\colon \mathbf{Coh}_0^{1,1} \to \mathbf{P}^1$ such that for all $x \in D$, $\sigma(x) = k_x^{(1,0)}$.

This follows directly from our explicit description of the map $\alpha\colon \mathbf{Coh}_0^{1,1}(\mathbf{F}_q) \to \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ (Proposition 6.19).

Remark 6.7. This theorem shows that the space of cusp forms on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ is q-dimensional; see Section 11 for a more elaborate discussion.

6.1. Proof of Theorem 6.2.

We start with a useful lemma.

Lemma 6.8. *Let $\mathcal{E}^\bullet = (\mathcal{O}(m)\oplus\mathcal{O}(n),(\ell_x)_{x\in S}) \in \mathrm{Bun}_{2,4}$ be a parabolic vector bundle with with $m \geq n$. Consider the following subset of D:*
$$I := \{p \in D : \ell_p = \mathcal{O}(m)|_p\}.$$
If
$$\#(D\setminus I) \leq m-n+1 \tag{6.1.1}$$
then $\mathcal{E}^\bullet$ is isomorphic to $(\mathcal{O}(m), I)\oplus(\mathcal{O}(n), D\setminus I)$.

The proof of this lemma uses a recurring idea that is useful to highlight separately from the proof.

Definition 6.9. For $(\mathcal{O}(m)\oplus\mathcal{O}(n),(\ell_x)) \in \mathrm{Bun}_{2,4}(\mathbf{F}_q)$ with $m,n \in \mathbf{Z}$, $m > n$, we say that the flags $(\ell_x)_{x\in I}$ for some $I \subset D$ *come from a global section*, if there exists a section $\sigma\colon \mathcal{O}(n) \to \mathcal{O}(m)$ such that for all $x \in I$, ℓ_x is the image of the map $\mathcal{O}(n)|_x \to (\mathcal{O}(m)\oplus\mathcal{O}(n))|_x$ induced by
$$(\sigma,\mathrm{id})\colon \mathcal{O}(n) \to \mathcal{O}(m)\oplus\mathcal{O}(n).$$

Remark 6.10. Let $(\mathcal{O}(m)\oplus\mathcal{O}(n),(\ell_x)_{x\in D}) \in \mathrm{Bun}_{2,4}$ with $m > n$ such that the flags at $I \subset D$ come from a global section. Then the isomorphism of vector bundles
$$\begin{pmatrix} 1 & -\sigma \\ 0 & 1 \end{pmatrix} : \mathcal{O}(m)\oplus\mathcal{O}(n) \xrightarrow{\sim} \mathcal{O}(m)\oplus\mathcal{O}(n)$$

induces an isomorphism of parabolic bundles
(6.1.2)
$$(\mathcal{O}(m)\oplus\mathcal{O}(n),(\ell_x)) \xrightarrow{\sim} (\mathcal{O}(m)\oplus\mathcal{O}(n),(\ell'_x)) \quad \text{with } \ell'_x=\mathcal{O}(n)|_x \text{ for } x\in I.$$

Remark 6.11. Let $(\mathcal{O}(m)\oplus\mathcal{O}(n),(\ell_x)_{x\in D})\in \mathrm{Bun}_{2,4}$ with $m>n$ and let $I\subset D$. If $\ell_x\neq\mathcal{O}(m)|_x$ for all $x\in I$ and $\#I\leq m-n+1$, then the flags at I come from a global section. Indeed, a global section $\sigma\colon\mathcal{O}(n)\to\mathcal{O}(m)$ corresponds to a polynomial of degree $m-n$, which can be chosen to interpolate any $m-n+1$ points.

Proof of Lemma 6.8. The condition (6.1.1) ensures that the flags at $\#(D\setminus I)$ come from a global section σ (Remark 6.11). The map
$$\begin{pmatrix}1 & -\sigma\\ 0 & 1\end{pmatrix} : \mathcal{E}\to(\mathcal{O}(m,I)\oplus(\mathcal{O}(n),S\setminus I))$$
(as in Remark 6.11) now provides the desired isomorphism, because the flags from the maximal destabilizing subbundle are preserved. □

At the beginning of Section 6, we explained that it suffices to prove Theorem 6.2 for $d=1$. That is the content of the following proposition.

Proposition 6.12. *Let $f\colon \mathrm{Bun}^1_{2,4}(\mathbf{F}_q)\to\mathbf{Q}_\ell$ be a function that satisfies the cusp conditions. Then:*

(1) f vanishes on parabolic bundles of which the underlying vector bundle is isomorphic to $\mathcal{O}(1+m)\oplus\mathcal{O}(-m)$ with $m>0$.

(2) f vanishes on parabolic bundles isomorphic to $(\mathcal{O}(1)\oplus\mathcal{O},(\ell_x))$ with either (2.1) $\ell_x=\mathcal{O}(1)|_x$ for at least two $x\in D$; or (2.2) $\ell_x=\mathcal{O}|_x$ for all $x\in D$.

Proof. Recall (see (5.2.2)) that the cusp condition for a function $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q)\to\mathbf{Q}_\ell$ says that for all
$$(\mathcal{L}^\bullet,\mathcal{M}^\bullet)\colon \operatorname{Spec} k\to \mathrm{Pic}_I\times\mathrm{Pic}_{S\setminus I},$$
we have
(6.1.3)
$$\sum_{\substack{(\mathcal{L}^\bullet\hookrightarrow\mathcal{E}^\bullet\twoheadrightarrow\mathcal{M}^\bullet)\\ \in\mathrm{Ext}^1(\mathcal{M}^\bullet,\mathcal{L}^\bullet)(\mathbf{F}_q)}} f(\mathcal{E}^\bullet)=0.$$
We now prove the theorem by finding for each bundle $\mathcal{E}^\bullet$ listed in Proposition 6.12 a pair $(\mathcal{L}^\bullet,\mathcal{M}^\bullet)$ such that $\mathcal{E}^\bullet$ is the only extension in $\mathrm{Ext}^1(\mathcal{M}^\bullet,\mathcal{L}^\bullet)$.

Let $\mathcal{E}^\bullet=(\mathcal{O}(1+m)\oplus\mathcal{O}(-m),(\ell_x)_{x\in D})$ with $m\in\mathbf{Z}_{\geq 0}$ be one of the bundles listed in Proposition 6.12. We define
$$I:=\{x\in D:\ell_x=\mathcal{O}(1+m)|_x\}.$$
Now any extension
$$\mathcal{F}^\bullet=(\mathcal{F},(\ell_{\mathcal{F},x})_{x\in D})\in\mathrm{Ext}^1((\mathcal{O}(-m),D\setminus I),(\mathcal{O}(1+m),I))$$

has $\mathcal{F} = \mathcal{O}(1+m) \oplus \mathcal{O}(-m)$ as its underlying vector bundle and satisfies

$$I = \{x \in D : \ell_{\mathcal{F},x} = \mathcal{O}(1+m)|_x\}.$$

Therefore, the condition 6.1.1 from Lemma 6.8 is the same for $\mathcal{E}^\bullet$ as for $\mathcal{F}^\bullet$ and it is

$$\#(S/I) \leq 2m + 2$$

with I as defined in this proof. If this equation is satisfied, the lemma tells us $\mathcal{E}^\bullet \cong \mathcal{F}^\bullet \cong (\mathcal{O}(1+m), I) \oplus (\mathcal{O}(-m), D \setminus I)$, which proves the proposition for this choice of $\mathcal{E}^\bullet$. We note that this equation is satisfied for all $\mathcal{E}^\bullet$ from part (1) and part (2.1) of the proposition.

The proof for $\mathcal{E}^\bullet = (\mathcal{O}(1), \emptyset) \oplus (\mathcal{O}, D)$ (the bundle in part (2.2) of the proposition) is similar: by the lemma or by direct inspection, all

$$\mathcal{F}^\bullet \in \mathrm{Ext}^1((\mathcal{O}, D), (\mathcal{O}(1), \emptyset))$$

are isomorphic to $\mathcal{E}^\bullet$. □

6.2. The relevant locus. We briefly analyze $\mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ to find a canonical parabolic vector bundle $\tilde{\mathcal{E}} \in \mathrm{Bun}_{2,4}^{2,\mathrm{r}}$. This bundle has the property that every bundle $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ is a length 1 lower modification of it; more precisely, there is a unique (up to automorphisms of $\tilde{\mathcal{E}}$) inclusion $\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}$ We use this to provide a description of $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ in terms of modifications of $\tilde{\mathcal{E}}$ (Proposition 6.19). Moreover, we show that for every $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$, there is a unique short exact sequence $0 \to \mathcal{E}^\bullet \to \tilde{\mathcal{E}} \to \mathcal{T}^\bullet \to 0$ with $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$, allowing us to establish a bijection $\mathbf{Coh}_0^{1,1}(\mathbf{F}_q) \to \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ (Corollary 6.23). This is the first step in the proof of Theorem 6.3, which we complete in Section 7.

As remarked earlier, we can use the elementary Hecke operators T_x with $x \in D$ to describe $\mathrm{Bun}_{2,4}^0$ in terms of $\mathrm{Bun}_{2,4}^1$. The following is a more direct description of the parabolic bundles in $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$.

Proposition 6.13. *The parabolic bundles in* $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}(\mathbf{F}_q)$ *are the following:*

- $\tilde{\mathcal{E}}(-1) := (\mathcal{O}(1), \emptyset) \oplus (\mathcal{O}(-1), D)$*;*
- $\hat{\mathcal{E}}(-1) := (\mathcal{O}(1) \oplus \mathcal{O}(-1), (\mathcal{O}(-1)|_x)_{x \in D\setminus\{0\}}, \ell_0 = (1:1))$
- *all parabolic vector bundles* $(\mathcal{O} \oplus \mathcal{O}, (\ell_x)_{x \in S})$ *that are not isomorphic to a parabolic bundle* $(\mathcal{O} \oplus \mathcal{O}, (\ell'_x)_{x \in S})$ *with at least three* ℓ'_x *equal to the line* $\mathcal{O}|_x$ *coming from the first summand* $\mathcal{O} \subset \mathcal{O} \oplus \mathcal{O}$*.*

Proof. Either apply an elementary Hecke operator T_x with $x \in D$ to all bundles in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$, or reason as in Section 6.1. □

Remark 6.14. The points $\tilde{\mathcal{E}}, \hat{\mathcal{E}} \in \mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ are the only points in $\mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ with underlying vector bundle isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}$. One can be distinguished from the other by the automorphism group: $\operatorname{Aut}\tilde{\mathcal{E}} \cong \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ (scaling either summand in $\mathcal{O}(2) \oplus \mathcal{O}$ independently) and $\operatorname{Aut}\hat{\mathcal{E}} \cong \mathbb{G}_{\mathrm{m}}$ (scaling all of $\mathcal{O}(2) \oplus \mathcal{O}$ by the same scalar). Another characterization is the following: any parabolic vector bundle $(\mathcal{O}(2) \oplus \mathcal{O}, (\ell_x)_x)$ with $\ell_x \neq \mathcal{O}(2)|_x$ for all $x \in D$, is isomorphic to $\tilde{\mathcal{E}}$ if and only if the flags ℓ_x come from a global section (Definition 6.9 and Remark 6.10); if the flags do not come from a global section, it is isomorphic to $\hat{\mathcal{E}}$, because the flags at $D \setminus \{0\}$ do come from a global section (Remark 6.11). In particular, in our definition of $\hat{\mathcal{E}}$, we chose to set the flag at the arbitrarily chosen point $0 \in D$ (and not at another $x \in D$) to an arbitrarily chosen flag that is neither $\mathcal{O}|_x$ nor $\mathcal{O}(2)|_x$; another choice would have given an isomorphic bundle.

Remark 6.15. The parabolic structure on $\tilde{\mathcal{E}}$ induces a canonical direct sum decomposition $\mathcal{O}(2) \oplus \mathcal{O}$ of the underlying vector bundle; the same is not true for $\hat{\mathcal{E}}$.

Both parabolic bundles will play an important role in the rest of this thesis.

We now continue with our goal of describing the points in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ as modifications of $\tilde{\mathcal{E}}$ (Proposition 6.19), but start with some remarks.

Remark 6.16. For $x \in \mathbf{P}^1$, the automorphism group $\operatorname{Aut}(\tilde{\mathcal{E}}) \cong \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ acts on $\mathbf{P}^1((\mathcal{O}(2) \oplus \mathcal{O})|_x)$ and lines in the same orbit correspond to isomorphic modifications. More specifically, the fiber decomposes into three orbits — $\{\mathcal{O}(2)|_x\}$, $\{\mathcal{O}|_x\}$ and the rest — and this means that for any two lines $\ell, \ell' \subset (\mathcal{O}(2) \oplus \mathcal{O})|_x$ different from $\mathcal{O}(2)|_x$ and $\mathcal{O}|_x$, there is an automorphism $\phi\colon \tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ that sends ℓ to ℓ'. This automorphism induces an isomorphism

$$\phi\colon T_x^{\ell}\tilde{\mathcal{E}} \xrightarrow{\sim} T_x^{\ell'}\tilde{\mathcal{E}}.$$

The same is true for $x \in D$ and $\ell, \ell' \subset (\mathcal{O}(2 - [x]) \oplus \mathcal{O})|_x$ different from $\mathcal{O}(2 - [x])|_x$ and $\mathcal{O}|_x$.

We first describe the the modifications of $\tilde{\mathcal{E}}$ that do not lie in the relevant locus.

Lemma 6.17. *The following modifications of $\tilde{\mathcal{E}}$ do not lie in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$:*

$$\begin{aligned} T_x^{\mathcal{O}(2)|_x}\tilde{\mathcal{E}} &= (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-x), D) && \text{for } x \in \mathbf{P}^1; \\ T_x^{\mathcal{O}|_x}\tilde{\mathcal{E}} &= (\mathcal{O}(2 - [x]), \emptyset) \oplus (\mathcal{O}, D) && \text{for } x \in \mathbf{P}^1 \setminus D; \text{ and} \\ {}^{\mathcal{O}|_x}T_x\tilde{\mathcal{E}} &= (\mathcal{O}(2 - [x]), \emptyset) \oplus (\mathcal{O}, D) && \text{for } x \in D. \end{aligned}$$

Proof. Both the equalities and the fact that the modifications do not lie in the relevant locus follow directly from the definitions. ∎

Remark 6.18. The modifications in the above lemma are exactly the kernels of maps
$$\tilde{\mathcal{E}} = (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D) \twoheadrightarrow \mathcal{T}^\bullet$$
with $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$, that factor through either $(\mathcal{O}(2), \emptyset)$ or $(\mathcal{O}, D)$.

The follow proposition states that all other modifications do lie in the relevant locus, and in fact, every point in the relevant locus is obtained as a modification of $\tilde{\mathcal{E}}$. We thus get a bijection between the $\mathrm{Aut}(\tilde{\mathcal{E}})$-orbits of the set
$$\{\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(\mathbf{F}_q),\ \tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet\} \setminus \left\{ \begin{array}{c} \tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet \text{ factoring through} \\ \tilde{\mathcal{E}} \to (\mathcal{O}(2), \emptyset) \text{ or } \tilde{\mathcal{E}} \to (\mathcal{O}, D) \end{array} \right\}$$
and the points in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$. It is the main result of this section.

Proposition 6.19. *The following is complete list of pairwise distinct points in* $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$*:*

(1) for $x \in S$,
   - *(a) the point $T_x\tilde{\mathcal{E}}$;*
   - *(b) the point $T_x^\ell\tilde{\mathcal{E}}$, where $\ell \subset (\mathcal{O}(2) \oplus \mathcal{O})|_x$ is different from $\mathcal{O}|_x$ and $\mathcal{O}(2)|_x$; and*
   - *(c) the point ${}^{\ell'}T_x\tilde{\mathcal{E}}$, where $\ell' \subset (\mathcal{O}(2-[x]) \oplus \mathcal{O})|_x$ is different from $\mathcal{O}|_x$ and $\mathcal{O}(2-[x])|_x$;*

(2) for $x \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$, the point $T_x^\ell\tilde{\mathcal{E}}$, where $\ell \subset (\mathcal{O}(2) \oplus \mathcal{O})|_x$ is different from $\mathcal{O}|_x$ and $\mathcal{O}(2)|_x$.

The proof of this proposition follows after a few remarks.

Remark 6.20. In Section 7, we prove that we have an isomorphism
$$\mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}, \qquad \mathcal{T}^\bullet \mapsto \ker(\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet),$$
where $\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet$ is a surjection taken from a certain open substack of the stack of all such surjections. The proposition above is essentially that statement, but on $\mathbf{F}_q$-points, and it is an essential ingredient in the proof.

Remark 6.21. Some of the points in the above proposition can be uniquely characterized. For $x_0 \in S$,

(a) the point $T_{x_0}\tilde{\mathcal{E}}$ is isomorphic the unique $(\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x \in S}) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ such that
   1. $\ell_{x_0} = \mathcal{O}(1)|_{x_0}$ and
   2. the flags at $x \in D \setminus \{x_0\}$ come from a global section;

(b) the point $T_{x_0}^\ell\tilde{\mathcal{E}}$ with ℓ different from $\mathcal{O}(2)|_{x_0}, \mathcal{O}|_{x_0}$ is isomorphic the unique $(\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x \in S}) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ such that

(1) $\ell_{x_0} = \mathcal{O}(1)|_{x_0}$ and

(2) the flags at $x \in D \setminus \{x_0\}$ *do not* come from a global section;

and

(c) the point ${}^{\ell'}T_{x_0}\tilde{\mathcal{E}}$ with ℓ' different from $\mathcal{O}(2-[x])|_{x_0}, \mathcal{O}|_{x_0}$ is isomorphic the unique $(\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x\in S}) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ such that

(1) $\ell_x \neq \mathcal{O}(1)|_x$ for all $x \in D$ and

(2) the flags at $x \in D \setminus \{x_0\}$ come from a global section;

Proof of Proposition 6.19. First, we note that all the listed points are in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$: the underlying vector bundle is $\mathcal{O}(1) \oplus \mathcal{O}$; the flag ℓ_x can only be $\mathcal{O}(1)|_x$ if we modify at x, so in particular, there is only one such flag; and lastly, not all lines come from a global section, as we now prove. If the lines do come from a global section, the modification is isomorphic to $(\mathcal{O}(1) \oplus \mathcal{O}, (\mathcal{O}|_x)_{x\in D})$. Any inclusion of this parabolic bundle into $\tilde{\mathcal{E}}$ is given on the underlying vector bundles by

$$\begin{pmatrix} \sigma & 0 \\ 0 & \mu \end{pmatrix} : \mathcal{O}(1) \oplus \mathcal{O} \to \mathcal{O}(2) \oplus \mathcal{O}$$

with $\sigma\colon \mathcal{O} \hookrightarrow \mathcal{O}(1)$ and $\mu \in \mathbf{F}_q^*$, because the fact that the map preserves the flags implies that the component $\mathcal{O} \to \mathcal{O}(2)$ has zeroes at all points in D, so is zero. The inclusion $(\mathcal{O}(1) \oplus \mathcal{O}, (\mathcal{O}|_x)_{x\in D}) \hookrightarrow \tilde{\mathcal{E}}$ is then a modification at the zero x_0 of σ and by considering the image, we see that this modification is in fact isomorphic to $T_{x_0}^{(0:1)}\tilde{\mathcal{E}} \hookrightarrow \mathcal{E}$ if $x_0 \notin D$ and ${}^{(0:1)}T_{x_0}\tilde{\mathcal{E}} \hookrightarrow \mathcal{E}$ otherwise. In other words, it is not of one of the modifications listed in the proposition.

We first identify the points $T_x\tilde{\mathcal{E}}$ and $T_x^{\ell}\tilde{\mathcal{E}}$ from points (1a) and (1b). Let $x_0 \in D$ and let $\mathcal{E}^\bullet = (\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x\in D}) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ with $\ell_{x_0} = \mathcal{O}(1)|_{x_0}$. There is only one ℓ_x with $\ell_x = \mathcal{O}(1)|_x$ by definition of $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ (Definition 6.1). If these flags at $D \setminus \{x_0\}$ come from a global section, then the bundle $\mathcal{E}^\bullet$ is isomorphic to

$$T_{x_0}\tilde{\mathcal{E}} = (\mathcal{O}(1), \{x_0\}) \oplus (\mathcal{O}, D \setminus \{x_0\})$$

(Remark 6.10). If the flags at $D \setminus \{x_0\}$ do not come from a global section, then $\mathcal{E}^\bullet$ is isomorphic to $T_x^{\ell}\tilde{\mathcal{E}}$ with $\ell \neq \mathcal{O}(2)|_x, \mathcal{O}|_x$.

Now we prove that all remaining points are of the form $T_x^{\ell}\tilde{\mathcal{E}}$ for $x \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$ or ${}^{\ell'}T_x\tilde{\mathcal{E}}$ for $x \in D$. Let $\mathcal{E}^\bullet = (\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x\in D}) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ be one of the remaining points, i.e.,, $\ell_x \neq \mathcal{O}(1)|_x$ for all $x \in D$. The idea of the proof is to show that for each such $\mathcal{E}^\bullet$, there is exactly one inclusion $\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}$, up to automorphisms of $\mathcal{E}^\bullet$. This uniqueness follows from condition (2) in Definition 6.1, which says that the flags do not come from a global section. This injection is then a modification at some $x \in \mathbf{P}^1$ and we complete the proof by showing that it is in fact the remaining modification at x listed in the proposition.

Let

$$\begin{pmatrix} \sigma & \tau \\ 0 & 1 \end{pmatrix} : \mathcal{O}(1) \oplus \mathcal{O} \hookrightarrow \mathcal{O}(2) \oplus \mathcal{O}$$

be a map of vector bundles with with $\sigma\colon \mathcal{O}(1) \to \mathcal{O}(2)$ not zero and $\tau\colon \mathcal{O} \to \mathcal{O}(2)$. The assumptions $\sigma \neq 0$ assure that this map is an injection. By setting the bottom right coordinate of the matrix to 1, we fix the scalar multiple of the matrix. This map induces a map of parabolic bundles

$$(6.2.1)\qquad \begin{pmatrix} \sigma & \tau \\ 0 & 1 \end{pmatrix} : (\mathcal{O}(1) \oplus \mathcal{O}, (\ell_x)_{x \in D}) \hookrightarrow (\mathcal{O}(2) \oplus \mathcal{O}, (\mathcal{O}|_x)_{x \in D}) = \tilde{\mathcal{E}}.$$

if and only if (σ, τ) is in the kernel of the following $\mathbf{F}_q$-linear map

$$(6.2.2)\qquad \mathrm{H}^0(\mathcal{O}(1) \oplus \mathcal{O}(2)) \to (\mathcal{O}(2)|_x)_{x \in D}, \qquad (\sigma', \tau') \mapsto (\sigma'\ \tau')\ell_x.$$

The space on the left has dimension 5, while the space on the right has dimension 4, so there is always at least one $(\sigma', \tau') \neq 0$ in the kernel. Such a pair (σ', τ') then defines a modification as in Equation (6.2.1), because the condition $(\sigma', \tau') \neq 0$ implies $\sigma' \neq 0$. Indeed, assume towards a contradiction $(\sigma', \tau') \neq 0$ is in the kernel, but $\sigma' = 0$. Then for all $x \in D$, we have $(0\ \tau')\ell_x = 0$, which implies $\tau'|_x = 0$, because $\ell_x \neq \mathcal{O}(1)|_x$. Since τ' is a map $\mathcal{O} \to \mathcal{O}(2)$, this in turn implies $\tau' = 0$, in contradiction to our assumption that (σ', τ') was non-trivial. Therefore, if we show that the kernel K of (6.2.2) has dimension at most 1, we have proven that there is a unique (up to scalar) inclusion $\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}$.

To show that the kernel of the map (6.2.2) has dimension at most 1, one could simply choose coordinates and write down the matrix. The conclusion will then be that the matrix has full rank if and only if the flags $(\ell_x)_{x \in D}$ do not come from a global section. We will however provide an alternative proof.

The flags ℓ_x are of the form $(r_x : 1)$ with $r_x \in \mathcal{O}(1)|_x$. The condition $(\sigma'\ \tau')\ell_x = 0$ can for those $x \in D$ can be rewritten as

$$\tau'_x = -r_x \sigma_x \quad \text{in } \mathcal{O}|_x.$$

Since there is a unique polynomial of degree 2 that interpolates any three points, this determines τ' uniquely. This implies that if the kernel K of (6.2.2) has dimension at least 2, there is injection

$$\mathrm{H}^0(\mathcal{O}(1)) \rightarrowtail \mathrm{H}^0(\mathcal{O}(2)), \qquad \sigma \mapsto \tau_\sigma$$

that maps σ to the unique τ_σ such that $(\sigma, \tau_\sigma) \in K$.

To see that this implies that the ℓ_x come from a global section, write $\chi = \tau_1$. This is a polynomial of degree 1 in X. For all $\sigma \in \mathrm{H}^0(\mathcal{O}(1))$, we now have

$$\tau_\sigma = \sigma\chi,$$

because $\sigma\chi$ takes the right values at $D \setminus \{x\}$ and there is only one degree 2 polynomial that takes the right values at three points. Therefore, we have

$$\sigma_x(r_x + \chi_x) = 0 \quad \text{in } \mathcal{O}|_x$$

for all σ and all $x \in D$. From this, it follows (by taking $\sigma = 1$ and $\sigma = X$, for example), that $r_x = -\chi_x$, which proves the contradiction that the flags ℓ_x come from the global section $-\chi$.

This shows $\dim K = 1$ and hence, for all $\mathcal{E}^\bullet = (\mathcal{E}, (\ell_x)_{x\in D}) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ with $\ell_x \neq \mathcal{O}(1)|_x$ for all $x \in D$, there is a unique (up to scaling) inclusion $\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}$. In particular, there is a unique $y \in \mathbf{P}^1$ (the zero of σ) such that $\mathcal{E}^\bullet$ is a modification of $\tilde{\mathcal{E}}$ at y. It remains to prove that the modification is isomorphic to $T_y^\ell\tilde{\mathcal{E}}$ with $\ell \neq \mathcal{O}(2)|_y, \mathcal{O}|_y$ if $y \notin D$ and ${}^{\ell'}T_y\tilde{\mathcal{E}}$ with $\ell' \neq \mathcal{O}(2-[y])|_y, \mathcal{O}|_y$ if $y \in D$. But this follows from the already known fact that all other length 1 lower modifications of $\tilde{\mathcal{E}}$ at y are either not in the relevant locus (Lemma 6.17) or have a flag that is equal to $\mathcal{O}(1)|_x$ (proven at the beginning of this proof). $\square$

Corollary 6.22. *Let $\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet$ be a length one lower modification of $\tilde{\mathcal{E}}$. Then $\mathcal{E}^\bullet$ lies in the relevant locus if and only if $\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet$ does not factor through one of the parabolic direct summands of $\tilde{\mathcal{E}} = (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D)$.*

Proof. Remark 6.18 states that when $\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet$ does factor through a parabolic summand, $\mathcal{E}^\bullet$ does not lie in the relevant locus. ByProposition 6.19, all other modifications do lie in the relevant locus. $\square$

The following corollary then summarizes what we have proven and concludes our proof of Theorem 6.3].

Corollary 6.23. *For every $x \in \mathbf{P}^1$, choose lines $\ell_x \subset \tilde{\mathcal{E}}|_x$ and (for $x \in D$) $\ell'_x \subset (T_x\tilde{\mathcal{E}})|_x$ as before, i.e., $\ell_x \neq \mathcal{O}(2)|_x, \mathcal{O}|_x$ and $\ell'_x \neq \mathcal{O}(2-[x])|_x, \mathcal{O}|_x$. The following defines a bijective map on $\mathbf{F}_q$-points:*

$$\begin{aligned} \mathbf{Coh}_0^{1,1}(\mathbf{F}_q) &\to \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q), \\ \textit{for } y \in \mathbf{P}^1 \setminus D\textit{:} \quad k_y &\mapsto T_y^{\ell_y}\tilde{\mathcal{E}} \\ \textit{for } x \in D\textit{:} \quad k_x^{(0,1)} &\mapsto T_x^{\ell_x}\tilde{\mathcal{E}} \\ k_x^{(1,0)} &\mapsto {}^{\ell'_x}T_x\tilde{\mathcal{E}} \\ k_x^{(0,0)} &\mapsto T_x\tilde{\mathcal{E}} \end{aligned}$$

that does not depend on the choice of lines ℓ_x, ℓ'_x. All other modifications of $\tilde{\mathcal{E}}$ do not lie in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$.

Lemma 6.24. *The automorphism group of the parabolic bundles $T_x\tilde{\mathcal{E}}$ for $x \in D$ is isomorphic to $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$. The automorphism group of any other parabolic bundle in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ is isomorphic to $\mathbb{G}_\mathrm{m}$.*

Proof. An automorphism of the underlying vector bundle $\mathcal{O}(1) \oplus \mathcal{O}$ is of the form

$$\begin{pmatrix} \lambda & \sigma \\ 0 & \mu \end{pmatrix}$$

for $\lambda, \mu \in k^*$ and $\sigma\colon \mathcal{O} \to \mathcal{O}(1)$ a morphism. To prove this lemma, we need to consider which of these automorphisms preserves the parabolic structure.

The points $T_x\tilde{\mathcal{E}}$ for $x \in S$ are isomorphic to $(\mathcal{O}(1) \oplus \mathcal{O}, (\ell_y)_{y\in S})$ with $\ell_y = \mathcal{O}$ for $y \neq x$ and $\ell_x = \mathcal{O}(1)|_x$. We see that λ, μ can be arbitrary scalars, but σ needs to be zero at all of $S \setminus \{x\}$, so is zero.

The other points are isomorphic to some $(\mathcal{O}(1) \oplus \mathcal{O}, (\ell_y)_{y\in S})$ where precisely two $y \in S$ are $\mathcal{O}|_y$. This implies $\sigma = 0$. Because there is one flag ℓ_x that is neither $\mathcal{O}(1)|_x$ or $\mathcal{O}|_x$, we need $\lambda = \mu$. □

Remark 6.25 (Alternative characterization of the parabolic bundles in the relevant locus)**.** The parabolic rank 2 vector bundles that are in the relevant locus, are exactly those with underlying vector bundle $\mathcal{O}(1) \oplus \mathcal{O}$ and either (1) automorphism group isomorphic to $\mathbb{G}_{\mathrm{m}}$; or (2) automorphism group isomorphic to $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ and stable under $T_D \circ (- \otimes \mathcal{O}(2))$.

6.3. Counting extensions. Recall (see (5.2.2)) that the cusp condition for a function $f\colon \operatorname{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ says that for all

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet)\colon \operatorname{Spec} k \to \operatorname{Pic}_I \times \operatorname{Pic}_{S\setminus I},$$

we have

$$\sum_{\substack{(\mathcal{L}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{M}^\bullet) \\ \in \operatorname{Ext}^1(\mathcal{M}^\bullet, \mathcal{L}^\bullet)(\mathbf{F}_q)}} f(\mathcal{E}^\bullet) = 0. \tag{6.3.1}$$

This equation can be rewritten as

$$\sum_{\mathcal{E}^\bullet \in \operatorname{Bun}^1_{2,4}(\mathbf{F}_q)} c_{\mathcal{E}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet} f(\mathcal{E}^\bullet) = 0$$

for suitably chosen coefficients $c_{\mathcal{E}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet} \in \mathbf{Z}$. The goal of this section is to understand these coefficients $c_{\mathcal{E}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet}$. The main result is Corollary 6.27, which states

$$c_{\mathcal{E}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet} = \frac{q-1}{\#\operatorname{Aut}(\mathcal{E}^\bullet)} \cdot \#\operatorname{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)$$

where $\operatorname{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)$ denotes the set of injective homomorphisms $\mathcal{L}^\bullet \to \mathcal{E}^\bullet$ of parabolic bundles that are saturated in every parabolic degree. Finally, in Corollary 6.29, we express $\#\operatorname{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)$ in terms of saturated homomorphisms of the underlying vector bundles of modifications of $\mathcal{L}^\bullet$ and $\mathcal{E}^\bullet$.

Let $(\mathcal{M}^\bullet_0, \mathcal{L}^\bullet_0) \in \operatorname{Pic}_I \times \operatorname{Pic}_{D\setminus I}$. Consider the map

$$\operatorname{Ext}^1(\mathcal{M}^\bullet_0, \mathcal{L}^\bullet_0)(\mathbf{F}_q)_{/\cong} \to \operatorname{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)_{/\cong} \tag{6.3.2}$$

where the $- / \cong$ means that we take the set of isomorphism classes in the groupoid of $\mathbf{F}_q$-points. Then for $\mathcal{E}^\bullet_0 \in \operatorname{Bun}_{2,4}$, the coefficient $c_{\mathcal{E}^\bullet_0, \mathcal{L}^\bullet_0, \mathcal{M}^\bullet_0}$ is the number of elements in the fiber of (6.3.2) over $\mathcal{E}^\bullet$. Indeed, this counts the number of isomorphism classes of the form $\mathcal{L}^\bullet_0 \hookrightarrow \mathcal{E}^\bullet_0 \twoheadrightarrow \mathcal{M}^\bullet_0$ in $\operatorname{Ext}^1(\mathcal{M}^\bullet_0, \mathcal{L}^\bullet_0)$. The following lemma gives an expression for this.

Lemma 6.26. *Let $\mathcal{E}_0^\bullet \in \mathrm{Bun}_{2,4}^1(\mathbf{F}_q)$ and $\mathcal{M}_0^\bullet \in \mathrm{Pic}_{D\setminus I}(\mathbf{F}_q)$, $\mathcal{L}_0^\bullet \in \mathrm{Pic}_I(\mathbf{F}_q)$. The fiber of*

$$\mathrm{Ext}^1(\mathcal{M}_0^\bullet, \mathcal{L}_0^\bullet)(\mathbf{F}_q)_{/\cong} \to \mathrm{Bun}_{2,4}^1(\mathbf{F}_q)_{/\cong}$$

over $\mathcal{E}_0^\bullet$ consists of

$$\frac{q-1}{\#\,\mathrm{Aut}(\mathcal{E}_0^\bullet)(\mathbf{F}_q)} \cdot \#\,\mathrm{Hom}_{\mathrm{par}}^{\mathrm{inj,sat}}(\mathcal{L}_0^\bullet, \mathcal{E}_0^\bullet)(\mathbf{F}_q)$$

points.

Proof. The number of points in this fiber is equal to the number of isomorphism classes in the groupoid of $\mathbf{F}_q$-points of the stack

$$\mathcal{X} := \left\langle (0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet \to \mathcal{M}^\bullet \to 0, \mathcal{L}^\bullet \xrightarrow{\sim} \mathcal{L}_0^\bullet, \mathcal{M}^\bullet \xrightarrow{\sim} \mathcal{M}_0^\bullet) : \ \exists \mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{E}_0^\bullet \ \right\rangle.$$

This stack is an $\mathrm{Aut}(\mathcal{M}_0^\bullet)$-torsor over the stack

$$\mathcal{Y} := \left\langle (0 \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet, \mathcal{L}^\bullet \xrightarrow{\sim} \mathcal{L}_0^\bullet) : \begin{array}{c} \exists \mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{E}_0^\bullet, \text{ and} \\ \mathcal{L}^\bullet \to \mathcal{E}^\bullet \text{ saturated} \\ \text{in all parabolic degrees} \end{array} \right\rangle$$

and $\mathrm{Aut}(\mathcal{M}_0^\bullet) = \mathbb{G}_{\mathrm{m}}$, so that the $\mathcal{X}$ has $q-1$ as many $\mathbf{F}_q$-points as $\mathcal{Y}$. The $\mathbf{F}_q$-points of $\mathcal{Y}$ can then be counted by remarking that the map

$$\mathrm{Hom}_{\mathrm{par}}^{\mathrm{inj,sat}}(\mathcal{L}_0^\bullet, \mathcal{E}_0^\bullet) \to \mathcal{Y}$$

is an $\mathrm{Aut}(\mathcal{E}_0^\bullet)$-torsor, which completes the proof. □

We have thus proven the following.

Corollary 6.27. *Let $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$. The cusp condition for the pair $(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \in \mathrm{Pic}_I^d(\mathbf{F}_q) \times \mathrm{Pic}_{D\setminus I}^{1-d}(\mathbf{F}_q)$ is equivalent to the condition*

$$\sum_{\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^1(\mathbf{F}_q)} \frac{q-1}{\#\,\mathrm{Aut}(\mathcal{E}^\bullet)} \#\,\mathrm{Hom}_{\mathrm{par}}^{\mathrm{inj,sat}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet) \cdot f(\mathcal{E}^\bullet) = 0.$$

Proof. This follows from the lemma and the observation that the coefficients $c_{\mathcal{E}_0^\bullet, \mathcal{L}_0^\bullet, \mathcal{M}_0^\bullet}$ are exactly the number of elements in the fiber counted in the lemma. □

To further continue our analysis, we would like to express the number of elements of $\mathrm{Hom}_{\mathrm{par}}^{\mathrm{inj,sat}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)$ in terms of the underlying vector bundles of $\mathcal{L}^\bullet$ and $\mathcal{E}^\bullet$. For $J \subset D$, we denote the underlying vector bundle of $T_J\mathcal{E}^\bullet$ by $T_J\mathcal{E}$, even though this is slightly misleading, as it depends on the parabolic structure $\mathcal{E}^\bullet$ on $\mathcal{E}$. Also note that a map of parabolic vector bundles is determined by the map on the underlying vector bundles.

Lemma 6.28. *Let $\mathcal{L}^\bullet = (\mathcal{L}, I)$ and let $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}$. The natural identification*

$$\mathrm{Hom}_{\mathrm{Coh}}^{\mathrm{inj}}(\mathcal{L}, T_I\mathcal{E}) = \mathrm{Hom}_{\mathrm{par}}^{\mathrm{inj}}((\mathcal{L}, I), \mathcal{E}^\bullet),$$
$$(\mathcal{L} \hookrightarrow T_I\mathcal{E}) \mapsto (\mathcal{L} \hookrightarrow T_I\mathcal{E} \hookrightarrow \mathcal{E})$$

restricts to an identification

$$\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_I\mathcal{E}) \setminus \bigcup_{x\in D} \mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_xT_I\mathcal{E}) = \mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}((\mathcal{L}, I), \mathcal{E}^\bullet)$$

where we embed

$$\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_xT_I\mathcal{E}) \hookrightarrow \mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_I\mathcal{E})$$

via composition with the inclusion $T_xT_I\mathcal{E} \hookrightarrow T_I\mathcal{E}$.

Proof. We first note that

$$\begin{gathered}\mathrm{Hom}^{\mathrm{inj}}_{\mathrm{Coh}}(\mathcal{L}, T_I\mathcal{E}) \to \mathrm{Hom}^{\mathrm{inj}}_{\mathrm{par}}((\mathcal{L}, I), \mathcal{E}^\bullet),\\ (\mathcal{L} \to T_I\mathcal{E}) \mapsto (\mathcal{L} \to T_I\mathcal{E} \hookrightarrow \mathcal{E})\end{gathered}$$

is a well-defined bijection: we have a commutative diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook] & \mathcal{E} \\ (T_I(\mathcal{L}, I))^0 = \mathcal{L} \arrow[u, hook] \arrow[r, hook] & T_I\mathcal{E} \arrow[u, hook] \end{tikzcd}$$

which shows that the underlying map of vector bundles $\mathcal{L} \to \mathcal{E}$ factors through $T_I\mathcal{E}$. It therefore remains to show that the saturatedness condition on the right corresponds to taking saturated maps on the left that do not lie in any of the subsets $\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_xT_I\mathcal{E}) \subset \mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_I\mathcal{E})$ for any $x \in D$.

A map $(\mathcal{L}, I) \to \mathcal{E}^\bullet$ is injective and saturated in all parabolic degrees if and only if the map

$$T_I(\mathcal{L}, I) = (\mathcal{L}, \emptyset) \to T_I\mathcal{E}^\bullet$$

is, since T_I only shifts the degrees of the parabolic chains. This latter map is injective and saturated in all parabolic degrees if and only if

(1) the map $\mathcal{L} \to T_I\mathcal{E}$ is injective and saturated; and

(2) for all $x \in D$, the map $\mathcal{L}(-x) \to T_xT_I\mathcal{E}$ is saturated at x

because under the assumption of (1), the maps in parabolic degree $(-1, x)$ are saturated outside of x. Moreover, if we assume condition (1), condition (2) is equivalent to:

(2') for all $x \in D$, the map $\mathcal{L} \to T_I\mathcal{E}$ doesn't factor through $\mathcal{L} \to T_xT_I\mathcal{E}$.

□

Corollary 6.29. *Let* $\mathcal{L}^\bullet = (\mathcal{L}, I)$ *a parabolic line bundle and* $\mathcal{E}^\bullet \in \mathrm{Bun}^1_{2,4}(\mathbf{F}_q)$. *Then*

$$\#\,\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}((\mathcal{L}, I), \mathcal{E}^\bullet) = \sum_{\emptyset\subseteq J\subseteq D} (-1)^{\#J}\,\#\,\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_JT_I\mathcal{E}).$$

Proof. This follows from the lemma and the inclusion-exclusion, because for $J, J' \subset D$, we have

$$\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_JT_I\mathcal{E}) \cap \mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_{J'}T_I\mathcal{E}) = \mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_{J\cup J'}T_I\mathcal{E})$$

as subsets of $\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_I\mathcal{E})$. $\square$

Therefore, the number of elements in the sets $\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_J T_I\mathcal{E})$ appearing above only depends on the isomorphism classes of $\mathcal{L}$ and $T_J T_I\mathcal{E}$ as vector bundles.

6.4. Sufficiency.

Proposition 6.30. *Let $f\colon \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \to \mathbf{Q}_\ell$ be a function such that*

$$\sum_{\mathcal{E}^\bullet \in \mathcal{P}} \frac{f(\mathcal{E}^\bullet)}{\#\,\mathrm{Aut}(\mathcal{E}^\bullet)} = 0$$

for all sets $\mathcal{P}$ defined in (2.1) to (2.3) in Theorem 6.4. Then f is cuspidal.

Proof. Let $I \subset J$ and $\mathcal{L}^\bullet \in \mathrm{Pic}_I$. We need to prove

$$\sum_{\mathcal{E}^\bullet \in \mathrm{Bun}^{;\mathrm{r}}_{2,4}1(\mathbf{F}_q)} \frac{q-1}{\#\,\mathrm{Aut}(\mathcal{E}^\bullet)} \#\,\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet) \cdot f(\mathcal{E}^\bullet) = 0$$

(Corollary 6.27). By our analysis of $\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{par}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)$ and the inclusion-exclusion principle (Corollary 6.29), it suffices to prove for all $J \subset D$,

$$\sum_{\mathcal{E}^\bullet \in \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)} \frac{q-1}{\#\,\mathrm{Aut}(\mathcal{E}^\bullet)} \#\,\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_J T_I\mathcal{E}) \cdot f(\mathcal{E}^\bullet) = 0. \tag{6.4.1}$$

Suppose that $\#J + \#I$ is even. Then for all $\mathcal{E}^\bullet \in \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)$, we have $T_J T_I\mathcal{E} \in \mathrm{Bun}^{d,\mathrm{r}}_{2,4}$ for some odd $d \in \mathbf{Z}$, so $T_J T_I \mathcal{E}$ is isomorphic to $\mathcal{O}(\lfloor d \rfloor) \oplus \mathcal{O}(\lceil d \rceil)$. Therefore, there exists $c \in \mathbf{Z}$ with

$$c = \#\,\mathrm{Hom}^{\mathrm{inj,sat}}_{\mathrm{Coh}}(\mathcal{L}, T_J T_I\mathcal{E}) \qquad \text{for all } \mathcal{E}^\bullet \in \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)$$

and we can rewrite (6.4.1) as

$$c \sum_{\mathcal{E}^\bullet \in \mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)} \frac{f(\mathcal{E}^\bullet)}{\#\,\mathrm{Aut}(\mathcal{E}^\bullet)(\mathbf{F}_q)} = 0.$$

Our function f does indeed satisfy this equation, because $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q)$ is a disjoint union of sets $\mathcal{P}$ from (2.1), (2.2) and (2.3) in Theorem 6.4.

Suppose that $d := \#J + \#I$ is odd. The reasoning is similar, but slightly more complicated, because there are two parabolic bundles with a different underlying vector bundle, but these two bundles lie in some $\mathcal{P}^e_x$. More precisely, there exist $x \in D$, $e \in \{(1,0),(0,1)\}$ and $\sigma\colon \mathbf{P}^1 \to \mathbf{Coh}^{1,1}_0$ a section of $\mathrm{Supp}\colon \mathbf{Coh}^{1,1}_0 \to \mathbf{P}^1$ that avoids $\alpha^{-1}(\mathcal{P}^e_x) \subset \mathbf{Coh}^{1,1}_0$, such that the following holds: letting

$$\mathcal{Q} := \{\mathcal{P}_\sigma\} \cup \{\mathcal{P}^\bullet_x\}_{x \in D} \setminus \{\mathcal{P}^e_x\}$$

we have a decomposition

$$\mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q) = \mathcal{P}^e_x \sqcup \bigsqcup_{\mathcal{P} \in \mathcal{Q}} \mathcal{P}$$

such that

(1) for $\mathcal{E}^\bullet \in \mathcal{P}_x^e$, the underlying bundle of $T_J T_I \mathcal{E}^\bullet$ is $\mathcal{O}((3-d)/2) \oplus \mathcal{O}((-1-d)/2)$; and

(2) for all $\mathcal{E}^\bullet \in \bigcup \mathcal{Q}$, the underlying bundle of $T_J T_I \mathcal{E}^\bullet$ is $\mathcal{O}((1-d)/2) \oplus \mathcal{O}((1-d)/2)$.

Therefore, there exist $c, d \in \mathbf{Z}$ such that we can rewrite (6.4.1) as

$$c \sum_{\mathcal{E}^\bullet \in \mathcal{P}_x^e} \frac{f(\mathcal{E}^\bullet)}{\#\operatorname{Aut}(\mathcal{E}^\bullet)(\mathbf{F}_q)} + d \sum_{\mathcal{E}^\bullet \in \bigcup \mathcal{Q}} \frac{f(\mathcal{E}^\bullet)}{\#\operatorname{Aut}(\mathcal{E}^\bullet)(\mathbf{F}_q)} = 0.$$

The function f does indeed satisfy this equation by assumption. □

6.5. Necessity.

Proposition 6.31. *Let $f\colon \operatorname{Bun}_{2,4}^1(\mathbf{F}_q) \to \mathbf{Q}_\ell$ be a function. If f is a cusp form, then f satisfies conditions (1) and (2) in Theorem 6.4.*

Proof. In Section 6.1, we already proved that f satisfies condition (1); this is Theorem 6.2.

By our description of the cusp condition in terms of injective, saturated, parabolic morphisms (Corollary 6.27), to prove condition (2), it suffices to find for each class $\mathcal{P}$ a parabolic line bundle $\mathcal{L}^\bullet \in \operatorname{Pic}_I$ such that the function

$$\operatorname{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q) \to \mathbf{Z}, \qquad \mathcal{E}^\bullet \mapsto \#\operatorname{Hom}_{\mathrm{par}}^{\mathrm{inj,sat}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)$$

is constant non-zero on $\mathcal{P}$ and zero outside of $\mathcal{P}$.

Using our explicit description of the classes $\mathcal{P}$ (Remark 6.6), we find that the following choices of $\mathcal{L}^\bullet$ work. Let $x \in D$. For $\mathcal{P}_x^{(1,0)}$, we can take $\mathcal{L}^\bullet = (\mathcal{O}(1), \{x\})$. For $\mathcal{P}_x^{(0,1)}$, we can take $\mathcal{L}^\bullet = (\mathcal{O}, D \setminus \{x\})$. For $\mathcal{P}_\sigma$, where $\sigma\colon \mathbf{P}^1 \to \mathbf{Coh}_0^{1,1}$ is the section of $\operatorname{Supp}\colon \mathbf{Coh}_0^{1,1} \to \mathbf{P}^1$ such that for all $x \in D$, $\sigma(x) = k_x^{(1,0)}$, we can take $\mathcal{L}^\bullet = (\mathcal{O}(1), \emptyset)$. This proves that condition (2.3) holds for every choice of σ (Remark 6.5). □

7. Chart

In this section we construct a canonical isomorphism of stacks

$$\alpha\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \operatorname{Bun}_{2,4}^{1,\mathrm{r}}$$

(Theorem 6.3). We have already seen that there is a natural bijection $\mathbf{Coh}_0^{1,1}(\mathbf{F}_q) \to \operatorname{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ ((6.23)). The idea behind the construction that we define a certain substack $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \subset \mathcal{H}$ that classifies only the "correct" modifications of $\tilde{\mathcal{E}}$. We will then restrict the maps $p\colon \mathcal{H}^2 \to \operatorname{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1}$ and $q\colon \mathcal{H}^2 \to \operatorname{Bun}_{2,4}^1$ to this substack $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}}$, which gives us an isomorphism $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \xrightarrow{\sim} \mathbf{Coh}_0^{1,1}$ (from the restriction of q) and an isomorphism $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \xrightarrow{\sim} \operatorname{Bun}_{2,4}^{1,\mathrm{r}}$ (from the restriction of p) that we use to construct α.

We first define a substack of the Hecke stack in degree 2, which in essence classifies length 1 lower modifications of $\tilde{\mathcal{E}}$.

Definition 7.1. We define a substack $\mathcal{H}_{\tilde{\mathcal{E}}} \subset \mathcal{H}^2$ as the pullback

$$\begin{tikzcd} \mathcal{H}_{\tilde{\mathcal{E}}} \arrow[r, hook] \arrow[d] & \mathcal{H}^2 \arrow[d, "p"] \\ \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[r, hook] & \operatorname{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1} \end{tikzcd}$$

of the map $p\colon \mathcal{H}^2 \to \operatorname{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1}$, $(\mathcal{E} \twoheadrightarrow \mathcal{T}) \mapsto (\mathcal{E}, [\mathcal{T}])$ along the inclusion $\mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \subset \operatorname{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1}$

An S-point of $\mathcal{H}_{\tilde{\mathcal{E}}}$ is a short exact sequence of parabolic sheaves on $\mathbf{P}^1 \times S$

$$0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to \mathcal{T}^\bullet \to 0$$

with $\mathcal{F}^\bullet \in \operatorname{Bun}_{2,4}^1(S)$, $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(S)$ and $\mathcal{E}^\bullet \in \operatorname{Bun}_{2,4}^2(S)$ locally over S isomorphic to the pullback of $\tilde{\mathcal{E}}$. Because $\tilde{\mathcal{E}}$ is canonically a direct sum of parabolic line bundles $(\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D)$, it has $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ as its automorphism group and for any $\mathcal{E}^\bullet$ as above, there are line bundles $\mathcal{N}_1, \mathcal{N}_2$ on S such that there exists an isomorphism

$$\mathcal{E}^\bullet \cong \big((\mathcal{O}(2), \emptyset) \boxtimes \mathcal{N}_1\big) \;\oplus\; \big((\mathcal{O}, D) \boxtimes \mathcal{N}_2\big).$$

In Section 6.2 (more precisely, Remark 6.18 and Proposition 6.19), we saw that, except for the modifications of $\tilde{\mathcal{E}}$ that factor through one of the parabolic direct summands, all length 1 lower modifications of $\tilde{\mathcal{E}}$ lie in the relevant locus. The following substack $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \subset \mathcal{H}_{\tilde{\mathcal{E}}}$ classifies exactly those modifications that lie in the relevant locus, as we prove in Lemma 7.3.

Definition 7.2. We define by

$$j^{\mathrm{rel}}\colon \mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \hookrightarrow \mathcal{H}_{\tilde{\mathcal{E}}}$$

the substack of $\mathcal{H}_{\tilde{\mathcal{E}}}$ whose S-points are S-points of $\mathcal{H}_{\tilde{\mathcal{E}}}$

$$0 \to \mathcal{F}^\bullet \to \big((\mathcal{O}(2), \emptyset) \boxtimes \mathcal{N}_1\big) \;\oplus\; \big((\mathcal{O}, D) \boxtimes \mathcal{N}_2\big) \to \mathcal{T}^\bullet \to 0$$

with the property that the surjection to $\mathcal{T}^\bullet$ does not factor through $(\mathcal{O}(2), \emptyset) \boxtimes \mathcal{N}_1$ or $(\mathcal{O}, D) \boxtimes \mathcal{N}_2$.

Lemma 7.3. *The restriction of $q\colon \mathcal{H} \to \operatorname{Bun}_{2,4}^1$ to $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}}$ induces an isomorphism*

$$q^{\mathrm{rel}}\colon \mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \xrightarrow{\sim} \operatorname{Bun}_{2,4}^{1,\mathrm{r}}, \qquad (\tilde{\mathcal{E}} \overset{\phi}{\twoheadrightarrow} \mathcal{T}) \mapsto (\ker \phi).$$

Proof. The map q^{rel} is representable (it is injective on automorphism groups). It is also smooth. Indeed, the map on tangent spaces at the point $(\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet) \in \mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}}$ is

$$\operatorname{Hom}(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \to \operatorname{Ext}^1(\mathcal{E}^\bullet, \mathcal{E}^\bullet). \tag{7.0.1}$$

This map appears in the long exact sequence obtained by applying $\mathbf{R}\mathrm{Hom}(\mathcal{E}^\bullet, -)$ to the short exact sequence $0 \to \mathcal{E}^\bullet \to \tilde{\mathcal{E}} \to \mathcal{T}^\bullet \to 0$. The next term in this long exact sequence is $\mathrm{Ext}^1(\mathcal{E}^\bullet, \tilde{\mathcal{E}})$. Because $\mathcal{E}^\bullet$ and $\tilde{\mathcal{E}}$ lie in the relevant locus and $\mathcal{E}^\bullet$ has one degree lower than $\tilde{\mathcal{E}}$, we know the degrees of the direct summands of the underlying vector bundles in every parabolic degree (i,x) (Proposition 6.12 and Proposition 6.13) and can use them to conclude that for every parabolic degree (i,x), the group $\mathrm{Ext}^1(\mathcal{E}^{(i,x)}, \tilde{\mathcal{E}}^{(i,x)})$ vanishes. It follows that Equation (7.0.1) is surjective and q^{rel} is smooth.

The map q^{rel} is also an isomorphism on K-points for any field extension K of $\mathbf{F}_q$, as follows from our calculations of the relevant locus in Section 6.2 (in particular, Corollary 6.23). Together with the representability and smoothness, this proves it is an isomorphism. □

We have now considered the restriction of $q\colon \mathcal{H} \to \mathrm{Bun}_{2,4}$ to $\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}}$. The following lemma considers the restriction of $p\colon \mathcal{H} \to \mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$ to $\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}}$.

The automorphisms $\mathbb{G}_\mathrm{m} = \mathrm{Aut}((\mathcal{O}(2), \emptyset))$ of the parabolic direct summand $(\mathcal{O}(2), \emptyset) \subset \tilde{\mathcal{E}}$ form a subgroup of the automorphisms of $\mathrm{Aut}(\tilde{\mathcal{E}})$. By $\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_\mathrm{m}$, we denote the classifying stack of the quotient $\mathrm{Aut}(\tilde{\mathcal{E}})/\mathrm{Aut}((\mathcal{O}(2), \emptyset))$.

Lemma 7.4. *The restriction of* $p\colon \mathcal{H} \to \mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$ *to* $\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}}$ *induces an isomorphism*

$$p^{\mathrm{rel}}\colon \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \xrightarrow{p} \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_\mathrm{m} \times \overline{\mathbf{Coh}}_0^{1,1},$$
$$(\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\tilde{\mathcal{E}}, \mathcal{T}^\bullet)$$

Proof. We prove this by constructing a map

$$\phi\colon \mathbf{Coh}_0^{1,1} \to \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}}$$

that descends along the cover $\mathbf{Coh}_0^{1,1} \to \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_\mathrm{m} \times \overline{\mathbf{Coh}}_0^{1,1}$ to an inverse of p^{rel}.

$$\begin{tikzcd} \operatorname{Spec} k \times \mathbf{Coh}_0^{1,1} \arrow[r, "\phi"] \arrow[d] & \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \\ \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_\mathrm{m} \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[ur, dotted] & \end{tikzcd}$$

Let S be an $\mathbf{F}_q$-scheme and let $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(S)$. The support of $\mathcal{T}$ defines a map

$$s\colon S \to \mathbf{P}^1, \qquad x \mapsto \operatorname{Supp} \mathcal{T}|_x.$$

We denote by

$$i\colon S \hookrightarrow \mathbf{P}^1 \times S$$

the graph of s. The support of $\mathcal{T}$ is the image of i and every sheaf $\mathcal{T}^{(i,x)}$ is the pushforward of a line bundle on S along i.

The parabolic torsion sheaf $\mathcal{T}^\bullet$ is determined by the maps

$$\mathcal{T}^{(-2,D)} = \mathcal{T} \otimes \mathrm{pr}_{\mathbf{P}^1}^* \mathcal{O}(-D) \to \mathcal{T}^{(-1,D)} \to \mathcal{T}$$

and this sequence is the pushforward along i of

$$\mathcal{N}_1 \otimes s^*(\mathcal{O}(-D)) \xrightarrow{\beta} \mathcal{N}_2 \xrightarrow{\alpha} \mathcal{N}_1 \tag{7.0.2}$$

where $\mathcal{N}_1, \mathcal{N}_2$ are line bundles on S.

We define ϕ by sending an S-point $\mathcal{T}^\bullet$ of $\mathbf{Coh}_0^{1,1}$ defined by (7.0.2) to the S-point of $\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}}(S)$

$$((\mathcal{O}(2), \emptyset) \boxtimes (\mathcal{N}_1 \otimes s^*\mathcal{O}(-2))) \oplus (\mathcal{O}, D) \boxtimes \mathcal{N}_2 \twoheadrightarrow \mathcal{T}^\bullet \tag{7.0.3}$$

that under the (i^*, i_*)-adjunction corresponds to the following commutative diagram, where the left column is the pullback along i^* of the maps of $((\mathcal{O}(2), \emptyset) \boxtimes (\mathcal{N}_1 \otimes s^*\mathcal{O}(-2)))$ from parabolic degrees $(-2, D)$ to $(-1, D)$ to 0, and the right column is the same maps for $\mathcal{T}^\bullet$ (as explained in (7.0.2))

$$\begin{tikzcd}
(s^*(\mathcal{O}(2)) \otimes (\mathcal{N}_1 \otimes s^*(\mathcal{O}(-2)))) \oplus \mathcal{N}_2 \arrow[r, "{(1,\alpha)}"] & \mathcal{N}_1 \\
(s^*(\mathcal{O}(2-D)) \otimes (\mathcal{N}_1 \otimes s^*(\mathcal{O}(-2)))) \oplus \mathcal{N}_2 \arrow[r, "{(\beta,1)}"] \arrow[u, hook] & \mathcal{N}_2 \arrow[u, "\alpha"'] \\
(s^*(\mathcal{O}(2-D)) \otimes (\mathcal{N}_1 \otimes s^*(\mathcal{O}(-2)))) \oplus (s^*(\mathcal{O}(-D)) \otimes \mathcal{N}_2) \arrow[r, "{(1,\alpha)}"] \arrow[u, hook] & \mathcal{N}_1 \otimes s^*(\mathcal{O}(-D)) \arrow[u, "\beta"']
\end{tikzcd}$$

Note that this S-point does indeed lie in $\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \subset \mathcal{H}_{\tilde{\mathcal{E}}}$, because the horizontal maps do not factor through one of the parabolic direct summands on the left (i.e., it doesn't factor through $(\mathcal{O}(2), \emptyset) \boxtimes \mathcal{N}_1 \otimes s^*(\mathcal{O}(-2))$ or $(\mathcal{O}, D) \boxtimes \mathcal{N}_2$).

This map ϕ descends to an inverse of p^{rel}. Since p^{rel} only forgets the map, this is indeed an inverse. $\square$

Definition 7.5. We define the isomorphism

$$\alpha\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$$

as the composition

$$\mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_{\mathrm{m}} \times \overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{(p^{\mathrm{rel}})^{-1}} \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \xrightarrow{q^{\mathrm{rel}}} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}.$$

By choosing $x_0 \in D$, we can identify $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ with $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ by applying the elementary Hecke operator T_{x_0} $1-d$ times. This allows us to identify $\mathbf{Coh}_0^{1,1}$ with $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ for arbitrary d. However, for even d, this depends on the choice of $x_0 \in D$. We will usually choose $x_0 = \infty$.

Definition 7.6. Let $d \in \mathbf{Z}$. We define

$$\pi_d := \mathrm{Supp} \circ \alpha^{-1} \circ T_\infty^{d-1}\colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathbf{Coh}_0^{1,1} \to \mathbf{P}^1.$$

Remark 7.7. The map $\pi_d\colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \to \mathbf{P}^1$ exhibits $\mathbf{P}^1$ as the coarse moduli space of $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$, because $\mathrm{Supp}\colon \mathbf{Coh}_0^{1,1} \to \mathbf{P}^1$ makes $\mathbf{P}^1$ the coarse moduli space of $\mathbf{Coh}_0^{1,1}$.

The following is a useful lemma that follows from calculations we did earlier.

Lemma 7.8. *Let $d \in \mathbf{Z}$ and $x \in D$. Then $(\pi_{2d+1}^{-1}(\{x\}))(\mathbf{F}_q)$ consists of the three isomorphism classes*

$$T_x\hat{\mathcal{E}}(d), \quad T_{D\setminus\{x\}}\hat{\mathcal{E}}(d+1), \quad \textit{and} \quad T_x\tilde{\mathcal{E}}(d)$$

and $(\pi_{2d}^{-1}(\{x\}))(\mathbf{F}_q)$ consists of the three isomorphism classes

$$T_\infty T_x\hat{\mathcal{E}}(d), \quad T_{D\setminus\{\infty,x\}}\hat{\mathcal{E}}(d), \quad \textit{and} \quad T_\infty T_x\tilde{\mathcal{E}}(d)$$

Proof. By construction of π_1 and α, $T_x\tilde{\mathcal{E}}$ and $T_x\hat{\mathcal{E}} \cong T_x^{(1:1)}\tilde{\mathcal{E}}$ are mapped to x under π_1. Since $T_D(2)$ preserves $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ and is the identity on the coarse moduli space $\mathbf{P}^1$ (Lemma 7.8), $T_DT_x\hat{\mathcal{E}}(2) \cong T_{D\setminus\{x\}}\hat{\mathcal{E}}(1)$ is also sent to x. In addition, since $T_x^{(1:1)}\tilde{\mathcal{E}} = \hat{\mathcal{E}}$ is the unique length 1 lower modification of $\tilde{\mathcal{E}}$ with respect to $k_x^{(0,1)}$ in the relevant locus, the parabolic bundle $T_{D\setminus\{x\}}\hat{\mathcal{E}}(1)$ is the unique modification of $\tilde{\mathcal{E}}$ with respect to $T_Dk_x^{(0,1)} \cong k_x^{(1,0)}$, so that the listed points are indeed the three inverse images in $\pi_1^{-1}(\{x\})$ (Corollary 6.23). This proves the lemma for $d = 1$.

For $d = 0$, we note that $\pi_0^{-1}(\{x\})$ is by construction $T_\infty(\pi_1^{-1}(\{x\}))$. For all other d, it follows from the $d = 1, 0$ cases and the fact that applying T_∞ $2n$ times is the same as tensoring with $\mathcal{O}(-n[\infty])$. $\square$

8. Symmetries on the relevant locus

We study some automorphism of the relevant locus $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$. In Section 8.1, we show that the duality $\mathcal{E}^\bullet \mapsto \overset{\vee}{\mathcal{E}^\bullet}$ introduced in Section 4.3 restricts to an automorphism of the relevant locus. In Section 8.2, we study the isomorphism

$$T_D(2)\colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^d, \qquad \mathcal{E}^\bullet \mapsto T_D\mathcal{E}^\bullet \otimes \mathcal{O}(2).$$

Using an isomorphism $T_D\tilde{\mathcal{E}}(2) \xrightarrow{\sim} \tilde{\mathcal{E}}$, we prove that this induces an automorphism $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$. Identifying $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ with $\mathbf{Coh}_0^{1,1}$, we find that this is the evident automorphism of $\mathbf{Coh}_0^{1,1}$ that shifts the parabolic structure, i.e., it is $T_D\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathbf{Coh}_0^{1,1}$. In Section 8.3, we show that pulling back parabolic bundles by a Möbius transformation $M\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ that preserves D defines an automorphism $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$, which is simply the pullback $M^*\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathbf{Coh}_0^{1,1}$ after the identification $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathbf{Coh}_0^{1,1}$. (Proposition 8.5). Lastly, we study the isomorphism $T_xT_y^{-1}\colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ for $x, y \in D$ with $x \neq y$. This turns out to be almost the same as pulling back along the unique Möbius transformation that preserves D and sends x to y — the only difference is that $T_xT_y^{-1}$ shifts the parabolic structure of the bundles lying over $D \setminus \{x, y\}$.

8.1. Symmetry from duality on vector bundles. In Section 4.3, we introduced maps

$$\mathrm{dual}\colon \mathrm{Bun}_{2,4}^{d} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{-d}, \qquad \mathcal{E}^\bullet \mapsto \overset{\vee}{\mathcal{E}^\bullet}$$

In this section, we investigate how this duality operates on the relevant locus. As always, we can reduce such questions to questions on the special bundle $\tilde{\mathcal{E}} \in \mathrm{Bun}_{2,4}^{2,\mathrm{r}}(\mathbf{F}_q)$. The essential ingredient in this case is that we have an isomorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \overset{\vee}{\tilde{\mathcal{E}}}(1) \xrightarrow{\sim} \tilde{\mathcal{E}}(-1) \tag{8.1.1}$$

where the matrix is with respect to the direct sum decomposition induced by $\tilde{\mathcal{E}} = (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D)$. the dual of a parabolic line bundle $(\mathcal{L}, I)$ is $(\mathcal{L}^\vee, D \setminus I)$ and therefore the source of the above isomorphism is $(\mathcal{O}(1)^\vee, D) \oplus (\mathcal{O}(-1)^\vee, \emptyset)$, while the target is $(\mathcal{O}(1), \emptyset) \oplus (\mathcal{O}(-1), D)$.

Proposition 8.1. *The dualizing map restricts to an isomorphism*

$$\mathrm{dual}\colon \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{-1,\mathrm{r}}.$$

The map

$$(\, - \otimes \mathcal{O}(1)) \circ \mathrm{dual}\colon \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q) \to \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$$

is the identity on isomorphism classes.

Proof. Every bundle $\mathcal{F}^\bullet$ in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ is uniquely up to scaling a modification $\mathcal{F}^\bullet \subset \tilde{\mathcal{E}}$ such that neither of the parabolic direct summands of $\tilde{\mathcal{E}}$ is contained in $\mathcal{F}^\bullet$. We thus have inclusions

$$\tilde{\mathcal{E}}(-x) \hookrightarrow \mathcal{F}^\bullet \hookrightarrow \tilde{\mathcal{E}}.$$

Dualizing this, we get $\overset{\vee}{\tilde{\mathcal{E}}}(x) \supset \overset{\vee}{\mathcal{F}^\bullet} \supset \overset{\vee}{\tilde{\mathcal{E}}}$. We then use that $\overset{\vee}{\tilde{\mathcal{E}}}$ is isomorphic to $\tilde{\mathcal{E}}(-2)$ (via (8.1.1)) to find

$$\tilde{\mathcal{E}}(x-2) \supset \overset{\vee}{\mathcal{F}^\bullet} \supset \tilde{\mathcal{E}}(-2).$$

This shows that $\overset{\vee}{\mathcal{F}^\bullet}$ is a modification of $\tilde{\mathcal{E}}(-1)$ at x and by construction, $\overset{\vee}{\mathcal{F}^\bullet}$ does not contain either parabolic direct summand of $\tilde{\mathcal{E}}(-1)$, so that $\overset{\vee}{\mathcal{F}^\bullet}$ does indeed lie in the relevant locus.

If $x \notin D$, then $\mathcal{F}^\bullet$ is indeed isomorphic to $\overset{\vee}{\mathcal{F}^\bullet}(1)$, because there is only one modification of $\tilde{\mathcal{E}}$ at $x \in D$ (Proposition 6.19). To prove that $\overset{\vee}{\mathcal{F}^\bullet}(1)$ is also isomorphic to $\mathcal{F}^\bullet$ if $x \in D$, we consider the cokernels of $\mathcal{F}^\bullet \hookrightarrow \tilde{\mathcal{E}}$ and $\overset{\vee}{\mathcal{F}^\bullet}(1) \hookrightarrow \tilde{\mathcal{E}}$: the modifications are isomorphic if and only if the cokernels are isomorphic (Proposition 6.19). Let $\mathcal{T}$ denote the cokernel of $\mathcal{F}^\bullet \hookrightarrow \tilde{\mathcal{E}}$. The cokernel of $\overset{\vee}{\tilde{\mathcal{E}}} \hookrightarrow \overset{\vee}{\mathcal{F}^\bullet}$ is isomorphic to $\mathscr{H}\!om(\mathcal{T}^\bullet, \mathcal{O})$ (Lemma 4.9), which is isomorphic to $T_x\mathcal{T}^\bullet$. From this, it follows that the cokernel of $\overset{\vee}{\mathcal{F}^\bullet} \hookrightarrow \overset{\vee}{\tilde{\mathcal{E}}} \cong \tilde{\mathcal{E}}$ is isomorphic to $\mathcal{T}^\bullet$. Indeed, if we have any length 1 lower modification

$$0 \to \mathcal{E}_1^\bullet \hookrightarrow \mathcal{E}_2^\bullet \twoheadrightarrow \mathcal{T}_1^\bullet \to 0,$$

we get a short exact sequence

$$0 \to \mathcal{E}_2^\bullet(-x) \hookrightarrow \mathcal{E}_1^\bullet \twoheadrightarrow \mathcal{T}_2^\bullet \to 0;$$

these can be combined into an exact sequence

$$0 \to \mathcal{T}_2^\bullet \to \mathcal{E}_2^\bullet|_x \to \mathcal{T}_1^\bullet \to 0$$

and from the commutative squares in this short exact sequence of chains, it then follows, for example by applying the snake lemma on the exact sequences in two adjacent parabolic degrees, that (1) $\mathcal{T}_2^\bullet$ is isomorphic to $k_x^{(0,1)}$ if $\mathcal{T}_1^\bullet$ is isomorphic to $k_x^{(1,0)}$ and vice versa; and (2) $\mathcal{T}_2^\bullet$ is isomorphic to k_x^0 if $\mathcal{T}_1^\bullet$ is isomorphic to k_x^0. (These isomorphisms are not canonical and the above statement is not true in families.) We conclude that $\mathcal{F}^\bullet$ and $\overset{\vee}{\mathcal{F}^\bullet}(1)$ are indeed isomorphic. □

Lemma 8.2. *Let $x \in D$. Then*

$$T_x^{-1} \circ \mathrm{dual} = \mathrm{dual} \circ T_x \colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{-d+1,\mathrm{r}}.$$

Proof. This immediately follows from the definitions: T_x shifts the parabolic structure at x and taking the dual reverses the order of the arrows in the chain defining the parabolic structure at x. □

Corollary 8.3. *The dualizing map restricts to an isomorphism*

$$\mathrm{dual}\colon \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{0,\mathrm{r}}.$$

The map

$$\mathrm{dual}\colon \mathrm{Bun}_{2,4}^{0,\mathrm{r}}(\mathbf{F}_q) \to \mathrm{Bun}_{2,4}^{0,\mathrm{r}}(\mathbf{F}_q)$$

is the identity on isomorphism classes.

Proof. By the previous lemma (Lemma 8.2) and Proposition 8.1, the map

$$T_\infty \circ (-\otimes \mathcal{O}(1)) \circ \mathrm{dual} \circ T_\infty^{-1} = (T_\infty)^2 \circ (-\otimes \mathcal{O}(1)) \circ \mathrm{dual}$$

satisfies the properties of this corollary. But $(T_\infty)^2 = (-\otimes \mathcal{O}(-\infty))$, so a choice of an isomorphism $\mathcal{O}(\infty - 1) \xrightarrow{\sim} \mathcal{O}$ defines a natural isomorphism of $(T_\infty)^2 \circ (-\otimes \mathcal{O}(1))$ to the identity. □

8.2. Symmetry from T_D. Let $\sigma\colon \mathcal{O}(4-D) \xrightarrow{\sim} \mathcal{O}(-2)$ be an isomorphism. We then get an isomorphism

$$\phi := \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} : T_D\tilde{\mathcal{E}}(2) = (\mathcal{O}(4-D), D)\oplus(\mathcal{O}(2), \emptyset) \xrightarrow{\sim} (\mathcal{O}(2), \emptyset)\oplus(\mathcal{O}, D) = \tilde{\mathcal{E}}$$

An automorphism of $\tilde{\mathcal{E}}$ is the same as an automorphism on $T_D\tilde{\mathcal{E}}$, since only the parabolic degrees have shifted, and this in turn is equivalent to an automorphism on $T_D\tilde{\mathcal{E}}(2)$. Composing this identification $\mathrm{Aut}(\tilde{\mathcal{E}}) \xrightarrow{\sim} \mathrm{Aut}(T_D\tilde{\mathcal{E}}(2))$ with the isomorphism $\mathrm{Aut}(T_D\tilde{\mathcal{E}}(2)) \xrightarrow{\sim} \mathrm{Aut}(\tilde{\mathcal{E}})$ induced by $\phi\colon T_D\tilde{\mathcal{E}}(2) \xrightarrow{\sim} \tilde{\mathcal{E}}$, we get

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} = \mathrm{Aut}(\tilde{\mathcal{E}}) \xrightarrow{\sim} \mathrm{Aut}(T_D\tilde{\mathcal{E}}(2)) \overset{\phi}{\xrightarrow{\sim}} \mathrm{Aut}(\tilde{\mathcal{E}}) = \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}.$$

Using this isomorphism, we can prove the following result. We denote by $T_D(2)$ the operator

$$T_D(2) = (\mathcal{F}^\bullet \mapsto T_D\mathcal{F}^\bullet(2))$$

that acts on parabolic coherent sheaves $\mathcal{F}^\bullet$.

Lemma 8.4. *The operator $T_D(2)$ preserves the relevant locus and the diagram*

$$\begin{tikzcd} \mathbf{P}^1 \arrow[d, equal] & \mathbf{Coh}_0^{1,1} \arrow[l, "\mathrm{Supp}"'] \arrow[r, "\sim", "\alpha"'] \arrow[d, "T_D(2)"', "\wr"] & \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \arrow[d, "T_D(2)"', "\wr"] \\ \mathbf{P}^1 & \mathbf{Coh}_0^{1,1} \arrow[l, "\mathrm{Supp}"'] \arrow[r, "\sim", "\alpha"'] & \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \end{tikzcd}$$

is commutative.

Proof. Every point $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ is a modification of $\tilde{\mathcal{E}}$, i.e., there is a parabolic torsion sheaf $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ and a short exact sequence

$$0 \to \mathcal{E}^\bullet \to \tilde{\mathcal{E}} \to \mathcal{T}^\bullet \to 0.$$

Applying $T_D(2)$ is exact and gives us a short exact sequence

$$0 \to T_D\mathcal{E}^\bullet(2) \to T_D\tilde{\mathcal{E}}(2) \to T_D\mathcal{T}^\bullet(2) \to 0.$$

Since $\mathrm{Supp}\,\mathcal{T}^\bullet = \mathrm{Supp}\,T_D\mathcal{T}^\bullet(2)$, the left square is commutative. Because $T_D\tilde{\mathcal{E}}(2)$ is isomorphic to $\tilde{\mathcal{E}}$, as explained above, and because this isomorphism is a direct sum of isomorphisms between the parabolic direct summands, the image of $T_D\mathcal{E}^\bullet(2)$ in $\tilde{\mathcal{E}}|_{\mathrm{Supp}\,\mathcal{T}^\bullet}$ is not one of the excluded flags of Proposition 6.19 — recall that these excluded modifications are the ones that factor through one of the parabolic direct summands and they are the only length 1 lower modifications of $\tilde{\mathcal{E}}$ that do not lie in the relevant locus. We therefore conclude $T_D\mathcal{E}^\bullet(2) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$. The commutativity of the right square of the diagram follows from these two exact sequences and our construction of α. $\square$

8.3. Pulling back by Möbius transformations. There are precisely four Möbius transformations $M\colon \mathbf{P}^1 \to \mathbf{P}^1$ that preserve $D \subset \mathbf{P}^1$. With composition as the multiplication map, they form the Klein four-group. In particular, except for the identity, these Möbius transformations have order 2 and induce an order 2 permutation on D. We denote by M_x the unique transformation that sends $\infty \in D$ to x. More explicitely, we have

$$\begin{aligned}
M_\infty &= (z \mapsto z) && \text{inducing the identity on } D,\\
M_0 &= \left(z \mapsto \frac{t}{z}\right) && \text{inducing } (\infty\ 0)\ (1\ t) \text{ on } D,\\
M_1 &= \left(z \mapsto \frac{z-t}{z-1}\right) && \text{inducing } (\infty\ 1)\ (0\ t) \text{ on } D, \text{ and}\\
M_t &= \left(z \mapsto \frac{t(z-1)}{z-t}\right) && \text{inducing } (\infty\ t)\ (0\ 1) \text{ on } D.
\end{aligned}$$

The pullback

$$M_x^*\tilde{\mathcal{E}} = (\mathcal{O}(2[x]), \emptyset) \oplus (\mathcal{O}, D)$$

is isomorphic to $\tilde{\mathcal{E}}$: a choice of an isomorphism $\sigma\colon \mathcal{O}(2[x]) \xrightarrow{\sim} \mathcal{O}(2)$ defines an isomorphism

$$\sigma \oplus \mathrm{id} = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} : M_x^*\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}.$$

We use this to prove the following proposition, which describes how M_x^* acts on the relevant locus.

Proposition 8.5. *Let $x \in D$. Pulling back parabolic bundles by M_x induces an isomorphism*

$$M_x^*\colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d,\mathrm{r}}.$$

We have commutative diagrams

$$\begin{tikzcd}
\mathbf{Coh}_0^{1,1} \arrow[r, "\sim", "\alpha"'] \arrow[d, "M_x^*"] & \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \arrow[d, "M_x^*"] \\
\mathbf{Coh}_0^{1,1} \arrow[r, "\sim", "\alpha"'] & \mathrm{Bun}_{2,4}^{1,\mathrm{r}}
\end{tikzcd} \tag{8.3.1}$$

and

$$\begin{tikzcd}
\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \arrow[r, "T_\infty"] \arrow[d, "T_x T_\infty^{-1} \circ M_x^*"'] & \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \arrow[d, "M_x^*"] \\
\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \arrow[r, "T_\infty"] & \mathrm{Bun}_{2,4}^{0,\mathrm{r}}
\end{tikzcd} \tag{8.3.2}$$

Proof. Let

$$0 \to \mathcal{E}^\bullet \to \tilde{\mathcal{E}} \to \mathcal{T}^\bullet \to 0$$

with $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^1$ and $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ be a length 1 lower modification of $\tilde{\mathcal{E}}$. Pulling this back along M_x^* gives us a short exact sequence

$$0 \to M_x^*\mathcal{E}^\bullet \to M_x^*\tilde{\mathcal{E}} \to M_x^*\mathcal{T}^\bullet \to 0.$$

The map $\tilde{\mathcal{E}} \xrightarrow{\sim} M_x^*\tilde{\mathcal{E}} \twoheadrightarrow M_x^*\mathcal{T}^\bullet$ factors through one of the parabolic direct summands of $\tilde{\mathcal{E}}$ if and only if the map $\tilde{\mathcal{E}} \to \mathcal{T}^\bullet$ does, so $\mathcal{E}^\bullet$ is in the relevant locus if and only if $M_x^*\mathcal{E}^\bullet$ is (Corollary 6.22). As a consequence, we get the commutative diagram in (8.3.1).

The commutative diagram (8.3.2) follows from the identity $M_x^* \circ T_\infty = T_x \circ M_x^*$. □

Lemma 8.6. *Let $x, y, z \in D$. Then*

$$z = M_x(y)$$

holds if and only if for all $x_1, x_2, x_3 \in D$ with $\{x_1, x_2, x_3\} = \{x, y, z\}$, the equality

$$x_1 = M_{x_2}(x_3)$$

holds.

Proof. We can rewrite $z = M_x(y)$ as

$$M_z(\infty) = M_x(M_y(\infty)).$$

Since the Möbius transformations have order 2, this is equivalent to

$$M_y M_x M_z(\infty) = \infty.$$

The lemma now follows from the fact that these Möbius transformations commute with each other. (Indeed, $\{M_{x_0}\}_{x_0 \in D}$ is the Klein four-group, which is abelian.) □

8.4. Symmetry from $T_x T_y^{-1}$. Let $x, y \in D$ with $x \neq y$. We have seen that both x and y define isomorphisms

$$\mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d-1,\mathrm{r}}$$

for all $d \in \mathbf{Z}$. In this section, we describe the isomorphism

$$T_x T_y^{-1} \colon \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}.$$

First note that $T_x T_y^{-1} \circ T_x T_y^{-1} = (-\otimes \mathcal{O}(y-x))$. Therefore, an isomorphism $\mathcal{O} \xrightarrow{\sim} \mathcal{O}(y-x)$ defines a (non-canonical) natural isomorphism between the functor $(T_x T_y^{-1})^2$ and the identity.

The following lemma describes the restriction of $T_x T_y^{-1}$ to $\pi^{-1}(D)$.

Lemma 8.7. *Let $M \colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ be a Möbius transformation that preserves D, let $x_0 \in D$ and let $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ with* $\mathrm{Supp}\,\mathcal{T}^\bullet \in D$. *The automorphism*

$$T_{x_0} T_{M(x_0)}^{-1} \colon \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$$

maps

$$\alpha(\mathcal{T}^\bullet) \mapsto \begin{cases} \alpha(M^*(\mathcal{T}^\bullet)) & \text{if } \operatorname{Supp}\mathcal{T}^\bullet \in \{x_0, M(x_0)\} \\ \alpha(T_D M^*(\mathcal{T}^\bullet)) & \text{if } \operatorname{Supp}\mathcal{T}^\bullet \notin \{x_0, M(x_0)\} \end{cases}.$$

Proof. Recall (Lemma 7.8) that for $z \in D$, the three isomorphism classes in $\pi_1^{-1}(\{z\})$ are $T_z\hat{\mathcal{E}} = \alpha^{-1}(k_z^{(0,1)})$, $T_{D\setminus\{z\}}\hat{\mathcal{E}}(1) = \alpha^{-1}(k_z^{(1,0)})$ and $T_z\tilde{\mathcal{E}} = \alpha^{-1}(k_z^0)$. Also note that $T_D k_z^{(1,0)} \cong k_z^{(0,1)}$, while $T_D k_z^0 \cong k_z^0$. The lemma follows from the above facts, combined with the identities $T_z^{-1} = T_z \circ (- \otimes \mathcal{O}(z))$ and the natural isomorphism $T_{x_0}T_{M(x_0)}^{-1} \cong T_{x_0}^{-1}T_{M(x_0)}$. ☐

Lemma 8.8. *Let $x, y \in D$ with $x \neq y$. Let $M\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ denote the Möbius transformation that preserves D and sends x to y. We have a commutative diagram*

$$\begin{tikzcd} \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \arrow[r, "T_xT_y^{-1}"] \arrow[d, "\pi_1"] & \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \arrow[d, "\pi_1"] \\ \mathbf{P}^1 \arrow[r, "M"] & \mathbf{P}^1 \end{tikzcd}$$

Proof. Because $\pi_1\colon \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \to \mathbf{P}^1$ is the map to the coarse moduli space, the automorphism $T_xT_y^{-1}$ of $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ induces an automorphism of $\mathbf{P}^1$. Every isomorphism $\mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ is determined by its restriction to D and $T_xT_y^{-1}$ induces M on $D \subset \mathbf{P}^1$ (Lemma 8.7). ☐

Corollary 8.9. *Let $M\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ be a Möbius transformation that preserves D and let $x \in D$. We write $y = M(x)$. We have commutative diagrams*

$$\begin{tikzcd} \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \arrow[r, "T_xT_y^{-1}"] \arrow[d, "\pi_0"] & \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \arrow[d, "\pi_0"] \\ \mathbf{P}^1 \arrow[r, "M"] & \mathbf{P}^1 \end{tikzcd} \tag{8.4.1}$$

and

$$\begin{tikzcd} \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \arrow[r, "M^*"] \arrow[d, "\pi_0"] & \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \arrow[d, "\pi_0"] \\ \mathbf{P}^1 \arrow[r, "\mathrm{id}"] & \mathbf{P}^1 \end{tikzcd} \tag{8.4.2}$$

Proof. The first commutative diagram, diagram 8.4.1, follows Lemma 8.8, and the fact that the elementary Hecke operators commute.

The second commutative diagram follows from the fact that M^* on $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ corresponds to $T_{M(\infty)}T_\infty^{-1} \circ M^*$ on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ (using T_∞ to identify $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ with $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$; see Equation (8.3.2)) and, as we have seen, on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$, both M^* and

$T_{M(\infty)}T_\infty$ induce M on the coarse moduli space $\mathbf{P}^1$. Since M has order two, the composition induces the identity on the coarse moduli space. $\square$

Remark 8.10. Even though $M^*\colon \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ induces the identity on the coarse moduli space, the map itself is not naturally isomorphic to the identity if M is not the identity. Indeed, for $x, y \in D$, the point $T_xT_y\hat{\mathcal{E}}$ is mapped to $T_{M(x)}T_{M(y)}\hat{\mathcal{E}}$. If $\{x,y\} \neq \{M(x), M(y)\}$ (there exists 4 such pairs), then $T_xT_y\hat{\mathcal{E}}$ is not isomorphic to $T_{M(x)}T_{M(y)}\hat{\mathcal{E}}$.

Remark 8.11. The above results are stated for $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ and $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$. The results for $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ are true for all $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ with d odd and likewise, the results for $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ are true for all $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ with d even, because we can canonically identify degrees of the same parity by tensoring with $\mathcal{O}(n)$ for some $n \in \mathbf{Z}$.

9. Calculation of the length 1 modifications

In this section, we explicitely determine the length 1 lower modifications of all parabolic bundles in the relevant locus, up to application of the elementary Hecke operators T_x with $x \in D$. This is essential in understanding the maps from the Hecke stack to $\mathrm{Bun}_{2,4}$ and $\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$.

More precisely, for every surjection $\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet$ with $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ and $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$, we would like to determine $\ker(\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet)$. If this kernel is in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$, we would like to describe it in terms of the identification $\alpha\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$, or equivalently, describe it as a length 1 lower modification of $\tilde{\mathcal{E}}$. Kernels in $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ with $d \in \mathbf{Z}$ and $d \neq 1$ can then be expressed using the isomorphism $T_\infty^{d-1}\colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$. The results in Section 8.4 allow us to identify a bundle given as $T_J\mathcal{E}^\bullet$ with $J \subset D$ with a bundle of the form $T_\infty^n\mathcal{F}^\bullet$, so in this section it suffices to determine the modifications up to application of any of the elementary Hecke operators T_x, $x \in D$.

In Section 6.2, we determined all length 1 lower modifications of $\tilde{\mathcal{E}} = (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D)$. In Section 9.2, we do the same for $\hat{\mathcal{E}} := (\mathcal{O}(2) \oplus \mathcal{O}, (\mathcal{O}|_x)_{x \in D\setminus\{0\}}, \ell_0 = (1:1))$. Knowing these modifications allows us to determine the length 1 lower modifications of all points in $\pi_d^{-1}(D) \subset \mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ for all $d \in \mathbf{Z}$, because every bundle in $\pi_d^{-1}(D)$ is of the form $T_J\hat{\mathcal{E}}(n)$ (if the automorphism group is $\mathbb{G}_\mathrm{m}$) or $T_J\tilde{\mathcal{E}}(n)$ (if the automorphism group is $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$) for some $n \in \mathbf{Z}$ and $J \subset D$ (Lemma 7.8). Lastly, we determine in Section 9.3 the modifications of bundles in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \setminus \pi^{-1}(D)$. We explain why these results suffice to determine all length 1 lower modifications of points in $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ in Section 9.1.4.

Since this section is quite technical, we start with a summary of the results.

9.1. Summary of the results.

9.1.1. *Modifications of $\tilde{\mathcal{E}}$.* We already determined all length one lower modifications of $\tilde{\mathcal{E}} := (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D)$ in Section 6.2. See Corollary 6.23 for the modifications that do lie in the relevant locus and Lemma 6.17 for the modifications that do not.

9.1.2. *Modifications of $\hat{\mathcal{E}}$.* This is done in Section 9.2. The main takeaway is the following.

THEOREM 9.1.

(1) Let $x \in \mathbf{P}^1 \setminus D$ and consider the map

$$\phi\colon \mathbf{P}((\mathcal{O}(2) \oplus \mathcal{O})|_x) \to \mathrm{Bun}^1_{2,4}, \qquad \ell \mapsto T^\ell_x \hat{\mathcal{E}}.$$

We decompose $\mathbf{P}^1 = \mathbf{P}((\mathcal{O}(2) \oplus \mathcal{O})|_x)$ into the singleton $\{\mathcal{O}(2)|_x\}$ and its complement $\mathbf{A}^1 = \mathbf{P}((\mathcal{O}(2) \oplus \mathcal{O})|_x)$.

(a) The restrictions of ϕ to these sets factor as indicated in the following commutative diagram:

$$\begin{tikzcd}
\{\mathcal{O}(2)|_x\} \arrow[r] \arrow[d, hook] & \{(\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D)\} \notin \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \arrow[d, hook] \\
\mathbf{P}^1 \arrow[r, "\phi"] & \mathrm{Bun}^1_{2,4} \\
\mathbf{A}^1 \arrow[r] \arrow[u, hook] & \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \setminus \pi_1^{-1}(\{x\}) \arrow[u, hook]
\end{tikzcd}$$

(b) The composition with the map $\pi_1\colon \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \to \mathbf{P}^1$ to the coarse moduli space defines an isomorphism

$$\pi_1 \circ \phi|_{\mathbf{A}^1}\colon \mathbf{A}^1 \xrightarrow{\sim} \mathbf{P}^1 \setminus \{x\}.$$

(c) All parabolic vector bundles in the image of $\mathbf{A}^1$ under ϕ have automorphism group $\mathbb{G}_{\mathrm{m}}$.

(2) Let $x \in D$. Recall that the modifications of $\hat{\mathcal{E}} = (\mathcal{O}(2) \oplus \mathcal{O}, (\ell_y)_{y \in D})$ at x are classified by

$$\mathbf{P}^1(\hat{\mathcal{E}}^0|_x) \cup_{\ell_x} \mathbf{P}^1((T^{\ell_x}_x \hat{\mathcal{E}})^0|_x).$$

We can decompose this set according to the isomorphism class of the quotient of the corresponding modification:

$$\begin{aligned} &\mathbf{P}^1(\hat{\mathcal{E}}^0|_x) \cup_{\ell_x} \mathbf{P}^1((T^{\ell_x}_x \hat{\mathcal{E}})^0|_x) \\ &= \underbrace{\mathbf{P}^1(\hat{\mathcal{E}}^0|_x) \setminus \{\ell_x\}}_{k_x^{(0,1)}} \ \sqcup \ \underbrace{\{\ell_x\}}_{k_x^0} \ \sqcup \ \underbrace{\mathbf{P}^1((T^{\ell_x}_x \hat{\mathcal{E}})^0|_x) \setminus \{\ell_x\}}_{k_x^{(1,0)}}. \end{aligned} \tag{9.1.1}$$

Let

$$\phi\colon \mathbf{P}^1(\hat{\mathcal{E}}^0|_x) \cup_{\ell_x} \mathbf{P}^1((T^{\ell_x}_x \hat{\mathcal{E}})^0|_x) \to \mathrm{Bun}^1_{2,4}$$

denote the map that sends an element on the left to the corresponding modification of $\hat{\mathcal{E}}$. The following statements describe the restriction of ϕ to the disjoint sets in (9.1.1).

(a) There is a unique $\ell' \in \mathbf{P}^1(\hat{\mathcal{E}}^0|_x) \setminus \{\ell_x\}$ *such that the restriction of* ϕ

$$\mathbf{P}^1(\hat{\mathcal{E}}^0|_x) \setminus \{\ell_x\} \xrightarrow{\phi} \mathrm{Bun}^1_{2,4}$$

is given by

$$\ell \mapsto \begin{cases} (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D) \notin \mathrm{Bun}^{1,\mathrm{r}}_{2,4} & \text{if } \ell = \mathcal{O}(2)|_x \\ T_x \tilde{\mathcal{E}} & \text{if } \ell = \ell' \\ T_x \hat{\mathcal{E}} & \text{otherwise} \end{cases}$$

(b) The image of ℓ_x *is* $T_x \hat{\mathcal{E}}$*.*

(c) The restriction of ϕ *to* $\mathbf{P}^1((T_x^{\ell_x} \hat{\mathcal{E}})^0|_x) \setminus \{\ell_x\}$ *factors through*

$$\phi' \colon \mathbf{P}^1((T_x^{\ell_x} \hat{\mathcal{E}})^0|_x) \setminus \{\ell_x\} \to \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \setminus \pi^{-1}(\{x\})$$

The composition with the map $\pi_1 \colon \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \to \mathbf{P}^1$ *to the coarse moduli space defines an isomorphism*

$$\pi_1 \circ \phi' \colon \mathbf{P}^1((T_x^{\ell_x} \hat{\mathcal{E}})^0|_x) \setminus \{\ell_x\} \xrightarrow{\sim} \mathbf{P}^1 \setminus \{x\}.$$

All parabolic bundles in the image of ϕ' *have isomorphism group* $\mathbb{G}_{\mathrm{m}}$*.*

9.1.3. *Modifications of normal points.* In Section 9.3, we prove the following.

THEOREM 9.2. *Let* $(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \in \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$ *be any point such that* $\mathcal{T}$ *is supported outside of* D *and* $\mathcal{E}^\bullet$ *is a modification of* $\tilde{\mathcal{E}}$ *at a point outside of* D*.*

(1) The map

$$q|_{p^{-1}(\mathcal{E}^\bullet, \mathcal{T}^\bullet)} \colon p^{-1}(\mathcal{E}^\bullet, \mathcal{T}^\bullet) = \mathbf{P}^1(\mathcal{E}|_{\operatorname{Supp} \mathcal{T}}) \to \mathrm{Bun}^0_{2,4}, \qquad \ell \mapsto T^{\ell}_{\operatorname{Supp} \mathcal{T}} \mathcal{E}^\bullet$$

factors through $\phi \colon \mathbf{P}^1(\mathcal{E}|_{\operatorname{Supp} \mathcal{T}}) \to \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$*:*

$$\begin{tikzcd} p^{-1}(\mathcal{E}^\bullet, \mathcal{T}^\bullet) = \mathbf{P}^1(\mathcal{E}|_{\operatorname{Supp} \mathcal{T}}) \arrow[r, "q"] \arrow[dr, "\phi"] & \mathrm{Bun}^0_{2,4} \\ & \mathrm{Bun}^{0,\mathrm{r}}_{2,4} \arrow[u, hook] \end{tikzcd}$$

(2) The composition of ϕ *with the map* $\pi_0 \colon \mathrm{Bun}^{0,\mathrm{r}}_{2,4} \to \mathbf{P}^1$ *to the coarse moduli space is a degree 2 map*

$$\pi_0 \circ \phi \colon \mathbf{P}^1(\mathcal{E}|_{\operatorname{Supp} \mathcal{T}}) \to \mathbf{P}^1.$$

We give a formula for the map $\pi_0 \circ \phi \colon \mathbf{P}^1 \to \mathbf{P}^1$ in Lemma 9.31 (in the general case) and Lemma 9.35 (when $\mathcal{E}^\bullet$ is a modification of $\tilde{\mathcal{E}}$ at $\operatorname{Supp} \mathcal{T}$). This formula is easy to determine given the above theorem, because we can easily calculate the preimage of $D \subset \mathbf{P}^1$ (Proposition 9.18). The operator

$T_D(2) \circ \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ induces a certain symmetry in the degree 2 formula, as we explain in Remark 9.21.

These remarks give us sufficient understanding of the map $\pi_0 \circ \phi$. Because the inverse image of $\pi_0^{-1}(D)$ under ϕ is easily understood (Proposition 9.18) and because $\pi_0|_{\pi_0^{-1}(\mathbf{P}^1\setminus D)}$ is a bijection on points, this means we have a good understanding of the map ϕ. The following addendum to the theorem is a further property of the map ϕ that we will need later.

Addendum 9.3. *Let $\ell \in \mathbf{P}^1(\mathcal{E}|_{\operatorname{Supp}\mathcal{T}})$ such that*

$$(\pi \circ \phi)(\ell) = x \in D.$$

Then $\phi(\ell)$ is the unique point in $\pi^{-1}(x)$ with automorphism group $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ if and only if $\pi \circ \phi$ is ramified over x.

Proof. This is Lemma 9.33. □

9.1.4. *How this gives all modifications.* The results summarized above allow us to determine the modifications of any $\mathcal{E} \in \mathrm{Bun}_{2,4}^{d,\mathrm{r}}$, up to twisting by $\mathcal{O}(n)$ and applying the elementary Hecke operators $T_x\colon \mathrm{Bun}_{2,4}^{d} \to \mathrm{Bun}_{2,4}^{d-1}$ for $x \in D$. In this section, we explain why.

So far, we have determined

(1) all modifications of $\tilde{\mathcal{E}}$;

(2) all modifications of $\hat{\mathcal{E}}$; and

(3) all modifications of $\pi^{-1}(\mathbf{P}^1 \setminus D)$ with respect to torsion sheaves $\mathcal{T}^\bullet \in \overline{\mathbf{Coh}}_0^{1,1}$ supported outside of D.

Let $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}$ and let $x \in D$. The map $(\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet) \mapsto (T_x\mathcal{F}^\bullet \hookrightarrow T_x\mathcal{E}^\bullet)$ establishes a bijective correspondence between length one lower modificattions of $\mathcal{E}^\bullet$ and length one lower modifications of $T_x\mathcal{E}^\bullet$. It therefore suffices to determine the length one lower modifications of all degree 1 parabolic bundles.

The bundles in $\pi^{-1}(D) \subset \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ are of the form $T_{D\setminus\{x\}}\hat{\mathcal{E}}(1)$, $T_x\hat{\mathcal{E}}$ or $T_x\tilde{\mathcal{E}}$ (Lemma 7.8), so their modifications can be determined from the modifications of $\tilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$.

It remains to determine the modifications of a bundle $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \setminus \pi_1^{-1}(D)$ with respect to $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ with $\operatorname{Supp}\mathcal{T} \in D$. Up to isomorphism, we can assume $\mathcal{E}^\bullet = T_y^{(1:1)}\tilde{\mathcal{E}}$ for some $y \in \mathbf{P}^1 \setminus D$ (Proposition 6.19).

Note the similarity to the modifications of $\hat{\mathcal{E}}$ at a point in $\mathbf{P}^1 \setminus D$ (part 1 of Theorem 9.1).

Proposition 9.4. *Let $y \in \mathbf{P}^1 \setminus D$ and $x \in D$. As in Theorem 9.1, we consider the restrictions of the modification map*

$$\phi\colon \mathbf{P}^1((T_y^{1:1}\tilde{\mathcal{E}})^0|_x) \cup_{\mathcal{O}|_x} \mathbf{P}^1((T_xT_y^{1:1}\tilde{\mathcal{E}})^0|_x) \to \mathrm{Bun}_{2,4}^0$$

to the flags corresponding to $k_x^{(0,1)}$, k_x^0 *and* $k_x^{(1,0)}$, *respectively, as in Equation* (9.1.1).

(1) *The image of* $\mathcal{O}|_x$ *is* $T_x T_y^{(1:1)} \tilde{\mathcal{E}}$.

(2) *Denote by* $\mathbf{A}^1$ *the complement of* $\{\mathcal{O}|_y\}$ *in* $\mathbf{P}^1((T_y^{1:1}\tilde{\mathcal{E}})^0|_x)$. *The restriction of* ϕ *to this* $\mathbf{A}^1$ *factors through a map*

$$\phi|_{\mathbf{A}^1} \colon \mathbf{A}^1 \to \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \setminus \pi_0^{-1}\{M_x(y)\}.$$

All parabolic vector bundles in the image of this map have automorphism group $\mathbb{G}_{\mathrm{m}}$ *and the composition with the map* $\pi_0 \colon \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \to \mathbf{P}^1$ *to the coarse moduli space defines an isomorphism*

$$\pi_0 \circ \phi|_{\mathbf{A}^1} \colon \mathbf{A}^1 \xrightarrow{\sim} \mathbf{P}^1 \setminus \{M_x(y)\}$$

where $M_x \colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ *is the unique Möbius transformation preserving* D *and sending* ∞ *to* x.

(3) *Denote by* $\tilde{\mathbf{A}}^1$ *the complement of* $\mathcal{O}|_y$ *in* $\mathbf{P}^1((T_x T_y^{(1:1)}\tilde{\mathcal{E}})^0|_x)$. *The restriction of* ϕ *to* $\tilde{\mathbf{A}}^1$ *satisfies the properties of* $\phi|_{\mathbf{A}^1}$ *that we listed in part 2 of this statement.*

(4) *The sets* $\mathrm{im}(\phi|_{\mathbf{A}^1}) \cap \pi_0^{-1}(D)$ *and* $\mathrm{im}(\phi_{\tilde{\mathbf{A}}^1}) \cap \pi_0^{-1}(D)$ *are disjoint.*

Remark 9.5. Above, we claim that $\pi_0 \circ \phi|_{\mathbf{A}^1}$ misses $M_x(y)$. Our proof will show that $\phi|_{\mathbf{A}^1}$ misses the points $T_x T_y^{(1:1)} \tilde{\mathcal{E}}$. By definition,

$$\pi_0(T_x T_y \tilde{\mathcal{E}}) = \mathrm{Supp} \circ \alpha^{-1}(T_\infty^{-1} T_x T_y \tilde{\mathcal{E}})$$

and we will prove only much later (Lemma 8.8) that this is $M_x(y)$.

Proof. If $\mathcal{T}^\bullet = k_x^0$, then the only modification is $T_x \mathcal{E}^\bullet$. The modifications of $\mathcal{E}^\bullet$ with respect to $\mathcal{T}^\bullet = k_x^{(1,0)}$ are of the form $T_x^\ell \mathcal{E}^\bullet$ with $\ell \subset \mathcal{E}|_x$ different from $\mathcal{O}|_x \subset \mathcal{E}|_x$. If, in addition to the necessary condition $\ell \neq \mathcal{O}|_x$, we have $\ell \neq \mathcal{O}(2)|_x$, then there is an isomorphism

$$\sigma \colon T_x^\ell \tilde{\mathcal{E}} \xrightarrow{\sim} T_x \hat{\mathcal{E}}.$$

This restricts to an isomorphism

$$T_x^\ell T_y^{(1:1)} \tilde{\mathcal{E}} \xrightarrow{\sim} T_x T_y^{\sigma|_y (1:1)} \hat{\mathcal{E}}$$

so that we have reduced the determination of this modification to the problem of determining $T_y^{(\sigma|_y(1:1))} \hat{\mathcal{E}}$, which we have already done.

If $\ell = \mathcal{O}(2)|_x$, then an easy calculation (Lemma 9.20) shows

$$T_y^{(1:1)} T_x^{\mathcal{O}(2)|_x} \tilde{\mathcal{E}} \cong \hat{\mathcal{E}}(-1).$$

Modifications of $\mathcal{E}^\bullet$ with respect to $k_x^{(0,1)}$, the last isomorphism class of $\mathcal{T}^\bullet$, are of the form of the form ${}^\ell T_x \mathcal{E}^\bullet$. To determine them, we can use the previous results and the symmetry T_D (Section 8.2): we have

$$T_D\, {}^\ell T_x T_y^{(1:1)} \tilde{\mathcal{E}} = T_x^\ell T_y^{(1:1)} T_D \tilde{\mathcal{E}} \cong T_x^{\sigma(\ell)} T_y^{(1:1)} \tilde{\mathcal{E}}(-2)$$

where $\sigma\colon T_D\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}(-2)$ is an isomorphism that we can choose in such a way that $\sigma|_y(1:1) = (1:1)$. $\square$

9.1.5. *Summary of maps to the relevant locus.* We now provide a shorter overview of the results explained in more detail above. For every $(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \in (\mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1})(\mathbf{F}_q)$, we consider the fiber $p^{-1}(\mathcal{E}^\bullet, \mathcal{T}^\bullet)$ and write down the restriction of $q\colon \mathcal{H} \to \mathrm{Bun}^0_{2,4}$ to the preimage of $\mathrm{Bun}^{0,\mathrm{r}}_{2,4}$

$$\phi_{\mathcal{E}^\bullet,\mathcal{T}^\bullet}\colon p^{-1}(\mathcal{E}^\bullet \mathcal{T}^\bullet) \cap q^{-1}(\mathrm{Bun}^{0,\mathrm{r}}_{2,4}) \xrightarrow{q} \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$$

up to automorphism of the source. This will be relevant later when we calculate $(\mathbb{H}\,\mathrm{Aut}^0_E)|_{\mathcal{E}^\bullet,\mathcal{T}^\bullet}$, where Aut^0_E is the purported eigensheaf for the local system E.

The table below uses the following notation.

- By
$$\mathring{D} \subset \overline{\mathbf{Coh}}_0^{1,1}$$
we denote the substack of torsion sheaves supported at D with automorphism group $\mathbb{G}_\mathrm{m}$, i.e., the sheaves $k_x^{(1,0)}$ and $k_x^{(0,1)}$ for all $x \in D$.
- By
$$D^{\mathrm{aut}} \subset \overline{\mathbf{Coh}}_0^{1,1}$$
we denote the substack of torsion sheaves supported at D with automorphism group $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$, i.e., the sheaves k_x^0 for all $x \in D$.
- The maps
$$\mathbf{A}^1 \xrightarrow{\sigma} \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$$
decorated with the symbol σ are restrictions of maps of the form
$$\mathbf{P}^1 \xrightarrow{\sigma'} \overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$$
where the isomorphism is $T_\infty \circ \alpha\colon \overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$ and σ' is a section $\sigma'\colon \mathbf{P}^1 \to \overline{\mathbf{Coh}}_0^{1,1}$ of the support map $\mathrm{Supp}\colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$. The symbol σ only signifies that the map is of this form; the actual section σ' depends on the pair $(\mathcal{E}^\bullet, \mathcal{T}^\bullet)$.
- The maps
$$\mathbb{G}_\mathrm{m} \xrightarrow{\tau} \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$$
decorated with the symbol τ are, up to automorphism of $\mathbb{G}_\mathrm{m}$, of the form
$$\mathbb{G}_\mathrm{m} \xrightarrow{\tau'} \overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}^{0,\mathrm{r}}_{2,4}$$
where the isomorphism is again $T_\infty \circ \alpha$ and τ' is given by either
$$\lambda \mapsto (k_x \xrightarrow{\lambda-1} k_x \xrightarrow{0} k_x)$$
or
$$\lambda \mapsto (k_x \xrightarrow{0} k_x \xrightarrow{\lambda-1} k_x)$$
for some $x \in D$. The symbol τ only denotes that it is one of these maps and the actual map depends on the pair $(\mathcal{E}^\bullet, \mathcal{T}^\bullet)$.

- For every point $(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \times \overline{\mathbf{Coh}_0^{1,1}}$, we write

$$x := \mathrm{Supp}(\alpha^{-1}(\mathcal{E}^\bullet)), \qquad \text{and}$$
$$y := \mathrm{Supp}\,\mathcal{T}^\bullet.$$

- Let $x \in D$. The map

$$M_x \colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$$

denotes the unique Möbius transformation that preserves D and sends ∞ to x.

$(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \in \dots$	$\phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$	degree of $\pi_0 \circ \phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$
$\alpha(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D$	$\mathbf{P}^1 \to \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$	2; formula: 9.31; ramification: Cor. 9.32
$\alpha(\mathbf{P}^1 \setminus D) \times \mathring{D}$	$\mathbf{A}^1 \xrightarrow{\sigma} \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \setminus \pi_0^{-1}\{M_y(x)\}$	1
$\alpha(\mathring{D}) \times \mathbf{P}^1 \setminus D$	$\mathbf{A}^1 \xrightarrow{\sigma} \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \setminus \pi_0^{-1}\{M_x(y)\}$	1
$\alpha(\mathbf{P}^1 \setminus D) \times D^{\mathrm{aut}}$	$\mathbb{G}_\mathrm{m} \to \{T_y T_x^{(1:1)} \tilde{\mathcal{E}}\} \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$	0
$\alpha(D^{\mathrm{aut}}) \times \mathbf{P}^1 \setminus D$	$\mathbb{G}_\mathrm{m} \to \{T_x T_y^{(1:1)} \tilde{\mathcal{E}}\} \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$	0
$\alpha(\mathring{D}) \times \mathring{D}$	2 possibilities: $\mathbb{G}_\mathrm{m} \xrightarrow{\tau} \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ or $\mathbf{A}^1 \xrightarrow{\sigma} \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \setminus \pi_0^{-1}(M_x(y))$ $(\mathrm{im}\,\tau \subset \pi_0^{-1}(M_x(y))$	 0 1
$\alpha(\mathring{D}) \times D^{\mathrm{aut}}$	$\mathbb{G}_\mathrm{m} \to \{T_y \mathcal{E}^\bullet\} \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$	0
$\alpha(D^{\mathrm{aut}}) \times \mathring{D}$	$\mathbb{G}_\mathrm{m} \to \{*\} \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ ($*$ is $T_x T_y^{(1:1)} \tilde{\mathcal{E}}$ or $T_x({}^{(1:1)} T_y \tilde{\mathcal{E}})$)	0
$\alpha(D^{\mathrm{aut}}) \times D^{\mathrm{aut}}$	$\mathbb{G}_\mathrm{m} \to \{T_x T_y \tilde{\mathcal{E}}\} \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$	0

Remark 9.6. Let $x, y \in D$ and fix $\mathcal{E}^\bullet \in \alpha(x)$. There are two isomorphism classes $\mathcal{T}^\bullet \in \mathring{D}$ and each isomorphism class corresponds to one of the two possibilities for $\phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$ listed in the table.

Remark 9.7. Let $x, y \in D$. Then $M_x(y) = M_y(x)$ and

$$\{M_x(y)\} := \begin{cases} D \setminus \{\infty, x, y\} & \text{if } x \neq \infty \text{ and } y \neq \infty \\ \{x, y\} \setminus \{\infty\} & \text{otherwise} \end{cases}.$$

Remark 9.8. The above table is almost symmetric in x and y. Except for a few edge cases that have to be checked individually, this follows from the fact that modifications at different points commute.

9.2. Modifications of $\hat{\mathcal{E}}$. The following proposition gives all length 1 lower modifications of the parabolic bundle $\hat{\mathcal{E}}$, which we defined as

$$\hat{\mathcal{E}} := (\mathcal{O}(2) \oplus \mathcal{O}, (\mathcal{O}|_x)_{x \in D \setminus \{0\}}, \ell_0 = (1:1)),$$

in terms of modifications of $\tilde{\mathcal{E}}$. This section is entirely devoted to its proof. A more easily digestible statement that leaves out some details, appears as Theorem 9.1.

The first part of the proposition lists all modifications of the form $T_x^{\mathcal{O}(2)|_x}\hat{\mathcal{E}}$ for $x \in D$. The next parts then list all remaining modifications of $\hat{\mathcal{E}}$ at different $x \in \mathbf{P}^1$. For $x \in \mathbf{P}^1 \setminus D$, these are all of the form $T_x^{(r:1)}\hat{\mathcal{E}}$ (see (9.2.1)), while for $x \in D$, the remaining modifications are of three different types, corresponding to the isomorphism classes of length 1 torsion sheaves supported at D: $T_x\hat{\mathcal{E}}$ (see (9.2.2)), $T_x^{(r:1)}\hat{\mathcal{E}}$ (see (9.2.3)), and ${}^{\ell'}T_x\hat{\mathcal{E}}$, where $\ell' \subset (T_x\hat{\mathcal{E}})|_x$ is not the canonical flag of $T_x\hat{\mathcal{E}}$ (see (9.2.4)).

As always, we identify $\mathcal{O}(n)$ with $n \in \mathbf{Z}$ with the subsheaf $\mathcal{O}(n[\infty])$ of the locally-constant sheaf $\mathcal{K}$ of rational functions on $\mathbf{P}^1$. We denote the coordinate on $\mathbf{P}^1$ by X.

Proposition 9.9.

(1) For all $x \in \mathbf{P}^1$, there is an isomorphism

$$T_x^{\mathcal{O}(2)|_x}\hat{\mathcal{E}} \cong (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D).$$

(2) Let $x \in \mathbf{P}^1 \setminus D$. We define

$$\psi_x \colon \mathbf{A}^1 \to \mathbf{P}^1 \setminus \{x\}, \qquad r \mapsto y = \frac{xrt}{rt - (x-1)(x-t)}.$$

For every $r \in \mathbf{F}_q$, there is an isomorphism

$$T_x^{(r:1)}\hat{\mathcal{E}} \xrightarrow{\sim} T_D T_{\psi_x(r)}^{(1:1)}\tilde{\mathcal{E}}(2). \tag{9.2.1}$$

(3) We consider the remaining modifications of $\hat{\mathcal{E}}$ at $t \in D$. We define

$$\psi_t \colon \mathbf{A}^1 \to \mathbf{P}^1 \setminus \{t\}, \qquad (r'(X-t):1) \mapsto \frac{r't}{r' - (t-1)/t}$$

where the source $\mathbf{A}^1$ denotes the space $\mathbf{P}((T_t\hat{\mathcal{E}})^0|_t) \setminus \{\mathcal{O}|_t\}$, which we parametrize by $r' \in \mathbf{A}^1$ as indicated.

For $r, r' \in k$, we have isomorphisms

$$T_t\hat{\mathcal{E}} \cong T_t^{(1:1)}\tilde{\mathcal{E}} \tag{9.2.2}$$

$$T_t^{(r:1)}\hat{\mathcal{E}} \cong \begin{cases} T_t\tilde{\mathcal{E}} & \textit{if } r = 1-t \\ T_t\hat{\mathcal{E}} \cong T_t^{(1:1)}\tilde{\mathcal{E}} & \textit{otherwise} \end{cases} \tag{9.2.3}$$

$${}^{(r'\pi_t:1)}T_t\hat{\mathcal{E}} \cong T_D T_{\psi_t(r')}\tilde{\mathcal{E}}(2) \tag{9.2.4}$$

(4) Let $x \in D \setminus \{t\}$. We define three functions

$$\psi_\infty\colon \mathbf{A}^1 \to \mathbf{A}^1, \qquad (rX:1) \mapsto -rt,$$

$$\psi_0\colon \mathbf{A}^1 \to \mathbf{P}^1 \setminus \{0\}, \qquad (1+rX:1) \mapsto \frac{1}{r+\frac{t+1}{t}}, \quad \textit{and}$$

$$\psi_1\colon \mathbf{A}^1 \to \mathbf{P}^1 \setminus \{1\}, \qquad (r(X-1):1) \mapsto \frac{r}{r-\frac{1-t}{t}}$$

where the source of these maps is a subspace of $\mathbf{P}((T_x\hat{\mathcal{E}})^0|_x)$ that we identify with $\mathbf{A}^1$ using the parameter $r \in \mathbf{A}^1$. Then Equations (9.2.2) *to* (9.2.4) *hold with t replaced by x and the exceptional value for r in* (9.2.3) *replaced by another exceptional value.*

Remark 9.10. The only length one lower modifications of $\hat{\mathcal{E}}$ that do not lie in the relevant locus, are the $T_x^{\mathcal{O}(2)|_x}\hat{\mathcal{E}}$ for $x \in \mathbf{P}^1$.

Remark 9.11. We recall that for $y \in D$, we have

$$T_D T_y^{(1:1)}\tilde{\mathcal{E}}(2) \cong \begin{cases} T_y^{(1:1)}\tilde{\mathcal{E}} & \text{if } y \notin D \\ {}^{(1:1)}T_y\tilde{\mathcal{E}} & \text{if } y \in D \end{cases}.$$

This can be substituted in equations (9.2.1) and (9.2.4).

Lemma 9.12. *For all $x \in \mathbf{P}^1$,*

$$T_x^{\mathcal{O}(2)|_x}\hat{\mathcal{E}} \cong (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D).$$

For $x \in D$,

$$T_x\hat{\mathcal{E}} \cong T_x^{(1:1)}\tilde{\mathcal{E}}.$$

PROOF. Let $x \in \mathbf{P}^1$. The modification $T_x^{(1:0)}\hat{\mathcal{E}}$ is isomorphic to $(\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D)$, because the underlying vector bundle is $\mathcal{O}(2) \oplus \mathcal{O}(-1)$ and none of the flags comes from $\mathcal{O}(2)$.

Let $x \in D$. Recall that by definition of $T_x^{(1:1)}$, we have

$$T_x^{(1:1)}(\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in D}) = T_x(\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in D\setminus\{x\}}, \ell'_x = (1:1)).$$

If we replace the flag at x of $\tilde{\mathcal{E}}$ by $(1:1)$, we get a parabolic bundle that is isomorphic to $\hat{\mathcal{E}}$. □

Lemma 9.13. *All length 1 lower modifications of $\hat{\mathcal{E}}$ different from the modifications mentioned in Lemma 9.12 lie in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$.*

PROOF. The underlying vector bundle of any such modification is $\mathcal{O}(1) \oplus \mathcal{O}$, so it remains to check that none of the flags is $\mathcal{O}(1)|_x$ and that the flags do not come from a global section. Because none of the flags of $\tilde{\mathcal{E}}$ come from the maximal destabilizing subbundle, the flag at x of a length 1 lower modification of $\tilde{\mathcal{E}}$ can only come from the maximal destabilizing subbundle

$\mathcal{O}(1)$ if the modification was at x; in particular, there can only be one such flag.

To prove that the flags of any such modification do not come from a global section, suppose towards a contradiction that there exists a length 1 lower modification

$$i\colon (\mathcal{O}(1)\oplus\mathcal{O}, (\mathcal{O}|_x)_{x\in D}) \hookrightarrow \hat{\mathcal{E}}.$$

The underlying map on vector bundles is then of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \mathcal{O}(1)\oplus\mathcal{O} \to \mathcal{O}(2)\oplus\mathcal{O}$$

with $a\colon \mathcal{O}(1)\to\mathcal{O}(2)$, $b\colon \mathcal{O}\to\mathcal{O}(2)$ and $d\in\mathbf{F}_q^*$. Since the flags at $x\in D\backslash\{0\}$ are mapped to $\mathcal{O}|_x\subset(\mathcal{O}(2)\oplus\mathcal{O})|_x$, we find that b has a zero at all $D\setminus\{0\}$. This means $b=0$, which in turns implies the contradiction that the flag at 0 is not mapped to $(1:1)\subset(\mathcal{O}(2)\oplus\mathcal{O})|_0$ □

Let

$$i\colon \mathcal{E}^\bullet \hookrightarrow \hat{\mathcal{E}}$$

be a length 1 lower modification of $\hat{\mathcal{E}}$ at a point $x\in\mathbf{P}^1$ that is different from the modifications we already calculated in Lemma 9.12. To prove the main proposition, Proposition 9.9, we want to describe $\mathcal{E}^\bullet$ as a modification of $\tilde{\mathcal{E}}$. Our assumption that this modification is not as in Lemma 9.12. implies that the image of $\mathcal{E}|_x$ in $(\mathcal{O}(2)\oplus\mathcal{O})|_x$ is of the form

$$(r:1) = \operatorname{im}(i|_x\colon \mathcal{E}|_x \to (\mathcal{O}(2)\oplus\mathcal{O})|_x) \qquad \text{wtih } r\in\mathcal{O}(2)|_x. \tag{9.2.5}$$

Since $\mathcal{E}^\bullet$ lies in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$ (Lemma 9.13), there is a unique (up to scalar multiple) injective parabolic map

$$j\colon \mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}$$

which exhibits $\mathcal{E}^\bullet$ as a length 1 lower modification of $\tilde{\mathcal{E}}$ at some point $y\in\mathbf{P}^1$ (Proposition 6.19). Our next immediate goal is to determine the point y. Afterwards, it only remains to determine which modification at y the map $i\colon \mathcal{E}^\bullet\hookrightarrow\tilde{\mathcal{E}}$ is.

The map $j\colon \mathcal{E}^\bullet\hookrightarrow\tilde{\mathcal{E}}$ from (9.2) induces a map

$$j'\colon \tilde{\mathcal{E}}(-1)\to\mathcal{E}^\bullet.$$

Composing this with i, we get a map of parabolic bundles

$$k := i\circ j'\colon \tilde{\mathcal{E}}(-1)\hookrightarrow\mathcal{E}^\bullet\hookrightarrow\hat{\mathcal{E}}. \tag{9.2.6}$$

The cokernel of this map has support $\{x,y\}$ and its image in $\hat{\mathcal{E}}|_x$ is contained in the image of $\mathcal{E}|_x$, which we assumed to be $(r:1)$ (equation (9.2.5)).

Lemma 9.14. *On the underlying vector bundles, after scaling by an appropriate scalar in $\mathbf{F}_q^*$ the map k defined in* (9.2.6) *is given by*

$$k = \begin{pmatrix} X-x & \lambda(X-1)(X-t)/t \\ 0 & \mu(X-y) \end{pmatrix} : \mathcal{O}(1)\oplus\mathcal{O}(-1)\to\mathcal{O}(2)\oplus\mathcal{O} \tag{9.2.7}$$

for some $\lambda \in \mathbf{F}_q$, $\mu \in \mathbf{F}_q^*$. *In addition, we have*

$$d(0) = \lambda \qquad \text{and} \tag{9.2.8}$$

$$rd(x) = \lambda(x-1)(x-t)/t. \tag{9.2.9}$$

Proof. Every map $\mathcal{O}(1) \oplus \mathcal{O} \to \mathcal{O}(2) \oplus \mathcal{O}$ is of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathcal{O}(2) \oplus \mathcal{O}$$

with $a \colon \mathcal{O}(1) \to \mathcal{O}(2)$, $b \colon \mathcal{O}(-1) \to \mathcal{O}(2)$ and $d \colon \mathcal{O}(-1) \to \mathcal{O}$. We use the fact that this map k preserves the parabolic structure to get conditions on the maps a, b, d.

Recall the definitions of $\tilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$: $\tilde{\mathcal{E}} := (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}, D)$ and $\hat{\mathcal{E}} := (\mathcal{O}(2) \oplus \mathcal{O}, (\mathcal{O}|_x)_{x \in D \setminus \{0\}}, \ell_0 = (1:1))$. To preserve these flags, b is required to have zeroes at $\infty, 1$ and t, so there is a $\lambda \in \mathbf{F}_q$ with

$$b(X) = \lambda(X-1)(X-t)/t.$$

Because the image of k in $\hat{\mathcal{E}}|_x$ is contained in $(r:1)$, the map a has a zero at x. After scaling, we can assume

$$a(X) = X - x.$$

The condition on the parabolic structure at 0 imposes Equation (9.2.8) and because the image of k in $\hat{\mathcal{E}}|_x$ is contained in $(r:1)$, we have Equation (9.2.9).

The matrix 9.2.7 representing k does not have full rank at all $p \in \mathbf{P}^1$ where either a or d vanishes, so the support of the cokernel is $V(a) \cup V(d)$. By construction, as we have already remarked, the cokernel of k is supported on $\{x, y\}$. It follows that y is the unique zero of d. □

To complete the proof of Proposition 9.9, it remains to (1) recover y from the equations in the above lemma; and (2) determine which modification at y the inclusion $j \colon \mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}$ is. This unfortunately requires some case distinctions, which we have split up into several lemmas.

Lemma 9.15. *With notation as above, suppose* $x \in \mathbf{P}^1 \setminus D$. *Then*

$$y = \frac{xrt}{rt - (x-1)(x-t)}.$$

Proof. First, suppose $x \neq 0$ and $r \neq 0$. This implies $\lambda \neq 0$: if λ were 0, then the conditions $d \neq 0$, $d(0) = 0$ (from Equation (9.2.8)) and $rd(x) = 0$ (from Equation (9.2.9)) would imply that $x = 0$ is the unique zero of d, in contradiction to our assumption.

Furthermore, the map d is a linear polynomial with slope

$$\frac{\lambda(x-1)(x-t)/rt - \lambda}{x}.$$

Using $d(0) = \lambda$, we find that y (the zero of d) is

$$y = -\lambda \cdot \frac{x}{\lambda(x-1)(x-t)/rt - \lambda} = \frac{xrt}{rt - (x-1)(x-t)}.$$

(This formula is also correct in the case that the slope of d is zero; we then have $y = \infty$, as the above equation would suggest.)

Now suppose $x \notin D$ and $r = 0$. Then $\lambda = 0$ follows from $rg(x) = \lambda(x-1)(x-t)/t$ and the condition $d(0) = \lambda = 0$ leads to $y = 0$. This can alternatively be verified by calculating

$$T_x^{(0:1)}\hat{\mathcal{E}} = {}^{(1:1)}T_0\tilde{\mathcal{E}}.$$

This is still in accordance with the given formula. □

Corollary 9.16. *With notation as above, suppose $x \notin D$. Then*

$$\mathcal{E}^\bullet \cong T_y^{(1:1)}\tilde{\mathcal{E}}.$$

Proof. We have already proven that $\mathcal{E}^\bullet$ is a length 1 lower modification of $\tilde{\mathcal{E}}$ at y and that $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{1,\mathrm{r}}(\mathbf{F}_q)$. The only such modification is $T_y^{(r:1)}\tilde{\mathcal{E}}$ with $r \neq 0$ (Proposition 6.19) and these are all isomorphic to $T_y^{(1:1)}\tilde{\mathcal{E}}$. □

We have now found all modifications of $\hat{\mathcal{E}}$ at a point $x \in \mathbf{P}^1 \setminus D$. We continue with $x = t$.

Proof of Item 3 of Proposition 9.9. We have already proven $T_t\hat{\mathcal{E}} = T_t^{(1:1)}\tilde{\mathcal{E}}$ (Lemma 9.12).

To find $T_t^{(r:1)}\hat{\mathcal{E}}$ with $r \in k^*$ (note that $r = 0$ is not allowed; this is the operator T_t), we can reason in the same way as the first part of the proof of Lemma 9.15, where we only assume $x \neq 0$, to find

$$y = \frac{xrt}{rt - (x-1)(x-t)} = t.$$

The flag of $T_t^{(r:1)}\hat{\mathcal{E}}$ at t is $\mathcal{O}(2-[t])|_t$, so this modification is either $T_t\tilde{\mathcal{E}}$ or $T_t^{(1:1)}\tilde{\mathcal{E}}$. It is the first if and only if the flags in $\begin{pmatrix} \mathcal{O}(2) & 0 & 1 & 0 & r \\ \mathcal{O} & 1 & 1 & 1 & 1 \end{pmatrix}$ come from a global section, i.e., if and only if $r = 1 - t$.

It remains to consider the modifications of the form ${}^{(r'\pi_t:1)}T_t\hat{\mathcal{E}}$ with $r' \in \mathbf{F}_q$. Then the results from Lemma 9.14 hold for $r = 0$. The condition $rd(x) = \lambda(x-1)(x-t)/t$ ((9.2.9)) then holds independently of d and λ.

However, our choice of r' does impose another condition. The inclusion $k\colon \tilde{\mathcal{E}}(-1) \hookrightarrow \hat{\mathcal{E}}$ we have been studying is

$$\begin{pmatrix} X-t & \lambda(X-1)(X-t)/t \\ 0 & d \end{pmatrix} : \tilde{\mathcal{E}}(-1) \hookrightarrow {}^{(r'\pi_t:1)}T_t\hat{\mathcal{E}} \hookrightarrow \hat{\mathcal{E}}.$$

Because the first of these inclusion, $\tilde{\mathcal{E}}(-1) \hookrightarrow {}^{(r'\pi_t:1)}T_t\hat{\mathcal{E}}$, is a map of parabolic vector bundles, we can recover $(r'\pi_t : 1)$ from the inclusion as the image of k in $(T_t\hat{\mathcal{E}})|_t$: this image is

$$(\lambda(t-1)/t\pi_t : d(t)) \subset T_t\hat{\mathcal{E}}|_t.$$

This then gives the additional condition

$$r'd(t) = \lambda(t-1)/t.$$

If $\lambda = 0$, then $r' = 0$ and you get

$$\begin{pmatrix} \mathcal{O}(1) & 0 & 1 & 0 & 0 \\ \mathcal{O} & 1 & 1 & 1 & 1 \end{pmatrix} \cong T_{D\setminus\{0\}}\tilde{\mathcal{E}}(1).$$

If $\lambda \neq 0$, then $r', g(t) \neq 0$. The slope of d is

$$\frac{\lambda(t-1)/tr' - \lambda}{t} = \lambda \cdot \frac{(t-1)/t - r'}{r't}$$

and so the zero of g is

$$-\lambda \cdot \lambda^{-1} \cdot \frac{r't}{(t-1)/t - r'} = \frac{r't}{r' - (t-1)/t}.$$

(This equation correctly gives ∞ as the zero of g when the slope is zero.)

We can determine which modification of $\tilde{\mathcal{E}}$ at y this is by reasoning in the same way as for $x \in \mathbf{P}^1 \setminus D$, using that none of the flags is $\mathcal{O}(1)|_x$. $\square$

Remark 9.17. To find the modifications for $x \in D \setminus \{t\}$, let $M : \mathbf{P}^1 \to \mathbf{P}^1$ be an isomorphism that preserves D and sends t to x. This is a Möbius transformation of order two that swaps x with t and also swaps the other two points in D Section 8.3. If $\mathcal{E} \hookrightarrow \hat{\mathcal{E}}$ is a lower modification of $\hat{\mathcal{E}}$ at x, then the pullback $M^*(\mathcal{E})$ is a lower modification of $M^*(\hat{\mathcal{E}}) \cong \hat{\mathcal{E}}$ at t. This can be used to determine the modifications at x. To calculate this quickly, we can use that we know we are going to get a degree one map $\psi_x : \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ and then use that it is easy to find the preimages of $D \subset \mathbf{P}^1$ of ψ_x.

An alternative way to prove the formulas for $x \in D \setminus \{t\}$, is to calculate them as limits. For example, we can deduce the formula for $x = t$ by taking the formula that holds in the generic case, writing r as a function of x with the right first and second order behavior around t, e.g.

$$r(x) = r'(X - t)$$

and then taking the limit of the resulting expression as x goes to t.

9.3. Modifications of normal points. We have already determined the modifications of $\hat{\mathcal{E}}$ and $\tilde{\mathcal{E}}$. In this section, we determine the modifications of the parabolic bundles in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \setminus \pi^{-1}(D)$ with respect to torsion sheaves supported outside of D, as summarized in Theorem 9.2. All other length 1 lower modifications of bundles in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ can then be determined from the modifications we already have determined, as explained in Section 9.1.4.

Every point in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \backslash \pi^{-1}(D)$ is isomorphic to $T_y^{(1:1)}\tilde{\mathcal{E}}$ for some $y \in \mathbf{P}^1$ (Proposition 6.19). In Section 9.3.1 we determine all the length 1 lower modifications of such a point at a point $x \in \mathbf{P}^1 \setminus D$ that is unequal to y. In Lemma 9.35, we treat the case $x = y$.

Throughout this section, we identify $(\mathcal{O}(2) \oplus \mathcal{O})|_x$ and $(\mathcal{O}(2) \oplus \mathcal{O})|_y$ with $\mathbf{A}^2$, using the natural global section 1 of $\mathcal{O}(2) = \mathcal{O}(2[\infty])$ and $\mathcal{O}$. In addition, we use projective coordinates to denote lines in each of these two-dimensional spaces, so that, for example, $(1:0) \subset (\mathcal{O}(2) \oplus \mathcal{O})|_x$ denotes the line $\mathcal{O}(2)|_x$.

9.3.1. *When $x \neq y$.* Let $x, y \in \mathbf{P}^1(\mathbf{F}_q) \backslash D$ with $x \neq y$. We will sometimes denote by $\tilde{\mathcal{E}}^0$ the underlying vector bundle of $\tilde{\mathcal{E}}$, which is $\mathcal{O}(2) \oplus \mathcal{O}$. The main goal of this section is to understand the map

$$\phi = \phi_{x,y} \colon \mathbf{P}((\tilde{\mathcal{E}}^0)|_x) \to \mathrm{Bun}_{2,4}^0 \qquad \ell \mapsto T_x^{\ell} T_y^{(1:1)} \tilde{\mathcal{E}} \tag{9.3.1}$$

which determines all length 1 lower modifications of $T_y^{(1:1)}\tilde{\mathcal{E}}$. (Note that the inclusion $T_y^{(1:1)}\tilde{\mathcal{E}} \hookrightarrow \tilde{\mathcal{E}}$ induces an identification $\mathbf{P}((\tilde{\mathcal{E}}^0)|_x) = \mathbf{P}((T_y^{(1:1)}\tilde{\mathcal{E}})^0|_x)$.) We will show, among other things, that this map ϕ factors through $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$, so that it fits into the following commutative diagram

$$\begin{tikzcd}
& \mathbf{P}^1(\tilde{\mathcal{E}}^0|_x) \arrow[dl] \arrow[d] \arrow[dr, bend left, "\phi"] & \\
\operatorname{Spec} k \arrow[d, "(T_y^{(1:1)}\tilde{\mathcal{E}}, k_x)"'] \arrow[r, phantom, "\square"] & \mathcal{H} \arrow[dl] \arrow[dr] & \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \arrow[d, hook] \\
\mathrm{Bun}_{2,4}^{1} \times \overline{\mathbf{Coh}}_0^{1,1} & & \mathrm{Bun}_{2,4}^{0}
\end{tikzcd}$$

We will first calculate the preimages of the points in $\pi_0^{-1}(D)$ (Proposition 9.18). Because of their special form, these preimages are particularly easy to calculate. We then prove in a lengthy calculation that most points in the image of ϕ have exactly two preimages (Proposition 9.22). It is then not difficult to see that ϕ factors through the relevant locus $\mathrm{Bun}_{2,4}^{0,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^{0}$ (Lemma 9.30). It follows that the map

$$\pi_0 \circ \phi \colon \mathbf{P}^1 \to \mathbf{P}^1$$

is of degree 2. We finish with an explicit formula for this map (Lemma 9.31), which is an easy task given that we already know the preimages of 4 different points, and we say something about its ramification behavior (Corollary 9.32 and Lemma 9.33).

We first determine the preimages of the parabolic bundles in $\pi_0^{-1}(D) \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ under ϕ.

Proposition 9.18. *Let $x, y \in \mathbf{P}^1 \setminus D$ with $x \neq y$. Write $\{0, 1, t\} = \{p_1, p_2, p_3\}$ and*

$$r = \frac{(x-p_1)(x-p_2)}{(y-p_1)(y-p_2)} \qquad \textit{and} \qquad s = \frac{x-p_3}{y-p_3}.$$

Then ϕ maps

$$\begin{aligned}
&& (1:0) &\mapsto \hat{\mathcal{E}}(-1)\\
&& (0:1) &\mapsto T_D\hat{\mathcal{E}}(1)\\
&\textit{if } r = s: & (r:1) &\mapsto T_{p_1}T_{p_2}\tilde{\mathcal{E}}(\cong T_{p_3}T_\infty\tilde{\mathcal{E}})\\
&\textit{if } r \neq s: & (r:1) &\mapsto T_{p_1}T_{p_2}\hat{\mathcal{E}}\\
&& (s:1) &\mapsto T_{p_3}T_\infty\hat{\mathcal{E}}
\end{aligned}$$

Remark 9.19. The points $T_{p_3}T_\infty\hat{\mathcal{E}}$, $T_{p_1}T_{p_2}\hat{\mathcal{E}}$ and $T_{p_1}T_{p_2}\tilde{\mathcal{E}} \cong T_{p_3}T_\infty\tilde{\mathcal{E}}$ are the three points in $\pi_0^{-1}(p_3)$. This follows almost immediately from the definitions, as we show in Lemma 7.8.

Proof. Let p_1, p_2, p_3 and r, s as in the statement. We define maps $\sigma, \tau\colon \mathcal{O} \to \mathcal{O}(2)$ as

$$\sigma = \frac{(X-p_1)(X-p_2)}{(y-p_1)(y-p_2)} \qquad \text{and} \qquad \tau = \frac{X-p_3}{y-p_3}$$

so that $r = \sigma(x)$, $s = \tau(x)$, and $\sigma(y) = \tau(y) = 1$ hold. We will now calculate the modification $T_x^{(s:1)}T_y^{(1:1)}\tilde{\mathcal{E}}$; the computation of $T_x^{(r:1)}T_y^{(1:1)}\tilde{\mathcal{E}}$ is analogous.

Using the map τ, we get an isomorphism

$$\begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} : \tilde{\mathcal{E}} \xrightarrow{\sim} (\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in D})$$

where

$$\ell_p := (-\tau|_p : 1) \subset (\mathcal{O}(2) \oplus \mathcal{O})|_p.$$

Because τ has zeroes at ∞ and p_3, we have $\ell_p = \mathcal{O}|_p$ for $p = \infty, p_3$, In addition, this isomorphism maps the line $(1:1)$ at y to $(0:1) = \mathcal{O}|_y$ and the line $(r:1)$ at x to $(0:1) = \mathcal{O}|_x$. Therefore, this induces an isomorphism

$$\begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} : T_x^{(s:1)}T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} T_x^{(0:1)}T_y^{(0:1)}(\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in S}).$$

Finally, we apply another isomorphism

$$\begin{aligned}
\begin{pmatrix} \frac{1}{(X-x)(X-y)} & 0 \\ 0 & 1 \end{pmatrix} : T_x^{(0:1)}T_y^{(0:1)}(\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in S}) &= (\mathcal{O}(2-x-y) \oplus \mathcal{O}, (\ell_p)_{p\in S})\\
&\xrightarrow{\sim} (\mathcal{O}^1 \oplus \mathcal{O}^2, (\ell'_p)_{p\in S})
\end{aligned}$$

where $\mathcal{O}^1$ and $\mathcal{O}^2$ are two copies of $\mathcal{O}$ and

$$\ell'_p := \left(\left(\frac{-\tau}{(X-x)(X-y)}\right)|_p : 1\right). \tag{9.3.2}$$

By our choice of τ, we have $\ell'_p = \mathcal{O}^2|_p$ for $p = \infty, p_3$, which shows that $(\mathcal{O} \oplus \mathcal{O}, (\ell'_p)_{p\in S})$ is either $T_\infty T_{p_3}\hat{\mathcal{E}}$ or $T_\infty T_{p_3}\tilde{\mathcal{E}}$. To distinguish these cases, we need to consider ℓ_{p_1} and ℓ_{p_2}: we get $T_\infty T_{p_3}\tilde{\mathcal{E}}$ if and only if $\ell_{p_1} = \ell_{p_2}$. (The equality $\ell_{p_1} = \ell_{p_2}$ makes sense: with the natural inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(2) := \mathcal{O}(2[\infty])$, we can identify $(\mathcal{O}(2) \oplus \mathcal{O})|_{p_1} = \mathcal{O}|_{p_1} \oplus \mathcal{O}|_{p_1}$ with $(\mathcal{O}(2) \oplus \mathcal{O})_{p_2} = \mathcal{O}|_{p_2} \oplus \mathcal{O}|_{p_2}$, because $p_1, p_2 \neq \infty$.) We now prove that $\ell_{p_1} = \ell_{p_2}$ holds if and only if $r = s$.

Suppose $r = s$. By definition, $\sigma(x) = r$ and $\tau(x) = s$, so this implies $\sigma(x) = \tau(x)$. In addition, $\sigma(y) = \tau(y) = 1$, so that we find

$$\sigma - \tau = \mu(X - x)(X - y)$$

for some $\mu \in \mathbf{F}_q$, because $\sigma - \tau$ is a degree two polynomial in X with zeroes at x and y. Because σ was constructed to have a zeroes at p_1 and p_2, we have for $p = p_1,\ p_2$

$$\left(\frac{-\tau}{(X-x)(X-y)}\right)|_p = \left(\frac{\sigma-\tau}{(X-x)(X-y)}\right)|_p = \mu.$$

By equation (9.3.2), this proves that $\ell'_{p_1} = \ell'_{p_2}$ follows from the assumption $r = s$.

To prove the converse, suppose $r \neq s$. We can then simply calculate that $\frac{-\tau}{(X-x)(X-y)}$ has a different value when evaluated at p_1 than when evaluated at p_2: we write

$$\sigma - \tau - \frac{s-r}{x-y}(X-y) = \mu(X-x)(X-y)$$

for some $\mu \in k$, so that

$$\mu(X-x)(X-y) = \sigma - \tau - (s-r)\frac{X-y}{x-y}$$

for some $\mu \in k$ and therefore for $p = p_1, p_2$

$$\left(\frac{-\tau}{(X-x)(X-y)}\right)(p) = \left(\frac{-\mu\tau}{\sigma-\tau-(s-r)\frac{X-y}{x-y}}\right)(p) = \frac{-\mu\tau(p)}{-\tau(p)-(s-r)\frac{X-y}{x-y}(p)}$$
$$= \frac{-\mu}{-1-\frac{s-r}{x-y}\frac{X-y}{\tau}(p)}$$

but because $\frac{X-y}{\tau}$ is a bijective map $\mathbf{P}^1 \to \mathbf{P}^1$, this is different for $p = p_1$ than for $p = p_2$. We conclude that if $r \neq s$, the two lines ℓ'_{p_1} and ℓ'_{p_2} do not come from a global section and (because τ is not zero at p_1, p_2) are not equal to $(0:1)$, so that $(\mathcal{O} \oplus \mathcal{O}, (\ell'_p)_{p \in S})$ lies in the relevant locus and is isomorphic to $T_\infty T_{p_3}\hat{\mathcal{E}}$.

Lastly, we have to calculate the modifications $T_x^{(1:0)}T_y^{(1:1)}\tilde{\mathcal{E}}$ and $T_x^{(0:1)}T_y^{(1:1)}\tilde{\mathcal{E}}$. In the next lemma, Lemma 9.20, we calculate that $T_x^{(1:0)}T_y^{(1:1)}\tilde{\mathcal{E}}$ is isomorphic to $\hat{\mathcal{E}}(-1)$. To calculate $T_x^{(0:1)}T_y^{(1:1)}\tilde{\mathcal{E}}$, we can choose an isomorphism $T_D\tilde{\mathcal{E}}(2) \xrightarrow{\sim} \tilde{\mathcal{E}}$ (Section 8.2), which induces an isomorphism

$$T_D T_x^{(1:0)}T_y^{(1:1)}\tilde{\mathcal{E}}(2) = T_x^{(1:0)}T_y^{(1:1)}(T_D\tilde{\mathcal{E}}) \xrightarrow{\sim} T_x^{(0:1)}T_y^{(1:1)}\tilde{\mathcal{E}}.$$

This shows that $(0:1)$ is indeed mapped to $T_D\hat{\mathcal{E}}(1)$. □

The following calculation holds and is used in a slightly more general context: it allows $x \in \mathbf{P}^1$.

Lemma 9.20. *Let $x \in \mathbf{P}^1$ and $y \in \mathbf{P}^1 \setminus D$ with $y \neq x$. Then there exists an isomorphism*

$$T_y^{(1:1)}T_x^{\mathcal{O}(2)|_x}\tilde{\mathcal{E}} \cong \hat{\mathcal{E}}(-1)$$

PROOF. By definition, $T_x^{\mathcal{O}(2)|_x}\tilde{\mathcal{E}} = (\mathcal{O}(2),\emptyset)\oplus(\mathcal{O}(-[x]),D)$. Let $\sigma\colon \mathcal{O}(-1) \to \mathcal{O}(2)$ be the unique section that vanishes at $D \setminus \{y\}$ and is y at 0, and let $\tau\colon \mathcal{O}(-1) \xrightarrow{\sim} \mathcal{O}(-[x])$ be the unique isomorphism with $\tau|_y = \sigma|_y$. (Note that $\sigma|_y \neq 0$, because the three zeroes of σ are at $D \setminus \{0\}$ and $y \notin D$.) Then the map

$$\begin{pmatrix} X-y & \sigma \\ 0 & \tau \end{pmatrix} : \mathcal{O}(1) \oplus \mathcal{O}(-1) \hookrightarrow \mathcal{O}(2) \oplus \mathcal{O}(-[x])$$

sends the flags of $\hat{\mathcal{E}}(-1)$ at D to the flags of $T_x^{\mathcal{O}(2)|_x}$ and the image of the map lies in $T_y^{(1:1)}(\mathcal{O}(2) \oplus \mathcal{O}(-[x]))$. It therefore induces an isomorphism

$$\hat{\mathcal{E}}(-1) \xrightarrow{\sim} T_y^{(1:1)} T_x^{\mathcal{O}(2)|_x}\tilde{\mathcal{E}}.$$

$\square$

We continue with our study of the map $\phi = \phi_{(x,y)}$ with $x, y \in \mathbf{P}^1 \setminus D$, $x \neq y$.

Remark 9.21. The symmetry $T_D(2)$ explained in Section 8.2 is reflected in a symmetry for ϕ (Equation (9.3.3) below), which we explain here. Let $r \in k$. Recall (Lemma 8.4) that $T_D(2)$ fixes the points in $\pi_0^{-1}(\mathbf{P}^1 \setminus D)$ and the points in $\pi_0^{-1}(D)$ with automorphism group $\mathbb{G}_m \times \mathbb{G}_m$. If $\phi(r:1)$ is one of these points, then

$$\phi(r:1) = \phi\big(x(x-1)(x-t) : y(y-1)(y-t)\cdot r\big). \tag{9.3.3}$$

Indeed, the isomorphism

$$\begin{pmatrix} 0 & 1 \\ \frac{y(y-1)(y-t)}{X(X-1)(X-t)} & 0 \end{pmatrix} : T_D\tilde{\mathcal{E}}(2) \xrightarrow{\sim} \mathcal{E},$$

sends the flag $(1:1)$ at y to $(1:1)$ and the flag $(r:1)$ at x to $(x(x-1)(x-t) : y(y-1)(y-t)\cdot r)$, so that it induces an isomorphism on the modifications with respect to those flags:

$$\begin{aligned} T_D T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}(2) &= T_x^{(r:1)} T_y^{(1:1)} T_D \tilde{\mathcal{E}}(2) \\ &\xrightarrow{\sim} T_x^{(x(x-1)(x-t):y(y-1)(y-t)\cdot r)} T_y^{(1:1)} \tilde{\mathcal{E}}. \end{aligned}$$

It follows from the above remark that most points in the image of $\phi \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ have at least two preimages under ϕ. The following proposition says that they also have at most these two preimages. Its proof takes up most of this section.

Proposition 9.22. *Let $r, s \in \mathbf{F}_q$ with $r \neq s$. Then there is an isomorphism*

$$\phi(r:1) = T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \xrightarrow{\sim} T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}} = \phi(s:1) \tag{9.3.4}$$

if and only if

$$rs = \frac{x(x-1)(x-t)}{y(y-1)(y-t)} \tag{9.3.5}$$

and

$$r, s \neq \frac{x-p}{y-p} \qquad \textit{for all } p \in \{0, 1, t\}. \tag{9.3.6}$$

Every such isomorphism is given on the underlying vector bundle by a non-zero multiple of the map

$$\frac{1}{(X-x)(X-y)}\begin{pmatrix} a & -\frac{X(X-1)(X-t)}{y(y-1)(y-t)} \\ 1 & d \end{pmatrix}$$

where a and d are the unique degree 2 polynomials in X satisfying the following equations

$$\begin{aligned} a(x) &= s & d(x) &= -r \\ a(y) &= 1 & d(y) &= -1 \\ a'(y) - d'(y) &= -b'(y) & ra'(x) - sd'(x) &= -b'(x) \end{aligned}$$

Remark 9.23. Suppose that $r, s \in \mathbf{F}_q$ with $r \neq s$ satisfy (9.3.5), but not (9.3.6), i.e., there is a $p \in D$ such that either r or s is equal to $\frac{x-p}{y-p}$ for some $p \in D$. Then we have already found that $\phi(s:1)$ and $\phi(r:1)$ lie in $\pi_0^{-1}(\{p\})$ and are not isomorphic (Proposition 9.18).

Let $\mathcal{K}$ denote the locally constant sheaf of rational functions on $\mathbf{P}^1$. By definition, the underlying vector bundle $\mathcal{O}(2) \oplus \mathcal{O} = \mathcal{O}(2[\infty] \oplus \mathcal{O})$ of $\tilde{\mathcal{E}}$ comes equipped with an inclusion in $\mathcal{K}^{\oplus 2}$, and hence so do its subbundles $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$ and $T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$.

Note that every map $\mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$ is of the form

$$\frac{1}{(X-x)(X-y)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$$

for some rational functions a', b', c', d' and a, b, c, d. The following lemma gives necessary and sufficient conditions for such a map $\mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$ to restrict to a map $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$. These conditions come from the following sources: (1) the image of the restriction of $\mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$ to $(T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}})^0$ should land in $\tilde{\mathcal{E}}^0$, which gives some conditions on the behavior of the poles of the matrix coefficients (conditions 1, 2 and 4); (2) the image should even land in the modification $(T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}})^0 \subset \tilde{\mathcal{E}}^0$, which gives additional conditions on the maps $\mathcal{K}|_x^{\oplus} \to \mathcal{K}|_x^{\oplus 2}$ and $\mathcal{K}|_y^{\oplus} \to \mathcal{K}|_y^{\oplus 2}$ (conditions 3 and 5); and (3) the parabolic structures should be preserved (condition 6) .

Lemma 9.24. *Let $r, s \in \mathbf{F}_q$ (we allow $r = s$ for the moment) and $x, y \in \mathbf{P}^1 \setminus D$ with $x \neq y$. Consider the following conditions on maps*

$$\frac{1}{(X-x)(X-y)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2} :$$

(1) $a \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2))$, $b \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(4))$, $c \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O})$ and $d \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2))$;

(2) $ra+b$ and $rc+d$ have a zero at x;

(3) we have

$$a(x) = sc(x),$$
$$b(x) = sd(x), \quad \textit{and}$$
$$ra'(x) + b'(x) = s(rc'(x) + d'(x));$$

(4) $a+b$ and $c+d$ have a zero at y;

(5) we have

$$a(y) = c(y),$$
$$b(y) = d(y), \quad \textit{and}$$
$$a'(y) + b'(y) = c'(y) + d'(y);$$

and lastly

(6) for all $p \in D$,

$$a(p) = 0, \qquad b(p) = 0.$$

Then the restriction map defines a bijection

$$\left\{ \begin{array}{c} \textit{maps } \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2} \\ \textit{satisfying the above conditions} \end{array} \right\} \xrightarrow{1-1} \left\{ \textit{maps } T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \right\}$$

Proof. Every map $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$ uniquely extends to a map $\mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$. We will prove that the given conditions on $\mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$ are necessary and sufficient for restricting to a map $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$.

Let $\phi\colon \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$. The first, second and fourth conditions are conditions on the poles of the map ϕ that are necessary and sufficient to ensure that ϕ maps $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}^0 \subset \mathcal{K}^{\oplus 2}$ (the underlying bundle of the source) into $\tilde{\mathcal{E}}^0$: the first condition provides the correct poles outside of $\{x, y\}$; the first and second provide the correct conditions on the poles at x; and the first and fourth provide the correct conditions on the poles at y.

Given the other conditions, the third and fifth condition are equivalent to requiring that the image is in fact contained in $(T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}})^0$, which give additional conditions on the fibers at x and y. The three equalities in the third condition say that the lines $(\pi_x : 0)$, $(0 : \pi_x)$ (with $\pi_y \in \mathcal{O}|_y$ a uniformizer) and $(r : 1)$ in $(T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}})^0|_x$ are mapped to $(s : 1) \subset \tilde{\mathcal{E}}^0|_x$. (We could leave out the equality for the line $(0 : \pi_x)$, as it follows from the other two.) Similarly the equalities in the fifth condition say that the lines $(\pi_y : 0)$, $(0 : \pi_y)$ and $(1 : 1)$ in $(T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}})^0|_y$ are mapped to $(1 : 1) \subset \tilde{\mathcal{E}}^0|_y$.

Lastly, the sixth condition is equivalent to the condition that the restriction of ϕ preserves the parabolic structure. $\square$

Suppose that an isomorphism $\phi\colon T_x^{(r:1)}T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} T_x^{(s:1)}T_y^{(1:1)}\tilde{\mathcal{E}}$ as in Equation (9.3.4) exists. This isomorphism can uniquely be extended to an isomorphism

$$A := \frac{1}{(X-x)(X-y)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{K}^{\oplus 2} \xrightarrow{\sim} \mathcal{K}^{\oplus 2}$$

with coefficients a, b, c, d rational functions on $\mathbf{P}^1$ satisfying the conditions in Lemma 9.24. The remainder of the proof of Proposition 9.22 is essentially just calculations using these conditions. This is split into the following lemmas.

First, in Lemma 9.25, we show that if $c = 0$, then A is a multiple of the identity matrix (possibly zero). In the case $c \neq 0$, we rewrite the conditions in Lemma 9.24, which results in the equations in Lemma 9.26. Then we try to find a solution to these equations in Lemma 9.27. We finally complete the proof of Proposition 9.22 on page 82 using these lemmas.

Lemma 9.25. *Let $x, y \in \mathbf{P}^1 \setminus D$ with $x \neq y$ and let $r, s \in \mathbf{F}_q$. Let*

$$\phi = \frac{1}{(X-x)(X-y)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus}$$

satisfy the conditions in Lemma 9.24. If c = 0, then ϕ is a non-zero multiple of the identity matrix. In particular, $r = s$.

Proof. Assume $c = 0$. First, the condition on the poles at x and y (conditions 2 and 4) implies that d has zeroes at x and y and is therefore some multiple of $(X-x)(X-y)$. We can scale the matrix so that we in fact have $d = (X-x)(X-y)$.

The conditions on the image of the fiber at y (conditions 3 and 5) now require b to have a zero at y (because d does). In addition, the conditions on the parabolic structure (condition 6) require b to have zeroes at D as well. Since b is a degree four polynomial with zeroes at $D \cup \{y\}$, we conclude $b = 0$.

It then follows from the conditions on the poles at x and y (conditions 2 and 4) that a is some multiply of $(X-x)(X-y)$. But then conditions on the image of the fiber at y (the equality with the derivatives in condition 5)) show that $a = d$, so we do indeed get the identity matrix. ☐

The following lemma is a further translation of the conditions, but in the case that $c \neq 0$.

Lemma 9.26. *Let $x, y \in \mathbf{P}^1 \setminus D$ with $x \neq y$ and let $r, s \in \mathbf{F}_q$. Suppose*

$$\frac{1}{(X-x)(X-y)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$$

is not *a multiple of the identity matrix that induces a map*

$$T_x^{(r:1)}T_y^{(1:1)}\tilde{\mathcal{E}} \to T_x^{(s:1)}T_y^{(1:1)}\tilde{\mathcal{E}},$$

i.e., it satisfies the conditions in Lemma 9.24. Then after scaling appropriately, it satisfies the following equalities:

$$c = 1, \tag{9.3.7}$$

$$b = \mu X(X-1)(X-t), \qquad \text{for some } \mu \in \mathbf{F}_q. \tag{9.3.8}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(y) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \tag{9.3.9}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \begin{pmatrix} s & -rs \\ 1 & -r \end{pmatrix}, \tag{9.3.10}$$

$$d'(y) = a'(y) + b'(y), \quad \text{and} \tag{9.3.11}$$

$$sd'(x) = ra'(x) + b'(x). \tag{9.3.12}$$

Proof. The condition that the matrix is not a non-zero multiply of the identity matrix implies $c \neq 0$, as we have just seen (Lemma 9.25), so we can scale c to be 1.

The condition that the flags are preserved (condition 6) implies (9.3.8).

The conditions at y (conditions 4) and 5)) imply (9.3.9). Similarly, at x, the conditions 2 and 3)) give us (9.3.10).

Since c is constant, the condition on the images of the fibers at y involving the derivatives (condition 5) implies (9.3.11). Similarly, at x we get (9.3.12) via condition 3. □

We now try to find a, b, c, d that satisfy the above equations.

Lemma 9.27. *Let $x, y \in \mathbf{P}^1 \setminus D$ with $x \neq y$ and let $r, s \in \mathbf{F}_q$. Define the rational functions a_0, b_0, d_0, f in X as follows. Let*

$$b_0 := -\frac{X(X-1)(X-t)}{y(y-1)(y-t)}, \tag{9.3.13}$$

$$f := (X-x)(X-y), \tag{9.3.14}$$

and let a_0, d_0 be the unique linear polynomials in X that satisfy

$$a_0(x) = s, \qquad a_0(y) = 1, \tag{9.3.15}$$

$$d_0(x) = -r, \qquad d_0(y) = -1. \tag{9.3.16}$$

Consider the equation

$$rs = \frac{x(x-1)(x-t)}{y(y-1)(y-t)}. \tag{9.3.17}$$

(1) If Equation (9.3.17) does not hold, there are no maps
$$T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}.$$
that are not a multiple of the identity.

(2) If Equation (9.3.17) does hold, then the maps

$$\frac{\nu}{(X-x)(X-y)} \begin{pmatrix} a_0 + \lambda f & b_0 \\ 1 & d_0 + \mu f \end{pmatrix} : \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2} \tag{9.3.18}$$

with $\nu \in \mathbf{F}_q^*$ and $(\lambda, \mu) \in \mathbf{F}_q^2$ satisfying

$$\begin{aligned} \lambda - \mu &= a_0'(y) - d_0'(y) + b'(y), \quad \text{and} \\ r\lambda - s\mu &= -ra_0'(x) + sd_0'(x) - b'(x). \end{aligned} \tag{9.3.19}$$

are exactly the morphisms $\mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$ that are not multiples of the identity and restrict to a map

$$T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}.$$

Proof. Suppose that

$$\frac{1}{(X-x)(X-y)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{K}^{\oplus 2} \to \mathcal{K}^{\oplus 2}$$

is a map that is not a multiple of the identity matrix and induces a map

$$T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}.$$

Then we are in the situation of Lemma 9.26, so after scaling by a $\nu \in \mathbf{F}_q^*$, we can assume that $c = 1$ and the other conditions Lemma 9.26 hold.

We already know b is a non-zero multiple of $X(X-1)(X-t)$ (Equation (9.3.8)). The condition

$$b(y) = -1$$

(Equation (9.3.9)) then determines b completely, giving $b = b_0$ Equation (9.3.13).

The condition $b(x) = -rs$ (Equation (9.3.10)) then determines that we indeed need Equation (9.3.17). This proves part (1) of the lemma.

The degree 2 polynomials a, d have to interpolate the right points at x and y (Equation (9.3.10) and Equation (9.3.9)), which determines a and d up to addition of some multiple of $f = (X-x)(X-y)$. This additions of f to a and d form the last degrees of freedom in choosing a and d; we will use them to ensure that the conditions on the derivatives at x and y hold.

We can therefore indeed write write

$$a = a_0 + \lambda f, \qquad d = d_0 + \mu f$$

with $\mu, \lambda \in \mathbf{F}_q$. The derivatives of a and d at x and y are then

$$\begin{aligned} a'(x) &= a_0'(x) + \lambda, & a'(y) &= a_0'(y) - \lambda, \\ d'(x) &= d_0'(x) + \mu, & d'(y) &= d_0'(y) - \mu. \end{aligned}$$

The conditions on the derivatives (Equation (9.3.12) and Equation (9.3.11)) can then be rewritten as

$$\begin{pmatrix} a_0'(y) + b'(y) - \lambda \\ d_0'(y) - \mu \end{pmatrix} \in (1:1), \qquad \begin{pmatrix} ra_0'(x) + b'(x) + r\lambda \\ d_0'(x) + \mu \end{pmatrix} \in (s:1).$$

which is equivalent to (9.3.19).

This proves that all such maps are of form given in Equation (9.3.18). The fact that all maps of this form restrict to maps $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \to T_x^{(s:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$, can then easily be checked on the fibers. $\square$

Proof of Proposition 9.22. First note that if there is an isomorphism as in Proposition 9.22, it is necessarily not the identity, because $r \neq s$. It is therefore of the form described in Lemma 9.26. The proof of Lemma 9.27 shows that we do indeed get the exact form described in Proposition 9.22. These proves the second half of the proposition.

The first half of the proposition is an equivalence. Assume for the first implication of that equivalence that we have an isomorphism $T_x^{(r:1)}T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} T_x^{(s:1)}T_y^{(1:1)}\tilde{\mathcal{E}}$. In the lemmas we are conclude that the condition $rs = \frac{x(x-1)(x-t)}{y(y-1)(y-t)}$ is necessary (Lemma 9.27). In Remark 9.23, we already explained that the other condition

$$r, s \neq \frac{x-p}{y-p} \qquad \text{for all } p \in \{0, 1, t\}.$$

(Equation (9.3.6)) is necessary.

It remains to prove the other direction. Since we assume $r \neq s$ in the proposition, there is always a unique solution (λ, μ) to Equation (9.3.19) in Lemma 9.27. This shows that we can indeed find map

$$\frac{1}{(X-x)(X-y)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : T_x^{(r:1)}T_y^{(1:1)}\tilde{\mathcal{E}} \to T_x^{(s:1)}T_y^{(1:1)}\tilde{\mathcal{E}}.$$

If this map is injective, then it is an isomorphism, because the bundles have the same degree. This injectivity is equivalent to $ad \neq b$. The assumption $ad = b$ implies that either r or s is equal to $\frac{x-p}{y-p}$ for some $p \in D \setminus \{\infty\}$ (Lemma 9.28), which is in contradiction to the second condition on this side of the equivalence (Equation (9.3.6)). We therefore conclude that the map we found is an isomorphism. □

Lemma 9.28. *Let the notations and assumptions be as in Lemma 9.27 and suppose that the conditions in the lemma hold. If $b = ad$, then there is $p \in D \setminus \{\infty\}$ with either*

$$r = \frac{x-p}{y-p} \quad \text{or} \quad s = \frac{x-p}{y-p}.$$

In other words, r and s are the special values whose image under ϕ is a point in $\pi_0^{-1}(D)$ that we have already calculated in Proposition 9.18.

Proof. Suppose that a, b, d satisfy $b = ad$. Because b has three irreducible factors (recall $b = -\frac{X(X-1)(X-t)}{y(y-1)(y-t)}$, Equation (9.3.13)) and because the scaling of a and d are already determined by the values they take at y, we can write

$$\left\{\frac{X}{y}, \frac{X-1}{y-1}, \frac{X-t}{y-t}\right\} =: \{f_1, f_2, f_3\}.$$

in such a way that

$$\{a, -d\} = \{f_1, f_2 f_3\}.$$

This gives 6 possibilities for a and hence for $s = a(x)$, $r = -d(x)$, which are exactly the six values for r and s whose images we have already calculated in our calculation of the preimage of D (Proposition 9.18). □

We have mostly applied the above lemmas in the case $r \neq s$, since this is what we needed for the proof of Proposition 9.22. We can also apply them with $r = s$, however, to determine the automorphism group of the points $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$.

Lemma 9.29. *Let $r \in \mathbf{F}_q$. Suppose that $\mathrm{Aut}(T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}})$ is larger than $\mathbb{G}_{\mathrm{m}}$. Then we can write $\{p_1, p_2, p_2\} = \{0, 1, t\}$ in such a way that*

$$r = \frac{x - p_1}{y - p_1} = \frac{x - p_2}{x - p_2} \frac{x - p_3}{x - p_3}. \tag{9.3.20}$$

We determined the modification $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$ for $r \in \mathbf{F}_q$ satisfying Equation (9.3.20) in Proposition 9.18: it is a bundle with $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ automorphisms in $\pi_0^{-1}(D)$.

Proof. Suppose that there exists an automorphism

$$T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \xrightarrow{\sim} T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$$

that is not a non-zero multiple of the identity. In Lemma 9.27, we explicitly described the set of all endomorphisms (not necessarily automorphisms) of $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$. The idea of this proof is to use the equations in that lemma, to find a, b, d as in Lemma 9.28, with $ad = b$, so that $r = \frac{x-p}{y-p}$ follows for some $p \in D \setminus \{\infty\}$. Then because we have

$$r^2 = \frac{x(x-1)(x-t)}{y(y-1)(y-p)}$$

(Equation (9.3.17)), we have completed the proof.

Choose a pair $(\lambda, \mu) \in \mathbf{F}_q^2$ that satisfies Equation (9.3.19); such a pair exists because we assumed the existence of an automorphism of $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$ that is not a multiple of the identity (Lemma 9.27). Because Equation (9.3.19) only depends on λ and μ via their difference $\lambda - \mu$, we can and do assume that one of λ and μ is zero.

Define a_0, d_0, f and $b_0 = b$ as in Lemma 9.27. We set $c = 1$ and

$$a := a_0 + \lambda f, \qquad d := d_0 + \mu f.$$

Then a, b, c and d satisfy the properties from Lemma 9.26 with $s = r$: the matrix $\frac{1}{(X-x)(X-y)} \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ defines an endomorphism of $T_x^{(r:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$.

Because λ or μ is zero, one of a and d is linear, so ad has degree at most 3. To show that $ad = b$ holds, we evaluate both degree ≤ 3 polynomials, ad and b, at x and y and compare their derivatives at x and y. We find

$$(ad)(x) = -r^2 = b(x)$$
$$(ad)(y) = -1 = b(y)$$

(this follows directly from the definition of a_0, b_0, d_0 and f; see Equation (9.3.15) and Equation (9.3.16)). The derivatives of ad at y and x are equal to

$$(ad)'(x) = a'(x)d(x) + a(x)d'(x) = -ra'(x) + rd'(x)$$
$$(ad)'(y) = a'(y)d(y) + a(y)d'(y) = -a'(y) + d'(y),$$

which are indeed the same as $b'(x)$ and $b'(y)$, respectively (indeed, in Lemma 9.27, we construct a and d to satisfy these properties; see Equation (9.3.12) and Equation (9.3.11)). Since there is a unique polynomial of degree ≤ 3 with these fixed values and first derivatives at x and y, this proves $ad = b$. □

Lemma 9.30. *Let $x, y \in \mathbf{P}^1 \setminus D$. Let $\mathcal{E}^\bullet$ be a modification of $T_y^{(1:1)}\tilde{\mathcal{E}}$ at x. Then $\mathcal{E}^\bullet$ lies in $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$.*

Proof. We already know that the modifications mentioned in Proposition 9.18 lie in the relevant locus. The other modifications have automorphism group $\mathbb{G}_\mathrm{m}$ (Lemma 9.29). Because all bundles outside of $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ have automorphism groups of higher dimension, this concludes the proof. (The fact that the bundles outside of $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ have larger automorphism groups is perhaps easier seen in $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^{1}$: all bundles outside of the relevant locus are direct sums of parabolic line bundles by Lemma 6.8, so their automorphism groups contain at least $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$.) □

In the following lemma, we give an explicit formula for $\pi_0 \circ \phi$. See Lemma 9.35 for a similar formula when $x = y$.

Lemma 9.31. *Let $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$, $(\ell \subset \mathbf{P}^1((\mathcal{O}(2) \oplus \mathcal{O})|_x) = \mathbf{P}^1) \mapsto T_x^\ell T_y^{(1:1)}\tilde{\mathcal{E}}$ be the map defined in Equation* (9.3.1)*. The map*

$$\pi_0 \circ \phi\colon \mathbf{P}^1 \to \mathbf{P}^1$$

is given by

$$\begin{aligned} (X : Y) &\mapsto \pi_0(T_x^{(X:Y)} T_y^{(1:1)}\tilde{\mathcal{E}}) \\ &= \big((yX - xY) \cdot ((y-1)(y-t)X - (x-1)(x-t)Y) : -(x-y)^2 XY\big). \end{aligned} \tag{9.3.21}$$

(Here ∞ in the target $\mathbf{P}^1$ corresponds to $(1:0)$*.)*

Proof. We already know that this map has degree 2 (Proposition 9.22) and what the preimages of the four points $D \subset \mathbf{P}^1$ are (Proposition 9.18). This uniquely determines the map, so we only have to verify that the preimage of each point in D under the given equation, Equation (9.3.21), equals the preimage we calculated earlier (Proposition 9.18). □

Corollary 9.32. *The ramification locus of $\pi_0 \circ \phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ is*

$$\left\{ \pm\sqrt{\frac{x(x-1)(x-t)}{y(y-1)(y-t)}} \right\}$$

Proof. Being a degree 2 map $\mathbf{P}^1 \to \mathbf{P}^1$, $\pi_0 \circ \phi$ has a ramification locus that consists of two points. By Proposition 9.22, we see that these two points are the only possibilities. □

Lemma 9.33. *Let $p \in D$ and let $M\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ be the unique Möbius transformation that preserves D and sends ∞ to p. As in the rest of this section, ϕ denotes $\phi_{(x,y)}$ (Equation (9.3.1)). The following are equivalent.*

1. *The map $\pi_0 \circ \phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ is ramified over p.*
2. *The intersection $\operatorname{im}\phi \cap \pi_0^{-1}(p)$ consists only of the unique point in $\pi_0^{-1}(p)$ with $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$-automorphisms.*
3. *We have $y = M(x)$.*

Remark 9.34. It follows from our treatment of the case $x = y$ (Lemma 9.35) that the above lemma also holds when $x = y$, in which case $\pi_0 \circ \phi$ is ramified over ∞ and not over any other point in D.

Proof of Lemma 9.33. Statements (1) and (2) are equivalent by our calculation of the inverse image of $\pi_0^{-1}(D)$ (Proposition 9.18).

The same calculation shows that statement (1) is also equivalent to

$$r = \frac{(x-p_1)(x-p_2)}{(y-p_1)(y-p_2)} = \frac{x-p}{y-p} \tag{9.3.22}$$

with $\{0, 1, t\} = \{p_1, p_2, p\}$. We now prove that this equality is equivalent to $x = M(y)$.

This can be checked by explicit calculation, but we can also rewrite the above equation Equation (9.3.22) as

$$\frac{x-p_2}{y-p_2} = \frac{x-p}{y-p}\frac{y-p_1}{x-p_1} \tag{9.3.23}$$

which is nothing other than

$$\mathrm{CR}\,(x, y;\ p_2, \infty) = \mathrm{CR}\,(x, y;\ p, p_1)$$

where CR denotes the cross ratio. If we consider the above equation as an equality in x, keeping y constant, then it has at most two solutions in x. One is $x = y$, but in the proposition we assumed $x \neq y$. If $M(y) \neq y$, then the other solution is $x = M(y)$: because Möbius transformations preserve cross ratios, we have

$$\begin{aligned} \mathrm{CR}\,(M(y), y;\ p_2, \infty) &= \mathrm{CR}\,(y, M(y);\ M(p_2), M(\infty)) = \mathrm{CR}\,(y, M(y);\ p_1, p) \\ &= \mathrm{CR}\,(M(y), y;\ p, p_1) \end{aligned}.$$

If, however, $M(y) = y$, then $x = y$ is the only solution: if we derive equation (9.3.23) with respect to x and evaluate at $x = y$, we get

$$\frac{1}{y - p_2} = \frac{p - p_1}{(y - p)(y - p_2)}$$

which is equivalent to

$$\frac{y - p}{y - p_1} = \frac{p - p_2}{y - p_2}.$$

This last equation holds when $M(y) = y$: we recognize that it is an equality of cross-ratios and use that M preserves cross-ratios to conclude

$$\mathrm{CR}\,(y, \infty;\ p, p_1) = \mathrm{CR}\,(M(y), M(\infty);\ M(p), M(p_1)) = \mathrm{CR}\,(y, p;\ \infty, p_2)\,.$$

□

9.3.2. *When $x = y$.* Let $y \in \mathbf{P}^1 \setminus D$. It remains to determine the modifications of the form

$$T_y^{\ell} T_y^{(1:1)} \tilde{\mathcal{E}}$$

with $\ell \subset (T_y^{(1:1)} \tilde{\mathcal{E}})^0|_y$. This can be deduced from the modifications we have already calculated.

Note that every line $\ell \subset (T_y^{(1:1)} \tilde{\mathcal{E}})^0|_y$ can be written as

$$\ell = (\pi_y : 0) \quad \text{or} \quad \ell = (1 + r\pi_y : 1) \quad \text{with } r \in \mathbf{F}_q$$

where $\pi_y \in \mathcal{O}|_y$ is a uniformizer.

Lemma 9.35. *Let $y \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$, let $r, s \in \mathbf{F}_q$. Let $\pi \in \mathcal{O}|_y$ denote a uniformizer.*

(1) Every modification of $T_y^{(1:1)} \tilde{\mathcal{E}}$ at y lies in the relevant locus.

(2) There exists an isomorphism

$$T_y^{(1+r\pi:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \xrightarrow{\sim} T_y^{(1+s\pi:1)} T_y^{(1:1)} \tilde{\mathcal{E}}$$

if and only if

$$r = s \qquad \text{or} \qquad r + s = -\frac{1}{y^2} - \frac{1}{(y-1)^2} - \frac{1}{(y-t)^2}.$$

(3) The map

$$\mathbf{P}((T_y^{(1:1)} \tilde{\mathcal{E}})|_y) \to \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \to \mathbf{P}^1, \qquad \ell \mapsto \pi_0(T_y^{\ell} T_y^{(1:1)} \tilde{\mathcal{E}})$$

has degree 2, is ramified at $(\pi : 0)$ and $(1 - \frac{1}{2}(\frac{1}{y^2} + \frac{1}{(y-1)^2} + \frac{1}{(y-t)^2}) : 1)$ and is given by

$$(1 + r\pi : 1) \mapsto ((ry - 1)\,((y-1)(y-t)r - (2y - (1+t))) : -1) \tag{9.3.24}$$

This statement follows from the previous cases with $x \neq y$ by using a certain symmetry (Lemma 9.36). In principle, we could also use this symmetry to determine the explicit formula 9.3.24, but it is easier to use the symmetry to get the relevant qualitative results and then find the formula simply by taking a limit: write $x = y + \varepsilon$ and $r = 1 + r'\varepsilon$. We will not make this limit procedure more precise; the resulting formulas can be verified using the symmetry. In

Lemma 9.37, we give an explicit description of the modifications of $T_y^{(1:1)}\tilde{\mathcal{E}}$ at y that lie in $\pi_0^{-1}(D)$, which can be used to verify the formulas.

Let $M\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ be a Möbius transformation that preserves D and is not the identity and let $y \in \mathbf{P}^1 \setminus D$. We choose an isomorphism

$$\Phi\colon T_\infty T_{M(\infty)}^{-1} M^* T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} T_y^{(1:1)}\tilde{\mathcal{E}}. \tag{9.3.25}$$

Since $M^*\colon \operatorname{Bun}_{2,4}^{2,\mathrm{r}} \xrightarrow{\sim} \operatorname{Bun}_{2,4}^{2,\mathrm{r}}$ induces the identity on the coarse moduli space $\mathbf{P}^1$ (Corollary 8.9), we know that there exists an isomorphism

$$M^* T_\infty^{-1} T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} T_\infty^{-1} T_y^{(1:1)}\tilde{\mathcal{E}}$$

because $T_\infty^{-1} T_y^{(1:1)}\tilde{\mathcal{E}}$ lies in the relevant locus in even degree. By applying to such an isomorphism the functor T_∞ and using $M^* T_\infty^{-1} = T_{M(\infty)}^{-1} M^*$, we obtain an isomorphism as in Equation (9.3.25).

We will not need this, but for future reference, we provide an explicit isomorphism Φ. Write $x = M(y)$, $p = M(\infty)$ and $\{p_1, p_2\} = D \setminus \{\infty, p\}$. We claim that the matrix of rational functions on $\mathbf{P}^1$

$$\frac{1}{X-y}\begin{pmatrix} -(X-p)(y-p_1)(y-p_2) & (X-p)(X-p_1)(X-p_2) \\ -(x-p)(y-p_1)(y-p_2) & (X-p)(x-p_2)(x-p_3) \end{pmatrix} \tag{9.3.26}$$

defines an isomorphism

$$T_\infty T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} T_p T_x^{(1:1)}\tilde{\mathcal{E}}.$$

Proof. The given matrix induces an isomorphism $\mathcal{K}^{\oplus 2} \xrightarrow{\sim} \mathcal{K}^{\oplus 2}$. If it indeed restricts to a map

$$T_\infty T_y^{(1:1)}\tilde{\mathcal{E}} \to T_p T_x^{(1:1)}\tilde{\mathcal{E}},$$

then since this map is an injective map between bundles of the same degree, it is an isomorphism. That it indeed restricts to such a map follows directly from the definitions and the equality

$$-(x-p)(y-p_1)(y-p_2) = -(y-p)(x-p_1)(x-p_2).$$

A quick way to see that this last equality is true, is to rewrite it as

$$\frac{x-p}{y-p}\frac{y-p_1}{x-p_1} = \frac{x-p_2}{y-p_2}$$

which is in fact an equality of two cross-ratios

$$\mathrm{CR}\left(x, y;\ p, p_1\right) = \mathrm{CR}\left(y, x;\ \infty, p_2\right).$$

This last equality follows from the fact that cross-ratios are preserved by Möbius transformations and the order 2 Möbius transformation M maps x to y (by definition of y), ∞ to p (by definition of M) and p_1 to p_2 (because it was defined to preserve D). $\square$

With this isomorphism, we can identify length one lower modifications of $T_y^{(1:1)}\tilde{\mathcal{E}}$ at y with length one lower modifications of $T_y^{(1:1)}\tilde{\mathcal{E}}$ at $M(y) \neq y$, which allows us to transfer our results for the $x \neq y$ case to the $x = y$ case.

Lemma 9.36. *Let $M\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ be a Möbius transformation that preserves D and is not the identity, let $y \in \mathbf{P}^1 \setminus D$ and denote by F the functor*

$$F = T_\infty T_{M(\infty)}^{-1} \circ M^*\colon\ \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{d,\mathrm{r}}.$$

Choose isomorphisms

$$\Phi\colon F(T_y^{(1:1)}\tilde{\mathcal{E}}) \xrightarrow{\sim} T_y^{(1:1)}\tilde{\mathcal{E}}$$

and

$$\Psi\colon F(k_y) \xrightarrow{\sim} k_x.$$

The map of modifications

$$\{\mathcal{E}^\bullet \hookrightarrow T_y^{(1:1)}\tilde{\mathcal{E}} \twoheadrightarrow k_y\} \to \{\mathcal{E}^\bullet \hookrightarrow T_y^{(1:1)}\tilde{\mathcal{E}} \twoheadrightarrow k_x\}$$

that sends the short exact sequence

$$0 \to \mathcal{E}^\bullet \xrightarrow{i} T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{q} k_y \to 0$$

to

$$0 \to F\mathcal{E}^\bullet \xrightarrow{\Phi\circ F(i)} T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\Psi\circ F(q)\circ\Phi^{-1}} k_x \to 0$$

is a bijection.

Proof. Because F is exact, this is well-defined. The bijectivity follows from the fact that F^2 is naturally isomorphic to the identity. $\square$

Define

$$\phi\colon \mathbf{P}((T_y^{(1:1)}\tilde{\mathcal{E}})^0|_y) \to \mathrm{Bun}_{2,4}^{0,\mathrm{r}}, \qquad \ell \mapsto T_y^\ell T_y^{(1:1)}\tilde{\mathcal{E}}.$$

The above lemma allows us in principle to determine an explicit formula for the map

$$\pi_0 \circ \phi\colon \mathbf{P}^1 \to \mathbf{P}^1$$

using our explicit formula for the case $x \neq y$ (Lemma 9.31) and an explicit choice of Φ (such as Equation (9.3.26)). However, it is easier to determine an explicit formula for $\pi_0 \circ \phi$ by calculating the preimage of $\pi_0^{-1}(D)$ under ϕ, as we do in the following lemma, and using the fact that it is a degree 2 map.

Lemma 9.37. *Let $\pi_y \in \mathcal{O}|_y$ be a uniformizer.*

(1) We have an isomorphism

$$T_y^{(\pi_y:0)} T_y^{(1:1)}\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}(-1).$$

(2) Let $p_1, p_2 \in D$ with $p_1 \neq p_2$ and let $\sigma = \sigma_{\{p_1,p_2\}}\colon \mathcal{O} \to \mathcal{O}(2)$ be the unique global section of $\mathcal{O}(2)$ with

$$\sigma(p_1) = \sigma(p_2) = 0 \quad \textit{and} \quad \sigma(y) = 1 \in \mathcal{O}|_y.$$

Then σ defines a flag

$$(\sigma|_y : 1) \in \mathbf{P}^1((T_y^{(1:1)}\tilde{\mathcal{E}})^0|_y)$$

and we have

$$T_y^{(\sigma:1)} T_y^{(1:1)}\tilde{\mathcal{E}} \cong \begin{cases} T_{p_1}T_{p_2}\hat{\mathcal{E}} & \textit{if } \sigma|_{2y} \neq \sigma_{D\setminus\{p_1,p_2\}}|_{2y} \\ T_{p_1}T_{p_2}\tilde{\mathcal{E}} & \textit{if } \sigma|_{2y} = \sigma_{D\setminus\{p_1,p_2\}}|_{2y} \end{cases}$$

(3) All other length 1 lower modifications of $T_y^{(1:1)}\hat{\mathcal{E}}$ *lie in* $\mathrm{Bun}_{2,4}^{0;\mathrm{r}} \setminus \pi_0^{-1}(D)$.

PROOF. The first isomorphism follows directly from the definitions.

Let $p_1, p_2 \in D$ with $p_1 \neq p_2$ and σ be as in the statement. This defines an isomorphism

$$\begin{pmatrix} 1 & -\sigma \\ 0 & 1 \end{pmatrix} : \tilde{\mathcal{E}} \xrightarrow{\sim} (\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in D})$$

where ℓ_p is the image of $\mathcal{O}|_p$. Since σ was defined to have zeroes at p_1 and p_2, we have $\ell_{p_1} = \mathcal{O}|_{p_1}$ and $\ell_{p_2} = \mathcal{O}|_{p_2}$. In addition, $(\sigma|_{2y}, 1) \in (\mathcal{O}(2) \oplus \mathcal{O})|_{2y}$ is mapped to $(0, 1)$, so that we get an isomorphism

$$\begin{pmatrix} 1 & -\sigma \\ 0 & 1 \end{pmatrix} : T_y^{(\sigma|_y:1)} T_y^{(1:1)} \tilde{\mathcal{E}} \xrightarrow{\sim} T_y^{(0:1)} T_y^{(0:1)} (\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in D}).$$

This last bundle is isomorphic to

$$T_y^{(0:1)} T_y^{(0:1)} (\mathcal{O}(2) \oplus \mathcal{O}, (\ell_p)_{p\in D}) = (\mathcal{O}(2-2y) \oplus \mathcal{O}, (\ell_p)_{p\in D}) \cong (\mathcal{O} \oplus \mathcal{O}, (\ell'_p)_{p\in D})$$

Because $\ell'_{p_1}, \ell'|_{p_2}$ both come from the second summand, this bundle is isomorphic to either $T_{p_1}T_{p_2}\tilde{\mathcal{E}}$ or $T_{p_1}T_{p_2}\hat{\mathcal{E}}$. We can now either calculate that it is $T_{p_1}T_{p_2}\hat{\mathcal{E}}$ by considering the flags at the other points or conclude this from symmetry consideration (Lemma 9.36) and our knowledge of the case $x \neq y$. This relationship with the $x \neq y$ case also allows us to conclude that these are in fact all modifications. $\square$

10. The eigenfunction is given by E

Proposition 10.1. *Let* $f\colon \mathrm{Bun}_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$ *be a cusp form that is a Hecke eigenform. For* $y \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$*, we denote by* λ_y *the eigenvalue of the Hecke operator* $\mathbb{H}_y$*; for* $x \in \mathbf{P}^1 \setminus D$*, we denote by* λ_x *the eigenvalue of* $\mathbb{H}^{\mathrm{r}}_x$ *and by* λ'_x *the eigenvalue of* $\mathbb{H}^{\mathrm{l}}_x$*. After scaling* f *such that* $f(\tilde{\mathcal{E}}) = q - 1$*, we have for every* $y \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$

$$f(T_y^{(1:1)}\tilde{\mathcal{E}}) = \lambda_y$$

and for every $x \in D$

$$\lambda_x = \lambda'_x = f(T_x^{(1:1)}\tilde{\mathcal{E}}) = f({}^{(1:1)}T_x\tilde{\mathcal{E}}).$$

PROOF. Let $y \in \mathbf{P}^1(\mathbf{F}_q) \setminus D$. By definition, we have

$$(\mathbb{H}_y f)(\tilde{\mathcal{E}}) = f(T_y^{(1:0)}\tilde{\mathcal{E}}) + \sum_{r\in\mathbf{F}_q} f(T_y^{(r:1)}\tilde{\mathcal{E}}). \tag{10.0.1}$$

We have also seen that cusp forms vanish at $T_y^{(1:0)}\tilde{\mathcal{E}}$ and $T_y^{(0:1)}\tilde{\mathcal{E}}$ (Lemma 6.17). The remaining terms $f(T_y^{(r:1)}\tilde{\mathcal{E}})$ with $r \in \mathbf{F}_q^*$ are all equal to $f(T_y^{(1:1)}\tilde{\mathcal{E}})$, because scaling the summand $\mathcal{O}(2)$ in $\tilde{\mathcal{E}}$ by r^{-1} gives an isomorphism $T_y^{(r:1)}\tilde{\mathcal{E}} \to T_y^{(1:1)}\tilde{\mathcal{E}}$. Therefore we get

$$\lambda_y f(\tilde{\mathcal{E}}) = (q-1) f(T_y^{(1:1)}\tilde{\mathcal{E}})$$

which completes the proof of the first part of the proposition.

Let $x \in D$. By definition, we have

$$(\mathbb{H}^{\mathrm{r}}_x f)(\tilde{\mathcal{E}}) = f(T_x^{(1:0)}\tilde{\mathcal{E}}) + \sum_{r \in \mathbf{F}_q^*} f(T_x^{(r:1)})$$

which is the same as Equation (10.0.1), but summing over $r \in \mathbf{F}_q^*$ instead of $r \in \mathbf{F}_q$. We can therefore reason in the same way to conclude $\lambda_x = f(T_x^{(1:1)}\tilde{\mathcal{E}})$. Because f is a cusp form, $f(T_x^{(1:1)}\tilde{\mathcal{E}}) = f({}^{(1:1)}T_x\tilde{\mathcal{E}})$ (Theorem 6.4).

For $\mathbb{H}^{\mathrm{l}}_x$, the definitions tell us that for any choice of uniformizer $\pi_x \in \mathcal{O}(2-[x])|_x$, we have

$$(\mathbb{H}^{\mathrm{l}}_x f)(\tilde{\mathcal{E}}) = \sum_{r \in \mathbf{F}_q} f({}^{(r\pi_x:1)}T_x\tilde{\mathcal{E}}).$$

The modification ${}^{(0:1)}T_x\tilde{\mathcal{E}}$ does not lie in the relevant locus and all other modifications are isomorphic to ${}^{(\pi_x:1)}T_x\tilde{\mathcal{E}}$, which again follows by scaling one of the parabolic direct summands of $\tilde{\mathcal{E}}$, so that in the same way as before, we have proven $\lambda'_x = f({}^{(1:1)}T_x\tilde{\mathcal{E}})$. □

11. Explicit description of the Hecke operators

Here we define a basis of the cusp forms on $\mathrm{Bun}^d_{2,4}$ for every $d \in \mathbf{Z}$ and use it to explicitly describe the action of the local Hecke operators on the cusp forms.

11.1. Basis of the cusp forms. We first define a basis $\{F_z\}_{z\in\mathbf{F}_q}$ of cusp forms on $\mathrm{Bun}^1_{2,4}$. It is immediate from our characterization of the cusp forms on $\mathrm{Bun}^1_{2,4}(\mathbf{F}_q)$ (Theorem 6.4) that this is indeed a basis.

Definition 11.1. Let $z \in \mathbf{F}_q$. We denote by

$$F_z\colon \mathrm{Bun}^1_{2,4}(\mathbf{F}_q) \to \mathbf{Q}_\ell$$

the unique cusp form satisfying for all $y \in \mathbf{F}_q$

$$F_z(T_y^{(1:1)}\tilde{\mathcal{E}}) = \begin{cases} 1 & \text{if } y = z \\ 0 & \text{otherwise} \end{cases}.$$

Remark 11.2. Let $z \in \mathbf{F}_q$. The cusp form F_z satisfies the following properties.

(1) F_z vanishes outside of the relevant locus $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}(\mathbf{F}_q) \subset \mathrm{Bun}^1_{2,4}(\mathbf{F}_q)$ (part (1) of Theorem 6.4).

(2) Because $\sum_{y\in\mathbf{P}^1} F_z(T_y^{(1:1)}\tilde{\mathcal{E}}) = 0$ (part (2.3) of Theorem 6.4), we have

$$F_z(T_\infty^{(1:1)}\tilde{\mathcal{E}}) = -1.$$

(3) For all $x \in D$, we have

$$F_z(T_x^{(1:1)}\tilde{\mathcal{E}}) = F_z({}^{(1:1)}T_x\tilde{\mathcal{E}}) = -\frac{1}{q-1}F_z(T_x\tilde{\mathcal{E}})$$

(parts (2.1) and (2.2) of Theorem 6.4).

(4) The support of F_z is $\{\pi_1^{-1}(z), \pi_1^{-1}(\infty)\}$.

Definition 11.3. Let $d \in \mathbf{Z}$ and $z \in \mathbf{F}_q$. We denote by F_z^d the composition

$$F_z^d := F_z \circ T_\infty^{d-1} \colon \operatorname{Bun}_{2,4}^d(\mathbf{F}_q) \to \mathbf{Q}_\ell.$$

Remark 11.4. For every $d \in \mathbf{Z}$, the F_z^d with $z \in \mathbf{F}_q$ form a basis of the cusp forms on $\operatorname{Bun}_{2,4}^d(\mathbf{F}_q)$.

11.2. Matrix coefficients of the Hecke operators. Let $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(\mathbf{F}_q)$. This defines a local Hecke operator $\mathbb{H}_{\mathcal{T}^\bullet}$ (Definition 4.4). Let $y \in \mathbf{F}_q$. There are coefficients $\alpha_{z,y}^{\mathcal{T}^\bullet} \in \mathbf{Q}$ indexed by $y \in \mathbf{F}_q$ such that

$$\mathbb{H}_{\mathcal{T}^\bullet} F_z^0 = \sum_{y \in \mathbf{F}_q} \alpha_{z,y}^{\mathcal{T}^\bullet} F_y,$$

i.e., these are the matrix coefficients corresponding to $\mathbb{H}_{\mathcal{T}^\bullet}$ with respect to the bases $\{F_z\}_{z \in \mathbf{F}_q}, \{F_z^0\}_{\mathbf{F}_q}$. In this section, we give a formula for these matrix coefficients. This uses our determination of the length 1 lower modifications of all points in the relevant locus. In particular, the formula in Lemma 9.31 describing the length 1 lower modifications of a point $T_y^{(1:1)} \tilde{\mathcal{E}}$ with $y \in \mathbf{P}^1 \setminus D$ at a different point $x \in \mathbf{P}^1 \setminus D$ is used to describe the generic case. In the non-generic cases, i.e., the cases when $x \in D$ or $y \in D$ holds, there is an easier explicit description.

In Proposition 11.5, we state the result using many case distinctions, with each case corresponding to one of the maps in as described in Section 9.1.5, where we describe the maps $\ell \mapsto T_x^\ell T_y^{(1:1)} \tilde{\mathcal{E}}$ for different values of $x, y \in \mathbf{P}^1$. This allows for quite an easy proof, since we are only counting inverse images of those maps. Then in Lemma 11.8 we show that the description of the non-generic cases mostly applies to the cases with x or y in D as well — we only need to introduce a few correction terms. This then results in Theorem 1.2 from the introduction.

We start with the case distinctions. For the definition of the parabolic torsion sheaves $k_x^{(1,0)}$ and $k_x^{(0,1)}$ in $\mathbf{Coh}_0^{1,1}$, see Section 2.4.1. For $x \in D$, $M_x \colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ denotes the Möbius transformation that preserves D and sends ∞ to x (see Section 8.3).

Proposition 11.5. *Let $z, y \in \mathbf{F}_q$ and let $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(\mathbf{F}_q)$. We write $\{x\} = \operatorname{Supp} \mathcal{T}^\bullet$.*

(1) Suppose $x \notin D$, which implies $\mathcal{T}^\bullet \cong k_x$.

(a) Suppose $y \notin D$ and $y \neq x$. Then

$$\alpha^{k_x}_{z,y} = \begin{cases} 1-q & \text{if } z \in D \text{ and } y = M_z(x) \\ \#\left\{ r \in \mathbf{F}_q^* : z = \frac{(yr-x)((y-1)(y-t)r-(x-1)(x-t))}{-(x-y)^2 r} \right\} & \text{otherwise} \end{cases}$$
$$-\,2$$

(b) Suppose $y = x$. Then

$$\alpha^{k_y}_{z,y} = \#\left\{ r \in \mathbf{F}_q : z = -\left(r - \frac{1}{y}\right)\left(r - \frac{1}{y-1} - \frac{1}{y-t}\right) \right\} - (1-q)$$

(c) Suppose $y \in D \setminus \{\infty\}$. Then

$$\alpha^{k_x}_{z,y} = \begin{cases} -1 & \text{if } z = M_y(x) \\ 0 & \text{otherwise} \end{cases}$$

(2) Suppose $x \in D$ and $\mathcal{T}^\bullet \cong k_x^{(1,0)}$ or $\mathcal{T}^\bullet \cong k_x^{(0,1)}$, i.e., $\mathcal{T}^\bullet$ is not isomorphic to k_x^0. Then

$$\alpha^{\mathcal{T}^\bullet}_{z,y} = \begin{cases} 1 & \text{if } x = y \\ -1 & \text{if } z = M_x(y) \\ 0 & \text{otherwise} \end{cases}$$

(3) Suppose $\mathcal{T}^\bullet \cong k_x^0$ for $x \in D$. Then

$$\alpha^{k_x^0}_{z,y} = \begin{cases} q-1 & \text{if } z = M_x(y) \\ 0 & \text{otherwise} \end{cases}.$$

The first step in the calculation of these coefficients is the following lemma, which follows directly from the definitions.

Lemma 11.6. *Let $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}(\mathbf{F}_q)$, $z, y \in \mathbf{F}_q$. Using the notation*

$$N(\mathcal{E}^\bullet) = N_z^{\mathcal{T}^\bullet}(\mathcal{E}^\bullet) := \#\{(\mathcal{E}^\bullet \hookrightarrow T_z^{(1:1)}\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet) \in \mathcal{H}(\mathbf{F}_q)\},$$

we have

$$\begin{aligned}\alpha^{\mathcal{T}^\bullet}_{z,y} = {} & N(T_\infty T_y^{(1:1)}\tilde{\mathcal{E}}) + \underbrace{N(T_\infty({}^{(1:1)}T_y\tilde{\mathcal{E}})) + (1-q)N(T_\infty T_y\tilde{\mathcal{E}})}_{\text{if } y \in D} \\ & - \Big(N(T_\infty T_\infty^{(1:1)}\tilde{\mathcal{E}}) + N(T_\infty({}^{(1:1)}T_\infty\tilde{\mathcal{E}})) + (1-q)N(T_\infty T_\infty\tilde{\mathcal{E}}) \Big).\end{aligned}$$

Proof. First remark that for every $y \in \mathbf{F}_q$, we have by construction of the basis functions $\{F_z\}_{z \in \mathbf{F}_q}$

$$\alpha^{\mathcal{T}^\bullet}_{z,y} = (\mathbb{H}_{\mathcal{T}^\bullet} F_z^0)(T_y^{(1:1)}\tilde{\mathcal{E}}),$$

so we can find $\alpha^{\mathcal{T}^\bullet}_{z,y}$ by calculating the right hand side. Applying the definitions we obtain

$$\begin{aligned}\alpha^{\mathcal{T}^\bullet}_{z,y} &= (\mathbb{H}_{\mathcal{T}^\bullet} F^0_z)(T^{(1:1)}_y \tilde{\mathcal{E}}) \\ &= \sum_{(\mathcal{E}^\bullet \hookrightarrow T^{(1:1)}_y \tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet) \in \mathcal{H}(\mathbf{F}_q)} F^0_z(\mathcal{E}^\bullet) \\ &= \sum_{(\mathcal{E}^\bullet \hookrightarrow T^{(1:1)}_y \tilde{\mathcal{E}} \twoheadrightarrow \mathcal{T}^\bullet) \in \mathcal{H}(\mathbf{F}_q)} F_z(T^{-1}_\infty \mathcal{E}^\bullet).\end{aligned}$$

Because F_z is supported on $\{\pi_1^{-1}(z),\ \pi_1^{-1}(\infty)\}$ and we know its values at the points in the support (Remark 11.2), we get the expression stated in the lemma. □

In Section 9, we explicitly calculated all the length one modifications of points in the relevant locus. The proof of the proposition is therefore nothing more than a careful analysis of the results in Section 9 to obtain the integers $N(\mathcal{E}^\bullet)$ for all relevant $\mathcal{E}^\bullet$. More precisely, we will use the summary in Section 9.1.5 of the behavior of the maps

$$\ell \mapsto T^\ell_x T^{(1:1)}_y \tilde{\mathcal{E}}.$$

Proof of Proposition 11.5. We prove the proposition case by case for each of the enumerated cases. In each case, there is a relevant map

$$\phi\colon \ell \mapsto \text{modification of } T^{(1:1)}_y \tilde{\mathcal{E}} \text{ with respect to } \ell$$

where ℓ is taken from the appropriate set of flags; in Section 9.1.5, we describe these maps. We recall that a point $\mathcal{E}^\bullet$ with $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$-automorphisms is in the image of this map ϕ if and only if $\pi_0 \circ \phi$ is ramified over $\pi_0(\mathcal{E}^\bullet)$ (Lemma 9.33).

The $N(T_\infty \mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ in the support of F_z are the number of inverse images of $T_\infty \mathcal{E}^\bullet$ under this map ϕ, by definition, and these are what we need to determine.

For part (1a), we can use the formula for the map

$$\phi_{x,y}\colon \mathbf{P}((T^{(1:1)}_y \tilde{\mathcal{E}})|_x) \to \mathbf{P}^1, \qquad \ell \mapsto \pi_0(T^\ell_x T^{(1:1)}_y \tilde{\mathcal{E}})$$

given in Lemma 9.31. The flags $(1:0)$ and $(0:1)$ always map to $\infty = (1:0) \in \mathbf{P}^1$; therefore (Lemma 9.33), $N(T_\infty T^{(1:1)}_\infty \tilde{\mathcal{E}}) + N(T_\infty\, {}^{(1:1)}T_\infty \tilde{\mathcal{E}}) = 2$ and $N(T_\infty T_\infty \tilde{\mathcal{E}}) = 0$, which gives the -2 in the expression.

Suppose that $z \in D$ and $y = M_z(x)$. Then $\pi_0 \circ \phi_{x,y}$ is ramified over z and $N(T_\infty T_z \tilde{\mathcal{E}}) = 1$, while $N(T_\infty T^{(1:1)}_z \tilde{\mathcal{E}}) = N(T_\infty({}^{(1:1)}T_z \tilde{\mathcal{E}})) = 0$ (Lemma 9.33) Otherwise, the expression

$$\#\left\{ r \in \mathbf{F}^*_q : z = \frac{(yr - x)((y-1)(y-t)r - (x-1)(x-t))}{-(x-y)^2 r} \right\}$$

counts the number of inverse images of $(z:1) \in \mathbf{P}^1$ and therefore counts $N(T_\infty T^{(1:1)}_z \tilde{\mathcal{E}})$ when $z \notin D$ and $N(T_\infty T^{(1:1)}_z \tilde{\mathcal{E}}) + N({}^{(1:1)}T_z \tilde{\mathcal{E}})$ if $z \in D$; in both cases, $N(T_\infty T_z \tilde{\mathcal{E}}) = 0$. Note that we can take $r \in \mathbf{F}^*_q$, because $(1:0)$

and $(0:1)$ are mapped to ∞ (hence not to $(z:1)$), as already remarked. This gives the other term in part (1a).

Part (1b) is similar to part (1a), but uses the formula in Lemma 9.35. This map is always ramified over ∞ and not over any of the other points in D, so that (1) $N(T_z^{(1:1)}\tilde{\mathcal{E}}) + N({}^{(1:1)}T_z\tilde{\mathcal{E}})$ is always given by the formula; (2) $N(T_\infty T_\infty \tilde{\mathcal{E}}) = 1$; and (3) the other $N(\mathcal{E}^\bullet)$ are zero.

Now suppose that $\mathcal{T}^\bullet$ is not isomorphic to k_x^0 and suppose that at least one of x and y lies in D. This holds for parts (1c) and (2). We prove that in this case, the formula given in part (2) holds; the formula of part (1c) is the same as (2), but since $x \notin D$, one of the three cases never occurs.

Under these assumptions, we define

$$y_0 := \begin{cases} M_y(x) & \text{if } y \in D \\ M_x(y) & \text{if } x \in D \end{cases}$$

(if x and y are both in D, then $M_x(y) = M_y(x)$) and find that the map ϕ is either

$$\mathbf{A}^1 \xrightarrow{\sigma} \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \backslash \pi_0^{-1}(y_0)$$

or

$$\mathbb{G}_{\mathrm{m}} \xrightarrow{\tau} \pi_0^{-1}(y_0) \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$$

(Section 9.1.5).

In the first case, the composition with π_0 is a map of degree 1 and the image is never a point with $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$-automorphisms (Section 9.1.5). Therefore, for all $y' \in \mathbf{P}^1$

$$N(T_\infty T_{y'}^{(1:1)}\tilde{\mathcal{E}}) + \underbrace{N(T_\infty ({}^{(1:1)}T_y \tilde{\mathcal{E}})}_{\text{if } y' \in D} = \begin{cases} 1 & \text{if } y' \neq y_0 \\ 0 & \text{if } y' = y_0 \end{cases},$$

and for all $y' \in D$,

$$N(T_\infty T_{y'}\tilde{\mathcal{E}}) = 0.$$

This proves the formula given in (2) in the case that ϕ is the map σ.

If the map ϕ is of the form $\mathbb{G}_{\mathrm{m}} \xrightarrow{\tau} \pi_0^{-1}(y_0) \subset \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$, then

$$N(T_\infty T_{y_0}\tilde{\mathcal{E}}) = 1$$

and

$$N(T_\infty T_{y_0}^{(1:1)}\tilde{\mathcal{E}}) + N(T_\infty ({}^{(1:1)}T_{y_0}\tilde{\mathcal{E}}) = q-2,$$

while all the other $N(\mathcal{E}^\bullet)$ are zero. If $z = y_0$, this gives a contribution of $(q-2)\cdot 1 + 1 \cdot (1-q) = -1$; if $\infty = y_0$, we get minus that. We conclude that the formula in (2) also holds in this case.

To prove part (3), note that every parabolic bundle $\mathcal{E}^\bullet$ has exactly $q-1$ modifications with respect to k_x^0 for any $x \in D$, and all these modifications are isomorphic to $T_x\mathcal{E}^\bullet$. Since $T_\infty T_x^{-1}$ induces M_x on the coarse moduli space $\mathbf{P}^1$ (Lemma 8.8), the result follows. $\square$

Remark 11.7. Let $z, y \in \mathbf{F}_q$ and let $x \in D$. It follows from the proposition that $\alpha_{z,y}^{k_x^{(1,0)}} = \alpha_{z,y}^{k_x^{(0,1)}}$. (This can also easily be proven using the symmetry $T_D(2)$ and the fact that cusp forms are invariant under this symmetry.)

In light of the above remark, it makes sense to write for $z, y \in \mathbf{F}_q$ and $x \in \mathbf{P}^1$ $\alpha_{z,y}^x$ for $\alpha_{z,y}^{k_x}$ if $x \notin D$ and for $\alpha_{z,y}^{k_x^{(1,0)}} = \alpha_{z,y}^{k_x^{(0,1)}}$ if $x \in D$.

The following lemma shows that the formulas for $x, y \notin D$ mostly apply to the cases with x or y in D as well, at least after introducing a few correction terms.

Lemma 11.8. *Let $z, y \in \mathbf{F}_q$ and $x \in \mathbf{P}^1 \setminus \{\infty\}$. Suppose that x or y lies in D.*

(1) If $x \neq y$ and $x \neq \infty$, then

$$\begin{aligned}\alpha_{z,y}^x = \#&\left\{r \in \mathbf{F}_q^* : z = \frac{(yr-x)((y-1)(y-t)r-(x-1)(x-t))}{-(x-y)^2 r}\right\}\\ &+ \underbrace{\begin{cases} 0 & \text{if } x, y \in D \\ -1 & \text{otherwise}\end{cases}}_{\text{contribution of } \infty}\\ &+ \underbrace{\begin{cases} -q & \text{if } x, y \in D \text{ and } z = M_x(y) \\ 0 & \text{otherwise}\end{cases}}_{\text{correction when one modification has automorphism group } \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}}\end{aligned}$$

If $x \neq y$ and $x = \infty$, then the above is true, but to obtain the correct limit of the formula, r should first be replaced by rx^2.

(2) If $x = y$,

$$\alpha_{z,y}^y = \#\left\{r \in \mathbf{F}_q : z = -(yr-1)((y-1)(y-t)r-(2y-(1+t)))\right\}$$

Proof. There are two ways to see this. The first is to simply verify that the equations in this lemma are in accordance with Proposition 11.5, i.e., by proving that $\alpha_{z,y}^x$ is -1 if $z = M_x(y)$ (with $x \in D$) or $z = M_y(x)$ (with $y \in D$), 1 if $x = y$ and 0 otherwise.

We write

$$\begin{aligned} f(r) &= \frac{(yr-x)((y-1)(y-t)r-(x-1)(x-t))}{-(x-y)^2 r} \qquad \text{and}\\ g(r) &= -(yr-1)((y-1)(y-t)r-(2y-(1+t)))\end{aligned}$$

for the large formulas appearing in the lemma. In essence, we only need to count the number of solutions to $\#\{r \in \mathbf{F}^* : z = f(r)\}$ and $\#\{r \in \mathbf{F}_q : z = g(r)\}$ when x or y lies in D. The following two tables show the formulas for f when x or y lies in D. The column labeled "missing point" labels the point in $\mathbf{P}^1$ that is not in the image of $\ell \mapsto \pi_0(T_x^\ell T_y^{(1:1)}\tilde{\mathcal{E}})$.

	formula	missing point $(r = 0)$
$x = 0$	$-\frac{((y-1)(y-t)r-t)}{y}$	$\frac{t}{y} = M_0(y)$
$x = 1$	$-\frac{(yr-1)(y-t)}{y-1}$	$\frac{y-t}{y-1} = M_1(y)$
$x = t$	$-\frac{(yr-t)(y-1)}{y-t}$	$\frac{t(y-1)}{y-t} = M_t(y)$
$x = \infty$	$y - ry(y-1)(y-t)$	$y = M_\infty(y)$

These formulas are obtained by simply substituting the value of x (but for $x = \infty$, we have to reparametrize the flag by replacing r with rx^2, which corresponds to the fact that we are parametrizing the flag $(rX^2 : 1)$, where $X \in \mathcal{O}(1)$ is the coordinate on $\mathbf{P}^1$) and then canceling the common factor $r(x-y)$ in the numerator and denominator.

For $y \in D \setminus \{\infty\}$, we substitute the value for y and divide out $-(x-y)$ to obtain:

	formula	missing point $(r = \infty)$
$y = 0$	$\frac{tr-(x-1)(x-t)}{xr}$	$\frac{t}{x} = M_0(x)$
$y = 1$	$\frac{(r-x)(x-t)}{r(x-1)}$	$\frac{x-t}{x-1} = M_1(x)$
$y = t$	$\frac{(tr-x)(x-1)}{r(x-t)}$	$\frac{t(x-1)}{x-t} = M_t(x)$

Suppose $x \in D$ and $y \notin D$. Then $f(r)$ is linear in r (and non-constant), so that for every $z \in \mathbf{F}_q$ except $z_0 = f(0)$, we have $\#\{r \in \mathbf{F}_q^* : z = f(r)\} = 1$; for z_0, this quantity is zero. We also know that $N(T_\infty T_\infty^{(1:1)}\tilde{\mathcal{E}}) + N(T_\infty(^{(1:1)}T_\infty)\tilde{\mathcal{E}}) = 1$, Because $z_0 := f(0)$ is in fact equal to $M_x(y)$, the result from Proposition 11.5. agrees with this formula.

Now suppose $x \in D$, $y \in D$, but $x \neq y$. Then $f(r)$ is in fact constant, so that $\#\{r \in \mathbf{F}_q^* : z = f(r)\}$ is $q-1$ for $z = f(1)$ and zero for all other values of $z \in \mathbf{F}_q$. In this case we have $f(1) = f(0) = M_x(y)$. The formula is also correct in this case.

Thirdly, suppose $y \in D$, $x \notin D$. Then $f(r)$ is linear in r and non-constant, so that $\{r \in \mathbf{F}_q^* : z = f(r)\}$ is empty when $z = f(\infty) = M_y(x)$ and a singleton otherwise.

Lastly, suppose $x = y \in D$, in which case the formula for $\alpha_{z,y}^y$ should always be equal to 1. In this case, $g(r)$ is linear (not constant), so $\#\{r \in \mathbf{F}_q : z = g(r)\}$ is 1 for every $z \in \mathbf{F}_q$.

Another way to prove this lemma would be to note that the degenerations of f and g to these cases correspond to the maps

$$\mathbf{P}((T_y^{(1:1)}\tilde{\mathcal{E}})|_x) \to \mathbf{P}^1, \qquad \ell \mapsto T_x^\ell T_y^{(1:1)}\tilde{\mathcal{E}}$$

and reason as in the proof of the proposition. □

Together, Proposition 11.5 and Lemma 11.8 prove Theorem 1.2 from the introduction. We also use the fact that for $x, y, z \in D$, the condition $z = M_x(y)$ is equivalent to $x = M_z(y)$ (Lemma 8.6).

CHAPTER 3

The Hecke eigensheaf

As before, let $D = \{\infty, 0, 1, t\} \subset \mathbf{P}^1(\mathbf{F}_q)$ be four distinct points and let E be an irreducible pure rank 2 local system on $\mathbf{P}^1 \setminus D$ with unipotent monodromy. In Section 10 we proved that the trace-of-Frobenius function defined by the intermediate extension of E to $\mathbf{P}^1$ is the Hecke eigenfunction for E. In this chapter, we prove a geometric upgrade: we define a perverse sheaf Aut_E on $\mathrm{Bun}_{2,4}$ whose trace-of-Frobenius function is the Hecke eigenfunction defined before and we prove that this perverse sheaf is the Hecke eigensheaf for E.

The construction and first properties of Aut_E are the goal of Section 12. In Section 13, we prove it is in fact a Hecke eigensheaf for E.

12. Definition and first properties of the eigensheaf

In this section, we construct a perverse sheaf Aut_E on $\mathrm{Bun}_{2,4}$ (Definition 12.8). associated to a pure irreducible rank 2 local system E on $\mathbf{P}^1 \setminus D$ with unipotent monodromy. In Section 13 we will prove that Aut_E is in fact the Hecke eigensheaf for E (Theorem 13.1). The main results of this section is the following theorem about its restriction $\mathrm{Aut}_E^d := \mathrm{Aut}_E |_{\mathrm{Bun}_{2,4}^d}$ to degree d. We denote by $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1}$ the natural inclusion and recall that we have constructed a canonical isomorphism $\alpha\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ (Definition 7.5). We write

$$\mathcal{L}_E := j_{!*}E$$

for the sheaf on $\overline{\mathbf{Coh}}_0^{1,1}$. There is a canonical isomorphism $\mathbf{Coh}_0^{1,1} \cong \overline{\mathbf{Coh}}_0^{1,1} \times \mathbf{B}\mathbb{G}_{\mathrm{m}}$. Because $\mathbb{G}_{\mathrm{m}}$ is connected, and ℓ-adic vector spaces are totally disconnected, every $\mathbb{G}_{\mathrm{m}}$-action on an ℓ-adic sheaf is trivial. We can therefore identify the derived category of ℓ-adic sheaves on $\mathbf{Coh}_0^{1,1}$ with the derived category of ℓ-adic sheaves on $\overline{\mathbf{Coh}}_0^{1,1}$ and will accordingly also denote by $\mathcal{L}_E$ the sheaf on $\mathbf{Coh}_0^{1,1}$. By $E|_\infty$ we denote the constant local system that is the pullback of the fiber $(j_{!*}E)|_{k_\infty^{(1,0)}}$. The complex Aut_E defined in Definition 12.8 then has the following properties and alternative description.

THEOREM 12.1. *Let $d \in \mathbf{Z}$. Aut_E^d is an irreducible perverse sheaf that is supported on $\mathrm{Bun}_{2,4}^{d,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^d$ and*

$$\mathrm{Aut}_E^d |_{\mathrm{Bun}_{2,4}^{d,\mathrm{r}}} = (T_\infty^{d-1})^* \left(\alpha_*(\mathcal{L}_E[1])\right) \otimes E|_\infty^{\otimes -d+1}.$$

This is proven in Corollary 12.11 and Corollary 12.12.

We start in Section 12.1 by describing some essential properties of the local system E, which correspond to the cusp conditions for the trace-of-Frobenius functions described in Theorem 6.4. In Section 12.2, we construct Aut_E, Roughly speaking, we consider $j_{!*}E$ as a sheaf on $\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \subset \mathrm{Bun}^2_{2,4,} \times \overline{\mathbf{Coh}}_0^{1,1}$ and then use the correspondence

$$\mathrm{Bun}^2_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} \xleftarrow{\;p\;} \mathcal{H} \xrightarrow{\;q\;} \mathrm{Bun}^1_{2,4}$$

to transport this to $\mathrm{Bun}^1_{2,4}$. The result is Aut^1_E, and we define Aut^d_E via the isomorphism $T_\infty^{d-1}\colon \mathrm{Bun}^1_{2,4} \xrightarrow{\sim} \mathrm{Bun}^d_{2,4}$.

The rest of Section 12 is concerned with the proof of Theorem 12.1. A rough sketch of the proof is the following. The main idea is that the construction of Aut^1_E is very closed tied to the construction of the isomorphism $\alpha\colon \mathbf{Coh}_0^{1,1} \xrightarrow{\sim} \mathrm{Bun}^{1,\mathrm{r}}_{2,4}$. Indeed, $\alpha = (p^{\mathrm{rel}})^{-1} \circ q^{\mathrm{rel}}$, where p^{rel} and q^{rel} are certain restrictions of p and q, respectively; these are restrictions to all modifications of $\tilde{\mathcal{E}}$, except those that lie outside of the relevant locus. This roughly explains why $\mathrm{Aut}^1_E|_{\mathrm{Bun}^{1,\mathrm{r}}_{2,4}}$ is $\alpha_*(j_{!*}E)$; a more careful explanation is given in Section 12.3. To show that Aut^1_E is supported on $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}$, we calculate in Section 12.4 that all the modifications that we excluded in our definition of p^{rel} and q^{rel}, do not in fact contribute anything to $\mathbf{R}q_!\mathbf{R}p_*\,\mathrm{Aut}^1_E$. This uses our explicit determination of the closure of $\tilde{\mathcal{E}}$ in $\mathrm{Bun}^2_{2,4}$ (Section 12.5). To show the perversity, we apply the decomposition theorem to q.

12.1. Properties of the local system E. Denote by $\bar{j}\colon \mathbf{P}^1 \setminus D \hookrightarrow \mathbf{P}^1$ the inclusion and by $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1}$ the inclusion defined in Section 2.4.2. In this section, we study (the cohomological properties of) the intermediate extensions $\bar{j}_{!*}E[1]$ and $j_{!*}E[1]$. Since $\bar{j}$ is an open embedding into a curve, we have $\bar{j}_{!*} = \bar{j}_*$.

Lemma 12.2. *The Euler-characteristic of $\bar{j}_{!*}E$ is zero.*

Proof. The Grothendieck-Ogg-Shafarevich formula ([**Gro77**, formula 7.2], or [**KR14**, theorem 9.1]) says

$$\chi_{\mathrm{c}}(\mathbf{P}^1_{\bar{k}} \setminus D, E) = \mathrm{rk}(E) \cdot \chi_{\mathrm{c}}(\mathbf{P}^1_{\bar{k}} \setminus D, \mathbf{Q}_\ell) - \sum_{x \in D} \mathrm{Sw}_x(E),$$

where χ_{c} is the alternating sum of the dimension of the cohomology groups with compact support and $\mathrm{Sw}_x(E)$ is the Swan conductor, which is zero because E is tamely ramified. Note that

$$\chi_{\mathrm{c}}(\mathbf{P}^1_{\bar{k}} \setminus D, E) = \chi(\mathbf{R}\bar{j}_!E).$$

Applying the additivity of the Euler characteristic to the distinguished triangle

$$\mathbf{R}\bar{j}_!E \to \bar{j}_{!*}E \to (\bar{j}_{!*}E)|_D \xrightarrow{+1},$$

we find

$$\begin{aligned}\chi(\mathbf{P}^1, \bar{j}_{!*}E) &= \chi(\mathbf{R}\bar{j}_! E) + \chi((\bar{j}_{!*}E)|_D) \\ &= \mathrm{rk}(E) \cdot \chi_{\mathrm{c}}(\mathbf{P}^1_{\bar{k}} \setminus D, \mathbf{Q}_\ell) + \chi((\bar{j}_{!*}E)|_D) \\ &= 2 \cdot (-2) + 4 = 0.\end{aligned}$$

□

Proposition 12.3. *All cohomology groups* $\mathrm{H}^i(\mathbf{P}^1, \bar{j}_{!*}E[1])$ *are zero.*

PROOF. Because E is irreducible, it has no global sections and thus we find $\mathrm{H}^{-1}(\mathbf{P}^1, \bar{j}_{!*}E[1]) = \mathrm{H}^0(\mathbf{P}^1 \setminus D, E) = 0$. The Verdier dual $\mathbb{D}\,E$ of E is also irreducible, and by applying the same reasoning as above to $\mathbb{D}\,E$, we conclude that the cohomology in degree 1 is also zero: letting $p\colon \mathbf{P}^1 \to \operatorname{Spec} k$ denote the map to the point, we calculate

$$\begin{aligned}\mathrm{H}^1(\mathbf{P}^1, \bar{j}_{!*}E[1]) &= \mathrm{H}^1\mathbf{R}p_*\bar{j}_{!*}E[1] \\ &= (\mathrm{H}^{-1}\,\mathbb{D}\,\mathbf{R}p_*\bar{j}_{!*}E[1])^\vee \\ &= (\mathrm{H}^{-1}\mathbf{R}p_*\bar{j}_{!*}(\mathbb{D}\,E)[1])^\vee = 0.\end{aligned}$$

Finally, because the Euler characteristic of $\bar{j}_{!*}E[1]$ is zero by Lemma 12.2 and the cohomology in degrees smaller than -1 and bigger than 1 vanishes, we find that all cohomology groups are zero. □

Let $x \in D$. Recall from Section 2.4.1 our notation $k_x^{(1,0)}$, $k_x^{(0,1)}$, and k_x^0 for three parabolic torsion sheaves in $\mathbf{Coh}_0^{1,1}$ representing the three distinct isomorphism classes of sheaves in $\mathbf{Coh}_0^{1,1}$ supported on x.

Lemma 12.4. *Let* $x \in D$.

(1) *The stalks of* $j_{!*}E[1]$ *at the torsion sheaves supported at* x *are given by*

$$(j_{!*}E[1])|_{k_x^{(1,0)}} = (j_{!*}E[1])|_{k_x^{(0,1)}} = (\bar{j}_*E[1])|_x$$

and

$$(j_{!*}E[1])|_{k_x^0} = (\bar{j}_*E[1])|_x \otimes \mathrm{H}^*(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell).$$

(2) *Let* $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ *be one of the two parabolic length 1 torsion sheaves with* $\mathbb{G}_{\mathrm{m}}$*-automorphisms that are supported on* $x \in D$. *Let*

$$\mathcal{P}_x := \{k_x^0, \mathcal{T}^\bullet\} \subset \overline{\mathbf{Coh}}_0^{1,1}$$

the substack that contains k_x^0 *and* $\mathcal{T}^\bullet$. *Then*

$$\mathrm{H}^*_{\mathrm{c}}(\mathcal{P}_x, (j_{!*}E)|_{\mathcal{P}_x}) = 0. \tag{12.1.1}$$

PROOF. This follows from [**Hei04**, corollary 4.5] (in particular the first few sentences of the proof). □

Remark 12.5. Let $x \in D$. Consider the map

$$k_x^{(\,-\,,0)} \colon \mathbf{A}^1 \to \overline{\mathbf{Coh}}_0^{1,1}, \qquad \lambda \mapsto (k_x \xrightarrow{\lambda} k_x \xrightarrow{0} k_x)$$

whose image in $\overline{\mathbf{Coh}}_0^{1,1}$ contains exactly the points $k_x^{(1,0)}$ and k_x^0, and also consider the analogous map

$$k_x^{(0,\,-\,)} \colon \mathbf{A}^1 \to \overline{\mathbf{Coh}}_0^{1,1}, \qquad \lambda \mapsto (k_x \xrightarrow{0} k_x \xrightarrow{\lambda} k_x).$$

The statements in Lemma 12.4 follow from the more explicit formula

$$(k_x^{(\,-\,,0)})^*(j_{!*}E) = (k_x^{(0,\,-\,)})^*(j_{!*}E) = (\bar{j}_{!*}E)|_x \otimes \mathbf{R}(\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbf{A}^1)_*\mathbf{Q}_\ell.$$

For part 2, this uses that the compact cohomology of $\mathbf{R}(\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbf{A}^1)_*\mathbf{Q}_\ell$ vanishes.

Remark 12.6. Let $s \colon \mathbf{P}^1 \to \overline{\mathbf{Coh}}_0^{1,1}$ be one of the sections of $\mathrm{Supp} \colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$ discussed in Section 2.4.2. Because $s^*(j_{!*}E[1]) = \bar{j}_{!*}E[1]$, we can apply the previous results in this subsection to understand $j_{!*}E[1]$.

Remark 12.7. Under the function-sheaf dictionary, Proposition 12.3 and Lemma 12.4 translate to the cusp conditions on functions $f \colon \mathrm{Bun}_{2,4}^1(\mathbf{F}_q) \to \mathbf{Q}_\ell$ defined in in Theorem 6.4. More precisely, if the results from those corollaries hold for a perverse sheaf F (playing the role of $j_{!*}E[1]$), then the associated trace-of-Frobenius function satisfies the cusp conditions.

12.2. Definition of Aut_E**.** As explained in Section 2.4.2, we have a map

$$j \colon \mathbf{P}^1 \setminus D \xrightarrow{\sim} \overline{\mathbf{Coh}}_0^{1,1} \setminus \bar{D} \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1}.$$

We use this map to define

$$\mathcal{L}_E := j_{!*}E.$$

We introduce the notation $\mathcal{L}_E^\alpha$ and $\tilde{\mathcal{L}}_E$ for two pullbacks of $\mathcal{L}_E$:

$$\tilde{\mathcal{L}}_E := (\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\mathrm{pr}} \overline{\mathbf{Coh}}_0^{1,1})^*\mathcal{L}_E, \quad \text{and}$$
$$\mathcal{L}_E^\alpha := (\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\alpha} \mathbf{Coh}_0^{1,1} \to \overline{\mathbf{Coh}}_0^{1,1})^*\mathcal{L}_E.$$

We recall that we have maps

$$\mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1} \xleftarrow{\;p\;} \mathcal{H} \xrightarrow{\;q\;} \mathrm{Bun}_{2,4}^1.$$

The point $\tilde{\mathcal{E}} \in \mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ defines a residual gerbe $\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \subset \mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ and we denote by

$$i \colon \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \hookrightarrow \mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1}$$

the inclusion.

Definition 12.8. We define the object Aut_E in the bounded derived category of constructible ℓ-adic sheaves on $\mathrm{Bun}_{2,4}$ by defining all its restrictions $\mathrm{Aut}_E^d := \mathrm{Aut}_E|_{\mathrm{Bun}_{2,4}^d}$ ($d \in \mathbf{Z}$) to the connected components of $\mathrm{Bun}_{2,4}$ indexed by the degree. We define

$$\mathrm{Aut}_E^1 := \mathbf{R}q_! p^* i_{!*}\tilde{\mathcal{L}}_E$$

and for $d \in \mathbf{Z}$,

$$\mathrm{Aut}_E^d := (T_\infty^{d-1})^* \mathrm{Aut}_E^1 \otimes E|_\infty^{\otimes -d+1}$$

where $T_\infty^{d-1} \colon \mathrm{Bun}_{2,4}^d \xrightarrow{\sim} \mathrm{Bun}_{2,4}^1$ denotes the $d-1$-fold application of the elementary Hecke operator T_∞ and $E|_\infty$ denotes the fiber $(j_{!*}E)|_{k_\infty^{(1,0)}}$ at one of the parabolic torsion sheaves with $\mathbb{G}_\mathrm{m}$-automorphisms supported at ∞.

Remark 12.9. Recall from Section 2.4.1 that for $x \in D$, k_x^0 denotes the parabolic torsion sheaf supported at x with parabolic structure given by the chain $\ldots \xrightarrow{0} k_x \xrightarrow{0} k_x \xrightarrow{0} k_x \xrightarrow{0} \ldots$. The elementary modification T_x coincides with modification with respect to k_x^0. The tensor product with $\mathcal{L}_E^\vee|_{k_\infty^0}$ ensures that the Hecke operators at ∞ act with the correct eigenvalue.

12.3. Perversity and support. This subsection contains the two main results of Section 12, Corollary 12.11 and Corollary 12.12, which together prove Theorem 12.1.

The main ingredient of the proof is the following proposition. Recall that we have defined Aut_E^1 as the sheaf obtained by applying $\mathbf{R}q_! p^*$ to $i_{!*}\tilde{\mathcal{L}}_E$, where $i \colon \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}_0}^{1,1} \hookrightarrow \mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}_0}^{1,1}$ is the natural inclusion. Denote by $\mathbf{A}$ the functor

$$\mathbf{A} := \mathbf{R}q_! p^* \colon D^\mathrm{b}(\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}_0}^{1,1}, \mathbf{Q}_\ell) \to D^\mathrm{b}(\mathrm{Bun}_{2,4}, \mathbf{Q}_\ell).$$

Proposition 12.10. *The natural map $\mathbf{R}i_! \to \mathbf{R}i_*$ induces an isomorphism*

$$\mathbf{A}\mathbf{R}i_!\tilde{\mathcal{L}}_E \xrightarrow{\sim} \mathbf{A}\mathbf{R}i_*\tilde{\mathcal{L}}_E$$

and hence we get isomorphisms

$$\mathbf{A}\mathbf{R}i_!\tilde{\mathcal{L}}_E \xrightarrow{\sim} \mathbf{A}i_{!*}\tilde{\mathcal{L}}_E = \mathrm{Aut}_E^1 \xrightarrow{\sim} \mathbf{A}\mathbf{R}i_*\tilde{\mathcal{L}}_E.$$

We prove this result at the end of this section, using some auxiliary results that we prove in the next two sections. We start, however, with the corollaries.

Corollary 12.11. *The sheaf Aut_E^1 is supported on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ and $\mathrm{Aut}_E^1|_{\mathrm{Bun}_{2,4}^{1,\mathrm{r}}} = \mathcal{L}_E^\alpha$. Likewise, Aut_E^0 is supported on $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ and we have $\mathrm{Aut}_E^0|_{\mathrm{Bun}_{2,4}^{0,\mathrm{r}}} = T_{\infty,*}\mathcal{L}_E^\alpha \otimes E|_\infty$.*

Proof. In light of the isomorphism $\mathrm{Aut}_E^1 \xrightarrow{\sim} \mathbf{R}q_! p^* \mathbf{R}i_! \tilde{\mathcal{L}}_E$ (Proposition 12.10), the following proof is a quite straightforward application of the proper base change theorem and an analysis of the fibers of q.

Recall (Definition 7.1) the substack $\mathcal{H}_{\tilde{\mathcal{E}}} \subset \mathcal{H}$ defined as the pullback of p along the inclusion i. The maps relevant to this situation are captured in the following commutative diagram

$$\begin{tikzcd}
\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[d, hook, "i"] \arrow[dr, phantom, "\square"] & \mathcal{H}_{\tilde{\mathcal{E}}} \arrow[l, "p_{\tilde{\mathcal{E}}}"'] \arrow[d, hook] \arrow[dr, "q_{\tilde{\mathcal{E}}}"] & \\
\mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1} & \mathcal{H} \arrow[l, "p"] \arrow[r, "q"'] & \mathrm{Bun}_{2,4}^1
\end{tikzcd}$$

where the left square is Cartesian. Applying the proper base change theorem to the left square, we find

$$\mathrm{Aut}_E^1 = \mathbf{R}q_{\tilde{\mathcal{E}},!}p_{\tilde{\mathcal{E}}}^*\tilde{\mathcal{L}}_E. \tag{12.3.1}$$

This already shows that $\mathrm{Aut}_E^1|_{\mathrm{Bun}_{2,4}^{1,\mathrm{r}}}$ is indeed $\mathcal{L}_E^\alpha$: above $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^1$, $q_{\tilde{\mathcal{E}}}$ is by definition q^{rel}, which is an isomorphism (Lemma 7.3), and it then follows directly from the construction of α and $\mathcal{L}_E^\alpha$.

Equation (12.3.1) also tells us that Aut_E^1 is supported on the image of $\mathcal{H}_{\tilde{\mathcal{E}}} \to \mathrm{Bun}_{2,4}^1$. In Section 6.2 (more specifically, Lemma 6.17 and Proposition 6.19), we determined this image: it is

$$\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \cup \{A, B\} \subset \mathrm{Bun}_{2,4}^1$$

where

$$A := (\mathcal{O}(1), \emptyset) \oplus (\mathcal{O}, D) \qquad B := (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D)$$

It therefore only remains to show $\mathrm{Aut}_E^1|_A = 0$ and $\mathrm{Aut}_E^1|_B = 0$. We prove the latter; the first is analogous. (The analogy can be made precise using an isomorphism $T_D A(2) \xrightarrow{\sim} B$.)

Using proper base change, we have

$$\mathrm{Aut}_E^1|_B = \mathrm{H}_c^*(q_{\tilde{\mathcal{E}}}^{-1}(\{B\}),\ p_{\tilde{\mathcal{E}}}^*\tilde{\mathcal{L}}_E).$$

Let

$$\beta\colon \mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1 \to q_{\tilde{\mathcal{E}}}^{-1}(\{B\})$$

denote the map that sends the S-point

$$\big(((\mathcal{O}(2), \emptyset) \boxtimes \mathcal{L}) \ \oplus\ ((\mathcal{O}, D) \boxtimes \mathcal{M}),\ s\colon S \to \mathbf{P}^1\big), \qquad \mathcal{L}, \mathcal{M} \text{ line bundles on } S$$

to

$$((((\mathcal{O}(2), \emptyset) \boxtimes \mathcal{L}) \ \oplus\ ((\mathcal{O}, D) \boxtimes \mathcal{M})) \twoheadrightarrow \Gamma_{s,*}s^*\mathcal{M}) \in \mathcal{H}_{\tilde{\mathcal{E}}}(S)$$

where

$$\Gamma_s\colon S \to \mathbf{P}^1 \times S$$

denotes the graph. Note that the kernel of this S-point is

$$(\mathcal{O}(2), \emptyset) \boxtimes \mathcal{L} \oplus ((\mathcal{O}, D) \boxtimes \mathcal{M}) \otimes \mathcal{O}_{\mathbf{P}^1 \times S}(-\Gamma_s),$$

so that the image does indeed lie in $q_{\tilde{\mathcal{E}}}(\{B\}) \subset \mathcal{H}_{\tilde{\mathcal{E}}}$.

This map is an isomorphism: the inverse is given by $(\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\mathcal{E}^\bullet, \operatorname{Supp} \mathcal{T}^\bullet)$. In addition, there is a section $s\colon \mathbf{P}^1 \to \overline{\mathbf{Coh}_0}^{1,1}$ of the map $\operatorname{Supp}\colon \overline{\mathbf{Coh}_0}^{1,1} \to \mathbf{P}^1$ such that β fits into the commutative diagram

$$\begin{tikzcd} q_{\tilde{\mathcal{E}}}^{-1}(\{B\}) \arrow[rr, "\sim", "\beta"'] \arrow[dr, "p_{\tilde{\mathcal{E}}}"'] & & \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1 \arrow[dl, "\mathrm{id}\times s"] \\ & \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}_0}^{1,1} & \end{tikzcd}$$

It follows that

$$\operatorname{Aut}^1_E|_B = \mathrm{H}^*_c(q_{\tilde{\mathcal{E}}}^{-1}(\{B\}),\ p_{\tilde{\mathcal{E}}}^* \tilde{\mathcal{L}}_E) = \mathrm{H}^*_c(\mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1,\ (\mathrm{id} \times s)^* \tilde{\mathcal{L}}_E).$$

It follows from the Künneth formula and our calculation $\mathrm{H}^*(\mathbf{P}^1, \bar{j}_{!*}E) = 0$ (Proposition 12.3; $\bar{j}$ denotes the inclusion $\mathbf{P}^1 \setminus D \hookrightarrow \mathbf{P}^1$) that this is zero.

The last statements in the lemma, on Aut^0_E, follow immediately from the fact that T_∞ restricts to an isomorphism $T_\infty\colon \operatorname{Bun}^{1,\mathrm{r}}_{2,4} \xrightarrow{\sim} \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$. □

Corollary 12.12. *Let $j\colon \operatorname{Bun}^{1,\mathrm{r}}_{2,4} \hookrightarrow \operatorname{Bun}^1_{2,4}$ denote the inclusion. Aut^1_E is a perverse sheaf and $\operatorname{Aut}^1_E = j_{!*}(\operatorname{Aut}^1_E|_{\operatorname{Bun}^{1,\mathrm{r}}_{2,4}}) = \mathbf{R}j_!(\operatorname{Aut}^1_E|_{\operatorname{Bun}^{1,\mathrm{r}}_{2,4}})$.*

Since $\mathbb{D}\operatorname{Aut}^1_E = \operatorname{Aut}^1_{E^\vee}$, this implies that the extension is clean, i.e., $\mathbf{R}j_!(\operatorname{Aut}^1_E|_{\operatorname{Bun}^{1,\mathrm{r}}_{2,4}}) = \mathbf{R}j_*(\operatorname{Aut}^1_E|_{\operatorname{Bun}^{1,\mathrm{r}}_{2,4}})$.

Proof. As before, we let $i\colon \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}_0}^{1,1} \hookrightarrow \operatorname{Bun}^2_{2,4} \times \overline{\mathbf{Coh}_0}^{1,1}$ denote the inclusion. First, note that $p^* i_{!*}\tilde{\mathcal{L}}_E[1]$ is perverse and pure. Since q is proper, we can apply the decomposition theorem to the map $\mathbf{R}q_!$ to find that $\operatorname{Aut}^1_E := \mathbf{R}q_! p^* i_{!*}\tilde{\mathcal{L}}_E[1]$ decomposes as a direct sum of semisimple shifted perverse sheaves. (In more detail, since q is proper and $p^* i_{!*}\tilde{\mathcal{L}}_E[1]$ is pure, its pushforward along q is pure ([**Del80**, prop. 6.2.6]); that pure sheaf is a direct sum of its perverse cohomology sheaves ([**BBD82**, thm. 5.4.5]) and each of the perverse summands is a direct sum of intersection complexes after the base change to the algebraic closure of the base field ([**BBD82**, thm. 5.3.8]).) By the previous corollary (Corollary 12.11), $j_{!*}(\operatorname{Aut}^1_E|_{\operatorname{Bun}^{1,\mathrm{r}}_{2,4}})$ is one of them. Because Aut^1_E is zero outside of $\operatorname{Bun}^{1,\mathrm{r}}_{2,4}$, this intermediate extension is all of Aut^1_E, proving $\operatorname{Aut}^1_E = j_{!*}(\operatorname{Aut}^1_E|_{\operatorname{Bun}^{1,\mathrm{r}}_{2,4}})$. □

We now start the proof of the main proposition of this section, Proposition 12.10.

Proposition 12.13. *Let $Z := \overline{\{\tilde{\mathcal{E}}\}} \subset \operatorname{Bun}^2_{2,4}$ denote the closure of $\tilde{\mathcal{E}}$ and let $F \in D^{\mathrm{b}}(\operatorname{Bun}^2_{2,4}, \mathbf{Q}_\ell)$ be a derived constructible sheaf with support on $Z \setminus \{\tilde{\mathcal{E}}\}$. Then*

$$\mathbf{A}(F \boxtimes \tilde{\mathcal{L}}_E) = 0.$$

We prove this lemma in the next section, but first, we use it to finish the proof of the proposition.

Proof of Proposition 12.10. There exist an $F \in D^{\mathrm{b}}(\mathrm{Bun}^2_{2,4}, \mathbf{Q}_\ell)$ with support on $Z \setminus \{\tilde{\mathcal{E}}\}$ and a distinguished triangle

$$\mathbf{R}i_!\tilde{\mathcal{L}}_E \to \mathbf{R}i_*\tilde{\mathcal{L}}_E \to F \boxtimes \tilde{\mathcal{L}}_E \xrightarrow{+1} . \tag{12.3.2}$$

Applying the functor $\mathbf{A}$ to this distinguished triangle gives a distinguished triangle

$$\mathbf{A}\mathbf{R}i_!\tilde{\mathcal{L}}_E \to \mathbf{A}\mathbf{R}i_*\tilde{\mathcal{L}}_E \to \mathbf{A}(F \boxtimes \tilde{\mathcal{L}}_E) \xrightarrow{+1} .$$

and since $\mathbf{A}(F \boxtimes \tilde{\mathcal{L}}_E)$ is zero (Proposition 12.13), this shows that $\mathbf{A}\mathbf{R}i_!\tilde{\mathcal{L}}_E \to \mathbf{A}\mathbf{R}i_*\tilde{\mathcal{L}}_E$ is an isomorphism, which completes the proof. □

12.4. Proof of Proposition 12.13. In this section, we prove Proposition 12.13, using our determination of the closure $Z := \overline{\{\tilde{\mathcal{E}}\}} \subset \mathrm{Bun}^2_{2,4}$ from Section 12.5. Namely, one of the results of Section 12.5 says that any $\mathcal{E} \in \overline{\{\tilde{\mathcal{E}}\}} \setminus \{\tilde{\mathcal{E}}\}$ is a direct sum of line bundles and can therefore be represented by a map $\operatorname{Spec} \mathbf{F}_p \xrightarrow{\mathcal{E}} \overline{\{\tilde{\mathcal{E}}\}} \subset \mathrm{Bun}^2_{2,4}$. When we write $\{\mathcal{E}\}$ below, we mean the scheme $\operatorname{Spec} \mathbf{F}_p$ and the restriction $(\mathbf{R}i_*\tilde{\mathcal{L}}_E)|_{\{\mathcal{E}\}\times\overline{\mathbf{Coh}}_0^{1,1}}$ should be interpreted to mean the pullback of $\mathbf{R}i_*\tilde{\mathcal{L}}_E$ along the map $\overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\mathcal{E},\mathrm{id}} \mathrm{Bun}^2_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$.

We first reduce the proof of the lemma to the following proposition. Recall that $Z = \overline{\{\tilde{\mathcal{E}}\}} \subset \mathrm{Bun}^2_{2,4}$ denotes the closure of $\tilde{\mathcal{E}}$.

Proposition 12.14. *Let $\mathcal{E}^\bullet \in Z \setminus \{\tilde{\mathcal{E}}\}$ and let $F \in D^{\mathrm{b}}(\mathrm{Bun}^2_{2,4}, \mathbf{Q}_\ell)$ be a skyscraper sheaf supported on $\mathcal{E}$. Then*

$$\mathbf{A}(F \boxtimes \tilde{\mathcal{L}}_E) = 0.$$

Reduction of Proposition 12.13 to Proposition 12.14. Let, as in Proposition 12.13, $F \in D^{\mathrm{b}}(\mathrm{Bun}^2_{2,4}, \mathbf{Q}_\ell)$ be a skyscraper sheaf supported on $Z \setminus \{\tilde{\mathcal{E}}\}$. We want to prove $\mathbf{A}(F \boxtimes \tilde{\mathcal{L}}_E) = 0$. We do this by showing for every $\mathcal{F}^\bullet \in \mathrm{Bun}^1_{2,4}$ that the stalk $\mathbf{A}(F \boxtimes \tilde{\mathcal{L}}_E)|_{\mathcal{F}^\bullet}$ is zero. So let $\mathcal{F}^\bullet \in \mathrm{Bun}^1_{2,4}$.

Since every $\mathcal{E}^\bullet \in Z$ is a direct sum of parabolic line bundles (Corollary 12.16; proven in the next section) and since an upper modification of $\mathcal{O}(1+d)\oplus\mathcal{O}(-d)$ is isomorphic to either $\mathcal{O}(2+d) \oplus \mathcal{O}(-d)$ or $\mathcal{O}(1+d) \oplus \mathcal{O}(1-d)$, there is a finite number of points $\mathcal{E}_i^\bullet \in |Z| \setminus \{\tilde{\mathcal{E}}\}$ $(i = 1, \dots, N)$ such that

$$p(q^{-1}(\mathcal{F}^\bullet)) \cap (Z \setminus \{\tilde{\mathcal{E}}\}) \subset \bigcup_{i=1}^N \mathbf{B}\operatorname{Aut}(\mathcal{E}_i^\bullet) \times \overline{\mathbf{Coh}}_0^{1,1}.$$

Hence, we can decompose $q^{-1}(\mathcal{F}^\bullet)$ into finitely many strata using the inverse images of the points $\mathcal{E}_i^\bullet$, i.e., we stratify it as

$$q^{-1}(\mathcal{F}^\bullet) = \bigcup_{i=1}^N q^{-1}(\mathcal{F}^\bullet) \cap p^{-1}(\mathbf{B}\operatorname{Aut}(\mathcal{E}_i^\bullet) \times \overline{\mathbf{Coh}}_0^{1,1}).$$

By definition of $\mathbf{A}$, we have

$$\mathbf{A}(F \boxtimes \mathcal{L}_E)|_{\mathcal{F}^\bullet} = \mathrm{H}_c^*(q^{-1}(\mathcal{F}^\bullet), p^*(F \boxtimes \mathcal{L}_E)|_{q^{-1}(\mathcal{F}^\bullet)}).$$

The cohomology of this sheaf restricted to the stratum indexed by i is

$$\begin{aligned}&\mathrm{H}_c^*(q^{-1}(\mathcal{F}^\bullet) \cap p^{-1}(\mathbf{B}\operatorname{Aut}(\mathcal{E}_i^\bullet) \times \overline{\mathbf{Coh}}_0^{1,1}), p^*(F \boxtimes \mathcal{L}_E)|_{q^{-1}(\mathcal{F}^\bullet)})\\ &= \mathrm{H}_c^*(q^{-1}(\mathcal{F}^\bullet) \cap p^{-1}(\mathbf{B}\operatorname{Aut}(\mathcal{E}_i^\bullet) \times \overline{\mathbf{Coh}}_0^{1,1}), p^*(F|_{\mathcal{E}_i^\bullet} \boxtimes \mathcal{L}_E))\end{aligned}$$

and, if we assume Proposition 12.14 is true, then this is zero. Since the restriction to all of the finitely many strata is zero, it follows, using successive open-closed decompositions for example, that the entire cohomology is zero, proving $\mathbf{A}(F \boxtimes \mathcal{L}_E)|_{\mathcal{F}^\bullet} = 0$. □

The rest of this section is concerned with the proof of Proposition 12.14. We start with an overview.

Overview of the proof of Proposition 12.14. Let $F \in D^{\mathrm{b}}(\mathrm{Bun}_{2,4}^2, \mathbf{Q}_\ell)$ be a skyscraper sheaf supported at a point $\mathcal{E}^\bullet \in Z \setminus \{\tilde{\mathcal{E}}\}$.

Then $\mathbf{A}(F \boxtimes \mathcal{L}_E)$ is supported on

$$q(p^{-1}(\mathbf{B}\operatorname{Aut}(\mathcal{E}^\bullet) \times \overline{\mathbf{Coh}}_0^{1,1})) \subset \mathrm{Bun}_{2,4}^1.$$

We determine the points in this set explicitly: by definitions, these points are length 1 lower modifications of $\mathcal{E}^\bullet$; $\mathcal{E}^\bullet$ and its length 1 lower modifications are all direct sums of parabolic line bundles (Corollary 12.16), so they are the finitely many points $\mathcal{F}_i^\bullet \in \mathrm{Bun}_{2,4}^1(\mathbf{F}_q)$, $i = 1, \ldots, N$ listed in Lemma 12.17. We prove that $\mathbf{A}(F \boxtimes \mathcal{L}_E)$ is zero by showing that all stalks $\mathbf{A}(F \boxtimes \mathcal{L}_E)|_{\mathcal{F}_i^\bullet}$ are zero.

Denote by

$$p_0 \colon p^{-1}(\{\mathcal{E}^\bullet\} \times \overline{\mathbf{Coh}}_0^{1,1}) \to \operatorname{Spec} k \times \overline{\mathbf{Coh}}_0^{1,1}$$

the pullback of $p \colon \mathcal{H} \to \mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1}$ along the map

$$(\mathcal{E}^\bullet, \mathrm{id}) \colon \operatorname{Spec} \mathbf{F}_q \times \overline{\mathbf{Coh}}_0^{1,1} \to \mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1}.$$

We then have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} k \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[d, "{(\mathcal{E}^\bullet,\mathrm{id})}"] \arrow[dr, phantom, "\square"] & p^{-1}(\mathcal{E}^\bullet \times \overline{\mathbf{Coh}}_0^{1,1}) \arrow[l, "p_0"'] \arrow[d] \arrow[dr, bend left, "q"] & \\
\mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1} & \mathcal{H} \arrow[l, "p"] \arrow[r, "q"] & \mathrm{Bun}_{2,4}^1
\end{tikzcd}$$

By definition of F as a skyscraper sheaf at $\mathcal{E}^\bullet$, and by the proper base change theorem, we have

$$\mathbf{A}(F \boxtimes \mathcal{L}_E)|_{\mathcal{F}_i^\bullet} = \mathrm{H}_c^*(p^{-1}(\mathcal{E}^\bullet \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}(\mathcal{F}_i^\bullet), p_0^*(F|_{\mathcal{E}^\bullet} \boxtimes \mathcal{L}_E)).$$

We then complete the proof by showing in Sections 12.4.1 to 12.4.4 (the different sections correspond to different isomorphism classes of $\mathcal{F}_i^\bullet$) that we have a space

$$p' \colon X \to \operatorname{Spec} \mathbf{F}_q \times \overline{\mathbf{Coh}}_0^{1,1}$$

and an isomorphism over $\operatorname{Spec}\mathbf{F}_q \times \overline{\mathbf{Coh}}_0^{1,1}$

$$\begin{tikzcd}
p^{-1}(\mathcal{E}^\bullet \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}(\mathcal{F}_i^\bullet) \arrow[rr, "\sim"] \arrow[dr, "p_0"'] & & X \arrow[dl, "p'"] \\
& \operatorname{Spec}\mathbf{F}_q \times \overline{\mathbf{Coh}}_0^{1,1} &
\end{tikzcd}$$

such that $(p')^*\mathcal{L}_E$ has vanishing cohomology by the cohomological properties of E (corresponding to the cusp conditions) established in Proposition 12.3 and Lemma 12.4. This proves that $\mathbf{A}(F \boxtimes \mathcal{L}_E)|_{\mathcal{F}_i^\bullet} = \mathrm{H}_c^*(X, F|_{\mathcal{E}^\bullet} \boxtimes (p')^*\mathcal{L}_E)$ is zero, completing the proof. $\square$

As explained in the overview above, we start with the determination of the length 1 lower modifications of the points $\mathcal{E}^\bullet \in Z \setminus \{\tilde{\mathcal{E}}\}$.

The following lemma is a small addition to Lemma 6.8: besides giving a criterion for parabolic rank 2 vector bundles to be a direct sum of parabolic line bundles, it also shows then all the length 1 lower modifications of such a vector bundle satisfy the same property of being a direct sum of parabolic line bundles.

Lemma 12.15. *Let $\mathcal{E}^\bullet := (\mathcal{O}(m) \oplus \mathcal{O}(n), (\ell_x)_{x\in D})$ be a parabolic vector bundle on $\mathbf{P}^1$ with $m > n$. Let*

$$I := \{x \in D : \ell_x \neq \mathcal{O}(m)|_x\}.$$

Suppose that

$$\#I \leq m - n + 1. \tag{12.4.1}$$

(1) $\mathcal{E}^\bullet$ is isomorphic to the direct sum of parabolic line bundles

$$\mathcal{E}^\bullet \cong (\mathcal{O}(m), D \setminus I) \oplus (\mathcal{O}(n), I).$$

(2) If the inequality (12.4.1) *is strict, then every length 1 lower modification of $\tilde{\mathcal{E}}^\bullet$ satisfies the inequality 12.15 possibly non-strictly, and is therefore a direct sum of parabolic line bundles.*

Proof. The first part is Lemma 6.8.

It remains to prove the second part concerning the case of a strict inequality. When taking a length 1 lower modification, the right hand side of the inequality (12.4.1) can decrease by at most 1, whereas the left hand side $\#\{x \in D : \ell_x \neq \mathcal{O}(m)|_x\}$ can increase by at most 1. Hence, it suffices to show both cannot occur at the same time. This follows from the observation that the left hand side $\#\{x \in D : \ell_x \neq \mathcal{O}(m)|_x\}$ can only increase when we take a modification of the form T_x or ${}^{\ell}T_x$ at a point $x \in D$ with $\ell_x = \mathcal{O}(m)|_x$, in which cases the right hand side actually increases by one. $\square$

Corollary 12.16. *Let $\mathcal{E} \in \overline{\{\tilde{\mathcal{E}}\}} \setminus \{\tilde{\mathcal{E}}\}$. Then $\mathcal{E}$ and any of its length 1 modifications are direct sums of parabolic line bundles.*

PROOF. Our determination of the closure of $\tilde{\mathcal{E}}$ (Proposition 12.19) shows that $\mathcal{E}$ satisfies the strict inequality in Lemma 12.15. $\square$

Lemma 12.17. *Let $(\mathcal{L}, I) \oplus (\mathcal{M}, D \setminus I)$ be a parabolic vector bundle on $\mathbf{P}^1$ such that all its length 1 lower modifications are also direct sums of line bundles. (Lemma 12.15 provides examples of such parabolic vector bundles.) Then each length 1 lower modification is isomorphic to one of the following bundles:*

$$\begin{aligned} &(\mathcal{L}, I) \oplus (\mathcal{M}(-1), D \setminus I), \\ &(\mathcal{L}(-1), I) \oplus (\mathcal{M}, D \setminus I), \\ &(\mathcal{L}(-1), I \cup \{x\}) \oplus (\mathcal{M}, D \setminus I \setminus \{x\}) && \textit{for } x \in D \setminus I, \\ \textit{or} \quad &(\mathcal{L}, I \setminus \{x\}) \oplus (\mathcal{M}(-1), D \setminus I \cup \{x\}) && \textit{for } x \in I. \end{aligned}$$

PROOF. Let $(\mathcal{L}', I') \oplus (\mathcal{M}', D \setminus I')$ be a length 1 lower modification of $(\mathcal{L}, I) \oplus (\mathcal{M}, D \setminus I)$. The assumption $\deg \mathcal{L} + 1 > \deg \mathcal{M}$ implies $\deg \mathcal{L}' \neq \deg \mathcal{M}'$, so we can assume $\deg \mathcal{L}' > \deg \mathcal{M}'$ without loss of generality. We have either $(\mathcal{L}', \mathcal{M}') = (\mathcal{L}, \mathcal{M}(-1))$ or $(\mathcal{L}', \mathcal{M}') = (\mathcal{L}(-1), \mathcal{M})$. If $I' = I$, which is the case when we modify outside of D (but can also be the case for modifications at $x \in D$), we get one of the first two isomorphism classes.

Suppose now that $I' \neq I$. If $(\mathcal{L}', I') \oplus (\mathcal{M}, D \setminus I')$ is a modification at $x \in D$, then $I' = I \sqcup \{x\}$ or $I' = I \setminus \{x\}$. The induced maps $(\mathcal{L}', I') \to (\mathcal{L}, I)$ and $(\mathcal{M}', D \setminus I') \to (\mathcal{M}, D \setminus I)$ cannot be zero, so they are injective. A morphism $(\mathcal{L}', I') \to (\mathcal{L}, I)$ is the same as a morphism of line bundles $\mathcal{L}' \to T_{I'}(\mathcal{L}, I) = \mathcal{L}(-I' \setminus I)$, so by considering the degrees, we see that $I' = I \sqcup \{x\}$ implies $\mathcal{L}' = \mathcal{L}(-1)$. Reasoning in the same way with $\mathcal{M}'$ shows that the assumption $I' = I \setminus \{x\}$ (so $D \setminus I' = (D \setminus I) \sqcup \{x\}$) implies $\mathcal{M}' = \mathcal{M}(-1)$. This gives the last two cases. $\square$

Remark 12.18. The last two isomorphism classes in the lemma above are both of the form $T_x((\mathcal{L}, I) \oplus (\mathcal{M}, D \setminus I))$ for $x \in D$.

Let $\mathcal{E}^\bullet \in Z \setminus \{\tilde{\mathcal{E}}\}$ and write

$$\mathcal{E}^\bullet = (\mathcal{L}, I) \oplus (\mathcal{M}, D \setminus I)$$

with $\deg \mathcal{L} > \deg \mathcal{M} + 1$. As explained in the overview of the proof of Proposition 12.14 on page 105, we would now like to describe the map

$$p^{-1}(\mathcal{E}^\bullet) \cap q^{-1}(\mathcal{F}^\bullet) \to \operatorname{Spec} \mathbf{F}_q \times \overline{\mathbf{Coh}}_0^{1,1}$$

for each of the finitely many $\mathcal{F}^\bullet \in q(p^{-1}(\mathcal{E}^\bullet))$, which are the length one lower modifications of $\mathcal{E}^\bullet$. In the following, we refer to the spaces $p^{-1}(\mathcal{E}^\bullet) \cap q^{-1}(\mathcal{F}^\bullet)$ as "the fibers (of q)". The following table summarizes the results. Here $\sigma, \tau \colon \mathbf{P}^1 \to \overline{\mathbf{Coh}}_0^{1,1}$ denote sections of the support map $\operatorname{Supp} \colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$

that satisfy for all $x \in D$

$$\sigma(x) = \begin{cases} k_x^{(1,0)} & \text{if } x \in I \\ k_x^{(0,1)} & \text{if } x \in D \setminus I \end{cases}, \quad \text{and}$$

$$\tau(x) = \begin{cases} k_x^{(0,1)} & \text{if } x \in I \\ k_x^{(1,0)} & \text{if } x \in D \setminus I \end{cases}.$$

number	isomorphism class of the modification $\mathcal{F}^\bullet$			isomorphism class of the fiber $p^{-1}(\mathcal{E}^\bullet) \cap q^{-1}(\mathcal{F}^\bullet)$
1	$(\mathcal{L}, I)$	$\oplus$	$(\mathcal{M}(-1), D \setminus I)$	$\mathbf{P}^1 \xrightarrow{\sigma} \overline{\mathbf{Coh}_0}^{1,1}$
2	$(\mathcal{L}(-1), I)$	$\oplus$	$(\mathcal{M}, D \setminus I)$	$\mathcal{M}^\vee \otimes \mathcal{L} \to \mathbf{P}^1 \xrightarrow{\tau} \overline{\mathbf{Coh}_0}^{1,1}$
3	$(\mathcal{L}, I \setminus \{y\})$	$\oplus$	$(\mathcal{M}(-y), (D \setminus I) \cup \{y\})$	$\mathbf{A}^1 \to \overline{\mathbf{Coh}_0}^{1,1},\ \lambda \mapsto (k_y \xrightarrow{\lambda} k_y \xrightarrow{0} k_y)$
4	$(\mathcal{L}(-y), I \cup \{y\})$	$\oplus$	$(\mathcal{M}, D \setminus (I \cup \{y\}))$	$\mathbf{A}^1 \to \overline{\mathbf{Coh}_0}^{1,1}$, $\lambda \mapsto (k_y \xrightarrow{0} k_y \xrightarrow{\lambda} k_y)$

Every length one lower modification $\mathcal{F}^\bullet$ of $\mathcal{E}^\bullet$ can also be described as $T_y^\ell \mathcal{E}^\bullet$, ${}^\ell T_y \mathcal{E}^\bullet$ or $T_y \mathcal{E}^\bullet$ for some $y \in \mathbf{P}^1$ and some flag ℓ. The following table shows which of the modifications $\mathcal{F}^\bullet$ occur as which of the above three types. In this table ℓ_∞ always denotes the appropriate flag coming from the maximal destabilizing subbundle, which in these cases is either $\mathcal{L}|_y$ or $\mathcal{L}(-y)|_y$, and ℓ_0 denotes any of the other flags.

number	isomorphism class of the modifications			modification at $y \in \mathbf{P}^1 \setminus D$	modification at $y \in I$	modification at $y \in D \setminus I$
1	$(\mathcal{L}, I)$	$\oplus$	$(\mathcal{M}(-1), D \setminus I)$	$T_y^{\ell_\infty}$	${}^{\ell_\infty}T_y$	$T_y^{\ell_\infty}$
2	$(\mathcal{L}(-1), I)$	$\oplus$	$(\mathcal{M}, D \setminus I)$	$T_y^{\ell_0}$	$T_y^{\ell_0}$	${}^{\ell_0}T_y$
3	$(\mathcal{L}, I \setminus \{y\})$	$\oplus$	$(\mathcal{M}(-y), (D \setminus I) \cup \{y\})$	none	the rest	none
4	$(\mathcal{L}(-y), I \cup \{y\})$	$\oplus$	$(\mathcal{M}, D \setminus (I \cup \{y\}))$	none	none	the rest

These results are perhaps easiest to visualize and remember using the description of $p^{-1}(\mathcal{E}^\bullet)$ as $\mathrm{Bl}_D(\mathbf{P}(\mathcal{E}))$ (Proposition 2.28; we left this unproven): the fibers numbered 1 and 2 make up a $\mathbf{P}(\mathcal{E}) \subset \mathrm{Bl}_D(\mathbf{P}(\mathcal{E}))$, the first fiber being a section $\mathbf{P}^1 \to \mathbf{P}(\mathcal{E})$ and the second fiber being the rest of the $\mathbf{P}(\mathcal{E})$; and the last two types of fibers are the exceptional divisors minus a point each.

We will now prove that these fibers are indeed as summarized in the above tables. For $s\colon S \to \mathbf{P}^1$, we denote by

$$\Gamma_s\colon S \to \mathbf{P}^1 \times S$$

its graph.

12.4.1. *First fiber.* We define an isomorphism

$$\mathbf{P}^1 \xrightarrow{\sim} p^{-1}(\{\mathcal{E}\} \times \overline{\mathbf{Coh}_0}^{1,1}) \cap q^{-1}((\mathcal{L}, I) \oplus (\mathcal{M}(-1), D \setminus I))$$

from $\mathbf{P}^1$ to the first fiber: on S-points it is given by

$$(s\colon S \to \mathbf{P}^1) \mapsto \left(\mathrm{pr}_{\mathbf{P}^1}^*((\mathcal{L}, I) \oplus (\mathcal{M}, D \setminus I)) \twoheadrightarrow \Gamma_{s,*}s^*(\mathcal{M}, D \setminus I)\right),$$

where the map in the image is the map on parabolic sheaves induced by the adjoint of the projection map

$$\Gamma_s^* \operatorname{pr}_{\mathbf{P}^1}^*(\mathcal{L} \oplus \mathcal{M}) \to s^*\mathcal{M}.$$

(Note that $\operatorname{pr}_{\mathbf{P}^1} \circ \Gamma_s = s$.) The inverse is given on S-points by

$$(\operatorname{pr}_{\mathbf{P}^1}^* \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto \operatorname{Supp} \mathcal{T}$$

where we consider $\operatorname{Supp} \mathcal{T}$ as a map $S \to \mathbf{P}^1$. (The sheaf $\mathcal{T}$ is supported on the graph of this map.)

This isomorphism fits into a commutative diagram

$$\begin{tikzcd}
\mathbf{P}^1 \arrow[r, "\cong"] \arrow[dr, bend right] & p^{-1}(\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}((\mathcal{L}, I) \oplus (\mathcal{M}(-1), D \setminus I)) \arrow[d, "p"] \\
 & \{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}
\end{tikzcd}$$

where the map from $\mathbf{P}^1$ to $\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}$ is one of the sections of $\operatorname{Supp}\colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$ defined in Section 2.4.2. The pullback of $\tilde{\mathcal{L}}_E$ along such a section has vanishing cohomology groups, because the intermediate extensions of E along $\mathbf{P}^1 \setminus D \hookrightarrow \mathbf{P}^1$ has vanishing cohomology groups (Proposition 12.3).

12.4.2. *Second fiber.* Now on to the second fiber, which is isomorphic to the total space of the line bundle $\mathcal{M}^\vee \otimes \mathcal{L}$ on $\mathbf{P}^1$. We simply denote this total space (defined as the relative spectrum of $\operatorname{Sym}^\bullet(\mathcal{M}^\vee \otimes \mathcal{L})^\vee$) by $\mathcal{M}^\vee \otimes \mathcal{L}$. An S-point of this total space is a pair

$$(s\colon S \to \mathbf{P}^1, \phi\colon s^*\mathcal{M} \to s^*\mathcal{L}).$$

It has a map to the second fiber, compatible with the structure map to $\mathbf{P}^1$, as in the diagram

$$\begin{tikzcd}
\mathcal{M}^\vee \otimes \mathcal{L} \arrow[r, "\cong"] \arrow[d] & p^{-1}(\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}((\mathcal{L}(-1), I) \oplus (\mathcal{M}, D \setminus I)) \arrow[d, "p"] \\
\mathbf{P}^1 \arrow[r] & \{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}
\end{tikzcd}$$

where the map from $\mathbf{P}^1$ to $\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}$ is again one of the sections of $\operatorname{Supp}\colon \overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$ defined in Section 2.4.2; the image of D under this section is disjoint from the image of D under the section we had for the first fiber.

On S-points, this isomorphism is defined by

$$(s\colon S \to \mathbf{P}^1, \phi\colon s^*\mathcal{M} \to s^*\mathcal{L}) \mapsto \left(\operatorname{pr}_{\mathbf{P}^1}^*((\mathcal{L}, I) \oplus (\mathcal{M}, D \setminus I)) \to \Gamma_{s,*} s^*\mathcal{L}\right),$$

where the map in the image is the parabolic map induced by the adjoint of the projection

$$(\mathrm{id}, \phi)\colon \Gamma_s^* \operatorname{pr}^*(\mathcal{L} \oplus \mathcal{M}) \to s^*\mathcal{L}.$$

The pullback of $\tilde{\mathcal{L}}_E$ along this map is zero for the same reason as the pullback of the first fiber.

12.4.3. *Third fiber.* The third kind of fiber appears once for each $y \in I$. The fiber is isomorphic to the total space of $\mathcal{M}(-y)^\vee|_y \otimes \mathcal{L}|_y$, which is isomorphic to $\mathbf{A}^1$. The isomorphism

$$\mathcal{M}(-y)|_y^\vee \otimes \mathcal{L}|_y \xrightarrow{\sim} p^{-1}(\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}((\mathcal{L}, I \setminus \{y\}) \oplus (\mathcal{M}(-y), (D \setminus I) \cup \{y\}))$$

is defined on S-points by

$$(\phi\colon S \to \mathcal{M}(-y)|_y^\vee \otimes \mathcal{L}|_y) \mapsto \left(\begin{array}{ccc} \mathrm{pr}^*(\mathcal{L} \oplus \mathcal{M}) & \xrightarrow{(0,1)} & \mathcal{M}|_y \\ {\scriptstyle (\mathrm{id}, \mathcal{O}(-y) \hookrightarrow \mathcal{O})}\big\uparrow & & {\scriptstyle 0}\big\uparrow \\ \mathrm{pr}^*(\mathcal{L} \oplus \mathcal{M}(-y)) & \xrightarrow{(1,\phi)} & \mathcal{L}|_y \\ {\scriptstyle (\mathcal{O}(-y) \hookrightarrow \mathcal{O}, \mathrm{id})}\big\uparrow & & {\scriptstyle \phi}\big\uparrow \\ \mathrm{pr}^*(\mathcal{L}(-y) \oplus \mathcal{M}(-y)) & \xrightarrow{(0,1)} & \mathcal{M}(-y)|_y \end{array}\right).$$

In the diagram on the right, the rows represent the different parabolic degrees (degree zero is on top, degree $(-y, 1)$ in the middle and degree $(-y, 2)$ at the bottom) and in the right column, $\mathcal{M}|_y$ and $\mathcal{L}|_y$ denote torsion sheaves on $\mathbf{P}^1 \times S$ with support on $\{y\} \times S$.

This map fits into a commutative diagram

$$\begin{tikzcd} \mathcal{M}(-y)^\vee \otimes \mathcal{L} \arrow[r] \arrow[dr, bend right] & p^{-1}(\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}((\mathcal{L}, I \setminus \{y\}) \oplus (\mathcal{M}(-y), (D \setminus I) \cup \{y\})) \arrow[d, "p"] \\ & \{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1} \end{tikzcd}$$

where the map $\mathcal{M}(-y)^\vee \otimes \mathcal{L} \to \{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}$ is given by (up to isomorphism of the source and target)

$$\mathbf{A}^1 \to \overline{\mathbf{Coh}}_0^{1,1}, \lambda \mapsto (k_y \xrightarrow{\lambda} k_y \xrightarrow{0} k_y).$$

The pullback of $\tilde{\mathcal{L}}_E$ along this map vanishes by Lemma 12.4.

12.4.4. *Fourth fiber.* The fourth kind of fiber is similar to the third. We have one for each $y \in D \setminus I$ and it is also isomorphic to $\mathcal{M}|_y^\vee \otimes \mathcal{L}|_y$. The map

$$\mathcal{M}|_y^\vee \otimes \mathcal{L}|_y \to p^{-1}(\{\mathcal{E}\} \times \overline{\mathbf{Coh}}_0^{1,1}) \cap q^{-1}((\mathcal{L}(-y), I \cup \{y\}) \oplus (\mathcal{M}, D \setminus (I \cup \{y\})))$$

is defined on S-points by

$$(\phi\colon S \to \mathcal{M}|_y^\vee \otimes \mathcal{L}|_y) \mapsto \left(\begin{array}{ccc} \mathrm{pr}^*(\mathcal{L} \oplus \mathcal{M}) & \xrightarrow{(1,\phi)} & \mathcal{L}|_y \\ {\scriptstyle (\mathcal{O}(-y) \hookrightarrow \mathcal{O}, \mathrm{id})}\big\uparrow & & {\scriptstyle \phi}\big\uparrow \\ \mathrm{pr}^*(\mathcal{L}(-y) \oplus \mathcal{M}) & \xrightarrow{(0,1)} & \mathcal{M}|_y \\ {\scriptstyle (\mathrm{id}, \mathcal{O}(-y) \hookrightarrow \mathcal{O})}\big\uparrow & & {\scriptstyle 0}\big\uparrow \\ \mathrm{pr}^*(\mathcal{L}(-y) \oplus \mathcal{M}(-y)) & \xrightarrow{(1,\phi)} & \mathcal{L}(-y)|_y \end{array}\right).$$

The diagram on the right should be interpreted in the same way as in the third fiber. It fits into a similar kind of commutative diagram, but the map to $\overline{\mathbf{Coh}}_0^{1,1}$ looks like

$$\lambda \mapsto (k_y \xrightarrow{0} k_y \xrightarrow{\lambda} k_y),$$

that is, it is the same map, but with the parabolic degree of the image shifted by one. The pullback of $\tilde{\mathcal{L}}_E$ along this map therefore vanishes by the same reason.

12.5. Appendix: determining the closure of $\tilde{\mathcal{E}}$. The proof of the following proposition is the main concern of this section.

Proposition 12.19. *Let $(\mathcal{E}, (\ell_x)_{x\in D})\colon \operatorname{Spec} K \to \mathrm{Bun}_{2,4}^2$ (with K some field over $\mathbf{F}_q$) be a parabolic bundle in the closure of $\tilde{\mathcal{E}}$ inside $\mathrm{Bun}_{2,4}^2$. Then the underlying vector bundle of $\mathcal{E}$ is isomorphic to $\mathcal{O}(2+n)\oplus\mathcal{O}(-n)$ for $n \in \mathbf{Z}_{\geq 0}$. If $n = 0$, one of the following conditions hold:*

- *$(\mathcal{E}, (\ell_x)_{x\in D}) \cong \tilde{\mathcal{E}}$; or*
- *$\ell_x = \mathcal{O}(2)|_x$ for at least 2 different $x \in D$.*

The first part of the proof of the proposition follows from the following lemma.

Lemma 12.20. *There are no parabolic bundles in $\overline{\{\tilde{\mathcal{E}}\}} \subset \mathrm{Bun}_{2,4}^2$ whose underlying vector bundle is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. The stack Bun_2^2 of rank 2, degree 2 vector bundles (without parabolic structure) has an open substack consisting of all vector bundles isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Taking the inverse image of this substack along the forgetful map $\mathrm{Bun}_{2,4}^2 \to \mathrm{Bun}_2^2$, we get an open substack containing all parabolic bundles isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$, but not containing $\tilde{\mathcal{E}}$.

$\square$

Proof of the second part of the proposition. By [**Sta18**, Tag 0CL2], for every point $\mathcal{E}$ in the closure of $\{\tilde{\mathcal{E}}\}$, there exists a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \arrow[d, hook] \\ \operatorname{Spec} A \arrow[r, "y_A"] & \mathrm{Bun}_{2,4}^{\mathcal{O}(2)\oplus\mathcal{O}} \end{tikzcd},$$

where A is a valuation ring with fraction field K, such that the closed point of A maps to $\mathcal{E}$. This result requires the map $\mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \hookrightarrow \mathrm{Bun}_{2,4}^{\mathcal{O}(2)\oplus\mathcal{O}}$ to be quasi-compact, which follows from the fact that both stacks are finite type over $\mathbf{F}_p$.

Moreover, we can assume that A is a complete DVR. We write $A = L[[\varpi]]$ (and hence $K = L((\varpi))$).

There exists
$$g_K := \begin{pmatrix} \lambda_K & \sigma_K \\ 0 & \mu_K \end{pmatrix} \in \mathrm{Aut}(\mathcal{O}(2) \oplus \mathcal{O})(K)$$
with $\lambda_K, \mu_K \in K^*$ and $\sigma_K \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes_k K$, such that the $\operatorname{Spec} A$-point y_A of $\mathrm{Bun}_{2,4}^{\mathcal{O}(2)\oplus\mathcal{O}}$ is the same K-point (really the same, not up to isomorphism) as
$$y_K := g_K \begin{pmatrix} \mathcal{O}(2) & 0 & 0 & 0 & 0 \\ \mathcal{O} & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(2) & \sigma_K(\infty) & \sigma_K(0) & \sigma_K(1) & \sigma_K(t) \\ \mathcal{O} & \mu_K & \mu_K & \mu_K & \mu_K \end{pmatrix}.$$
(Here "the same" means that if we appropriately scale the flags so that they lie in A, we recover y_A. The closed point can then be obtained by modding out ϖ.) (For $x \in D$, we denote by $\sigma_K(x)$ the value of σ_K in the fiber $\mathcal{O}(2)|_x$.)

If $\sigma_K = 0$, then $\mathcal{E}$ is simply $\tilde{\mathcal{E}}$. We are interested in discovering the other points in the closure over $\{\tilde{\mathcal{E}}\}$, so we can assume $\sigma_K \neq 0$ for the rest of the proof.

Write
$$\sigma_K = \sum_{i >> -\infty} \sigma_i \varpi^i$$
with $\sigma_i \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes_k L$. By scaling the flags (i.e., the columns) by a suitably chosen unit in K, we get a matrix that represents the same K-point and is of the form

(12.5.1)
$$\begin{pmatrix} \mathcal{O}(2) & \sum_{i\geq i_0} \tau_i(\infty)\varpi^i & \sum_{i\geq i_0} \tau_i(0)\varpi^i & \sum_{i\geq i_0} \tau_i(1)\varpi^i & \sum_{i\geq i_0} \tau_i(t)\varpi^i \\ \mathcal{O} & \varpi^{i_1} & \varpi^{i_1} & \varpi^{i_1} & \varpi^{i_1} \end{pmatrix}$$
with $\tau_i \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes_k L$, i_0 such that $\tau_{i_0} \neq 0$ and $\min\{i_0, i_1\} = 0$. This matrix also represents the A-point $y_A \colon \operatorname{Spec} A \to \mathrm{Bun}_{2,4}^{\mathcal{O}(2)\oplus\mathcal{O}}$, so we can now determine the image of the closed point of $\operatorname{Spec} A$ by modding out ϖ in the flags.

Suppose $i_1 \leq i_0$, so in particular $i_1 = 0$. Then by multiplying the matrix above from the left by the element of $G(A)$ represented by

$$\begin{pmatrix} 1 & -\sum_{i\geq i_0} \tau_i \\ 0 & 1 \end{pmatrix} \tag{12.5.2}$$

we see that the image of the closed point of $\operatorname{Spec} A$ lies in the same G-orbit as the point

$$\begin{pmatrix} \mathcal{O}(2) & 0 & 0 & 0 & 0 \\ \mathcal{O} & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{12.5.3}$$

Therefore, $y' = \tilde{\mathcal{E}}$ (as elements of $|\mathrm{Bun}_{2,4}^{\mathcal{O}(2)\oplus\mathcal{O}}|$).

Suppose now $i_1 > i_0$, which implies $i_0 = 0$. For each $x \in D$, denote by j_x the smallest value of $i \in \mathbf{Z}_{\geq 0}$ such that $\sigma_i(x) \neq 0$ (or ∞ if all $\sigma_i(x)$ are zero). The associated closed point has the following flags, obtained by modding out ϖ in the flags of the A-point:

- if $j_x < i_1$ (so in particular if $j_x = 0$), the flag at x is $\mathcal{O}(2)|_x$;
- if $j_x > i_1$, the flag at x is $\mathcal{O}|_x$; and finally,

- if $j_x = i_1$, the flag at x is $(\sigma_{j_x}(x) : 1)$.

Since σ_0 is not zero (because $i_0 = 0$) and has at most two zeroes (being an element of $\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes_k L$), there are at most 2 $x \in D$ with $j_x \neq 0$, so the first case listed above occurs for at least two $x \in D$. As a consequence, the third and second case occur for at most two times, so that y' is isomorphic to a parabolic vector bundle $(\mathcal{O}(2) \oplus \mathcal{O}, (\ell_x)_{x \in D})$ with $\ell_x = \mathcal{O}|_x$ for all $x \in D$ with $j_x \geq i_1$ and $\ell_x = \mathcal{O}(2)|_x$ for the other $x \in D$, of which there are at least two. This concludes the proof. □

We will not need the following result, but include it for completeness

Lemma 12.21. *The closure $\overline{\{\tilde{\mathcal{E}}\}} \subset \mathrm{Bun}^2_{2,4}$ contains all parabolic bundles with underlying vector bundle isomorphic to $\mathcal{O}(2+n) \oplus \mathcal{O}(-n)$ for $n \in \mathbf{Z}_{\geq 1}$.*

PROOF. By Corollary 12.16, every such parabolic bundle $\mathcal{E}$ is isomorphic to a parabolic of the form $(\mathcal{O}(2+n) \oplus \mathcal{O}(-n), (\ell_x)_{x \in D})$ with $\ell_x = \mathcal{O}(-n)|_x$ for all x in some $I \subset D$ and $\ell_x = \mathcal{O}(2+n)|_x$ for all $x \in D \setminus I$. Hence we can construct a filtration

$$0 \subset (\mathcal{O}(-n), I) \xrightarrow{(\alpha_1 \colon \mathcal{O}(-n) \to \mathcal{O}(2),\ \alpha_2 \colon \mathcal{O}(-n) \to \mathcal{O})} \tilde{\mathcal{E}}$$

where α_1, α_2 are injective maps such that α_1 has zeroes at I and not at $D \setminus I$, while α_2 is not zero at I. Then the associated graded object for this filtration is a parabolic bundle isomorphic to $\mathcal{E}$ and by the Artin-Rees construction, this implies that $\mathcal{E}$ is a specialization of $\tilde{\mathcal{E}}$. □

13. Proof of the Hecke property

The main result in this section and one of the main results of this thesis is the following theorem. For the definition of the global Hecke operator $\mathbb{H}$ and of a Hecke eigensheaf, see Definition 4.2 and Definition 4.3.

THEOREM 13.1. *Let E be a rank 2 pure irreducible local system on $\mathbf{P}^1 \setminus D$ with unipotent monodromy. Then the sheaf Aut_E (Definition 12.8, or Theorem 12.1 for an alternative characterization) is a Hecke eigensheaf for E, i.e., there is an isomorphism in the derived constructible category $D^{\mathrm{b}}(\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1},\ \mathbf{Q}_\ell)$*

$$\mathbb{H}\,\mathrm{Aut}_E \cong \mathrm{Aut}_E \boxtimes j_{!*}E[1].$$

We only need to prove this theorem in one degree, i.e., we can reduce it to the following theorem.

THEOREM 13.2. *Let the notation be as in Theorem 13.1 and for $d \in \mathbf{Z}$, we write $\mathrm{Aut}_E^d := \mathrm{Aut}_E|_{\mathrm{Bun}^d_{2,4}}$. There is an isomorphism*

$$\mathbb{H}\,\mathrm{Aut}_E^0 \cong \mathrm{Aut}_E^1 \boxtimes j_{!*}E[1].$$

Reduction of Theorem 13.1 to Theorem 13.2. Assume Theorem 13.2 holds and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Bun}^2_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[d, "{(T_\infty,T_\infty)}"] & \mathcal{H}^2 \arrow[l, "p_2"'] \arrow[r, "q_2"] \arrow[d, "T_\infty"] & \mathrm{Bun}^1_{2,4} \arrow[d, "T_\infty"] \\
\mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} & \mathcal{H}^1 \arrow[l, "p_1"'] \arrow[r, "q_1"] & \mathrm{Bun}^0_{2,4}
\end{tikzcd}$$

Pulling back and pushing forward Aut^0_E along the various maps gives us

$$\begin{tikzcd}
T^*_\infty(\mathrm{Aut}^1_E) \boxtimes T^*_\infty(j_{!*}E)[-1] & & \mathrm{Aut}^1_E \otimes E|_\infty \\
\mathrm{Aut}^1_E \boxtimes j_{!*}E[-1] \arrow[u, mapsto, "{(T_\infty,T_\infty)^*}"] & & \mathrm{Aut}^0_E \arrow[ll, mapsto, "\mathbf{R}p_{1,!}q_1^*"] \arrow[u, mapsto, "T^*_\infty"']
\end{tikzcd}$$

where $\mathbf{R}p_{1,!}q_1^* \mathrm{Aut}^0_E = \mathbb{H}[-2]\mathrm{Aut}^0_E = \mathrm{Aut}^1_E \boxtimes j_{!*}E[-1]$ by assumption. Because $T_\infty \colon \overline{\mathbf{Coh}}_0^{1,1} \to \overline{\mathbf{Coh}}_0^{1,1}$ is the identity on $\mathbf{P}^1 \setminus D$, we have $T^*_\infty(j_{!*}E) = j_{!*}E$ and therefore

$$T^*_\infty(\mathrm{Aut}^1_E) \boxtimes T^*_\infty(j_{!*}E)[-1] = (\mathrm{Aut}^2_E \otimes E|_\infty) \boxtimes j_{!*}E[-1]$$

The commutativity of the diagram hence implies

$$\mathbb{H}\,\mathrm{Aut}^1_E = \mathbf{R}p_{2,!}q_2^* \mathrm{Aut}^0_E[2] = \mathrm{Aut}^2_E \boxtimes j_{!*}E.$$

We can repeat this argument in the other degrees to conclude. □

After this reduction, the next step is to prove the following theorem.

Theorem 13.3. *The sheaf* $\mathbb{H}\,\mathrm{Aut}^0_E$ *is supported on* $\mathrm{Bun}^{1,\mathrm{r}}_{2,4} \subset \mathrm{Bun}^1_{2,4}$ *and is the intermediate extension of a rank 4 local system along*

$$\pi_1^{-1}(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D \hookrightarrow \mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}.$$

The proof of this theorem is spread across three subsections. In Section 13.1, we show that $\mathbb{H}\,\mathrm{Aut}^0_E$ decomposes as a direct sum of shifted perverse sheaves (Proposition 13.4) by compactifying the map $p\colon \mathcal{H} \to \mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$ (recall that $\mathbb{H} := \mathbf{R}p_!q^*$) and using the decomposition theorem. In Section 13.2, we prove that the restriction of $\mathbb{H}\,\mathrm{Aut}^0_E$ to $\pi_1^{-1}(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D$ is a rank 4 local system. This is a calculation on the cohomology of Aut^0_E that uses our explicit determination of the length 1 lower modifications from Section 9. A dimension counting argument in Section 13.4 then shows that the intermediate extension of the restriction to $\pi_1^{-1}(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D$ is the only shifted perverse sheaf that appears in that direct sum decomposition, which completes the proof of Theorem 13.3.

Finally, in the last sections (Sections 13.5 to 13.7), we complete the proof of Theorem 13.2 by using the symmetry and checking the restriction to $\mathrm{Bun}^{1,\mathrm{r}}_{2,4} \times \{k^0_x\}$.

13.1. Decomposition theorem and compactification. In this section, we prove the following proposition.

Proposition 13.4. *The sheaf* $\mathbb{H}\,\mathrm{Aut}_E^0$ *decomposes as a direct sum of shifted simple perverse sheaves.*

PROOF. Recall that the global Hecke operator $\mathbb{H} := \mathbf{R}p_! q^*[2]$ is defined using the correspondence

$$\mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} \xleftarrow{p} \mathcal{H} \xrightarrow{q} \mathrm{Bun}^0_{2,4}\,.$$

The map p is not proper over the points in $\mathrm{Bun}^1_{2,4} \times \mathrm{Supp}^{-1}(D)$: for example, for $x \in D$, the fiber over a point $(\mathcal{E}^\bullet, k_x^0)$ is the set of surjections $\mathcal{E}^\bullet \twoheadrightarrow k_x^0$ and this is isomorphic to $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ — the kernel is $T_x\mathcal{E}^\bullet$, which determines the map up to scaling, and we can scale the map in even and odd degree independently.

In Section 13.1.1, we introduce a compactification $\bar{p}$ of p: we embed $\mathcal{H}$ into another stack $\overline{\mathcal{H}}$

$$j\colon \mathcal{H} \subset \overline{\mathcal{H}}$$

and extend p along this embedding to a proper map

$$\bar{p}\colon \overline{\mathcal{H}} \to \mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$$

giving a commutative diagram

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, bend right, "\bar{p}"'] & \\ \mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} & \mathcal{H} \arrow[l, "p"] \arrow[r, "q"'] \arrow[u, hook, "j"] & \mathrm{Bun}^0_{2,4}\,. \end{tikzcd}$$

We then show in Section 13.1.2 that $q^*\,\mathrm{Aut}_E^0$ extends cleanly along j, i.e.,

$$\mathbf{R}j_! q^*\,\mathrm{Aut}_E^0 = j_{!*}q^*\,\mathrm{Aut}_E^0\,.$$

(This is then also equal to $\mathbf{R}j_* q^*\,\mathrm{Aut}_E^0$ by duality.) Since q is smooth (Lemma 4.12), this shows that $\mathbf{R}j_! q^*\,\mathrm{Aut}_E^0$ is perverse (up to shift) and pure. We can therefore use the equality

$$\mathbb{H}\,\mathrm{Aut}_E^0 = \mathbf{R}\bar{p}_! \mathbf{R}j_! q^*\,\mathrm{Aut}_E^0$$

and apply the decomposition theorem to the map $\mathbf{R}\bar{p}_!$ and the shifted perverse sheaf $\mathbf{R}j_! q^*\,\mathrm{Aut}_E^0$ to complete the proof. $\square$

13.1.1. *Compactification of the Hecke stack.* In this section, we compactify the map

$$p\colon \mathcal{H} \to \mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}, \qquad (\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\mathcal{E}^\bullet, [\mathcal{T}^\bullet]).$$

The result is stated in Lemma 13.7.

It turns out that it is easier to compactify the dual map:

$$p^\vee\colon \mathcal{H} \to \mathrm{Bun}^{-1}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}, \qquad (\mathcal{E}^\bullet \overset{\phi}{\twoheadrightarrow} \mathcal{T}^\bullet) \mapsto (\ker(\phi), [\mathcal{T}^\bullet]).$$

(See Section 4.3 for an analysis of the dual maps.) This is unfortunate from an expositional standpoint, but does not matter for the proofs: we have a commutative diagram

$$\begin{tikzcd} \mathrm{Bun}_{2,4}^{-1} \times \overline{\mathbf{Coh}_0}^{1,1} \arrow[d, "\mathrm{dual}"', "\wr"] & \mathcal{H}^0 \arrow[l, "p^\vee"'] \arrow[r, "q^\vee"] \arrow[d, "\mathrm{dual}"', "\wr"] & \mathrm{Bun}_{2,4}^0 \arrow[d, "\mathrm{dual}"', "\wr"] \\ \mathrm{Bun}_{2,4}^{1} \times \overline{\mathbf{Coh}_0}^{1,1} & \mathcal{H}^1 \arrow[l, "p"] \arrow[r, "q"'] & \mathrm{Bun}_{2,4}^0 \end{tikzcd}$$

(Lemma 4.9), so by compactifying the map $p^\vee$, we have also found a compactification of the map p.

Therefore, we consider the stack $\mathcal{H} = \mathcal{H}^0$ defined by

$$\mathcal{H} := \left\langle 0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to \mathcal{T}^\bullet \to 0 : \begin{array}{c} \mathcal{F}^\bullet \in \mathrm{Bun}_{2,4}^{-1}, \\ \mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{0}, \\ \mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1} \end{array} \right\rangle.$$

This naturally embeds into the stack $\overline{\mathcal{H}}$ that classifies similar extensions, but we allow $\mathcal{E}^\bullet$ to be in a slightly larger stack $\mathrm{Bun}_{2,4}^{+,0} \supset \mathrm{Bun}_{2,4}^0$: we set

$$\overline{\mathcal{H}} := \left\langle 0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to \mathcal{T}^\bullet \to 0 : \begin{array}{c} \mathcal{F}^\bullet \in \mathrm{Bun}_{2,4}^{-1}, \\ \mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^{+,0}, \\ \mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1} \end{array} \right\rangle \tag{13.1.1}$$

where $\mathrm{Bun}_{2,4}^{+,0}$ denotes the stack

$$\mathrm{Bun}_{2,4}^{+,0} := \left\langle \mathcal{E}^\bullet \in \mathbf{Coh}_{2,4}^0 : \begin{array}{c} \text{the torsion part } \mathcal{T}^\bullet \text{ of } \mathcal{E}^\bullet \\ \text{is such that} \\ \mathcal{T}^0 \oplus \mathcal{T}^{(-1,D)} \\ \text{has length 1} \end{array} \right\rangle. \tag{13.1.2}$$

Remark 13.5. We have inclusions

$$\mathrm{Bun}_{2,4}^0 \subset \mathrm{Bun}_{2,4}^{+,0} \subset \mathbf{Coh}_{2,4}^0$$

and $\mathrm{Bun}_{2,4}^{+,0}$ is the smallest substack that contains all $\mathcal{E}^\bullet \in \mathbf{Coh}_{2,4}^0$ appearing in extensions

$$0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to \mathcal{T}^\bullet \to 0$$

with $\mathcal{F}^\bullet \in \mathrm{Bun}_{2,4}^{-1}$ and $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$ such that the extension is non-trivial in at least one parabolic degree.

Remark 13.6. Let $\mathcal{T}^\bullet$ be a non-zero parabolic torsion sheaf on $\mathbf{P}^1$. The condition that $\mathcal{T}^0 \oplus \mathcal{T}^{(-1,D)}$ has length 1 implies that $\mathcal{T}^\bullet$ is supported on a point in D. Indeed, if a torsion sheaf $\mathcal{T}^\bullet$ is supported on points away from D, then $\mathcal{T}^0 = \mathcal{T}^{(-1,D)}$, so that the length is even.

Lemma 13.7. *The map*

$$\bar{p} \colon \overline{\mathcal{H}} \to \mathrm{Bun}_{2,4}^{-1} \times \overline{\mathbf{Coh}_0}^{1,1}, \qquad (\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\mathcal{F}^\bullet, [\mathcal{T}^\bullet])$$

is proper.

Proof. The moduli stack

$$\underline{\mathrm{Ext}}^1(\mathcal{T}^\bullet_{\mathrm{univ}}, \mathcal{F}^\bullet_{\mathrm{univ}}) \to \mathrm{Bun}_{2,4}^{-1} \times \mathbf{Coh}_0^{1,1}$$

is a vector bundle. The fibers of this map are $\mathbf{A}^2$. The compactification $\overline{\mathcal{H}}$ is the complement in $\underline{\mathrm{Ext}}^1(\mathcal{T}^\bullet_{\mathrm{univ}}, \mathcal{F}^\bullet_{\mathrm{univ}})$ of the zero section

$$\overline{\mathcal{H}} = \underline{\mathrm{Ext}}^1(\mathcal{T}^\bullet_{\mathrm{univ}}, \mathcal{F}^\bullet_{\mathrm{univ}}) \setminus \mathrm{Bun}_{2,4}^{-1} \times \mathbf{Coh}_0^{1,1}$$

and by mapping to $\overline{\mathbf{Coh}}_0^{1,1}$ instead of $\mathbf{Coh}_0^{1,1}$, we divide out the scaling, so that $\overline{\mathcal{H}}$ is a projective bundle over $\mathrm{Bun}_{2,4}^{-1} \times \overline{\mathbf{Coh}}_0^{1,1}$. □

It is immediate from the definition that this map extends $p^\vee \colon \mathcal{H} \to \mathrm{Bun}_{2,4}^{-1} \times \overline{\mathbf{Coh}}_0^{1,1}$ and can therefore be considered a compactification.

13.1.2. *Clean extension.* The goal of this section is to prove the following proposition, which is one of the ingredients of the proof of Proposition 13.4.

Proposition 13.8. *Let $j \colon \mathcal{H} \hookrightarrow \overline{\mathcal{H}}$ the natural inclusion and let $q^\vee \colon \mathcal{H} \to \mathrm{Bun}_{2,4}^0$ the map given by $(\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto \mathcal{E}^\bullet$. Then*

$$j_!(q^{\vee,*}\,\mathrm{Aut}_E^0) = j_{!*}(q^{\vee,*}\,\mathrm{Aut}_E^0).$$

Remark 13.9. The above proposition is true for $q^\vee$ if and only if it is true for q. (In the statement for q instead of $q^\vee$, the compactification we use is $j \circ \mathrm{dual}$.) Indeed, because $\mathrm{dual} \colon \mathrm{Bun}_{2,4}^{0,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ is the identity on $\mathbf{F}_q$-points (Corollary 8.3) and Aut_E^0 is supported on the relevant locus, we have $\mathrm{dual}^*\,\mathrm{Aut}_E^0 = \mathrm{Aut}_E^0$.

To prove this proposition, we extend the map $q^\vee$ to the following map $\bar{q}^\vee$.

Lemma 13.10. *The map*

$$\bar{q}^\vee \colon \overline{\mathcal{H}} \to \mathrm{Bun}_{2,4}^{+,0}, \qquad (\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto \mathcal{E}^\bullet.$$

is smooth and surjective and fits into the Cartesian square

$$\begin{tikzcd} \mathcal{H} \arrow[r, hook, "j"] \arrow[d, "q^\vee"] & \overline{\mathcal{H}} \arrow[d, "\bar{q}^\vee"] \\ \mathrm{Bun}_{2,4}^0 \arrow[r, hook] & \mathrm{Bun}_{2,4}^{+,0} \end{tikzcd}$$

Proof. We prove the smoothness by examining the tangent spaces: for every $(\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \in \overline{\mathcal{H}}$, we show that the map on tangent spaces

$$\mathrm{Ext}^1(\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet, \mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet) \to \mathrm{Ext}^1(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$$

is surjective.

Let

$$(0 \to \mathcal{E}^\bullet \to \tilde{\mathcal{E}}^\bullet \to \mathcal{E}^\bullet \to 0) \in \mathrm{Ext}^1(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$$

be arbitrary. We construct a preimage as follows. First, we take the pushout of the inclusion $\mathcal{E}^\bullet \hookrightarrow \tilde{\mathcal{E}}^\bullet$ along $\mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet$ to get a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^\bullet \arrow[r] \arrow[d, two heads] & \tilde{\mathcal{E}}^\bullet \arrow[r] \arrow[d] & \mathcal{E}^\bullet \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{T}^\bullet \arrow[r] & \mathcal{M}^\bullet \arrow[r] & \mathcal{E}^\bullet \arrow[r] & 0 \end{tikzcd} \tag{13.1.3}$$

The bottom row is the pullback of an extension

$$0 \to \mathcal{T}^\bullet \to \mathcal{K}^\bullet \to \mathcal{T}^\bullet \to 0. \tag{13.1.4}$$

Indeed, in the long exact sequence obtained by applying $\operatorname{Ext}^\bullet(-,\mathcal{T}^\bullet)$ to the short exact sequence $0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to \mathcal{T}^\bullet \to 0$, we find the exact sequence

$$\operatorname{Ext}^1(\mathcal{T}^\bullet,\mathcal{T}^\bullet) \to \operatorname{Ext}^1(\mathcal{E}^\bullet,\mathcal{T}^\bullet) \to \operatorname{Ext}^1(\mathcal{F}^\bullet,\mathcal{T}^\bullet);$$

since $\mathcal{F}^\bullet$ is torsion-free by definition of $\overline{\mathcal{H}}$, $\operatorname{Ext}^1(\mathcal{F}^\bullet,\mathcal{T}^\bullet)$ is zero and therefore every extension in $\operatorname{Ext}^1(\mathcal{E}^\bullet,\mathcal{T}^\bullet)$ lifts to an extension in $\operatorname{Ext}^1(\mathcal{T}^\bullet,\mathcal{T}^\bullet)$.

Combining (13.1.3) and (13.1.4), we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^\bullet \arrow[r] \arrow[d, two heads] & \tilde{\mathcal{E}}^\bullet \arrow[r] \arrow[d] & \mathcal{E}^\bullet \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \mathcal{T}^\bullet \arrow[r] & \mathcal{K}^\bullet \arrow[r] & \mathcal{T}^\bullet \arrow[r] & 0 \end{tikzcd} \tag{13.1.5}$$

By taking the kernels of the vertical maps, we find a preimage of the extension $(\mathcal{E}^\bullet \to \tilde{\mathcal{E}}^\bullet \to \mathcal{E}^\bullet) \in \operatorname{Ext}^1(\mathcal{E}^\bullet,\mathcal{E}^\bullet)$ in $\operatorname{Ext}^1(\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet, \mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet)$, which proves the smoothness.

For any $\mathcal{E}^\bullet \in \operatorname{Bun}_{2,4}^{+,0}$, the short exact sequence

$$(T_x\mathcal{E}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow (\mathcal{E}^\bullet/T_x\mathcal{E}^\bullet)) \in \overline{\mathcal{H}}$$

is an inverse image of $\mathcal{E}^\bullet$. This proves the surjectivity.

It follows immediately from the definitions that the square is Cartesian. □

Proof of Proposition 13.8. Because of Lemma 13.10, we can calculate the intermediate extension of $q^{\vee,*}\operatorname{Aut}_E^0$ along j by calculating the intermediate extension of Aut_E^0 along the bottom map and pulling it back along $\bar{q}^\vee$. In Proposition 13.12, we will show that the intermediate extension of Aut_E^0 along $\operatorname{Bun}_{2,4}^0 \hookrightarrow \operatorname{Bun}_{2,4}^{+,0}$ is the extension by zero, which completes the proof. □

Corollary 13.11. *Up to shift, $j_! q^{\vee,*}\operatorname{Aut}_E^0$ is perverse and pure.*

Proof. We know that Aut_E^0 is perverse (Theorem 12.1). Since $q^\vee$ is smooth (Lemma 4.12 and Lemma 4.9), $q^{\vee,*}\operatorname{Aut}_E^0$ is perverse up to shift. Lastly, because $j_{!*}$ preserves perversity and purity, the corollary now follows from the proposition. □

The rest of this section is devoted to the proof of the following proposition.

Proposition 13.12. *Let* $j\colon \mathrm{Bun}^0_{2,4} \hookrightarrow \mathrm{Bun}^{+,0}_{2,4}$ *the natural inclusion. Then*

$$j_! \operatorname{Aut}^0_E = j_{!*} \operatorname{Aut}^0_E .$$

We prove this on smooth charts of $\mathrm{Bun}^{+,0}_{2,4}$. This is possible because intermediate extensions can be calculated on smooth charts. Let $x \in D$ and let

$$\mathrm{Bun}^{+,0}_{2,4,x} \subset \mathrm{Bun}^{+,0}_{2,4}$$

be the substack consisting of those $\mathcal{E}^\bullet \in \mathrm{Bun}^{+,0}_{2,4}$ whose torsion part lives in even degree and is supported at x, i.e., $\mathcal{E}^{(2i+1,x)}$ is torsion-free for all $i \in \mathbf{Z}$. We can do the same when the torsion part lives in odd degree, but will leave this out of the proof.

Consider the stack $\mathcal{H}^+_{D\setminus\{x\}}$ defined as

$$\mathcal{H}^+_{D\setminus\{x\}} := \left\langle 0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to k_x \to 0 : \begin{array}{c} \mathcal{F}^\bullet \in \mathrm{Bun}^{-1}_{2,D\setminus\{x\}}, \\ \mathcal{E}^\bullet \in \mathbf{Coh}^0_{2,D\setminus\{x\}}, \\ k_x \in \mathbf{Coh}^1_0 \text{ length 1, supported on } x \end{array} \right\rangle .$$

The isomorphisms in this stack are isomorphisms of exact sequence. We emphasize that the parabolic sheaves $\mathcal{F}^\bullet, \mathcal{E}^\bullet$ that $\mathcal{H}^+_{D\setminus\{x\}}$ classifies only have a parabolic structure at $D \setminus \{x\}$ and not at x, and that $\mathcal{E}^\bullet$ can have torsion, while $\mathcal{F}^\bullet$ can not. If we have such an exact sequence

$$(0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to k_x \to 0) \in \mathcal{H}^+_{D\setminus\{x\}},$$

then we can equip $\mathcal{E}^\bullet$ with a parabolic structure at x by setting

$$\mathcal{E}^{0,x} := \mathcal{E} \qquad \text{and} \qquad \mathcal{E}^{(-1,x)} := \mathcal{F}.$$

With this structure, we have $\mathcal{E}^\bullet \in \mathrm{Bun}^{+,0}_{2,4}$. This construction defines an isomorphism

$$\pi_x \colon \mathcal{H}^+_{D\setminus\{x\}} \xrightarrow{\sim} \mathrm{Bun}^{+,0}_{2,4,x} .$$

with inverse given by

$$\mathcal{E}^\bullet \mapsto (0 \to \mathcal{E}^{\bullet,(-1,x)} \to \mathcal{E}^{\bullet,(0,x)} \to \mathcal{Q} \to 0)$$

where $\mathcal{Q}$ is the quotient and the other sheaves are the sheaves $\mathcal{E}^{(-1,x)}$ and $\mathcal{E}^{(0,x)}$ equipped with their parabolic structure at $D \setminus \{x\}$.

Let $\widetilde{\operatorname{Aut}}^0_E$ be the pullback of Aut^0_E to $\pi_x^{-1}(\mathrm{Bun}^0_{2,4})$ and let $j_x \colon \pi_x^{-1}(\mathrm{Bun}^0_{2,4}) \hookrightarrow \mathcal{H}^+_{D\setminus\{x\}}$ be the inclusion. We have thus reduced the proof to showing

$$j_{x,!*}\widetilde{\operatorname{Aut}}^0_E = \mathbf{R}j_{x,!}\widetilde{\operatorname{Aut}}^0_E. \tag{13.1.6}$$

In fact, we prove the slightly stronger statement

$$\mathbf{R}j_{x,*}\widetilde{\operatorname{Aut}}^0_E = \mathbf{R}j_{x,!}\widetilde{\operatorname{Aut}}^0_E \tag{13.1.7}$$

by showing that the stalks of $\mathbf{R}j_{x,*}\widetilde{\operatorname{Aut}}^0_E$ at all points in the complement of j are zero. This follows from the following two lemmas.

Lemma 13.13. *Let*

$$\xi = (0 \to \mathcal{F}^\bullet \to \mathcal{E}^\bullet \to k_x \to 0) \in \mathcal{H}^+_{D\setminus\{x\}} \setminus \operatorname{im} j_x.$$

Then we have a map

$$\big(\operatorname{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus 0\big) \to \operatorname{im} j_x \subset \mathcal{H}^+_{D\setminus\{x\}}$$

and

$$(\mathbf{R} j_{x,*}\widetilde{\operatorname{Aut}}^0_E)|_\xi = \mathrm{H}^*(\operatorname{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus 0,\ \widetilde{\operatorname{Aut}}^0_E). \tag{13.1.8}$$

Proof. The stack $\mathcal{H}^+_{D\setminus\{x\}}$ comes equipped with a map

$$f\colon \mathcal{H}^+_{D\setminus\{x\}} \to \operatorname{Bun}^{-1}_{2,D\setminus\{x\}} \times \mathbf{B}\operatorname{Aut}(k_x), \qquad (\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow k_x) \mapsto (\mathcal{F}^\bullet, k_x).$$

This map is a vector bundle ([**Hei04**, Remark 6.3 (1)]) and its zero section

$$\operatorname{Bun}^{-1}_{2,D\setminus\{x\}} \times \mathbf{B}\operatorname{Aut}(k_x) \to \mathcal{H}^+_{D\setminus\{x\}}$$

is exactly the complement of $\operatorname{im} j_x$. A general lemma ([**Hei04**, Lemma 0.3] is the statement we use; the calculation appears in [**FGV02**] and [**Bry86**]) then says that because $\widetilde{\operatorname{Aut}}^0_E$ is $\mathbb{G}_\mathrm{m}$-equivariant for the natural $\mathbb{G}_\mathrm{m}$-action on the vector bundle $\mathcal{H}^+_{D\setminus\{x\}} \to \operatorname{Bun}^{-1}_{2,D\setminus\{x\}} \times \mathbf{B}\operatorname{Aut}(k_x)$, we have

$$s^*\left(\mathbf{R} j_{x,*}\widetilde{\operatorname{Aut}}^0_E\right) = \mathbf{R} f_*\widetilde{\operatorname{Aut}}^0_E.$$

The statement then follows by pulling back $\mathbf{R} f_*\widetilde{\operatorname{Aut}}^0_E$ along $(\mathcal{F}^\bullet, k_x)\colon \operatorname{Spec} k \to \operatorname{Bun}^{-1}_{2,D\setminus\{x\}} \times \mathbf{B}\operatorname{Aut}(k_x)$. $\square$

Lemma 13.14. *Let $\mathcal{F}^\bullet \in \operatorname{Bun}^{-1}_{2,D\setminus\{x\}}$. Then*

$$\mathrm{H}^*(\operatorname{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus 0,\ \widetilde{\operatorname{Aut}}^0_E) = 0.$$

Proof. Let $\mathcal{F}^\bullet \in \operatorname{Bun}^{-1}_{2,D\setminus\{x\}}$ and consider the map

$$\begin{gathered} \operatorname{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus \{0\} \to \mathcal{H}^+_{D\setminus\{x\}} \xrightarrow{\pi_x} \operatorname{Bun}^0_{2,4}, \\ (\mathcal{F}^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto (\mathcal{E}^\bullet, \operatorname{im}(\mathcal{F}|_x \to \mathcal{E}|_x)) \end{gathered}$$

where we denote by $\operatorname{im}(\mathcal{E}^\bullet, \mathcal{F}|_x \to \mathcal{E}|_x)$ the parabolic sheaf in $\operatorname{Bun}^0_{2,4}$ whose parabolic structure at x is given by the flag $\operatorname{im}(\mathcal{F}|_x \to \mathcal{E}|_x) \subset \mathcal{E}|_x$, while the parabolic structure at $D \setminus \{x\}$ is given by the parabolic structure of $\mathcal{E}^\bullet$. We can factorize this map as

$$\operatorname{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus \{0\} \to \mathbf{P}(\mathcal{F}|_x) \to \operatorname{Bun}^{-1}_{2,4} \xrightarrow{T_x^{-1}} \operatorname{Bun}^0_{2,4}$$

where the first map is

$$\rho\colon \operatorname{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus \{0\} \to \mathbf{P}(\mathcal{F}|_x), \quad (\mathcal{F}^\bullet \overset{i}{\hookrightarrow} \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto \ker(i|_x\colon \mathcal{F}|_x \to \mathcal{E}|_x)$$

and the second is

$$\phi_{\mathcal{F}^\bullet}\colon \mathbf{P}(\mathcal{F}|_x) \to \operatorname{Bun}^{-1}_{2,4}, \qquad (\ell \subset \mathcal{F}|_x) \mapsto (\mathcal{F}^\bullet, \ell \subset \mathcal{F}|_x). \tag{13.1.9}$$

We will now show that for all $\mathcal{F}^\bullet \in \mathrm{Bun}_{2,D\setminus\{x\}}^{-1}$, we have

$$\mathrm{H}^*(\mathbf{P}^1(\mathcal{F}|_x), \phi_{\mathcal{F}^\bullet}^* T_{x,*} \mathrm{Aut}_E^0) = 0. \tag{13.1.10}$$

The lemma follows from this statement, because by the projection formula, the cohomology of the pullback to $\mathrm{Ext}^1(k_x, \mathcal{F}^\bullet) \setminus \{0\}$ is equal to

$$\mathrm{H}^*(\mathbf{P}^1(\mathcal{F}|_x), \phi_{\mathcal{F}^\bullet}^* T_{x,*} \mathrm{Aut}_E^0 \otimes \mathbf{R}\rho_* \mathbf{Q}_\ell)$$

and $\mathbf{R}\rho_*\mathbf{Q}_\ell$, being the cohomology of $\mathbb{G}_\mathrm{m}$, is the extension of two constant local systems.

To prove (13.1.10), we distinguish several cases and in each case, the pullback of $T_{x,*} \mathrm{Aut}_E^0$ vanishes because of one of the cusp conditions (Theorem 6.4), or more precisely, because of the cohomological vanishing properties of the local system E (Section 12.1) that imply most of the cusp conditions.

Recall what we know about $T_{x,*} \mathrm{Aut}_E^0$: it is supported on $\mathrm{Bun}_{2,4}^{-1,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^{-1}$ and there is an isomorphism $\psi = T_x T_\infty \colon \mathrm{Bun}_{2,4}^{1,\mathrm{r}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{-1,\mathrm{r}}$ such that the restriction to $\mathrm{Bun}_{2,4}^{-1,\mathrm{r}}$ is isomorphic to $\psi_*\alpha_*(j_{!*}E) \otimes E|_\infty$. (In this argument, we can forget about the constant local system $E|_\infty$.)

Let $\mathcal{F}^\bullet \in \mathrm{Bun}_{2,D\setminus\{x\}}^{-1}$. Suppose that the underlying vector bundle $\mathcal{F}$ is not isomorphic to $\mathcal{O} \oplus \mathcal{O}(-1)$. Then it immediately follows from the definition of the relevant locus (Definition 6.1) that the image of $\phi_{\mathcal{F}^\bullet}$ lies outside of $\mathrm{Bun}_{2,4}^{-1,\mathrm{r}}$, so that the pullback of $T_{x,*} \mathrm{Aut}_E^0$ is zero. Likewise, if the underlying vector bundle $\mathcal{F}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-1)$, but at least two of the flags of $\mathcal{F}^\bullet$ are of the form $\mathcal{O}|_y$, then the image of $\phi_{\mathcal{F}^\bullet}$ also misses the relevant locus and the pullback is zero again.

Write $\mathcal{F}^\bullet = (\mathcal{O} \oplus \mathcal{O}(-1), (\ell_y)_{y\in D\setminus\{x\}})$ and suppose there is exactly one $y_0 \in D \setminus \{x\}$ with $\ell_{y_0} = \mathcal{O}|_y$. Up to isomorphism, we can then assume that $\ell_y = \mathcal{O}(-1)|_y$ for $y \in D \setminus \{x, y_0\}$. We then have for all $\ell \in \mathbf{P}(\mathcal{F}|_x)$,

$$(\mathcal{F}^\bullet, \ell) = {}^{\ell}T_x \left(T_{y_0} T_x \tilde{\mathcal{E}}\right)$$

where ${}^{\mathcal{O}(-1)|_x}T_x$ should be interpreted as T_x. Since we understand all length 1 lower modifications of points in the relevant locus, we can now conclude (in this case, Corollary 6.23 tells us all we need) that $\mathcal{O}|_x$ is mapped to a point outside of the relevant locus, while the remaining $\mathbf{A}^1$ is mapped as

$$T_x T_\infty \circ k_x^{(\,-\,,0)} \colon \mathbf{A}^1 \to \mathrm{Bun}_{2,4}^{-1,\mathrm{r}}, \qquad \lambda \mapsto T_x T_\infty \alpha(k_x \xrightarrow{\lambda} k_x \xrightarrow{0} k_x)$$

(or its shift, $T_x T_\infty \circ k_x^{(0,\,-\,)}$), so that the pullback has vanishing cohomology by Lemma 12.4.

Lastly, suppose that none of the flags ℓ_y are equal to $\mathcal{O}|_y$. If they come from a global section, then ϕ is given by

$$\ell \mapsto {}^{\ell}T_x \tilde{\mathcal{E}}(-1)$$

(this follows immediately from the definitions) and the calculation then proceeds as the previous one. If, however, they do not come from a global section, then ϕ is given by

$$\ell \mapsto {}^{\ell}T_x(\hat{\mathcal{E}}(-1))$$

as one immediately sees by writing this down explicitly. In this case, our calculation of the modifications of $\hat{\mathcal{E}}$ (Theorem 9.1; part (2c)) tells us that there is a section $\sigma\colon \mathbf{P}^1 \to \overline{\mathbf{Coh}_0}^{1,1}$ of the support map $\mathrm{Supp}\colon \overline{\mathbf{Coh}_0}^{1,1} \to \mathbf{P}^1$ such that ϕ is equal to $T^2_\infty \circ \alpha \circ \sigma$, up to automorphism on source and target. The pullback of $T_{x,*}\operatorname{Aut}^0_E$ along this map is zero (Proposition 12.3 and Remark 12.6). $\square$

13.2. Local system outside of D. In this section we prove the following proposition.

Proposition 13.15. *The complex*

$$(\mathbb{H}\operatorname{Aut}^0_E)|_{\pi_1^{-1}(\mathbf{P}^1\setminus D)\times\mathbf{P}^1\setminus D} \tag{13.2.1}$$

is a local system of rank 4 concentrated in degree -1.

Proof. We first consider the fibers of $(\mathbb{H}\operatorname{Aut}^0_E)|_{\pi_1^{-1}(\mathbf{P}^1\setminus D)\times\mathbf{P}^1\setminus D}$: we prove that for any point $(\mathcal{E}^\bullet,\mathcal{T}^\bullet)\in\pi_1^{-1}(\mathbf{P}^1\setminus D)\times\mathbf{P}^1\setminus D$, the fiber $(\mathbf{R}p_!q^*\operatorname{Aut}^0_E)|_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$ is concentrated in degree 1 and of constant rank 4. (Recall $\mathbb{H}=\mathbf{R}p_!q^*[2]$.) We denote by

$$\phi\colon p^{-1}((\mathcal{E}^\bullet,\mathcal{T}^\bullet))\to\mathrm{Bun}^{0,\mathrm{r}}_{2,4}$$

the restriction of q; we know the image lies in $\mathrm{Bun}^{0,\mathrm{r}}_{2,4}\subset\mathrm{Bun}^0_{2,4}$ (Theorem 9.2) and $p^{-1}((\mathcal{E}^\bullet,\mathcal{T}^\bullet))$ is isomorphic to $\mathbf{P}^1$. By definition, we have

$$(\mathbf{R}p_!q^*\operatorname{Aut}^0_E)|_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}=\mathrm{H}^*_c(p^{-1}((\mathcal{E}^\bullet,\mathcal{T}^\bullet)),\ \phi^*\operatorname{Aut}^0_E). \tag{13.2.2}$$

Because we know that Aut^0_E is the intermediate extension of the local system $\alpha^*(E\otimes E|_\infty)$ on $\alpha(\mathbf{P}^1\setminus D)$ (Theorem 12.1) and that ϕ gives a degree 2 map $\mathbf{P}^1\to\mathbf{P}^1$ when composed with $\pi_0\colon\mathrm{Bun}^{0,\mathrm{r}}_{2,4}\to\mathbf{P}^1$ (Theorem 9.2) whose ramification behavior we understand well enough (Addendum 9.3), it is manageable to calculate these cohomology groups.

We first prove that the pullback along ϕ is still irreducible.

Lemma 13.16. *Let $f\colon C'\to C$ be a finite map of degree 2 between two curves (not necessarily complete) and let F be an irreducible rank 2 local system on C such that the monodromy around at least one point is unipotent and not semisimple. Then the pullback f^*F is irreducible.*

Proof. Suppose towards a contradiction that f^*F is not irreducible. Then there exists a rank 1 local system L on C' and a non-zero map $f^*F\to L$, which induces a non-zero map $F\to f_*L$. Since both local systems are rank 2 and F is irreducible, this map $F\to f_*L$ is an isomorphism. However, for every $\gamma\in\pi_1(C)$, the square γ^2 acts semisimply on f_*L, contradiction the assumption that the monodromy of F is not semisimple. $\square$

As a result, $\phi^* \operatorname{Aut}^0_E$ has no global sections.

Corollary 13.17. *Let* $D' := (\pi_0 \circ \phi)^{-1}(D) \subset \mathbf{P}^1$*. Then* $(\phi^* \operatorname{Aut}^0_E)|_{\mathbf{P}^1 \setminus D'}$ *is irreducible and therefore*

$$\mathrm{H}^0(\mathbf{P}^1, \phi^* \operatorname{Aut}^0_E) = 0.$$

Proof. We have

$$\operatorname{Aut}^0_E|_{\mathrm{Bun}^{0,\mathrm{r}}_{2,4}} = (\pi_0^{-1}(\mathbf{P}^1 \setminus D) \hookrightarrow \mathrm{Bun}^{0,\mathrm{r}}_{2,4})_{!*}\, \pi_0^{-1}(E) \otimes E|_\infty$$

(Theorem 12.1) and as a consequence, we find

$$(\phi^* \operatorname{Aut}^0_E)|_{\mathbf{P}^1 \setminus D'} = (\pi_0 \circ \phi)^*(E) \otimes E|_\infty.$$

By Lemma 13.16, this is an irreducible rank 2 local system. □

By duality, its H^2 is zero as well.

Corollary 13.18. *We have*

$$\mathrm{H}^2(\mathbf{P}^1, \phi^* \operatorname{Aut}^0_E) = 0.$$

Proof. This follows by applying Verdier duality: we have $(\mathbb{D}\, \phi^* \operatorname{Aut}^0_E)|_{\mathbf{P}^1 \setminus D'} = (\pi_0 \circ \phi)^*(E^\vee \times E|^\vee_\infty)$ (up to shift) and $E^\vee$ is also an irreducible rank 2 local system with unipotent monodromy. □

Lastly, we calculate the Euler characteristic.

Lemma 13.19 (The Euler characteristic of $\phi^* E$)**.** *Let* $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \overline{\mathbf{Coh}}^{1,1}_0$ *denote the inclusion and let*

$$\phi\colon \mathbf{P}^1 \to \overline{\mathbf{Coh}}^{1,1}_0$$

be a map such that

$$\phi_0 := \operatorname{Supp} \circ \phi\colon \mathbf{P}^1 \to \mathbf{P}^1 \quad \text{has degree 2.}$$

Then the Euler characteristic of $\phi^* j_{!*} E$ *is -4.*

Proof. Let $S \subset \mathbf{P}^1$ denote the ramification locus of the degree 2 map ϕ_0. It consists of two points. As before, we write $D' = \phi_0^{-1}(D)$. We define

$$s := \#(\phi_0^{-1}(D) \cap S).$$

This implies

$$\#D' = 8 - s.$$

Let

$$k\colon \mathbf{P}^1 \setminus D' \hookrightarrow \mathbf{P}^1$$

denote the inclusion. This gives a distinguished triangle

$$\mathbf{R}k_!(\phi^* j_{!*} E) \to \phi^* j_{!*} E \to (\phi^* j_{!*} E)|_{D'} \xrightarrow{+1}$$

The additivity of the Euler characteristic χ then implies

$$\chi(\mathbf{P}^1, \phi^* j_{!*}E) = \chi_c(\mathbf{P}^1 \setminus D', \phi^* j_{!*}E) + \chi(D', \phi^* j_{!*}E)$$

where χ_c denotes the Euler-characteristic with compact support. We now calculate these two terms on the right.

To calculate $\chi_c(\mathbf{P}^1 \setminus D', \phi^* j_{!*}E)$, first note that

$$\phi^*(j_{!*}E)|_{\mathbf{P}^1 \setminus D'} = (\phi_0|_{\mathbf{P}^1 \setminus D})^* E$$

is a local system of rank 2. The Grothendieck-Ogg-Shafarevich formula ([**Gro77**, formula 7.2], or [**KR14**, theorem 9.1]) then tells us

$$\chi_c(\mathbf{P}^1 \setminus D', \phi^* j_{!*}E) = 2 \cdot \chi_c(\mathbf{P}^1 \setminus D', \mathbf{Q}_\ell) = 2 \cdot (2 - \#D') = -12 + 2s$$

where χ_c denotes the Euler characteristic using compact cohomology and the 2 is the rank of the local system.

To calculate the second term, we use

$$\chi(D', \phi^* j_{!*}E) = \sum_{x \in D'} \chi(\operatorname{Spec} k,\ (j_{!*}E)|_{\phi(x)}).$$

For $x \in S \cap D'$, we have $\phi(x) = k^0_{x_0}$ (the bundle with $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ automorphisms), while for $x \in D' \setminus S$, the image $\phi(x)$ is either $k^{(1,0)}_{x_0}$ or $k^{(0,1)}_{x_0}$ for some $x_0 \in D$ (Lemma 9.33). Using our calculation of the stalks of $j_{!*}E$ (Lemma 12.4), we then find

$$\begin{aligned} \chi(\operatorname{Spec} k, (j_{!*}E)|_{\phi(x)}) &= 0 \qquad && \text{for } x \in D' \cap S, \text{ and} \\ \chi(\operatorname{Spec} k, (j_{!*}E)|_{\phi(x)}) &= 1 \qquad && \text{for } x \in D' \setminus S. \end{aligned}$$

Summing over these gives us

$$\chi(D', \phi^* j_{!*}E) = 8 - 2s.$$

The sum of the two Euler characteristics is then $-12 + 2s + 8 - 2s = -4$, which completes the proof. □

We have now proven that $\mathrm{H}^i(\mathbf{P}^1, \phi^* \operatorname{Aut}^0_E) = \mathbf{R}^i q_! p^* \operatorname{Aut}^0_E$ is zero when $i \neq 1$ and has dimension 4 when $i = 1$. Since $\mathbb{H} \operatorname{Aut}^0_E$ decomposes as a direct sum of shifted simple perverse sheaves (Proposition 13.4), this completes the proof. □

13.3. Vanishing outside of $\operatorname{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$. In this section, we prove that $\mathbb{H} \operatorname{Aut}^0_E$ is zero outside of $\operatorname{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1} \subset \operatorname{Bun}^1_{2,4} \times \overline{\mathbf{Coh}}_0^{1,1}$ (Corollary 13.25). It is easy to show that $\mathbb{H} \operatorname{Aut}^0_E$ vanishes on codimension 3 points, but for the complete statement, we have to determine the length 1 lower modifications of the eight parabolic bundles in $\operatorname{Bun}^1_{2,4} \setminus \operatorname{Bun}^{1,\mathrm{r}}_{2,4}$ with $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ as their automorphism group.

Consider the following parabolic bundle:

$$\mathcal{E}^\bullet_{\mathrm{irrel}} := (\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-1), D) \in \operatorname{Bun}^1_{2,4} \setminus \operatorname{Bun}^{1,\mathrm{r}}_{2,4}. \tag{13.3.1}$$

Lemma 13.20. *There are 8 parabolic bundles $\mathcal{E}^\bullet \in \mathrm{Bun}_{2,4}^1 \setminus \mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ with $\dim \mathrm{Aut}(\mathcal{E}^\bullet) = 2$: $\mathcal{E}^\bullet_{\mathrm{irrel}}$, $T_D \mathcal{E}^\bullet_{\mathrm{irrel}}(2)$ and $T_{x_1} T_{x_2} \mathcal{E}^\bullet_{\mathrm{irrel}}(1)$ with $x_1, x_2 \in D$ and $x_1 \neq x_2$.*

Proof. This is an easy calculation. □

Remark 13.21. The scalars $\mathbb{G}_\mathrm{m}$ are in the automorphism group of any parabolic bundle and all parabolic bundles that have an automorphism group of dimension 1 lie in the relevant locus.

We now determine the modifications of these points, which will be necessary to prove that $\mathbb{H}\,\mathrm{Aut}_E^0$ is supported on the relevant locus. For $x_0 \in D$, we introduce the following notation for maps we have already seen before:

$$k_{x_0}^{(\,-\,,0)} \colon \mathbf{A}^1 \to \overline{\mathbf{Coh}}_0^{1,1}, \qquad \lambda \mapsto (k_{x_0} \xrightarrow{\lambda} k_{x_0} \xrightarrow{0} k_{x_0}).$$

and

$$k_{x_0}^{(0,\,-\,)} \colon \mathbf{A}^1 \to \overline{\mathbf{Coh}}_0^{1,1}, \qquad \lambda \mapsto (k_{x_0} \xrightarrow{0} k_{x_0} \xrightarrow{\lambda} k_{x_0}).$$

Lemma 13.22. *Let $\mathcal{E}^\bullet$ be one of the eight points outside of the relevant locus with* $\dim \mathrm{Aut}(\mathcal{E}^\bullet) = 2$.

(1) *Let $x \in \mathbf{P}^1 \setminus D$. Consider the map*

$$\mathbf{P}(\mathcal{E}|_x) \to \mathrm{Bun}_{2,4}^0, \qquad \ell \mapsto T_x^\ell \mathcal{E}^\bullet.$$

This map sends the flag from the maximal destabilizing subbundle to a point outside of $\mathrm{Bun}_{2,4}^{0,\mathrm{r}}$ and the restriction to the complement is either $k_{x_0}^{(\,-\,,0)}$ or $k_{x_0}^{(0,\,-\,)}$ for some $x_0 \in D$.

(2) *Let $x \in D$. Consider the modification map*

$$\phi \colon \mathbf{P}(\mathcal{E}|_x) \cup_{\ker(\mathcal{E}|_x \to \mathcal{E}^{(1,x)}|_x)} \mathbf{P}((T_x\mathcal{E})|_x) \to \mathrm{Bun}_{2,4}^0$$

that sends a flag of $\mathcal{E}^\bullet$ at x to the corresponding modification. The restriction of ϕ to either $\mathbf{P}(\mathcal{E}|_x)$ or $\mathbf{P}((T_x\mathcal{E})|_x)$ has image in $\mathrm{Bun}_{2,4}^0 \setminus \mathrm{Bun}_{2,4}^{0,\mathrm{r}}$. The restriction to the complement, which is isomorphic to $\mathbf{A}^1$, is either $k_{x_0}^{(\,-\,,0)}$ or $k_{x_0}^{(0,\,-\,)}$ for some $x_0 \in D$.

Proof. It suffices to prove this for the point $\mathcal{E}^\bullet_{\mathrm{irrel}}$: the other 7 points are obtained by applying elementary Hecke operators to $\mathcal{E}^\bullet_{\mathrm{irrel}}$ (Lemma 13.20) and this allows us to find their modifications as explained in Section 9.1.4. The proof is essentially a few simple calculations.

First we prove this for $x \in \mathbf{P}^1 \setminus D$. Modifying $\mathcal{E}^\bullet_{\mathrm{irrel}}$ with respect to $\mathcal{O}(2)|_x$ gives $(\mathcal{O}(2), \emptyset) \oplus (\mathcal{O}(-2), D)$, which is not in the relevant locus. Let $r \in \mathbf{A}^1$ and let $\sigma \colon \mathcal{O}(-1) \to \mathcal{O}(2)$ be the unique section with zeroes at $D \setminus \{0\}$ and $\sigma|_x = r \in \mathcal{O}|_x$. Then we have an isomorphism

$$\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \colon (\mathcal{O}(2-y) \oplus \mathcal{O}(-1), (\ell_x)_{x \in D}) \xrightarrow{\sim} T_y^{(r:1)} \mathcal{E}^\bullet_{\mathrm{irrel}}$$

where

$$\ell_x = \mathcal{O}|_x \qquad \text{for } x \in D \setminus \{0\}$$

and

$$\ell_0 = (-\sigma|_0 : 1) \in \mathbf{P}((\mathcal{O} \oplus \mathcal{O})|_0).$$

If $r = 0$, then $\sigma|_0 = 0$ and we get $\tilde{\mathcal{E}}(-1)$; otherwise, $\sigma|_0 \neq 0$ and we get $\hat{\mathcal{E}}(-1)$.

The calculations for $x \in D$ follow directly from the definitions. $\square$

Corollary 13.23. *Let $\mathcal{E}^\bullet$ be one of the eight points outside of the relevant locus with* $\dim \operatorname{Aut}(\mathcal{E}^\bullet) = 2$ *and let $\mathcal{T}^\bullet \in \overline{\mathbf{Coh}}_0^{1,1}$. By $\phi_{(\mathcal{E}^\bullet, \mathcal{T}^\bullet)}$ we denote, as before, the following restriction of q:*

$$\phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)} \colon p^{-1}(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \cap q^{-1}(\operatorname{Bun}_{2,4}^{0,\mathrm{r}}) \xrightarrow{q} \operatorname{Bun}_{2,4}^{0,\mathrm{r}}.$$

Then

$$(\mathbb{H} \operatorname{Aut}_E^0)|_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)} = \mathrm{H}_c^*(\phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}^* \operatorname{Aut}_E^0[2]) = 0.$$

Proof. This follows from the fact that for all $x_0 \in D$, the pullback of Aut_E^0 along $k_{x_0}^{(-,0)}$ or $k_{x_0}^{(0,-)}$ is isomorphic to

$$(j_{!*}E)|_{x_0} \otimes E|_\infty \otimes \mathbf{R}(\mathbb{G}_\mathrm{m} \hookrightarrow \mathbf{A}^1)_* \mathbf{Q}_\ell$$

(Theorem 12.1 and Lemma 12.4) which has vanishing compact cohomology. $\square$

Lemma 13.24. *Let $\mathcal{E}^\bullet \in \operatorname{Bun}_{2,4}^{1,\mathrm{r}}$ be a parabolic bundle with* $\dim \operatorname{Aut}(\mathcal{E}^\bullet) \geq 3$. *Then no length 1 lower modification of $\mathcal{E}^\bullet$ lies in* $\operatorname{Bun}_{2,4}^{0,\mathrm{r}}$.

Proof. If $\mathcal{E}^\bullet$ had a length 1 lower modification $\mathcal{F}^\bullet \in \operatorname{Bun}_{2,4}^{0,\mathrm{r}}$, then $\mathcal{F}^\bullet \in \operatorname{Bun}_{2,4}^{0,\mathrm{r}}$ has a length 1 lower modification with automorphism group of dimension 3 (viz., $\mathcal{E}^\bullet(-1)$). This is in contradiction to our calculation of the length 1 lower modifications of points in $\operatorname{Bun}_{2,4}^{0,\mathrm{r}}$. (The verification of this fact can be simplified by remarking that any length one lower modification of $\mathcal{E}^\bullet$ has an automorphism group of dimension at least 2.) $\square$

Corollary 13.25. $\mathbb{H} \operatorname{Aut}_E^0$ *is supported on* $\operatorname{Bun}_{2,4}^{1,\mathrm{r}} \times \overline{\mathbf{Coh}}_0^{1,1}$.

Proof. Let $\mathcal{E}^\bullet \in \operatorname{Bun}_{2,4}^1 \setminus \operatorname{Bun}_{2,4}^{1,\mathrm{r}}$; we want to show $\operatorname{Bun}_{2,4}^1|_{\mathbf{B}\operatorname{Aut}(\mathcal{E}^\bullet) \times \overline{\mathbf{Coh}}_0^{1,1}} = 0$. This follows from Corollary 13.23 if $\operatorname{codim} \mathcal{E}^\bullet = 2$; if $\dim \operatorname{Aut}(\mathcal{E}^\bullet) \neq 2$, then $\operatorname{codim} \mathcal{E}^\bullet \geq 3$ (because all codimension 1 points are contained in $\operatorname{Bun}_{2,4}^{1,\mathrm{r}}$) and the statement follows from Lemma 13.24. $\square$

13.4. Proof of Theorem 13.3. The local system $E^\vee$ that is dual to E is also a pure rank 2 irreducible local system with unipotent monodromy, so that the results we have proven for E also hold for $E^\vee$. The proof of Theorem 13.3 uses this, together with explicit calculations of the degrees in which

$$(\mathbb{H}\operatorname{Aut}_E^0)|_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$$

lives for all $(\mathcal{E}^\bullet,\mathcal{T}^\bullet) \in \operatorname{Bun}_{2,4}^1 \times \overline{\mathbf{Coh}}_0^{1,1}$. (We have already done this for $(\mathcal{E}^\bullet,\mathcal{T}^\bullet) \in \pi_0^{-1}(\mathbf{P}^1 \setminus D) \times (\mathbf{P}^1 \setminus D)$ to show that $\mathbb{H}\operatorname{Aut}_E^0$ is a local system on that subspace (Proposition 13.15). Proposition 13.27 determines the stalks for the remaining points.) These calculations rely on our results on the modifications of the parabolic bundles in $\operatorname{Bun}_{2,4}^{1,\mathrm{r}}$ (Section 9).

Lemma 13.26. *We have*

$$\mathbb{D}\,\mathbb{H}\operatorname{Aut}_E^0 \cong \mathbb{H}\operatorname{Aut}_{E^\vee}^0 .$$

Proof. Let $\bar{p}\colon \overline{\mathcal{H}} \to \operatorname{Bun}_{2,4}^1 \times \overline{\mathbf{Coh}}_0^{1,1}$ denote the compactification of $p\colon \mathcal{H} \to \operatorname{Bun}_{2,4}^1 \times \overline{\mathbf{Coh}}_0^{1,1}$ constructed in Section 13.1.1. Denote by $j\colon \mathcal{H} \hookrightarrow \overline{\mathcal{H}}$ the inclusion. One of the main arguments in the previous section was that we can calculate $\mathbb{H}\operatorname{Aut}_E^0$ via this compactification. More precisely, we have

$$\mathbb{H}\operatorname{Aut}_E^0 = \mathbf{R}\bar{p}_! j_{!*} q^* \operatorname{Aut}_E^0[2] \tag{13.4.1}$$

(Proposition 13.8). Because the functors $\mathbf{R}\bar{p}_!$, $j_{!*}$ and $q^*[2]$ all commute with $\mathbb{D}$ (q is smooth of relative dimension 2) and because $\mathbb{D}\operatorname{Aut}_E^0 = \operatorname{Aut}_{E^\vee}^0$, we find

$$\mathbb{D}\,\mathbb{H}\operatorname{Aut}_E^0 = \mathbf{R}\bar{p}_! j_{!*} q^* \operatorname{Aut}_{E^\vee}^0[2].$$

This expression is in turn equal to $\mathbb{H}\operatorname{Aut}_{E^\vee}^0$, because Equation (13.4.1) also holds for $E^\vee$ instead of E. This completes the proof. $\square$

In the following proposition, j denotes the inclusion $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \mathbf{P}^1$. We determine the stalks of $\mathbb{H}\operatorname{Aut}_E^0$ on the points $(\mathcal{E}^\bullet,\mathcal{T}^\bullet) \in \operatorname{Bun}_{2,4}^{1,\mathrm{r}} \times \overline{\mathbf{Coh}}_0^{1,1}$ that do not lie in $\pi_0^{-1}(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D$. When reading this proposition, the reader is advised to keep in mind that $\mathbb{H}\operatorname{Aut}_E 0$ is supposed to be isomorphic to $\operatorname{Aut}_E^1 \boxtimes j_{!*}E$, which in turn looks like $j_{!*}E \boxtimes j_{!*}E$ after identifying $\operatorname{Bun}_{2,4}^{1,\mathrm{r}}$ with $\mathbf{Coh}_0^{1,1}$. A description of the stalks of $j_{!*}E$ is in Lemma 12.4. For $x \in D$, we denote by $M_x\colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ the unique Möbius transformation preserving D and sending ∞ to x.

Proposition 13.27. *Let* $(\mathcal{E}^\bullet,\mathcal{T}^\bullet) \in \operatorname{Bun}_{2,4}^{1,\mathrm{r}} \times \mathbf{Coh}_0^{1,1}$. *Write*

$$\{x,y\} = \{\operatorname{Supp}(\alpha^{-1}(\mathcal{E}^\bullet)),\ \operatorname{Supp}(\mathcal{T}^\bullet)\}$$

and assume $x \in D$, *i.e., we have* $\mathcal{E}^\bullet \in \pi_0^{-1}(D)$, $\mathcal{T}^\bullet \in \operatorname{Supp}^{-1}(D)$ *or both.*

(1) *If* $\mathcal{E}^\bullet$ *and* $\mathcal{T}^\bullet$ *both have automorphism group* $\mathbb{G}_\mathrm{m}$, *then*

$$(\mathbb{H}\operatorname{Aut}_E^0)|_{(\mathcal{E}^\bullet,[\mathcal{T}^\bullet])} = (j_{!*}E[1])|_{M_x(y)} \otimes E|_\infty.$$

(2) If either $\mathcal{E}^\bullet$ or $\mathcal{T}^\bullet$, but not both, has $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ as its automorphism group, then

$$(\mathbb{H}\operatorname{Aut}^0_E)|_{(\mathcal{E}^\bullet,[\mathcal{T}^\bullet])} = (j_{!*}E[2])|_{M_x(y)} \otimes \mathrm{H}^*_c(\mathbb{G}_\mathrm{m}, \mathbf{Q}_\ell) \otimes E|_\infty.$$

(3) If both $\mathcal{E}^\bullet$ and $\mathcal{T}^\bullet$ have automorphism group $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$, then

$$(\mathbb{H}\operatorname{Aut}^0_E)|_{(\mathcal{E}^\bullet,[\mathcal{T}^\bullet])} = (j_{!*}E[2])|_{M_x(y)} \otimes \mathrm{H}^*_c(\mathbb{G}_\mathrm{m}, \mathbf{Q}_\ell) \times \mathrm{H}^*(\mathbb{G}_\mathrm{m}, \mathbf{Q}_\ell) \otimes E|_\infty.$$

Proof. In Section 9.1.5, we summarized some properties of the maps $\phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$ defined for $\mathbf{F}_q$-points $(\mathcal{E}^\bullet, [\mathcal{T}^\bullet]) \in \operatorname{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}^{1,1}_0$ as the following restriction of q

$$\phi_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)} \colon p^{-1}(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \cap q^{-1}(\operatorname{Bun}^{0,\mathrm{r}}_{2,4}) \xrightarrow{q} \operatorname{Bun}^{0,\mathrm{r}}_{2,4}.$$

These maps are relevant because we have

$$(\mathbb{H}\operatorname{Aut}^0_E)|_{(\mathcal{E}^\bullet,[\mathcal{T}^\bullet])} = \mathrm{H}^*_c(p^{-1}(\mathcal{E}^\bullet, [\mathcal{T}^\bullet]) \cap q^{-1}(\operatorname{Bun}^{0,\mathrm{r}}_{2,4}),\ \phi^*_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)} \operatorname{Aut}^0_E)[2].$$

This equality follows from the definitions and the fact that Aut^0_E is supported on $\operatorname{Bun}^{0,\mathrm{r}}_{2,4}$, so that we only need to restrict ourselves to the inverse image of $\operatorname{Bun}^{0,\mathrm{r}}_{2,4}$ under q. The summary in Section 9.1.5, gives us enough information to show that these cohomology groups are as claimed in the statement of the proposition.

For the purpose of this proof, we divide the maps appearing in the summary into the following four classes (we will repeat their definitions in the course of the proof): (1) maps labeled σ; (2) maps labeled τ; (3) constants maps with image in D^{aut}; and (4) constant maps with image outside of D^{aut}. For each such map ϕ, we determine $\mathrm{H}^*_c(\phi^* \operatorname{Aut}^0_E[2])$. The summary in Section 9.1.5 shows which points $(\mathcal{E}^\bullet, \mathcal{T}^\bullet)$ correspond to which of these maps.

Suppose ϕ is of the first kind, labeled by σ. Then it is a map

$$\phi \colon \mathbf{A}^1 \to \operatorname{Bun}^{0,\mathrm{r}}_{2,4} \backslash \pi_0^{-1}(M_x(y))$$

that is a restriction to the complement of a point, of the map

$$T_\infty \circ \alpha \circ \sigma' \colon \mathbf{P}^1 \xrightarrow{\sigma'} \overline{\mathbf{Coh}}^{1,1}_0 \xrightarrow{\sim} \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$$

where $\sigma' \colon \mathbf{P}^1 \to \overline{\mathbf{Coh}}^{1,1}_0$ is a section of $\operatorname{Supp} \colon \overline{\mathbf{Coh}}^{1,1}_0 \to \mathbf{P}^1$. The pullback of Aut^0_E along this map is $j_{!*}E \otimes E|_\infty$. The cohomology groups $\mathrm{H}^*_c(\mathbf{P}^1, j_{!*}E)$ vanish (Proposition 12.3) and $E|_\infty$ is a constant local system, so the distinguished triangle coming from the open-closed decomposition

$$\mathbf{A}^1 \hookrightarrow \mathbf{P}^1 \hookleftarrow \{M_x(y)\}$$

shows that the compact cohomology of $\phi^* \operatorname{Aut}^0_E$ is indeed $(j_{!*}E[-1])|_{M_x(y)} \otimes E|_\infty$.

Suppose ϕ is the second kind of map, labeled by τ, i.e., it is a map

$$\mathbb{G}_\mathrm{m} \xrightarrow{\tau} \pi_0^{-1}(M_x(y)) \subset \operatorname{Bun}^{0,\mathrm{r}}_{2,4},$$

up to automorphism of $\mathbb{G}_\mathrm{m}$ given by either

$$\lambda \mapsto (T_\infty \circ \alpha)(k_{M_x(y)} \xrightarrow{\lambda-1} k_{M_x(y)} \xrightarrow{0} k_{M_x(y)})$$

or

$$\lambda \mapsto (T_\infty \circ \alpha)(k_{M_x(y)} \xrightarrow{0} k_{M_x(y)} \xrightarrow{\lambda - 1} k_{M_x(y)}).$$

Let $\psi\colon \mathbf{A}^1 \to \pi_0^{-1}(M_x(y))$ denote the obvious extension of this map to $\mathbf{A}^1$. Then $\psi^* \operatorname{Aut}^0_E$ has vanishing compact cohomology groups (Lemma 12.4), so again using the distinguished triangle coming from the open-closed decomposition

$$\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbf{A}^1 \hookleftarrow \{0)\} \ ,$$

we find that the compact cohomology of $\phi^* \operatorname{Aut}^0_E$ is $(j_{!*}E[-1])|_{M_x(y)} \otimes E|_\infty$ (using our calculation Lemma 12.4 of the stalk of $j_{!*}E$ at $k^{(1,0)}_{M_x(y)}$ and $k^{(1,0)}_{M_x(y)}$).

Lastly, the compact cohomology of the pullback along a constant map is easy to calculate, since we know all the fibers of Aut^0_E (Lemma 12.4). $\square$

Recall that $\mathbb{H} \operatorname{Aut}^0_E$ is supported on $\operatorname{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}^{1,1}_0$ (Corollary 13.25). The following corollary shows in which cohomological degrees it lives.

Corollary 13.28. *The derived constructible sheaf* $\mathbb{H} \operatorname{Aut}^0_E$ *is supported on* $\operatorname{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}^{1,1}_0$ *has the following cohomology sheaves:*

$$\mathcal{H}^i(\mathbb{H} \operatorname{Aut}^0_E) = 0 \qquad \textit{for } i < -1,$$

$$\mathcal{H}^{-1}(\mathbb{H} \operatorname{Aut}^0_E) \textit{ is supported on an open of } \operatorname{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}}^{1,1}_0,$$

$$\mathcal{H}^{0}(\mathbb{H} \operatorname{Aut}^0_E) \textit{ is supported on an codimension 2 set, and}$$

$$\mathcal{H}^{1}(\mathbb{H} \operatorname{Aut}^0_E) \textit{ is supported on an codimension 4 set.}$$

Proof. This follows from the previous proposition and the fact that a point $\mathcal{E}^\bullet \in \operatorname{Bun}^1_{2,4}$ lies in codimension $\dim(\operatorname{Aut}(\mathcal{E}^\bullet))$. $\square$

The following table is a variation of the table in Section 9.1.5 and summarizes the results obtained above. We recall that x denotes $\pi_1(\mathcal{E}^\bullet)$ and y is $\operatorname{Supp} \mathcal{T}^\bullet$.

$(\mathcal{E}^\bullet, \mathcal{T}^\bullet) \in \dots$	$\phi_{(\mathcal{E}^\bullet, \mathcal{T}^\bullet)}$	$\mathrm{H}^*_c(\phi^*_{(\mathcal{E}^\bullet, \mathcal{T}^\bullet)} \operatorname{Aut}^0_E[2])$
$\alpha(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D$	$\mathbf{P}^1 \to \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$	in degree -1; see Proposition 13.15
$\alpha(\mathbf{P}^1 \setminus D) \times \mathring{D}$ $\alpha(\mathring{D}) \times \mathbf{P}^1 \setminus D$	$\mathbf{A}^1 \xrightarrow{\sigma} \operatorname{Bun}^{0,\mathrm{r}}_{2,4} \setminus \pi_0^{-1}\{M_x(y)\}$ $\mathbf{A}^1 \xrightarrow{\sigma} \operatorname{Bun}^{0,\mathrm{r}}_{2,4} \setminus \pi_0^{-1}\{M_y(x)\}$	$(j_{!*}E[1])\|_{M_x(y)} \otimes E\|_\infty$ $(j_{!*}E[1])\|_{M_y(x)} \otimes E\|_\infty$
$\alpha(\mathbf{P}^1 \setminus D) \times D^{\mathrm{aut}}$ $\alpha(D^{\mathrm{aut}}) \times \mathbf{P}^1 \setminus D$	$\mathbb{G}_{\mathrm{m}} \to \{T_y T_x^{(1:1)} \tilde{\mathcal{E}}\} \subset \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$ $\mathbb{G}_{\mathrm{m}} \to \{T_x T_y^{(1:1)} \tilde{\mathcal{E}}\} \subset \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$	$(j_{!*}E[2])\|_{M_y(x)} \otimes \mathrm{H}^*_c(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell) \otimes E\|_\infty$ $(j_{!*}E[2])\|_{M_x(y)} \otimes \mathrm{H}^*_c(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell) \otimes E\|_\infty$
$\alpha(\mathring{D}) \times \mathring{D}$	2 possibilities: $\mathbb{G}_{\mathrm{m}} \xrightarrow{\tau} \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$ or $\mathbf{A}^1 \xrightarrow{\sigma} \operatorname{Bun}^{0,\mathrm{r}}_{2,4} \setminus \pi_0^{-1}(M_x(y))$ $(\operatorname{im} \tau \subset \pi_0^{-1}(M_x(y))$	 $(j_{!*}E[1])\|_{M_x(y)} \otimes E\|_\infty$ $(j_{!*}E[1])\|_{M_x(y)} \otimes E\|_\infty$
$\alpha(\mathring{D}) \times D^{\mathrm{aut}}$ $\alpha(D^{\mathrm{aut}}) \times \mathring{D}$	$\mathbb{G}_{\mathrm{m}} \to \{T_y \mathcal{E}^\bullet\} \subset \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$ $\mathbb{G}_{\mathrm{m}} \to \{*\} \subset \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$ $(*\text{ is } T_x T_y^{(1:1)} \tilde{\mathcal{E}} \text{ or } T_x({}^{(1:1)} T_y \tilde{\mathcal{E}})) \otimes E\|_\infty$	$(j_{!*}E[2])\|_{M_y(x)} \otimes \mathrm{H}^*_c(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell) \otimes E\|_\infty$ $(j_{!*}E[2])\|_{M_x(y)} \otimes \mathrm{H}^*_c(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell) \otimes E\|_\infty$
$\alpha(D^{\mathrm{aut}}) \times D^{\mathrm{aut}}$	$\mathbb{G}_{\mathrm{m}} \to \{T_x T_y \tilde{\mathcal{E}}\} \subset \operatorname{Bun}^{0,\mathrm{r}}_{2,4}$	$(j_{!*}E[2])\|_{M_x(y)} \otimes \mathrm{H}^*_c(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell) \times \mathrm{H}^*(\mathbb{G}_{\mathrm{m}}, \mathbf{Q}_\ell) \otimes E\|_\infty$

We can now prove that $\mathbb{H} \operatorname{Aut}^0_E$ is supported on $\operatorname{Bun}^{1,\mathrm{r}}_{2,4}$ and is the intermediate extension of a local system on $\pi_1^{-1}(\mathbf{P}^1 \setminus D) \times (\mathbf{P}^1 \setminus D)$.

Proof of Theorem 13.3. The sheaf $\mathbb{H}\operatorname{Aut}^0_E$, decomposes into a direct sum of finitely many simple shifted perverse sheaves on $\mathrm{Bun}_{2,4} \times \overline{\mathbf{Coh}_0}^{1,1}$. (Proposition 13.4). Because the restriction of $\mathbb{H}\operatorname{Aut}^0_E$ to the open $\pi_1^{-1}(\mathbf{P}^1 \setminus D) \times \mathrm{Supp}^{-1}(\mathbf{P}^1 \setminus D)$ is a rank 4 local system, some of those simple shifted perverse sheaves constitute the intermediate extension of this local system. To prove the theorem, it only remains to show that there are no other shifted perverse sheaves in the decomposition. These other direct summands, if they exist, are necessarily supported on a subspace of codimension at least 1 outside of $\pi_0^{-1}(\mathbf{P}^1 \setminus D) \times \mathbf{P}^1 \setminus D$.

Assume towards a contradiction that there is a locally closed $i\colon Z \hookrightarrow \mathrm{Bun}^1_{2,4} \times \overline{\mathbf{Coh}_0}^{1,1}$, a local system L on Z of rank at least 1 and $r \in \mathbf{Z}$ such that $i_{!*}L[r]$ is one of the direct summands in the decomposition. The key idea is that we know in which cohomological degrees the sheaf $i_{!*}L[r]$ lives (Corollary 13.28), and we also now that if $i_{!*}L[r]$ is a non-zero direct summand of $\mathbb{H}\operatorname{Aut}^0_E$, then $\mathbb{D}(i_{!*}L[r]) = i_{!*}L^\vee[-r]$ is a non-zero direct summand of $\mathbb{D}\,\mathbb{H}\operatorname{Aut}^0_E = \mathbb{H}\operatorname{Aut}^0_{E^\vee}$ (Lemma 13.26). This dual sheaf satisfies the same properties on the cohomological degrees, and we can use this to derive a contradiction if Z has codimension at least 1.

Let $m \in \mathbf{Z}$ denote the perverse degree of $i_{!*}L[r]$. By definition of the perverse t-structure, we have

$$r = \dim Z - m.$$

We know values $s_0, s_1 \in \mathbf{Z}$ such that $(\mathbb{H}\operatorname{Aut}^0_E)|_Z$, and hence its subsheaf $(i_{!*}L[r])|_Z$, lies in cohomological degree at least s_0 and at most s_1 (Corollary 13.28). Specifically, we have

$$(s_0, s_1) = \begin{cases} (-1,-1) & \text{if } \dim Z = 1, 0 \\ (-1,0) & \text{if } \dim Z = -1,-2 \\ (-1,1) & \text{if } \dim Z \leq -3 \end{cases}.$$

We therefore have

$$-s_1 \leq \dim Z - m \leq -s_0.$$

Applying the same reasoning to the duals, as explained in the previous paragraph, we find

$$-s_1 \leq \dim Z + m \leq -s_0.$$

Adding these two inequalities gives

$$-s_1 \leq \dim Z \leq -s_0.$$

We see that this only has a solution for $\dim Z = 1$, which proves that every direct perverse summand is the intermediate extension of a local system on an open of $\mathrm{Bun}^{1,\mathrm{r}}_{2,4} \times \overline{\mathbf{Coh}_0}^{1,1}$. $\square$

13.5. Applying the Hecke operator twice. In this section, we express in terms of Hecke correspondences the phenomenon that modifications at different points commute.

The following diagram shows the composition of two Hecke correspondences. We denote by $\tilde{\mathcal{H}}_2$ the fiber product of the indicated square.

(13.5.1)

$$\begin{tikzcd}
 & & \tilde{\mathcal{H}}_2 \arrow[dl] \arrow[dr] & & \\
 & \mathcal{H} \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[dl, "p\times \mathrm{id}"'] \arrow[dr, "q\times \mathrm{id}"] \arrow[rr, phantom, "\square"] & & \mathcal{H} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1} & & \mathrm{Bun}_{2,4}^1 \times \overline{\mathbf{Coh}}_0^{1,1} & & \mathrm{Bun}_{2,4}^0
\end{tikzcd}$$

We define Δ^+ as the fiber product

$$\Delta^+ := \overline{\mathbf{Coh}}_0^{1,1} \times_{\mathbf{P}^1} \overline{\mathbf{Coh}}_0^{1,1} \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1}$$

where the maps $\overline{\mathbf{Coh}}_0^{1,1} \to \mathbf{P}^1$ are given by taking the support. We denote by $\tilde{\mathcal{H}}_2^{\setminus\Delta} \subset \tilde{\mathcal{H}}_2$ the inverse image of $\mathrm{Bun}_{2,4}^2 \times (\overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1} \setminus \Delta^+)$ and by $p_2 \colon \tilde{\mathcal{H}}_2^{\setminus\Delta} \to \mathrm{Bun}_{2,4}^2 \times \overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1}$ and $q_2 \colon \tilde{\mathcal{H}}_2^{\setminus\Delta} \to \mathrm{Bun}_{2,4}^0$ the compositions appearing in the diagram above. Then by restricting the above diagram to the complement of Δ^+, we get

(13.5.2)

$$\begin{tikzcd}
 & \tilde{\mathcal{H}}_2^{\setminus\Delta} \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & \\
\mathrm{Bun}_{2,4}^2 \times (\overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1} \setminus \Delta^+) & & \mathrm{Bun}_{2,4}^0
\end{tikzcd}$$

Since we have now restricted to the stack $\tilde{\mathcal{H}}_2^{\setminus\Delta}$ that classifies two length 1 lower modifications at different points and such modifications commute, we can define an involution

$$\tau \colon \tilde{\mathcal{H}}_2^{\setminus\Delta} \xrightarrow{\sim} \tilde{\mathcal{H}}_2^{\setminus\Delta} \tag{13.5.3}$$

that changes the order of the two Hecke operators that $\tilde{\mathcal{H}}_2^{\setminus\Delta}$ classifies. We define

$$\sigma \colon \overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\sim} \overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1}$$

as the map that swaps the two factors; this map restricts to an automorphism of $\overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1} \setminus \Delta^+$. The following diagram displays the maps we have introduced:

(13.5.4)

$$\begin{tikzcd}
 & \tilde{\mathcal{H}}_2^{\setminus\Delta} \arrow[dl, "p_2"'] \arrow[dr, "q_2"] \arrow[loop right, "\tau"] & \\
\mathrm{Bun}_{2,4}^2 \times (\overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1} \setminus \Delta^+) \arrow[loop left, "\mathrm{id}\times\sigma"] & & \mathrm{Bun}_{2,4}^0
\end{tikzcd}$$

By construction, the map p_2 is equivariant under the $\mathbf{Z}/2\mathbf{Z}$-action given by τ above and $\mathrm{id} \times \sigma$ below, and the map q_2 is σ-invariant. As a consequence, we derive the following lemma. Recall that the global Hecke operator $\mathbb{H}$ is defined as $\mathbf{R}p!q^*[2]$. We denote by $\mathbb{H}\times\mathrm{id}$ the functor

$$\mathbb{H}\times\mathrm{id} = \mathbf{R}(p\times\mathrm{id})_!(q\times\mathrm{id})^*[2].$$

Lemma 13.29. *The sheaf*

$$F := ((\mathbb{H}\times\mathrm{id})\mathbb{H}\operatorname{Aut}^0_E)|_{\mathrm{Bun}^2_{2,4}\times(\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}\setminus\Delta^+)}$$

is symmetric under $\mathrm{id}\times\sigma$*, i.e., there exists an isomorphism*

$$(\mathrm{id}\times\sigma)^*F \xrightarrow{\sim} F.$$

Proof. Since

$$((\mathbb{H}\times\mathrm{id})\mathbb{H}\operatorname{Aut}^0_E)|_{\mathrm{Bun}^2_{2,4}\times(\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}\setminus\Delta^+)} = \mathbf{R}p_{2!}q_2^*\operatorname{Aut}^0_E[4],$$

this follows from the τ-invariance of q_2 and the equivariance of p_2. ☐

13.6. Symmetry. Let

$$\sigma\colon \overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1} \xrightarrow{\sim} \overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}$$

denote the automorphism that swap the two factors. When we say that a (constructible derived) sheaf F on $\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}\setminus\Delta^+$ is symmetric, we mean that there exists an isomorphism

$$\sigma^*(F|_{\Delta^+}) \xrightarrow{\sim} F|_{\Delta^+}.$$

In this section we prove the following proposition.

Proposition 13.30. *The sheaf*

$$(\alpha\times\mathrm{id})^*((\mathbb{H}\operatorname{Aut}^0_E)|_{\mathrm{Bun}^{1,\mathrm{r}}_{2,4}\times\overline{\mathbf{Coh}}_0^{1,1}}) \qquad \textit{on } \mathbf{Coh}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}$$

descends to a sheaf on $\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}$ *whose restriction to* $\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}\setminus\Delta^+$ *is symmetric.*

We denote the sheaf on $\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}$ from the proposition by

$$\mathbb{H}_\alpha\operatorname{Aut}^0_E.$$

Proof. First note that ℓ-adic sheaves on $\mathbf{Coh}_0^{1,1}$ can be identified with ℓ-adic sheaves on $\overline{\mathbf{Coh}}_0^{1,1}$, because there is a natural isomorphism $\mathbf{Coh}_0^{1,1}\cong\overline{\mathbf{Coh}}_0^{1,1}\times\mathbf{B}\mathbb{G}_\mathrm{m}$ and the connected group $\mathbb{G}_\mathrm{m}$ acts trivially on every ℓ-adic sheaf, since ℓ-adic vector spaces are totally disconnected.

We claim

$$\big((\mathbb{H}\times\mathrm{id})\mathbb{H}\operatorname{Aut}^0_E\big)|_{\tilde{\mathcal{E}}\times(\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}\setminus\Delta^+)}$$

pulls back to $(\alpha\times\mathrm{id})^*\mathbb{H}\operatorname{Aut}^0_E$. Since the restriction of $(\mathbb{H}\times\mathrm{id})\mathbb{H}\operatorname{Aut}^0_E$ to $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}\times(\overline{\mathbf{Coh}}_0^{1,1}\times\overline{\mathbf{Coh}}_0^{1,1}\setminus\Delta^+)$ is symmetric (Lemma 13.29), the proposition follows.

We now prove the claim. The main idea is that α is defined in terms of modifications of $\tilde{\mathcal{E}}$, so that applying $\mathbb{H} \times \mathrm{id}$ and then restricting to $\tilde{\mathcal{E}}$ is in fact the same as pulling back by $\alpha \times \mathrm{id}$. In Lemma 13.31, we will prove that applying $\mathbb{H}$ to a sheaf supported on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ is indeed the same as pulling back by α. It then follows that applying $\mathbb{H} \times \mathrm{id}$ to a sheaf on $\mathrm{Bun}_{2,4}^{1} \times \mathbf{Coh}_0^{1,1}$ supported on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \times \mathbf{Coh}_0^{1,1}$ (and $\mathbb{H}\,\mathrm{Aut}_E^0$ is such a sheaf (Corollary 13.25)) is the same as pulling that sheaf back along $\alpha \times \mathrm{id}$, which completes the proof. $\square$

Lemma 13.31. *Let F be a (derived) constructible sheaf on $\mathrm{Bun}_{2,4}^{1}$ that is supported on $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^{1}$. Then*

$$(\mathbb{H}F)|_{\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})\times\mathbf{P}^1\setminus D}$$

descends to $\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_{\mathrm{m}} \times \mathbf{P}^1 \setminus D$ and this descended sheaf is the restriction of the sheaf $\alpha^*(F[2]|_{\mathrm{Bun}_{2,4}^{1,\mathrm{r}}})$ on $\mathbf{Coh}_0^{1,1} = \mathbf{B}\mathbb{G}_{\mathrm{m}} \times \overline{\mathbf{Coh}}_0^{1,1}$.

Proof. Recall that $\mathbb{H}$ is defined as $\mathbf{R}p_!q^*[2]$. As we did in Section 7, we denote by $\mathcal{H}_{\tilde{\mathcal{E}}} \subset \mathcal{H}$ the inverse image of $\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1}$, i.e., we have a Cartesian square

$$\begin{tikzcd}
\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \overline{\mathbf{Coh}}_0^{1,1} \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{H}_{\tilde{\mathcal{E}}} \arrow[l, "\tilde{p}"'] \arrow[d, hook] \\
\mathrm{Bun}_{2,4}^{2} \times \overline{\mathbf{Coh}}_0^{1,1} & \mathcal{H} \arrow[l, "p"']
\end{tikzcd}$$

In this lemma, we are only interested in the subspace $\mathrm{Bun}_{2,4}^{2} \times \mathbf{P}^1 \setminus D \subset \mathrm{Bun}_{2,4}^{2} \times \overline{\mathbf{Coh}}_0^{1,1}$. We denote by $\mathcal{H}_{\setminus D}$ and $\mathcal{H}_{\tilde{\mathcal{E}},\setminus D}$ the inverse images of this subspace in $\mathcal{H}$ and $\mathcal{H}_{\tilde{\mathcal{E}}}$, respectively:

$$\begin{tikzcd}
\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1 \setminus D \arrow[d, hook] & \mathcal{H}_{\tilde{\mathcal{E}},\setminus D} \arrow[l, "\tilde{p}"'] \arrow[d, hook] \\
\mathrm{Bun}_{2,4}^{2} \times \mathbf{P}^1 \setminus D & \mathcal{H}_{\setminus D} \arrow[l, "p"']
\end{tikzcd} \tag{13.6.1}$$

The map $p\colon \mathcal{H}_{\setminus D} \to \mathrm{Bun}_{2,4}^{2} \times \mathbf{P}^1 \setminus D$ is proper, so by proper base change, we have for any sheaf G on $\mathcal{H}_{\setminus D}$

$$(\mathbf{R}p_!G)|_{\mathbf{B}\,\mathrm{Aut}(\tilde{\mathcal{E}})\times\mathbf{P}^1\setminus D} = \mathbf{R}\tilde{p}_!(G|_{\mathcal{H}_{\tilde{\mathcal{E}},\setminus D}}). \tag{13.6.2}$$

We can in particular apply this to the sheaf $G = q^*F$.

In Section 7, we found that the map

$$q\colon \mathcal{H}_{\tilde{\mathcal{E}}} \to \mathrm{Bun}_{2,4}^{1}$$

restricts to an isomorphism over $\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \subset \mathrm{Bun}_{2,4}^{1}$: we denoted the preimage of $\mathrm{Bun}_{2,4}^{1,\mathrm{r}}$ under q by $\mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \subset \mathcal{H}_{\tilde{\mathcal{E}}}$ and we denoted the map by

$$q^{\mathrm{rel}}\colon \mathcal{H}_{\tilde{\mathcal{E}}}^{\mathrm{rel}} \xrightarrow{\sim} \mathrm{Bun}_{2,4}^{1,\mathrm{r}}.$$

These maps fit into the commutative diagram

$$
\begin{tikzcd}
\mathcal{H}_{\tilde{\mathcal{E}}} \arrow[d, hook] & \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \arrow[l, hook'] \arrow[dr, "\sim", "q^{\mathrm{rel}}"'] & \\
\mathcal{H} \arrow[dr, "q"] & & \\
& \mathrm{Bun}^{1}_{2,4} & \mathrm{Bun}^{1,\mathrm{r}}_{2,4} \arrow[l, hook']
\end{tikzcd}
\tag{13.6.3}
$$

Because F is supported on $\mathrm{Bun}^{1,\mathrm{r}}_{2,4}$, we have

$$(q^*F)|_{\mathcal{H}_{\tilde{\mathcal{E}}}} = \mathbf{R}(\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \hookrightarrow \mathcal{H}_{\tilde{\mathcal{E}}})_!\, q^{\mathrm{rel},*}(F|_{\mathrm{Bun}^{1,\mathrm{r}}_{2,4}}).$$

Restricting this sheaf further to the stack $\mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}},\setminus D} \subset \mathcal{H}_{\tilde{\mathcal{E}},\setminus D}$ and applying Equation (13.6.2), thereby combining our results on the left side of the Hecke correspondence (diagram (13.6.1)) with that on the right side of the Hecke correspondence (diagram (13.6.3)), we find that the restriction of $\mathbb{H}F$ to $\mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1 \setminus D$ is obtained via the correspondence

$$\mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1 \setminus D \xleftarrow{p^{\mathrm{rel},\prime}} \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}},\setminus D} \xrightarrow{q^{\mathrm{rel}}} \mathrm{Bun}^{1,\mathrm{r}}_{2,4}$$

where $p^{\mathrm{rel},\prime}\colon \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}},\setminus D} \to \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}}) \times \mathbf{P}^1 \setminus D$ denotes the restriction of $\tilde{p}$. More precisely, we have

$$\mathbb{H}F = \mathbf{R}p^{\mathrm{rel},\prime}_!\, q^{\mathrm{rel},*}F[2]. \tag{13.6.4}$$

In Section 7, we also proved that the map

$$p^{\mathrm{rel}}\colon \mathcal{H}^{\mathrm{rel}}_{\tilde{\mathcal{E}}} \xrightarrow{\sim} \mathbf{B}\operatorname{Aut}(\tilde{\mathcal{E}})/\mathbb{G}_{\mathrm{m}} \times \mathbf{P}^1 \setminus D,$$

which is the composition of $p^{\mathrm{rel},\prime}$ with the natural map to the rigidification, is in fact an isomorphism. We then defined $\alpha = q^{\mathrm{rel}} \circ p^{\mathrm{rel},-1}$. Therefore, if we push $\mathbb{H}F$ forward along the rigidification map, Equation (13.6.4) shows that we get $\alpha^*F[2]$. $\square$

Corollary 13.32. *Let $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \mathbf{P}^1$ denote the inclusion. There are irreducible local systems $F_i, G_i, H_{i'}$ on $\mathbf{P}^1 \setminus D$ such that the sheaf $\mathbb{H}_\alpha \operatorname{Aut}^0_E$ on $\overline{\mathbf{Coh}}_0^{1,1} \times \overline{\mathbf{Coh}}_0^{1,1}$ is isomorphic to*

$$(j \times j)_{!*}\left(\left(\bigoplus_i (F_i \boxtimes G_i) \oplus (G_i \boxtimes F_i)\right) \oplus \left(\bigoplus_i H_i \boxtimes H_i\right)\right).$$

Proof. Because E was assumed to be pure, $\mathbb{H}_\alpha \operatorname{Aut}^0_E$ is semisimple. $\square$

13.7. Proof of Theorem 13.2.

Lemma 13.33. *Let $x \in D$ and let $\mathcal{T}^\bullet$ be either $k_x^{(1,0)}$ or $k_x^{(0,1)}$. Then*

$$(\mathbb{H}\operatorname{Aut}_E^0)|_{\mathrm{Bun}_{2,4}^{1,\mathrm{r}} \times \{\mathcal{T}^\bullet\}} \cong \alpha_*(j_{!*}M_x^*(E)[1]) \otimes E|_\infty.$$

Likewise,

$$(\mathbb{H}\operatorname{Aut}_E^0)|_{\alpha(\mathcal{T}^\bullet)\times\overline{\mathbf{Coh}}_0^{1,1}} \cong \alpha_*(j_{!*}M_x^*(E)[1]) \otimes E|_\infty.$$

Proof. This follows from our calculations of all the stalks $(\mathbb{H}\operatorname{Aut}_E^0)|_{(\mathcal{E}^\bullet,\mathcal{T}^\bullet)}$ when either $\mathcal{E}^\bullet$ or $\mathcal{T}^\bullet$ lies over D (Proposition 13.27). $\square$

Proof of the main theorem. Let F_i, G_i be irreducible local systems on $\mathbf{P}^1 \setminus D$ such that

$$\mathbb{H}_\alpha \operatorname{Aut}_E^0 = (j \times j)_{!*}\left(\bigoplus_i F_i \boxtimes G_i\right). \tag{13.7.1}$$

and such that if F_i is not isomorphic to G_i, then $G_i \boxtimes F_i$ is also one of the summands in the decomposition (Corollary 13.32).

Because the restriction of $\mathbb{H}_\alpha \operatorname{Aut}_E^0$ to $\overline{\mathbf{Coh}}_0^{1,1} \times k_x^{(1,0)}$ (for any $x \in D$) is isomorphic to (a Möbius pullback of) $j_{!*}E$ (Lemma 13.33), which is the intermediate extension of an irreducible local system, there is at least one direct summand $F \boxtimes G$ in the decomposition of 13.7.1 of rank at least 2. If we can prove it has rank 4, then it is the only direct summand.

Suppose that it has rank 3. Then F and G are not isomorphic, so the rank 3 summand $G \boxtimes F$ also appears in 13.7.1, but this leads to the contradiction that the restriction of $\mathbb{H}_\alpha \operatorname{Aut}_E^0$ to $\mathbf{P}^1 \setminus D \times \mathbf{P}^1 \setminus D$, which is a local system of rank 4, contains a rank 6 local system $(F \boxtimes G) \oplus (G \boxtimes F)$.

Suppose that $F \boxtimes G$ has rank 2. Then again F and G are not isomorphic, so $G \boxtimes F$ is one of the other summands in the decomposition, and in fact the only other summand. But this would imply that $(\mathbb{H}_\alpha \operatorname{Aut}_E^0)|_{(k_x^{(1,0)},k_x^{(1,0)})}$ has rank 2, in contradiction to Lemma 13.33 and the fact that $(j_{!*}E)|_{k_x^{(1,0)}}$ has rank 1 for all $x \in D$ (Lemma 12.4).

We conclude that there exists an irreducible local system F of rank 2 such that $\mathbb{H}_\alpha \operatorname{Aut}_E^0$ is isomorphic to $(j \times j)_{!*}(F \boxtimes F)$. The restriction to $\overline{\mathbf{Coh}}_0^{1,1} \times \{k_\infty^{(1,0)}\}$ is therefore $j_{!*}F \otimes (j_{!*}F)_{k_\infty^{(1,0)}}$; but by our calculations (Lemma 13.33), this is also equal to $j_{!*}E \otimes (j_{!*}E)|_{k_\infty^{(1,0)}}$. We therefore conclude $F = E$, which completes the proof. $\square$

Notation

We list some of the notation used throughout the thesis.

π_d a universal map $\pi_d\colon \mathrm{Bun}_{2,4}^{d,\mathrm{r}} \to \mathbf{P}^1$ to the coarse moduli space.
Definition 7.6, page 52.

Aut_E the proposed Eigensheaf corresponding to the local system E.
Theorem 12.1, page 97.

Aut_E^d the restriction of Aut_E to $\mathrm{Bun}_{2,4}^d$.
Theorem 12.1, page 97.

$\mathrm{Bun}_{2,4}$ the moduli space of rank 2 vector bundles with a parabolic structure at D.
Section 3, page 19.

$\mathrm{Bun}_{2,4}^d$ the degree d part of $\mathrm{Bun}_{2,4}$.
Section 3, page 19.

$\mathrm{Bun}_{2,4}^{\mathrm{rel}}$, $\mathrm{Bun}_{2,4}^{d,\mathrm{r}}$ the open substacks of $\mathrm{Bun}_{2,4}$, $\mathrm{Bun}_{2,4}^d$ on which the cusp forms are supported. We call this the *relevant locus*.
Definition 6.1, page 35.

$\mathbf{Coh}_0^{1,1}$ the moduli space of torsion sheaves on $\mathbf{P}^1$ with length 1 in every parabolic degree.
Definition 2.10, page 12.

$\overline{\mathbf{Coh}}_0^{1,1}$ the rigidification of $\mathbf{Coh}_0^{1,1}$ with respect to the diagonal automorphisms. There is a natural isomorphism $\mathbf{Coh}_0^{1,1} \cong \overline{\mathbf{Coh}}_0^{1,1} \times \mathbf{B}\mathbb{G}_{\mathrm{m}}$.
Definition 2.10, page 12.

D the subset $\{\infty, 0, 1, t\} \subset \mathbf{P}^1(\mathbf{F}_q)$ of four distinct points in $\mathbf{P}^1$.
Section 1.1, page 1.

E a pure irreducible rank 2 local system on $\mathbf{P}^1 \setminus D$ with unipotent monodromy.
Section 1.1, page 1.

$E|_\infty$ the constant local system with fiber $(j_{!*}E)_{k_x^{(1,0)}}$, where $j\colon \mathbf{P}^1 \setminus D \hookrightarrow \overline{\mathbf{Coh}}_0^{1,1}$ is the inclusion.
Section 12, page 97.

$\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ our usual notation for vector bundles on $\mathbf{P}^1$ with a parabolic structure at D. Their underlying vector bundle is denoted by $\mathcal{E}, \mathcal{F}$ and

the bundle in parabolic degree (i,x) is denoted by $\mathcal{E}^{(i,x)}, \mathcal{F}^{(i,x)}$.
Definition 2.1, page 7.

$\tilde{\mathcal{E}}$ a canonically defined bundle in $\mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ with automorphism group $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$.
Proposition 6.13, page 39.

$\hat{\mathcal{E}}$ a canonically defined bundle in $\mathrm{Bun}_{2,4}^{2,\mathrm{r}}$ with automorphism group $\mathbb{G}_\mathrm{m}$.
Proposition 6.13, page 39.

$\mathcal{H}$ the Hecke stack of length 1. We sometimes denote by $\mathcal{H}^d$ with $d \in \mathbf{Z}$ a connected component of this Hecke stack that classifies only modifications of parabolic bundles of degree d.
Definition 4.1, page 20.

$\mathbb{H}$ the global Hecke operator.
Definition 4.2, page 20.

$\mathbb{H}_{\mathcal{T}^\bullet}$ the local Hecke operator with respect to $\mathcal{T}^\bullet \in \mathbf{Coh}_0^{1,1}$.
Definition 4.4, page 21.

$\mathbb{H}_x$, $\mathbb{H}_x^\mathrm{r}$, $\mathbb{H}_x^\mathrm{l}$ local Hecke operators at $x \in \mathbf{P}^1$. (The latter two are only defined for $x \in D$.)
Notation 4.5, page 21.

k_x for $x \in \mathbf{P}^1 \setminus D$, this denotes the skyscraper sheaf of length 1 supported at x, usually considered as an element of $\mathbf{Coh}_0^{1,1}$.
Section 2.4.1, page 12.

$k_x^{(1,0)}$, $k_x^{(0,1)}$, k_x^0 for $x \in D$, these are representatives of the three isomorphism classes of parabolic torsion sheaves of length one (points in $\mathbf{Coh}_0^{1,1}$) supported at x.
Section 2.4.1, page 12.

$(\mathcal{L}, I)$ for $\mathcal{L}$ a line bundle on $\mathbf{P}^1$ and $I \subset D$, $(\mathcal{L}, I)$ denotes a line bundle on $\mathbf{P}^1$ with parabolic structure given by I.
Notation 2.7, page 10.

M_x for $x \in D$, $M_x \colon \mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1$ denotes the unique Möbius transformation that sends D to D and ∞ to x.
Section 8.3, page 57.

T_x for $x \in D$, this denotes the elementary Hecke operator that shifts the parabolic structure at x.
Example 2.17, page 15.

T_J for $J \subset D$, this denotes the composition of the operators T_x with $x \in J$.
Notation 2.25, page 17.

T_x^ℓ, ${}^\ell T_x$ denote modifications of a (parabolic) vector bundle with respect to the given ℓ.
Notation 2.23, page 16.

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Bibliography

[ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers. *Communications in Algebra*, 31(8):3547–3618, 2003. Special issue in honor of Steven L. Kleiman.

[BBD82] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[Bry86] Jean-Luc Brylinski. Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. *Astérisque*, (140-141):3–134, 251, 1986. Géométrie et analyse microlocales.

[Del80] Pierre Deligne. La conjecture de weil : Ii. *Publications Mathématiques de l'IHÉS*, 52:137–252, 1980.

[DOPW00] Ron Donagi, Burt A. Ovrut, Tony Pantev, and Dan Waldram. Standard-model bundles. Standard-model bundles. *Adv. Theor. Math. Phys.*, 5(math/0008010. CERN-TH-2000-203-A. UPR-894-T. RU-00-5-B):563–615. 45 p, Aug 2000.

[Dri87] V. G. Drinfel'd. Two-dimensional l-adic representations of the galois group of a global field of characteristic p and automorphic forms on gl(2). *Journal of Soviet Mathematics*, 36(1):93–105, Jan 1987.

[FGV02] E. Frenkel, D. Gaitsgory, and K. Vilonen. On the geometric Langlands conjecture. *J. Amer. Math. Soc.*, 15(2):367–417, 2002.

[Gro77] Alexandre Grothendieck. *Séminaire de géométrie algébrique du Bois-Marie 1965-66, Cohomologie l-adique et fonctions L, SGA5*, volume 589 of *Springer Lecture Notes*. Springer-Verlag, 1977.

[Hei04] Jochen Heinloth. Coherent sheaves with parabolic structure and construction of Hecke eigensheaves for some ramified local systems. *Ann. Inst. Fourier (Grenoble)*, 54(7):2235–2325 (2005), 2004.

[HN75] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Mathematische Annalen*, 212(3):215–248, Sep 1975.

[Kon09] Maxim Kontsevich. *Notes on Motives in Finite Characteristic*, pages 213–247. Birkhäuser Boston, Boston, 2009.

[KR14] L. Kindler and K. Rülling. Introductory course on ℓ-adic sheaves and their ramification theory on curves, September 2014. Preprint.

[Lau87] Gérard Laumon. Transformation de fourier, constantes d'équations fonctionnelles et conjecture de weil. *Publications Mathématiques de l'IHÉS*, 65:131–210, 1987.

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.