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OPTIMAL CONTROL OF ELLIPTIC VARIATIONAL INEQUALITIES WITH BOUNDED AND UNBOUNDED OPERATORS

L. BETZ* AND I. YOUSEPT*

Abstract. This paper examines optimal control problems governed by elliptic variational inequalities of the second kind with bounded and unbounded operators. To tackle the bounded case, we exploit the dual formulation of the governing variational inequality, which turns out to be an obstacle-type variational inequality featuring a test set with a polyhedric structure. Based thereon, we are able to prove the directional differentiability of the associated solution operator, which leads to a strong stationary optimality system. These results correspond to the ones obtained recently by De los Reyes and Meyer [10]. Differently from their work, our results benefit from the L^2-boundedness property such that we do not require any additional regularity or structural assumption on the unknown solution and the slack variable. The second part of the paper deals with the unbounded case. Due to the non-smoothness of the variational inequality and the unboundedness of the governing elliptic operator, the directional differentiability of the solution operator becomes highly difficult to handle. Our strategy is to apply the Yosida approximation to the unbounded operator, while the non-smoothness of the variational inequality is still preserved. Based on the developed strong stationary result for the bounded case, we are able to derive optimality conditions for the unbounded case by passing to the limit in the Yosida approximation. Finally, we apply the developed results to Maxwell-type variational inequalities arising in superconductivity.

Key words. variational inequality of the second kind, directional differentiability, strong stationarity, dual formulation, Yosida approximation, Maxwell variational inequality

1. Introduction. Deriving first-order necessary optimality conditions for optimal control problems governed by variational inequalities (VIs) is a challenging issue, which is mainly complicated by the lack of the Gâteaux-differentiability of the corresponding solution operator. In the past decades, two main strategies have been developed for the derivation of optimality conditions. The first strategy was introduced by Barbu [2,3], which is based on a regularization approach for the non-smooth variational inequalities. His method has been applied and extended by many authors to various problems (see e.g. [9, 11, 12, 14, 16, 18, 19, 29]). In all these contributions, a regularized problem featuring a Gâteaux differentiable solution operator is introduced. This allows the derivation of a necessary optimality system after passing to the limit with respect to the regularization parameter.

Without any use of regularization, Mignot and Puel [23] introduced a direct method of proving necessary optimality conditions for the optimal control of elliptic obstacle problems (VIs of the first kind). The main tool used in their direct approach is based on the conical differentiability property developed by Mignot [22]. We note that necessary optimality conditions obtained by Mignot and Puel [23] are stronger than those by Barbu [3] and equivalent to necessary optimality conditions in the primal form. For this reason, in the literature, they are also called strong stationary conditions. In general, strong stationary conditions are more difficult to derive than necessary optimality conditions by [3], since specific characteristics such as ample controls (cf. [15]) are required. Furthermore, the direct approach mainly relies on the directional differentiability property of the governing control-to-state mapping. For $H^1(\Omega)$-elliptic VIs of the second kind, this property requires some regularity and structural assumptions on the unknown state and slack variable. See the recent work

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by De los Reyes and Meyer [10] and its extension [6]. Regarding other directional differentiability and strong stationarity results for non-smooth problems, we refer to the contributions [1, 7, 8, 17, 21].

The first goal of this paper is to analyze optimal control problems governed by elliptic VIs of the second kind involving *bounded* operators. The problem we investigate reads as follows:

$$\left.\begin{aligned} \min_{\boldsymbol{f}\in\boldsymbol{U}} \quad & J(\boldsymbol{y},\boldsymbol{f}) \\ \text{s.t.} \quad & -B\boldsymbol{y}+\boldsymbol{f}\in\partial\varphi(\boldsymbol{y}) \quad \text{in } \boldsymbol{L}^2(\Omega), \end{aligned}\right\} \tag{P$_b$}$$

where $\boldsymbol{U}$ is a Hilbert space, so that $\boldsymbol{U}\overset{d}{\hookrightarrow}\boldsymbol{L}^2(\Omega)$ and $J:\boldsymbol{L}^2(\Omega)\times\boldsymbol{U}\to\mathbb{R}$ is Fréchet differentiable. The operator $B:\boldsymbol{L}^2(\Omega)\to\boldsymbol{L}^2(\Omega)$ is linear, bounded and coercive, i.e., there exists a constant $\alpha>0$ such that

$$(B\boldsymbol{x},\boldsymbol{x})_{\boldsymbol{2}}\geq\alpha\|\boldsymbol{x}\|_{\boldsymbol{2}}^2 \quad \forall \boldsymbol{x}\in\boldsymbol{L}^2(\Omega).$$

Furthermore, the non-smooth functional $\varphi:\boldsymbol{L}^2(\Omega)\to\mathbb{R}$ is defined as follows:

$$\varphi(\boldsymbol{v}) := \int_\Omega \sum_{i=1}^n \mathfrak{g}_i(x)|\boldsymbol{v}_i(x)|\,dx, \tag{1.1}$$

with $\mathfrak{g}\in L^\infty(\Omega;\mathbb{R}^n)$ satisfying $\mathfrak{g}_i\geq 0$ a.e. in Ω for all $i=1,\dots,n$.

Our aim is to derive a strong stationary optimality system for (P$_b$) in the spirit of Mignot and Puel [23]. To this end, we shall address the directional differentiability of the control-to-state operator for (P$_b$). The key idea here is to make use of the dual formulation of the inclusion constraint in (P$_b$), which turns out to be an obstacle-type variational inequality featuring a test set with a polyhedric structure. This property allows us to derive the directional differentiability of the solution operator, which leads to a strong stationary optimality system. Let us underline that our results correspond to the ones established recently in the work [10] on the optimal control of elliptic VIs of the second kind in the $H_0^1(\Omega)$–$H^{-1}(\Omega)$ duality. Differently from [10], our differentiability result benefits from the L^2-boundedness of (P$_b$) and does not require any additional regularity or structural assumption on the unknown state and slack variable.

The second goal of this paper is to investigate an optimal control problem, where the governing elliptic operator appearing in the constraint is *unbounded*, namely:

$$\left.\begin{aligned} \min_{\boldsymbol{u}\in\boldsymbol{U}} \quad & \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{y_d}\|_{\boldsymbol{2}}^2+\frac{\kappa}{2}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 \\ \text{s.t.} \quad & -\boldsymbol{\nu}\boldsymbol{y}-A\boldsymbol{y}+\boldsymbol{u}\in\partial\varphi(\boldsymbol{y}) \quad \text{in } \boldsymbol{L}^2(\Omega). \end{aligned}\right\} \tag{P}$$

In the setting of (P), the function $\boldsymbol{y}_d\in\boldsymbol{L}^2(\Omega)$ denotes the desired state, and $\kappa>0$ the control cost term. Moreover, $\boldsymbol{\nu}\in L^\infty(\Omega;\mathbb{R}^{n\times n}_{sym})$ is a uniformly positive definite function and $A:\mathcal{D}(A)\overset{d}{\hookrightarrow}\boldsymbol{L}^2(\Omega)\to\boldsymbol{L}^2(\Omega)$ is a linear, unbounded and skew-adjoint operator. We point out that the governing operator $\boldsymbol{\nu}+A$ arises e.g. from the time discretization of first-order hyperbolic wave equations such as linear wave acoustic equations or Maxwell's equations. The Maxwell case will be considered in the final part of the paper. The precise assumptions for the data involved in (P) will be specified in Section 3.

The optimal control problem (P) features two main difficulties: the non-smooth character arising from the VI-structure and the unboundedness of the elliptic operator. In particular, differently from (P_b), the unboundedness of the operator A makes the directional differentiability of the control-to-state mapping difficult to tackle. To be more precise, it is not clear if the difference quotients associated with the state are uniformly bounded in suitable spaces. For this reason, we reach out to the Yosida approximation of the unbounded operator A. This gives rise to an optimal control problem governed by:

$$-\boldsymbol{\nu}\boldsymbol{y} - A_\lambda \boldsymbol{y} + \boldsymbol{u} \in \partial\varphi(\boldsymbol{y}) \quad \text{in } \boldsymbol{L}^2(\Omega), \quad \boldsymbol{u} \in \boldsymbol{U}. \tag{1.2}$$

Here, $\lambda > 0$ is a fixed parameter, while A_λ is the Yosida approximation of A (see Definition 3.2). We should underline that the variational inequality structure of the second kind is preserved in (1.2), and thus the resulting optimal control problem still has a non-smooth character. This is different from the regularization method by Barbu [3]. By employing the developed results for (P_b), we obtain an optimality system of strong stationary type for the optimal control problem governed by (1.2). Then, passing to the limit $\lambda \searrow 0$ in the strong stationary optimality system, we derive optimality conditions for local minimizers of the original problem (P).

1.1. Preliminaries. Throughout this paper, C denotes a generic positive constant. For a given Hilbert space V, we use the notation $\|\cdot\|_V$ and $(\cdot,\cdot)_V$ for the standard norm and the standard scalar product in V, respectively. By V^*, we denote the dual space of V and for the associated duality pairing we write $\langle .,. \rangle_{V^*,V}$. Let X and Y be two normed linear spaces. If X is continuously embedded in Y, we write $X \hookrightarrow Y$. Furthermore, if X is compactly embedded in Y, we write $X \hookrightarrow\hookrightarrow Y$, and $X \overset{d}{\hookrightarrow} Y$ indicates that X is dense in Y. The open ball in X around $x \in X$ with radius $R > 0$ is denoted by $B_X(x, R)$. Throughout this paper, Ω is a bounded subset of $\mathbb{R}^3$ and $n \in \mathbb{N}$ is fixed. $\boldsymbol{L}^2(\Omega)$ stands for $L^2(\Omega;\mathbb{R}^n)$, $(\cdot,\cdot)_{\boldsymbol{2}}$ is the associated standard scalar product, and $\|\cdot\|_{\boldsymbol{2}}$ denotes the standard norm in $\boldsymbol{L}^2(\Omega)$. Similarly, a bold typeface is used to indicate an n-dimensional vector function or a Hilbert space of n-dimensional vector functions. For a convex function $F : \boldsymbol{L}^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ we denote by $\partial F(\boldsymbol{x})$ the convex subdifferential of F at $\boldsymbol{x} \in \boldsymbol{L}^2(\Omega)$, i.e.,

$$\partial F(\boldsymbol{x}) := \{\boldsymbol{\xi} \in \boldsymbol{L}^2(\Omega) : (\boldsymbol{\xi}, \boldsymbol{v} - \boldsymbol{x})_{\boldsymbol{2}} \le F(\boldsymbol{v}) - F(\boldsymbol{x}) \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}.$$

The polar cone of a set $M \subset \boldsymbol{L}^2(\Omega)$ is denoted by

$$M^\circ := \{\boldsymbol{x}^* \in \boldsymbol{L}^2(\Omega) : (\boldsymbol{x}^*, \boldsymbol{x})_{\boldsymbol{2}} \le 0 \quad \forall \boldsymbol{x} \in M\}. \tag{1.3}$$

The radial cone of $M \subset \boldsymbol{L}^2(\Omega)$ is defined as

$$\operatorname{cone} M := \cap\{A \subset \boldsymbol{L}^2(\Omega) : M \subset A,\ A \text{ is a convex cone}\}. \tag{1.4}$$

For the indicator functional of $M \subset \boldsymbol{L}^2(\Omega)$, we write $\mathcal{I}_M$.

2. Strong Stationarity for (P_b). We begin by noticing that the inclusion constraint of (P_b)

$$-B\boldsymbol{y} + \boldsymbol{f} \in \partial\varphi(\boldsymbol{y}) \quad \text{in } \boldsymbol{L}^2(\Omega)$$

is equivalent to

$$(B\,\boldsymbol{y}, \boldsymbol{v} - \boldsymbol{y})_{\boldsymbol{2}} + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{y}) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{y})_{\boldsymbol{2}} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \tag{VI}$$

LEMMA 2.1. *For every* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, (VI) *admits a unique solution* $\boldsymbol{y} \in \boldsymbol{L}^2(\Omega)$. *The associated solution operator* $S : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$, $\boldsymbol{f} \mapsto \boldsymbol{y}$, *fulfills*

$$\|S(\boldsymbol{f}_1) - S(\boldsymbol{f}_2)\|_{\boldsymbol{2}} \leq 1/\alpha \, \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{2}} \quad \forall \, \boldsymbol{f}_1, \boldsymbol{f}_2 \in \boldsymbol{L}^2(\Omega), \tag{2.1}$$

where $\alpha > 0$ *denotes the coercivity constant of* B.

Proof. Since the functional φ is convex and continuous, the well-posedness for (VI) follows from the classical result [20]. Let $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{y}_1 := S(\boldsymbol{f}_1)$, $\boldsymbol{y}_2 := S(\boldsymbol{f}_2)$. Testing the VI for $\boldsymbol{y}_1$ with $\boldsymbol{y}_2$ and vice versa implies

$$(B(\boldsymbol{y}_1 - \boldsymbol{y}_2), \boldsymbol{y}_1 - \boldsymbol{y}_2)_{\boldsymbol{2}} \leq (\boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{y}_1 - \boldsymbol{y}_2)_{\boldsymbol{2}}, \tag{2.2}$$

which by the coercivity of B then gives (2.1). ∎

LEMMA 2.2. *Let* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. *Then, the variational inequality* (VI) *is equivalent to*

$$\boldsymbol{j}_i(x)\boldsymbol{y}_i(x) = \mathfrak{g}_i(x)|\boldsymbol{y}_i(x)|, \tag{2.3a}$$

$$|\boldsymbol{j}_i(x)| \leq \mathfrak{g}_i(x) \quad \text{for a.e. } x \in \Omega, \ \forall \, i \in \{1, ..., n\}, \tag{2.3b}$$

with $\boldsymbol{y} := S(\boldsymbol{f})$ *and* $\boldsymbol{j} := -B\,\boldsymbol{y} + \boldsymbol{f}$.

Proof. Testing in (VI) with 0 and $2\boldsymbol{v}$ implies that (VI) is equivalent to

$$\begin{cases} (\boldsymbol{j}, \boldsymbol{y})_{\boldsymbol{2}} = \varphi(\boldsymbol{y}), \\ (\boldsymbol{j}, \boldsymbol{v})_{\boldsymbol{2}} \leq \varphi(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \end{cases} \tag{2.4}$$

We first focus on the implication (VI)⇒(2.3). To this end, let $w \in L^2(\Omega)$ with $w \geq 0$ a.e. in Ω. Testing with the vector $(w, 0, 0, ..., 0)$ in the inequality in (2.4) yields $\boldsymbol{j}_1(x) \leq \mathfrak{g}_1(x)$ a.e. in Ω. By choosing $w \in L^2(\Omega)$ with $w \leq 0$ a.e. in Ω and testing in the exact same way, we arrive at $|\boldsymbol{j}_1(x)| \leq \mathfrak{g}_1(x)$ a.e. in Ω. Completely analogously, we obtain $|\boldsymbol{j}_i(x)| \leq \mathfrak{g}_i(x)$ a.e. in Ω for all $i \in \{1, ..., n\}$, i.e., the inequality (2.3b). From $\boldsymbol{j}_i(x) \in [-\mathfrak{g}_i(x), \mathfrak{g}_i(x)]$, it follows that $\boldsymbol{j}_i(x)\boldsymbol{y}_i(x) \leq \mathfrak{g}_i(x)|\boldsymbol{y}_i(x)|$ a.e. in Ω for all $i \in \{1, ..., n\}$. By defining for $i \in \{1, ..., n\}$ the set $M_i := \{x \in \Omega : \boldsymbol{j}_i(x)\boldsymbol{y}_i(x) < \mathfrak{g}_i(x)|\boldsymbol{y}_i(x)|\}$ (up to a set of measure zero), one then has

$$(\boldsymbol{j}, \boldsymbol{y})_{\boldsymbol{2}} - \varphi(\boldsymbol{y}) = \sum_{i=1}^{n} \int_{M_i} \underbrace{\boldsymbol{j}_i(x)\boldsymbol{y}_i(x) - \mathfrak{g}_i(x)|\boldsymbol{y}_i(x)|}_{<0} \, dx.$$

The identity (2.3a) now follows from the identity in (2.4). This proves (VI)⇒(2.3).

In order to show the opposite implication, we argue as follows. Let $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ be arbitrary, but fixed. From (2.3b), we deduce $\boldsymbol{j}_i(x)\boldsymbol{v}_i(x) \leq \mathfrak{g}_i(x)|\boldsymbol{v}_i(x)|$ a.e. in Ω for all $i \in \{1, ..., n\}$. This together with (2.3a) immediately implies that (2.4), and thus, (VI) holds true. The proof is now complete. ∎

DEFINITION 2.3 (Conjugate functional). *For a functional* $F : \boldsymbol{L}^2(\Omega) \to [-\infty, \infty]$, *we define its conjugate functional* $F^* : \boldsymbol{L}^2(\Omega) \to [-\infty, \infty]$ *as*

$$F^*(\boldsymbol{\psi}) = \sup_{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)} (\boldsymbol{\psi}, \boldsymbol{v})_{\boldsymbol{2}} - F(\boldsymbol{v}).$$

LEMMA 2.4. *Let* $F : \boldsymbol{L}^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ *be a positive homogeneous functional, i.e.,* $F(\lambda \boldsymbol{v}) = \lambda F(\boldsymbol{v})$ *for all* $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ *and for all* $\lambda \geq 0$. *Then, its conjugate* F^* *coincides* *with the indicator functional of the set* $K := \{\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega) : (\boldsymbol{\zeta}, \boldsymbol{v})_{\mathbf{2}} \le F(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$*, i.e.,* $F^* = \mathcal{I}_K$*.*

Proof. For $\boldsymbol{\psi} \in K$, it holds that

$$F^*(\boldsymbol{\psi}) = \sup_{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)} (\boldsymbol{\psi}, \boldsymbol{v})_{\mathbf{2}} - F(\boldsymbol{v}) \le 0.$$

Thus, since $F(\mathbf{0}) = 0$, we obtain that $F^*(\boldsymbol{\psi}) = 0$ for all $\boldsymbol{\psi} \in K$.
If $\boldsymbol{\psi} \notin K$, then there is some $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ so that $(\boldsymbol{\psi}, \boldsymbol{w})_{\mathbf{2}} > F(\boldsymbol{w})$. Then, by the definition, it follows that

$$F^*(\boldsymbol{\psi}) \ge \sup_{\gamma > 0} (\boldsymbol{\psi}, \gamma \boldsymbol{w})_{\mathbf{2}} - F(\gamma \boldsymbol{w}) = \sup_{\gamma > 0} \gamma \underbrace{[(\boldsymbol{\psi}, \boldsymbol{w})_{\mathbf{2}} - F(\boldsymbol{w})]}_{>0} = \infty$$

□

In the upcoming lemma, we compute the conjugate functional of φ; see (1.1) for its definition.

LEMMA 2.5 (Conjugate functional of φ). *For every* $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$*, it holds that*

$$\varphi^*(\boldsymbol{\psi}) = \begin{cases} 0, & \textit{if } |\boldsymbol{\psi}_i(x)| \le \mathfrak{g}_i(x) \textit{ for a.e. } x \in \Omega \textit{ and for all } i \in \{1, ..., n\}, \\ \infty, & \textit{otherwise.} \end{cases}$$

In other words, $\varphi^* = \mathcal{I}_{\mathcal{K}}$*, where*

$$\mathcal{K} := \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : |\boldsymbol{w}_i(x)| \le \mathfrak{g}_i(x) \textit{ for a.e. } x \in \Omega \textit{ and for all } i \in \{1, ..., n\}\}. \tag{2.5}$$

Proof. According to Lemma 2.4, we only have to show that the set $\mathcal{K}$ coincides with the set $M := \{\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega) : (\boldsymbol{\zeta}, \boldsymbol{v})_{\mathbf{2}} \le \varphi(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$. Firstly, let us verify the inclusion $M \subset \mathcal{K}$. To this end, consider $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$ with

$$(\boldsymbol{\psi}, \boldsymbol{v})_{\mathbf{2}} \le \int_\Omega \sum_{i=1}^n \mathfrak{g}_i(x) |\boldsymbol{v}_i(x)| \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

In the same way as in the proof of Lemma 2.2, we deduce that $|\boldsymbol{\psi}_i(x)| \le \mathfrak{g}_i(x)$ for a.e. $x \in \Omega$ and for all $i \in \{1, ..., n\}$, i.e., $M \subset \mathcal{K}$. Let us now show $\mathcal{K} \subset M$. To this aim, let $\boldsymbol{\psi} \in \mathcal{K}$ and $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$. From $|\boldsymbol{\psi}_i(x)| \le \mathfrak{g}_i(x)$ for a.e. $x \in \Omega$ and for all $i \in \{1, ..., n\}$, we have

$$\boldsymbol{\psi}_i(x) \boldsymbol{v}_i(x) \le \mathfrak{g}_i(x) |\boldsymbol{v}_i(x)| \text{ for a.e. } x \in \Omega \text{ and for all } i \in \{1, ..., n\}.$$

This proves $\mathcal{K} \subset M$, in view of the definition of M and φ. □

2.1. Directional differentiability. As pointed out in the introduction, we shall establish the directional differentiability of the solution operator S by exploiting the dual formulation of (VI), which turns out to be an obstacle-type variational inequality featuring a polyhedric structure. In the following, let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and we set

$$\boldsymbol{y} := S(\boldsymbol{f}) \quad \text{and} \quad \boldsymbol{j} := -B\boldsymbol{y} + \boldsymbol{f}. \tag{2.6}$$

Since φ is convex and continuous, a well known convex analysis result, see e.g. [25], yields that the variational inequality (VI) is equivalent to

$$\boldsymbol{y} \in \partial \varphi^*(\boldsymbol{j}) \quad \text{in } \boldsymbol{L}^2(\Omega). \tag{2.7}$$

By Lemma 2.5, (2.7) is nothing but

$$\boldsymbol{j} \in \mathcal{K}, \quad (\boldsymbol{y}, \boldsymbol{v} - \boldsymbol{j})_{\boldsymbol{2}} \leq 0 \quad \forall \boldsymbol{v} \in \mathcal{K}. \tag{VI_D}$$

We recall here that the set $\mathcal{K}$ is given by

$$\mathcal{K} := \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : |\boldsymbol{w}_i(x)| \leq \mathfrak{g}_i(x) \text{ for a.e. } x \in \Omega \text{ and for all } i \in \{1, ..., n\}\}. \tag{2.8}$$

Note that $\mathcal{K}$ is a convex and closed subset of $\boldsymbol{L}^2(\Omega)$. This set turns out to be polyhedric, as we shall show in the upcoming lemma. To this aim, in all what follows, we use for simplicity the following abbreviations:

$$\begin{aligned} R_{\mathcal{K}}(\boldsymbol{x}) &:= \operatorname{cone}(\mathcal{K} - \boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathcal{K}, \\ [\boldsymbol{\mu}]^{\perp} &:= \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : (\boldsymbol{\mu}, \boldsymbol{w})_{\boldsymbol{2}} = 0\} \text{ for all } \boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega). \end{aligned} \tag{2.9}$$

Note that according to (1.4), it holds that $R_{\mathcal{K}}(\boldsymbol{x}) = \{\beta(\mathcal{K} - \boldsymbol{x}) \,|\, \beta > 0\}$ for all $\boldsymbol{x} \in \mathcal{K}$.

Proposition 2.6 ([4, Proposition 6.35]). *Let $m \in \mathbb{N}$ and $\mathcal{M}$ denotes the power set of $\{1, \ldots, m\}$. Furthermore, let $I : \Omega \to \mathcal{M}$ be a measurable mapping and*

$$P(x) := \{\boldsymbol{w} \in \mathbb{R}^n | \boldsymbol{a_i}(x) \cdot \boldsymbol{w} \leq b_i(x), \ i \in I(x)\} \quad \text{for a.e. } x \in \Omega,$$

with

$$\boldsymbol{a_i} \in L^\infty(\Omega; \mathbb{R}^n) \text{ and } b_i \in L^\infty(\Omega) \quad \forall i \in \{1, \ldots, m\}.$$

Then, the set

$$\mathcal{P} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,|\, \boldsymbol{v}(x) \in P(x) \text{ for a.e. } x \in \Omega\}$$

satisfies the polyhedricity condition, i.e., $\operatorname{cone}(\mathcal{P} - \boldsymbol{x}) \cap [\boldsymbol{\mu}]^{\perp}$ is dense in $\overline{\operatorname{cone}(\mathcal{P} - \boldsymbol{x})} \cap [\boldsymbol{\mu}]^{\perp}$ for all $\boldsymbol{x} \in \mathcal{P}$ and all $\boldsymbol{\mu} \in \overline{\operatorname{cone}(\mathcal{P} - \boldsymbol{x})}^{\circ}$.

The next lemma is crucial for proving the directional differentiability of the solution operator of (VI).

Lemma 2.7. *The set $\mathcal{K}$ is polyhedric, i.e., it satisfies*

$$\overline{R_{\mathcal{K}}(\boldsymbol{x})} \cap [\boldsymbol{\mu}]^{\perp} = \overline{R_{\mathcal{K}}(\boldsymbol{x}) \cap [\boldsymbol{\mu}]^{\perp}} \quad \forall \boldsymbol{x} \in \mathcal{K}, \forall \boldsymbol{\mu} \in \overline{R_{\mathcal{K}}(\boldsymbol{x})}^{\circ}. \tag{2.10}$$

where $\overline{R_{\mathcal{K}}(\boldsymbol{x})}^{\circ}$ denotes the polar cone of $\overline{R_{\mathcal{K}}(\boldsymbol{x})}$; see (1.3) for its definition.

Proof. We observe that

$$\mathcal{K} = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}(x) \in K(x) \text{ a.e. in } \Omega\}, \tag{2.11}$$

with

$$K(x) := \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v}_i \in [-\mathfrak{g}_i(x), \mathfrak{g}_i(x)] \text{ for all } i = 1, ..., n\} \quad \text{a.e. in } \Omega.$$

The assertion follows immediately from Proposition 2.6 with $I(x) = \{1, ..., 2n\}$ a.e. in Ω, in combination with $\mathfrak{g} \in L^\infty(\Omega; \mathbb{R}^n)$. ◻

Now, let $\boldsymbol{\delta f} \in \boldsymbol{L}^2(\Omega)$ and $\{\tau_k\} \subset \mathbb{R}^+$ with $\tau_k \searrow 0$. For every $k \in \mathbb{N}$, we set

$$\boldsymbol{y}_k := S(\boldsymbol{f} + \tau_k \boldsymbol{\delta f}) \quad \text{and} \quad \boldsymbol{j}_k := -B\boldsymbol{y}_k + \boldsymbol{f} + \tau_k \boldsymbol{\delta f}. \tag{2.12}$$

As an immediate consequence of (2.1), there exists a constant $C > 0$, independent of k, such that

$$\left\| \frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} \right\|_{\boldsymbol{2}} \leq C \quad \forall\, k \in \mathbb{N}. \tag{2.13}$$

Hence, there exists a (not relabeled) subsequence of $\{\tau_k\}$ and $\boldsymbol{\delta y} \in \boldsymbol{L}^2(\Omega)$ such that

$$\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} \rightharpoonup \boldsymbol{\delta y} \quad \text{in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty. \tag{2.14}$$

By this weak convergence together with (2.6) and (2.12), we infer that

$$\frac{\boldsymbol{j}_k - \boldsymbol{j}}{\tau_k} \rightharpoonup -B\boldsymbol{\delta y} + \boldsymbol{\delta f} =: \boldsymbol{\delta j} \quad \text{in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty, \tag{2.15}$$

where we have employed the weak sequential continuity of the operator B.

Lemma 2.8. *The weak limits in* (2.14) *and* (2.15) *are strong and satisfy*

$$\boldsymbol{\delta y} \in \partial \mathcal{I}_{\mathcal{C}(\boldsymbol{f})}(\boldsymbol{\delta j}), \tag{2.16}$$

where

$$\begin{aligned} \mathcal{C}(\boldsymbol{f}) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \; & \boldsymbol{v}_i(x) \leq 0 \;\; \textit{if } \boldsymbol{j}_i(x) = \mathfrak{g}_i(x), \\ & \boldsymbol{v}_i(x) \geq 0 \;\; \textit{if } \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x), \\ & \boldsymbol{v}_i(x)\boldsymbol{y}_i(x) = 0 \;\; \textit{a.e. in } \Omega, \; i = 1, ..., n\}. \end{aligned} \tag{2.17}$$

Proof. The proof is divided into two steps: First, we pass to the limit by employing the dual formulation (VI_D). This will lead to (2.18) below. As a consequence, strong convergence in (2.14) is also obtained. In the second part of the proof we precisely characterize the test set in (2.18).

1. Step: We show that

$$\boldsymbol{\delta j} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp}, \; (\boldsymbol{\delta y}, \boldsymbol{v} - \boldsymbol{\delta j})_{\boldsymbol{2}} \leq 0 \quad \forall\, \boldsymbol{v} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp}. \tag{2.18}$$

We follow the ideas by [22] (cf. also [27, Theorem 5.2]). Let $k \in \mathbb{N}$ be arbitrary, but fixed. Then, by (VI_D), we know that $\boldsymbol{j}_k \in \mathcal{K}$, and thus

$$\frac{\boldsymbol{j}_k - \boldsymbol{j}}{\tau_k} \in R_{\mathcal{K}}(\boldsymbol{j}) \quad \forall\, k \in \mathbb{N}.$$

The weak sequential closedness of $\overline{R_{\mathcal{K}}(\boldsymbol{j})}$ together with (2.15) then gives that $\boldsymbol{\delta j} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})}$. Let now $\boldsymbol{\psi} \in R_{\mathcal{K}}(\boldsymbol{j})$ be arbitrary, but fixed. By setting $\boldsymbol{v} = \gamma\boldsymbol{\psi} + \boldsymbol{j} \in \mathcal{K}$ with $\gamma > 0$ in (VI_D), we have $(\boldsymbol{y}, \boldsymbol{\psi})_{\boldsymbol{2}} \leq 0$. Therefore, $\boldsymbol{y} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})}^{\circ}$, by a density argument employed in $(\boldsymbol{y}, \boldsymbol{\psi})_{\boldsymbol{2}} \leq 0$ for all $\boldsymbol{\psi} \in R_{\mathcal{K}}(\boldsymbol{j})$. Altogether it holds that

$$\boldsymbol{y} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})}^{\circ}, \quad \boldsymbol{\delta j} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})}, \quad (\boldsymbol{y}, \boldsymbol{\delta j})_{\boldsymbol{2}} \leq 0. \tag{2.19}$$

We now consider $\boldsymbol{\xi} \in \mathcal{K}$ with $(\boldsymbol{y}, \boldsymbol{\xi})_{\boldsymbol{2}} = (\boldsymbol{y}, \boldsymbol{j})_{\boldsymbol{2}}$. In view of ($VI_D$), we arrive at

$$\left(\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k}, \boldsymbol{\xi} - \boldsymbol{j}_k\right)_{\boldsymbol{2}} \leq \left(\frac{-\boldsymbol{y}}{\tau_k}, \boldsymbol{\xi} - \boldsymbol{j}_k\right)_{\boldsymbol{2}} = \left(\frac{-\boldsymbol{y}}{\tau_k}, \boldsymbol{j} - \boldsymbol{j}_k\right)_{\boldsymbol{2}} \quad \forall\, k \in \mathbb{N}.$$

Passing to the limit $k \to \infty$ then yields

$$(\boldsymbol{\delta y}, \boldsymbol{\xi} - \boldsymbol{j})_{\mathbf{2}} \le (\boldsymbol{y}, \boldsymbol{\delta j})_{\mathbf{2}}, \tag{2.20}$$

where we have used (2.14)-(2.15) and the strong convergence $\boldsymbol{j}_k \to \boldsymbol{j}$ in $\boldsymbol{L}^2(\Omega)$ due to (2.1). By inserting $\boldsymbol{\xi} = \boldsymbol{j}$ in the above inequality, we obtain from (2.19) that

$$\boldsymbol{\delta j} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp}. \tag{2.21}$$

On the other hand, (2.20) together with the inequality in (2.19) implies that

$$(\boldsymbol{\delta y}, \boldsymbol{\zeta})_{\mathbf{2}} \le 0 \quad \forall\, \boldsymbol{\zeta} \in (\mathcal{K} - \boldsymbol{j}) \cap [\boldsymbol{y}]^{\perp}. \tag{2.22}$$

By (2.9), it follows that $\overline{(\mathcal{K} - \boldsymbol{j}) \cap [\boldsymbol{y}]^{\perp}}^{\circ} = \overline{R_{\mathcal{K}}(\boldsymbol{j}) \cap [\boldsymbol{y}]^{\perp}}^{\circ}$, which yields

$$(\boldsymbol{\delta y}, \boldsymbol{\zeta})_{\mathbf{2}} \le 0 \quad \forall\, \boldsymbol{\zeta} \in \overline{R_{\mathcal{K}}(\boldsymbol{j}) \cap [\boldsymbol{y}]^{\perp}}. \tag{2.23}$$

In view of (2.21), the polyhedricity of $\mathcal{K}$ (Lemma 2.7) implies $\boldsymbol{\delta j} \in \overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp} = \overline{R_{\mathcal{K}}(\boldsymbol{j}) \cap [\boldsymbol{y}]^{\perp}}$. Thus, by (2.23), we obtain

$$(\boldsymbol{\delta j}, \boldsymbol{\delta y})_{\mathbf{2}} \le 0.$$

Further, we note that the operator B induces a norm on $\boldsymbol{L}^2(\Omega)$, which we denote by $\|\cdot\|_B := \sqrt{(B\cdot,\cdot)_{\mathbf{2}}}$. Relying on (2.15) and (2.14) one has

$$\begin{aligned} 0 \le -(\boldsymbol{\delta j}, \boldsymbol{\delta y})_{\mathbf{2}} = (B\boldsymbol{\delta y} - \boldsymbol{\delta f}, \boldsymbol{\delta y})_{\mathbf{2}} &\le \liminf_{k\to\infty} (B\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} - \boldsymbol{\delta f}, \frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k})_{\mathbf{2}} \\ &\le \limsup_{k\to\infty} (B\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} - \boldsymbol{\delta f}, \frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k})_{\mathbf{2}} \le 0, \end{aligned} \tag{2.24}$$

in light of (2.2). Thus, $(\boldsymbol{\delta j}, \boldsymbol{\delta y})_{\mathbf{2}} = 0$, which combined with (2.21), (2.23) and the polyhedricity of $\mathcal{K}$ gives in turn (2.18). Moreover, from (2.24) we infer that

$$\Big\| \frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} \Big\|_B \to \|\boldsymbol{\delta y}\|_B \quad \text{as } k \to \infty.$$

The weak convergence (2.14) and the coercivity of B then gives

$$\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} \to \boldsymbol{\delta y} \quad \text{in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty.$$

By the above convergence together with (2.6) and (2.12), we conclude the strong convergence in (2.15).
2. Step: Now we prove that

$$\boldsymbol{\delta y} \in \partial \mathcal{I}_{\mathcal{C}(\boldsymbol{f})}(\boldsymbol{\delta j}),$$

where $\mathcal{C}(\boldsymbol{f})$ is defined as in (2.17). To this aim, let us note that (2.18) yields

$$\boldsymbol{\delta y} \in \partial \mathcal{I}_{\overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp}}(\boldsymbol{\delta j}).$$

Thus, we only need to verify that $\overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp}$ coincides with the set $\mathcal{C}(\boldsymbol{f})$. In view of (2.11) and (2.19), [4, Lemma 6.34] implies that

$$\overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}(x) \in \overline{\operatorname{cone}(K(x) - \boldsymbol{j}(x))},\ \boldsymbol{v}(x) \cdot \boldsymbol{y}(x) = 0 \text{ a.e. in } \Omega\}.$$

Moreover, it holds that

$$\overline{\operatorname{cone}(K(x)-\boldsymbol{j}(x))} = \{\boldsymbol{w}\in\mathbb{R}^n : \boldsymbol{w}_i \le 0 \text{ if } \boldsymbol{j}_i(x) = \mathfrak{g}_i(x), \\ \boldsymbol{w}_i \ge 0 \text{ if } \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x),\ i=1,...,n\} \quad \text{for a.e. } x\in\Omega,$$

and thus,

$$\begin{aligned}\overline{R_{\mathcal{K}}(\boldsymbol{j})}\cap[\boldsymbol{y}]^\perp = \{\boldsymbol{v}\in \boldsymbol{L}^2(\Omega) : \ & \boldsymbol{v}_i(x)\le 0 \text{ if } \boldsymbol{j}_i(x) = \mathfrak{g}_i(x),\\ & \boldsymbol{v}_i(x)\ge 0 \text{ if } \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x),\ i=1,...,n,\\ & \boldsymbol{v}(x)\cdot\boldsymbol{y}(x) = 0 \text{ a.e. in } \Omega\}.\end{aligned} \tag{2.25}$$

In particular, it follows that $\mathcal{C}(\boldsymbol{f})\subset\overline{R_{\mathcal{K}}(\boldsymbol{j})}\cap[\boldsymbol{y}]^\perp$. Let now $\boldsymbol{v}\in\overline{R_{\mathcal{K}}(\boldsymbol{j})}\cap[\boldsymbol{y}]^\perp$ be arbitrary, but fixed. From (2.3a) we know that

$$\boldsymbol{j}_i(x) = \operatorname{sgn}\boldsymbol{y}_i(x)\mathfrak{g}_i(x) \text{ a.e. in } \{x\in\Omega : \boldsymbol{y}_i(x)\neq 0\} \quad \forall i=1,...,n.$$

Hence, by (2.25) we obtain

$$\boldsymbol{v}_i(x)\boldsymbol{y}_i(x)\le 0 \text{ a.e. in } \Omega \quad \forall\, i=1,...,n. \tag{2.26}$$

Assume now that there exists a set $\omega\subset\Omega$ with $|\omega|>0$ such that

$$\boldsymbol{v}_j(x)\boldsymbol{y}_j(x) < 0 \text{ a.e. in } \omega,$$

for some $j\in\{1,\dots,n\}$. Thus, (2.26) implies that

$$\boldsymbol{v}(x)\cdot\boldsymbol{y}(x) = \sum_{i=1}^n \boldsymbol{v}_i(x)\boldsymbol{y}_i(x) < 0 \text{ a.e. in } \omega,$$

which is in contradiction with $\boldsymbol{v}\in\overline{R_{\mathcal{K}}(\boldsymbol{j})}\cap[\boldsymbol{y}]^\perp$ (see (2.25)). Therefore, (2.26) holds as an equality, which means that $\boldsymbol{v}\in\mathcal{C}(\boldsymbol{f})$ (see (2.17)). This completes the proof.

□

Lemma 2.9. *The polar cone $\mathcal{C}(\boldsymbol{f})^\circ$ of (2.17) is given by*

$$\begin{aligned}\mathcal{Q}(\boldsymbol{f}) := \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega) : \ & \boldsymbol{v}_i(x) = 0 \ \textit{ if } |\boldsymbol{j}_i(x)| < \mathfrak{g}_i(x),\\ & \boldsymbol{v}_i(x)\boldsymbol{j}_i(x)\ge 0 \ \textit{ if } |\boldsymbol{j}_i(x)| = \mathfrak{g}_i(x) \textit{ and } \boldsymbol{y}_i(x) = 0\\ & \textit{a.e. in } \Omega,\ i=1,...,n\}.\end{aligned} \tag{2.27}$$

Proof. (i) We first show that $\mathcal{Q}(\boldsymbol{f})\subset\mathcal{C}(\boldsymbol{f})^\circ$. To this end, let $\boldsymbol{v}\in\mathcal{Q}(\boldsymbol{f})$ and $\boldsymbol{\xi}\in\mathcal{C}(\boldsymbol{f})$ be arbitrary, but fixed. For every $i\in\{1,...,n\}$, we set

$$\begin{aligned} U_i &:= \{x\in\Omega : |\boldsymbol{j}_i(x)| < \mathfrak{g}_i(x)\},\\ V_i^+ &:= \{x\in\Omega : \boldsymbol{j}_i(x) = \mathfrak{g}_i(x),\ \boldsymbol{y}_i(x) = 0\},\\ V_i^- &:= \{x\in\Omega : \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x),\ \boldsymbol{y}_i(x) = 0\},\\ W_i &:= \{x\in\Omega : |\boldsymbol{j}_i(x)| = \mathfrak{g}_i(x),\ \boldsymbol{y}_i(x)\neq 0\}.\end{aligned} \tag{2.28}$$

Since $\boldsymbol{j}\in\mathcal{K}$, cf. ($VI_D$), we have $\Omega = U_i\cup V_i^+\cup V_i^-\cup W_i$. Note that $\boldsymbol{v}_i(x)\ge 0$ a.e. in $V_i^+\cap\{\mathfrak{g}_i>0\}$, while $\boldsymbol{\xi}_i(x)\le 0$ a.e. in $V_i^+\cap\{\mathfrak{g}_i>0\}$ and $\boldsymbol{\xi}_i(x) = 0$ a.e. in

$V_i^+ \cap \{\mathfrak{g}_i = 0\}$. Moreover, $\boldsymbol{v}_i(x) \leq 0$ a.e. in $V_i^- \cap \{\mathfrak{g}_i > 0\}$, while $\boldsymbol{\xi}_i(x) \geq 0$ a.e. in $V_i^- \cap \{\mathfrak{g}_i > 0\}$ and $\boldsymbol{\xi}_i(x) = 0$ a.e. in $V_i^- \cap \{\mathfrak{g}_i = 0\}$. Thus

$$\boldsymbol{v}_i(x)\boldsymbol{\xi}_i(x) \leq 0 \quad \text{a.e. in } V_i^+ \cup V_i^-. \tag{2.29}$$

Since $\boldsymbol{y}_i(x) \neq 0$ a.e. in W_i, $\boldsymbol{\xi}_i(x) = 0$ must hold a.e. in W_i, due to the identity in the definition of $\mathcal{C}(\boldsymbol{f})$, see (2.17). From the above and by employing the definition of $\mathcal{Q}(\boldsymbol{f})$ we have

$$\begin{aligned}(\boldsymbol{v}, \boldsymbol{\xi})_{\mathbf{2}} = \sum_{i=1}^{n} \int_{U_i} \underbrace{\boldsymbol{v}_i(x)}_{=0} \boldsymbol{\xi}_i(x)\, dx + \int_{V_i^+ \cup V_i^-} \underbrace{\boldsymbol{v}_i(x)\boldsymbol{\xi}_i(x)}_{\leq 0}\, dx + \int_{W_i} \boldsymbol{v}_i(x) \underbrace{\boldsymbol{\xi}_i(x)}_{=0}\, dx \\ \leq 0,\end{aligned} \tag{2.30}$$

from which $\boldsymbol{v} \in \mathcal{C}(\boldsymbol{f})^\circ$ follows.
(ii) Let now $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{f})^\circ$ be arbitrary, but fixed and consider $i \in \{1, ..., n\}$. Our goal is to prove that $\boldsymbol{z} \in \mathcal{Q}(\boldsymbol{f})$. To this end, we first show that

$$\boldsymbol{z}_i(x) = 0 \ \text{ a.e. in } \{x \in \Omega : |\boldsymbol{j}_i(x)| < \mathfrak{g}_i(x)\}. \tag{2.31}$$

We define the sets

$$\begin{aligned} U_i &:= \{x \in \Omega : |\boldsymbol{j}_i(x)| = \mathfrak{g}_i(x)\} \cup \{x \in \Omega : |\boldsymbol{j}_i(x)| < \mathfrak{g}_i(x),\, \boldsymbol{z}_i(x) = 0\}, \\ V_i^+ &:= \{x \in \Omega : |\boldsymbol{j}_i(x)| < \mathfrak{g}_i(x),\, \boldsymbol{z}_i(x) > 0\}, \\ V_i^- &:= \{x \in \Omega : |\boldsymbol{j}_i(x)| < \mathfrak{g}_i(x),\, \boldsymbol{z}_i(x) < 0\} \end{aligned} \tag{2.32}$$

and the function $\boldsymbol{\xi}_i : \Omega \to \mathbb{R}$ as

$$\boldsymbol{\xi}_i(x) := 0 \text{ a.e. in } U_i, \quad \boldsymbol{\xi}_i(x) := 1 \text{ a.e. in } V_i^+, \quad \boldsymbol{\xi}_i(x) := -1 \text{ a.e. in } V_i^-.$$

We notice $\Omega = U_i \cup V_i^+ \cup V_i^-$, since $\boldsymbol{j} \in \mathcal{K}$. Note that the resulting vector valued function $\boldsymbol{\xi} : \Omega \to \mathbb{R}^n$ is measurable, since U_i, V_i^+ and V_i^- are measurable (as $\boldsymbol{j}_i$, $\mathfrak{g}_i$ and $\boldsymbol{z}_i$ do so). Further, $\boldsymbol{\xi}$ satisfies a.e. in Ω

$$\begin{aligned} &\boldsymbol{\xi}_i(x) \leq 0 \ \text{ a.e. in } \{x \in \Omega : \boldsymbol{j}_i(x) = \mathfrak{g}_i(x)\}, \\ &\boldsymbol{\xi}_i(x) \geq 0 \ \text{ a.e. in } \{x \in \Omega : \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x)\}, \\ &\boldsymbol{\xi}_i(x)\boldsymbol{y}_i(x) = 0 \text{ a.e. in } \Omega \quad (\boldsymbol{y}_i(x) = 0 \ \text{ a.e. in } V_i^+ \cup V_i^-, \text{due to (2.3a)}). \end{aligned}$$

Hence, $\boldsymbol{\xi} \in \mathcal{C}(\boldsymbol{f})$. Then, $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{f})^\circ$ leads to

$$0 \geq (\boldsymbol{z}, \boldsymbol{\xi})_{\mathbf{2}} = \sum_{i=1}^{n} \int_{V_i^+} \underbrace{\boldsymbol{z}_i(x)}_{>0} \underbrace{\boldsymbol{\xi}_i(x)}_{=1}\, dx + \int_{V_i^-} \underbrace{\boldsymbol{z}_i(x)}_{<0} \underbrace{\boldsymbol{\xi}_i(x)}_{=-1}\, dx.$$

If the set V_j^+ or V_j^- has positive measure for some $j \in \{1, ..., n\}$, the term on the above right-hand side is strictly positive, which is in contradiction with $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{f})^\circ$. Thus, V_i^+ or V_i^- have measure zero for all $i \in \{1, ..., n\}$ and in view of (2.32), we can now deduce (2.31) $\forall\, i \in \{1, ..., n\}$.
It remains to show $\boldsymbol{z}_i(x)\boldsymbol{j}_i(x) \geq 0$ a.e. in $\{x \in \Omega : |\boldsymbol{j}_i(x)| = \mathfrak{g}_i(x) \,\text{and}\, \boldsymbol{y}_i(x) = 0\}$. First we observe that

$$\boldsymbol{z}_i(x)\boldsymbol{j}_i(x) \geq 0 \ \text{ a.e. in } \{x \in \Omega : |\boldsymbol{j}_i(x)| = \mathfrak{g}_i(x),\, \mathfrak{g}_i(x) = 0 \,\text{and}\, \boldsymbol{y}_i(x) = 0\} \tag{2.33}$$

is automatically fulfilled. In the following we prove

$$\boldsymbol{z}_i(x) \geq 0 \ \text{ a.e. in } \{x \in \Omega : \boldsymbol{j}_i(x) = \mathfrak{g}_i(x), \mathfrak{g}_i(x) > 0 \,\text{and}\, \boldsymbol{y}_i(x) = 0\}. \tag{2.34}$$

Let us point out that (2.34) does not necessarily hold true on the set $\{x \in \Omega : \boldsymbol{j}_i(x) = \mathfrak{g}_i(x) = 0 \text{ and } \boldsymbol{y}_i(x) = 0\}$, see (2.33). To prove (2.34), we define

$$W_i := \{x \in \Omega : \boldsymbol{j}_i(x) = \mathfrak{g}_i(x)\,, \mathfrak{g}_i(x) > 0,\, \boldsymbol{y}_i(x) = 0, \boldsymbol{z}_i(x) < 0\} \tag{2.35}$$

and $\boldsymbol{\xi}_i : \Omega \to \mathbb{R}$ as

$$\boldsymbol{\xi}_i(x) := -1 \text{ a.e. in } W_i, \quad \boldsymbol{\xi}_i(x) := 0 \text{ a.e. in } \Omega \setminus W_i.$$

Note that $\boldsymbol{\xi} \in \boldsymbol{L}^2(\Omega)$ satisfies:

$$\begin{aligned}
&\boldsymbol{\xi}_i(x) \leq 0 \ \text{ a.e. in } \{x \in \Omega : \boldsymbol{j}_i(x) = \mathfrak{g}_i(x)\},\\
&\boldsymbol{\xi}_i(x) \geq 0 \ \text{ a.e. in } \{x \in \Omega : \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x)\} \quad (W_i \cap \{\boldsymbol{j}_i = -\mathfrak{g}_i\} = \emptyset),\\
&\boldsymbol{\xi}_i(x)\boldsymbol{y}_i(x) = 0 \text{ a.e. in } \Omega \quad (\boldsymbol{y}_i(x) = 0 \ \text{ a.e. in } W_i).
\end{aligned}$$

Hence, $\boldsymbol{\xi} \in \mathcal{C}(\boldsymbol{f})$, and due to $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{f})^\circ$ one has

$$0 \geq (\boldsymbol{z}, \boldsymbol{\xi})_{\boldsymbol{2}} = \sum_{i=1}^{n} \int_{W_i} \underbrace{\boldsymbol{z}_i(x)}_{<0} \underbrace{\boldsymbol{\xi}_i(x)}_{=-1} \, dx.$$

If the set W_j has positive measure for some $j \in \{1, ..., n\}$, the term on the above right-hand side is strictly positive, which is in contradiction with $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{f})^\circ$. Thus, W_i has measure zero for all $i \in \{1, ..., n\}$ and in view of (2.35), it follows that (2.34) holds true for all $i \in \{1, ..., n\}$. Its counterpart, namely

$$\boldsymbol{z}_i(x) \leq 0 \ \text{ a.e. in } \{x \in \Omega : \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x),\, \mathfrak{g}_i(x) > 0 \,\text{and}\, \boldsymbol{y}_i(x) = 0\}, \tag{2.36}$$

follows by defining

$$\begin{gathered}
W_i := \{x \in \Omega : \boldsymbol{j}_i(x) = -\mathfrak{g}_i(x), \mathfrak{g}_i(x) > 0,\, \boldsymbol{y}_i(x) = 0, \boldsymbol{z}_i(x) > 0\},\\
\boldsymbol{\xi}_i(x) := 1 \text{ a.e. in } W_i, \quad \boldsymbol{\xi}_i(x) := 0 \text{ a.e. in } \Omega \setminus W_i
\end{gathered}$$

and arguing as above. The inclusion $\mathcal{C}(\boldsymbol{f})^\circ \subset \mathcal{Q}(\boldsymbol{f})$ is now given by (2.31), (2.33), (2.34) and (2.36). This completes the proof. ∎

We have now all ingredients to prove the (strong) directional differentiability of S.

THEOREM 2.10. *The solution operator $S : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ of (VI) is directionally differentiable. For all $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\delta f} \in \boldsymbol{L}^2(\Omega)$, its directional derivative $\boldsymbol{\eta} := S'(\boldsymbol{f}; \boldsymbol{\delta f})$ is the unique solution of the following VI of the first kind*

$$\boldsymbol{\eta} \in \mathcal{Q}(\boldsymbol{f}), \ \ (B\boldsymbol{\eta} - \boldsymbol{\delta f}, \boldsymbol{v} - \boldsymbol{\eta})_{\boldsymbol{2}} \geq 0 \quad \forall\, \boldsymbol{v} \in \mathcal{Q}(\boldsymbol{f}), \tag{2.37}$$

where $\mathcal{Q}(\boldsymbol{f})$ was defined in (2.27).

Proof. Let $\boldsymbol{f}, \boldsymbol{\delta f} \in \boldsymbol{L}^2(\Omega)$ and $\{\tau_k\} \subset \mathbb{R}^+$ with $\tau_k \searrow 0$. We set $\boldsymbol{y} = S(\boldsymbol{f})$ and $\boldsymbol{y}_k = S(\boldsymbol{f} + \tau_k \boldsymbol{\delta f})$ for all $k \in \mathbb{N}$. According to Lemma 2.8, there is a (not relabeled) subsequence of $\{\tau_k\}$ and $\boldsymbol{\delta y} \in \boldsymbol{L}^2(\Omega)$ such that

$$\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k} \to \boldsymbol{\delta y} \quad \text{ in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty.$$

We also know from (2.16) that

$$\boldsymbol{\delta y} \in \partial \mathcal{I}_{\mathcal{C}(\boldsymbol{f})}(-B\boldsymbol{\delta y} + \boldsymbol{\delta f}). \tag{2.38}$$

Moreover, in the second part of the proof of Lemma 2.8 we established that $\mathcal{C}(\boldsymbol{f}) = \overline{R_{\mathcal{K}}(\boldsymbol{j})} \cap [\boldsymbol{y}]^{\perp}$, where $\boldsymbol{j} := -B\boldsymbol{y} + \boldsymbol{f}$. Hence, $\mathcal{C}(\boldsymbol{f})$ is a nonempty, convex and closed cone, see (2.9). Therefore, $\mathcal{I}_{\mathcal{C}(\boldsymbol{f})}$ is a proper, convex and lower semicontinuous functional, and so (2.38) is equivalent to

$$-B\boldsymbol{\delta y} + \boldsymbol{\delta f} \in \partial \mathcal{I}^*_{\mathcal{C}(\boldsymbol{f})}(\boldsymbol{\delta y}).$$

On the other hand, Lemma 2.4 implies that $\mathcal{I}^*_{\mathcal{C}(\boldsymbol{f})} = \mathcal{I}_K$, where $K := \{\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega) : (\boldsymbol{\zeta}, \boldsymbol{v})_{\boldsymbol{2}} \leq 0 \quad \forall\, \boldsymbol{v} \in \mathcal{C}(\boldsymbol{f})\} = \mathcal{C}(\boldsymbol{f})^{\circ}$. Hence, on account of Lemma 2.9, (2.38) is equivalent to

$$-\, B\boldsymbol{\delta y} + \boldsymbol{\delta f} \in \partial \mathcal{I}_{\mathcal{Q}(\boldsymbol{f})}(\boldsymbol{\delta y}). \tag{2.39}$$

Thus, $\boldsymbol{\delta y}$ is the unique solution of (2.37) (see e.g. [13] for the existence and uniqueness result). Therefore, if we consider another subsequence of the original sequence $\{\tau_k\}$, such that the associated sequence $\left\{\frac{\boldsymbol{y}_k - \boldsymbol{y}}{\tau_k}\right\}$ convergences strongly, then its limit will be the same $\boldsymbol{\delta y}$. Altogether, we conclude

$$\frac{S(\boldsymbol{f} + \tau_k \boldsymbol{\delta f}) - S(\boldsymbol{f})}{\tau_k} \to \boldsymbol{\eta} \quad \text{ in } \boldsymbol{L}^2(\Omega) \quad \forall\, \tau_k \searrow 0,$$

where $\boldsymbol{\eta}$ is characterized by (2.37). This completes the proof. ∎

Remark 2.11. *The operator S is Hadamard directionally differentiable, since it is directionally differentiable and Lipschitz continuous, see e.g. [4, Proposition 2.49].*

2.2. Strong Stationarity. Exploiting the directional differentiability of the solution operator S, we shall prove a strong stationary optimality system for (P_b), i.e., optimality conditions which are equivalent to the necessary optimality condition in primal form (cf. [26]).

Theorem 2.12. *Let $\bar{\boldsymbol{u}}$ be a local optimum of (P_b) with associated state $\bar{\boldsymbol{y}}$. Then there is a unique adjoint state $\boldsymbol{p} \in \boldsymbol{L}^2(\Omega)$ and a unique multiplier $\boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)$ so that the following strong stationary optimality system is fulfilled*

$$-B\bar{\boldsymbol{y}} + \bar{\boldsymbol{u}} \in \partial\varphi(\bar{\boldsymbol{y}}), \tag{2.40a}$$
$$\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) - B^*\boldsymbol{p} + \boldsymbol{\mu} = \boldsymbol{0}, \tag{2.40b}$$
$$\boldsymbol{\mu} \in \mathcal{C}(\overline{\boldsymbol{u}}), \tag{2.40c}$$
$$\boldsymbol{p} \in \mathcal{Q}(\overline{\boldsymbol{u}}), \tag{2.40d}$$
$$\boldsymbol{p} + \partial_{\boldsymbol{u}} J(\overline{\boldsymbol{y}}, \overline{\boldsymbol{u}}) = \boldsymbol{0}, \tag{2.40e}$$

where $\mathcal{C}(\overline{\boldsymbol{u}})$ and $\mathcal{Q}(\overline{\boldsymbol{u}})$ are defined as in (2.17) and (2.27), respectively.

Proof. Let us introduce the control-reduced objective functional of (P_b) by

$$f : \boldsymbol{U} \to \mathbb{R}, \quad f(\boldsymbol{u}) := J(S(\boldsymbol{u}), \boldsymbol{u}).$$

As $J : \boldsymbol{L}^2(\Omega) \times \boldsymbol{U} \to \mathbb{R}$ is Fréchet-differentiable and $S : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is directionally differentiable (Theorem 2.10), it follows that $f : \boldsymbol{U} \to \mathbb{R}$ is directional differentiable (cf. [15, Lemma 3.9]). Its directional derivative at $\bar{\boldsymbol{u}}$ in the direction $\boldsymbol{h} \in \boldsymbol{U}$ is given by $\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) S'(\bar{\boldsymbol{u}}; \boldsymbol{h}) + \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) \boldsymbol{h}$. Thus, $\bar{\boldsymbol{u}}$ satisfies the following necessary optimality condition

$$\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) S'(\bar{\boldsymbol{u}}; \boldsymbol{h}) + \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) \boldsymbol{h} \geq 0 \quad \forall\, \boldsymbol{h} \in \boldsymbol{U}. \tag{2.41}$$

We define $\boldsymbol{p} := -\partial_{\boldsymbol{u}} J(\overline{\boldsymbol{y}}, \overline{\boldsymbol{u}})$. By testing (2.37) with $\boldsymbol{0}$ and $2S'(\bar{\boldsymbol{u}}; \boldsymbol{h})$, respectively, we obtain

$$\alpha \|S'(\bar{\boldsymbol{u}}; \boldsymbol{h})\|_{\boldsymbol{2}}^2 \leq (BS'(\bar{\boldsymbol{u}}; \boldsymbol{h}), S'(\bar{\boldsymbol{u}}; \boldsymbol{h}))_{\boldsymbol{2}} = (\boldsymbol{h}, S'(\bar{\boldsymbol{u}}; \boldsymbol{h}))_{\boldsymbol{2}} \quad \forall\, \boldsymbol{h} \in \boldsymbol{L}^2(\Omega). \tag{2.42}$$

From (2.41)-(2.42) it follows that

$$\langle \boldsymbol{p}, \boldsymbol{h} \rangle_{\boldsymbol{U}^*, \boldsymbol{U}} \leq c \|\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})\|_{\boldsymbol{2}} \|\boldsymbol{h}\|_{\boldsymbol{2}} \quad \forall\, \boldsymbol{h} \in \boldsymbol{U},$$

and so Hahn-Banach's theorem implies that $\boldsymbol{p} \in \boldsymbol{L}^2(\Omega)$. Then, as $\boldsymbol{U} \overset{d}{\hookrightarrow} \boldsymbol{L}^2(\Omega)$, (2.41) yields that

$$\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) S'(\bar{\boldsymbol{u}}; \boldsymbol{h}) - (\boldsymbol{p}, \boldsymbol{h})_{\boldsymbol{2}} \geq 0 \quad \forall\, \boldsymbol{h} \in \boldsymbol{L}^2(\Omega), \tag{2.43}$$

where we also employed that $S'(\bar{\boldsymbol{u}}; \cdot) : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is continuous, due to [4, Proposition 2.49].

By Theorem 2.10, we have that $S'(\bar{\boldsymbol{u}}; \boldsymbol{h}) \in \mathcal{Q}(\overline{\boldsymbol{u}})$, and so inserting $h \in \mathcal{Q}(\overline{\boldsymbol{u}})^\circ$ in (2.42) implies $S'(\bar{\boldsymbol{u}}; \boldsymbol{h}) = \boldsymbol{0}$ for all $\boldsymbol{h} \in \mathcal{Q}(\overline{\boldsymbol{u}})^\circ$. It follows therefore from (2.43) that

$$(\boldsymbol{p}, \boldsymbol{h})_{\boldsymbol{2}} \leq 0 \quad \forall \boldsymbol{h} \in \mathcal{Q}(\overline{\boldsymbol{u}})^\circ \quad \Rightarrow \quad \boldsymbol{p} \in (\mathcal{Q}(\overline{\boldsymbol{u}})^\circ)^\circ = \mathcal{Q}(\overline{\boldsymbol{u}}),$$

since $\mathcal{Q}(\overline{\boldsymbol{u}})$ is a nonempty closed convex cone (Lemma 2.9).

Let us now set

$$\boldsymbol{\mu} := -\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) + B^* \boldsymbol{p}. \tag{2.44}$$

It remains to prove that $\boldsymbol{\mu} \in \mathcal{C}(\overline{\boldsymbol{u}})$. By Theorem 2.10, it holds that

$$\alpha \|S'(\bar{\boldsymbol{u}}; B\boldsymbol{v}) - \boldsymbol{v}\|_{\boldsymbol{2}}^2 \leq (BS'(\bar{\boldsymbol{u}}; B\boldsymbol{v}) - B\boldsymbol{v}, S'(\bar{\boldsymbol{u}}; B\boldsymbol{v}) - \boldsymbol{v})_{\boldsymbol{2}} \leq 0 \quad \forall \boldsymbol{v} \in \mathcal{Q}(\overline{\boldsymbol{u}}),$$

where the first inequality is again a result of the coercivity of B. Thus, $S'(\bar{\boldsymbol{u}}; B\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{Q}(\overline{\boldsymbol{u}})$. Testing in (2.43) with $\boldsymbol{h} \in B\mathcal{Q}(\overline{\boldsymbol{u}})$ now gives in turn

$$(\boldsymbol{\mu}, \boldsymbol{v})_{\boldsymbol{2}} \underbrace{=}_{(2.44)} -\partial_{\boldsymbol{y}} J(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) \boldsymbol{v} + (B^* \boldsymbol{p}, \boldsymbol{v})_{\boldsymbol{2}} \leq 0 \quad \forall\, \boldsymbol{v} \in \mathcal{Q}(\overline{\boldsymbol{u}}) \quad \Rightarrow \quad \boldsymbol{\mu} \in \mathcal{Q}(\overline{\boldsymbol{u}})^\circ = \mathcal{C}(\overline{\boldsymbol{u}}),$$

where we have used Lemma 2.9 and the fact that $\mathcal{C}(\overline{\boldsymbol{u}})$ is a nonempty closed convex cone. In conclusion, (2.40c) is valid.

Let us finally show that (2.40) is strong stationary, i.e., it is equivalent to (2.41). Indeed, this follows from (2.40c), $\mathcal{C}(\overline{\boldsymbol{u}}) = \mathcal{Q}(\overline{\boldsymbol{u}})^\circ$, (2.37) and (2.40d), which imply

$$(\boldsymbol{\mu}, S'(\bar{\boldsymbol{u}}; \boldsymbol{h}))_{\boldsymbol{2}} \leq 0 \leq (BS'(\bar{\boldsymbol{u}}; \boldsymbol{h}) - \boldsymbol{h}, \boldsymbol{p})_{\boldsymbol{2}} \quad \forall\, \boldsymbol{h} \in \boldsymbol{L}^2(\Omega).$$

Thus, the inequality (2.41) is an immediate result of the above inequality in combination with (2.40b) and (2.40e). The proof is now complete. ∎

Remark 2.13. *Let us compare our results with the ones obtained in [10] for $H_0^1(\Omega)$-elliptic VIs of the second kind. Firstly, we notice that our set $\mathcal{Q}(\boldsymbol{f})$ corresponds to the one in [10, Eq. (3.28)]. Moreover, Theorems 2.10 and 2.12 correspond to [10, Theorems 3.19 and 5.4] with the exception that [10] requires an additional regularity and structural assumptions on the unknown solution. This is due to the fact that $\mathcal{K}$ is polyhedric in $\boldsymbol{L}^2(\Omega)$, cf. Lemma 2.7, while in [10], the corresponding set $\{\xi \in H^{-1}(\Omega) : \langle \xi, v\rangle \leq \|v\|_1 \;\forall\, v \in H_0^1(\Omega)\}$ is not polyhedric in $H^{-1}(\Omega)$ (see the recent contribution [8]). Thus, additional assumptions have to be imposed in [10] to guarantee the polyhedricity. We also note that the authors of [10] show only weak directional differentiability.*

3. Optimality system for (P).

In this section, we analyze the optimal control problem (P) involving an unbounded operator:

$$\left.\begin{aligned} &\min_{\boldsymbol{u}\in\boldsymbol{U}} && \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 \\ &\text{s.t.} && -\boldsymbol{\nu}\boldsymbol{y} - A\boldsymbol{y} + \boldsymbol{u} \in \partial\varphi(\boldsymbol{y}). \end{aligned}\right\} \tag{P}$$

We summarize all mathematical assumptions for the data involved in (P):

Assumption 3.1. *For the quantities in (P) we require the following:*

1. *$\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain.*
2. *$\boldsymbol{\nu} : \Omega \to \mathbb{R}^{n\times n}_{sym}$ is a function of class $L^\infty(\Omega; \mathbb{R}^{n\times n}_{sym})$. Moreover, $\boldsymbol{\nu}$ is uniformly coercive, in the sense that there exists $\nu_0 > 0$ so that*
$$\boldsymbol{w}^T\boldsymbol{\nu}(x)\boldsymbol{w} \geq \nu_0|\boldsymbol{w}|^2 \text{ a.e. in } \Omega, \forall\, \boldsymbol{w} \in \mathbb{R}^n.$$
3. *$A : \mathcal{D}(A) \overset{d}{\hookrightarrow} \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is a linear, unbounded and skew-adjoint operator, i.e., $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^* = -A$.*
4. *$\boldsymbol{U}$ is a Hilbert space, so that $\boldsymbol{U} \overset{d}{\hookrightarrow} \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{U} \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$. Moreover, $\boldsymbol{U} \subset \mathcal{D}(A)$.*
5. *The desired state $\boldsymbol{y_d}$ belongs to $\boldsymbol{L}^2(\Omega)$ and $\kappa > 0$.*

As the operator A is skew-adjoint, a standard result implies that A is maximal monotone. For this reason, the Yosida approximation of A is well-defined as follows:

Definition 3.2. *For every $\lambda > 0$, the Yosida approximation of A is given by the operator $A_\lambda : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ defined as $A_\lambda := \frac{1}{\lambda}(I - (I + \lambda A)^{-1})$.*

For convenience of the reader we enumerate here some properties of A_λ which will be needed throughout this paper:

$$(A_\lambda \boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{2}} \geq 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \text{ and } \forall\, \lambda > 0, \tag{3.1a}$$

$$A_\lambda \boldsymbol{v} \overset{\lambda\to 0}{\to} A\boldsymbol{v} \;\text{ in } \boldsymbol{L}^2(\Omega),\ \forall\, \boldsymbol{v} \in \mathcal{D}(A), \tag{3.1b}$$

$$\|A_\lambda \boldsymbol{v}\|_{\boldsymbol{2}} \leq \|A\boldsymbol{v}\|_{\boldsymbol{2}} \;\; \forall\, \boldsymbol{v} \in \mathcal{D}(A) \text{ and } \forall\, \lambda > 0. \tag{3.1c}$$

Proposition 3.3. *Let $F : \boldsymbol{L}^2(\Omega) \to [0, \infty)$ be a convex, lower semicontinuous and positive homogeneous functional, i.e., $F(\beta\boldsymbol{w}) = \beta F(\boldsymbol{w})$ for all $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ and for all $\beta \geq 0$. Then, for any $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ the equation*

$$-\boldsymbol{\nu}\boldsymbol{y} - A\boldsymbol{y} + \boldsymbol{u} \in \partial F(\boldsymbol{y}) \tag{3.2}$$

admits a unique solution $\boldsymbol{y} \in \mathcal{D}(A)$.

Proof. Let $\lambda > 0$ be fixed and define $\boldsymbol{y}_\lambda := \mathcal{S}_\lambda^F(\boldsymbol{u})$, where $\mathcal{S}_\lambda^F : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is the solution operator of

$$-\boldsymbol{\nu}\boldsymbol{y} - A_\lambda \boldsymbol{y} + \boldsymbol{u} \in \partial F(\boldsymbol{y}).$$

Note that its existence is due to [13]. Since $\mathcal{S}_\lambda^F(\boldsymbol{0}) = \boldsymbol{0}$, we deduce by the Lipschitz continuity of $\mathcal{S}_\lambda^F : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ that $\|\boldsymbol{y}_\lambda\|_{\boldsymbol{2}} \le L \|\boldsymbol{u}\|_{\boldsymbol{2}}$, where $L > 0$ is independent of λ. Thus, we can extract a weakly convergent subsequence, denoted by the same symbol such that

$$\boldsymbol{y}_\lambda \rightharpoonup \hat{\boldsymbol{y}} \quad \text{in } \boldsymbol{L}^2(\Omega) \quad \text{as } \lambda \searrow 0. \tag{3.3}$$

Further, we define $\boldsymbol{j}_\lambda := -\boldsymbol{\nu}\boldsymbol{y}_\lambda - A_\lambda \boldsymbol{y}_\lambda + \boldsymbol{u}$ and $\hat{\boldsymbol{j}} : \mathcal{D}(A) \to \mathbb{R}$ as

$$\hat{\boldsymbol{j}}(\boldsymbol{v}) := -(\boldsymbol{\nu}\hat{\boldsymbol{y}}, \boldsymbol{v})_{\boldsymbol{2}} - (\hat{\boldsymbol{y}}, A^*\boldsymbol{v})_{\boldsymbol{2}} + (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{2}}. \tag{3.4}$$

By Lemma 5.1, (3.1b) and (3.3) we have for all $\boldsymbol{v} \in \mathcal{D}(A)$

$$(A_\lambda \boldsymbol{y}_\lambda, \boldsymbol{v})_{\boldsymbol{2}} = (\boldsymbol{y}_\lambda, A_\lambda^* \boldsymbol{v})_{\boldsymbol{2}} = (\boldsymbol{y}_\lambda, \underbrace{A_\lambda^*\boldsymbol{v} - A^*\boldsymbol{v}}_{\to 0})_{\boldsymbol{2}} + (\boldsymbol{y}_\lambda, A^*\boldsymbol{v})_{\boldsymbol{2}} \to (\hat{\boldsymbol{y}}, A^*\boldsymbol{v})_{\boldsymbol{2}},$$

which together with (3.3) gives

$$\boldsymbol{j}_\lambda(\boldsymbol{v}) \to \hat{\boldsymbol{j}}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathcal{D}(A) \quad \text{as } \lambda \searrow 0. \tag{3.5}$$

Since $\boldsymbol{j}_\lambda \in \partial F(\boldsymbol{y}_\lambda)$, we have in view of the positive homogeneity of F the following

$$\begin{cases} (\boldsymbol{j}_\lambda, \boldsymbol{y}_\lambda)_{\boldsymbol{2}} = F(\boldsymbol{y}_\lambda), \\ (\boldsymbol{j}_\lambda, \boldsymbol{v})_{\boldsymbol{2}} \le F(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \end{cases} \tag{3.6}$$

Passing to the limit in the inequality in (3.6) gives that $\hat{\boldsymbol{j}}(\boldsymbol{v}) \le F(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathcal{D}(A)$, due to (3.5). By Hahn-Banach's theorem we then infer that there exists $\widetilde{\boldsymbol{j}} \in \boldsymbol{L}^2(\Omega)$ so that

$$\begin{cases} \widetilde{\boldsymbol{j}}(\boldsymbol{v}) = \hat{\boldsymbol{j}}(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathcal{D}(A), \\ (\widetilde{\boldsymbol{j}}, \boldsymbol{v})_{\boldsymbol{2}} \le F(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \end{cases} \tag{3.7}$$

Since A is a maximal monotone operator, there exists a unique $\widetilde{\boldsymbol{y}} \in \mathcal{D}(A)$ such that

$$\widetilde{\boldsymbol{y}} + A\widetilde{\boldsymbol{y}} + \boldsymbol{\nu}\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}} - \boldsymbol{u} = -\widetilde{\boldsymbol{j}}. \tag{3.8}$$

Combined with the identity in (3.7) and (3.4), this yields for all $\boldsymbol{v} \in \mathcal{D}(A)$

$$(-\widetilde{\boldsymbol{y}} - A\widetilde{\boldsymbol{y}} - \boldsymbol{\nu}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}} + \boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{2}} = -(\boldsymbol{\nu}\hat{\boldsymbol{y}}, \boldsymbol{v})_{\boldsymbol{2}} - (\hat{\boldsymbol{y}}, A^*\boldsymbol{v})_{\boldsymbol{2}} + (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{2}}. \tag{3.9}$$

From the above identity and Assumption 3.1.3 we deduce

$$(\widetilde{\boldsymbol{y}} + A\widetilde{\boldsymbol{y}}, \boldsymbol{v})_{\boldsymbol{2}} - (\hat{\boldsymbol{y}}, \boldsymbol{v}) = (\hat{\boldsymbol{y}}, A^*\boldsymbol{v})_{\boldsymbol{2}} = -(\hat{\boldsymbol{y}}, A\boldsymbol{v})_{\boldsymbol{2}} \quad \forall\, \boldsymbol{v} \in \mathcal{D}(A). \tag{3.10}$$

Thus, $\hat{\boldsymbol{y}} \in \mathcal{D}(A^*) = \mathcal{D}(A)$, see e.g. [24]. Relying on $\mathcal{D}(A) \overset{d}{\hookrightarrow} \boldsymbol{L}^2(\Omega)$ and (3.10), we arrive at $(I + A)(\widetilde{\boldsymbol{y}} - \hat{\boldsymbol{y}}) = \boldsymbol{0}$ in $\boldsymbol{L}^2(\Omega)$, since $-(\hat{\boldsymbol{y}}, A\boldsymbol{v})_{\boldsymbol{2}} = (A\hat{\boldsymbol{y}}, \boldsymbol{v})_{\boldsymbol{2}}$. The maximal monotonicity of A gives in turn that $\hat{\boldsymbol{y}} = \widetilde{\boldsymbol{y}}$. Therefore, in view of (3.8),

$$-\boldsymbol{\nu}\hat{\boldsymbol{y}} - A\hat{\boldsymbol{y}} + \boldsymbol{u} = \widetilde{\boldsymbol{j}}. \tag{3.11}$$

By the weak sequential lower semicontinuity of F and (3.3), the identity in (3.6), the definition of $\boldsymbol{j}_\lambda$ and (3.1a), we further obtain

$$\begin{aligned} F(\hat{\boldsymbol{y}}) &\le \liminf_{\lambda\to 0} F(\boldsymbol{y}_\lambda) = \liminf_{\lambda\to 0} (\boldsymbol{j}_\lambda, \boldsymbol{y}_\lambda)_{\mathbf{2}} \\ &= \liminf_{\lambda\to 0} (-\boldsymbol{\nu}\boldsymbol{y}_\lambda - A_\lambda \boldsymbol{y}_\lambda + \boldsymbol{u}, \boldsymbol{y}_\lambda)_{\mathbf{2}} \\ &\le \limsup_{\lambda\to 0} (-\boldsymbol{\nu}\boldsymbol{y}_\lambda + \boldsymbol{u}, \boldsymbol{y}_\lambda)_{\mathbf{2}} \\ &\le (-\boldsymbol{\nu}\hat{\boldsymbol{y}} + \boldsymbol{u}, \hat{\boldsymbol{y}})_{\mathbf{2}}. \end{aligned}$$

Note that for the last inequality we employed the weak lower semicontinuity of $\boldsymbol{L}^2(\Omega) \ni \boldsymbol{v} \mapsto (\boldsymbol{\nu}\boldsymbol{v}, \boldsymbol{v})_{\mathbf{2}}$. In view of $(A\hat{\boldsymbol{y}}, \hat{\boldsymbol{y}})_{\mathbf{2}} = 0$ and (3.11), the above inequality can be continued as

$$F(\hat{\boldsymbol{y}}) \le (\widetilde{\boldsymbol{j}}, \hat{\boldsymbol{y}})_{\mathbf{2}} \le F(\hat{\boldsymbol{y}}), \tag{3.12}$$

where the last inequality is due to the inequality in (3.7). Hence, $(\widetilde{\boldsymbol{j}}, \hat{\boldsymbol{y}})_{\mathbf{2}} = F(\hat{\boldsymbol{y}})$. Thanks to the inequality in (3.7) and in view of (3.11), we can now conclude that there exists $\hat{\boldsymbol{y}} \in \mathcal{D}(A)$ so that $-\boldsymbol{\nu}\hat{\boldsymbol{y}} - A\hat{\boldsymbol{y}} + \boldsymbol{u} \in \partial F(\hat{\boldsymbol{y}})$. To show the uniqueness, let $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{D}(A)$ be two solutions of (3.2). Testing the VI for $\boldsymbol{y}_1$ with $\boldsymbol{y}_2$ and viceversa then gives

$$(\boldsymbol{\nu}(\boldsymbol{y}_1 - \boldsymbol{y}_2), \boldsymbol{y}_1 - \boldsymbol{y}_2)_{\mathbf{2}} + \underbrace{(A(\boldsymbol{y}_1 - \boldsymbol{y}_2), \boldsymbol{y}_1 - \boldsymbol{y}_2)_{\mathbf{2}}}_{=0} \le 0,$$

whence $\boldsymbol{y}_1 = \boldsymbol{y}_2$ follows, by the uniform coercivity of $\boldsymbol{\nu}$. This completes the proof. □

Since φ satisfies the assumptions of Proposition 3.3 (see (1.1)), the variational inequality associated with (P) admits a unique solution. In the following, we denote by $\mathcal{S} : \boldsymbol{L}^2(\Omega) \to \mathcal{D}(A), \boldsymbol{u} \mapsto \boldsymbol{y}$, the solution operator for

$$-\boldsymbol{\nu}\boldsymbol{y} - A\boldsymbol{y} + \boldsymbol{u} \in \partial\varphi(\boldsymbol{y}). \tag{3.13}$$

Remark 3.4. *As in the proof of Lemma 2.1, by employing the fact that $A(\boldsymbol{v}, \boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \mathcal{D}(A)$ and the uniform coercivity of $\boldsymbol{\nu}$, it can be shown that the solution operator $\mathcal{S}$ is Lipschitz continuous from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{L}^2(\Omega)$. Then, due to the compact embedding $\boldsymbol{U} \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$, the direct method of calculus of variations yields that* (P) *admits at least one solution.*

Next, for every $\lambda > 0$, let $\mathcal{S}_\lambda : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ denote the solution operator to

$$-\boldsymbol{\nu}\boldsymbol{y} - A_\lambda\boldsymbol{y} + \boldsymbol{u} \in \partial\varphi(\boldsymbol{y}). \tag{3.14}$$

From Assumption 3.1.2 and (3.1a) we deduce that $\boldsymbol{\nu}I + A_\lambda$ is linear, bounded and coercive with coercivity constant ν_0. Therefore, in view of Lemma 2.1, the following result holds:

Lemma 3.5. *For every $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ the equation (3.14) admits a unique solution $\boldsymbol{y} \in \boldsymbol{L}^2(\Omega)$. The associated solution operator $\mathcal{S}_\lambda : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is Lipschitz continuous with Lipschitz constant independent of λ.*

Lemma 3.6. *If $\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{U}$ as $\lambda \searrow 0$, then*

$$\mathcal{S}_\lambda(\boldsymbol{u}_\lambda) \to \mathcal{S}(\boldsymbol{u}) \quad \text{in } \boldsymbol{L}^2(\Omega), \text{ as } \lambda \searrow 0.$$

Proof. Let $\{\boldsymbol{u}_\lambda\}_\lambda \subset \boldsymbol{U}$ be a fixed sequence such that $\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{U}$ as $\lambda \searrow 0$. Let $\lambda > 0$ be arbitrary, but fixed. We define $\boldsymbol{y}_\lambda := \mathcal{S}_\lambda(\boldsymbol{u}_\lambda)$ and $\boldsymbol{y} := \mathcal{S}(\boldsymbol{u})$. Testing (3.13) with $\boldsymbol{y}_\lambda$ and (3.14) with $\boldsymbol{y}$ and adding the resulting inequalities then yields

$$(\boldsymbol{\nu}(\boldsymbol{y}-\boldsymbol{y}_\lambda), \boldsymbol{y}-\boldsymbol{y}_\lambda)_{\boldsymbol{2}} + (A\boldsymbol{y} - A_\lambda \boldsymbol{y}_\lambda, \boldsymbol{y}-\boldsymbol{y}_\lambda)_{\boldsymbol{2}} + (\boldsymbol{u}_\lambda - \boldsymbol{u}, \boldsymbol{y}-\boldsymbol{y}_\lambda)_{\boldsymbol{2}} \leq 0.$$

The uniform coercivity of $\boldsymbol{\nu}$, cf. Assumption 3.1.2, and (3.1a) then give in turn

$$\nu_0\|\boldsymbol{y}-\boldsymbol{y}_\lambda\|_{\boldsymbol{2}}^2 + (A\boldsymbol{y} - A_\lambda\boldsymbol{y}, \boldsymbol{y}-\boldsymbol{y}_\lambda)_{\boldsymbol{2}} + \underbrace{(A_\lambda(\boldsymbol{y}-\boldsymbol{y}_\lambda), \boldsymbol{y}-\boldsymbol{y}_\lambda)_{\boldsymbol{2}}}_{\geq 0} \leq \|\boldsymbol{u}_\lambda - \boldsymbol{u}\|_{\boldsymbol{2}}\|\boldsymbol{y}-\boldsymbol{y}_\lambda\|_{\boldsymbol{2}}$$

By bringing the second term on the right-hand side and by dividing with $\|\boldsymbol{y}-\boldsymbol{y}_\lambda\|_{\boldsymbol{2}}$ (we assume $\boldsymbol{y} \neq \boldsymbol{y}_\lambda$), we then have

$$\nu_0\|\boldsymbol{y}-\boldsymbol{y}_\lambda\|_{\boldsymbol{2}} \leq \|\boldsymbol{u}_\lambda - \boldsymbol{u}\|_{\boldsymbol{2}} + \|A\boldsymbol{y} - A_\lambda\boldsymbol{y}\|_{\boldsymbol{2}}.$$

Assumption 3.1.4 and (3.1b) combined with $\boldsymbol{y} \in \mathcal{D}(A)$ imply the assertion. ◻

For the rest of this section let $\bar{\boldsymbol{u}} \in \boldsymbol{U}$ be an arbitrary, but fixed local minimizer of (P). For a given $\lambda > 0$, we consider the following optimization problem:

$$\left.\begin{aligned} \min_{\boldsymbol{u}\in\boldsymbol{U}} \quad & \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 \\ \text{s.t.} \quad & -\boldsymbol{\nu}\boldsymbol{y} - A_\lambda\boldsymbol{y} + \boldsymbol{u} \in \partial\varphi(\boldsymbol{y}). \end{aligned}\right\} \tag{P_λ}$$

We emphasize that (P_λ) is still a non-smooth problem. Here, the non-smooth variational-inequality-structure is preserved, while the unbounded operator A is approximated using the Yosida approximation. In the following, we shall make use of the reduced cost functionals of (P) and (P_λ), denoted respectively by

$$\mathcal{J} : \boldsymbol{U} \to \mathbb{R}, \quad \boldsymbol{u} \mapsto \frac{1}{2}\|\mathcal{S}(\boldsymbol{u}) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2, \tag{3.15a}$$

$$\mathcal{J}_\lambda : \boldsymbol{U} \to \mathbb{R}, \quad \boldsymbol{u} \mapsto \frac{1}{2}\|\mathcal{S}_\lambda(\boldsymbol{u}) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2. \tag{3.15b}$$

Proposition 3.7 (Convergence of the minimizers). *Let $\bar{\boldsymbol{u}} \in \boldsymbol{U}$ be a local minimizer of (P). Then, there exists a sequence $\{\boldsymbol{u}_\lambda\}_{\lambda>0}$ of local minimizers of (P_λ) such that*

$$\boldsymbol{u}_\lambda \to \bar{\boldsymbol{u}} \quad \text{in } \boldsymbol{U} \text{ as } \lambda \searrow 0. \tag{3.16}$$

Moreover,

$$\mathcal{S}_\lambda(\boldsymbol{u}_\lambda) \to \mathcal{S}(\bar{\boldsymbol{u}}) \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \lambda \searrow 0. \tag{3.17}$$

Proof. Let $B(\bar{\boldsymbol{u}}, \rho) := \overline{B_{\boldsymbol{U}}(\bar{\boldsymbol{u}}, \rho)}$ with some $\rho > 0$ be the closed ball of local optimality of $\bar{\boldsymbol{u}}$, i.e.,

$$\mathcal{J}(\bar{\boldsymbol{u}}) \leq \mathcal{J}(\boldsymbol{u}) \quad \forall\, \boldsymbol{u} \in B(\bar{\boldsymbol{u}}, \rho). \tag{3.18}$$

For every $\lambda > 0$, we consider the following auxiliary optimal control problem:

$$\left.\begin{aligned} \min \quad & \mathcal{J}_\lambda(\boldsymbol{u}) \\ \text{s.t.} \quad & \boldsymbol{u} \in B(\bar{\boldsymbol{u}}, \rho). \end{aligned}\right\} \tag{P_λ^ρ}$$

The existence of global minimizers for (P_λ^ρ) follows by standard arguments thanks to the compactness embedding $\boldsymbol{U} \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$ and the Lipschitz continuity of $\mathcal{S}_\lambda : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ (Lemma 3.5). In the sequel, for every $\lambda > 0$, let $\boldsymbol{u}_\lambda$ denote a global minimizer of (P_λ^ρ). As $\{\boldsymbol{u}_\lambda\} \subset B(\bar{\boldsymbol{u}}, \rho)$, we can select a weakly converging subsequence, which we denote by the same symbol, i.e.,

$$\boldsymbol{u}_\lambda \rightharpoonup \tilde{\boldsymbol{u}} \quad \text{in } \boldsymbol{U} \quad \text{as } \lambda \searrow 0. \tag{3.19}$$

Note that since $B(\bar{\boldsymbol{u}}, \rho)$ is weakly closed, $\tilde{\boldsymbol{u}} \in B(\bar{\boldsymbol{u}}, \rho)$ follows. By the weak lower semicontinuity of the squared norm, we arrive at

$$\begin{aligned}
&\mathcal{J}(\bar{\boldsymbol{u}}) + \frac{1}{2}\|\tilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 \le \frac{1}{2}\|\mathcal{S}(\tilde{\boldsymbol{u}}) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\tilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 \quad \text{((3.15a) and (3.18))}\\
&\le \liminf_{\lambda\to 0} \frac{1}{2}\|\mathcal{S}_\lambda(\boldsymbol{u}_\lambda) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\boldsymbol{u}_\lambda - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 \quad \text{((3.19) and Lemma 3.6)}\\
&\le \limsup_{\lambda\to 0} \frac{1}{2}\|\mathcal{S}_\lambda(\boldsymbol{u}_\lambda) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\boldsymbol{u}_\lambda - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2\\
&\le \limsup_{\lambda\to 0} \frac{1}{2}\|\mathcal{S}_\lambda(\bar{\boldsymbol{u}}) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 \quad (\boldsymbol{u}_\lambda \text{ global optimal for } (\mathrm{P}_\lambda^\rho) \text{ and (3.15b))}\\
&= \mathcal{J}(\bar{\boldsymbol{u}}) \quad \text{(Lemma 3.6 and (3.15a))},
\end{aligned}$$

whence $\tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}}$ follows. Using this information in the above series of inequalities we further obtain

$$\begin{aligned}
\lim_{\lambda\to 0} \frac{1}{2}\|\mathcal{S}_\lambda(\boldsymbol{u}_\lambda) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\boldsymbol{u}_\lambda - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 &= \mathcal{J}(\bar{\boldsymbol{u}})\\
&= \frac{1}{2}\|\mathcal{S}(\bar{\boldsymbol{u}}) - \boldsymbol{y_d}\|_{\boldsymbol{2}}^2 + \frac{\kappa}{2}\|\bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2.
\end{aligned}$$

Since Lemma 3.6 yields the strong convergence $\mathcal{S}_\lambda(\boldsymbol{u}_\lambda) \to \mathcal{S}(\bar{\boldsymbol{u}})$ in $\boldsymbol{L}^2(\Omega)$, it follows that

$$\lim_{\lambda\to 0} \frac{\kappa}{2}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{U}}^2 + \frac{1}{2}\|\boldsymbol{u}_\lambda - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2 = \frac{\kappa}{2}\|\bar{\boldsymbol{u}}\|_{\boldsymbol{U}}^2,$$

which can be written as

$$\|(\sqrt{\kappa}\,\boldsymbol{u}_\lambda, \boldsymbol{u}_\lambda - \bar{\boldsymbol{u}})\|_{\boldsymbol{U}\times\boldsymbol{U}}^2 \to \|(\sqrt{\kappa}\,\bar{\boldsymbol{u}}, \boldsymbol{0})\|_{\boldsymbol{U}\times\boldsymbol{U}}^2 \quad \text{as } \lambda \searrow 0.$$

Thanks to $\boldsymbol{u}_\lambda \rightharpoonup \bar{\boldsymbol{u}}$ in $\boldsymbol{U}$ and since $\boldsymbol{U}$ is a Hilbert space (Assumption 3.1.4), this gives in turn $\boldsymbol{u}_\lambda \to \bar{\boldsymbol{u}}$ in $\boldsymbol{U}$. Since (3.17) is a direct result of Lemma 3.6, it only remains to prove that $\boldsymbol{u}_\lambda$ is a local minimizer for (P_λ). To this end, let $\boldsymbol{v} \in B_{\boldsymbol{U}}(\boldsymbol{u}_\lambda, \rho/2)$. Then, for sufficiently small $\lambda > 0$, (3.16) leads to

$$\|\boldsymbol{v} - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}} \le \|\boldsymbol{u}_\lambda - \bar{\boldsymbol{u}}\|_{\boldsymbol{U}} + \|\boldsymbol{v} - \boldsymbol{u}_\lambda\|_{\boldsymbol{U}} < \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

This yields $\boldsymbol{v} \in B(\bar{\boldsymbol{u}}, \rho)$, and the global optimality of $\boldsymbol{u}_\lambda$ for (P_λ^ρ) implies $\mathcal{J}_\lambda(\boldsymbol{u}_\lambda) \le \mathcal{J}_\lambda(\boldsymbol{v})$. Since $\boldsymbol{v} \in B_{\boldsymbol{U}}(\boldsymbol{u}_\lambda, \rho/2)$ was choosen arbitrarily, the claims follows. ∎

Theorem 3.8. *Let $\bar{\boldsymbol{u}}$ be a local optimum of* (P) *with the associated state $\bar{\boldsymbol{y}}$. Then there is a unique adjoint state $\boldsymbol{p} \in \boldsymbol{U}$ and a unique multiplier $\boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)$ so that the* *following optimality system is fulfilled*

$$-\boldsymbol{\nu}\bar{\boldsymbol{y}} - A\bar{\boldsymbol{y}} + \bar{\boldsymbol{u}} \in \partial\varphi(\bar{\boldsymbol{y}}), \tag{3.20a}$$
$$\bar{\boldsymbol{y}} - \boldsymbol{y_d} - \boldsymbol{\nu}\boldsymbol{p} - A^*\boldsymbol{p} + \boldsymbol{\mu} = \boldsymbol{0}, \tag{3.20b}$$
$$\boldsymbol{p} + \kappa\overline{\boldsymbol{u}} = \boldsymbol{0}, \tag{3.20c}$$
$$\boldsymbol{\mu}_i(x)\bar{\boldsymbol{y}}_i(x) = 0 \ \textit{a.e. in } \Omega, \ \forall\, i = 1, ..., n, \tag{3.20d}$$
$$\boldsymbol{\mu}_i(x)\boldsymbol{p}_i(x) \leq 0 \ \textit{a.e. in } \Omega, \ \forall\, i = 1, ..., n. \tag{3.20e}$$

Proof. Let $\{\boldsymbol{u}_\lambda\}_{\lambda>0}$ be the sequence of local minimizers of (P_λ) established in Proposition 3.7, which converges strongly to $\overline{\boldsymbol{u}}$ as $\lambda \to 0$. For every $\lambda > 0$, we set $\boldsymbol{y}_\lambda = \mathcal{S}_\lambda(\boldsymbol{u}_\lambda)$. As $A_\lambda : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is a linear bounded operator, we see that (P_λ) is a special case of (P_b) with $B = \boldsymbol{\nu} + A_\lambda$. Thus, we may apply Theorem 2.12 to (P_λ) to deduce that

$$\boldsymbol{y}_\lambda - \boldsymbol{y_d} - (\boldsymbol{\nu}I + A_\lambda)^*\boldsymbol{p}_\lambda + \boldsymbol{\mu}_\lambda = \boldsymbol{0}, \tag{3.21a}$$
$$\boldsymbol{\mu}_\lambda \in \mathcal{C}(\boldsymbol{u}_\lambda), \tag{3.21b}$$
$$\boldsymbol{p}_\lambda \in \mathcal{Q}(\boldsymbol{u}_\lambda), \tag{3.21c}$$

where $\mathcal{C}(\boldsymbol{u}_\lambda)$ and $\mathcal{Q}(\boldsymbol{u}_\lambda)$ are defined as in (2.17) and (2.27).

By the definition of $\boldsymbol{p}_\lambda$ and (3.16), it follows that

$$\boldsymbol{p}_\lambda \to -\kappa\bar{\boldsymbol{u}} =: \boldsymbol{p} \ \text{ in } \boldsymbol{U} \text{ as } \lambda \searrow 0. \tag{3.22}$$

Moreover, (3.22) combined with $\boldsymbol{U} \subset \mathcal{D}(A)$, Lemma 5.1, (3.1b) and (3.1c) imply that

$$\begin{aligned} \|A_\lambda^*\boldsymbol{p}_\lambda - A^*\boldsymbol{p}\|_{\boldsymbol{2}} &\leq \|A_\lambda^*(\boldsymbol{p}_\lambda - \boldsymbol{p})\|_{\boldsymbol{2}} + \|(A_\lambda^* - A^*)\boldsymbol{p}\|_{\boldsymbol{2}} \\ &\leq \|A^*(\boldsymbol{p}_\lambda - \boldsymbol{p})\|_{\boldsymbol{2}} + \|(A_\lambda^* - A^*)\boldsymbol{p}\|_{\boldsymbol{2}} \to 0 \text{ as } \lambda \searrow 0. \end{aligned} \tag{3.23}$$

Passing to the limit $\lambda \to 0$ in (3.21a) together with (3.17), (3.22) and (3.23) yields

$$\boldsymbol{\mu}_\lambda \to \boldsymbol{\mu} \ \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \lambda \searrow 0, \tag{3.24}$$

where $-\boldsymbol{\mu} = \bar{\boldsymbol{y}} - \boldsymbol{y_d} - \boldsymbol{\nu}\boldsymbol{p} - A^*\boldsymbol{p}$.

It remains to prove (3.20d)-(3.20e). Due to (3.21b),

$$\boldsymbol{\mu}_\lambda^i(x)\boldsymbol{y}_\lambda^i(x) = 0 \text{ a.e. in } \Omega, \ \forall\, i = 1, ..., n$$

holds. Furthermore, from (3.24) and (3.17) we have for all $i = 1, ..., n$ that

$$\boldsymbol{\mu}_\lambda^i\boldsymbol{y}_\lambda^i \to \boldsymbol{\mu}^i\bar{\boldsymbol{y}}^i \ \text{ in } L^1(\Omega) \text{ as } \lambda \searrow 0,$$

whence (3.20d) follows. To prove (3.20e), we define for every $\lambda > 0$ and $i \in \{1, ..., n\}$ the following sets (up to sets of measure zero):

$$\begin{aligned} U_\lambda^i &:= \{x \in \Omega : |\boldsymbol{j}_\lambda^i(x)| < \mathfrak{g}_i(x)\}, \\ V_{\lambda,+}^i &:= \{x \in \Omega : \boldsymbol{j}_\lambda^i(x) = \mathfrak{g}_i(x),\, \mathfrak{g}_i(x) > 0,\, \boldsymbol{y}_\lambda^i(x) = 0\}, \\ V_{\lambda,-}^i &:= \{x \in \Omega : \boldsymbol{j}_\lambda^i(x) = -\mathfrak{g}_i(x),\, \mathfrak{g}_i(x) > 0,\, \boldsymbol{y}_\lambda^i(x) = 0\}, \\ W_\lambda^i &:= \{x \in \Omega : |\boldsymbol{j}_\lambda^i(x)| = 0,\, \boldsymbol{y}_\lambda^i(x) = 0\} \cup \{x \in \Omega : |\boldsymbol{j}_\lambda^i(x)| = \mathfrak{g}_i(x),\, \boldsymbol{y}_\lambda^i(x) \neq 0\}. \end{aligned}$$

Clearly, $\Omega = U^i_\lambda \cup V^i_{\lambda,+} \cup V^i_{\lambda,-} \cup W^i_\lambda$. In view of (3.21b) and (3.21c), we obtain that

$$\begin{aligned} \boldsymbol{p}^i_\lambda(x) = 0 \quad & \text{a.e. in } U^i_\lambda, \\ \boldsymbol{p}^i_\lambda(x) \geq 0 \text{ and } \boldsymbol{\mu}^i_\lambda(x) \leq 0 \quad & \text{a.e. in } V^i_{\lambda,+}, \\ \boldsymbol{p}^i_\lambda(x) \leq 0 \text{ and } \boldsymbol{\mu}^i_\lambda(x) \geq 0 \quad & \text{a.e. in } V^i_{\lambda,-}, \\ \boldsymbol{\mu}^i_\lambda(x) = 0 \quad & \text{a.e. in } W^i_\lambda. \end{aligned}$$

Altogether, this implies $\boldsymbol{\mu}^i_\lambda(x)\boldsymbol{p}^i_\lambda(x) \leq 0$ a.e. in Ω. By means of (3.22) and (3.24), we infer as above that (3.20e) holds. This completes the proof. ∎

4. Application to the Bean critical-state model. Throughout this section, we set $n = 6$, and thus $\boldsymbol{L}^2(\Omega)$ stands for $L^2(\Omega;\mathbb{R}^6)$. We consider the following optimal control problem:

$$\left.\begin{aligned} \min_{(\boldsymbol{f},\boldsymbol{g})\in\mathcal{H}} \quad & \frac{1}{2}\|\boldsymbol{e}-\boldsymbol{e_d}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}\|\boldsymbol{h}-\boldsymbol{h_d}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{\kappa}{2}\|(\boldsymbol{f},\boldsymbol{g})\|^2_{\mathcal{H}} \\ \text{s.t.} \quad & \begin{cases} \epsilon\boldsymbol{e} - \mathbf{curl}\,\boldsymbol{h} + \boldsymbol{j} = \boldsymbol{f} & \text{in } \Omega, \\ \mu\boldsymbol{h} + \mathbf{curl}\,\boldsymbol{e} = \boldsymbol{g} & \text{in } \Omega, \\ \boldsymbol{e}\times\boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{j}_i(x)\boldsymbol{e}_i(x) = j_c(x)|\boldsymbol{e}_i(x)| & \text{for a.a. } x\in\Omega,\ \forall\, i = 1,2,3, \\ |\boldsymbol{j}_i(x)| \leq j_c(x) & \text{for a.a. } x\in\Omega,\ \forall\, i = 1,2,3. \end{cases} \end{aligned}\right\} \tag{4.1}$$

Here, $(\boldsymbol{e_d},\boldsymbol{h_d}) \in \boldsymbol{L}^2(\Omega)$ and $\kappa > 0$ are fixed. The electric permittivity $\epsilon : \Omega \to \mathbb{R}^{n\times n}_{sym}$ and the magnetic permeability $\mu : \Omega \to \mathbb{R}^{n\times n}_{sym}$ are assumed to be of class $L^\infty(\Omega;\mathbb{R}^{n\times n}_{sym})$. There exist constants $\epsilon_0 > 0$ and $\mu_0 > 0$ such that

$$\boldsymbol{w}^T\epsilon(x)\boldsymbol{w} \geq \epsilon_0|\boldsymbol{w}|^2 \quad \text{and} \quad \boldsymbol{w}^T\mu(x)\boldsymbol{w} \geq \mu_0|\boldsymbol{w}|^2 \quad \text{a.e. in } \Omega,\ \forall\,\boldsymbol{w}\in\mathbb{R}^n.$$

The function $j_c : \Omega \to \mathbb{R}$ is Lebesgue measurable, nonnegative and essentially bounded. We notice that the PDE-constraint in (4.1) arises from the time-discretization of Bean's critical-state model for type-II superconductivity (cf. [28,29]). Let us now reformulate (4.1) as a problem of the type (P). To this aim, we introduce the following Hilbert spaces:

$$\begin{aligned} \boldsymbol{H}(\mathbf{curl}) &:= \Big\{\boldsymbol{v}\in L^2(\Omega;\mathbb{R}^3) \;\Big|\; \mathbf{curl}\,\boldsymbol{v}\in L^2(\Omega;\mathbb{R}^3)\Big\}, \\ \boldsymbol{H}_0(\mathbf{curl}) &:= \Big\{\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl}) \;\Big|\; \boldsymbol{v}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\Big\}, \\ \boldsymbol{H}(\mathrm{div}) &:= \Big\{\boldsymbol{v}\in L^2(\Omega;\mathbb{R}^3) \;\Big|\; \mathrm{div}\,\boldsymbol{v}\in L^2(\Omega)\Big\}, \\ \boldsymbol{H}_0(\mathrm{div}) &:= \Big\{\boldsymbol{v}\in \boldsymbol{H}(\mathrm{div}) \;\Big|\; \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\Big\}, \\ \mathcal{H} &:= (\boldsymbol{H}_0(\mathbf{curl})\cap\boldsymbol{H}(\mathrm{div}))\times(\boldsymbol{H}(\mathbf{curl})\cap\boldsymbol{H}_0(\mathrm{div})), \end{aligned}$$

where the **curl**- and div-operators, as well as the tangential and normal traces are understood in the sense of distributions. We set

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega), \quad \mathcal{A} := \begin{pmatrix} 0 & -\mathbf{curl} \\ \mathbf{curl} & 0 \end{pmatrix},$$

where the domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) := \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}). \tag{4.2}$$

We note that the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is skew-adjoint, i.e., it holds for the corresponding adjoint operator that $\mathcal{A}^* = -\mathcal{A}$ and $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A}) = \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})$. Thus, $\mathcal{A}$ satisfies Assumption 3.1.3.

Next, we specify $\varphi : \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ to be

$$\varphi(\boldsymbol{v}) := \int_\Omega j_c(x) \sum_{i=1}^{3} |\boldsymbol{v}_i(x)| \, dx. \tag{4.3}$$

Then, the relation

$$\begin{cases} \boldsymbol{j}_i(x)\boldsymbol{e}_i(x) = j_c(x)|\boldsymbol{e}_i(x)| & \text{for a.a. } x \in \Omega,\ \forall\, i = 1,2,3, \\ |\boldsymbol{j}_i(x)| \le j_c(x) & \text{for a.a. } x \in \Omega,\ \forall\, i = 1,2,3. \end{cases} \tag{4.4}$$

can be equivalently written as $(\boldsymbol{j}, \mathbf{0}) \in \partial\varphi(\boldsymbol{e}, \boldsymbol{h})$; see the proof of Lemma 2.2. Finally, by introducing

$$\boldsymbol{\nu} := \begin{pmatrix} \epsilon & \mathbf{0} \\ \mathbf{0} & \mu \end{pmatrix},$$

we conclude that (4.1) is equivalent to

$$\left.\begin{aligned} \min_{(\boldsymbol{f},\boldsymbol{g})\in\mathcal{H}} \quad & \frac{1}{2}\|\boldsymbol{e} - \boldsymbol{e_d}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}\|\boldsymbol{h} - \boldsymbol{h_d}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{\kappa}{2}\|(\boldsymbol{f},\boldsymbol{g})\|^2_{\mathcal{H}} \\ \text{s.t.} \quad & -\boldsymbol{\nu}(\boldsymbol{e},\boldsymbol{h}) - \mathcal{A}(\boldsymbol{e},\boldsymbol{h}) + (\boldsymbol{f},\boldsymbol{g}) \in \partial\varphi(\boldsymbol{e},\boldsymbol{h}). \end{aligned}\right\} \tag{P_{Bean}}$$

We observe that (P_{Bean}) is a special case of (P), where Assumption 3.1 is fulfilled. Thus, we can apply Theorem 3.8 to (P_{Bean}) and obtain the following result:

Corollary 4.1. *Let $(\bar{\boldsymbol{f}}, \bar{\boldsymbol{g}}) \in \mathcal{H}$ be a local optimum of (P_{Bean}) with the associated state $(\bar{\boldsymbol{e}}, \bar{\boldsymbol{h}}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})$. Then, there is a unique adjoint state $\boldsymbol{p} \in \mathcal{H}$ and a unique multiplier $\boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)$ so that the following optimality system is fulfilled*

$$-\boldsymbol{\nu}(\bar{\boldsymbol{e}}, \bar{\boldsymbol{h}}) - \mathcal{A}(\bar{\boldsymbol{e}}, \bar{\boldsymbol{h}}) + (\bar{\boldsymbol{f}}, \bar{\boldsymbol{g}}) \in \partial\phi(\bar{\boldsymbol{e}}, \bar{\boldsymbol{h}}), \tag{4.5a}$$

$$(\bar{\boldsymbol{e}}, \bar{\boldsymbol{h}}) - (\boldsymbol{e_d}, \boldsymbol{h_d}) - \boldsymbol{\nu}\boldsymbol{p} - \mathcal{A}^*\boldsymbol{p} + \boldsymbol{\mu} = \mathbf{0}, \tag{4.5b}$$

$$\boldsymbol{p} + \kappa(\bar{\boldsymbol{f}}, \bar{\boldsymbol{g}}) = \mathbf{0}, \tag{4.5c}$$

$$\boldsymbol{\mu}_i(x)\bar{\boldsymbol{e}}_i(x) = 0 \ \textit{a.e. in } \Omega,\ \forall\, i = 1,2,3, \tag{4.5d}$$

$$\boldsymbol{\mu}_i(x)\boldsymbol{p}_i(x) \le 0 \ \textit{a.e. in } \Omega,\ \forall\, i = 1,2,3, \tag{4.5e}$$

$$\boldsymbol{\mu}_i(x) = 0 \ \textit{a.e. in } \Omega,\ \forall\, i = 4,5,6. \tag{4.5f}$$

Proof. The system (4.5a)-(4.5e) is a direct consequence of Theorem 3.8. To prove (4.5f), let us consider the sequence $\{(\boldsymbol{f}_\lambda, \boldsymbol{g}_\lambda)\}_\lambda$ associated to $(\bar{\boldsymbol{f}}, \bar{\boldsymbol{g}})$ from Proposition 3.7. Moreover, let $\boldsymbol{e}_\lambda, \boldsymbol{j}_\lambda$ and $\boldsymbol{\mu}_\lambda$ denote the corresponding quantities. In view of the definition of φ, it follows that

$$\begin{aligned} \mathcal{C}(\boldsymbol{f}_\lambda, \boldsymbol{g}_\lambda) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \ & \boldsymbol{v}_i(x) \le 0 \ \text{ if } \boldsymbol{j}_i^\lambda(x) = j_c(x), \\ & \boldsymbol{v}_i(x) \ge 0 \ \text{ if } \boldsymbol{j}_i^\lambda(x) = -j_c(x), \\ & \boldsymbol{v}_i(x)\boldsymbol{e}_i^\lambda(x) = 0 \text{ a.e. in } \Omega,\ i = 1, ..., 3, \\ & \boldsymbol{v}_i(x) = 0 \text{ a.e. in } \Omega,\ i = 4, ..., 6\}, \end{aligned} \tag{4.6}$$

which together with (3.21b) yields that $\boldsymbol{\mu}_i^\lambda(x) = 0$ a.e. in Ω, $\forall\, i = 4, ..., 6$. Thus, by (3.24), we have (4.5f). ∎

Remark 4.2. *Note that the structure of (4.1) allows us to improve the system in Proposition 3.8. To be more precise, the fact that the current density has only three components (instead of $n = 6$) allows us to choose $\mathfrak{g}_4 = \mathfrak{g}_5 = \mathfrak{g}_6 = 0$ a.e. in Ω (see the definition of φ). This yields the additional information in (4.5f).*

5. Appendix. Lemma 5.1. *Let H be a Hilbert space and $A : \mathcal{D}(A) \subset H \to H$ be a maximal monotone operator. Then, for any $\lambda > 0$, it holds $(A^*)_\lambda = (A_\lambda)^*$, where $(A^*)_\lambda$ is the Yosida approximation of A^*.*

Proof. Let us begin by noticing that the maximal monotonicity of A ensures the maximal monotonicity of A^*, in view of [5, page 194], see also [24, Lemma 10.2, page 38]. Hence, one can indeed define the Yosida approximation of A^*. To show the desired result, we argue as in the proof of [5, Proposition 7.6]. Let $\lambda > 0$ be arbitrary, but fixed. For $x, y \in H$, we define $w := (I + \lambda A)^{-1}x$ and $z := (I + \lambda A^*)^{-1}y$. Note that $w \in \mathcal{D}(A)$ and $z \in \mathcal{D}(A^*)$, in view of the maximal monotonicity of A and A^*, respectively. Moreover, we have the identities

$$\begin{aligned} w + \lambda A w &= x, \\ z + \lambda A^* z &= y. \end{aligned}$$

Testing the above identities with z and w, respectively, implies that

$$(w, z)_H = (x, z)_H - \lambda(Aw, z)_H = (y, w)_H - \lambda(A^* z, w)_H. \tag{5.1}$$

On the other hand, $w \in \mathcal{D}(A)$ and $z \in \mathcal{D}(A^*)$ yields $(Aw, z)_H = (A^* z, w)_H$. Then, by the definition of w and z, we deduce from (5.1) the following

$$(x, (I + \lambda A^*)^{-1} y)_H = ((I + \lambda A)^{-1} x, y)_H.$$

Hence,

$$\big(x, \frac{y}{\lambda}\big)_H - \big(x, \frac{1}{\lambda}(I + \lambda A^*)^{-1} y\big)_H = \big(x, \frac{y}{\lambda}\big)_H - \big(y, \frac{1}{\lambda}(I + \lambda A)^{-1} x\big)_H.$$

This gives in turn

$$\big(x, \underbrace{\frac{1}{\lambda}(I - (I + \lambda A^*)^{-1})}_{(A^*)_\lambda} y\big)_H = \big(\underbrace{\frac{1}{\lambda}(I - (I + \lambda A)^{-1})}_{A_\lambda} x, y\big)_H,$$

in view of Definition 3.2. Since $x, y \in H$ were arbitrary, the proof is now complete. ∎

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